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GRAPH COLORINGS AND REALIZATION OF MANIFOLDS AS LEAVES

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GRAPH COLORINGS AND REALIZATION OF MANIFOLDS AS LEAVES

Ramón Barral Lijó

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Não me esqueci de nada, mãe.
Guardo a tua voz dentro de mim.
E deixo-te as rosas.

Eugénio de Andrade

Si conobbero. Lui conobbe lei e se stesso, perché in verità non s'era mai saputo. E lei conobbe lui e se stessa, perché pur essendosi saputa sempre, mai s'era potuta riconoscere così.

Italo Calvino

Canta, rula:
alto, quedo.
Na nenez
do meu pai.

Rodolfo Alonso

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Contents

Chapter 1

Introduction

This thesis has two main parts. The first one is devoted to show that, for any infinite connected (repetitive) graph X with finite maximum vertex degree $\deg X < \infty$, there exists a (repetitive) limit-aperiodic coloring by at most $\deg X$ colors. Several direct consequences of this theorem are also derived, like the existence of (repetitive) limit-aperiodic colorings of any (repetitive) tiling of a Riemannian manifold. The second part is devoted to prove that any (repetitive) Riemannian manifold of bounded geometry can be isometrically realized as leaf of a Riemannian (minimal) matchbox manifold, whose leaves have no holonomy. This also uses the previous result about colorings, but it also requires much more technical work concerning the space of pointed Riemannian manifolds with the topology defined by the C^∞ convergence. The following sections contain more precise descriptions.

1.1 Graph colorings

The results of this section will be in the publication [7].

1.1.1 Estimates of the distinguishing number

Let $X \equiv (X, E)$ be a simple (undirected countable) graph (with finite vertex degrees). Assume that X is connected and consider its natural distance. The *degree* of X, denoted $\deg X$, is the supremum of its vertex degrees.

Consider a (vertex) coloring $\phi\colon X \to F$ (the set of colors F is usually assumed to be a subset of $\mathbb{N}$). It is said that ϕ (or (X, ϕ)) is *aperiodic* or *distinguishing* if there is no nontrivial automorphism of (X, ϕ). The *distinguishing number* of X is

$$D(X) = \min\{\, n \in \mathbb{Z}^+ \mid X \text{ has some aperiodic coloring by } n \text{ colors} \,\} .$$

This concept was introduced by Albertson and Collins [2], and the calculation of $D(X)$ (or bounds thereof) for many families of graphs has been the subject of much research in recent years (see e.g. [41, 42]). This ended up with following sharp estimate for finite graphs, where K_n, $K_{n,n}$ and C_n denote the complete graph on n vertices, the (n, n)-bipartite graph and the cyclic graph with n vertices, respectively.

Theorem 1.1.1 (Collins-Trenk [27], Klavžar-Wong-Zhu [46]). *If X is a finite connected simple graph different from K_n, $K_{n,n}$ and C_5 ($n \geq 2$), then $D(X) \leq \deg X$. If X is K_n, $K_{n,n}$ or C_5 ($n \geq 2$), then $D(X) = \deg X + 1$.*

For infinite graphs, the following result has been recently proved. It is easy to check that the bound it provides is sharp.

Theorem 1.1.2 (Lehner-Pilśniak-Stawiski [47]). *Let X be an infinite connected simple graph with $\deg X \geq 3$. Then $D(X) \leq \deg X - 1$.*

It is clear that $D(X) = 2$ if X is an infinite graph of degree two. Also very recently, Hüning et al. have provided a complete classification of all connected graphs X with $\deg X = D(X) = 3$ [40].

1.1.2 Space of colored graphs

Consider pointed connected colored simple graphs, (X, x, ϕ), with colors in $\mathbb{N}$. Their isomorphism classes, $[X, x, \phi]$, form a Polish space $\widehat{\mathcal{G}}_*$ with a canonical topology (Section 2.1.4). For any such graph (X, ϕ), there is a canonical map $\hat{\iota}_{X,\phi} : X \to \widehat{\mathcal{G}}_*$ defined by $\hat{\iota}_{X,\phi}(x) = [X, x, \phi]$. The images of the maps $\hat{\iota}_{X,\phi}$, denoted by $[X, \phi]$, form a canonical partition of $\widehat{\mathcal{G}}_*$. We have $[X, \phi] \equiv \operatorname{Aut}(X, \phi) \backslash X$, where $\operatorname{Aut}(X, \phi)$ is the group of color-preserving automorphisms of (X, ϕ). Every closure $\overline{[X, \phi]}$ is saturated. It is said that (X, ϕ) is:

aperiodic when $\operatorname{Aut}(X, \phi) = \{\mathrm{id}_X\}$ ($\hat{\iota}_{X,\phi}$ is injective);

limit aperiodic when $\hat{\iota}_{(Y,\psi)}$ is injective for all $[Y, \psi, y] \in \overline{[X, \phi]}$; and,

repetitive if, roughly speaking, every colored disk of (X, ϕ) is repeated uniformly in M.

The closure $\overline{[X, \phi]}$ is compact if and only if $\deg X, |\operatorname{im} \phi| < \infty$ (Proposition 2.1.21). Moreover $\overline{[X, \phi]}$ is minimal if (X, ϕ) is repetitive, and the reciprocal holds when $\overline{[X, \phi]}$ is compact.

By forgetting the colorings ϕ, we get a Polish space $\mathcal{G}_*$, with a partition defined by the images of maps $\iota_X : X \to \mathcal{G}_*$, obtaining obvious versions without colorings of the above properties. In this way, limit aperiodicity and repetitivity become similar to the definitions of strong aperiodicity and strong limit aperiodicity of colorings on groups. But, in Theorem 1.1.4, the minimality does not follow directly from the limit aperiodicity, like in Theorem 1.1.3, because $\overline{[X]}$ may contain elements $[Y, y]$ with $Y \ncong X$.

1.1.3 Strongly aperiodic colorings of groups

Let G be a countable group, and F a finite set equipped with the discrete topology. Then the F-valued colors on G form the compact second countable space F^G, which has a canonical left action of G, defined by $(g \cdot \phi)(h) = \phi(g^{-1}h)$. This G-space is called a *shift* (space), and any non-empty G-invariant closed subset of F^G is called a *subshift* (space). In particular, the orbit closure $\overline{G \cdot \phi}$ of any $\phi \in F^G$ is a subshift. If the action of G on $\overline{G \cdot \phi}$ is free (respectively, minimal), then ϕ is said to be *strongly aperiodic* (respectively, *strongly repetitive*). The existence of such colorings is guaranteed by the following sharp result.

Theorem 1.1.3 (Gao-Jackson-Seward [31]; see also [12]). *Every countable group admits a strongly aperiodic and strongly repetitive coloring by 2 colors.*

Indeed, the original statement in [31] only gives strong aperiodicity, but then strong repetitivity follows immediately with the following short argument. The existence of a strongly aperiodic coloring on G means that G acts freely on some subshift $X \subset \{0,1\}^G$. Then there is a minimal subset $Y \subset X$, and any coloring in Y is strongly aperiodic and strongly repetitive.

Suppose from now on that G is finitely generated, and let S be a minimal set of generators such that all elements of $S \cap S^{-1}$ are of order two. Consider the (left-invariant) Cayley graph defined by S, also denoted by G, where the degree of every vertex is $|S|$. Up to isomorphisms, the only possible limit of the graph G is G. Thus F^G is closed by taking limits of colors in the sense of Section 1.1.4. But, in this setting, it is natural to modify the definition of a limit of a coloring $\phi \in F^G$ by using only graph isomorphisms between disks given by left translations of G. The "limits by left translations" obtained in this way are just the elements of $\overline{G \cdot \phi}$, and the corresponding notion of "limit aperiodicity by left translations" means strong aperiodicity. Similarly, we can also define "repetitivity by left translations," which turns out to be strong repetitivity. By definition, limit aperiodicity is stronger than "limit aperiodicity by left translations" (strong aperiodicity), whereas repetitivity is weaker than "repetitivity by left translations" (strong repetitivity).

The Cayley graph of G induced by S is also equipped with a G-invariant edge coloring ψ_0 by colors in S, assigning to an edge between vertices $a, b \in G$ the unique element $s \in S$ satisfying $as^{\pm 1} = b$. Moreover, if the order of s is not 2, then the choice of ± 1 in the above exponent defines an orientation of the edge. This defines a canonical partial G-invariant direction $\mathcal{O}_0$ of G. The left translations are just the graph isomorphisms of G that preserve ψ_0 and $\mathcal{O}_0$. Consider the obvious extensions of the concepts of limit aperiodicity and repetitivity to triples $(\phi, \psi, \mathcal{O})$, where ϕ is a vertex coloring, ψ an edge coloring and $\mathcal{O}$ a partial direction. Then, using the interpretation of strong aperiodicity and strong repetitivity as "limit aperiodicity by left translations" and "repetitivity by left translations", we get that a coloring $\phi \in \{0,1\}^G$ is strongly aperiodic (respectively, strongly repetitive) if and only if $(\phi, \psi_0, \mathcal{O}_0)$ is limit aperiodic (respectively, repetitive). Thus, in this case, Theorem 1.1.3 can be restated by saying that G admits a coloring $\phi \in \{0,1\}^G$ such that $(\phi, \psi_0, \mathcal{O}_0)$ is limit aperiodic and repetitive.

1.1.4 Main theorem about colorings

The distinguishing number can be refined as follows. The *limit distinguishing number* of X is

$$D_L(X) = \inf\{\, n \in \mathbb{Z}^+ \mid X \text{ has a limit aperiodic coloring by } n \text{ colors} \,\}\,.$$

When X is repetitive, its *repetitive limit distinguishing number* is

$$D_{RL}(X) = \inf\{\, n \in \mathbb{Z}^+ \mid X \text{ has a repetitive limit aperiodic coloring by } n \text{ colors} \,\}\,.$$

It only makes sense to consider these concepts when X is infinite because, if X is finite, then limit aperiodicity means aperiodicity, and repetitivity always holds, obtaining $D_{RL}(X) = D_L(X) = D(X)$. Our main result is the following estimate of $D_L(X)$ and $D_{RL}(X)$, which can be considered as a refined version of Theorem 1.1.1.

Theorem 1.1.4. *If X is an infinite connected simple graph, then $D_L(X) \leq \deg X$. If moreover X is repetitive, then $D_{RL}(X) \leq \deg X$.*

With this generality, the estimates of Theorem 1.1.4 are sharp, as shown by the Cayley graph of $\mathbb{Z}$ (defined with the generating set $\{1\}$). For $\deg X \geq 3$, the estimates of Theorem 1.1.4 might not be optimal, according to Theorem 1.1.2. An obvious approach to get that optimal estimate would be to try to somehow incorporate the idea of the proof of Theorem 1.1.2 in [47] into our techniques. However, we divide X into finite pieces and work locally. This becomes a problem since the assumption that X is infinite is crucial in their proof, as they make use of a geodesic ray going to infinity. In any case, like in Theorem 1.1.2, it is obvious that the optimal estimates in Theorem 1.1.4 are at least $\deg X - 1$ if $\deg X \geq 3$.

Theorem 1.1.4 will be derived from Theorem 2.1.26, which is actually stronger in the following sense. The conditions of being limit aperiodic and repetitive can be restated quantitatively, so that the coloring has to satisfy certain statements for some choice of constants. We prove that these constants can be chosen "uniformly", depending on Δ and not on the particular choice of X, which does not follow from Theorem 1.1.4. The precise statement of this dependence can be found in Theorem 2.1.26. The same can be said for finite graphs, where the analogue of Theorem 2.1.26 would give a quantitative result stronger than Theorem 1.1.1.

In the case of a group G finitely generated by S (Section 1.1.3), Theorem 1.1.4 states that G has a repetitive limit aperiodic vertex coloring by $|S|$ colors. Since the total number of colors of $(\phi, \psi_0, \mathcal{O}_0)$ is $2 + |S|$, without taking into account the additional values of $\mathcal{O}_0$, it can be said that Theorem 1.1.4 somehow improves Theorem 1.1.3 in this case.

1.1.5 An idea of the proof

We have to prove that, if $\deg X < \infty$, then X has a limit aperiodic coloring ϕ by $\deg X$ colors, which is repetitive if X is repetitive.

