ROBERT A. WOLAK

GEOMETRIC STRUCTURES ON FOLIATED MANIFOLDS

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Preface

This volume presents mostly my own results concerning geometric structures on foliated manifolds. Some of them have been obtained in colaboration with Luis A. Cordero. Most results are quite new; either published in the last three years or in some cases not published yet and available only in preprints.

The material is presented from the most general to the most particular. Thus general definitions of geometric structures come first in the second chapter and we study transversely affine foliations in the last one.

Since Riemannian foliations are covered in great detail and depth from various points of view in two excelent books:

P. Molino, *Riemannian Foliations,* Progress in Math., Birkhäuser 1988,

Ph. Tondeur, *Foliations on Riemannian Manifolds*, Springer 1988,

I have decided to leave out this class of foliations from my study. At the present moment it is difficult to add anything of interest to these two volumes. The exclusion of the topics covered by these books means also that the reader will not find in this volume any results on Lie and transversely parallelisable foliations.

This book is divided into chapters, sections and subsections, thus Subsection III.2.3 is Subsection 3 of Section 2 of Chapter III. Theorems, propositions and the like are numbered within each chapter, thus Example IV.3 refers to Example 3 of Chapter IV.

In Chapter I we have gathered some very useful definition and properties. First of all we give various definitions of foliations and establish their equivalence. Next we recall basic facts about pseudogroups and define the holonomy pseudogroup of a foliation. Finally, we provide some classical constructions of foliations and a characterization of developable foliations due to A. Haefliger.

Chapter II contains the general theory of geometric structures on foliated manifolds. We distinguish three types of them: foliated, transverse and associated. Let us take as an example a very simple object: a global vector field preserving the foliation. Such a vector field is an associated geometric structure in our sense. This vector field defines a global section (a foliated vector field) of the normal bundle of the foliation. This section is a foliated structure in our sense. In its turn a foliated vector field defines a holonomy invariant vector field on any transverse manifold of the foliation. This vector field is a transverse structure. It is not difficult to see that there is one–to–one correpondence between foliated vector fields and holonomy invariant ones. To get a vector field on the manifold corresponding to a foliated vector field we have to choose a supplementary subbundle to the tangent bundle to leaves of the foliation.

We use the notion of a natural bundle to define these geometric structures. Thanks to this most of objects considered on foliated manifolds fall into one of these three classes of geometric structures and we show how some very well–known objects fit into our definitions. To complete this chapter we explain relations between these three types of structures.

In our search for this very general definition we have been motivated by the similarity of proofs of some properties for different geometric structures. The relations (between various geometric structures) demonstrated in this chapter allow us to obtain results valid for large classes of these structures and which generalize those proved for many particular classes of foliated manifolds.

The third chapter presents the theory of foliated systems of differential equations (FSDE). Studying Riemannian foliations one can quite easily notice that many facts about these foliations can be obtained using only properties of geodesics of the bundle–like metric. One of the basic properties of these geodesics is the following:

> a geodesic orthogonal to leaves of the foliation at one point of its domain is orthogonal to leaves of the foliation at any point of its domain

This property even characterizes Riemannian foliations. It is not difficult to notice that the equation of the geodesic of the Levi–Civita connection of the bundle–like metric has very particular form. In the local representation for an adapted chart the coefficients corresponding to the transverse part depend only on the 'transverse' coordinates; in our 'language' it is a foliated system of ordinary DE (FSODE). Other examples of these foliations provide transversely affine and $\nabla - G$–foliations which are considered in more detail in Chapter VI and VII.

The main aim of this chapter is to show that foliations admitting an FSODE which satisfies some natural assumptions have many properties similar to those of Riemannian foliations on a compact manifold. The most important assumption on an FSODE is (transverse) completeness, i.e. the global existence of 'transverse' solutions. In the case of a Riemannian foliation on a compact manifold the Hopf–Rinow theorem assures the global existence (completeness) of geodesics, and it is not a coincidence that for non–compact manifolds many authors simply assume that the bundle–like metric is complete. In general this condition is rather difficult to verify. Even in such a simple case as that of a foliation by points and a flat connection on a compact manifold it is not always true. Another difficulty poses the dependence on the choice of a subbundle supplementary to the tangent bundle to leaves of the foliation. G. Hector has shown that there are codimension

1 transversely affine foliations for which transverse completeness depends on this choice. We study this dependence in the fifth section.

The first section is dedicated to basic definitions and some properties of solutions of FSODE. In the following section we continue to study solutions of FSODE and their relations with solutions of the corresponding SODE on the transverse manifold. The third and forth sections contain proofs of the fundamental properties of foliated manifolds with FSODEs. The last section (sixth) deals with a particular class of FSODEs which admit a sufficiently great number of global foliated fields of initial conditions. Foliations of non-compact manifolds admitting such an FSODE with some additional natural properties behave similarly to complete TP foliations.

At the end we should mention that FSDEs appear quite naturally in the investigation of SDE, cf. [OL] and [OV].

Chapter IV presents a self-contained study of geometric properties of G-foliations or rather of foliated G-structures. In the first section we give some preliminary definitions including that of the structure tensor and prove fundamental properties of the defined objects. We pay particular attention to conditions under which transversely projectable connections exist. The second section is dedicated to foliated G-structures of finite type which we study using properties of prolongations of these structures. Among others we obtain a stability theorem for foliations with all leaves compact. In the last two chapters we shall improve considerably this result for $\nabla - G-$ and transversely affine foliations. In the third section we apply a theorem proved in the first one to obtain strong vanishing theorems for characteristic classes of flag structures with additional 'adapted' foliated G-structures. In particular, we get a generalization of a Carreras–Naveira vanishing theorem, cf. Canad. Math. Bull. 28 (1985), 77–83. G-foliations of finite type are once again studied in the last section. This time we pay attention to the properties of their leaves. A careful review of some results of R. A. Blumenthal reveals that their proofs can be done according to a general 'scheme' which is also valid in many other situations (taking into account the bijective correspondence between foliated and transverse structures). Thanks to this 'proof scheme' we simplify the proofs of R. A. Blumenthal and obtain some new interesting theorems on G-foliations of finite type.

The fifth chapter is dedicated to a special class of G-structures – transversely Hermitian ones. These foliations have been investigated in great detail by many authors. In the first section we restrict ourselves to give some examples of such foliation having very precise transverse structures; for example

1. transversely symplectic but never transversely Kähler;
2. transversely symplectic and holomorphic but never transversely Kähler, on complex and non-complex nilmanifolds;

3. transversely symplectic but not transversely holomorphic.

Base–like cohomology of transversely Hermitian foliations is studied in the second section. We define the basic Frölicher spectral sequence. For transversely Kähler foliations this spectral sequence collapses at the first level thanks to the properties proved by A. El Kacimi. Next we present some examples of transversely Hermitian foliations for which the basic Frölicher spectral sequence does not collapse at the first level. At the end of this section we prove that the minimal model for the base–like cohomology of a transverse Kähler foliation is formal and demonstrate that our examples do not have formal minimal models for the base–like cohomology. In the third section we present a new method of studying Sasakian manifolds. A Sasakian manifold is a foliated manifold with a very particular foliated structure. Using the correspondence between foliated and transverse structures, we reduce many theorems about geometrical objects in Sasakian manifolds to theorems about corresponding objects in Kähler manifolds. In fact, the 1–dimensional foliation of a Sasakian manifold generated by the characteristic vector field is a transversely Kähler isometric flow. We call this foliation the characteristic foliation. We consider two books of K.Yano and M.Kon, cf. [YK1,YK2], and demonstrate that most results on the local structure of Sasakian manifolds can be derived from the corresponding ones for Kähler manifolds. To complete this paper we present some new local properties of Sasakian manifolds obtained applying our foliated method.

$\nabla - G$–foliations form a very particular class of G–foliations. They are these G–foliations which admit a transversely projectable connection. Although they seem to be very similar to Riemannian foliations they differ in at least two basic aspects. Their base–like cohomology can be infinite dimensional and the closures of leaves needn't be submanifolds. Examples of such $\nabla - G$–foliations are presented in Chapter VII. In Chapter VI we concentrate our attention on the following problem: a $\nabla - G$–foliation, when is it a Riemannian one? First we establish when a pseudogroup of local affine transformations is a pseudogroup of local isometries. Then we use this result to demonstrate that a transversely complete $\nabla - G$ foliation with a particular commuting sheaf ('of compact type') must be Riemannian. In Section 4, for flows, we improve this theorem using a totally different method. In the fifth section we apply our considerations to prove Ghys' conjecture, cf. Appendix E in Molino's book, for $\nabla - G$–flows. The last section of this chapter is dedicated to foliations with all leaves compact in which we refine the stability theorem.

Transversely affine foliations (TAF) are studied in the last chapter. These foliations admit a transversely projectable connection and at the same time they are developable. This allows us to define two notions of completeness (complete and transversely geodesically complete) which are equivalent for foliations by points. For general foliations it is not the case. However it is easy to check that a transversely geodesically complete TAF is complete. The completeness is one of the most important properties and it has a very deep influence on the behaviour of the foliation. The first section is almost totally devoted to the investigation of these notions. Additionally we describe the commuting sheaf of a complete TAF. The next three sections are dedicated to the comparison between TAFs and affine manifolds. As our basis we take a series of papers by D. Fried, W. Goldman and M. W. Hirsch. Using the radiance obstruction of the affine holonomy representation of the foliation and properties of the algebraic hull of its affine holonomy group we prove a series of results which are counterparts of the similar facts now well-known for affine manifolds. In the case when foliation behave differently we provide examples which illustrate these differences. In Section 5 we estimate the growth of leaves. Fortunately, as we prove the growth of leaves of a TAF and of the corresponding foliation on the total space of the bundle of transverse frames is the same for a good choice of adapted atlases. It allows us to apply results of Y. Carrière as this lifted foliation is a Lie one. In spite of this we cannot obtain the same strong relation between the structure algebra and the degree of growth. Leaves of TAFs behave more 'wildly' as it is well illustrated by two examples which can be found at the end of this section. The promised examples of $\nabla - G$-foliations (in this case TAFs) in which there are leaves whose closures are not submanifolds are given in the last section. Moreover we prove that some properties of the affine holonomy group ensure that the closures of leaves form a singular foliation.

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I am most grateful to Universidad de Santiago de Compostela for its hospitality and for providing me with all the facilities and an enviroment most conducive to work. In this respect I am particularly indebted to members of the staff of Departamento de Geometría y Topología and most of all to its director Prof. Luis A. Cordero. Due to them my stay in Santiago has been very fruitful and resulted in a series of papers of which this volume is an elaboration. Some of them have been written jointly with Prof. Luis A. Cordero. I am very grateful to him for this collaboration in course of which I have learnt a great deal.

I would like to mention and express my deep gratitude to several mathematicians who influenced my mathematical thinking and whose ideas can be discerned in this volume.

First my thanks and respect are due to Prof. Andrzej Zajtz who introduced me to foliations and guided my first steps in this field.

The definition of a transverse structure of a foliation presented in this volume cristalized itself during several discussions with Prof. André Haefliger, for which I am most grateful. His influence can be easily recognized in many places.

Prof. Pierre Molino has showed great patience towards me and to him I owe many improvements in the presentation of results on $\nabla - G$–foliations and foliated systems of DEs. In my presentation of foliated geometric structures and foliated G–structures I am heavily indebted to him.

I have learnt a great deal about foliations and TAFs in particular during many conversations with Prof. Gilbert Hector. Some of the papers on TAFs has been written in response to the problems posed by him during the VIth International Colloquium on Differential Geometry in Santiago in 1988. I am most grateful for his help and encouragement.

Last but not the least my deep gratitude is due to Prof. José L. Viviente with whom I had the priviledge of discussing the nature of transverse properties of foliations several times. The importance of these conversations cannot be

overestimated.

Finally I would like to thank Professor Luis A. Cordero for introducing me to TEX. Without his help, patience and kindness this volume would not have such a form as it has.

Santiago de Compostela, 16th september, 1989

Chapter I

Preliminaries

In this introductory chapter we present various definitions and basic examples of foliations.

Definition 1 *A foliation $\mathcal{F}$ of dimension p and codimension q on a manifold M is a partition $\{L_\alpha\}_{\alpha\in A}$ of M consisting of connected submanifolds having the following property:*

for any point of M there exists a chart (U,φ) at this point

$$\varphi = (x_1, ...x_p, y_1, ...y_q): U \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$$

such that the sets

$$\varphi(c) = \{z \in M: (y_1(z), ...y_q(z)) = c \in \mathbb{R}^q\}$$

are the connected components of $U \cap L_\alpha$.

The elements of the partition are called leaves. The chart (U,φ) is called an adapted or distinguished chart and the sets $\varphi^{-1}(c)$ plaques. The pair $(M,\mathcal{F})$ is called a foliated manifold.

Let us denote by Γ_q^{p+q} the pseudogroup of all local diffeomorphisms φ of $\mathbb{R}^p \times \mathbb{R}^q$, $\varphi = (\varphi_1,\varphi_2)$, $\varphi_1: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, $\varphi_2: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ such that the mapping φ_2 is independent of the first p coordinates, i.e. $\varphi_2(x,y) = \varphi_2(y)$ for any $(x,y) \in dom\varphi \subset \mathbb{R}^p \times \mathbb{R}^q$. Then, if we take any two adapted charts (U_i,φ_i) and (U_j,φ_j), the transformation $\varphi_j \circ \varphi_i^{-1}: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$ is an element of the pseudogroup Γ_q^{p+q}. The above considerations provide us with another definition of a foliation or rather of a foliated manifold.

Definition 2 *An n-dimensional manifold M is foliated by a codimension q foliation if there exists a Γ_q^n-atlas $\mathcal{U}$ on this manifold. A maximal Γ_q^n-atlas is called a codimension q foliation on M.*

The partition of M can be recovered from this atlas by considering the sets $L_i(c) = \{z \in U_i: \varphi_i^{p+1}(z) = c_1, ...\varphi_i^n(z) = c_q\}$ where $(U_i, \varphi_i) \in \mathcal{U}$, $(c_1, ...c_q) \in \mathbb{R}^q$. The sets $L_i(c)$ define a topology on the set M. The connected components in this topology are submanifolds of dimension $p = n - q$ and form a partition of M. Equivalent atlases define the same topology and therefore the same partition.

As we are only interested in the last q–coordinates of any adapted chart we can define a foliation in the following way.

Let $\mathcal{V} = \{V_i, f_i, g_{ij}\}_{i \in I}$ be a cocycle modellled on a q–manifold N_0 such that:

1 $\{V_i\}$ is an open covering of M,

2 $f_i : V_i \longrightarrow N$ are submersions with connected fibres,

3 $g_{ij} : f_j(V_i \cap V_j) \longrightarrow f_j(V_i \cap V_j)$ are local diffeomorphisms of N_0 for which $f_i|V_i \cap V_j = g_{ij} \circ f_j|V_i \cap V_j$,

The third condition ensures that $g_{ki} = g_{kj} \circ g_{ji}$ whenever defined.

If $\{(U_i, \varphi_i)\}$ is an adapted atlas then we can define a cocycle modelled on $\mathbb{R}^q$ as follows: $V_i = U_i$, $f_i = \varphi_i^2$, $g_{ij} = \varphi_i \circ \varphi_j^{-1}|\{0\} \times \mathbb{R}^q$. For a given cocycle $\mathcal{V}$ the sets $f^{-1}(c)$, $c \in \mathbb{R}^q$, define the foliation of M. By taking a finer covering, for any cocycle $\mathcal{V}$, we can find another cocycle $\mathcal{U} = \{U_\alpha, h_\alpha, k_{\alpha\beta}\}$ modelled on N_0 defining the foliation $\mathcal{F}$ such that

1' $\{U_\alpha\}$ is an open locally finite covering of M by relatively compact sets,

2' the covering $\{U_\alpha\}$ is finer than $\{V_i\}$, i.e. for any α there exists $i(\alpha)$ for which $U_\alpha \subset V_{i(\alpha)}$ and $\bar{U}_\alpha \subset V_{i(\alpha)}$,

3' $h_\alpha: U_\alpha \longrightarrow N_0$ has connected fibers and $h_\alpha = f_{i(\alpha)}|U_\alpha$,

4' $k_{\alpha\beta} = g_{i(\alpha)i(\beta)}|f_{i(\alpha)}(U_\alpha) \cap f_{i(\beta)}(U_\beta)$.

The cocycle $\mathcal{U}$ is called relatively compact. If we start with a cocycle $\mathcal{V}$ satisfying the condition $1'$ then the cocycle $\mathcal{U}$ can have the same set of indices.

We say that two cocycles $\mathcal{U} = \{U_i, f_i, g_{ij}\}_I$ and $\mathcal{V} = \{V_\alpha, f_\alpha, g_{\alpha,\beta}\}_A$ modelled on N_0 are equivalent if there exists a third cocycle $\mathcal{W} = \{W_s, f_s, g_{st}\}_B$ modelled on N_0 such that for any $s \in B$ there exists $i(s) \in I$ or $a(s) \in A$ and

i) $W_s \subset U_{i(s)}$ or $W_s \subset V_{a(s)}$, respectively;

ii) $f_s = f_{i(s)}|W_s$ or $f_s = f_{a(s)}|W_s$, respectively.

The conditions i) and ii) ensure that the fibres of the submersions of cocycles $\mathcal{U}, \mathcal{V}, \mathcal{W}$ define the same partion of M. Therefore we can put forward the following definition.

Definition 3 *An equivalence class of cocycles on M modelled on a q–manifold N_0 is called a codimension q foliation of M (modelled on N_0).*

Finally, we give another classical definition of a foliation.

Definition 4 *An involutive subbundle of constant dimension of the tangent bundle TM is called a foliation on the manifold M.*

The famous Frobenius theorem, cf. [NR], ensures that this definition is equivalent to our first one. A foliation can be also defined using differential forms. The second version of the Frobenius theorem, cf. [NR], ensures that an integrable Pfaff system of constant rank is a foliation, i.e. let $\alpha_1, ...\alpha_k$ be an intgrable system of 1–forms such that the subbundle

$$ker\alpha = \{X \in TM: \alpha_1(X) = ... = \alpha_k(X) = 0\}$$

is of constant dimension. Then the subbundle $ker\alpha$ is involutive iff

$$d\alpha_i = \sum \beta_{ij} \wedge \alpha_j$$

for some 1–forms β_{ij}, $i, j = 1, ...k$.

Let N be a q–manifold and $j: N \longrightarrow M$ be an injective immersion transverse to the leaves of the foliation $\mathcal{F}$. N is called a complete transverse manifold of $\mathcal{F}$ if any leaf of $\mathcal{F}$ meets $j(N)$ at at least one point.

Let $\mathcal{U} = \{(U_i, \varphi_i)\}$ be an adapted atlas such that $\varphi_i(U_i) = (-\epsilon, \epsilon)^n$. Then the q–manifold $N = \coprod N_i$ where $\varphi_i^2(U_i) = N_i$ is a complete transverse manifold of M. Indeed, let us put $j_i: N_i \longrightarrow M$, $j_i(y) = \varphi_i^{-1}(0, y)$. Then $j = \coprod j_i$, if necessary deformed a little, is the injective immersion we have been looking for.

To any cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_I$ defining the foliation $\mathcal{F}$ we can naturally associate the following q–manifold $N_{\mathcal{U}} = \coprod N_i$ where $f_i(U_i) = N_i$. In general this manifold is not a complete transverse manifold of the foliation $\mathcal{F}$ – the required injective submersion does not always exists. But the constant rank theorem ensures that for some covering $\{\tilde{N}_\alpha\}_A$ of N finer than $\{N_i\}_I$ the manifold $\tilde{N} = \coprod \tilde{N}_\alpha$ is a complete transverse manifold. This implies that we can refine any cocycle $\mathcal{U}$ to a cocycle $\mathcal{V}$ for which the corresponding manifold $N_{\mathcal{V}}$ is a complete transverse manifold. Moreover, as we will be able to judge later, the manifold $N_{\mathcal{U}}$ fulfills well all the duties of a complete transverse manifold. Therefore in spite of the above restrictions, we shall call the manifold $N_{\mathcal{U}}$ a (complete) transverse manifold of $\mathcal{F}$ associated to the cocycle $\mathcal{U}$. As we always talk about complete transverse manifolds we drop "complete" whenever possible and $\mathcal{U}$ whenever no confusion arises.

Now we are going to define the holonomy pseudogroup of a foliation. Let $\alpha: [0, a] \longrightarrow M$ be a leaf curve, i.e. the image of α is contained in a leaf of $\mathcal{F}$ and let $\gamma: [0, \epsilon_0) \longrightarrow M$ be a curve transverse to the foliation such that $\alpha(a) = \gamma(0)$. Then there exist $0 < \epsilon < \epsilon_0$ and a smooth mapping $\sigma: [0, a] \times [0, \epsilon) \longrightarrow M$ such that

1. for any $t \in [0,\epsilon), \sigma|[0,a] \times \{t\} = \sigma_t$ is a leaf curve called the holonomy lift of α to $\gamma(t)$ and $\sigma_0 = \alpha$;

2. for any $s \in [0,a], \sigma|\{s\} \times [0,\epsilon) = \sigma^s$ is a curve transverse to the foliation and $\sigma^0 = \gamma$.

This property ensures that for any leaf curve $\alpha: [0,a] \to M$ such that $\alpha(0) \in j(N)$ and $\alpha(a) \in j(N)$ there exist open neighbourhoods U and V of $\alpha(0)$ and $\alpha(a)$, respectively, in $j(N)$ and a diffeomorphism $h_\alpha: U \to V$ assigning to any point of U the end of the curve σ_t starting at this point; we can always arrange that the end points of these curves belong to V. Moreover, for any two leaf curves α and β which are homotopic relative to its ends in the leaf the corresponding local diffeomorphisms h_α and h_β have the same germ at $\alpha(0) = \beta(0)$. h_α is called the holomomy mapping along α and the germ of h_α at $\alpha(0)$ the holonomy of α. Taking various leaf curves with ends in $j(N)$ we obtain a collection of local diffeomorphisms of the transverse manifold N which generates a pseudogroup called the holonomy pseudogroup (representative) of $\mathcal{F}$ on N.

On the transverse manifold N associated to the cocycle $\mathcal{U}$ the local diffeomorphisms g_{ij} generate a pseudogroup $\mathcal{H}_\mathcal{U}$ which is called the holonomy pseudogroup representative on N associated to $\mathcal{U}$. If the manifold N is a 'real' complete transverse manifold, then the pseudogroup $\mathcal{H}_\mathcal{U}$ is equal to the holonomy pseudogroup defined on this manifold by leaf curves. Therefore we can use in this case the term 'holonomy' pseudogroup representative without any restriction.

These definitions raise a very important question of relations between various holonomy pseudogroup representatives. In fact, all these pseudogroups are equivalent in the following sense; the definition is due to A. Haefliger, cf. [HA3,HA4].

Let $\mathcal{H}$ and $\mathcal{H}'$ be two pseudogroups of local diffeomorphisms of manifolds N and N', respectively. A morphism $\Phi: \mathcal{H} \to \mathcal{H}'$ is a collection Φ of diffeomorphisms of open sets of N on open sets of N' such that:

i) the sources of φ belonging to Φ cover N;

ii) if $h \in \mathcal{H}$ and φ_1, $\varphi_2 \in \Phi$, then $\varphi_1 o h o \varphi_2^{-1} \in \mathcal{H}'$;

iii) if $h \in \mathcal{H}$, $h' \in \mathcal{H}', \varphi \in \Phi$, then $h' o \varphi o h \in \Phi$;

iv) Φ is closed under unions.

Any collection $\hat{\Phi}$ such that

a) the $\mathcal{H}$–orbit of each point of N intersects the source of an elements of $\hat{\Phi}$;

b) if $h \in \mathcal{H}$ and $\varphi_1, \varphi_2 \in \hat{\Phi}$ then $\varphi_1 o h o \varphi_2^{-1} \in \mathcal{H}'$

can be uniquely extended to a collection Φ satisfying the conditions i)–iv) by considering all unions of elements of the form $h' \circ \varphi \circ h$, $\varphi \in \hat{\Phi}, h \in \mathcal{H}$, $h' \in \mathcal{H}'$. Such a collection $\hat{\Phi}$ is called an atlas generating the morphism Φ.

Let Φ' is a morphism of $\mathcal{H}'$ in $\mathcal{H}''$, then the collection of all $\varphi' \circ \varphi$, $\varphi \in \Phi$, $\varphi' \in \Phi'$ generates a morphism of $\mathcal{H}$ into $\mathcal{H}''$. Therefore pseudogroups with their morphisms form a category. A collection Φ of local diffeomorphisms satisfying the conditions a) and b) generates an isomorphism (or an equivalence) of $\mathcal{H}$ on $\mathcal{H}'$ iff the union of targets of elements of Φ intersects each orbit of $\mathcal{H}'$ and for any $\varphi_1, \varphi_2 \in \Phi, h' \in \mathcal{H}'$ $\varphi_2^{-1} \circ h' \circ \varphi_1 \in \mathcal{H}$. In that case we say that $\mathcal{H}$ is equivalent to $\mathcal{H}'$. For instance, let U be an open subset of N and let $\mathcal{H}_U$ be the pseudogroup of local diffeomorphisms of U whose elements are the restrictions to U of elements of $\mathcal{H}$. Then the inclusion of U in N generates a morphism of $\mathcal{H}_U$ in $\mathcal{H}$, and an isomorphism iff U meets each orbit of $\mathcal{H}$. In the case when the space $N/\mathcal{H}$ of $\mathcal{H}$–orbits is a differentiable manifold, the natural projection $p: N \to N/\mathcal{H}$ being locally a diffeomorphism, the pseudogroup $\mathcal{H}$ is equivalent to the trivial pseudogroup on $N/\mathcal{H}$, i.e. the pseudogroup generated by the identity transformation.

To complete this short résumé on pseudogroups we recall Haefliger's definition of a complete pseudogroup, cf. [HA3,SA2].

Definition 5 *A pseudogroup $\mathcal{H}$ on a manifold N is complete if for any two points x and y of N there exist open neighbourhoods U and V of x and y, respectively, such that any element h of $\mathcal{H}$ with domain in U and target in V can be extended to an element of $\mathcal{H}$ defined on the whole U.*

The following lemma linking leaves of $\mathcal{F}$ with orbits of the holonomy pseudogroup $\mathcal{H}$ seems to be well–known.

Lemma 1 *Let L be a leaf of the foliation $\mathcal{F}$ and x_0 be a point of $L \cap U_i$. Then a point $x \in U_j$ belongs to the leaf L iff the point $f_j(x)$ belongs to the $\mathcal{H}$–orbit of the point $f_i(x_0)$.*

Let us take two transverse manifolds (N_1, j_1) and (N_2, j_2). The holonomy transformations h_α defined by leaf curves $\alpha: [0, a] \to M$ with $\alpha(0) \in j_1(N_1)$ and $\alpha(a) \in j_2(N_2)$ generate the equivalence between the holonomy pseudogroup representatives on N_1 and N_2. Similarly for cocycles. Let $\mathcal{U}$ and $\mathcal{V}$ be two equivalent cocycles and $\mathcal{W}$ be the third one realising this equivalence. Then the transformations g_{st} with $W_s \subset V_{a(s)}$ and $W_t \subset U_{i(t)}$ generate an equivalence between the pseudogroups $\mathcal{H}_\mathcal{U}$ and $\mathcal{H}_\mathcal{V}$. Our previous considerations ensure that all holonomy pseudogroup representatives for a given foliation are equivalent. Therefore the equivalence class of these pseudogroups we call the holonomy pseudogroup of the foliation $\mathcal{F}$. Moreover one can easily prove the following lemma:

Lemma 2 *Let $\mathcal{H}$ be a representative of the holonomy pseudogroup of a foliation $\mathcal{F}$. For any pseudogroup $\mathcal{H}'$ equivalent to $\mathcal{H}$ there exists a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_I$ defining $\mathcal{F}$ with transformation g_{ij} being elements of the pseudogroup $\mathcal{H}'$.*

Lemma 2 can be formulated very neatly using the notion of a $\mathcal{K}$–foliation, cf. [HA1]. Let $\mathcal{K}$ be any pseudogroup of local diffeomorphisms on a q–manifold N_0. A $\mathcal{K}$–foliation is a foliation defined by a cocycle $\mathcal{U}$ modelled on N_0 with g_{ij} being elements of the pseudogroup $\mathcal{K}$. Then the holonomy pseudogroup associated to $\mathcal{U}$ is equivalent to a subpseudogroup of $\mathcal{K}$. With this in mind we have the following.

Lemma 3 *Let $\mathcal{F}$ be a foliation defined by a cocycle $\mathcal{U}$ and let $(\mathcal{H}', N')$ be a pseudogroup equivalent to $(\mathcal{H}, N)$. Then $\mathcal{F}$ is an $\mathcal{H}'$–foliation.*

Remark If the manifold M is a compact one, we can consider only adapted atlases and cocycles consisting of a finite number of elements and for whose the sets U_i are relatively compact.

Now we are going to give various examples of foliations and their holonomy pseudogroups.

Example 1 Simple foliation

Let M and N be two smooth manifolds and $f: M \to N$ be a surjective submersion with connected fibres. The fibres constitute a foliation whose space of leaves is N. A representative of the holonomy pseudogroup on N is the trivial pseudogroup.

Example 2 Fibre bundles with discrete structure group

Let B and T be two smooth manifolds and $h: \pi_1(B) \to \mathit{Diff}(T)$ be a representation of the fundamental group $\pi_1(B)$ into the group of diffeomorphisms of T. Consider the simple foliation $\tilde{\mathcal{F}}$ of the product $\tilde{B} \times T$ given by the projection $\tilde{B} \times T \to T$ where $\tilde{B}$ is the universsal covering of B. The representation h defines an equivalence relation R_h on $\tilde{B} \times T$, namely $(x', t) R_h (x, t')$ iff $x' = x\alpha$ and $t' = h(\alpha)t$. We denote by $\tilde{B} \times_h T$ the quotient manifold of $\tilde{B} \times T$ by the equivalence relation R_h. It is a fibre bundle over B and with the standard fibre T. The foliation $\tilde{\mathcal{F}}$ is invariant by the relation R_h and thus projects to the foliation $\mathcal{F}$ of the manifold $\tilde{B} \times_h T$.

It is evident from the construction that T is a transverse manifold of $\mathcal{F}$ and the holonomy pseudogroup on T is generated by the group imh.

This construction provides us with some very interesting examples.

a) **Linear foliations**

Let $B = \mathbb{T}^p$ and $h_1 = (\alpha_1^1, ...\alpha_q^1), ...h_p = (\alpha_1^p, ...\alpha_q^p)$. The reductions of $h_i \ mod \ \mathbb{Z}^q$ represent rotations of $\mathbb{T}^q = S^1 \times ...S^1$. The choice of $h_1, ...h_p$ determines a representation h of $\pi_1(\mathbb{T}^p) = \mathbb{Z}^p$ into $\mathit{Diff}(\mathbb{T}^q)$ defined by putting $h(e_i) = h_i, i = 1, ...p$, where e_i are the generators of the abelian group $\mathbb{Z}^p$. In this case $\tilde{B} \times_h T$ is diffeomorphic to the torus $\mathbb{T}^{p+q} = \mathbb{T}^p \times \mathbb{T}^q$ and the foliation $\mathcal{F}$ is the linear foliation.

b) Suspension of a diffeomorphism

Let $B = S^1$, then a representation of the fundamental group $\pi_1(S^1) = \mathbb{Z}$ is defined by a choice of a diffeomorphism h of T, and the construction leads to a one-dimensional foliation on $\mathbb{R} \times_h T$. In particular, let us take $T = \mathbb{T}^2$ and a diffeomorphism h_A defined by a matrix $A \in SL(2, \mathbb{Z})$. We obtain a one-dimensional foliation $\mathcal{F}_A$ on $R \times_{h_A} \mathbb{T}^2 = \mathbb{T}^3_A$, cf. [GSS].

To obtain more examples we can modify the construction of Example 2 in the following way.

Example 3 Let us assume that on the manifold T there is a foliation $\mathcal{F}_0$ and the representation h has the image in the group $Diff(T, \mathcal{F}_0)$ of global diffeomorphisms of T preserving $\mathcal{F}_0$. Thus on the manifold $\tilde{B} \times T$ there are two new foliations: $\mathcal{F}_0$ whose leaves are just $\{b\} \times L$, and $\tilde{\mathcal{F}} \times \mathcal{F}_0$ whose leaves are $\tilde{B} \times L$, where L is a leaf of $\mathcal{F}_0$. Both foliations are invariant by the relation R_h and, therefore, define foliations on the quotient manifold $\tilde{B} \times_h T$.

As an example of the above construction we consider the following, cf. [GSS]. Let us take the space $\mathbb{T}^3_A$ for a matrix A whose trace is strictly greater that 2. Then the manifold $\mathbb{T}^3_A$ is called the hyperbolic torus. The matrix A has two irrational eigenvalues λ and $1/\lambda$. The corresponding eigenvectors v_1 and v_2 define linear flows $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ on the torus $\mathbb{T}^2$. The matrix A having integral coefficients defines a diffeomorphism of $\mathbb{T}^2$ which preserves the foliations $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$. The one dimensional foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathbb{T}^3_A$ defined by $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ are called the proper flows corresponding to eigenvectors v_1 and v_2. The two-dimensional foliation $\mathcal{F}^1$ defined by $\tilde{\mathcal{F}}_1$ is transverse to the flow $\mathcal{F}_2$ and vice versa.

$\mathbb{R} \times S^1$ can be considered as a transverse manifold of $\mathcal{F}_1$. In fact, the corresponding foliation of $\mathbb{R} \times \mathbb{T}^2$ admits $\mathbb{R} \times S^1$ as a transverse manifold, where S^1 is a transverse manifold of the foliation $\tilde{\mathcal{F}}_1$ of $\mathbb{T}^2$. Thus $\mathbb{R} \times S^1$ is a transverse manifold of $\mathcal{F}_1$ and the holonomy pseudogroup is generated by the mapping $id \times R_\alpha$ where R_α denotes the rotation of S^1 by the angle α.

In the case of $\mathcal{F}^1$ we can take as a transverse manifold $\mathbb{R}$ immersed as $\mathbb{R}v_2$ into $\mathbb{T}^3_A$ and the holonomy pseudogroup contains the homothety $1/\lambda$. In the fibre $\mathbb{T}^2$ $\mathbb{R}v_2$ is a transverse manifold of the linear foliation $\mathcal{F}_1$. The holonomy pseudogroup representative of this foliation on $\mathbb{R}v_2$ is just a translation. Therefore the holonomy pseudogroup of $\mathcal{F}^1$ is generated by these two transformations.

Example 4 Actions of Lie groups provide us with a whole family of examples. Let $\alpha: G \times M \to M$ be a smooth action of a Lie group G on a manifold M. The orbits of this action are submanifolds of M. Vectors of the Lie algebra $Lie(G)$ of G define global vector fields on M which are tangent to orbits of G. Therefore the action α defines singular (Stephan) foliation on M, cf. [DZ,ST,SM]. But we are only interested in regular foliation, i.e. with leaves of constant dimension. It is the case, for example, if the action of G is locally free. The action of a Lie subgroup H of G on G by left translations gives such an example. Foliations obtained in this way can have very interesting geometrical properties, cf. [TY]. For more details on actions of Lie groups see [VA].

Example 5 Developable foliations

Definition 6 *A foliation $\mathcal{F}$ on a manifold M is said to be developable if there exists a covering $\hat{M}$ of M and a submersion $f: \hat{M} \to N$ such that the connected components of fibres of f are the leaves of the lifted foliation $\hat{\mathcal{F}}$.*

One can assume a little less restrictive definition, namely:

Definition 7 *A foliation $\mathcal{F}$ on a manifold M is said to be developable if there exists a covering $\hat{M}$ of M and a surjective submersion $f: \hat{M} \to N$ of connected fibres onto a non–necessarily Hausdorff q–manifold N such that the fibres of f are the leaves of the lifted foliation $\hat{\mathcal{F}}$.*

A classical result of R. A. Palais ensures that a developable foliation in the first sense is developable in the second, cf. [PA1].

The following proposition characterizing developable foliations is due to A. Haefliger, cf. [HA4]. We recall that a group G acts quasi–analytically if for any $g \in G$ the fact that the transformation g restricted to an open subset is the identity implies that it is the identity on the whole manifold.

Proposition 1 *(Haefliger) Let $\mathcal{F}$ be a foliation on a connected manifold M.*

i) *If the foliation $\mathcal{F}$ is developable, then its holonomy pseudogroup has a representative which is a pseudogroup generated by a group of transformations G of a connected non–necessarily Hausdorff manifold N. There is a Galois covering $\hat{M}$ of M with its Galois group isomorphic to G and a G–equivariant submersion $f: \hat{M} \to N$ (called the development) such that the leaves of the lifted foliation $\hat{\mathcal{F}}$ of $\hat{M}$ are the fibres of f.*

ii) *If the holonomy pseudogroup has a representative generated by a group G acting quasi–analytically on a connected, non–necessarily Hausdorff manifold N, then there exists a Galois covering $\hat{M}$ of the manifold M with the Galois group G and a G–equivariant surjective submersion of connected fibres $f: \hat{M} \to N$ such that the leaves of the lifted foliation $\hat{\mathcal{F}}$ of $\hat{M}$ are the fibres of f.*

Proof i) In virtue of the hypothesis the space of leaves N of the lifted foliation $\tilde{\mathcal{F}}$ of the universal covering space $\tilde{M}$ of M is a connected, non-necessarily Hausdorff manifold and the canonical projection $\tilde{f}: \tilde{M} \to N$ is a submersion with connected fibres. The fundamental group $\tilde{G}$ of M acts by deck transformations on $\tilde{M}$ and preserves the foliation $\tilde{\mathcal{F}}$. So each element of the group $\tilde{G}$ projects, relative to f, onto a diffeomorphism of N. Let G_0 be the normal subgroup of $\tilde{G}$ consisting of elements of $\tilde{G}$ projecting onto the identity of N. Then the quotient manifold $\hat{M}$ of $\tilde{M}$ by G_0 is a Galois covering of M with the Galois group $G = \tilde{G}/G_0$, and the factor mapping $f: \hat{M} \to N$ of $\tilde{f}$ is a G–equivariant submersion with connected fibres.

ii) Let us assume that the holonomy pseudogroup has as its representative a pseudogroup generated by a group G of transformations acting quasi–analytically on some manifold N. According to Lemma 2 there exists a cocycle $\{U_i, f_i, g_{ij}\}$ modelled on N such that the transformations g_{ij} are restrictions of elements of G.

Let $\mathcal{S}$ be the space of germs of submersions $g \circ f_i$ where g is an element of G:

$$\mathcal{S} = \{(g \circ f_i)_x : g \in G\}$$

where $(f)_x$ denotes the germ of a mapping f at the point x. We endow the space $\mathcal{S}$ with the sheaf topology. Let $\hat{M}$ be a connected component of $\mathcal{S}$. Then $\hat{M}$ with the projection assigning to each germ its source is a Galois covering with the Galois group isomorphic to G. The evaluation mapping $h: \hat{M} \to N, h((g \circ f_i)_x) = g(f_i(x))$ is a submersion into N constant along the leaves of the lifted foliation $\hat{F}$. □

Remarks 1) The part i) of the proposition asserts that a developable foliation is a (G, N)–structure, cf. [TH2], [CAG].

2) The part ii) asserts that any (G, N)–structure with the group G acting quasi–analytically is a developable foliation.

3) If $\mathcal{F}$ is developable in the sense of Definition 6, it does not mean that it is a (G, N)–structure for the manifold N of Definition 6.

4) It should be stressed that in the part ii) of the proposition the manifold N (from the (G, N)–structure) could be different from the q–manifold which appears in Definition 7. The submersion constructed in the proof does not need to be either surjective or of connected fibres. Later we shall find examples illustrating well these points.

Example 6 Let $(M, \mathcal{F})$ be a foliated manifold. Assume that a Lie group G acts freely and properly on M and that the foliation $\mathcal{F}$ is G-invariant. Let T be a smooth manifold on which the group G acts as well. Then the diagonal action of G on $M \times T$ is also free and proper. The product foliation whose leaves are of the form $L \times \{t\}$ where $t \in T$ and L is a leaf of $\mathcal{F}$, is G–invariant. Therefore it induces a foliation on the quotient manifold $M \times T/G$.

Chapter II

Geometric structures, general approach

In this chapter we study geometric structures from the general point of view. Three types of structures are distinguished: foliated, transverse and associated ones. We give formal definitions as well as many examples. Moreover, we explain relations between these structures.

II.1 Foliated and transverse geometric structures

We present many examples of geometric objects which have been studied on foliated manifolds. Then we give a formal definition of a foliated geometric structure. Next we show that various well-known geometric objects fit into this general definition. We complete the chapter with the definition and examples of transverse geometric structures of a foliation as well as with the precise description of relations between foliated and transverse structures.

Let $T\mathcal{F}$ be the tangent bundle to the leaves of $\mathcal{F}$. It is a subbundle of TM of constant dimension $p = n - q$. The bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$ is called the normal bundle of $\mathcal{F}$. $N(M, \mathcal{F})$ is isomorphic to any supplementary subbundle Q to $T\mathcal{F}$, and very often it has been identified with such a subbundle; e.g. in the case of a Riemannian foliation many authors identify $N(M, \mathcal{F})$ with the orthogonal complement of $T\mathcal{F}$. Let $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ be a cocycle defining $\mathcal{F}$. Then on U_i the differential df_i defines a linear mapping $\bar{f}_i\colon N(M, \mathcal{F})|U_i \to TN_i$ which is an isomorphism on each fibre. Therefore $V_i = N(M, \mathcal{F})|U_i \cong f_i^* TN_i$. Moreover $\{V_i, \bar{f}_i, dg_{ij}\}$ is a cocycle defining a foliation $\mathcal{F}_N$ of dimension p and codimension $2q$ on the total space of the normal bundle. Leaves of this foliation are covering spaces of leaves of $\mathcal{F}$. It is not difficult to verify that equivalent cocycles lead to equivalent cocycles and thus give the same foliation $\mathcal{F}_N$.

The tensor product bundles

$$\bigotimes_s^r N(M,\mathcal{F}) = \underbrace{N(M,\mathcal{F}) \otimes \ldots \otimes N(M,\mathcal{F})}_{s} \otimes \underbrace{N^*(M,\mathcal{F}) \otimes \ldots \otimes N^*(M,\mathcal{F})}_{r}$$

also admit a foliation $\mathcal{F}_N^{r,s}$ whose leaves are covering spaces of leaves of $\mathcal{F}$.

Let $L(M,\mathcal{F})$ be the bundle of linear frames of the nomal bundle $N(M,\mathcal{F})$. It is a principal fibre bundle with the structure group $GL(q)$ and its elements are called transverse linear frames. The bundle $L(M,\mathcal{F})$ is also called the bundle of transverse linear frames of $\mathcal{F}$. Over U_i the differential df_i defines the mapping $\tilde{f}_i: L(M,\mathcal{F})|U_i \to L(N_i)$ which is an isomorphism on each fibre. Therefore $L_i = L(M,\mathcal{F})|U_i$ is isomorphic to $f_i^* L(N_i)$ and $\{L_i, \tilde{f}_i, \tilde{g}_{ij}\}$ form a cocycle defining a foliation $\mathcal{F}_L$ of dimension p and codimension $q + q^2$ of the total space of the bundle $L(M,\mathcal{F})$. As previously, equivalent cocycles define the same foliation. Thus we have associated in a canonical way to $\mathcal{F}$ a foliation $\mathcal{F}_L$ of the total space of $L(M,\mathcal{F})$ whose leaves are covering spaces of leaves of $\mathcal{F}$.

Any associated fibre bundle to the bundle of transverse linear frames inherits a foliation whose leaves are covering spaces of leaves of $\mathcal{F}$. In this way by taking as the standard fibre $F = \mathbb{R}^q$ we recover the normal bundle $N(M,\mathcal{F})$ with its foliation $\mathcal{F}_N$.

Now we shall present some more examples of fibre bundles which can be constructed on a foliated manifold.

Example 1 Transverse (s,r)–velocities (s^r –jets)

Let m be a point of the manifold M and $f, g: (\mathbb{R}^s, 0) \to (M, m)$ be any local smooth mappings of $\mathbb{R}^s$ into M mapping 0 on m. Let us choose an adopted chart (U,φ) at m, $\varphi = (\varphi_1, \varphi_2) = (x_1, \ldots x_p, y_1, \ldots y_q)$. We say that the mappings f and g are equivalent if $j_0^r \varphi_2 f = j_0^r \varphi_2 g$. This is equivalent to $\partial^{|\nu|}/\partial t^\nu (y_i f)(0) = \partial^{|\nu|}/\partial t^\nu (y_i g)(0)$ for any multiindex $\nu \in \mathbb{N}^s$, $|\nu| \leq r$, $i = 1, \ldots q$. We denote the set of such indices by $N(s,r)$ and their number by $s(r)$. This equivalence relation does not depend on the choice of an adapted chart at the point m. The equivalence class of a mapping f is denoted by $[f]_s^r$. The set of all equivalence classes at a point m we denote by $N_m^{s,r}(M,\mathcal{F})$, and the space $\bigcup_{m\in M} N_m^{s,r}(M,\mathcal{F})$ by $N^{s,r}(M,\mathcal{F})$. By π_s^r let us denote the natural projection of $N^{s,r}(M,\mathcal{F})$ onto M, i.e. $\pi_s^r([f]_s^r) = f(0)$. One can easily check that for any adapted chart (U,φ) the set $\bigcup_{m\in U} N_m^{s,r}(M,\mathcal{F})$ is isomorphic to $U \times \mathbb{R}^{q\cdot s(r)-q}$ and that the isomorphism is given by the correspondence

$$[f]_s^r \longmapsto (f(0), \partial^{|\nu|}/\partial t^\nu (y_i f))_{i=1,\ldots q}^{\nu, |\nu| \geq 1}.$$

This mapping defines a chart φ_s^r, $\varphi_s^r: (\pi_s^r)^{-1}(U) \to \mathbb{R}^{n-q} \times \mathbb{R}^{q\cdot s(r)}$ and the collection of all such φ_s^r given by an adapted atlas on M, defines an atlas on the space $N^{s,r}(M,\mathcal{F})$. To see that one has only to notice that if φ_i, φ_j are two adapted charts the composition $\varphi_i \circ \varphi_j^{-1}: \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^{n-q} \times \mathbb{R}^q$ is of the form $(f_1(x,y), f_2(y))$ and then $\varphi_{is}^r \circ (\varphi_{js}^r)^{-1}: \mathbb{R}^{n-q} \times \mathbb{R}^{q\cdot s(r)} \to \mathbb{R}^{n-q} \times \mathbb{R}^{q\cdot s(r)}$ is equal to $(f_1, T_s^r(f_2))$ where $T_s^r(f_2)$ is the mapping of $T_s^r(\mathbb{R}^q) \cong \mathbb{R}^{q\cdot s(r)}$ induced by f_2, cf. [MM]. Thus $\{\varphi_{is}^r\}$ form an adapted atlas of a foliation of codimension $q \cdot s(r)$. Moreover, equivalent atlases give the same foliation.

Summing up, we have proved that $N^{s,r}(M,\mathcal{F})$ is a locally trivial fibre bundle whose total space admits a codimension $q\cdot s(r)$ foliation $\mathcal{F}_s^r$ projecting by π_s^r onto the initial foliation $\mathcal{F}$. Leaves of $\mathcal{F}_s^r$ are covering spaces of leaves of $\mathcal{F}$. For $s = 1$ the bundle $N^{1,r}(M,\mathcal{F})$ is denoted by $N^r(M,\mathcal{F})$ and is called the normal bundle of order r of the foliation $\mathcal{F}$. If $s = q$ and we take only mappings transverse to the foliation the above construction gives a bundle $L^r(M,\mathcal{F})$ called the bundle of transverse frames of order r of the foliated manifold $(M,\mathcal{F})$. $L^r(M,\mathcal{F})$ is a principal fibre bundle with the fibre L_s^r. Sometimes this bundle is called the bundle of transverse r–frames.

In the same way as $N^r(M,\mathcal{F})$ we define the space $J^r(\mathbb{R}, M;\mathcal{F})$ – the space of transverse r–jets of mappings of $\mathbb{R}$ into M. $J^r(\mathbb{R}, M;\mathcal{F})$ is a fibre bundle over both $\mathbb{R}$ and M. Its fibre over any point v of $\mathbb{R}$ is diffeomorphic to $N^r(M,\mathcal{F})$. On the total space of $J^r(\mathbb{R}, M;\mathcal{F})$ there is a foliation $\mathcal{F}_r$ which induces on each fibre the foliation $\mathcal{F}^r$.

Remarks 1) Similar constructions can be carried out using a subbundle Q supplementary to $\mathcal{F}$. By Q^r we shall denote the subset (subbundle) of $T^r(M)$, the r–tangent bundle of the manifold M, consisting of r–jets of curves tangent to Q. The bundle Q^r is isomorphic to $N^r(M,\mathcal{F})$. The bundle $J^r(\mathbb{R}, Q)$ of r–jets of mappings f from $\mathbb{R}$ into M tangent to Q is isomorphic to $J^r(\mathbb{R}, M;\mathcal{F})$ and thus admit a foliation $\mathcal{F}_r$ of the same dimension as $\mathcal{F}$.

2) A leaf curve $\alpha: [0,1] \to M, \alpha(0) = x, \alpha(1) = y$, defines a holonomy isomorphism $T_\alpha: N(M,\mathcal{F})_x \to N(M,\mathcal{F})_y$ of the fibres of the normal bundle as well as $N^r(T_\alpha): N^r(M,\mathcal{F})_x \to N^r(M,\mathcal{F})_y$. In fact let γ be a transverse curve at x, and let α_t be the curve starting at $\gamma(t)$, the holonomy lift of α to $\gamma(t)$. Then the curve $t \mapsto \alpha_t(1)$ is a curve at y transverse to $\mathcal{F}$. The equivalence class of its tangent vector at 0 in $N(M,\mathcal{F})_y$ does not depend on the choice of γ and therefore defines the mapping.

In the light of our previous considerations these mappings can be interpreted as follows:

Let $\xi \in N^r(M,\mathcal{F})_x$ and $\tilde{\alpha}$ be the lift of the leaf curve α to the leaf of the foliation $\mathcal{F}^r$ passing through ξ then the vector $N^r(T_\alpha)(\xi) \in N^r(M,\mathcal{F})_y$ is the end of the curve $\tilde{\alpha}$.

All these bundles are particular examples of the following general notion: a foliated natural bundle.

Let $\underline{Fol}_q$ be the category of foliated manifolds with codimension q foliations. Global mappings which preserve foliations and which are transverse to them are

the morphisms in this category, i.e.
$f \in Mor((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$ iff $df(T\mathcal{F}_1) \subset T\mathcal{F}_2$ and $f^*\mathcal{F}_2 = \mathcal{F}_1$.

Definition 1 *A covariant (resp. contravariant) functor F on the category $\underline{Fol}_q$ into the category of locally trivial fibre bundles and their fibre mappings is called a foliated natural bundle if the following conditions are satisfied:*

i) *for any foliated manifold $(M, \mathcal{F})$, $F(M, \mathcal{F})$ is a locally trivial fibre bundle over M;*

ii) *let $f \in Mor((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$. Then the fibre mapping $F(f)$ has the following properties:*

a) *covariant case:*

- *$F(f)$ covers f, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} F(M_1, \mathcal{F}_1) & \xrightarrow{F(f)} & F(M_2, \mathcal{F}_2) \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array} \quad ,$$

- *for any point x of M_1, the mapping $F(f)_x : F(M_1, \mathcal{F}_1)_x \longrightarrow F(M_2, \mathcal{F}_2)_{f(x)}$ is a diffeomorphism;*

b) *contravariant case:*

- *for any point x of M_1, the mapping $F(f)_x : F(M_2, \mathcal{F}_2)_{f(x)} \longrightarrow F(M_1, \mathcal{F}_1)_x$ is a diffeomorphism,*
- *the following diagram is commutative:*

$$\begin{array}{ccccc} F(M_2, \mathcal{F}_2) & \xleftarrow{\tilde{f}} & f^*F(M_2, \mathcal{F}_2) & \xrightarrow{F(f)} & F(M_1, \mathcal{F}_1) \\ \downarrow & & \downarrow & & \downarrow \\ M_2 & \xleftarrow{f} & M_1 & \xrightarrow{id} & M_1 \end{array} \quad ;$$

iii) *the functor F is regular.*

Remarks 1) In [WO7] we have called such functors transverse natural bundles.

2) If a foliated natural bundle is a principal fibre bundle as well, then any associated fibre bundle to this foliated natural bundle is also a foliated natural bundle, cf. Example I.6. Therefore the tensor bundles $\bigotimes_s^r N(M,\mathcal{F})$ are foliated natural bundles, as they are associated fibre bundles of the bundle $L(M,\mathcal{F})$ of transverse frames. The same is also true for the exterior product bundles $\bigwedge^k N(M,\mathcal{F})^*$. In this way these bundles can be considered as covariant foliated natural bundles.

The use of the adjective 'foliated' is best explained by the following. Let F be any covariant foliated natural bundle functor. If the foliation $\mathcal{F}$ is defined by a cocycle $\mathcal{U}$ then $\{F(U_i,\mathcal{F}), F(f_i), F(g_{ij})\}_I$ is a cocycle defining a foliation $\mathcal{F}_F$ on the total space of the bundle $F(M,\mathcal{F})$. The leaves of the foliation $\mathcal{F}_F$ are covering spaces of leaves of $\mathcal{F}$. In the contravariant case we define this cocycle as follows:

We start with the commutative diagram:

$$\begin{array}{ccccc} F(N_i) & \xleftarrow{\tilde{f}_i} & f_i^*F(N_i) & \xrightarrow{F(f_i)} & F(U_i,\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ N_i & \xleftarrow{f_i} & U_i & \xrightarrow{id} & U_i \ . \end{array}$$

Then take $\tilde{F}(f_i)\colon F(U_i,\mathcal{F}) \longrightarrow F(N_i)$ equal to $\tilde{f}_i F(f_i)^{-1}$ and $\tilde{g}_{ij} = F(g_{ji})$. Indeed, $\{F(U_i,\mathcal{F}), \tilde{F}(f_i), \tilde{g}_{ij}\}_I$ is a cocycle defining a foliation $\mathcal{F}_F$ on the total space of the bundle $F(M,\mathcal{F})$ whose leaves are covering spaces of leaves of $\mathcal{F}$. It is not difficult to verify that to equivalent cocycles defining the foliation $\mathcal{F}$ correspond equivalent cocycles on the total space of $F(M,\mathcal{F})$. Therefore the foliation $\mathcal{F}_F$ does not depend on the choice of a cocycle defining $\mathcal{F}$.

The following lemma explains the action of morphisms on fibres of foliated natural bundles. Let f,g be two morphisms of $(M_0,\mathcal{F}_0)$ into $(M_1,\mathcal{F}_1)$ such that $f(x_0) = g(x_0)$, $dim M_0 = n$, $dim M_1 = m$. Let (U,φ) and (V,ψ) be adapted charts at x_0 and $f(x_0)$, respectively. Since f and g preserve the foliations, the mappings $\bar{f} = \psi\circ f\circ\varphi^{-1}, \bar{g} = \psi\circ g\circ\varphi^{-1}\colon \mathbb{R}^{n-q}\times\mathbb{R}^q \to \mathbb{R}^{m-q}\times\mathbb{R}^q$ are of the form $\bar{f}(x,y) = (f_1(x,y), f_2(y))$, $\bar{g}(x,y) = (g_1(x,y), g_2(y))$ where $f_1,\ g_1\colon \mathbb{R}^n \to \mathbb{R}^{m-q}$, and $f_2,\ g_2\colon \mathbb{R}^n \to \mathbb{R}^q$.

