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Á memoria dos meus avós Antonio e Carmen
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Abstract

Riemannian Geometry was introduced for the first time by Bernhard Riemann in the 19th century. Nowadays, a Riemannian manifold is defined as a smooth manifold endowed with a smooth positive definite quadratic form in each tangent space, a so-called Riemannian metric. A Riemannian metric induces a distance on the manifold, and the transformations of the space that preserve this distance are called isometries. The symmetry of an object, from the point of view of Riemannian Geometry, is the invariance of that object under the action of a certain subgroup of isometries.

Kähler manifolds are Riemannian manifolds that have a parallel complex structure. A complex space form is a complete simply connected Kähler manifold with constant holomorphic sectional curvature, and it is known that such a manifold is one of the following: a complex projective space $\mathbb{C}P^n$ (for positive curvature), a complex Euclidean space $\mathbb{C}^n$ (for vanishing curvature), or a complex hyperbolic space $\mathbb{C}H^n$ (for negative curvature).

An isometric action on a Riemannian manifold is the action of a group of isometries of that manifold. An orbit of an isometric action is called an (extrinsically) homogeneous submanifold. The cohomogeneity of an isometric action is the codimension of an orbit of maximal dimension. The study of isometric actions in its full generality is in principle difficult; thus, certain types of isometric actions have been studied. A polar action is an isometric action such that there exists a totally geodesic submanifold, called section, that intersects all the orbits orthogonally. The cohomogeneity of a polar action coincides with the dimension of its section. Polar actions on $\mathbb{C}P^n$ have been classified by Podestà and Thorbergsson [93], whereas for $\mathbb{C}H^n$ these have been classified by Díaz-Ramos, Domínguez-Vázquez and Kollross [39] .

One outstanding problem in submanifold geometry is to characterize homogeneous submanifolds, or more generally, submanifolds with many symmetries, by means of geometric properties. In this thesis we focus, on the one hand, on the study of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with a high degree of symmetry. More specifically, we study cohomogeneity one hypersurfaces that are induced by polar actions of cohomogeneity two on $\mathbb{C}P^2$ and $\mathbb{C}H^2$. On the other hand, we are interested in the characterization of the principal orbits of these actions by means of certain geometric criteria, such as isoparametric submanifolds.

In what follows we explain our main motivation for this thesis. We try to explain how an open problem that we decided to tackle led us to the research project that is carried out in this memoir.

A classic theorem in the geometry of surfaces of $\mathbb{R}^3$ establishes that a totally umbilical

surface is an open part of a sphere or a plane; in particular, it is an open part of a homogeneous surface. This result is known to be true in higher dimensions. This suggests to tackle the following problem: to what extent having a small number of distinct principal curvatures imposes restrictions on the geometry of a hypersurface of a Riemannian manifold. Tashiro and Tachibana [100] showed that there are no umbilical hypersurfaces in nonflat complex space forms. Later, Cecil and Ryan [24] for the projective case, and Montiel [86] for the hyperbolic case, carried out the classification of real hypersurfaces with two distinct principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$, $n \geq 3$. All these examples are Hopf (that is, the Reeb vector field is a principal curvature vector everywhere) and have constant principal curvatures, and thus, are open parts of homogeneous hypersurfaces in view of the results by Kimura [71] and Berndt [8]. Niebergall and Ryan [88, Open problem 9.2] state the interest of addressing this question for $n = 2$, as the methods applied for $n \geq 3$ do not work in this situation. We have shown in [41], a result that is part of this thesis, that the principal curvatures of a real hypersurface with two distinct principal curvatures in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ do not have to be constant; we also classify these hypersurfaces, thus completing the classification for $n = 2$. In particular, there are inhomogeneous examples.

The examples that appear in the previous classification, some non-Hopf homogeneous hypersurfaces in $\mathbb{C}H^n$, and the examples of cohomogeneity one hypersurfaces with constant mean curvature constructed by Gorodski and Gusevskii in [54], satisfy certain geometric properties that we have encoded in a definition. Thus, a real hypersurface in a complex space form is called strongly 2-Hopf if the smallest distribution invariant under the shape operator that contains the Reeb vector field is two-dimensional, integrable, and the principal curvatures of the hypersurface along the integral submanifolds of this distribution are constant. These hypersurfaces turn out to be intrinsically of cohomogeneity one, and in fact, they are obtained by the equivariant geometry method applied to cohomogeneity two polar actions. We impose further assumptions to get concrete examples and more specific classifications. For example, in order to characterize the examples obtained by Gorodski and Gusevskii [54], we study strongly 2-Hopf hypersurfaces with constant mean curvature. The results obtained under these conditions underline the fact that the method of equivariant geometry in conjunction with the strongly 2-Hopf condition gives rise to a better understanding of these hypersurfaces and suggests a powerful approach to generate new examples.

A Levi-flat hypersurface is a real hypersurface foliated by complex hypersurfaces. It is an open problem to determine the existence of compact Levi-flat hypersurfaces in $\mathbb{C}P^2$. Motivated by this question and the previous results, we have tackled the study of Levi-flat strongly 2-Hopf hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. We have shown the existence of a great deal of examples, although only locally. If we further assume that the hypersurface has constant mean curvature, then it turns out to be austere. Austere hypersurfaces, introduced by Harvey and Lawson [56] in their study of special Lagrangian submanifolds, are defined as hypersurfaces whose shape operators are invariant under multiplication by -1. The classification of austere hypersurfaces whose Reeb vector field has $h = 1$ or $h = 2$ nontrivial projections onto the principal curvature spaces led us to the classification of three important types of ruled hypersurfaces: Lohnherr hypersurfaces, Clifford cones and bisectors.

The fundamental role of the principal orbits of polar actions in the study of strongly 2-Hopf hypersurfaces suggests the characterization of these submanifolds by means of a suitable geometric property. Motivated by the recent classification of isoparametric hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ we have addressed the study of isoparametric submanifolds in 2-dimensional complex space forms as a first step to understand these objects in arbitrary dimension.

A real hypersurface is called isoparametric if its sufficiently close equidistant hypersurfaces have constant mean curvature. Terng [101] generalized this concept for higher codimension in manifolds of constant curvature. Nowadays the concept of isoparametric submanifold is credited to Heintze, Liu and Olmos [57], who define it as a submanifold with flat normal bundle, whose sufficiently close parallel submanifolds have constant mean curvature in radial directions, and such that there is a section through every point of the submanifold.

The study of isoparametric hypersurfaces in complex projective spaces can be deduced from the classification of isoparametric hypersurfaces in spheres via a careful study of the Hopf map. The classification of isoparametric hypersurfaces in spheres is actually an outstanding open problem to this date. The works of Cecil, Chi and Jensen [23], Immervoll [65], Chi [31], [32] and Miyaoka [84] have completed the classification except for 4 principal curvatures with multiplicities $(7, 8)$. Domínguez-Vázquez [49] obtained from here the classification of isoparametric hypersurfaces in $\mathbb{C}P^n$, $n \neq 15$. It is shown that there is a number of inhomogeneous examples, but not for $n = 2$. The classification of isoparametric hypersurfaces in complex hyperbolic spaces has recently been obtained by Díaz-Ramos, Domínguez-Vázquez and Sanmartín-López in [40]. In this classification there are also inhomogeneous examples, but none of them arises in $\mathbb{C}H^2$.

For higher codimension, the classification of isoparametric submanifolds of $\mathbb{C}P^n$, $n \neq 15$, follows from [49] as well, and again, inhomogeneous examples exist. For $\mathbb{C}H^n$, the problem seems to be much more difficult and is still open. In [42], whose results are part of this thesis, we have carried out the classification of isoparametric submanifolds in $\mathbb{C}H^2$ as a first step to understand the problem in spaces of nonconstant, nonpositive curvature. Indeed, we address both Heintze, Liu and Olmos' and Terng's definition, since the latter had not been previously studied in spaces of nonconstant curvature.

So far we have presented our research from a chronological point of view. Nonetheless, this thesis is structured in a more logical rather than historical order for the sake of narrative coherence. In what follows we summarize the results obtained in this project.

Isoparametric submanifolds

Motivated by the classification of isoparametric hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$, we have tackled this classification in higher codimension. Moreover, in a space of nonconstant sectional curvature it is not clear whether the definitions of Terng's and of Heintze, Liu and Olmos' agree. Thus, in Chapter 2 we carry out the classification of both isoparametric submanifolds of $\mathbb{C}H^2$ and Terng-isoparametric submanifolds of $\mathbb{C}P^2$ and $\mathbb{C}H^2$.

The key result for this classification is obtained in Section 2.2. This result allows us to establish that flat Lagrangian submanifolds with parallel mean curvature have parallel second fundamental form and are isoparametric, Terng-isoparametric and coincide with open parts of principal orbits of cohomogeneity two polar actions on $\mathbb{C}P^2$ or $\mathbb{C}H^2$ (Theorem 2.1).

In Section 2.3 we present the classification of isoparametric and Terng-isoparametric submanifolds in complex projective and hyperbolic planes. We show that an isoparametric submanifold in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ is congruent to an open part of a principal orbit of a cohomogeneity two polar action (Theorem 2.2). On the other hand, we show that a Terng-isoparametric submanifold is either isoparametric, a Chen's surface in $\mathbb{C}H^2$ or a circle (Theorem 2.3). The Chen's surface, constructed in [26], turns out to be homogeneous, but not an orbit of a polar action. In Subsection 2.3.1 we give a Lie theoretic description different from the definition given by Chen.

Strongly 2-Hopf hypersurfaces

As a means to generalize the examples obtained in [41], [54], and some examples of non-Hopf homogeneous hypersurfaces in $\mathbb{C}H^n$ [12], we introduce in Chapter 3 the concept of strongly 2-Hopf hypersurface in a complex space form. We apply the method of equivariant geometry to cohomogeneity two polar actions in order to characterize these hypersurfaces.

The idea behind the construction of strongly 2-Hopf hypersurfaces is as follows. We take a cohomogeneity two polar action on $\mathbb{C}P^2$ or $\mathbb{C}H^2$ and a curve in the regular part of a section; we take the union of all the orbits through that curve. This gives a hypersurface with at least two distinct principal curvatures that is generically strongly 2-Hopf.

Conversely, assume that we have a strongly 2-Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$. Some calculations show that there exist two perpendicular integrable distributions on that hypersurface, one of which is 2-dimensional and the other one is 1-dimensional. The integral surfaces of the 2-dimensional distribution are equidistant, flat, totally real, with parallel second fundamental form and flat normal bundle. Each integral curve of the other distribution is contained in a totally real, totally geodesic submanifold of the ambient space. Therefore, the results of Section 2.2 show that the hypersurface is foliated orthogonally by principal orbits of a cohomogeneity two polar action, and thus, it can be built as explained above.

A particular case of this setting is the construction of hypersurfaces with two distinct principal curvatures. In this case the curve in the section simply satisfies certain ordinary differential equation. The existence of solutions to ordinary differential equations guarantees that such hypersurfaces exist. Since they are generically non-Hopf, the examples obtained in this way are different from the standard ones. In particular we respond positively to the question posed by Niebergall and Ryan in [88]. We give a detailed description and a classification of these examples in Section 3.4.

In Section 3.5 we focus on austere hypersurfaces. After proving that there are no Hopf examples, we calculate their Levi-Civita connection explicitly and show that austere hypersurfaces whose Hopf vector field has nontrivial projection onto $h = 2$ principal curvature spaces are ruled, that is, their maximal complex distribution is integrable and their integral submanifolds are totally geodesic in the ambient space. The result now follows from the classification of minimal ruled hypersurfaces obtained by Lohnherr and Reckziegel in [79]. The examples arising under these hypotheses are: a Lohnherr's hypersurface in $\mathbb{C}H^2$, a Clifford cone in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, or a bisector in $\mathbb{C}H^2$.

Finally, we apply these results to study strongly 2-Hopf hypersurfaces that are Levi-flat or have constant mean curvature. Any of these two extra conditions is equivalent for the curve in the section to satisfy an ordinary differential equation. The existence and uniqueness of solutions to ordinary differential equations allows us to characterize these hypersurfaces. If the strongly 2-Hopf hypersurface is simultaneously Levi-flat and has constant mean curvature, then it is austere, and we apply the previous results to obtain a full description of them.

Structure of this thesis

In Chapter 1 the basic terminology and conventions are introduced. We explain the following concepts by order: semi-Riemannian manifolds (§1.1), submanifold geometry (§1.2) singular Riemannian foliations (§1.3), isoparametric submanifolds (§1.4), isometric actions (§1.5), complex space forms (§1.6) and polar actions (§1.7).

The core of the thesis is contained in chapters 2 and 3, where the main results are presented.

In Chapter 2 we carry out the proof of, arguably, the pivotal result of this thesis (§3.1), which guarantees that a flat Lagrangian surface with parallel mean curvature in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ is an open part of a principal orbit of a cohomogeneity two polar action. Then, a description of Chen's surface as an orbit of an isometric action is given in Section 2.3.1. In sections 2.3.2 and 2.3.3 we obtain the classification of isoparametric submanifolds and Terng-isoparametric submanifolds in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, respectively.

Finally, Chapter 3 is focused on the study of certain types of real hypersurfaces that are built using the method of equivariant geometry. We explain the construction method in Section 3.1. Then we obtain the Levi-Civita connection of a real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ whose Reeb vector field has $h = 2$ nontrivial projections onto the principal curvature spaces in Section 3.2. Last, but not least, we obtain the following results for real hypersurfaces in complex projective and hyperbolic planes: characterization of strongly 2-Hopf hypersurfaces (§3.3), classification of hypersurfaces with two distinct principal curvatures (§3.4), classification of austere hypersurfaces with $h \leq 2$ nontrivial projections of the Hopf vector field onto the principal curvature spaces (§3.5), and characterization of strongly 2-Hopf hypersurfaces that have constant mean curvature, that are Levi-flat or satisfy both conditions simultaneously (§3.6).

Contents

Chapter 1

Preliminaries and conventions

In this chapter the basic notions and terminology needed for this thesis are introduced. Throughout this work we will use the notations and conventions described here unless otherwise mentioned.

In Section 1.1, the concept of semi-Riemannian manifold is introduced and the sign convention for the curvature tensor is settled. In Section 1.2 all the basic concepts and equations of submanifold geometry needed for this thesis are explained. In Section 1.3 the definitions of singular and polar Riemannian foliations are introduced. In Section 1.4 we recall the notions of isoparametric submanifold according to Heintze, Liu and Olmos, and according to Terng. Section 1.5 is devoted to presenting the basic terminology related to isometric actions. In Section 1.6 we provide the construction of complex projective and hyperbolic spaces (in Subsections 1.6.2 and 1.6.3), we characterize them as irreducible Hermitian symmetric spaces (in Subsection 1.6.1), and we provide a model of the complex hyperbolic space as a solvable Lie group. Finally, we introduce polar actions in Section 1.7 and recall their classification in the complex projective and hyperbolic spaces (in Subsections 1.7.1 and 1.7.2).

1.1 Semi-Riemannian manifolds

Although this thesis deals mainly with geometric objects in the Riemannian setting, at some points we will make use of arguments concerning the more general semi-Riemannian geometry. That is why in this section we focus not only on the Riemannian case, but also on semi-Riemannian geometry.

Let M be an n-dimensional smooth manifold. For each $p \in M$, we denote by T_pM the tangent space of M at p. The tangent bundle is denoted by TM. If $\mathcal{D}$ is a distribution along M, we will denote by $\Gamma(\mathcal{D})$ the module of sections of such distribution, that is, those vector fields X on M such that $X_p \in \mathcal{D}_p$ for all $p \in M$.

If T denotes a bilinear tensor of type $(0, 2)$ defined on a vector space, we say that T is symmetric if $T(x, y) = T(y, x)$ for all x, y and nondegenerate if $T(x, y) = 0$ for all y implies $x = 0$. The signature of the tensor is a pair (r, s) such that the tensor T is linearly congruent to a diagonal matrix $\operatorname{diag}(1, \overset{r}{\cdots}, 1, -1, \overset{s}{\cdots}, -1)$.

A *semi-Riemannian manifold* is a pair $(M, \langle\cdot,\cdot\rangle)$, where M is a manifold and $\langle\cdot,\cdot\rangle$ is a nondegenerate symmetric bilinear tensor field of type $(0,2)$ and constant signature. In particular, each tangent space T_pM is endowed with a nondegenerate symmetric bilinear tensor $\langle\cdot,\cdot\rangle_p$. If the tensor has signature (r,s) we will say that M has signature (r,s). Semi-Riemannian manifolds with signature $(n,0)$ are called *Riemannian manifolds*, while *Lorentzian manifolds* are those with signature $(n-1,1)$. If M is a Riemannian manifold and X is a vector field on M, then we will denote by $\|X\|$ its norm, $\|X\| = \sqrt{\langle X, X\rangle}$. The Riemannian exponential map of M will be denoted by exp.

The study of curvature plays an important role in semi-Riemannian geometry. The *curvature tensor* R of a semi-Riemannian manifold M provides the curvature information. It is a tensor of type $(1,3)$ that we define with the following sign convention:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X, Y, Z \in \Gamma(TM),$$

where ∇ is the *Levi-Civita connection* of M, that is, the unique torsion-free metric connection on M. When $R = 0$ we say that the manifold is flat.

Let $G_2^*(TM)$ be the Grassmann bundle over M consisting of all nondegenerate 2-dimensional linear subspaces σ of T_pM, for $p \in M$. If σ is a nondegenerate subspace of T_pM, then we define the *sectional curvature* of σ as

$$K(\sigma) = \frac{\langle R(U,V)V, U\rangle}{\langle U,U\rangle\langle V,V\rangle - \langle U,V\rangle^2},$$

where $\{U, V\}$ is a basis for σ. The function $K\colon G_2^*(TM) \longrightarrow \mathbb{R}$ is called the sectional curvature function of M. A semi-Riemannian manifold M is said to have *constant curvature* if the sectional curvature function is constant. In this case, the curvature tensor can be written as $R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$ for any vector fields X, Y and Z on M and some constant c. It is known that the only connected, complete, simply connected Riemannian manifolds of constant curvature are Euclidean spaces $\mathbb{R}^n$ ($c = 0$), spheres S^n ($c > 0$) and real hyperbolic spaces $\mathbb{R}H^n$ ($c < 0$). These are the so-called *(real) space forms.*

1.2 Submanifold geometry

In this section we introduce the basic terminology and fundamental formulas for the study of submanifolds of Riemannian manifolds. More information can be found in [9, Chapters 2 and 8] and [25]. For semi-Riemannian manifolds of arbitrary signature, a detailed reference is [91, Chapter 4] .

Let $\bar{M}$ be a semi-Riemannian manifold with metric $\langle\cdot,\cdot\rangle$ and M an embedded submanifold of $\bar{M}$. The restriction of $\langle\cdot,\cdot\rangle$ to M induces a symmetric bilinear tensor on M that can be degenerate. If it is nondegenerate then M is a semi-Riemannian manifold by itself and M is called a *semi-Riemannian submanifold* or a *nondegenerate submanifold* of $\bar{M}$. If $\bar{M}$ is Riemannian, every submanifold of $\bar{M}$ is a Riemannian submanifold. In this work we will assume that submanifolds are embedded and equipped with the induced semi-Riemannian metric (whenever the induced metric is nondegenerate). All the concepts and terminology that we will explain below involve local geometry, so it also applies to immersed submanifolds, since immersed submanifolds are locally embedded.

From now on, $\bar{M}$ will be a Riemannian manifold and M will be a Riemannian submanifold of $\bar{M}$. Each tangent space T_pM is endowed with the metric $\langle\cdot,\cdot\rangle_p$. Hence, we can consider the bundle of vectors that are orthogonal to the tangent space, which is called the *normal bundle* of M and that is denoted by νM. By $\Gamma(\nu M)$ we denote the module of all normal vector fields to M. At each point $p \in M$, the canonical isomorphism $T_p\bar{M} = T_pM \oplus \nu_pM$ holds. In this work, the symbol $\oplus$ will always denote direct sum (not necessarily orthogonal direct sum). Given a vector field X on $\bar{M}$ along M we denote by $X^\top$ the orthogonal projection of X onto TM and by $X^\perp$ the orthogonal projection onto νM.

If V is a vector space with symmetric bilinear form $\langle\cdot,\cdot\rangle$ and $W \subset V$ is a vector subspace, we denote $V \ominus W = \{v \in V : \langle v, w\rangle = 0, \forall w \in W\}$. If $\langle\cdot,\cdot\rangle$ is positive definite, this notation stands for the orthogonal complement of W in V. We will use the same notation for distributions on M or subbundles of $\bar{M}$ defined along M.

It is known that the curvature tensor R of a Riemannian submanifold M depends only on the metric of M, thus the curvature tensor is an important intrinsic geometric invariant. One may study the intrinsic geometry of both $\bar{M}$ and M. Nonetheless, one can also investigate the geometry of M in relation to the geometry of $\bar{M}$. This is called the extrinsic geometry of M.

We denote by $\bar{R}$ and R the curvature tensors of $\bar{M}$ and M, and by $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\bar{M}$ and M, respectively. We decompose $\bar{\nabla}_X Y$ in its tangent part $(\bar{\nabla}_X Y)^\top$ and its normal part $(\bar{\nabla}_X Y)^\perp$, for any X, $Y \in \Gamma(TM)$. Then, the Levi-Civita connection of M is given by the tangent part, $\nabla_X Y = (\bar{\nabla}_X Y)^\top$, and we define the *second fundamental form* as the normal part, $I\!I(X,Y) = (\bar{\nabla}_X Y)^\perp$. Hence, we have an orthogonal descomposition

$$\bar{\nabla}_X Y = \nabla_X Y + I\!I(X,Y)$$

for any X, $Y \in \Gamma(TM)$, which is called the *Gauss formula* of M. Let $\xi \in \Gamma(\nu M)$ be a unit normal vector field. The *shape operator* of M associated with ξ is the self-adjoint operator $\mathcal{S}_\xi$ on M defined by $\langle \mathcal{S}_\xi X, Y\rangle = \langle I\!I(X,Y), \xi\rangle$, where X, $Y \in \Gamma(TM)$. The eigenvalues and eigenspaces of $\mathcal{S}_\xi$ are called the *principal curvatures* and the *principal curvature spaces* of M with respect to ξ, respectively. Moreover, denote by $\nabla^\perp$ the normal connection of M, that is, $\nabla^\perp_X \xi = (\bar{\nabla}_X \xi)^\perp$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Then, we have an orthogonal decomposition

$$\bar{\nabla}_X \xi = -\mathcal{S}_\xi X + \nabla^\perp_X \xi,$$

which is called the *Weingarten formula*.

We can relate the curvature tensors of $\bar{M}$ and M applying the previous information, via the second fundamental form. This relation is known as the *Gauss equation*, and for $X, Y, Z, W \in \Gamma(TM)$ it is written as follows:

$$\langle \bar{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle - \langle I\!I(Y,Z), I\!I(X,W)\rangle + \langle I\!I(X,Z), I\!I(Y,W)\rangle.$$

We also obtain the *Codazzi equation* which is very important in this work,

$$(\bar{R}(X,Y)Z)^{\perp} = (\nabla^{\perp}_X I\!I)(Y,Z) - (\nabla^{\perp}_Y I\!I)(X,Z),$$

where the covariant derivative of the second fundamental form is given by

$$(\nabla^{\perp}_X I\!I)(Y,Z) = \nabla^{\perp}_X I\!I(Y,Z) - I\!I(\nabla_X Y, Z) - I\!I(Y, \nabla_X Z).$$

The Codazzi equation can also be written using the shape operator of M as follows

$$\begin{aligned}\langle \bar{R}(X,Y)Z,\xi\rangle &= \langle \nabla_X \mathcal{S}_\xi Y, Z\rangle - \langle \nabla_X Y, \mathcal{S}_\xi Z\rangle - \langle \mathcal{S}_{\nabla^{\perp}_X \xi} Y, Z\rangle \\ &\quad - \langle \nabla_Y \mathcal{S}_\xi X, Z\rangle + \langle \nabla_Y X, \mathcal{S}_\xi Z\rangle + \langle \mathcal{S}_{\nabla^{\perp}_Y \xi} X, Z\rangle,\end{aligned}$$

where ξ is a unit normal vector field on M.

The last of the three fundamental equations of second order in submanifold theory is the *Ricci equation*

$$\langle R^{\perp}(X,Y)\xi,\eta\rangle = \langle \bar{R}(X,Y)\xi,\eta\rangle + \langle [\mathcal{S}_\xi,\mathcal{S}_\eta]X, Y\rangle,$$

where X, $Y \in \Gamma(TM)$, ξ, $\eta \in \Gamma(\nu M)$ and $R^{\perp}$ is the curvature tensor of the normal bundle of M, defined by $R^{\perp}(X,Y)\xi = [\nabla^{\perp}_X, \nabla^{\perp}_Y]\xi - \nabla^{\perp}_{[X,Y]}\xi$.

We say that a submanifold is *totally geodesic* if its second fundamental form vanishes identically, $I\!I = 0$. This is equivalent to saying that every geodesic in M is also a geodesic in $\bar{M}$. If M is complete and totally geodesic we have that $M = \exp_p(T_pM)$ for any $p \in M$.

The fact that $\xi \in \Gamma(\nu M)$ is a *parallel normal vector field* to M means that ξ is parallel with respect to the normal connection of M, i.e. $\nabla^{\perp}\xi = 0$. When for any point p in M and any normal vector $\xi \in \nu_p M$ there exists a neighbourhood of p in M such that ξ can be extended to a parallel normal vector field on that neighbourhood, then M is said to have *flat normal bundle*. A submanifold has flat normal bundle if and only if the normal curvature tensor $R^{\perp}$ vanishes identically.

The *mean curvature vector* H of a Riemannian submanifold M is defined as the trace of the second fundamental form. Hence, with respect to a local orthonormal frame $\{E_i\}$ of M we may write $H = \sum_i I\!I(E_i, E_i)$. If $\xi \in \Gamma(\nu M)$, then the *mean curvature* of M with respect to ξ is the trace of the shape operator $\mathcal{S}_\xi$. We will say that M has *parallel mean curvature* if the mean curvature vector field is parallel with respect to the normal connection of M. The *mean curvature function* of M is the norm of the mean curvature vector field $h = \|H\|$. A submanifold is said to be *minimal* if and only if its mean curvature function vanishes. Minimal submanifolds appear in a natural way as the critical points of the volume functional and they are a topic of current interest in Differential Geometry. A submanifold M is called *austere* if, for each unit normal vector ξ, the eigenvalues of $\mathcal{S}_\xi$ are invariant under multiplication by -1. Austere submanifolds constitute an interesting subclass of minimal submanifolds. These objects were introduced by Harvey and Lawson [56] in their study of special Lagrangian submanifolds.

A submanifold M is said to be *totally umbilical* if there exists a normal vector field ξ such that $I\!I(X,Y) = \langle X,Y\rangle\xi$ for all tangent vector fields X and Y. In this case ξ is proportional to the mean curvature vector field H by a factor $\dim M$. It is clear that every one-dimensional submanifold is totally umbilical, as well as any totally geodesic submanifold.

A smooth curve $\sigma\colon I \to \bar{M}$ parametrized by arc length is called a *circle* in $\bar{M}$ if it satisfies the third order differential equation $\bar{\nabla}_{\dot\sigma}\bar{\nabla}_{\dot\sigma}\dot\sigma + \langle \bar{\nabla}_{\dot\sigma}\dot\sigma, \bar{\nabla}_{\dot\sigma}\dot\sigma\rangle\dot\sigma = 0$, where $\dot\sigma$ denotes the tangent vector to the curve σ. Circles are precisely the unit speed parametrizations of curves with parallel mean curvature. In particular, unit speed geodesics are circles. More information on circles can be found in [9, §8.4].

We say that M has *parallel second fundamental form* if $\nabla^{\perp}_X II = 0$ for all $X \in \Gamma(TM)$. Submanifolds with parallel second fundamental form have parallel mean curvature.

Two Riemannian submanifolds M_1 and M_2 of $\bar{M}$ are said to be *congruent* if there exists an isometry of $\bar{M}$ that takes M_1 into M_2.

Assume now that M is a hypersurface of $\bar{M}$, that is, an embedded submanifold of codimension one. Then, locally and up to sign, there exists a unique unit normal vector field $\xi \in \Gamma(\nu M)$. Hence the second fundamental form II is a multiple of ξ.

We will denote by $\mathcal{S} = \mathcal{S}_\xi$ the shape operator with respect to ξ. The Gauss and Weingarten formulas can now be written as

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + \langle \mathcal{S}X, Y\rangle \xi, \\ \bar{\nabla}_X \xi &= -\mathcal{S}X. \end{aligned}$$

Then, the Gauss and Codazzi equations reduce to

$$\begin{aligned} \langle \bar{R}(X,Y)Z, W\rangle &= \langle R(X,Y)Z, W\rangle - \langle \mathcal{S}Y, Z\rangle\langle \mathcal{S}X, W\rangle + \langle \mathcal{S}X, Z\rangle\langle \mathcal{S}Y, W\rangle, \\ \langle \bar{R}(X,Y)Z, \xi\rangle &= \langle (\nabla_X \mathcal{S})Y - (\nabla_Y \mathcal{S})X, Z\rangle, \end{aligned}$$

whereas the Ricci equation does not give further information for hypersurfaces.

Since the mean curvature vector H is the trace of the second fundamental form, which is a multiple of ξ, H is proportional to the vector ξ. Thus, one usually talks about the *mean curvature* of the hypersurface, which is defined as the trace of its shape operator $\mathcal{S}$.

Let ξ be a unit normal vector field defined on an open subset U of the hypersurface M. We say that $\lambda\colon U \subset M \to \mathbb{R}$ is a *principal curvature* of M (associated with ξ) if there exists a vector field $X \in \Gamma(TU)$ such that $\mathcal{S}X = \lambda X$. If λ is a principal curvature we denote by $T_\lambda(p)$ the eigenspace of $\lambda(p)$ and call it the *principal curvature space* associated with $\lambda(p)$. If $X \in T_\lambda(p)$, $X \neq 0$, we say that X is a *principal curvature vector* of λ at p. In general, the dimension of the principal curvature spaces associated with a principal curvature λ does not need to coincide at different points. This dimension, that is, $\dim\ker(\mathcal{S}_p - \lambda(p)\,\mathrm{Id})$, is called the *multiplicity* of λ at p.

A connected hypersurface is said to have *constant principal curvatures* if the eigenvalues of the shape operator are the same at every point. In this case the principal curvature spaces associated with an eigenvalue λ have the same dimension at any point.

1.3 Singular Riemannian foliations

In this work we will deal with some examples of singular Riemannian foliations, a topic that is an active field of research nowadays. Singular Riemannian foliations were introduced by Molino [85] in his study of Riemannian foliations. See the articles [6], [7], [82] and [105] for more information.

Let $\mathcal{F}$ be a decomposition of a Riemannian manifold $\bar{M}$ into connected injectively immersed submanifolds, called *leaves*, which may have different dimensions. We say that $\mathcal{F}$ is a *singular Riemannian foliation* if the following conditions are satisfied:

(i) $\mathcal{F}$ is a *singular foliation*, that is, $T_pL = \{X_p : X \in \mathcal{X}_{\mathcal{F}}\}$ for every leaf L in $\mathcal{F}$ and every $p \in L$, where $\mathcal{X}_{\mathcal{F}}$ is the module of smooth vector fields on the ambient manifold that are everywhere tangent to the leaves of $\mathcal{F}$, and

(ii) $\mathcal{F}$ is a *transnormal system*, that is, every geodesic orthogonal to one leaf remains orthogonal to all the leaves that it intersects.

If $\bar{M}$ is complete, the transnormality condition implies that the leaves are equidistant to each other.

The leaves of maximal dimension are called *regular* and the other ones are *singular*. The points of $\bar{M}$ are said to be regular or singular according to the leaves through them. A singular Riemannian foliation is called *regular* if all leaves are regular, that is, if it is a Riemannian foliation. The *dimension* of $\mathcal{F}$ is the maximal dimension of the leaves and its *codimension* is $\dim \bar{M} - \dim \mathcal{F}$.

An important example are *polar foliations*, also called *singular Riemannian foliations with sections* in the terminology of Alexandrino [5]. Let $\mathcal{F}$ be a foliation on $\bar{M}$. Then $\mathcal{F}$ is said to be polar if, for each point $p \in \bar{M}$, there is an immersed submanifold Σ_p, called *section*, that passes through p and that meets all the leaves and always perpendicularly. It follows that Σ_p is totally geodesic and that the dimension of Σ_p is equal to the codimension of $\mathcal{F}$. When the sections of a polar foliation are flat submanifolds, the foliation is called *hyperpolar*.

If the ambient manifold $\bar{M}$ is complete, the fact that a singular Riemannian foliation be polar is equivalent to the fact that the distribution generated by the normal spaces to the regular leaves is integrable. In this case, the sections are complete. Moreover, the leaves of a polar foliation on a complete, simply connected Riemannian manifold are always closed submanifolds with globally flat normal bundle (see [83, Theorem 1.2]). Note that, in a complete ambient manifold, codimension one foliations are always polar.

Polar foliations are related to the so-called polar actions, whose definition will be introduced in Section 1.7. One important question in the study of polar foliations is to decide whether polar foliations are orbit foliations of isometric actions. In this case, such homogeneous polar foliations are precisely the orbit foliations of polar actions.

1.4 Isoparametric submanifolds

Let M be a submanifold of a Riemannian manifold $\bar{M}$. If the normal bundle of M is flat, for each parallel normal vector field ξ and each sufficiently small $r > 0$, we can consider the set $M^{r,\xi} = \{\exp(r\xi_p) : p \in M\}$. If such a set is a submanifold, then we call it a *parallel submanifold* of M determined by the vector field ξ. Locally and for r sufficiently small, $M^{r,\xi}$ is always a parallel submanifold. Indeed, every $p \in M$ admits an open neighbourhood $\mathcal{U}$ where every normal vector can be extended to a parallel normal field. By restricting $\mathcal{U}$ further if necessary, we can assume that there is an $s > 0$ such that for all $r < s$ and for every parallel normal field ξ on $\mathcal{U}$, the set $\mathcal{U}^{r,\xi} = \{\exp(r\xi_p) : p \in \mathcal{U}\}$ is an embedded parallel submanifold of $\mathcal{U} \subset M$.

The submanifold M is said to be *almost isoparametric* [57] if its normal bundle νM is flat and if, locally, the parallel submanifolds of M have constant mean curvature in radial directions.

A submanifold M is said to admit sections if for any point $p \in M$ there is a totally geodesic submanifold Σ_p, called the *section* through p, such that $T_p\Sigma_p = \nu_p M$. Then, we say M is *isoparametric* (or more precisely, isoparametric according to Heintze, Liu and Olmos [57]) if it is almost isoparametric and admits sections. Throughout this thesis whenever we consider an isoparametric submanifold, we understand that it is isoparametric according to this definition.

In particular, for the codimension one case, the definition of isoparametric submanifold above simplifies. Let M be a hypersurface of $\bar{M}$. Then, the normal bundle of M is flat. Let ξ be a parallel unit normal vector field and $r > 0$ sufficiently small. We can consider the set $M^r = \{\exp(r\xi) : \xi \in \nu M\}$. If such a set is a hypersurface, then we call it an *equidistant hypersurface* to M. Locally and for r sufficiently small, M^r is always an equidistant hypersurface. Thus, a hypersurface in a Riemannian manifold $\bar{M}$ is isoparametric if, locally, it and its sufficiently close equidistant hypersurfaces have constant mean curvature. More information about isoparametric hypersurfaces can be found in [103].

A submanifold M is said to have *constant principal curvatures* if for any curve $\sigma\colon I \to M$ and any parallel unit normal vector field $\xi \in \Gamma(\sigma^*\nu^1 M)$ along σ the eigenvalues of the shape operator $\mathcal{S}_{\xi(t)}$ with respect to $\xi(t)$ are constant along σ. Then, M is called *Terng-isoparametric* (or isoparametric according to Terng [101]) if it has constant principal curvatures and flat normal bundle.

1.5 Isometric actions

In this section, we present the basic concepts and terminology for the study of isometric actions on Riemannian manifolds. More information can be found in [9, Chapter 3].

Let $\bar{M}$ be a Riemannian manifold and G a Lie group. We say that G acts isometrically on $\bar{M}$ if there exists a smooth map

$$\varphi\colon G \times \bar{M} \to \bar{M}, \qquad (g, p) \mapsto gp,$$

satisfying $(gg')p = g(g'p)$ for all g, $g' \in G$ and $p \in \bar{M}$, and such that the map

$$\varphi_g \colon \bar{M} \to \bar{M}, \qquad p \mapsto gp,$$

is an isometry of $\bar{M}$ for every $g \in G$. The map φ is called an *isometric action* on $\bar{M}$. If we denote by $I(\bar{M})$ the isometry group of $\bar{M}$, which is known to be a Lie group [87], then we have a Lie group homomorphism $\rho \colon G \to I(\bar{M})$ given by $\rho(g) = \varphi_g$. For each point $p \in \bar{M}$, the *orbit* of the action of G through p is

$$G \cdot p = \{gp : g \in G\}$$

and the *isotropy group* or *stabilizer* at p is

$$G_p = \{g \in G : gp = p\}.$$

If we consider another isometric action $\varphi' \colon G \times \bar{M}' \to \bar{M}'$, then φ and φ' are said to be *conjugate* or *equivalent* if there is a Lie group isomorphism $\psi \colon G \to G'$ and an isometry $f \colon \bar{M} \to \bar{M}'$ such that $f(gp) = \psi(g)f(p)$ for all $p \in \bar{M}$ and $g \in G$. We say that both isometric actions, φ and φ', are *orbit equivalent* if there is an isometry $f \colon \bar{M} \to \bar{M}'$ that maps the orbits of the G-action on $\bar{M}$ to the orbits of the G'-action on $\bar{M}'$. Clearly, two conjugate actions are orbit equivalent.

If $G \cdot p = \bar{M}$ for some $p \in \bar{M}$, and hence for each $p \in \bar{M}$, the G-action is said to be *transitive* and $\bar{M}$ is a *homogeneous G-space*. The action is said to be *trivial* if each point in $\bar{M}$ is a fixed point. An action is called *effective* if the associated map ρ above is injective, which means that G is isomorphic to a subgroup of $I(\bar{M})$. If for every $p \in \bar{M}$ and every g, $h \in G$, the equality $gp = hp$ implies $g = h$, then the action is *free*. If a G-action on $\bar{M}$ is free and transitive we say that G acts *simply transitively* on $\bar{M}$.

We will be mostly interested in studying the extrinsic geometry of the orbits of isometric actions. Let M be a submanifold of a Riemannian manifold $\bar{M}$. We say that M is an *(extrinsically) homogeneous submanifold* of $\bar{M}$ if M is an orbit of an isometric action on $\bar{M}$. In general, these submanifolds will only be immersed submanifolds of $\bar{M}$. With respect to the induced metric, each orbit $G \cdot p$ is a Riemannian homogeneous space $G \cdot p = G/G_p$, on which G acts transitively by isometries.

Each isometric action induces certain orthogonal representations in a natural way. Recall that a representation of a Lie group G on a vector space V is a Lie group homomorphism $\rho \colon G \to GL(V)$ or, equivalently, an action $G \times V \to V$ given by automorphisms of V; when V is a Euclidean space and the automorphisms $\rho(g)$, $g \in G$, are orthogonal transformations of V, we have an orthogonal representation $\rho \colon G \to O(V)$.

Let $\varphi \colon G \times \bar{M} \to \bar{M}$ be an isometric action on a Riemannian manifold $\bar{M}$, and let $p \in \bar{M}$. Since the isotropy group G_p fixes p and G_p leaves the orbit $G \cdot p$ invariant, the action of the isotropy group G_p acting on $T_p\bar{M}$ by

$$G_p \times T_p\bar{M} \to T_p\bar{M}, \qquad (g, X) \mapsto (\varphi_g)_{*p}X,$$

leaves the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$ invariant, where $(\varphi_g)_{*p}$ denotes the differential of φ_g at the point p. The restriction of this action to the tangent space to the orbit $T_p(G \cdot p)$

$$G_p \times T_p(G \cdot p) \to T_p(G \cdot p), \qquad (g, X) \mapsto (\varphi_g)_{*p} X,$$

is called the *isotropy representation* of the action φ at p, while the restriction to the normal space to the orbit $\nu_p(G \cdot p)$

$$G_p \times \nu_p(G \cdot p) \to \nu_p(G \cdot p), \qquad (g, \xi) \mapsto (\varphi_g)_{*p} \xi,$$

is called the *slice representation* of the action φ at p.

Let $\bar{M}/G$ be the set of orbits of the action of G on $\bar{M}$, and equip $\bar{M}/G$ with the quotient topology relative to the canonical projection $\bar{M} \to \bar{M}/G$, $p \mapsto G \cdot p$. In general, $\bar{M}/G$ is not a Hausdorff space. To avoid this behaviour, the so-called proper actions were introduced. Thus, the action of G on $\bar{M}$ is *proper* if the map

$$G \times \bar{M} \to \bar{M} \times \bar{M}, \qquad (g, p) \mapsto (p, gp),$$

is a proper map, i.e. the inverse image of each compact set in $\bar{M} \times \bar{M}$ is compact in $G \times \bar{M}$. Every compact Lie group action is proper. If $\bar{M}$ is a Riemannian manifold and G is a subgroup of $I(\bar{M})$, then the G-action is proper if and only if G is closed in $I(\bar{M})$. Moreover, if G acts properly on $\bar{M}$, then $\bar{M}/G$ is a Hausdorff space, each isotropy group G_p is compact, and each orbit $G \cdot p$ is closed in $\bar{M}$ and hence an embedded submanifold. In fact, the orbits of an isometric action are closed if and only if the action is orbit equivalent to a proper isometric action, see [36].

We can distinguish three different kinds of orbits of a proper isometric action: principal, exceptional and singular orbits. A *principal orbit* is an orbit $G \cdot p$ such that for each $q \in \bar{M}$ the isotropy group G_p at p is conjugate in G to some subgroup of G_q. The union of all principal orbits is a dense and open subset of $\bar{M}$ and any orbit $G \cdot p$ of a proper action is principal if and only if the slice representation at p is trivial. Each principal orbit is an orbit of maximal dimension. The codimension of any principal orbit is the *cohomogeneity* of the action. If an orbit has maximal dimension but is non-principal then it is called an *exceptional orbit*. Finally, a *singular orbit* is an orbit whose dimension is less than the dimension of a principal orbit or, equivalently, an orbit whose codimension is greater than the cohomogeneity.

If $G \cdot p$ is a principal orbit and $\xi \in \nu_p(G \cdot p)$, define $\tilde{\xi}_{gp} = g \cdot \xi$ for all $g \in G$. The vector field $\tilde{\xi}$ is a well-defined normal vector field on $G \cdot p$ called the *equivariant normal vector field determined by* ξ. Hence, if $G \cdot p$ is a principal orbit and $\xi_1, \cdots, \xi_k$ is an orthonormal basis of $\nu_p(G \cdot p)$, then $\tilde{\xi}_1, \cdots, \tilde{\xi}_k$ is a global smooth orthonormal frame of the normal bundle of $G \cdot p$ (thus, the normal bundle of a principal orbit is flat and equivariant vector fields are parallel [9, Corollary 3.2.5]). Moreover, from a given principal orbit $G \cdot p$, one can determine all nearby orbits by using equivariant normal vector fields of $G \cdot p$. Since $\exp_{gp}(\tilde{\xi}_{gp}) = \exp_{gp}(g \cdot \tilde{\xi}_p) = g \exp(\tilde{\xi}_p)$ then $M_\xi = \{\exp_q(\tilde{\xi}_q) : q \in G \cdot p\} = G \cdot \exp_p \tilde{\xi}_p$, that is, M_ξ is the orbit through $\exp_p(\tilde{\xi}_p)$.

An important kind of isometric actions are polar actions. Let H be a connected closed subgroup of $I(\bar{M})$. The standard action of H on $\bar{M}$ is called *polar* if there exists an immersed submanifold Σ of $\bar{M}$ such that:

1. Σ intersects all the orbits of the H-action, and

2. for each $p \in \Sigma$, the tangent space of Σ at p, $T_p\Sigma$, and the tangent space of the orbit through p at p, $T_p(H \cdot p)$, are orthogonal.

In such a case, the submanifold Σ is called a *section* of the H-action, which is always a totally geodesic submanifold of $\bar{M}$. In this case, the orbit foliation of $\bar{M}$ by the polar action is a polar foliation. Any polar action admits sections through any given point. The action of H is called *hyperpolar* if the section Σ is flat in its induced Riemannian metric. Every cohomogeneity one action is automatically hyperpolar.