First, we divide the graph $X = X_{-1}$ into finite connected clusters of bounded size, such that their centers form a Delone set $X_0 \subset X_{-1}$. Moreover X_0 can be endowed with a connected graph structure with $\deg X_0 < \infty$. On every cluster with center $x \in X_0$, the method of the proof of Theorem 1.1.1 is used to construct a large enough amount of different colorings $\psi^i_{0,x}$ by $\deg X$ colors breaking its symmetry. Any assignment of such colorings, $x \mapsto \psi^i_{0,x}$, is considered as a coloring, $x \mapsto i$, of X_0. For these colorings of X_0, we have enough avaliable colors to be able to proceed in the same way. Thus X_0 is divided into clusters, defining a graph $X_1 \subset X_0$. The above type of colorings of X_0 are considered in the new clusters. Again, for every $x \in X_1$, we can break the symmetry of the corresponding cluster with a large enough amount of different colors $\psi^i_{1,x}$ of the above kind. Any assignment of such colorings, $x \mapsto \psi^i_{1,x}$, is considered a coloring, $x \mapsto i$, of X_1. This process is continued indefinitely, producing a sequence of graphs X_n, divided into clusters whose centers form X_{n+1}, and colorings $\psi^i_{n+1,x}$ breaking the symmetry in the cluster of X_n with center $x \in X_{n+1}$. We use these data for $0 \leq n \leq N$ to define a coloring ϕ^N preventing isomorphisms between disks centered at points within a certain distance; namely, given any $\varepsilon \in \mathbb{Z}^+$, there is some $N, \delta \in \mathbb{Z}^+$ such that

$$0 < d(x,y) < \varepsilon \Longrightarrow [D(x,\delta), x, \phi^N] \neq [D(y,\delta), y, \phi^N] \tag{1.1.1}$$

for all $x, y \in X$. By taking a subsequence if necessary, we can assume that the sequence ϕ^N is eventually constant on finite sets, converging in this sense to a coloring ϕ. This coloring ϕ is limit aperiodic because it satisfies (1.1.1). Indeed δ depends only on Δ and ε in (1.1.1), as stated in Theorem 2.1.26, the indicated refinement of Theorem 1.1.4.

The definition of every X_n resembles very much the notion of a *shallow minor* of X_{n-1} at certain *depth* (see [52] and other references therein).

In the above process, there is a sequence of integers r_n that provides a lower bound for the "radii" of the clusters in X_{n-1}. Two crucial quantities that one needs to control are the number of suitable aperiodic colorings on each cluster, which depends exponentially on the cardinality of the cluster, and the number of clusters that are close to each other (depending on ε_n), which is always lower than the maximum cardinality of a disk of radius $O(r_n)$. If our graph has a uniform growth function, then we can choose r_n large enough so that there are enough different colorings on each cluster compared to the number of neighbouring clusters. At first glance, a similar argument could not work if the growth of the graph is not uniform, since for any choice of r_n there could be points $x \in X_n$ such that there are not enough colorings compared to the number of nearby clusters. However, the crucial observation is that, if there are many neighbouring clusters, then the disk of radius $O(r_n)$ has large enough cardinality, and we can construct sufficiently many aperiodic colorings on a cluster containing the disk. This observation makes the argument more involved, since we need to divide every X_n into two subsets, $X_n^{\pm}$, and different definitions and estimates are used in each of them. Besides this difficulty, the proof becomes quite complex with the arguments about repetitivity. It may be interesting to focus in the limit aperiodicity at first reading, omitting the arguments about repetitivity (Section 2.3 and its further use).

For the sake of brevity, a preliminary part of the construction of X_n, concerning repetitivity, is shown in the companion paper [5]; actually, a version for Riemannian manifolds is proved there, and the case of graphs involves simpler arguments.

Despite its complexity, the proof only uses elementary tools, and it would be much simpler without achieving the optimal number of colors.

1.1.6 First applications

As first straightforward applications, we derive some versions of Theorem 1.1.4 for edge colorings and for more general graphs, and the existence of limit aperiodic and repetitive tilings. In Chapter 3, we will give a more involved application of Theorem 1.1.4 concerning the realization of manifolds as leaves of compact foliated spaces.

1.1.6.1 Limit aperiodic and repetitive edge colorings

The notions of *aperiodicity*, *limit aperiodicity* and *repetitivity* have obvious analogues for edge colorings of a connected simple graph X. The analogue of $D(X)$ for edge colorings is called the *distinguishing index* [15], and denoted by $DI(X)$. When X is infinite, it makes sense to consider the obvious versions of $D_L(X)$ and $D_{RL}(X)$ for edge colorings, denoted by $DI_L(X)$ and $DI_{RL}(X)$, and called (*repetitive*) *limit distinguishing index.*

Recall that the *line graph* X' of X is defined as follows: the vertices of X' are the edges of X, and two vertices of X' are joined by an edge if they are edges of X meeting at some vertex; thus the edges of X' can be also identified to the vertices of X. Note that X' is connected and simple, $\deg X' \le 2(\deg X - 1)$, and

$$DI(X) = D(X')\ , \quad DI_L(X) = D_L(X')\ , \quad DI_{RL}(X) = D_{RL}(X')\ .$$

Then the following is a direct consequence of Theorem 1.1.4.

Corollary 1.1.5. *If X is an infinite connected simple graph, then $DI_L(X), DI_{RL}(X) \le 2(\deg X - 1)$.*

However, Corollary 1.1.5 is not very satisfactory. Its estimate can be surely improved by adapting the proof of Theorem 1.1.4, probably obtaining $DI_L(X), DI_{RL}(X) \le \deg X$. We hope to prove this in another publication.

1.1.6.2 Extension to general graphs

Now, let Y be a (countable) *general graph* (with finite vertex degrees); namely, Y may have a partial direction, multiple edges, and loops. Assuming that Y is connected, there are obvious extensions of the concepts of Sections 1.1.4 and 1.1.6.1 to this general setting. There is an induced undirected simple graph $\overline{Y}$ with the same vertex set, where the partial orientation and loops are forgotten, and with a single edge between every pair of adjacent vertices in Y. Clearly, $D(Y) \le D(\overline{Y})$ and $D_L(Y) \le D_L(\overline{Y})$.

Corollary 1.1.6. *If Y is an infinite connected general graph, then $D_L(Y), D_{RL}(Y) \le \deg \overline{Y}$.*

The inequality $D_L(Y) \le \deg \overline{Y}$ is a direct consequence of Theorem 1.1.4 since $D_L(Y) \le D_L(\overline{Y})$.

The inequality $D_{RL}(Y) \le \deg \overline{Y}$ follows with a small modification of the proof of Theorem 2.1.27. Namely, the sets Ω_n must be defined using isometries between disks of $\overline{Y}$ induced by isomorphisms between subgraphs of Y. Then the isometries $\mathfrak{h}_{n,x}$ between disks of $\overline{Y}$, constructed according to Section 2.3, can be assumed to be induced by isomorphisms between subgraphs of Y. The rest of the proof can be obviously adapted.

For example, with the notation of Section 1.1.3, we can consider the *Schreier graph* Y defined by G, S and any subgroup $H < G$. It is a general graph whose vertex set is $H\backslash G$, where the edges between vertices Ha and Hb are given by the elements $s \in S$ with $Has^{\pm 1} = Hb$. By Corollary 1.1.6, Y has some limit aperiodic vertex coloring by $\deg \overline{Y}$ colors. Note that $\deg \overline{Y} \le |S|$.

1.1.6.3 Limit aperiodic and repetitive tilings

Let us recall the general definition of tiling given in [14] (see also [28]). We use the term *n-complex* for a connected topological space with a simplicial complex structure of dimension n. A *set of prototiles* $\mathcal{T} \equiv (\mathcal{T}, \mathcal{F})$ consists of a finite collection $\mathcal{T}$ of compact metric n-complexes, called *prototiles*, and a collection $\mathcal{F}$ of subcomplexes of dimension

$< n$, called *faces*, together with an *opposition* involution $o : \mathcal{F} \to \mathcal{F}$. A *tiling* or *tessellation* α of a metric space X *by* $\mathcal{T}$ is a collection of isometries $a_\lambda : t_\lambda \subset X \to t'_\lambda \in \mathcal{T}$, where every t_λ is called a *tile* with *faces* defined via a_λ, such that:

- $X = \bigcup_\lambda t_\lambda$;
- the complement in t_λ of its faces is $\mathrm{Int}(t_\lambda)$ in X;
- if $\mathrm{Int}(t_\lambda \cup t_{\lambda'}) \neq \mathrm{Int}(t_\lambda) \cup \mathrm{Int}(t_{\lambda'})$, then t_λ and $t_{\lambda'}$ intersect along a face, f in t_λ and $o(f)$ in $t_{\lambda'}$; and
- there are no free faces of t_λ.

Similarly, we can define a *set of colored prototiles* by endowing $\mathcal{T}$ with a coloring ϕ, and a *set of prototiles with colored faces* by endowing $\mathcal{F}$ with a coloring ψ preserved by the opposition map. Then we get the corresponding definitions of (*tile-*) *colored tiling by* $(\mathcal{T}, \phi) \equiv (\mathcal{T}, \mathcal{F}, \phi)$ and *face-colored tiling by* $(\mathcal{T}, \psi) \equiv (\mathcal{T}, \mathcal{F}, \psi)$. These concepts can be also described by colorings of $\{t_\lambda\}$, and colorings of the set of intersections $t_\lambda \cap t_{\lambda'}$ along faces. Like $\mathcal{G}_*$ and $\widehat{\mathcal{G}}_*$ (Section 1.1.2), the sets of tilings of X by $\mathcal{T}$, colored tilings of X by $(\mathcal{T}, \phi)$ and face-colored tilings of X by $(\mathcal{T}, \psi)$ can be endowed with topologies after choosing a distinguished point of X, and there are obvious versions of *aperiodicity*, *limit aperiodicity* and *repetitivity* for tilings, colored tilings and face-colored tilings, using isometries of the ambient metric spaces [13, 28, 56]. Like in the case of groups (Section 1.1.3), refined versions of these concepts can be given using some subgroup of isometries, obtaining a weaker version of (limit) aperiodicity and a stronger version of repetitivity; for instance, if X is a Lie group, it is natural to use its left translations.

Every tiling α of X by $\mathcal{T}$ defines a connected undirected simple graph G whose vertices are the tiles of α, with an edge between two tiles if they meet along a face. Thus G is infinite just when X is not compact, and $\deg G$ is bounded by the maximum number of faces of the prototiles in $\mathcal{T}$, which is bounded by $|\mathcal{F}|$. Therefore the following is a direct consequence of Theorem 1.1.4 and Corollary 1.1.5.

Corollary 1.1.7. *Suppose that X is not compact, and let Δ denote the maximum number of faces of the prototiles in $\mathcal{T}$. Then any (repetitive) tiling of X by $\mathcal{T}$ has a (repetitive) limit aperiodic tile-coloring by Δ colors, and a (repetitive) limit aperiodic face-coloring by $2(\Delta - 1)$ colors.*

Since the face-colorings can be geometrically realized by dovetailing the faces, we get the following.

Corollary 1.1.8. *With the notation and conditions of Corollary 1.1.7, if X has a (repetitive) tiling by $\mathcal{T}$, then it has a (repetitive) limit aperiodic tiling by at most $|\mathcal{T}|\Delta 2(\Delta - 1)$ prototiles.*

For example, let $\widetilde{M}$ be any regular covering of a compact Riemannian n-manifold M, let Γ denote its group of deck transformations, and let t be a fundamental domain. Then the Γ-translates of t form a repetitive periodic tiling of $\widetilde{M}$ by the prototile t. Here, every face f of t corresponds to an element $\gamma_f \in \Gamma$ such that $t \cap \gamma_f t = f$. These elements γ_f form a generating set S of Γ. By Corollary 1.1.8, it follows that $\widetilde{M}$ has a repetitive limit aperiodic tiling by at most $2|S|(|S|-1)$ prototiles; in particular, every hyperbolic space $\mathbf{H}^n$ has a repetitive limit aperiodic tiling by finitely many prototiles (cf. [14, 28]).

With more generality, let Γ be a discrete group acting by isometries properly and cocompactly on a metric space X. For any fixed $x \in X$, the orbit Γx is a Delone set in X, and the corresponding *Voronoi cells*,

$$V_{\gamma x} = \{\, y \in X \mid d(y, \gamma x) \leq d(y, \Gamma x) \,\} \quad (\gamma \in \Gamma)\ ,$$

form a repetitive periodic tiling of X by one prototile (all tiles are isometric). Let Δ denote the number of faces of these tiles. Then, by Corollary 1.1.7, X has a repetitive limit aperiodic tiling by at most $2\Delta(\Delta-1)$ prototiles (cf. [28]).

In Corollaries 1.1.7 and 1.1.8, and in the previous examples, the number of colors or prototiles would be improved by the expected improvement of Corollary 1.1.5.

1.2 Realizability of manifolds as leaves

The results of this section will be also in the publication [5], which are the main application of Theorem 1.1.4.

1.2.1 Realization of manifolds as leaves

Sondow [64] and Sullivan [65] began the fundamental study of which connected manifolds can be realized as leaves of foliations on compact manifolds. A manifold is called a *leaf* or *non-leaf* if the answer is positive or negative, respectively. In codimension one, Cantwell and Conlon [22] have shown that any open connected surface is a leaf, whereas Ghys [32], Inaba *et al.* [43], and Schweitzer and Souza [61] constructed non-leaves of dimension 3 and higher. Other non-leaves in codimension one, with exotic differential structures, were constructed by Meniño Cotón and Schweitzer [50].

Any leaf of a foliation on a compact Riemannian manifold M is of bounded geometry, and its quasi-isometry type is independent of the metric on the ambient manifold. Thus it is also natural to study which connected Riemannian manifolds of bounded geometry are quasi-isometric to leaves of foliations on compact manifolds. This metric version of the realization problem was studied by Phillips and Sullivan [54], Januszkiewicz [44], Cantwell and Conlon [19–21], Cass [23], Schweitzer [59, 60], Attie and Hurder [11], and Zeghib [66], constructing examples of non-leaves in codimension one and higher.