Lemma 1 *If the germs of the mappings f_2 and g_2 at $\varphi^{-1}(x_0)$ are equal, then the mappings $F(f)$ and $F(g)$ define the same mapping on the fibre $F(M_0,\mathcal{F})_{x_0}$.*

Proof We can take U, V so small that $f_2|\varphi(U) = g_2|\varphi(U)$ and $\varphi(U) = D^{n-q}\times$

D^q, $\psi(V) = D^{m-q} \times D^q$ where D^k denotes the k–disc. Assume that $\varphi(m) = 0$ and $\psi(f(m)) = 0$. Then the following diagram is commutative.

$$\begin{array}{ccc}
U & \xrightarrow{\quad f|U,\, g|U \quad} & V \\
\varphi \downarrow & & \downarrow \psi \\
D^{n-q} \times D^q & \xrightarrow{\quad \bar{f},\, \bar{g} \quad} & D^{m-q} \times D^q \\
i_0 \uparrow & & \downarrow p \\
D^q & \xrightarrow{\quad f_2,\, g_2 \quad} & D^q
\end{array}$$

where $i_0\colon D^q \longrightarrow D^{n-q} \times D^q$ is given by $i_0(y) = (0, y)$ and $p\colon D^{m-q} \times D^q \longrightarrow D^q$ by $p(x, y) = y$.

Since F is a functor it is sufficient to show that the mappings $\bar{f}$ and $\bar{g}$ induce the same mapping in the fibre over 0. As $p\bar{f}i_0 = f_2$ and $p\bar{g}i_0 = g_2$, $F(f_2) = F(p) \circ F(\bar{f}) \circ F(i_0) = F(g_2) = F(p) \circ F(\bar{g}) \circ F(i_0)$. But $F(p)$ and $F(i_0)$ induce isomorphisms on the fibre, hence the mappings $F(\bar{f})$ and $F(\bar{g})$ are equal on the fibre over 0, which ends the proof.□

Definition 2 *A foliated natural bundle F is of finite order r if for any two morphisms $f, g\colon (M_0, \mathcal{F}_0) \longrightarrow (M_1, \mathcal{F}_1)$ the integer r is the smallest one for which the following implication is true:*

$$j^k_x f_2 = j^k_x g_2 \Rightarrow F(f)(y) = F(g)(y)$$

for any point y of the fibre $F(M_0, \mathcal{F}_0)_x$.

Having this definition we can formulate the following theorem.

Theorem 1 *Let F be a foliated covariant natural bundle. Then there exists an integer r and an L^r_q–space W such that F is isomorphic to the fibre bundle associated with the transverse r–frame bundle with standard fibre W. The smallest such integer r is the order of the foliated natural bundle F.*

The proof of the theorem is easy but technical. All details can be found in [WO7]. In fact one can easily verify that any foliated natural bundle is determined by its values on q–manifolds. In fact, the bundle $\mathcal{F}_{\mathcal{U}} = \coprod f_i^* F(N_i)/\sim$ where $(x, i, v) \sim (x', j, v')$ iff $x = x'$ and $v' = F(g_{ji})(v)$ (resp. $v = F(g_{ji})(v')$ in the contravariant case), is a well–defined locally trivial fibre bundle over M. Two equivalent cocycles define isomorphic bundles. Moreover this isomorphism preserves the natural foliations which, locally, are defined by projections $f_i^* F(N_i) \longrightarrow F(N_i)$). The bundle $F(M,\mathcal{F})$ is isomorphic to $F_{\mathcal{U}}$ and this isomorphism is foliation preserving. Therefore any natural functor can be uniquely extended to a foliated natural one.

We have noticed that any foliated natural bundle $F(M,\mathcal{F})$ admits a foliation $\mathcal{F}_F$ of the same dimension as $\mathcal{F}$. Therefore we can talk about foliated subbundles of $F(M,\mathcal{F})$, i.e. those whose total space is saturated for $\mathcal{F}_F$, cf. [KT1,MO1,M11]. This leads us to the following definition of a foliated geometric structure.

Definition 3 *A foliated subbundle E of a foliated natural bundle $F(M,\mathcal{F})$ is called a foliated geometric structure.*

Now we proceed to give some examples of these structures. First of all we notice that a foliated section of a foliated natural bundle is a foliated geometric structure, i.e. a section which as a mapping of foliated manifolds is foliation preserving. This gives us a whole family of vary interesting structures.

A foliated section of the normal bundle is called a foliated vector field. It is a section of the bundle $TM/T\mathcal{F}$, which is a local adapted chart $\varphi = (x_1,...x_p,y_1,...y_q)$ can be represented as $\sum a_i(y)\partial/\partial y_i$ since the equivalence classes of vector fields $\partial/\partial y_i$ span $TM/T\mathcal{F}$ and the foliation $\mathcal{F}_N$ is given by

$$(x_1,...x_p,y_1,...y_q,v_1,...v_q) \longmapsto (y_1,...y_q,v_1,...v_q)$$

where $v_1,..v_q$ are the coordinates of a vector with respect to the transverse basis $\partial/\partial y_1,...\partial/\partial y_q$. Therefore a foliated vector field can be considered as an equivalence class of infinitesimal automorphisms of the foliation $\mathcal{F}$ relative to the vector fields tangent to leaves. In other words two infinitesimal automophisms X_1 and X_2 define the same foliated vector field iff their difference $X_1 - X_2$ is tangent to the leaves, or iff they have the same transverse part, i.e. if, locally, in an adapted chart φ $X_\epsilon = \sum b_\alpha^\epsilon(x,y)\partial/\partial x_\alpha + \sum a_i^\epsilon(y)\partial/\partial y_i$, $\epsilon = 1,2$, then they define the same foliated vector field iff $a_i^1 = a_i^2$ for $i = 1,...q$.

A foliated section α of the bundle $\bigwedge^k N^*(M,\mathcal{F})$ is called a k–base–like form. It can be characterized as a k–form α such that $i_X\alpha = i_X d\alpha = 0$ for any vector X tangent to $\mathcal{F}$. Locally in an adapted chart such a form can be written as $\sum a_{i_1,...i_k}(y)dy_{i_1} \wedge ...dy_{i_k}$. A base–like 0–form is a function constant along the leaves. It is called a foliated or base–like function. Base–like forms constitute a complex denoted by $(A^*(M,\mathcal{F}),d)$. The cohomology of this complex, $H^*(M/\mathcal{F})$, is called the base–like cohomology of the foliated manifold $(M,\mathcal{F})$.

A G–reduction $B(M,G;\mathcal{F})$ of the bundle of transverse frames $L(M;\mathcal{F})$ is called a foliated G–structure if $B(M,G;\mathcal{F})$ is a foliated subbundle of $L(M;\mathcal{F})$.

A foliated section of the tensor product bundle $\bigotimes_s^r N(M,\mathcal{F})$ is called a foliated tensor field of type (r,s). Later we shall demonstrate a relation between foliated tensor fields and foliated G–structures which is well–known for the non–foliated case.

As a foliated section of the associated fibre bundle to $B(M, G; \mathcal{F})$ with the standard fibre $\underline{g} = Lie(G)$ we get transversely projectable connections, i.e. a connection in $B(M, G; \mathcal{F})$ whose connection form is a base-like form. We shall discuss connections in foliated G-structures in Chapter IV.

It is well-known that many properties of a foliation can be read from the properties of its holonomy pseudogroup. Therefore it is of importance to know whether some representative of the holonomy pseudogroup is a pseudogroup of local automorphisms of a geometric structure. With this in mind we introduce the following definition.

Definition 4 *Let N be a transverse manifold of the foliation $\mathcal{F}$ and $\mathcal{H}$ a representative of the holonomy pseudogroup on N. An $\mathcal{H}$-invariant subbundle E of a natural fibre bundle $F(N)$ is called a transverse geometric structure of the foliation $\mathcal{F}$.*

It is not difficult to see that the definition does not depend on the choice of a cocycle $\mathcal{U}$ defining the foliation, i.e. on the choice of a transverse manifold and holonomy pseudogroup. Let $(\mathcal{H}', N')$ be a pseudogroup equivalent to $(\mathcal{H}, N)$ and let $\Phi = (\phi_\alpha)$ be the equivalence. Then the subbundle

$$E_\Phi = \{v' \in F(N') : v' = F(\phi_\alpha)(v),\ v \in E\}$$

(resp. $E_\Phi = \{v' \in F(N') : F(\phi_\alpha)(v') \in E\}$ for a contravariant functor) is an $\mathcal{H}'$-invariant subbundle of $F(N')$ which is locally isomorphic to the subbundle E. Therefore, we can talk about holonomy invariant subbundles on the transverse manifold. Moreover, when solving a particular problem, we can choose a cocycle making our foliation a $\mathcal{K}$-foliation for a suitable pseudogroup $\mathcal{K}$.

Example 2 1. Let L be the functor associating to a manifold its bundle of linear frames. Then any $\mathcal{H}$-invariant G-reduction of $L(N)$ is a transverse geometric structure. Such a foliation is called a G-foliation, cf. [KT2,MO1,DU,RM].

2. Any $\mathcal{H}$-invariant section of a fibre bundle $F(N)$ is a transverse geometric structure. Thus, if we take the functor of the tangent bundle we get holonomy invariant vector fields on the transverse manifold. For the contravariant functor of the cotangent bundle we get $\mathcal{H}$-invariant 1-forms and as $\mathcal{H}$-invariant sections of the tensor products of these bundles (functors) we get $\mathcal{H}$-invariant tensor fields.

Using a suitable associated fibre bundle to the bundle of linear frames (such functors are natural fibre bundles), we obtain that any $\mathcal{H}$-invariant linear connection in an $\mathcal{H}$-invariant G-structure is a transverse geometric structure. Such foliations we have called $\nabla - G$-foliations, cf. [WO3]. Riemannian and transversely affine foliations belong to this class.

We obtain holonomy invariant connections of higher order or Cartan connections as holonomy invariant sections of associated fibre bundles to the bundle of frames of higher order. Such connections exist for transversely conformal or transversely projective foliations, cf. [BL6].

Many geometric constructions provide useful ways of constructing new transverse geometric structures. In fact, the holonomy pseudogroup is a pseudogroup of local automorphisms of a given transverse geometric structure. Any geometric object related to this structure and invariant under the action of the pseudogroup of local automorphisms of the initial structure is itself a transverse geometric structure. For example, take a holonomy invariant G–structure $B(N,G)$. Then its prolongations and their structure tensors are also holonomy invariant, cf. [KO2,SB]. If ∇ is a holonomy invariant connection in $B(N,G)$, then its torsion and curvature tensor fields T and R, respectively, are holonomy invariant as well as $\nabla^k T$ and $\nabla^k R$.

Let us look closer at the relation between foliated and transverse structures. Let $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ be a cocycle defining $\mathcal{F}$ and N and $\mathcal{H}$ the transverse manifold and the holonomy pseudogroup representative associated to $\mathcal{U}$, respectively. Let B be the total space of a foliated structure $B(M,\mathcal{F})$ which is a foliated subbundle of a foliated natural fibre bundle $F(M,\mathcal{F})$. Then the image $F(f_i)(B)$ is a submanifold B_i of $F(N_i) \subset F(N)$. Moreover as $f_j = g_{ji}f_i$ over $U_i \cap U_j$ $F(f_j)(B) = B_j = F(g_{ji}f_i)(B) = F(g_{ji}) \circ F(f_i)(B) = F(g_{ji})(B_i)$. Therefore $B_N = \coprod B_i$ form an $\mathcal{H}$–invariant subbundle of $F(N)$, thus a transverse geometric structure of $\mathcal{F}$. Vice versa, to any $\mathcal{H}$–invariant subbundle of $F(N)$ corresponds a foliated subbundle of $(M,\mathcal{F})$. Therefore foliated geometric structures are in one-to-one correspondence with transverse ones. This correspondence can be presented in the following dictionary:

Dictionary

foliated	**holonomy invariant**
normal bundle of order r	tangent bundle of order r
bundle of transverse (s,r)–velocities	bundle of (s,r)–velocities
bundle of transverse A–points	bundle of A–points
foliated natural bundle	natural bundle
foliated vector field	vector field
base–like r–form (on the normal bundle)	r–form
foliated tensor field of type (s,r)	tensor field of type (s,r)
foliated G–structure	G–structure
fundamental form of " "	fundamental form of "
structure tensor of " "	structure tensor of "
kth prolongation of " "	kth prolongation of "
transversely projectable G–connection	G–connection
torsion tensor of " "	torsion tensor of "
curvature tensor of " "	curvature tensor of "
bundle–like metric (on the normal bundle)	Riemannian metric
transverse sectional curvature	sectional curvature
foliated Cartan connection	Cartan connection
curvature of " " "	curvature of " "

In [MO2] P.Molino considers foliations admitting transversely projectable connections. The correspondence between structure tensors and prolongations have been proved in [MO1,WO4]. Foliated Cartan connections were introduced by M.Takeuchi in [TA] and R.A.Blumenthal in [BL6] and transverse sectional curvature in [BL5].

Having established the correspondence between foliated and transverse structures we return to foliated tensor fields. We are going to prove the proposition mentioned earlier.

The vector bundle $\otimes^s_r N(M,\mathcal{F})$ is the fibre bundle associated to the bundle $L(M,\mathcal{F})$ of transverse frames with the standard fibre $\otimes^s_r\mathbb{R}^q$. Therefore to any section t of $\otimes^s_r N(M,\mathcal{F})$ corresponds a mapping $\tilde{t}: L(M,\mathcal{F}) \rightarrow \otimes^s_r\mathbb{R}^q$ such that $\tilde{t}\circ R_g = h(g^{-1})\tilde{t}$ and h is the natural representation of the group $GL(q)$ on the vector space $\otimes^s_r\mathbb{R}^q$.

A transverse frame v at a point x, considered as an linear isomorphism $v: \mathbb{R}^q \rightarrow N(M,\mathcal{R})_x$, defines an isomorphism $\bar{v}^{-1}: \otimes^s_r N(M,\mathcal{F})_x \longrightarrow \otimes^s_r\mathbb{R}^q$. For any vector $w \in \otimes^s_r N(M,\mathcal{F})_x$ the vector $\bar{v}^{-1}(w)$ is called the representation of w in the transverse frame v. A foliated tensor field t of type (s,r) is said to be 0–deformable if there exists an element u of $\otimes^s_r\mathbb{R}^q$ such that for any point x of the manifold M there exists a transverse frame v at this point in which the representation of t is equal to u.

Proposition 1 *Let t be a foliated 0–deformable tensor field of type (s,r) on a foliated manifold $(M,\mathcal{F})$. The subspace*

$$B(t) = \{v \in L(M,\mathcal{F}): \bar{v}^{-1}(t) = u\}$$

is a foliated subbundle of $L(M,\mathcal{F})$ and a principal fibre bundle with the structure group $GL(u) = \{g \in GL(q): h(g)u = u\}$.

Proof A foliated 0–deformable tensor field t defines a 0-deformable tensor field t_N on the transverse manifold N. The tensor field t_N is $\mathcal{H}$–invariant. The space $B(t_N) = \{v \in L(N): \bar{v}^{-1}(t_N) = u\}$ is a subbundle of $L(N)$ with the structure group $GL(u)$. This subbundle is preserved by $\mathcal{H}$. The corresponding foliated subbundle of $L(M,\mathcal{F})$ is precisely $B(t)$.□

The dictionary reduces the study of "foliated" problems to holonomy invariant ones and sometimes to the right choice of a cocycle defining the foliation. This, owing to Lemma I.2, is equivalent to a good choice of a representative of the holonomy pseudogroup. It is a very powerful tool. We shall show its strength by analyzing the results of R. A. Blumenthal presented in [BL1,BL2,BL3,BL4,BL6], cf. Chapter IV.

II.2 Associated structures

On a foliated manifold there are other structures than foliated ones. Let us consider the following two examples.

Example 3 A foliated vector field, which is a foliated section of the normal bundle, is a foliated structure. On the other hand it is an equivalence class of global infitesimal automorphisms of the foliation $\mathcal{F}$. Such a global infinitesimal automorphism is a foliated section of the tangent bundle TM with the foliation $\mathcal{F}_{TM}$ defined by the atlas of TM associated to the adapted atlas of $(M, \mathcal{F})$. The foliation $\mathcal{F}_{TM}$ has codimension $2q$. Its leaves project onto the leaves of $\mathcal{F}$ and the natural projection $p_N: TM \longrightarrow N(M; \mathcal{F})$ is a morphism in $\underline{Fol}_{2q}$.

Example 4 Let $L_{\mathcal{F}}(M)$ be the bundle of linear frames adapted to $\mathcal{F}$, i.e. $(v_i)_1^n \in L_{\mathcal{F}}(M)_x$ iff $v_1, ... v_{n-q}$ span the subspace tangent to $\mathcal{F}$ at the point x. There is a natural mapping p_L from this bundle into the bundle of transverse linear frames $L(M; \mathcal{F})$ assigning to any frame $(v_1, ... v_n)$ the frame $(\tilde{v}_1, ... \tilde{v}_q)$ corresponding to the vectors $v_{n-q+1}, ... v_n$. As in the previous example the adapted atlas of $\mathcal{F}$ defines a foliation $\mathcal{F}_L$ of codimension $q + q^2$ on the total space of $L_{\mathcal{F}}(M)$. The leaves of $\mathcal{F}_L$ project onto the leaves of $\mathcal{F}$. The mapping p_L is a morphism in the category $\underline{Fol}_{q^2+q}$.

In the above examples we have presented two structures which are not foliated in our sense, but which are closely related to the structure of a foliated manifold. These two structure are, what we call, associated geometric structures. Now we are going to present a formal definition.

Let $\underline{Fol}_q^*$ be a category of foliated manifolds with foliations of codimension q. The morphisms in this category are the following:
$f \in Mor((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$ iff

1. $\dim M_1 = \dim M_2$, then $f: M_1 \longrightarrow M_2$ is an embedding and $f^*\mathcal{F}_2 = \mathcal{F}_1$;

2. $\dim M_1 > \dim M_2$, then $\dim M_2 = q$, $\mathcal{F}_2$ is the foliation by points and $f: M_1 \longrightarrow M_2$ is a submersion defining $\mathcal{F}_1$.

Definition 5 An associated natural bundle is a functor defined on $\underline{Fol}_q^*$ with values in the category of locally trivial fibre bundles such that:

i) the bundle $F(M, \mathcal{F})$ is a locally trivial bundle over M;

ii) for any morphism $f \in Mor((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$, $F(f)$ is a bundle mapping such that:

a) covariant case: the diagram

$$\begin{array}{ccc} F(M_1,\mathcal{F}_1) & \xrightarrow{F(f)} & F(M_2,\mathcal{F}_2) \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is commutative. Moreover, for any $x \in M_1$, if $\dim M_1 = \dim M_2$ the mapping $F(f)_x \colon F(M_1,\mathcal{F}_1)_x \to F(M_2,\mathcal{F}_2)_{f(x)}$ is a diffeomorphism, and if $\dim M_1 > \dim M_2 = q$ then this mapping is a surjective submersion;

b) contravariant case: the diagram

$$\begin{array}{ccccc} F(M_2,\mathcal{F}_2) & \xleftarrow{\tilde{f}} & f^*F(M_2,\mathcal{F}_2) & \xrightarrow{F(f)} & F(M_1,\mathcal{F}_1) \\ \downarrow & & \downarrow & & \downarrow \\ M_2 & \xleftarrow{f} & M_1 & \xrightarrow{id} & M_1 \end{array}$$

is commutative, where $\tilde{f}$ is the natural projection induced by f. Moreover, for any points $y \in M_2$, $x \in f^{-1}(y)$, if $\dim M_1 = \dim M_2$ the mapping $F(f)_y \colon F(M_2,\mathcal{F}_2)_y \to F(M_1,\mathcal{F}_1)_x$ is a diffeomorphism, and if $\dim M_1 > \dim M_2 = q$ then this mapping is an embedding;

iii) the functor F is regular.

Remarks 1) If we consider the category $\underline{Man}_q$ as a subcategory of $\underline{Fol}^*_q$, the restriction of any associated natural bundle functor to this subcategory is a natural bundle functor. In general, it does not seem to be possible to reconstruct an associated natural bundle functor from its values on $\underline{Man}_q$. But owing to the considerations of the previous section, any associated natural bundle defines a foliated natural bundle, e.g. the passage from the bundle of linear frames adapted to the foliation to the bundle of transverse linear frames. Moreover, any foliated natural bundle functor is an associated one.

2) Assume that an associated natural bundle F is a principal fibre bundle as well. It means that if $f \in Mor((M_1,\mathcal{F}_1),(M_2,\mathcal{F}_2))$ then $F(f)$ is a mapping in the

category of principal fibre bundles: let $F(M_1, \mathcal{F}_1)$ be a $G(M_1, \mathcal{F}_1)$ principal fibre bundle and $F(M_2, \mathcal{F}_2)$ be a $G(M_2, \mathcal{F}_2)$ principal fibre bundle, then there exists a morphism h_f of Lie groups $G(M_1, \mathcal{F}_1)$ and $G(M_2, \mathcal{F}_2)$ such that

$$F(f)(p \cdot g) = F(f)(p) \cdot h_f(g)$$

Assume that we have a corresponding system T of G–spaces i.e. $T(M_1, \mathcal{F}_1)$ is a $G(M_1, \mathcal{F}_1)$–space and for any $f \in Mor((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$ $T(f)(t \cdot g) = T(f)(t) \cdot h_f(g)$. Then the functor $\mathcal{F}_T$:

$$(M_1, \mathcal{F}_1) \longmapsto E(F(M_1, \mathcal{F}_1), T(M_1, \mathcal{F}_1)),$$

the associated fibre bundle to $F(M_1, \mathcal{F}_1)$ with the standard fibre $T(M_1, \mathcal{F}_1)$, is an associated natural bundle. Therefore, if we choose a supplementary subbundle to $T\mathcal{F}$, the bundles $\bigotimes_s^r TM^*$ and $\bigwedge^k TM^*$ can be considered as associated bundles in the above sense and therefore they are covariant associated natural bundles.

Definition 6 *Let f be a morphism in $\underline{Fol}_q^*$, $f \in Mor((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$. Two subbundles B_1 of $F(M_1, \mathcal{F}_1)$ and B_2 of $F(M_2, \mathcal{F}_2)$ are said to be f–related if the fibre mapping $F(f)$ restricted to B_1 is a surjective submersion onto B_2 (resp. it is a diffeomorphism of f^*B_2 onto B_1 in the contravariant case).*

Let $\mathcal{U}$ be a cocycle defining the foliation $\mathcal{F}$ modelled on N. In the covariant case this cocycle defines a cocycle $\mathcal{U}_F$ on the total space of the bundle $F(M, \mathcal{F})$, namely: $\{V_i, \tilde{f}_i, \tilde{g}_{ij}\}_I$ where $F(M, \mathcal{F})|U_i = V_i$, $\tilde{f}_i = F(f_i)$, $\tilde{g}_{ij} = F(g_{ij})$. The foliation $\mathcal{F}_F$ defined by this cocycle (equivalent cocycles of $\mathcal{F}$ give equivalent ones) is not of the same dimension as $\mathcal{F}$ but it projects onto $\mathcal{F}$. The codimension of $\mathcal{F}_F$ is equal to $dim F(N)$. In the contravariant case, we obtain a subbundle $\tilde{F}(M, \mathcal{F})$ of the fibre bundle $F(M, \mathcal{F})$. Over U_i, it is isomorphic to $f_i^* F(N)$ and therefore it is naturally foliated. This subbundle and its foliation $\mathcal{F}_F$ does not depend on the choice of the cocycle $\mathcal{U}$. The foliation has codimension equal to $dim F(N)$ and is of the same dimension as $\mathcal{F}$.

Definition 7 *A foliated subbundle E of $F(M, \mathcal{F})$ (resp. of $\tilde{F}(M, \mathcal{F})$ in the contravariant case) is called an associated geometric structure on the foliated manifold $(M, \mathcal{F})$.*

It is not difficult to see that for a given associated natural bundle $F(M, \mathcal{F})$, associated geometric structures on $(M, \mathcal{F})$ which are foliated subbundles of $F(M, \mathcal{F})$ define holonomy invariant subbundles of $F(N)$ to which they are $\mathcal{U}$–related, i.e. f_i–related for any $i \in I$.

Example 5 1. A global infinitesimal automorphism of the foliation $\mathcal{F}$ is an associated geometric structure; a base–like form is such a structure as well.

2. Let us consider the bundle $L_{\mathcal{F}}(M)$ of adapted linear frames to the foliation $\mathcal{F}$. It is a reduction of the linear frame bundle $L(M)$ to the structure group $GL(n, n-q)$. The foliation $\mathcal{F}_L$ of $L_{\mathcal{F}}(M)$, locally, is given by the following submersion: let (U, ϕ), $\phi = (\phi_1, \phi_2): U \longrightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$ be an adapted chart, then the mapping $\tilde{\phi}: L_{\mathcal{F}}(M)|U \longrightarrow L(\mathbb{R}^q)$,

$$\tilde{\phi}(v_1, ...v_n) = (d\phi_2(v_1), ...d\phi_2(v_n)) = (d\phi_2(v_{n-q+1}), ...d\phi_2(v_n))$$

is a submersion defining $\mathcal{F}_L$ over U.

Associated G–structures are foliated reductions of $L_{\mathcal{F}}(M)$ to groups consisting of matrices of the form $\begin{pmatrix} * & 0 \\ & A \end{pmatrix}$ where A is a matrix of a given Lie subgroup G_0 of $GL(q)$.

3. A linear connection w in the bundle $L_{\mathcal{F}}(M)$ is given by a G–invariant section S_w of the sheaf $A^1(L_{\mathcal{F}}(M), E(L_{\mathcal{F}}(M), \underline{gl}(n, n-q)))$ of 1–forms on $L_{\mathcal{F}}(M)$ with values in the associated fibre bundle $E(L_{\mathcal{F}}(M), \underline{gl}(n, n-q))$ (to be precise we take the pull–back of this bundle to the total space of $L_{\mathcal{F}}(M)$). Such a connection is an associated geometric structure (or an associated connection) if in the bundle L(N) there exists a connection w_N given by a section S_N which is $\mathcal{U}$–related to the $\underline{gl}(q)$–component of S_w. Locally, it means that for any submersion $f: U \longrightarrow N$ defining the foliation $\mathcal{F}$ the following diagram is commutative:

$$\begin{array}{ccc}
A^1(\tilde{U}, E(L_{\mathcal{F}}(M), \underline{gl}(q))) & \xrightarrow{\quad \bar{f}^* \quad} & A^1(\tilde{U}, \tilde{f}^* E(L(N), \underline{gl}(q))) \\
\uparrow \rho^* & & \uparrow \tilde{f}_* \\
A^1(\tilde{U}, E(L_{\mathcal{F}}(M), \underline{gl}(n, n-q))) & & A^1(L(N), E(L(N), \underline{gl}(q))) \\
S_w \uparrow \; \downarrow & & \downarrow \; \uparrow S_N \\
\tilde{U} & \xrightarrow{\quad \tilde{f} \quad} & L(N)
\end{array}$$

where $\tilde{U} = L_{\mathcal{F}}(M)|U$, $\tilde{f}$ is the mapping of frame bundles induced by f, $\tilde{f}_*$ and $\bar{f}$ the corresponding mappings of the sheaves and associated fibre bundles, respectively, and ρ^* the mapping induced by the homomorphism $\rho: \underline{gl}(n, n-q) \longrightarrow$

$\underline{gl}(q)$ defined by the correspondence

$$\underline{gl}(n, n-q) \ni \begin{pmatrix} * & 0 \\ & A \end{pmatrix} \longmapsto A \in \underline{gl}(q).$$

Thus

$$\tilde{f}^* \rho^* S_w | \tilde{U} = \tilde{f}_* S_N \tilde{f}$$

and therefore

$$\rho w = \tilde{f}^* w_N.$$

To be absolutely precise, such a connection itself is not an associated structure, but rather a class of connections in $L_{\mathcal{F}}(M)$ having the same $\underline{gl}(q)$–component.

In terms of geometric properties it means that the parallel transport on M defined by the connection w projects onto the parallel transport on the transverse manifold N defined by the connection w_N. Such pairs of connections were studied by R. A. Blumenthal in [BL7,BL8].

4. Let us consider associated $\{e\}$–structures. An associated $\{e\}$–structure is a foliated section of the bundle $L_{\mathcal{F}}(M)$; thus at each point x we have a linear frame $(v_1, ...v_n)$ of $L_{\mathcal{F}}(M)$. Moreover, at any two points x_1 and x_2 of U_i such that $f_i(x_1) = f_i(x_2)$, the value of the mapping $\tilde{f}_i$ on these frames is the same, i.e. $df_i(v_j(x_1)) = df_i(v_j(x_2))$. Therefore, the global vector fields $X_j, X_j(x) = v_{j+n-q}(x)$, $j = 1, ...q$, are infinitesimal automorphisms defining a transverse parallelism of $\mathcal{F}$.

It is easy to verify that on the total space of $L_{\mathcal{F}}(M)$ an associated connection defines an associated $\{e\}$–structure for the foliation $\mathcal{F}_L$.

5. Prolongations of associated G–structures provide other examples of associated geometric structures. Let us consider the vector space $\mathbb{R}^n$ as the product $\mathbb{R}^p \times \mathbb{R}^q$, $p = n - q$, the natural projection $\rho_0 : \mathbb{R}^n \to \mathbb{R}^q$ and the natural inclusion $s_0 : \mathbb{R}^q \to \mathbb{R}^n$. Then let us take the Lie subgroup G of $GL(n,p)$ consisting of matrices of the form $\begin{pmatrix} * & 0 \\ & A \end{pmatrix}$ with A from a Lie subgroup G' of $GL(q)$. The correspondence $\rho : \begin{pmatrix} * & 0 \\ & A \end{pmatrix} \longmapsto A$ defines a homomorphism of Lie groups $\rho : G \longrightarrow G'$ and of Lie algebras $\rho : \underline{g} \longrightarrow \underline{g}'$. The mapping $s : \underline{g}' \longrightarrow \underline{g}$ defined as $s(B) = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ is a section of ρ.

It is straight–forward to verify that the mapping ρ induces surjective homomorphisms ρ_k of the subsequent prolongations of the Lie algebras of $\underline{g}$ and $\underline{g}'$; $\rho_k : \underline{g}^{(k)} \longrightarrow \underline{g}'^{(k)}$.

As the result of these considerations we obtain the following lemma.

Lemma 2 *If the Lie algebra $\underline{g}$ is of type k, the Lie algebra $\underline{g}'$ is of type k', $k' \leq k$.*

Let $f: M \longrightarrow N$ be a surjective submersion. Let $B(M, G, \pi)$ be a G–structure on the n–manifold M and $B'(N, G', \pi')$ a G'–structure on the q–manifold N which are f–related. Then for the fundamental forms θ and θ' of $B(M, G, \pi)$ and $B'(N, G', \pi')$, respectively, we have the following:

$$\text{(II.1)} \qquad \theta' \circ d\tilde{f} = \rho_0 \circ \theta$$

Let $X \in T_pB$, then $\theta'(d\tilde{f}(X)) = \tilde{f}(p)^{-1}d\pi'(d\tilde{f}(X)) = \tilde{f}(p)^{-1} \circ df \circ d\pi(X)$. As $\tilde{f}(p)^{-1} \circ d_x f = \rho_0 p^{-1}$ where $\pi(p) = x$, we have

$$\theta'(d\tilde{f}(X)) = \rho_0 p^{-1} d\pi(X) = \rho_0 \theta(X).$$

We would like to show that the consecutive prolongations of the f–related structures $B(M, G, \pi)$ and $B'(N, G', \pi')$ are f–related. First of all, we shall demonstrate that their structure tensors are $\tilde{f}$–related.

The structure tensor c of $B(M, G, \pi)$ takes values in

$$Hom(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)/\partial Hom(\mathbb{R}^n, \underline{g}) = H^{0,2}(\underline{g}).$$

The mappings ρ and ρ_0 induce the mappings

$$\hat{\rho}: Hom(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n) \longrightarrow Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q)$$

and

$$\tilde{\rho}: H^{0,2}(\underline{g}) \longrightarrow H^{0,2}(\underline{g}'),$$

the second one being the quotient of the first. The mappings s and s_0 define the corresponding sections of $\hat{\rho}$ and $\tilde{\rho}$, respectively, i.e.

$$\hat{s}: Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q) \longrightarrow Hom(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n) \text{ and } \tilde{s}: H^{0,2}(\underline{g}') \longrightarrow H^{0,2}(\underline{g}).$$

The equality (1) ensures that

$$\text{(II.2)} \qquad c' \circ \tilde{f} = \tilde{\rho} \circ c.$$

Take a subspace S of $Hom(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)$ supplementary to $\partial Hom(\mathbb{R}^n, \underline{g})$ such that the space $S \cap \hat{s}(Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q))$ is supplementary to $\hat{s}(\partial Hom(\mathbb{R}^q, \underline{g}'))$ in $\hat{s}(Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q))$. Then $\hat{\rho}(S) = S'$ is supplementary to $\partial Hom(\mathbb{R}^q, \underline{g}')$ in $Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q)$.

Let V be a horizontal subspace of T_pB such that $C_V \in S$. The equality (1) ensures that for the horizontal subspace $V' = d\tilde{f}(V)$, $C_{V'} \in S'$, as $\rho C_V = C_{V'}$. This implies that the correspondence $V \longmapsto d\tilde{f}(V) = V'$ defines a transformation f_1 of the first prolongation $B^{(1)}(M, G, \pi)$ of the G–structure $B(M, G, \pi)$ onto the first prolongation $B'^{(1)}(N, G', \pi')$ of the G'–structure $B'(N, G', \pi')$. f_1 covers $\tilde{f}$ and makes these two structures $\tilde{f}$–related, and thus the bundles $B^{(1)} \longrightarrow M$ and $B'^{(1)} \longrightarrow N$ f–related. Repeating the construction we obtain mappings

$f^k: B^{(k)}(M,G,\pi) \longrightarrow B'^{(k)}(N,G',\pi')$ covering the mappings f^{k-1} and f, and making these structures f–related. If the group G is of type $k+1$, the total spaces of the bundles $B^{(k)}(M,G,\pi)$ and $B'^{(k)}(N,G',\pi')$ are parallelisable and these parallelisms are f^k–related.

In the case of a foliation $\mathcal{F}$ defined by a cocycle $\mathcal{U}$, the above construction yields structures $B^{(k)}(M,G,\pi)$ and $B'^{(k)}(N,G',\pi')$ which are $\mathcal{U}$–related. For a group G of type $k+1$, the foliation $\mathcal{F}_{k+1}$ of the total space B^k of the bundle $B^{(k)}(M,G,\pi)$ is transversely parallelisable. The parallelisms of the manifolds B^k and B'^k are $\mathcal{U}_{k+1}$–related where $\mathcal{U}_{k+1}$ is the cocycle defining $\mathcal{F}_{k+1}$ derived from the cocycle $\mathcal{U}$.

Before we can formulate our main theorem on associated geometric structures we need the following definitions.

Definition 8 *A G–structure $B(M,G,\pi)$ of type $k+1$ is complete if the parallelism of the total space B^k of the kth–prolongation $B^{(k)}(M,G,\pi)$ is complete.*

Definition 9 *1. An associated geometric structure $E(M,\mathcal{F})$ is of finite type if the total space E of this bundle is parallelisable and this parallelism is $\mathcal{U}_E$–related to a parallelism of the total space of the corresponding bundle on the transverse manifold.*

2. An associated geometric structure $E(M,\mathcal{F})$ of finite type is complete if the vector space spanned by the vector fields of the parallelism of E consists of complete vector fields.

3. An associated geometric structure $E(M,\mathcal{F})$ is a Serre structure if the projection in the bundle $E(N)$ is a Serre fibration.

Example 6 Associated G–structures of finite type and G–structures of higher order, cf. [OC,BL9], are Serre associated geometric structures of finite type.

Theorem 2 *Let the foliation $\mathcal{F}$ be an (N,K)–structure on a manifold M. If $(M,\mathcal{F})$ admits a complete Serre associated geometric structure $E(M,\mathcal{F})$ of finite type, then the natural projection $p: \tilde{M} \longrightarrow \tilde{M}/\tilde{\mathcal{F}}$ of the universal covering space $\tilde{M}$ of M onto the space of leaves $\tilde{M}/\tilde{\mathcal{F}}$ of the lifted foliation $\tilde{\mathcal{F}}$ is a Serre fibration and the space $\tilde{M}/\tilde{\mathcal{F}}$ is a Hausdorff manifold.*

Proof From the very beginning we can assume that the manifold N is simply connected. We can always take its universal covering and the group generated by the action of K on it. Since elements of K lift to diffeomorphisms preserving the parallelism of the total space of $E(N)$, K acts quasi–analytically on N. Therefore $\mathcal{F}$ is developable. Let $h: \tilde{M} \longrightarrow N$ be the developing mapping. Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{E(h)} & E_N \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{h} & N \end{array}$$

where $\tilde{E}(\tilde{M}, \tilde{\mathcal{F}})$ is the lift of the bundle $E(M, \mathcal{F})$ to $\tilde{M}$. The parallelisms of the total spaces $\tilde{E}$ and E_N of the bundles $\tilde{E}(\tilde{M}, \tilde{\mathcal{F}})$ and $E(M, \mathcal{F})$, respectively, are $E(h)$-related. Therefore, the lifted foliation $\tilde{\mathcal{F}}_E$ of $\tilde{E}$ which is defined by the global submersion $E(h)$ is a complete transversely parallelisable foliation. Thus the submersion $E(h)$ is a locally trivial fibre bundle, and the developing mapping is surjective. The fact that the projection $E_N \longrightarrow N$ is a Serre fibration ensures that the developing mapping itself is a Serre fibration. Since the manifold N is simply connected, its fibres must be connected and the space of leaves of the foliation $\tilde{\mathcal{F}}$ is just the manifold N.□

Remarks This theorem generalizes results of R. A. Blumenthal contained in [BL7,BL8,BL9]. In its main outline the proof is the same as the one presented by him and can be considered as a kind of 'proof scheme'. By adopting various assumptions on geometric structures and imposing the completeness conditions we can prove a series of results on associated geometric structures. It is worth stressing that the completeness of an associated structure does not imply the completeness of the corresponding foliated structure.

This kind of theorems is quite useful in the study of homotopy and homology groups of leaves and their relations with the corresponding groups of the ambient manifold, cf. [BL7,BL8,BL9]. As an example we can give the following corollary.

Corollary 1 *Let the model space N be contractible and the holonomy pseudogroup equivalent to the one generated by a group K. Then for any leaf L of the foliation $\mathcal{F}$ the homotopy groups $\pi_i(L)$ inject into the corresponding homotopy groups of the ambient manifold M.*

Notes Most of the results of this chapter have been published in [WO7] and [WO13].

For another approach to the transverse structures of foliations see [MO3] and [MO8]. There are very few papers concerned with general theory of geometric structures of foliations. In addition to the ones mentioned earlier we should add a very interesting paper by P. Libermann, cf. [LIB], and the book of P. Molino, cf. [MO11]. One should also mention a thesis by A. Mba written under supervision of Woufo Kamga.

Chapter III

Foliations admitting transverse systems of differential equations

Riemannian foliations have been for a long time the subject of particular attention and at present we know a lot about their properties. It turns out that many of these properties are the consequence of two facts. First, that the geodesics are global on compact manifolds, and secondly that if a geodesic is orthogonal to the foliation at one point then it is orthogonal to the foliation at any point of its domain. A quick look at the equation of the geodesic of the Levi–Civita connection of a bundle–like metric reveals that this equation is of a special form which we shall call a foliated system of differential equations.

Foliations admitting 'foliated' differential equations have many properties of Riemannian foliations. Unfortunately we have to assume some additional properties like completeness of the equation and smooth dependence on the initial condition. For example, the equation of the geodesic is complete iff the geodesics are global. It is obvious that on compact Riemannian manifolds the equation of the Levi–Civita connection is complete, and it is not coincidental that for non–compact manifolds one assumes that the metric is complete. For other connections the completeness of its geodesics must be proved. Even for the flat connection of a compact affine manifold it is a non–trivial matter, cf. [FR2].

In addition to Riemannian foliations foliated equations admit, among others, transversely affine, transversely homogeneous, conformal and $\nabla - G$–foliations. Until now they have been considered separately. In all these cases we take the equation of the geodesic of some connection and the completeness of this equation means precisely that geodesics are global.

In this chapter we are going to study foliations admitting such 'foliated' systems of ordinary differential equations (SODE). We show that, in many respects, their properties are similar to those of Riemannian foliations. In other chapters dealing with $\nabla - G$– and transversely affine foliations we shall stress the differences.

III.1 Preliminaries

At the very beginning we recall some definitions concerning systems of ordinary differential equations.

Definition 1 *A subbundle* $\mathbf{E}$ *of* $J^k(\mathbb{R}, M)$ *is called a system of ordinary differential equations of order* k *on the manifold* M.

A mapping $f: \mathbb{R} \to M$ *of connected domain is a solution of the system* $\mathbf{E}$ *if the mapping* $\tilde{f}: dom f \ni t \mapsto j^k_t f \in J^k(\mathbb{R}, M)$ *is a section of* $\mathbf{E}$

Let r *be an integer smaller or equal to* k, $0 \leq r \leq k$. *For each such an* r *the system* $\mathbf{E}$ *defines a subset* $\mathbf{E}^r_0$ *of* $T^r(M)$;

$$\mathbf{E}^r_0 = \{j^r_0 f \circ \tau_t: f \text{ is a solution of } \mathbf{E} \text{ at } t \in \mathbb{R}\}$$

where τ_t *is a translation in* $\mathbb{R}$ *by the vector* t. *The set* $\mathbf{E}^r_0$ *is called the set of initial conditions of order* r *of the system* $\mathbf{E}$.

A system $\mathbf{E}$ *is called a USP (Unique Solution Property) system if there exists* $0 \leq r \leq k$ *such that the set* $\mathbf{E}^r_0$ *is a subbundle of* $T^r(M)$ *and for any pair* $(t, \xi) \in \mathbb{R} \times \mathbf{E}^r_0$ *there exists exactly one solution* f *in a neighbourhood of* t *such that* $j^r_o f \circ \tau_t = \xi$. *Moreover, we assume that the solutions depend smoothly on the initial condition.*

Let r *be the smallest integer having the above property, then the bundle* $\mathbf{E}^r_0$ *is called the bundle of initial conditions of the system* $\mathbf{E}$.

We say that solutions of the system $\mathbf{E}$ *depends smoothly on the initial condition if for any smooth mapping* $f: W \to \mathbf{E}^r_0$, W *an open subset of* $\mathbb{R}^m$, *the mapping* $\varphi: W \times \mathbb{R} \to M$ *defined as* $\varphi(t, v) = \varphi_t(v)$, φ_t *the solution with the initial condition* $f(t)$, *is a smooth mapping.*

A system $\mathbf{E}$ *is called transitive if for any tangent vector* X *there exists a solution* f *of the system* $\mathbf{E}$ *such that* $X \in imd_0 f$.

It is our aim to explain the influence of systems of differential equations on the structure of foliations. To have any relation between properties of these two objects on the manifold they must be in some way compatible, i.e. the system should be 'adapted' or 'foliated'. We shall work with the following definition of a foliated system of differential equations.

Definition 2 *A SODE* $\mathbf{E}$ *is called foliated if there exists a subbundle* Q *supplementary to* $T\mathcal{F}$ *such that the set* $J^k(\mathbb{R}, Q) \cap \mathbf{E} = \mathbf{E}_Q$ *is a foliated subbundle of* $J^k(\mathbb{R}, Q)$.

The foliated subbundle $\mathbf{E}_Q$ defines a system $\mathbf{E}_N$ of ODE on the transverse manifold N, i.e. $\mathbf{E}_N \subset J^k(\mathbb{R}, N)$, and $\mathbf{E}_Q$ is the subbundle of $J^k(\mathbb{R}, Q)$ corresponding to $\mathbf{E}_N$. The holonomy pseudogroup $\mathcal{H}$ is a pseudogroup of local automorphisms of the system $\mathbf{E}_N$.

The following lemma can help to characterize FSODE.

Lemma 1 *Let $(M, \mathcal{F})$ be a foliated manifold, Q a supplementary subbundle to $T\mathcal{F}$ and $\mathbf{E} \subset J^k(\mathbf{R}, M; Q)$ be SODE. Then the system $\mathbf{E}$ is foliated for $\mathcal{F}$ iff for any vector field X tangent to $\mathcal{F}$ its flow consists of automorphisms of $\mathbf{E}$.*

Proof. The foliation $\mathcal{F}_k$ of $J^k(\mathbf{R}, M; Q)$ is defined by the lifted action of the sheaf $\mathcal{S}_{\mathcal{F}}$ of germs of vector fields tangent to $\mathcal{F}$. The condition means precisely that the subbundle $\mathbf{E}$ is foliated in $J^k(\mathbf{R}, M; Q)$. □

Since we are interested in the transverse structure of the foliation, therefore we shall look only at solutions transverse to the foliation. The following definitions will be very useful.

Definition 3 *A solution $f: \mathbf{R} \to M$ of $\mathbf{E}$ is said to be tangent to Q if $im d_t f \subset Q$ for any $t \in dom f$.*

Let the system $\mathbf{E}_N$ be USP and $\mathbf{E}_0^r(N)$ be its bundle of initial conditions. The corresponding subbundle of Q^r we denote by $\mathbf{E}_0^r(Q)$ and it is called the bundle of transverse initial conditions. A foliated system $\mathbf{E}$ is called a TUSP (Transverse Unique Solution Property) system if solutions with initial conditions from the bundle $\mathbf{E}_0^r(Q)$ are unique and depend smoothly on the initial condition.

A solution is said to be transverse if its initial condition belongs to the bundle of transverse initial conditions.

A system $\mathbf{E}$ is (transversely) complete if any (transverse) solution can be extended to a global one.

A foliated system $\mathbf{E}$ is transversely transitive if for any vector X of the bundle Q there exists a transverse solution f of the system $\mathbf{E}$ such that $im f \ni X$.

A curve $\gamma: (-\epsilon, \epsilon) \to M$ is called a solution curve of the system $\mathbf{E}$ if there exists a solution f of the system $\mathbf{E}$ at $x = \gamma(0)$ and a curve $\tilde{\gamma}: (-\epsilon, \epsilon) \to \mathbf{R}$, $\tilde{\gamma}(0) = 0$ such that $\gamma = f \circ \tilde{\gamma}$.

A curve $\gamma: [0, 1] \to M$ is called a piecewise solution curve of the system $\mathbf{E}$ if there exists a sequence of numbers $t_0 < t_1 < \dots < t_{m+1} = 1$ such that for $i = 0, \dots m$ the curve $\gamma \mid [t_i, t_{i+1}]$ is a solution curve.

Remark If the system $\mathbf{E}$ is transversely transitive then any two leaves of the connected manifold M can be joined by a piecewise solution curve.

To complete the introduction we provide a method of producing examples of foliations with FSODE.

Example 1 Let $\mathcal{F}$ be a foliation of the manifold $B \times_h N$ constructed in Example I.2 . If the group imh is a group of automorphisms of a SODE E_N on N then the foliation $\mathcal{F}$ admits an FSODE.

Example 2 Let N be a transverse manifold of the foliation $\mathcal{F}$ and $\mathcal{H}$ be the holonomy pseudogroup representative on N. Let us assume that there exists a G–connection on N of which the pseudogroup $\mathcal{H}$ is a pseudogroup of local affine transformations. Thus in the induced foliated G–structure there is a transversely projectable connection. Let us take a supplementary subbundle Q. The transversely projectable connection defines a covariant differentiation in Q. We can extend this operation to the whole tangent bundle by choosing any covariant differentation on $T\mathcal{F}$. The equation of the geodesic of this connection is a foliated one.

This class of foliations also includes transversely parallelisable foliations. Let us choose a subbundle Q as before and vector fields $X_1, ..X_q$, sections of Q, defining the transverse parallelism. The connection making the vector fields X_i parallel is a transversely projectable one and segments of the flows of the vector fields X_i are geodesics. The equation of the geodesic of this connection is transversely complete iff the vector fields $\sum a_i X_i$ are complete.

We can do the same for connections of higher order.

Let us return to our general considerations.

On any transverse manifold N of $(M, \mathcal{F})$ an FSODE $\mathbf{E}$ defines a holonomy invariant SODE $\mathbf{E}_N$. Vice versa, for any choice of a supplementary subbundle Q, a holonomy invariant SODE $\mathbf{E}_N$ defines a foliated system $\mathbf{E}_Q$. Having chosen two supplementary subbundles Q and Q' we get two different FSODE $\mathbf{E}_Q$ and $\mathbf{E}_{Q'}$. They are isomorphic as fibre bundles, but this isomorphism does not need to be holonomic. Therefore these FSODE can have very different properties. There is no problem with local existence of solutions. However, the global existence of solutions (transverse completeness) of $\mathbf{E}_Q$ does not insure that for some other subbundle Q' the system $\mathbf{E}_{Q'}$ has the same property as the following example shows.

Example 3 Let us consider the set $V = \mathbb{R}^3 \setminus \{(x,y,z) | y = z = 0\}$ and the projection $p_1 \colon V \longrightarrow \mathbb{R}$, $(x,y,z) \mapsto x$. The homothety $h_\lambda \colon (x,y,z) \mapsto (\lambda x, \lambda y, \lambda z)$, $0 < \lambda < 1$, preserves V and the foliation of V defined by p_1. The induced foliation $\mathcal{F}_\lambda$ of the quotient manifold $V_\lambda = V/h_\lambda$ is transversely affine. The transversely projectable flat connection of this foliation is transversely complete for the supplementary subbundle Q generated by the vector field $\partial/\partial x$ of V. For subbundles generated by vector fields of the form $\partial/\partial x + a\partial/\partial y + b\partial/\partial z$, $a^2 + b^2 \neq 0$, the corresponding transversely projectable flat connections are not transversely complete.

The transverse completeness plays a crucial role in the study of foliations admitting FSODE. In Section 5 we are going to look into this particular question, and, additionally, we give some conditions on the holonomy pseudogroup and the transverse system $\mathbf{E}_N$ which ensure that for any supplementary subbundle Q the system $\mathbf{E}_Q$ is transversely complete.

III.2 Basic properties of solutions

In this section we establish basic properties of transverse solutions. In particular we look into relations between transverse solutions of an FSODE and solutions of the corresponding system on the transverse manifold.

Lemma 2 *Let $\mathcal{F}$ be a simple foliation given by a submersion $p: M \longrightarrow N$ and $\mathbf{E}$ be foliated TSUP system of differential equations on $(M, \mathcal{F})$. Denote by $\mathbf{E}_N$ the induced system on N. Let ξ_0 be an element of the bundle $\mathbf{E}_0^r(N)$ of the initial conditions of the system $\mathbf{E}_N$ over a point x_0. If f is a solution of the system $\mathbf{E}_N$ such that $j_0^r f \circ \tau_t = \xi_0$, then for any $x \in p^{-1}(x_0)$ $\xi = (N^r(p)_x)^{-1}(\xi_0) \in \mathbf{E}_0^r(Q)$ there exists a solution f_x of the system $\mathbf{E}_Q$ such that $j_0^r f_x \circ \tau_t = \xi$ and $p \circ f_x = f$ in a neighbourhood of t.*

Proof The mapping f is the solution of the system $\mathbf{E}_N$ with the initial condition ξ_0, so its lift f_x at x tangent to Q is a solution of the system $\mathbf{E}_Q$. As $p \circ f_x = f$ in a neighbourhood of t and $N^r(p)_x(\xi) = \xi_0$ the r–jet of $f_x \circ \tau_t$ at 0 must be ξ.□

Corollary 1 *Let $\mathcal{F}$ be a simple foliation defined by a submersion $p: M \longrightarrow N$ and let $\mathbf{E}$ be a transversely complete, TUSP foliated system. Let f_1 and f_2 be two transverse solutions of $\mathbf{E}$ such that $pf_1(0) = pf_2(0)$ and $N^r(p)(j_0^r f_1) = N^r(p)(j_0^r f_2)$. Then for any t of the intersection $dom f_1 \cap dom f_2$ the points $f_1(t)$ and $f_2(t)$ belong to the same fibre of p.*

Proof The mappings pf_1 and pf_2 are solutions of the system $\mathbf{E}_N$ with the initial condition $N^r(p)(j_0^r f_1)$. From Lemma 2 it follows that pf_1 and pf_2 are equal on the intersection of domains.□

Lemma 3 *Let $\mathbf{E}$ be a TSUP foliated SDE. Let $\alpha: [0, s_0] \rightarrow \mathbf{E}_0^r(Q)$ be a leaf curve and f_s the solution of the system $\mathbf{E}$ with the initial condition $\alpha(s)$ at 0. If for any $s \in [0, s_0]$ the solution f_s is defined on a compact connected neighbourhood W of 0 in $\mathbb{R}$, then for any t of W the points $f_s(t)$, $s \in [0, s_0]$ belong to the same leaf of the foliation $\mathcal{F}$.*

Proof Since $\mathbf{E}$ is a TUSP system, the mapping $F: [0, s_0] \times W \longrightarrow M$, $F(s,t) = f_s(t)$, is a smooth mapping. The set $F([0, s_0] \times W)$ is compact and we can cover it by a finite number of adapted charts. Let us choose an $s_1 \in [0, s_0]$ and adapted charts $(U_1, \varphi_1), \ldots (U_m, \varphi_m)$ covering the set $f_{s_1}(W)$. Then there exists $\epsilon > 0$ and compact sets $K_1, ..K_v$ covering W such that each set $F([s_1 - \epsilon, s_1 + \epsilon] \times K_i)$ is contained in some U_j for some $j = 1, \ldots m$. Corollary 1 ensures that for any $t \in K_i$ the points $f_s(t)$, $s \in [s_1 - \epsilon, s_1 + \epsilon]$, belong to the same leaf of the foliation $\mathcal{F}$. Thus for any $t \in W$ the points $f_s(t)$, $s \in [s_1 - \epsilon, s_1 + \epsilon]$, belong to the same leaf of $\mathcal{F}$. As we can cover the interval $[0, s_0]$ with intervals $[s_1 - \epsilon, s_1 + \epsilon]$ having the required property, the lemma has been proved. □

The best known example of a foliated SODE is the equation of the geodesic of the Levi–Civita connection of a bundle–like metric on a foliated manifold. Bundle–like metrics are characterized by the following property, cf. [MO11,RE,YO]:

> on a foliated manifold $(M, \mathcal{F})$ a Riemannian metric g is bundle–like iff any geodesic of g orthogonal to $\mathcal{F}$ at one point is orthogonal to $\mathcal{F}$ at any point of its domain.

We have showed that for a TUSP system foliated for a subbundle Q supplementary to $T\mathcal{F}$ any solution tangent to Q at one point remains tangent to Q at any point of its domain, cf. Lemma 2. However, this property does not characterize foliated systems as the following example illustrates.

Example 4 Let $\mathcal{F}$ be a transversely oriented codimension 1 foliation on a compact manifold. Any global non–vanishing vector field X transverse to $\mathcal{F}$ defines a system $\mathbf{E}$ of ODE on this manifold. Its solutions are integral curves of this vector field. Let Q be the subbundle generated by X. Any solution of $\mathbf{E}$ is tangent to Q. But the system $\mathbf{E}$ is foliated only if X is an infinitesimal automorphism of $\mathcal{F}$. There are many examples of foliations which do not admit such infinitesimal automorphisms, e.g. the Reeb foliation of S^3.

We continue our considerations with the study of properties of the transverse system $\mathbf{E}_N$ and of the holonomy pseudogroup $\mathcal{H}$. The transverse manifold N is rarely connected. Therefore, we propose the following definition of completeness for transverse SODE.

Definition 4 *The system $\mathbf{E}_N$ of ODE on the transverse manifold N is called complete if*

i) *for any solution $\gamma: (a_0, b_0) \longrightarrow N$, there exist solutions $\gamma_i: (a_i, b_i) \longrightarrow N$, $i \in Z$, $a_i \longrightarrow -\infty$ as $i \longrightarrow -\infty$, $b_i \longrightarrow +\infty$ as $i \longrightarrow +\infty$, and local diffeomorphisms h_i of the pseudogroup $\mathcal{H}$ such that $a_i \in (a_{i-1}, b_{i-1}), b_i \in (a_{i+1}, b_{i+1})$ and $h_i\gamma_{i-1}|(a_i, b_{i-1}) = \gamma_i|(a_i, b_{i-1})$;*

ii) *let γ_1 and γ_2 be two solutions of $\mathbf{E}_N$ with initial conditions ξ_1 and ξ_2, respectively. If there exists an element h of $\mathcal{H}$ such that $j^r h(\xi_1) = \xi_2$, then for any t of the common domain, there exists an element h_t of $\mathcal{H}$ such that $j^r h_t(j^r_t \gamma_1) = j^r_t \gamma_2$.*

It is easy to verify that for complete pseudogroups the condition (ii) of Definition 4 is always satisfied.