1.6 Complex space forms

This thesis is devoted to the study of geometric objects in two families of symmetric spaces of rank one: complex projective and hyperbolic spaces. Thus, in this section we present their construction and some properties of these two spaces. The reader is referred to [46], [48], [88] and [25] for more information about this topic. At the end of this section, we describe some important classes of submanifolds in Kähler manifolds that will be important to our classifications.

Firstly, let us recall some definitions concerning complex, Hermitian and Kähler manifolds. See [109] for more details and proofs.

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. In this work, an orthogonal transformation J of V such that $J^2 = -\operatorname{Id}$ will be known a *complex structure* on the vector space V . Thus, any endomorphism J of V is a complex structure if and only if the following properties are satisfied: (i) $\langle Jv, Jw \rangle = \langle v, w \rangle$ for all v, $w \in V$, that is, $J \in O(V)$, and (ii) $J^2 = -\operatorname{Id}$ for all v, $w \in V$, which implies that $\langle Jv, w \rangle = -\langle v, Jw \rangle$, that is, $J \in \mathfrak{so}(V)$.

A *complex manifold* is a manifold that admits charts with their image onto open subsets of $\mathbb{C}^n$ and such that the coordinate transitions are holomorphic. This induces an almost complex structure J on M, i.e. an endomorphism of the tangent bundle of M such that $J^2 = -\operatorname{Id}$. Then, M is called a *Hermitian manifold* if M is Riemannian and complex, and the complex structure J is orthogonal (equivalently, J restricts to a complex structure of each tangent space T_pM, $p \in M$). A *Kähler manifold* is a Hermitian manifold M satisfying $\nabla J = 0$, where ∇ is the Levi-Civita connection of M. The endomorphism J is known as the *Kähler structure* or the *complex structure* of M.

In Section 1.1 we have said that the only connected, simply connected Riemannian manifolds of constant curvature are the real space forms. However, in Kähler geometry, spaces of constant curvature are not very relevant because Kähler manifolds of constant curvature and dimension greater than two are necessarily flat. Thus, for Kähler manifolds a new concept is introduced. Let $\bar{M}$ be a Kähler manifold with complex structure J and curvature tensor $\bar{R}$. The restriction of the sectional curvature $\bar{K}$ to J-invariant 2-dimensional subspaces of the tangent space is called the *holomorphic sectional curvature* $\bar{K}_{hol}$ of $\bar{M}$. Since these subspaces are generated by pairs of the form $\{v, Jv\}$, with $v \in T_p\bar{M}$, $p \in \bar{M}$, $\bar{K}_{hol}$ can be regarded as a function that maps each unit tangent vector $v \in T\bar{M}$ to the real number $\bar{K}_{hol}(v) = \bar{K}(v, Jv) = \langle \bar{R}(v, Jv)Jv, v\rangle$.

A Kähler manifold is said to have *constant holomorphic curvature* if $\bar{K}_{hol}$ is constant for any unit tangent vector of $\bar{M}$. A complete, simply-connected Kähler manifold of constant holomorphic curvature c is isometric to one of the following spaces: a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, a complex projective space $\mathbb{C}P^n$ if $c > 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. These are the so-called *complex space forms*. Furthermore, if $\bar{M}$ has constant holomorphic curvature c then its curvature tensor can be written as

$$\bar{R}(X,Y)Z = \frac{c}{4}\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ\right).$$

In this work we will focus on complex projective and hyperbolic spaces. Thus we give a description of their construction.

1.6.1 Irreducible semisimple Hermitian symmetric spaces

Complex projective and hyperbolic spaces are precisely the irreducible semisimple Hermitian symmetric spaces of rank one. Symmetric spaces constitute a particularly nice class of homogeneous spaces. Here we give a quick introduction to this topic. Basic facts about these spaces and more information can be found in the standard references [59], [80], [81] and [110].

Let M be a Riemannian manifold. Let $o \in M$. Take $r > 0$ sufficiently small so that normal coordinates are defined on the open ball $B_r(o)$. We define the local geodesic symmetry at o as the map $s_o \colon B_r(o) \to B_r(o)$ given by $s_o(\exp_o(v)) = \exp_o(-v)$ for $v \in T_oM$, $\|v\| < r$. In general, this map is defined only locally. A connected Riemannian manifold M is called a *(Riemannian) symmetric space* if each local geodesic symmetry s_o can be extended to a global isometry $s_o \colon M \to M$.

Let M be a symmetric space and $\widetilde{M}$ its universal covering. Then the De Rham theorem guarantees that $\widetilde{M}$ can be decomposed as $\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k$. Here $\widetilde{M}_0$ is the Euclidean factor, that is, $\widetilde{M}_0$ is isometric to a Euclidean space, and each $\widetilde{M}_i$, $i = 1, \ldots, k$, is a simply connected, irreducible symmetric space. Recall that M is *irreducible* if and only if its universal covering $\widetilde{M}$ is irreducible, that is, $\widetilde{M}$ does not split as a product of manifolds, and is *reducible* otherwise.

A *semisimple* symmetric space is one for which the Euclidean factor of its universal covering space has dimension zero. In this case, the Lie algebra of the isometry group of $\widetilde{M}$ is semisimple. A semisimple symmetric space is said to be of *compact type* if all the De Rham factors of its universal covering are compact. It is said to be of *noncompact type* if all the De Rham factors of its universal covering are non-Euclidean and noncompact.

Simply connected, irreducible Riemannian symmetric spaces have been classified by Cartan. One can find a list of them in [59, p. 515–520].

The *rank* of a symmetric space M is, by definition, the dimension of a maximal flat, totally geodesic submanifold of M.

A symmetric space M is *Hermitian* if it is a Hermitian manifold and the geodesic symmetries s_p, $p \in M$, are holomorphic transformations. It occurs that every Hermitian symmetric space is Kähler.

The classification of the irreducible semisimple Hermitian symmetric spaces is shown in the following Table 1.1.

Compact	Noncompact	Dimension	Rank
$SU(p+q)/S(U(p)U(q))$	$SU(p,q)/S(U(p)U(q))$	$2pq$	$min\{p,q\}$
$SO(2+q)/SO(2)SO(q)$	$SO^o(2,q)/SO(2)SO(q)$	$2q$	$min\{2,q\}$
$SO(2n)/U(n)$	$SO^*(2n)/U(n)$	$n(n-1)$	$[n/2]$
$Sp(n)/U(n)$	$Sp(n,\mathbb{R})/U(n)$	$n(n+1)$	n
$E_6/U(1)\cdot Spin(10)$	$E_6^{-14}/U(1)\cdot Spin(10)$	32	2
$E_7/U(1)\cdot E_6$	$E_7^{-25}/U(1)\cdot E_6$	54	3

Table 1.1: Irreducible semisimple Hermitian symmetric spaces

The first row of Table 1.1, taking $p = 1$, contains the simply connected rank one irreducible semisimple Hermitian symmetric spaces. These are precisely the nonflat complex space forms: complex projective spaces $\mathbb{C}P^n$ and complex hyperbolic spaces $\mathbb{C}H^n$. In what follows we will denote by $\bar{M}^n(c)$ an n-dimensional complex space form of constant holomorphic sectional curvature $c \neq 0$. Thus, $\bar{M}^n(c)$ is a complex projective space $\mathbb{C}P^n$ if $c > 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. Now we will give a more detailed description of these spaces.

1.6.2 The complex projective space $\mathbb{C}P^n$

The complex projective space of complex dimension n (real dimension $2n$) is defined as the space of complex lines of $\mathbb{C}^{n+1}$ through the origin, or equivalently, as the quotient manifold of a sphere, $S^{2n+1}(r)/\sim$, by the equivalence relation given by $z \sim \lambda z$, $z \in S^{2n+1}(r)$, $\lambda \in S^1 \subset \mathbb{C}$. We denote by π the canonical projection of $\sim$ from the sphere $S^{2n+1}(r)$ onto the complex projective space, $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$. Then π is a smooth surjective submersion called the *Hopf map*. We will consider the metric on $\mathbb{C}P^n$ that makes $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$ a Riemannian submersion.

In what follows, we give a more detailed description of the construction presented above. Consider a complex structure J on $\mathbb{R}^{2n+2}$, which allows us to identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, where the multiplication by the imaginary unit i is induced by J. Now we consider the scalar product on $\mathbb{C}^{n+1}$ given by

$$\langle z, w\rangle = \mathrm{Re}\left(\sum_{k=0}^{n} z_k \bar{w}_k\right),$$

for each $z = (z_0, z_1, \dots, z_n)$, $w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1}$. This scalar product yields the standard Euclidean metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n+2}$.

The $(2n+1)$-dimensional sphere of radius r is $S^{2n+1}(r) = \{z \in \mathbb{C}^{n+1} : \langle z, z\rangle = r^2\}$. Its tangent space at $z \in S^{2n+1}(r)$ is $T_z S^{2n+1}(r) = \{w \in \mathbb{C}^{n+1} : \langle z, w\rangle = 0\}$.

The restriction of the above inner product yields a Riemannian metric of constant sectional curvature $1/r^2$ on $S^{2n+1}(r)$. A unit normal vector field ξ along $S^{2n+1}(r)$ is given by $\xi_z = \frac{1}{r} z$.

We consider the equivalence relation on $S^{2n+1}(r)$ generated by $z \sim \lambda z$ with $\lambda \in S^1 \subset \mathbb{C}$. This defines a principal fiber bundle over $\mathbb{C}P^n$ with total space $S^{2n+1}(r)$, fiber S^1 and projection map $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$.

Define $V = J\xi$. Obviously, V is a unit tangent vector field to $S^{2n+1}(r)$ and we can write

$$TS^{2n+1}(r) = \mathbb{R}V \oplus V^{\perp},$$

where $V^{\perp}$ is the orthogonal complement of V with respect to the metric on $S^{2n+1}(r)$. Actually, if $z \in S^{2n+1}(r)$, then $\mathbb{R}V_z$ is the kernel of π_{*z}, where π_* denotes the differential of π. Hence, π_{*z} maps $V_z^{\perp}$ isomorphically onto $T_{\pi(z)}\mathbb{C}P^n$, and for each $X \in T_{\pi(z)}\mathbb{C}P^n$ we can define the horizontal lift X_z^L of X to z as the unique tangent vector in $V_z^{\perp}$ such that $\pi_* X_z^L = X$. The map $t \mapsto \varphi_t(z) = e^{it} z$ is exactly the geodesic on $S^{2n+1}(r)$ starting at z with initial speed $Jz = iz = rV_z$. We have $\pi \circ \varphi_t = \pi$, and thus $X^L_{\varphi_t(z)} = (\varphi_t)_{*z} X_z^L$.

The complex structure J on $\mathbb{C}P^n$ is then defined by

$$JX = \pi_*(JX^L)$$

for each $X \in T\mathbb{C}P^n$, whereas the metric on $\mathbb{C}P^n$ is given by

$$\langle X, Y\rangle = \left\langle X^L, Y^L \right\rangle$$

for all X, $Y \in T\mathbb{C}P^n$. This metric, called the *Fubini-Study metric* of $\mathbb{C}P^n$, makes $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$ a Riemannian submersion. It also satisfies $\langle JX, JY\rangle = \langle X, Y\rangle$ for any tangent vectors X and Y. By virtue of the formulas for Riemannian submersions [90], the Levi-Civita connection of $\mathbb{C}P^n$ is given by

$$\bar{\nabla}_X Y = \pi_*\left(\widetilde{\nabla}_{X^L} Y^L\right),$$

for tangent vector fields X, Y on $\mathbb{C}P^n$. Using this formula one can show that J is Kähler.

The theory of semi-Riemannian submersions [90] also allows to calculate the holomorphic sectional curvature of $\mathbb{C}P^n$, which turns out to be $\bar{K}_{hol}(X) = 4/r^2$ for every unit $X \in T\mathbb{C}P^n$. Therefore, $\mathbb{C}P^n$ is a space of constant holomorphic curvature $c = 4/r^2$.

The unitary group $U(n+1) = \{A \in GL(n+1, \mathbb{C}) : AA^* = \mathrm{Id}\}$, where A^* denotes the conjugate transpose matrix of A, $A^* = \bar{A}^T$, preserves the standard metric of $\mathbb{R}^{2n+2} \equiv \mathbb{C}^{n+2}$. Since it preserves complex lines through the origin of $\mathbb{C}^{n+1}$ and acts transitively on them, $U(n+1)$ acts transitively by isometries on $\mathbb{C}P^n$ by $A(p) = \pi(Az)$ where $p = \pi(z) \in \mathbb{C}P^n$, and $A \in U(n+1)$. However, the action is not effective, as all transformations of the form $z\,\mathrm{Id}$ with $|z| = 1$, act trivially on $\mathbb{C}P^n$. The subgroup $SU(n+1)$ of those matrices in $U(n+1)$ with determinant one keeps acting transitively on $\mathbb{C}P^n$ but with finite kernel constituted by the matrices $z\,\mathrm{Id}$ with z an $(n+1)$-th root of the unit.

Therefore $\mathbb{C}P^n$ is a homogeneous space. The isotropy group at, for example, the point $p = \pi(r, 0, \ldots, 0) \in \mathbb{C}P^n$ is $S(U(1)U(n))$, which is isomorphic to $U(n)$. Thus, the complex projective space turns out to be the Hermitian symmetric space of rank one given by

$$\mathbb{C}P^n = SU(n+1)/S(U(1)U(n)).$$

The fact that $\mathbb{C}P^n$ has rank one follows, for instance, from the following classification of totally geodesic submanifolds, which implies that any totally geodesic, flat submanifold of maximal dimension in $\mathbb{C}P^n$ is a geodesic [107].

Theorem 1.1. *Let M be a totally geodesic submanifold of $\mathbb{C}P^n$. Then M is holomorphically congruent to an open part of a real projective space $\mathbb{R}P^k$ for some $k \in \{1, \ldots, n\}$ or to a complex projective space $\mathbb{C}P^k$ for some $k \in \{0, \ldots, n\}$. Any two totally geodesic submanifolds of $\mathbb{C}P^n$ are locally holomorphically congruent to each other if and only if they are locally isometric.*

1.6.3 The complex hyperbolic space $\mathbb{C}H^n$

The complex hyperbolic space, as a smooth manifold, is the quotient manifold $\mathbb{C}H^n = H_1^{2n+1}(r)/\sim$ of an anti-De Sitter space $H_1^{2n+1}(r)$ by the equivalence relation given by $z \sim \lambda z$, $z \in H_1^{2n+1}(r)$, $\lambda \in S^1 \subset \mathbb{C}$. Equivalently, $\mathbb{C}H^n$ is the space of timelike complex lines through the origin of $\mathbb{C}^{n+1}$. The canonical projection is denoted by $\pi\colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ and is called the *Hopf map* of $\mathbb{C}H^n$. As a Riemannian manifold, the metric on $\mathbb{C}H^n$ will be induced by the metric on the anti-De Sitter space through the map π.

Remark 1.2. Anti-De Sitter spaces are Lorentzian manifolds of constant negative sectional curvature. Then, in order to construct the complex hyperbolic space we will need the theory of semi-Riemannian submanifolds. The theory of Riemannian submanifolds presented in Section 1.2 can be adapted to the semi-Riemannian setting with minor changes as follows.

Let $\bar{M}$ be a semi-Riemannian manifold and assume now that M is a semi-Riemannian hypersurface of $\bar{M}$, that is, an embedded Riemannian submanifold of codimension one and with nondegenerate induced metric. Then, locally and up to sign, there exists a unique unit normal vector field $\xi \in \Gamma(\nu M)$, such that $\langle \xi, \xi \rangle = \varepsilon \in \{1, -1\}$. Hence the second fundamental form $I\!I$ is a multiple of ξ.

We will denote by $\mathcal{S} = \mathcal{S}_\xi$ the shape operator with respect to ξ. The Gauss and

Weingarten formulas can now be written as

$$\bar{\nabla}_X Y = \nabla_X Y + \varepsilon\langle \mathcal{S}X, Y\rangle\xi,$$
$$\bar{\nabla}_X \xi = -\mathcal{S}X.$$

The construction of the complex hyperbolic space is very close to the construction of the complex projective space. Hence, in this subsection we summarize the basic facts of this construction, following the steps in the description of §1.6.2. However, we will see that their geometries turn out to be very different.

As above, take a complex structure J on $\mathbb{R}^{2n+2}$, and identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$. Now we consider the scalar product on $\mathbb{C}^{n+1}$ given by

$$\langle z, w\rangle = \operatorname{Re}\left(-z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k\right),$$

for each $z = (z_0, z_1, \dots, z_n)$, $w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1}$. This scalar product does no longer induce the usual inner product of $\mathbb{R}^{2n+2}$, but a standard semi-Riemannian metric of signature $(2, 2n)$. Hence, we consider the *anti-De Sitter space* (of radius r), which can be regarded as the Lorentzian analogue of the real hyperbolic space, and that is defined as $H_1^{2n+1}(r) = \{z \in \mathbb{C}^{n+1} : \langle z, z\rangle = -r^2\}$. Its tangent space at $z \in H_1^{2n+1}(r)$ is $T_zH_1^{2n+1}(r) = \{w \in \mathbb{C}^{n+1} : \langle z, w\rangle = 0\}$. The restriction of the above inner product to H_1^{2n+1} yields a Lorentzian metric of constant sectional curvature $-1/r^2$ on $H_1^{2n+1}(r)$. A unit normal vector field ξ along $H_1^{2n+1}(r)$ is given by $\xi_z = \frac{1}{r}z$, but in this case it satisfies $\langle \xi, \xi\rangle = -1$.

We define the equivalence relation on $H_1^{2n+1}(r)$ given by $z \sim \lambda z$ with $\lambda \in S^1 \subset \mathbb{C}$. This defines a principal fiber bundle over $\mathbb{C}H^n$ with total space H_1^{2n+1}, fiber S^1 and projection map $\pi\colon H_1^{2n+1}(r) \to \mathbb{C}H^n$. Note that, as smooth manifolds, we have that $\mathbb{C}H^n \subset \mathbb{C}P^n$.

Define $V = J\xi$. Then V is a unit tangent vector field to $H_1^{2n+1}(r)$, where now "unit" means, similarly as for ξ, that $\langle V, V\rangle = -1$. Hence, both ξ and V are timelike vector fields. We can write

$$TH_1^{2n+1}(r) = \mathbb{R}V \oplus V^\perp,$$

where $V^\perp$ is the orthogonal complement of V with respect to the Lorentzian metric on $H_1^{2n+1}(r)$. Actually, if $z \in H^{2n+1}(r)$, then $\mathbb{R}V_z$ is the kernel of π_{*z}, where π_* denotes the differential of π. Hence, π_{*z} maps $V_z^\perp$ isomorphically onto $T_{\pi(z)}\mathbb{C}H^n$, and for each $X \in T_{\pi(z)}\mathbb{C}H^n$ we can define the horizontal lift X_z^L of X to z as the unique tangent vector in $V_z^\perp$ such that $\pi_* X_z^L = X$. The map $t \mapsto \varphi_t(z) = e^{it}z$ is exactly the geodesic on $H^{2n+1}(r)$ starting at z with initial speed $Jz = iz = rV_z$. We have $\pi \circ \varphi_t = \pi$, and thus $X^L_{\varphi_t(z)} = (\varphi_t)_{*z}X_z^L$.

The complex structure J on $\mathbb{C}H^n$ is then defined by

$$JX = \pi_*(JX^L)$$

for each $X \in T\mathbb{C}H^n$, whereas the metric on $\mathbb{C}H^n$ is given by

$$\langle X, Y\rangle = \left\langle X^L, Y^L\right\rangle$$

for all X, $Y \in T\mathbb{C}H^n$. An important point here is the fact that the metric of $H_1^{2n+1}(r)$ is positive definite on $V_z^\perp$ and, hence, the metric on $\mathbb{C}H^n$ is positive definite. Thus $\mathbb{C}H^n$ becomes a Riemannian manifold. This metric, called the *Bergman metric* of $\mathbb{C}H^n$, makes $\pi\colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ a semi-Riemannian submersion. Moreover, the Bergman metric is Hermitian, i.e. it satisfies $\langle JX, JY\rangle = \langle X, Y\rangle$ for any tangent vectors X and Y. By virtue of the formulas for semi-Riemannian submersions (see [90] or [91, p. 213]), the Levi-Civita connection of $\mathbb{C}H^n$ is given by

$$\bar{\nabla}_X Y = \pi_* \left(\widetilde{\nabla}_{X^L} Y^L\right),$$

for tangent vector fields X, Y on $\mathbb{C}H^n$. Using this formula one can show that J is Kähler.

Again, the theory of semi-Riemannian submersions allows to calculate the holomorphic sectional curvature of $\mathbb{C}H^n$, which turns out to be $\bar{K}_{hol}(X) = -4/r^2$ for every $X \in T\mathbb{C}H^n$. Therefore, $\mathbb{C}H^n$ is a space of constant holomorphic sectional curvature $c = -4/r^2$.

The indefinite unitary group $U(1,n) = \{A \in GL(n,\mathbb{C}) : AI_{1,n}A^* = I_{1,n}\}$, where A^* denotes conjugate transpose and $I_{1,n}$ is the diagonal matrix $\operatorname{diag}(-1, 1, \ldots, 1)$, leaves invariant the metric of $\mathbb{R}^{2n+2} \equiv \mathbb{C}^{n+2}$ with signature $(2, 2n)$ considered above. It also preserves timelike complex lines through the origin of $\mathbb{C}^{n+1}$ and acts transitively on them. Then it follows that $U(1,n)$ acts transitively by isometries on $\mathbb{C}H^n$ and, like with $\mathbb{C}P^n$, we can restrict to $SU(1,n)$, the group of the matrices of $U(1,n)$ with determinant one, which still acts transitively on $\mathbb{C}H^n$. This shows that $\mathbb{C}H^n$ is a homogeneous space. Even more, the complex hyperbolic space is a Hermitian symmetric space that has the following expression as coset space:

$$\mathbb{C}H^n = SU(1,n)/S(U(1)U(n)).$$

The following result completely explains both the intrinsic and extrinsic geometry of totally geodesic submanifolds of $\mathbb{C}H^n$, and implies that complex hyperbolic spaces have rank one as symmetric spaces. Note the analogy with Theorem 1.1, which can be obtained using duality of symmetric spaces (cf. [9, §9.1]).

Theorem 1.3. *Let M be a totally geodesic submanifold of $\mathbb{C}H^n$. Then M is holomorphically congruent to an open part of a real hyperbolic space $\mathbb{R}H^k$ for some $k \in \{1, \ldots, n\}$ or to a complex hyperbolic space $\mathbb{C}H^k$ for some $k \in \{0, \ldots, n\}$. Any two totally geodesic submanifolds of $\mathbb{C}H^n$ are locally holomorphically congruent to each other if and only if they are locally isometric.*

1.6.4 The complex hyperbolic space as a symmetric space and as a solvable Lie group

In Subsection 1.6.1 we have said that the simply connected rank one irreducible semisimple Hermitian symmetric spaces of noncompact type are the complex hyperbolic spaces $\mathbb{C}H^n$. Every symmetric space of noncompact type is a solvable Lie group and its metric is left-invariant with respect to the Lie group structure (the proof of this fact is analogous to the one we sketch below for the case of $\mathbb{C}H^n$). Thus, in this subsection, we will give an explicit description of $\mathbb{C}H^n$ as a symmetric space and as a solvable Lie group AN equipped with a left-invariant metric. Our exposition will make use of the Iwasawa descomposition of a real semisimple Lie algebra. A complete description of the Iwasawa descomposition of the Lie algebra of the isometry group of $\mathbb{C}H^n$ appears in [46, Chapter 2]; see also [73, §6.4]. Here, we will give the construction without providing the proofs.

Now, let us consider some basic notation. We denote the Lie algebra of a Lie group G with the corresponding gothic letter, in this case, $\mathfrak{g}$. Exp will be the notation for the Lie exponential map. The map $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$, $g \to (I_g)_*$ is the Lie group adjoint map, where $\mathrm{Aut}(\mathfrak{g})$ is the group of automorphisms of the Lie algebra $\mathfrak{g}$, i.e. those linear transformations $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ such that $\varphi[X,Y] = [\varphi X, \varphi Y]$ for all X, $Y \in \mathfrak{g}$, and $(I_g)_*$ is the differential at the identity element $e \in G$ of the conjugation map $I_g\colon G \to G$, $h \mapsto ghg^{-1}$. The differential of Ad at e yields the Lie algebra adjoint map $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $X \mapsto \mathrm{ad}(X) = [X, \cdot\,]$. The bilinear form $\mathcal{B} = \mathcal{B}_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X,Y) \mapsto \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$ is the Killing form of the real Lie algebra $\mathfrak{g}$.

As we have seen, $\mathbb{C}H^n$ is a rank one Hermitian symmetric space of noncompact type. From now on we denote by $G = SU(1,n)$ the identity connected component of the isometry group $I(\mathbb{C}H^n)$ and by $\mathfrak{g}$ the Lie algebra of G. Let $o \in \mathbb{C}H^n$ and let s_o be the geodesic symmetry at o. The isotropy group of G at o is denoted by K, that is, $K = G_o = S(U(1)U(n))$, which is compact. The coset space G/K is diffeomorphic to $\mathbb{C}H^n$ by means of the map $\Phi\colon G/K \to \mathbb{C}H^n$, $gK \mapsto g(o)$. Φ is an isometry and the metric $\langle \cdot, \cdot \rangle$ is G-invariant, that is, the map $gK \to hgK$ is an isometry for each $h \in G$, where $\langle \cdot, \cdot \rangle$ denotes the metric obtained by pulling back the metric of $\mathbb{C}H^n$. The *isotropy representation* of the symmetric space $\mathbb{C}H^n \cong G/K$ at o is the orthogonal representation defined by $K \times T_o\mathbb{C}H^n \to T_o\mathbb{C}H^n$, $(k,v) \mapsto k_*v$. The isotropy representation of a semisimple symmetric space is called an *s-representation.*

The map $\sigma\colon G \to G$, $g \mapsto s_o g s_o$, is an involutive automorphism of G, and $G_\sigma^0 \subset K \subset G_\sigma$, where s_o is the local geodesic symmetry at o, $G_\sigma = \{g \in G : \sigma(g) = g\}$, and G_σ^0 is the connected component of the identity of G_σ . Let θ be the differential of σ at the identity. The Lie algebra of K is given by $\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}$, and we define $\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}$. Let $\mathcal{B}$ be the Killing form of $\mathfrak{g}$. Since G/K is of noncompact type, this is equivalent to the fact that $\mathcal{B}|_{\mathfrak{p}}$ is positive definite. The space $\mathfrak{p}$ may be identified with $T_o\mathbb{C}H^n$ by using the map Φ and taking into account that $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\mathcal{B}$. Thus, $\mathfrak{p}$ inherits an inner product from $T_o\mathbb{C}H^n$ which turns out to be $\mathrm{Ad}(K)$-invariant. In fact, the isotropy representation of G/K is equivalent to the adjoint representation of K on $\mathfrak{p}$, $K \times \mathfrak{p} \to \mathfrak{p}$, $(k,X) \mapsto \mathrm{Ad}(k)X$. Moreover, we have the Lie bracket relations $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to the involution θ (or the point $o \in \mathbb{C}H^n$), and θ is called the *Cartan involution.*

The Cartan involution θ is defined by $\theta(X) = X$ for all $X \in \mathfrak{k}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$. It turns out that $\mathcal{B}_\theta(X,Y) = -\mathcal{B}(\theta X, Y)$ defines a positive definite inner product on $\mathfrak{g}$ satisfying the relation $\mathcal{B}_\theta(\mathrm{ad}(X)Y, Z) = -\mathcal{B}_\theta(Y, \mathrm{ad}(\theta X)Y)$ for all X, Y, $Z \in \mathfrak{g}$.

We consider now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Recall that we defined the rank of a symmetric space $G/K = M$ as the dimension of a maximal flat, totally geodesic submanifold of M. The isotropy representation of a semisimple symmetric space M is a polar action on the Euclidean space $T_oM \cong \mathfrak{p}$, and its cohomogeneity is precisely the rank of M. In fact, any maximal abelian subspace of $\mathfrak{p}$ is a section of this representation. Then, the definition of the rank of a symmetric space $G/K = M$ is equivalent to defining the rank as the dimension of a maximal abelian subspace of $\mathfrak{p}$. Thus, since $G/K = \mathbb{C}H^n$ is a symmetric space of rank one, the dimension of $\mathfrak{a}$ is one.

The set $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ is a family of commuting self-adjoint endomorphisms of $\mathfrak{g}$ with respect to the inner product $\mathcal{B}_\theta$, and hence simultaneously diagonalizable with real eigenvalues. Denoting by $\mathfrak{a}^*$ the dual vector space of $\mathfrak{a}$, we define for each $\lambda \in \mathfrak{a}^*$

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$, these are called the *restricted roots* and the *restricted root spaces* of the simple Lie algebra $\mathfrak{g}$, respectively. These restricted root spaces provide us the following descomposition, which is called the *restricted root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$$

where α is a certain covector, $\alpha \in \mathfrak{a}^*$. In particular, -2α, $-\alpha$, α and 2α are precisely the roots of $\mathfrak{g}$ and $\mathfrak{g}_{-2\alpha}$, $\mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ are the root spaces, which are $\mathcal{B}_\theta$-orthogonal subspaces. Moreover, $\mathfrak{a} \subset \mathfrak{g}_0$, and for every $\lambda, \mu \in \mathfrak{a}^*$, we have that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ and $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$.

In our case, $\mathfrak{g} = \mathfrak{su}(1, n) = \{A \in \mathfrak{gl}(n, \mathbb{C}) : AI_{1,n} + I_{1,n}A^* = 0, \quad \mathrm{tr}(A) = 0\}$, where $\mathrm{tr}(A)$ denotes the trace of the matrix A. Then, one can obtain the root spaces making matrix calculations and it can be shown that $\dim \mathfrak{g}_{2\alpha} = \dim \mathfrak{g}_{-2\alpha} = \dim \mathfrak{a} = 1$ and $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 2n-2$. Furthermore, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k} \cong \mathfrak{u}(n-1)$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$. The root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ are both normalized by $\mathfrak{k}_0$.

We now fix the following criterion of positivity in the set of roots: we will say that α is a positive root. We define the subset of the set of roots $\Sigma^+ = \{\alpha, 2\alpha\}$ as the set of positive roots. We also define $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ as the sum of the root spaces corresponding to all positive roots. Due to the properties of the root space decomposition, $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ with center $\mathfrak{g}_{2\alpha}$; in fact $\mathfrak{n}$ is isomorphic to the $(2n-1)$-dimensional Heisenberg algebra (see [18, Chapter 3] for a description of generalized Heisenberg algebras). Then $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, since $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$ is nilpotent.

The vector space direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is called the *Iwasawa decomposition* of the semisimple Lie algebra $\mathfrak{g}$. It is important to mention that this descomposition of $\mathfrak{g}$ is not an orthogonal decomposition ($\mathfrak{k}$ and $\mathfrak{n}$ are not orthogonal spaces), or a direct sum of Lie algebras (for example, $[\mathfrak{a}, \mathfrak{n}] \neq 0$), in spite of the fact that $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$. We would also like to mention that $\mathfrak{a} \oplus \mathfrak{n}$ is a semidirect sum of Lie algebras.

Let A, N and AN be the connected subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$, respectively. The Iwasawa decomposition theorem at the Lie group level ensures that the product map $(k, a, m) \in K \times A \times N \mapsto kam \in G$ is a diffeomorphism. Again, it is important to mention that $K \times A \times N$ is a product of smooth manifolds but not a direct product of

Lie groups. It follows from the Iwasawa decomposition that the solvable group AN acts simply transitively on $\mathbb{C}H^n$.

Consider now the differentiable map

$$\phi\colon h \in G \mapsto h(o) \in \mathbb{C}H^n.$$

Note that $\ker \phi_* = \mathfrak{k}$. Since AN acts simply transitively on $\mathbb{C}H^n$, the map $\phi|_{AN}\colon AN \to \mathbb{C}H^n$ is a diffeomorphism, and one can identify $\mathfrak{a} \oplus \mathfrak{n}$ with the tangent space $T_o\mathbb{C}H^n$ via ϕ_*. The Bergman metric g of the complex hyperbolic space $\mathbb{C}H^n$ induces a metric $\phi^* g$ on AN. The Riemannian manifolds $(AN, \phi^* g)$ and $(\mathbb{C}H^n, g)$ are then trivially isometric. Let us denote by L_h the left translation in G by the element $h \in G$. As the metric g on $\mathbb{C}H^n$ is invariant under isometries (and then under elements of G), it follows that

$$L_h^*(\phi^* g) = L_h^* \phi^* (h^{-1})^* g = (h^{-1} \circ \phi \circ L_h)^* g = \phi^* g, \quad \text{for all } h \in G,$$

because $(h^{-1} \circ \phi \circ L_h)(h') = h^{-1}(hh'(o)) = h'(o) = \phi(h')$ for all $h' \in G$. Therefore the metric $\phi^* g$ on AN is left-invariant. From now on, we will denote by $\langle \cdot, \cdot \rangle$ the metric $\phi^* g$ on AN. Thus, we have obtained that $\mathbb{C}H^n$ can be seen as a solvable Lie group AN endowed with a left-invariant metric.

The Lie group AN can be equiped with a Kähler structure induced by the Kähler structure in $\mathbb{C}H^n$ via $\phi|_{AN}$. Then, we obtain a complex structure that we will denote by J on AN and also on $\mathfrak{a} \oplus \mathfrak{n}$. It can be proved using matricial calculations that the complex structure on $\mathfrak{a} \oplus \mathfrak{n}$ leaves $\mathfrak{g}_\alpha$ invariant and $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$.

Thus, we have obtained a model for the complex hyperbolic space $\mathbb{C}H^n$ as a solvable Lie group AN endowed with a left-invariant Riemannian metric whose Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ can be identified with the tangent space $T_o\mathbb{C}H^n$, and such that $\mathfrak{g}_\alpha$ can be seen as a complex vector space $\mathbb{C}^{n-1}$.

Let $B \in \mathfrak{a}$ be a vector such that $\langle B, B \rangle = 1$ and define $Z = JB \in \mathfrak{g}_{2\alpha}$. Then $\langle Z, Z \rangle = 1$. Let now a, b, x, y be real numbers and U, $V \in \mathfrak{g}_\alpha$. One can show that the Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$ is given by

$$[B, Z] = \sqrt{-c}\, Z, \quad 2[B, U] = \sqrt{-c}\, U, \quad [U, V] = \sqrt{-c}\, \langle JU, V \rangle Z,$$

where c is the constant holomorphic sectional curvature of $\mathbb{C}H^n$. Furthermore, the Levi-Civita connection $\bar{\nabla}$ of $(AN, \langle \cdot, \cdot \rangle)$ can be calculated by the expression (cf. [12, §2]):

$$(1.1)\quad \begin{aligned} \frac{1}{\sqrt{-c}} \bar{\nabla}_{aB+U+xZ}(bB + V + yZ) = {} & \left(xy + \frac{1}{2}\langle U, V \rangle \right) B - \frac{1}{2}(bU + yJU + xJV) \\ & + \left(-bx + \frac{1}{2}\langle JU, V \rangle \right) Z. \end{aligned}$$

Now, we give an idea of the geometric interpretation of the groups A and N that appear in the Iwasawa decomposition of $G = SU(1, n)$. Details can be found in [46, §2.2]; see also [50, Chapter 1].

Two unit speed curves γ and σ in a nonpositively curved, complete, simply connected Riemannian manifold $\bar{M}$ are called asymptotic if there is a positive constant C such that $\bar{d}(\gamma(t), \sigma(t)) \leq C$ for all $t \geq 0$, where $\bar{d}$ denotes the Riemannian distance in $\bar{M}$. In particular, for symmetric spaces of rank 1 we have $\lim_{t\to\infty} \bar{d}(\gamma(t), \sigma(t)) = 0$. This definition establishes an equivalence relation in the collection of complete geodesics of $\bar{M}$. Each equivalence class is called *point at infinity* of $\bar{M}$. The set of the points at infinity of $\bar{M}$ is the *ideal boundary* of $\bar{M}$ and is denoted by $\bar{M}(\infty)$.

In our case $\bar{M} = \mathbb{C}H^n$, so we denote by $\mathbb{C}H^n(\infty)$ the ideal boundary of $\mathbb{C}H^n$. It is possible to endow $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ with a topology (the so-called cone topology) that makes $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ homeomorphic to the closed unit ball of $\mathbb{R}^{2n}$ in such a way that $\mathbb{C}H^n(\infty)$ corresponds to the unit sphere of $\mathbb{R}^{2n}$. In this model, two geodesics in $\mathbb{C}H^n$ are asymptotic if they converge to the same point of the unit sphere. Moreover, for each $p \in \mathbb{C}H^n$ and $x \in \mathbb{C}H^n(\infty)$ there is a unique geodesic $\gamma_{px} \colon \mathbb{R} \to \mathbb{C}H^n$ such that $\|\dot{\gamma}_{px}\| = 1$, $\gamma_{px}(0) = p$ and $\lim_{t\to\infty} \gamma_{px}(t) = x$.

The Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is a 1-dimensional abelian subspace of $\mathfrak{p}$. In $\mathfrak{p} \equiv T_o\mathbb{C}H^n$, the Riemannian exponential map and the Lie group exponential map coincide, that is, $\operatorname{Exp}(tX) \cdot o = \exp_o(tX)$ for all $X \in \mathfrak{p}$ and $t \in \mathbb{R}$. It follows that the orbit of the group A through o is the geodesic through o with tangent space at o given by $\mathfrak{a} \subset \mathfrak{p} \equiv T_o\mathbb{C}H^n$. This geodesic determines two points at infinity; let x be one of them. Thus, the submanifold A of AN corresponds to $\gamma_{ox}(\mathbb{R})$ under the isometry $\phi|_{AN} \colon AN \to \mathbb{C}H^n$. In other words, $\gamma_{ox}(\mathbb{R})$ is the orbit $A \cdot o$ of the action of A on $\mathbb{C}H^n$, while the rest of the orbits are equidistant curves to $A \cdot o$.

Now, let us comment on the action of the nilpotent part N of the Iwasawa decomposition. First notice that N has dimension $2n - 1$. This, together with the fact that AN acts simply transitively on $\mathbb{C}H^n$, implies that N acts isometrically with cohomogeneity one on $\mathbb{C}H^n$. It turns out that the orbits of this action are hypersurfaces in $\mathbb{C}H^n$ which are orthogonal at every point to the integral curves of the left-invariant vector field $B \in \mathfrak{a}$. These integral curves are all geodesics with a common point at infinity (x, according to the notation above).

More specifically, the orbits of the N-action are the *horospheres* of $\mathbb{C}H^n$ determined by the point at infinity x. In order to define this concept, consider a unit speed geodesic γ in $\mathbb{C}H^n$. The real function $f \colon \mathbb{C}H^n \to \mathbb{R}$ given by $f_\gamma(p) = \lim_{t\to\infty} \big(\bar{d}(\gamma(t), p) - t\big)$ is said to be the *Busemann function* with respect to γ. Then, horospheres are defined as the level sets of a Busemann function, and these are parallel real hypersurfaces of $\mathbb{C}H^n$ defining a regular Riemannian foliation, each of whose leaves has a unique adherent point at infinity. Thus, it turns out that the orbits of the N-action on $\mathbb{C}H^n$ are the horospheres determined by the geodesic γ_{ox} or, in other words, the horospheres adherent to x.

1.6.5 Important classes of submanifolds in Kähler manifolds

Let $\bar{M}$ be a Kähler manifold with complex structure J. We define the following classes of submanifolds.

Lagrangian submanifolds

Symplectic manifolds and their Lagrangian submanifolds appear naturally in the areas of Classical Mechanics and Physics. A $2n$-dimensional smooth manifold N is called a *symplectic manifold* if it admits a nondegenerate closed 2-form Ω, called the *symplectic form* (see [1] for more information on symplectic manifolds). An n-dimensional submanifold L of N is called a *Lagrangian submanifold* of the symplectic manifold N if the restriction of the symplectic form to the tangent bundle of L vanishes identically. Thus, one has $\Omega(X, Y) = 0$ for X, $Y \in \Gamma(TL)$. Lagrangian submanifolds are included in a list of special geometries with mathematically rich properties which are interesting for string theory [2].

The study of Lagrangian submanifolds of Kähler manifolds from the Riemannian geometric point of view was initiated in the early 1970s. The complex structure J of a Kähler manifold $\bar{M}$ defines a symplectic form Ω on $\bar{M}$ by $\Omega(X, Y) = \langle JX, Y\rangle$, with X and Y vector fields on $\bar{M}$. Then, a submanifold M is Lagrangian if and only if it is a submanifold of maximum possible dimension such that Ω vanishes identically on each tangent space of M. Recall also that a submanifold M is said to be *totally real* if JT_pM is perpendicular to T_pM for any $p \in M$. It is easy to see that in a Kähler manifold of complex dimension n $\bar{M}$, n-dimensional submanifolds are Lagrangian precisely if they are totally real. See [27] for more information on the Riemannian geometry of Lagrangian submanifolds.

Hopf hypersurfaces

A submanifold of a Kähler manifold is called a *real hypersurface* if it has real codimension one (as opposed to complex codimension one). Let M be a real hypersurface in a Kähler manifold $\bar{M}$, and ξ a (locally defined) unit normal vector field on M. Then, the tangent vector $J\xi$ is called the *Hopf* or *Reeb vector field* of M. Moreover, M is said to be Hopf at a point $p \in M$ if $J\xi_p$ is an eigenvector of the shape operator $\mathcal{S}$ of M at p, and M is called a *Hopf hypersurface* if it is Hopf at every point. However, in general, $J\xi$ can have nontrivial projections onto several principal curvature spaces. Thus, we define the integer-valued function h on M as the number of principal curvature spaces where $J\xi$ has nontrivial projection or, equivalently, as the dimension of the smallest subspace of the tangent space to M that contains $J\xi$ and is invariant under the shape operator $\mathcal{S}$. Thus, M is Hopf at a point p if $h(p) = 1$, and is a Hopf hypersurface if $h = 1$ on M, that is, if $J\xi$ is a principal curvature vector field everywhere. If h is constantly equal to an integer number k, then there is a smooth distribution $\mathcal{D}$ of rank k on M that consists of the maximal subspace of the tangent space to M at each point that contains $J\xi$ and is $\mathcal{S}$-invariant. If $\mathcal{D}$ is integrable, then M is said to be k-Hopf [25].

Ruled hypersurfaces

Let M be a real hypersurface in $\bar{M}$. We say that M is a *ruled hypersurface* if the maximal complex distribution $(J\xi)^\perp$ of M is integrable and its integral submanifolds are totally geodesic complex hypersurfaces of the ambient space $\bar{M}$ [25, §8.5.1].

Levi-flat hypersurfaces

The Levi form of a real hypersurface M in a Kähler manifold is the symmetric bilinear map $L\colon (J\xi)^\perp \times (J\xi)^\perp \to \nu M$ defined by

$$L(X,Y) = I\!I(X,Y) + I\!I(JX,JY),$$

where $(J\xi)^\perp$ is the maximal complex distribution of M. Then M is called *Levi-flat* if its Levi form vanishes identically. It is easy to check that M is Levi-flat if and only if the maximal complex distribution of M is integrable. Then, a real hypersurface is Levi-flat if it is foliated by complex hypersurfaces. Thus, ruled hypersurfaces are a very particular case of Levi-flat hypersurfaces. See [60] for more information on Levi-flat hypersurfaces.

1.7 Polar actions

Let $\bar{M}$ be a Riemannian manifold and H a connected Lie group that acts on $\bar{M}$ via isometries. The action of H on $\bar{M}$ is called polar if there exists a section Σ, that is, an immersed submanifold of $\bar{M}$ such that Σ intersects all the orbits of the H-action and for each $p \in \Sigma$, the tangent space of Σ at p, $T_p\Sigma$, and the tangent space of the orbit through p at p, $T_p(H \cdot p)$, are orthogonal. The action of H is called hyperpolar if the section Σ is flat in its induced Riemannian metric.

Polar actions are much more rigid than arbitrary isometric actions. For complete, simply connected ambient manifolds $\bar{M}$, the orbits of a polar H-action are always closed submanifolds, and the image of the group H on the isometry group $I(\bar{M})$ is closed (see [83, Corollary 1.3]). This, in particular, implies that polar actions on complete, simply connected manifolds are orbit equivalent to proper actions, and it turns out that none of the orbits of a proper polar action is exceptional. Furthermore, if φ is a polar action of a connected group H on $\bar{M}$, $p \in \bar{M}$ and Σ is a section through p, then the slice representation of such action at p is polar with section $T_p\Sigma$.

A first important objective in the study of polar actions is to classify them on certain Riemannian manifolds of particular interest, as is the case of Riemannian symmetric spaces. In this thesis we are interested in this classification for $\mathbb{C}P^2$ and $\mathbb{C}H^2$. From the geometric viewpoint, that is, if one is mainly interested in the geometry of the orbits, it is enough to obtain such classifications up to orbit equivalence.