This realization problem can be also considered using compact (Polish) foliated spaces. On foliated spaces, differentiable structures or Riemannian metrics refer to the leafwise direction, keeping continuity on the ambient space. Like in the case of foliations, any leaf of a compact Riemannian foliated space is of bounded geometry. The converse statement is also true, in contrast with the case of foliations; actually, any connected Riemannian manifold of bounded geometry is isometric to a leaf without holonomy in some compact Riemannian foliated space [6, Theorem 1.1] (see also [8, Theorem 1.5]). Another interesting realization of hyperbolic surfaces as leaves of compact foliated spaces was achieved in [4].

1.2.2 Space of pointed connected complete Riemannian manifolds and their smooth functions

Let us recall some concepts and properties used in our main results and their proofs, and already used in [6]. They are manifold versions of the concepts used in Section 1.1.2 for graphs. Consider triples (M, x, f), where M is a complete connected Riemannian n-manifold, $x \in M$, and $f : M \to \mathfrak{H}$ is a C^∞ function to a fixed separable (real) Hilbert space (of finite or infinite dimension). An equivalence $(M, x, f) \sim (M', x', f')$ is defined when there is a pointed isometric bijection $\phi : (M, x) \to (M', x')$ with $\phi^* f' = f$. Let $\widehat{\mathcal{M}}^n_*$ be the Polish space of equivalence classes $[M, x, f]$ of triples (M, x, f), with the topology induced by the C^∞ convergence of pointed Riemannian manifolds and C^∞ topology on smooth functions (Section 3.1.3). For any M and f as above, there is a map $\hat\iota_{M,f} : M \to \widehat{\mathcal{M}}^n_*$ defined by $\hat\iota_{M,f}(x) = [M, x, f]$. The images $[M, f]$ of all possible maps $\hat\iota_{M,f}$ form a canonical partition of $\widehat{\mathcal{M}}^n_*$, which is considered when using saturations or minimal sets in $\widehat{\mathcal{M}}^n_*$. The saturation of any open subset of $\widehat{\mathcal{M}}^n_*$ is open, and therefore the closure of any saturated subset of $\widehat{\mathcal{M}}^n_*$ is saturated. It is said that (M, f) (or f) is:

aperiodic if $\hat\iota_{M,f}$ is injective (id_M is the only isometry of M that preserves f);

limit aperiodic if (M', f') is aperiodic for all $[M', x', f'] \in \overline{[M, f]}$; and

repetitive if, roughly speaking, every ball with the restriction of f is approximately repeated uniformly in M.

When $\overline{[M, f]}$ is compact, the repetitivity of (M, f) means that $\overline{[M, f]}$ is minimal (Proposition 3.1.10).

If we only use immersions $f : M \to \mathfrak{H}$, we get a subspace $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}} \subset \widehat{\mathcal{M}}^n_*$, which is a Riemannian foliated space with the canonical partition such that the maps $\hat\iota_{M,f} : M \to [M, f]$ are local isometries. Moreover these maps are the holonomy covers of the leaves.

If $\mathfrak{H}$ is of finite dimension, then $\overline{[M, f]}$ is a compact subspace of $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$ if and only if M is of bounded geometry, $|\nabla^m f|$ is uniformly bounded for all $m \in \mathbb{N}$, and $|\nabla f|$ is uniformly bounded away from 0 (Propositions 3.1.12 and 3.1.15).

Different versions of this space can be defined with other structures, with similar basic properties. For instance, by forgetting the functions f in the construction of $\widehat{\mathcal{M}}^n_*$, we get a partitioned Polish space $\mathcal{M}^n_*$. In [1], a partitioned Polish space $\mathcal{CM}^n_*$ is defined like $\mathcal{M}^n_*$ by using distinguished closed subsets of the Riemannian manifolds, whose topology also involves the Chabauty (or Fell) topology on the families of closed subsets. An easy refined version $\widehat{\mathcal{CM}}^n_*$ of $\mathcal{CM}^n_*$ can be defined by using locally constant colorings of closed subsets. In Section 2.1.4, we have also used similar partitioned Polish spaces, $\mathcal{G}_*$ and $\widehat{\mathcal{G}}_*$, defined with connected simple (colored) graphs. In this sense, we will also use (limit) aperiodicity and repetitiveness for complete connected Riemannian manifolds, for their (colored) Delone subsets, and for (colored) graphs.

1.2.3 Main results about realization of manifolds as leaves

In this paper, we realize manifolds as leaves of matchbox manifolds, which are the compact connected foliated spaces with zero-dimensional local transversals. Moreover we trivialize the holonomy group of all leaves, and characterize the possibility of minimality. The following is our main result.

Theorem 1.2.1. *Any (repetitive) connected Riemannian manifold of bounded geometry is isometric to a leaf in a (minimal) Riemannian matchbox manifold without holonomy.*

Besides achieving realization in matchbox manifolds, Theorem 1.2.1 improves [6, Theorem 1.1] by removing holonomy from all leaves, and achieving minimality in the case of repetitive manifolds. Thus Theorem 1.2.1 implies the converse of the following implication: in any minimal compact Riemannian foliated space, all leaves without holonomy are repetitive (Proposition 3.1.16).

For example, Theorem 1.2.1 can be applied to any complete connected hyperbolic manifold with positive injectivity radius. It can be also applied to any connected Lie group with a left invariant metric. Some of them are not coarsely quasi-isometric to any finitely generated group [24, 30], obtaining compact, minimal, Riemannian matchbox manifolds without holonomy whose leaves are isometric to each other, but not coarsely quasi-isometric to any finitely generated group.

Since any smooth C^∞ manifold admits a metric of bounded geometry [34], it follows from Theorem 1.2.1 that any C^∞ connected manifold can be realized as a leaf of a C^∞ matchbox manifold without holonomy. For instance, this is true for the exotic 4-manifolds that are non-leaves in codimension one [50].

In Theorem 1.2.1, the realization of leaves in smooth matchbox manifolds without holonomy is relevant because they are homeomorphic to a projective limit of maps between compact branched manifolds [3, 26]. This was generalized to arbitrary matchbox manifolds in [48], but the proof has a gap, even though the result might be correct.

In the following consequences of Theorem 1.2.1, the realization of a Riemannian manifold as a leaf is achieved with some additional properties, but losing the density of that leaf.

Corollary 1.2.2. *Any non-compact connected Riemannian manifold of bounded geometry is isometric to a leaf in some Riemannian matchbox manifold without holonomy that has a complete transversal homeomorphic to a Cantor space.*

Since minimal matchbox manifolds have complete Cantor transversals, Corollary 1.2.2 is a direct consequence of Theorem 1.2.1 if the manifold is repetitive. Otherwise its proof needs some work.

Corollary 1.2.3. *Let M be a connected Riemannian manifold of bounded geometry, and let $\widetilde{M}$ be a regular covering of M. Then M is isometric to a leaf with holonomy covering $\widetilde{M}$ in a compact Riemannian matchbox manifold.*

A more difficult problem is the description the pairs $(M, \widetilde{M})$ that satisfy the statement of Corollary 1.2.3 with a minimal compact foliated space. In this sense, Cass [23] has given a quasi-isometric property satisfied by the leaves of compact minimal foliated spaces without restriction on the holonomy.

Additional properties have been considered in the realization problem: Schweitzer and Souza [62] constructed connected Riemannian manifolds of bounded geometry that are not quasi-isometric to leaves in compact equicontinuous foliated spaces; Hurder and

Lukina used a coarse quasi-isometric invariant, the coarse entropy, to estimate the Hausdorff dimension of local transversals when applied to leaves of compact foliated spaces; and Lukina [49] has studied the Hausdorff dimension of local transversals in a foliated space.

1.2.4 Ideas of the proofs

The proof of Theorem 1.2.1 has two steps. In the first one (Theorem 3.4.1), we realize M as a dense leaf of a (minimal) compact Riemannian foliated space $\mathfrak{X}$ without holonomy. According to Section 1.2.2, this is achieved with $\mathfrak{X} = \overline{[M, f]}$ for some (repetitive) limit aperiodic C^∞ function $f : M \to \mathfrak{H}$, where $\mathfrak{H}$ is of finite dimension, such that $|\nabla^m f|$ is bounded for all $m \in \mathbb{N}$, and $|\nabla f|$ is bounded away from zero. This idea was already used in the proof of [6, Theorem 1.1], with less conditions on f. In the construction of f (Proposition 3.4.3), an important role is played by a Delone subset $X \subset M$, which becomes a (repetitive) connected graph of finite degree by attaching an edge between any pair of close enough points. Then f is defined using normal coordinates at the points of X, and a (repetitive) limit aperiodic coloring ϕ of X by finitely many colors. The existence of ϕ is guaranteed by Theorem 1.1.4. Actually, (M, X, ϕ) must be repetitive when M is repetitive, which requires a closer look at the proof of Theorem 1.1.4 for this particular graph X (Proposition 3.4.2).

At this point, there is an interdependence between this chapter and Chapter 2, kept for the sake of brevity. The proof of Proposition 3.4.2 uses Theorem 1.1.4 (its graph version) and some preliminary results about repetitivity on Riemannian manifolds (Section 3.2). Graph versions of those preliminary results are also needed in Chapter 2, but their proofs are simpler than in the manifold versions (Section 3.3). Therefore those proofs are only given in this chapter for manifolds.

In the second step of the proof, we construct a (minimal) matchbox manifold $\mathfrak{M}$ without holonomy and a foliated projection $\pi : \mathfrak{M} \to \mathfrak{X}$ whose restrictions to the leaves are diffeomorphisms (Theorem 3.4.4). Then $\mathfrak{X}$ can be replaced with $\mathfrak{M}$ by considering the lift of the Riemannian metric of $\mathfrak{X}$ to $\mathfrak{M}$. The construction of $\mathfrak{M}$ uses simple expressions of the local transversals of $\mathfrak{X}$ as quotients of zero-dimensional spaces. This idea is implemented by using again the space $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$.

The proofs of Corollaries 1.2.2 and 1.2.3 use the following common procedure. Given a compact foliated space $\mathfrak{X}$ and a Polish flat bundle E over some leaf M with non-compact locally compact fibers, we can attach E to $\mathfrak{X}$, obtaining a new compact foliated space $\mathfrak{X}'$ (Section 3.4.3). This is applied to the matchbox manifold $\mathfrak{M}$ given by Theorem 1.2.1, using an aproprite choice of E to get the additional property stated in each corollary.

Chapter 2

Graph colorings

2.1 Preliminaries

Let us recall some basic definitions and elementary results about graphs and its metric properties. Short proofs are indicated for completeness.

2.1.1 Partitioned spaces

Let X be a topological space equipped with an equivalence relation $\mathcal{R}$. It may be said that $(X, \mathcal{R})$ is a *partitioned space*.

Lemma 2.1.1. *If the saturation of any open subset of X is open, then the closure of any saturated subset of X is saturated.*

Proof. For any saturated $A \subset X$, let $x \in \overline{A}$ and $y \in \mathcal{R}(x)$. For every open neighborhood U of y, its saturation $\mathcal{R}(U)$ is an open neighborhood of x, and therefore $\mathcal{R}(U) \cap A \neq \emptyset$. Since A is saturated, it follows that $U \cap A \neq \emptyset$. This shows that $y \in \overline{A}$, and therefore $\overline{A}$ is saturated. □

The properties indicated in Lemma 2.1.1 are well known for the equivalence relations defined by continuous group actions or foliated structures.

Like in the case of group actions or foliations, a *minimal set* A in X is a non-empty closed saturated subset that is minimal among the sets with these properties—this minimality is achieved just when every equivalence class in A is dense in A.

Given another partitioned space $(Y, \mathcal{S})$, a map $f : X \to Y$ is said to be *relation-preserving* if $f(\mathcal{R}(x)) \subset \mathcal{S}(f(x))$ for all $x \in X$. The notation $f : (X, \mathcal{R}) \to (Y, \mathcal{S})$ is used in this case.

2.1.2 Metric spaces

Let X be a metric space. For $x \in X$ and $r \in \mathbb{R}$, let $S(x,r) = \{\, y \in X \mid d(x,y) = r \,\}$, $B(x,r) = \{\, y \in X \mid d(x,y) < r \,\}$ and $D(x,r) = \{\, y \in X \mid d(x,y) \leq r \,\}$ (the sphere, and the open and closed balls of center x and radius r) (the sphere and disk of center x and radius r). For $s \geq r \geq 0$, the set $C(x,r,s) = D(x,s) \setminus D(x,r)$ is called a *corona*.

For $Q \subset X$, its *closed penumbra*[1] of *radius* r is $\mathrm{CPen}(Q,r) = \{\, y \in X \mid d(Q,y) \le r \,\}$; in particular, $\mathrm{CPen}(B(x,r),t) \subset B(x,r+t)$ and $\mathrm{CPen}(D(x,r),t) \subset D(x,r+t)$ for all $r,t > 0$, and the equalities hold when X is a length space [16, 36]. We may add X as a subindex to all of this notation if necessary. It is said that Q is (K-) *separated* if there is some $K > 0$ such that $d(x,y) \ge K$ for all $x \ne y$ in Q. On the other hand, Q is said to be (C-) *relatively dense*[2] in X if there is some $C > 0$ such that $\mathrm{CPen}(Q,C) = X$. A separated relatively dense subset is called a *Delone* subset.