Lemma 4 *Let $\mathcal{H}$ be a complete pseudogroup of local diffeomorphisms and $\mathbf{E}$ an $\mathcal{H}$-invariant USP system of ODE on N. Then the condition (ii) of Definition 4 is always satisfied.*

Proof Let $\gamma_1\colon [0,a] \to N$ and $\gamma_2\colon [0,a] \to N$ be two solutions of $\mathbf{E}$ with initial conditions ξ_1 and ξ_2 at $x_1 = \gamma_1(0)$ and $x_2 = \gamma_2(0)$, respectively. Assume that there exists an element h_0 of $\mathcal{H}$ such that $j^r h_0(\xi_1) = \xi_2$.

Let us consider the set:

$$A = \{t \in [0,a]\colon \exists h \in \mathcal{H}\colon j^r h(j^r_t \gamma_1) = j^r_t \gamma_2\}.$$

Since the system $\mathbf{E}$ is USP the solutions $h_0\gamma_1$ and γ_2 are equal at a neighbourhood of 0, and thus this neighbourhood is contained in A. The same considerations ensure that A is open. We shall prove that it is also closed. Let us assume that the interval $[0,s)$ is contained in A. Consider two points $x = \gamma_1(s)$ and $y = \gamma_2(s)$, and take open neighbourhoods U and V of x and y, respectively, from the definition of completeness. For t sufficiently close to s, $\gamma_1(t) \in U$ and $\gamma_2(t) \in V$. Moreover, as $t \in A$, there exists an element h_t of the holonomy pseudogroup $\mathcal{H}$ such that $j^r h_t(j^r_t\gamma_1) = j^r_t\gamma_2$. Thus $h_t\gamma_1 = \gamma_2$ whenever both curves are defined. But the mapping h_t is defined on the whole set U, thus also in an open neighbourhood of $\gamma_1(s)$. Hence $h_t\gamma_1 = \gamma_2$ as well in a neighbourhood of s, so $j^r h_t(j^r_s\gamma_1) = j^r_s\gamma_2$ and $s \in A$. Therefore the set A is closed and thus equal to [0,a]. □

The next lemma elucidates the relation between solutions of the foliated system $\mathbf{E}_Q$ on $(M,\mathcal{F})$ and solutions of the transverse system $\mathbf{E}_N$ on the transverse manifold N.

Lemma 5 *Let $\mathbf{E}_Q$ be an FSODE on $(M,\mathcal{F})$ such that the system $\mathbf{E}_N$ is USP and the holonomy pseudogroup $\mathcal{H}$ is complete. Let $y \in U_i$ and $f_i(y) = x$. Let us take an r-vector $\xi_0 \in \mathbf{E}^r_0(N)_x$ of the bundle of initial conditions of $\mathbf{E}_N$ and let ξ be the corresponding transverse r-vector at y, i.e. $j^r f_i(\xi) = \xi_0$. If $\gamma_0\colon [0,a] \to N$ and $\gamma\colon [0,a] \to M$ are the solutions of the systems $\mathbf{E}_N$ and $\mathbf{E}_Q$ with the initial conditions ξ_0 and ξ, respectively, then for any $t \in [0,a]$ there exists an element h_t of $\mathcal{H}$ such that $j^r h_t j^r_t(f_j\gamma) = j^r_t\gamma_0$ for some j.*

Proof Let us consider the set:

$$A = \{t \in [0,a]\colon \exists h \in \mathcal{H}\colon j^r h(j^r_t(f_j\gamma)) = j^r_t\gamma_0\}.$$

It is obvious that the definition does not depend on the choice of j. The set A is non-empty as $0 \in A$. Moreover, some open neighbourhood of 0 belongs to A. We shall prove that $A = [0,a]$. In fact, there exits $t_0, 0 < t_0 \leq a$ such that $\gamma|[0,t_0) \in U_i$ and $\gamma(t_0) \in \partial U_i$. If it were not the case $\gamma([0,a]) \subset U_i$ and immediately $[0,a] = A$ as the $\mathbf{E}_N$ is USP. Thus we have $f_i\gamma|[0,t_0) = \gamma_0|[0,t_0)$ and

$[0, t_0) \subset A$. Let us show that $t_0 \in A$. There exists U_j for which $\gamma(t_0) \in U_j$. Then we can find t_1 and t_2, $0 \leq t_1 < t_0 < t_2 \leq a$ such that $\gamma((t_1, t_2)) \subset U_j$. Hence $\gamma((t_1, t_0)) \subset U_i \cap U_j$ and $f_j\gamma|(t_1, t_0) = g_{ji}f_i\gamma|(t_1, t_0)$. Lemma 4 ensures that for any $t \in (t_1, t_2)$ there exists $h_t \in \mathcal{H}$ such that $j^r h_t(j^r_t(f_j\gamma)) = j^r_t\gamma_0$. This means precisely that $(t_1, t_2) \subset A$ and $t_0 \in A$. As the set A is always open, and we have just demontrated that it is closed, $A = [0, a]$. □

Remark The statement of Lemma 5 can be shortened using the notion of a Q–horizotal lift of a curve in the transverse manifold. In fact, in Lemma 5, the solution γ is a Q–horizontal lift of γ_0.

Definition 5 *A curve $\gamma: [0, a] \longrightarrow M$ tangent to Q is called a Q-horizontal lift of $\gamma_0: [0, a] \longrightarrow N$ if for any $t \in [0, a]$ there exists $h_t \in \mathcal{H}$ such that $j^r f_j(j^r_t\gamma) = j^r h_t(j^r_t\gamma_0)$ for some j.*

The same considerations as in the proof of Lemma 5 show that if the system $\mathbf{E}_Q$ is transversely complete, then the condition i) of Definition 4 is satisfied. As we have already proved that for a complete holonomy pseudogroup the condition ii) is always fulfilled, cf. Lemma 4, we have the following.

Proposition 1 *Let $\mathbf{E}_Q$ be a transversely complete, TUSP, FSODE on a foliated manifold $(M, \mathcal{F})$. If its holonomy pseudogroup is complete, then the transverse system $\mathbf{E}_N$ complete as well.*

III.3 Properties of transversely complete SODE

To obtain more information about a foliation admitting an FSODE we must know that this system is sufficiently rich in 'good' solutions, namely we assume that it is transversely complete and transversely transitive. In this section we study the properties of leaves of foliations with such FSODE.

Let us consider the bundle $\mathbf{E}^r_0(Q) \times \mathbb{R}$. This manifold is foliated by the product foliaton $\mathcal{F}^r$ and the foliation by points of $\mathbb{R}$. To any pair $(\xi, t) \in \mathbf{E}^r_0(Q) \times \mathbb{R}$ we can associate the point $f_\xi(t) \in M$, where f_ξ is the solution of the system $\mathbf{E}$ with the initial condition ξ at 0. This correspodence defines a smooth mapping $Exp: \mathbf{E}^r_0(Q) \times \mathbb{R} \longrightarrow M$. In the case of the equation of the geodesic this mapping becomes the real exponential mapping.

Lemma 6 *Let x be a point of M, $\xi \in \mathbf{E}^r_0(Q)_x$ and L_ξ be the leaf of the foliation $\mathcal{F}^r$ passing through ξ. If L_t is the leaf of $\mathcal{F}$ passing through $Exp(\xi, t)$, then for any $t \in \mathbb{R}$ the mapping $Exp|L_\xi \times \{t\}: L_\xi \longrightarrow L_t$ is a covering.*

Proof Lemma 3 ensures that $Exp(L_\xi \times \{t\})$ is contained in a leaf of $\mathcal{F}$. It is sufficient to show that the mapping in question is a local diffeomorphism and that it has the property of lifting curves. Let y be any point of L_t and α be a leaf curve linking $x_t = Exp(\xi, t)$ to y. Let $\xi_t = j_0^r(f_\xi \circ \tau_t) \in \mathbf{E}_0^r(Q)_{x_t}$ and α_t be the lift of α to ξ_t. Then the correspondence $s \mapsto j_0^r f_{\alpha_t(s)}$ is a curve in the leaf L_ξ as $j_0^r f_{\alpha_t(0)} = \xi$ ($f_{\alpha_t(s)}$ is the solution of $\mathbf{E}$ with the initial condition $\alpha_t(s)$ at t). The fact that transverse solutions project onto solutions of the system $\mathbf{E}_N$ ensures that the mapping is a local diffeomorphism. □

Actually, we have proved the following.

Corollary 2 *(of the proof) Let x be a point M, $\xi \in \mathbf{E}_0^r(Q)_x$, L_ξ and L_{ξ_t} be the leaves of the foliation $\mathcal{F}^r$ passing through ξ and ξ_t, respectively. Then for any $t \in \mathbb{R}$ the leaves L_ξ and L_{ξ_t} are diffeomorphic.*

From the above corollary we immediately obtain the following proposition.

Proposition 2 *Let $(M, \mathcal{F})$ be a foliated manifold admitting a transversely complete, transversely transitive, foliated TUSP system of differential equations. Then the leaves of the foliation $\mathcal{F}$ have the common universal covering space.*

Proof Corollary 2 asserts that two leaves joined by a solution curve have the same universal covering space. This fact coupled with the remark that for a transversely transitive system any two leaves can be linked by a piecewise solution curve completes the proof.□

Lemma 6 leads us to the formulation of the following proposition.

Proposition 3 *Let $\mathbf{E}$ be a transversely complete, foliated TUSP system on a foliated manifold $(M, \mathcal{F})$. Then the mapping $Exp: \mathbf{E}_0^r(Q) \times \mathbb{R} \to M$ is smooth and foliated.*

It results from Proposition 3 that any global section X of the bundle $\mathbf{E}_0^r(Q)$, for any $t \in \mathbb{R}$, defines a global foliated mapping $exp_{X,t}: M \to M$, $exp_{X,t}(x) = Exp(X(x), t)$. We denote the semigroup generated by mappings of this form by $\mathrm{Map}_{\mathbf{E}}(M, \mathcal{F})$. Then, as one can easily show, we obtain the following proposition.

Proposition 4 *Let $\mathbf{E}$ be a transversely complete, foliated TUSP system of differential equations on a foliated manifold $(M, \mathcal{F})$. Then the leaf of $\mathcal{F}$ passing through a point x is a covering space of any leaf passing through the $\mathrm{Map}_{\mathbf{E}}(M, \mathcal{F})$-orbit of the point x.*

Let $X_1, ... X_w$ be foliated sections of $\mathbf{E}_0^r(Q)$ over an open subset U. Then we can define the following mapping:

$$exp_X^w: \mathbb{R}^w \times U \to M,$$

$$exp^w_X(t_1, ...t_w, x) = Exp(X_w, t_w) \circ ... \circ Exp(X_1, t_1)(x)$$

where $Exp(X_i, t_i)(x) = Exp(X_i(x), t_i)$.

The smooth mapping exp^w_X is foliated for the product foliation of $\mathcal{F}|U$ and the foliation by points of $\mathbf{R}^w$.

Before going further we need the following notation. Let $\pi^r: T^r(M) \to T(M)$ be the natural projection. Then π^r maps $\mathbf{E}^r_0(Q)$ into Q. We conclude our considerations with the following theorem which is a generalization of Herman's theorem about Riemannian submersions, cf. [HN].

Theorem 1 *Let $h: M \longrightarrow N$ be a submersion with connected fibres of a manifold M of dimension n into a connected manifold N of dimension q. If for the foliation $\mathcal{F}$ defined by the submersion h there exists a transversely complete, transversely transitive, foliated TSUP system $\mathbf{E}$ of differential equations, then the submersion h is a locally trivial fibre bundle.*

Proof Let us consider two mappings

$$Exp_Q: \mathbf{E}^r_0(Q) \times \mathbf{R} \longrightarrow M \quad \text{and} \quad Exp_{TN}: \mathbf{E}^r_0(N) \times \mathbf{R} \longrightarrow N$$

defined as above. The second mapping Exp_{TN} is defined by the induced system $\mathbf{E}_N$ which is, obviously, a foliated system for the foliation by points. Lemma 2 ensures that the following diagram is commutative.

$$\begin{array}{ccc} E^r_0(Q) \times \mathbf{R} & \xrightarrow{\quad Exp_Q \quad} & M \\ {\scriptstyle N^r(h) \times id} \downarrow & & \downarrow {\scriptstyle h} \\ E^r_0(N) \times \mathbf{R} & \xrightarrow{\quad Exp_{TN} \quad} & N \end{array}$$

Since the system $\mathbf{E}$ is transversely transitive, the system $\mathbf{E}_N$ is also transitive. Therefore for any point $x_0 \in N$ there exist a neighbourhood U and sections $\hat{X}_1, ...\hat{X}_q$ of $\mathbf{E}^r_0(N)$ over U such that for any point $x \in U$ the vectors $\pi^r(\hat{X}_1(x)), ...\pi^r(\hat{X}_q(x))$ span TN_x. Thus for some neighbourhood W of 0 of $\mathbf{R}^q$ the mapping $exp^q_{\hat{X}}|W \times \{x_0\}$ is a diffeomorphism on the image $\tilde{W}$. As the foliation defined by h is without holonomy, the sections $\hat{X}_1, ...\hat{X}_q$ define the foliated sections $X_1, ...X_q$ of $\mathbf{E}^r_0(Q)$ over $h^{-1}(U)$. The image of the mapping $exp^r_X|W \times h^{-1}(x_0)$ is precisely $h^{-1}(\tilde{W})$. For each $t \in W$, the mapping $exp^r_X|\{t\} \times h^{-1}(x_0)$ is a diffeomorphism of $h^{-1}(x_0)$ onto $h^{-1}(exp^q_{\hat{X}}(t, x_0))$ as the leaves of the foliation $\mathcal{F}^r$ of $\mathbf{E}^r_0(Q)$ are diffeomorphic to the corresponding leaves of $\mathcal{F}$. The fact that $exp^q_{\hat{X}}|W \times \{x_0\}$ is a diffeomorphism on the image insures that the mapping $exp^r_X|W \times h^{-1}(x_0)$ is itself a diffeomorphism on the image which precisely means that the submersion h is a locally trivial fibre bundle. □

Remark It is possible to drop the assumption 'with connected fibres'. Then the system $\mathbf{E}$ must be induced by a SODE $\mathbf{E}_N$, i.e. $\mathbf{E} = p^*\mathbf{E}_N$.

Having proved a theorem about submersions we go back to study of foliated manifolds and, in particular, of the universal covering of the manifold itself. But first some preparatory explanations are necessary.

Let $\alpha: [0,1] \to M$ be any leaf curve, and ξ_0 be an element of the fibre $\mathbf{E}_0^r(Q)_{\alpha(0)}$. The bundle $\mathbf{E}_0^r(Q)$ is foliated by $\mathcal{F}^r$. Thus the curve α admits a lift $\tilde{\alpha}$ to ξ_0 such that $\tilde{\alpha}$ is a leaf curve. In this way we have obtained a differentiable field of initial conditions along α.

Let us assume that the TUSP foliated system $\mathbf{E}$ is transversely complete. If at a point x_0 of M we have a pair of curves $\alpha: [0,1] \to M, \sigma: [0,\epsilon] \to M$ where α is a leaf curve and σ is a solution curve, then there exists a mapping $\kappa: [0,1] \times [0,\epsilon] \to M$ such that $\kappa|[0,1] \times \{0\} = \alpha$ and $\kappa|\{0\} \times [0,\epsilon] = \sigma$, for any $t \in [0,\epsilon]$ $\kappa|[0,1] \times \{t\}$ is a leaf curve, and for any $v \in [0,1]$, $\kappa|\{v\} \times [0,\epsilon]$ is a solution curve tangent to Q. Since σ is a solution curve there is a solution $f: \mathbb{R} \to M$ of the system $\mathbf{E}$ at 0 and a curve $\gamma: [0,\epsilon] \to \mathbb{R}$ for which $\sigma = f \circ \gamma$. Denote by ξ_0 the initial condition of the solution f, i.e. $\xi_0 = j_0^r f$. Let $\tilde{\alpha}$ be the lift of the curve α to ξ_0. Then the mapping $\kappa: [0,1] \times [0,\epsilon] \to M$, $\kappa(v,t) = f_v \circ \gamma(t)$, where f_v is the solution of the system $\mathbf{E}$ with the initial condition $\tilde{\alpha}(v)$, has the required properties. Moreover, if we take at a point x_0 a pair of curves $\alpha: [0,1] \to M, \sigma: [0,\epsilon] \to M$ such that α is a leaf curve and σ is a piecewise solution curve, i.e. there is a sequence $t_0 = 0 < t_1 < ... < t_{m+1} = \epsilon$ for which $\sigma|[t_i, t_{i+1}], i = 0, ...m$, is a solution curve of the system $\mathbf{E}$ tangent to Q, then there exists a mapping $\kappa: [0,1] \times [0,\epsilon] \to M$ with the same properties as above but with the following change: for any $v \in [0,1]$ the curve $\kappa|\{v\} \times [t_i, t_{i+1}]$ is a solution curve ot the system $\mathbf{E}$ tangent to the bundle Q.

Now we shall deal with the universal covering space of a foliated manifold admitting a foliated system of differential equations. First of all we shall prove a preparatory lemma.

Lemma 7 *Let $\sigma: [0,1] \to M$ be a curve. Then σ is a homotopic, relative to its ends, to a curve of the form $\beta * \alpha$ such that α is a leaf curve and β is a piecewise solution curve of the system $\mathbf{E}$ tangent to the bundle Q.*

Proof For any point x of the manifold M there exist foliated sections $X_1, ...X_q$ defined on a neighbourhood V of x in the leaf passing through x and a neighbourhood W of 0 in $\mathbb{R}^q$ such that the mapping $exp_X^q|W \times V$ is a diffeomorphism on the image. By taking smaller W and V we can assume that both sets are contractible. Then it is obvious that the lemma is true for curves contained in $exp_X^q(W \times V)$.

The lemma results easily from the following two facts:

i) any curve σ can be covered by a finite number of sets of the form as above;

ii) a curve of the form $\alpha * \beta$ is homotopic relative to its ends to a curve of the form $\beta' * \alpha'$ where α, α' are leaf curves and β, β' are piecewise solution curves. (It is a consequence of the considerations preceding the lemma).□

Using a standard method, cf. [BH1], we can prove the following proposition.

Proposition 5 *Let $(M, \mathcal{F})$ be a foliated manifold. Let* $\mathbf{E}$ *be a transversely complete, transversely transitive, foliated TSUP system of differential equations. If the bundle Q is integrable, then the universal covering space $\tilde{M}$ of the manifold M is diffeomorphic to $\tilde{L} \times \tilde{K}$ where $\tilde{L}$ is the universal covering space of a leaf L of the foliation $\mathcal{F}$, and $\tilde{K}$ is the universal covering space of a leaf K of the foliation Q.*

Corollary 3 *If the foliation $\mathcal{F}$ is of codimension 1, then the universal covering space $\tilde{M}$ is diffeomorphic to $\tilde{L} \times \mathbf{R}$ where $\tilde{L}$ is the universal covering space of leaves of the foliation.*

Corollary 4 *If M is a compact manifold and the foliation $\mathcal{F}$ is of codimension 1, then the fundamental group $\pi_1(M)$ of the manifold M is infinite.*

Proof If the group $\pi_1(M)$ were finite, then the universal covering space $\tilde{M}$ would be compact and homoemorphic to $\tilde{L} \times \mathbf{R}$; contradiction.□

If the foliation $\mathcal{F}$ is Riemannian and Q a supplementary foliations, then the leaves of $\mathcal{F}$ and Q intersect one another; this can be proved by a well-known method, cf. [BH2], also in our case.

Proposition 6 *Let $(M, \mathcal{F})$ be a foliated manifold with a transversely complete, transversely transitive, foliated TUSP system* $\mathbf{E}$ *of differential equations. If the normal bundle Q is integrable, then any leaf L of the foliation $\mathcal{F}$ intersect any leaf K of the foliation Q.*

III.4 The graph

Foliated manifolds with tranversely complete FSODE have another very important property: the source projection of the graph of such a foliation is a locally trivial fibre bundle. This fact is of great consequence for the C^*–algebra associated to the foliation, but this goes beyond the scope of our study. [MS] can provide the reader with more information and details about C^*–algebras associated to foliations.

The graph $GR(\mathcal{F})$ of the foliation $\mathcal{F}$ is the space of equivalence classes of triples (y, α, x) where x and y are points of the same leaf L of $\mathcal{F}$ and α is a path in L linking x to y. Two triples (y, α, x) and (y', α', x') are equivalent iff

$x = x'$, $y = y'$ and the holonomy of the curve $\alpha^{-1} * \alpha'$ is trivial. A neighbourhood of $\langle y, \alpha, x\rangle$ consists of elements represented by the triples of the form (y', α', x') where x' belongs to some neighbourhood of x is a transverse manifold passing through x, y' belongs to some neighbourhood of y is a transverse manifold passing through y and α' is the holonomy lift of α to x', cf. [WI1]. In the same paper the author proved that the graph of a foliation is a manifold of dimension $n+p$ but in general non–Hausdorff. Moreover, if the elements of the holonomy pseudogroup are determined by their jets the graph is a Hausdorff topological space.

In our case we can prove that the elements of the holonomy pseudogroup are determined by their r–jets.

Lemma 8 *Let* $\mathbf{E}$ *be a transitive USP SDE of order* k *on a manifold* N*. If* f *and* g *are two automorphisms of* $\mathbf{E}$ *such that* $j^k_x f = j^k_x g$ *for some point* x *of* N*, then they are equal in some neighbourhood of* x*.*

Proof Let h be a solution of the system $\mathbf{E}$ with the initial condition ξ at x. Then fh and gh are two solutions of $\mathbf{E}$ at $f(x)$ with the initial conditions $j^k_x f(\xi)$ and $j^k_x g(\xi)$, respectively. As $j^k_x f(\xi) = j^k_x g(\xi)$ $fh = gh$ in some neighbourhood of $h^{-1}(x)$. Then as the solutions of the system $\mathbf{E}$ cover a neighbourhood of x the germs of f and g at x must be equal.□

In this way we have obtained the following proposition.

Proposition 7 *Let* $\mathbf{E}$ *be a FSODE of order* k *on a foliated manifold* $(M, \mathcal{F})$*. If* $\mathbf{E}$ *is transversely transitive TUSP system, then the graph* $GR(\mathcal{F})$ *of the foliation* $\mathcal{F}$ *is a Hausdorff manifold of dimension* $n+p$*.*

The correspondences

$$p_1 \colon \langle y, \alpha, x\rangle \longmapsto x,$$

the source projection, and

$$p_2 \colon \langle y, \alpha, x\rangle \longmapsto y,$$

the target projection, define two submersions

$$p_1 \colon GR(\mathcal{F}) \longrightarrow M \quad \text{and} \quad p_2 \colon GR(\mathcal{F}) \longrightarrow M.$$

In local coordinates they can be written as follows. Let $\langle y, \alpha, x\rangle$ be a point of $GR(\mathcal{F})$ and (U, φ), (V, ψ) and $(U \times_\alpha V, \varphi \times_\alpha \psi)$ be adapted charts at x, y and $\langle y, \alpha, x\rangle$, respectively, where

$$\begin{aligned} U \times_\alpha V \;=\; & \{\langle y', \alpha', x'\rangle \in GR(\mathcal{F}) \colon x' \in U, \varphi(x') = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^q, \\ & y' \in V,\ \psi(y') = (y_1, y_2) \in \mathbb{R}^p \times \mathbb{R}^q,\ h_\alpha(x_2) = y_2 \\ & \text{and } \alpha' \text{ is the holonomy lift of } \alpha \text{ to } x'\} \end{aligned}$$

and

$$\varphi \times_\alpha \psi(\langle y, \alpha, x\rangle) = (x_1, x_2, x_3) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p.$$

Then

$$\varphi \circ p_1 \circ (\varphi \times_\alpha \psi)^{-1}: \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$$

$$(x_1, x_2, x_3) \longmapsto (x_1, x_2)$$

and

$$\psi \circ p_1 \circ (\varphi \times_\alpha \psi)^{-1}: \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p \longrightarrow \mathbb{R}^q \times \mathbb{R}^p$$

$$(x_1, x_2, x_3) \longmapsto (x_3, h_\alpha(x_2)).$$

On $GR(\mathcal{F})$ there are three foliations:

i) $\mathcal{F}_1$ defined by the fibres of p_1;

ii) $\mathcal{F}_2$ defined by the fibres of p_2;

iii) $\mathcal{F}_1 \oplus \mathcal{F}_2 = p_1^{-1}\mathcal{F} = p_2^{-1}\mathcal{F}$.

The fibres of the submersions p_1 and p_2 are the holonomy coverings of leaves of $\mathcal{F}$. Let Q be a subbundle of TM supplementary to $T\mathcal{F}$ and $\tilde{Q}$ be the subbundle of $TGR(\mathcal{F})$ defined as follows:

$$\tilde{Q} = \{X \in TGR(\mathcal{F}): dp_1(X) \in Q,\ dp_2(X) \in Q\}.$$

The tangent bundle of $GR(\mathcal{F})$ admits the following decomposition:

$$TGR(\mathcal{F}) = T\mathcal{F}_1 \oplus T\mathcal{F}_2 \oplus \tilde{Q}.$$

A curve γ in $GR(\mathcal{F})$ is tangent to $\tilde{Q}$ iff the curves $p_1\gamma$ and $p_2\gamma$ are tangent to Q. Moreover, the fibre bundle $\tilde{Q}$ is isomorphic to p_1^*Q and p_2^*Q, and $\tilde{Q} = p_1^{-1}Q \cap p_2^{-1}Q$.

Let $\mathbf{E}_Q$ be the transverse part of the system $\mathbf{E}$. We would like to lift this system to $GR(\mathcal{F})$. First we describe the bundle $J^k(\mathbb{R}, GR(\mathcal{F}); \tilde{Q})$ as

$$\{j^k f \in J^k(\mathbb{R}, GR(\mathcal{F})): j^k p_1 f \in J^k(\mathbb{R}, M; Q) \text{ and } j^k p_2 f \in J^k(\mathbb{R}, M; Q)\}.$$

Then the set

$$\mathbf{E}_{\tilde{Q}} = \{j^k f \in J^k(\mathbb{R}, GR(\mathcal{F})): j^k p_1 f \in \mathbf{E}_Q \text{ and } j^k p_2 f \in \mathbf{E}_Q\}$$

is isomorphic to both $p_1^*\mathbf{E}_Q$ and $p_2^*\mathbf{E}_Q$. It is a subbundle of $J^k(\mathbb{R}, GR(\mathcal{F}); \tilde{Q})$. The bundle $\mathbf{E}_{\tilde{Q}}$ is a foliated subbundle of $J^k(\mathbb{R}, GR(\mathcal{F}); \tilde{Q})$ for the foliation $\mathcal{F}_1 \oplus \mathcal{F}_2$. The bundle of initial conditions $\mathbf{E}_0^r(\tilde{Q})$, a foliated subbundle of $J^k(\mathbb{R}, GR(\mathcal{F}); \tilde{Q})$, is isomorphic to $p_1^*\mathbf{E}_0^r(Q)$. In fact a mapping $f: \mathbb{R} \to GR(\mathcal{F})$ is a solution of the system $\mathbf{E}_{\tilde{Q}}$ iff $p_1 f$ and $p_2 f$ are solutions of the system $\mathbf{E}_Q$. Therefore if the system $\mathbf{E}_Q$ is TUSP so is $\mathbf{E}_{\tilde{Q}}$. It is not so simple with transverse completeness.

Let $\tilde{\xi}$ be an r–jet from $\mathbf{E}_0^r(\tilde{Q})$ and put $\xi_1 = N^r(p_1)(\tilde{\xi})$, $\xi_2 = N^r(p_2)(\tilde{\xi})$. If $\tilde{\xi}$ is in the fibre over the point $\langle y, \alpha, x\rangle$, then $\xi_2 = N^r(T_\alpha)(\xi_1)$, or equivalently ξ_2 is the end of the lift $\tilde{\alpha}$ of the curve α to the vector ξ_1 in $\mathbf{E}_0^r(\tilde{Q})$. Let f_t be the solutions of the system $\mathbf{E}$ with the initial condition $\tilde{\alpha}(t)$. Then for any vector v of $\mathbb{R}$ the curve $\alpha_v: [0,1] \to M$, $\alpha_v(t) = f_t(v)$, $t \in [0,1]$, is a leaf curve. Moreover the mapping $\tilde{f}: \mathbb{R} \longrightarrow GR(\mathcal{F})$, $\tilde{f} = \langle f_1(v), \alpha_v, f_0(v)\rangle$, is smooth. It is a solution of the system $\mathbf{E}_{\tilde{Q}}$ with the initial condition $\tilde{\xi}$ and tangent to $\tilde{Q}$. It is a $\tilde{Q}$–horizontal lift of f relative to the both projections p_1 and p_2. Thus we have proved the following lemma.

Lemma 9 *If the system* $\mathbf{E}$ *is transversely complete and TUSP, so is the system* $\mathbf{E}_{\tilde{Q}}$. *Moreover, the transverse solutions of* $\mathbf{E}_{\tilde{Q}}$ *are* $\tilde{Q}$ *horizontal lifts of transverse solutions of the system* $\mathbf{E}$.

To show that the submersion p_1 is a locally trivial fibre bundle it is sufficient to find a family of curves in M having the following properties:

i) there exists a subbundle S of $TGR(\mathcal{F})$ transverse to the fibres of p_1 such that the curves of this family can be lifted S–horizontally to $GR(\mathcal{F})$;

ii) for any point and any parametrized family of points of M it is possible to find a family of curves linking this point to the points of the family in such a way that the resulting mapping is continuous.

Our family of curves will consists of curves whose pieces are either leaf curves or solution curves. We take $T\mathcal{F}_2 \oplus \tilde{Q}$ as the subbundle S. We have already proved that we can lift S–horizontally transverse solutions, so we can also lift solution curves. A leaf curve $\gamma: [0, s] \to M$ we can lift in the following way. Let $\langle y, \alpha, x\rangle$ be a point over $x = \gamma(0)$. For any $0 \le t \le s$ we define $\gamma_t: [0, t] \to M$ as $\gamma_t(u) = \gamma(u)$ for $0 \le u \le t \le s$. Then we put $\tilde{\gamma}(t) = \langle y, \alpha * \gamma_t^{-1}, \gamma(t)\rangle$. The curve $\tilde{\gamma}$ is the S–horizontal lift of γ.

For any point x of M there exists a neighbourhood U of this point which can be parametrized by such curves. Let $X_1, ...X_q$ be foliated sections of $\mathbf{E}_0^r(Q)$ over some neighbourhood U of x. Then there exists a neighbourhood P of x in the leaf such that the mapping $exp_X^q: \mathbb{R}^q \times P \to M$ is a diffeomorphism of some neighbourhood of $(0, x)$ on U. Therefore to any point z of U corresponds a path $\gamma_z: [0, q+1] \to M$

$$\gamma_z(s) = \begin{cases} exp_X^q(sy(z), 0) & s \in [0,1] \\ exp_X^q(y(z), t_1(z), ...t_{i-1}(z), s_i t_i(z), ...) & s \in [i, i+1] \end{cases}$$

where $(exp_X^q)^{-1}(z) = (y(z), t_1(z), ...t_q(z))$ and $s_i = s - i$ for $s \in [i, i+1]$.

Lifts of these curve gives us a trivialisation of p_1 over U. Thus we have proved the following theorem.

Theorem 2 *Let* $\mathbf{E}$ *be a transversely complete, transversely transitive, foliated TUSP SODE on a foliated manifold* $(M, \mathcal{F})$. *Then the source projection of its graph is a locally trivial fibre bundle.*

The graph $GR(\mathcal{F}) = \mathcal{G}$ is a groupoid, cf. [HA2], thus we can construct its classifying space. From our theorem results the following.

Corollary 5 *(cf. [HA2]) The mapping* i *of* M *into* $B\mathcal{G}$ *classifying* $GR(\mathcal{F})$ *as a* $\mathcal{G}$*–principal fibre bundle is homotopy equivalent to a locally trivial fibre bundle over* $B\mathcal{G}$.

H. Winkelnkemper informed the author that his results of [WI2] are also true in our case. For example:

Theorem 3 *(Winkelnkemper) Let* $(M, \mathcal{F})$ *be a foliated manifold with a transversely complete, transversely transitive, TUSP FSODE* $\mathbf{E}$. *If* M *is compact and simply connected then the universal leaf of* $\mathcal{F}$ *has at most two ends.*

III.5 Transverse completeness

We turn our attention to transverse completeness of FSODE. We give conditions on the holonomy pseudogroup and the transverse SODE $\mathbf{E}_N$ which, in particular cases, ensure that the system $\mathbf{E}_Q$ is transversely complete.

At first, we shall deal with a foliation given by a smooth action of a Lie group G on the manifold M. We assume that this action admits an almost connection, cf. [MO4], i.e. there exits a supplementary subbundle Q to $T\mathcal{F}$ which is invariant by the action of G. For such foliations the following theorem is true:

Theorem 4 *Let* $\mathcal{F}$ *be a foliation given by a smooth action of a Lie group* G *on a complete Riemannian* M *whose Atiyah–Molino class vanishes. If on its transverse manifold* N *there exists a complete USP system* $\mathbf{E}_N$ *of ODE invariant by the holonomy pseudogroup* $\mathcal{H}$, *then there exists a supplementary subbundle* Q *to* $T\mathcal{F}$ *for which the system* $\mathbf{E}_Q$ *of ODE is transversely complete.*

Proof The vanishing of the Atiyah–Molino class of the action defining the foliation $\mathcal{F}$ ensures that there exists a supplementary subbundle Q to $T\mathcal{F}$ which is invariant by this action. We shall demontrate that the system $\mathbf{E}_Q$ is transversely complete. Let $\alpha: [0, a) \to M$ be a solution of the system $\mathbf{E}_Q$, $a \in \mathbb{R}$. We have to show that we can extend it to $[0, a]$. It can be always done if the length of α is finite. Let us assume that the length of α is infinite. We can cover $\alpha([0, a))$ by a sequence of sets U_n, $n \in \mathbb{N}$, and there exists a sequence of numbers $a_n \to a$ such that $\alpha([a_n, a_{n+1}]) \in U_n$ for $n = 0, 1, \ldots$ Let us consider the solution $\alpha_0 = f_0\alpha|[a_0, a_1]$ of the system $\mathbf{E}_N$ and the corresponding sequence α_i, $i \in \mathbb{Z}$, of solutions of $\mathbf{E}_N$ from Definition 4. For any $t \in [a_n, a_{n+1}]$ there exists an element h_t of $\mathcal{H}$ such that $j^r h_t j^r_t(f_n\alpha) = j^r_t\alpha_i$, for i suitably chosen. Let us take the curve α_i whose domain contains a. Then the point $\alpha_i(a)$ belongs to some N_j and consider any point y of $f_j^{-1}(\alpha_i(a))$. There exist $\epsilon > 0$ and a solution $\tilde{\alpha}$ of $\mathbf{E}_Q$ defined on $(a-\epsilon, a+\epsilon)$, $\tilde{\alpha}(a) = y$, such that $f_j\tilde{\alpha} = \alpha_i|(a-\epsilon, a+\epsilon)$. Lemma I.1 ensures that for any $t \in (a-\epsilon, a)$ the points $\tilde{\alpha}(t)$ and $\alpha(t)$ belong to the same leaf of $\mathcal{F}$. The curves $g\tilde{\alpha}$, $g \in G$, are solutions of the system $\mathbf{E}_Q$ and they are the only solutions whose projections on N have the initial condition of the form $j^r h(j^r_t\alpha)$ at $t \in (a-\epsilon, a)$. Therefore, choosing $t \in (a-\epsilon, a) \cap [a_n, a_{n+1}]$, non-empty for n sufficiently large, we get that $\alpha|[a_n, a_{n+1}] = g\tilde{\alpha}|[a_n, a_{n+1}]$ for some $g \in G$. Hence we can extend the solution α to $[0, a+\epsilon)$; contradiction.□

Remark The foliation of Example 3 is of the type considered in Theorem 4. The transversely projectable flat connection is transversely complete only for some choices of the supplementary subbundle.

Transversely affine foliations form a very important class of foliations. We are going to give two theorems concerning transverse completeness of their transversely projectable flat connections, for the discussion of this property and its importance see Chapter VII.

Let $\mathcal{F}$ be a transversely affine foliation defined by a cocycle $\mathcal{V}$ and let $\mathcal{U}$ be a relatively compact cocycle which can be derived from $\mathcal{V}$. Then the connected components of its transverse manifolds N and N' corresponding to cocycles $\mathcal{U}$ and $\mathcal{V}$, respectively, are open subsets of $\mathbf{R}^q$. Let g be the Riemannian metric on N' induced by the immersion of N' into $\mathbf{R}^q$. The assumption of completeness of the holonomy pseudogroup has important consequences.

Theorem 5 *Let $\mathcal{F}$ be a transversely affine foliation on a compact manifold M. If for some relatively compact cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_1^k$ the holonomy pseudogroup $\mathcal{H}$ associated to it and the transverse system $\mathbf{E}_N$ are complete, then for any supplementary subbundle Q the corresponding transversely projectable connection is transversely complete.*

Proof First we are going to look at the consequences of the completeness assumption. Let x, y be two points of N and U, V be open neighbourhoods of x and y, respectively, of Definition I.5. By taking a smaller set $U_1 \subset U$ we can assume that there exists $\delta > 0$ such that for any $z \in U_1$ $B(z,\delta) = exp_z(B(0_z,\delta)) \subset U$ where $B(0_z,\delta) = \{v \in TN_z : \|v\|_g < \delta\}$. Let $h \in \mathcal{H}$ be defined on U. Then

$$h \circ exp_z|B(0_z,\delta) = exp_{h(z)} \circ d_z h|B(0_z,\delta).$$

The set V is contained in some N_i which is relatively compact in $\mathbf{R}^q$. Therefore $d_z h(B(0_z,\delta)) \subset B(0_{h(z)},\epsilon)$ for some $\epsilon > 0$. The same ϵ can be chosen for all $z \in U_1$. This means that the set $\{\|d_z h\|_g : h \in \mathcal{H}, z \in U_1, h(z) \in V\}$ is bounded.

Let us denote by $\mathbf{E}_N$ the equation of the geodesic of the flat connection on N, and let us choose a supplementary subbundle Q to $T\mathcal{F}$. The corresponding transversely projectable connection defines a FSODE $\mathbf{E}_Q$, the equation of its geodesic. We have to demonstrate that its transverse solutions, i.e. geodesics tangent to Q, are globally defined. We are going to show that it is possible to lift Q-horizontally any solution of the system $\mathbf{E}_N$. This fact coupled with the completeness of the system $\mathbf{E}_N$ ensures that the system $\mathbf{E}_Q$ is transversely complete.

Let us assume that there is a solution $\gamma: [0, s_0) \longrightarrow N$ of $\mathbf{E}_N$ which does not admit a Q-horizontal lift. Let $\tilde{\gamma}: [0, s) \longrightarrow M$ be the lift of γ which cannot be extended to $[0, s]$, $s_0 > s$. Thus the curve $\tilde{\gamma}$ must have infinite length. Let y_0 be an accumulation point of the sequence $\tilde{\gamma}(t_n)$ for some sequence of numbers $\{t_n\}$ tending to s. As M is compact such a point exists and we can assume that y_0 is the limit of the sequence $\tilde{\gamma}(t_n)$. The point y_0 belongs to some U_i, and let us denote by y the point $f_i(y_0) \in N$. Then there exists a sequence of elements h_n of the pseudogroup $\mathcal{H}$ for which $j^1 f_i(j^1_{t_n}\tilde{\gamma}) = j^1 h_n(j^1_{t_n}\gamma)$. Therefore $h_n(\gamma(t_n)) \to y$. The completeness of $\mathcal{H}$ assures that for n sufficiently large the mappings h_n are defined for $\gamma(s) = x_s$. There exist $\delta > 0$ and a curve $\xi: [s - \delta, s] \longrightarrow T_{x_s}N$ such that $exp\xi = \gamma|[s - \delta, s]$. Thus $h_n(\gamma(t)) = h_n exp(\xi(t)) = exp\,(j^1 h_n(\xi(t)))$. Therefore for a sufficiently large n we have a curve $\xi_n: t \mapsto j^1 h_n(\xi(t)) \in T_{x_n}N$ where $x_n = h_n(x_s)$. The length of these curves is bounded from above. Thus the curves $\gamma_n: t \mapsto h_n(\gamma(t))$, $t \in [s - \delta, s]$, have bounded length. This ensures that for n sufficiently large we can lift the whole curve $\gamma_n|[t_n, s]$ to any point in the fibre $f_i^{-1}(h_n(\gamma(t_n)))$, in particular to $\tilde{\gamma}(t_n)$. Contradiction.□

Let $\mathbf{E}_q$ be a SODE on $\mathbb{R}^q$ and $Aut(\mathbf{E}_q)$ the group of its global automorphisms. Let us consider an $(\mathbb{R}^q, Aut(\mathbf{E}_q))$-foliation $\mathcal{F}$ given by a locally free action of the group $\mathbb{R}^p$ on a manifold M and assume that the universal covering $\tilde{M}$ of M is the product $\mathbb{R}^p \times \mathbb{R}^q$ with the lifted foliation $\tilde{\mathcal{F}}$ given by the projection onto the second factor. We would like to apply the theory of differential inequalities developed in [SZ].

The tangent bundle to $\mathcal{F}$ is trivial, thus we can take 1-forms $\omega^1, \ldots, \omega^p$ on $\mathbb{R}^p \times \mathbb{R}^q$ which are invariant by the deck transformations and which at each point define a basis of the cotangent bundle of the lifted foliation. The forms ω^i can be represented as follows:

$$\omega^i = \sum_{j=1}^{p} a^i_j(x, y)dx^j + \sum_{k=1}^{q} b^i_k(x, y)dy^k$$

where $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ and $det(a^i_j) \neq 0$, $i, j = 1, \ldots, p$.

Thus for any deck transformation ϕ the form $\phi^*\omega^i$ is the following:

$$\begin{aligned}\phi^*\omega^i_{(x,y)} &= \textstyle\sum_{j=1,k=1}^{p}(a^i_j(\phi(x, y))\partial\phi^j/\partial x_k)dx^k \\ &+ \textstyle\sum_{j=1}^{p}\sum_{k,s=1}^{q}(a^i_j(\phi(x, y))\partial\phi^j/\partial y_k + b^i_s(\phi(x, y))\partial\phi^s/\partial y_k)dy^k\end{aligned}$$

as $\partial\phi^k/\partial x_j = 0$ for $k = 1, \dots, q$, $j = 1, \dots, p$, since ϕ preserves $\tilde{\mathcal{F}}$.

Since the forms ω^i are ϕ–invariant we get

$$\text{(III.1)} \qquad \begin{cases} a^i_j(\phi(x,y)) &= \sum_{s=1}^p a^i_s(x,y)\partial(\phi^{-1})^s/\partial x_j(\phi(x,y)), \\ b^i_k(\phi(x,y)) &= \sum_{s=1}^q b^i_s(x,y)\partial(\phi^{-1})^s/\partial y_k(\phi(x,y)) \\ & \quad + \sum_{j=1}^p a^i_j(x,y)\partial(\phi^{-1})^j/\partial y_k(\phi(x,y)). \end{cases}$$

Let $\gamma_0\colon \mathbb{R} \to \mathbb{R}^q$ be a solution of the system $\mathbf{E}_q$. The solution $\tilde{\gamma}$ of the system $\mathbf{E}_Q$ on $\tilde{M}$ corresponding to γ is a curve $\tilde{\gamma}\colon \mathbb{R} \to \mathbb{R}^p \times \mathbb{R}^q$, $\tilde{\gamma} = (\gamma, \gamma_0)$. The tangent field along $\tilde{\gamma}$ must be annihilated by ω^i, $i = 1, \dots, p$; thus

$$\omega^i(\gamma(t)) = \sum_{j=1}^p a^i_j(\gamma(t), \gamma_0(t)) d\gamma^j/dt(t) + \sum_{k=1}^q b^i_k(\gamma(t), \gamma_0(t)) d\gamma_0^k/dt(t) = 0.$$

Hence the curve γ is a solution of the following equation:

$$\text{(III.2)} \qquad d\gamma^j/dt = -\sum \hat{a}^j_i b^i_k(\gamma, \gamma_0) d\gamma_0^k/dt$$

where $(\hat{a}^j_i)$ is the inverse matrix of (a^j_i).

Since the manifold M is compact, there exists a compact subset K of $\mathbb{R}^p \times \mathbb{R}^q$ such that $\bigcup_{\phi\in D} \phi(K) = \mathbb{R}^p \times \mathbb{R}^q$ where D is the group of deck transformations. Combining (1) and (2) we get the following:

$$\begin{aligned} d\gamma^j/dt = &-\sum \partial\phi^j/\partial x_k \hat{a}^k_i a^i_l(\phi^{-1}(\gamma, \gamma_0))\partial(\phi^{-1})^l/\partial y_s(\gamma, \gamma_0) d\gamma_0^s/dt \\ &-\sum \partial\phi/\partial x_k \hat{a}^k_i b^i_w(\phi^{-1}(\gamma, \gamma_0))\partial(\phi^{-1})^w/\partial y_s(\gamma, \gamma_0) d\gamma_0^s/dt. \end{aligned}$$

Hence

$$\text{(III.3)} \qquad \begin{aligned} d\gamma^j/dt = &-\sum \partial\phi^j/\partial x_k \phi^{-1} \partial(\phi^{-1})^k/\partial y_s(\gamma, \gamma_0) d\gamma_0^s/dt \\ &-\sum \partial\phi^j/\partial x_k \phi^{-1} \hat{a}^k_i \phi^{-1} b^i_w \phi^{-1} \partial(\phi^{-1})^w/\partial y_s(\gamma, \gamma_0) d\gamma_0^s/dt. \end{aligned}$$

We would like to find a comparison system for the system (2). On the compact set $\phi(K)$, $\phi \in D$, we have the following estimation:

$$\begin{aligned} |d\gamma^j/dt| \le &\sum |\partial\phi^j/\partial x_k \phi^{-1}||\partial(\phi^{-1})^k/\partial y_s| \, |d\gamma_0^s/dt| \\ &+ \sum |\partial\phi^j/\partial x_k \phi^{-1}| \, |\hat{a}^k_i \phi^{-1}| \, |b^i_w \phi^{-1}| \, |\partial(\phi^{-1})^w/\partial y_s| \, |d\gamma_0^s/dt| \\ \le &\sum |\partial\phi^j/\partial x_k \phi^{-1}| \, |\partial(\phi^{-1})^k/\partial y_s| \, |d\gamma_0^s/dt| \\ &+ \sum pC^2 |\partial\phi^j/\partial x_k \phi^{-1}| \, |\partial(\phi^{-1})^w/\partial y_s| \, |d\gamma_0^s/dt| \end{aligned}$$

where C is an upper bound on K of coefficients $\hat{a}^i_j$ and b^i_j.

About the partial derivatives of ϕ ($\in D$) we know that:

i) $\partial\phi^k/\partial y_s$ depends only on the variable y for $k, s = 1, \dots q$;

ii) $\partial\phi^k/\partial y_s \equiv 0$ for $k = 1, \ldots p$, $s = 1, \ldots q$ if the transverse bundle is integrable;

iii) $\partial\phi^i/\partial x_s = \delta^i_s$, $i, s = 1, \ldots p$ for a suitable choice of adapted charts.

If M is compact, the group D of deck transformations is finitely generated. Let $\{\phi_1, \ldots, \phi_k\}$ be a symmetric set of generators. If ϕ_i have bounded partial derivatives, we get a following comparison system for the norm:

$$dr/dt = ce^{g(t,r)}$$

where g is a smooth function and $c \in \mathbb{R}$. However, even in a very simple case: $g(t,r) = t + r$, this system does not have global solutions. It is clear from the comparisons theorems that if we can find a comparison system depending only on the independent variable t, then the system (2) must have global solutions. Thus from the above considerations we can deduce the following:

Theorem 6 *Let $\mathcal{F}$ be an $(\mathbb{R}^q, Diff(\mathbb{R}^q))$-foliation given by a locally free action of the group $\mathbb{R}^p$ on a manifold M. If the foliation $\mathcal{F}$ admits a transversely transitive, TUSP system* **E** *of ODE foliated and transversely complete for an integrable subbundle, then the system $\mathbf{E}_Q$ is transversely complete for any supplementary subbundle Q.*

Proof Since the system **E** is transversely complete for an interable subbundle, the universal covering $\tilde{M}$ of M is the product $\mathbb{R}^p \times \mathbb{R}^q$ and the lifted foliation is given by the projection onto the second factor. Moreover, in this case we can find a comparison system which depends only on the variable t. Then the comparison theorems ensure that solutions γ of (2) have the same domain as γ_0. Since the transverse solutions of **E** are global the system $\mathbf{E}_q$ must have global solutions, so the system $\mathbf{E}_Q$ is transversely complete.□

Corollary 6 *Let $\mathcal{F}$ be a transversely affine foliation defined by a locally free action of $\mathbb{R}^p$. If $\mathcal{F}$ is transversely geodesically complete for an integrable supplementary subbundle, so is it for any supplementary subbundle.*

III.6 Transversely parallelisable systems

In this section we are going to consider a special class of foliated systems of differential equations called transversely parallelisable. Foliations admitting such systems have properties similar to transversely parallelisable (TP) foliations.

A foliated system **E** is transversely parallelisable if there exist foliated sections $X_i: M \longrightarrow \mathrm{E}^r_0(Q)$, $i = 1, ...q$, such that the vector fields $\pi^r X_i, i = 1, ...q$, form a transverse parallelism of the foliation $\mathcal{F}$.

A foliated system $\mathbf{E}$ is locally transversely parallelisable if there exist foliated sections $X_i, i = 1, ...m$, of the bundle of transverse initial conditions such that the vector fields $\pi^r X_i, i = 1, ...m$, span the vector bundle Q. Although we restrict our attention to transversely parallelisable systems, all the results of this paper are true for locally transversely parallelisable ones.

The interesting case is that of a non–compact manifold as in the compact case we have Molino's structure theorem, cf. [MO5,MO11].

Any foliated field X of initial condition, (a foliated section of $\mathbf{E}^r_0(Q)$), defines, for any $t \in \mathbb{R}$, a smooth foliated mapping $exp_{X,t}: M \longrightarrow M$. Since our foliation is without holonomy, the restriction of this mapping to any leaf is a diffeomorphism onto the image which is also a leaf of the foliation $\mathcal{F}$.

Now we shall study the semigroup $Map^\infty(M, \mathcal{F})$ of global smooth mappings of M preserving the foliation. For transversely complete and transversely parallelisable systems the semigroup $Map^\infty(M, \mathcal{F})$ is transversely transitive in the following sense. For any two leaves L and L' of the foliation $\mathcal{F}$ there exist a sequence of leaves $L_0, ...L_{k+1}, L_0 = L$ and $L' = L_{k+1}$, and a sequence of mappings $f_0, ...f_k$ of $Map^\infty(M, \mathcal{F})$ such that $f_{2i}(L_{2i}) = L_{2i+1}$ and $f_{2i-1}(L_{2i}) = L_{2i-1}, i = 1, ...[k/2]$. This results from the fact that any two leaves can be joined by a piecewise solution curve whose pieces are solution curves with initial conditions given by the fields X_i and that the system is transversely complete.

Lemma 10 *If the system* $\mathbf{E}$ *is transversely parallelisable and transversely complete then the semigroup* $Map^\infty(M, \mathcal{F})$ *is transversely transitive.*

If the subbundle Q of the definition of a foliated system is integrable, we can consider the semigroup $Map^\infty(M, \mathcal{F}; Q)$ of mappings preserving both foliations. Lemma 10 is also true for this semigroup, or for the semigroup $Map^\infty_D(M, \mathcal{F})$ whose elements are additionally diffeomorphisms when restricted to any leaf.

For further considerations we need the following lemma whose proof is classical.

Lemma 11 *Let* $F: W \times M \rightarrow M$ *be a smooth mapping. Denote by* $F_w: M \rightarrow M$ *the mapping* $F_w(x) = F(w, x)$. *If for some point* (w_0, x_0) *there exists a neighbourhood* V_0 *of* x_0 *such that the mapping* $F_{w_0}|V_0$ *is a diffeomorphism onto the image, then there exist neighbourhoods* W_0 *of* w_0 *and* V *of* x_0 *having the property that for any* w *of* W_0 *the mapping* $F_w|V$ *is a diffeomorphism onto the image.*

Let us consider the space $C^\infty(M, \mathcal{F})$ of basic functions. The mapping $rank_{\mathcal{F}}$ assigning to each point x of the manifold M the dimension of the space

$$\{d_x f: f \in C^\infty(M, \mathcal{F})\}$$

is semi–continuous from above. Moreover, for any mapping ϕ of $Map^\infty(M, \mathcal{F})$ and basic function f, the function $f\phi$ is basic. Therefore Lemma 11 ensures that the function $rank_{\mathcal{F}}$ is also semi–continuous from below, and hence it is a continuous function. As it takes only integer values, it is a constant function. The space of vectors annihilated by basic functions

$$\mathcal{F}_b = \{X \in TM: df(X) = 0, f \in C^\infty(M, \mathcal{F})\}$$

is an involutive vector subbundle of TM of constant rank. It defines a foliation $\mathcal{F}_b$, called the basic foliation, whose leaves contain leaves of the foliation $\mathcal{F}$.

We shall investigate foliations with a closed leaf on non–compact manifolds admitting a transversely complete, transversely parallelisable TUSP system of differential equations.

For complete TP foliations the existence of a closed leaf ensures that all leaves are closed and that the natural projection onto the leaf space is a locally trivial fibre bundle. In our case it is not so simple. The main reason is that the mappings $exp_{X,t}$, although globally defined, are not diffeomorphisms as it is in the case of vector fields.

First of all we are going to consider the mapping exp^q_X defined by $\{X_1, ...X_q\}$ in a neighbourhood of a closed leaf. We would like to show that such a leaf has a basis of open saturated neighbourhoods as it is in the case of complete TP foliations. In general, it seems not to be true. One has to impose the following condition, which links the length of a solution curve with the initial condition and parametrization. We assume that the manifold M is a Riemannian manifold.

1) Let $\gamma: [0, a] \longrightarrow M$ be a solution curve given by $\gamma(t) = Exp(\xi, t)$ for some r–vector ξ. Then the length of γ is smaller than $cf(\|\xi\|, a)$, where $c \in \mathbb{R}^+$ and $f: \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function non–decreasing in each variable with $f(., 0) = 0$

The condition (1) is a natural one. It is verified by the equation of the geodesic of a Riemannian metric and on compact manifolds, any linear SODE satisfies it.

Lemma 12 *Let E be a system of differential equations satisfying the condition (1) and there exists an open saturated neighbourhood of L on which the norms of the sections X_i are bounded. Then is a neighbourhood W of 0 in $\mathbb{R}^q$ for which the mapping $exp^q_X|W \times L$ is a diffeomorphism onto the image.*

Proof The proof is essentially the same as in [MO5]. The main difficulty is the fact that for some neighbourhood W of 0 the mapping $exp^q_X|W \times L$ is only a local diffeomorphism, and, à priori, the leaves can meet any $exp^q_X(W \times \{x\})$ at more than one point. Having demonstrated this fact we proceed as in [MO5] with only minor changes which the reader can easily do.

For some small open neighbourhood W_0 of 0 the mapping $exp^q_X|W_0 \times L$ is a local diffeomorphism. Let us assume that for any open neighbourhood W contained in W_0 there exists a leaf L' which meets the transverse manifold $exp^q_X(W \times \{x_0\})$,

$x_0 \in L$, at more than one point. Let us take a sequence of balls $W_n = B(0, 1/n)$ in $\mathbb{R}^q$. For each n there exists a leaf L_n such that $L_n \cap exp^q_X(W_n \times \{x_0\}) = \{x^n_1, x^n_2, ...\}$. Thus there are $t^n_1, t^n_2 \in W_n$ for which $exp^q_X(t^n_1, x_0) = x^n_1$ and $exp^q_X(t^n_2, x_0) = x^n_2$. As the mapping exp^q_X restricted to any leaf is a diffeomorphism, we can find a point x^n of the leaf L such that $exp^q_X(t^n_1, x^n) = x^n_2$. Since $t^n_1 \to 0$ and $t^n_2 \to 0$ with $n \to \infty$, the sequence of points x^n of the leaf L has x_0 as its limit. As L is closed there exists an adapted chart (U, φ) at x_0 meeting L at one plaque P only such that for n sufficiently large $x^n \in U$. By taking a smaller U we can assume that $exp^q_X|V \times P$ is a diffeomorphism on U for some neighbourhood V of 0 in $\mathbb{R}^q$. Contradiction.□

Example 5 Let us consider the foliation on $\mathbb{R}^2 \setminus \{(0,0)\}$ defined by a global vector field with one hyberbolic singularity at the origin. It convinces us that in order to have the space of leaves Hausdorff we have to assume much more than in Lemma 12.