We now give a brief idea of the historical evolution of the study of polar actions until the classification for $\mathbb{C}P^n$ and $\mathbb{C}H^n$ was obtained. More detailed information and references can be found in [104], [105] and [37].

Polar actions have their origin in polar coordinates, that is, the well-known way of representing a point of the plane given the angle with a fixed axis going through the origin and the distance to the origin. Szenthe [97] and Palais and Terng [92] were the first mathematicians to study the fundamental properties of polar actions on Riemannian manifolds. Then Dadok [35] achieved the first classification on a concrete manifold, namely the classification of polar representations on Euclidean spaces up to orbit equivalence, which implies the classification on spheres.

Several years later, a great interest in classifying polar actions on symmetric spaces of compact type appeared [58]. In particular, it became interesting to classify polar actions on compact symmetric spaces of rank one. Since the complex projective space has rank one, it is clear that a hyperpolar action on $\mathbb{C}P^n$ must be of cohomogeneity one, that is, the minimum codimension of an orbit is one, and this codimension is precisely the dimension of the section. Hence, the classification of hyperpolar actions on $\mathbb{C}P^n$ follows from Takagi's classification [98] of cohomogeneity one actions on $\mathbb{C}P^n$. Takagi's result can be considered the first step towards the classification of polar actions on complex projective spaces. The complete classification of polar actions up to orbit equivalence on compact symmetric spaces of rank one was obtained by Podestà and Thorbergsson [93] (see also [55] for a missing example in the Cayley projective plane). In particular, they obtained the classification for $\mathbb{C}P^n$.

Many authors, such as Biliotti [19], Kollross [74], [75] and, Kollross and Lytchak [77], made great advances in the study of polar actions on compact symmetric spaces of higher rank, but these classifications are of no concern to us in this work. The situation in the noncompact case is still open nowadays. Wu [108] classified polar actions on $\mathbb{R}H^n$ and showed that, up to orbit equivalence, the groups acting upon are products of a noncompact factor (which is either the isometry group of a lower dimensional real hyperbolic space or the nilpotent part of its Iwasawa decomposition), and a compact factor (which comes from the isotropy representation of a symmetric space). Berndt and Díaz-Ramos obtained in [14] the classification of polar actions on the complex hyperbolic plane $\mathbb{C}H^2$. The complete classification of polar actions on complex hyperbolic spaces of arbitrary dimension was obtained by Díaz-Ramos, Domínguez-Vázquez and Kollross in [39].

An important fact to keep in mind is the duality between compact and noncompact symmetric spaces. Using duality Díaz-Ramos and Kollross in [43] classified polar actions with a fixed point on symmetric spaces. However, in general, duality cannot be applied to derive classifications of polar actions on noncompact symmetric spaces from the corresponding classifications in the compact setting. For example, a horosphere foliation on a real hyperbolic space is polar but cannot be obtained from duality. Nevertheless, there are certain situations where duality can be really useful to obtain partial classifications, as in [76].

Berndt and Tamaru [16] classified cohomogeneity one actions on complex hyperbolic spaces, the quaternionic hyperbolic plane, and the Cayley hyperbolic plane. The classification remains open in quaternionic hyperbolic spaces $\mathbb{H}H^n$, $n \geq 3$, and in most symmetric spaces of higher rank. See [15] and [17] for more information on cohomogeneity one actions on symmetric spaces of noncompact type.

1.7.1 Polar actions on $\mathbb{C}P^2$

As we have commented above, polar actions on irreducible symmetric spaces of compact type are nowadays well understood. In this subsection, we recall the classification of polar actions on complex projective spaces.

The complete classification of polar actions on $\mathbb{C}P^n$ was obtained by Podestà and Thorbergsson in [93]. This classification follows from the following result.

Theorem 1.4. *If H acts polarly on a complex projective space $\mathbb{C}P^n$, then the action of H is, up to orbit equivalence, induced by the isotropy representation of a Hermitian symmetric space.*

Let us explain the statement of Theorem 1.4.

Recall that a Hermitian symmetric space $M = G/K$ is a Riemannian symmetric space endowed with a complex structure J that is invariant under the geodesic symmetries; see Table 1.1 for the list of irreducible Hermitian symmetric spaces. Assume M has complex dimension $n+1$. Then, the tangent space at o has a complex structure J_o, which commutes with the isometries of K, and which turns T_oM into a complex vector space $\mathbb{C}^{n+1}$. It turns out that the isotropy representation of the semisimple symetric space G/K, $K \times T_oM \to T_oM$, is a polar action on the Euclidean space $T_oM \cong \mathfrak{p}$, and its cohomogeneity is precisely the rank of M. In fact, any maximal abelian subspace of $\mathfrak{p}$ is a section of this representation. Since G/K is Hermitian, the section $\mathfrak{a}$ is totally real (that is, $J\mathfrak{a}$ is orthogonal to $\mathfrak{a}$) and the action of K induces a polar action on the unit sphere S^{2n+1} of $\mathbb{C}^{n+1}$. A complex projective space can be defined as $\mathbb{C}P^n = S^{2n+1}/S^1$, and since J_o is invariant by the isometries of K, the action of K on $T_oM \cong \mathbb{C}^{n+1}$ descends via the Hopf map $\pi\colon S^{2n+1} \to \mathbb{C}P^n$ to an isometric action on $\mathbb{C}P^n$. Using the fact that $\mathfrak{a}$ is totally real, it is not difficult to see that $\pi(\mathfrak{a} \cap S^{2n+1})$ is a section of the action induced on $\mathbb{C}P^n$. Theorem 1.4 implies that any polar action on $\mathbb{C}P^n$ can be obtained, up to orbit equivalence, in this way. Note as well that the cohomogeneity of the polar action on $\mathbb{C}P^n$ induced by the symmetric space G/K, coincides with the rank of G/K minus one.

In particular, the classification of polar actions on complex projective planes reduces to the following corollary.

Corollary 1.5. *If H acts polarly on a complex projective plane $\mathbb{C}P^2$, then the action of H is, up to orbit equivalence, induced by the connected subgroup of $SU(3)$ whose Lie algebra $\mathfrak{h}$ is one of*

(i) *Actions of cohomogeneity one - the section Σ is a totally geodesic real projective line $\mathbb{R}P^1 \subset \mathbb{C}P^2$:*

  (a) $\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \cong \mathfrak{u}(2)$*; this action has a fixed point o and its cut locus (which is a totally geodesic $\mathbb{C}P^1$) as singular orbits, while the principal orbits are geodesic spheres centered at o, or equivalently, tubes around the other singular orbit.*

  (b) $\mathfrak{h} = \mathfrak{so}(2) \oplus \mathfrak{so}(3)$*; this action has a totally geodesic real projective plane $\mathbb{R}P^2$ and the complex quadric $\{[z] \in \mathbb{C}P^2 : z_0^2 + z_1^2 + z_2^2 = 0\}$ as singular orbits, and the principal orbits are tubes around any of the singular orbits.*

(ii) *Actions of cohomogeneity two - the section Σ is a totally geodesic real projective plane $\mathbb{R}P^2 \subset \mathbb{C}P^2$:*

(a) $\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1))$; *the singular orbits are three fixed points and an uncountable number of circles, whereas the principal orbits are 2-dimensional tori which are contained in the geodesic spheres around each one of the three fixed points.*

1.7.2 Polar actions on $\mathbb{C}H^2$

As we have mentioned above, Díaz-Ramos, Domínguez-Vázquez and Kollross obtained in [39] the complete classification of polar actions on complex hyperbolic spaces $\mathbb{C}H^n$.

Let us briefly recall the notation introduced in §1.6.4, which will be important to understand the statements of the results below. Let $\mathbb{C}H^n = G/K$ be the complex hyperbolic space, where $G = SU(1,n)$, and $K = S(U(1)U(n))$ is the isotropy group of G at some point o. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to o, with associated Cartan involution θ. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the root space decomposition with respect to $\mathfrak{a}$. Set $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0 \cong \mathfrak{u}(n-1)$. Since $\mathfrak{k}_0$ acts on the root space $\mathfrak{g}_\alpha$, the center of $\mathfrak{k}_0$ induces a complex structure J on $\mathfrak{g}_\alpha$ which makes it isomorphic to $\mathbb{C}^{n-1}$. We will say that a subset of $\mathfrak{g}_\alpha$ is a real subspace of $\mathfrak{g}_\alpha$ if it is a linear subspace of $\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is viewed as a real vector space. The solvable Lie algebra $\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is endowed with certain inner product $\langle \cdot, \cdot \rangle$ which is induced naturally from the metric on $\mathbb{C}H^n$. A real subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ is said to be totally real if $\langle \mathfrak{w}, J(\mathfrak{w}) \rangle = 0$.

Theorem 1.6. *For each of the Lie algebras $\mathfrak{h}$ below, the corresponding connected subgroup of $U(1,n)$ acts polarly on $\mathbb{C}H^n$:*

(i) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{so}(1,k) \subset \mathfrak{u}(n-k) \oplus \mathfrak{su}(1,k)$, $k \in \{0, \dots, n\}$, *where $\mathfrak{q}$ is a subalgebra of $\mathfrak{u}(n-k)$ such that the corresponding subgroup Q of $U(n-k)$ acts polarly with a totally real section on $\mathbb{C}^{n-k}$.*

(ii) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{su}(1,n)$, *where $\mathfrak{b}$ is a linear subspace of $\mathfrak{a}$, $\mathfrak{w}$ is a real subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{q}$ is a subalgebra of $\mathfrak{k}_0$ which normalizes $\mathfrak{w}$ and such that the connected subgroup of $SU(1,n)$ with Lie algebra $\mathfrak{q}$ acts polarly with a totally real section on the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$.*

Conversely, every nontrivial polar action on $\mathbb{C}H^n$ is orbit equivalent to one of the actions above.

The classification of polar actions on the complex hyperbolic plane $\mathbb{C}H^2$ had previously been obtained by Berndt and Díaz-Ramos [14] and reduces to the following theorem (in the statement, $\mathfrak{g}_\alpha^{\mathbb{R}}$ is any one-dimensional real subspace of $\mathfrak{g}_\alpha$).

Theorem 1.7. *For each of the subalgebras $\mathfrak{h}$ of $\mathfrak{su}(1,2)$ listed below the connected closed subgroup H of $SU(1,2)$ with Lie algebra $\mathfrak{h}$ acts polarly on $\mathbb{C}H^2$:*

(i) *Actions of cohomogeneity one - the section Σ is a totally geodesic real hyperbolic line $\mathbb{R}H^1 \subset \mathbb{C}H^2$:*

(a) $\mathfrak{h} = \mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \cong \mathfrak{u}(2)$; *the orbits are a fixed point* o *and the geodesic spheres centered at* o*.*

(b) $\mathfrak{h} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha} = \mathfrak{s}(\mathfrak{u}(1,1) \oplus \mathfrak{u}(1)) \cong \mathfrak{u}(1,1)$; *the orbits are a totally geodesic complex hyperbolic line* $\mathbb{C}H^1 \subset \mathbb{C}H^2$ *and the tubes around it.*

(c) $\mathfrak{h} = \theta(\mathfrak{g}_\alpha^{\mathbb{R}}) \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}} \cong \mathfrak{so}(1,2)$; *the orbits are a totally geodesic real hyperbolic plane* $\mathbb{R}H^2 \subset \mathbb{C}H^2$ *and the tubes around it.*

(d) $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ *or* $\mathfrak{h} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$; *the orbits form a foliation of* $\mathbb{C}H^2$ *by horospheres.*

(e) $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$; *the orbits form a foliation of* $\mathbb{C}H^2$; *there is exactly one minimal leaf, which is the so-called Lohnherr hypersurface* W^3 *of* $\mathbb{C}H^2$, *and the other leaves are its equidistant hypersurfaces.*

(ii) *Actions of cohomogeneity two - the section* Σ *is a totally geodesic real hyperbolic plane* $\mathbb{R}H^2 \subset \mathbb{C}H^2$*:*

(a) $\mathfrak{h} = \mathfrak{k} \cap (\mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha}) = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)) \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1)$; *the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (a) *and* (b) *in* (i)*: the action has one fixed point* o*, and on each geodesic sphere centered at* o *the orbits are two circles as singular orbits and* 2*-dimensional tori as principal orbits.*

(b) $\mathfrak{h} = \mathfrak{g}_0$; *the action leaves a totally geodesic* $\mathbb{C}H^1 \subset \mathbb{C}H^2$ *invariant. On this* $\mathbb{C}H^1$ *the action induces a foliation by a totally geodesic real hyperbolic line* $\mathbb{R}H^1 \subset \mathbb{C}H^1$ *and its equidistant curves in* $\mathbb{C}H^1$*. The other orbits are* 2*-dimensional cylinders whose axis is one of the curves in that* $\mathbb{C}H^1$*.*

(c) $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$; *the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (b) *and* (d) *in* (i)*: the action leaves a horosphere foliation invariant, and on each horosphere the orbits consist of a complex horocycle and the tubes around it.*

(d) $\mathfrak{h} = \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$; *the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (d) *and* (e) *in* (i)*: the action leaves a horosphere foliation invariant, and on each horosphere the action induces a foliation for which a minimally embedded Euclidean plane and its equidistant surfaces are the leaves.*

Every polar action on $\mathbb{C}H^2$ *is either trivial, transitive, or orbit equivalent to one of the polar actions described above.*

This was the first such classification in a noncompact symmetric space of nonconstant curvature. Apart from the trivial and transitive actions, there are exactly nine orbit equivalence classes of polar actions on $\mathbb{C}H^2$.

Chapter 2

Isoparametric submanifolds

In this chapter we study and classify isoparametric submanifolds in the complex projective and hyperbolic planes. We will do so by proving that they are induced by polar actions. In particular, it will follow from our arguments that Lagrangian flat surfaces with parallel mean curvature in these spaces are open parts of principal orbits of polar actions. This result will be fundamental to achieve the results of Chapter 3. Moreover, we also classify Terng-isoparametric submanifolds in $\mathbb{C}P^2$ and $\mathbb{C}H^2$.

The motivation for this chapter comes from the study of isoparametric submanifolds in symmetric spaces. The history of isoparametric submanifolds can be traced back at least to the works of Somigliana [96], Levi-Civita [78] and Segre [94] who classified isoparametric hypersurfaces in Euclidean spaces. All the examples in this classification are homogeneous. Thorbergsson showed in [102] that compact, full and irreducible isoparametric submanifolds of codimension greater than 2 in Euclidean spaces are homogeneous, which implies, with a bit more work, that such submanifolds are open parts of principal orbits of polar actions, which in turn correspond to isotropy representations of symmetric spaces [35].

Thorbergsson's remarkable result [102] readily implies the classification of isoparametric submanifolds of codimension ≥ 2 in spheres. However, the classification of isoparametric hypersurfaces in spheres is open and still an active topic of research. See [105] for a recent survey on this and other related topics.

Isoparametric hypersurfaces in real hyperbolic spaces were classified by Cartan [21], and all such examples are homogeneous. For higher codimension, Wu [108] reduced the classification problem of isoparametric submanifolds in $\mathbb{R}H^n$ to the classification in spheres. Moreover, the classification of polar actions on $\mathbb{R}H^n$ follows from Wu's results. It is important to mention that, in real space forms, homogeneous isoparametric submanifolds are always principal orbits of polar actions.

The general study of isoparametric submanifolds was started by Terng [101], whose definition was given for spaces of constant curvature. Nowadays the general definition of isoparametric submanifold is credited to Heintze, Liu and Olmos [57]. This is the definition that we use in this chapter, although we also consider Terng's definition which turns out to be less rigid in our context. The definitions of isoparametric and Terng-isoparametric submanifolds have been introduced in Section 1.4.

Isoparametric submanifolds of complex projective spaces $\mathbb{C}P^n$ have been studied by Domínguez-Vázquez in [49], who gave a classification if $n \neq 15$. It turns out that inhomogeneous isoparametric submanifolds are relatively common. In this chapter we also study Terng-isoparametric submanifolds of $\mathbb{C}P^2$ and conclude that no new interesting examples arise.

The classification of isoparametric hypersurfaces in complex hyperbolic spaces has recently been obtained in [40]. It also turns out that there are inhomogeneous examples. For higher codimension the problem seems to be much more complicated. We restrict to $\mathbb{C}H^2$ in this chapter and show that all examples are open parts of principal orbits of polar actions on $\mathbb{C}H^2$. Surprisingly, there is a Terng-isoparametric submanifold of codimension 2 that is not isoparametric; this submanifold is homogeneous but not an orbit of a polar action.

This chapter is organized as follows. We present our main results in Section 2.1. Next, in Section 2.2, we give the relation between isoparametric submanifolds, Terng-isoparametric submanifolds and principal orbits of cohomogeneity two polar actions on a nonflat two-dimensional complex space form (Theorem 2.1). Finally, in subsections 2.3.2 and 2.3.3, we proceed with the classification of isoparametric submanifolds and Terng-isoparametric submanifolds in the two-dimensional nonflat complex space forms (theorems 2.2 and 2.3), respectively.

2.1 Main results

Let us recall from Section 1.4 that a submanifold M of a Riemannian manifold is called isoparametric if its normal bundle νM is flat, all nearby parallel submanifolds to M have constant mean curvature in the radial directions, and for any $p \in M$ there exists a totally geodesic submanifold Σ_p through p such that $T_p\Sigma_p = \nu_p M$. We also recall that a submanifold is Terng-isoparametric if it has constant principal curvatures and flat normal bundle (see Section 1.4).

We use the notation given in Section 1.6. Then, we denote by $\bar{M}^2(c)$ a nonflat 2-dimensional complex space form of constant holomorphic sectional curvature $c \neq 0$. In Section 2.2 we prove the following fundamental result that establishes a relationship between different properties of surfaces in $\bar{M}^2(c)$.

Theorem 2.1. *Let M be a 2-dimensional submanifold of $\bar{M}^2(c)$, $c \neq 0$. Then the following conditions are equivalent:*

(i) *M is Lagrangian, flat and with parallel mean curvature.*

(ii) *M is Lagrangian, flat and with parallel second fundamental form.*

(iii) *M is Lagrangian and Terng-isoparametric.*

(iv) *M is isoparametric.*

(v) *M is an open part of a principal orbit of a cohomogeneity two polar action on $\bar{M}^2(c)$.*

The study of submanifolds with parallel mean curvature is an active field of research nowadays; see [28] for a survey. In particular, the case of surfaces with parallel mean curvature in 2-dimensional complex space forms deserves special attention, and is the subject of important recent advances, e.g. [51]. As a consecuence of these relations between isoparametric submanifolds, Terng isoparametric submanifolds and principal orbits of polar actions on $\bar{M}^2(c)$, we will prove in Section 2.3 the following characterization:

Theorem 2.2. *An isoparametric submanifold of $\bar{M}^2(c)$, $c \neq 0$, is congruent to an open part of a principal orbit of a polar action on $\bar{M}^2(c)$.*

The previous theorem has been obtained in much greater generality for complex projective spaces $\mathbb{C}P^n$, $n \neq 15$, using a different method [49]. Here we deal with the projective and hyperbolic cases simultaneously and obtain the result for $\mathbb{C}H^2$. Recall that in Section 1.7 we have presented the classification of polar actions on complex projective and hyperbolic spaces. Thus, our result implies the classification of isoparametric submanifolds of $\bar{M}^2(c)$. Note that the classification of isoparametric submanifolds in $\bar{M}^2(0) = \mathbb{C}^2$ is known, in view of the results of Cartan [21]. In this case, all examples are open parts of principal orbits of polar actions on $\mathbb{C}^2 \equiv \mathbb{R}^4$.

In our context Terng's definition is less rigid than Heintze, Liu and Olmos', and thus, a new example appears in codimension two:

Theorem 2.3. *A submanifold of $\bar{M}^2(c)$, $c \neq 0$, is Terng-isoparametric if and only if it is congruent to an open part of:*

(i) *an isoparametric submanifold of $\bar{M}^2(c)$, or*

(ii) *a Chen's surface in $\mathbb{C}H^2$, or*

(iii) *a circle.*

The proof of Theorem 2.3 is given in Section 2.3.3. Apart from circles, which are trivial examples of Terng-isoparametric submanifolds, we do not get new examples in complex projective spaces. However, there exists a Terng-isoparametric submanifold in $\mathbb{C}H^2$ that is neither a circle nor a principal orbit of a polar action. We have called this new example a Chen's surface, which is homogeneous and unique up to isometric congruence (see §2.3.1). This new example was introduced by Chen in [26], and a geometric characterization was given in [30]. In Subsection 2.3.1 we present a new Lie group theoretic description of this submanifold in terms of the root space decomposition of the Lie algebra of the isometry group of $\mathbb{C}H^2$. Theorems 2.2 and 2.3 have given rise to the paper [42].

2.2 Proof of Theorem 2.1

Let M be a Lagrangian submanifold of $\bar{M}^2(c)$, $c \neq 0$. Let J be the complex structure of $\bar{M}^2(c)$. Then M is totally real, that is, $\langle JX, Y\rangle = 0$ for all tangent vector fields X, Y on M.

Let us start with an easy lemma.

Lemma 2.4. *If M is a Lagrangian submanifold of a Kähler manifold, then M is flat if and only if M has flat normal bundle.*

Proof. Let U, $X \in \Gamma(TM)$. Since M is Lagrangian we have

$$\nabla^\perp_X(JU) = (\bar\nabla_X JU)^\perp = (J\bar\nabla_X U)^\perp = (J(\nabla_X U + I\!I(X,U)))^\perp = J\nabla_X U.$$

Hence, U is parallel if and only if JU is $\nabla^\perp$-parallel. In particular, M is flat if and only if the normal bundle of M is flat. □

Theorem 2.1 provides a geometric characterization of the principal orbits of polar actions of cohomogeneity two on $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Now, we proceed with its proof.

Proof of (ii) ⇒ (i). We prove that an arbitrary submanifold with parallel second fundamental form has parallel mean curvature. Let $\{E_i\}$ be an orthonormal frame of M. Then, $\langle \nabla_X E_i, E_j\rangle = -\langle \nabla_X E_j, E_i\rangle$ for any $X \in \Gamma(TM)$. Since $I\!I$ is symmetric,

$$\begin{aligned}\sum_i I\!I(\nabla_X E_i, E_i) &= \sum_{i,j}\langle \nabla_X E_i, E_j\rangle I\!I(E_j,E_i) = -\sum_{i,j}\langle \nabla_X E_j, E_i\rangle I\!I(E_j,E_i)\\ &= -\sum_{i,j}\langle \nabla_X E_i, E_j\rangle I\!I(E_i,E_j) = -\sum_i I\!I(\nabla_X E_i, E_i).\end{aligned}$$

Thus, $\sum_i I\!I(\nabla_X E_i, E_i) = 0$. Since M has parallel second fundamental form, the previous equation yields

$$\nabla^\perp_X H = \sum_i \nabla^\perp_X I\!I(E_i,E_i) = \sum_i \Big((\nabla^\perp_X I\!I)(E_i,E_i) + 2I\!I(\nabla_X E_i, E_i)\Big) = 0,$$

for all $X \in \Gamma(TM)$, that is, M has parallel mean curvature. □

Proof of (i) ⇒ (iii). Let M be a Lagrangian, flat surface with parallel mean curvature. By Lemma 2.4, M has flat normal bundle. Let $\lambda_i\colon \nu M \to \mathbb{R}$ be the principal curvature functions. Let $\{\xi,\eta\}$ be a local parallel orthonormal frame of νM, and let $\{U_1,U_2\}$ be an orthonormal frame of TM such that $\mathcal{S}_\xi U_i = \lambda_i U_i$, $i = 1,2$. Without loss of generality, we assume that η is perpendicular to the mean curvature vector.

Lemma 2.5. *There are smooth functions b_1, $b_2\colon M \to \mathbb{R}$ with $b_1^2 + b_2^2 = 1$, such that*

$$\begin{aligned} J\xi &= b_1U_1 + b_2U_2, \qquad J\eta = -b_2U_1 + b_1U_2,\\ JU_1 &= -b_1\xi + b_2\eta, \qquad JU_2 = -b_2\xi - b_1\eta, \end{aligned} \tag{2.1}$$

Proof. Since $J\xi$ is a unit vector field tangent to M and $\{U_1,U_2\}$ constitutes an orthonormal frame on M we can write $J\xi = b_1U_1 + b_2U_2$, where b_1, b_2 are smooth functions on M satisfying $b_1^2 + b_2^2 = 1$. Furthermore, since η is a locally parallel unit vector field on νM orthogonal to ξ, we can write, changing the sign of η if necessary, $J\eta = -b_2U_1 + b_1U_2$.

As $-\xi = J^2\xi = b_1JU_1 + b_2JU_2$ and $-\eta = J^2\eta = -b_2JU_1 + b_1JU_2$, taking inner product with U_2 we get $b_1\langle JU_1, U_2\rangle = 0$ and $b_2\langle JU_1,U_2\rangle = 0$. Since $b_1 \neq 0$ or $b_2 \neq 0$, we have that

$\langle JU_1, U_2\rangle = 0$. This implies that JU_1, $JU_2 \in \operatorname{span}\{\xi, \eta\}$. Now, $\langle JU_1, \xi\rangle = -\langle U_1, J\xi\rangle = -b_1$ and $\langle JU_1, \eta\rangle = -\langle U_1, J\eta\rangle = b_2$. A similar argument shows that $JU_2 = -b_2\xi - b_1\eta$, from where the result follows. □

Since νM is flat, the Ricci equation gives

$$-\frac{c}{4} = \langle \bar{R}(U_1, U_2)\xi, \eta\rangle = \langle \mathcal{S}_\xi U_1, \mathcal{S}_\eta U_2\rangle - \langle \mathcal{S}_\eta U_1, \mathcal{S}_\xi U_2\rangle = (\lambda_1 - \lambda_2)\langle \mathcal{S}_\eta U_1, U_2\rangle. \tag{2.2}$$

Since $c \neq 0$, this readily implies $\lambda_1 \neq \lambda_2$ on M. As η is taken to be perpendicular to the mean curvature vector we have that $\operatorname{tr} \mathcal{S}_\eta = 0$ and, thus, the shape operator $\mathcal{S}_\eta$ with respect to $\{U_1, U_2\}$ can be written as

$$\mathcal{S}_\eta = \begin{pmatrix} \mu & -\frac{c}{4(\lambda_1-\lambda_2)} \\ -\frac{c}{4(\lambda_1-\lambda_2)} & -\mu \end{pmatrix}, \tag{2.3}$$

for some function $\mu\colon M \to \mathbb{R}$.

Now, using the Codazzi equation we get

$$0 = \langle \bar{R}(U_1, U_2)U_1, \xi\rangle = (\lambda_2 - \lambda_1)\langle \nabla_{U_1} U_2, U_1\rangle - U_2(\lambda_1),$$
$$0 = \langle \bar{R}(U_1, U_2)U_2, \xi\rangle = (\lambda_2 - \lambda_1)\langle \nabla_{U_2} U_1, U_2\rangle + U_1(\lambda_2).$$

Since $\{U_1, U_2\}$ is an orthonormal frame of the tangent bundle we obtain

$$\nabla_{U_i} U_i = \frac{U_j(\lambda_i)}{\lambda_i - \lambda_j} U_j, \qquad \nabla_{U_i} U_j = \frac{U_j(\lambda_i)}{\lambda_j - \lambda_i} U_i, \qquad i, j \in \{1, 2\}, i \neq j. \tag{2.4}$$

Now we compute the derivatives of the b_i:

$$\begin{aligned} U_i(b_i) &= U_i\langle U_i, J\xi\rangle = \langle \bar{\nabla}_{U_i} U_i, b_i U_i + b_j U_j\rangle + \langle U_i, \bar{\nabla}_{U_i} J\xi\rangle \\ &= b_j\langle \nabla_{U_i} U_i, U_j\rangle - \lambda_i\langle U_i, JU_i\rangle = b_j \frac{U_j(\lambda_i)}{\lambda_i - \lambda_j}, \\ U_i(b_j) &= U_i\langle U_j, J\xi\rangle = \langle \bar{\nabla}_{U_i} U_j, b_i U_i + b_j U_j\rangle + \langle U_j, \bar{\nabla}_{U_i} J\xi\rangle \\ &= b_i\langle \nabla_{U_i} U_j, U_i\rangle - \lambda_i\langle U_j, JU_i\rangle = b_i \frac{U_j(\lambda_i)}{\lambda_j - \lambda_i}. \end{aligned} \tag{2.5}$$

Moreover, we have

$$\begin{aligned} 0 &= U_1\langle U_1, JU_2\rangle = \langle \bar{\nabla}_{U_1} U_1, -b_2\xi - b_1\eta\rangle + \langle U_1, \bar{\nabla}_{U_1} JU_2\rangle \\ &= -b_2\lambda_1 - b_1\mu - \langle -b_1\xi + b_2\eta, \bar{\nabla}_{U_1} U_2\rangle = -b_2\lambda_1 - b_1\mu + \frac{cb_2}{4(\lambda_1 - \lambda_2)}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} 0 &= U_2\langle U_1, JU_2\rangle = \langle \bar{\nabla}_{U_2} U_1, -b_2\xi - b_1\eta\rangle + \langle U_1, \bar{\nabla}_{U_2} JU_2\rangle \\ &= \frac{cb_1}{4(\lambda_1 - \lambda_2)} - \langle -b_1\xi + b_2\eta, \bar{\nabla}_{U_2} U_2\rangle = \frac{cb_1}{4(\lambda_1 - \lambda_2)} + b_1\lambda_2 + b_2\mu. \end{aligned} \tag{2.7}$$

Without restriction of generality let us assume that $b_1 \neq 0$ on an open subset of M. Henceforth working on this subset, from (2.6) we get

$$\mu = \frac{b_2(c - 4\lambda_1^2 + 4\lambda_1\lambda_2)}{4b_1(\lambda_1 - \lambda_2)}. \tag{2.8}$$

Substituting (2.8) into (2.7) we deduce

$$0 = b_1^2(c + 4\lambda_2(\lambda_1 - \lambda_2)) + b_2^2(c - 4\lambda_1(\lambda_1 - \lambda_2)). \tag{2.9}$$

This equation together with $b_1^2 + b_2^2 = 1$ form a linear system in the unknowns b_1^2, b_2^2. Since we know that $\lambda_1 \neq \lambda_2$, this system is compatible only if $\lambda_1 + \lambda_2 \neq 0$, or $\lambda_2 = -\lambda_1$ and $\lambda_1^2 = \frac{c}{8}$. If the second relation holds on an open set, then μ vanishes therein in view of (2.8), and by (2.3), this open set is Terng-isoparametric.

Hence, from now on we assume that $\lambda_1 + \lambda_2 \neq 0$. Then, solving the system we have

$$b_1^2 = -\frac{c + 4\lambda_1(\lambda_2 - \lambda_1)}{4(\lambda_1^2 - \lambda_2^2)}, \qquad b_2^2 = \frac{c + 4\lambda_2(\lambda_1 - \lambda_2)}{4(\lambda_1^2 - \lambda_2^2)}. \tag{2.10}$$

Since the mean curvature in the direction of ξ is constant by assumption, we have $U_i(\lambda_1) + U_i(\lambda_2) = 0$, $i = 1, 2$. Therefore, using (2.5)

$$0 = U_1(b_1^2) - U_1\Big(-\frac{c + 4\lambda_1(\lambda_2 - \lambda_1)}{4(\lambda_1^2 - \lambda_2^2)}\Big) = 2b_1b_2\frac{U_2(\lambda_1)}{\lambda_1 - \lambda_2} - \frac{c + 2(\lambda_1 - \lambda_2)^2}{2(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)}U_1(\lambda_1),$$
$$0 = U_2(b_1^2) - U_2\Big(-\frac{c + 4\lambda_1(\lambda_2 - \lambda_1)}{4(\lambda_1^2 - \lambda_2^2)}\Big) = -2b_1b_2\frac{U_1(\lambda_1)}{\lambda_1 - \lambda_2} - \frac{c + 2(\lambda_1 - \lambda_2)^2}{2(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)}U_2(\lambda_1).$$

This gives a homogeneous linear system in the unknowns $U_1(\lambda_1)$ and $U_2(\lambda_1)$. Using (2.10) one easily gets that this system is determined if and only if $2c + (\lambda_1 + \lambda_2)^2 \neq 0$. Hence, in the open set where $2c + (\lambda_1 + \lambda_2)^2 \neq 0$ holds, we have that $U_1(\lambda_1) = U_2(\lambda_1) = 0$, which implies that λ_1 and λ_2 are constant, and by (2.10), (2.8) and (2.3), we deduce that the open set is Terng-isoparametric. Finally, assume that $2c + (\lambda_1 + \lambda_2)^2 = 0$ on some open set. Then $c < 0$ and $\lambda_2 = \sqrt{-2c} - \lambda_1$. Then, using (2.10) we have

$$b_1^2 = \frac{(2\sqrt{2}\lambda_1 - \sqrt{-c})^2}{8\sqrt{-c}(\sqrt{2}\lambda_1 - \sqrt{-c})}, \qquad b_2^2 = -\frac{(3\sqrt{-c} - 2\sqrt{2}\lambda_1)^2}{8\sqrt{-c}(\sqrt{2}\lambda_1 - \sqrt{-c})}.$$

Since $b_1^2 \geq 0$ we get $b_2^2 \leq 0$ from the previous equation. Thus, $b_2 = 0$. But then λ_1 is constant and, as above, we get that any open set where $2c + (\lambda_1 + \lambda_2)^2 = 0$ holds is Terng-isoparametric. □

Proof of (iii) ⇒ (v). Let now M be a Lagrangian, Terng-isoparametric surface in $\bar{M}^2(c)$, $c \neq 0$. We first consider the case that M is not minimal. Then, the notations above apply, and equations (2.4), (2.8) and (2.10), together with the constancy of λ_1, λ_2, hold. In particular, $\{U_1, U_2\}$ is a parallel orthonormal frame on M, and λ_1, λ_2, μ, b_1 and b_2 are constant on M.

We first restrict ourselves to the case $c < 0$. Let then $\pi\colon H_1^5 \to \mathbb{C}H^2$ be the Hopf map from the anti-De Sitter space $H_1^5 \subset \mathbb{C}^3$ of constant sectional curvature $c/4$ to the complex hyperbolic plane. Denote by J the complex structure of $\mathbb{C}^3$, and by N the outward normal vector field to H_1^5 with $\langle N, N\rangle = -1$, that is, $N_q = \sqrt{-c}\,q/2$, for each $q \in H_1^5$. Then $V = JN$ is a unit timelike vector field tangent to the fibers of π. From the fundamental equations of semi-Riemannian submersions [90], [91] (see also [40]) and the umbilicity of H_1^5 with principal curvature $-\sqrt{-c}/2$, we have the following relations between the Levi-Civita connections D and $\bar\nabla$ of $\mathbb{C}^3$ and $\mathbb{C}H^2$, respectively:

$$\begin{aligned} D_{X^L}Y^L &= (\bar\nabla_X Y)^L + \frac{\sqrt{-c}}{2}\langle X, Y\rangle N,\\ D_{X^L}N &= \frac{\sqrt{-c}}{2}X^L, \end{aligned} \tag{2.11}$$

where X, $Y \in \Gamma(T\mathbb{C}H^2)$ satisfy $\langle JX, Y\rangle = 0$, and $(\cdot)^L$ denotes the horizontal lift of a vector field.

Now, using (2.1) and (2.3) we can easily compute the following derivatives on the surface M:

$$\begin{aligned} \bar\nabla_{J\xi}J\xi &= J\bar\nabla_{b_1U_1+b_2U_2}\xi = -b_1J\mathcal{S}_\xi U_1 - b_2J\mathcal{S}_\xi U_2 = -b_1\lambda_1 JU_1 - b_2\lambda_2 JU_2 =\\ &= (b_1^2\lambda_1 + b_2^2\lambda_2)\xi + b_1b_2(\lambda_2-\lambda_1)\eta,\\ \bar\nabla_{J\xi}J\eta &= J\bar\nabla_{b_1U_1+b_2U_2}\eta = -b_1J\mathcal{S}_\eta U_1 - b_2J\mathcal{S}_\eta U_2 =\\ &= \Big(-b_1\mu + \frac{cb_2}{4(\lambda_1-\lambda_2)}\Big)JU_1 + \Big(b_2\mu + \frac{cb_1}{4(\lambda_1-\lambda_2)}\Big)JU_2 =\\ &= \left((b_1^2-b_2^2)\mu - \frac{cb_1b_2}{2(\lambda_1-\lambda_2)}\right)\xi + \left(-2b_1b_2\mu + \frac{c(b_2^2-b_1^2)}{4(\lambda_1-\lambda_2)}\right)\eta,\\ \bar\nabla_{J\eta}J\xi &= J\bar\nabla_{-b_2U_1+b_1U_2}\xi = b_2J\mathcal{S}_\xi U_1 - b_1J\mathcal{S}_\xi U_2 = b_2\lambda_1JU_1 - b_1\lambda_2JU_2 =\\ &= b_1b_2(\lambda_2-\lambda_1)\xi + (b_1^2\lambda_2 + b_2^2\lambda_1)\eta,\\ \bar\nabla_{J\eta}J\eta &= J\bar\nabla_{-b_2U_1+b_1U_2}\eta = b_2J\mathcal{S}_\eta U_1 - b_1J\mathcal{S}_\eta U_2 =\\ &= \Big(b_2\mu + \frac{cb_1}{4(\lambda_1-\lambda_2)}\Big)JU_1 + \Big(b_1\mu - \frac{cb_2}{4(\lambda_1-\lambda_2)}\Big)JU_2 =\\ &= \left(\frac{c(b_2^2-b_1^2)}{4(\lambda_1-\lambda_2)} - 2b_1b_2\mu\right)\xi + \left(\frac{cb_1b_2}{2(\lambda_1-\lambda_2)} + \mu(b_2^2-b_1^2)\right)\eta. \end{aligned} \tag{2.12}$$

Consider the lifted submanifold $\pi^{-1}M$ of H_1^5, which is invariant under the Hopf S^1-action. The tangent bundle of $\pi^{-1}M$ is spanned by $J\xi^L$, $J\eta^L$ and V. Let A be the complex (3×3)-matrix whose columns are the derivatives $D_{J\xi^L}N$, $D_{J\xi^L}J\xi^L$, $D_{J\xi^L}J\eta^L$ expressed in the pseudo-unitary basis $\{N, J\xi^L, J\eta^L\}$ of $\mathbb{C}^3$; recall that the complex structure J of $\mathbb{C}^3$ is given by multiplication by the imaginary unit i. Define B analogously as the matrix giving the D-derivatives in the direction of $J\eta^L$. Then, using (2.11), (2.12), (2.8) and (2.10), we get that

$$A = \begin{pmatrix} 0 & \frac{\sqrt{-c}}{2} & 0 \\ \frac{\sqrt{-c}}{2} & i\frac{c-4(\lambda_1^2+\lambda_2^2)}{4(\lambda_1+\lambda_2)} & ib_1b_2(\lambda_1-\lambda_2) \\ 0 & ib_1b_2(\lambda_1-\lambda_2) & -i\frac{c+8\lambda_1\lambda_2}{4(\lambda_1+\lambda_2)} \end{pmatrix},$$

$$(2.13)\qquad B = \begin{pmatrix} 0 & 0 & \frac{\sqrt{-c}}{2} \\ 0 & ib_1b_2(\lambda_1-\lambda_2) & -i\frac{c+8\lambda_1\lambda_2}{4(\lambda_1+\lambda_2)} \\ \frac{\sqrt{-c}}{2} & -i\frac{c+8\lambda_1\lambda_2}{4(\lambda_1+\lambda_2)} & ib_1b_2(\lambda_2-\lambda_1) \end{pmatrix}.$$

It is easy to check that both matrices A and B belong to the Lie algebra $\mathfrak{u}(1,2)$ of the pseudo-unitary group $U(1,2)$.

Let now E be the complex (3×3)-matrix whose columns are the coordinates of N, $J\xi^L$ and $J\eta^L$ in the canonical pseudo-unitary basis of $\mathbb{C}^3$, along the lifted submanifold $\pi^{-1}M \subset H_1^5$. Observe that, as well as A and B, E is an endomorphism-valued tensor field on $\pi^{-1}M$. Indeed, the value of E at each point is a matrix in $U(1,2)$. Moreover, since E is the transition matrix from $\{N, J\xi^L, J\eta^L\}$ to the canonical basis of $\mathbb{C}^3$, we must have

$$(2.14)\qquad D_{J\xi^L}E = EA, \qquad D_{J\eta^L}E = EB, \qquad D_V E = i\frac{\sqrt{-c}}{2}E.$$

The last equation follows from the umbilicity of H_1^5 with principal curvature $-\sqrt{-c}/2$, and from the equation of semi-Riemannian submersions [90], [40]

$$D_V X^L = D_{X^L}V = \frac{\sqrt{-c}}{2}JX^L, \qquad X \in \Gamma(TM).$$

It follows from (2.11) and (2.12) that $J\xi^L$, $J\eta^L$ are parallel vector fields on $\pi^{-1}M$. This and the previous equation imply that $\{V, J\xi^L, J\eta^L\}$ is a parallel orthonormal frame on $\pi^{-1}M$. Moreover, it is also easy to check, using (2.10), that A and B commute. Hence, due to the constancy of the entries of the matrices A and B, each equation in the partial differential system (2.14) can be integrated separately to deduce that E is parametrized as

$$E(t,u,v) = e^{it\frac{\sqrt{-c}}{2}}E(0,0,0)e^{uA}e^{vB},$$

where $E(0,0,0)$ is any matrix in $U(1,2)$. Therefore, since N is the normalized position vector field, it follows that the first column of E, multiplied by $2/\sqrt{-c}$, gives the parametrization of $\pi^{-1}M$. In particular, $\pi^{-1}M$ is an open part of an orbit of the action of the abelian Lie subgroup $\{e^{it\frac{\sqrt{-c}}{2}}e^{uA}e^{vB} : t,u,v \in \mathbb{R}\}$ of $U(1,2)$ on H_1^5.

Now, since $\{e^{it\frac{\sqrt{-c}}{2}} : t \in \mathbb{R}\}$ acts trivially on $\mathbb{C}H^2$, it follows that M is contained in an orbit of the action of the abelian Lie subgroup of $U(1,2)$ with Lie algebra $\mathbb{R}A \oplus \mathbb{R}B$. We can define the trace-free matrices $A' = A - \frac{\operatorname{tr} A}{3}\mathrm{Id} = A + \frac{i(\lambda_1+\lambda_2)}{3}\mathrm{Id}$ and $B' = B - \frac{\operatorname{tr} B}{3}\mathrm{Id} = B$, so that $A', B' \in \mathfrak{su}(1,2)$. Therefore, $\mathfrak{h} = \mathbb{R}A' \oplus \mathbb{R}B'$ is an abelian Lie subalgebra of $\mathfrak{su}(1,2)$, and M is an open part of an orbit of the action of the Lie subgroup H of $SU(1,2)$ with Lie algebra $\mathfrak{h}$.

We just have to prove that the H-action on $\mathbb{C}H^2$ is polar. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the simple Lie algebra $\mathfrak{g} = \mathfrak{su}(1,2)$. This decomposition is orthogonal with respect to the inner product on $\mathfrak{g}$ defined by the Killing form of $\mathfrak{g}$. Note also that $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \subset \mathfrak{su}(1,2)$ and $\mathfrak{p}$ consists of all matrices

$$\begin{pmatrix} 0 & \bar{z}_1 & \bar{z}_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(1,2), \qquad \text{with } z_1, z_2 \in \mathbb{C}.$$

Recall that there is an isomorphism between $\mathfrak{p}$ and the tangent space $T_o\mathbb{C}H^2$, where $o = \pi(1,0,0) \in \mathbb{C}H^2$ is the only fixed point of the isotropy group $K = S(U(1)U(2))$. Define the two-dimensional totally real subspace $\mathfrak{s}$ of $\mathfrak{p}$ given by the relation $z_1,\ z_2 \in i\mathbb{R}$. Now we define the totally geodesic submanifold $\Sigma = \{(\operatorname{Exp} X)(o) : X \in \mathfrak{s}\}$ of $\mathbb{C}H^2$, where Exp is the Lie group exponential map. In particular, the isomorphism between $T_o\mathbb{C}H^2$ and $\mathfrak{p}$ allows us to identify $T_o\Sigma$ with $\mathfrak{s}$. We show that the action of H on $\mathbb{C}H^2$ is polar with section Σ. We use the criterion of polarity that can be found in [14, Corollary 3.2].