Lemma 2.1.2. *If $X = \bigcup_{n=0}^{\infty} Q_n$, where $Q_0 \subset Q_1 \subset \cdots$ and every Q_n is K-separated, then X is K-separated.*

Proof. Given $x \ne y$ in X, we have $x, y \in Q_n$ for some n, and therefore $d(x,y) \ge K$. □

Lemma 2.1.3 (Álvarez-Candel [9, Proof of Lemma 2.1])**.** *A maximal K-separated subset of X is K-relatively dense.*

Lemma 2.1.3 has the following easy consequence using Zorn's lemma.

Corollary 2.1.4 (Cf. [10, Lemma 2.3 and Remark 2.4])**.** *Any K-separated subset of X is contained in some maximal K-separated K-relatively dense subset.*

Recall that X is said to be *proper* is its bounded sets are relatively compact; i.e., the map $d(x,\cdot) : X \to [0,\infty)$ is proper for any $x \in X$.

Definition 2.1.5. For $A \subset X$ and $\varepsilon > 0$, a subset $B \subset X$ is called an *ε-perturbation* of A if there is a bijection $h \colon A \to B$ such that $d(x,h(x)) \le \varepsilon$ for every $x \in A$.

The following result is an elementary consequence of the triangle inequality.

Lemma 2.1.6. *Let $A \subset X$ and let $B \subset X$ be an ε-perturbation of A. If A is η-relatively dense in X for $\eta > 0$, then B is $(\eta + \varepsilon)$-relatively dense in X. If A is τ-separated for $\tau > 2\varepsilon$, then B is $(\tau - 2\varepsilon)$-separated.*

2.1.3 Graphs

An (*undirected*) *simple graph* $X \equiv (X,E)$ is a set X and a family E of subsets $e \subset X$ with cardinality $|e| = 2$. The term "simple" refers to the existence of no loops and of at most one edge joining any pair of vertices. The elements of X and E are called *vertices* and *edges*, respectively. If an edge e contains a vertex x, it is said that e *connects* to x (or e *meets* x, or e and x are *incident*). The *degree* (or *valency*) $\deg x$ of a vertex x is the number of edges connecting to x. The *degree* of X is $\deg X = \sup_{x \in X} \deg x$. Two different vertices are *adjacent* if they define an edge. Two different edges are *consecutive* if they have a common vertex. For $n \in \mathbb{N}$ (we assume that $0 \in \mathbb{N}$), a *path* of *length* n from x to y in X is a sequence of n consecutive edges joining x to y; in terms of their vertices, it can be considered as a sequence $(z_0, \dots, z_n)$, where $z_0 = x$, $z_n = y$, and z_{i-1}

[1] The penumbra $\mathrm{Pen}(Q,r)$ usually has a similar definition with an strict inequality. On graphs it is more practical to use non-strict inequalities.

[2] A *C-net* is similarly defined with the penumbra. If reference to C is omitted, both concepts are equivalent.

and z_i are adjacent vertices for all $i = 1, \dots, n$. If any two vertices of X can be joined by a path, then X is called *connected*. The topological and geometric properties of X indeed refer to its geometric realization.

On any $Y \subset X$, we get the induced graph structure $E|_Y = \{\, \{x, y\} \in E \mid x, y \in Y \,\}$. Then $Y \equiv (Y, E|_Y)$ is called a *subgraph* of X. By Zorn's lemma, there are maximal connected subgraphs of X, called *connected components*, which form a partition of X. Any connected subgraph of X is contained in some connected component of X.

Let $X' \equiv (X', E')$ be another graph. An bijection $X \to X'$ is an *isomorphism* (*of graphs*) if it induces a bijection $E \to E'$. Given distinguished points, $x_0 \in X$ and $x'_0 \in X'$, a (*pointed*) *isomorphism* $f : (X, x_0) \to (X', x'_0)$ is an isomorphism $f : X \to X'$ satisfying $f(x_0) = x'_0$. If there is an isomorphism $X \to X'$ (respectively, $(X, x_0) \to (X', x'_0)$), then these structures are called *isomorphic*, and the notation $X \cong X'$ (respectively, $(X, x_0) \cong (X', x'_0)$) may be used. The composition of isomorphisms is another isomorphism. An isomorphism $X \to X$ (respectively, $(X, x_0) \to (X, x_0)$) is called an *automorphism* of X (respectively, (X, x_0)). The group of automorphisms of X (respectively, (X, x_0)) is denoted by $\mathrm{Aut}(X)$ (respectively, $\mathrm{Aut}(X, x_0)$).

Assume from now on that X is connected. Then we get a metric space $X \equiv (X, d)$, where d is the $\mathbb{N}$-valued metric defined by declaring $d(x, y)$ to be the minimum length of paths in X from x to y. The following property is easily verified:

$$\forall x, y \in X,\ \forall m, n \in \mathbb{N},\ d(x, y) = m + n \Longrightarrow \exists z \in X \mid d(x, z) = m,\ d(y, z) = n\ . \quad (2.1.1)$$

Note that $E = \{\, \{x, y\} \mid x, y \in X,\ d(x, y) = 1 \,\}$. Therefore E and d are equivalent objects; in fact, this correspondence defines a bijection between the families of connected graph structures and $\mathbb{N}$-valued metrics satisfying (2.1.1). Thus an isomorphism between connected graphs is the same as an isometry, and both of these terms will be indistinctly used. A path $(u_0, \dots, u_n)$ in X is called a *minimizing geodesic segment* if $d(u_0, u_n) = n$. By (2.1.1), there exists a minimizing geodesic segment joining any pair of vertices.

The terminology and notation of Section 2.1.2 is adopted here. Now $\mathrm{CPen}(D(x, r), t) = D(x, r + t)$ for $r, t \in \mathbb{N}$ by (2.1.1). Note that $D(x, r)$ is connected. More generally, $\mathrm{CPen}(Q, r)$ is connected if Q is connected. Note also that $|S(x, 0)| = 1$ and $|S(x, 1)| = \deg x$. Since the metric is now $\mathbb{N}$-valued, Lemma 2.1.3 and Corollary 2.1.4 can be restated for graphs as follows.

Lemma 2.1.7. *A maximal K-separated subset Q is $(K - 1)$-relatively dense in X.*

Corollary 2.1.8. *Any K-separated subset of X is contained in some maximal K-separated $(K - 1)$-relatively dense subset.*

On any connected $Y \subset X$, two canonical metrics can be considered, d_Y (defined by $E|_Y$) and the restriction of d_X. Clearly, $d_X \leq d_Y$ on Y.

Lemma 2.1.9. *Let $Y = \mathrm{CPen}(Y_0, r)$ for a connected $Y_0 \subset X$ and $r \in \mathbb{N}$. Then $d_Y(x, y) = d_X(x, y)$ for all $x, y \in Y_0$ with $d_X(x, y) \leq 2r$.*

Proof. Let $(u_0, \dots, u_n)$ be a minimizing geodesic segment of X between $x, y \in Y_0$ of length $n \leq 2r$. Then $d_X(x, u_i), d_X(y, u_j) \leq r$ if $i, n - j \leq r$, yielding $u_0, \dots, u_n \in Y$. So $(u_0, \dots, u_n)$ is a path in Y, and therefore $d_Y(x, y) \leq n = d_X(x, y)$. $\square$

Corollary 2.1.10. *With the notation of Lemma 2.1.9, let $A \subset Y_0$ and $2r \geq K \in \mathbb{Z}^+$. Then $A \subset Y_0$ is K-separated with respect to d_Y if and only if it is K-separated in d_X.*

Definition 2.1.11. For connected $Y, Z \subset X$ and $m \in \mathbb{N}$, a map $f : X \to Y$ is called an *m-short scale isometry* if $d_Z(f(x), f(y)) = d_Y(x, y)$ for all $x, y \in Y$ with $d_Y(x, y) \leq m$.

The above definition is also valid for maps between arbitrary metric spaces.

Corollary 2.1.12. *Let $Y = \operatorname{CPen}(Y_0, r)$ and $Z = \operatorname{CPen}(Z_0, r)$ for connected $Y_0, Z_0 \subset X$ and $r \in \mathbb{N}$, and let $2r \geq m \in \mathbb{N}$. If $f : Y \to Z$ is a graph isomorphism with $f(Y_0) = Z_0$, then $f : Y_0 \to Z_0$ is an m-short scale isometry with respect to the restrictions of d_X.*

Proof. For $x, y \in Y_0$ with $d_X(x, y) \leq m \leq 2r$, we have $d_Y(x, y) = d_X(x, y)$ by Lemma 2.1.9. So $d_Y(x, y) = d_Z(f(x), f(y))$ since $f : Y \to Z$ is an isomorphism. Thus $d_Z(f(x), f(y)) \leq 2r$, and therefore $d_Z(f(x), f(y)) = d_X(f(x), f(y))$ by Lemma 2.1.9 because $f(x), f(y) \in Z_0$. Finally, we get $d_X(x, y) = d_X(f(x), f(y))$. □

Corollary 2.1.13. *For $x, y \in X$ and $r \in \mathbb{N}$, if $h\colon (D(x, 2r), x) \to (D(y, 2r), y)$ is a pointed isomorphism, then $h\colon D(x, r) \to D(y, r)$ is an isometry with respect to the restrictions of d_X.*

Lemma 2.1.14. *If every vertex of X is adjacent to a countable set of vertices, then X is countable.*

Proof. Given any $x \in X$, since $X = \bigcup_{r=0}^{\infty} S(x, r)$, it is enough to prove that $S(x, r)$ is countable for all $r \in \mathbb{N}$. This is done by induction on r. We have $S(x, 0) = \{x\}$, and $S(x, 1)$ is countable by hypothesis. If $S(x, r)$ is countable for some $r \in \mathbb{N}$, then $S(s, r+1)$ is also countable because it is contained in $\bigcup_{y \in S(x,r)} S(y, 1)$. □

Lemma 2.1.15. *The vertices of X have finite degree if and only if its disks are finite.*

Proof. The "if" part is true because $|D(x, 1)| = 1 + \deg x$ for all $x \in X$. Now, assume that the vertices have finite degree, and let us show that $|D(x, r)| < \infty$ for all $x \in X$ and $r \in \mathbb{Z}^+$. This follows by induction on r using that $D(x, r + 1) = \operatorname{CPen}(D(x, r), 1)$ by (2.1.1). □

The disks of X are finite just when X is a *proper* metric space, in the sense that its closed disks are compact.

Lemma 2.1.16. *If X is unbounded, then $|S(x, r)| \geq 1$ for all $x \in X$ and $r \in \mathbb{N}$.*

Proof. By (2.1.1) and since X is unbounded, we have $S(x, r) \neq \emptyset$ for all $r \in \mathbb{N}$. □

Corollary 2.1.17. *If X is unbounded, then $|D(x, r)| \geq r + 1$ and $|C(x, r, s)| \geq s - r$ for all $x \in X$ and $r < s$ in $\mathbb{N}$.*

Proof. Apply Lemma 2.1.16 to the expressions[3] $D(x, r) = \dot{\bigcup}_{i=0}^{r} S(x, i)$ and $C(x, r, s) = \dot{\bigcup}_{i=r+1}^{s} S(x, i)$. □

Now, suppose also that $\Delta := \deg X < \infty$. Since X is connected, it is a singleton if $\Delta = 0$, and it has two vertices if $\Delta = 1$. Thus assume $\Delta \geq 2$.

[3] A dotted union symbol is used for unions of disjoint sets.

Lemma 2.1.18. *$|S(x,r)| \le \Delta(\Delta-1)^{r-1}$ for all $x \in X$ and $r \in \mathbb{Z}^+$.*

Proof. The vertex x is adjacent with at most Δ vertices, which form $S(x,1)$. For all $r \in \mathbb{Z}^+$, any $y \in S(x,r)$ is adjacent with at least one vertex in $S(x,r-1)$ by (2.1.1), and therefore y is adjacent to at most $\Delta - 1$ vertices in $S(x,r+1)$. Then the inequality $|S(x,r)| \le \Delta(\Delta-1)^{r-1}$ follows easily by induction on r. □

Corollary 2.1.19. *Let $x \in X$ and $r \in \mathbb{Z}^+$. Then*

$$|D(x,r)| \le \begin{cases} 1+2r & \text{if } \Delta = 2 \\ 3 \cdot 2^r & \text{if } \Delta = 3 \\ 4(\Delta-1)^r & \text{if } \Delta > 3 \,. \end{cases}$$

Proof. Applying Lemma 2.1.18 to the disjoint union $D(x,r) = \dot\bigcup_{i=0}^{r} S(x,i)$, we get $|D(x,r)| \le 1+2r$ if $\Delta = 2$, and

$$|D(x,r)| \le 1 + \frac{\Delta((\Delta-1)^r - 1)}{\Delta - 2} = \frac{\Delta(\Delta-1)^r - 2}{\Delta - 2}$$

if $\Delta \ge 3$. But

$$\frac{\Delta(\Delta-1)^r - 2}{\Delta - 2} = 3 \cdot 2^r - 2 < 3 \cdot 2^r$$

if $\Delta = 3$, and

$$\frac{\Delta(\Delta-1)^r - 2}{\Delta - 2} < 2\frac{\Delta(\Delta-1)^r - 1}{\Delta - 1} < 4\frac{\Delta(\Delta-1)^r}{\Delta} = 4(\Delta-1)^r$$

if $\Delta > 3$ because

$$u \ge v \ge 1 \Longrightarrow 2\frac{u+1}{v+1} > \frac{u}{v} \,.$$

□

Lemma 2.1.20. *If A is a K-separated $(K-1)$-relatively dense subset of X for some $K \in \mathbb{Z}^+$, then $|A| > |X|/\Delta^K$.*

Proof. We have $X \subset \bigcup_{a\in A} D(a,K-1)$, yielding $|X| \le \sum_{a\in A} |D(a,K-1)|$. By Corollary 2.1.19, for $a \in A$,

$$\begin{aligned} |D(a,K-1)| &\le 1 + 2(K-1) < 2^K && \text{if } \Delta = 2 \,, \\ |D(a,K-1)| &\le 3 \cdot 2^{K-1} < 3^K && \text{if } \Delta = 3 \,, \\ |D(a,K-1)| &\le 4(\Delta-1)^{K-1} \le \Delta(\Delta-1)^{K-1} < \Delta^K && \text{if } \Delta > 3 \,. \end{aligned}$$

□

2.1.4 Colorings

A *coloring* of a set X (by a set F "of *colors*") is a map $\phi : X \to F$. The pair (X,ϕ) is called a *colored set*. The sets of colors F will usually be a finite initial segmen[4] of $\mathbb{N}$, denoted by $[M] = \{0,\dots,M-1\}$ for some $M \in \mathbb{N}$.