Lemma 13 *Let* **E** *be a transversely complete system of differential equations satisfying the condition (1). Assume that all leaves of $\mathcal{F}$ are closed. If for a leaf L the following conditions are satisfied:*

i) *there exists an open neighbourhood W of 0 in $\mathbb{R}^q$ such that the set of mappings $W \ni w \longmapsto exp_X(w, x)$, $x \in L$, is equicontinuous,*

ii) *there exists an open saturated neighbourhood of L on which the norms of the sections X_i are bounded,*

then the leaf L has a basis of closed saturated neighbourhoods.

Proof Lemma 12 ensures that for some neighbourhood W of 0 in $\mathbb{R}^q$ the mapping $exp^q_X|W \times L$ is a diffeomorphism on the image. Taking a decreasing sequence of neighbourhoods W_n of 0 in $\mathbb{R}^q$, $\cap W_n = \{0\}$ we obtain a sequence of foliated neighbourhoods $\tilde{W}_n = exp^q_X(W_n \times L)$ of L with the intersection equal to L. Therefore the proof would be complete if we manage to show that for a sufficiently small closed neighbourhood W of 0 the set $exp^q_X(W \times L)$ is closed in M.

Let $\{y_n\} \subset exp^q_X(W \times L)$ be a sequence of points with the limit $y \in M$. We must show that $y \in \tilde{W}$. Let us take W equal to a small closed ball $B(0, r)$, then $y_n = exp^q_X(t_n, x_n)$ where $t_n \in B(0, r)$ and $x_n \in L$. By taking a subsequence we can assume that $t_n \to t_0$. The points $x^0_n = exp^q_X(t_0, x_n)$ form a sequence of points of the leaf $exp^q_X(t_0, L) = L_0$. The assumptions i) and ii) ensure that as $n \to \infty$, the distance between y_n and x^0_n tends to 0, thus the point y is also the limit of $\{x^0_n\}$. Since the leaf L_0 is closed, $y \in L_0$, and it also belongs to the set $\tilde{W}$.□

Lemma 13 leads us to the formulation of a proposition for which we need the following definition.

Definition 6 *A transversely parallelisable system* $\mathbf{E}$ *is called locally bounded if for any leaf there exists an open saturated neighbourhood of this leaf on which the sections of the transverse parallelism of* $\mathbf{E}$ *have bounded norm.*

Proposition 8 *Let* $\mathbf{E}$ *be a transversely parallelisable, transversely complete, foliated TUSP system of differential equations on a foliated manifold* $(M, \mathcal{F})$. *If the following conditions are satisfied:*

i) *all leaves of the foliation* $\mathcal{F}$ *are closed,*

ii) *the system* $\mathbf{E}$ *has the property (1),*

iii) *(condition (E)) for any leaf* L *of* $\mathcal{F}$ *there exists an open neighbourhood* W *of* 0 *in* $\mathbb{R}^q$ *such that the set of mappings* $W \ni w \longmapsto exp_X(w, x)$, $x \in L$, *is equicontinuous,*

iv) *the system* $\mathbf{E}$ *is locally bounded,*

then the space of leaves is a Hausdorff manifold and the natural projection onto the space of leaves is a locally trivial fibre bundle.

Proof According to [PA1] the space of leaves is a T_1–manifold and Lemma 13 ensures that it is a Hausdorff manifold. Theorem 1 tells us that the natural projection onto the space of leaves is a locally trivial fibre bundle.□

In Proposition 8 we have assumed from the very beginning that all leaves are closed. It is quite a strong assumption. We would like to weaken it in the following sense. We want to find some conditions which ensure that if one leaf is closed, then all leaves are closed. For example, it is easy to verify that if one leaf is compact, then all leaves are compact. As the foliation is without holonomy the assertion of Proposition 8 is classical. Moreover, the hypotheses of this proposition are satisfied in this case. If the subbundle Q is integrable we can obtain a much stronger result.

Lemma 14 *If the supplementary subbundle* Q *is integrable and has a compact leaf, then the foliation* $\mathcal{F}$ *has a closed leaf iff all leaves of* $\mathcal{F}$ *are closed.*

Proof Let L be a closed leaf and L' any leaf of $\mathcal{F}$. Under the hypothesis of Lemma 14 the semigroup $Map^\infty_D(M, \mathcal{F}; Q)$ is transversely transitive. Thus there exist a sequence of leaves $L_0, \ldots L_{k+1}$, $L_0 = L$ and $L_{k+1} = L'$ and a sequence of mappings $f_0, \ldots f_k$ of $Map^\infty_D(M, \mathcal{F}; Q)$ such that $f_{2i}(L_{2i}) = L_{2i+1}$ and $f_{2i+1}(L_{2(i+1)}) = L_{2i+1}$. Therefore to prove that L' is closed it is sufficient to show that

1. if L is a closed leaf, so is $f(L)$ for any $f \in Map^\infty_D(M, \mathcal{F}; Q)$,

2. if for some mapping $f \in Map^{\infty}_{\mathcal{D}}(M, \mathcal{F}; Q)$ $f(L)$ is closed, so is L.

Let K be a compact leaf of Q. Proposition 6 ensures that the leaf K is a complete transverse manifold of $\mathcal{F}$. Therefore a leaf L of $\mathcal{F}$ is closed iff its intersection with K is finite. One can easily check that as the mapping f preserves the foliation Q, $f(K \cap L) = f(K) \cap f(L) = f(L) \cap K$. Moreover, since f restricted to any leaf of $\mathcal{F}$ is bijective, the set $f(L) \cap K$ is finite iff $L \cap K$ is finite.

This lemma allows us to formulate the following corollary to Proposition 8.

Corollary 7 *Let* E *be a system of differential equations satisfying the conditions ii) –iv) of Proposition 8. Assume that the subbundle Q is integrable and the foliation Q has a compact leaf. Then if the foliation $\mathcal{F}$ has a closed leaf, all leaves of $\mathcal{F}$ are closed and the natural projection onto the space of leaves of $\mathcal{F}$ is a locally trivial fibre bundle.*

Now we shall present a condition which together with (1) will ensure that the union A of points of closed leaves is an open–closed subset of the manifold. Since we have assumed that the manifold is connected the set A is either the whole manifold or empty.

2) There exist an open covering $\mathcal{U}$ of the manifold M by adapted relatively compact charts $\{(U_i, \psi_i)\}$ and a number $\delta > 0$ such that for any plaque P of the covering $\mathcal{U}$ the set $P(\delta) = \{y \in M: d(y, P) > \delta\}$ is contained in some relatively compact adapted chart.

Let us assume that the conditions (1) and (2) are satisfied and that the norms of the sections X_i are locally bounded. First of all we shall prove that the set A is closed. The closure $\bar{A}$ of A is a saturated set and let L be a leaf of $\bar{A} \setminus A$. Then there exist a curve $\gamma: [0, a] \longrightarrow M, a > 0$, starting at a point of L transverse to the foliation, and a sequence of numbers $t_n \to 0$ such that the leaf L_n passing through the point $\gamma(t_n)$ is a closed one. There exists a leaf L_0, different from L, contained in $\bar{L}$. Let $U_1 \subset U_2$ be the pair of chart domains at a point x of L_0 satisfying the condition (2). Since the leaf L_0 is in the closure of L, there is an infinite number of plaques P_m of U_1 belonging to L. For n sufficiently large there is an infinite number of points y_m, $y_m = exp^q_{X,t_n}(x_m)$, $x_m \in P_m$, of the leaf L_n at the distance not greater than δ from the point x. As the leaf L_n is closed, it meets U_2 at a finite number of plaques, and one of the plaques contains an infinite number of points of $\{y_m\}$. Since the plaques are relatively compact these points are contained is a compact set K. However, the points $x_m = (exp^q_{X,t_n}|L)^{-1}(y_m)$ belong to different plaques of U_2 and the image of K by $(exp^q_{X,t_n}|L)^{-1}$ is not compact. Contradiction.

Now, we shall demonstrate that the set A is open. Actually, we shall prove that if L is a closed leaf, then for a sufficiently small neighbourhood W of 0 in $\mathbf{R}^q$

the set $exp^q_X(W \times L)$ contains only closed leaves. Namely, let us take W so small that for any $t \in W$ the curve $s \longmapsto exp_{X_i, s-t^{i-1}}(x_{i-1})$ where $s \in [t^{i-1}, t^i], t^i = \sum_{j \leq i} t_j$, $t = (t_1, ...t_q)$, $x_i = exp_{X_i,t_i} \circ ... \circ exp_{X_1,t_1}(x)$, $x \in L$, is of the length smaller than $\delta/2$.

Let us assume to the contrary that the leaf $L' = exp^q_{X,t_0}(L)$, $t_0 \in W$, is not closed. Thus there exists a leaf $L_0 \subset \bar{L}' \setminus L'$. Let us choose a point x_0 of L_0 and an adapted chart (U_1, ψ_1) at x_0 satisfying the condition (2). Then the leaf L' meets U_1 at an infinite number of plaques, say P_m. Using a similar argument as previously, this time for the points of the leaf L we obtain a contradiction.

We can summarize our considerations in the follwing theorem.

Theorem 7 *Let $(M, \mathcal{F})$ be a foliated connected manifold admitting a transversely parallelisable, locally bounded, transversely complete, TUSP system of differential equations. If the foliation $\mathcal{F}$ has a closed leaf and the conditions (1),(2) and (E) are satisfied, then all leaves of $\mathcal{F}$ are closed and the natural projection onto the space of leaves is a locally trivial fibre bundle.*

In attemps to prove a structure theorem along similar lines as for complete TP foliations one meets several difficulties. Since global foliated vector fields of the foliation $\mathcal{F}$ are also foliated for the basic foliation, their flows map leaves of the basic foliation onto themselves. It is not so in the case of equations. In general, we cannot prove that the image of a leaf of the basic foliation by $exp_{X,t}$ is a leaf of this foliation. It is a consequence of the fact that we do not know whether the system is foliated for the basic foliation. In our study we would like to point out when it is, in fact, true.

The projections of the section X_i, $i = 1, ...q$, define a transverse parallelism $\tilde{X}_i$, $i = 1, ...q$. Thus the normal bundle $N(M, \mathcal{F})$, isomorphic to Q, is a trivial bundle, i.e. $N(M, \mathcal{F}) \cong M \times \mathbb{R}^q$ and the natural foliation of $N(M, \mathcal{F})$ goes to the product foliation of $\mathcal{F}$ with the foliation by points of $\mathbb{R}^q$. This induces a global trivialisation of the fibre bundle $J^k(\mathbb{R}, M; \mathcal{F}) = J^k(\mathbb{R}, Q)$, i.e. $J^k(\mathbb{R}, M; \mathcal{F}) \cong \mathbb{R} \times M \times J^k_0(\mathbb{R}, \mathbb{R}^q)$. Namely, to $[f]_t \in J^k(\mathbb{R}, M; \mathcal{F})$ corresponds the point $(t, f(t), j^{k-1}_t \xi)$ where $\frac{df}{dt} = \sum_{i=1}^q \xi^i \tilde{X}_i$. Then the natural foliation $\mathcal{F}_k$ is isomorphic to the product foliation of $\mathcal{F}$ and the foliation by points. Therefore the subbundle $\mathbf{E}_Q$ of $J^k(\mathbb{R}, Q)$ corresponds to a field of submanifolds of $J^k_0(\mathbb{R}, \mathbb{R}^q)$, all of them being diffeomorphic.

Let us consider a system of linear differential equations of constant rank, i.e. the subbundle $\mathbf{E}_Q$ is a vector subbundle of $J^k(\mathbb{R}, Q)$ of constant rank m. To such a subbundle corresponds a field of vector subspaces of $J^k_0(\mathbb{R}, \mathbb{R}^q) = V$ of constant dimension m. Thus the subbundle $\mathbf{E}_Q$ is given by a smooth mapping f_E of $\mathbb{R} \times M$ into the Grassmann manifold $GR(V)$. The property that the bundle $\mathbf{E}_Q$ is foliated translates itself into the fact that f_E is constant on leaves of the foliation $\mathcal{F}$. Thus the mapping f_E is constant on leaves of the basic foliation as well.

The vector fields $\tilde{X}_i$, $i = 1, ...q$, define a bundle–like metric g on the normal bundle, i.e. $g(\tilde{X}_i, \tilde{X}_j) = \delta^i_j$. We denote the orthogonal complement of $T\mathcal{F}_b$ by Q_0. The bundle $J^k(\mathsf{R}, Q_0)$ can be considered as a foliated vector subbundle of $J^k(\mathsf{R}, Q)$. Thus it can be represented as a smooth mapping

$$f^0_Q: M \times \mathsf{R} \longrightarrow GR_p(V),$$

$p =$ dim $J^k(\mathsf{R}, Q_0) = V_0$, constant on leaves of $\mathcal{F}$, and, therefore on leaves of the basic foliation as well. For this reason the mapping

$$\mathsf{R} \times M \ni x \longmapsto f_E(x) \cap f^0_Q(x) \in \bigcup_{r \leq m} GR_r(V_0)$$

is constant on leaves of $\mathcal{F}_b$. This mapping defines a system of differential equations on Q_0, to be precise, it describes the system $\mathbf{E} \cap J^k(\mathsf{R}, Q_0) = \mathbf{E}_Q \cap J^k(\mathsf{R}, Q_0)$. Locally, the intersection of $\mathbf{E}_Q$ with $J^k(\mathsf{R}, Q_0)$, to the system $A_\alpha(x, \xi) = 0$, $\alpha = 1, ..v$, adds the system $\sum_i c^i_j \xi_i = 0, j = 1, ..v_1$, the condition of orthogonality to $T\mathcal{F}_b$ of $\xi_1, ...\xi_q$ and its prolongations up to order $k - 1$. As the functions A_α and c_i are constant on the leaves of $\mathcal{F}_b$, the system thus obtained is foliated for the basic foliation. It is not difficult to verify that although $\mathbf{E}_{Q_0}$ is not a subbundle of $J^k(\mathsf{R}, Q_0)$, it is a transversely complete, transversely transitive, TUSP foliated system on $(M, \mathcal{F}_b)$. Since the orthogonal projections of the vector fields $\tilde{X}_1, ...\tilde{X}_q$ define a local transverse parallelism of the basic foliation, the corresponding sections of the bundle $\mathbf{E}^r_0(Q_0) = \mathbf{E}^r_0(Q) \cap Q^r_0$ ensure that the system $\mathbf{E}_{Q_0}$ is a locally transversely parallelisable system on the foliated manifold $(M, \mathcal{F}_b)$. Therefore we have the following theorem.

Theorem 8 *Let* $\mathbf{E}$ *be a transversely parallelisable, transversely complete, foliated TUSP linear system of differential equations of constant rank on a connected foliated manifold* $(M, \mathcal{F})$. *If the conditions (1) and (E) are satisfied and the system is locally bounded then the closures of leaves are fibres of a locally trivial fibre bundle. The foliation induced on each fibre is a Lie foliation.*

Proof The previous considerations ensure that we can apply Proposition 8, which asserts that the leaves of the basic foliation are fibres of a locally trivial fibre bundle. The proof that the closures of leaves are leaves of the basic foliation and that of the last statement is made in the same way as for TP foliations, cf. [MO5,MO11].

Let us look at foliated equations from the point of view of holonomy pseudogroups. The fact that the system $\mathbf{E}$ is locally bounded means that the holonomy pseudogroup is of compact type. The condition (1) and local boundedness together with the assumption that the system is transversely complete ensure that the holonomy pseudogroup is a complete pseudogroup. The fact that this pseudogroup preserves a parallelism of the transverse manifold allows us to apply the theory developed in [WO15], cf. also Section VI.2. Thus we have the following.

Theorem 9 *Let* **E** *be a transversely parallelisable, transversely complete, foliated TUSP system of differential equations on a foliated manifold* $(M, \mathcal{F})$. *If the system* **E** *is locally bounded and the condition (1) is satisfied then the closures of leaves of the foliation* $\mathcal{F}$ *are submanifolds of* M.

At the end we formulate a technical lemma which can be useful in the study of flows admitting transverse systems of differential equations.

Lemma 15 *Let* Ψ *be a dense flow on a connected manifold* M *of dimension greater that 1. If* Ψ *admits a transversely parallelisable, transversely complete, foliated TUSP system* **E** *of differential equations which is locally bounded and the condition (1) is satified then the manifold* M *is diffeomorphic to the torus* T^k, $k > 1$.

Proof The tools which we have developed allow us to follow the method of Ghys and Molino, cf. [GH2,MO10].□

Remarks In our considerations we did not assume anything about the Riemannian structure of the manifold. The nontrivial case, not covered by P.Molino's results, cf. [MO5,MO11], is that of a non–compact manifold, or even non–complete Riemannian manifold. If the Riemannian manifold is complete the assumption of only local boundedness only ensures that Theorem 7 is true, cf. Section IV.2 for a discussion and a proof of a very similar fact. For the other results completeness is not sufficient.

The condition (1) is quite a natural one. It means that at infinity solutions does not behave 'wildly', i.e. the norm of the tangent vector to them does not grow too steeply.

The condition (E) is very important. It is a 'foliated' condition, saying that the exponential mapping associated to the system behaves topologically in the same way along any chosen leaf.

'Local boundedness' is a technical assumption which together with the condition (1) allows us to estimate the length of solutions emanating from the leaf.

All these conditions together with the completeness of the system of differential equations, in fact, replace the assumption that the bundle–like Riemannian metric defined by the transverse parallelism is transversely complete. In the TP case the transverse completeness of this Riemannian metric, or of the corresponding equation of the geodesic, is equivalent to the fact that the vector fields of the transverse parallelism are complete. In this case the condition (1) and (E) are satisfied. Moreover, the vector fields are of constant norm.

The condition (2) is purely topological and very well–known; essentially saying that the 'diameter' of charts is bounded away from 0 at infinity. It is a purely technical assumption.

We can meet the equations considered in this section in the following context. Let us consider a non–compact parallelisable manifold with vector fields $X_1, \ldots X_n$

of the parallelism being non–complete. Then assume that on this manifold we have a complete linear connection, ([FR2] provides an interesting condition ensuring the completeness of a flat connection). Then any global infinitesimal affine transformation X such that $[X, X_i] = f_i X$ for $i = 1, \dots n$ defines a foliation we are interested in. More generally, we can take any system of ordinary differential equations on a parallelisable manifold, and look for an infinitesimal symmetry X of this system for which $[X, X_i] = f_i X$ for $i = 1, \dots n$. Any such infinitesimal symmetry defines a foliation of the required type, cf. [OL,OV]. It is not, usually, too difficult to find such a infinitesimal symmetry as the conditions (equations) we have to solve are linear ones. Then we have to check whether our conditions are satisfied, but in any particular case this should not be too tedious.

Notes The chapter contains results of four papers. Sections 1 and 2 are based on [WO8] and [WO14]. Section 3 presents facts published in [WO8]. The results of Section 4 correspond to [WO10]. Section 5 contains the rest of the paper [WO14] and Section 6 is based on [WO9].

Chapter IV

G–foliations

In this chapter we are going to study G–foliations, or equivalently foliations admitting a foliated G–structure. We start by demonstrating some general properties of these structures. Then we turn our attention to G–foliations with the group G of finite type. The chapter is completed with a review of some results of R. A. Blumenthal. We provide a 'proof scheme' based on our considerations from Chapter II. In this way we simplify the proofs and get some interesting generalizations of these results.

The theory of foliated G–structures has been developed by P. Molino in [MO1,MO11]. The characteristic classes of G–foliations have been studied by F. W. Kamber, Ph. Tondeur and Th. Duchamp, cf. [DU,KT2].

IV.1 Preliminaries

First we recall some basic definitions:

Definition 1 i) *A transverse G–structure $B(M, G; \mathcal{F})$ is a G–reduction of the $GL(q)$–principal bundle $L(M, \mathcal{F})$.*

ii) *A foliated G–structure $B(M, G; \mathcal{F})$ is a G–reduction of $L(M, \mathcal{F})$ whose total space is $\mathcal{F}_1$–saturated (i.e. with a point it contains the whole leaf of $\mathcal{F}_1$ passing through this point).*

Definition 2 i) *A connection in a transverse G–structure is called a transverse connection.*

ii) *A connection in a foliated G–structure is called:*

a) *basic if its connection form vanishes on vectors tangent to the foliation $\mathcal{F}_1$;*

b) *foliated (or transversely projectable) if its connection form is base–like.*

In Chapter II we have already mentioned the following fact which is due to P. Molino, [MO1,MO11] .

Lemma 1 *A foliation $\mathcal{F}$ admits a foliated G–structure iff on the transverse manifold N there exists a G–structure of which the holonomy pseudogroup $\mathcal{H}$ is a pseudogroup of local automorphisms.*

It is well known that in any foliated G–structure there exists a basic G–connection, cf. [MO1,MO11].

Lemma 2 *(Molino) In any foliated G–structure $B(M,G;\mathcal{F})$ there exists a basic connection.*

Proof On any open set U_i the foliated G–structure $B(M,G;\mathcal{F})$ is isomorphic to $f_i^*B(N_i,G)$. In $B(N_i,G)$ there exists a G–connection with the connection form $\bar{\omega}_i$. Then the form $f_i^*\bar{\omega}_i = \omega_i$ defines a foliated connection on U_i. Taking a partition of unity $\{h_i\}$ subordinated to $\{U_i\}$ and putting $\omega = \sum h_i \pi\omega_i$ we get a connection form of a basic connection.□

The necessary and sufficient algebraic condition for the existence of foliated connections have been found by P. Molino, see [MO2,MO11]. The following is easy.

Lemma 3 *(Molino) A foliated G–structure $B(M,G;\mathcal{F})$ admits a foliated G-connection iff the corresponding G–structure on the transverse manifold admits an $\mathcal{H}$–invariant G–connection.*

Now we are going to define the Atiyah–Molino class of a foliated G–structure whose vanishing ensures that in this G–structure there exists a foliated connection, cf. [MO2,MO11].

Let ω be a k–form on B with values in a G–vector space (V,ρ). We assume that ω vanishes on the vectors tangent to the foliation $\mathcal{F}_1$ and $R_g^*\omega = \rho(g^{-1})\omega$ for any $g \in G$.

Let us take the associated fibre bundle $E(M,G;V;\mathcal{F})$ to $B(M,G;\mathcal{F})$ with the standard fibre V. It is a foliated vector bundle. Then the k–form ω can be considered as a k–form on B with values in $\pi^*E(M,G;V;\mathcal{F})$, the pull–back of the bundle $E(M,G;V;\mathcal{F})$ onto B. Assume that the $k+1$–form $d_{\mathcal{F}_1}\omega = \pi^*\alpha$ for some $k+1$–form α on M. The form α is $d_{\mathcal{F}}$–closed as $\pi^*d_{\mathcal{F}}\alpha = d_{\mathcal{F}_1}\pi^*\alpha = (d_{\mathcal{F}_1})^2\omega = 0$. Thus the form α defines a cohomology class in $H^{k,1}(M,E)$, cf. [MO11], [RG2].

If the class $[\alpha]$ vanishes, then there exists a form β of type $(k,0)$ on the manifold M with values in E such that $d_{\mathcal{F}}\beta = \alpha$. The form $\pi^*\beta$ is a k–form on B with values in π^*E. Let us take the form $\omega - \pi^*\beta$, then $d_{\mathcal{F}_1}(\omega - \pi^*\beta) = d_{\mathcal{F}_1}\omega - d_{\mathcal{F}_1}\pi^*\beta = d_{\mathcal{F}_1}\omega - \pi^*d_{\mathcal{F}}\beta = d_{\mathcal{F}_1}\omega - \pi^*\alpha = 0$. This precisely means that $\omega - \pi^*\beta$ is a base–like form on B of type (V,ρ).

Definition 3 *We say that two k–forms ω and ω' on B with values in π^*E are congruent if the form $\omega' - \omega$ is a projectable form, that is there exists a k–form ω_0 on M with values in E such that $\pi^*\omega_0 = \omega' - \omega$.*

The cohomology class defined above depends only on the congruency class of the form ω. In fact, let α and α' be two forms on M such that $d_{\mathcal{F}_1}\omega = \pi^*\alpha$ and $d_{\mathcal{F}_1}\omega' = \pi^*\alpha'$. Then $d_{\mathcal{F}_1}(\omega' - \omega) = \pi^*(\alpha' - \alpha) = d_{\mathcal{F}_1}\pi^*\omega_0 = \pi^* d_{\mathcal{F}}\omega_0$. Thus $\alpha' - \alpha = d_{\mathcal{F}}\omega_0$. We call the cohomology class $[\alpha]$ the Atiyah–Molino class of the form ω.

Theorem 1 *Let $B(M, G; \mathcal{F})$ be a foliated principal fibre bundle. Let ω be a k–form on B of type (V, ρ) annihilated by vectors tangent to the foliation $\mathcal{F}_1$. Then the Atiyah–Molino class of the form ω exists and it is zero iff the congruency class of this form there is a base–like form.*

The theorem is a consequence of the preceding considerations.

Let $B(M, G; \mathcal{F})$ be a foliated G–structure on the foliated manifold $(M, \mathcal{F})$. It is not difficult to verify that any the connection forms of two basic connections in this G–structure are congruent. Therefore these forms define just one cohomology class in $H^{0,1}(M, E(B(M, G; \mathcal{F}); \underline{g}))$ which is called the Atiyah–Molino class of the foliated G–structure $B(M, G; \mathcal{F})$.

Now we can formulate the following corollary of Theorem 1.

Corollary 1 *(Molino) Let $B(M, G; \mathcal{F})$ be a foliated G–structure. Then the bundle $B(M, G; \mathcal{F})$ admits a transversly projectable connection iff its Atiyah–Molino class is zero.*

Having made this small detour about foliated connections we return to the study of G-structures.

On the total space L of $L(M, \mathcal{F})$ or B of any transverse G–structure $B(M, G; \mathcal{F})$ we define an $\mathbb{R}^q$–valued 1–form θ, the fundamental form, as follows:

$$T_pL \xrightarrow{d\pi} T_xM \xrightarrow{p_N} N_x(M, \mathcal{F}) \xrightarrow{p^{-1}} \mathbb{R}^q$$

where $\pi: L \longrightarrow M$ and $p_N: TM \longrightarrow N(M, \mathcal{F})$ are the natural projections, and $x = \pi(p)$; or $\theta_p = p^{-1}p_N d_p\pi$.

It is not difficult to check that if a transverse G–structure $B(M, G; \mathcal{F})$ is foliated, then its fundamental form θ is base–like, i.e. over U_i $\theta = \tilde{f}_i^*\bar{\theta}$ where $\bar{\theta}$ is the fundamental form of the induced G–structure on the transverse manifold, and $\tilde{f}_i$ the mapping on the level of linear frames defined by f_i. Moreover, if the transverse G–structure $B(M, G; \mathcal{F})$ is foliated then the foliation $\mathcal{F}_1$ of the total space B of $B(M, G; \mathcal{F})$ can be defined in yet another way, namely:

$$T\mathcal{F}_1 = \{X \in TB : i_X\theta = i_X d\theta = 0\}.$$

Let ω be a connection form of a transverse G–connection. As in the case of G–structures on manifolds we can define fundamental horizontal vector fields $B(\xi)$. Let Q be a supplementary subbundle to $T\mathcal{F}$ in TM. Then there exists a G–invariant subbundle Q^1 of $ker\omega$ such that $d\pi|Q^1$ is an isomorphism on each fibre of Q^1 onto Q. Let $\xi \in \mathbb{R}^q$. Then for any point p of B there exists precisely one vector $B(\xi)_p$ of the tangent space of B at p such that $B(\xi)_p \in Q^1$ and $\theta_p(B(\xi)_p) = \xi$. We denote the vector field obtained in this way by $B(\xi)$. One can easily check that if $B(M,G;\mathcal{F})$ is a foliated G–structure and ω a foliated connection, then the vector fields $B(\xi)$ are infinitesimal automorphisms of the foliation $\mathcal{F}_1$ and, locally, they project onto the corresponding fundamental horizontal vector fields of the G–structure on the transverse manifold. Moreover, $R_a^*B(\xi) = B(a^{-1}\xi)$ for any $a \in G$ and $[A^*, B(\xi)] = -B(A\xi)$ for any $A \in \underline{g}$.

The torsion tensor field T of a transverse connection ω is defined as follows:

$$T(X,Y) = \nabla_X p_N Y - \nabla_Y p_N X - p_N[X,Y],$$

where ∇ is the covariant differentiation operator associated to ω and X,Y are vector fields on M. The tensor field T is a section of $T^*M \otimes T^*M \otimes N(M,\mathcal{F})$. It is not difficult to verify that the form Θ, $\Theta = d\theta - \omega \wedge \theta$ is the tensor form associated to T and it is called the torsion form of ω. If ω is basic, then its torsion tensor field T factorizes to a section of $N^*(M,\mathcal{F}) \otimes N^*(M,\mathcal{F}) \otimes N(M,\mathcal{F})$. The corresponding tensor to T (or Θ) with values in $Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q)$ we denote t_ω (or t if no confusion arises). Moreover, if ω is foliated then Θ is a base–like form, t_ω is constant along the leaves (foliated) and T, considered as a section of $N^*(M,\mathcal{F}) \otimes N^*(M,\mathcal{F}) \otimes N(M,\mathcal{F})$, is foliated. In the case of a transverse connection to define well the tensor t_ω with values in $Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q)$ we have to choose a supplementary subbundle Q to $T\mathcal{F}$ in TM and restrict the tensor field T to $Q^* \otimes Q^*$. Unfortunately, this operation is not independent of the choice of Q.

We will define the structure tensor of a transverse G–structure $B(M,G;\mathcal{F})$. Let us fix a subbundle Q of the tangent bundle TM supplementary to $T\mathcal{F}$. Let V be any q–dimensional subspace of the tangent space T_pB. We shall call this subspace horizontal if $d\pi(V) = Q_{\pi p}$. From now on we shall consider only horizontal subspaces.

For any horizontal subspace V we define the mapping

$$C_V : \mathbb{R}^q \wedge \mathbb{R}^q \longrightarrow \mathbb{R}^q;$$

$$C_V(u \wedge v) = \langle X \wedge Y, \, d\theta \rangle$$

where $X,Y \in V$ and $\theta(X) = u$, $\theta(Y) = v$. As in the standard theory (cf. [SB]) two such mappings defined for different horizontal subspaces at a given point differ by an element of $\partial Hom(\mathbb{R}^q, \underline{g})$, thus for any point p of B the mappings C_V define the unique class $c(p)$ in $Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q)/\partial Hom(\mathbb{R}^q, \underline{g})$. We shall call the tensor obtained in this way the structure tensor of the transverse G–structure $B(M,G;\mathcal{F})$ and we denote it by c.

We assume the standard Spencer notation, cf. [FU,SP]:

$$C^{1,1}(\underline{g}) = Hom(\mathbb{R}^q,\underline{g}),\ a_1 = i^*\otimes ad;$$

$$C^{0,2}(\underline{g}) = Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}),\ a_2 = \wedge^2 i^*\otimes i;$$

$$H^{0,2}(\underline{g}) = Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)/\partial Hom(\mathbb{R}^q,\underline{g}), a_3 q_0 = q_0 a_2;$$

where $a_i, i = 1,2,3$, are representations of $\underline{g}$ on the respective vector spaces, i is the natural representation of $\underline{g}$ on $\mathbb{R}^q$, and q_0 is the projection of $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$ onto $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)/\partial Hom(\mathbb{R}^q,\underline{g})$. Then, we have the following proposition.

Proposition 1 i) *Let ω be a transverse connection in a transverse G–structure, then its torsion tensor is a tensor of type $(a_2, C^{0,2}(\underline{g}))$.*

ii) *The structure tensor of a transverse G–structure is of type $(a_3, H^{0,2}(\underline{g}))$.*

The following lemma is easy but important:

Lemma 4 *If the transverse G–structure $B(M,G;\mathcal{F})$ is foliated, then its structure tensor is foliated.*

Proof Since $B(M,G;\mathcal{F})$ is foliated, there exists a G–structure $B(N,G)$ on the transverse manifold N such that $B(M,G;\mathcal{F})|U_i = f_i^*B(N,G)$. Therefore the horizontal subspaces at a point p of B over U_i are in one–to–one correspondence with horizontal subspaces at the point $\tilde{f}_i(p)$. Namely, if V is a horizontal subspace at the point p, then $d_p\tilde{f}_i(V) = \bar{V}$ is a horizontal subspace at $\tilde{f}_i(p)$. Moreover, as $\theta|\pi^{-1}(U_i) = \tilde{f}_i^*\bar{\theta}$, $C_V(u\wedge v) = C_{\bar{V}}(u\wedge v)$ for any $u,v\in\mathbb{R}^q$. If V_1 and V_2 are two horizontal subspaces of $B(M,G;\mathcal{F})$ at points p_1 and p_2, respectively, over U_i such that $d\tilde{f}_i(V_1) = d\tilde{f}_i(V_2)$, it results from the above relation that $C_{V_1} = C_{V_2}$. In particular, it means that $c(p_1) = c(p_2)$; thus indeed the structure tensor is foliated.□

The relation between the torsion tensor and the structure tensor is expressed by the following proposition, which results directly from the definition of the structure tensor.

Proposition 2 *Let t be the torsion tensor of a transverse G–connection in a transverse G–structure $B(M,G;\mathcal{F})$. Then for any point p of the total space of $B(M,G;\mathcal{F})$ $q_0 t(p) = c(p)$.*

Remark The structure tensor is functorial, i.e. if $f\colon M'\longrightarrow M$ is a mapping transverse to the foliation $\mathcal{F}$, then the structure tensor c' of the G–structure $f^*B(M,G;\mathcal{F})$ on the foliated manifold $(M', f^*\mathcal{F})$ is equal to $\tilde{f}^*c$.

Using the methods of [FU] we can prove the following.

Proposition 3 *Let $(M,\mathcal{F})$ be a foliated manifold with a transverse G–structure $B(M,G;\mathcal{F})$. If the structure tensor of every transverse G–structure on $(M,\mathcal{F})$ vanishes identically, then the mapping*

$$\partial: Hom(\mathbb{R}^q,\underline{g}) \longrightarrow Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$$

is surjective.

For the consequences of this proposition in the standard case and examples of Lie algebras for which ∂ is surjective see [FU].

If we assume that the first prolongation $\underline{g}^{(1)}$ of the Lie algebra $\underline{g}$ is trivial, we get a much stronger relation between basic G–connections and their torsion tensors.

Proposition 4 *If the first prolongation $\underline{g}^{(1)}$ of the Lie algebra $\underline{g}$ is trivial, then basic connections in any foliated G–structure are determined by their torsion tensors.*

Proof Let ω and ω' be two basic connections and Θ and Θ' be their torsion forms, respectively. From the structure equations we get that $\Theta-\Theta'=\partial t_\psi(\theta\wedge\theta)$ where $\omega-\omega'=\psi$ and t_ψ is the tensor corresponding to ψ. Thus, if $\Theta=\Theta'$, $\partial t_\psi=0$, and as $ker\partial=\underline{g}^{(1)}=0, \psi=0$. Therefore $\omega=\omega'$.□

Remark For transverse connections the same proof yields a weaker result:

Proposition 5 *If the first prolongation $\underline{g}^{(1)}$ of the Lie algebra $\underline{g}$ is trivial, then the transverse part (i.e. the restriction to any supplementary subbundle Q) of any transverse G–connection in a transverse G–structure is determined by its torsion tensor.*

Now we shall investigate the existence of transverse G–connections with a prescribed torsion tensor. With the standard Spencer notation (dropping $\underline{g}$) we have the following exact sequence of G–vector spaces:

$$(1)\ \ 0 \longrightarrow \underline{g}^{(1)} \longrightarrow C^{1,1} \xrightarrow{\partial} C^{0,2} \xrightarrow{q_0} H^{0,2} \longrightarrow 0$$

and then of foliated G–vector bundles over the foliated manifold $(M,\mathcal{F})$:

$$\begin{aligned}(2)\quad 0 \to \underline{g}^{(1)}B(M,G;\mathcal{F}) &\to C^{1,1}B(M,G;\mathcal{F}) \to \\ C^{0,2}B(M,G;\mathcal{F}) &\to H^{0,2}B(M,G;\mathcal{F}) \to 0.\end{aligned}$$

Let us assume that we have a G–invariant subspace S of $C^{1,1}$ supplementary to $i(\underline{g}^{(1)})$. Then the sequence:

$$0 \longrightarrow \underline{g}^{(1)} \longrightarrow C^{1,1} \xrightarrow{\partial} kerq_0 \longrightarrow 0$$

admits a G–invariant splitting $s: kerq_0 \longrightarrow C^{1,1}$, as the spaces $kerq_0$ and S are isomorphic as G–vector spaces.

Now, we are in a position to prove the following theorem:

Theorem 2 *Let $(M, \mathcal{F})$ be a foliated manifold with a transverse G–structure $B(M, G; \mathcal{F})$. Assume that the Lie algebra $\underline{g}$ has the following property:*
(*) there exists a G–invariant subspace S of $C^{1,1}(\underline{g})$ supplementary to $\underline{g}^{(1)}$. Then:

i) *a tensor t of type $(a_2, C^{0,2})$ on $B(M, G; \mathcal{F})$ is the torsion tensor of a transverse G–connection iff $q_0 t = c$;*

ii) *let $B(M, G; \mathcal{F})$ be a foliated G–structure. Then a tensor t of type $(a_2, C^{0,2})$ on $B(M, G; \mathcal{F})$ is the torsion tensor of a basic G–connection iff $q_0 t = c$;*

iii) *let $B(M, G; \mathcal{F})$ be a foliated G–structure admitting a foliated G–connection. Then a foliated tensor t on $B(M, G; \mathcal{F})$ of type $(a_2, C^{0,2})$ is the torsion tensor of a foliated G–connection iff $q_0 t = c$.*

Proof The necessity has already been proved. We shall demonstrate the case iii) of the theorem. The other ones are simpler and the proofs are similar. Let ω be a foliated connection in $B(M, G; \mathcal{F})$ and t_ω its torsion tensor. Then as $q_0 t_\omega = c$, $q_0(t_\omega - t) = 0$. Since the tensors are foliated, we can descend to the level of the transverse manifold N, and consider $\mathcal{H}$–invariant objects only. The corresponding objects will be underlined. When the condition (*) is fulfilled the vector space $C^{1,1}/\underline{g}^{(1)}$ is isomorphic to S as a G–vector space. Because of (*) $\underline{t}_\omega - \underline{t}$ defines a tensor $\underline{s}\colon \underline{B}(N, G) \longrightarrow S$ which is also $\mathcal{H}$–invariant such that $\partial \underline{s} = \underline{t}_\omega - \underline{t}$. Let $\underline{\psi}$ be the corresponding tensorial 1-form with values in $\underline{g}$. It is an $\mathcal{H}$–invariant 1-form. Therefore, it defines a foliated tensorial 1-form ψ on B. Then, $\omega' = \omega - 2\psi$ is a connection form such that $t_{\omega'} - t_\omega = -(t_\omega - t)$. Thus $t_{\omega'} = t$.□

Corollary 2 *In a foliated G–structure there exists a torsion–free basic G–connection iff the structure tensor vanishes identically.*

Corollary 3 *Let $B(M, G; \mathcal{F})$ be a foliated G–structure. Assume that there exists a foliated G–connection and that the condition (*) is fulfilled. Then there exists a torsion–free foliated G–connection iff the structure tensor vanishes identically.*

Using the above result we can demonstrate the existence of foliated connections in some other cases, (cf. [FU],[WO3]).

Proposition 6 *Let $B(M, G; \mathcal{F})$ be a foliated G-structure on a foliated manifold $(M, \mathcal{F})$ and $\underline{g}^{(1)} = 0$. Then, if the structure tensor of $B(M, G; \mathcal{F})$ vanishes, there exists a torsion–free foliated connection in $B(M, G; \mathcal{F})$.*

Remark The assumptions of the proposition are fulfilled by the bundle of transverse orthonormal frames of a Riemannian foliation. The torsion-free connection obtained in this way is the transverse Riemannian connection, (cf. [LP1], [LP2], [MO9], [MO11]).

We can also prove the existence of foliated connections in a more general situation.

Lemma 5 *Let $B(M,G;\mathcal{F})$ be a foliated G–structure on $(M,\mathcal{F})$. Assume that there exists a subspace S of $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$ invariant by the natural representation of the group G such that $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q) = S\oplus im\partial$. If the first prolongation $\underline{g}^{(1)}$ is trivial, then there exists a basic connection for which any automorphism of $B(M,G;\mathcal{F})$ is an affine transformation.*

Proof Let c be the structure tensor of $B(M,G;\mathcal{F})$. It takes values in

$$Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)/\partial Hom(\mathbb{R}^q,\underline{g}).$$

Let t_c be the tensor with values in S corresponding to the structure tensor. For any automorphism f of $B(M,G;\mathcal{F}), c\tilde{f} = c$. Therefore $t_c\tilde{f} = t_c$ as well. From Proposition 4 it results that there exists at most one basic G–connection whose torsion tensor is equal to t_c. On the other hand, from Theorem 2 ii) we obtain that such a connection exists. Therefore the automorphisms of $B(M,G;\mathcal{F})$ have to be affine transformations of this connection.□

As an easy consequence of this lemma we get the following:

Proposition 7 *Let G be a closed subgroup of $GL(q)$ such that the first prolongation $\underline{g}^{(1)}$ of its Lie algebra is trivial and that there exists a subspace S of $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$ invariant by the natural representation of G and supplementary to $\partial Hom(\mathbb{R}^q,\underline{g})$in $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$. Then there exists a foliated connection in any foliated G–structure $B(M,G;\mathcal{F})$ on a foliated manifold $(M,\mathcal{F})$.*

IV.2 G–structures of finite type

In this section we define prolongations of foliated G–structures and use their properties to obtain some more information about G–foliations for the group G of finite type.

Let Q be a subbundle of the tangent bundle TM supplementary to $T\mathcal{F}$. We have shown that any horizontal subspace V of the tangent space T_pB defines an element C_V of $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$, and that for any two such subspaces V_1 and V_2 at a given point their difference belongs to $\partial Hom(\mathbb{R}^q,\underline{g})$, i.e. $C_{V_1}-C_{V_2} = \partial S_{V_2V_1}$. Let us choose a subspace S of $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$ supplementary to $\partial Hom(\mathbb{R}^q,\underline{g})$. This choice, at each point p distinguishes a family of horizontal subspaces V for which $C_V\in S$. Then if V_1 and V_2 are two such subspaces $C_{V_1} - C_{V_2} = \partial S_{V_2V_1} = 0$ as $C_{V_1}-C_{V_2}$ belongs to S and $\partial Hom(\mathbb{R}^q,\underline{g})$ at the same time. Therefore $S_{V_2V_1}\in\underline{g}^{(1)}$.

Thus this choice of a subspace C defines a $G^{(1)}$–reduction of $L(B,\mathcal{F}_1)$ where $G^{(1)} = \left\{\begin{pmatrix} id & 0 \\ h & id\end{pmatrix} \in \mathrm{GL}(\mathbb{R}^q+\underline{\mathrm{g}})\colon \mathrm{h}\in\underline{\mathrm{g}}^{(1)}\right\}$. We call this $G^{(1)}$–structure the first prolongation of the foliated G–structure $B(M,G;\mathcal{F})$ and denote it by $B^1(M,G;\mathcal{F})$. The bundle $B^1(M,G;\mathcal{F})$ is a foliated subbundle of $L(B,\mathcal{F}_1)$ corresponding to the first prolongation $B^1(N,G)$ of the G–structure $B(N,G)$. The foliation of $B^1(M,G;\mathcal{F})$ we denote by $\mathcal{F}_2$. By the $k+1$st–prolongation of the foliated G–structure $B(M,G;\mathcal{F})$ we understand the first prolongation of the foliated $G^{(k)}$–structure $B^k(M,G;\mathcal{F}_{k+1})$ –the kth prolongation of $B(M,G;\mathcal{F})$. Its foliation we denote by $\mathcal{F}_{k+2}$.

Lemma 6 *If the group G is of type k, then the foliation $\mathcal{F}_k$ is transversely parallelisable.*

Proof The fact that the group G is of type k means that the group $G^{(k)}$ is trivial. Therefore the foliated structure $B^k(M,G;\mathcal{F})$ is an $\{e\}$–structure, the bundle $B^k(M,G;\mathcal{F})$ is a reduction of the bundle $L(B^{k-1},\mathcal{F}_k)$. Thus, indeed, the foliation $\mathcal{F}_k$ is transversely parallelisable. □

Definition 4 *Let $\mathcal{F}$ be a G–foliation for a Lie group G of type k. The foliation $\mathcal{F}$ is called transversely complete if the transverse parallelism of the foliation $\mathcal{F}_k$ is complete.*

Theorem 3 *Let G be a Lie group of type k and $\mathcal{F}$ be a transversely complete G–foliation. Then leaves of the foliation $\mathcal{F}$ have the same universal covering space.*

Proof Let us consider the foliation $\mathcal{F}_k$ of the total space of $B^{k-1}(M,G;\mathcal{F})$. It is a complete transversely parallelisable foliation. Therefore the leaves of $\mathcal{F}_k$ are diffeomorphic. It results from the construction that leaves of $\mathcal{F}_k$ are covering spaces of leaves of $\mathcal{F}_{k-1}$. Hence leaves of $\mathcal{F}_k$ are covering spaces of leaves of $\mathcal{F}$, and therefore leaves of $\mathcal{F}$ have the same covering space. □

Corollary 4 *(Blumenthal) Leaves of a transversely complete conformal foliation of codimension $q \geq 3$ have the same universal covering space*

Proof The conformal group $CO(q)$ is of type 2 for $q \geq 3$, cf. [BL5]. □

As an application of this method we consider the following foliations.

Definition 5 *Let N be a q–manifold and $\mathcal{K}$ a pseudogroup of local diffeomorphisms of N. For any open subset U of N $\mathcal{K}(U) = \{f \in \mathcal{K}: dom f = U\}$. We say that the pseudogroup $\mathcal{K}$ has the property E_k, $k \in \mathbb{N}$, if for any point x of N there exists a sequence of open subsets $\{U_n\}$ such that $U_0 = N$, $\bigcap U_n = \{x\}$ and the spaces $\{j^k_x f: f \in \mathcal{K}(U_n)\}$ are equal.*

Example 1 1) Any pseudogroup generated by a group of global diffeomorphisms has the property E_k for any k.

2) Let $B(N,G)$ be a regular G–structure on a simply connected compact manifold N for a group G of finite type k. The pseudogroup $\mathcal{K}$ generated by the

flows of infinitesimal automorphisms of this G–structure has the property E_k. In fact, the sheaf of germs of infinitesimal automorphisms of this G–structure is constant, cf. [FU]. Thus any germ of an infinitesimal automorphism can be extended to a global infinitesimal automorphism of $B(N,G)$, hence any local diffeomorphism from some flow with the domain small enough can be extended to a global diffeomorphism, so the pseudogroup has the property E_k.

Let G be a closed linear group of type k, $B(N,G)$ be a G–structure on N, and $\mathcal{K}$ a pseudogroup of local automorphisms of $B(N,G)$ having the property E_k. Let $\mathcal{F}$ be a $\mathcal{K}$–foliation on M. Then:

Lemma 7 *The restriction mapping $\mathcal{K}(U_m) \to \mathcal{K}(U_{m+1})$ is bijective. In particular, any element of $\mathcal{K}$ can be extended to a global automorphism of $B(N,G)$.*

Proof Let $f \in \mathcal{K}(U_{m+1})$. Then the lift $\bar{f}$ of f to $B^{k-1}(N,G)$ preserves the $\{e\}$–structure on $B^{k-1}(N,G)$, i.e. the parallelism $\{X_1,...X_r\}$. Hence $\bar{f}exp(tX_i) = exp(tX_i)\bar{f}$ for any t and $i = 1,...r$ whenever $exp(tX_i)$ is defined.

Let g be an element of $\mathcal{K}(U_m)$ such that $j^k_x g = j^k_x f$. Then for the mapping $\bar{g}$ we have the equality $\bar{g}(p) = \bar{f}(p)$ for any p over x. The set $A = \{p \in B^k(N,G) \colon \bar{f}(p) = \bar{g}(p)\}$ is non–empty and closed. Our previous considerations ensure that it is open as well. Hence over U_{m+1} $f = g$. □

Lemma 7 ensures that our foliation is developable. Therefore as a corollary we get the following fact.

Proposition 8 *Let $\mathcal{F}$ be a $\mathcal{K}$-foliation on a compact manifold M with the pseudogroup $\mathcal{K}$ having the property E_k. Then the foliation $\mathcal{F}$ is developable.*

The following theorem is true for transversely complete $\mathcal{K}$–foliations.

Theorem 4 *Let $\mathcal{F}$ be a transversely complete $\mathcal{K}$-foliation modelled on a q–manifold N with the pseudogroup $\mathcal{K}$ having the property E_k. Then the leaves of $\mathcal{F}$ have the common universal covering space and the space of leaves of $\mathcal{F}$ is homeomorphic to the orbit space of some group action on a covering space of the manifold N.*

Proof According to Proposition 8 there exists a covering $\hat{M}$ of M such that the lifted foliation $\hat{\mathcal{F}}$ is defined by a global submersion $\hat{f} \colon \hat{M} \to N$. We restrict our attention to $im\hat{f}$ which we denote also by N. Thus the foliation $\hat{\mathcal{F}}_k$ of $B^{k-1}(\hat{M}, G; \mathcal{F})$ is defined by the submersion $\hat{f}^k$

$$\begin{array}{ccc} B^{k-1}(\hat{M},G;\hat{\mathcal{F}}) & \xrightarrow{\hat{f}^k} & B^{k-1}(N,G) \\ \Big\downarrow & & \Big\downarrow \\ \hat{M} & \xrightarrow{\hat{f}} & N \end{array}$$

The total space of the bundle $B^{k-1}(\hat{M},G;\hat{\mathcal{F}})$ is a covering space of the total space of the bundle $B^{k-1}(M,G;\mathcal{F})$ and the foliation $\hat{\mathcal{F}}_k$ is the lifted foliation of $\mathcal{F}_k$. $\mathcal{F}_k$ is a complete transversely parallelisable foliation, so is $\hat{\mathcal{F}}_k$. Moreover, as leaves of $\hat{\mathcal{F}}_k$ are closed, the projection onto the leaf space $B^k/\mathcal{F}$ is a locally trivial fibre bundle. Thus indeed leaves of $\mathcal{F}$ have the same universal covering space.

The transverse parallelism of $\hat{\mathcal{F}}_k$ projects onto a complete parallelism of $B^k/\mathcal{F}$ and it is mapped by the induced mapping $\bar{f}_k$ onto the natural parallelism of the total space B^{k-1}_N of $B^{k-1}(N,G)$. Hicks' Theorem, cf. [HK], ensures that the mapping $\bar{f}_k$ is a covering mapping. Therefore we have the following commutative diagram.

$$\begin{array}{ccccc} B^{k-1} & \longrightarrow & B^k/\mathcal{F} & \xrightarrow{\bar{f}_k} & B^{k-1}_N \\ \Big\downarrow{\scriptstyle \hat{\pi}} & & \Big\downarrow{\scriptstyle \bar{\pi}} & & \Big\downarrow{\scriptstyle \pi_N} \\ \hat{M} & \longrightarrow & \hat{M}/\hat{\mathcal{F}} & \xrightarrow{\bar{f}} & N \end{array}$$

where $\hat{M}/\hat{\mathcal{F}}$ is a T_1 manifold according to [PA1]. Hence the mapping $\bar{f}$ is a local homeomorphism. The next step is to show that $\bar{f}$ is a covering. To prove this need only to show that $\bar{f}$ has the property of lifting curves. Since $B^{k-1}(N,G) \to N$ is a tower of principal fibre bundles, for any curve γ in N there is a horizontal lift $\tilde{\gamma}$ of γ to $B^{k-1}(N,G)$. Because $\bar{f}_k$ is a covering mapping, the curve $\tilde{\gamma}$ can be lifted to a curve $\bar{\gamma}$ in $B^k/\mathcal{F}$. Thus the curve $\bar{\pi} \circ \bar{\gamma}$ is a lift of γ to $\hat{M}/\hat{\mathcal{F}}$. The choice of a point y in $\bar{f}^{-1}(\gamma(0))$ forces the following choices in the liftings executed: $\bar{\pi}^{-1}(y) \ni \bar{y}$ and $\bar{f}_k(\bar{y}) \in \pi_N^{-1}(\gamma(0))$. Therefore the manifold $\hat{M}/\hat{\mathcal{F}}$ is a covering of a Hausdorff manifold. We denote it by $\hat{N}$.

Since the mapping $\bar{f}$ is a covering mapping we can lift $\mathcal{K}$ to a pseudogroup $\hat{\mathcal{K}}$ of $\hat{N}$. The pseudogroup $\hat{\mathcal{K}}$ has the property E_k as well. Thus $\mathcal{F}$ being a $\hat{\mathcal{K}}$–foliation is a developable one. Therefore there exists a homomorphism $h\colon \pi_1(M,x) \to Aut(B(\hat{N},G))$, cf. Proposition I.1. The space of leaves of $\mathcal{F}$ is homeomorphic to the space of orbits of the group $im h$ acting on $\hat{N}$. □

To complete this section we turn our attention to foliations with all leaves compact. First we establish a simple but useful lemma.

Lemma 8 *A leaf of $\mathcal{F}_k$ is compact iff the corresponding leaf of $\mathcal{F}$ is compact and has finite holonomy group.*

Proof Let $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ be a cocycle defining the foliation $\mathcal{F}$ with the following properties:

i) the open covering $\{U_i\}$ is locally finite,

ii) the open sets U_i are relatively compact,

iii) the submersions f_i take values in a q–manifold N admitting a G–structure $B(N,G)$ and have connected fibres,

iv) the local diffeomorphisms g_{ij} of N are local automorphisms of the G–structure $B(N,G)$.

Then the foliation $\mathcal{F}_k$ of the total space B^{k-1} of $B^{k-1}(M,G,\mathcal{F})$ is given by a cocycle

$$\mathcal{U}^k = \{V_i, B^{k-1}(f_i), B^{k-1}(g)\}$$

where V_i is the total space of $B^{k-1}(M,G;\mathcal{F})|U_i$, $B^{k-1}(f_i)$ is the fibre mapping corresponding to f_i of $B^{k-1}(M,G;\mathcal{F})$ into $B^{k-1}(N,G)$ and $B^{k-1}(g_{ij})$ the corresponding local diffeomorphism of the total space B^{k-1}_N of $B^{k-1}(N,G)$. The holonomy pseudogroup $\mathcal{H}^k$ of the foliation $\mathcal{F}_k$ given by the cocycle $\mathcal{U}^k$ consists of local diffeomorphisms of B^{k-1}_N preserving a parallelism of this manifold, cf. [KO2,SB]. Therefore, they are determined by their values at any point, and the elements of the holonomy pseudogroup $\mathcal{H}$ generated by the cocycle $\mathcal{U}$ are determined by their k–jets. The holonomy group of a leaf L of the foliation $\mathcal{F}$ at a point x of U_i, up to conjugation, is the group

$$\mathbf{H}_{\bar{x}} = \{(h)_{\bar{x}}: h(\bar{x}) = \bar{x}, h \in \mathcal{H}\}$$

where $\bar{x} = f_i(x)$. The leaf L is compact iff the $\mathcal{H}$–orbit of $\bar{x}$ is finite. Therefore, the lemma is equivalent to the following one:

an $\mathcal{H}$–orbit of $\bar{x}$ is finite and the group $\mathbf{H}_x$ is finite iff for any point p over $\bar{x}$ of B^{k-1}_N the $\mathcal{H}^k$–orbit of p is finite.