Proposition 2.6. *Let $\bar{M} = G/K$ be a Riemannian symmetric space of noncompact type, and let Σ be a connected totally geodesic submanifold of $\bar{M}$ with $o \in \Sigma$. A connected closed subgroup H of $I(\bar{M})$ acts polarly on $\bar{M}$ with section Σ if and only if*

1. *$T_o\Sigma \subset \nu_o(H \cdot o)$,*
2. *$T_o\Sigma$ is a section of the slice representation of H_o on $\nu_o(H \cdot o)$,*
3. *$\langle [v, w], X\rangle = 0$ for all $v,\ w \in T_o\Sigma \subset \mathfrak{p}$ and all $X \in \mathfrak{h}$.*

Note first that, since $\mathfrak{h}$ is abelian and $G = SU(1,2)$ has rank 2, H is a closed subgroup of $G = SU(1,2)$. According to the Proposition 2.6, it is sufficient to show that $T_o\Sigma \subset \nu_o(H \cdot o)$, $T_o\Sigma$ is a section for the slice representation of $H_o = \{h \in H : h(o) = o\}$ on $\nu_o(H \cdot o)$, and $\langle [\mathfrak{s}, \mathfrak{s}], \mathfrak{h}\rangle = 0$. It is straightforward to check that $\langle \mathfrak{s}, \mathfrak{h}\rangle = 0$, which implies that $T_o\Sigma$ is orthogonal to $T_o(H \cdot o)$. The second condition is also clearly satisfied, since the orbit of H through o is principal by (2.13) and, hence, its slice representation is trivial and thus polar with section $\nu_o(H \cdot o) = T_o\Sigma$. Finally, again some elementary calculations show that $\langle [\mathfrak{s}, \mathfrak{s}], \mathfrak{h}\rangle = 0$. This proves that the action of H on $\mathbb{C}H^2$ is polar, and hence M is an open part of the principal orbit $H \cdot o$ of the action of H.

For the case $c > 0$, with M nonminimal, we argue analogously as above, adapting the arguments to the Hopf map $\pi \colon S^5 \to \mathbb{C}P^2$. Thus, one shows that $\pi^{-1}M$ is an open part of an orbit of the connected Lie subgroup $\tilde{H}$ of $U(3)$ with Lie algebra $\tilde{\mathfrak{h}} = \mathbb{R}A \oplus \mathbb{R}B \oplus i\frac{\sqrt{c}}{2}\mathrm{Id}$, where now

$$A = \begin{pmatrix} 0 & -\frac{\sqrt{c}}{2} & 0 \\ \frac{\sqrt{c}}{2} & i\frac{c-4(\lambda_1^2+\lambda_2^2)}{4(\lambda_1+\lambda_2)} & ib_1b_2(\lambda_1-\lambda_2) \\ 0 & ib_1b_2(\lambda_1-\lambda_2) & -i\frac{c+8\lambda_1\lambda_2}{4(\lambda_1+\lambda_2)} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{c}}{2} \\ 0 & ib_1b_2(\lambda_1-\lambda_2) & -i\frac{c+8\lambda_1\lambda_2}{4(\lambda_1+\lambda_2)} \\ \frac{\sqrt{c}}{2} & -i\frac{c+8\lambda_1\lambda_2}{4(\lambda_1+\lambda_2)} & ib_1b_2(\lambda_2-\lambda_1) \end{pmatrix}.$$

One can check easily that $\tilde{\mathfrak{h}}$ is an abelian Lie subalgebra of $\mathfrak{u}(3)$ of dimension 3. Since all maximal abelian subalgebras of a compact Lie algebra are conjugate, we have that $\tilde{\mathfrak{h}}$ is conjugate to the standard embedding of $\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)$ in $\mathfrak{u}(3)$. Similarly, $\tilde{H}$ is a maximal torus of $U(3)$, so its action on S^5 is conjugate to the standard action of $U(1)\times U(1)\times U(1)$ on S^5. In particular, the action of $\tilde{H}$ on S^5 is polar, and it induces a polar action on $\mathbb{C}P^2$ that has an orbit containing M (see Subsection 1.7.1).

Finally, we are just left with the case of a minimal surface M. It follows from (2.9) that this case appears only if $c>0$. Moreover, we can put $\lambda_2 = -\lambda_1 = -\frac{\sqrt{c}}{2\sqrt{2}}$ and $\mu = 0$. Now, the matrices A and B are

$$A = \begin{pmatrix} 0 & -\frac{\sqrt{c}}{2} & 0 \\ \frac{\sqrt{c}}{2} & i\frac{\sqrt{c}}{2\sqrt{2}}(b_2^2-b_1^2) & ib_1b_2\frac{\sqrt{c}}{\sqrt{2}} \\ 0 & ib_1b_2\frac{\sqrt{c}}{\sqrt{2}} & i\frac{\sqrt{c}}{2\sqrt{2}}(b_1^2-b_2^2) \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{c}}{2} \\ 0 & ib_1b_2\frac{\sqrt{c}}{\sqrt{2}} & i\frac{\sqrt{c}}{2\sqrt{2}}(b_1^2-b_2^2) \\ \frac{\sqrt{c}}{2} & i\frac{\sqrt{c}}{2\sqrt{2}}(b_1^2-b_2^2) & -ib_1b_2\frac{\sqrt{c}}{\sqrt{2}} \end{pmatrix}.$$

The argument to conclude is then the same as in the previous paragraph. □

Proof of (v) ⇒ (iv). It is well-known that any principal orbit of a polar action on a Riemannian manifold is isoparametric; see [57, p. 152], [9, Corollary 3.2.5]. □

Proof of (iv) ⇒ (iii). Let M be an isoparametric submanifold of $\bar{M}^2(c)$, $c\neq 0$. By definition, M has a section at every point, that is, for each $p\in M$ there exists a totally geodesic submanifold Σ_p such that $T_p\Sigma_p = \nu_p M$. Totally geodesic submanifolds of complex space forms are known to be either complex or totally real.

First we assume that the section is complex. Then, M is almost complex, and it is well-known that an almost complex submanifold in a Kähler manifold is Kähler. Since the normal bundle of M is flat, [4, Theorem 19] implies that M is either a point or an open part of $\bar{M}^2(c)$.

Hence, we may assume from now on that sections are totally real. Since M has codimension 2, sections are totally geodesic real projective planes $\mathbb{R}P^2$ in $\mathbb{C}P^2$ or real hyperbolic planes $\mathbb{R}H^2$ in $\mathbb{C}H^2$. Since in this case sections are totally real, it follows that TM and νM are both totally real. Indeed, M is Lagrangian as $JT_pM = \nu_p M$ for each $p \in M$.

If M is totally umbilical, then it follows from [29] that M is an open part of a totally geodesic real projective plane $\mathbb{R}P^2$ in $\mathbb{C}P^2$ or a totally geodesic real hyperbolic plane $\mathbb{R}H^2$ in $\mathbb{C}H^2$. However, these are not isoparametric because their normal bundle is not flat. Indeed, by the Ricci equation

$$\langle R^\perp(X,Y)JX, JY\rangle = \langle \bar{R}(X,Y)JX, JY\rangle - \langle [\mathcal{S}_{JX}, \mathcal{S}_{JY}]X, Y\rangle = -\frac{c}{4}$$

for X, $Y \in TM$, where we have used that JX, $JY \in \nu M$.

Since the normal bundle of M is flat, for each parallel normal vector field ξ and each sufficiently small $r > 0$, we can consider the parallel submanifolds determined by the vector field ξ, $M^{r,\xi} = \{\exp(r\xi_p) : p \in M\}$. Our objective is the study of local geometric properties of the displacement of M in the direction given by ξ at a certain distance r.

We denote by $\nu^1 M$ the unit normal bundle of M. By assumption νM is flat. For a given parallel unit normal vector field $\xi \in \Gamma(\nu^1 M)$ and $r > 0$ we define $\Phi^{r,\xi} \colon M \to \bar{M}^2(c)$, $p \mapsto \exp_p(r\xi)$. Let γ_{ξ_p} be the geodesic of $\bar{M}^2(c)$ with initial conditions $\gamma_{\xi_p}(0) = p$, $\gamma_{\xi_p}'(0) = \xi_p$. We also define the vector field η^r along $\Phi^{r,\xi}$ by $\eta^r(p) = \gamma_{\xi_p}'(r)$ for each $p \in M$. Parallel submanifolds to M are of the form $M^{r,\xi} = \Phi^{r,\xi}(M)$. Clearly, $M^{r,\xi}$ is an immersed submanifold of $\bar{M}^2(c)$ if and only if $\Phi^{r,\xi}$ is an immersion. It may happen, however, that $M^{r,\xi}$ is a focal submanifold, that is, the codimension of $M^{r,\xi}$ is greater than the codimension of M . The fact that $M^{r,\xi}$ has higher codimension depends on the rank of $\Phi^{r,\xi}$. But for r small enough $M^{r,\xi}$ is a parallel submanifold with the same dimension as M. Our aim is then to calculate the mean curvature of $M^{r,\xi}$ at $\Phi^{r,\xi}(p)$ in the direction of $\eta^r(p)$.

We denote by $\lambda_1, \lambda_2 \colon \nu^1 M \to \mathbb{R}$ the principal curvature functions, which are given by the fact that $\lambda_1(\xi)$ and $\lambda_2(\xi)$ are the eigenvalues of the shape operator $\mathcal{S}_\xi$ for a fixed parallel unit normal vector field $\xi \in \Gamma(\nu^1 M)$. We have already seen that M cannot be umbilical, so we may assume that there exists $\xi \in \nu^1 M$ such that $\lambda_1(\xi) \neq \lambda_2(\xi)$. By continuity, the principal curvature functions are thus different on an open neighbourhood of ξ in $\nu^1 M$. In the sequel we assume that calculations take place in such a neighbourhood. We also denote by $U_1(\xi)$ and $U_2(\xi)$ a (local) orthonormal frame of TM consisting of principal curvature vectors associated with $\lambda_1(\xi)$ and $\lambda_2(\xi)$.

One can study the geometric behaviour of a submanifold when this is moved along normal directions. This is an important method in submanifold theory which is based on Jacobi vector field theory. More features of this method can be found in [9, Chapter 8].

Let $p \in M$. Let c be a curve in M such that $c(0) = p$ and $c'(0) = v$. Then, $F(s,t) = \exp_{c(s)}(t\xi_{c(s)})$ is a variation of γ_{ξ_p} through geodesics. Let X_v be the variational vector field of F along γ_{ξ_p}. Then, X_v is a Jacobi vector field, that is, a vector field along γ_{ξ_p} satisfying the initial value problem

$$X_v'' + \bar{R}(X_v, \gamma_{\xi_p}')\gamma_{\xi_p}' = 4X_v'' + cX_v + 3c\langle X_v, J\gamma_{\xi_p}'\rangle J\gamma_{\xi_p}' = 0, \quad X_v(0) = v, \quad X_v'(0) = -\mathcal{S}_\xi(v).$$

Here $(\cdot)'$ stands for covariant derivative along γ_ξ. A straightforward calculation shows that $\Phi^{r,\xi}_{*p}(v) = X_v(r)$ for each $v \in T_pM$.

In order to simplify notation we define $u_i = U_i(\xi_p)$, $i = 1, 2$, and we set $v = u_i$ in the previous calculations. Then

$$X_{u_i}(t) = f_{\lambda_i}(t)\mathcal{P}^\xi_{u_i}(t) + \langle u_i, J\xi\rangle g_{\lambda_i}(t) J\gamma'_\xi(t),$$

where $\mathcal{P}^\xi_v(t)$ denotes parallel transport of $v \in T_pM$ along the geodesic γ_ξ. The functions f_λ and g_λ are defined by

$$f_\lambda(t) = \cosh\Big(\frac{t\sqrt{-c}}{2}\Big) - \frac{2\lambda}{\sqrt{-c}}\sinh\Big(\frac{t\sqrt{-c}}{2}\Big),$$
$$g_\lambda(t) = \Big(\cosh\Big(\frac{t\sqrt{-c}}{2}\Big) - 1\Big)\Big(1 + 2\cosh\Big(\frac{t\sqrt{-c}}{2}\Big) - \frac{2\lambda}{\sqrt{-c}}\sinh\Big(\frac{t\sqrt{-c}}{2}\Big)\Big).$$

(For $c > 0$ one would have to replace hyperbolic trigonometric functions by standard trigonometric functions.) In other words, X_{u_i} is the parallel transport along γ_ξ of the tangent vector $f_{\lambda_i}u_i + \langle u_i, J\xi\rangle g_{\lambda_i} J\xi$.

Since the normal Riemannian exponential map is a local diffeomorphism, it is clear that $\Phi^{r,\xi}$ is a local diffeomorphism for sufficiently small values of r. Thus, we will take, if necessary, a sufficiently small neighbourhood of p and sufficiently small values of r so that $\Phi^{r,\xi}$ is a diffeomorphism.

At this point we recall that M has totally real tangent and normal bundles. Thus, $J\xi$ is tangent to M and can be written as $J\xi = \langle U_1(\xi), J\xi\rangle U_1(\xi) + \langle U_2(\xi), J\xi\rangle U_2(\xi)$. Moreover, since $T_{\Phi^{r,\xi}(p)}M^{r,\xi} = \Phi^{r,\xi}_{*p}(T_pM)$ and $\Phi^{r,\xi}$ is a diffeomorphism, it is then clear that $T_{\Phi^{r,\xi}(p)}M^{r,\xi} = \mathcal{P}^\xi_{T_pM}(r)$, that is, the tangent space of $M^{r,\xi}$ at $\Phi^{r,\xi}(r)$ is obtained by parallel translation of T_pM along the geodesic γ_ξ from $p = \gamma_\xi(0)$ to $\Phi^{r,\xi}(r) = \gamma_\xi(r)$.

The previous considerations allow us to define the endomorphism-valued map of the tangent space $D_\xi(t)\colon T_{\Phi^{t,\xi}(p)}M^{t,\xi} \to T_{\Phi^{t,\xi}(p)}M^{t,\xi}$ by $D_\xi(t)(\mathcal{P}^\xi_v(t)) = X_v(t)$, where $v \in T_pM$. As we are assuming that r is sufficiently small, $D_\xi(r)$ is actually an isomorphism of the tangent space. We denote now by $\mathcal{S}^{r,\xi}_{\eta^r}$ the shape operator of $M^{r,\xi}$ with respect to the radial vector η^r. It follows from Jacobi field theory that $\mathcal{S}^{r,\xi}_{\eta^r}(\Phi^{r,\xi}_{*p}(v)) = -X'_v(r)^\top$, where $(\cdot)^\top$ denotes the orthogonal projection onto the tangent space $T_{\Phi^{r,\xi}(p)}M^{r,\xi}$. By the previous calculations, $X'_{u_i}(t) = f'_{\lambda_i}(t)\mathcal{P}^\xi_{u_i}(t) + \langle u_i, J\xi\rangle g'_{\lambda_i}(t)J\gamma'_\xi(t) \in T_{\Phi^{r,\xi}(p)}M^{r,\xi}$. This implies that $\mathcal{S}^{r,\xi}_{\eta^r} = -D'_\xi(r)D_\xi(r)^{-1}$. Finally, the mean curvature in radial directions is the function $h^{r,\xi}\colon M^{r,\xi} \to \mathbb{R}$ determined by

$$h^{r,\xi}(\Phi^{r,\xi}(p)) = \operatorname{tr}\mathcal{S}^{r,\xi}_{\eta^r(p)} = -\operatorname{tr} D'_\xi(r)D_\xi(r)^{-1} = -\frac{\frac{d}{dr}\det D_\xi(r)}{\det D_\xi(r)}.$$

It is easy to check that $\det D_\xi = f_{\lambda_1}f_{\lambda_2} + \langle U_1(\xi), J\xi\rangle^2 f_{\lambda_2}g_{\lambda_1} + \langle U_2(\xi), J\xi\rangle^2 f_{\lambda_1}g_{\lambda_2}$. The function $h^{r,\xi} \circ \Phi^{r,\xi}$ can be calculated explicitly, but for our purpose it suffices to calculate its Taylor power series expansion. After some relatively long but elementary calculations, and using $\langle U_1(\xi), J\xi\rangle^2 + \langle U_2(\xi), J\xi\rangle^2 = \langle J\xi, J\xi\rangle = 1$, we get

$$\begin{aligned}(h^{r,\xi} \circ \Phi^{r,\xi})(p) = \lambda_1(\xi_p) + \lambda_2(\xi_p) + r\Big(\frac{5c}{4} + \lambda_1(\xi_p)^2 + \lambda_2(\xi_p)^2\Big) \\ + \frac{r^2}{4}\Big(c\big(\lambda_1(\xi_p) + \lambda_2(\xi_p)\big) + 4\big(\lambda_1(\xi_p)^3 + \lambda_2(\xi_p)^3\big) \\ + 3c\big(\lambda_1(\xi_p)\langle U_1(\xi_p), J\xi_p\rangle^2 + \lambda_2(\xi_p)\langle U_2(\xi_p), J\xi_p\rangle^2\big)\Big) + O(r^3).\end{aligned}$$

Since M is isoparametric, $h^{r,\xi}$ is constant. Since $\Phi^{r,\xi}$ is a diffeomorphism, this is equivalent to requiring that $(h^{r,\xi} \circ \Phi^{r,\xi})(p)$ does not depend on p. More precisely, $(h^{r,\xi} \circ \Phi^{r,\xi})(p)$ depends on r and the choice of parallel unit normal vector field $\xi \in \Gamma(\nu^1 M)$, but not on the base point p of ξ_p. Therefore, the above power series expansion implies that the functions $p \mapsto \lambda_i(\xi)(p) = \lambda_i(\xi_p)$, $p \mapsto \langle U_i(\xi), J\xi\rangle(p) = \langle U_i(\xi_p), J\xi_p\rangle$, $i = 1, 2$, are constant for a fixed parallel $\xi \in \Gamma(\nu^1 M)$. By linearity this readily implies that an isoparametric submanifold of $\bar{M}^2(c)$ has constant principal curvatures and is Terng-isoparametric. Then (iv) $\Rightarrow$ (iii). □

Proof of (iii) $\Rightarrow$ (ii). Let M be a Lagrangian and Terng-isoparametric submanifold. Then, by Lemma 2.4 M is flat. Moreover, M has parallel mean curvature, since M has flat normal bundle and the principal curvatures of M are constant for any parallel normal vector field. Thus, the arguments and notation used in the proof of (i) $\Rightarrow$ (iii) above can be applied to M. In particular, there are parallel orthonormal frames $\{\xi, \eta\}$ of νM, and $\{U_1, U_2\}$ of TM, such that $\langle \mathcal{S}_\xi U_j, U_k\rangle$ and $\langle \mathcal{S}_\eta U_j, U_k\rangle$ are constant on M, for $j, k \in \{1, 2\}$. Thus, for all $i, j, k \in \{1, 2\}$ we have:

$$\begin{aligned}(\nabla^\perp_{U_i} I\!I)(U_j, U_k) &= \nabla^\perp_{U_i} I\!I(U_j, U_k) - I\!I(\nabla_{U_i} U_j, U_k) - I\!I(U_j, \nabla_{U_i} U_k) \\ &= \nabla^\perp_{U_i}(\langle \bar{\nabla}_{U_j} U_k, \xi\rangle\xi + \langle \bar{\nabla}_{U_j} U_k, \eta\rangle\eta) \\ &= \nabla^\perp_{U_i}(\langle \mathcal{S}_\xi U_j, U_k\rangle\xi + \langle \mathcal{S}_\eta U_j, U_k\rangle\eta) = 0.\end{aligned}$$

Hence, M has parallel second fundamental form. Thus, (iii) implies (ii). □

We have thus concluded the proof of Theorem 2.1. As a corollary, we obtain the following result:

Corollary 2.7. *A Lagrangian submanifold of $\bar{M}^2(c)$ is isoparametric if and only if it is Terng-isoparametric.*

Remark 2.8. There is a shorter alternative proof of the implication (iv) $\Rightarrow$ (iii) that does not require working with Jacobi fields. Indeed, in the proof of this implication, once the problem was reduced to the case of an isoparametric Lagrangian surface M, we could have used Lemma 2.4 to show that M is flat. Since by assumption M is Lagrangian and has parallel mean curvature, then (i) holds and, since we had already proved that (i) $\Rightarrow$ (iii), we can conclude that (iv) $\Rightarrow$ (iii).

Remark 2.9. Surfaces with parallel mean curvature in two-dimensional complex space forms have been studied by Ogata [89], Kenmotsu and Zhou [70], Kenmotsu and Masuda [69], and Hirakawa [61], [62]. In [41] the equivalence (i)⇔(v) was proved using the strong results in the previously mentioned papers. However, in this thesis we have presented a direct approach that does not make use of any of these articles.

2.3 Classification

In this section we classify isoparametric submanifolds and Terng-isoparametric submanifolds in the 2-dimensional nonflat complex space forms. A special surface called the Chen's surface appears in this classification of Terng-isoparametric submanifolds.

2.3.1 Chen's surface

In this subsection we give a Lie group theoretic description of the surface introduced by Chen in [26] that arises in Theorem 2.3 (ii).

First we recall the characterizing properties of this surface according to [26]. A surface M in $\mathbb{C}H^2$ is called slant if its tangent space has constant Kähler angle (called Wirtinger angle or slant angle in [26]), that is, if for each nonzero vector $v \in T_pM$ the angle between Jv and T_pM is independent of $p \in M$ and $v \in T_pM$. Such surface is called proper slant if it is neither complex nor totally real, that is, if the Kähler angle is neither 0 nor $\pi/2$. The Chen's surface that appears in Theorem 2.3 (ii) is a proper slant surface of $\mathbb{C}H^2$ with Kähler angle $\theta = \arccos(1/3)$ and satisfying $\langle H, H\rangle = 8K - 2c(1 + 3\cos^2\theta)$, where K is the Gaussian curvature of M. It is proved in [30, Theorem 5.1] that K is constant, and that Chen's surface is unique up to isometric congruence.

The Chen's surface turns out to be homogeneous, although not an orbit of a polar action (by the classification in Theorem 1.7), and the aim of this subsection is to give a subgroup of the isometry group of $\mathbb{C}H^2$ one of whose orbits is precisely the Chen's surface. In Subsection 1.6.4 we have described $\mathbb{C}H^n$ as a symmetric space and as a solvable Lie group. Let $\mathfrak{a} \oplus \mathfrak{n}$ be the solvable part of the Iwasawa descomposition of $\mathfrak{g} = \mathfrak{su}(1, 2)$ and AN the connected subgroup of $SU(1, 2)$ whose Lie algebra is $\mathfrak{a} \oplus \mathfrak{n}$. Let B be a unit vector in $\mathfrak{a}$ and define $Z = JB \in \mathfrak{g}_{2\alpha}$.

Now assume that $U \in \mathfrak{g}_\alpha$ is a unit vector. We have $\mathfrak{g}_\alpha = \mathbb{R}U \oplus \mathbb{R}JU$. We define the following subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$:

$$\mathfrak{h} = \mathbb{R}U_1 \oplus \mathbb{R}U_2, \quad \text{with} \quad U_1 = \frac{1}{\sqrt{3}}\Big(\sqrt{2}B + JU\Big), \quad \text{and} \quad U_2 = \frac{1}{\sqrt{3}}\Big(U + \sqrt{2}Z\Big).$$

Let H be the connected subgroup of AN whose Lie algebra is $\mathfrak{h}$, and $M = H \cdot o$ the orbit through the unique fixed point $o \in \mathbb{C}H^2$ of $K = S(U(1)U(2))$. Since AN acts simply transitively on $\mathbb{C}H^2$ we may identify H with M for the calculations that follow.

First notice that $\{U_1, U_2\}$ is an orthonormal basis of the tangent space of M, and $\langle JU_1, U_2\rangle = 1/3$. By homogeneity we conclude that M is a proper slant surface with

Kähler angle $\theta = \arccos(1/3)$. Using (1.1) we get the mean curvature vector and the Gaussian curvature

$$H = \frac{2\sqrt{-c}}{3}\Big(B - \sqrt{2}JU\Big), \quad \text{and} \quad K = \frac{c}{6}.$$

It readily follows from this equation that $\langle H, H\rangle = 8K - 2c(1 + 3\cos^2\theta)$ and hence, [26, Theorem A] and [30, Theorem 5.1] imply that M is isometrically congruent to Chen's surface.

2.3.2 Proof of Theorem 2.2

Let M be an isoparametric submanifold of $\bar{M}^2(c)$, $c \neq 0$, and Σ_p a section for M through $p \in M$. Recall that Σ_p is a totally geodesic submanifold, and totally geodesic submanifolds of complex space forms are known to be either complex or totally real.

As we argued in the proof of the implication (iv) $\Rightarrow$ (iii) of Theorem 2.1, if we assume that the section is complex, M is either a point or an open part of $\bar{M}^2(c)$.

Hence, we may assume from now on that sections are totally real. In this case, sections are either geodesics or totally geodesic real projective planes $\mathbb{R}P^2$ in $\mathbb{C}P^2$ or real hyperbolic planes $\mathbb{R}H^2$ in $\mathbb{C}H^2$. If the section is a geodesic, M is an isoparametric hypersurface. The classification of isoparametric hypersurfaces in $\mathbb{C}P^2$ follows from [49], and all examples are open parts of orbits of cohomogeneity one actions. Isoparametric hypersurfaces in $\mathbb{C}H^n$ have been classified in [40, Corollary 1.2] and it follows from this paper that M is an open part of a principal orbit of a cohomogeneity one action on $\mathbb{C}H^2$.

Finally, if M has codimension 2 it follows from Theorem 2.1 that an isoparametric submanifold of $\bar{M}^2(c)$ is an open part of a principal orbit of a cohomogeneity two polar action on $\bar{M}^2(c)$. This concludes the proof of Theorem 2.2.

This result implies that the classification of isoparametric submanifolds in $\bar{M}^2(c)$ is equivalent to the classification of polar actions on $\bar{M}^2(c)$, whose description appears in Section 1.7.

2.3.3 Proof of Theorem 2.3

Let M now be a Terng-isoparametric submanifold of $\bar{M}^2(c)$. In particular, the normal bundle of M is flat, and we have already seen in the proof of the implication (iv) $\Rightarrow$ (iii) of Theorem 2.1 that, if the normal bundle of M is complex, then M is either a point or an open part of $\bar{M}^2(c)$. Thus we may assume that the normal bundle of M is not complex.

If the normal bundle of M is totally real, M is either a hypersurface or a Lagrangian submanifold. In the first case, M is a hypersurface of $\bar{M}^2(c)$ with constant principal curvatures. These were classified in [106] for $\mathbb{C}P^2$ and in [11] for $\mathbb{C}H^2$ where it was shown that such hypersurfaces are open parts of homogeneous hypersurfaces. In particular they are open parts of orbits of cohomogeneity one actions, which are polar.

If the normal bundle is totally real and has rank 2, then M is Lagrangian. Hence, it follows from Theorem 2.1 that M is an open part of a principal orbit of a cohomogeneity two polar action on $\bar{M}^2(c)$.

Therefore, we can assume from now on that the normal bundle of M is neither complex nor totally real. If M is 1-dimensional, then M has to be a geodesic or a circle (Section 1.2), so we also assume that M is 2-dimensional.

Hence, we take, at least locally, a parallel orthonormal basis $\{\xi, \eta\}$ of the normal bundle of M, and let $\{U_1, U_2\}$ be an orthonormal basis of the tangent space of M such that $\mathcal{S}_\xi U_i = \lambda_i U_i$, $i = 1, 2$. Since ξ is parallel, λ_1 and λ_2 are constant by assumption. At this moment we observe that the mean curvature vector field of M is parallel because the normal bundle is flat and the principal curvatures are constant (and hence the trace of each shape operator with respect to a parallel normal vector field is constant). Therefore, we may further assume that $\{\xi, \eta\}$ is chosen so that η is perpendicular to the mean curvature vector field.

Using the fact that the tangent and normal spaces are neither complex nor totally real we can write $J\xi = b_1 U_1 + b_2 U_2 + a\eta$, where $a, b_1, b_2 \colon M \to \mathbb{R}$ are smooth functions with $b_1^2 + b_2^2 + a^2 = 1$, and $b_1^2 + b_2^2 \neq 0$, $a \neq 0$. Since $\{U_1, U_2, \xi, \eta\}$ is an orthonormal frame of $T\mathbb{C}H^2$ along M we can write

$$\begin{aligned}
-\xi = J^2\xi &= b_1 JU_1 + b_2 JU_2 + aJ\eta \\
&= b_1(\langle JU_1, U_2\rangle U_2 - b_1\xi + \langle JU_1, \eta\rangle\eta) + b_2(-\langle JU_1, U_2\rangle U_1 - b_2\xi + \langle JU_2, \eta\rangle\eta) \\
&\quad + a(-\langle JU_1, \eta\rangle U_1 - \langle JU_2, \eta\rangle\eta - a\xi) \\
&= (-b_2\langle JU_1, U_2\rangle - a\langle JU_1, \eta\rangle)U_1 + (b_1\langle JU_1, U_2\rangle - a\langle JU_2, \eta\rangle)U_2 \\
&\quad + (b_1\langle JU_1, \eta\rangle + b_2\langle JU_2, \eta\rangle)\eta - \xi.
\end{aligned}$$

Thus, $-b_2\langle JU_1, U_2\rangle - a\langle JU_1, \eta\rangle = b_1\langle JU_1, U_2\rangle - a\langle JU_2, \eta\rangle = b_1\langle JU_1, \eta\rangle + b_2\langle JU_2, \eta\rangle = 0$. Using these equalities and $b_1^2 + b_2^2 + a^2 = 1$, it is easy to show that we can write (up to a choice of orientation)

$$\begin{aligned}
J\xi &= b_1 U_1 + b_2 U_2 + a\eta, & J\eta &= -b_2 U_1 + b_1 U_2 - a\xi, \\
JU_1 &= -aU_2 - b_1\xi + b_2\eta, & JU_2 &= aU_1 - b_2\xi - b_1\eta.
\end{aligned}$$

For $i \in \{1, 2\}$, using the Codazzi equation, taking into account that ξ is parallel and that λ_1 and λ_2 are constant, we get

$$-\frac{3cab_i}{4} = \langle \bar{R}(U_1, U_2)U_i, \xi\rangle = (\lambda_2 - \lambda_i)\langle \nabla_{U_1} U_2, U_i\rangle - (\lambda_1 - \lambda_i)\langle \nabla_{U_2} U_1, U_i\rangle.$$

Since $a \neq 0$ and $b_1^2 + b_2^2 \neq 0$, we readily get $\lambda_1 \neq \lambda_2$. Since $\{U_1, U_2\}$ is an orthonormal basis of the tangent space we obtain

$$\nabla_{U_i} U_i = -\frac{3cab_i}{4(\lambda_1 - \lambda_2)} U_j, \qquad \nabla_{U_i} U_j = \frac{3cab_i}{4(\lambda_1 - \lambda_2)} U_i, \qquad i, j \in \{1, 2\}, i \neq j. \tag{2.15}$$

Now, since νM is flat, the Ricci equation implies

$$\begin{aligned}\frac{c}{4}(-b_1^2 - b_2^2 + 2a^2) = \langle \bar{R}(U_1, U_2)\xi, \eta\rangle &= \langle \mathcal{S}_\xi U_1, \mathcal{S}_\eta U_2\rangle - \langle \mathcal{S}_\eta U_1, \mathcal{S}_\xi U_2\rangle \\ &= (\lambda_1 - \lambda_2)\langle \mathcal{S}_\eta U_1, U_2\rangle.\end{aligned}$$

Recall that, since η is perpendicular to the mean curvature vector, we have $\operatorname{tr} \mathcal{S}_\eta = 0$, and thus, with respect to the orthonormal basis $\{U_1, U_2\}$ the shape operator $\mathcal{S}_\eta$ can be written as

$$\mathcal{S}_\eta = \begin{pmatrix} \mu & -\frac{c(1-3a^2)}{4(\lambda_1-\lambda_2)} \\ -\frac{c(1-3a^2)}{4(\lambda_1-\lambda_2)} & -\mu \end{pmatrix}, \tag{2.16}$$

for some function $\mu\colon M \to \mathbb{R}$.

By assumption, the eigenvalues of $\mathcal{S}_\eta$ are constant, or equivalently, the functions

$$\operatorname{tr} \mathcal{S}_\eta = 0 \quad \text{and} \quad \operatorname{tr} \mathcal{S}_\eta^2 = 2\mu^2 + \frac{c^2(1-3a^2)^2}{8(\lambda_1-\lambda_2)^2} \tag{2.17}$$

are constant.

Now we calculate the derivatives of the functions b_1, b_2 and a. We take $i, j \in \{1, 2\}$, $i \neq j$. Using (2.15) and (2.16) we obtain

$$\begin{aligned}
U_i b_i &= U_i\langle U_i, J\xi\rangle = \langle \bar{\nabla}_{U_i} U_i, b_i U_i + b_j U_j + a\eta\rangle + \langle U_i, \bar{\nabla}_{U_i} J\xi\rangle \\
&= b_j\langle \nabla_{U_i} U_i, U_j\rangle + a\langle U_i, \mathcal{S}_\eta U_i\rangle - \lambda_i\langle U_i, JU_i\rangle = -\frac{3cab_1b_2}{4(\lambda_1-\lambda_2)} - a(-1)^i\mu, \\
U_i b_j &= U_i\langle U_j, J\xi\rangle = \langle \bar{\nabla}_{U_i} U_j, b_i U_i + b_j U_j + a\eta\rangle + \langle U_j, \bar{\nabla}_{U_i} J\xi\rangle \\
&= b_i\langle \nabla_{U_i} U_j, U_i\rangle + a\langle U_j, \mathcal{S}_\eta U_i\rangle - \lambda_i\langle U_j, JU_i\rangle \\
&= \frac{3cab_i^2}{4(\lambda_1-\lambda_2)} - \frac{ca(1-3a^2)}{4(\lambda_1-\lambda_2)} - a(-1)^i\lambda_i, \\
U_i a &= U_i\langle J\xi, \eta\rangle = \langle \bar{\nabla}_{U_i} J\xi, \eta\rangle + \langle b_i U_i + b_j U_j + a\eta, \bar{\nabla}_{U_i}\eta\rangle \\
&= -\lambda_i\langle JU_i, \eta\rangle - b_i\langle U_i, \mathcal{S}_\eta U_i\rangle - b_j\langle U_j, \mathcal{S}_\eta U_i\rangle \\
&= b_j(-1)^i\lambda_i + b_i(-1)^i\mu + \frac{cb_j(1-3a^2)}{4(\lambda_1-\lambda_2)}.
\end{aligned} \tag{2.18}$$

In order to get a relation for the derivatives of μ, we use the Codazzi equation together with (2.15), (2.16) and (2.18) to get, after some calculations

$$\begin{aligned}
-\frac{3c(-1)^i ab_j}{4} &= \langle \bar{R}(U_1, U_2)U_i, \eta\rangle \\
&= \langle \nabla_{U_1}\mathcal{S}_\eta U_2, U_i\rangle - \langle \nabla_{U_1} U_2, \mathcal{S}_\eta U_i\rangle - \langle \nabla_{U_2}\mathcal{S}_\eta U_1, U_i\rangle + \langle \nabla_{U_2} U_1, \mathcal{S}_\eta U_i\rangle \\
&= -U_j\mu - \frac{3ca(b_j\lambda_i + 2b_i\mu)}{2(\lambda_1-\lambda_2)}.
\end{aligned}$$

Thus, we obtain

$$U_i\mu = \frac{3ca}{4(\lambda_1-\lambda_2)}(b_i\lambda_i - 3b_i\lambda_j - 4b_j\mu), \quad i,j\in\{1,2\}, i\neq j. \tag{2.19}$$

The aim of the argument that follows is to show that the functions b_1, b_2, a and μ are constant. We first have

Lemma 2.10. *If the function $a\colon M\to\mathbb{R}$ is constant, then b_1, b_2 and μ are also constant.*

Proof. If a is constant, it readily follows from (2.17) that μ is constant. Hence, from (2.19) we get $(\lambda_1-3\lambda_2)b_1 - 4\mu b_2 = -4\mu b_1 + (\lambda_2 - 3\lambda_1)b_2 = 0$. This is a homogeneous linear system in the variables b_1 and b_2, whose coefficients are constant. It cannot have a unique solution because $b_1=b_2=0$ is not possible, and thus the rank of the matrix of the system cannot be 2. The rank cannot be 0 because that would imply $\lambda_1=\lambda_2=0$. Thus, it has rank one and we can write $b_2=\nu b_1$ for some constant $\nu\in\mathbb{R}$. Then $1-a^2=b_1^2+b_2^2=(1+\nu^2)b_1^2$ implies that b_1 is constant, and hence also b_2. □

In view of Lemma 2.10, the calculations that follow aim at proving that a is constant. Recall from (2.17) that $\operatorname{tr}\mathcal{S}_\eta^2$ is constant. Hence there is $k\in\mathbb{R}$ such that

$$\mu^2 = k - \frac{c^2(1-3a^2)}{16(\lambda_1-\lambda_2)^2}. \tag{2.20}$$

Taking derivatives with respect to U_i, using (2.18) and (2.19) and substituting μ^2 by (2.20) we get, after some calculations

$$\begin{aligned} 0 &= b_j\big((-1)^j c^2(1-3a^2)^2 + 4c(1-3a^2)\lambda_i(\lambda_1-\lambda_2) + 32(-1)^i k(\lambda_1-\lambda_2)^2\big) \\ &\quad + 4b_i(\lambda_1-\lambda_2)\big(c(1-3a^2) - 2(-1)^i(\lambda_1-\lambda_2)(\lambda_i-3\lambda_j)\big)\mu. \end{aligned} \tag{2.21}$$

If $c(1-3a^2)+2(\lambda_1-\lambda_2)(\lambda_1-3\lambda_2)$ or $c(1-3a^2)-2(\lambda_1-\lambda_2)(\lambda_2-3\lambda_1)$ is zero in an open set, then the function a is constant and it follows from Lemma 2.10 that b_1, b_2 and μ are also constant. As a consequence, we may assume that there is a point in M where these two functions do not vanish, and thus, they do not vanish in an open set. Moreover, if $b_i=0$ in an open set, then it follows from the first equation in (2.18) that $\mu=0$, so by (2.20), a is constant, and thus also b_j. Hence, we also assume that b_i, $i=1,2$, is not zero on an open set. Thus, from (2.21) we get two possible expressions for μ, and combining this with (2.20) yields

$$\begin{aligned} 0 &= \Big(k - \frac{c^2(1-3a^2)^2}{16(\lambda_1-\lambda_2)^2}\Big) - \Big(-b_2\frac{c^2(1-3a^2)^2+4c(1-3a^2)\lambda_1(\lambda_1-\lambda_2)-32k(\lambda_1-\lambda_2)^2}{4b_1(\lambda_1-\lambda_2)(c(1-3a^2)+2\lambda_1^2-8\lambda_1\lambda_2+6\lambda_2^2)}\Big)\cdot \\ &\qquad\cdot\Big(b_1\frac{c^2(1-3a^2)^2-4c(1-3a^2)\lambda_2(\lambda_1-\lambda_2)-32k(\lambda_1-\lambda_2)^2}{4b_2(\lambda_1-\lambda_2)(c(1-3a^2)+6\lambda_1^2-8\lambda_1\lambda_2+2\lambda_2^2)}\Big) \\ &= \frac{-c^3(1-3a^2)^3-3c^2(1-3a^2)^2(4k+(\lambda_1-\lambda_2)^2)+16k(\lambda_1-\lambda_2)^2(16k+3\lambda_1^2-10\lambda_1\lambda_2+3\lambda_2)}{4(c(1-3a^2)+2\lambda_1^2-8\lambda_1\lambda_2+6\lambda_2^2)(c(1-3a^2)+6\lambda_1^2-8\lambda_1\lambda_2+2\lambda_2^2)}. \end{aligned}$$

This equation implies that $1-3a^2$ is constant, and hence, by Lemma 2.10 we get that b_1, b_2 and μ are also constant.

Using (2.18) we get

$$0 = U_1 b_1 + U_2 b_2 = -\frac{3cab_1b_2}{2(\lambda_1 - \lambda_2)},$$

and since $a \neq 0$ we get $b_1 = 0$ or $b_2 = 0$. We may assume $b_1 \neq 0$, $b_2 = 0$, $a^2 = 1 - b_1^2$. Then, by (2.18) we obtain $0 = U_2 b_2 = -a\mu$, so $\mu = 0$. Next, equation (2.19) implies that $0 = U_1\mu = 3cab_1(\lambda_1 - 3\lambda_2)/(4(\lambda_1 - \lambda_2))$, and thus, $\lambda_1 = 3\lambda_2 \neq 0$. Finally, using (2.18) once more,

$$0 = U_1 b_2 = \frac{3cab_1^2 - ca(1-3a^2)}{4(\lambda_1 - \lambda_2)} + a\lambda_1 = \frac{a(c + 12\lambda_2^2)}{4\lambda_2}.$$

Hence, if $c > 0$ we get a contradiction, which yields

Proposition 2.11. *A Terng-isoparametric surface of $\mathbb{C}P^2$ is isoparametric.*

Otherwise, if $c < 0$ we have $\lambda_2 = \pm\sqrt{-3c}/6$. By changing the orientation if necessary, we may assume $\lambda_2 > 0$. Finally, (2.18) yields $0 = U_2 a = cb_1(9b_1^2 - 8)/(4\sqrt{-3c})$. Altogether we have obtained

$$\mathcal{S}_\xi = \begin{pmatrix} \frac{\sqrt{-3c}}{2} & 0 \\ 0 & \frac{\sqrt{-3c}}{6} \end{pmatrix}, \qquad \mathcal{S}_\eta = \begin{pmatrix} 0 & \frac{\sqrt{-3c}}{6} \\ \frac{\sqrt{-3c}}{6} & 0 \end{pmatrix}, \qquad a = \frac{1}{3}, \qquad b_1 = \frac{2\sqrt{2}}{3}, \qquad b_2 = 0.$$

Finally, it follows from [30, Theorem 5.1(vi)] that M is an open part of a Chen's surface, as we wanted to show.

Chapter 3

Real hypersurfaces

The interest of studying real hypersurfaces in Kähler manifolds appeared in the field of Complex Analysis. In the theory of several complex variables, an important problem is to understand the relation between holomorphic functions defined on a domain of the complex space $\mathbb{C}^n$, and the boundary of such domain. When this boundary is smooth, it becomes a real hypersurface, that is, a submanifold of the Euclidean space $\mathbb{R}^{2n}$ with real codimension one. See [68] for a survey on real hypersurfaces from the viewpoint of Complex Analysis.

From the point of view of Differential Geometry, a problem that has attracted the attention of many mathematicians over the last few decades is the classification of real hypersurfaces in terms of different geometric conditions. The case of real hypersurfaces in nonflat complex space forms deserves special attention, for these spaces are the nonflat Kähler manifolds with the simplest curvature tensor.

The method of equivariant differential geometry has shown to be a powerful tool for the construction of submanifolds with specific geometric properties, see for example [63], [64]. Given a proper isometric action of a Lie group H on a Riemannian manifold $\bar{M}$, the idea of the method is to find a curve in the orbit space $\bar{M}/H$ such that the union of the corresponding orbits in $\bar{M}$ yields a submanifold M with the desired geometric property. It turns out that for many interesting properties, finding such a curve is equivalent to solving certain ordinary differential equation. Thus, existence and uniqueness of this curve is guaranteed for given initial conditions. The resulting submanifolds M are, intrinsically, manifolds of cohomogeneity one, that is, they admit an isometric action whose principal orbits have codimension one in M.

In [54], Gorodski and Gusevskii constructed many examples of complete constant mean curvature hypersurfaces of cohomogeneity one in complex hyperbolic spaces $\mathbb{C}H^n$ by applying the equivariant method to several cohomogeneity two polar actions on $\mathbb{C}H^n$. We recall that a proper isometric action on a Riemannian manifold is called polar if there is a submanifold intersecting all the orbits of the action perpendicularly; such a submanifold must be totally geodesic, and is called a section of the action. Thus, the resulting hypersurfaces appear as the union of orbits through some curve in the 2-dimensional section.

In the context of real hypersurfaces in Kähler manifolds, the class of Hopf hypersurfaces has been studied thoroughly. Recall that if M is a real hypersurface in a Kähler manifold with complex structure J, and ξ is a (locally defined) unit normal vector field on M, we have that M is a Hopf hypersurface if $J\xi$ is an eigenvector of the shape operator $\mathcal{S}$ of M at every point p (see Subsection 1.6.5). For example, homogeneous hypersurfaces in $\mathbb{C}P^n$ (that is, those which are orbits of a cohomogeneity one isometric action on $\mathbb{C}P^n$) happen to be Hopf, and Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ are open parts of homogeneous hypersurfaces and, thus, are classified; see [8] and [71] (see [47] for a survey). However, as we have said in Subsection 1.6.5, the Hopf vector field $J\xi$ can have nontrivial projection onto several principal curvature spaces; the number of these nontrivial projections is denoted by h. In particular, the examples constructed in [54] are generically non-Hopf, that is, they usually satisfy $h > 1$. For more information on real hypersurfaces in complex space forms we refer to [25] and [88].