[4] Recall that a subset S of an ordered set $(Z,\le)$ is called an *initial segment* if, for all $s \in S$ and $z \in Z$, $z \le s$ implies $z \in S$.

Let X be a simple graph. A coloring of its vertex set, $\phi : X \to F$, is called an (F-) (*vertex*) *coloring* of X, and (X, ϕ) is called an (F-) *colored graph*. If $x_0 \in Y \subset X$, then the simplified notation $(Y, \phi) = (Y, \phi|_Y)$ will be used. The following concepts for colored graphs are the obvious extensions of their graph versions: (*pointed*) *isomorphisms*, denoted by $f : (X, \phi) \to (X', \phi')$ and $f : (X, x_0, \phi) \to (X', x'_0, \phi')$, *isomorphic* (pointed) colored graphs, denoted by $(X, \phi) \cong (X', \phi')$ and $(X, x_0, \phi) \cong (X', x'_0, \phi')$, and *automorphism* groups of (pointed) colored graphs, denoted by $\operatorname{Aut}(X, \phi)$ and $\operatorname{Aut}(X, x_0, \phi)$.

Consider only colorings by F. Let $\widehat{\mathcal{G}}_* = \widehat{\mathcal{G}}_*(F)$ be the set[5] of isomorphism classes, $[X, x, \phi]$, of pointed connected colored graphs, (X, x, ϕ), whose vertices have finite degree. For each $R \in \mathbb{Z}^+$, let

$$\widehat{U}_R = \{ ([X, x, \phi], [Y, y, \psi]) \in \widehat{\mathcal{G}}_*^2 \mid (D_Y(y, R), y, \psi) \cong (D_X(x, R), x, \phi) \} .$$

These sets form a base of entourages of a uniformity on $\widehat{\mathcal{G}}_*$, which is easily seen to be complete. Moreover this uniformity is metrizable because this base is countable.

Note that the *degree map* $\deg : \widehat{\mathcal{G}}_* \to \mathbb{Z}^+$, $[X, x, \phi] \mapsto \deg x$, and the *evaluation map* $\operatorname{ev} : \widehat{\mathcal{G}}_* \to F$, $[X, x, \phi] \mapsto \phi(x)$, are continuous. Suppose that F is countable. Then $\widehat{\mathcal{G}}_*$ is separable because the elements $[X, x, \phi]$, where X is finite, form a countable dense subset. Thus $\widehat{\mathcal{G}}_*$ becomes a Polish space.

For any connected simple colored graph (X, ϕ), there is a canonical map $\hat{\imath}_{X,\phi} : X \to \widehat{\mathcal{G}}_*$ defined by $\hat{\imath}_{X,\phi}(x) = [X, x, \phi]$. its image, denoted by $[X, \phi]$, has an induced connected colored graph structure, and all of these images form a canonical partition of $\widehat{\mathcal{G}}_*$. As we will see in Lemma 3.1.8 for a similar space, we get that the saturation of any open subset of $\widehat{\mathcal{G}}_*$ is open, and therefore the closure operation preserves saturated subsets of $\widehat{\mathcal{G}}_*$; in particular, $\overline{[X, \phi]}$ is saturated. The following result indicates the role played by graphs with finite degrees, colored by finitely many colors.

Proposition 2.1.21. *The closure* $\overline{[X, \phi]}$ *is compact if and only if* $\deg X, |\operatorname{im} \phi| < \infty$.

Proof. The "if" part follows using that, if $\deg X, |\operatorname{im} \phi| < \infty$, then, for each $R \in \mathbb{Z}^+$, the pointed colored disks $(D_X(x, R), x, \phi)$ $(x \in X)$ represent finitely many pointed isomorphism classes $[D_X(x, R), x, \phi]$. The "only if" part follows using the continuity of $\deg : \widehat{\mathcal{G}}_* \to \mathbb{Z}^+$ and $\operatorname{ev} : \widehat{\mathcal{G}}_* \to F$. □

It is said that (X, ϕ) (or ϕ) is *aperiodic* (or *non-periodic*) if $\operatorname{Aut}(X, \phi) = \{\mathrm{id}_X\}$, which means that $\hat{\imath}_{X,\phi}$ is injective; otherwise, it is said that (X, ϕ) (or ϕ) is *periodic*. More strongly, (X, ϕ) (or ϕ) is called *limit aperiodic* if (Y, ψ) is aperiodic for all $[Y, y, \psi] \in \overline{[X, \phi]}$. If X is finite, aperiodicity is equivalent to its limit aperiodicity, and an aperiodic coloring of X by finitely many colors can be easily given. If X is infinite, limit aperiodic colorings by finite finitely many colors are much more difficult to construct. The following lemma will be useful for that purpose.

Lemma 2.1.22. (X, ϕ) *is limit aperiodic if and only if, for all sequences,* x_i, y_i *in* X *and* $R_i, S_i \uparrow \infty$ *in* $\mathbb{Z}^+$*, and pointed isomorphisms,*

$$f_i : (D(x_i, R_i), x_i, \phi) \to (D(x_{i+1}, R_i), x_{i+1}, \phi) ,$$
$$h_i : (D(x_i, S_i), x_i, \phi) \to (D(y_i, S_i), y_i, \phi) ,$$

[5] The graphs X are countable (Lemma 2.1.14), and therefore we can assume that their underlying sets are contained in $\mathbb{N}$. In this way, $\widehat{\mathcal{G}}_*$ becomes a well defined set.

such that $d(x_i, y_i) + S_i \leq R_i$*,* $f_i(y_i) = y_{i+1}$*, and the diagram*

$$\begin{array}{ccc} D(x_{i+1}, S_{i+1}) & \xrightarrow{h_{i+1}} & D(y_{i+1}, S_{i+1}) \\ {\scriptstyle f_i}\uparrow & & \uparrow{\scriptstyle f_i} \\ D(x_i, S_i) & \xrightarrow{h_i} & D(y_i, S_i) \end{array} \tag{2.1.2}$$

is commutative, we have that, either $x_i = y_i$ *for* i *large enough, or* $\limsup_i d(x_i, y_i) = \infty$*.*

Proof. This follows easily from the definition of the topology of $\widehat{\mathcal{G}}_*$. □

Remark 1. In Lemma 2.1.22, the case of bounded sequences x_i, y_i characterizes the aperiodicity of X. Thus the case of unbounded sequences x_i, y_i describes when (Y, ψ) is aperiodic for all $[Y, y, \psi] \in \overline{[X, \phi]} \setminus [X, \phi]$.

On the other hand, (X, ϕ) (or ϕ) is called *repetitive* if there is some point $p \in X$ and a sequence $R_i \uparrow \infty$ in $\mathbb{Z}^+$ such that the sets

$$\{\, x \in X \mid [D(p, R_i), p, \phi] = [D(x, R_i), x, \phi] \,\}$$

are relatively dense in X. This property is clearly independent of the choice of p. If (X, ϕ) is repetitive, then $\overline{[X, \phi]}$ is minimal, and the reciprocal also holds when $\overline{[X, \phi]}$ is compact, as we will see in Proposition 3.1.10 for an analogous setting.

There are obvious versions without colorings of the above definitions and properties, which can be also described by taking $|F| = 1$. Namely, we get: a Polish space $\mathcal{G}_*$, canonical continuous maps $\iota_X : X \to \mathcal{G}_*$, $\iota_X(x) = [X, x]$, whose images, denoted by $[X]$, define a canonical partition of $\mathcal{G}_*$, and the concepts of *non-periodic* (or *aperiodic*), *limit aperiodic*) and *repetitive* graphs. The forgetful (or underlying) map $\mathfrak{u} : \widehat{\mathcal{G}}_* \to \mathcal{G}_*$, $\mathfrak{u}([X, x, \phi]) = [X, x]$, is continuous. If X is repetitive, then $\overline{[X]}$ is minimal, and the reciprocal also holds when $\overline{[X]}$ is compact. The closure $\overline{[X]}$ is compact if and only if $\deg X < \infty$. Then, as we will see in Proposition 3.1.12 for a similar setting, we obtain that $\overline{[X, \phi]}$ is compact if and only if $\deg X < \infty$ and $\overline{\operatorname{im} \phi}$ is compact. By Lemma 2.1.15, the space $\mathcal{G}_*$ is a subspace of the Gromov space of isomorphism classes of pointed proper metric spaces [35], [36, Chapter 3]. The obvious versions of Lemma 2.1.22 and Proposition 2.1.21 in this setting follow by considering a constant coloring.

For $R \geq 0$ and $\lambda \geq 1$, an (R, λ)*-pointed partial quasi-isometry* (shortly, an (R, λ)-p.p.q.i.) between pointed graphs, (X, x) and (Y, y), is a λ-bilipschitz pointed partial map $h\colon (X, x) \rightarrowtail (Y, y)$ such that $D(x, R) = \operatorname{dom} h$ and $D(y, R/\lambda) \subset \operatorname{im} h$.

Proposition 2.1.23. *Let* $h\colon (X, x) \rightarrowtail (X, y)$ *be an* (R, λ)*-p.p.q.i. and* $h'\colon (X, x) \rightarrowtail (X, y')$ *an* (R', λ')*-p.p.q.i. Then* $h^{-1}\colon (X, y) \rightarrowtail (X, x)$ *is n* $(\lambda^{-1}R, \lambda)$*-p.p.q.i. If* $R\lambda + d(x, y) \leq R'$*, then* $h' \circ h\colon (X, x) \rightarrowtail (X, h'(y))$ *is an* $(R, \lambda\lambda')$*-p.p.q.i.*

The following is a simple consequence of the fact that graph metrics take integer values.

Proposition 2.1.24. *Let* $1 \leq \lambda < 2$ *and* $R \geq 0$*. Any* (R, λ)*-p.p.q.i.* $h\colon (X, x) \rightarrowtail (Y, y)$ *between pointed graphs defines a pointed graph isomorphism* $h : (\operatorname{dom} h, x) \to (\operatorname{im} h, y)$*. In particular, it defines an* $(R/\lambda, 1)$*-p.p.q.i.* $(X, x) \rightarrowtail (Y, y)$*.*

Corollary 2.1.25. *A colored graph (X,ϕ) is repetitive if and only if (M,f) (or f) is said to be* repetitive *if, given any $p\in X$, for all $R>0$ and $\lambda>1$, the set*

$$\{\, x\in X \mid \exists \text{ a color preserving } (R,\lambda)\text{-p.p.q.i. } h\colon (X,p,\phi)\rightarrowtail (M,x,\phi)\,\}$$

is relatively dense in M.

2.1.5 A refinement of the main theorem

Using Lemma 2.1.22, Theorem 1.1.4 follows from the following.

Theorem 2.1.26. *Let X be an infinite connected simple graph with $\Delta:=\deg X<\infty$. Then the following properties hold for any sequence $\varepsilon_n\uparrow\infty$ in $\mathbb{Z}^+$:*

(i) There are:

- *a sequence δ_n in $\mathbb{Z}^+$, with every δ_n depending only on Δ, ε_m for $m\le n$, and δ_m for $m<n$; and*
- *a coloring ϕ of X by Δ colors, depending on the sequence ε_n;*

such that, for all $x,y\in X$ and $n\in\mathbb{N}$,

$$0<d(x,y)<\varepsilon_n \Longrightarrow [D(x,\delta_n),x,\phi]\neq[D(y,\delta_n),y,\phi]\ .$$

(ii) Suppose that, for some $p\in X$ and some sequences $\mathfrak{r}_n\uparrow\infty$ and ω_n in $\mathbb{Z}^+$, with every $\mathfrak{r}_n$ large enough depending on Δ and ε_m for $m\le n$, the sets

$$\Omega_n=\{\, x\in X \mid [D(x,\mathfrak{r}_n),x,d_X]=[D(p,\mathfrak{r}_n),p,d_X]\,\}$$

are ω_n-relatively dense in X. Then there are:

- *a sequence $r_n\uparrow\infty$ in $\mathbb{Z}^+$, with every r_n depending on Δ, ε_m and ω_m for $m\le n$, and r_m for $m<n$;*
- *a sequence α_n in $\mathbb{Z}^+$, with every α_n depending on Δ, ε_m and ω_m for $m\le n$, and r_m and α_m for $m<n$; and*
- *a coloring ϕ by Δ colors, depending on the sequences ε_n and $\mathfrak{r}_n$;*

such that ϕ satisfies (i) with some sequence δ_n, and the sets

$$\widehat{\Omega}_n=\{\, x\in X \mid [D(x,\textstyle\sum_{i=0}^n r_i),x,\phi]=[D(p,\sum_{i=0}^n r_i),p,\phi]\,\}$$

are α_n-relatively dense in X.