The statement that the $\mathcal{H}^k$–orbit of p is finite means precisely that the set $\mathbf{H}^k_{\bar{x}} = \{j^k_{\bar{x}}h: h \in \mathcal{H}\}$ is finite. The fact that local automorphisms of the G–structure $B(N,G)$ are determined by their k–jets ensures that the set $\mathbf{H}^k_{\bar{x}}$ is finite iff the group $\mathbf{H}_{\bar{x}}$ is finite and the $\mathcal{H}$–orbit of the point $\bar{x}$ is finite.□

Let us assume that all the leaves of $\mathcal{F}_k$ are compact. According to Lemma 8 it is so if the leaves of $\mathcal{F}$ are compact and have finite holonomy. Then the Reeb Stability Theorem, cf. [HH,LA], ensures that the space of leaves $M/\mathcal{F}$ of the foliation $\mathcal{F}$ is a Satake manifold.

The converse is also true. Let $\mathcal{F}$ be a foliation on a compact manifold. If the space of leaves is a Satake manifold the foliation cannot have a non-compact leaf. Otherwise this leaf could not be separated from any other leaf contained in its closure. Moreover, the structure of a Satake manifold ensures that these compact leaves cannot have infinite holonomy. Therefore we have proved the following:

Corollary 5 *All the leaves of the foliation $\mathcal{F}_k$ are compact iff the space of leaves of the foliation $\mathcal{F}$ is a Satake manifold.*

This corollary leads us to the following theorem.

Theorem 5 *Let $\mathcal{F}$ be a transversely complete G–foliation of finite type k on a compact manifold M. All the leaves of $\mathcal{F}$ are compact iff the space of leaves of $\mathcal{F}$ is a Satake manifold.*

Proof The set of points of generic leaves (i.e. without holonomy) is open and dense in M, cf. [HC,EP]. Thus according to Lemma 8 the foliation $\mathcal{F}_k$ has compact leaves. Moreover, as it is a complete T.P. foliation, all its leaves are compact and the theorem results from Corollary 5.□

As a corollary we obtain the following fact due to P.Molino, cf. [MO9,MO11].

Corollary 6 *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold with all leaves compact. Then the space of leaves of $\mathcal{F}$ is a Satake manifold.*

Proof A Riemannian foliation on a compact manifold is a transversely complete foliation of type 1.□

As a corollary of the proof of Theorem 5 we get:

Corollary 7 *Let $\mathcal{F}$ be a transversely complete G–foliation of finite type. If the foliation $\mathcal{F}$ has a compact leaf with finite holonomy then all the leaves of the foliation $\mathcal{F}$ are compact with finite holonomy and the space of leaves of $\mathcal{F}$ is a Satake manifold.*

The assumption of transverse completeness which ensures that all the leaves of $\mathcal{F}_k$ are compact can be weakened in many ways. In fact, for us, it is enough to show that the union of the compact leaves of $\mathcal{F}_k$ is open and closed, or just closed as it is always open. For example, we have the following.

Proposition 9 *Let $\mathcal{F}$ be a G–foliation of finite type k with all leaves compact on a compact manifold M. If one of the following conditions is satisfied:*

a) *the foliation $\mathcal{F}_k$ is a complete Riemannian foliation,*

b) *on the manifold B^{k-1} there exists a complete Riemannian metric in which the vector fields of the transverse parallelism have locally bounded norms,*

c) *the foliation $\mathcal{F}$ is given by an action of a Lie group K such that:*

> **i)** *there exists a supplementary subbundle Q to $T\mathcal{F}$ invariant by K,*
>
> **ii)** *the group K acts by automorphisms of $B(M, G; \mathcal{F})$,*

then the space of leaves of $\mathcal{F}$ is a Satake manifold.

The proof of the part a) of the proposition is based on the following lemma for T.P. foliations.

Lemma 9 *If a transversely parallelisable foliation $\mathcal{F}$ has a compact leaf and admits a complete bundle–like metric for the foliation $\mathcal{F}$ then the leaves of $\mathcal{F}$ are compact.*

Proof At the very beginning let us stress that there is no connection between the T.P. and the bundle–like metric. Let us pass to the bundle of transverse orthonormal frames. Its foliation is a complete T.P. one as the bundle–like metric is complete. It has also a compact leaf. Thus all leaves of this foliation are compact which implies that all leaves of $\mathcal{F}$ are compact. □

Part a) of the proposition is a consequence of Lemmas 8 and 9 and the fact that the foliation has a compact leaf without holonomy. The proof of the parts b and c) depends on the fact that under these assumptions, using similar arguments as in Lemma 9, we can prove that all leaves of the foliation $\mathcal{F}_k$ are compact.□

Remarks

1) The assumption that the manifold M is compact and that all the leaves of the foliation $\mathcal{F}$ are compact can be replaced by "there exists a compact leaf with finite holonomy".

2) If the group K of c) is compact a K–invariant subbundle Q supplementary to $T\mathcal{F}$ always exists.

3) The above considerations are also valid for $\nabla - G$–foliations as the foliation $\mathcal{F}_1$ of the total space of $B(M, G; \mathcal{F})$ is T.P. More generally, it is also true for foliations admitting foliated Cartan connections, cf. [BL6].

4) The lemma for T.P. foliations corresponding to the point b is the weakest possible as indicates the following example:

Example 2 Let $T^2 = S^1 \times S^1$ be the 2–torus foliated by circles $\{x\} \times S^1$. When we puncture it, we obtain a T.P.foliation of the punctured torus whose all but one leaves are compact. We can make it a complete manifold, but then the vector field defining the T.P. will not have locally bounded norm.

IV.3 Applications: characteristic classes of flag structures

A flag structure $\mathcal{F}$ on a manifold M is a system of foliations

$$(\mathcal{F}_1, ...\mathcal{F}_k),\ \mathcal{F}_i \subseteq \mathcal{F}_{i+1},\ i = 1,..k-1.$$

The characteristic classes of these structures were studied by many authors, cf. [CB,CMA,DO,WO1,WO2]. We are going to apply Theorem 2 to obtain vanishing theorems for characteristic classes of flag structures with an adapted transverse geometric structure. In particular, as a corollary we obtain the vanishing theorem of F. J. Carreras and A. M. Naveira, cf. [CN]. For simplicity's sake we assume k = 2. Let $q_1 = codim\mathcal{F}_1$, $q = codim\mathcal{F}_2$ and $p = q_1 - q$.

Definition 6 *A transverse G–structure adapted to the flag $\mathcal{F}$ is a $G_1{\times}G_2$–reduction $B(M, G_1{\times}G_2; \mathcal{F}_1)$ of the bundle $L(M;\mathcal{F}_1)$ where $G_1 \subset GL(p)$, $G_2 \subset GL(q)$, such that if $p = (v_1, ...v_p, v_{p+1}, ...v_{p+q})$ is a transverse frame of $B(M, G_1{\times}G_2; \mathcal{F}_1)$ at a point x, then the vectors $v_1, ...v_p$ span the vector space $T\mathcal{F}_2/T\mathcal{F}_{1x}$.*

If we identify the normal bundle of the foliation $\mathcal{F}_1$ with a subbundle Q of TM supplementary to $T\mathcal{F}_1$, the choice of a $G_1{\times}G_2$–reduction of $L(M;\mathcal{F}_1)$ as above means that we have chosen subbundles Q_1 and Q_2 of TM such that Q_1 is supplementary to $T\mathcal{F}_1$ in $T\mathcal{F}_2$, Q_2 is supplementary to $T\mathcal{F}_2$ in TM and $Q = Q_1 \oplus Q_2$. Moreover, we have a G_1–reduction $B(M, G_1; \mathcal{F})$ of the frame bundle $L(Q_1)$ of Q_1 and G_2–reduction $B(M, G_2; \mathcal{F}_2)$ of the frame bundle $L(Q_2)$ of Q_2.

Let $\pi_1: B(M, G_1{\times}G_2; \mathcal{F}_1) \longrightarrow B(M, G_1; \mathcal{F})$ be the projection defined by the correspondence

$$(v_1, ...v_p, v_{p+1}, ...v_{p+q}) \longmapsto (v_1, ...v_p)$$

and $\pi_2: B(M, G_1{\times}G_2; \mathcal{F}_1) \longrightarrow B(M, G_2; \mathcal{F}_2)$ be the projection defined by the correspondence

$$(v_1, ...v_p, v_{p+1}, ...v_{p+q}) \longmapsto (v_{p+1}, ...v_{p+q}).$$

Moreover, the principal fibre bundle $B(M, G_1{\times}G_2; \mathcal{F}_1)$ is the fibre product of the bundles $B(M, G_1; \mathcal{F})$ and $B(M, G_2; \mathcal{F}_2)$, i.e.

$$B(M, G_1{\times}G_2; \mathcal{F}_1) = B(M, G_1; \mathcal{F}){\times}_M B(M, G_2; \mathcal{F}_2).$$

Let B, B_1 and B_2 be the total spaces of the bundles $B(M, G_1{\times}G_2; \mathcal{F}_1)$, $B(M, G_1; \mathcal{F})$ and $B(M, G_2; \mathcal{F}_2)$, respectively.

Let ω be a linear connection in the bundle $B(M, G_1{\times}G_2; \mathcal{F}_1)$. Then ω is a 1-form on B with values in $Lie(G_1{\times}G_2) = \underline{g}_1 \oplus \underline{g}_2$. Let

$$p_1: \underline{g}_1 \oplus \underline{g}_2 \longrightarrow \underline{g}_1 \text{ and } p_2: \underline{g}_1 \oplus \underline{g}_2 \longrightarrow \underline{g}_2$$

be the natural projections. Then there exists exactly one linear connection ω_1 in $B(M, G_1; \mathcal{F})$ (resp. ω_2 in $B(M, G_2; \mathcal{F}_2)$) such that $p_1\omega = \pi_1^*\omega_1$ (resp. $p_2\omega = \pi_2^*\omega_2$). Conversely, if ω_1 and ω_2 are G_1- and G_2- connections in the bundles $B(M, G_1; \mathcal{F})$ and $B(M, G_2; \mathcal{F}_2)$, respectively, then the form $\omega = \pi_1^*\omega_1 \oplus \pi_2^*\omega_2$ defines a $G_1 \times G_2$-connection in $B(M, G_1 \times G_2; \mathcal{F}_1)$. Moreover, the canonical form θ of $B(M, G_1 \times G_2; \mathcal{F}_1)$ which takes values in $\mathbb{R}^p \oplus \mathbb{R}^q = \mathbb{R}^{p+q}$ decomposes itself with respect to this direct sum into $\theta = (\theta^1, \theta^2)$ and $\theta^1 = \pi_1^*\theta_1$, $\theta^2 = \pi_2^*\theta_2$ where θ_1 and θ_2 are the fundamental forms of the structures $B(M, G_1; \mathcal{F})$ and $B(M, G_2; \mathcal{F}_2)$, respectively.

Now, we are going to study the relationship between the structure tensors c, c_1 and c_2 of $B(M, G_1 \times G_2; \mathcal{F}_1)$, $B(M, G_1; \mathcal{F})$ and $B(M, G_2; \mathcal{F}_2)$, respectively. Let p be a point of B and V a horizontal space at p. Then there exist (unique) subspaces V_1 and V_2 of V such that $d_p\pi(V_1) = Q_{1x}$ and $d_p\pi(V_2) = Q_{2x}$. Thus the subspaces V_1 and V_2 are horizontal. The mapping C_V is an element of $Hom(\mathbb{R}^{p+q} \wedge \mathbb{R}^{p+q}, \mathbb{R}^{p+q})$ which decomposes itself into:

$$\begin{array}{rl} & Hom(\mathbb{R}^p \wedge \mathbb{R}^p, \mathbb{R}^p) \oplus Hom(\mathbb{R}^p \wedge \mathbb{R}^p, \mathbb{R}^q) \oplus Hom(\mathbb{R}^p \wedge \mathbb{R}^q, \mathbb{R}^p) \\ \oplus & Hom(\mathbb{R}^p \wedge \mathbb{R}^q, \mathbb{R}^q) \oplus Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^p) \oplus Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q) \end{array}$$

The subspaces of this decomposition we denote V_1, V_2, V_3, V_4, V_5 and V_6, respectively. The corresponding components of C_V we denote by C_V^i, $i = 1, \ldots 6$, respectively, i.e. $C_V = C_V^1 \oplus \ldots \oplus C_V^6$.

We are going to calculate C_V taking into account this decomposition. Let $u, v \in \mathbb{R}^{p+q}$, $\bar{p}_1: \mathbb{R}^{p+q} \longrightarrow \mathbb{R}^p$, $\bar{p}_2: \mathbb{R}^{p+q} \longrightarrow \mathbb{R}^q$ be the natural projections and $X(u), X(v)$ be the unique vectors of V such that $\theta(X(u)) = u$ and $\theta(X(v)) = v$. We can extend these vectors to vector fields, denoted by the same letters, defined on an open subset of B having the same properties, i.e. $\theta(X(u)) \equiv u$ and $d\pi(X(u)) \in Q$. Then

$$\begin{array}{rcl} C_V(u \wedge v) & = & d\theta(X(u), X(v)) = d\theta^1(X(u), X(v)) \oplus d\theta^2(X(u), X(v)) \\ & = & d(\pi_1^*\theta_1)(X(u), X(v)) \oplus d(\pi_2^*\theta_2)(X(u), X(v)) \\ & = & d\theta_1(d\pi_1(X(u), d\pi_1(X(v))) \oplus d\theta_2(d\pi_2(X(u), d\pi_2(X(v))) \end{array}$$

and as $d\theta_1(d\pi_1(X(u))) \equiv \bar{p}_1(u)$ and $d\theta_2(d\pi_2(X(u)) \equiv \bar{p}_2(u)$

$$= -1/2\{\theta_1([d\pi_1(X(u)), d\pi_1(X(v))]) \oplus \theta_2([d\pi_2(X(u)), d\pi_2(X(v))])\}$$

If $u, v \in \mathbb{R}^p$ then the mapping

$$u \wedge v \longmapsto -1/2\theta_1(d\pi_1([X(u), X(v)]))$$

is the component C_V^1, and the mapping

$$u \wedge v \longmapsto -1/2\theta_2(d\pi_2([X(u), X(v)]))$$

is the component C_V^2. If $u, v \in \mathbb{R}^q$, then the mapping

$$u \wedge v \longmapsto -1/2\theta_1(d\pi_1([X(u), X(v)]))$$

is the component C_V^5, and the mapping

$$u \wedge v \longmapsto -1/2\theta_2(d\pi_2([X(u), X(v)]))$$

is the component C_V^6.

The structure tensor takes values in the space

$$Hom(\mathbb{R}^{p+q} \wedge \mathbb{R}^{p+q}, \mathbb{R}^{p+q})/\partial Hom(\mathbb{R}^{p+q}, \underline{g}_1 \oplus \underline{g}_2)$$

which decomposes itself in the following way:

$$\begin{array}{ccccc} & V^1/\partial Hom(\mathbb{R}^p, \underline{g}_1) & \oplus \; V^2 & \oplus & V^3/\partial Hom(\mathbb{R}^q, \underline{g}_1) \\ \oplus & V^4/\partial Hom(\mathbb{R}^p, \underline{g}_2) & \oplus \; V^5 & \oplus & V^6/\partial Hom(\mathbb{R}^q, \underline{g}_2) \end{array}$$

The corresponding components of the structure tensor c we denote by $c^1, ...c^6$, respectively. It results from this decomposition that $c^2(p) = C_V^2$ and $c^5(p) = C_V^5$ for any horizontal space V at p. Since $\theta_1(d\pi_1(X(u))) = \bar{p}_1(u)$ and $\theta_2(d\pi_2(X(u))) = \bar{p}_2(u)$, $c_1\pi_1 = c^1$ and $c_2\pi_2 = c^6$. Thus we have identified the components c^1 and c^6 of the structure tensor.

Now, we shall look at the component c^2. The vanishing of c^2 means that

$$\theta_2(d\pi_2([X(u), X(v)])) = 0$$

for any $u, v \in \mathbb{R}^p$, which is equivalent to

$$\theta^2([X(u), X(v)]) = 0.$$

Let Γ be the horizontal bundle of a connection ω. Relative to the decomposition $T\mathcal{F}_1 \oplus Q_1 \oplus Q_2$ of TM, the bundle Γ decomposes itself into $T\mathcal{F}_1^1 \oplus Q^1 \oplus Q^2$ where all the three bundles are $G_1 \times G_2$–invariant. Then we can assume that the vector fields $X(u)$ and $X(v)$ are local sections of Q^1 and $\theta^2([X(u), X(v)]) \equiv 0$ is equivalent to the fact that

$$[X(u), X(v)] \in T\mathcal{F}_1^1 \oplus Q^1 \oplus \mathcal{V}$$

($\mathcal{V}$ – the vertical bundle). In its turn, this is equivalent to the condition that for any two sections of Q^1 its bracket is a section of $T\mathcal{F}_1^1 \oplus Q^1 \oplus \mathcal{V}$. This condition on the level of the manifold M means precisely that for any two sections of the bundle Q_1, their bracket is a section of $T\mathcal{F}_1 \oplus Q_1$; i.e. the subbundle $T\mathcal{F}_2$ is integrable. Summing up, $c^2 \equiv 0$ iff $\mathcal{F}_2$ is a foliation.

Similar considerations lead to the following conclusion: $c^5 \equiv 0$ iff $\mathcal{F}_1 \oplus Q_2$ is a foliation.

The restriction of the G_1–structure $B(M, G_1; \mathcal{F})$ to a leaf L of the foliation $\mathcal{F}_2$ is a transverse G_1–structure $B(M, G_1; \mathcal{F}_1^L)$ for the foliation $\mathcal{F}_1^L$ of L by the leaves of $\mathcal{F}_1$. The restriction ω_1^L of the 1-form ω_1 to the total space B_L of $B(M, G_1; \mathcal{F}_1^L)$ defines a transverse G_1–connection in this G_1–structure. Its structure tensor c_1^L is the restriction of the tensor c_1 to B_L.

Let us assume that there exists a supplementary subspace $S = \oplus_1^6 S_i$

$$\partial Hom(\mathbb{R}^{p+q}, \underline{g}_1 \oplus \underline{g}_2) \oplus S = Hom(\mathbb{R}^{p+q} \wedge \mathbb{R}^{p+q}, \mathbb{R}^{p+q}),$$

where S_i is a supplementary subspace of A^i in V^i, $A^1 = \partial Hom(\mathbb{R}^p, \underline{g}_1)$, $A^2 = 0$, $A^3 = \partial Hom(\mathbb{R}^q, \underline{g}_1)$, $A^4 = \partial Hom(\mathbb{R}^p, \underline{g}_2,)$, $A^5 = 0$ and $A^6 = \partial Hom(\mathbb{R}^q, \underline{g}_2)$; moreover the subspaces S_i are invariant with respect to the natural representation of $G_1 \times G_2$ on $Hom(\mathbb{R}^{p+q} \wedge \mathbb{R}^{p+q}, \mathbb{R}^{p+q})$.

Let ω be a $G_1 \times G_2$–connection in $B(M, G_1 \times G_2; \mathcal{F}_1)$ whose torsion tensor takes values in S. Then the torsion tensor of the connection ω_1^L takes values in S_1 and of the connection ω_2 in S_6. Therefore the results of Section 1 ensure the following:

Lemma 10 a) *If $\underline{g}_1^{(1)} = 0$ and the G_1–structure $B(L, G_1; \mathcal{F}_1^L)$ is foliated then the connection ω_1^L is foliated.*

b) *If $\underline{g}_1^{(1)} = 0$ and the G_1–structure $B(M, G_1; \mathcal{L})$ is foliated then the connection ω_1 is foliated and $i_L^* \omega_1 = \omega_1^L$.*

c) *If $\underline{g}_2^{(1)} = 0$ and the G_2–structure $B(M, G_2; \mathcal{F}_2)$ is $\mathcal{F}_i$–foliated ($i = 1, 2$), then the connection ω_2 is $\mathcal{F}_i$–foliated.*

Summing up, we have proved the following:

Theorem 6 *Let $\mathcal{F}$ be a flag structure on a manifold M with an adapted transverse $G_1 \times G_2$–structure $B(M, G_1 \times G_2; \mathcal{F}_1)$.*

i) *If the G_1–structure $B(M, G_1; \mathcal{F})$is foliated, $\underline{g}_1^{(1)} = 0$ and there exists a G_1–invariant subspace S of $Hom(\mathbb{R}^p \wedge \mathbb{R}^p, \mathbb{R}^p)$ supplementary to $\partial Hom(\mathbb{R}^p, \underline{g}_1)$, then there exists a transverse connection ω in the transverse $G_1 \times GL(q)$–structure extending $B(M, G_1 \times G_2; \mathcal{F}_1)$ such that the connection ω_1 induced by ω on $B(M, G_1; \mathcal{F})$ is the unique foliated G_1–connection whose torsion tensor takes values in the subspace S and ω_2 is a basic connection for the foliation $\mathcal{F}_2$.*

ii) *If the G_2–structure $B(M, G_2; \mathcal{F}_2)$ is foliated, $\underline{g}_2^{(1)} = 0$ and there exists a G_2–invariant subspace S of $Hom(\mathbb{R}^q \wedge \mathbb{R}^q, \mathbb{R}^q)$ supplementary to $\partial Hom(\mathbb{R}^q, \underline{g}_2)$, then there exists a transverse connection ω in the transverse $GL(p) \times G_2$–structure extending $B(M, G_1 \times G_2; \mathcal{F}_1)$ such that the connection ω_2 induced by ω on $B(M, G_2; \mathcal{F}_2)$ is the unique foliated G_2–connection whose torsion tensor takes values in S and ω_1 is a basic connection for $\mathcal{F}_1$.*

iii) *If the G_1–structure $B(M,G_1;\mathcal{F})$ is $\mathcal{F}_1$–foliated, the G_2–structure $B(M,G_2;\mathcal{F}_2)$ $\mathcal{F}_2$–foliated, $\underline{g}_1^{(1)} = \underline{g}_2^{(1)} = 0$ and there exist G_i–invariant subspaces S_i of $Hom(\mathbb{R}^p\wedge\mathbb{R}^p,\mathbb{R}^p)$ and $Hom(\mathbb{R}^q\wedge\mathbb{R}^q,\mathbb{R}^q)$ supplementary to $\partial Hom(\mathbb{R}^p,\underline{g}_1)$ and $\partial Hom(\mathbb{R}^q,\underline{g}_2)$, $i = 1,2$, respectively, then there exists a transverse connection ω in the $G_1\times G_2$–structure $B(M,G_1\times G_2;\mathcal{F}_1)$ such that the induced connection ω_1 in $B(M,G_1;\mathcal{F})$ is the unique foliated G_1–connection whose torsion tensor has values in S_1 and the induced connection ω_2 in $B(M,G_2;\mathcal{F}_2)$ is the unique foliated G_2–connection whose torsion tensor has values in S_2.*

iv) *If there exist a $G_1\times G_2$–subspace S of $Hom(\mathbb{R}^{p+q}\wedge\mathbb{R}^{p+q},\mathbb{R}^{p+q})$ supplementary to $\partial Hom(\mathbb{R}^{p+q},\underline{g}_1\oplus\underline{g}_2)$ which agrees with the natural decomposition of $Hom(\mathbb{R}^{p+q}\wedge\mathbb{R}^{p+q},\mathbb{R}^{p+q})$ and $(\underline{g}_1\oplus\underline{g}_2)^{(1)} = 0$, then there exists the unique foliated connection ω in $B(M,G_1\times G_2;\mathcal{F}_1)$ whose structure tensor has values in S and which induces the unique foliated connections ω_1 and ω_2 in $B(M,G_1;\mathcal{F})$ and $B(M,G_2;\mathcal{F}_2)$ whose torsion tensors have values in S_1 and S_6, respectively.*

As a consequence of this theorem, using the standard methods, we get the following corollary.

Corollary 8 i) *If the assumptions of Theorem 6 i) are satisfied then the Chern–Weil homomorphim for the foliation $\mathcal{F}_1$ vanishes on*

$$c = c_1\otimes c_2 \in I^r(\underline{g}_1)\otimes I^s(\underline{gl}(q))$$

for $2(r+s) > p+2q$.

ii) *If the assumptions of Theorem 6 ii) are satisfied then the Chern–Weil homomorphism for the foliation $\mathcal{F}_1$ vanishes on*

$$c = c_1\otimes c_2 \in I^r(\underline{gl}(p)) \oplus I^s(\underline{g}_2)$$

for $r+2s > p+q$ or $s > [q/2]$.

iii) *If the assumptions of Theorem 6 iii) are satisfied then the Chern–Weil homomorphism for the foliation $\mathcal{F}_1$ vanishes on*

$$c = c_1\otimes c_2 \in I^r(\underline{g}_1) \oplus I^s(\underline{g}_2)$$

for $r+2s > p/2+q$.

Remarks 1) Corollary 8 i) generalizes the result of F. J. Carreras and A. M. Naveira, cf. [CN].

2) By imposing the condition '$\mathcal{F}_1$–foliated' on $B(M, G_1 \times G_2; \mathcal{F}_1)$ and using Corollary 8 we obtain algebraic obstructions to the existence of structures with the above mentioned properties.

3) For examples of flag structures and their characteristic classes see [DO], and also [GM,MI,NA].

4) The secondary characteristic classes involving the unique connection obtained from Theorem 6 behave under deformations in the same way as the secondary classes of Riemannian foliations, cf. [LP1,LP2], or, more generally, as those of $\nabla - G$–foliations, cf. [WO3].

5) For most considerations of this paper we do not need the assumption of the integrability of the bundle $T\mathcal{F}_2$. We could have just considered the almost product structure $T\mathcal{F}_1 \oplus Q_1 \oplus Q_2$ and its characteristic classes or the characteristic classes of the foliation $\mathcal{F}_1$ obtained from the reduction of the normal bundle to the structure group $G_1 \times G_2$.

6) For example, in Theorem 6 ii) and iii) we can replace the assumption '$\mathcal{F}_2$–foliated' by '$\mathcal{F}_1$–foliated'. Then we obtain the following:

Corollary 8 iv) *If the above assumptions are satisfied, then the Chern–Weil homomorphism for the foliation $\mathcal{F}_1$ vanishes on*

$$c_1 \otimes c_2 \in I^r(\underline{g}_1) \otimes I^s(\underline{g}_2) \text{ for } r > [p + q/2] \text{ or } s > [p + q/2].$$

IV.4 Proof scheme

To complete this chapter we look at the results of R. A. Blumenthal in the light of our considerations on foliated structures. In [BL1,BL2,BL3,BL4] he considers foliations admitting a transversely projectable connection whose curvature and torsion tensors have some additional properties. These properties, the connection being transversely projectable, can be read as the properties of the curvature and torsion tensors, respectively, of the corresponding holonomy invariant connection on the transverse manifold. In Blumenthal's case well–known theorems ensure that we can choose a very good representative of the holonomy pseudogroup. It is precisely what Blumenthal did in each case. In fact, as a holonomy pseudogroup representative we can take a subpseudogroup of the pseudogroup obtained as the localization of an quasi–analitical action of a Lie group K on a connected simply–connected manifold N_0. Therefore, our foliation is an (N_0, K)–structure and a developable foliation. The completeness assumptions ensure that the developing mapping is a locally trivial fibre bundle, and this fact yields the most important results.

At the basis of these four papers is the following scheme:

- Realize that the considered geometric objects on the foliated manifold are foliated and pass to the holonomy invariant ones.
- Using theorems on local equivalence of geometric structures choose a good representative of the holonomy pseudogroup.
- Check that any element of the holonomy pseudogroup can be uniquely extended to a global one.
- Verify that "completeness" assumptions ensure that the developing mapping is a locally trivial fibre bundle; for example the foliated differential equation defined by the foliated geometric object is a transversely complete foliated differential equation and it projects onto a differential equation on the developing image.

The same scheme has been applied to foliations admitting Cartan connections in [BL6].

Having formulated our proof scheme we can prove the results contained in [BL1,BL2,BL3,BL4] quite easily. We only need some results on local equivalence of reductive and locally symmetric spaces, see for example [KN,WF]. In the case of Cartan connections we should take into account Lemma 11.10 and Proposition 11.1 of [OC]. Then the equation of the geodesic of the transversely projectable connection is a tranversely complete foliated differential equation corresponding to the equation of the geodesic of the connection of the transverse manifold. Therefore the developing mapping must be locally trivial.

The theory of foliated Cartan connections is very similar to its classical counterpart, cf. [OC]. Let us restrict our attention to G–structures of second order modelled on a semi–simple homogeneous space. Let $B(M,G;\mathcal{F})$ be such a foliated G–structure. Its total space B is foliated by a foliation $\mathcal{F}_B$. As in the case of 1st order G–structures foliated Cartan connections do not always exist. The standard construction (using the partition of unity and local existence) ensures the existence of basic Cartan connections, i.e. 1–forms on B with values in $Lie(L) = \underline{l}$ vanishing on vectors tangent to the foliation $\mathcal{F}_B$. It is also possible to introduce the notion of basic admissible Cartan connections. The vanishing of the Spencer cohomology group $H^{2,1}(\underline{l})$ ensures the existence of a foliated Cartan connection, cf. Theorem 1.1 of [TA]. In fact, the vanishing of this cohomology group makes sure that in the corresponding G–structure on the transverse manifold there exists the normal Cartan connection and that this connection is holonomy invariant. Thus it defines a foliated Cartan connection in $B(M,G;\mathcal{F})$ which we call the normal foliated Cartan connection of this G–structure. Next one can define its Weyl tensor, and it is not difficult to verify that it is a foliated tensor. In some cases its vanishing ensures that the model G–structure is flat. This leads us to formulate the following theorem which is a generalization of Theorem 2 of [BL6].

Theorem 7 *Let $B(M,G;\mathcal{F})$ be a foliated G–structure of second order modelled on a semi–simple flat homogeneous space L/L_0 such that $H^{2,1}(\underline{l}) = H^{2,2}(\underline{l}) = H^{1,1}(\underline{l}) = 0$. If the normal foliated Cartan connection of this structure is complete, and the Weyl tensor of $B(M,G;\mathcal{F})$ vanishes, then $\mathcal{F}$ is an $(L/L_0, L)$–structure and the developing mapping $h: \tilde{M} \longrightarrow \widetilde{L/L_0}$ ($\widetilde{L/L_0}$ is the universal covering space of L/L_0) is a locally trivial fibre bundle whose fibres are the leaves of the lifted foliation $\tilde{\mathcal{F}}$.*

Proof We follow our proof scheme. Since the Weyl tensor is foliated, the Weyl tensor of the corresponding G–structure on the transverse manifold vanishes. Ochai's theorem (cf. [OC] Theorem 12.1) ensures that the normal Cartan connection is flat. Then according to Proposition 11.1 and Lemma 11.10 of the same paper the foliation $\mathcal{F}$ is an $(L/L_0, L)$–structure. The completeness of the normal foliated Cartan connection means that the foliation $\mathcal{F}_B$ is a complete transversely parallelisable foliation, cf. [BL6], and that the equation of the geodesic of this connection is complete. This first fact ensures that the holonomy coverings of leaves of $\mathcal{F}$ are diffeomorphic, and the second that the developing mapping is a locally trivial fibre bundle.□

Let us provide some more background material for our 'proof scheme'. Local equivalence of geometric structures have been studied for many years. The best account can be found in [AM], see also [MO7]. Geometers looked for a set of invariants of $\mathcal{K}$–structures ($\mathcal{K}$ a Lie pseudogroup) which would ensure that any two $\mathcal{K}$–structures having the same invariants are locally equivalent. Having defined the structure tensors, it was necessary to determine whether the formal integrability (i.e. all structure tensors vanish) is equivalent to the integrability (i.e. the $\mathcal{K}$–structure is locally equivalent to the corresponding canonical flat structure on $\mathbb{R}^n$). It is true for any G–structure, as well as for many other $\mathcal{K}$–structures, cf. [AM]. We can say a little more about G–structures of finite type: two G–structures with the same constant structure tensors are locally equivalent, cf. [KO2,SB]. Therefore transverse G–structures of foliated G–structures of finite type with the same constant structure tensors are locally equivalent. Thus any foliated G–structure of finite type with vanishing structure tensors can be modelled on the canonical flat G–structure of $\mathbb{R}^q$. The group of automorphisms of such a G–structure $B(N_0, G)$ of finite type is a Lie group, and its elements are determined by their finite jets, cf. [KO2]. Therefore, this group acts quasi–analytically on the manifold N_0.

Having chosen a model $(\mathcal{K}, N)$ of the transverse structure of the foliation $\mathcal{F}$, with N being a connected manifold, we would like to know whether the pseudogroup $\mathcal{K}$ is generated by a group, i.e. whether any element of $\mathcal{K}$ can be uniquely extended to a global diffeomorphism of N. It is a very well known problem, and there are many theorems of this type. We have already used some of them. They are based on the following principle:

There exists a fibre bundle $B(N)$ over N whose total space is parallelisable and which has the following properties:

- the vector space spanned by vector fields of the parallelism consists of complete vector fields,
- the group generated by the flows of these vector fields acts freely (i.e. $f(x) = x$ implies $f = id$),
- elements of the pseudogroup $\mathcal{K}$ lift to local diffeomorphisms of the total space of $B(N)$ which commute with the parallelism,
- any local diffeomorphism commuting with the parallelism is, locally, the lift of an element of $\mathcal{K}$.

Then, of course, any element of $\mathcal{K}$ can be uniquely extended to a global diffeomorphism of N. In some cases it is possible to verify directly that the pseudogroup of automorphisms of a given geometric structure has this extension property.

In general, it is easier to solve the infinitesimal version of this problem, (as it only concerns solutions of systems of linear differential equations): can any local $\mathcal{K}$–vector field be extended to a global one?, cf. [NO2,LO,AR]. Having a positive answer to this question does not solve the extension problem for the pseudogroup $\mathcal{K}$. First of all, we have to know whether any element of $\mathcal{K}$, at least locally, can be represented as the composition of a finite number of elements of flows of $\mathcal{K}$–vector fields and whether global $\mathcal{K}$–vector fields are complete. The first one have been studied thoroughly in the framework of Lie pseudogroups and we have a definite answer, cf. [RO] Propositions 3.6 and 3.7. The second one is just the question whether a certain differential equation has global solutions.

The above considerations lead to several interesting results concerning G–foliations.

Theorem 8 *Let N be a simply connected compact analytic manifold with an analytic G–structure $B(N,G)$ of finite type. Let $\mathcal{H}$ be a connected regular pseudogroup of local analytic automorphisms of $B(N,G)$. Then any $\mathcal{H}$–foliation $\mathcal{F}$ is developable. Moreover, if the G–foliation $\mathcal{F}$ is transversely complete, then the developing mapping is a locally trivial fibre bundle whose fibres are the leaves of the lifted foliation.*

Proof Our proof scheme takes care of everything but the fact that the foliation $\mathcal{F}$ is an (N,K)–structure. Proposition 3.6 or 3.7 of [RO] ensures that any element of the pseudogroup $\mathcal{H}$, locally, can be represented as the composition of a finite number of local diffeomorphisms from flows of $\mathcal{H}$–vector fields. The theorem of Amores, cf. [AR], makes sure that any local (analytic) infinitesimal automorphism of the G–structure $B(N,G)$ can be extended to a global one, and as the manifold

N is compact, these vector fields are complete. Thus $\mathcal{F}$ is an $(N, Aut(B(N,G)))$–structure. The local triviality of the developing mapping results from the fact that the transversely complete TUSP foliated system of differential equations on $(M,\mathcal{F})$ defined by the transverse parallelism of $\mathcal{F}_k$ projects via the developing mapping onto a system of differential equations defined by the parallelism of $B^{(k-1)}(N,G)$.□

The results of A.Y.Ledger and M.Obata lead us to the formulation of the following theorem, cf. [LO].

Theorem 9 *Let $(M,\mathcal{F})$ be a conformal but non–Riemannian transversely analytic foliation of codimension q $(q > 2)$. Then:*

1. *Let the foliation $\mathcal{F}$ be modelled on a compact Riemannian analytic manifold with finite fundamental group whose pseudogroup $\mathcal{C}$ of local conformal transformations is a regular Lie pseudogroup. If the holonomy pseudogroup of $\mathcal{F}$ is contained in the connected component of id in $\mathcal{C}$ then the foliation $\mathcal{F}$ is developable and the developing mapping is into the q–sphere S^q.*

2. *If, additionally, the foliation $\mathcal{F}$ is transversely complete, then the developing mapping is a locally trivial fibre bundle with fibres being leaves of the lifted foliation.*

Taking into account the results of A.Y.Ledger and M.Obata the proof of this theorem is the same as that of Theorem 8.

Theorem 10 *Let $(M,\mathcal{F})$ be a G–foliation of type 1 with vanishing structure tensors. If the holonomy pseudogroup $\mathcal{H}$ on the transverse manifold N is contained in the connected component of id of the pseudogroup of local automorphisms of the G–structure $B(N,G)$, then the foliation $\mathcal{F}$ is developable. Moreover, if $\mathcal{F}$ is transversely complete, then the developing mapping is a locally trivial fibre bundle over $\mathbb{R}^q$ with fibres being the leaves of the lifted foliation.*

Proof The vanishing of the structure tensors of the foliated G–structure $B(M,G;\mathcal{F})$ ensures that the structure tensors of $B(N,G)$ also vanish. Therefore the G–structure $B(N,G)$ is integrable, cf. [KO2,SB], and the foliation $\mathcal{F}$ is modelled on the canonical flat G–structure of $\mathbb{R}^q$. Since the group G is of type 1, only the vector fields of the form $\sum_{j=1}^q a_i^j \partial_j$ or $\sum_{i,j=1}^q a_i^j x_j \partial_i$ $((a_j^i) \in \underline{g})$ are infinitesimal automorphisms of this flat G–structure, and any local infinitesimal automorphism can be extended to a global one. It is not difficult to see that these vector fields are complete. The rest of the proof is standard.□

Using our proof scheme we can produce some other theorems of this kind. For example, by imposing conditions on the transverse sectional curvature of a Riemannian foliation we obtain:

Theorem 11 *Let $(M, \mathcal{F}, g)$ be a complete Riemannian foliation. If the transverse sectional curvature is constant or depends only on the point of the manifold M, then the foliation $\mathcal{F}$ is developable, and the developing mapping is a locally trivial fibre bundle over $\mathbb{R}^q$, S^q or $\mathbb{H}^q$ with fibres being the leaves of the lifted foliation.*

Proof The second condition ensures that the transverse sectional curvature is constant (a foliated version of the Schur lemma which can be easily proved or deduced from the original one). Then the theorem on local isometries of such Riemannian manifolds, cf. [KN,WF], and our proof scheme take care of the rest.□

There are many applications of these theorems. We shall give only some of them, see also [BL1,BL2,BL4,BL6]. We assume that $\mathcal{F}$ is transversely complete and that the assumptions of one of the theorems are satisfied.

Corollary 9 *Let N be a contractible q–manifold and K a Lie group acting quasi–analytically on it. If the foliation $\mathcal{F}$ is an (N, K)–structure then the universal covering space $\tilde{M}$ of the manifold M is the product $\tilde{L} \times N$, where $\tilde{L}$ is the common universal covering space of leaves of the foliation $\mathcal{F}$.*

Proof In this case the developing mapping is a trivial bundle.□

Corollary 10 *On a compact manifold with a finite fundamental group there are no G–foliations satisfying the assumption of Theorem 10.*

Proof In this case, the universal covering space would be both compact and diffeomorphic to $\tilde{L} \times \mathbb{R}^q$; contradiction.□

Corollary 11 *Let Φ be a flow on a compact manifold with finite fundamental group. If the flow Φ admits a foliated structure satisfying the assumptions of one of the theorems then its leaves have finite holonomy. If all its orbits are closed then Φ is a Riemannian flow.*

Proof Assume the contrary. Let L be a leaf with infinite holonomy. Its holonomy covering cannot be compact. Thus the developing mapping is a locally trivial fibre bundle whose total space is compact but whose fibres are diffeomorphic to $\mathbb{R}$. Contradition. Then the second assertion follows from [EP], see also Chapter VI.□

Notes The results of this chapter has been published in four papers of ours. Additionally, we have included some basic results of P. Molino from [MO1], [MO2] and [MO11]. Section 1 is based on [WO11]. Section 2 presents results of [WO4] and a part of [WO11]. Section 3 contains the most important part of [WO11]. The last section reproduces a part of [WO13].

There have been published many papers on the theory of G-structures, in particular on characteristic classes of these foliations. Let us mention only some of them: [BL5], [BL7], [CO1], [DU], [HU1], [KT1], [KT2], [MN], [MY1], [MT2], [NT], [RM], [RG1], [TA], [VN1], [VN2]. At the end we would like to add that in [WO6] and [WO7] we developed the theory of liftings of foliated tensor fields.

Chapter V

Transversely Hermitian and transversely Kähler foliations

In this chapter we are going to study transversely Kähler foliations. They form a very interesting class of foliations studied by many authors. First we present some non-trivial examples of which show that the classes of transversely Hermitian, transversely Kähler and transversely symplectic foliations are, in fact, disjoint. Then we study the base-like cohomology of manifolds foliated by a transversely Kähler foliation, which is illustrated by more examples. The work presented in Sections 1 and 2 was done jointly with Luis A. Cordero.

V.1 Examples

Let N be a simply connected nilpotent Lie group and Γ a torsionfree, finitely generated subgroup of N. Then according to [MA], or [RA], Theorems 2.11 and 2.18, there exist a simply connected nilpotent Lie group U containing Γ as a uniform subgroup and a homomorphism $u : U \longrightarrow N$ such that u is the identity on Γ (if we identify the subgroups of U and N isomorphic to Γ). So we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma \subset U & & \\ \uparrow \cong & \searrow^{u} & \\ \Gamma & \longrightarrow & N \end{array}$$

The homomorphism u is a surjective submersion with connected fibres since both manifolds U and N are contractible. The foliation defined by the submersion u is Γ–invariant and therefore it projects to a foliation $\mathcal{F}(\Gamma, U, u)$ on the compact manifold $M(\Gamma) = \Gamma \setminus U$. The foliation $\mathcal{F}(\Gamma, U, u)$ is an (N, Γ)–structure, a developable one, and the submersion u is its developing mapping. Therefore, foliated geometric structures on $(M(\Gamma), \mathcal{F}(\Gamma, U, u))$ correspond bijectively to Γ–invariant ones on N.

If the subgroup Γ contains a uniform subgroup Γ_0 of N, then any foliated geometric structure on $(M(\Gamma), \mathcal{F}(\Gamma, U, u))$ defines a geometric structure of the same type on the compact manifold $E(\Gamma_0) = \Gamma_0 \setminus N$. The following diagram presents this correspondence:

$$\begin{array}{ccc} \Gamma \subset U & \xrightarrow{\quad u \quad} & N \supset \Gamma \supset \Gamma_0 \\ \downarrow & & \downarrow \\ \Gamma \setminus U & \dashrightarrow & \Gamma_0 \setminus N \end{array}$$

In fact, any foliated geometric structure on $(M(\Gamma), \mathcal{F}(\Gamma, U, u))$ lifts to a Γ–invariant foliated structure on U. This one, in its turn, defines a Γ–invariant structure on N which projects to a geometric structure on $E(\Gamma_0) = \Gamma_0 \setminus N$.

For example, if $(M(\Gamma), \mathcal{F}(\Gamma, U, u))$ is

– transversely symplectic then $E(\Gamma_0)$ is symplectic,

– transversely holomorphic then $E(\Gamma_0)$ is complex,

– transversely Kähler then $E(\Gamma_0)$ is Kähler,

– transversely Hermitian then $E(\Gamma_0)$ is Hermitian,

and so on.

We are now going to present examples of foliations on compact nilmanifolds which are

1. transversely symplectic but never transversely Kähler;

2. transversely symplectic and holomorphic but never transversely Kähler, on complex and non–complex nilmanifolds;

3. transversely symplectic but not transversely holomorphic;

4. transversely Sasakian but never transversely cosymplectic.

Our examples are based on the Luis A. Cordero work with M. Fernández, A. Gray and others on geometric structures on compact nilmanifolds.

The general scheme for the constructions is the following. First, we consider a simply connected nilpotent group N of uppertriangular matrices and $\Gamma_0 \subset N$ the uniform subgroup of matrices with integral entries. Next, we a take a subgroup Γ of N whose matrices have some entries of the form $a_i + s\, b_i$ where $s \notin \mathbb{Q}$ and $a_i, b_i \in \mathbb{Z}$. This subgroup Γ can be represented as a uniform subgroup of some group of uppertriangular matrices, which will be the group U of the construction described above.

Let us pass to the precise examples.

V.1.1 A transversely symplectic but not transversely Kähler foliation

Let us consider $N = (\mathbb{R}^4, *)$ with the following group operation:

$$(a,b,c,d) * (x,y,z,t) = (a+x, b+y, c+z+ay, d+t) \quad .$$

In the matrix form the group $N = (\mathbb{R}^4, *)$ can be represented as follows:

$$\begin{pmatrix} 1 & x & t & z \\ & 1 & 0 & y \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} .$$

As the group Γ_0 we take $(\mathbb{Z}^4, *)$, and as $\Gamma \supset \Gamma_0$ we take the group of matrices of the form

$$\begin{pmatrix} 1 & x_1 + s\, x_2 & t & z_1 + s\, z_2 \\ & 1 & 0 & y \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} ,$$

where $s \notin \mathbb{Q}$ and $x_1, x_2, y, z_1, z_2, t \in \mathbb{Z}$.

In this case the group U is $\mathbb{R}^6$ with the following group operation:

$$(a_1, a_2, b, c_1, c_2, d) \square (x_1, x_2, y, z_1, z_2, t)$$
$$= (a_1 + x_1, a_2 + x_2, b + y, c_1 + z_1 + a_1 y, c_2 + z_2 + a_2 y, d + t) \ ,$$

and the subgroup Γ is isomorphic to $(\mathbb{Z}^6, \square)$.

The submersion $u : U = (\mathbb{R}^6, \square) \longrightarrow N = (\mathbb{R}^4, *)$ is given by the correspondence

$$(x_1, x_2, y, z_1, z_2, t) \mapsto (x_1 + s\, x_2, y, z_1 + s\, z_2, t) \quad .$$

The foliation obtained in this way cannot be transversely Kähler because $E(\Gamma_0) = \Gamma_0 \backslash N$ is Kähler if and only if N is commutative (cf. [BG,CFG3]). This foliation is transversely symplectic as the form

$$\Omega = dx \wedge (dz - x\, dy) + dy \wedge dt$$

is a closed left invariant 2–form on N of maximal rank.

Summing up, we have constructed a transversely symplectic foliation $\mathcal{F}(\Gamma, U, u)$ of codimension 4 on a real compact manifold $M(\Gamma)$ of dimension 6, and such a foliation cannot be made transversely Kähler.

V.1.2 Transversely symplectic and transversely holomorphic but not transversely Kähler foliations on real and complex manifolds

Let $E(\Gamma_0) = \Gamma_0 \setminus N$ be the Iwasawa manifold, i.e. N is the complex Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_3 \\ & 1 & z_2 \\ & & 1 \end{pmatrix} ,$$

and Γ_0 is the subgroup of N of those matrices whose entries are Gauss integers.

A basis of holomorphic left invariant 1-forms on N is given by

$$\begin{aligned} \alpha &= dz_1 , \\ \beta &= dz_2 , \\ \gamma &= dz_3 - z_1\, dz_2 , \end{aligned}$$

and it verifies

$$d\alpha = 0 \ , \quad d\beta = 0 \ , \quad d\gamma = -\alpha \wedge \beta \ .$$

Let us put

$$\begin{aligned} \alpha &= \alpha_1 + \sqrt{-1}\,\alpha_2 , \\ \beta &= \beta_1 + \sqrt{-1}\,\beta_2 , \\ \gamma &= \gamma_1 + \sqrt{-1}\,\gamma_2 . \end{aligned}$$

Then

$$\Omega = \alpha_1 \wedge \gamma_1 - \alpha_2 \wedge \gamma_2 + \beta_1 \wedge \beta_2$$

is a left invariant symplectic form on N.

Let $\Gamma_1 \supset \Gamma_0$ be the subgroup of N of matrices of the form

$$\begin{pmatrix} 1 & x_1 + \sqrt{-1}\,(y_1 + s\, y_1') & x_3 + s\, x_3' + \sqrt{-1}\,(y_3 + s\, y_3') \\ & 1 & x_2 + \sqrt{-1}\, y_2 \\ & & 1 \end{pmatrix} ,$$

where $s \notin \mathbb{Q}$, and $x_i, x_i', y_i, y_i' \in \mathbb{Z}$.

Then Γ_1 can be considered as a uniform subgroup of $\mathbb{R}^9$ with the group operation:

$$\begin{aligned} (a_1, \ldots, a_9)\Box(x_1, \ldots, x_9) = (&a_i + x_i, a_6 + x_6 + a_1x_4 - a_2x_5, \\ &a_7 + x_7 - a_3x_5, a_8 + x_8 + a_1x_5 + a_2x_4, a_9 + x_9 + a_3x_4) \ . \end{aligned}$$

The matricial form of $U = (\mathbb{R}^9, \square)$ is the following:

$$\begin{pmatrix} 1 & x_3 & x_7 & x_9 & x_1 & x_2 & x_6 & x_8 \\ & 1 & -x_5 & x_4 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & x_4 & x_5 \\ & & & & & 1 & -x_5 & x_4 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix} .$$

The submersion $u : U = (\mathbb{R}^9, \square) \longrightarrow N$ is given by the correspondence:

$$(x_1, \dots, x_9) \mapsto (x_1 + \sqrt{-1}(x_2 + s\,x_3), x_4 + \sqrt{-1}x_5, x_6 + s\,x_7 + \sqrt{-1}(x_8 + s\,x_9)) .$$

In this case, the resulting foliation $\mathcal{F}(\Gamma_1, U, u)$ is transversely symplectic and transversely holomorphic of complex codimension 3 on a real compact manifold $M(\Gamma_1)$ of real dimension 9, and it cannot be made transversely Kähler (cf. [CFG]).

A completely different example arises if we consider the group $\Gamma_2 \supset \Gamma_0$ of matrices of the form

$$\begin{pmatrix} 1 & z_1 + s\,z_1' & z_3 + s\,z_3' \\ & 1 & z_2 \\ & & 1 \end{pmatrix} ,$$

where $s \notin \mathbb{Q}$ and $z_1, z_1', z_2, z_3, z_3'$ are Gauss integers.

The subgroup Γ_2 can be considered as a uniform subgroup of $\mathbb{C}^5$ with the following group operation:

$$(u_1, \dots, u_5)\square(z_1, \dots, z_5) = (z_i + u_i, z_4 + u_4 + u_1 z_3, z_5 + u_5 + u_2 z_3) .$$

The submersion $u : U = (\mathbb{C}^5, \square) \longrightarrow N$ is given by the correspondence

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1 + s\,z_2, z_3, z_4 + s\,z_5) .$$

The foliation $\mathcal{F}(\Gamma_2, U, u)$ obtained in this way is holomorphic and transversely symplectic of complex codimension 3 on a compact complex nilmanifold $M(\Gamma_2)$ of complex dimension 5, and it cannot be made transversely Kähler either.

The group $(\mathbb{C}^5, \square)$ can be represented in the matrix form as

$$\begin{pmatrix} 1 & z_1 & z_4 & z_2 & z_5 \\ & 1 & z_3 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & z_3 \\ & & & & 1 \end{pmatrix} ,$$

and the submersion $u : U = (\mathbb{C}^5, \square) \longrightarrow N$ is given by

$$\begin{pmatrix} 1 & z_1 & z_4 & z_2 & z_5 \\ & 1 & z_3 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & z_3 \\ & & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & z_1 + s\, z_2 & z_4 + s\, z_5 \\ & 1 & z_3 \\ & & 1 \end{pmatrix} .$$

Remark Examples of this type have been considered, from another point of view, by A. El Kacimi and M. Nicolau in [EN1]; there, only the non–existence of transverse Kähler structures is studied. Nevertheless, the existence of foliated symplectic structures has a significant influence on the cohomological structure of the foliated manifold (cf. [BE]).

V.1.3 A transversely symplectic but not transversely Kähler foliation on a compact non–complex nilmanifold

Let N be the real nilpotent Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & \bar{z}_1 & z_2 \\ & 1 & z_1 \\ & & 1 \end{pmatrix}$$

where $z_1, z_2 \in \mathbb{C}$, and let Γ_0 be the uniform subgroup of matrices with Gauss integer entries. Then $E(\Gamma_0) = \Gamma_0 \backslash N$ is the well known Kodaira-Thurston manifold (cf. [FGG]).

Let us consider the left invariant 1–forms over N given by:

$$\begin{aligned} \alpha &= dz_1 \ , \\ \beta &= dz_2 - \bar{z}_1\, dz_2 \ ; \end{aligned}$$

they define real 1-forms

$$\begin{aligned} \alpha_1 &= dx_1 \ , \\ \alpha_2 &= dy_1 \ , \\ \beta_1 &= dx_2 - x_1\, dx_1 - y_1\, dy_1 \ , \\ \beta_2 &= dy_2 - x_1\, dy_1 + y_1\, dx_1 \ , \end{aligned}$$

where $z_1 = x_1 + \sqrt{-1}\, y_1$, $z_2 = x_2 + \sqrt{-1}\, y_2$. Then

$$d\alpha_1 = d\alpha_2 = d\beta_1 = 0 \ , \quad d\beta_2 = -2\alpha_1 \wedge \alpha_2 \ .$$

Therefore, the 2–form

$$\Omega = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2$$

is a left invariant symplectic form on N.

As the group $\Gamma \supset \Gamma_0$ we take the group of matrices of the form

$$\begin{pmatrix} 1 & \bar{z}_1 + s\,\bar{z}_1' & z_2 + s\,z_2' + s^2\,z_2'' \\ & 1 & z_1 + s\,z_1' \\ & & 1 \end{pmatrix} ,$$

where $s \notin \mathbb{Q}$ and z_1, z_1', z_2, z_2' and z_2'' are Gauss integers.

The group Γ can be considered as the uniform subgroup of Gauss integers 5-tuples in $(\mathbb{C}^5, \square)$, where $\square$ is the following group operation:

$$(a_1, \ldots, a_5)\square(z_1, \ldots, z_5) = (a_i + z_i, a_3 + z_3 + \bar{a}_1 z_1, \\ a_4 + z_4 + \bar{a}_1 z_2 + \bar{a}_2 z_1, a_5 + z_5 + \bar{a}_2 z_2) .$$

The group $(\mathbb{C}^5, \square)$ can be represented in matricial form by

$$\begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 & z_3 & z_4 & z_5 \\ & 1 & 0 & z_1 & z_2 & 0 \\ & & 1 & 0 & z_1 & z_2 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} .$$

The submersion $u : U = (\mathbb{C}^5, \square) \longrightarrow N$ is given by the correspondence

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1 + s\,z_2, z_3 + s\,z_4 + s^2\,z_5) .$$

The foliation $\mathcal{F}(\Gamma, U, u)$ constructed in this example is transversely symplectic and transversely holomorphic of complex codimension 2 on a complex manifold $M(\Gamma)$ of complex dimension 5 (which is a real nilmanifold but not a complex nilmanifold, cf. [CFG3]).

V.1.4 A transversely symplectic but not transversely holomorphic foliation

To construct this example we shall consider a compact 4-dimensional nilmanifold which is symplectic but does not admit any complex structure (cf. [FGG] and [CM]).

Let N be the 4-dimensional Lie group of real matrices of the form

$$\begin{pmatrix} 1 & y & 2x & -(z/n) & -(t/q) \\ & 1 & 0 & 0 & 0 \\ & & 1 & 2y & ny^2/2 \\ & & & 1 & ny/2 \\ & & & & 1 \end{pmatrix} ,$$

where $n, q \in \mathbb{Z}$ are nonzero and fixed. Then $N = (\mathbb{R}^4, *)$, where

$$(a,b,c,d) * (x,y,z,t) = (a+x, b+y, c+z-2n\,ay, d+t-nq\,ay^2+q\,cy)\ .$$

As the subgroup Γ_0 we take the integer lattice, and then the compact nilmanifold $E(\Gamma_0) = \Gamma_0 \setminus N$ is symplectic but not complex (cf. [FGG]).

Now, let us consider the group Γ of the matrices of the form

$$\begin{pmatrix} 1 & b & 2(a+s\,a') & -2(c+s\,c')/n & -(d+s\,d')/q \\ & 1 & 0 & 0 & 0 \\ & & 1 & 2b & nb^2/2 \\ & & & 1 & nb/2 \\ & & & & 1 \end{pmatrix}\ ,$$

where $s \notin \mathbb{Q}$ and $a, a', b, c, c', d, d' \in \mathbb{Z}$.