The main idea of this chapter is to use the equivariant method to study certain types of real hypersurfaces in the two-dimensional nonflat complex space forms. In particular, we find the first examples of real hypersurfaces with exactly two distinct principal curvatures in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ and, then, classify all such examples by proving that they must be constructed using the equivariant method applied to a polar action. Then, we introduce the notion of strongly 2-Hopf real hypersurface and we prove that, roughly speaking, this notion characterizes the cohomogeneity one real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ constructed by the equivariant method applied to a polar action. Finally, we also obtain a partial classification of austere real hypersurfaces in the complex projective and hyperbolic planes, as well as some other applications.

This chapter is organized as follows. In Section 3.1 we explain how one can use the equivariant method applied to a cohomogeneity two polar action on a two-dimensional nonflat complex space form. In Section 3.2 we prove some formulas for the Levi-Civita connection of a hypersurface satisfying $h = 2$, that will allow us to classify real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with certain geometric properties. In Section 3.3 we introduce the notion of strongly 2-Hopf real hypersurface and investigate its relation to those cohomogeneity one real hypersurfaces constructed by the equivariant method applied to a polar action. In Section 3.4 we classify real hypersurfaces with exactly two distinct principal curvatures in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Section 3.5 is devoted to the classification of austere hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ that satisfy $h \leq 2$. Finally, in Section 3.6 we apply these results to the study of strongly 2-Hopf real hypersurfaces that have constant mean curvature or are Levi-flat.

3.1 Method of equivariant geometry

In this section we present the method of equivariant differential geometry to construct real hypersurfaces in nonflat complex space forms. We use the basic notations and formulas established in Chapter 1. For the construction of these hypersurfaces we make use of the notion of polar action introduced in Section 1.7. Let H be a group acting polarly on $\bar{M}^2(c)$, $c \neq 0$, with cohomogeneity two and section $\Sigma \subset \bar{M}^2(c)$, where $\bar{M}^2(c)$ denotes a nonflat complex space form. It is known that the section Σ is a totally geodesic real projective plane $\mathbb{R}P^2$ if $c > 0$, or a totally geodesic real hyperbolic plane $\mathbb{R}H^2$ if $c < 0$.

The idea behind our construction is rather simple. We start with a polar action of a group H acting with cohomogeneity two on $\bar{M}^2(c)$. In one of its 2-dimensional sections we find a (locally defined) curve σ such that, if we attach to each point $\sigma(t)$ of σ the H-orbit through $\sigma(t)$, we obtain a 3-dimensional real hypersurface M of $\bar{M}^2(c)$.

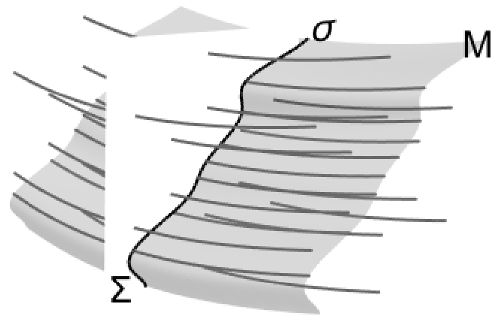

Figure 3.1: Idea of the construction

In Figure 3.1, the orbits of the polar action, which are 2-dimensional, are (unfaithfully) represented by grey lines. The section Σ is a real projective or hyperbolic plane and the black curve σ is contained in the section. Then, the real hypersurface we are looking for is (at least locally) the union of the orbits through the points of σ.

We will check that this hypersurface has at least 2 different principal curvatures which are constant along the principal orbits of the polar action we consider. Moreover, we will see that, generically, such a hypersurface is non-Hopf.

Let H be the connected subgroup of the isometry group of $\bar{M}^2(c)$ acting polarly and with cohomogeneity two on $\bar{M}^2(c)$. Let Σ be a section, and Σ_{reg} the set of regular points of Σ for the H-action.

Let $\sigma\colon t \in (-\varepsilon, \varepsilon) \mapsto \sigma(t) \in \Sigma_{reg}$ be a curve in the regular part of Σ. Then, the subset

$$M = H \cdot \sigma = \{h(\sigma(t)) : t \in (-\varepsilon, \varepsilon),\, h \in H\}$$

is a 3-dimensional hypersurface in $\bar{M}^2(c)$ that is foliated by equidistant H-orbits, and orthogonally, by the curves $h \circ \sigma\colon t \in (-\varepsilon, \varepsilon) \mapsto (h \circ \sigma)(t) = h(\sigma(t)) \in \Sigma_{reg}$ for each $h \in H$. Note that $H \cdot \sigma$ is intrinsically a cohomogeneity one manifold.

Let $p \in \bar{M}^2(c)$ be any regular point for the H-action, and take $\{\xi, \eta\}$ an orthonormal basis of the normal space $\nu_p(H \cdot p)$. We denote by λ_1, λ_2 the principal curvatures of $H \cdot p$ with respect to ξ, and by $\{U_1, U_2\}$ the corresponding basis of $T_p(H \cdot p)$ of principal curvature vectors. Since $H \cdot p$ is Lagrangian, JU_1, JU_2 are orthogonal vectors of the normal space. Since $H \cdot p$ has flat normal bundle, the Ricci equation reads

$$\begin{aligned}\frac{c}{4}(\langle JU_2, \xi\rangle\langle JU_1, \eta\rangle - \langle JU_1, \xi\rangle\langle JU_2, \eta\rangle) &= \langle \bar{R}(U_1, U_2)\xi, \eta\rangle \\ &= \langle R^{\perp}(U_1, U_2)\xi, \eta\rangle - \langle [\mathcal{S}_\xi, \mathcal{S}_\eta]U_1, U_2\rangle \\ &= (\lambda_1 - \lambda_2)\langle \mathcal{S}_\eta U_1, U_2\rangle.\end{aligned}$$

The left-hand side of this equation is nonzero because JU_1 and JU_2 are linearly independent. Thus, $\lambda_1 \neq \lambda_2$. Since ξ was taken arbitrarily, the principal curvatures are always different.

Fix now $p \in \Sigma_{reg}$, and $w \in T_p\Sigma_{reg}$. Consider a (locally defined) curve σ in Σ_{reg}, parametrized by arc-length, and such that $\sigma(0) = p$, and $\dot{\sigma}(0) = w$. Fix a unit vector field ξ along σ tangent to Σ_{reg}, and such that $\langle \xi(t), \dot{\sigma}(t) \rangle = 0$ for all t where σ is defined. Thus, $\xi(t)$ is a unit normal vector field to $H \cdot \sigma$ along σ. Consider also a local chart $\mathcal{U}$ for Σ_{reg} around p, with coordinates (x_1, x_2). Let α, $\beta \colon T\mathcal{U} \to \mathbb{R}$ be the principal curvature functions of the principal orbits of H intersecting $\mathcal{U}$ at the intersection points. As explained above, we know that $\alpha(\eta) \neq \beta(\eta)$ for any vector $\eta \in T\mathcal{U}$, and thus, α and β are smooth functions.

The shape operator of $H \cdot \sigma$ at $\sigma(t)$ with respect to the unit normal vector $\xi(t)$ has the following eigenvalues:

$$\alpha(\xi(t)),\ \beta(\xi(t)),\ \text{and } \langle \mathcal{S}_{\xi(t)}\dot{\sigma}(t), \dot{\sigma}(t) \rangle = -\langle \bar{\nabla}_{\dot{\sigma}(t)}\xi, \dot{\sigma}(t) \rangle = \langle \bar{\nabla}_{\dot{\sigma}(t)}\dot{\sigma}(t), \xi \rangle.$$

This last eigenvalue is precisely the curvature of the curve σ in $\bar{M}^2(c)$, or equivalently, since Σ is totally geodesic, the curvature of σ (with respect to the orientation determined by the normal field ξ) as a curve in Σ.

Moreover, we check now that $H \cdot \sigma$ has constant principal curvatures along the H-orbits. The integrable distributions associated with the two foliations of $M = H \cdot \sigma$ are invariant under the shape operator of M. Indeed, let ξ be an equivariant unit normal vector field on M. Then, the principal curvatures (resp. principal curvature spaces) of some orbit $H \cdot q$ at q with respect to ξ are also principal curvatures (resp. principal curvature spaces) of M at q. This follows from the fact that H-equivariant normal fields along principal orbits of a polar action are parallel with respect to the normal connection of the orbits [9, Corollary 3.2.5]. An H-equivariant unit normal vector field to $H \cdot \sigma$ is given by $h_*\xi(t)$, for any $h \in H$ and any possible t. Note that this is a well-defined vector field because $H \cdot \sigma(t)$ is a principal orbit. Since H acts by isometries of $\bar{M}^2(c)$, the principal curvatures of M at $h(\sigma(t))$ with respect to $h_*\xi(t)$ are the same as the principal curvatures at $\sigma(t)$ with respect to $\xi(t)$. Therefore, the principal curvatures of M along an H-orbit are constant.

Now, we see that $H \cdot \sigma$ is generically non-Hopf. Fix a point $p \in \Sigma_{reg}$. Then we know that for every unit $w \in T_p\Sigma_{reg}$, there is a locally defined curve σ_w such that $\sigma_w(0) = p$, $\dot{\sigma}_w(0) = w$, and $H \cdot \sigma_w$ has at least two distinct principal curvatures. Let $\mathfrak{w}_p$ be the subset of the unit sphere $S^1(T_p\Sigma_{reg})$ of $T_p\Sigma_{reg}$ consisting of those vectors w such that the real hypersurface $H \cdot \sigma_w$ is Hopf at p. Note that if $w \in \mathfrak{w}_p$, then $-w \in \mathfrak{w}_p$. Assume that $\mathfrak{w}_p$ is an infinite set. We will get a contradiction with this assumption. For each hypersurface $H \cdot \sigma_w$, $w \in \mathfrak{w}_p$, let ξ_w be a unit normal vector field along $H \cdot \sigma_w$, which we know is H-equivariant along the principal H-orbits that foliate $H \cdot \sigma_w$. Note that the subindex w in ξ_w denotes only that the normal vector field depends on the initial value w for σ_w; in particular, $\langle \dot{\sigma}_w(t), (\xi_w)_{\sigma_w(t)} \rangle = 0$ for each possible t. The assumption that $H \cdot \sigma_w$ is Hopf at p means that $(J\xi_w)_p$ is an eigenvector of the shape operator of $H \cdot \sigma_w$, and hence, $(J\xi_w)_p$ is also an eigenvector of the shape operator $\mathcal{S}_{(\xi_w)_p}$ of the principal orbit $H \cdot p$ with respect to the normal vector $(\xi_w)_p$. In particular, the map

$$\Phi\colon w \in S^1(T_p\Sigma_{reg}) \mapsto \langle \mathcal{S}_{(\xi_w)_p}(J\xi_w)_p, Jw\rangle \in \mathbb{R}$$

vanishes in $\mathfrak{w}_p$. Since this map is the restriction of an analytic map to $S^1(T_p\Sigma_{reg})$ and $\mathfrak{w}_p$ has an accumulation point in $S^1(T_p\Sigma_{reg})$, Φ vanishes identically, which means that $(J\xi_w)_p$ is an eigenvector of $\mathcal{S}_{(\xi_w)_p}$ for every unit $w \in T_p\Sigma_{reg}$. Since Jw is perpendicular to $(J\xi_w)_p$ and $H \cdot p$ is 2-dimensional, we have that Jw is also an eigenvector of $\mathcal{S}_{(\xi_w)_p}$ for each w. But now, if we fix any w and take unit normal vectors $\xi = (\xi_w)_p$ and $\eta = w$ at p, then $\{J\xi, J\eta\}$ is a common basis of eigenvectors for the shape operators $\mathcal{S}_\xi$ and $\mathcal{S}_\eta$ of the principal orbit $H \cdot p$ at p with respect to ξ and η. This means that the shape operators $\mathcal{S}_\xi$ and $\mathcal{S}_\eta$ commute. Using this and the fact that the principal orbit $H \cdot p$ has flat normal bundle [9, Corollary 3.2.5], the Ricci equation of $H \cdot p$ applied to $J\xi$, $J\eta$, ξ, and η reads

$$0 = \langle R^\perp(J\xi, J\eta)\xi, \eta\rangle = \langle \bar{R}(J\xi, J\eta)\xi, \eta\rangle + \langle [\mathcal{S}_\xi, \mathcal{S}_\eta]J\xi, J\eta\rangle = -\frac{c}{4},$$

where $R^\perp$ is the normal curvature tensor of $H \cdot p$. This gives the desired contradiction. Therefore, the real hypersurfaces $H \cdot \sigma_w$ are Hopf at p at most for a finite collection of vectors of $S^1(T_p\Sigma_{reg})$. We keep denoting by $\mathfrak{w}_p$ the smallest subset of $S^1(T_p\Sigma)$ such that, if $w = \dot{\sigma}(0) \notin \mathfrak{w}_p$, then $M = H \cdot \sigma$ is not Hopf at p. We have just proved that $\mathfrak{w}_p$ is finite.

3.2 Real hypersurfaces with $h = 2$

Let M be a real hypersurface of $\bar{M}^n(c)$. Let ξ be a (local) unit normal vector field to M, and as above denote by J the complex structure of $\bar{M}^2(c)$. Recall that the tangent vector field $J\xi$ is called the Hopf or Reeb vector field of M. Moreover, we have defined the integer-valued function h on M as the number of principal curvature spaces where $J\xi$ has nontrivial projection. In particular, if h is constantly equal to 2, then there is a smooth distribution $\mathcal{D}$ of rank 2 on M that consists of the maximal subspace of the tangent space to M at each point that contains $J\xi$ and is $\mathcal{S}$-invariant (see Subsection 1.6.5).

Now, we calculate the Levi-Civita connection of a real hypersurface M in $\bar{M}^2(c)$, $c \neq 0$, satisfying $h = 2$. This information will be used several times throughout this chapter.

Let α, β and γ be the three principal curvatures of M. For each principal curvature λ, we denote by T_λ the corresponding principal curvature distribution; note that, in principle, this distribution might be singular.

For the following two propositions we only assume that M satisfies that the Hopf vector field $J\xi$ of M has nontrivial projections onto exactly $h = 2$ principal curvature spaces, say onto T_α and T_β. This implies that $\alpha \neq \beta$ at every point. Recall that $\Gamma(T_\lambda)$ denotes the module of smooth vector fields X on M such that $X_p \in T_\lambda(p)$ for every point p.

Proposition 3.1. *There are positive smooth functions b_1, $b_2\colon M \to \mathbb{R}$ with $b_1^2 + b_2^2 = 1$, and an orthonormal frame $\{U_1, U_2, A\}$ on M with $U_1 \in \Gamma(T_\alpha)$, $U_2 \in \Gamma(T_\beta)$, $A \in \Gamma(T_\gamma)$,* *such that*

$$J\xi = b_1U_1 + b_2U_2, \qquad JU_1 = -b_2A - b_1\xi,$$
$$JU_2 = b_1A - b_2\xi, \qquad JA = b_2U_1 - b_1U_2.$$

Proof. Since $J\xi$ is a unit vector field tangent to M that has nontrivial projection onto T_α and T_β, we can write $J\xi = b_1U_1 + b_2U_2$, where $U_1 \in \Gamma(T_\alpha)$, $U_2 \in \Gamma(T_\beta)$ are unit vector fields, and b_1, b_2 are smooth functions on M satisfying $b_1^2 + b_2^2 = 1$, and b_1, $b_2 > 0$. Let A be a unit vector field perpendicular to U_1 and U_2. In what follows we always assume that $\dim\Gamma(T_\alpha) = 1$. If $\dim\Gamma(T_\beta) = 2$, then $\gamma = \beta$ and A is a unit vector field in $\Gamma(T_\beta)$ orthogonal to U_2. If $\dim\Gamma(T_\beta) = 1$, then $\gamma \neq \beta$ and $A \in \Gamma(T_\gamma)$ is a unit vector field perpendicular to U_1 and U_2. Then, $\{U_1, U_2, A\}$ always constitutes an orthonormal frame on an open set of M.

As $-\xi = J^2\xi = b_1JU_1 + b_2JU_2$, and $b_1 \neq 0$, taking inner product with U_2 we get that $\langle JU_1, U_2\rangle = 0$. This implies that JU_1, $JU_2 \in \operatorname{span}\{A, \xi\}$. Now, $\langle JU_1, \xi\rangle = -\langle U_1, J\xi\rangle = -b_1$, and since U_1 has unit length, we obtain $\langle JU_1, A\rangle = \pm b_2$. By changing the sign of A if necessary, we can assume that $JU_1 = -b_2A - b_1\xi$. A similar argument shows that $JU_2 = b_1A - b_2\xi$. Finally, these expressions imply $\langle JA, U_1\rangle = b$, $\langle JA, U_2\rangle = -b_1$, and $\langle JA, \xi\rangle = 0$, from where the result follows. □

Proposition 3.2. *Assume that $\gamma \neq \alpha \neq \beta$ at every point. Then the Levi-Civita connection of M in terms of the basis $\{U_1, U_2, A\}$ is given by the following equations:*

$$\nabla_{U_1}U_1 = \frac{U_2\alpha}{\alpha-\beta}U_2 + \frac{1}{\alpha-\gamma}\Big(A\alpha - \frac{3b_1b_2c}{4}\Big)A,$$
$$\nabla_{U_2}U_1 = \frac{U_1\beta}{\alpha-\beta}U_2 + \Big(\frac{c}{4(\alpha-\gamma)} + \frac{\alpha(\beta-\gamma)}{\alpha-\gamma} + \frac{3b_1^2c(\beta-\gamma)}{4(\alpha-\gamma)^2} - \frac{b_1(\beta-\gamma)}{b_2(\alpha-\gamma)^2}A\alpha\Big)A,$$
$$\nabla_{U_1}U_2 = -\frac{U_2\alpha}{\alpha-\beta}U_1 + \Big(\alpha - \frac{b_1}{b_2(\alpha-\gamma)}\Big(A\alpha - \frac{3b_1b_2c}{4}\Big)\Big)A,$$
$$\nabla_{U_2}U_2 = -\frac{U_1\beta}{\alpha-\beta}U_1 - \Big(\frac{b_1\beta}{b_2} + \frac{b_1c}{4b_2\alpha-\gamma)} + \frac{b_1\alpha(\beta-\gamma)}{b_2(\alpha-\gamma)} + \frac{3b_1^3c(\beta-\gamma)}{4b_2(\alpha-\gamma)^2} - \frac{b_1^2(\beta-\gamma)}{b_2^2(\alpha-\gamma)^2}A\alpha\Big)A,$$
$$\nabla_{U_1}A = -\frac{1}{\alpha-\gamma}\Big(A\alpha - \frac{3b_1b_2c}{4}\Big)U_1 - \Big(\alpha - \frac{b_1}{b_2(\alpha-\gamma)}\Big(A\alpha - \frac{3b_1b_2c}{4}\Big)\Big)U_2,$$
$$\begin{aligned}\nabla_{U_2}A = &-\Big(\frac{c}{4(\alpha-\gamma)} + \frac{\alpha(\beta-\gamma)}{\alpha-\gamma} + \frac{3b_1^2c(\beta-\gamma)}{4(\alpha-\gamma)^2} - \frac{b_1(\beta-\gamma)}{b_2(\alpha-\gamma)^2}A\alpha\Big)U_1\\ &+\Big(\frac{b_1\beta}{b_2} + \frac{b_1c}{4b_2(\alpha-\gamma)} + \frac{b_1\alpha(\beta-\gamma)}{b_2(\alpha-\gamma)} + \frac{3b_1^3c(\beta-\gamma)}{4b_2(\alpha-\gamma)^2} - \frac{b_1^2(\beta-\gamma)}{b_2^2(\alpha-\gamma)^2}A\alpha\Big)U_2,\end{aligned}$$
$$\nabla_AU_1 = \Big(\gamma - \frac{Ab_2}{b_1}\Big)U_2 + \frac{U_1\gamma}{\alpha-\gamma}A, \qquad \nabla_AU_2 = -\Big(\gamma - \frac{Ab_2}{b_1}\Big)U_1 - \frac{b_1U_1\gamma}{b_2(\alpha-\gamma)}A,$$
$$\nabla_AA = -\frac{U_1\gamma}{\alpha-\gamma}U_1 + \frac{b_1U_1\gamma}{b_2(\alpha-\gamma)}U_2,$$

Moreover,

$$U_1 b_1 = \frac{b_2 U_2 \alpha}{\alpha - \beta}, \qquad U_2 b_1 = \frac{b_2 U_1 \beta}{\alpha - \beta}, \qquad A b_1 = -\frac{b_2 A b_2}{b_1},$$
$$U_1 b_2 = -\frac{b_1 U_2 \alpha}{\alpha - \beta}, \quad U_2 b_2 = -\frac{b_1 U_1 \beta}{\alpha - \beta}, \quad U_2 \gamma = -\frac{b_1(\beta - \gamma)}{b_2(\alpha - \gamma)} U_1 \gamma.$$
$$A b_2 = b_1 \gamma + \frac{b_1 c(b_1^2 - 2b_2^2)}{4(\alpha - \beta)} - \frac{b_1 \alpha(\beta - \gamma)}{(\alpha - \beta)} - \frac{3b_1^3 c(\beta - \gamma)}{4(\alpha - \beta)(\alpha - \gamma)} + \frac{b_1^2(\beta - \gamma)}{b_2(\alpha - \beta)(\alpha - \gamma)} A\alpha,$$
$$A\beta = -\frac{3b_1 b_2 c}{4} - \frac{a\beta(\beta - \gamma)}{b_2} - \frac{b_1 c(\beta - \gamma)}{4b_2(\alpha - \gamma)} - \frac{b_1 \alpha(\beta - \gamma)^2}{b_2(\alpha - \gamma)} - \frac{3b_1^3 c(\beta - \gamma)^2}{4b_2(\alpha - \gamma)^2} + \frac{b_1^2(\beta - \gamma)^2}{b_2^2(\alpha - \gamma)^2} A\alpha.$$

Proof. Using the fact that U_1 and A are orthogonal eigenvectors of $\mathcal{S}$ associated with the eigenvalues α and γ respectively, and the symmetry of $\mathcal{S}$ with respect to the inner product, we get

$$\begin{aligned}\langle (\nabla_{U_1} \mathcal{S}) A, U_1 \rangle &= \langle \nabla_{U_1} \mathcal{S} A - \mathcal{S} \nabla_{U_1} A, U_1 \rangle = \langle \nabla_{U_1} (\gamma A), U_1 \rangle - \langle \nabla_{U_1} A, \mathcal{S} U_1 \rangle \\ &= (U_1 \gamma) \langle A, U_1 \rangle + \gamma \langle \nabla_{U_1} A, U_1 \rangle - \alpha \langle \nabla_{U_1} A, U_1 \rangle = (\alpha - \gamma) \langle \nabla_{U_1} U_1, A \rangle.\end{aligned}$$

As U_1 is a unit vector field we have $\langle \nabla_A U_1, U_1 \rangle = 0$. Thus, proceeding as before, we get $\langle (\nabla_A \mathcal{S}) U_1, U_1 \rangle = A\alpha$. Moreover, the expression of the curvature tensor of a complex space form yields $\langle \bar{R}(U_1, A) U_1, \xi \rangle = -3b_1 b_2 c/4$. Hence, the Codazzi equation applied to the triple (U_1, A, U_1) implies

$$\langle \nabla_{U_1} U_1, A \rangle = \frac{1}{\alpha - \gamma} \left(A\alpha - \frac{3b_1 b_2 c}{4} \right).$$

Applying the Codazzi equation to the triples (U_1, U_2, U_1), (U_1, U_2, U_2), (U_1, U_2, A), (U_1, A, U_1), (U_1, A, U_2), (U_1, A, A), (A, U_2, U_2) and (A, U_2, A) we obtain in a similar way:

$$\begin{aligned}
&\langle \nabla_{U_1} U_1, U_2 \rangle = \frac{U_2 \alpha}{\alpha - \beta}, \quad \langle \nabla_{U_2} U_1, A \rangle = \frac{c + 4(\beta - \gamma) \langle A, \nabla_{U_1} U_2 \rangle}{4(\alpha - \gamma)}, \\
&\langle \nabla_{U_2} U_1, U_2 \rangle = \frac{U_1 \beta}{\alpha - \beta}, \quad \langle \nabla_{U_1} U_1, A \rangle = \frac{1}{\alpha - \gamma} \Big(A\alpha - \frac{3b_1 b_2 c}{4} \Big), \\
&\langle \nabla_A U_1, A \rangle = \frac{U_1 \gamma}{\alpha - \gamma}, \quad \langle \nabla_A U_1, U_2 \rangle = \frac{c(2b_2^2 - b_1^2) - 4(\beta - \gamma) \langle U_2, \nabla_{U_1} A \rangle}{4(\alpha - \beta)}, \\
&A\beta = -\frac{3b_1 b_2 c}{4} - (\beta - \gamma) \langle U_2, \nabla_{U_2} A \rangle, \\
&U_2 \gamma = (\beta - \gamma) \langle A, \nabla_A U_2 \rangle.
\end{aligned} \tag{3.1}$$

Since J is parallel with respect to the connection $\bar{\nabla}$ of $\bar{M}^2(c)$, we have $\bar{\nabla}_{U_1} J\xi = J \bar{\nabla}_{U_1} \xi = -J\mathcal{S} U_1 = -\alpha J U_1$. Taking this into account, and using Proposition 3.1 and (3.1) we get

$$\begin{aligned}0 &= U_1\langle A, J\xi\rangle = \langle \bar\nabla_{U_1}A, J\xi\rangle + \langle A, \bar\nabla_{U_1}J\xi\rangle\\ &= b_1\langle \nabla_{U_1}A, U_1\rangle + b_2\langle \nabla_{U_1}A, U_2\rangle + \alpha b_2\langle A, A\rangle + \alpha b_1\langle A, \xi\rangle\\ &= -\frac{b_1}{\alpha-\gamma}\Big(A\alpha - \frac{3b_1b_2c}{4}\Big) + b_2\langle \nabla_{U_1}A, U_2\rangle + \alpha b_2,\end{aligned}$$

from where we obtain $\langle \nabla_{U_1}A, U_2\rangle$.

An analogous argument with $U_2\langle A, J\xi\rangle = 0$, gives the following result:

$$\langle U_2, \nabla_{U_2}A\rangle = \frac{b_1\beta}{b_2} + \frac{b_1(c - 4(\beta-\gamma)\langle \nabla_{U_1}A, U_2\rangle)}{4b_2(\alpha-\gamma)}.$$

Using the expression for $\langle \nabla_{U_1}A, U_2\rangle$ we also get $\langle U_2, \nabla_{U_2}A\rangle$. This and an analogous argument with $A\langle A, J\xi\rangle = 0$, yields

$$\begin{aligned}&\langle \nabla_{U_1}A, U_2\rangle = -\alpha + \frac{b_1}{b_2(\alpha-\gamma)}\Big(A\alpha - \frac{3b_1b_2c}{4}\Big), \qquad \langle \nabla_A U_2, A\rangle = -\frac{b_1U_1\gamma}{b_2(\alpha-\gamma)},\\ &\langle U_2, \nabla_{U_2}A\rangle = \frac{b_1\beta}{b_2} + \frac{b_1c}{4b_2(\alpha-\gamma)} + \frac{b_1\alpha(\beta-\gamma)}{b_2(\alpha-\gamma)} + \frac{3b_1^3c(\beta-\gamma)}{4b_2(\alpha-\gamma)^2} - \frac{b_1^2(\beta-\gamma)}{b_2^2(\alpha-\gamma)^2}A\alpha.\end{aligned} \tag{3.2}$$

The next step is to calculate the derivatives of the functions b_1 and b_2 in terms of the derivatives of α and β. For example, using Proposition 3.1, (3.1) and (3.2) we get

$$U_1b_1 = U_1\langle U_1, J\xi\rangle = \langle \bar\nabla_{U_1}U_1, J\xi\rangle + \langle U_1, \bar\nabla_{U_1}J\xi\rangle = b_2\langle \nabla_{U_1}U_1, U_2\rangle - \alpha\langle U_1, JU_1\rangle = b_2\frac{U_2\alpha}{\alpha-\beta}.$$

We directly give the results for the other derivatives, whose calculations are similar to those of U_1b_1. Note that we use the fact that $b_1^2 + b_2^2 = 1$.

$$\begin{aligned}&U_1b_1 = \frac{b_2U_2\alpha}{\alpha-\beta}, \qquad U_2b_1 = \frac{b_2U_1\beta}{\alpha-\beta}, \qquad Ab_1 = -\frac{b_2Ab_2}{b_1},\\ &U_1b_2 = -\frac{b_1U_2\alpha}{\alpha-\beta}, \quad U_2b_2 = -\frac{b_1U_1\beta}{\alpha-\beta},\\ &Ab_2 = b_1\gamma + \frac{b_1c(b_1^2-2b_2^2)}{4(\alpha-\beta)} - \frac{b_1\alpha(\beta-\gamma)}{(\alpha-\beta)} - \frac{3b_1^3c(\beta-\gamma)}{4(\alpha-\beta)(\alpha-\gamma)} + \frac{b_1^2(\beta-\gamma)}{b_2(\alpha-\beta)(\alpha-\gamma)}A\alpha.\end{aligned} \tag{3.3}$$

Putting together (3.1), (3.2) and (3.3), we obtain Proposition 3.2, as desired. □

Remark 3.3. Since $b_1^2 + b_2^2 = 1$ we can write $b_1 = \cos\phi$, $b_2 = \sin\phi$, for a smooth function $\phi\colon M \to (0, \pi/2)$. This will be more convenient in some calculations, for example in Proposition 3.4.

Assume now that M has two distinct principal curvatures. The Levi-Civita connection of M is summarized in the following proposition:

Proposition 3.4. *Assume that $\alpha \neq \beta = \gamma$ at every point. Then the Levi-Civita connection of M in terms of the basis $\{U_1, U_2, A\}$ is given by the following equations:*

$$\nabla_{U_1}U_1 = -\frac{b_2(c-4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)}A, \qquad \nabla_{U_1}U_2 = \frac{c}{4(\alpha-\beta)}A,$$
$$\nabla_{U_2}U_1 = \frac{c}{4(\alpha-\beta)}A, \qquad \nabla_{U_2}U_2 = -\frac{b_1(c+4\beta(\alpha-\beta))}{4b_2(\alpha-\beta)}A,$$
$$\nabla_A U_1 = -\frac{(b_1^2-2b_2^2)c}{4(\alpha-\beta)}U_2, \qquad \nabla_A U_2 = \frac{(b_1^2-2b_2^2)c}{4(\alpha-\beta)}U_1,$$
$$\nabla_{U_1}A = \frac{b_2(c-4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)}U_1 - \frac{c}{4(\alpha-\beta)}U_2, \qquad \nabla_A A = 0,$$
$$\nabla_{U_2}A = -\frac{c}{4(\alpha-\beta)}U_1 + \frac{b_1(c+4\beta(\alpha-\beta))}{4b_2(\alpha-\beta)}U_2.$$

Furthermore, we have $U_1\phi = U_2\phi = U_1\alpha = U_2\alpha = U_1\beta = U_2\beta = 0$, and

$$\begin{aligned} A\phi &= \beta + \frac{c(1-3\sin^2\phi)}{4(\alpha-\beta)}, \\ A\alpha &= \frac{1}{4}\big(c(2-3\sin^2\phi)+4\alpha(\alpha-\beta)\big)\tan\phi, \\ A\beta &= -\frac{3c}{8}\sin 2\phi. \end{aligned} \tag{3.4}$$

Proof. In order to prove this proposition, we need to calculate $U_1\beta$, but this will take some effort. Using Proposition 3.2 with $\gamma = \beta$ we get

$$\begin{aligned} [A,U_2]\beta = (\nabla_A U_2 - \nabla_{U_2}A)\beta &= -\frac{c(2b_2^2-b_1^2)}{4(\alpha-\beta)}U_1\beta + \frac{3b_1b_2c}{4}\frac{b_1}{b_2(\alpha-\beta)}U_1\beta + \frac{c}{4(\alpha-\beta)}U_1\beta \\ &= \frac{c(1+4b_1^2-2b_2^2)}{4(\alpha-\beta)}U_1\beta. \end{aligned}$$

On the other hand, since $U_2\beta = 0$ by Proposition 3.2 with $\gamma = \beta$, it is clear that $AU_2\beta = 0$. Using Proposition 3.2 with $\gamma = \beta$, we also obtain

$$U_2A\beta = -\frac{3c}{4}U_2(b_1b_2) = -\frac{3c}{4}((U_2b_1)b_2 + b_1(U_2b_2)) = \frac{3c(b_1^2-b_2^2)}{4(\alpha-\beta)}U_1\beta.$$

Taking all this together we have

$$0 = ([A,U_2] - AU_2 + U_2A)\beta = \frac{c(1+7b_1^2-5b_2^2)}{4(\alpha-\beta)}U_1\beta. \tag{3.5}$$

Assume momentarily that there exists a point $p \in M$ such that $1+7b_1(p)^2-5b_2(p)^2=0$. Taking the derivative with respect to U_2 in the previous equation and evaluating at p yields

$$\begin{aligned} 0 &= (U_2)_p\Big(\frac{c(1+7b_1^2-5b_2^2)}{4(\alpha-\beta)}U_1\beta\Big) \\ &= \frac{c}{4}\Big(\frac{U_2(1+7b_1^2-5b_2^2)}{\alpha-\beta}U_1\beta-(1+7b_1^2-5b_2^2)\frac{U_2(\alpha-\beta)}{(\alpha-\beta)^2}U_1\beta+\frac{1+7b_1^2-5b_2^2}{\alpha-\beta}U_2U_1\beta\Big)\Big|_p \\ &= \frac{c}{4(\alpha(p)-\beta(p))}(U_2)_p(1+7b_1^2-5b_2^2)((U_1)_p\beta). \end{aligned}$$

Now, using Proposition 3.2 with $\gamma=\beta$, we get

$$0=\frac{c}{4(\alpha(p)-\beta(p))}\Big(14b_1\frac{b_2}{\alpha-\beta}+10b_2\frac{b_1}{\alpha-\beta}\Big)(p)((U_1)_p\beta)^2=\frac{6b_1(p)b_2(p)c}{(\alpha(p)-\beta(p))^2}((U_1)_p\beta)^2,$$

and since by assumption $b_1, b_2 \neq 0$, we finally get $(U_1)_p\beta=0$. Therefore, (3.5) implies $U_1\beta=0$ and thus we have from Proposition 3.2, setting $\gamma=\beta$,

$$U_1\beta=U_2\beta=0, \qquad A\beta=-\frac{3b_1b_2c}{4}. \tag{3.6}$$

Now, by Proposition 3.2 and (3.6) it follows that $[A,U_1]\beta=0$, $AU_1\beta=0$ and

$$0=([A,U_1]-AU_1+U_1A)\beta=U_1\Big(-\frac{3b_1b_2c}{4}\Big)=-\frac{3c(b_2^2-b_1^2)}{4(\alpha-\beta)}U_2\alpha.$$

Arguing as before, if $p \in M$ is such that $b_2(p)^2-b_1(p)^2=0$, then using again Proposition 3.2 with $\gamma=\beta$, we get

$$0=(U_1)_p\Big(-\frac{3c(b_2^2-b_1^2)}{4(\alpha-\beta)}U_2\alpha\Big)=\frac{3b_1(p)b_2(p)c}{(\alpha(p)-\beta(p))^2}((U_2)_p\alpha)^2,$$

and the two previous equations readily imply $U_2\alpha=0$ on M.

Proposition 3.2 with $\gamma=\beta$ and equation (3.6) yield

$$0=([U_1,U_2]-U_1U_2+U_2U_1)\beta=-\frac{3b_1b_2c}{4}\Big(\Big(\alpha-\frac{b_1}{b_2(\alpha-\beta)}\Big(A\alpha-\frac{3b_1b_2c}{4}\Big)\Big)-\frac{c}{4(\alpha-\beta)}\Big),$$

from where we can obtain $A\alpha$. Indeed, we have

$$U_2\alpha=0, \qquad A\alpha=\frac{b_2}{4b_1}\Big(c(3b_1^2-1)+4\alpha(\alpha-\beta)\Big). \tag{3.7}$$

In order to obtain $U_1\alpha$ we need the Gauss equation for the tuple (U_1,U_2,U_1,A).

Proposition 3.1 with $\gamma=\beta$ implies $\langle \bar{R}(U_1,U_2)U_1,A\rangle=0$, and $\langle \mathcal{S}U_1,U_1\rangle\langle \mathcal{S}U_2,A\rangle-\langle \mathcal{S}U_1,A\rangle\langle \mathcal{S}U_2,U_1\rangle=0$.

Using Proposition 3.2 with $\gamma = \beta$ together with (3.6) and (3.7) we get

$$\begin{aligned}\langle \nabla_{U_1}\nabla_{U_2}U_1, A\rangle =& \langle \nabla_{U_1}\Big(\frac{c}{4(\alpha-\beta)}A\Big), A\rangle = U_1\Big(\frac{c}{4(\alpha-\beta)}\Big) = -\frac{cU_1(\alpha-\beta)}{4(\alpha-\beta)^2}\\ =& -\frac{c}{4(\alpha-\beta)^2}U_1\alpha.\end{aligned}$$

Now, substituting (3.7) in Proposition 3.2 with $\gamma = \beta$ yields

$$\nabla_{U_1}U_1 = -\frac{b_2(c-4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)}A.$$

Taking into account that $U_2\alpha = U_2\beta = U_2b_1 = U_2b_2 = 0$ by equations (3.7) and (3.6), and Proposition 3.2 with $\gamma = \beta$, we get

$$\langle \nabla_{U_2}\nabla_{U_1}U_1, A\rangle = \langle \nabla_{U_2}\Big(-\frac{b_2(c-4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)}A\Big), A\rangle = U_2\Big(-\frac{b_2(c-4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)}\Big) = 0.$$

Finally, since $U_1\beta = U_2\alpha = 0$, Proposition 3.2 with $\gamma = \beta$ implies that $[U_1, U_2]$ is a multiple of A. Since $\langle \nabla_A U_1, A\rangle = 0$, it readily follows that $\langle \nabla_{[U_1,U_2]}U_1, A\rangle = 0$.

Altogether this means that the Gauss equation is equivalent to

$$-\frac{c}{4(\alpha-\beta)^2}U_1\alpha = 0,$$

from where it follows that $U_1\alpha = 0$.

Putting together all the results of this section, and taking into account that $b_1 = \cos\phi$, $b_2 = \sin\phi$, we obtain Proposition 3.4, as desired. □

3.3 Strongly 2-Hopf real hypersurfaces

As we have stated, the method of equivariant differential geometry has shown to be a powerful tool for the construction of submanifolds with specific geometric properties. In particular, in [54], Gorodski and Gusevskii constructed many examples of complete constant mean curvature hypersurfaces of cohomogeneity one in complex hyperbolic spaces $\mathbb{C}H^n$, which are generically non-Hopf, by applying the equivariant method to several cohomogeneity two polar actions on $\mathbb{C}H^n$.

Moreover, in $\mathbb{C}H^n$, $n \geq 2$, there are examples of non-Hopf homogeneous hypersurfaces [12]. The observation that motivates the results in this section is that most of the examples in [54], and some examples in [12], share the following geometric properties:

(C1) The smallest $\mathcal{S}$-invariant distribution $\mathcal{D}$ of M that contains $J\xi$ has rank 2.

(C2) $\mathcal{D}$ is integrable.

(C3) The spectrum of $\mathcal{S}|_{\mathcal{D}}$ is constant along the integral submanifolds of $\mathcal{D}$.

Here, as usual, $\mathcal{S}$ stands for the shape operator of M. Recall that a real hypersurface M satisfying (C1) and (C2) has been called 2-Hopf in [25] and [67]. Motivated by this terminology, we will say that a real hypersurface M in a Kähler manifold is *strongly* 2*-Hopf* if it satisfies conditions (C1), (C2) and (C3) above. The generalization of these definitions to k-Hopf and strongly k-Hopf hypersurfaces, for any positive integer k, is straightforward. It is important to mention that the notions of Hopf, 1-Hopf and strongly 1-Hopf real hypersurfaces agree when the ambient manifold is a nonflat complex space form $\mathbb{C}P^n$ or $\mathbb{C}H^n$ (see [88]). Also, note that condition (C1) has been investigated in the context of real hypersurfaces with constant principal curvatures in nonflat complex space forms [38]. Finally, observe that we have defined h as the number of principal curvature spaces of M onto which the Hopf vector field has nontrivial projection. Then M is Hopf precisely when $h = 1$, and condition (C1) is equivalent to $h = 2$.

The main result of this section is a characterization of the non-Hopf cohomogeneity one hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ constructed via the equivariant method applied to a polar action of cohomogeneity two. Such characterization is achieved in terms of the strongly 2-Hopf property. Then, our main result in this section can be stated as follows.

Theorem 3.5. *Consider a polar action of a group H acting with cohomogeneity two and with section Σ on a nonflat complex space form $\bar{M}^2(c)$.*

Let $p \in \Sigma$ be a regular point, and $\sigma\colon (-\varepsilon, \varepsilon) \to \Sigma$ a unit speed curve in Σ with $\sigma(0) = p$. Define the subset $H \cdot \sigma = \{h(\sigma(t)) : h \in H, t \in (-\varepsilon, \varepsilon)\}$ of $\bar{M}^2(c)$. Then, for ε small enough, there exists a finite set $\mathfrak{w}_p \subset S^1(T_p\Sigma)$ such that, if $\dot{\sigma}(0) \notin \mathfrak{w}_p$, the set $H \cdot \sigma$ is a strongly 2-Hopf hypersurface of $\bar{M}^2(c)$, whereas if $\dot{\sigma}(0) \in \mathfrak{w}_p$, then $H \cdot \sigma$ is a real hypersurface of $\bar{M}^2(c)$ that is Hopf at p.

Conversely, any strongly 2*-Hopf real hypersurface in $\bar{M}^2(c)$ is locally congruent to a hypersurface constructed as above.*

Theorem 3.5 guarantees that a real hypersurface under the above mentioned assumptions is foliated by orbits of maximal dimension of a polar action of cohomogeneity two on $\bar{M}^2(c)$. Interestingly, the proof of the Theorem 3.5 relies on the geometric characterization of the principal orbits of polar actions of cohomogeneity two on $\bar{M}^2(c)$, $c \neq 0$, achieved in Theorem 2.1 of Chapter 2.

Now, we investigate the structure of strongly 2-Hopf hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. The first part of the Theorem 3.5 has already been proved in Section 3.1. Then, we only have to prove the classification part, that is, we show that a strongly 2-Hopf real hypersurface in $\bar{M}^2(c)$, $c \neq 0$, must be locally congruent to a hypersurface constructed as in Section 3.1.

The Levi-Civita connection of a strongly 2-Hopf real hypersurface

From now on we assume that M is a strongly 2-Hopf real hypersurface, we denote by ξ a (local) unit normal vector field of M, and let $\mathcal{D}$ be the smallest $\mathcal{S}$-invariant distribution containing $J\xi$. We will use the notation given in Proposition 3.1, that is, $J\xi = b_1U_1 + b_2U_2$, $JU_1 = -b_2A - b_1\xi$, $JU_2 = b_1A - b_2\xi$ and $JA = b_2U_1 - b_1U_2$, where b_1 and b_2 are positive smooth functions with $b_1^2 + b_2^2 = 1$, and $\{U_1, U_2, A\}$ is an orthonormal frame on M. So, in particular $\mathcal{D} = \mathbb{R}U_1 \oplus \mathbb{R}U_2$. In the following proposition we determine the Levi-Civita connection of M.

Proposition 3.6. *The Levi-Civita connection of M in terms of the frame $\{U_1, U_2, A\}$ is given by the following equations:*

$$\nabla_{U_1} U_1 = -\frac{b_2(c - 4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)} A, \qquad \nabla_{U_1} U_2 = \frac{c}{4(\alpha-\beta)} A,$$
$$\nabla_{U_2} U_1 = \frac{c}{4(\alpha-\beta)} A, \qquad \nabla_{U_2} U_2 = -\frac{b_1(c + 4\beta(\alpha-\beta))}{4b_2(\alpha-\beta)} A,$$
$$\nabla_A U_1 = \left(\frac{c(\beta-\gamma)}{4(\alpha-\beta)^2} - \frac{c\,(b_1^2 - 2b_2^2)}{4(\alpha-\beta)}\right) U_2, \quad \nabla_A U_2 = \left(-\frac{c(\beta-\gamma)}{4(\alpha-\beta)^2} + \frac{c\,(b_1^2 - 2b_2^2)}{4(\alpha-\beta)}\right) U_1,$$
$$\nabla_{U_1} A = \frac{b_2(c - 4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)} U_1 - \frac{c}{4(\alpha-\beta)} U_2,$$
$$\nabla_{U_2} A = -\frac{c}{4(\alpha-\beta)} U_1 + \frac{b_1(c + 4\beta(\alpha-\beta))}{4b_2(\alpha-\beta)} U_2,$$
$$\nabla_A A = 0.$$

Furthermore, we have $\mathcal{D}b_1 = \mathcal{D}b_2 = \mathcal{D}\alpha = \mathcal{D}\beta = \mathcal{D}\gamma = 0$.