As indicated in Section 1.1.4, Theorem 2.1.26 is stronger than Theorem 1.1.4 because δ_n, r_n and α_n are independent of the choice of X satisfying the hypothesis.

In Theorem 2.1.26, the assumption that X is infinite can be disposed of. The same ideas work with minor tweaks when X is a finite graph large enough depending on $\deg X$, refining also Theorem 1.1.1. Since the proof is already quite involved, we leave the details to the interested reader.

Theorem 2.1.26 is equivalent to the following finitary version, where every coloring can be explicitly constructed in a finite number of steps.

Theorem 2.1.27. *Let X be a connected infinite simple graph with $\Delta := \deg X < \infty$. Then the following properties hold for any sequence $\varepsilon_n \uparrow \infty$ in $\mathbb{Z}^+$:*

(i) There are:

- *a sequence δ_n in $\mathbb{Z}^+$, with every δ_n depending only on Δ, ε_m for $m \leq n$, and δ_m for $m < n$; and*
- *a sequence of colorings ϕ^N of X by Δ colors, with every ϕ^N depending on ε_m for $m \leq N$;*

such that, for all $x, y \in X$, $N \in \mathbb{N}$ and $0 \leq n \leq N$,

$$0 < d(x,y) < \varepsilon_n \Longrightarrow [D(x,\delta_n), x, \phi^N] \neq [D(y,\delta_n), y, \phi^N] .$$

(ii) Suppose that, for some $p \in X$ and some sequence $\mathfrak{r}_n \uparrow \infty$ and ω_i in $\mathbb{Z}^+$, with every $\mathfrak{r}_n$ large enough depending on Δ and ε_m for $m \leq n$, the sets

$$\Omega_n = \{\, x \in X \mid [D(x,\mathfrak{r}_n), x, d_X] = [D(p,\mathfrak{r}_n), p, d_X] \,\}$$

are ω_n-relatively dense in X. Then there are:

- *a sequence $r_n \uparrow \infty$ in $\mathbb{Z}^+$, with every r_n depending on Δ, ε_m and ω_m for $m \leq n$, and r_m for $m < n$;*
- *a sequence α_n in $\mathbb{Z}^+$, with every α_n depending on Δ, ε_m and ω_m for $m \leq n$, and r_m and α_m for $m < n$; and*
- *a sequence of colorings ϕ^N by Δ colors, with every ϕ^N depending on ε_m and $\mathfrak{r}_m$ for $m \leq N$;*

such that ϕ^N satisfies (i) with some sequence δ_n, and the sets

$$\widehat{\Omega}_n = \{\, x \in X \mid [D(x, \textstyle\sum_{i=0}^n r_i), x, \phi^N] = [D(p, \textstyle\sum_{i=0}^n r_i), p, \phi^N] \,\}$$

are α_n-relatively dense in X for $0 \leq n \leq N$.

Let us derive Theorem 2.1.26 from Theorem 2.1.27. Let X be a graph and ε_n be an increasing sequence of positive integers satisfying the conditions of Theorem 2.1.27. Then this result gives a sequence of colorings ϕ^N. The set of colorings of X by Δ colors is endowed with the topology of convergence over finite subsets of X. Since the set $[\Delta]$ of colors is finite, possibly passing to a subsequence, we can suppose that the sequence of colorings ϕ^N converges to some coloring ϕ. This means that, on any finite $A \subset X$, the colorings ϕ and ϕ^N coincide for N large enough. Let us prove that ϕ satisfies Theorem 2.1.26.

Assume by absurdity that there are some $n \in \mathbb{N}$ and $x, y \in X$ so that $0 < d(x,y) < \varepsilon_n$ and $[D(x,\delta_n), x, \phi] \neq [D(y,\delta_n), y, \phi]$. By the convergence of ϕ^N, there is some $N \geq n$ such that $[D(x,\delta_n), x, \phi] = [D(x,\delta_n), x, \phi^N]$ and $[D(y,\delta_n), y, \phi] = [D(y,\delta_n), y, \phi^N]$, contradicting Theorem 2.1.27 (i). Therefore ϕ satisfies Theorem 2.1.26 (i), with the same choice of sequence δ_n.

Suppose that, additionally, the family ϕ^N satisfies the conditions of Theorem 2.1.27 (ii), with a distinguished point p. Then, for any $n \leq N$ and $x \in X$, there is some $y \in X$ such that $d(x,y)\le\alpha_n$ and $[D(y,\sum_{i=0}^n r_i),y,\phi^N]=[D(p,\sum_{i=0}^n r_i),p,\phi^N]$. Assume by absurdity that there are some $n\in\mathbb{N}$ and $x\in X$ such that $[D(y,\sum_{i=0}^n r_i),y,\phi]\ne[D(p,\sum_{i=0}^n r_i),p,\phi]$ for all $y\in D(x,\alpha_n)$. By the convergence of ϕ^N, we have that ϕ and ϕ^N coincide over $D(p,\sum_{i=0}^n r_i)$ and $D(y,\sum_{i=0}^n r_i)$ for every $y\in D(x,\alpha_n)$ and N large enough, contradicting Theorem 2.1.27 (ii). Therefore the sets

$$\{\,x\in X\mid[D(x,\textstyle\sum_{i=0}^n r_j),x,\phi]=[D(p,\sum_{i=0}^n r_j),p,\phi]\,\}$$

are α_n-relatively dense in X. So ϕ satisfies Theorem 2.1.26 (ii), with the same choice of sequence α_n.

The rest of the paper is devoted to prove Theorem 2.1.27.

2.2 Constants

In order to prove our result, we need to define quantities depending on the sequences appearing in the statements of Theorem 2.1.26 that will function as a priori upper bounds for parameters that arise in the definition of ϕ. They depend on each other in non-trivial ways, so their definitions are quite involved, which makes this section rather technical.

Let X be a graph satisfying the conditions of Theorem 2.1.26, and let ε_n be an increasing sequence of positive integers. By induction on $n\in\mathbb{N}$, we are going to define sequences of positive integers, s_n, $\hat r_n$, $\hat r_n^\pm$, $\bar r_n$ and $\bar r_n^\pm$, and sequences of functions, $\bar\eta_n,\boldsymbol{R}_n^\pm,\boldsymbol{\lambda}_n,\boldsymbol{K}_n,\overline{\boldsymbol{K}}_n\colon\mathbb{N}\to\mathbb{N}$ and $\boldsymbol{\Lambda}_n,\boldsymbol{\Gamma}_n^\pm,\boldsymbol{\Delta}_n\colon\mathbb{N}^{n+1}\to\mathbb{N}$. First, set

$$s_0=27+\varepsilon_0\;,\quad\Delta_{-1}=\deg X=\Delta\;.\tag{2.2.1}$$

The notation $\deg X$, Δ and Δ_{-1} will be used indistinctly, depending on the convenience. Define $\bar\eta_0\colon\mathbb{N}\to\mathbb{Q}$ by

$$\bar\eta_0(a)=\exp_2\left(\lfloor(a-\Delta^{11}-1)/\Delta^3\rfloor\right)\;,\tag{2.2.2}$$

where we use the notation $\exp_2(r)=2^r$ for $r\in\mathbb{R}$. Let $\hat r_0$ be the smallest positive integer such that

$$\bar\eta_0\left(\sqrt{\bar\eta_0(\hat r_0)}-6\right)>\left(4(\Delta-1)^{\hat r_0 s_0^2(3s_0+1)}+6\right)^2\;.\tag{2.2.3}$$

Note that this is well-defined since there is a double exponential in the left-hand side of the inequality, whereas there is a single exponential on the right-hand side. Observe also that (2.2.2) and (2.2.3) yield

$$\hat r_0>2^{11}\tag{2.2.4}$$

because $\Delta\ge2$ since X is infinite. Let

$$\bar r_0=\hat r_0(3s_0+1)\;.\tag{2.2.5}$$

From (2.2.3) and the fact that $\bar\eta_0$ is an increasing function we get

$$\begin{aligned}\bar\eta_0\left(\sqrt{\bar\eta_0(\bar r_0)}-6\right)&>\bar\eta_0\left(\sqrt{\bar\eta_0(\hat r_0)}-6\right)\\&>\left(4(\Delta-1)^{\hat r_0 s_0^2(3s_0+1)}+6\right)^2=\left(4(\Delta-1)^{\bar r_0 s_0^2}+6\right)^2\;.\end{aligned}\tag{2.2.6}$$

Define the remaining functions for $n = 0$ as follows:

$$\left.\begin{array}{lll} \boldsymbol{R}_0^-(a) = 4a - 1 \ , & \boldsymbol{R}_0^+(a) = a(2s_0 + 3) \ , & \boldsymbol{\lambda}_0(a) = 2\boldsymbol{R}_0^+(a) + 1 \ , \\ \boldsymbol{\Delta}_0(a) = 4(\Delta - 1)^{2\boldsymbol{R}_0^+(a)} \ , & \boldsymbol{\Lambda}_0(a) = \boldsymbol{\lambda}_0(a) \ , & \boldsymbol{\Gamma}_0^\pm(a) = \boldsymbol{R}_0^\pm(a) \ . \end{array}\right\} \quad (2.2.7)$$

Now, given $n > 0$, suppose that we have defined the desired constants and functions for integers $0 \leq m < n$. Let $\bar{\boldsymbol{r}}_{n-1}$ denote the n-tuple $(\bar{r}_0, \dots, \bar{r}_{n-1})$. Then define

$$s_n = 27 + 10\boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) + 2\boldsymbol{\Gamma}_{n-1}^+(\bar{\boldsymbol{r}}_{n-1}) + \varepsilon_n \ . \quad (2.2.8)$$

Let $\bar{\eta}_n \colon \mathbb{N} \to \mathbb{Q}$ be defined by

$$\bar{\eta}_n(a) = \exp_2\left(\left\lfloor \left(a - \boldsymbol{\Delta}_{n-1}^{11}(\bar{\boldsymbol{r}}_{n-1}) - 1\right) / \boldsymbol{\Delta}_{n-2}^{\bar{r}_{n-1}^2 s_{n-1}}(\bar{\boldsymbol{r}}_{n-2}) \right\rfloor\right) \ . \quad (2.2.9)$$

Then, let $\hat{r}_n$ be the smallest positive integer so that

$$\bar{\eta}_n\left(\sqrt{\bar{\eta}_n(\hat{r}_n)} - 6\right) > \left(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) - 1)^{\hat{r}_n s_n^2(3s_n+1)} + 6\right)^2 \ . \quad (2.2.10)$$

This is well-defined like in the case of $\hat{r}_0$. Let

$$\bar{r}_n = \hat{r}_n(3s_n + 1) \ . \quad (2.2.11)$$

From (2.2.3), (2.2.10) and the fact that $\bar{\eta}_n$ is an increasing function, we get

$$\begin{aligned} \bar{\eta}_n\left(\sqrt{\bar{\eta}_n(\bar{r}_n)} - 6\right) &> \bar{\eta}_n\left(\sqrt{\bar{\eta}_n(\hat{r}_n)} - 6\right) \\ &> \left(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) - 1)^{\hat{r}_n s_n^2(3s_n+1)} + 6\right)^2 \\ &= \left(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) - 1)^{\bar{r}_n s_n^2} + 6\right)^2 \ . \end{aligned} \quad (2.2.12)$$

For $n \in \mathbb{N}$, let $\boldsymbol{a}_n$ and $\boldsymbol{a}_{n-1}$ denote the $(n+1)$ and n-tuples $(a_0, \dots, a_n)$ and $(a_0, \dots, a_{n-1})$. Let

$$\left.\begin{array}{c} \boldsymbol{R}_n^-(a) = 4a - 1 \ , \quad \boldsymbol{R}_n^+(a) = a(2s_n + 3) \ , \quad \boldsymbol{\lambda}_n(a) = 2\boldsymbol{R}_n^+(a) + 1 \ , \\ \boldsymbol{\Delta}_n(\boldsymbol{a}_n) = 4\left(\boldsymbol{\Delta}_{n-1}(\boldsymbol{a}_{n-1}) - 1\right)^{2\boldsymbol{R}_n^+(a_n)} \ , \\ \boldsymbol{\Lambda}_n(\boldsymbol{a}_N) = \prod_{i=0}^{n} \boldsymbol{\lambda}_i(a_i) \ , \quad \boldsymbol{\Gamma}_n^\pm(\boldsymbol{a}_N) = \boldsymbol{R}_n^\pm(a_n) \cdot \boldsymbol{\Lambda}_{n-1}(\boldsymbol{a}_{N-1}) + \boldsymbol{\Gamma}_{n-1}^+(\boldsymbol{a}_{N-1}) \ . \end{array}\right\} \quad (2.2.13)$$