Then Γ can be imbedded as a uniform subgroup of $\mathbb{R}^7$ with the following group operation:

$$\begin{aligned}(a_1,\ldots,a_7)\square(x_1,\ldots,x_7) &= (a_i+x_i, a_4+x_4-2n\,a_1x_3, a_5+x_5-2n\,a_2x_3,\\ &\quad a_6+x_6-nq\,a_1x_3^2+q\,a_4x_3, a_7+x_7-nq\,a_2x_3^2+q\,a_5x_3)\ .\end{aligned}$$

The group $U = (\mathbb{R}^7, \square)$ can be represented as the following group of matrices:

$$\begin{pmatrix} 1 & x_3 & 2x_1 & 2x_2 & -2x_4/n & -2x_5/n & -x_6/q & -x_7/q \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 2x_3 & 0 & nx_3^2/2 & 0 \\ & & & 1 & 0 & 2x_3 & 0 & nx_3^2/2 \\ & & & & 1 & 0 & nx_3/2 & 0 \\ & & & & & 1 & 0 & nx_3/2 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}\ .$$

The submersion $u : U \longrightarrow N$ is given by the correspondence

$$(x_1,\ldots,x_7) \mapsto (x_1+s\,x_2, x_3, x_4+s\,x_5, x_6+s\,x_7)\ .$$

The foliation $\mathcal{F}(\Gamma, U, u)$ is, in this case, transversely symplectic of real codimension 4 on a real compact nilmanifold $M(\Gamma)$ of real dimension 7, and it cannot be made transversely holomorphic (since the results in [FGG] ensure that $E(\Gamma_0) = \Gamma_0 \setminus N$ is never a complex manifold).

V.1.5 A transversely Sasakian but not transversely cosymplectic foliation

Let $N = H(r,1)$, $r \geq 1$, be the Heisenberg group of real matrices of the form

$$\begin{pmatrix} 1 & A & c \\ & I_r & B \\ & & 1 \end{pmatrix}$$

where A is a $1 \times r$ matrix, B is an $r \times 1$ matrix and c is a real number. As the subgroup Γ_0 we take the group of matrices of the same form with integer entries, and as the subgroup Γ the group of matrices of the form

$$\begin{pmatrix} 1 & A + s\,A' & c + s\,c' \\ & I_r & B \\ & & 1 \end{pmatrix}$$

where A, A', B are matrices with integer entries, c, c' are integers too, and $s \notin \mathbb{Q}$.

The group Γ can be identified with the integer lattice of the group U of real matrices of the form

$$\begin{pmatrix} 1 & a_1 & \cdots & a_r & a'_1 & \cdots & a'_r & c & c' \\ & 1 & \cdots & 0 & 0 & \cdots & 0 & b_1 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & 0 & \cdots & 0 & b_r & 0 \\ & & & & 1 & \cdots & 0 & 0 & b_1 \\ & & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & & 1 & 0 & b_r \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{pmatrix} .$$

The submersion $u : U \longrightarrow N$ is given by the correspondence

$$(a_i, a'_i, c, c', b_i) \mapsto (a_i + s\,a'_i, c + s\,c', b_i) \quad .$$

From [CFL] it is known that $E(\Gamma_0) = \Gamma_0 \setminus N$ admits a Sasakian structure. Then the resulting foliation $\mathcal{F}(\Gamma, U, u)$ is transversely Sasakian of codimension $2r+1$ on a compact real manifold of dimension $4r+1$; it cannot be made transversely cosymplectic (cf. [CFL]).

V.1.6 Other examples

Let be $N = H(1,r)$, $r \geq 1$, be the Heisenberg group of real matrices of the form

$$\begin{pmatrix} I_r & A & C \\ & 1 & b \\ & & 1 \end{pmatrix} ,$$

where A, C are $r \times 1$ matrices and b is a real number.

Let Γ_0 be the integer lattice of N, and let Γ be the subgroup of matrices of the form

$$\begin{pmatrix} I_r & A + s\,A' & C + s\,C' \\ & 1 & b \\ & & 1 \end{pmatrix} ,$$

where A, A', C, C' are integer matrices, b is an integer too, and $s \notin \mathbb{Q}$.

Then the group Γ can be considered as a uniform subgroup of the group U of matrices of the form

$$\begin{pmatrix} I_{2r} & \begin{matrix} A \\ A' \end{matrix} & \begin{matrix} C \\ C' \end{matrix} \\ & 1 & b \\ & & 1 \end{pmatrix} .$$

The submersion $u : U \longrightarrow N$ is given by the correspondence

$$\begin{pmatrix} I_{2r} & \begin{matrix} A \\ A' \end{matrix} & \begin{matrix} C \\ C' \end{matrix} \\ & 1 & b \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} I_r & A + s\,A' & C + s\,C' \\ & 1 & b \\ & & 1 \end{pmatrix} .$$

According to the results of [CFL] the foliation $\mathcal{F}(\Gamma, U, u)$ of $M(\Gamma)$ admits

1. a foliated nonnormal almost cosymplectic structure, but no foliated cosymplectic structure (for $r \geq 1$);
2. if $r = 2p$ or $r = 4p + 1$, no foliated Sasakian structure;
3. if $r = 2p$, a foliated semi–cosymplectic normal structure;
4. if $r = 2p + 1$ $(p \geq 0)$, a foliated normal structure.

In particular, if $r = 1$ the foliation modeled on $H(1,1)$ always admit a foliated Sasakian structure.

Remark Let us recall the following result of E. Macías (cf. [MC]):

Theorem *If a dense subgroup Γ of a simply connected nilpotent Lie group N contains a uniform subgroup Γ_0 then the mapping*

$$H^*(N) \cong H^*(\Gamma_0 \setminus N) \longrightarrow H^*(\Gamma \setminus U) \cong H^*(U) \ ,$$

induced by the mapping $u : U \longrightarrow N$ defined by Γ, is injective. (U is the Malc'ev completion of Γ.)

This theorem ensures that any invariant symplectic form on N is mapped to a non–zero cohomology class in $H^*(\Gamma \setminus U)$. This means that the characteristic classes defined by the elements ω^k of the complex $W(sp\,(q),2)_{2q}$, cf. [BE], are non–zero for such foliations.

Therefore the example constructed in I.1.3 is a new example of a transversely symplectic foliation with non–trivial characteristic classes. The other examples can be reworked so that the subgroups Γ would be dense, and therefore they would provide examples of transversely symplectic foliations with non–trivial characteristic classes.

V.2 Base–like cohomology

V.2.1 Preliminaries

Base–like forms on the foliated manifold $(M,\mathcal{F})$ are in one–to–one correspondence with $\mathcal{H}$–invariant forms on the transverse manifold N. Moreover, base–like k–forms can be considered as foliated sections of $\Lambda^k N(M,\mathcal{F})^*$, the kth exterior product of the conormal bundle of $\mathcal{F}$. If the foliation $\mathcal{F}$ is transversely holomorphic the normal bundle $N(M,\mathcal{F})$ of $\mathcal{F}$ has a complex structure corresponding to the complex structure of N. Therefore any complex valued base–like k–form can be represented as a sum of the k–forms of pure type (r,s) corresponding to the decomposition of k–forms on the complex manifold N. We can obtain the same decomposition by looking at the decomposition of sections of the complex bundle $\Lambda^k_{\mathbb{C}} N(M,\mathcal{F})^*$, i.e. a base–like k–form α is of pure type (r,s) if for any point of M there exists an adapted chart $(x_1,\ldots,x_{n-2q},z_1,\ldots,z_q)$ such that

$$\alpha = \sum f_{IJ} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_s} \ .$$

where $1 \le i_1 < \ldots < i_r \le q$, $1 \le j_1 < \ldots < j_s \le q$, $I = (i_1,\ldots,i_r)$, $J = (j_1,\ldots,j_s)$.

Let us denote by $\Lambda^k_{\mathbb{C}}(M,\mathcal{F})$ the space of complex valued base–like k–forms on the foliated manifold $(M,\mathcal{F})$, and by $\Lambda^{r,s}_{\mathbb{C}}(M,\mathcal{F})$ the space of complex valued base–like forms of pure type (r,s). Then

$$\Lambda^k_{\mathbb{C}}(M,\mathcal{F}) = \sum_{r+s=k} \Lambda^{r,s}_{\mathbb{C}}(M,\mathcal{F}) \ ,$$

for short $\Lambda^k = \sum_{r+s=k} \Lambda^{r,s}$.

The exterior derivative $d\colon \Lambda^k_{\mathbb{C}}(M,\mathcal{F}) \longrightarrow \Lambda^{k+1}_{\mathbb{C}}(M,\mathcal{F})$ decomposes itself into two components $d = \partial + \bar{\partial}$, where ∂ is of bidegree $(1,0)$ and $\bar{\partial}$ is of bidegree $(0,1)$, i.e.

$$\partial\colon \Lambda^{r,s} \to \Lambda^{r+1,s} \quad \text{and} \quad \bar{\partial}\colon \Lambda^{r,s} \to \Lambda^{r,s+1} \ .$$

Now we are going to recall some results of A. El Kacimi, cf. [EK], concerning transversely Hermitian and transversely Kähler foliations.

Let us assume that $\mathcal{F}$ is a transversely Hermitian foliation. The operator

$$*: \Lambda^k(M,\mathcal{F}) \longrightarrow \Lambda^{2q-k}(M,\mathcal{F})$$

defined in [EH] via the transverse part of the bundle–like metric of $\mathcal{F}$ extends to an operator

$$\bar{*}: \Lambda^k_{\mathbf{C}}(M,\mathcal{F}) \longrightarrow \Lambda^{2q-k}_{\mathbf{C}}(M,\mathcal{F}) .$$

Let $B(M,\mathrm{SO}(2q),\pi,\mathcal{F})$ be the bundle of transverse orthonormal frames of $\mathcal{F}$. Let κ be the form defining the volume form on each fibre of the bundle $B(M,\mathrm{SO}(2q),\pi,\mathcal{F})$. The foliation $\mathcal{F}_1$ on the total space B of $B(M,\mathrm{SO}(2q),\pi,\mathcal{F})$ is transversely parallelisable and therefore the closures of the leaves are the fibres of the basic fibration $p: B \longrightarrow W$, cf. [MO11].

We define the scalar product on $\Lambda^*_{\mathbf{C}}(M,\mathcal{F}) = \sum_{k=1}^{2q} \Lambda^k_{\mathbf{C}}(M,\mathcal{F})$ as

$$\text{(V.1)} \quad \begin{aligned} \langle\alpha,\beta\rangle &= 0 \text{ if } \alpha\in\Lambda^k_{\mathbf{C}}(M,\mathcal{F}) ,\ \beta\in\Lambda^l_{\mathbf{C}}(M,\mathcal{F}) \text{ and } k\neq l ,\\ \langle\alpha,\beta\rangle &= \textstyle\int_W \bar{I}(\pi^*(\alpha\wedge\bar{*}\beta)\wedge\kappa) \text{ for } \alpha,\beta\in\Lambda^r_{\mathbf{C}}(M,\mathcal{F}) \end{aligned}$$

where $\bar{I}$ is the integration along the fibres of the basic fibration.

The operator $\delta: \Lambda^k_{\mathbf{C}}(M,\mathcal{F}) \longrightarrow \Lambda^{k-1}_{\mathbf{C}}(M,\mathcal{F})$ defined as $\delta = \bar{*}^{-1}d\bar{*}$ is the adjoint operator of d relative to the scalar product $\langle\, ,\rangle$.

Similarly, the "foliated" Laplacian operator is defined by $\Delta = d\delta + \delta d$; it is, in this case, an auto–adjoint foliated (transversely) elliptic operator.

Let us consider the following differential complex

$$\text{(V.2)} \qquad 0 \longrightarrow \Lambda^{r,0} \xrightarrow{\bar{\partial}} \Lambda^{r,1} \longrightarrow \cdots \xrightarrow{\bar{\partial}} \Lambda^{r,q} \longrightarrow 0 .$$

We denote its cohomology $H^{r,*}(M,\mathcal{F}) = \sum_{s=0}^{q} H^{r,s}(M,\mathcal{F})$ and we call it the base–like Dolbeault cohomology of the foliation $\mathcal{F}$.

The operator $\bar{*}$ induces isomorphisms $\bar{*}: \Lambda^{r,s} \longrightarrow \Lambda^{q-r,q-s}$. Let us put $\bar{\delta} = -\bar{*}\bar{\partial}\bar{*}$. Then the operator $\bar{\delta}$ is the adjoint of $\bar{\partial}$ relative to the inner product $\langle\, ,\rangle$ defined in (I.1). Moreover, the operator $\Delta'' = \bar{\partial}\bar{\delta} + \bar{\delta}\bar{\partial}$ is an auto–adjoint foliated (transversely) elliptic operator.

Now, let $\mathcal{F}$ be transversely Kähler. The Kähler form of N defines a base–like $(1,1)$–form ω on $(M,\mathcal{F})$ which we call the transverse Kähler form of the foliation $\mathcal{F}$. This form allows to define the following operator:

$$L: \Lambda^k_{\mathbf{C}}(M,\mathcal{F}) \longrightarrow \Lambda^{k+2}_{\mathbf{C}}(M,\mathcal{F}) \ , \quad L\alpha = \alpha\wedge\omega ,$$

and its adjoint $\Lambda = -\bar{*}L\bar{*}$.

Then for a transversely Kähler foliation $\mathcal{F}$ on a compact manifold, the following relations hold:

$$\text{(V.3)} \qquad \begin{cases} \Lambda\partial - \partial\Lambda \;=\; -\sqrt{-1}\,\bar\delta\,, \\ \Lambda\bar\partial - \bar\partial\Lambda \;=\; \sqrt{-1}\,\delta\,, \\ \partial\bar\delta + \bar\delta\partial \;=\; \bar\partial\delta + \delta\bar\partial \;=\; 0\,, \\ \Delta \;=\; 2\Delta''\,, \\ \Delta L \;=\; L\Delta \;, \quad \Delta\Lambda = \Lambda\Delta\,. \end{cases}$$

These identities lead to the following theorem, cf. [EK]:

Theorem 1 *(El Kacimi) Let $\mathcal{F}$ be a transversely Kähler foliation on a compact manifold M. If $\mathcal{F}$ is homologically oriented, then*

i) *a base–like k–form $\alpha = \sum_{r+s=k}\alpha_{r,s}$, $\alpha_{r,s} \in \Lambda^{r,s}$, is harmonic if and only if the forms $\alpha_{r,s}$ are harmonic; thus*

$$H^k_{\mathbb{C}}(M,\mathcal{F}) \cong \sum_{r+s=k} H^{r,s}(M,\mathcal{F})\,.$$

ii) *the conjugation induces isomorphisms $H^{r,s}(M,\mathcal{F}) \cong H^{s,r}(M,\mathcal{F})$.*

iii) *for any $0 \le r \le q$, the form ω^r is harmonic, thus $H^{r,r}(M,\mathcal{F}) \neq 0$.*

V.2.2 Basic Frölicher spectral sequence

Now we turn our attention to the basic Frölicher spectral sequence.

Let us consider the complex $(\Lambda = \sum_{r,s}\Lambda^{r,s}, d)$ of complex valued base–like forms of the foliated manifold $(M,\mathcal{F})$. We can filtrate it as follows

$$F^k\Lambda = \sum_{r\ge k}\Lambda^{r,*}\,.$$

This filtration is compatible with the bigradation of the complex. The spectral sequence associated to this filtration is called the basic Frölicher spectral sequence of the transversely holomorphic foliation $\mathcal{F}$, cf. [FL]. It can be easily shown that it converges to the complex base–like cohomology of $(M,\mathcal{F})$.

The terms $E_1^{r,s}$ of this spectral sequence are the cohomology groups of the differential complex (I.2), i.e. $E_1^{r,s} = H^{r,s}(M,\mathcal{F})$, the (r,s)–th base–like Dolbeault cohomology group.

When the foliation $\mathcal{F}$ is homologically oriented and transversely Kähler, Theorem 1 ensures that $E_1^{r,s} \simeq \mathcal{H}^{r,s}$, $\mathcal{H}^{*,*}$ – the complex of complex valued base–like harmonic forms; therefore the differential operator $d_1\colon E_1^{r,s} \to E_1^{r+1,s}$ vanishes and the spectral sequence collapses at the level E_1.

Theorem 2 *Let $\mathcal{F}$ be a homologically oriented transversely Kähler foliation on a compact manifold M. The basic Frölicher spectral sequence of $\mathcal{F}$ collapses at the first term, i.e. $E_1 \simeq E_2 \simeq \cdots \simeq E_\infty$.*

Now, we shall present examples of homologically oriented transversely Hermitian foliations on compact nilmanifolds whose basic Frölicher spectral sequence behaves in a markedly different manner. The examples are based on the examples of [CFG2] of compact complex nilmanifolds whose Frölicher spectral sequences have similar properties.

Example 1. *A transversely Hermitian foliation for which $E_1 \not\cong E_2$.*

Let us consider the 3–dimensional complex Heisenberg group $N = (\mathbb{C}^3, *)$ where

$$(a_1, a_2, a_3) * (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3 + a_1 b_2) .$$

The compact complex nilmanifold $\Gamma_0\backslash\mathbb{C}^3$, where Γ_0 is the lattice of Gauss integers, is the well known Iwasawa manifold for which $E_1 \not\cong E_2$, cf. [CFG2].

The second example of V.1.2 provides us with the foliation $\mathcal{F}(\Gamma_2, U, u)$ which is a holomorphic and transversely symplectic of complex codimension 3 on a compact complex nilmanifold $\Gamma_2\backslash U$ of complex dimension 5, and it cannot be made transversely Kähler either. Its base–like forms are in one–to–one correspondence with the Γ_2–invariant forms on $(\mathbb{C}^3, *)$. The same considerations as in [CFG2,GRH] ensure that the basic Frölicher spectral sequence of $\mathcal{F}(\Gamma_2, U, u)$ has the property of being $E_1 \not\cong E_2$.

Example 2. *A transversely Hermitian foliation for which $E_2 \not\cong E_3$.*

Let us consider the group $N = (\mathbb{C}^4, *)$ with the following group operation:

$$(a_1, a_2, a_3, a_4) * (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2,$$
$$a_3 + b_3 + (a_2 + \bar{a}_2)b_1, a_4 + b_4 - \bar{a}_1 b_2) .$$

This is a real nilpotent Lie group with a left invariant complex structure. The compact complex nilmanifold $\Gamma_0\backslash\mathbb{C}^4$, where Γ_0 is the lattice of Gauss integers, has the required property, cf. [CFG2].

Now, let us consider the following finitely generated subgroup Γ_4 of $(\mathbb{C}^4, *)$:

$$\{(n_1, n_2 + s\, m_2, n_3 + s\, m_3, n_4 + s\, m_4) : n_i, m_i \text{ Gauss integers}, \ s \notin \mathbb{Q}\} .$$

Γ_4 can be embedded as a uniform subgroup of the group $U = (\mathbb{C}^7, \square)$ with the following group operation:

$$(a_1, \ldots, a_7) \square (b_1, \ldots, b_7)$$
$$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + (a_2 + \bar{a}_2)b_1,$$
$$a_5 + b_5 + (a_3 + \bar{a}_3)b_1, a_6 + b_6 - \bar{a}_1 b_2, a_7 + b_7 - \bar{a}_1 b_3) .$$

It is a real nilpotent Lie group with a left invariant complex structure. The Γ_4–equivariant submersion $u: U \longrightarrow N$ is given by the formula:

$$(a_1, \ldots a_7) \longmapsto (a_1, a_2 + s a_3, a_4 + s a_5, a_6 + s a_7)$$

The foliation $\mathcal{F}(U, \Gamma_4, u)$ is tranversely Hermitian. Its base-like forms are in one-to-one correspondence with the Γ_4-invariant forms on $(\mathbb{C}^4, *)$. The same considerations as in [CFG2] ensure that the basic Frölicher spectral sequence of $\mathcal{F}(\Gamma_4, U, u)$ has the property $E_2 \not\cong E_3$.

Example 3. *A transversely Hermitian foliation for which $E_3 \not\cong E_4$.*

Let us consider the group $N = (\mathbb{C}^6, *)$ with the following group operation:

$$\begin{aligned}(a_1, \ldots, a_6) * (b_1, \ldots, b_6) \\ = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + (a_2 + \bar{a}_2)b_1, \\ a_5 + b_5 - \bar{a}_1 b_2, a_6 + b_6 + (1/2)(a_2 + \bar{a}_2)b_1^2 + a_4 b_1 + \bar{a}_3 b_1) \ .\end{aligned}$$

The manifold $\Gamma_0 \backslash \mathbb{C}^6$, where Γ_0 is the lattice of Gauss integers, has the required property, cf. [CFG2].

Let us consider the following finitely generated subgroup Γ_6 of $(\mathbb{C}^6, *)$:

$$\{(n_1, n_2 + sm_2, n_3, n_4 + sm_4, n_5 + sm_5, n_6 + sm_6) \colon n_i, m_i \text{ Gauss integers}, \ s \notin \mathbb{Q}\}.$$

As in the previous examples, we can find a simply connected nilpotent Lie group U containing Γ_6 as a uniform subgroup and a surjective homomorphism of Lie groups $u \colon U \longrightarrow (\mathbb{C}^6, *)$ which is the identity on Γ_6. The resulting foliation $\mathcal{F}(\Gamma_6, U, u)$ of the manifold $\Gamma_6 \backslash U$ is transversely Hermitian. The same considerations as in [CFG2] ensure that the basic Frölicher spectral sequence of $\mathcal{F}(\Gamma_6, U, u)$ has the property $E_3 \not\cong E_4$.

The group U can be represented as $(\mathbb{C}^{10}, \Box)$ with the following group operation:

$$\begin{aligned}(a_1, \ldots, a_{10}) \Box (b_1, \ldots, b_{10}) \\ = (a_i + b_i, a_5 + b_5 + (a_2 + \bar{a}_2)b_1, a_6 + b_6 + (a_3 + \bar{a}_3)b_1, a_7 + b_7 - \bar{a}_1 b_2, \\ a_8 + b_8 - \bar{a}_1 b_3, a_9 + b_9 + (1/2)(a_2 + \bar{a}_2)b_1^2 + a_5 b_1 + \bar{a}_4 b_1, \\ a_{10} + b_{10} + (1/2)(a_3 + \bar{a}_3)b_1^2 + a_6 b_1) \ .\end{aligned}$$

The submersion u is the following:

$$u(a_1, \ldots, a_{10}) = (a_1, a_2 + s\, a_3, a_4, a_5 + s\, a_6, a_7 + s\, a_8, a_9 + s\, a_{10}) \ .$$

V.2.3 Complex conjugation

Theorem 1 asserts that for homologically oriented transversely Kähler foliations the complex conjugation induces an isomorphism of the base-like Dolbeault cohomology. We are going to give a simple example of a homologically oriented transversely Hermitian foliation for which this is not true.

Example 4. Let us consider the Example V.1.3. The foliation $\mathcal{F}(\Gamma, U, u)$ constructed in this example is transversely symplectic and transversely holomorphic of complex codimension 2 on a complex manifold $M(\Gamma)$ of complex dimension 5 (which is a real nilmanifold but not a complex nilmanifold, cf. [CFG3]). The base–like forms of $\mathcal{F}(\Gamma, U, u)$ are in one–to–one correspondence with Γ–invariant forms on N. Since Γ is dense in N the base–like forms can be identified with left invariant forms. As Γ contains Γ_0 the base–like cohomology of $\mathcal{F}(\Gamma, U, u)$ is isomorphic to the cohomology of the Kodaira–Thurston manifold. The computations of [C] show that $\dim H^{1,0}(\Gamma_0\backslash N) = 1$ and $\dim H^{0,1}(\Gamma_0\backslash N) = 2$, which means that in this case the complex conjugation does not induce an isomorphism in the Dolbeault cohomology. However its basic Frölicher spectral sequence collapses at the first level.

V.2.4 Formality of the minimal model

We are going to look at the minimal model of the complex base–like cohomology of a homologically oriented transversely Kähler foliation. In fact, as for compact Kähler manifolds, the minimal model for the complex base–like cohomology is formal, and hence all Massey products must vanish, cf. [DGM].

Lemma 1 *The $dd^{\mathbb{C}}$–lemma is true in the algebra of complex valued base–like forms of a homologically oriented transversely Kähler foliation on a compact manifold.*

Proof The identities (I.3) and Theorem 1 ensure that we can repeat the proof of the $dd^{\mathbb{C}}$–lemma for compact Kähler manifolds cf. [5.11] of [DGM].□

As the theorem on the formality of the minimal model is a purely "formal" consequence of the $dd^{\mathbb{C}}$–lemma, cf. [Sect. 6] of [DGM], we have the following theorem:

Theorem 3 *Let $\mathcal{F}$ be a transversely Kähler foliation on a compact manifold M. If $\mathcal{F}$ is homologically oriented then the minimal model of the complex base–like cohomology of $\mathcal{F}$ is formal and thus Massey products of complex valued base–like forms vanish.*

To show the non–triviality of this result we present a homologically oriented transversely Hermitian foliation whose complex base–like cohomology possesses non–vanishing Massey products, thus whose minimal model of the base–like cohomology cannot be formal. The following proposition for Lie foliations asserts that Examples 1, 2 and 3 have this property.

Proposition 1 *Let $\mathcal{F}$ be a Lie foliation on a compact manifold modelled on a nilpotent Lie group N with a left invariant complex structure,* $\dim_{\mathbb{C}} N = q$. *If the* *holonomy group Γ of $\mathcal{F}$ contains a uniform subgroup Γ_0 of N then the following conditions are equivalent:*

i) *$\mathcal{F}$ is transversely Kähler;*

ii) *the complex base-like cohomology of $(M, \mathcal{F})$ has no non-trivial Massey products;*

iii) *the group N is commutative.*

Proof We can assume that the group N is simply connected. As the manifold M is compact, the developing mapping is surjective and has connected fibres. Therefore base-like forms on $(M, \mathcal{F})$ are in one-to-one correspondence with Γ-invariant forms on N. On the manifold N we can consider three complexes of complex valued forms:

$\Lambda^*_{\mathbb{C}}(N, \Gamma_0)$ – the complex of Γ_0-invariant forms,

$\Lambda^*_{\mathbb{C}}(N, \Gamma)$ – the complex of Γ-invariant forms,

$\Lambda^*_{\mathbb{C}}(N, N)$ – the complex of N-invariant forms.

Of course, $\Lambda^*_{\mathbb{C}}(N, N) \subset \Lambda^*_{\mathbb{C}}(N, \Gamma) \subset \Lambda^*_{\mathbb{C}}(N, \Gamma_0)$. Then Nomizu's theorem, cf. [NO1], ensures that in cohomology we have

$$H^*_{\mathbb{C}}(N, N) \hookrightarrow H^*_{\mathbb{C}}(N, \Gamma) \twoheadrightarrow H^*_{\mathbb{C}}(N, \Gamma_0) \tag{V.4}$$

as $H^*_{\mathbb{C}}(N, N) \simeq H^*_{\mathbb{C}}(N, \Gamma_0)$.

Theorem 3 ensures that i) $\Longrightarrow$ ii).

Let us look at the second implication ii) $\Longrightarrow$ iii). Assume that the group N is not commutative. In [CFG3], the authors proved that in this case there exist non-trivial Massey products in $H^*_{\mathbb{C}}(N, N)$. In view of (I.4), the Massey product of the same cohomology classes considered in $H^*_{\mathbb{C}}(N, \Gamma)$ must be also non-trivial. Contradiction.

The third implication is trivial as N is just $\mathbb{C}^q$.□

V.3 Sasakian manifolds

We complete the chapter with a quick look at Sasakian manifolds. In this section we present a new method of studing Sasakian manifolds. A Sasakian manifold is a foliated manifold with a very particular foliated structure. Using the correspondence between foliated and transverse structures, we reduce many theorems about geometrical objects in Sasakian manifolds to theorems about corresponding objects in Kähler manifolds. In fact, the 1-dimensional foliation of a Sasakian manifold generated by the characteristic vector field is a transversely Kähler isometric flow. We call this foliation the characteristic foliation. We consider two books of K.Yano and M.Kon, cf. [YK1,YK2], and demonstrate that most results on the local structure of Sasakian manifolds can be derived from the corresponding ones for Kähler manifolds. To complete this section we present some new local properties of Sasakian manifolds obtained applying our foliated method.

V.3.1 Preliminaries

Now let us recall the definition of a Sasakian manifold. Let M be a smooth manifold of dimension $2n+1$. The manifold M is called an almost contact metric manifold if there exist on M:

1. a non–vanishing vector field ξ and a 1– form η such that $\eta(\xi) \equiv 1$;

2. a tensor field φ of type $(1,1)$ such that $\varphi^2 = -Id + \eta \otimes \xi$, this implies that $\varphi(\xi) = 0$ and $\eta \circ \varphi \equiv 0$;

3. a Riemannian metric g such that $g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X)\eta(Y)$.

An almost contact metric manifold is called Sasakian if, additionally, it satisfies the following condition, cf. [SH,BL2],

5. $(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$,

and hence $\nabla_X \xi = -\varphi(X)$, $d\eta(X,Y) = g(X, \varphi Y)$ and the $2n+1$–form $\eta \wedge d\eta^n$ does not vanish.

The last condition ensures that the vector field ξ is a Killing vector field for the metric g. Therefore, this vector field defines a Riemannian foliation $\mathcal{F}$ of dimension 1 which is an isometric flow, cf. [CA1,CA3]. The vector field ξ is called the characteristic vector field and $\mathcal{F}$ the characteristic foliation of the Sasakian manifold M. It is not difficult to verify that $L_\xi \varphi | ker\eta \equiv 0$. Therefore the tensors g and φ induce foliated tensors in the normal bundle of the characteristic foliation which can be identified with the bundle $ker\eta$. Let $\bar{g}$ and $\bar{J}$ be the corresponding tensors on the transverse manifold N (of the characteristic foliation). The Riemannian connection ∇ of M induces a transversely projectable connection in $ker\eta$ and which projects onto the Riemannian connection of $(N, \bar{g})$. The condition (5) ensures that the almost complex structure $\bar{J}$ is integrable, thus $(N, \bar{J}, \bar{g})$ is a Hermitian manifold. The equality $d\eta(X,Y) = g(X, \varphi Y)$ means that the 2–form $d\eta$ is base–like. The corresponding 2–form Φ on the transverse manifold is its Kähler form. The holonomy pseudogroup is a pseudogroup of Kähler transformations and $\mathcal{F}$ is transversely Kähler.

There are many transversely Kähler isometric flows which are not given by any Sasakian structure. Let Ψ be an isometric flow defining a transversely Kähler foliation. The transverse manifold N of this foliation admits a holonomy invariant Kähler structure $(\bar{g}, \bar{J})$. Let ξ be the vector field tangent to the flow Ψ, g the Riemannian metric for which the flow is isometric, and let Q be the orthogonal complement of ξ in the metric g. Then we put:

η: $\eta(\xi) \equiv 1$ and $\eta | Q \equiv 0$;
φ: $\varphi(\xi) \equiv 0$ and $df_i(\varphi(X)) = \bar{J}(df_i(X))$ for any $X \in TU_i$.

One can easily check that the structure (g, φ, ξ, η) defined above satisfies the conditions 1)– 4), i.e. it is an almost contact metric structure. The condition

(5) is not a transverse one which can be deduced easily from the Boothby–Wang theorem, cf. [BL2,BW] and also [OG,TN,TB].

Theorem 4 *Let M be a Sasakian compact manifold. If the characteristic vector field ξ is regular, then the manifold M is the total space of a S^1-bundle over a Kähler manifold N which is the space of orbits of the vector field ξ. Conversely, let M be the total space of an S^1-bundle over a Kähler manifold N. If the Euler class of this S^1-bundle is cohomologous to the Kähler form of N, then there exists a Sasakian structure (g, φ, ξ, η) on M inducing on N its Kähler structure with ξ being the vertical vector field of the S^1-bundle.*

Let us look at the consequences of this very well-known result. The fundamental vertical vector field of any S^1–bundle over a Kähler manifold is a transversely Kähler isometric flow. However such a flow comes from a Sasakian structure iff the Euler class of this S^1–bundle is cohomologous to the Kähler form of the base manifold, or equivalently iff there exists a 1–form η with $\eta(\xi) \equiv 1$ and whose exterior differential projects onto the Kähler form of the base manifold N. This has two interesting consequences:

a) the vertical vector field of the trivial S^1–bundle (equivalent to the vanishing of its Euler class according to [KO1]) cannot be the characteristic vector field of a Sasakian structure, as the cohomology class of the Kähler form is non–trivial.

b) if the cohomology class of the Kähler form of the base manifold N is not integral the vertical vector field of any S^1–bundle over N cannot be the characteristic vector field of a Sasakian structure inducing this Kähler structure.

Now let us look closer at the condition (5). We have the following well known lemma:

Lemma 2 *Let ξ be a non–vanishing vector field on a Riemannian manifold (M, g) defining a transversely Hermitian flow. With the notation as above, any two of the following three conditions imply the third one:*

1. *ξ is a Killing vector field,*
2. *$\varphi(X) = -\nabla_X \xi$,*
3. *$g(X, \varphi(Y)) = d\eta(X, Y)$.*

This lemma leads to the following characterization of Sasakian structures.

Theorem 5 *Let ξ be a non–vanishing Killing vector field defining a transversely Kähler isometric flow on a Riemannian manifold (M, g). The corresponding almost contact metric structure (g, φ, ξ, η) is Sasakian iff $\varphi(X) = -\nabla_X \xi$.*

As a corollary we get:

Corollary 1 *Let ξ be a transversely Kähler flow. If there exists a 1-form η such that $\eta(\xi) > 0$ and the 2-form $d\eta$ projects onto the Kähler form of the transverse manifold, then we can reparametrize the vector field ξ to obtain a vector field ξ' such that the corresponding almost contact metric structure is Sasakian.*

Proof The reparametrisation ξ' of ξ such that $\eta(\xi') = 1$ is a Killing vector field, cf. [GL]. On the other hand we know that $d\eta(X,Y) = g(X, \varphi Y)$ as the 2-form $d\eta$ is base-like and projects onto the Kähler form of the transverse manifold. Therefore according to Lemma 2 $\nabla_X \xi = -\varphi X$. Then Theorem 5 ensures the rest.□

V.3.2 Sasakian versus Kähler

First of all we are going to compare various curvature tensors of the manifolds (M, g, φ) and $(N, \bar{g}, \bar{J})$. For any $X \in T_x N_i$ and $y \in f_i^{-1}(x)$ denote by X^* the only vector of $ker \eta_y$ such that $df_i(X^*) = X$. The considerations of [YK1], Chapter VI, yield the following relations:

1. $(\bar{J}X)^* = \varphi(X^*)$;
2. $g(X^*, Y^*) = \bar{g}(X, Y)$;
3. $(\bar{\nabla}_X Y)^* = \nabla_{X^*} Y^* + g(Y^*, \varphi X^*)\xi$, where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of g and $\bar{g}$, respectively;
4. $(\bar{R}(X,Y)Z)^*$ $= R(X^*, Y^*)Z^* + g(Z^*, \varphi Y^*)\varphi X^* - g(Z^*, \varphi X^*)\varphi Y^* - 2g(Y^*, \varphi X^*)\varphi Z^*$ where R and $\bar{R}$ are the curvature tensors of ∇ and $\bar{\nabla}$, respectively;
5. $\bar{S}(X,Y) = S(X^*, Y^*) + 2g(X^*, Y^*)$ where S and $\bar{S}$ are the Ricci curvature tensors of (M, g) and $(N, \bar{g})$, respectively;
6. $\bar{r} = r + 2n$ where r and $\bar{r}$ are the scalar curvatures of (M, g) and $(N, \bar{g})$, respectively;
7. $\bar{K}(X, \bar{J}Y) = K(X^*, \varphi Y^*) + 3$ where K and $\bar{K}$ are the sectional curvature tensors of (M, g) and $(N, \bar{g})$, respectively;
8. $\bar{g}(\bar{B}(X,Y)Z, W) = g(B(X^*, Y^*)Z^*, W^*)$ where B and $\bar{B}$ are the contact Bochner curvature and Bochner curvature tensors of (M, g) and $(N, \bar{g})$, respectively. Moreover, B vanishes iff $\bar{B}$ does.

As an example we shall prove the following theorem, cf. [BR2]:

Theorem 6 *The φ-sectional curvature determines completely the sectional curvature of a Sasakian manifold.*

Proof It is well-known that for any plane tangent to the characteristic vector field the sectional curvature is equal to 1. The formula (4) establishes the relation between the sectional curvature in the transverse direction in a Sasakian manifold and the corresponding sectional curvature in the transverse manifold. The formula (7) gives the precise relation between φ-sectional curvature and the holomorphic sectional curvature of M and N, respectively. As N is a Kähler manifold, its holomorphic sectional curvature determines its sectional curvature, so the φ-sectional curvature determines the sectional curvature of a Sasakian manifold M.□

The formula (7) leads to the following proposition:

Proposition 2 *The characteristic foliation of a Sasakian space form $M(c)$ is a transversely Kähler isometric flow modelled on a Kähler space form $N(c-3)$.*

Now let us turn our attention to submanifolds. Let W be an $m+1$ dimensional submanifold of M tangent to ξ, i.e. for any $x \in W$ $\xi(x) \in T_xW$. For any point of this submanifold we can find a very special adapted chart at this point.

Lemma 3 *Let x be a point of a submanifold W tangent to the characteristic vector field of a Sasakian manifold M. Then there exists an adapted chart $\psi: V \longrightarrow \mathbb{R}^{2n+1}, \psi = (\psi_1, ..., \psi_{2n+1})$, at x such that the set*

$$U = \{y \in V : \psi_{m+2}(y) = ... = \psi_{2n+1}(y) = 0\}$$

is a connected component of $V \cap W$ containing x and

$$(\psi_1|U, ...\psi_{m+1}|U): U \to \mathbb{R}^{m+1}$$

is an adapted chart for the induced foliation of W.

Proof It is a simple generalization of the classical result for submanifolds; we have to start with adapted charts, and then proceed as in the standard case.□

This lemma leads us to the following proposition:

Proposition 3 *Let W be a submanifold tangent to the characteristic foliation of a Sasakian manifold M. Then for any point x of W there exist neighbourhoods U and V of x in W and M, respectively, having the following properties:*

i) *U is a connected component of $V \cap W$ containing x;*

ii) *U is a foliated subset of V (for the characteristic foliation);*

iii) *there exists a Riemannian submersion with connected fibres $f: V \to N_0$ onto a Kähler manifold N_0 defining the characteristic foliation;*

iv) *there exists a submanifold W_0 of N_0 such that $U = f^{-1}(W_0)$.*

Proof Let U and V be neighbourhoods of the point x from Lemma 3. Then the submersion f we define as $p_{2n} \circ \psi \colon V \longrightarrow \mathbb{R}^{2n}$ where p_{2n} is the propjection $(x_1, ... x_{2n+1}) \mapsto (x_2, ... x_{2n+1})$. On the set $imf \subset \mathbb{R}^{2n}$ the Sasakian structure of M induces a Kähler structure for which the submersion is a Riemannian submersion. Since the characteristic foliation restricted to V is defined by this submersion and the set U is saturated, there exists a submanifold W_0 of N_0 satisfying the condition (iv). □

If the submanifold W is foliated for the characteristic foliation Lemma 3 yields the following stronger result. For example it is the case if W is a complete submanifold.

Proposition 4 *Let W be a submanifold of a Sasakian manifold M. If W is foliated for the characteristic foliation of M, then for every cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ defining the characteristic foliation there exists a holonomy invariant submanifold W_0 of the transverse manifold such that $W \cap U_i = f_i^{-1}(W_0 \cap N_i)$.*

Proposition 3 ensures that the study of local properties of submanifolds tangent to the characteristic vector field of a Sasakian manifold can be reduced to the study of a foliated submanifold of a Sasakian manifold with its characteristic foliation given by a global Riemannian submersion with connected fibres. Then the properties of submanifolds related via a Riemannian submersion, cf. Appendix, further reduce it to the study of properties of the corresponding submanifolds of a Kähler manifold. This method applied to anti–invariant submanifolds of a Sasakian manifold studied in [YK1] ensures that their properties are immediate consequences of the properties of anti–invariant submanifolds of Kähler manifolds. We present a list of corresponding theorems. Of course this list does not pretend to be exhaustive.

Sasakian (Chapter IV of [YK1])	**Kähler (Chapter III of [YK1])**
Proposition 2.3, Theorem 2.1, Cor. 2.1	Lemma 2.1 plus curvature estimates
Lemma 1.1	Lemma 2.1
Corollary 8.1	Corollary 5.1
Corollary 8.2	Corollary 5.2
Lemma 8.1	Lemma 5.2
Proposition 8.2	Proposition 6.5
Proposition 8.3 [1]	Proposition 6.6
Proposition 8.4	Proposition 6.7
Theorem 12.1	Theorem 10.2

[1] In the statement of Proposition 8.3 the authors forgot to include the assumption "the normal f–structure is parallel".

V.3.3 Contact CR–submanifolds

First we recall the definition of a contact CR–submanifold, cf. [YK2].

Definition 1 *Let W be a submanifold of a Sasakian manifold M tangent to the characteristic vector field. W is called a contact CR-submanifold of M if there exists a differentiable distribution D on W of constant dimension, $D: x \mapsto D_x \subset T_xW$, satisfying the following conditions:*

i) *D is invariant with respect to φ, i.e. for any $x \in W$ $\varphi D_x \subset D_x$;*

ii) *the complementary orthogonal distribution $D^\perp: x \mapsto D_x^\perp \subset T_xW$ is anti–invariant with respect to φ, i.e. for any $x \in W$ $\varphi D_x^\perp \subset T_xW^\perp$.*

The distribution D can be described in the following way:

$$\varphi(T_xW) = \varphi(D_x \oplus D_x^\perp) = \varphi(D_x) \oplus \varphi(D_x^\perp) \subset T_xW \oplus T_xW^\perp.$$

Thus

$$\varphi(D_x) \subset T_xW \cap \varphi(T_xW) \text{ and } \varphi(D_x^\perp) \subset T_xW^\perp \cap \varphi(T_xW).$$

Since $\varphi^2 = -id + \eta \otimes \xi$, $\varphi(T_xW \cap \varphi(T_xW)) \subset \varphi(T_xW) \cap T_xW$. Therefore $D_0 = TW \cap \varphi(TW) \subset D$. Similarly $\varphi(D^\perp) = \varphi(TW) \cap TW^\perp$. Thus the distribution D_0 has constant dimension and $D_0 = D$ or $D = D_0 \oplus T\mathcal{F}$, and $D^\perp = D_0^\perp \oplus T\mathcal{F}$ or $D_0^\perp$, respectively, where $D_0^\perp$ is the orthogonal complement of $D_0 \oplus T\mathcal{F}$. This means that the tangent bundle TW of W admits the following decomposition: $T\mathcal{F} \oplus D_0 \oplus D_0^\perp$. Moreover the distributions D_0 and $D_0^\perp$ define the decomposition of the subbundle $ker\eta = im\varphi$. For the rest of the paper we assume that $D = D_0 \oplus T\mathcal{F}$.

The above description of the distributions D and $D^\perp$ coupled with the fact that the tensors g and φ induce foliated tensors on $ker\eta$ yield the following; cf. [YK2]:

Proposition 5 *Let W be a submanifold tangent to the characteristic vector field of a Sasakian manifold. Then W is a contact CR–submanifold iff the corresponding submanifolds in the transverse manifold are CR–submanifolds.*

Having described in detail the distributions D and $D^\perp$ we turn our attention to their properties. The argument in the proof of Theorem III.3.1 of [YK2] ensures only that the distribution $D^\perp \oplus T\mathcal{F}$ is integrable. Thus the correct version of Theorem III.3.1 is the following:

Theorem 7 *Let W be a contact CR–submanifold of a Sasakian manifold M. Then the distribution $D^\perp \oplus T\mathcal{F}$ is completely integrable and its integral submanifolds are anti–invariant submanifolds (tangent to the characteristic vector field).*

For the same reason Theorem III.3.2 of [YK2] is not exact. It should read as follows:

Theorem 8 *Let W be a contact CR–submanifold of a Sasakian manifold M. Then the distribution D is integrable iff $B(X, PY) = B(Y, PX)$ for any $X, Y \in D$. Its integral submanifolds are invariant submanifolds of M.*

Remark As the properties described by the above theorems are local, they can be derived from the corresponding theorems for CR–submanifolds of Kähler manifolds, compare Theorems IV.4.1 and IV.4.2 of [YK2].

Proposition 6 *Let W be a contact CR–submanifold tangent to the characteristic vector field of a Sasakian manifold M. If $g(B(X,Y), FZ) = 0$ for any $X, Y \in D_0$, $Z \in D_0^\perp$ then any geodesic of W tangent to D_0 at one point remains tangent to D_0 at any point of its domain.*

Proof Since the foliation $\mathcal{F}|W$ is a Riemannian foliation a geodesic orthogonal to $\mathcal{F}$ at one point is orthogonal to $\mathcal{F}$ at any point of its domain, and it is a $D_0 \oplus D_0^\perp$ horizontal lift of the corresponding geodesic in the transverse manifold, cf. [RE,YO]. Let us consider a geodesic $\alpha\colon (a,b) \to W$ tangent to D_0 at 0 and the set $A = \{t \in (a,b)\colon \dot{\alpha}(t) \in D_0\}$. The set A is closed and $0 \in A$. We shall show that it is also open. As the problem is local we can reduce our considerations to a foliated submanifold of a Sasakian manifold with the characteristic foliation given by a global submersion with connected fibres, cf. Appendix. The relations from Appendix ensure that $\bar{g}(\bar{B}(X,Y), \bar{F}Z) = 0$ for any $X, Y \in \bar{D}$ and $Z \in \bar{D}^\perp$. Then Proposition IV.4.2 of [YK2] ensures that $\bar{D}$ is a totally geodesic foliation of W_0. Let $\bar{\alpha}$ be the geodesic in W_0 corresponding to α. If α is tangent to D_0 at $t \in (a,b)$, then $\bar{\alpha}$ is tangent to $\bar{D}$ at this point. Since the foliation $\bar{D}$ is totally geodesic $\bar{\alpha}$ must be contained in some leaf of $\bar{D}$. Hence α being the $D_0 \oplus D_0^\perp$ horizontal lift of $\bar{\alpha}$, it must be tangent to D_0. Therefore the set A is open, and thus $A = (a,b)$.□

Taking as a model Kähler manifolds we can introduce the following notions:

Definition 2 *We say that a contact CR–submanifold W is:*

i) *D_0–totally geodesic iff $B(X,Y) = 0$ for any $X, Y \in D_0$;*

ii) *contact mixed foliate if $B(X,Y)$ for any $X \in D$ and $Y \in D^\perp$, and $B(PX,Y) = B(X, PY)$ for any $X, Y \in D_0$.*

Proposition 7 *Let W be a contact CR–submanifold tangent to the characteristic vector field of a Sasakian manifold M. If W is D_0–totally geodesic, then D is a foliation and any geodesic of W tangent to D_0 at one point remains tangent to D_0 at any point of its domain.*

Proof It is a consequence of Appendix (8), Corollary IV.4.3 of [YK2] and of the considerations similar to those of the second part of the proof of Proposition 6. □

Proposition 8 *If W is a contact mixed foliate non-trivial contact CR-submanifold of a Sasakian manifold space form $M(c)$, then $c \leq -3$.*

Proof The transverse manifold of the characteristic foliation has the constant holomorphic sectional curvature equal to $c+3$. The problem is local and Appendix (8) together with Proposition IV.4.3 of [YK2] ensures that $c+3 \leq 0$. Thus $c \leq -3$. □

Corollary 2 *Let W be a contact mixed foliate contact CR-submanifold of a Sasakian space form $M(c)$. If $c > -3$, then W is either an invariant submanifold or an anti-inavriant submanifold of $M(c)$.*

It is a counterpart of Corollary IV.4.4 of [YK2]. Appendix 6 and Theorem IV.6.1 of [YK2] or [BJ] yield the following.

Theorem 9 *Let W be a contact totally umbilical non-trivial contact CR-submanifold of a Sasakian manifold M. If $dim D_0^{\perp} > 1$, then a geodesic orthogonal to ξ and tangent to W at one point has this property on an open subset of its domain.*

Proof Since the characteristic foliation is Riemannian we have to show that the geodesic is tangent to W on an open subset of its domain. This property is a local one and therefore we can reduce our considerations to the situation described in Appendix. The geodesic is the $ker\eta$ horizontal lift of a geodesic in N. Therefore it is sufficient to know that the submanifold W_0 is totally geodesic. This is precisely the fact which Bejancu's theorem ensures. □

Theorem 10 *Let W be a totally geodesic contact CR-submanifold of a Sasakian manifold M. Then D and $D^{\perp} \oplus T\mathcal{F}$ are Riemannian foliations, and locally:*

i) *W is diffeomorphic to $\mathbb{R} \times W_0$,*

ii) *$\mathcal{F}$ is given by the projection $\mathbb{R} \times W_0 \to W_0$,*

iii) *W_0 is a Riemannian product of $N_0 \times N_1$ of a totally geodesic invariant submanifold N_0 and a totally geodesic anti-invariant submanifold N_1 of N.*

Proof The properties i), ii) and iii) are a consequence of Proposition 2, A.6 and Theorem IV.6.2 of [YK2]. Therefore it remains to prove that the foliations D and $D^{\perp} \oplus T\mathcal{F}$ are Riemannian. i.e. that a geodesic of W which is tangent to D_0 (resp. $D_0^{\perp}$) at one point remains tangent to D_0 (resp. $D_0^{\perp}$) at any point of its domain. A similar argument as in the proof of Proposition 5 ensures that the problem is local and that we can reduce our considerations to the case described in Appendix. Since the distributions D_0 and $D_0^\perp$ are orthogonal to $\mathcal{F}$ this property is a simple consequence of the fact that this geodesic, being a geodesic of M, is a $ker\eta$ horizontal lift of a geodesic in N and that the corresponding distributions $\bar{D}$ and $\bar{D}^\perp$, respectively, on W_0 are totally geodesic foliations. □

Final remarks 1. The same method can be applied to submanifolds transverse to the characteristic vector field of a Sasakian manifold.

2. This method is also applicable to the $\mathcal{S}$–structures of D. E. Blair, cf. [BR1].

Appendix. Riemannian submersions and Sasakian manifolds

Let M be a Sasakian manifold with the characteristic foliation given by a global submersion $f\colon M \to N$ with connected fibres. The manifold N has an induced Kähler structure and the submersion f is Riemannian for these Riemannian structures. Any foliated submanifold W of M is of the form $f^{-1}(W_0)$ where W_0 is a submanifold of N; we say that the submanifolds W and W_0 correspond via the Riemannian submersion f. We are going to compare the properties of W and W_0, cf. Chapter VI of [YK1] and [YK2]; most of these properties are proved in these two books.

1. Let ∇^0 and $\bar{\nabla}^0$ be the induced connections on W and W_0, respectively. They are the Levi–Civita connections of the induced metrics g_0 and $\bar{g}_0$, respectively. Moreover

$$(\bar{\nabla}^0_X Y)^* = -\varphi^2(\nabla^0_{X^*} Y^*).$$

2. Let D and $\bar{D}$ be the induced connections in the normal bundles of W and W_0, respectively. Then

$$(\bar{D}_X V)^* = D_{X^*} V^* \text{ where } X \in TW_0 \text{ and } V \in T^\perp W_0.$$

3. Let B and $\bar{B}$ be the second fundamental forms of W and W_0, respectively. Then

i) $B(X^*, Y^*) = \bar{B}(X, Y)^*$;

ii) the second fundamental form of W is commutative iff the second fundamental form of W_0 is;

iii) let $\mathcal{S}$ and $\bar{\mathcal{S}}$ be the square of the length of the second fundamental forms of W and W_0, respectively, then $\mathcal{S} = \bar{\mathcal{S}} + 2\sum_{i=1}^m \bar{g}(\bar{J}e_i^\perp, \bar{J}e_i^\perp)$; thus $\mathcal{S} \leq \bar{\mathcal{S}} + 2m$.

4. Let m and $\bar{\mathrm{m}}$ be the mean curvature vectors of W and W_0, respectively, then

i) $\bar{\mathrm{m}}^* = \frac{m+1}{m}\mathrm{m}$;

ii) if the means curvature vector of W is parallel, so is the mean curvature vector of W_0.

5. Let P, F, f, f, t and $\bar{P}$, $\bar{F}$, $\bar{f}$, $\bar{t}$ be the corresponding $(1,1)$–tensors associated with the submanifolds W and W_0, respectively. Then

$$\bar{P}(X)^* = P(X^*),\ \bar{F}(X)^* = F(X^*),\ \bar{f}(X)^* = f(X^*),\ \bar{t}(X)^* = t(X^*).$$

The following properties are not difficult to verify:

i) if f is parallel, so is $\bar{f}$;

ii) if P is parallel, so is $\bar{P}$;

iii) if $PA_V = A_V P$, so $\bar{P}\bar{A}_V = \bar{A}_V\bar{P}$ for any $V \in TW^{\perp}$ and $V \in TW_0^{\perp}$, respectively.

6. The previous considerations ensure that:

i) W is minimal iff W_0 is;

ii) W is anti–invariant iff W_0 is;

iii) W is invariant iff W_0 is, cf. [HR];

iv) if W is totally geodesic, so is W_0;

v) W is contact totally umbilical iff W_0 is totally umbilical, cf. [KON];

vi) if W is anti–invariant, then

- a) W is flat iff W_0 is;
- b) if $m = n$, then the normal connection of W is flat iff the normal connection of W_0 is flat;
- c) $(\bar{R}_0(X,Y)Z)^* = R_0(X^*,Y^*)Z^*$ where R_0 and $\bar{R}_0$ are the curvature tensors of W and W_0, respectively. Thus

  $\bar{K}_0(p) = K_0(p^*)$ where K_0 and $\bar{K}_0$ are the corresponding sectional curvatures;

  $\bar{S}_0(X,Y) = S_0(X^*,Y^*)$ where S_0 and $\bar{S}_0$ are the corresponding Ricci curvatures;

  $\bar{r}_0 = r_0$ where r_0 and $\bar{r}_0$ are the corresponding scalar curvatures;

vii) from (4) we get that:

- a) W is anti–ivariant iff $\mathcal{S} = \bar{\mathcal{S}} + 2m$;
- b) W is invariant iff $\mathcal{S} = \bar{\mathcal{S}}$.

viii) if W is invariant then W is totally geodesic iff W_0 is;

7. The submanifold W is a contact CR–submanifold iff W_0 is a CR–submanifold.

Let us denote by $\bar{D}$ and $\bar{D}^{\perp}$ the distributions on W_0 corresponding to D and $D^{\perp}$, respectively. The distribution $D^{\perp}$ is said to be totally geodesic if any geodesic of W tangent to $D^{\perp}$ at one point remains tangent to $D^{\perp}$. Then we have the following:

Proposition 9 **i)** *The distribution $D^{\perp}$ is totally geodesic iff the foliation $\bar{D}^{\perp}$ is totally geodesic;*

ii) *the distribution $D^{\perp}$ is totally geodesic iff $B(X,Y) \in fTW^{\perp}$ for any $X \in D_0^{\perp}$, $Y \in D_0$;*

iii) *if $B(X,Y) = 0$ for any $X \in D_0$, $Y \in D_0^{\perp}$, then the distribution $D^{\perp}$ is totally geodesic;*

iv) *a generic submanifold is contact mixed totally geodesic iff the distribution $D^{\perp}$ is totally geodesic.*

Proof First let us remark that if we prove (i) then the other points will follow easily from the corresponding properties of Kähler manifolds as $\bar{D}^{\perp *} = D_0^{\perp}$ and $\bar{D}^* = D_0$. Let us return to (i). Leaves of $D^{\perp} \oplus T\mathcal{F}$ are anti–invariant submanifolds of M. The corresponding submanifolds of N are leaves of $\bar{D}^{\perp}$. A geodesic α tangent to $D_0^{\perp}$ is a geodesic ortogonal to $\mathcal{F}|W$. Therefore it must be a $ker\eta$ horizontal lift of a geodesic $\bar{\alpha}$ in W_0. But the distribution $\bar{D}^{\perp}$ defines a totally geodesic foliation, so $\bar{\alpha}$ remains tangent to $\bar{D}^{\perp}$ and α to $D_0^{\perp}$.□

8. For the notions introduced in Definition 2 we have the following:

i) W is a D_0–totally geodesic iff W_0 is $\bar{D}$–totally geodesic;

ii) W is contact mixed foliate iff W_0 is mixed foliate.