Proof. First of all, note that, in case that γ equals one of the other two principal curvatures in an open set of M, then the relations above hold, according to Proposition 3.4.

Therefore, it is enough to prove Proposition 3.6 if M has three distinct principal curvatures at every point. In particular, we use the results provided by Proposition 3.2.

By definition of strongly 2-Hopf hypersurface we have $U_1\alpha = U_1\beta = U_2\alpha = U_2\beta = 0$. Then, Proposition 3.2 implies $U_1 b_1 = U_1 b_2 = U_2 b_1 = U_2 b_2 = 0$.

Since the distribution $\mathcal{D} = \operatorname{span}\{U_1, U_2\}$ is integrable due to the strongly 2-Hopf assumption, we must have $\langle \nabla_{U_1} U_2 - \nabla_{U_2} U_1, A\rangle = 0$. Using Proposition 3.2, this allows us to obtain, after some calculations,

$$\begin{aligned} A\alpha &= \frac{\alpha b_2(\alpha-\gamma)}{b_1} + \frac{b_2 c(\alpha-\gamma)}{4b_1(\beta-\alpha)} + \frac{3b_1 b_2 c}{4}, \\ A\beta &= -\frac{\beta b_1(\beta-\gamma)}{b_2} - \frac{b_1 c(\beta-\gamma)}{4b_2(\alpha-\beta)} - \frac{3b_1 b_2 c}{4}, \\ Ab_2 &= b_1\left(\frac{c\,(b_1^2 - 2b_2^2)}{4(\alpha-\beta)} - \frac{c(\beta-\gamma)}{4(\alpha-\beta)^2} + \gamma\right). \end{aligned} \tag{3.8}$$

The last step is to show that $U_1\gamma = 0$. Proposition 3.2, (3.8), and $U_1\alpha = 0$, easily imply

$$[U_1, A]\alpha = (\nabla_{U_1} A - \nabla_A U_1)\alpha = -\frac{1}{\alpha-\gamma}\left(\frac{\alpha b_2(\alpha-\gamma)}{b_1} + \frac{b_2 c(\alpha-\gamma)}{4b_1(\beta-\alpha)} + \frac{3b_1 b_2 c}{4}\right) U_1\gamma,$$
$$U_1 A\alpha = \frac{b_2(c - 4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)} U_1\gamma, \qquad AU_1\alpha = 0.$$

Thus,

$$0 = ([U_1, A] - U_1 A + A U_1)\alpha = -\frac{3 b_1 b_2 c}{4(\alpha - \gamma)} U_1 \gamma,$$

which yields $U_1\gamma = 0$, as desired. Finally, by Proposition 3.2, we get $U_2\gamma = U_1\gamma = 0$. Altogether we obtain Proposition 3.6. □

Two perpendicular integrable distributions

In order to conclude the proof of the Theorem 3.5 we need first to extract certain geometric information on the integrable distributions $\mathcal{D} = \mathbb{R}U_1 \oplus \mathbb{R}U_2$ and $\mathbb{R}A$, and then use this information to show that M can be constructed as in the statement of the Theorem 3.5. We assume the notation and results obtained so far.

Proposition 3.7. *The leaves of the integrable distribution $\mathcal{D}$ are Lagrangian, flat, totally real submanifolds of $\bar{M}^2(c)$ with parallel second fundamental form and flat normal bundle.*

Proof. It follows from Proposition 3.6, using the Gauss formula, that

$$\begin{aligned} &\bar{\nabla}_{U_1} U_1 = -\frac{b_2(c - 4\alpha(\alpha-\beta))}{4b_1(\alpha-\beta)} A + \alpha\xi, \quad \bar{\nabla}_{U_2} U_2 = -\frac{b_1(c + 4\beta(\alpha-\beta))}{4b_2(\alpha-\beta)} A + \beta\xi, \\ &\bar{\nabla}_{U_2} U_1 = \bar{\nabla}_{U_1} U_2 = \frac{c}{4(\alpha-\beta)} A. \end{aligned} \tag{3.9}$$

As M is a strongly 2-Hopf hypersurface, $\mathcal{D}$ is integrable. Let L be any integral submanifold of the distribution $\mathcal{D}$, and denote by $\widetilde{\nabla}$ its Levi-Civita connection. The normal space $\nu_p L$ of L at $p \in L$, as a submanifold of $\bar{M}^2(c)$, is generated by A_p and ξ_p. Hence, using (3.9), we get $\widetilde{\nabla}_{U_1} U_1 = \widetilde{\nabla}_{U_1} U_2 = \widetilde{\nabla}_{U_2} U_1 = \widetilde{\nabla}_{U_2} U_2 = 0$. Therefore, U_1 and U_2 are parallel vector fields on L with respect to the Levi-Civita connection $\widetilde{\nabla}$. In particular, L is flat as a 2-dimensional Riemannian manifold of $\bar{M}^2(c)$. By Proposition 3.1 L is a totally real submanifold of $\bar{M}^2(c)$ and hence a Lagrangian submanifold of $\bar{M}^2(c)$. Lagrangian surfaces of $\bar{M}^2(c)$ are flat if and only if they have flat normal bundle, see Lemma 2.4.

Finally, the functions α, β, γ, b_1 and b_2, are constant along the integral curves of U_1 and U_2 by Proposition 3.6. Therefore, they are constant along L. This, the fact that U_1 and U_2 are parallel with respect to $\widetilde{\nabla}$, and the fact L has flat normal bundle, imply that the second fundamental form of L is parallel. □

Thus, the real hypersurface M is foliated orthogonally by the leaves of the 2-dimensional distribution $\mathcal{D}$, and by the integral curves of the vector field A. Observe that the relation $\nabla_A A = 0$ in Proposition 3.6 implies that the integral curves of A are geodesics of M and, by the Gauss formula, they have curvature γ as a curve in $\bar{M}^2(c)$. Moreover, these curves are, locally, intersections of M with totally geodesic, totally real surfaces in $\bar{M}^2(c)$. More precisely, we have:

Proposition 3.8. *Let σ be an integral curve of A through a point $p \in M$. Let $Q_p = \exp_p(\mathbb{R}A_p \oplus \mathbb{R}\xi_p)$, where $\exp_p$ denotes the Riemannian exponential map of $\bar{M}^2(c)$ at p. Then, Q_p is a totally real, totally geodesic surface of $\bar{M}^2(c)$, and σ is contained in Q_p.*

Furthermore, the curve σ is determined by the initial conditions $\sigma(0) = p$, $\dot{\sigma}(0) = A_p$, and the fact that σ is a unit speed curve in $Q_p = \exp_p(\mathbb{R}A_p \oplus \mathbb{R}\xi_p)$ with curvature γ with respect to ξ.

Proof. From Proposition 3.1 it is clear that $\langle JA, \xi\rangle = 0$, and hence $\mathbb{R}A_p \oplus \mathbb{R}\xi_p$ is a totally real subspace of $T_p\bar{M}^2(c)$, that is, $J(\mathbb{R}A_p \oplus \mathbb{R}\xi_p)$ is orthogonal to $\mathbb{R}A_p \oplus \mathbb{R}\xi_p$. Then, it is well-known that Q_p is a totally real, totally geodesic submanifold of $\bar{M}^2(c)$. We now prove that the curve σ is contained in Q_p.

From the Gauss equation and Proposition 3.6 it follows that

$$\bar{\nabla}_A A = \gamma\xi, \quad \bar{\nabla}_A\xi = -\gamma A.$$

As a consequence, the curvature of σ with respect to the vector field ξ is given by $\kappa[\sigma](t) = \langle\bar{\nabla}_{\dot{\sigma}(t)}\dot{\sigma}, \xi_{\sigma(t)}\rangle = (\gamma\circ\sigma)(t)$. Thus, the curvature of σ is given by the function $\tilde{\gamma}(t) = (\gamma\circ\sigma)(t)$. Therefore, the curve σ is determined by the differential equation

$$\bar{\nabla}_{\dot{\sigma}}\dot{\sigma} = \tilde{\gamma}\tilde{\xi}, \quad \bar{\nabla}_{\dot{\sigma}}\tilde{\xi} = -\tilde{\gamma}\dot{\sigma}, \quad \sigma(0) = p, \quad \dot{\sigma}(0) = A_p, \quad \tilde{\xi}(0) = \xi_p. \tag{3.10}$$

Let $(\mathcal{U}, \psi = (y_1, y_2))$ be a normal coordinate chart of Q_p around p with $\partial_1(p) = A_p$, $\partial_2(p) = \xi_p$, and $\{\eta_1, \eta_2\}$ an orthonormal frame of $\nu\mathcal{U}$. We consider the Fermi coordinates $\varphi = (x_1, x_2, x_3, x_4)$ of $\bar{M}^2(c)$ on a neighbourhood of p associated with $(\mathcal{U}, \psi)$ and $\{\eta_1, \eta_2\}$, which are defined as

$$\begin{aligned} x_i(\exp_q(a_1\eta_1(q) + a_2\eta_2(q))) &= y_i(q), && i = 1, 2,\\ x_i(\exp_q(a_1\eta_1(q) + a_2\eta_2(q))) &= a_{i-2}, && i = 3, 4, \end{aligned}$$

for each $q \in \mathcal{U}$. If $q \in \mathcal{U}$, then $\varphi(q) = (y_1(q), y_2(q), 0, 0)$ and

$$\partial_1(q) = \frac{\partial}{\partial y_1}(q),\ \partial_2(q) = \frac{\partial}{\partial y_2}(q) \in \Gamma(T\mathcal{U}), \qquad \partial_3(q) = \eta_1(q),\ \partial_4(q) = \eta_2(q) \in \Gamma(\nu\mathcal{U}).$$

We denote by Γ^k_{ij} the Christoffel symbols with respect to these coordinates, which are defined by $\bar{\nabla}_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$. See [53, Chapter 2] for more information on Fermi coordinates.

Now we write σ and $\tilde{\xi}$ in Fermi coordinates as

$$\sigma(t) = (x_1(t), x_2(t), x_3(t), x_4(t)), \qquad \tilde{\xi}(t) = \xi_1(t)\partial_1 + \xi_2(t)\partial_2 + \xi_3(t)\partial_3 + \xi_4(t)\partial_4.$$

Then, (3.10) becomes

$$\begin{aligned} &x_k'' + \sum_{i,j=1}^{4} x_i' x_j' \Gamma^k_{ij} = \tilde{\gamma}\xi_k, \quad \xi_k' + \sum_{i,j=1}^{4} x_i'\xi_j\Gamma^k_{ij} = -\tilde{\gamma}x_k', \quad k = 1, 2, 3, 4,\\ &x_1(0) = y_1(p), \quad x_2(0) = y_2(p), \quad x_1'(0) = 1, \quad x_2'(0) = 0,\\ &\xi_1(0) = 0, \qquad \xi_2(0) = 1,\\ &x_3(0) = x_4(0) = x_3'(0) = x_4'(0) = \xi_3(0) = \xi_4(0) = 0. \end{aligned} \tag{3.11}$$

Since Q_p is totally geodesic, the second fundamental form $I\!I$ of $\mathcal{U}$ satisfies

$$0 = I\!I(\partial_i, \partial_j) = \Big(\sum_{k=1}^{4} \Gamma_{ij}^k \partial_k\Big)^{\perp} = \Gamma_{ij}^3 \partial_3 + \Gamma_{ij}^4 \partial_4,$$

with $i,\ j \in \{1,2\}$. Thus, along $\mathcal{U}$, $\Gamma_{ij}^k = 0$ if $i,\ j \in \{1,2\}$ and $k \in \{3,4\}$. Furthermore, since Q_p is totally geodesic, the Christoffel symbols Γ_{ij}^k, $i,j,k \in \{1,2\}$, of φ coincide with the Christoffel symbols of ψ.

We consider the new differential equation on $\mathcal{U}$

$$\hat{x}_k'' + \sum_{i,j=1}^{2} \hat{x}_i' \hat{x}_j' \Gamma_{ij}^k = \tilde{\gamma} \hat{\xi}_k, \quad \hat{\xi}_k' + \sum_{i,j=1}^{2} \hat{x}_i' \hat{\xi}_j \Gamma_{ij}^k = -\tilde{\gamma} \hat{x}_k', \quad k = 1,2,$$

$$\hat{x}_1(0) = y_1(p), \quad \hat{x}_2(0) = y_2(p), \quad \hat{x}_1'(0) = 1, \quad \hat{x}_2'(0) = 0, \quad \hat{\xi}_1(0) = 0, \quad \hat{\xi}_2(0) = 1,$$

and let $(\hat{x}_1(t), \hat{x}_2(t))$, $(\hat{\xi}_1(t), \hat{\xi}_2(t))$ be the unique solution to this initial value problem. Taking into account that, along $\mathcal{U}$, the Christoffel symbols Γ_{ij}^k, $i,j,k \in \{1,2\}$, of ψ and φ coincide, and that $\Gamma_{ij}^k = 0$ if $i,\ j \in \{1,2\}$, $k \in \{3,4\}$, it turns out that $(\hat{x}_1(t), \hat{x}_2(t), 0, 0)$, $(\hat{\xi}_1(t), \hat{\xi}_2(t), 0, 0)$ is a solution to the initial value problem (3.11). Since the solution to such initial value problem is unique, it turns out that $\sigma(t) = (\hat{x}_1(t), \hat{x}_2(t), 0, 0)$ and $\tilde{\xi}(t) = \hat{\xi}_1(t)\partial_1 + \hat{\xi}_2(t)\partial_2$. Therefore, σ is contained in Q_p and $\tilde{\xi}$ is tangent to Q_p. (Another way of proving this fact is to use [45, Theorem 3.4].)

Finally, since σ is contained in Q_p, and Q_p is a complete 2-dimensional Riemannian manifold of constant sectional curvature (because it is totally real and totally geodesic in $\bar{M}^2(c)$), then the curve σ is determined by its curvature, an initial point, an initial tangent vector, and a choice of orientation given in this case by ξ. Hence, Proposition 3.8 follows. □

We study some further properties of the integral submanifolds of the distribution $\mathcal{D}$.

Lemma 3.9. *Let $p \in M$, $Q_p = \exp_p(\mathbb{R}A_p \oplus \mathbb{R}\xi_p)$, and σ an integral curve of A through p. Then, Q_p intersects the integral submanifolds of $\mathcal{D}$ through $\sigma(t)$ perpendicularly.*

Proof. By Proposition 3.8, σ is contained in Q_p. Clearly, $A_{\sigma(t)} = \dot{\sigma}(t)$ is tangent to Q_p along σ. We now show that $\xi_{\sigma(t)}$ is tangent to Q_p. Let η be a vector field along σ such that $\eta_p \in \nu_p Q_p$ and that is parallel with respect to the normal connection $D^{\perp}$ of Q_p. Then, since Q_p is totally geodesic, the Weingarten formula implies $\bar{\nabla}_{\dot{\sigma}} \eta = D_{\dot{\sigma}}^{\perp} \eta = 0$. Hence,

$$\frac{d}{dt}\langle \xi, \eta \rangle = \langle \bar{\nabla}_{\dot{\sigma}} \xi, \eta \rangle + \langle \xi, \bar{\nabla}_{\dot{\sigma}} \eta \rangle = -\gamma \langle \dot{\sigma}, \eta \rangle = 0,$$

and since $\langle \xi_{\sigma(0)}, \eta_{\sigma(0)} \rangle = 0$, and η is arbitrary, we conclude that $\xi_{\sigma(t)}$ is tangent to Q_p for all t.

Altogether this implies that $T_{\sigma(t)} Q_p = \operatorname{span}\{A_{\sigma(t)}, \xi_{\sigma(t)}\}$. Therefore, by construction we have $Q_p = Q_{\sigma(t)}$ for all t, and in particular, Q_p is perpendicular to the leaf of $\mathcal{D}$ through $\sigma(t)$, as we wanted to show. □

We are now in position to state the key result of this subsection.

Proposition 3.10. *We have:*

(i) *The integral submanifolds of $\mathcal{D}$ are equidistant surfaces of $\bar{M}^2(c)$.*

(ii) *Let L be an integral surface of the distribution $\mathcal{D}$, and let L_t be an integral surface of $\mathcal{D}$ whose distance to L is a sufficiently small number t. Then, in a neighbourhood $\mathcal{U}$ of a point of L there exists a parallel normal vector field η_t such that*

$$L_t = \{\exp_p(\eta_t(p)) : p \in \mathcal{U}\}.$$

Proof. Let L be a leaf of the distribution $\mathcal{D}$. We continue to denote by $\nabla^\perp$ the normal connection of L. Recall from Proposition 3.7 that $\nabla^\perp$ is flat, and indeed, $\{A, \xi\}$ constitutes a parallel basis of the normal bundle νL of L as a submanifold of $\bar{M}^2(c)$.

Let $p \in L$, and let τ_p be an integral curve of A through p. We denote by L_t the integral manifold of $\mathcal{D}$ through $\tau_p(t)$. Since A is a geodesic vector field in M by Proposition 3.6, τ_p is a unit speed geodesic in M. Since τ_p is perpendicular to L and L_t, $d_M(L, L_t) = \mathcal{L}(\tau_{p|[0,t]})$, where d_M denotes the Riemannian distance of M, and $\mathcal{L}(\cdot)$ is the length of a curve. The point p is arbitrary, and thus the integral submanifolds of $\mathcal{D}$ are equidistant in M.

For a point $p \in L$, we consider the geodesic ρ_p of $\bar{M}^2(c)$ that minimizes the distance between $\rho_p(0) = p$ and $\rho_p(1) = \tau_p(t)$. Since $Q_p = \exp_p(\mathbb{R}A_p \oplus \mathbb{R}\xi_p)$ is totally geodesic, τ_p is contained in Q_p (see Proposition 3.8), and t is sufficiently small, it follows that ρ_p is contained in Q_p. Since Q_p intersects L and L_t orthogonally by Lemma 3.9, so does ρ_p and we conclude that ρ_p is a minimizing geodesic of $\bar{M}^2(c)$ between these two submanifolds. We define $\eta_t(p) = \dot{\rho}_p(0) \in \mathbb{R}A_p \oplus \mathbb{R}\xi_p$ for each $p \in L$.

Now let $q \in L$ be another point. By the previous argument the curve τ_q, if defined for time t, realizes the distance in M between L and L_t. The normal bundle of L in $\bar{M}^2(c)$ is flat with respect to the normal connection $\nabla^\perp$, and $\{A, \xi\}$ is parallel. Thus, Q_q is obtained by the parallel transport of Q_p to q along L. There is a unique holomorphic isometry g of $\bar{M}^2(c)$ such that $g(p) = q$, $g_{*p}(A_p) = A_q$, and $g_{*p}(\xi_p) = \xi_q$, where g_* denotes the differential of g (note that $\mathbb{R}A \oplus \mathbb{R}\xi$ is totally real). Since g is an isometry, and the curvature of τ_q is given by the same function as that of τ_p due to Proposition 3.8, it follows that the curve $g \circ \tau_p$ satisfies the differential equation (3.10) with q instead of p. By uniqueness, we have that $g \circ \tau_p = \tau_q$. Hence, the geodesic ρ_q that minimizes the distance between q and $\tau_q(t)$ coincides with $g \circ \rho_p$, and thus it satisfies that $\dot{\rho}_q(0) = g_{*p}(\dot{\rho}_p(0))$. Since parallel transport is a linear isometry, and there is a unique isometry between $\nu_p L$ and $\nu_q L$ mapping A_p to A_q and ξ_p to ξ_q, it follows that $\dot{\rho}_q(0)$ is precisely the $\nabla^\perp$-parallel transport of $\dot{\rho}_p(0)$ to q. Therefore, η_t is a normal parallel vector field along L wherever it is defined, and we conclude that $L_t = \{\exp_p(\eta_t(p)) : p \in L\}$ (at least locally). This proves Proposition 3.10. □

Then M is foliated orthogonally by the integral submanifolds of the integrable 2-dimensional distribution $\mathcal{D}$, and by the integral curves of the vector field A. Now we are in position to prove the final part of our Theorem 3.5.

Proof of Theorem 3.5

It follows directly from Proposition 3.7 that the integral submanifolds of $\mathcal{D}$ are 2-dimensional Lagrangian, flat surfaces with parallel second fundamental form and flat normal bundle in $\bar{M}^2(c)$. Theorem 2.1 then guarantees that each integral submanifold of $\mathcal{D}$ is an open part of a principal orbit of a polar action of cohomogeneity two on $\bar{M}^2(c)$. In order to conclude the proof of Theorem 3.5 we have to show that all integral surfaces of $\mathcal{D}$ are open parts of principal orbits of the *same* polar action.

Let H be a subgroup of the isometry group of $\bar{M}^2(c)$ acting polarly and with cohomogeneity two on $\bar{M}^2(c)$. Let $H \cdot p$ be a principal orbit of the action of H, and let η be an H-equivariant normal vector field along $H \cdot p$. If $\nabla^\perp$ denotes now the normal connection of $H \cdot p$, then $\nabla^\perp \eta = 0$, as equivariant vector fields are parallel, see Section 1.5. In fact, we have seen that equivariant vector fields along $H \cdot p$ are in one-to-one correspondence with $\nabla^\perp$-parallel normal vector fields along $H \cdot p$. If $q \in \bar{M}^2(c)$, then there exists a minimizing geodesic ρ from p to $H \cdot q$ which intersects both $H \cdot p$ and $H \cdot q$ orthogonally. We may assume that $\rho(0) = p$, $\rho(1) = q$, define $\eta_p = \dot{\rho}(0)$, and extend η_p to a normal parallel vector field η along $H \cdot p$. Using that η is also equivariant, it is then easy to obtain

$$H \cdot q = \{h(\exp_p(\eta_p)) : h \in H\} = \{\exp_{h(p)}(\eta_{h(p)}) : h \in H\} = \{\exp_x(\eta_x) : x \in H \cdot p\}. \tag{3.12}$$

We saw above that any fixed integral submanifold L of the distribution $\mathcal{D}$ is an open part of a principal orbit of the action of some group H acting polarly and with cohomogeneity two on $\bar{M}^2(c)$. According to Proposition 3.10, the rest of the sufficiently close integral submanifolds of $\mathcal{D}$ are obtained locally as $\{\exp_q(\eta_t(q)) : q \in \mathcal{U}\}$, where $\mathcal{U}$ is an open subset of L, and η_t is a suitable parallel normal vector field along $\mathcal{U} \subset L$. Hence, by (3.12), all the integral submanifolds of $\mathcal{D}$ are open parts of principal orbits of the same polar action of a group H. Moreover, for each $p \in M$ the integral curve of A through p is contained in the totally geodesic submanifold $Q_p = \exp_p(\mathbb{R}A_p \oplus \mathbb{R}\xi_p)$, which is perpendicular to the leaf of $\mathcal{D}$ through p and then, must be a section for the H-action. If σ is an integral curve of the vector field A, then it is obvious that in a neighbourhood of the point p, the hypersurface M is obtained as $M = H \cdot \sigma$, as stated in Theorem 3.5. This concludes the proof of the Theorem 3.5.

3.4 Real hypersurfaces with 2 principal curvatures

The main reference for the study of real hypersurfaces in complex space forms is the influential survey [88] by Niebergall and Ryan, where the authors reviewed the basic terminology and results in the field, and included a list of open problems that has motivated research over the last years. One of the outstanding problems that was still pending, in spite of the efforts of several geometers, is Question 9.2 in [88]:

> *Are there hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ that have two principal curvatures, other than the standard examples?*

Let us explain the motivation of this question. Tashiro and Tachibana [100] proved that there are no umbilical real hypersurfaces (i.e. real hypersurfaces with exactly one principal curvature) in nonflat complex space forms $\bar{M}^n(c)$, $c \neq 0$. Cecil and Ryan [24, Proposition 5.2] showed that a real hypersurface with two principal curvatures in $\mathbb{C}P^n$, $n \geq 3$, has constant principal curvatures, and is an open part of a geodesic sphere. Their classification needs the assumption $n > 2$. This hypothesis implies that a real hypersurface M has at least dimension 5. They assume that the Reeb vector field has nontrivial projection onto 2 different principal curvature spaces. Since one of them has at least dimension 3, using the Codazzi equation, they conclude that both spaces need to have the same dimension. Then, M has to be even-dimensional, which is a contradiction. Thus, M is Hopf. As a consequence, M has constant principal curvatures and Hopf real hypersurfaces with constant principal curvatures are homogeneous [8], [71]. Montiel [86] obtained an analogous result for $\mathbb{C}H^n$, $n \geq 3$, showing that real hypersurfaces with two distinct principal curvatures must be open parts of geodesic spheres, tubes around a totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$, tubes of radius $\frac{1}{\sqrt{-c}}\log(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$, or horospheres. All these examples are homogeneous, that is, orbits of isometric actions on $\bar{M}^n(c)$.

As we have said, the methods used by Cecil and Ryan, and Montiel, do not work if $n = 2$. The question above states the interest of extending the classification of real hypersurfaces with two distinct principal curvatures to $\mathbb{C}P^2$ and $\mathbb{C}H^2$, and poses the problem of the existence of examples with nonconstant principal curvatures. Here, we answer Question 9.2 in [88] affirmatively for $\mathbb{C}P^2$ and $\mathbb{C}H^2$, and obtain a complete description of all the examples.

We state now the main contribution of this section.

Theorem 3.11. *Consider a polar action of a group H acting with cohomogeneity two and with section Σ on a nonflat complex space form $\bar{M}^2(c)$.*

Then, for any regular point $p \in \Sigma$ and any unit tangent vector $w \in T_p\Sigma$, there are exactly two different locally defined unit speed curves $\gamma_i\colon (-\epsilon, \epsilon) \to \Sigma$, $i = 1, 2$, with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = w$, such that the set $H \cdot \gamma_i = \{h(\gamma_i(t)) : h \in H, t \in (-\epsilon, \epsilon)\}$ is a real hypersurface with two principal curvatures in $\bar{M}^2(c)$. Generically, such hypersurface is non-Hopf and with nonconstant principal curvatures.

Conversely, let M be a real hypersurface of $\bar{M}^2(c)$ with two principal curvatures and that is non-Hopf at every point. Then M is locally congruent to an open part of a real hypersurface constructed as above.

Theorem 3.11 guarantees that a real hypersurface under the assumptions mentioned above is foliated by orbits of maximal dimension of a polar action of cohomogeneity two on $\bar{M}^2(c)$. Indeed, it follows that such hypersurfaces are generically strongly 2-Hopf. Interestingly, the proof of Theorem 3.11 relies on the characterization of the principal orbits of polar actions of cohomogeneity two on $\bar{M}^2(c)$, $c \neq 0$, which are precisely the Lagrangian and flat surfaces with parallel mean curvature in $\bar{M}^2(c)$, in the same way as the proof of the Theorem 3.5.

These are the first examples of real hypersurfaces with exactly two distinct nonconstant principal curvatures in the complex projective and hyperbolic planes, $\mathbb{C}P^2$ and $\mathbb{C}H^2$, thus answering the open question posed by Niebergall and Ryan in [88]. These new examples are, again, constructed using the equivariant method applied to cohomogeneity two polar actions on $\mathbb{C}P^2$ and $\mathbb{C}H^2$. The results of this sections can also be found in [41].

Remark 3.12. Some weeks after we finished the article [41], Ivey and Ryan derived a construction of the same examples and an equivalent result using the method of moving frames in [67].

Remark 3.13. It is possible to construct the curves σ_i, $i = 1, 2$, mentioned in Theorem 3.11 in terms of a solution to an explicit system of ordinary differential equations, which we present now. Given $p \in \Sigma$ and $w \in T_p\Sigma$, let $\xi_p \in T_p\Sigma$ be a unit vector orthogonal to w. Let α_0, β_0 be the principal curvatures of the surface $H \cdot p$ with respect to the normal vector ξ_p, $(U_1)_p$ the orthogonal projection of $J\xi_p$ onto the principal curvature space associated with α_0, and $\phi_0 \in [0, \pi/2]$ the angle between $J\xi_p$ and $(U_1)_p$. Let us assume that $\phi_0 \in (0, \pi/2)$. By reversing the sign of ξ_p if necessary, we can further assume that $\langle Jw, (U_1)_p \rangle > 0$. Consider a local solution $(\tilde{\alpha}, \tilde{\beta}, \tilde{\phi})\colon (-\epsilon, \epsilon) \to \mathbb{R}^3$ to the ODE

$$\begin{aligned}
\tilde{\alpha}' &= \frac{1}{4}\big(c(2 - 3\sin^2\tilde{\phi}) + 4\tilde{\alpha}(\tilde{\alpha} - \tilde{\beta})\big)\tan\tilde{\phi}, \\
\tilde{\beta}' &= -\frac{3c}{8}\sin 2\tilde{\phi}, \\
\tilde{\phi}' &= \tilde{\beta} + \frac{c(1 - 3\sin^2\tilde{\phi})}{4(\tilde{\alpha} - \tilde{\beta})},
\end{aligned}$$

with initial conditions $\tilde{\alpha}(0) = \alpha_0$, $\tilde{\beta}(0) = \beta_0$, and $\tilde{\phi}(0) = \phi_0$. Then, one of the two curves mentioned in the Theorem 3.11 is the unique curve $\sigma_\beta\colon (-\epsilon, \epsilon) \to \Sigma$ with curvature function $\tilde{\beta}$ and orientation given by ξ_p (i.e. $\bar{\nabla}_{\dot{\sigma}_\beta(t)}\dot{\sigma}_\beta = \tilde{\beta}(t)\xi_{\sigma_\beta(t)}$ for all t, where $\bar{\nabla}$ is the Levi-Civita connection of Σ, and ξ is a smooth normal vector field along σ_β extending ξ_p and tangent to Σ). The corresponding hypersurface $H \cdot \sigma_\beta$ has two principal curvatures α and β (this one with multiplicity 2) such that $(\alpha \circ \sigma_\beta)(t) = \tilde{\alpha}(t)$, and $(\beta \circ \sigma_\beta)(t) = \tilde{\beta}(t)$ for all t. The other possible curve σ_α is obtained by interchanging the roles of α_0 and β_0 in the description above.

This Section is organized as follows. In Subsection 3.4.1 we present the construction of the new examples of real hypersurfaces with two principal curvatures. Then, in Subsection 3.4.2 we focus on the classification problem. Using the information given in Propositions 3.1 and 3.4 we show that our hypersurface is foliated orthogonally by equidistant, Lagrangian, flat surfaces with parallel second fundamental form, and by the integral curves of certain vector field and finally, we conclude the proof of the Theorem 3.11.

3.4.1 Construction

We assume the notation and terminology introduced in Section 3.1. Thus, let M be a hypersurface constructed as in Section 3.1. Then, the fact that M has two principal curvatures is equivalent to the fact that σ satisfies certain ordinary differential equation.

Using the theorem of existence of solutions to ODEs in canonical form, we derive the existence of our hypersurfaces. Let us be more specific.

We recall that for a connected subgroup H of isometries of $\bar{M}^2(c)$ acting polarly and with cohomogeneity two and section Σ, we take a curve $\sigma\colon t \in (-\varepsilon, \varepsilon) \mapsto \sigma(t) \in \Sigma_{reg}$ in the regular part of Σ. Then, the 3-dimensional hypersurface $H \cdot \sigma$ in $\bar{M}^2(c)$ given by the subset $M = H \cdot \sigma$ is orthogonally foliated by equidistant H-orbits, and by the curves $h \circ \sigma\colon t \in (-\varepsilon, \varepsilon) \mapsto (h \circ \sigma)(t) = h(\sigma(t)) \in \Sigma_{reg}$ for each $h \in H$.

In Section 3.1 we proved that the two principal curvatures (with respect to any unit normal vector) of a principal orbit of the cohomogeneity two action of the group H on $\bar{M}^2(c)$ are always different. The curvature of the curve σ in $\bar{M}^2(c)$ is given by

$$\langle \mathcal{S}_{\xi(t)}\dot{\sigma}(t), \dot{\sigma}(t)\rangle = -\langle \bar{\nabla}_{\dot{\sigma}(t)}\xi, \dot{\sigma}(t)\rangle = \langle \bar{\nabla}_{\dot{\sigma}(t)}\dot{\sigma}(t), \xi\rangle.$$

Hence, $H \cdot \sigma$ will have two principal curvatures at the points of σ if and only if the curvature of σ (with respect to ξ) as a curve in Σ coincides with one of the two functions $\alpha(\xi(t))$ or $\beta(\xi(t))$. If we write σ in local coordinates as $\sigma(t) = (x_1(t), x_2(t))$, we have:

$$\begin{aligned}\bar{\nabla}_{\dot{\sigma}}\dot{\sigma} &= x_1''\partial_1 + x_1'\bar{\nabla}_{\dot{\sigma}}\partial_1 + x_2''\partial_2 + x_1'\bar{\nabla}_{\dot{\sigma}}\partial_2 \\ &= (x_1'' + f(x_1, x_2, x_1', x_2'))\partial_1 + (x_2'' + g(x_1, x_2, x_1', x_2'))\partial_2,\end{aligned}$$

where f, g are smooth functions depending on x_1, x_2, x_1', x_2' and the Christoffel symbols of Σ. The fact that the curvature of σ coincides with $\alpha(\xi(t))$ means that $\bar{\nabla}_{\dot{\sigma}}\dot{\sigma} = \alpha(\xi(t))\xi_{\sigma(t)}$. Hence, this condition is equivalent to the existence of smooth functions F_α, G_α such that

$$x_1'' = F_\alpha(x_1, x_2, x_1', x_2'), \qquad\qquad x_2'' = G_\alpha(x_1, x_2, x_1', x_2').$$

This is a second order system of ordinary differential equations that has a unique solution for given initial conditions $\sigma(0) = p$ and $\dot{\sigma}(0) = w$. Therefore, the hypersurface $H \cdot \sigma$ has two principal curvatures at the points of σ if and only if σ is a solution to one of the two possible systems of ODEs constructed as explained above. A completely analogous argument applies for β instead of α, and it is obvious that the two possible choices that we have for σ, depending on whether we choose α or β to be the principal curvature with multiplicity two, provide indeed two different curves, say $\sigma_\alpha \neq \sigma_\beta$, and thus, two different hypersurfaces $H\cdot\sigma_\alpha$ and $H\cdot\sigma_\beta$. Moreover, as proved in Section 3.1, the principal curvatures of the hypersurface $H \cdot \sigma$ are constant along the H-orbits. Thus, $H \cdot \sigma$ has exactly two distinct principal curvatures at every point, not only on the points of σ.

Finally, we have to show that the examples we have just constructed are indeed new, or equivalently, that their two principal curvatures are nonconstant. Fix a point $p \in \Sigma_{reg}$. Then we know that for every unit $w \in T_p\Sigma_{reg}$, there is a locally defined curve σ_w such that $\sigma_w(0) = p$, $\dot{\sigma}_w(0) = w$, and $H \cdot \sigma_w$ has two principal curvatures. In fact, there are exactly two such curves, but the argument that follows applies to any of them. In Section 3.1, we have proved that there exists a finite set $\mathfrak{w}_p \subset S^1(T_p\Sigma)$ such that, if $\dot{\sigma}(0) \notin \mathfrak{w}_p$, the hypersurface $M = H \cdot \sigma$ is non-Hopf at any point of M. But since M has two principal curvatures, these cannot be constant, because all hypersurfaces in $\bar{M}^2(c)$ with two constant principal curvatures are Hopf, as follows from a well-known classification result (see [10] and [99]). Thus, we have proved the first part of the Theorem 3.11.

The previous construction can be made more explicit for complex projective planes. The rest of this subsection is devoted to this alternative description. It is interesting to point out that our construction ultimately relies on the use of classical geometric objects of surfaces of $\mathbb{R}^3$. This alternative description has been published in [44].

Recall from Corollary 1.5 that the only polar action of cohomogeneity two on $\mathbb{C}P^2$ is induced by the isotropy representation of the Hermitian symmetric space $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$. This isotropy representation is equivalent to the action of $K = U(1) \times U(1) \times U(1)$ on $\mathbb{C}^3$.

Isotropy representations of symmetric spaces are known to be polar, and any maximal flat is a section. For Hermitian symmetric spaces, sections are also totally real. Let $\{e_1, e_2, e_3\}$ be a unitary basis of $\mathbb{C}^3$ and $\{\lambda_1, \lambda_2, \lambda_3\}$ its dual basis. Then $\mathfrak{a} = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ is a section of the action of K on $\mathbb{C}^3$. Given an orbit of K there exists an element $\mathbf{x} \in \mathfrak{a}$ where the orbit meets the section. Assume that the orbit is principal, so that $\mathbf{x} = \sum_j x_j e_j$ with $x_j \neq 0$, that is, $\mathbf{x}$ is a regular point. We are going to calculate the shape operator of the orbit $K \cdot \mathbf{x}$. It suffices to do so at $\mathbf{x}$ by homogeneity of $K \cdot \mathbf{x}$. Let $\xi \in \nu_{\mathbf{x}}(K \cdot \mathbf{x})$ be a normal vector of $K \cdot \mathbf{x}$. Since the orbit is principal, $\nu_{\mathbf{x}}(K \cdot \mathbf{x}) = \mathfrak{a}$, and we write $\xi = \sum_j \xi_j e_j$. We define $\mathfrak{p}_j = \mathbb{R}ie_j$. Then, using the root space decomposition of a real semisimple Lie algebra [9, Example 3.4] it has been shown that the shape operator of $K \cdot \mathbf{x}$ with respect to ξ is given by

$$\tilde{\mathcal{S}}_\xi|_{\mathfrak{p}_j} = -\frac{\lambda_j(\xi)}{\lambda_j(\mathbf{x})} \operatorname{Id}_{\mathfrak{p}_j}, \quad j \in \{1, 2, 3\}. \tag{3.13}$$

Let $S^5(r)$ be the sphere of radius $r > 0$ in $\mathbb{C}^3$. Since $K \subset O(6)$, it is clear that the orbits of K are contained in spheres. We denote by $\pi \colon S^5(r) \to \mathbb{C}P^2$ the Hopf map, which is a Riemannian submersion. In fact, given $\mathbf{x} \in S^5(r)$, $\ker \pi_{*\mathbf{x}} = \mathbb{R}i\mathbf{x}$ and the restriction $\pi_{*\mathbf{x}} \colon T_{\mathbf{x}} S^5(r) \ominus \mathbb{R}i\mathbf{x} \to T_{\pi(\mathbf{x})}\mathbb{C}P^2$ is an isometry of vector spaces, where by $\ominus$ we denote orthogonal complement.

Let $\mathbf{x} \in \mathfrak{a} \cap S^5(r)$ be a regular point. Since $\mathfrak{a}$ is totally real (that is, $i\mathfrak{a}$ is orthogonal to $\mathfrak{a}$), the action of K descends to $\mathbb{C}P^2$, and it follows that $\Sigma = \pi(\mathfrak{a} \cap S^5(r))$ is a section of the induced action of K on $\mathbb{C}P^2$ [93]. Moreover, Σ and $\mathfrak{a} \cap S^5(r)$ are locally isometric, and $K \cdot \mathbf{x}$ is a principal orbit if and only if the induced orbit $K \cdot \pi(\mathbf{x})$ is. Now we take $\xi \in T_{\mathbf{x}} S^5(r) \ominus T_{\mathbf{x}}(K \cdot \mathbf{x}) = \mathfrak{a} \ominus \mathbb{R}\mathbf{x}$ a vector that is normal to the orbit $K \cdot \mathbf{x}$ at $\mathbf{x}$, but tangent to the sphere. Let $\zeta = \pi_{*\mathbf{x}}\xi$. Our aim in what follows is to calculate the shape operator $\mathcal{S}_\zeta$ of $K \cdot \pi(\mathbf{x})$ at $\pi(\mathbf{x})$ with respect to $\zeta = \pi_{*\mathbf{x}}\xi$.

The restriction $\pi_{*\mathbf{x}} \colon T_{\mathbf{x}}(K \cdot \mathbf{x}) \ominus \mathbb{R}i\mathbf{x} \to T_{\pi(\mathbf{x})}(K \cdot \pi(\mathbf{x}))$ is a linear isometry. Hence, if $v \in T_{\mathbf{x}}(K \cdot \mathbf{x}) \ominus \mathbb{R}i\mathbf{x}$, the shape operator is determined by the equation $\mathcal{S}_\zeta(\pi_{*\mathbf{x}} v) = \pi_{*\mathbf{x}}(\tilde{\mathcal{S}}_\xi(v))^\top$, where $(\cdot)^\top$ denotes orthogonal projection onto $T_{\mathbf{x}}(K \cdot \mathbf{x}) \ominus \mathbb{R}i\mathbf{x} = i(\mathfrak{a} \ominus \mathbb{R}\mathbf{x})$. Let $\{A_1, A_2\}$ be an orthonormal basis of $\mathfrak{a} \ominus \mathbb{R}\mathbf{x}$. Writing $\xi = \xi_1 A_1 + \xi_2 A_2$, $A_j = \sum_l a_{jl} e_l$, and using (3.13), we get that the (j,k) component of $\mathcal{S}_\zeta$ is

$$\langle \mathcal{S}_{\pi_*\xi}(\pi_*(iA_j)), \pi_*(iA_k) \rangle = \langle \tilde{\mathcal{S}}_\xi(iA_j), iA_k \rangle = -\sum_{l=1}^{2} \xi_l \sum_{s=1}^{3} \frac{a_{js} a_{ks} a_{ls}}{x_s}. \tag{3.14}$$

We denote by $\bar{\nabla}$ the Levi-Civita connection of $\mathbb{C}P^2$, and by D the usual connection of a Euclidean space.

We start with a curve $\sigma\colon I \to \Sigma$ contained in the section Σ. We denote by ζ a unit normal vector field of σ in Σ (recall that Σ is a two-dimensional totally geodesic real projective plane). We assume that σ is parametrized by arc length. We consider, at least in a sufficiently small neighbourhood, the submanifold $M = K \cdot \sigma = \cup_{t\in I} K \cdot \sigma(t)$. For a fixed t, the principal curvatures of M along $K \cdot \sigma(t)$ are constant, as $K \cdot \sigma(t)$ is an orbit of an isometric action. The other principal curvature of M at $\sigma(t)$ is $\gamma(t) = -\langle \bar{\nabla}_{\dot{\sigma}(t)}\zeta(t), \dot{\sigma}(t)\rangle$, because Σ is totally geodesic and ζ has unit length. The idea is to make $\gamma(t)$ equal to one of the principal curvatures of $K \cdot \sigma(t)$.