Note that $\boldsymbol{R}_n^-$ is independent of n. Also, by a simple induction argument, we get, for $l = 0, \dots, N$,

$$\boldsymbol{\Gamma}_n^\pm(\boldsymbol{a}_N) \geq \boldsymbol{R}_l^\pm(a_l) \ . \quad (2.2.14)$$

Lemma 2.2.1. *Let $n \in \mathbb{N}$, and let $\boldsymbol{a} = (a_0, \dots, a_n)$ be an $(n+1)$-tuple such that, for $0 \leq m \leq n$, we have $a_m \leq \bar{r}_m$. Then*

$$a_n s_n \geq 2\boldsymbol{\Gamma}_n^-(\boldsymbol{a}_n) + \varepsilon_n \ , \quad a_n s_n^2 \geq 2\boldsymbol{\Gamma}_n^+(\boldsymbol{a}_n) + \varepsilon_n \ .$$

Proof. By definition of s_n, we have

$$\begin{aligned} a_n s_n &= a_n(10\boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) + 2\boldsymbol{\Gamma}^+_{n-1}(\bar{\boldsymbol{r}}_{n-1}) + \varepsilon_n) \\ &> 10 a_n \boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) + 2\boldsymbol{\Gamma}^+_{n-1}(\bar{\boldsymbol{r}}_{n-1}) + \varepsilon_n \ . \end{aligned}$$

On the other hand, using (2.2.13) and the fact that $\boldsymbol{\Lambda}_{n-1}$ and $\boldsymbol{\Gamma}^\pm_{n-1}$ are monotone increasing functions on every coordinate, we have

$$\boldsymbol{\Gamma}^\pm_n(\boldsymbol{a}_n) \leq \boldsymbol{R}^\pm_n(a_n) \cdot \boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) + \boldsymbol{\Gamma}^+_{n-1}(\bar{\boldsymbol{r}}_{n-1}) \ .$$

Then the proof follows by showing that $10a_n > 2\boldsymbol{R}^-_n(a_n)$ and $10a_n s_n > 2\boldsymbol{R}^+_n(a_n)$, which is an easy consequence of the definitions. □

Let $\overline{\mathbf{K}}_{-1} = \mathbf{K}_{-1} \equiv \overline{K}_{-1} = K_{-1} = 0$, and continue defining $\overline{\mathbf{K}}_n$ and $\mathbf{K}_n$ by induction on $n \in \mathbb{N}$ as follows:

$$\overline{\boldsymbol{K}}_n(\boldsymbol{a}_n) = \boldsymbol{K}_{n-1}(\boldsymbol{a}_{n-1}) + \boldsymbol{\Lambda}_n(\boldsymbol{a}_n)(a_n s_n^2 + a_n(2s_n + 1)) \ , \tag{2.2.15}$$

$$\boldsymbol{K}_n(\boldsymbol{a}_n) = \overline{\boldsymbol{K}}_n(\boldsymbol{a}_n) + \boldsymbol{\Lambda}_n(\boldsymbol{a}_n)(s_{n+1}\boldsymbol{R}^+_{n+1}(\bar{\boldsymbol{r}}_{n+1}) + \boldsymbol{\Gamma}^+_n(\bar{\boldsymbol{r}}_n) + 2\boldsymbol{R}^+_n(\bar{\boldsymbol{r}}_n)) \ . \tag{2.2.16}$$

Finally, for all $n \in \mathbb{N}$, let

$$\bar{r}^-_n = \bar{r}_n \ , \quad \bar{r}^+_n = s_n \bar{r}_n \ , \quad \hat{r}^-_n = \hat{r}_n \ , \quad \hat{r}^+_n = s_n \hat{r}_n \ .$$

2.3 Construction of $\mathfrak{X}_n$

This section is devoted to the construction of subsets $\mathfrak{X}_n \subset X$, which will be used later to achieve the repetitiveness of ϕ under the assumptions of Theorem 2.1.26 (ii). Hence we suppose that X satisfies the hypothesis of Theorem 2.1.26 (ii) throughout this section. Therefore we have a distinguished point $p \in X$, some sequences $\mathfrak{r}_n \uparrow \infty$ and ω_n in $\mathbb{Z}^+$, with every $\mathfrak{r}_n$ large enough depending on Δ and ε_m for $m \leq n$, such that every set

$$\Omega_n = \{\, x \in X \mid [D(p, \mathfrak{r}_n), p, d_X] = [D(x, \mathfrak{r}_n), x, d_X] \,\}$$

is ω_n-relatively dense in X. Thus, for each $x \in \Omega_n$, there is a pointed isometry $\mathfrak{f}_{n,x} : (D(p, \mathfrak{r}_n), p) \to (D(x, \mathfrak{r}_n), x)$.

Some of the results of the present section will be direct applications of the results in [5, Section 2]. For notational convenience, let also $\mathfrak{r}_{-1} = \mathfrak{s}_{-1} = \mathfrak{t}_{-1} = \omega_{-1} = 0$. We will now define sequences $\mathfrak{s}_n, \mathfrak{t}_n \uparrow \infty$ and $2 > \lambda_n \downarrow 1$. Assuming this divergence is fast enough, and possible taking a subsequence of $\mathfrak{r}_n$, we can assume that they satisfy

$$\mathfrak{r}_n > \boldsymbol{K}_n(\bar{\boldsymbol{r}}_n) + \mathfrak{s}_n^2 4\boldsymbol{\Lambda}_n(\bar{\boldsymbol{r}}_n)(\boldsymbol{\Gamma}^+_n(\bar{\boldsymbol{r}}_n) + n) \ , \tag{2.3.1}$$

$$\mathfrak{s}_n > \boldsymbol{\Lambda}_{n-1}(\bar{\boldsymbol{r}}_{n-1})(2\mathfrak{r}_n + \boldsymbol{K}_{n-1}(\bar{\boldsymbol{r}}_{n-1})), 3\boldsymbol{\Lambda}_n(\bar{\boldsymbol{r}}_n)\boldsymbol{\Gamma}^+_{n+1}(\bar{\boldsymbol{r}}_{n+1}) \ , \tag{2.3.2}$$

$$\mathfrak{t}_n > \boldsymbol{K}_n(\bar{\boldsymbol{r}}_n) \ , \tag{2.3.3}$$

in addition to the following inequalities, which are the graph versions of Eqs. (3.2.1) to (3.2.6) of Section 3.2:

$$\begin{aligned}
\mathfrak{r}_n &> \frac{\lambda_0^5}{\lambda_0 - 1}(\mathfrak{r}_{n-1} + \mathfrak{s}_{n-1} + t_{i-1} + 2\omega_{n-1} + 1) \;, \\
\mathfrak{s}_n &> 2\lambda_0^5(\mathfrak{r}_n + \mathfrak{s}_{n-1} + \omega_n) \;, \\
\mathfrak{t}_n &> \lambda_0^3(5\mathfrak{t}_{n-1} + \mathfrak{r}_n + \mathfrak{s}_{n-1} + 2\omega_{n-1} + 1) \;, \\
\mathfrak{t}_n &> 4\frac{\lambda_n^4 + \lambda_n^2 - 1}{\lambda_n^2}\mathfrak{r}_n + \mathfrak{t}_{n-1} + \Lambda_n(\mathfrak{s}_{n-1} + 2\omega_{n-1} + \omega_n) \;, \\
\lambda_n^2 &< \lambda_{n-1} \;, \\
2^{2^{-n}} &> \frac{\mathfrak{r}_n(\lambda_n^5 - 1)\lambda_{n-1}^2}{\mathfrak{r}_{n-1}(\lambda_{n-1}^5 - 1)\lambda_n^2}, \frac{\mathfrak{r}_n(\lambda_n^6 - 1)\lambda_{n-1}^2}{\mathfrak{r}_{n-1}(\lambda_{n-1}^6 - 1)\lambda_n^2} \;.
\end{aligned}$$

We also assume

$$\prod_{n=0}^{\infty} \lambda_n < 2 \;.$$

For $n \in \mathbb{N}$, define $\mathfrak{X}_n^n = \{p\}$ and $\mathfrak{h}_{n,p}^n = \mathrm{id}_{D(x,\mathfrak{r}_n)}$. In Proposition 3.2.2, we will continue defining subsets $\mathfrak{X}_n^m \subset X$ for $0 \le n < m$, and pointed isometries $\mathfrak{h}_{n,z}^m \colon (D(p,\mathfrak{r}_n), p) \to (D(z,\mathfrak{r}_n), z)$ for $z \in \mathfrak{X}_n^m$. We will use the following notation:

$$\mathfrak{P}_m^m = \{\, (l,z) \in \mathbb{N} \times X \mid n < l < m, \ z \in \mathfrak{X}_l^m \,\} \;.$$

Let $<$ denote the binary relation on $\mathfrak{P}_m^m$ defined by declaring $(l,z) < (l',z')$ if $l < l'$ and $z \in \mathfrak{h}_{l',z'}^m(\mathfrak{X}_l^{l'})$, and let $\le$ denote its reflexive closure of $<$. This is actually a partial order relation, as explained in Section 3.3 with more generality. Let $\overline{\mathfrak{P}}_n^m$ denote the subset of maximal elements of $\mathfrak{P}_n^m$. For every $(l,z) \in \mathfrak{P}_n^m$, there is a unique $(l',z') \in \overline{\mathfrak{P}}_n^m$ such that $(l,z) \le (l',z')$ (see Section 3.3). The following result is a particular case of Proposition 3.3.1, which will be proved independently of this chapter.

Proposition 2.3.1. *For $0 \le n < m$, there are sets $\mathfrak{X}_n^m \subset X$, and for each $z \in \mathfrak{X}_n^m$ there is a pointed isometry $\mathfrak{h}_{n,z}^m \colon (D(p,\mathfrak{r}_n), p) \to (D(z,\mathfrak{r}_n), z)$, satisfying the following properties:*

(i) *The set $\mathfrak{X}_n^m$ is an $\mathfrak{s}_n$-separated subset of $\Omega_n \cap D(p, \mathfrak{r}_n - \mathfrak{t}_n)$.*

(ii) *For every $(l,z) \in \mathfrak{P}_m^m$ and $x \in \mathfrak{X}_n^m \cap D(z,\mathfrak{r}_l)$ we have $\mathfrak{h}_{n,x}^m = \mathfrak{h}_{l,z}^m \circ \mathfrak{h}_{n,x'}^l$ on $D(p,\mathfrak{r}_n)$, where $x' := (\mathfrak{h}_{l,z}^m)^{-1}(x)$.*

(iii) *For any $(l,z) \in \mathfrak{P}_m^m$, one has $\mathfrak{X}_n^m \cap D(z, \mathfrak{r}_{l'} + \mathfrak{s}_n) = \mathfrak{h}_{l,z}^m(\mathfrak{X}_n^l)$.*

(iv) *For any $x \in \mathfrak{X}_n^m$ and $(l,z) \in \mathfrak{P}_m^m$, either $d(x,z) \ge \mathfrak{r}_l + \mathfrak{s}_n$, or $x \in \mathfrak{h}_{l,z}^m(\mathfrak{X}_n^l)$.*

(v) *Consider integers $0 \le k \le l$ such that either $l < m$ and $k \ge n$, or $l = m$ and $k > n$. Then $\mathfrak{X}_k^l \subset \mathfrak{X}_n^m$, and for any $z \in \mathfrak{X}_k^l$ we have $\mathfrak{h}_{n,z}^m = \mathfrak{h}_{k,z}^l|_{D(p,\mathfrak{r}_n)}$.*

(vi) *We have $p \in \mathfrak{X}_n^m$ and $\mathfrak{h}_{n,p}^m = \mathrm{id}_{D(p,\mathfrak{r}_n)}$.*

For $n < m$, let $\mathfrak{c}_n^m \colon D(p, \mathfrak{r}_m) \to \{n+1, \dots, m\}$ be defined by

$$\mathfrak{c}_n^m(x) = \min\{\, n \in \mathbb{Z} \mid n < l \leq m,\ \exists z \in \mathfrak{X}_l^m,\ x \in D(z, \mathfrak{r}_l) \,\} .$$

Since the set $\mathfrak{X}_l^m$ is $2\mathfrak{r}_l$-separated by Proposition 2.3.1 (i) and (3.2.2), if $x \in D(z, \mathfrak{r}_l)$ for some $z \in \mathfrak{X}_l^m$, then z is the unique point in $\mathfrak{X}_l^m$ that satisfies this condition. Let $\mathfrak{p}_n^m \colon \mathfrak{X}_n^m \to X$ be defined by assigning to every $x \in \mathfrak{X}_n^m$ the unique point $\mathfrak{p}_n^m(x)$ in $\mathfrak{X}_{\mathfrak{c}_n^m(x)}^m$ satisfying $x \in D(\mathfrak{p}_n^m(x), \mathfrak{r}_{\mathfrak{c}_n^m(x)})$.

For $n \in \mathbb{N}$, let $\preceq_n^n$ be the trivial order relation on $\mathfrak{X}_n^n = \{p\}$.