The proof of both equivalences is a simple calculation.

Notes The chapter contains results of three papers which correspond to three section, cf. [CW1], [CW2] and [WO19], respectively.

It is almost impossible to mention all papers on transversely symplectic, holomorphic or Kähler foliations. Various authors turned their attention to different aspects of the theory, from characteristic classes to the theory of deformations. The reader can easily find many papers on these foliations in the Bibliography.

Chapter VI

$\nabla - G$–foliations

The notion of a $\nabla - G$–foliation the author introduced in [WO3]. They are these G–foliations on whose transverse manifold there is a holonomy invariant G–connection. In Chapter II we showed that this property is equivalent to the existence of a transversely projectable G–connection. This type of foliations was studied earlier by P. Molino, cf. [MO2]. In Chapter IV we dedicated a lot of space to find some conditions ensuring the existence of transversely projectable G–connections. Under some assumptions $\nabla - G$–foliations behave very much like Riemannian foliations, which, of course, admit a transversely projectable connection. In Chapter III we showed that these foliations admit a FSODE and proved some similarities. But we would like to know which properties distinguish between these two classes of foliations. Let us take two other fundamental properties of Riemannian foliations on compact manifolds:

a) the closures of leaves are submanifolds;

b) the base–like cohomology is of finite dimension.

In the next chapter we shall show that both properties do not hold for $\nabla - G$–foliations on a compact manifold, even if the corresponding FSODE (i.e. the transversely projectable connection) is transversely complete, cf. Examples VII.10, VII.11 and VII.2.3 . In this chapter we concentrate our attention on the converse problem: a $\nabla - G$–foliation, when is it a Riemannian one?

VI.1 Preliminaries

We begin with the structure theorem for our foliations. Let ω be the corresponding transversely projectable connection in the foliated G–structure $B(M, G; \mathcal{F})$. The choice of a supplementary subbundle Q to $T\mathcal{F}$ fixes our choice of a supplementary subundle $\tilde{Q}$ to $T\mathcal{F}_1$, i.e. $\tilde{Q} = (d\pi)^{-1}(Q)$. Therefore the corresponding fundamental horizontal vector fields $B(\xi)$ and the fundamental vertical vector fields A^* form a transverse parallelism of $\mathcal{F}_1$. This transverse parallelism is complete iff the connection ω is transversely complete, i.e. its geodesics tangent to Q are globally defined. This results from the following simple lemma.

Lemma 1 *The projections onto M of integral curves of the vector fields $B(\xi)$ are geodesics tangent to Q of the connection ω.*

Proof Let $\tilde{\omega}$ be the extension of the connection ω as in Example III.2. $\tilde{\omega}$ is a connection in the $GL(p)\times G$–structure $B(M, GL(p)\times G)$ which can be written as the fibre product $L(T\mathcal{F}) \times_M B(M, G; \mathcal{F})$. The geodesics of $\tilde{\omega}$ which are tangent to Q are precisely the "transverse geodesics" of ω, i.e. solutions of the FSODE of $(M, \mathcal{F})$.

The fundamental horizontal vector fields $B(\xi)$, $\xi \in \mathbb{R}^q$, of $B(M, G; \mathcal{F})$ can be lifted to $B(M, GL(p) \times G)$. The lift of $B(\xi)$ is precisely the vector field $B((0,\xi))$ for $(0,\xi) \in \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$. The projection of an integral curve of $B((0,\xi))$ is a geodesic of $\tilde{\omega}$, cf. [LIC], which must be tangent to Q. Since $B((0,\xi))$ is the lift of $B(\xi)$ the projections on M of integral curves of these vector fields are the same. □

The above considerations lead to the following definition.

Definition 1 *A $\nabla - G$-foliation is transversely complete if for some choice of a supplementary subbundle Q the corresponding FSODE is transversely complete, or equivalently if the corresponding transverse parallelism is complete.*

With this definition in mind we have the following structure theorem for $\nabla - G$–foliations:

Theorem 1 *let $\mathcal{F}$ be a transversely complete $\nabla - G$-foliation on a manifold M. Then the closures of leaves of the foliation $\mathcal{F}_1$ of the foliated G-structure $B(M, G; \mathcal{F})$ are fibres of a locally trivial fibre bundle, called the basic fibration. The foliation of the closure of a leaf of $\mathcal{F}_1$ by leaves of $\mathcal{F}_1$ is a Lie foliation with the same model Lie group for any leaf.*

Proof It is a direct consequence of our considerations and Molino's structure theorem for complete T.P. foliations, cf. [MO5,MO11].□

For $\nabla - G$–foliations we can define, following P. Molino, the commuting sheaf, cf. [MO6,MO9,MO11]. Let $\mathcal{C}_1$ be the sheaf of germs of foliated vector fields X on B commuting with all global foliated vector fields of $(B, \mathcal{F}_1)$, thus in particular the transverse parallelism of $\mathcal{F}_1$. This last condition is equivalent to $L_X\theta = L_X\omega = 0$. Let $\bar{X}$ be the corresponding vector field on the total space of $B(N, G)$. Then $L_{\bar{X}}\bar{\theta} = L_{\bar{X}}\bar{\omega} = 0$ where $\bar{\omega}$ is the connection form of ∇. This means that $\bar{X}$ is the lift of a local infinitesimal affine transformation of ∇. Thus the sheaf $\mathcal{C}_1$ defines the sheaf $\mathcal{C}$ of germs of foliated vector fields which are also local infinitesimal affine transformations of the transversely projectable connection ω. We call $\mathcal{C}$ the commuting sheaf of $\mathcal{F}$.

Definition 2 *We say that the commuting sheaf $\mathcal{C}$ is of compact type if the orbits of the sheaf $\mathcal{C}_1$ are compact.*

The following proposition is an immediate consequence of the definition and of the properties of T.P. foliations, cf. [MO5,MO9,MO11].

Proposition 1 *Let $\mathcal{F}$ be a transversely complete $\nabla - G$-foliation. If its commuting sheaf is of compact type, then the closures of leaves are compact and they are integral submanifolds of a regular distribution of non-constant dimension defined by the commuting sheaf.*

Let us assume that the foliation $\mathcal{F}$ is a transversely complete $\nabla - G$-foliation. Since the foliation $\mathcal{F}_1$ is complete T.P., it determines a locally trivial fibration (the basic fibration) $p: B \longrightarrow W$ whose fibres are the closures of leaves of the foliation $\mathcal{F}_1$, cf. [MO5,MO9,MO11]. As this foliation is G-invariant, the group G acts on the basic manifold W. This leads us to the following lemma whose proof is trivial.

Lemma 2 *If the commuting sheaf $\mathcal{C}$ is of compact type, then the action of the group G on the basic manifold W is proper.*

Using the properties of proper actions of Lie groups, cf. [PA2,KS,DA], we can define types of closures of leaves, cf. [DA,HA3]. In the foliations considered by us the closures of leaves correspond bijectively to orbits of the action of the group G on the basic manifold W. We say that two leaves have the same closure type if the corresponding (to the closures) orbits are of the same normal orbit type. Let α be a normal orbit type. Then the space W_α of all points of W whose G-orbit is of type α is a proper submanifold of W, the space V_α of G-orbits of W_α is a manifold and the natural projection $\kappa: W_\alpha \longrightarrow V_\alpha$ is a locally trivial fibre bundle. The same can be proved for leaves.

Proposition 2 *Let $\mathcal{F}$ be a $\nabla - G$-foliation with the commuting sheaf $\mathcal{C}$ of compact type and let α be a closure type of leaves of the foliation $\mathcal{F}$. Then the space M_α of all points of leaves of the closure type α is a proper submanifold of M. The space of closures of leaves of M_α is a Hausdorff manifold V_α and the natural projection $\bar{p_\alpha}: M_\alpha \longrightarrow V_\alpha$ is a locally trivial fibre bundle.*

Proof Let α be a closure type of leaves of the foliation $\mathcal{F}$. The corresponding G-orbit type we denote by the same letter α. The stratum W_α of the basic manifold is a proper submanifold of W. Thus $p^{-1}(W_\alpha)$ is a proper submanifold of B. Since the set M_α is equal to the quotient $p^{-1}(W_\alpha)/G$, it is a proper submanifold of M. The space of closures of leaves of M_α is precisely the space of G-orbits of W_α, thus it is a Hausdorff manifold. The submersion is a locally trivial fibre bundle, as its fibres, the closures of leaves of $\mathcal{F}$, are compact.

Corollary 1 *If γ is the principal normal orbit type of the action of the group G on the basic manifold W, then the corresponding stratum M_γ is open and dense in M.*

Remark In the Riemannian case the notion of the α–stratum was introduced by A. Haefliger, cf. [HA3,HA4]; compare also [PI].

VI.2 The case of a pseudogroup

Let $\mathcal{H}$ be a pseudogroup of local affine transformations of a connection ∇ in a G–structure $B(S,G)$. To any element h of $\mathcal{H}$ corresponds a local diffeomorphism h^1 of B which preserves the parallelism of B. And vice-versa, any such a local diffeomorphism of B of connected domain is defined by a local affine transformation. The correspondence $j^1\colon h \longmapsto h^1$ associates to the pseudogroup $\mathcal{H}$ a pseudogroup $j^1\mathcal{H}$ of local diffeomorphisms of B preserving the parallelism.

There is a natural one-to-one correspondence between pseudogroups and groupoids of germs of local diffeomorphisms. The groupoid defined by a pseudogroup $\mathcal{H}$ we denote by $\underline{\mathcal{H}}$.

Definition 3 *We say that a pseudogroup $\mathcal{H}$ of local affine transformations of a connection in $B(S,G)$ is closed if the groupoid $\underline{j^1\mathcal{H}}$ is closed in the groupoid of germs of local diffeomorphisms preserving the parallelism of B.*

For our purposes we need also the following definition.

Definition 4 *We say that a pseudogroup $\mathcal{H}$ of local affine transformations of a connection in a G–structure $B(S,G;\pi)$ is of compact type if for any compact subset K of S and any point x of B the set $j^1\mathcal{H}x \cap \pi^{-1}(K)$ is relatively compact.*

Remark Both notions are 'invariant' under compactly generated equivalences of pseudogroups.

Having formulated these definitions we can prove the following:

Proposition 3 *Let $\mathcal{H}$ be a complete pseudogroup of local affine transformations of a connection ∇ in a G–structure $B(S,G)$. Then there exists the unique pseudogroup $\overline{\mathcal{H}}$ of local affine transformations of the connection ∇ called the closure of $\mathcal{H}$ such that $\underline{j^1\overline{\mathcal{H}}}$ is the closure of $\underline{j^1\mathcal{H}}$. The pseudogroup $\overline{\mathcal{H}}$ is also complete and $p : \overline{\mathcal{H}}/\mathcal{H} \longrightarrow S$ is a covering. The pseudogroup $\overline{\mathcal{H}}$ is equivalent to a covering of $\mathcal{H}$. If $\mathcal{H}$ is of compact type, so is $\overline{\mathcal{H}}$. Moreover, the closures of orbits of $\mathcal{H}$ are the orbits of the pseudogroup $\overline{\mathcal{H}}$ and the space of orbits of $\overline{\mathcal{H}}$ is Hausdorff.*

Proof The results concerning complete pseudogroups can be derived easily from the known results on pseudogroups of local isometries, cf. [HA3,HA4], as the pseudogroup $j^1\mathcal{H}$ is a pseudogroup of local diffeomorphisms preserving a parallelism, thus, in fact, a pseudogroup of local isometries.

If the pseudogroup $\mathcal{H}$ is of compact type, so must be $\overline{\mathcal{H}}$, as $j^1\overline{\mathcal{H}}$ is the closure of $j^1\mathcal{H}$. The orbits of the pseudogroup $j^1\overline{\mathcal{H}}$ are closed; they are the closures of orbits of $j^1\mathcal{H}$. Since the pseudogroup $\overline{\mathcal{H}}$ is of compact type, their projections onto manifold S are closed and they are the closures of orbits of the pseudogroup $\mathcal{H}$. Moreover, from the very definition these projections are the orbits of the pseudogroup $\overline{\mathcal{H}}$. Using the same methods as in the proof of Proposition 3.1 of [HA4] we can show that the space of orbits of $\overline{\mathcal{H}}$ is Hausdorff.□

In our case, Salem's theorem, (cf. [SA1,SA2]), yields the following.

Theorem 2 *A closed complete pseudogroup $\mathcal{H}$ of local affine transformations of a connection ∇ in a G–structure $B(S,G)$ is a Lie pseudogroup i.e. for each point of the manifold S, there is an open neighbourhood U and a finite dimensional Lie algebra $\underline{g}(U)$ of infinitesimal affine transformations of the connection ∇ on U such that for any relatively compact subset V of U ; $\bar{V} \subset U$, the elements of $\mathcal{H}$ close to the identity of the domain V are of the form* $\exp\xi$ *for some* $\xi \in \underline{g}(U)$.

The closure of the $\mathcal{H}$–orbit of a point x_0 of S is a submanifold S_0. Denote by N its normal bundle. Both pseudogroups $\mathcal{H}$ and $\overline{\mathcal{H}}$ act on this bundle. The pseudogroup defined by these actions we denote by $\mathcal{H}_N$ and $\overline{\mathcal{H}}_N$, respectively. Then:

Lemma 3 *The pseudogroup $\overline{\mathcal{H}}_N$ is differentiably equivalent to the restriction of $\overline{\mathcal{H}}$ to a small tubular neighbourhood of S_0.*

Proof The submanifold S_0 is the $\overline{\mathcal{H}}$–orbit of the point x_0. The isotropy group $\overline{\mathcal{H}}_{x_0}$ at x_0 can be identified with a compact subgroup of the group of linear transformations of the tangent space of S at x_0. Therefore the action of $\overline{\mathcal{H}}_{x_0}$ is semi-simple. Since the action of $\overline{\mathcal{H}}$ preserves the tangent bundle to the submanifold S_0, there exists a subbundle Q of TS on S_0 supplementary to TS_0 which is $\overline{\mathcal{H}}$ –invariant. The exponential mapping defined by the connection restricted to the subbundle Q provides the equivalence we have been looking for.□

This lemma and the facts we have proved earlier (cf. Proposition 3 and Theorem 2) ensure that Theorems 4.3 and 5.6.1 of [HA4] are also valid in this case. This implies that for any closure $\overline{\mathcal{H}x_0}$ of an orbit $\mathcal{H}x_0$ of the pseudogroup $\mathcal{H}$ there exists a tubular neighbourhood U of this submanifold and a Riemannian metric g_U on this neighbourhood such that the restriction $\mathcal{H}_U$ of this pseudogroup to the open subset U is a pseudogroup of local isometries of g_U. Let $\mathcal{O}$ be an open covering by $\mathcal{H}$–saturated open subsets of S as above. Using the same method as in [WO21] we can construct an $\mathcal{H}$ –invariant partition of unity subordinated to $\mathcal{O}$. Piecing together the Riemannian metrics g_U with the help of this partition of unity we get a Riemannian metric on the whole manifold S of which the pseudogroup $\mathcal{H}$ is a pseudogroup of local isometries. Lemma 3 ensures that the pseudogroup generated by restrictions of $\mathcal{H}$ to open sets of $\mathcal{O}$ is the pseudogroup $\mathcal{H}$ itself. Therefore we have proved the following:

Theorem 3 *Let $\mathcal{H}$ be a pseudogroup of local affine transformations of a linear connection on a manifold S. If the pseudogroup $\mathcal{H}$ is complete and of compact type, then $\mathcal{H}$ is a pseudogroup of local isometries of some Riemannian metric on the manifold S.*

VI.3 The case of a foliation

In this section we shall apply the results of Section 2 to $\nabla - G$–foliations. Let $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ be a cocycle defining the foliation $\mathcal{F}$ for which the sets U_i are relatively compact and such that the covering $\{U_i\}$ is locally finite. For such a cocycle the leaves of the foliation $\mathcal{F}$ correspond to the orbits of the pseudogroup $\mathcal{H}$ and the closures of leaves correspond to the closures of orbits of this pseudogroup. Moreover, it is not difficult to verify that if the closures of leaves of the foliation $\mathcal{F}_1$ are compact, the pseudogroup $\mathcal{H}$ is of compact type. Thus, if the commuting sheaf $\mathcal{C}$ of the foliation $\mathcal{F}$ is of compact type the holonomy pseudogroup $\mathcal{H}$ associated to the cocycle $\mathcal{U}$ is of compact type as well. Therefore to apply Theorem 3 to our $\nabla - G$–foliations we have to find some natural condition which would ensure that $\mathcal{H}$ is complete.

Lemma 4 *If the commuting sheaf is of compact type then the linear holonomy of any leaf is distal.*

Proof The bundle $N(M;\mathcal{F})$ is the associated fibre bundle to $L(M;\mathcal{F})$ (or $B(M,G;\mathcal{F})$) with the standard fibre $\mathbb{R}^q$. The foliation $\mathcal{F}_N$ of $N(M;\mathcal{F})$ is the corresponding foliation to the foliation $\mathcal{F}_1$ of $L(M;\mathcal{F})$. If the commuting sheaf of $\mathcal{F}$ is of compact type it is not difficult to see that the closures of leaves of the foliation $\mathcal{F}_N$ are compact and that the saturation by the closures of leaves of any compact subset of $N(M;\mathcal{F})$ is also compact.

Having said that it is easy to verify that any eigenvalue of the linear holonomy map must have absolute value equal to 1. This means precisely that this linear mapping is distal and therefore the linear holonomy of any leaf must be distal, cf. [CG].□

Let us look at the consequences of this property. Let v be a point of the fibre $N(M;\mathcal{F})_m$ over a point m. The leaf L_v of the foliation $\mathcal{F}_N$ of $N(M;\mathcal{F})$ passing through this point v cannot approach the zero section of $N(M;\mathcal{F})$. Assume that it is the case. Then $\bar{L_v} \cap \bar{L_m} \neq \emptyset$, where L_m is the leaf passing through m. By taking the leaf from this intersection we can assume that $\bar{L_v} \supset \bar{L_m}$. The projection L' of the leaf L onto M also contains L_m in its closure. Then there exists a leaf L'' of $\mathcal{F}_N$ over L_m such that $exp(L'') = L'$. Therefore there exists a sequence of points v_n in $N(M;\mathcal{F})_m \cap L''$ which tends to m. Let α_n be a leaf curve linking v_n to v_{n+1}. It is contained in $N(M;\mathcal{F})|L_m$. Its projection γ_n is a curve in the leaf L_m. Consider the holonomy mapping h_n defined by γ_n. Its linear part must map v_n to v_{n+1} as it is the end of the lift of the curve γ_n to the vector v_n This means precisely that the linear holonomy of the leaf of $\mathcal{F}$ passing though m does not have distal linear holonomy at m. Contradiction.

Lemma 5 *Let $\mathcal{F}$ be a transversely complete $\nabla - G$–foliation with the commuting sheaf of compact type. Then its holonomy pseudogroup is complete.*

Proof Let us consider a cocycle $\mathcal{U}$ definig the foliation $\mathcal{F}$ as at the beginning of this section and the holonomy pseudogroup $\mathcal{H}$ defined by this cocycle. Let us chose two points x and y of the transverse manifold N belonging to N_i and N_j, respectively. We can choose two points x' and y' and sets U' and V' such that $f_i|U'$ and $f_j|V'$ are diffeomorphisms onto open neighbourhoods U and V of x and y, respectively.

We shall try to demonstrate that the neighbourhoods U and V or some their open subsets are the ones we are looking for.

Let h be an element of the holonomy pseudogroup such that for some point z of U $h(z) \in V$. To this local diffeomorphism corresponds a local diffeomorphism of U' into V' obtained as the holonomy along a leaf curve β linking the point z' of U' to $h(z)'$ of V'; where $f_i(z') = z$ and $f_j(h(z)') = h(z)$. Let us choose complete vector fields $\tilde{B}(e_i)$ correponding to the foliated vector fields $B(e_i), i = 1, ...q$. The vector subbundle spanned by these vector fields projects onto the manifold M and forms a subbundle Q supplementary to the tangent bundle to leaves of the foliation. The projections of integral curves of the vector fields $\sum \alpha_i B(e_i) = B(\sum \alpha_i e_i)$ are geodesics tangent to the subbundle Q of the transversely projectable connection. Therefore these geodesics are globally defined. For any point m of the manifold M let us denote by exp_m the exponential mapping defined by the connection: $exp_m: Q_m \longrightarrow M$. By $\epsilon(m)$ we denote the greatest number for which $exp_m|B(m, \epsilon(m))$ is an embedding, where $B(m, r)$ is the open ball in Q_m at 0 with radius r. Since the foliation $\mathcal{F}$ is of compact type, for any leaf L of $\mathcal{F}$ there exist an open saturated neighbourhood P of this leaf and a positive number ϵ such that for any point m of P $exp_m|B(m, \epsilon)$ is an embedding. We can assume that both U' and V' are contained in $exp_{x'}(B(x', \epsilon))$ and $exp_{y'}(B(y', \epsilon))$, respectively.

Let us describe in detail the way one can obtain the holonomy along any leaf curve α from x' to y'. Let w be any point of U'. There exists precisely one geodesic γ_w linking x' to w in U' with the initial condition $exp_{x'}^{-1}(w) = \xi_w$. Let

α_w be the lift of the curve α to the leaf of the foliation $\mathcal{F}_N$ passing through the point ξ_w. The curve $t \longmapsto exp(\alpha_w(t))$ is a leaf curve passing through the point w. Its end, $h_\alpha(w)$, is the value at w of the holonomy diffeomorphism h_α defined by the curve α. To have the mapping well defined on the whole U' we must know that the norm of vectors of curves α_w is always smaller than ϵ.

We shall prove that there exists $\delta > 0$ such that the saturation of $B(x', \delta)$ does not have vectors with the norm greater or equal to ϵ. Assume the contrary. Then there exists a sequence of vectors ξ_n of the norm ϵ which belong to the closures of leaves passing through points of $B(x', 1/n)$, respectively. Thus the saturation of $\overline{\{\xi_n\}}$ is a compact set. Hence it must contain L; contradiction.

Let us return to our point z' and the holonomy mapping h_β defined by the leaf curve β. Assume that this point belongs to $exp(B(x', \delta))$. The previous considerations ensure that to complete the proof we need to show that h_β is defined by some holonomy mapping at x'. Let $\xi_{x'} = exp_{z'}^{-1}(x')$ and $\tilde{\beta}$ be the lift of the curve β to the leaf of the foliation $\mathcal{F}_N$ passing through $\xi_{x'}$. Then the leaf curve $\alpha : t \longmapsto exp(\tilde{\beta}(t))$ defines the holonomy mapping we have been looking for.

The subsets $U = f_i(exp_{x'}(B(x', \delta)))$ and $V = f_i(exp_{x'}(B(x', \epsilon)))$ with δ chosen as above satisfy the condition of Definition I.5.□

Our considerations together with these of Section 1 lead to the following theorem.

Theorem 4 *Let $\mathcal{F}$ be a transversely complete $\nabla - G$–foliation. If its commuting sheaf is of compact type, then the foliation $\mathcal{F}$ is a Riemannian one.*

Proof Lemma 5 ensures that the holonomy pseudogroup of $\mathcal{F}$ is complete. Moreover this pseudogroup is also of compact type as our previous considerations indicate. Then the theorem is a consequence of Theorem 3.

VI.4 The case of a flow

In this section we pay particular attention to flows admitting foliated G–structures. Using a different method we refine the results obtained in the previous section.

Let the foliation $\mathcal{F}$ be given by a flow $\Phi = (\phi_t)$. We say that the flow admits an almost connection, cf. [MO4], if there exists a supplementary subbundle Q to the tangent bundle to $\mathcal{F}$ such that for any t

a) $d\phi_t(Q) \subset Q$.

Now we shall look at the consequences of this condition for flows admitting a transversely projectably connection $\nabla^{\mathcal{F}}$. This connection can be considered as a connection in the vector bundle Q. It defines a system of ordinary differential equations of order 2, the equation of the geodesic. The projections of geodesics of $\nabla^{\mathcal{F}}$ tangent to Q onto the transverse manifold are geodesics of the corresponding connection ∇ of this manifold. Moreover, a curve tangent to Q and projecting onto a geodesic of ∇ is itself a geodesic of $\nabla^{\mathcal{F}}$.

For any leaf curve $\alpha: [a, b] \longrightarrow M$ and geodesic $\gamma: [0, \epsilon] \longrightarrow M$ tangent to Q, $\alpha(a) = \gamma(0)$, there exists the unique mapping $\sigma: [a, b] \times [0, \epsilon] \longrightarrow M$ such that

1. for any $t \in [0, \epsilon], \sigma|[a, b] \times \{t\} = \sigma_t$ is leaf curve and $\sigma_0 = \alpha$;
2. for any $s \in [a, b], \sigma|\{s\} \times [0, \epsilon] = \sigma^s$ is a geodesic tangent to Q and $\sigma^a = \gamma$.

Let us take α equal to a segment of the flow Φ, i.e. $\alpha(s) = \Phi(x, s)$ for some $x \in M$. Then the condition (a) ensures that the curves σ^s are equal to $\phi_s \circ \gamma$, as the curve $\phi_s \circ \gamma$ has the same projection onto the transverse manifold, modulo the action of the holonomy pseudogroup, as γ. Thus, in fact, it is a geodesic of $\nabla^{\mathcal{F}}$ tangent to Q. Therefore the curve σ_ϵ is a segment of the flow Φ, i.e. $\sigma_\epsilon(s) = \Phi(\gamma(\epsilon), s)$. Reversing, if for any pair of curves α and γ, as above, the curve σ_ϵ is a segment of the flow Φ, then the curves $\phi_s \circ \gamma$ are geodesics tangent to Q and, therefore, the flow Φ satisfies the condition (a).

We have just proved that the condition (a) is equivalent to the following one:

$a')$ the mappings ϕ_s send geodesics of $\nabla^{\mathcal{F}}$ tangent to Q onto geodesics of $\nabla^{\mathcal{F}}$ tangent to Q.

The choice of a supplementary subbundle Q allows us to define global vector-fields $B(\xi)$ for any $\xi \in \mathbb{R}^q$. It is easy to check that the condition (a') is equivalent to the following one:

$a'')$ the vector fields $B(\xi)$ commute with the lifted flow Φ_1 on the total space B of the principal fibre bundle $B(M, G; \mathcal{F})$.

The condition (a) is equivalent to the vanishing of a cohomology class with values in $\mathbb{R}$ associated to the action of the additive group $\mathbb{R}$, called the Atiyah–Molino class of the action (cf. [MO4]).

We continue the study of properties of $\nabla - G$–flows.

Theorem 5 *Let $(M, \mathcal{F})$ be a $\nabla - G$-flow with non-zero commuting sheaf. Then, if it admits an almost-connection there exists a parametrization of $\mathcal{F}$ making it an isometric flow.*

Proof Let $\Phi = (\phi_t)$ be the parametrization of the flow $\mathcal{F}$ satisfying the condition (a). Let X be the vector field defined by the flow. First of all, let us remark that there is a 1-form w such that $w(X) > 0$ and $i_X dw = 0$. In fact, let us put $w(X) \equiv 1$ and $w|Q \equiv 0$. We have to verify only the second condition. Let Y be any vector of Q at a point x. The condition (a) ensures that there exists a vector field $\tilde{Y}$, a section of Q, defined on a small neighbourhood of x extending the vector Y and commuting with X. Then $dw(X, \tilde{Y}) = 0$.

Taking into account the criterion of Sullivan–Gluck (cf. [GL]) we have to show the existence of a bundle–like metric for the foliation $\mathcal{F}$. Unfortunately, we cannot use Theorem 4 as we do not assume that the foliation is a transversely complete $\nabla - G$–foliation. We have to do it in another way.

The lifted flow Φ_1 on the total space B commutes with the vector fields $B(\xi)$ and A^*. Using the methods developed by P.Molino, cf. [MO5,MO11], one can verify that although the vector fields $B(\xi)$ need not to be complete, the space of leaves of Φ_1 is a Hausdorff manifold W and the natural submersion $p: B \longrightarrow W$ is a locally trivial fibre bundle. It results from the fact that integral curves of the vector field $B(\xi)$ are defined on the same interval for any point of a given orbit. As the commuting sheaf is non-zero, the fibres of the basic fibration are the tori with dense flows, cf. [MO10,GH2]. We can choose the standard fibre T^k and a dense flow Φ_0 in T^k. The structure group of the basic fibration commutes with the flow Φ_0 as the vector fields A^* and $B(\xi)$ commute with Φ_1. The flow Φ_0 is a reparametrization of a linear flow of T^k .

The action of the group G on W is proper. Therefore any point v of W has a G–invariant neighbourhood of the form $V \times_H G$, where H is the isotropy subgroup at v, and V is a small transverse submanifold at v identified with $T_x W / T_x xG$. The transverse submanifold V is contractible; thus the basic fibration over V is trivial, i.e. $B|V \cong V \times \mathrm{T}^k$. Therefore, over $V \times_H G$, there exists a diffeomorphism h commuting with the actions of the group G,

$$h: p^{-1}(V \times_H G) = V' \longrightarrow (V \times \mathrm{T}^k) \times_H G.$$

The diffeomorphism h maps the lifted flow Φ_1 onto the flow of $(V \times \mathrm{T}^k) \times_H G$ defined by the standard flow Φ_0 of T^k. The foliation of $V \times \mathrm{T}^k$ defined by the flow Φ_0 is a Riemannian foliation. Since this foliation is H–invariant, it admits an H–invariant bundle–like metric. Thus it is possible to find a G–invariant bundle–like metric on V'. In fact, it is sufficient to take a right G–invariant and left H–invariant metric on G. These two metrics define an H–invariant metric for the diagonal action of H on $V \times \mathrm{T}^k \times G$. Thus this metric induces a Riemannian metric $g_{V'}$ on V' . The Riemannian metric $g_{V'}$ is a bundle-like metric for the foliation $\mathcal{F}_1$ restricted to V' and, moreover, it is a G–invariant one.

Let $\mathcal{V} = \{V_\alpha\}$ be an open locally finite covering of B by G–invariant sets of type $(V \times \mathrm{T}^k) \times_H G$. We would like to construct a partition of unity $\{\lambda_\alpha\}$ subordinated to the covering $\mathcal{V}$ by G–invariant basic functions. Choose a set

$V' \cong (V \times \mathrm{T}^k) \times_H G$. For any compact subset K of V we shall construct a G–invariant basic function which is equal to 1 on K and when restricted to V has compact support. In fact, for any compact $K \subset V$ there exists a function $f: V \longrightarrow [0,1]$ of compact support such that $f|K \equiv 1$. As the group H is compact, we can assume that f is H–invariant. Then the function fp_1, $p_1: V \times \mathrm{T}^k \times G \longrightarrow V$, is G–invariant and H–invariant for the diagonal action of H. Therefore it projects to a G–invariant function on V' with the required properties. Eventually, by taking a refined covering we obtain a partition of unity for which we have been looking for.

Using a G–invariant partition of unity $\{\lambda_\alpha\}$ and G–invariant bundle–like metrics $\{g_{V_\alpha}\}$ constructed above, we obtain a G–invariant bundle–like metric $\tilde{g} = \sum \lambda_\alpha g_{V_\alpha}$ on B. The restriction of this metric to the horizontal space Γ of the connection induces a bundle–like metric on the manifold M. This ends the proof of the theorem.□

Corollary 2 *If the holonomy pseudogroup is 1–connected, then the flow $\mathcal{F}$ is isometric.*

Proof Let $\{U_i, f_i, g_{ij}\}$ be a cocycle making the foliation $\mathcal{F}$ a Riemannian flow. Since the fundamental groups of representatives of the holonomy pseudogroup are isomorphic (cf. [HA4,HS,SA2]), the representative $\mathcal{H}$ of the holonomy pseudogroup defined by this cocycle is also 1–connected. This implies that the commuting sheaf is constant, which in its turn ensures that the flow $\mathcal{F}$ can be parametrized in such a way as to make it isometric, cf. [MOS].

VI.5 Equicontinuous $\nabla - G$–foliations

In Appendix E to [MO11] E. Ghys suggested the study of foliations whose holonomy pseudogroup consists of equicontinuous transformations for some metric on the transverse manifold. Moreover he conjectured that the closures of orbits of a flow whose holonomy pseudogroup is equicontinuous are orbits of some abelian groups. We prove this conjecture for foliations admitting a transversely projectable connection. In fact we show something much stronger; that such foliations must be Riemannian.

For general pseudogroups the notion of a complete pseudogroup is not invariant under equivalences as the following example illustrates well this fact.

Example 1 Let $N = \mathbb{R}$ and $\mathcal{H}$ be a pseudogroup generated by the homothety $h_\lambda: x \mapsto \lambda x$ for $0 < \lambda < 1$ and the translation $\tau_1: x \mapsto x + 1$. $(N, \mathcal{H})$ is a complete pseudogroup. It is equivalent to its restriction $\mathcal{H}'$ to the interval $(-1/2, 1/2)$. However this second pseudogroup is not complete.

To obtain a notion which is invariant under equivalences of pseudogroups we should demand more.

Definition 5 *A pseudogroup $(N, \mathcal{H})$ is strongly complete if for any two points x and y of N and any neighbourhood V_0 of y there exist neighbourhoods U and V of x and y, respectively, $V \subset V_0$ such that any element h of $\mathcal{H}$ with domain in U and target in V can be extended to an element h' of $\mathcal{H}$ defined on the whole set U and whose image is contained in V_0.*

For pseudogroups of local isometries and for "equicontinuous" pseudogroups the notions of completeness and strong completeness are equivalent. It is not difficult to check that a pseudogroup equivalent to a strongly complete pseudogroup is itself strongly complete. Moreover it is obvious that a strongly complete pseudogroup is complete. The pseudogroup $(N, \mathcal{H})$ of Example 1 is not strongly complete although it is complete.

The assumption of strong completeness seems to impose very strong restrictions on the pseudogroup. Let us look into this problem. Let $\mathcal{V}$ be any cocycle defining $\mathcal{F}$ and $\mathcal{U}$ be a relatively compact cocycle associated to it, cf. Chapter I. We denote the transverse manifold associated to the cocycle $\mathcal{U}$ by N' and the holonomy pseudogroup representative on N' by $\mathcal{H}'$. The transverse manifold associated to the cocycle $\mathcal{V}$ is denoted by N and the holonomy pseudogroup representative on N by $\mathcal{H}$. The subset N' of N is relatively compact and the pseudogroup $\mathcal{H}'$ is the restriction of $\mathcal{H}$ to N'.

Since N' is a relatively compact subset of N there exists $\epsilon > 0$ such that for any point $x \in N'$ there is an open neighbourhood V_x of x with the following properties:

i) for any $z \in V_x$ $exp_z|B(0_z, \epsilon)$ is a diffeomorphism onto the image;

ii) $exp_z(S(0_z, \epsilon)) \cap V_x = \emptyset$;

where $B(0_z, \epsilon) = \{v \in TN_z : \|v\| < \epsilon\}$ and $S(0_z, \epsilon) = \{v \in TN_z : \|v\| = \epsilon\}$, for some Riemannian metric on N. This fact results easily from a slight refinement of the clasical argument about geodesically convex neighbourhoods, cf. [KN].

Having established these technical details we return to the strong completeness. Let us take a pair of points x and y with $V_0 = V_y$. Then the set U can be equal to $B(x, \delta) = exp_x(B(0_x, \delta))$ for some $\delta > 0$, and $V \subset V_y$. Then for any element h of $\mathcal{H}$ defined on U we have

$$h \circ exp_x|B(0_x, \delta) = exp_{h(x)} \circ d_x h|B(0_x, \delta).$$

As $h(U) \subset V_y$, the set $d_x h(B(0_x, \delta))$ must be contained in $B(0_{h(x)}, \epsilon)$. This means precisely that the set $J^1\mathcal{H}(x, V) = \{j^1_x h : h \in \mathcal{H},\ h(x) \in V\}$ is bounded (relatively compact). Hence for any relatively compact subset K of N the set $J^1\mathcal{H}(x, K) = \{j^1_x h : h \in \mathcal{H},\ h(x) \in K\}$ is relatively compact. Thus we have proved the following lemma.

Lemma 6 *A strongly complete pseudogroup of local affine transformations of an affine connection is of compact type.*

Combining Lemma 6 with Theorem 3 we get the following theorem.

Theorem 6 *Let $\mathcal{F}$ be a $\nabla - G$–foliation on a compact manifold. If the holonomy pseudogroup of $\mathcal{F}$ is strongly complete, then $\mathcal{F}$ is a Riemannian foliation.*

As a corollary we obtain:

Corollary 3 *A $\nabla - G$–foliation $\mathcal{F}$ of a compact manifold whose holonomy pseudogroup is equicontinuous is a Riemannian foliation.*

Theorem 7 *Let $\mathcal{F}$ be a flow on a compact manifold admitting a transversely projectable connection. If $\mathcal{F}$ has a representative of the holonomy pseudogroup which is equicontinuous for some metric inducing the natural topology on the transverse manifold, then the closures of its orbits are diffeomorphic to tori.*

Proof It is a consequence of Corollary 3 and Carrière's result on minimal Riemannian flows, cf. [CA1,CA3].□

For transversely affine flows we have an even stronger result.

Corollary 4 *Let $\mathcal{F}$ be a transversely affine flow on a compact manifold. If for some finite cocycle defining $\mathcal{F}$ the representative of its holonomy pseudogroup is complete, then the closures of its orbits are diffeomorphic to tori.*

Proof It is a consequence of Theorem IV.5 and Theorem 3.□

Remarks 1. It is impossible to prove that the flow is distal and then apply the result of Ellis, cf. [EL]. There are 1–dimensional transversely affine flows with distal holonomy group which are induced by both distal and non–distal flows. Such an example has been constructed by E. Ghys.
2. It can be easily proved that the condition

> $\forall x, y \in N\ \exists U, V$ neighbourhoods of x and y, respectively, such that $J^1\mathcal{H}(U,V) = \{j^1_z h \colon z \in U,\ h(U) \subset V\}$ is bounded

is sufficient for strong completeness of the holonomy pseudogroup.

VI.6 Foliations with all leaves compact

We complete the chapter with a glance at $\nabla - G$ foliations with all leaves compact. The results presented in this section complement our considerations of Section IV.2. The main idea is to linearize the holonomy of any leaf.

Using the exponential mapping defined by transverse geodesics of the transversely projectable connection we can try to linearize the holonomy of any leaf L of the foliation, i.e. we would like to know whether there exists an open neighbourhood D of L in the normal bundle $N(L)$ of the leaf L and an open saturated neighbourhood U of L in M such that $exp|D: D \to M$ is a diffeomorphism of D onto U. If it is true, then for any point $x \in L$ and any element h of the holonomy group of L at x we have: $exp \circ h \circ (exp|D)^{-1}|D_x = d_x h|D_x$. Let us assume that we can linearize holonomy. If the leaves are compact, then any orbit of the linear holonomy group is finite. In this case it is not difficult to see that any element of the linear holonomy group is of finite order and the Schur Theorem, cf. [CG,SU], ensures that this group is finite and that the space of leaves of $\mathcal{F}$ is a Satake manifold.

The converse is, in fact, also true. The results of D.B.A.Epstein, cf. [EP], ensure that any leaf has a basis of open saturated neighbourhoods. As the leaves are compact, for any leaf L there exists an $\epsilon > 0$ such that $exp|D_\epsilon(L): D_\epsilon(L) \to M$ is a diffeomorphism onto the image, where $D_\epsilon(L) = \{v \in N(L): \|v\| < \epsilon\}$ for some Riemannian metric on $N(\mathcal{F})$. Then we can find an open saturated neighbourhood U of L contained in $exp(D_\epsilon(L))$. Thus $exp|exp^{-1}(U)$ defines the required linealization of the holonomy group of L. We have proved the following theorem.

Theorem 8 *Let $\mathcal{F}$ be a $\nabla - G$-foliation with all leaves compact. Then the holonomy of any leaf of $\mathcal{F}$ can be linearized iff the space of leaves of $\mathcal{F}$ is a Satake manifold.*

As a corollary we get the fact that the space of leaves of a Riemannian foliation with all leaves compact on a non–compact manifold is a Satake manifold. It is so since the holonomy of any compact leaf of a Riemannian foliation is linearisable.

These considerations can be summarized as follows:

Corollary 5 *Let $\mathcal{F}$ be a $\nabla - G$-foliation with all leaves compact on a compact manifold M. Then the following conditions are equivalent:*

1. *the holonomy of each leaf is linearisable;*
2. *$\mathcal{F}$ is Riemannian;*
3. *the holonomy of each leaf is finite;*
4. *the space of leaves is a Satake manifold;*

5. *the volume function is locally bounded;*

6. *the foliation $\mathcal{F}$ is minimal.*

Corollary 5 is a consequence of our considerations and of [EMS,EP4,RU], see also Appendix B by V.Sergiescu in [MO11]. Conditions 2, 3, 4, 5 and 6) are equivalent for any foliation with all leaves compact on a compact manifold.

Corollary 6 *If the foliation $\mathcal{F}$ is of codimension 2, then the holonomy of any leaf is linearisable.*

Proof It is a consequence of Theorem 2 of [EMS].□

Notes The chapter contains results published in four papers. Section 1 is based on [WO3] and [WO15]. Sections 2,3 and 4 present the rest of [WO15]. Section 5 contains the main result of [WO20]. A part of [WO12] dealing with $\nabla - G$-foliations is reproduced in Section 6.

Only a few authors have been interested in transversely projectable connections and $\nabla - G$-foliations. The basic facts have been established by P. Molino in [MO1], [MO2] and [MO11]. The book of F. Kamber and Ph. Tonder, cf. [KT1], contains very important results on characteristic classes of foliated bundles with foliated connections. We should also mention a series of results of R. A. Blumenthal about which we have talked in Section IV.4 as well as a paper by I. V. Belko on the group of global affine transformations of a transversely projectable connection, cf. [BK], and a paper by H. Suzuki, cf. [SK].

Chapter VII

Transversely affine foliations

Let M be a connected manifold of dimension n and $\mathcal{F}$ a codimension q foliation on M. The foliation $\mathcal{F}$ is called transversely affine (for short TAF) if there exists a cocycle $\mathcal{U}$ defining $\mathcal{F}$, $\mathcal{U} = \{U_i, f_i, g_{ij}\}$, modelled on a q-dimensional affine space $\mathbf{E}$ such that the transformations g_{ij} are restrictions of elements of the affine group $\mathit{Aff}(\mathbf{E})$; in other words $\mathcal{F}$ is an $(\mathit{Aff}(\mathbf{E}), \mathbf{E})$-structure.

The pair $(M, \mathcal{F})$ where $\mathcal{F}$ is a TAF is called a transversely affine foliated manifold, for short TAFM. TAFs are developable, i.e. there exist a covering $\hat{M}$ of M (called the holonomy covering of $(M, \mathcal{F})$) and a submersion $D : \hat{M} \longrightarrow \mathbf{E}$ called the developing mapping of $\mathcal{F}$ such that the leaves of the lifted foliation $\hat{\mathcal{F}}$ on $\hat{M}$ are the connected components of the fibres of the submersion D. In general, the developing mapping D is neither surjective nor locally trivial, and neither of these conditions implies the other.

Moreover there exist homomorphisms: $\alpha: \pi_1(M) \to \mathit{Aff}(\mathbf{E})$ called the affine holonomy representation and $im\alpha = \Gamma$ the affine holonomy group, $\lambda: \pi_1(M) \to GL(\mathbf{E})$ called the linear holonomy representation (just the linear part of α) and $im\lambda$ the linear holonomy group, and the mapping $u: \pi_1(M) \longrightarrow \mathbf{E}$ the translational part of α which is a cocycle with values in $\mathbf{E}$ (for the group $\pi_1(M)$ with the linear representation λ). The group of the deck transformations of $\hat{M}$ can be identified with the affine holonomy group.

Although a TAF is developable, it does not mean that $D(\hat{M})$ is a transverse manifold. The fibres of the developing mapping needn't be connected and therefore the holonomy pseudogroup of such a foliation is not the pseudogroup obtained as the localisation of the action of the affine holonomy group on $D(\hat{M})$.

Example 1 *(Hopf foliation)*

Let us consider the submersion $p_1: \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}, p_1: (x,y,z) \mapsto x$. The quotient of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ by the homothety h_λ with $|\lambda| < 1$ is $S^2 \times S^1$. The foliation given by the submersion p_1 projects to a TAF of $S^2 \times S^1$. Its affine holonomy group is generated by the homothety h_λ of $\mathbb{R}$. This foliation is not complete, although the developing mapping is surjective. Therefore $\mathbb{R}$ is not a (complete) transverse manifold of the Hopf foliation. In fact, we need two copies of $\mathbb{R}$. The pseudogroup obtained as the localization of the action of the affine holonomy group is a proper subpseudogroup of the holonomy pseudogroup on this transverse manifold.

VII.1 Completeness

In the case of TAFM we have the following two notions of completeness:

Definition 1 *A TAF $\mathcal{F}$ is:*

i) complete if the universal covering of the manifold M is of the form $\tilde{L} \times \mathbf{E}$ where $\tilde{L}$ is a leaf of the lifted foliation $\tilde{\mathcal{F}}$ and the foliation $\tilde{\mathcal{F}}$ is defined by the projection onto the second factor;

ii) transversely geodesically complete if for some choice of a supplementary subbundle Q transverse geodesics tangent to Q of the flat transversely projectable connection are global.

If the foliation $\mathcal{F}$ is complete $\mathbf{E}$ can be considered as a (complete) transverse manifold and its holonomy pseudogroup is the pseudogroup obtained by the localisation of the action of the affine holonomy group.

Before embarking on the discussion of completeness we prove a very important result concerning the affine holonomy group of a complete TAF.

Theorem 1 *Let $\mathcal{F}$ be a complete transversely affine foliation on a compact manifold M. Then the affine holonomy representation is irreducible.*

Proof The manifold M is the quotient of its covering space $\hat{M}$ by the group π of the deck transformations. Thus $M = \pi\backslash\hat{L}\times\mathbf{E}$ and the group π is isomorphic to the affine holonomy group. Let us assume that $\mathbf{F}$ is an affine subspace of $\mathbf{E}$ invariant under π. Then the inclusion of $\pi\backslash\hat{L} \times \mathbf{F}$ into $\pi\backslash\hat{L} \times \mathbf{E}$ induces isomorphisms of the homotopy groups and therefore it is a homotopy equivalence of compact manifolds $\pi\backslash\hat{L} \times \mathbf{F}$ and M. Thus they must be of the same dimension. Hence $\mathbf{F} = \mathbf{E}$ and the affine holonomy representation is irreducible.□

From the considerations of Chapter III result that transverse geodesics via the developing mapping (after being lifted to $\tilde{M}$) project onto straight lines in $\mathbf{E}$. The developing mapping of a transversely geodesically complete foliation is surjective and a locally trivial fibre bundle, cf. Theorem III.1. And as $\mathbf{E}$ is contractible the developing mapping is a trivial bundle, thus the foliation $\mathcal{F}$ is complete. Unfortunately, it is not known whether these two conditions are equivalent for TAFs on compact manifolds. Moreover, it is unknown whether transverse geodesic completeness depends on the choice of a subbundle Q. The geodesic completeness is not equivalent to completeness and it depends on the choice of a supplementary subbundle, cf. Example III.3. for an example on a non–compact manifold. For compact manifolds this problem has been recently solved by G. Hector. However, for a TAF of a compact manifold with all leaves compact the following is true.

Proposition 1 *Let $\mathcal{F}$ be a TAF with all leaves compact. If $\mathcal{F}$ is transversely geodesically complete for one supplementary subbundle or more generally, if $\mathcal{F}$ is complete, then the foliation $\mathcal{F}$ is transversely geodesically complete for any supplementary subbundle.*

Proof The developing mapping of $\mathcal{F}$ is a trivial bundle over $\mathbf{E}$. Its fibre being the holonomy covering of leaves of $\mathcal{F}$ must be compact as the foliation $\mathcal{F}$ has leaves without holonomy, cf. [EP,HC]. Therefore for any horizontal bundle of the developing mapping we can lift horizontally any straight line in $\mathbf{E}$. This ensures that our foliation is transversely geodesically complete for any supplementary subbundle.□

Unfortunately, we cannot prove that a foliation with all leaves compact is complete. It is sufficient to take an example of Goldman, cf. [GO2], of a non-complete compact affine manifold and take the product of this manifold with S^1. The foliation given by the projection onto the non-complete flat manifold is not complete. Although the completeness is not a transverse property some assumptions on the affine holonomy group can ensure that the foliation is complete.

The flat connection on $\mathbf{E}$ defines a transversely projectable connection ∇ in the normal bundle of the foliation. We can consider its linear holonomy : for any point x of M there is a homomorphism $h_x: \pi_1(M,x) \to GL(q)$. When M is connected, by changing the point x we get conjugate homomorphisms. For simplicity's sake we identify the normal bundle $N(M,\mathcal{F})$ with a subbundle Q of TM supplementary to $T\mathcal{F}$.

We say that the foliation $\mathcal{F}$ is distal if the action of the linear holonomy group $H_x = imh_x$ on $\mathbb{R}^q \equiv Q_x$ is distal (cf. [CG,FR2]). This condition does not depend on the choice of a point x. The condition equivalent to the distality ensures that the group H_x preserves a flag of subspaces $V_0 = \{0\} \subset V_1 ... \subset V_{k+1} = Q_x$. Since the connection ∇ is without torsion the corresponding subbundles $T\mathcal{F} \subset Q_1 \subset ...Q_k \subset TM$ of the tangent bundle are involutive, thus they define a flag of foliations $\mathcal{F}_0 = \mathcal{F} \subset \mathcal{F}_1 \subset ...\mathcal{F}_k$. The foliations $\mathcal{F}_i$, $i = 1,...k$, are totally geodesic with respect to the connection ∇. Moreover, we can find a Riemannian metric on M adapted to the flag $(\mathcal{F}_0, \mathcal{F}_1, ...\mathcal{F}_k)$ such that the foliations of leaves of $\mathcal{F}_{i+1}$ by leaves of $\mathcal{F}_i$ are Riemannian (cf. [FR2]). The foliations $\mathcal{F}_i$ correspond via the developing mapping to the foliations $\mathcal{F}_i^0$ of $\mathbb{R}^q$ which are totally geodesic with respect to the flat connection of $\mathbb{R}^q$. Thus these foliations are by parallel affine subspaces of $\mathbb{R}^q$. From this fact it results that the foliations of leaves of $\mathcal{F}_{i+1}$ by leaves of $\mathcal{F}_i$ are complete Riemannian foliations modelled on $\mathbb{R}^{q_i}$ where q_i is the codimension of $\mathcal{F}_i$ in $\mathcal{F}_{i+1}$. Let $D: \hat{M} \to \mathbb{R}^q$ be the developing map and $\hat{\mathcal{F}}_i$ be the lift of $\mathcal{F}_i$ to $\hat{M}$. Let L_{i+1} be a leaf of $\hat{\mathcal{F}}_{i+1}$ and L_i a leaf of $\hat{\mathcal{F}}_i$ contained in L_{i+1}. The foliation of L_{i+1} by leaves of $\hat{\mathcal{F}}_i$ is a complete Riemannian foliation modelled on $\mathbb{R}^{q_i}$ and it is given by a global submersion $D_i: L_i \to \mathbb{R}^{q_i}$ defined as follows

$$L_{i+1} \longrightarrow \mathbb{R}^{\tilde{q}_i} \longrightarrow \mathbb{R}^{q_i} \cong \mathbb{R}^{\tilde{q}_i}/\mathbb{R}^{\tilde{q}_{i-1}}$$

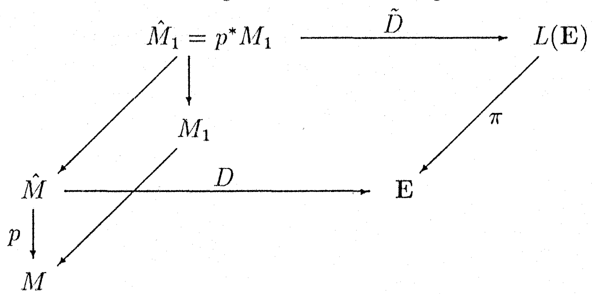
where $\tilde{q}_i = \sum_{j=0}^{i} q_j$.

A well-known Herman's theorem, cf. [HN] or Theorem III.1, ensures that D_i is a locally trivial fibre bundle, and as $\mathbb{R}^{q_i}$ is contractible it is a trivial fibre bundle, i.e. $L_{i+1} \cong L_i \times \mathbb{R}^{q_i}$, and therefore $\hat{M} \cong \hat{L} \times \mathbb{R}^q$ where $\hat{L}$ is a leaf of the lifted foliation $\hat{\mathcal{F}}$. We have just proved the following proposition:

Proposition 2 *A distal transversely affine foliation on a compact manifold is complete.*

To complete this section we describe the commuting sheaf of a TAF.

A TAF $\mathcal{F}$ is a $\nabla - G$–foliation with ∇ being the canonical flat connection of $\mathbf{E}$. Thus we have the following commutative diagram:



where M_1 is the total space of the bundle of transverse frames of $(M, \mathcal{F})$. M_1 admits a canonical foliation $\mathcal{F}_1$ of the same dimension as $\mathcal{F}$ and whose leaves are coverings of leaves of $\mathcal{F}$. $\mathcal{F}_1$ is a developable foliation modelled on the total space L of the bundle $L(\mathbf{E})$, the bundle of linear frames of $\mathbf{E}$; thus it is an $\mathit{Aff}(\mathbf{E})$–Lie foliation.

In Chapter VI we have defined the commuting sheaf of a $\nabla - G$–foliation. The lift $\hat{\mathcal{C}}$ of the sheaf $\mathcal{C}$ to $\hat{M}$ consists of germs of foliated vector fields of $(\hat{M}, \hat{\mathcal{F}})$ whose lifts to $\hat{M}_1$, forming a sheaf $\hat{\mathcal{C}}_1$, commute with all Γ–invariant global foliated vector fields. If $\mathcal{F}$ is complete $\hat{\mathcal{C}}_1$ projects to a sheaf $\mathcal{C}_A$ on $\mathit{Aff}(\mathbf{E})$ whose elements commute with all global (left) Γ–invariant vector fields on $\mathit{Aff}(\mathbf{E})$, thus with all $K = \bar{\Gamma}$–invariant vector fields. This means that the vector fields of the sheaf $\mathcal{C}_A$ must be tangent to the fibres of the K–fibre bundle $\mathit{Aff}(\mathbf{E}) \to K\backslash \mathit{Aff}(\mathbf{E}) = W$. Additionally, they must commute with the fundamental vertical vector fields k^*, $k \in Lie(K) = \underline{k}$, of this bundle as well as with any vector field of the form $\sum f_i k_i^*$ where $k_i \in \underline{k}$, $f_i \in C^\infty(W)$. Thus each stalk of $\mathcal{C}_A$ is isomorphic to the conjugated algebra $\underline{k}^-$. We call this Lie algebra the structure algebra of the transversely affine foliation $\mathcal{F}$. We have proved the following.

Proposition 3 *Let $\mathcal{F}$ be a complete transversely affine foliation with the affine holonomy group Γ. Then its commuting sheaf is a locally constant sheaf of Lie algebras whose stalk is isomorphic to the conjugated Lie algebra of $Lie(K)$, $K = \bar{\Gamma} \subset Aff(\mathbf{E})$.*

VII.2 Radiance obstruction and tensor fields

The radiance obstruction of an affine manifold proved to be a very useful tool in the study of these manifolds. In this section we define the radiance obstruction of a TAFM and look at the influence of its properties on foliated tensor fields.

Definition 2 *The radiance obstruction c_α of the affine holonomy representation α is called the radiance obstruction $c_\mathcal{F}$ of the foliation $\mathcal{F}$. If $c_\mathcal{F} = 0$, the foliation $\mathcal{F}$ is called radiant.*

It is the cohomology class $c_\alpha = c_\mathcal{F} = [u] \in H^1(G; \mathbf{E}_\lambda)$, where $G = \pi_1(M)$ and $H^*(G; \mathbf{E}_\lambda)$ denotes the cohomology of the group G with values in the G–module $\mathbf{E}_\lambda$ obtained from $\mathbf{E}$ via the representation λ (of G onto $\mathbf{E}$), (cf. [FGH,GH2] for general properties of the radiance obstruction of an affine representation).