We now make this more explicit by lifting the geometric data to $S^5(r)$. Thus, we choose a unit-speed curve $\mathbf{x}\colon I \to S^5(r) \cap \mathfrak{a}$ such that $\sigma = \pi \circ \mathbf{x}$. Since $\mathbf{x}$ is contained in a sphere, $\dot{\mathbf{x}}(t) \in \mathfrak{a} \ominus \mathbb{R}\mathbf{x}(t)$, and a unit normal of $\mathbf{x}$ in $S^5(r)$ is $\xi(t) = \frac{1}{r}\mathbf{x}(t) \times \dot{\mathbf{x}}(t) \in \mathfrak{a} \ominus \mathbb{R}\mathbf{x}(t)$, where $\times$ denotes vector product of $\mathfrak{a} \cong \mathbb{R}^3$. Thus, we have that the curvature of $\mathbf{x}$ is

$$\begin{aligned}\gamma(t) &= -\langle \bar{\nabla}_{\dot{\sigma}(t)}\zeta(t), \dot{\sigma}(t)\rangle = -\langle \bar{\nabla}_{\pi_*\dot{\mathbf{x}}(t)}\pi_*\xi(t), \pi_*\dot{\mathbf{x}}(t)\rangle = -\langle D_{\dot{\mathbf{x}}(t)}\xi(t), \dot{\mathbf{x}}(t)\rangle \\ &= -\frac{1}{r}\langle \mathbf{x}(t) \times \ddot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)\rangle = \frac{1}{r}\det(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)).\end{aligned}$$

Now we turn to the explicit calculation of (3.14). We will take here a simple approach and use spherical coordinates on a sphere, which, in particular, gives a system of coordinates whose coordinate vectors are orthogonal. Thus, let $\phi(u, v) = r(\cos u \cos v, \sin u \cos v, \sin v)$, and write $\mathbf{x}(t) = \phi(u(t), v(t))$. Since $\mathbf{x}$ is parametrized by arc length, we obtain the equation $r^2(u'(t)^2 \cos^2 v(t) + v'(t)^2) = 1$. We also get

$$\gamma = r^2((u'v'' - u''v')\cos v + (u')^3 \cos^2 v \sin v + 2u'(v')^2 \sin v).$$

A suitable orthonormal basis of $\mathfrak{a} \ominus \mathbb{R}\mathbf{x}(t)$ is $A_1 = \partial_u\phi/\|\partial_u\phi\|$, $A_2 = \partial_v\phi/\|\partial_v\phi\|$ at $(u(t), v(t))$. Thus, $\xi = -rv'A_1 + ru'\cos v A_2$, and substituting in (3.14) we obtain

$$\mathcal{S}_\zeta = \begin{pmatrix} u'\sin v + 2v'\frac{\cot 2u}{\cos v} & -v'\tan v \\ -v'\tan v & -u'\frac{\cos 2v}{\sin v}\end{pmatrix},$$

whose characteristic polynomial is

$$\begin{aligned}p(\lambda) = \lambda^2 - \lambda\left(u'\left(\sin v - \frac{\cos 2v}{\sin v}\right) + v'\frac{2\cot 2u}{\cos v}\right) \\ - 4u'v'\cot 2u \cot 2v - (u')^2\cos 2v - (v')^2\tan^2 v.\end{aligned}$$

All in all this means that, in order to construct a real hypersurface with exactly two principal curvatures we have to solve the ODE:

$$\begin{aligned}&r^2((u')^2\cos^2 v + (v')^2) = 1, \\ &p\left(r^2((u'v'' - u''v')\cos v + (u')^3\cos^2 v \sin v + 2u'(v')^2 \sin v)\right) = 0,\end{aligned}$$

where the first equation is the condition that $\mathbf{x}(t)$ is parametrized by arc length, and the second states that the curvature of $\mathbf{x}$ coincides with one of the principal curvatures of $K \cdot \pi(\mathbf{x}(t))$. It is proved above that, for given initial conditions, there are two curves (one for each eigenvalue of $\mathcal{S}_\zeta$) satisfying the above ODE. Thus, this method produces a real hypersurface with two principal curvatures in $\mathbb{C}P^2$.

3.4.2 Classification

A real hypersurface of $\bar{M}^2(c)$ has, generically, three distinct principal curvatures. As we have said, Tashiro and Tachibana [100] proved that there are no umbilical real hypersurfaces in nonflat complex spaces forms. If $n \geq 3$, Cecil and Ryan [24] in $\mathbb{C}P^n$, and Montiel [86] in $\mathbb{C}H^n$ proved that real hypersurfaces with two distinct principal curvatures are open parts of homogeneous Hopf hypersurfaces. The methods used by Cecil and Ryan, and Montiel, do not work if $n = 2$, and we have indeed provided in Section 3.4.1 examples of real hypersurfaces in $\bar{M}^2(c)$ with two distinct nonconstant principal curvatures. The classification of real hypersurfaces with two distinct *constant* principal curvatures in $\bar{M}^2(c)$ is not very difficult to obtain and can be found in [99] for $\mathbb{C}P^2$ as a particular case of the classification in arbitrary dimensions, and in [10] for $\mathbb{C}H^2$. Again, all the examples are open parts of homogeneous Hopf real hypersurfaces. The classification of Hopf real hypersurfaces with constant principal curvatures in nonflat complex space forms is due to Kimura [71] in $\mathbb{C}P^n$, and to Berndt [8] in $\mathbb{C}H^n$.

The aim of this section is to prove the second part of the Theorem 3.11, that is, to show that a real hypersurface in $\bar{M}^2(c)$, $c \neq 0$, must be locally congruent to a hypersurface constructed as in the previous subsection.

From now on we assume that M is a real hypersurface of $\bar{M}^2(c)$, $c \neq 0$, with two nonconstant principal curvatures α and β and that is non-Hopf at every point. In particular, we will assume without loss of generality that $\dim T_\alpha = 1$ and $\dim T_\beta = 2$, and that $\{U_1, U_2, A\}$ is an orthonormal frame of M where $U_1 \in \Gamma(T_\alpha)$ and $U_2, A \in \Gamma(T_\beta)$. We will use the notation given in Proposition 3.1 and Remark 3.3 with $\gamma = \beta$, and where b_1, $b_2 \neq 0$ along M. Then, since M has two principal curvatures, Proposition 3.4 gives us the Levi-Civita connection of M. In particular, the relations in Proposition 3.4 imply that M is a strongly 2-Hopf real hypersurface with two principal curvatures in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, that is, $\mathcal{D} = \mathbb{R}U_1 \oplus \mathbb{R}U_2$, the smallest $\mathcal{S}$-invariant distribution of M that contains $J\xi$, has rank 2 and is integrable, and α and β are constant along the integral submanifolds of $\mathcal{D}$. Furthermore α, β and ϕ satisfy the following ODE,

$$
\begin{aligned}
A\phi &= \beta + \frac{c(1 - 3\sin^2\phi)}{4(\alpha - \beta)}, \\
A\alpha &= \frac{1}{4}\big(c(2 - 3\sin^2\phi) + 4\alpha(\alpha - \beta)\big)\tan\phi, \qquad (3.15) \\
A\beta &= -\frac{3c}{8}\sin 2\phi.
\end{aligned}
$$

This concludes the classification part of the Theorem 3.11.

We still have to justify the assertions made in Remark 3.13, which basically follow from Proposition 3.4 where it is said that the functions ϕ, α, and β are constant along the leaves of the distribution $\mathcal{D}$, and thus, (3.15) translates into the differential equation given in Remark 3.13. Consider a group H acting polarly with cohomogeneity two and section Σ on $\bar{M}^2(c)$, and fix a regular point $p \in \Sigma$, and a unit vector $w \in T_p\Sigma$. Let σ be one of the two locally defined unit speed curves on Σ with $\sigma(0) = p$ and $\dot{\sigma}(0) = w$ such that $H \cdot \sigma$ has two principal curvatures (cf. Section 3.4.1). We have to show, under the assumption that $H \cdot \sigma$ is everywhere non-Hopf, that the curve σ can be obtained by means of the initial value problem stated in Remark 3.13. We start by regarding $M = H \cdot \sigma$ as a real hypersurface with two principal curvatures in $\bar{M}^2(c)$, so that we can apply to it the arguments presented above in this subsection. Let ξ be a unit normal vector field along M. Recall from Section 3.1 that the principal curvatures α_0 and β_0 of $H \cdot p$ at p with respect to ξ_p coincide with the two principal curvatures of M at p. Assume without restriction of generality that β_0 has multiplicity two as principal curvature of M. In view of Proposition 3.8, it is enough to show that the tangent vector field to σ is given by the vector field A introduced in Proposition 3.1. The Hopf vector $J\xi$ is orthogonal to $\dot{\sigma}$ along σ, since Σ is totally real and $\xi_{\sigma(t)}$ is tangent to Σ for all t. Moreover, as explained in Section 3.1, $\dot{\sigma}(t)$ is a principal curvature vector of the hypersurface M for all t. Therefore, it follows that $\dot{\sigma}$ is collinear with A. Since both A and $\dot{\sigma}$ have unit length, reversing the sign of ξ if necessary, we have that $\dot{\sigma}(t) = A_{\sigma(t)}$ for all t. This proves Remark 3.13.

Remark 3.14. Let us explain why we make the assumption that the real hypersurface M is non-Hopf at every point. On the one hand, if it were Hopf (at every point), it is known (it follows for example from [88, Theorem 2.1 and Corollary 2.3]) that M would have constant principal curvatures, thus leading to well-known examples. On the other hand, we should consider the case when M is Hopf only along a subset of M whose complement in M is open and dense. In this situation, the vector fields U_2 and A in Proposition 3.1 with $\gamma = \beta$ might not be well-defined in the points where M is Hopf. However, in any case we know that there is an open and dense subset of M that has the structure described in Theorem 3.11.

3.5 Austere hypersurfaces

In this section we investigate austere hypersurfaces in two-dimensional nonflat complex space forms. Recall that a hypersurface is called austere if its principal curvature functions are invariant under multiplication by -1 (see Section 1.2). The classification of austere hypersurfaces in spheres, or in the complex projective and hyperbolic planes, is not known [34], [66]. In this sense we prove the following result.

Theorem 3.15. *Let M be a real hypersurface of $\bar{M}^2(c)$, $c \neq 0$, whose Hopf vector field has nontrivial projection onto at most two principal curvature spaces (i.e. $h \leq 2$). Then M is austere if and only if it is an open part of one of the following examples:*

(i) *a Lohnherr hypersurface in $\mathbb{C}H^2$, or*

(ii) *a Clifford cone in* $\mathbb{C}P^2$ *or* $\mathbb{C}H^2$*, or*

(iii) *a bisector in* $\mathbb{C}H^2$*.*

In particular, M *is strongly* 2*-Hopf on the open and dense subset of nonumbilic points.*

All the examples in this classification are ruled, in the sense that their maximal complex distribution is integrable and its integral submanifolds are totally geodesic in the ambient space. We briefly describe the examples in Theorem 3.15. The Lohnherr hypersurface is the only complete ruled hypersurface of $\mathbb{C}H^n$ with constant principal curvatures, up to congruence [79]. It is also the unique minimal homogeneous hypersurface of $\mathbb{C}H^n$ [12]; indeed it is the only minimal orbit of the action in Theorem 1.7 (i)-(e). A Clifford cone is a minimal hypersurface which is constructed as follows (see also [3], [52] and [72] for alternative descriptions). The Lie group $H = U(1) \times U(1)$ acts on $\bar{M}^2(c)$ polarly with cohomogeneity two. This action has three fixed points in $\mathbb{C}P^2$, and only one in $\mathbb{C}H^2$. Let p be one of these fixed points, and S^r any geodesic sphere centered at p. Then a Clifford cone with vertex p is the (singular) hypersurface made by all geodesic rays starting from p and hitting the only 2-dimensional H-orbit that is minimal as a submanifold of S^r. Finally, a bisector in $\mathbb{C}H^n$ is a minimal hypersurface of cohomogeneity one defined as the set of points in $\mathbb{C}H^n$ that are at the same distance from two fixed points [52].

Now, we proceed with the study of austere real hypersurfaces in $\bar{M}^2(c)$, $c \neq 0$, under the only assumption that the Hopf vector field does not have nontrivial projection onto three principal curvature spaces. In other words, we just assume $h \leq 2$. We prove first that h must be constantly equal to 2 in an open dense subset.

Proposition 3.16. *There are no Hopf austere hypersurfaces in* $\bar{M}^2(c)$*,* $c \neq 0$*.*

Proof. Austere hypersurfaces have, by definition, vanishing mean curvature. Then, by Theorem 3.20 below in Subsection 3.6.1, a Hopf austere hypersurface in $\bar{M}^2(c)$, $c \neq 0$, must be an open part of a Hopf homogeneous hypersurface (see Subsection 3.6.1 for the complete list of such hypersurfaces, known as Tagaki's and Montiel's examples). But by direct inspection of the principal curvatures of the examples in Takagi's and Montiel's lists [88, §3] one can check that the only Hopf, homogeneous, minimal hypersurfaces in $\bar{M}^2(c)$, $c \neq 0$, are geodesic spheres or tubes around a totally geodesic $\mathbb{R}P^2$ of certain fixed radius in $\mathbb{C}P^2$. But none of these examples is austere. □

Hence, if M is an austere hypersurface of $\bar{M}^2(c)$, $c \neq 0$, with $h \leq 2$, then there is an open and dense subset of M where $h = 2$. In what follows we will assume that calculations take place in this subset. Note that the assumption that M is austere implies that its principal curvatures are α, $-\alpha$ and 0, for some smooth function α on M. We will use the notation established in Proposition 3.1.

Proposition 3.17. *Let* M *be an austere hypersurface of* $\bar{M}^2(c)$*,* $c \neq 0$*, with* $h = 2$*, and three distinct principal curvatures* α*,* $-\alpha$ *and* 0*. Then* M *is strongly 2-Hopf, the Hopf vector field has nontrivial projections onto* T_α *and* $T_{-\alpha}$*, and the norm of both projections is* $b_1 = b_2 = 1/\sqrt{2}$*.*

Proof. Assume first that $J\xi$ has nontrivial projection onto T_α and T_0. Thus, we put $\beta = 0$ and $\gamma = -\alpha$ in the results of Section 3.2. In particular, from Proposition 3.2 we get:

$$(3.16)\qquad \begin{aligned} \nabla_{U_2}U_1 &= -\frac{3b_2^2c}{4\alpha}A, & \nabla_A U_1 &= \frac{(3b_1^2-5)c}{4\alpha}U_2 - \frac{U_1\alpha}{2\alpha}A,\\ \nabla_{U_2}U_2 &= \frac{3b_1b_2c}{4\alpha}A, & \nabla_A U_2 &= -\frac{(3b_1^2-5)c}{4\alpha}U_1 + \frac{b_1U_1\alpha}{2b_2\alpha}A,\\ \nabla_{U_2}A &= \frac{3b_2^2c}{4\alpha}U_1 - \frac{3b_1b_2c}{4\alpha}U_2, & \nabla_A A &= \frac{U_1\alpha}{2\alpha}U_1 - \frac{b_1U_1\alpha}{2b_2\alpha}U_2.\\ \nabla_{U_1}U_1 &= -\frac{b_1U_1\alpha}{2b_2\alpha}U_2 + \frac{b_2((7-6b_1^2)c+4\alpha^2)}{4b_1\alpha}A, \\ \nabla_{U_1}U_2 &= \frac{b_1U_1\alpha}{2b_2\alpha}U_1 + \frac{(6b_1^2-7)c}{4\alpha}A, \\ \nabla_{U_1}A &= -\frac{b_2((7-6b_1^2)c+4\alpha^2)}{4b_1\alpha}U_1 - \frac{(6b_1^2-7)c}{4\alpha}U_2, \end{aligned}$$

$$(3.17)\qquad \begin{aligned} U_1b_1 &= -\frac{b_1}{2\alpha}U_1\alpha, \quad U_2b_1 = 0, \quad Ab_1 = -\frac{b_2}{4\alpha}(5c-4\alpha^2-3cb_1^2),\\ U_1b_2 &= \frac{b_1^2}{2b_2\alpha}U_1\alpha, \quad U_2b_2 = 0, \quad Ab_2 = \frac{b_1}{4\alpha}(5c-4\alpha^2-3cb_1^2), \end{aligned}$$

$$(3.18)\qquad U_2\alpha = -\frac{b_1}{2b_2}U_1\alpha, \qquad A\alpha = \frac{b_2}{4b_1}(5c+8\alpha^2+9b_2^2c).$$

In order to obtain $U_1\alpha$, $U_2\alpha$, U_1b_1 and U_1b_2 we need the Gauss equation. Thus, let us use the Gauss equation for the tuple (U_1, U_2, U_1, A). Proposition 3.1 (with $\beta = 0$ and $\gamma = -\alpha$), implies $\langle \bar{R}(U_1,U_2)U_1, A\rangle = 0$, and $\langle \mathcal{S}U_1,U_1\rangle\langle \mathcal{S}U_2,A\rangle - \langle \mathcal{S}U_1,A\rangle\langle \mathcal{S}U_2,U_1\rangle = 0$.

Using (3.16), together with (3.17) and (3.18), we get

$$\begin{aligned} \langle \nabla_{U_1}\nabla_{U_2}U_1, A\rangle &= \langle \nabla_{U_1}\Big(-\frac{3b_2^2c}{4\alpha}A\Big), A\rangle = U_1\Big(-\frac{3b_2^2c}{4\alpha}\Big) = -\frac{3b_2c}{4}\Big(\frac{2\alpha U_1b_2 - b_2U_1\alpha}{\alpha^2}\Big)\\ &= -\frac{3c(b_1^2-b_2^2)}{4\alpha^2}U_1\alpha,\\ \langle \nabla_{U_2}\nabla_{U_1}U_1, A\rangle &= \langle \nabla_{U_2}\Big(-\frac{b_1U_1\alpha}{2b_2\alpha}U_2 + \frac{b_2((7-6b_1^2)c+4\alpha^2)}{4b_1\alpha}A\Big), A\rangle\\ &= -\frac{b_1U_1\alpha}{2b_2\alpha}\langle \nabla_{U_2}U_2, A\rangle + U_2\Big(\frac{b_2((7-6b_1^2)c+4\alpha^2)}{4b_1\alpha}\Big)\\ &= -\frac{3b_1^2c}{8\alpha^2}U_1\alpha + \frac{7c-6b_1^2c+4\alpha^2}{4b_1\alpha}U_2b_2\\ &\quad - \frac{b_2(7c+6b_1^2c+4\alpha^2)}{4b_1^2\alpha}U_2b_1 - \frac{b_2(7c-6b_1^2c-4\alpha^2)}{4b_1\alpha^2}U_2\alpha\\ &= \frac{(7-9b_1^2)c-4\alpha^2}{8\alpha^2}U_1\alpha, \end{aligned}$$

$$\begin{aligned}\langle\nabla_{\nabla_{U_1}U_2}U_1,A\rangle &= \langle\nabla_{\frac{b_1U_1\alpha}{2b_2\alpha}U_1+\frac{(-7+6b_1^2)c}{4\alpha}A}U_1,A\rangle = \frac{b_1U_1\alpha}{2b_2\alpha}\langle\nabla_{U_1}U_1,A\rangle + \frac{(-7+6b_1^2)c}{4\alpha}\langle\nabla_A U_1,A\rangle\\ &= \frac{(7-6b_1^2)c+2\alpha^2}{4\alpha^2}U_1\alpha,\\ \langle\nabla_{\nabla_{U_2}U_1}U_1,A\rangle &= -\frac{3b_2^2c}{4\alpha}\langle\nabla_A U_1,A\rangle = \frac{3b_2^2c}{8\alpha^2}U_1\alpha.\end{aligned}$$

Altogether this means that the Gauss equation is equivalent to

$$\frac{3(-7+5b_1^2+3b_2^2)c}{8\alpha^2}U_1\alpha = 0.$$

Since the expression $-7+5b_1^2+3b_2^2 = 2(-2+b_1^2)$ cannot vanish on M, we deduce that $U_1\alpha = 0$. Therefore, the Gauss equation for the tuple (U_1,U_2,U_1,A) implies $U_1\alpha = 0$. Then, using (3.17) and (3.18), we get

$$U_1\alpha = U_2\alpha = U_1b_1 = U_1b_2 = 0. \tag{3.19}$$

Similarly, using again Proposition 3.1 (with $\beta = 0$ and $\gamma = -\alpha$), (3.16), (3.17), (3.18), and (3.19), together with the Gauss equation applied to (U_1,U_2,U_1,U_2), we obtain $\alpha^2 = \frac{1}{4}(2+3b_2^2)c$. Again, by the Gauss equation applied to (A,U_2,U_1,A) we obtain $\alpha^2 = \frac{(-8+9b_2^2+27b_2^4)c}{4(-7+3b_2^2)}$. But both expressions for α^2 are incompatible for $b_2 \in \mathbb{R}$. This contradiction implies the nonexistence of austere hypersurfaces whose Hopf vector field has nontrivial projections onto T_α and T_0.

Since α and $-\alpha$ are interchangeable, we just have to deal with the case where $J\xi$ has nontrivial projection onto T_α and $T_{-\alpha}$. Thus, we put $\beta = -\alpha$ and $\gamma = 0$ in the results of Section 3.2. In particular, Proposition 3.2 yields

$$\begin{aligned}&\nabla_{U_1}U_1 = \frac{U_2\alpha}{2\alpha}U_2 - \frac{b_1b_2(c-8\alpha^2)}{4\alpha}A, \quad \nabla_{U_2}U_1 = -\frac{U_1\alpha}{2\alpha}U_2 + \frac{4\alpha^2+b_2^2(c-8\alpha^2)}{4\alpha}A,\\ &\nabla_{U_2}U_2 = \frac{U_1\alpha}{2\alpha}U_1 - \frac{b_1b_2(c-8\alpha^2)}{4\alpha}A, \quad \nabla_{U_1}U_2 = -\frac{U_2\alpha}{2\alpha}U_1 + \frac{4\alpha^2+b_1^2(c-8\alpha^2)}{4\alpha}A,\\ &\nabla_A U_1 = \frac{(-1+2b_2^2)(c-2\alpha^2)}{4\alpha}U_2, \qquad \nabla_A U_2 = -\frac{(-1+2b_2^2)(c-2\alpha^2)}{4\alpha}U_1,\\ &\nabla_{U_1}A = \frac{b_1b_2(c-8\alpha^2)}{4\alpha}U_1 - \frac{4\alpha^2+b_1^2(c-8\alpha^2)}{4\alpha}U_2,\\ &\nabla_{U_2}A = -\frac{4\alpha^2+b_2^2(c-8\alpha^2)}{4\alpha}U_1 + \frac{b_1b_2(c-8\alpha^2)}{4\alpha}U_2,\\ &\nabla_A A = 0.\end{aligned} \tag{3.20}$$

$$\begin{aligned}&U_1b_1 = \frac{b_2U_2\alpha}{2\alpha}, \quad U_2b_1 = -\frac{b_2U_1\alpha}{2\alpha}, \quad Ab_1 = \frac{b_2(-1+2b_2^2)(c-2\alpha^2)}{4\alpha},\\ &U_1b_2 = -\frac{b_1U_2\alpha}{2\alpha}, \quad U_2b_2 = \frac{b_1U_1\alpha}{2\alpha}, \quad Ab_2 = -\frac{b_1(-1+2b_2^2)(c-2\alpha^2)}{4\alpha},\end{aligned} \tag{3.21}$$

$$A\alpha = \frac{b_1 b_2}{2}(c + 4\alpha^2). \tag{3.22}$$

Then, by applying the Gauss equation to (A, U_2, A, U_1), and using Proposition 3.1 (with $\beta = -\alpha$ and $\gamma = 0$), (3.20), and (3.22), we obtain $b_1 b_2 c(b_1^2 - b_2^2)(c + 4\alpha^2) = 0$. If $b_1 \neq b_2$ on a nonempty subset $\mathcal{U}$ of M, we deduce that $\mathcal{U}$ is a real hypersurface with constant principal curvatures $\pm\frac{\sqrt{-c}}{2}$ and 0 in $\bar{M}^2(c)$, $c < 0$. By the classification in [11], $\mathcal{U}$ must be an open part of a Lohnherr hypersurface, but this example satisfies $b_1 = b_2$ everywhere, which is a contradiction. Therefore we must have $b_1 = b_2$ on M. Since $b_1^2 + b_2^2 = 1$, we deduce that $b_1 = b_2 = \frac{1}{\sqrt{2}}$. But then (3.21) yields $U_1\alpha = U_2\alpha = 0$. This, together with (3.20), implies that $\nabla_{U_1} U_2 - \nabla_{U_2} U_1 = 0$. Hence, M is strongly 2-Hopf, as we wanted to show. □

In order to conclude the proof of Theorem 3.15 we will make use of the notion of ruled hypersurface. Recall that a real hypersurface M in a complex space form is called ruled if the maximal complex distribution $(J\xi)^\perp$ of M is integrable and its integral submanifolds are totally geodesic complex hypersurfaces of the ambient space (see Subsection 1.6.5).

Proof of Theorem 3.15. Observe that $(J\xi)^\perp = \mathbb{R}JA \oplus \mathbb{R}A = \mathbb{R}(b_2 U_1 - b_1 U_2) \oplus \mathbb{R}A$. By Proposition 3.17 we have $\mathcal{S}JA = \alpha b_2 U_1 + \alpha b_1 U_2 = \frac{\alpha}{\sqrt{2}}(U_1 + U_2) = \alpha J\xi$ and $\mathcal{S}A = 0$, which implies that $\mathcal{S}(J\xi)^\perp \subset \mathbb{R}J\xi$. By [25, Proposition 8.27], M is a ruled hypersurface. In particular, M is a minimal ruled hypersurface in $\bar{M}^2(c)$, $c \neq 0$. Lohnherr and Reckziegel [79] proved that there is at most one minimal ruled hypersurface in $\mathbb{C}P^2$ up to local congruence, and at most three in $\mathbb{C}H^2$.

Kimura [72] proved that a cone over a Clifford torus in $\mathbb{C}P^2$ is austere and ruled. Since ruled hypersurfaces satisfy $h \leq 2$ everywhere (indeed $h = 2$ on an open and dense subset), Kimura's example gives the only possibility of an austere hypersurface with $h \leq 2$ in $\mathbb{C}P^2$.

In $\mathbb{C}H^2$ there are three known (noncongruent) examples of minimal ruled hypersurfaces: Clifford cones [3, §3], Lohnherr hypersurfaces [79, §4], and bisectors [25, p. 447]; see also [52, p. 253]. All of them are known to be austere with $h \leq 2$. Therefore, these are precisely the examples of austere hypersurfaces with $h \leq 2$ in $\mathbb{C}H^2$. □

Remark 3.18. It is known that a ruled hypersurface M in a complex space form is locally constructed by attaching to an integral curve τ of $J\xi$ the complex totally geodesic hypersurfaces that are normal to $\dot{\tau}$. It was also shown in [79] that a ruled hypersurface in $\bar{M}^2(c)$, $c \neq 0$, is minimal if and only if τ is a circle contained in a totally geodesic and totally real submanifold of $\bar{M}^2(c)$. Moreover, in the projective case, any two such circles give rise to the same ruled hypersurface, up to congruence, whereas in the hyperbolic case, two circles τ_1, τ_2 give rise to congruent ruled hypersurfaces if and only if their curvatures $\|\bar{\nabla}_{\dot{\tau}_1}\dot{\tau}_1\|$, $\|\bar{\nabla}_{\dot{\tau}_2}\dot{\tau}_2\|$ are both greater, equal, or less than $\frac{\sqrt{-c}}{2}$. It follows from our study above that $\bar{\nabla}_{J\xi}J\xi = \alpha A$ for an austere hypersurface with $h = 2$. Note that from (3.22) with $b_1 = b_2 = \frac{1}{\sqrt{2}}$, we have that $\alpha - \frac{\sqrt{-c}}{2}$ has constant sign. It turns out that the cases $\alpha > \frac{\sqrt{-c}}{2}$, $\alpha = \frac{\sqrt{-c}}{2}$ and $\alpha < \frac{\sqrt{-c}}{2}$ correspond, respectively, to Clifford cones, Lohnherr hypersurfaces and bisectors.

3.6 Applications

In this section we derive some characterizations of strongly 2-Hopf hypersurfaces that satisfy some additional properties. In Subsection 3.6.1 we use the strongly 2-Hopf assumption to characterize certain cohomogeneity one real hypersurfaces with constant mean curvature (Corollary 3.19). In Subsection 3.6.2 we describe Levi-flat strongly 2-Hopf real hypersurfaces in $\bar{M}^2(c)$, $c \neq 0$ (Corollary 3.21). Finally, in Subsection 3.6.3, we classify strongly 2-Hopf, Levi-flat real hypersurfaces with constant mean curvature in $\bar{M}^2(c)$, $c \neq 0$, which turn out to be austere (Theorem 3.22).

Theorem 3.5 guarantees that strongly 2-Hopf hypersurfaces in $\bar{M}^2(c)$, $c \neq 0$, are constructed locally as a set $H \cdot \sigma = \{h(\sigma(t)) : t \in (-\varepsilon, \varepsilon),\, h \in H\}$, where H is a connected group of isometries acting polarly and with cohomogeneity two on $\bar{M}^2(c)$, and σ is a smooth curve in the regular part of a section Σ of the H-action. Our purpose is to determine which curves σ give rise to a real hypersurface with one or several additional properties.

3.6.1 Strongly 2-Hopf hypersurfaces with constant mean curvature

A first consequence of the classification theorem of strongly 2-Hopf hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ is a local characterization of the examples of constant mean curvature hypersurfaces constructed by Gorodski and Gusevskii [54].

Corollary 3.19. *Let H and Σ be as in the Theorem 3.5, and let $\eta \in \mathbb{R}$. Then, for any regular point $p \in \Sigma$ and any unit $w \in T_p\Sigma$, there is exactly one locally defined curve σ on Σ with $\sigma(0) = p$, $\dot{\sigma}(0) = w$, and such that the hypersurface $H \cdot \sigma$ has constant mean curvature η. Conversely, any strongly 2-Hopf real hypersurface with constant mean curvature in $\bar{M}^2(c)$ is locally congruent to a hypersurface constructed in this way.*

It is interesting to point out here that, in the family of constant mean curvature hypersufaces in $\bar{M}^2(c)$, the wealth of strongly 2-Hopf examples contrasts with the rigidity of those that are Hopf. Indeed, we have the following result.

Theorem 3.20. *Let M be a connected Hopf real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with constant mean curvature. Then M is an open part of a standard homogeneous hypersurface.*

Here, by standard homogeneous hypersurface we refer to the Hopf homogeneous hypersurfaces known as the examples in Takagi's and Montiel's lists [88]. In the case of $\mathbb{C}P^2$ these are geodesic spheres and tubes around a totally geodesic $\mathbb{R}P^2$, whereas in $\mathbb{C}H^2$ they are geodesic spheres, tubes around a totally geodesic $\mathbb{R}H^2$, tubes around a totally geodesic $\mathbb{C}H^1$, and horospheres (compare with the classification of cohomogeneity one polar actions on $\mathbb{C}P^2$ and $\mathbb{C}H^2$ in Section 1.7).

As an application of well-known results about Hopf real hypersurfaces in nonflat complex space forms, we prove Theorem 3.20.

Proof of Theorem 3.20. Let M be a Hopf real hypersurface in $\bar{M}^2(c)$, $c \neq 0$, with constant mean curvature. Let α denote the principal curvature of the Hopf vector field. By [88, Theorem 2.1] we know that α is constant on M. Now, by [88, Corollary 2.3(ii)], if β and γ denote the other principal curvatures of M, we have $2\alpha(\beta+\gamma)-4\beta\gamma+c=0$. This equation, together with the constancy of α and $\alpha+\beta+\gamma$, implies that β and γ are also constant. Hence, M is a Hopf hypersurface in $\bar{M}^2(c)$, $c \neq 0$, with constant principal curvatures. According to the classification by Kimura [71] and Berndt [8], we conclude that M must be an open part of a standard homogeneous hypersurface. □

Proof of Corollary 3.19. Assume that the mean curvature of the resulting hypersurface $H \cdot \sigma$ is constant. Thus, let $p \in \Sigma$ be a regular point, $w \in T_p\Sigma$ a tangent vector, and σ a smooth curve in the regular part of Σ such that $\sigma(0) = p$ and $\dot{\sigma}(0) = w$. Let ξ be one of the two unit normal vector fields along σ that are tangent to Σ, and let γ denote the curvature of σ with respect to ξ. We also denote by ξ the unique extension to a smooth unit normal vector field along $H \cdot \sigma$; note that such extension is H-equivariant. Observe also that, by equivariance, the principal curvatures of $H \cdot \sigma$ with respect to ξ are constant along each H-orbit. Then the mean curvature of $H \cdot \sigma$ with respect to ξ will have a constant value $\eta \in \mathbb{R}$ if and only if the curvature function γ satisfies $\gamma(t) = \eta - \alpha(\xi(t)) - \beta(\xi(t))$ for all t where σ is defined, where here $\alpha(\xi(t))$ and $\beta(\xi(t))$ are the principal curvatures of the orbit $H \cdot \sigma(t)$ with respect to $\xi_{\gamma(t)}$ at the point $\gamma(t)$. In other words, we need $(\bar{\nabla}_{\dot{\sigma}}\dot{\sigma})(t) = (\eta - \alpha(\xi(t)) - \beta(\xi(t)))\xi_{\sigma(t)}$ for all t. But, in local coordinates, this yields an ordinary differential equation of second order in normal form, so it admits a unique local solution σ for initial conditions $\sigma(0) = p$ and $\dot{\sigma}(0) = w$. This, together with the Theorem 3.5, proves Corollary 3.19. □

Observe that, by the Theorem 3.5, the hypersurface with constant mean curvature constructed above is generically strongly 2-Hopf.

3.6.2 Levi-flat strongly 2-Hopf hypersurfaces

Another application of Theorem 3.5 concerns the existence of Levi-flat hypersurfaces of cohomogeneity one. Recall that a real hypersurface of a complex manifold is called Levi-flat if it is foliated by complex hypersurfaces, or equivalently, if the Levi form vanishes identically (see Subsection 1.6.5). This notion is important in the study of holomorphic foliations, and indeed, an outstanding problem is the existence of complete, smooth Levi-flat hypersurfaces in the complex projective plane; nonexistence has been proved for $\mathbb{C}P^n$, $n \geq 3$ [95]. Note that the following result contrasts with the nonexistence of Levi-flat, Hopf real hypersurfaces in nonflat complex space forms [33].

Corollary 3.21. *Let H and Σ be as in the Theorem 3.5. Then, for any regular point $p \in \Sigma$ and any unit $w \in T_p\Sigma$, there is exactly one locally defined curve σ on Σ with $\sigma(0) = p$, $\dot{\sigma}(0) = w$, and such that the hypersurface $H \cdot \sigma$ is Levi-flat. Conversely, any strongly 2-Hopf, Levi-flat real hypersurface in $\bar{M}^2(c)$ is constructed locally in this way.*

Proof. Consider a real hypersurface M in $\bar{M}^2(c)$, $c \neq 0$, satisfying $h = 2$. We will use the notation established in Section 3.3. Assume that M is Levi-flat. Then, its Levi form vanishes. Since $A, JA \in (J\xi)^\perp$ by Proposition 3.1, we have $I\!I(A, A) + I\!I(JA, JA) = 0$ or, equivalently, $\langle \mathcal{S}A, A\rangle + \langle \mathcal{S}JA, JA\rangle = 0$. Using Proposition 3.1 again, this condition reads

$$\gamma + b_2^2\alpha + b_1^2\beta = 0.$$

Now, in order to prove Corollary 3.21, assume that the resulting hypersurface $H \cdot \sigma$ is Levi-flat. Thus, let $p \in \Sigma$ be a regular point, $w \in T_p\Sigma$ a tangent vector, and σ a smooth curve in the regular part of Σ such that $\sigma(0) = p$ and $\dot{\sigma}(0) = w$. Let ξ be one of the two unit normal vector fields along σ that are tangent to Σ, and let γ denote the curvature of σ with respect to ξ. We also denote by ξ the unique extension to a smooth unit normal vector field along $H \cdot \sigma$, which is H-equivariant. Observe also that, by equivariance, the principal curvatures of $H \cdot \sigma$ with respect to ξ are constant along each H-orbit. Then $H \cdot \sigma$ will be Levi-flat if and only if the curvature function γ satisfies $\gamma(t) = -b_2(\xi(t))^2\alpha(\xi(t)) - b_1(\xi(t))^2\beta(\xi(t))$ for all t where σ is defined, where $b_1(\xi(t))$ and $b_2(\xi(t))$ are the norms of the orthogonal projections of $J\xi(t)$ onto the principal curvature spaces $T_{\alpha(\xi(t))}$ and $T_{\beta(\xi(t))}$ of the surface $H \cdot \sigma(t)$ with respect to $\xi(t)$, at each point $\sigma(t)$. In other words, we need $(\bar{\nabla}_{\dot{\sigma}}\dot{\sigma})(t) = (-b_2(\xi(t))^2\alpha(\xi(t)) - b_1(\xi(t))^2\beta(\xi(t)))\xi_{\sigma(t)}$ for all t. But, in local coordinates, this yields an ordinary differential equation of second order in normal form, so it admits a unique local solution σ for initial conditions $\sigma(0) = p$ and $\dot{\sigma}(0) = w$. This, together with Theorem 3.5, proves Corollary 3.21. Observe that, by Theorem 3.5, the Levi-flat hypersurface constructed above is generically strongly 2-Hopf. □

3.6.3 Levi-flat strongly 2-Hopf hypersurfaces with constant mean curvature

One can wonder to what extent imposing some additional geometric conditions restricts the class of Levi-flat hypersurfaces. In this sense, Bryant [20] classified Levi-flat minimal hypersurfaces in 2-dimensional complex space forms. It follows from his result that, for $\mathbb{C}P^2$ and $\mathbb{C}H^2$, each example in his classification is invariant under a one-dimensional subgroup of the ambient isometry group. By weakening the minimality condition, and adding the strongly 2-Hopf assumption, we can obtain the following result.

Theorem 3.22. *Let M be a connected, Levi-flat, strongly 2-Hopf real hypersurface in $\bar{M}^2(c)$, $c \neq 0$. Then M has constant mean curvature if and only if it is an open part of*

(i) *a Lohnherr hypersurface in $\mathbb{C}H^2$, or*

(ii) *a Clifford cone in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, or*

(iii) *a bisector in $\mathbb{C}H^2$.*

In particular, M is austere and ruled.

Proof. Let M be a Levi-flat strongly 2-Hopf real hypersurface in $\bar{M}^2(c)$, $c \neq 0$, with constant mean curvature η. By Subsections 3.6.1 and 3.6.2, we have that $\gamma = \eta - \alpha - \beta$ and $\gamma = -b_2^2\alpha - b_1^2\beta$. Since $b_1^2 + b_2^2 = 1$, we deduce $b_1^2\alpha + b_2^2\beta = \eta$. If we take derivatives in this expression with respect to the vector field A we obtain

$$2b_1\alpha Ab_1 + b_1^2 A\alpha + 2b_2\beta Ab_2 + b_2^2 Ab_2 = 0.$$

From (3.8) in Section 3.3 and from the relation $b_1Ab_1 + b_2Ab_2 = 0$, we deduce the expressions of Ab_1, Ab_2, $A\alpha$ and $A\beta$ in terms of b_1, b_2 and the principal curvatures. Thus, substituting into the previous equation we obtain, after some calculations, $3\gamma = \alpha + \beta$. Since $\gamma = \eta - \alpha - \beta$, then $\gamma = \frac{\eta}{4}$ and $\alpha + \beta = \frac{3}{4}\eta$.

From the equations $b_1^2 + b_2^2 = 1$ and $\alpha b_2^2 + \beta b_1^2 = -\gamma$ we get the expressions $b_1^2 = \frac{\gamma+\beta}{\beta-\alpha}$ and $b_2^2 = \frac{\alpha+\gamma}{\alpha-\beta}$. Since $\alpha + \beta$ is constant, we have $A(\alpha + \beta) = 0$. This, together with (3.8), the previous expressions for b_1^2 and b_2^2, and the relations $\gamma = \frac{\eta}{4}$ and $\beta = \frac{3}{4}\eta - \alpha$, imply after some calculations that

$$0 = \eta(8\alpha^2 - 6\eta\alpha + 3\eta^2 - 4c).$$

We distinguish between the minimal and non-minimal cases. Thus, if $\eta \neq 0$, the previous equation implies that α is constant, and then M has constant principal curvatures. But real hypersurfaces with constant principal curvatures in the complex projective and hyperbolic planes have been classified [11], [106]. On the one hand, in $\mathbb{C}P^2$ there do not exist non-Hopf hypersurfaces with constant principal curvatures. On the other hand, in $\mathbb{C}H^2$ the only non-Hopf hypersurfaces with constant principal curvatures are the Lohnherr hypersurface (which is minimal), and its equidistant hypersurfaces (which are non-minimal). All of them are strongly 2-Hopf, as follows from [12, §4.1] (cf. [38]). However, only the Lohnherr hypersurface is Levi-flat: it is the only one that satisfies the relation $\gamma = -b_2^2\alpha - b_1^2\beta$, as can be checked from [38, Theorem 3.12]. Hence, the case $\eta \neq 0$ is impossible.

Assume now that $\eta = 0$. Then $\gamma = 0$, $\beta = -\alpha$ and $b_1^2 = b_2^2 = \frac{1}{2}$. In particular, M is an austere strongly 2-Hopf hypersurface in $\bar{M}^2(c)$, $c \neq 0$. By the classification achieved in Section 3.5, we deduce that M must be an open part of a Lohnherr hypersurface, or a Clifford cone, or a bisector. Finally, all these examples are Levi-flat, since they are ruled. This concludes the proof of Theorem 3.22. □

Conclusions and open problems

This thesis deals with the classification of isoparametric submanifolds and real hypersurfaces with certain geometric properties in two-dimensional nonflat complex space forms. The investigation carried out allows us to present the following conclusions.

- Any Lagrangian flat surface with parallel mean curvature in a complex projective or hyperbolic plane is congruent to an open part of a principal orbit of a polar action (Theorem 2.1).

- Any isoparametric submanifold of $\mathbb{C}P^2$ or $\mathbb{C}H^2$ is congruent to an open part of a principal orbit of a polar action on $\mathbb{C}P^2$ or $\mathbb{C}H^2$ (Theorem 2.2).

- Any Terng-isoparametric submanifold of $\mathbb{C}P^2$ or $\mathbb{C}H^2$ is an open part of an isoparametric submanifold of $\mathbb{C}P^2$ or $\mathbb{C}H^2$, or a Chen's surface in $\mathbb{C}H^2$, or a circle (Theorem 2.3).

- Any strongly 2-Hopf real hypersurface in a complex projective or hyperbolic plane can be constructed using the equivariance method applied to a polar action of cohomogeneity two on $\mathbb{C}P^2$ or $\mathbb{C}H^2$ (Theorem 3.5).

- There are non-Hopf real hypersurfaces with two distinct principal curvatures in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. This provides an affirmative answer to Question 9.2 in [88].

- Any austere real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ whose Hopf vector field has $h \leq 2$ nontrivial projections onto the principal curvature spaces is an open part of a Lohnherr hypersurface in $\mathbb{C}H^2$, or a Clifford cone in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, or a bisector in $\mathbb{C}H^2$ (Theorem 3.15).

- We have obtained a characterization of strongly 2-Hopf real hypersurfaces with constant mean curvature and of Levi-flat strongly 2-Hopf real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ (Corollaries 3.19 and 3.21, respectively).

- Any Levi-flat strongly 2-Hopf real hypersurface with constant mean curvature in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ is an open part of a Lohnherr hypersurface in $\mathbb{C}H^2$, or a Clifford cone in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, or a bisector in $\mathbb{C}H^2$ (Theorem 3.22).

In view of these results, there are still some open problems to be solved concerning isoparametric submanifolds and real hypersurfaces in nonflat complex space forms. We emphasize the following ones, which are directly related to the investigation carried out in this memoir.

- Surfaces with parallel mean curvature and constant Gaussian curvature in the two-dimensional nonflat complex space forms are relatively well understood after the work of Hirakawa [62]. It would be interesting to investigate surfaces with parallel mean curvature without the assumption on the Gaussian curvature.

- Complete the classification of isoparametric submanifolds in $\mathbb{C}H^n$. This problem seems unapproachable in its full generality, but it would be interesting to obtain some partial classifications.

- Can one characterize some particular class of isoparametric (or Terng-isoparametric) submanifolds in higher dimensional nonflat complex space forms as principal orbits of polar actions?

- For each polar action on $\mathbb{C}P^2$ and $\mathbb{C}H^2$, determine the tangent directions $\mathfrak{w}$ to the section that give rise, via the equivariant method, to real hypersurfaces that are Hopf at some point (see the end of Section 3.1). Are there inhomogeneous Hopf real hypersurfaces of cohomogeneity one?

- Classify the surfaces with constant principal curvatures in the two dimensional nonflat complex space forms. In view of Theorem 2.3 it just remains to consider those surfaces with nonflat normal bundle.

- Complete the classification of austere real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. In view of Theorem 3.15 it just remains to consider the case $h = 3$.

- Prove that there cannot exist compact Levi-flat strongly 2-Hopf real hypersurfaces in $\mathbb{C}P^2$ without singularities.

- Find a geometric condition to characterize cohomogeneity one real hypersurfaces obtained via de equivariant method applied to a cohomogeneity two polar action on a nonflat complex space form of dimension $n \geq 3$.

Resumo en galego

O concepto de xeometría de Riemann foi proposto por primeira vez de xeito xeral por Bernhard Riemann no século XIX. Esta xeometría baséase no estudo das variedades diferenciables dotadas dunha métrica de Riemann (forma cuadrática definida positiva). En particular, céntrase no estudo das propiedades métricas e da curvatura desas variedades. Unha métrica de Riemann determina unha distancia. O estudo do grupo daquelas transformacións do espazo ambiente que preserva dita distancia foi un problema moi frutífero dentro das matemáticas. Aquí é onde aparece o interese polo concepto de simetría dentro da xeometría de Riemann. Tal grupo de transformacións preservando as distancias é coñecido como o grupo de isometrías.