Since the affine holonomy representation of a complete TAF on a compact manifold is irreducible, cf. Theorem 1, as a corollary we get the following.

Proposition 4 *A complete TAF on a compact manifold is not radiant.*

Now, we shall express the radiance obstruction in various cohomology theories (these considerations are based on [GH2]). The normal bundle $N(M, \mathcal{F})$ of $\mathcal{F}$ can be considered as a bundle in three different ways:

i) a flat affine bundle $N^{aff}(M; \mathcal{F})$ – the affine bundle with cocycle α;

ii) a flat vector bundle identified with the derived bundle of $N^{aff}(M; \mathcal{F})$ (notation: $N^{aff}(M; \mathcal{F})^L$) – the vector bundle with the cocycle λ;

iii) a vector bundle $TM/T\mathcal{F}$.

Then:

i) the identity mapping $N(M; \mathcal{F}) \longrightarrow N^{aff}(M; \mathcal{F})$ is an isomorphism of affine bundles;

ii) the identity mapping $N^{aff}(M;\mathcal{F})^L \longrightarrow N^{aff}(M;\mathcal{F})$ is an isomorphism of affine bundles, but not as flat bundles unless $\mathcal{F}$ is radiant.

Notation For convenience sake we shall denote the normal bundle $N(M;\mathcal{F})$ of the foliation $\mathcal{F}$, considered as a flat vector bundle, by $\mathcal{E}_{\mathcal{F}}$ or just $\mathcal{E}$; to stress that it is the 'inverse image' of the tangent vector bundle to the affine space $\mathbf{E}$ by the cocycle $\mathcal{U}$.

Let $\{U_i, f_i, g_{ij}\}$ be a cocycle modelled on $\mathbf{E}$ defining $\mathcal{F}$. Then for each i and each $x \in U_i$ we define an affine isomorphism:

$$\theta_{i,x}: N_x(M,\mathcal{F}) \longrightarrow \mathbf{E},$$

$$v \longmapsto f_i(x) + d_x f_i(v)$$

and the natural affine trivializations $\{(U_i, \theta_i)\}$;

$$\theta_i: N(M,\mathcal{F})|U_i \longrightarrow U_i \times \mathbf{E},$$

$$v \longmapsto (\pi(v), \theta_{i,\pi(v)}(v))$$

where $\pi: N(M,\mathcal{F}) \longrightarrow M$ is the bundle projection, form an atlas for a flat affine bundle structure on $N(M,\mathcal{F})$ which is completely determined by the transverse affine structure of $\mathcal{F}$. The resulting flat affine bundle is called $N^{aff}(M,\mathcal{F})$ (the flat affine normal bundle of $(M,\mathcal{F})$).

The bundle $N^{aff}(M,\mathcal{F})|U_i$ has the flat foliation given by $p_2\theta_i = const$, thus taking account of $N(M;\mathcal{F})|U_i = U_i \times \mathbf{E}$ (as vector bundles) and identifying $f_i: U_i \longrightarrow \mathbf{E}$ with the projection $p_i: U_i^0 \times V_i^0 \longrightarrow V_i^0 \in \mathbf{E}$, where $U_i = U_i^0 \times V_i^0$, we get that the foliation of $N^{aff}(M;\mathcal{F})|U_i$ is given by the level sets of the mappings $(x,y;v) \longmapsto y+v$, thus by the translations of the graph of the mapping $(x,y) \longmapsto -y$. Theorem 2.4 of [GH2] ensures that for any section s of $N^{aff}(M,\mathcal{F})$ the cohomology class of ∇s in $H^1(M,\mathcal{E})$ is the radiance obstruction of $(M,\mathcal{F})$. If we take as s the zero section of $N(M,\mathcal{F})$, our considerations together with those of Section 1.3 of [GH2] show that ∇s is equal to the natural projection $TM \longrightarrow N(M,\mathcal{F})$. Therefore we have proved the following: (compare Theorem 2.4 of [GH2]).

Theorem 2 *The radiance obstruction of a TAF $\mathcal{F}$ on a manifold M is the de Rham class $c_{\mathcal{F}} \in H^1(M,\mathcal{E};\mathcal{F})$ represented by the natural projection p_{N} of TM onto the normal bundle of $\mathcal{F}$.*

Let us look at the radiance obstruction of codimension 1 TAFs. Such a transversely orientable foliation $\mathcal{F}$ is given by the following data, cf. [BS]:

i) (ω, ω_1) a couple of 1–forms on M, where ω is a Pfaff form defining $\mathcal{F}$,

ii) $d\omega = \omega \wedge \omega_1$, and $d\omega_1 = 0$.

Assume that $c_{\mathcal{F}} = 0$. Then two cases are possible. The first one: the developing mapping is surjective. Then the radiant vector field defines a global foliated vector field with isolated singularities , i.e. it is tangent to the foliation along compact leaves corresponding to the fixed point of the affine holonomy representation. If the holonomy representation is not trivial there is only a finite number of compact leaves, cf. Example 1; compare [FGH].

The second case is that of a non-surjective developing mapping with the fixed point of the affine holonomy representation lying outside the developing image. Then the radiant foliated vector field is without singularity and the foliation can be defined by a closed 1–form. Thus there exist a different transverse affine structure making our foliation a complete TAF. This corresponds to a diffeomorphism $log: (0, \infty) \to (-\infty, +\infty)$ which does not preserve the affine structure of the manifolds. Therefore there exist TAFM admitting two different transverse affine structures one of which is non–complete and the other is complete. This means that we cannot expect any relation between $\wedge^q c_{\mathcal{F}} \neq 0$ and the existence of a transverse volume form. In codimension 1 a transverse volume form is precisely a closed 1–form defining the foliation. Of course, there are codimension 1 TAFs on compact manifold which are not defined by closed 1–forms.

Foliated tensor fields are in one–to–one correspondence with holonomy invariant tensor fields on the transverse manifold. Therefore if the developing mapping has connected fibres it means that they are in one–to–one correspondence with $im\alpha$–invariant tensor fields on $D(\hat{M})$. We say that a foliated tensor field is parallel (resp. polynomial) if locally, the corresponding local tensor field on $D(\hat{M})$ is parallel (resp. polynomial), i.e. its coefficients with respect to the natural basis are constant (resp. polynomial). Therefore they induce $im\alpha$–invariant tensor fields on $D(\hat{M})$. Parallel (resp. polynomial) base–like forms constitute a subcomplex of the complex of base–like forms on $(M, \mathcal{F})$. Its cohomology we denote by $H^*_{para}(M, \mathcal{F})$ (resp. $H^*_{poly}(M, \mathcal{F})$). Moreover, parallel base–like k–forms are in one–to–one correspondence with $im\lambda$–invariant linear mappings $\wedge^k \mathbf{E} \to \mathbb{R}$. This means that the space of parallel base–like k–forms on $(M, \mathcal{F})$ is equal to $H^0(M, \wedge^k \mathcal{E}^*_{\mathcal{F}})$. The complex of base–like forms corresponding to $im\alpha$–invariant forms on $D(\hat{M})$ we propose to call the reduced complex of base–like forms and its cohomology the reduced base–like cohomology (notation: $A^*_r(M, \mathcal{F})$ and $H^*_r(M, \mathcal{F})$, respectively). In the case of foliations with the developing mapping of connected fibres the reduced complex of base–like forms is equal to the complex of base–like forms, for example in the case of a complete TAF.

The natural pairing $\wedge^k \mathcal{E}^*_{\mathcal{F}} \times \wedge^k \mathcal{E}_{\mathcal{F}} \longrightarrow M \times \mathbb{R}$ induces in cohomology the mapping $\langle , \rangle: H^0(M, \wedge^k \mathcal{E}^*_{\mathcal{F}}) \times H^k(M, \wedge^k \mathcal{E}_{\mathcal{F}}; \mathcal{F}) \longrightarrow H^k(M; \mathcal{F})$ which allows to formulate the following proposition (compare 2.6 of [GH2]) ; $H^k(M, \wedge^k \mathcal{E}_{\mathcal{F}}; \mathcal{F})$ – the kth cohomology group of the complex of foliated forms on $(M, \mathcal{F})$ with values in the flat bundle $\wedge^k \mathcal{E}_{\mathcal{F}}$.

Proposition 5 *Let $\omega \in H^0(M, \wedge^k \mathcal{E}^*_{\mathcal{F}})$ be a parallel base-like k-form. Then $\langle \omega, \wedge^k c_{\mathcal{F}} \rangle = [\omega], [\omega]$ – the base-like cohomology class of the form ω.*

The equality results directly from the definition of the pairing $\langle , \rangle$ and the fact that $\wedge^k c_{\mathcal{F}}$ is represented by the projection $\wedge^k p_N : \wedge^k TM \longrightarrow \wedge^k \mathcal{E}_{\mathcal{F}}$.

Locally the projection p_N can be written as $\sum v_i^* \otimes v_i$ where v_i are vector fields corresponding to constant vector fields on $\mathbf{E}$ and v_i^* are the duals of v_i. Then, locally, $\wedge^q p_N$ is equal to $v_1^* \wedge ... v_q^* \otimes v_1 \wedge ... v_q$. The form $v_1^* \wedge ... v_q^*$ is the inverse image of the parallel volume form on $\mathbf{E}$. If the holonomy group in contained in $SL(q) \times \mathbf{E}$, the qth exterior power of the radiance obstruction can be expressed as the cohomology class of $\omega \otimes w$ where ω is the parallel transverse volume form and w is the corresponding flat section of $\wedge^q \mathcal{E}$. Therefore the base-like cohomology class of ω is zero iff $\wedge^q c_{\mathcal{F}} = 0$.

A simple translation allows us to prove the following properties, cf. [GH2], sections 2.6–2.8.

Theorem 3 *Let $(M, \mathcal{F})$ be a TAFM. Then*

1. *If there exists a parallel base-like k-form defining a non-zero base-like cohomology class, then*

   a) *the k-th exterior power of the radiance obstruction $\wedge^k c_{\mathcal{F}}$ is non-zero.*

   b) *the affine holonomy group cannot preserve an affine subspace of $\mathbf{E}$ of dimension smaller than k.*

2. *If $\mathcal{F}$ admits a parallel base-like volume form defining a non-zero base-like cohomology class, then its affine holonomy representation is irreducible.*

3. *Let $\mathcal{F}$ be transversely orientable. If the linear holonomy representation of $\mathcal{F}$ factorizes through a group G which admits no nontrivial homomorphisms to the group $\mathbb{R}$ of real numbers, then the foliation $\mathcal{F}$ has a parallel transverse volume form. In particular, if the first Betti number of M is zero, then $\mathcal{F}$ has a parallel transverse volume form.*

Now, let us turn our attention to polynomial tensor fields. Using methods of [GH1,GH2] we can prove the following:

Proposition 6 *Let $(M, \mathcal{F})$ be a TAFM.*

1. *If $D(\hat{M}) \subseteq h(C)$ where $C \subset \mathbf{E}$ is a non-empty open cone and $h: \mathbf{E} \longrightarrow \mathbf{E}$ is a polynomial mapping whose Jacobian is not identically zero, then every bounded polynomial foliated function on M is constant.*

2. *If M is compact and $\mathcal{F}$ radiant, then every polynomial closed base-like 1-form is zero.*

3. *If $\mathcal{F}$ is radiant, then every closed polynomial base-like k-form ($k > 0$) is exact (i.e. $H^k_{poly}(M, \mathcal{F}) = 0$ for $k > 0$).*

4. *If the affine holonomy group is nilpotent and the foliation admits a parallel base-like k-form defining a non-zero base-like cohomology class, then for any $j \leq k$ there exists a non-zero parallel base-like j-form.*

Remark The results of [FR1] are also valid for TAFMs as the author himself indicated.

VII.3 Algebraic hull of affine holonomy group

In this section we shall look into the influence of the properties of the algebraic hull of the affine holonomy group Γ of a TAFM $(M, \mathcal{F})$. Let us denote by $A(\Gamma)$ the algebraic hull of Γ and its unipotent radical by $UA(\Gamma)$. The basic reference for the properties of the algebraic hulls of affine groups, their radicals and radiance obstructions is [GH3].

Combining Theorem 3 with 1.9 of [GH3] we get:

Proposition 7 *Let $(M, \mathcal{F})$ be a TAFM which admits a parallel base-like k-form defining a non-zero base-like cohomology class. Then every orbit of $UA(\Gamma)$ is of dimension $\geq k$.*

Corollary 1 *Let $(M, \mathcal{F})$ have a parallel transverse volume form defining a non-zero base-like cohomology class. Then the group $A(\Gamma)$ acts transitively.*

The proof of Theorem 2.6 of [GH3] we can be shortened and in the foliated case it yields the following:

Theorem 4 *Let $(M, \mathcal{F})$ be a compact complete TAFM. Then the group $A(\Gamma)$ acts transitively on* $\mathbf{E}$.

Proof Let us represent $A(\Gamma)$ as a semidirect product of its unipotent radical $UA(\Gamma)$ and a maximal reductive subgroup $\mathcal{R}$, cf. [GH3,HO]. It is sufficient to show that the group $UA(\Gamma)$ acts transitively. The group $\mathcal{R}$, being reductive, fixes a point b of $\mathbf{E}$. Then the $UA(\Gamma)$-orbit $UA(\Gamma)b$ is invariant under the action of $A(\Gamma)$.

Let $G = UA(\Gamma)$ and H be its isotropy subgroup at b. As G is a connected unipotent algebraic group, its orbits are closed, cf. [HO]. Moreover, the exponential mapping induces a diffeomorphism $Lie(G)/Lie(H) \to G/H$. Therefore G/H is a closed contractible submanifold $\mathbf{E}_0$ of $\mathbf{E}$. Let $\hat{M}_0$ be the inverse image of $\mathbf{E}_0$ by the developing mapping, i.e. $\hat{M}_0 = \hat{L} \times \mathbf{E}_0$. The group Γ acts freely and properly discontinuously on $\hat{M}_0$ and the inclusion $\hat{M}_0 \to \hat{M}$ is Γ-equivariant.

Thus $M_0 = \Gamma\backslash\hat{M}$ is a compact foliated submanifold of M of the same homotopy type. This means that M_0 is homotopy equivalent to M, and thus is equal to M. Hence $\mathbf{E}_0 = \mathbf{E}$ and the unipotent radical $UA(\Gamma)$ acts transitively on $\mathbf{E}$.

Remark For other conditions equivalent to the transitiveness of the action of $A(\Gamma)$, cf. [GH3], 1.11.

Using the considerations of Theorem 2.10 of [GH3] we can prove:

Theorem 5 *Let $(M, \mathcal{F})$ be a connected TAFM. If either $\mathcal{F}$ is complete and M compact or there exists a parallel transverse volume form defining a non–zero base–like cohomology class, then*

i) *M admits no nonconstant basic rational function;*

ii) *every rational foliated tensor field of type (r, s) on $(M, \mathcal{F})$ is polynomial of degree $\leq (s + r)A_q$ where $A_q = (2q - 2)!/(2^{q-1}(q - 1)!)$.*

Furthermore, if $A(\Gamma)$ acts transitively (cf. Theorem 4) then any $A(\Gamma)$–invariant tensor field of type (r, s) (thus a polynomial tensor field, cf. [GH3] Lemma 2.11) is determined by its value at a given point. Therefore the space of $A(\Gamma)$–invariant tensor fields of type (r, s) is of finite dimension and its dimension is smaller than the dimension of the space $\otimes^r_s\mathbf{E}$. This implies that the space of polynomial base–like forms is of finite dimension and thus the spaces $H^*_{poly}(M, \mathcal{F})$ are of finite dimension, and in particular dim $H^q_{poly}(M, \mathcal{F}) \leq 1$. As a lot of properties depend on the fact that some parallel form defines a non–zero base–like cohomology class, it is important to know the properties of the mappings

$$H^*_{poly}(M, \mathcal{F}) \longrightarrow H^*(M, \mathcal{F}) \quad \text{and} \quad H^*_{para}(M, \mathcal{F}) \longrightarrow H^*(M, \mathcal{F}).$$

Let K be the closure of the affine holonomy group and H the isotropy subgroup at 0 of $A(\Gamma)$. Then the reduced complex of base–like forms is isomorphic to the complex of K–invariant forms on the homogeneous space $A(\Gamma)/H$ and the complex of polynomial base–like forms is isomorphic to the complex of $A(\Gamma)$–invariant forms on the homogeneous space $A(\Gamma)/H$.

Definition 3 *We say that a TAF $\mathcal{F}$ is amenable if the Zariski closure $A(\Gamma)$ of the affine holonomy group Γ acts transitively on $\mathbf{E}$ and the homogeneous space $A(\Gamma)/K$ is compact and amenable, i.e. there exists a linear continuous $A(\Gamma)$–invariant form m on the Banach space of bounded complex uniformly continuous functions on $A(\Gamma)/K$ with the uniform convergence topology such that:*

1. $m(1) = 1$;
2. $m(\bar{f}) = \overline{m(f)}$;

3. $m(f) \geq 0$ *for any real positive function* f *on* $A(\Gamma)/K$.

Theorem 0.1 of [EN2] ensures that the following is true:

Theorem 6 *Let* $\mathcal{F}$ *be an amenable TAF, then the mapping*

$$H^*_{poly}(M, \mathcal{F}) \longrightarrow H^*_r(M, \mathcal{F})$$

induced by the inclusion of the complexes is injective.

Example 2

1. Any complete TAF on a manifold with nilpotent fundamental group is amenable, cf. Theorem 2.1 of [RA].
2. A complete TAF with solvable holonomy group is amenable iff $A(\Gamma)/K$ is compact, cf. Theorem 3.1 of [RA].
3. (cf. [GS3]) Let A be a matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $SL(2, Z)$. It defines a diffeomorphism A of the torus $\mathbb{T}^2$. Suspending this diffeomorphism A over S^1 we obtain a 1–dimensional TAF $\mathcal{F}$ of the compact manifold $\mathbb{T}^3_A$ whose holonomy group is generated by A and the translations by vectors $(1,0)$ and $(0,1)$. E. Ghys demonstrated that 1–base–like forms correspond to forms on $\mathbb{R}^2$ of the form: $a(x)dx$ and 2–base–like forms to forms on $\mathbb{R}^2$ of the form $b(x)dx \wedge dy$. Therefore, the space $H^2(\mathbb{T}^3_A; \mathcal{F}_A)$ is infinite dimensional. Moreover, one can easily check that polynomial forms are the following: adx, $bdx \wedge dy$ where $a, b \in \mathbb{R}$, compare [EGS].
4. Example 2 of [GHL] gives us an example of a complete codimension 2 TAF $\mathcal{F}$ of a compact manifold M for which the mapping

   $$H^2_{para}(M, \mathcal{F}) \to H^2(M)$$

   is not injective but the mapping

   $$H^2_{poly}(M, \mathcal{F}) \to H^2(M, \mathcal{F})$$

   is injective according to our theorem. The foliation $\mathcal{F}$ is given by the vector field $\partial/\partial x$. In particular, the parallel volume form of $\mathcal{F}$ defines a non–zero base–like cohomology class but is cohomologous to 0 in the de Rham cohomology.
5. Example 5 of [GHL] provides us with an example of a codimension 2 TAF $\mathcal{F}$ of a compact manifold for which the mapping

   $$H^2_{para}(M, \mathcal{F}) \longrightarrow H^2(M, \mathcal{F})$$

   is not injective.

Let us look at the implications of Theorem 6. Assume that the foliation $\mathcal{F}$ is amenable and that it admits a parallel transverse volume form. This transverse volume form defines a cohomology class in $H^q_{poly}(M, \mathcal{F})$. It is non–zero iff $H^q_{poly}(M, \mathcal{F}) \neq 0$. Therefore we have the following.

Corollary 2 *Let $\mathcal{F}$ be an amenable TAF admitting a parallel volume form. Then its reduced base–like cohomology class is non–zero iff $H^q_{poly}(M, \mathcal{F}) \neq 0$.*

VII.4 Nilpotent affine holonomy group

In this section we pay particular attention to TAFs with nilpotent holonomy. The results of [FGH] proved for affine manifolds can be generalized without many difficulties to the case of TAFMs. In fact the following theorem holds, compare Theorem 4.1 of [GH3] as well.

Theorem 7 *Let $(M, \mathcal{F})$ be a compact TAFM with nilpotent affine holonomy group. Then the following conditions are equivalent:*

a) *$(M, \mathcal{F})$ is complete;*

b) *the developing mapping is surjective;*

c) *the linear holonomy is unipotent;*

d) *the affine holonomy is irreducible;*

e) *the affine holonomy is indecomposable;*

f) *M has a parallel transverse volume form;*

g) *$A(\Gamma)$ acts transitively;*

If $H^q_{poly}(M, \mathcal{F}) \neq 0$, then the above conditions are equivalent to:

h) $\wedge^q c_{\mathcal{F}} \neq 0$.

Proof The equivalence of a)–f) can be proved as in [FGH]. It is not difficult to see that most theorems have their corresponding versions for transversely affine foliations. In general, the terms 'vector field' and 'form' on the manifold M are replaced by 'foliated vector field' and 'base–like form' on the foliated manifold $(M, \mathcal{F})$. We leave to the reader the statements and proofs of the theorems for transversely affine foliations corresponding to the following ones of [FGH]: 3.2-3,4.1,4.3-4,6.1-4,6.6, 6.8-9.

Theorem 4 states that from a) follows g). The fact that $A(\Gamma)$ acts transitively implies that the affine holonomy is irreducible, and thus a), cf. Theorem 1.

The condition h) always imply g) (cf. Theorem 1.9 of [GH3]). Let us prove the converse. Any volume form on **E** is of the form $fdy_1 \wedge ...dy_q$ and for a polynomial transverse volume form the function f is a polynomial. Transverse volume forms correspond to Γ-invariant volume forms. As the linear holonomy is unipotent, the form $fdy_1 \wedge ...dy_q$ is Γ-invariant, iff f is. Since f is a polynomial function, it must be constant, and this parallel volume form defines a non-zero base-like cohomology class. It results from Proposition 5 that $\wedge^q c_{\mathcal{F}} \neq 0$.

Example 3 The Hopf foliation of $S^2 \times S^1$ is not complete, but its developing mapping is surjective, compare Theorem 6.11 of [FGH].

In the affine case the Nomizu theorem can be applied to obtain the representation of cohomology classes of the affine manifold, cf. [FGH]. In the case of foliations it is more complex.

As Example 2 has demonstrated we cannot expect Theorems 8.1 and 8.4 of [GH3] to be true for TAFs. However, when the closure of the affine holonomy group Γ in *Aff*(**E**) is connected the following is true: (for nilpotent groups the above condition means precisely that the closure of the group Γ is equal to its Zariski-closure (cf. [RA])).

Theorem 8 *Let $(M, \mathcal{F})$ be a compact complete TAFM with nilpotent affine holonomy group whose closure in Aff*(**E**) *is connected. Then the inclusion of the complex of polynomial base-like forms in the reduced complex of base-like forms induces an isomorphism in cohomology, and the reduced base-like cohomology is finite dimensional.*

Proof According to Theorem 7 the closure of the affine holonomy group acts transitively on **E**. Thus any holonomy invariant tensor or form is polynomial, (cf. Lemma 2.11 of [GH3]), and hence these two complexes are, in fact, equal.

Remark Theorem 8 is also true if the closure of the affine holonomy group has a finite number of connected components. It is easy to see that the affine holonomy group of the foliation of Example 2.3 is discrete and denumerable.

Example 4 The space $H^1_{para}(M, \mathcal{F})$ is the space of parallel 1-base-like forms on $(M, \mathcal{F})$. For a complete TAF on a compact manifold with abelian fundamental group it results from Theorem 7 that its holonomy group is unipotent. So is its algebraic closure $A(\Gamma)$. The group $A(\Gamma)$ acts transitively on **E**, cf. Theorem 4, so according to Theorem 4.b of [FR2] it acts simply transitively. Moreover translations in $A(\Gamma)$ correspond to $A(\Gamma)$-invariant parallel vector fields on **E**, i.e. to parallel foliated vector fields and hence to parallel 1-base-like forms, [FGH]. Using the example of Fried, cf. [FR2], and the construction presented in Chapter V we can find a complete TAF on a compact manifold admitting a parallel foliated vector field but for which the algebraic closure of the affine holonomy group does not contain any translation.

The foliated version of Theorem 4.2 of [GH3] is the following:

Theorem 9 *Let $(M, \mathcal{F})$ be a compact TAFM with nilpotent affine holonomy group. Then the largest k such that $\wedge^k c_{\mathcal{F}}$ is non-zero is smaller than the minimal dimension of $A(\Gamma)$-orbits on* $\mathbf{E}$. *Furthermore, if M is incomplete, then the unique orbit of minimal dimension lies outside the developing image.*

Proposition 4.3 and Theorem 4.6 of [GH3] are also true for TAFs.

Proposition 8 *Let $(M, \mathcal{F})$ be a compact TAFM with nilpotent affine holonomy group Γ. Then:*

i) there exist a unique Γ- invariant affine subspace $\mathbf{E_U} \subset \mathbf{E}$ upon which Γ acts unipotently and a unique Γ-invariant affine projection $\pi : \mathbf{E} \longrightarrow \mathbf{E_U}$;

ii) $\pi D : \hat{M} \longrightarrow \mathbf{E_U}$ is a surjective fibration and if $\mathbf{E} \neq \mathbf{E_U}$ $\mathbf{E_U}$ is disjoint from the developing image;

iii) there exists $\gamma \in \Gamma$ whose linear part $L(\gamma)$ restricted to a fibre of π is an expansion.

iv) the group $A(\Gamma)$ acts transitively on $\mathbf{E_U}$;

v) every Γ-invariant Zariski-closed non-empty subspace of $\mathbf{E}$ contains $\mathbf{E_U}$.

Remark The space $\mathbf{E_U}$ is called the Fitting subspace of Γ or $(M, \mathcal{F})$.

Unfortunately, the base-like cohomology class of the volume form of the Fitting subspace does not need to be non-zero which implies that the inequality in Theorem 9 can be sharp, cf. Example 2.4 and compare [GHL].

Now let us consider a compact TAFM $(M, \mathcal{F})$ with nilpotent affine holonomy group of rank smaller that the codimension of the foliation.

Theorem 10 *Let $(M, \mathcal{F})$ be a compact TAFM with nilpotent affine holonomy group. If the foliation $\mathcal{F}$ is complete, then the rank of the group $\pi_1(M)$ must be greater or equal to the codimension of $\mathcal{F}$. Moreover, if the equality holds, the foliation $\mathcal{F}$ is given by a global submersion onto a nilmanifold.*

Proof Assume that it is not true. Let U be the simply connected nilpotent Lie group containing Γ as a uniform subgroup. Then $dim U = rank \Gamma \leq q$. According to Theorem 7 the group Γ is unipotent, thus the group Γ is a subgroup of a maximal unipotent subgroup A_U of affine transformations. Therefore there exists a homomorphism $h: U \longrightarrow A_U$ which is identity on Γ. The image $h(U)$ of the group U is a Zariski-closed (connected) subgroup of A_U. Hence it contains the Zariski-closure of Γ in A_U. Thus $dim A(\Gamma) \leq dim h(U) \leq dim U = rank \Gamma \leq$

$codim\mathcal{F}$. Then, as the foliation $\mathcal{F}$ is complete, the group $A(\Gamma)$ acts transitively on $\mathbf{E}$; hence $codim\mathcal{F} \leq dimA(\Gamma)$. Thus $codim\mathcal{F} = dimA(\Gamma) = dimU = rank\Gamma$. The simply connected group U must therefore act transitively on $\mathbf{E}$, and hence simply transitively. From this fact it results that the group Γ acts freely and uniformly on $\mathbf{E}$. Therefore the foliation $\mathcal{F}$ is defined by a global submersion with connected fibres onto the nilmanifold $\Gamma\backslash\mathbf{E}$.

Definition 4 *Let M be a manifold whose universal covering space is an algebraic variety. Then a TAF $\mathcal{F}$ on M is said to be algebraic if its developing mapping is a morphism of algebraic varieties.*

Theorem 11 *Let $(M, \mathcal{F})$ be a compact complete TAFM with nilpotent fundamental group. If M is a complete affine manifold and $\mathcal{F}$ an algebraic foliation, then the universal covering $\tilde{M}$ of M is a simply-connected nilpotent Lie group N and the lifted foliation is given by a transitive action of N on the affine space $\mathbf{E}$.*

Proof It is well-known, cf. [FGH], that M is diffeomorphic to $\Gamma_0\backslash N$ where N is a simply connected nilpotent Lie group and Γ_0 is a uniform subgroup of N. Let $D\colon N \longrightarrow \mathbf{E}$ be the development of $(M, \mathcal{F})$ such that $D(e) = 0$. The holonomy group Γ is unipotent, its Zariski-closure $A(\Gamma)$ acts transitively on $\mathbf{E}$ and it is a subgroup of a maximal unipotent subgroup A_U. Then we can extend the homomorphism $\alpha\colon \pi_1(M) = \Gamma_0 \longrightarrow \Gamma \subset Aff(\mathbf{E})$ to a homomorphism $h\colon N \longrightarrow A_U$ which is also a morphism of algebraic varieties. Thus the action of N via h on $\mathbf{E}$ is also algebraic. Let us consider the orbit mapping of h_0 of N at 0 and the developing mapping D. They are both morphisms of algebraic varieties and are equal on Γ_0. As the subgroup Γ_0 is Zariski-dense in N, these two mappings are equal and the foliation $\mathcal{F}$ is just the quotient of the foliation given by the submersion h_0. Its space of leaves is $\Gamma\backslash N/H$, where H is the isotropy group of N at 0.

The result of D. Fried, cf. Theorem 4.b of [FR2], has a very interesting consequence for algebraic complete TAFs on compact manifolds with abelian fundamental group.

Corollary 3 *Let $\mathcal{F}$ be an algebraic complete TAF on a compact affine manifold with abelian fundamental group. Then the foliation $\mathcal{F}$ is a Lie foliation modelled on an abelian Lie group.*

Proof The considerations of Example 4 ensure that the group $A(\Gamma)$ acts simply-transitively on $\mathbf{E}$. Therefore the mapping h_0 of Theorem 11 considered as a mapping into $A(\Gamma)$ must be a surjective submersion making $\mathcal{F}$ a Lie foliation.

Example 5 As a curiosity we can add that on a non–complete compact AM we can have a complete algebraic TAF. In fact the Goldman example of [GO2] provides us with such a foliated manifold. The projection $O_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ defines a complete algebraic TAF of a non–complete compact AM $O_+ \times \mathbb{R}/\Gamma$.

Polynomial Riemannian metrics have been used in [GH0] to characterize compact affine manifolds with virtually nilpotent fundamental group and to obtain in this way a generalization of the Bieberbach theorem. The foliated case is much different. Let us consider the following three conditions:

1. $(M, \mathcal{F})$ admits a complete polynomial bundle–like metric;
2. there exist a compact affine manifold M_1, a complete transversely affine foliation $\mathcal{F}_1$ of M_1 with all leaves compact, a finite covering M' of M and a mapping $h : M' \longrightarrow M_1$ with connected fibres such that $\mathcal{F}' = h^*\mathcal{F}_1$;
3. the fundamental group of M contains a nilpotent subgroup of finite index.

We have the following relations between them:

Proposition 9 *Let $(M, \mathcal{F})$ be a TAFM. Then 1) and 2) imply 3).*

Proof Since there exists a polynomial bundle–like metric, the developing mapping is a Riemannian submersion and therefore any curve of finite length can be lifted horizontally to $\hat{M}$ (the holonomy covering of $(M, \mathcal{F})$). Proceding step by step we can lift any straight line in $\mathbf{E}$. Thus the foliation $\mathcal{F}$ is geodesically complete.

As the fundamental group of a finite covering can be considered a subgroup of finite index of $\pi_1(M)$, we can assume that $M' = M$. The affine holonomy group of the foliation $\mathcal{F}_1$ is equal to the affine holonomy group of $\mathcal{F}$. Since the leaves of $\mathcal{F}_1$ are compact, the affine holonomy group Γ of $\mathcal{F}_1$ must be a discrete subgroup of *Aff*($\mathbf{E}$). Even more: the space of leaves of $\mathcal{F}_1$, which is equal to the space of orbits of Γ on $\mathbf{E}$, is a Satake manifold according to Proposition 1. Then following the considerations of [GH0] p.4 we obtain 3).

Example 6 It is worth mentioning that, contrary to the affine case, neither 1) ensures 3) nor 3) implies 1). In fact, let us take a compact complete AM M with virtually nilpotent fundamental group and the product $L \times M$ with a compact manifold L whose fundamental group is neither nilpotent nor finite. Then the foliation given by the projection onto the second factor is transversely affine and admits a polynomial bundle–like metric. To construct the second example let us consider a TAF modelled on a nilpotent simply connected Lie group N with a dense finitely generated subgroup Γ, cf. Section V.1. We can easily find a transitive action of the group N on $\mathbb{R}^p$ with $p < q = dim N$ such that the isotropy groups of the induced action of Γ are not relatively compact. Therefore there does not exist any Γ–invariant Riemannian metric on $\mathbb{R}^p$, and the codimension p foliation on $\Gamma\backslash U$ defined by this action cannot admit a polynomial bundle–like metric.

Proposition 10 *Let $(M, \mathcal{F})$ be a TAFM. If the foliation $\mathcal{F}$ is algebraic and the conditions 1) and 3) are satisfied, then $\mathcal{F}$ is a Lie foliation modelled on a unipotent Lie group.*

Proof By passing to a finite covering we can assume that the manifold has the nilpotent fundamental group. Therefore as $\mathcal{F}$ is complete its affine holonomy group must be a unipotent group of isometries of a polynomial metric on $\mathbf{E}$. Moreover, its Zariski closure acts transitively on $\mathbf{E}$ and consists also of isometries of this metric. Thus any isotropy subgroup of $A(\Gamma)$ is compact. These isotropy groups consist of unipotent isometries, hence they must be trivial and $A(\Gamma)$ acts simply transitively on $\mathbf{E}$. Therefore there exists a mapping $\hat{D}: \hat{M} \longrightarrow A(\Gamma)$ which composed with the orbit mapping at 0 is the developing mapping. The existence of the mapping $\hat{D}$ ensures that the foliation $\mathcal{F}$ is a Lie foliation.

Proposition 11 *Let $(M, \mathcal{F})$ be a complete TAFM with all leaves compact. Then the condition 3) ensures 1).*

Proof By passing to a finite covering we can assume that M has nilpotent fundamental group. As the foliation $\mathcal{F}$ is complete and the leaves of $\mathcal{F}$ are compact, the holonomy of each leaf is finite and the space of leaves is Hausdorff. Therefore, the group Γ acts properly discontinuously on $\mathbf{E}$. Γ being unipotent, it is also torsion–free. Thus it acts freely on $\mathbf{E}$, cf. [FG], and $\Gamma\backslash\mathbf{E}$ is a compact complete flat manifold with nilpotent fundamental group. Then according to Theorem 7.1 of [FGH] it is a nilmanifold, i.e. there exists a simply connected nilpotent Lie group N containing Γ as a uniform subgroup and acting simply transitively on $\mathbf{E}$ in such a way that $\Gamma\backslash\mathbf{E}$ is affinely diffeomorphic to $\Gamma\backslash N$. Hence any left invariant Riemannian metric on N defines a polynomial Γ–invariant Riemannian metric on $\mathbf{E}$, and hence induces a polynomial bundle–like Riemannian metric on $(M, \mathcal{F})$.

VII.5 Growth of leaves

The very particular structure of TAFs allows us to estimate the growth of these foliations. Using the methods of Y. Carrière developed for Lie and Riemannian foliations combined with some peculiar properties of TAFs we obtain estimates similar to those for Riemannian foliations. However, in this case TAFs 'behave' more disorderly than Riemannian ones as Examples 7 and 8 illustrate well.

Since the foliation $\mathcal{F}$ is developable, we have the following estimate due to R. A. Blumenthal, cf. [BL].

Proposition 12 *(Blumenthal) Let $\mathcal{F}$ be a transversely affine foliation of a compact manifold M. Then $gr(\mathcal{F}) \leq gr(\pi_1(M))$, or to be precise, $gr(\mathcal{F}) \leq gr(\Gamma)$.*

Combining this result with the estimate of H. Bass, cf. [BA], we get:

Corollary 4 *Let $\mathcal{F}$ be a transversely affine foliation of a compact manifold M. If the fundamental group $\pi_1(M)$ is nilpotent, then $gr(\mathcal{F}) \leq d(\Gamma)$.*

This corollary together with Plante's theorem, cf. [PL1], ensure that any transversely affine foliation with nilpotent, or virtually nilpotent, affine holonomy group admits a holonomy invariant measure, cf. [GHL].

The following result proved by Y. Carrière for Lie foliations is also valid for a larger class of foliations, among them for transversely affine foliations, cf. [CA2].

Proposition 13 *(Carrière) Let $\mathcal{F}$ be a complete transversely affine foliation of a compact manifold M. Then the growth type of a leaf L of $\mathcal{F}$ can be read as the local growth of the corresponding orbit of the affine holonomy group on* $\mathbf{E}$.

To get more information about the growth type of $\mathcal{F}$ we compare the growth type of leaves of $\mathcal{F}$ and $\mathcal{F}_1$. However, as M_1 is not compact, the growth type of a leaf is not well–defined; it depends on the choice of an adapted atlas, cf. [PL1]. Fortunately, the foliated manifold $(M_1, \mathcal{F}_1)$ has a class of adapted atlases which are particularly well–suited to the comparative study of the growth types of leaves of $\mathcal{F}$ and $\mathcal{F}_1$. We have in mind the atlases which are derived from adapted atlases of $(M, \mathcal{F})$; we call them preferred adapted atlases. If M is compact, it can be easily shown that the growth type of a leaf of $\mathcal{F}_1$ does not depend on the choice of a preferred adapted atlas. Therefore, for any leaf $\hat{L}$ of $\mathcal{F}_1$, we denote by $gr(\hat{L})$ the growth type of $\hat{L}$ for any preferred adapted atlas of $(M_1, \mathcal{F}_1)$. Then:

Lemma 1 *Let $\hat{L}$ be a leaf of $\mathcal{F}_1$ covering a leaf L of $\mathcal{F}$. Then $gr(L) \leq gr(\hat{L})$.*

Proof It is a simple consequence of the definition.□

Considerations of the proof of Proposition 13 applied to the foliation $\mathcal{F}_1$ yields the following:

Lemma 2 *The growth type of any leaf of $\mathcal{F}_1$ can be read as the growth type of the corresponding orbit of the action of the affine holonomy group restricted to the set $\pi^{-1}(U)$ where U is a relatively compact subset of* $\mathbf{E}$.

Proof It results from the considerations of Y. Carrière, cf. [CA2], and the fact that we calculate the growth type relative to a preferred adapted atlas of $(M_1, \mathcal{F}_1)$. □

Having described the growth type of leaves of both foliations, we can show that, in fact, $gr(\mathcal{F}) = gr(\mathcal{F}_1)$.

Proposition 14 *Let $\mathcal{F}$ be a complete transversely affine foliation on a compact manifold M, then $gr(\mathcal{F}) = gr(\mathcal{F}_1)$.*

Proof As the growth type of a foliation is the supremum of the growth type of its leaves it would be sufficient to show that there exists a leaf L of $\mathcal{F}$ such that $gr(L) = gr(\mathcal{F})$ and that for any leaf $\hat{L}$ of $\mathcal{F}_1$ covering L $gr(L) = gr(\hat{L}) = gr(\mathcal{F}_1)$. Owing to Proposition 13 and Lemma 2 it would be sufficient to find a point v of $\mathbf{E}$ such that the correspondence $\Gamma \ni a \mapsto a(v)$ is one-to-one, or equivalently, to find a leaf without holonomy. Therefore the following lemma completes the proof.

Lemma 3 *Any complete transversely affine foliation $\mathcal{F}$ of a compact manifold M has a leaf without holonomy.*

Proof Leaves of $\mathcal{F}$ correspond to orbits of the affine holonomy group Γ on $\mathbf{E}$. Let us assume that for any point v of $\mathbf{E}$ there exists $a \in \Gamma$, $a \neq id$, such that $a(v) = v$, i.e. any leaf of $\mathcal{F}$ has holonomy. For any element $a \neq id$ of Γ consider the set $\mathbf{E}_a = \{v \in \mathbf{E}: a(v) = v\}$, thus $\bigcup_{a \in \Gamma \setminus \{id\}} \mathbf{E}_a = \mathbf{E}$. As M is compact the set Γ is denumerable. Therefore the Baire property of $\mathbf{E}$ ensures that for some $a \in \Gamma \setminus \{id\}$ $int\mathbf{E}_a \neq \emptyset$. But as the group $Aff(\mathbf{E})$ acts quasi-analytically, $a = id$; contradiction. □

Now we would like to find a lower bound for the growth of leaves of $\mathcal{F}$ depending on the structure of the affine holonomy group. Proposition 14 makes our task easier, as we can work with the Lie foliation $\mathcal{F}_1$. The closure of a leaf $\hat{L}$ correspond via D to the orbit of the group $K = \bar{\Gamma} \subset Aff(\mathbf{E})$. Let us assume that the affine holonomy group Γ is nilpotent. We have proved that in this case Γ and therefore K must be unipotent, cf. Theorem 7. Thus the connected component K_0 of e in K is a simply connected closed nilpotent Lie subgroup of $Aff(\mathbf{E})$. The group $\Gamma_0 = K_0 \cap \Gamma$ is a finitely generated dense subgroup of K_0. Using the Malcev theorem , cf. [MA,RA] we can construct a Lie foliation $\mathcal{F}_0$ with dense leaves of a compact manifold for which the group Γ_0 is the holonomy group. In [CA2] Y. Carrière showed that in such a case $gr(\mathcal{F}_0) \geq \delta(K_0)$ where $\delta(K_0)$ is the degree of nilpotency of K_0. The growth of leaves of $\mathcal{F}_0$ can be read as the local growth of orbits of Γ_0 in K_0. On the other hand the growth of leaves of $\mathcal{F}_1$ can be read as the growth of orbits of Γ in K relative to some open subset of K. Thus $gr(\mathcal{F}_1) \geq gr(\mathcal{F}_0)$ and $\delta(K_0) \leq gr(\mathcal{F}_0) \leq gr(\mathcal{F}_1) = gr(\mathcal{F})$. The structure algebra of $\mathcal{F}$ is the conjugated algebra of $Lie(K_0)$. Therefore we have proved the following proposition, compare Proposition 3.1 of [CA2].

Proposition 15 *Let $\mathcal{F}$ be a complete transversely affine foliation on a compact manifold M. Then $gr(\mathcal{F}) \geq \delta(\underline{k})$ where $\delta(\underline{k})$ is the degree of nilpotency of the structure algebra of $\mathcal{F}$.*

The above considerations together with results of [SR,ZM] yield the following theorem, compare Theorem A of [CA2].

Theorem 12 *Let $\mathcal{F}$ be a complete transversely affine foliation of a compact manifold M with the structure algebra $\underline{k}^-$ of degree of nilpotency $\delta(\underline{k})$. Then:*

i) *if $\mathcal{F}$ has polynomial growth then the structure algebra is nilpotent;*

ii) *if the affine holonomy group is nilpotent, then $\mathcal{F}$ has polynomial growth and $gr(\mathcal{F}) \geq \delta(\underline{k})$.*

Proof The point ii) combines the assertions of Proposition 15 and Corollary 4. If we assume that the growth of $\mathcal{F}$ is polynomial, then $\mathcal{F}_1$ has polynomial growth as well and it is amenable according to [SR]. Then the result of Zimmer, cf. [ZM], ensures that the group K_0 is solvable. If K_0 is not nilpotent then according to [CA2] the foliation $\mathcal{F}_0$ has "suprapolynomial" growth type. Thus $\mathcal{F}$ cannot have polynomial growth type; contradiction.□

Corollary 5 *If the growth type of $\mathcal{F}$ is linear then the structure algebra is abelian. In particular, for any complete transversely affine flow its structure algebra is abelian, which is equivalent to the fact that the group Γ_0 is abelian.*

To illustrate that our Theorem 12 cannot be improved we provide two examples.

Let us consider the 3-dimensional solvable group S_1, cf. [AGH], which can be represented in the matrix form as follows:

$$\begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where x, y and z are real numbers and k is a fixed real number such that $e^k + e^{-k}$ is an integer different from 2. This group acts simply transitively on $\mathbb{R}^3$ and admits a uniform discrete subgroup Γ_1. The manifold $\Gamma_1 \backslash S_1 = S_1(\Gamma_1)$ is an affine manifold which can be identified with the hyperbolic torus T^3_A, $A \in SL(2, \mathbb{Z}), trA > 2$, cf. [GSS].

Example 7 The projection $p_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(x, y, z) \longmapsto (x, y)$, is S_1-equivariant for the natural actions of this group. Therefore p_2 defines a complete transversely affine flow $\mathcal{F}^2$ of the compact manifold $S_1(\Gamma_1)$. The flow $\mathcal{F}^2$ is diffeomorphic to the flow of T^3_A given by suspention of the matrix A. The growth type of the foliation $\mathcal{F}^2$ is linear as it is a flow with non-compact leaves. The affine holonomy group Γ can be identified with a subgroup of $Aff(\mathbb{R}^2)$ generated by A and the integer translations. Since Γ is a discrete subgroup, the structure algebra of $\mathcal{F}^2$ is 0-dimensional. However, the group Γ is solvable, but not nilpotent. Therefore we cannot improve the implication (i).

Example 8 The projection $p_1: \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(x, y, z) \longmapsto y$, is S_1-equivariant for the natural actions of this group. Therefore p_1 defines a codimension 1 complete transversely affine foliation $\mathcal{F}^1$ of the compact manifold $S_1(\Gamma_1)$. The leaves of this foliation are dense, and only the leaves corresponding to rational numbers have holonomy. The foliation $\mathcal{F}^1$ is diffeomorphic to the proper foliation of $\mathbf{T}^3_A$ corresponding to one of the eigenvalues of A, cf. [GSS]. The affine holonomy group of $\mathcal{F}^1$ is solvable, the structure algebra is isomorphic to $\mathbb{R}$, but the leaves of $\mathcal{F}_1$ have exponential growth. In fact, as all leaves are dense, these corresponding to points of $\mathbf{T}^2$ with rational coefficients are resilient. Hence they must have exponential growth, cf. [GSS]. This implies that a complete transversely affine foliation with nilpotent structure algebra can have the growth type even exponential. Thus we cannot weaken the assumption that the affine holonomy group is nilpotent.

VII.6 Closures of leaves

Geodesically complete TAFs have many properties of Riemannian foliations. We have considered these similarities in Chapter III. However when we look at the closures of leaves we find the first main difference. In Riemannian foliations the closures of leaves form a singular foliation, cf. [MO11]. In TAF it is not the case. Examples of non-complete TAF on compact manifolds and of complete ones on non-compact manifolds have been well-known, cf. the 1-dimensional Hopf foliation of $S^2 \times S^1$. However, even in transversely geodesically complete TAF leaves can behave very strangely. The following example is due to E. Ghys, cf. [GS3].

Example 9 Let us take once again Example 2.3. The leaves of $\mathcal{F}_A$ correspond to orbits of A on $\mathbf{T}^2$. Thus the leaves corresponding to points with the first coefficient rational are compact, the closures of other leaves are 2-tori, cf. [GSS]. Hence the closures of leaves of $\mathcal{F}_A$, although submanifolds, they do not form a singular foliation.

Using the same suspension procedure, cf. Example I.2, we can construct transversely geodesically complete TAF on compact manifolds in which the closures of leaves are not necessarily submanifolds.

Example 10 Let us consider a linear Anosov diffeomorphism A of a q–torus $\mathbb{T}^q$. We would like to impose some additional conditions on A which would ensure that the foliation $\mathcal{F}_A$ obtained by suspending A has some leaves whose closures are not submanifolds. First, according to a result of M. Hirsch this cannot occur if $q = 2$, cf. [HI]. Let us choose an irreducible primitive matrix $A \in SL(q, \mathbb{Z}), q > 2$, cf. [GA]. Assume that the closure of an orbit of such a linear Anosov diffeomorphism is a submanifold. By passing to a finite covering, which correspond to the suspension of the diffeomorphism A^k for some $k > 0$, we can assume that the closure of the orbit is a connected submanifold. Then Theorem A of [MN] ensures that it must be a torus, but from Proposition of [HN] it results that our submanifold is either the torus $\mathbb{T}^q$ or a point. S. G. Hancock and F. Przytycki constructed very complicated invariant subsets for any linear Anosov diffeomorphism, cf. [HN,PR]. Thus the closure of any non–periodic orbit contained in such an invariant subset cannot be a submanifold.

Having shown that on compact manifolds there exist transversely geodesically complete TAF with leaves whose closures are not submanifolds we would like to find out whether under some additional assumptions it is possible to demonstrate that the closures of leaves are submanifolds. First we must reduce the study of the closures of leaves to the study of some more manageable objects.

Leaves of $\mathcal{F}$ correspond to orbites of Γ on $\mathbf{E}$. Let L be a leaf of $\mathcal{F}$ and Γv ($v \in \mathbf{E}$) the corresponding orbit of Γ. Then $\bar{L} = \Gamma\backslash D^{-1}(\overline{\Gamma v})$. This equality leads to the following lemma.

Lemma 4 *Let $\mathcal{F}$ be a complete TAF. Then the closure of a leaf L is a submanifold iff the closure of the corresponding orbit of the affine holonomy group is a submanifold.*

Thus we can concentrate our attention on the study of orbits of finitely generated subgroups of *Aff*($\mathbf{E}$). We have proved that for a complete TAF the algebraic closure $A(\Gamma)$ of the affine holonomy group Γ must act transitively on $\mathbf{E}$, i.e. $\mathbf{E}$ can be considered as a homogeneous space $A(\Gamma)\backslash H$ where H is an isotropy group of the natural representation of $A(\Gamma)$ on $\mathbf{E}$. Then we have:

Lemma 5 *The closure of an orbit Γv is a submanifold of $\mathbf{E}$ iff the set $\overline{\Gamma \cdot H}$ is a submanifold of $A(\Gamma)$ where H is the isotropy group of $A(\Gamma)$ at v.*

Proof Consider $\mathbf{E}$ as the homogeneous space $A(\Gamma)\backslash H$. Then the orbit Γv correspond to the orbit Γe_H where $e_H = eH \in A(\Gamma)\backslash H$. The closure of Γe_H in $A(\Gamma)\backslash H$ is equal to $\overline{\Gamma \cdot H}\backslash H$. Thus it is a submanifold iff $\overline{\Gamma \cdot H}$ is a submanifold. □

The lemmas lead to the following theorem.

Theorem 13 *Let $\mathcal{F}$ be a complete TAF on a compact manifold M with abelian fundamental group. Then the closures of leaves of $\mathcal{F}$ form a singular foliation.*

Proof The affine holonomy group Γ of $\mathcal{F}$ is abelian, so is its algebraic closure $A(\Gamma)$. Since $\mathcal{F}$ is complete, the group $A(\Gamma)$ acts transitively on $\mathbf{E}$. As $A(\Gamma)$ is abelian, the isotropy groups of the representation of $A(\Gamma)$ on $\mathbf{E}$ are equal. We denote it by H. Lemma 5 ensures that the closures of orbits of Γ are the orbits of the group $\overline{H \cdot \Gamma} = H(\Gamma)$ which is a Lie subgroup of $A(\Gamma) \subset \mathit{Aff}(\mathbf{E})$. Thus these orbits are submanifolds, and therefore the closures of leaves are submanifolds as well. Elements of the Lie algebra of $H(\Gamma)$ define vector fields which span the tangent space to the orbits of $H(\Gamma)$. As they are Γ–invariant, these vector fields induce global foliated vector fields on M. Their foliated orbits are precisely the closures of leaves of $\mathcal{F}$. Thus, in fact, the closures of leaves form a singular foliation in the sense of Stefan, cf. [ST,SM].□

Foliations with nilpotent affine holonomy group form another more general and very interesting class of TAF. For them we can prove the following:

Theorem 14 *Let $\mathcal{F}$ be a complete TAF of a compact manifold with nilpotent affine holonomy group Γ. If the group $K = \overline{\Gamma}$ has a finite number of components, then the closures of leaves forms a singular foliation and they are the orbits of the commuting sheaf.*

Proof We have proved that the group Γ must be unipotent. Thus the connected component K_0 of K is an algebraic group, and according to [RO] the orbits of K_0 are closed. Since K has a finite number of connected components, its orbits are closed, and thus equal to the closures of orbits of Γ. Therefore the closures of leaves are submanifolds. The description of the commuting sheaf ensures that these closures are the orbits of this sheaf. Hence the closures of leaves of $\mathcal{F}$ form a singular foliation.□

Example 7 shows that Theorem 14 is false if the group K has an infinite number of connected components, but we can still hope that the closures of leaves are submanifolds. We have seen that in the same example the space of closures of leaves have been a very irregular topological space. If we impose some separability condition on the space of orbits of the group K we can relax a little our other assumptions.

Proposition 16 *Let $\mathcal{F}$ be a complete TAF on a compact manifold. If*

a) *the affine holonomy group Γ is distal;*

b) *the group $K = \overline{\Gamma}$ has a finite number of connected components;*

c) *the space $K \backslash \mathbf{E}$ is T_0,* *then the closures of leaves are the orbits of the commuting sheaf of $\mathcal{F}$ and they form a singular foliation.*

Proof Glimm's theorem, cf. [Gl], ensures that orbits of K are relatively open in their closures. On the other hand these closures are minimal, cf. [MR]. Therefore the orbits of K must be closed. The rest follows as in the proof of Theorem 14.□

To complete this section we give an example of a TAF having a solvable affine holonomy group and with some closures of leaves not being submanifolds.

Example 11 As in the previous examples we suspend an Anosov diffeomorphism; this time of a non–toral nilmanifold, cf. [SM,BR].

Let H be a 3–dimensional real Heisenberg group, and let $G = H \times H$. The group G is diffeomorphic to $\mathbb{R}^6$. G admits the following uniform subgroup Γ_0:

$$\Gamma_0 = \{(a_1, ..a_6) \in \mathbb{R}^6 : a_i \in \mathbb{Z}(\sqrt{3}) \text{ and } a_{i+3} = \bar{a}_i, i = 1,2,3\}$$

where if $a = m+n\sqrt{3}$, $m,n \in \mathbb{Z}$, then $\bar{a} = m-n\sqrt{3}$. The space G/Γ_0 is a compact non–toral nilmanifold. Let $\lambda = 2+\sqrt{3}$, $\nu = (2-\sqrt{3})^2$, $\mu = \lambda\nu = 2-\sqrt{3}$. Then the transformation $\phi: \mathbb{R}^6 \to \mathbb{R}^6$,

$$\phi(x_1, ..x_6) = (\lambda x_1, \mu x_2, \nu x_3, \bar{\lambda} x_4, \bar{\mu} x_5, \bar{\nu} x_6)$$

preserves the lattice Γ_0, and therefore defines an Anosov diffeomorphism of G/Γ_0. The suspension of ϕ defines a 1–dimensional TAF $\mathcal{F}_\phi$ on the total space of $\mathbb{R} \times_\phi G/\Gamma_0$. The affine holonomy group Γ of $\mathcal{F}_\phi$ is the subgroup of $Aff(\mathbb{R}^6)$ generated by the group Γ_0 and ϕ. It is a solvable group. It is not difficult to verify that the closure of the Γ–orbit of the point $(x, 0, ...0)$, $x \neq 0$, is not a submanifold. Thus, indeed, the foliation $\mathcal{F}_\phi$ has the property we have been looking for.

Other examples of this sort can be constructed from the examples of H. L. Porteous, cf. [PO]. Moreover, E. Ghys has informed the author that similar examples can be constructed using the work of M. Morse on dymanics on tori, which is of course previous to the results of S. G. Hancock and F. Przytycki.

Notes This chapter presents results from four papers. Parts of [WO12] and [WO20] form Section 1. Sections 2,3 and 4 are derived from [WO16]. Section 5 is based on [WO17] and Section 6 on [WO18].

The most important work has been done on codimension 1 TAFs. We should mention papers by E. Fedida, P. M. D. Furness and Bobo Seke, and in particular unpublished results of G. Hector concerning the existence of exeptional minimal sets for TAFs. There are some other papers concerned with TAFs, for example [GHL], [ME1] and [ME2].

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