Dentro da xeometría de Riemann atopámonos cun tipo especial de variedades, coñecidas como variedades Kähler. Unha variedade Kähler é unha variedade de Riemann dotada dunha estrutura complexa paralela. O estudo das variedades Kähler atraeu a múltiples matemáticos ó longo da historia, os cales, engadindo certas propiedades xeométricas intentaron caracterizalas. Hai que destacar dentro desas propiedades a de ter curvatura seccional holomorfa constante. As variedades Kähler con curvatura seccional holomorfa constante redúcense a un dos seguintes espazos: o espazo proxectivo complexo, $\mathbb{C}P^n$ (para curvatura positiva), o espazo euclidiano complexo, $\mathbb{C}^n$ (para curvatura nula), e o espazo hiperbólico complexo, $\mathbb{C}H^n$ (para curvatura negativa). Referímonos a estes espazos como os espazos modelo complexos. Centrarémonos no estudo de subvariedades con certas propiedades xeométricas dentro dos espazos modelo complexos non chans, é dicir, en $\mathbb{C}P^n$ e $\mathbb{C}H^n$.

A acción dun subgrupo do grupo de isometrías dunha variedade dada coñécese como acción isométrica. Cada unha das órbitas dunha acción isométrica recibe o nome de subvariedade extrinsecamente homoxénea ou, simplemente, subvariedade homoxénea. O estudo de tales accións na súa total xeneralidade é un problema matemático complicado, polo que se introdúce o concepto de acción polar. Unha acción polar non é mais que unha acción isométrica que admite unha subvariedade totalmente xeodésica que interseca a todas as órbitas da acción ortogonalmente. As accións polares foron completamente clasificadas en $\mathbb{C}P^n$ e en $\mathbb{C}H^n$ grazas a Podestà e Thorbergsson [93] (para o caso proxectivo), e a Díaz-

Ramos, Domínguez-Vázquez e Kollross [39] (para o caso hiperbólico). Ditas clasificacións serán fundamentais para o noso traballo. Coñécese como cohomoxeneidade dunha acción isométrica á menor codimensión das súas órbitas. En particular, debemos destacar que a cohomoxeneidade dunha acción polar coincide coa dimensión da súa sección.

Ó longo da historia moitos matemáticos intentaron caracterizar certos tipos de subvariedades homoxéneas, ou máis xeneralmente, de subvariedades con alto grao de simetría, a través de diferentes propiedades xeométricas. Así, nesta tese centrarémonos, por un lado, no estudo de hipersuperficies nos planos proxectivo e hiperbólico complexos, $\mathbb{C}P^2$ e $\mathbb{C}H^2$, dotadas dun alto grao de simetría, máis especificamente, estudaremos certas hipersuperficies que admiten unha acción de cohomoxeneidade un inducida por unha acción polar de cohomoxeneidade dous no espazo ambiente. Por outro lado, interésanos caracterizar precisamente as órbitas principais de tales accións polares mediante propiedades como a de ser unha subvariedade isoparamétrica.

A continuación presentamos a motivación matemática deste estudo, tratando de explicar como un problema aberto que inicialmente decidimos abordar acabou conducindo a realizar a investigación que se expón nesta memoria.

Un teorema clásico dentro da xeometría de superficies en $\mathbb{R}^3$ establece que unha superficie totalmente umbílica (isto é, unha superficie con exactamente unha curvatura principal) en $\mathbb{R}^3$ é parte aberta dunha esfera e dun plano; en particular, é parte aberta dunha superficie homoxénea. Tamén é coñecido que este resultado pode ser estendido a dimensións maiores. Isto suxire o problema de determinar en que medida, a propiedade de ter un pequeno número fixo de curvaturas principais distintas impón restricións sobre a xeometría dunha hipersuperficie. No contexto de espazos modelo complexos, Tashiro e Tachibana [100] amosaron que non hai hipersuperficies reais totalmente umbílicas en $\mathbb{C}P^n$ e $\mathbb{C}H^n$. Máis tarde, Cecil e Ryan [24] (para o caso proxectivo), e Montiel [86] (para o caso hiperbólico), levaron a cabo a clasificación de hipersuperficies reais con exactamente dúas curvaturas principais en $\mathbb{C}P^n$ e $\mathbb{C}H^n$, $n \geq 3$. Todos os exemplos que aparecen en ditas clasificacións son Hopf, é dicir, o campo de Reeb de tales hipersuperficies é campo de curvatura principal, e teñen curvaturas principais constantes, co cal, son partes abertas de hipersuperficies homoxéneas, en vista dos traballos de Kimura [71] e Berndt [8]. Non obstante, esta clasificación non está completa. Tal e como establecen Niebergall e Ryan no [88, Problema aberto 9.2], os argumentos de Cecil, Ryan e Montiel deixan aberta a cuestión da existencia e clasificación das hipersuperficies reais con dúas curvaturas principais non constantes en dimensión $n = 2$. No artigo [41], cuxos resultados forman parte desta tese, amosamos que unha hipersuperficie real con dúas curvaturas principais en $\mathbb{C}P^2$ e $\mathbb{C}H^2$ pode ter curvaturas principais non constantes; ademais, posteriormente, clasificamos tales hipersuperficies. Deste modo complétase a clasificación de hipersuperficies reais dentro dos espazos proxectivo e hiperbólico complexos con dúas curvaturas principais e amósase que non todos os exemplos son homoxéneos.

Os exemplos que aparecen nesta clasificación verifican unha serie de propiedades interesantes. Ademais dos nosos exemplos, estas propiedades están presentes en certas hipersuperficies homoxéneas non Hopf de $\mathbb{C}H^n$, así como nos exemplos de hipersuperficies de cohomoxeneidade un con curvatura media constante construídos por Gorodski e Gusevskii en [54]. Polo tanto, é natural a idea de establecer un novo concepto que resuma estas características, ó cal chamaremos hipersuperficie fortemente 2-Hopf. Diremos que unha hipersuperficie M é fortemente 2-Hopf se verifica que a menor distribución invariante baixo o operador de configuración de M que contén ó campo de Reeb ten dimensión 2, é integrable e ademais as curvaturas principais de M son constantes ó longo das subvariedades integrais de tal distribución.

Grazas a este novo concepto, intentamos xeneralizar a clasificación de hipersuperficies reais con dúas curvaturas principais non constantes, xunto cos demais exemplos mencionados anteriormente. Deste xeito levamos a cabo a caracterización das hipersuperficies fortemente 2-Hopf nos planos proxectivo e hiperbólico complexos. Estas hipersuperficies resultan ser, intrinsecamente, de cohomoxeneidade un e, de feito, obtéñense mediante o denominado método de xeometría equivariante aplicado a unha acción polar de cohomoxeneidade dous. Co fin de ter clasificacións explícitas e obter exemplos concretos, impóñense certas condicións adicionais. En particular, para caracterizar os exemplos obtidos por Gorodski e Gusevskii en [54], impuxemos a propiedade de ter curvatura media constante.

Estes resultados puxeron de manifesto que a combinación do método equivariante xunto coa propiedade de ser fortemente 2-Hopf daba lugar a unha mellor comprensión de todos estes exemplos, e se presentaba como un método con certa potencia á hora de xerar novos exemplos. Motivados por isto, e polo problema aberto da existencia de exemplos de hipersuperficies Levi-chás (é dicir, foliadas por hipersuperficies complexas) compactas sen singularidades en $\mathbb{C}P^2$, abordamos o estudo das hipersuperficies fortemente 2-Hopf que son Levi-chás en $\mathbb{C}P^2$ e $\mathbb{C}H^2$, podendo demostrar a existencia de múltiples exemplos, pero só a nivel local. Como mencionamos anteriormente, o noso obxectivo é o de obter exemplos concretos de dita clasificación, co cal impoñemos ambas as propiedades, é dicir, ter curvatura media constante e ser Levi-chás. Chegados a este punto, un dáse conta de que unha hipersuperficie real, fortemente 2-Hopf, Levi-chan con curvatura media constante resulta ser austera. Este concepto foi introducido por Harvey e Lawson [56] no seu estudo de variedades lagrangianas especiais, e foi definido como aquelas hipersuperficies cuxas curvaturas principais son invariantes baixo multiplicación por -1. Polo tanto, procedemos á clasificación das hipersuperficies austeras en $\mathbb{C}P^2$ e $\mathbb{C}H^2$ cuxo vector de Reeb proxecta en $h = 1$ ou $h = 2$ espazos de curvaturas principais. Este resultado proporciónanos unha interesante caracterización de tres importantes tipos de hipersuperficies regradas (as hipersuperficies de Lohnherr, os conos de Clifford, e os bisectores) mediante a propiedade de ser austeras.

A importancia da xeometría das órbitas principais das accións polares nos exemplos de hipersuperficies fortemente 2-Hopf construídos levounos a preguntarnos se sería posible caracterizar tales órbitas mediante algunha propiedade xeométrica adecuada. Así, motivados por clasificacións recentes de hipersuperficies isoparamétricas en $\mathbb{C}P^n$ e $\mathbb{C}H^n$, abordamos o estudo das subvariedades isoparamétricas nos espazos modelo complexos de dimensión dous, como primeiro paso de cara a unha comprensión destes obxectos en codimensión arbitraria.

Unha hipersuperficie denomínase isoparamétrica se as súas hipersuperficies equidistantes próximas teñen curvatura media constante. A xeneralización deste concepto a codimensión arbitraria foi desenvolvida por Terng [101], aínda que a súa definición foi dada para espazos de curvatura constante. Hoxe en día, tomamos como definición xeral de subvariedade isoparamétrica, nunha variedade de Riemann calquera, a establecida por Heintze, Liu e Olmos [57], que di que unha subvariedade é isoparamétrica se ten fibrado normal chan, as súas subvariedades paralelas suficientemente próximas teñen curvatura media paralela e ademais admite seccións.

A clasificación das hipersuperficies isoparamétricas nos espazos proxectivos complexos dedúcese da clasificación das hipersuperficies isoparamétricas nas esferas mediante un detallado estudo da correspondente fibración de Hopf. A clasificación de hipersuperficies isoparamétricas nas esferas é un problema aínda aberto. Os traballos de Cecil, Chi e Jensen [23], Immervoll [65], Chi [31], [32] e Miyaoka [84] completan a clasificación nas esferas salvo o caso no que a hipersuperficie isoparamétrica teña catro curvaturas principais con multiplicidades $(7, 8)$. Utilizando estes resultados, Domínguez-Vázquez [49] obtivo a clasificación de hipersuperficies isoparamétricas en $\mathbb{C}P^n$, $n \neq 15$. En dita clasificación amósase que existen numerosos exemplos non homoxéneos de hipersuperficies isoparamétricas en $\mathbb{C}P^n$, pero non para $n = 2$. A clasificación das hipersuperficies isoparamétricas no espazo hiperbólico complexo foi obtida recentemente por Díaz-Ramos, Domínguez-Vázquez e Sanmartín-López en [40]. Nesta clasificación tamén se amosa que existen exemplos non homoxéneos, pero de novo, todos os exemplos en $\mathbb{C}H^2$ son homoxéneos.

Para codimensión maior, as subvariedades isoparamétricas en $\mathbb{C}P^n$ foron clasificadas por Domínguez-Vázquez tamén no traballo [49]. Novamente, aparecen exemplos non homoxéneos. Para o caso de $\mathbb{C}H^n$, o problema de clasificar subvariedades isoparamétricas de codimensión maior que un vólvese moito máis complicado e permanece completamente aberto. En [42], cuxos resultados forman parte desta tese, levamos a cabo a clasificación de subvariedades isoparamétricas restrinxíndonos ó caso $n = 2$, como primeiro paso para unha mellor comprensión destes obxectos en espazos de curvatura non positiva. Ademais, estudamos tamén a definición de isoparametricidade de Terng que, ó noso entender, aínda nunca fora estudado en subvariedades de codimensión maior que un nos espazos de curvatura non constante.

Nesta introdución presentamos a evolución cronolóxica das investigacións que vimos desenvolvendo nestes anos. Non obstante, ó longo da tese e na procura dunha maior coherencia expositiva, usaremos unha orde diferente. A continuación resumimos os principais resultados obtidos.

Subvariedades isoparamétricas en $\mathbb{C}P^2$ e $\mathbb{C}H^2$

Motivados pola clasificación de hipersuperficies isoparamétricas en $\mathbb{C}P^n$ e $\mathbb{C}H^n$, xórdenos a idea de xeneralizar esta clasificación a codimensión maior. Como xa mencionamos, para o caso de $\mathbb{C}P^n$ tal clasificación foi realizada por Domínguez-Vázquez en [49]. Non obstante para o caso hiperbólico o problema está aberto. Ademais, non está clara a relación entre as definicións de isoparametricidade de Terng por un lado, e de Heintze, Liu e Olmos por outro, en espazos de curvatura non constante. No Capítulo 2 levamos a cabo a clasificación de subvariedades isoparamétricas en $\mathbb{C}H^2$, así como das subvariedades Terng-isoparamétricas en $\mathbb{C}P^2$ e $\mathbb{C}H^2$.

A clave para estas clasificacións é o resultado principal obtido na Sección 2.2. Dito resultado permítenos establecer unha relación entre certas propiedades que cumpren as superficies lagrangianas chás con curvatura media paralela en $\mathbb{C}P^2$ e $\mathbb{C}H^2$. Estas superficies verifican ter segunda forma fundamental paralela, ser Terng-isoparamétricas, isoparamétricas e ser parte aberta dunha órbita principal dunha acción polar de cohomoxeneidade dous en $\mathbb{C}P^2$ ou $\mathbb{C}H^2$.

Deste xeito, na Sección 2.3 podemos establecer a clasificación de subvariedades isoparamétricas e Terng-isoparamétricas nos planos proxectivo e hiperbólico complexos. En tal clasificación obtemos que unha subvariedade isoparamétrica en $\mathbb{C}P^2$ e $\mathbb{C}H^2$ é congruente a unha parte aberta dunha órbita principal dunha acción polar. Para o caso de subvariedades Terng-isoparamétricas, obtemos que unha subvariedade é Terng-isoparamétrica en $\mathbb{C}P^2$ ou $\mathbb{C}H^2$ se, e só se, é unha subvariedade isoparamétrica, unha superficie de Chen en $\mathbb{C}H^2$, ou un círculo.

A superficie de Chen, construída por primeira vez en [26], é unha subvariedade homoxénea que non é órbita de ningunha acción polar, e para a cal, na Sección 2.3, damos unha descrición mediante grupos de Lie, distinta á proporcionada por Chen.

Hipersuperficies fortemente 2-Hopf en $\mathbb{C}P^2$ e $\mathbb{C}H^2$

Como xa mencionamos, co fin de xeneralizar certos exemplos obtidos en [41], [54], e certos exemplos de hipersuperficies homoxéneas non Hopf [12], no Capítulo 3 levamos a cabo a caracterización das hipersuperficies fortemente 2-Hopf en $\mathbb{C}P^2$ e $\mathbb{C}H^2$. Para iso faremos uso do método de xeometría equivariante aplicado a accións polares de cohomoxeneidade dous sobre $\mathbb{C}P^2$ e $\mathbb{C}H^2$.

En primeiro lugar, e co obxectivo de construír os nosos exemplos, debemos considerar unha acción polar sobre a nosa variedade ambiente, de cohomoxeneidade dous. A idea detrás desta construción é a seguinte: tomamos unha curva na parte regular da sección da acción polar e consideramos a unión das órbitas principais da acción polar a través de dita curva. Esta unión dá lugar a unha hipersuperficie con polo menos dúas curvaturas principais distintas. Xenericamente, esta hipersuperficie terá tres curvaturas principais e será fortemente 2-Hopf.

Agora supoñamos que temos unha hipersuperficie fortemente 2-Hopf en $\mathbb{C}P^2$ ou $\mathbb{C}H^2$. Tras unha serie de cálculos, probamos a existencia de dúas distribucións perpendiculares sobre a hipersuperficie, as cales son integrables e de dimensións 1 e 2. A distribución 2-dimensional verifica que as súas subvariedades integrais son subvariedades da variedade ambiente totalmente reais, equidistantes e chás, con segunda forma fundamental paralela e fibrado normal chan. Á súa vez, cada unha das curvas integrais da outra distribución está contida nunha subvariedade totalmente xeodésica e totalmente real do espazo ambiente. Utilizando entón o resultado obtido na Sección 2.2 pódese probar que a hipersuperficie está foliada por órbitas principais dunha acción polar de cohomoxeneidade dous, e así, pode ser construída polo método que describimos anteriormente.

A construción de hipersuperficies reais con dúas curvaturas principais en $\mathbb{C}P^2$ e $\mathbb{C}H^2$ é un caso particular da construción de hipersuperficies fortemente 2-Hopf. Así, na Sección 3.4, usamos o método da xeometría equivariante aplicado a accións polares de cohomoxeneidade dous sobre $\mathbb{C}P^2$ e $\mathbb{C}H^2$, e observamos que o feito de que a hipersuperficie resultante teña exactamente dúas curvaturas principais distintas é equivalente a que a curva escollida na sección satisfaga unha certa ecuación diferencial ordinaria. Deste xeito proporciónase un método para a construción de hipersuperficies con dúas curvaturas principais. Xenericamente, tales hipersuperficies son non Hopf, e polo tanto, constitúen exemplos novos. A continuación, tras extraer a suficiente información das ecuacións fundamentais da xeometría de subvariedades, conseguimos probar que unha hipersuperficie con dúas curvaturas principais distintas e que é non Hopf en todo punto é fortemente 2-Hopf e, polo tanto, pode ser construída polo método de xeometría equivariante a partir de certa curva na sección dunha acción polar.

Como xa mencionamos, estes son os primeiros exemplos de hipersuperficies reais con exactamente dúas curvaturas principais distintas e non constantes en $\mathbb{C}P^2$ e $\mathbb{C}H^2$, polo que respondemos á pregunta presentada por Niebergall e Ryan en [88]. Ademais, na Sección 3.4 tamén daremos unha construción explícita destas hipersuperficies para o caso do plano proxectivo complexo.

A continuación, na Sección 3.5 centrarémonos no estudo das hipersuperficies austeras nos planos proxectivo e hiperbólico complexos. Despois de probar que non existen exemplos Hopf, e tras obter unha descrición explícita da conexión de Levi-Civita da hipersuperficie, conseguimos probar que todos os exemplos posibles de hipersuperficies austeras cuxo vector de Reeb proxecta en dous espazos de curvaturas principais en $\mathbb{C}P^2$ e $\mathbb{C}H^2$ son regrados. Neste caso, dicimos que unha hipersuperficie é regrada no sentido de que a súa distribución complexa maximal é integrable e as súas subvariedades integrais son totalmente xeodésicas no espazo ambiente. Polo tanto, a clasificación séguese da clasificación de hipersuperficies regradas minimais obtida por Lohnherr e Reckziegel en [79]. Os exemplos que aparecen nesta clasificación son os seguintes: a denominada hipersuperficie de Lohnherr en $\mathbb{C}H^2$, un cono de Clifford en $\mathbb{C}P^2$ ou $\mathbb{C}H^2$ e un bisector en $\mathbb{C}H^2$.

Finalmente, aplicamos os resultados obtidos para estudar as hipersuperficies fortemente 2-Hopf que ademais teñen curvatura media constante ou ben son Levi-chás. Que a nosa hipersuperficie verifique algunha destas propiedades é equivalente ó feito de que a curva usada no proceso de construción dos exemplos de hipersuperficies fortemente 2-Hopf satisfaga unha certa ecuación diferencial ordinaria. O resultado de existencia e unicidade para ecuacións diferenciais ordinarias permite caracterizar estes exemplos.

Por último, probamos que unha hipersuperficie real fortemente 2-Hopf, Levi-chá e con curvatura media constante en $\mathbb{C}P^2$ e $\mathbb{C}H^2$ é austera. Deste xeito, o resultado de clasificación de hipersuperficies austeras obtido previamente permítenos concluír este estudo.

Estrutura da tese

Esta memoria organízase como segue.

No Capítulo 1 introdúcense os conceptos básicos, terminoloxía e convenios necesarios para a realización desta tese. Dito capítulo divídese nos seguientes temas: variedades semi-riemannianas (§1.1), xeometría de subvariedades (§1.2), foliacións riemannianas singulares

(§1.3), subvariedades isoparamétricas (§1.4), accións isométricas (§1.5), espazos modelo complexos (§1.6) e accións polares (§1.7).

Os seguientes capítulos conteñen as clasificacións e contribucións principais desta tese.

No Capítulo 2 lévase a cabo a demostración do que é, tal vez, o resultado central desta tese (§2.2), que en particular garante que unha superficie lagrangiana chá con curvatura media paralela en $\mathbb{C}P^2$ ou $\mathbb{C}H^2$ é parte aberta dunha acción polar de cohomoxeneidade dous. Posteriormente, dáse unha descrición da superficie de Chen como órbita dunha acción isométrica (§2.3.1), e obtense a clasificación de subvariedades isoparamétricas (§2.3.2) e Terng-isoparamétricas (§2.3.3) en $\mathbb{C}P^2$ e $\mathbb{C}H^2$.

Finalmente, o Capítulo 3 céntrase no estudo de certo tipo de hipersuperficies reais construídas mediante o método de xeometría equivariante aplicado a accións polares en $\mathbb{C}P^2$ e $\mathbb{C}H^2$. Así, primeiro descríbese a construción de tales hipersuperficies mediante este método (§3.1). Posteriormente procédese a realizar os cálculos precisos co fin de determinar explicitamente a conexión de Levi-Civita dunha hipersuperficie real cuxo campo de Hopf ten exactamente $h = 2$ proxeccións non triviais nos espazos de curvatura principais (§3.2). Finalmente obtéñense os seguintes resultados relativos a hipersuperficies reais nos planos proxectivo e hiperbólico complexos: a caracterización das hipersuperficies fortemente 2-Hopf (§3.3), a clasificación das hipersuperficies con dúas curvaturas principais (§3.4), a clasificación das hipersuperficies austeras con $h \leq 2$ (§3.5), así como a caracterización das hipersuperficies fortemente 2-Hopf con curvatura media constante, a caracterización das hipersuperficies fortemente 2-Hopf e Levi-chás, e a clasificación das hipersuperficies fortemente 2-Hopf, Levi-chás con curvatura media constante (§3.6).

Resumen en castellano

El concepto de geometría de Riemann fue propuesto por primera vez de manera general por Bernhard Riemann en el siglo XIX. Esta geometría se basa en el estudio de las variedades diferenciables dotadas de una métrica de Riemann (forma cuadrática definida positiva). En particular, se centra en el estudio de las propiedades métricas y de curvatura de esas variedades. Una métrica de Riemann determina una distancia. El estudio del grupo de aquellas transformaciones del espacio ambiente que preservan dicha distancia ha sido un problema muy fructífero dentro de las matemáticas. Aquí es donde surge el interés por el concepto de simetría dentro de la geometría de Riemann. Tal grupo de transformaciones preservando las distancias es conocido como el grupo de isometrías.

Dentro de la geometría de Riemann nos encontramos con un tipo especial de variedades, conocidas como variedades Kähler. Una variedad Kähler es una variedad de Riemann dotada de una estructura compleja paralela. El estudio de las variedades Kähler ha atraído a múltiples matemáticos a lo largo de la historia, los cuales, añadiendo ciertas propiedades geométricas han intentado caracterizarlas. Cabe destacar dentro de esas propiedades la de tener curvatura seccional holomorfa constante. Las variedades Kähler con curvatura seccional holomorfa constante se reducen a uno de los siguientes espacios: el espacio proyectivo complejo, $\mathbb{C}P^n$ (para curvatura positiva), el espacio euclidiano complejo, $\mathbb{C}^n$ (para curvatura nula), y el espacio hiperbólico complejo, $\mathbb{C}H^n$ (para curvatura negativa). Nos referimos a estos espacios como los espacios modelo complejos. En esta tesis nos centraremos en el estudio de subvariedades con ciertas propiedades geométricas dentro de los espacios modelo complejos no llanos, es decir, en $\mathbb{C}P^n$ y $\mathbb{C}H^n$.

La acción de un subgrupo del grupo de isometrías de una variedad dada se conoce como acción isométrica. Cada una de las órbitas de una acción isométrica recibe el nombre de subvariedad extrínsecamente homogénea o, simplemente, subvariedad homogénea. El estudio de tales acciones en su total generalidad es un problema matemático complicado, por ello se introduce el concepto de acción polar. Una acción polar no es más que una acción isométrica que admite una subvariedad totalmente geodésica que interseca a todas las órbitas de la acción ortogonalmente. Las acciones polares han sido completamente clasificadas en $\mathbb{C}P^n$ y en $\mathbb{C}H^n$ gracias a Podestà y Thorbergsson [93] (para el caso proyectivo), y a Díaz-Ramos, Domínguez-Vázquez y Kollross [39] (para el caso hiperbólico). Dichas clasificaciones serán fundamentales para nuestro trabajo. Se conoce como cohomogeneidad de una acción isométrica a la menor codimensión de sus órbitas. En particular, debemos destacar que la cohomogeneidad de una acción polar coincide con la dimensión de su sección.

A lo largo de la historia muchos matemáticos han intentado caracterizar ciertos tipos de subvariedades homogéneas o, más generalmente, de subvariedades con alto grado de simetría a través de diferentes propiedades geométricas. Así, en esta tesis nos centraremos, por un lado, en el estudio de hipersuperficies en los planos proyectivo e hiperbólico complejos, $\mathbb{C}P^2$ y $\mathbb{C}H^2$, dotadas de un alto grado de simetría; más específicamente, estudiaremos ciertas hipersuperficies que admiten una acción de cohomogeneidad uno inducida por una acción polar de cohomogeneidad dos en el espacio ambiente. Por otro lado, nos interesará caracterizar precisamente las órbitas principales de tales acciones polares mediante propiedades como la de ser una subvariedad isoparamétrica.

A continuación presentamos la motivación matemática de este estudio, tratando de explicar cómo un problema abierto que inicialmente decidimos abordar nos acabó conduciendo a realizar la investigación que se expone en esta memoria.

Un teorema clásico dentro de la geometría de superficies en $\mathbb{R}^3$ establece que una superficie totalmente umbílica (esto es, una superficie con exactamente una curvatura principal) en $\mathbb{R}^3$ es parte abierta de una esfera o de un plano; en particular, es parte abierta de una superficie homogénea. También es bien conocido que este resultado puede ser extendido a dimensiones mayores. Esto sugiere el problema de determinar en qué medida la propiedad de tener un pequeño número fijo de curvaturas principales distintas impone restricciones sobre la geometría de una hipersuperficie. En el contexto de los espacios modelo complejos, Tashiro y Tachibana [100] mostraron que no hay hipersuperficies reales totalmente umbílicas en $\mathbb{C}P^n$ y $\mathbb{C}H^n$. Más tarde, Cecil y Ryan [24] (para el caso proyectivo), y Montiel [86] (para el caso hiperbólico), llevaron a cabo la clasificación de hipersuperficies reales con exactamente dos curvaturas principales en $\mathbb{C}P^n$ y $\mathbb{C}H^n$, $n \geq 3$. Todos los ejemplos que aparecen en dichas clasificaciones son Hopf, es decir, el campo de Reeb de tales hipersuperficies es campo de curvatura principal, y tienen curvaturas principales constantes, con lo cual, son partes abiertas de hipersuperficies homogéneas, en vista de los trabajos de Kimura [71] y Berndt [8]. Sin embargo, esta clasificación no está completa. Tal y como establecen Niebergall y Ryan en [88, Problema abierto 9.2], los argumentos de Cecil, Ryan y Montiel dejaban abierta la cuestión de la existencia y clasificación de las hipersuperficies reales con dos curvaturas principales no constantes en dimensión $n = 2$. En el artículo [41], cuyos resultados forman parte de esta tesis, mostramos que una hipersuperficie real con dos curvaturas principales en $\mathbb{C}P^2$ y $\mathbb{C}H^2$ puede tener curvaturas principales no constantes; además, posteriormente, clasificamos tales hipersuperficies. De este modo se completa la clasificación de hipersuperficies reales dentro de los espacios proyectivo e hiperbólico complejos con dos curvaturas principales y se muestra que no todos los ejemplos son homogéneos.

Los ejemplos que aparecen en esta clasificación verifican una serie de propiedades interesantes. Además de en nuestros ejemplos, estas propiedades están presentes en ciertas subvariedades homogéneas no Hopf de $\mathbb{C}H^n$, así como en los ejemplos de hipersuperficies de cohomogeneidad uno con curvatura media constante construidos por Gorodski y Gusevskii en [54]. Por tanto, es natural la idea de establecer un concepto nuevo que resuma estas características, al cual llamaremos hipersuperficie fuertemente 2-Hopf. Diremos que una hipersuperficie M es fuertemente 2-Hopf si verifica que la menor distribución invariante bajo el operador de configuración de M que contiene al campo de Reeb tiene dimensión 2, es integrable y además las curvaturas principales de M son constantes a lo largo de las subvariedades integrales de tal distribución.

Gracias a este nuevo concepto, intentamos generalizar la construcción de hipersuperficies reales con dos curvaturas principales no constantes, junto con los demás ejemplos mencionados anteriormente. De esta manera llevamos a cabo la caracterización de las hipersuperficies fuertemente 2-Hopf en los planos proyectivo e hiperbólico complejos. Estas hipersurperficies resultan ser, intrínsecamente, de cohomogenidad uno y, de hecho, se obtienen mediante el denominado método de geometría equivariante aplicado a una acción polar de cohomogeneidad dos. Con el fin de tener clasificaciones explícitas y obtener ejemplos concretos, se imponen ciertas condiciones adicionales. En particular, para caracterizar los ejemplos obtenidos por Gorodski y Gusevskii en [54], impusimos la propiedad de tener curvatura media constante.

Estos resultados pusieron de manifiesto que la combinación del método equivariante junto con la propiedad de ser fuertemente 2-Hopf daba lugar a una mejor comprensión de todos estos ejemplos, y se presentaba como un método con cierta potencia a la hora de generar nuevos ejemplos. Motivados por esto, y por el problema abierto de la existencia de ejemplos de hipersuperficies Levi-llanas (es decir, foliadas por hipersuperficies complejas) compactas sin singularidades en $\mathbb{C}P^2$, abordamos el estudio de las hipersuperficies fuertemente 2-Hopf que son Levi-llanas en $\mathbb{C}P^2$ y $\mathbb{C}H^2$, pudiendo demostrar la existencia de múltiples ejemplos, pero sólo a nivel local. Como mencionamos anteriormente, nuestro objetivo es el de obtener ejemplos concretos, con lo cual imponemos ambas propiedades, es decir, tener curvatura media constante y ser Levi-llana. Llegados a este punto, uno se da cuenta de que una hipersuperficie real, fuertemente 2-Hopf, Levi-llana con curvatura media constante resulta ser austera. Este concepto fue introducido por Harvey y Lawson [56] en su estudio de variedades lagrangianas especiales, y fue definido como aquella hipersuperficie cuyas curvaturas principales son invariantes bajo multiplicación por -1. Por tanto, procedemos a la clasificación de las hipersuperficies austeras en $\mathbb{C}P^2$ y $\mathbb{C}H^2$ cuyo vector de Reeb proyecta en $h = 1$ o $h = 2$ espacios de curvaturas principales. Este resultado nos proporcionó una interesante caracterización de tres importantes tipos de hipersuperficies regladas (las hipersuperficies de Lohnherr, los conos de Clifford, y los bisectores) mediante la propiedad de ser austeras.

La importancia de la geometría de las órbitas principales de las acciones polares en los ejemplos de hipersuperficies fuertemente 2-Hopf construidos nos llevó a preguntarnos si sería posible caracterizar tales órbitas mediante alguna propiedad geométrica adecuada. Así, motivados por clasificaciones recientes de hipersuperficies isoparamétricas en $\mathbb{C}P^n$ y $\mathbb{C}H^n$, abordamos el estudio de las subvariedades isoparamétricas en los espacios modelo complejos de dimensión dos, como primer paso hacia una comprensión de estos objetos en codimensión arbitraria.

Una hipersuperficie se denomina isoparamétrica si sus hipersuperficies equidistantes próximas tienen curvatura media constante. La generalización de este concepto a codimensión arbitraria fue desarrollada por Terng [101], aunque su definición fue dada para espacios de curvatura constante. Hoy en día, tomamos como definición general de subvariedad isoparamétrica, en una variedad de Riemann cualquiera, la establecida por Heintze, Liu y Olmos [57], que dice que una subvariedad es isoparamétrica si tiene fibrado normal llano, sus subvariedades paralelas suficientemente próximas tienen curvatura media paralela y además admite secciones.

El estudio de hipersuperficies isoparamétricas en los espacios proyectivos complejos se deduce de la clasificación de hipersuperficies isoparamétricas en las esferas mediante un detallado estudio de la correspondiente fibración de Hopf. La clasificación de hipersuperficies isoparamétricas en las esferas es un problema aún abierto. Los trabajos de Cecil, Chi y Jensen [23], Immervoll [65], Chi [31], [32] y Miyaoka [84] completan la clasificación en las esferas salvo el caso en que la hipersuperficie isoparamétrica tenga cuatro curvaturas principales con multiplicidades $(7, 8)$. Utilizando estos resultados, Domínguez-Vázquez [49] obtuvo la clasificación de hipersuperficies isoparamétricas en $\mathbb{C}P^n$, $n \neq 15$. En dicha clasificación se muestra que existen numerosos ejemplos no homogéneos de hipersuperficies isoparamétricas en $\mathbb{C}P^n$, pero no para $n = 2$. La clasificación de las hipersuperficies isoparamétricas en el espacio hiperbólico complejo ha sido obtenida recientemente por Díaz-Ramos, Domínguez-Vázquez y Sanmartín-López en [40]. En esta clasificación también se muestra que existen ejemplos no homogéneos, pero de nuevo, todos los ejemplos en $\mathbb{C}H^2$ son homogéneos.

Para codimensión mayor, las subvariedades isoparamétricas en $\mathbb{C}P^n$ han sido clasificadas por Domínguez-Vázquez también en el trabajo [49]. Nuevamente, aparecen ejemplos no homogéneos. Para el caso de $\mathbb{C}H^n$, el problema de clasificar subvariedades isoparamétricas de codimensión mayor que uno se vuelve mucho más complicado y permanece completamente abierto. En [42], cuyos resultados forman parte de esta tesis, llevamos a cabo la clasificación de subvariedades isoparamétricas restringiéndonos al caso de dimensión $n = 2$, como primer paso para una mejor comprensión de estos objetos en espacios de curvatura no positiva. Además, estudiamos también la definición de isoparametricidad de Terng que, a nuestro entender, aún no había sido nunca estudiada en subvariedades de codimensión mayor que uno en espacios de curvatura no constante.

En esta introducción hemos presentado la evolución cronológica de las investigaciones que hemos venido desarrollando en estos años. Sin embargo, a lo largo de la tesis y en busca de una mayor coherencia expositiva, usaremos un orden diferente. A continuación resumimos los principales resultados obtenidos.

Subvariedades isoparamétricas en $\mathbb{C}P^2$ y $\mathbb{C}H^2$

Motivados por la clasificación de hipersuperficies isoparamétricas en $\mathbb{C}P^n$ y $\mathbb{C}H^n$, nos surge la idea de generalizar esta clasificación a codimensión mayor. Como ya mencionamos, para el caso de $\mathbb{C}P^n$ tal clasificación fue realizada por Domínguez-Vázquez en [49]. Sin embargo para el caso hiperbólico el problema está abierto. Además, no está clara la relación entre las definiciones de isoparametricidad de Terng por un lado, y de Heintze, Liu y Olmos por otro, en espacios de curvatura no constante. En el Capítulo 2 llevamos a cabo la clasificación de subvariedades isoparamétricas en $\mathbb{C}H^2$, así como la de las subvariedades Terng-isoparamétricas en $\mathbb{C}P^2$ y $\mathbb{C}H^2$.

La clave para estas clasificaciones es el resultado principal obtenido en la Sección 2.2. Dicho resultado nos permite establecer una relación entre ciertas propiedades que cumplen las superficies lagrangianas llanas con curvatura media paralela en $\mathbb{C}P^2$ y $\mathbb{C}H^2$. Estas superficies verifican tener segunda forma fundamental paralela, ser Terng-isoparamétricas, isoparamétricas y ser parte abierta de una órbita principal de una acción polar de cohomogeneidad dos en $\mathbb{C}P^2$ y $\mathbb{C}H^2$.

De este modo, en la Sección 2.3 podemos establecer la clasificación de subvariedades isoparamétricas y Terng-isoparamétricas en los planos proyectivo e hiperbólico complejos. En tal clasificación obtenemos que una subvariedad isoparamétrica en $\mathbb{C}P^2$ o $\mathbb{C}H^2$ es congruente a una parte abierta de una órbita principal de una acción polar. Para el caso de subvariedades Terng-isoparamétricas, obtenemos que una subvariedad es Terng-isoparamétrica en $\mathbb{C}P^2$ o $\mathbb{C}H^2$ si, y sólo si, es una subvariedad isoparamétrica, una superficie de Chen en $\mathbb{C}H^2$, o un círculo.

La superficie de Chen, construida por primera vez en [26], es una subvariedad homogénea que no es órbita de ninguna acción polar, y para la cual, en la Sección 2.3 damos una descripción mediante grupos de Lie, distinta a la proporcionada por Chen.

Hipersuperficies fuertemente 2-Hopf en $\mathbb{C}P^2$ y $\mathbb{C}H^2$

Como ya mencionamos, con el fin de generalizar ciertos ejemplos obtenidos en [41], [54], y ciertos ejemplos de hipersuperficies homogéneas no Hopf [12], en el Capítulo 3 llevamos a cabo la caracterización de las hipersuperficies fuertemente 2-Hopf en $\mathbb{C}P^2$ y $\mathbb{C}H^2$. Para ello haremos uso del método de geometría equivariante aplicado a acciones polares de cohomogeneidad dos sobre $\mathbb{C}P^2$ y $\mathbb{C}H^2$.

En primer lugar, y con el objetivo de construir nuestros ejemplos, debemos considerar una acción polar sobre nuestra variedad ambiente, de cohomogeneidad dos. La idea detrás de esta construcción es la siguiente: tomamos una curva en la parte regular de la sección de la acción polar y consideramos la unión de las órbitas principales de la acción polar a través de dicha curva. Esta unión da lugar a una hipersuperficie con al menos dos curvaturas principales distintas. Genéricamente, esta hipersuperficie tendrá tres curvaturas principales y será fuertemente 2-Hopf.

Ahora supongamos que tenemos una hipersuperficie fuertemente 2-Hopf en $\mathbb{C}P^2$ o $\mathbb{C}H^2$. Tras una serie de cálculos, probamos la existencia de dos distribuciones perpendiculares sobre la hipersuperficie, las cuales son integrables y de dimensiones 1 y 2. La distribución 2-dimensional verifica que sus subvariedades integrales son subvariedades de la variedad ambiente totalmente reales, equidistantes y llanas, con segunda forma fundamental paralela y fibrado normal llano. A su vez, cada una de las curvas integrales de la otra distribución está contenida en una subvariedad totalmente geodésica y totalmente real del espacio ambiente. Utilizando entonces el resultado obtenido en la Sección 2.2 se puede probar que la hipersuperficie está foliada por órbitas principales de una acción polar de cohomogeneidad dos, y así, puede ser construida por el método que describimos anteriormente.

La construcción de hipersuperficies reales con dos curvaturas principales en $\mathbb{C}P^2$ y $\mathbb{C}H^2$ es un caso particular de la construcción de hipersuperficies fuertemente 2-Hopf. Así, en la Sección 3.4, usamos el método de geometría equivariante aplicado a acciones polares de cohomogeneidad dos sobre $\mathbb{C}P^2$ y $\mathbb{C}H^2$, y observamos que el hecho de que la hipersuperficie resultante tenga exactamente dos curvaturas principales distintas es equivalente a que la curva escogida en la sección satisfaga una cierta ecuación diferencial ordinaria. De este modo se proporciona un método para la construcción de hipersuperficies con dos curvaturas principales. Genéricamente, tales hipersuperficies son no Hopf, y por tanto, constituyen ejemplos nuevos. A continuación, tras extraer la suficiente información de las ecuaciones fundamentales de la geometría de subvariedades, conseguimos probar que una hipersuperficie con dos curvaturas principales distintas y que es no Hopf en todo punto es fuertemente 2-Hopf y, por tanto, puede ser construida por el método de geometría equivariante a partir de cierta curva en la sección de una acción polar.

Como ya mencionamos, estos son los primeros ejemplos de hipersuperficies reales con exactamente dos curvaturas principales distintas y no constantes en $\mathbb{C}P^2$ y $\mathbb{C}H^2$, por lo que respondemos a la pregunta presentada por Niebergall y Ryan en [88]. Además, en la Sección 3.4 también daremos una construcción explícita de estas hipersuperficies para el caso del plano proyectivo complejo.

A continuación, en la Sección 3.5 nos centramos en el estudio de las hipersuperficies austeras en los planos proyectivo e hiperbólico complejos. Después de probar que no existen ejemplos Hopf, y tras obtener una descripción explícita de la conexión de Levi-Civita de la hipersuperficie, conseguimos probar que todos los ejemplos posibles de hipersuperficies austeras cuyo vector de Reeb proyecta en dos espacios de curvaturas principales en $\mathbb{C}P^2$ y $\mathbb{C}H^2$ son reglados. En este caso, decimos que una hipersuperficie es reglada en el sentido de que su distribución compleja maximal es integrable y sus subvariedades integrales son totalmente geodésicas en el espacio ambiente. Por tanto, la clasificación se sigue de la clasificación de hipersuperficies regladas minimales obtenida por Lohnherr y Reckziegel en [79]. Los ejemplos que aparecen en esta clasificación son los siguientes: la denominada hipersuperficie de Lohnherr en $\mathbb{C}H^2$, un cono de Clifford en $\mathbb{C}P^2$ o $\mathbb{C}H^2$ y un bisector en $\mathbb{C}H^2$.

Finalmente, aplicamos los resultados obtenidos para estudiar las hipersuperficies fuertemente 2-Hopf que además tienen curvatura media constante o bien son Levi-llanas. Que nuestra hipersuperficie verifique alguna de estas propiedades es equivalente al hecho de que la curva usada en el proceso de construcción de los ejemplos de hipersuperficies fuertemente 2-Hopf satisfaga una cierta ecuación diferencial ordinaria. El resultado de existencia y unicidad para ecuaciones diferenciales ordinarias permite caracterizar estos ejemplos.

Por último, probamos que una hipersuperficie real fuertemente 2-Hopf, Levi-llana y con curvatura media constante en $\mathbb{C}P^2$ o $\mathbb{C}H^2$ es austera. De este modo, el resultado de clasificación de hipersuperficies austeras obtenido previamente nos permite concluir este estudio.

Estructura de la tesis

Esta memoria se organiza como sigue.

En el Capítulo 1 se introducen los conceptos básicos, terminología y convenios necesarios para la realización de esta tesis. Dicho capítulo se divide en los siguientes temas: variedades semi-riemannianas (§1.1), geometría de subvariedades (§1.2), foliaciones riemannianas singulares (§1.3), subvariedades isoparamétricas (§1.4), acciones isométricas (§1.5), espacios modelo complejos (§1.6) y acciones polares (§1.7).

Los siguientes capítulos contienen las clasificaciones y contribuciones principales de esta tesis.

En el Capítulo 2 se lleva a cabo la demostración del que es, tal vez, el resultado central de esta tesis (§2.2), que en particular garantiza que una superficie lagrangiana llana con curvatura media paralela en $\mathbb{C}P^2$ o $\mathbb{C}H^2$ es una parte abierta de una acción polar de cohomogeneidad dos. Posteriormente, se da una descripción de la superficie de Chen como órbita de una acción isométrica (§2.3.1), y se obtienen la clasificación de subvariedades isoparamétricas (§2.3.2) y Terng-isoparamétricas (§2.3.3) en $\mathbb{C}P^2$ y $\mathbb{C}H^2$.

Finalmente, el Capítulo 3 se centra en el estudio de cierto tipo de hipersuperficies reales construidas mediante el método de geometría equivariante aplicado a acciones polares en $\mathbb{C}P^2$ y $\mathbb{C}H^2$. Así, primero se describe la construcción de tales hipersuperficies mediante este método (§3.1). Posteriormente se procede a realizar los cálculos necesarios con el fin de determinar explícitamente la conexión de Levi-Civita de una hipersuperficie real cuyo campo de Hopf tiene exactamente $h = 2$ proyecciones no triviales en los espacios de curvatura principal (§3.2). Finalmente se obtienen los siguientes resultados relativos a hipersuperficies reales en los planos proyectivo e hiperbólico complejos: la caracterización de las hipersuperficies fuertemente 2-Hopf (§3.3), la clasificación de las hipersuperficies con dos curvaturas principales (§3.4), la clasificación de las hipersuperficies austeras con $h \leq 2$ (§3.5), así como la caracterización de las hipersuperficies fuertemente 2-Hopf con curvatura media constante, la caracterización de las hipersuperficies fuertemente 2-Hopf y Levi-llanas, y la clasificación de las hipersuperficies fuertemente 2-Hopf, Levi-llanas con curvatura media constante (§3.6).

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