Proposition 2.3.2. *For $0 \leq n < m$, there is an order*[6] *relation $\preceq_n^m$ on $\mathfrak{X}_n^m$ such that:*

(i) *p is the least element of $(\mathfrak{X}_n^m, \preceq_n^m)$;*

(ii) *for $x, y \in \mathfrak{X}_n^m$, if $\mathfrak{c}_n^m(x) < \mathfrak{c}_n^m(y)$, then $x \prec_n^m y$ (meaning $x \preceq_n^m y$ and $x \neq y$); and,*

(iii) *for any $(l, z) \in \mathfrak{P}_m^m$, the map $\mathfrak{h}_{l,z}^m \colon (\mathfrak{X}_n^l, \preceq_n^l) \to (\mathfrak{X}_n^m \cap D(z, \mathfrak{r}_l), \preceq_n^m)$ is order preserving.*

Proof. We proceed by induction. Let $\preceq_n^{n+1}$ be an arbitrary ordering of $\mathfrak{X}_n^{n+1}$ whose least element is p. For $m = n+1$, we have $\mathfrak{c}_n^m(x) = m$ for every $x \in \mathfrak{X}_n^m$ if $\mathfrak{P}_m^m = \emptyset$. Thus (ii) and (iii) are trivially satisfied in this case.

Suppose now that we have defined $\preceq_k^l$ when either $l > n$, or $l = n$ and $k < m$. Let $\trianglelefteq_n^m$ be an arbitrary ordering of $D(p, \mathfrak{r}_n) \setminus \bigcup_{(l,z) \in \mathfrak{P}_m^m} D(z, \mathfrak{r}_l)$. Then we define $\preceq_n^m$ using several cases as follows:

(a) if $\mathfrak{c}_n^m(x) < \mathfrak{c}_n^m(y)$, then $x \prec_n^m y$;

(b) if $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) < m$ and $\mathfrak{p}_n^m(x) = \mathfrak{p}_n^m(y)$, then $x \preceq_n^m y$ if and only if

$$\left(\mathfrak{h}_{\mathfrak{c}_n^m(x), \mathfrak{p}_n^m(x)}^m\right)^{-1}(x) \preceq_n^{\mathfrak{c}_n^m(x)} \left(\mathfrak{h}_{\mathfrak{c}_n^m(x), \mathfrak{p}_n^m(x)}^m\right)^{-1}(y) ;$$

(c) if $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) < m$ and $\mathfrak{p}_n^m(x) \neq \mathfrak{p}_n^m(y)$, then $x \prec_n^m y$ if and only if $\mathfrak{p}_n^m(x) \prec_{\mathfrak{c}_n^m(x)}^m \mathfrak{p}_n^m(y)$; and,

(d) if $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) = m$, then $x \preceq_n^m y$ if and only if $x \trianglelefteq_n^m y$.

It can be easily checked that this is indeed an order relation, and it is obvious that it satisfies (i) and (ii). Let us prove that it also satisfies (iii). Suppose first that $(l, z) \in \overline{\mathfrak{P}}_n^m$. For any $x, y \in D(z, \mathfrak{r}_l)$ we have $\mathfrak{c}_n^m(x) = \mathfrak{c}_n^m(y) = l$ and $\mathfrak{p}_n^m(x) = \mathfrak{p}_n^m(y) = z$, and therefore $\mathfrak{h}_{l,z}^m$ is order preserving by (b).

Suppose now that $(l, z) \in \mathfrak{P}_m^m \setminus \overline{\mathfrak{P}}_n^m$, and let $(l', z') \in \overline{\mathfrak{P}}_n^m$ be the unique maximal element such that $(l, z) < (l', z')$. Let $z'' = (\mathfrak{h}_{l',z'}^m)^{-1}(z)$. By the induction hypothesis, the map

$$\mathfrak{h}_{l,z''}^m \colon \left(\mathfrak{X}_n^l, \preceq_n^l\,\right) \to \left(\mathfrak{X}_n^{l'} \cap D(z'', \mathfrak{r}_l), \preceq_n^{l'}\,\right)$$

is order preserving, and

$$\mathfrak{h}_{l',z'}^m \colon \left(\mathfrak{X}_n^{l'}, \preceq_n^{l'}\,\right) \to \left(\mathfrak{X}_n^m \cap D(z', \mathfrak{r}_{l'}), \preceq_n^m\,\right)$$

[6] In the order relations, it is assumed that any pair of elements is comparable. When this property is not satisfied, we use the term partial order relation.

is order preserving because $(l',z') \in \overline{\mathfrak{P}}_n^m$. Therefore

$$\mathfrak{h}_{l,z}^m = \mathfrak{h}_{l',z'}^m \circ \mathfrak{h}_{l,z''}^m : \left(\mathfrak{X}_n^l, \preceq_n^l\right) \to \left(\mathfrak{X}_n^m \cap D(z,\mathfrak{r}_l), \preceq_n^m\right)$$

is also order preserving. □

Define

$$\begin{aligned}\mathfrak{X}_n &= \bigcup_{m\geq n} \mathfrak{X}_n^m\,,\\ \mathfrak{P}_n &= \bigcup_{m\geq n} \mathfrak{P}_m^m = \{\,(m,x)\in\mathbb{N}\times X \mid n<m,\ x\in\mathfrak{X}_m\,\}\,.\end{aligned} \tag{2.3.4}$$

For $n \in \mathbb{N}$ and $x \in \mathfrak{X}_n$, there is some $m \geq n$ such that $x \in \mathfrak{X}_n^m$. Let $\mathfrak{h}_{n,x} = \mathfrak{h}_{n,x}^m : (D(p,\mathfrak{r}_n),p) \to (D(x,\mathfrak{r}_n),x)$, which is independent of m by Proposition 2.3.1 (v).

Let $<$ be the binary relation on $\mathfrak{P}_n$ defined by declaring $(m,x) < (m',x')$ if $m < m'$ and $D(x,\mathfrak{r}_n) \subset D(x',\mathfrak{r}_{m'})$, and let $\leq$ be the reflexive closure of $<$.

Consider the choice of $\mathfrak{r}_n$, $\mathfrak{s}_n$ and $\mathfrak{t}_n$ given at the beginning of the present section. The following is a particular case of Proposition 3.3.2, which will be proved independently of this chapter.

Proposition 2.3.3. *For $n \in \mathbb{N}$, the following properties hold:*

(i) *The set $\mathfrak{X}_n$ is an $\mathfrak{s}_n$-separated subset of Ω_n containing p.*

(ii) *For any $(l,z) \in \mathfrak{P}_n$, we have $\mathfrak{X}_n \cap D(z,\mathfrak{r}_l) = \mathfrak{h}_{l,z}(\mathfrak{X}_n^l)$.*

(iii) *For any $x \in \mathfrak{X}_n$ and $(l,z) \in \mathfrak{P}_n$ such that $x \in \mathfrak{X}_n \cap D(z,\mathfrak{r}_l)$, we have $\mathfrak{h}_{n,x} = \mathfrak{h}_{l,z} \circ \mathfrak{h}_{n,x'}$ for $x' = \mathfrak{h}_{l,z}^{-1}(x)$.*

(iv) *For any $x \in \mathfrak{X}_n$ and $(l,z) \in \mathfrak{P}_n$, either $d(x,z) \geq \mathfrak{r}_l + \mathfrak{s}_n$, or $x \in h_{l,x}(\mathfrak{X}_n^l)$.*

(v) *For $n \leq m$, we have $\mathfrak{X}_m \subset \mathfrak{X}_n$, and $\mathfrak{h}_{n,x} = \mathfrak{h}_{m,x}|_{D(p,\mathfrak{r}_n)}$ for $x \in \mathfrak{X}_m$.*

(vi) *We have $p \in \mathfrak{X}_n$ and $\mathfrak{h}_{n,p} = \mathrm{id}_{D(p,\mathfrak{r}_n)}$.*

Remark 2. As we will see in Remark 23, if we choose relatively dense subsets

$$\Omega_n \subset \{\, x \in X \mid [D(p,\mathfrak{r}_n),p,d_X] = [D(x,\mathfrak{r}_n),x,d_X]\,\}$$

and, for every $x \in \Omega_n$, we choose a pointed isometry $f_{n,x} : (D(x,\mathfrak{r}_n),p) \to (D(x,\mathfrak{r}_n),p)$, then we may assume that $\mathfrak{X}_n \subset \Omega_n$ and every map $\mathfrak{h}_{n,x}$ is a composition of the form $f_{n_m,x_m} \cdots f_{n_1,x_1}$. Note that in this case the constant ω_n may change. The following is a particular case of Proposition 3.3.3, which will be proved independently of this chapter.

Proposition 2.3.4. *$\mathfrak{X}_n$ is relatively dense in X and the implied constant depends only on Δ, ε_m and ω_m for $m \leq n$, and $\mathfrak{r}_n$ for $m < n$.*

By Propositions 2.3.1 (vi) and 2.3.2 (iii), the order relations $\preceq_n^m$, $m \geq n$, define an order relation $\preceq_n$ on $\mathfrak{X}_n$. The following is a consequence of Proposition 2.3.2.

Proposition 2.3.5. *For $n \in \mathbb{N}$, the following properties hold:*

(i) *The point p is the least element.*

(ii) *For $x, y \in \mathfrak{X}_n$, if $\mathfrak{c}_n(x) < \mathfrak{c}_n(y)$, then $x \prec_n y$.*

(iii) *For any $(l, z) \in \mathfrak{P}_n$, the map $\mathfrak{h}_{l,z} \colon (\mathfrak{X}_n^l, \preceq_n^l) \to (\mathfrak{X}_n \cap D(z, \mathfrak{r}_l), \preceq_n)$ is order preserving.*

For $m \in \mathbb{N}$, let

$$\begin{aligned} \mathfrak{P}_{-1}^m &= \{\, (l, z) \in \mathbb{N} \times X \mid 0 \le l < m,\ z \in \mathfrak{X}_l^m \,\} , \\ \mathfrak{P}_{-1} &= \{\, (m, x) \in \mathbb{N} \times X \mid x \in \mathfrak{X}_m \,\} . \end{aligned} \tag{2.3.5}$$

We can define on both of these sets the relation $<$ by declaring $(l, z) < (l', z')$ if $l < l'$ and $D(z, \mathfrak{r}_l) \subset D(z', \mathfrak{r}_{l'})$. The induced reflexive closures $\le$ are partial order relations. Let $\overline{\mathfrak{P}}_{-1}^m$ denote the subset of maximal elements of $\mathfrak{P}_{-1}^m$. For every $(l, z) \in \mathfrak{P}_{-1}^m$, there is a unique $(l', z') \in \overline{\mathfrak{P}}_{-1}^m$ such that $(l, z) \le (l', z')$.

2.4 Construction of X_n

In this section we define a sequence of nested subsets $X_n \subset X$ that will constitute the centers of the clusters used in the construction of the colorings ϕ^N, as explained in Section 1.1.5. This will be used to prove Theorem 2.1.26 in full generality, so we will assume that X satisfies the hypothesis of Theorem 2.1.26 (ii). If X does not satisfy this, the same proof applies to Theorem 2.1.26 (i) by taking $\mathfrak{X}_n = \emptyset$, and therefore omitting the use of the sets $\mathfrak{P}_n$, numbers $\mathfrak{r}_n$, and maps $\mathfrak{h}_{n,z}^m$ and $\mathfrak{h}_{n,x}$.

For notational convenience, let

$$(X_{-1}, E_{-1}) = (X, E) , \quad d_{-1} = d , \quad r_{-1} = s_{-1} = R_{-1}^{\pm} = 0 , \quad \lambda_{-1} = \Lambda_{-1} = 1 . \tag{2.4.1}$$

For $n \in \mathbb{N}$, we will continue defining constants r_n, subsets $X_n \subset X$ containing $\mathfrak{X}_n$, and a connected graph structure E_n on every X_n with induced metrics d_n. Also, for $x \in X_n$ and $l \in \mathbb{N}$, let $D_n(x, l)$ and $S_n(x, l)$ denote the disks and spheres of center x and radius l in (X_n, d_n). (Recall that, in connected graphs, we use disks defined with non-strict inequalities.) With this notation, let $\eta_n \colon \mathbb{N} \to \mathbb{Q}$ be given by

$$\eta_n(a) = \begin{cases} \exp_2\big(\big\lfloor\big(a - (\deg X_{-1})^{11} - 1\big)/(\deg X_{-1})^3\big\rfloor\big) & \text{if } n = 0 \\ \exp_2\big(\big\lfloor\big(a - (\deg X_{n-1})^{11} - 1\big)/(\deg X_{n-2})^{r_{n-1}^2 s_{n-1}}\big\rfloor\big) & \text{if } n > 0 . \end{cases} \tag{2.4.2}$$

Suppose that, for $n \in \mathbb{N}$, the graphs (X_m, E_m) and constants r_m have been defined for integers $-1 \le m < n$. Then let r_n be defined as follows:

(A) If there is some $x \in D_{n-1}(p, \hat{r}_n(2s_n + 1))$ such that

$$(|D_{n-1}(x, \hat{r}_n s_n)| + 6)^2 \ge \eta_n(|D_{n-1}(x, \hat{r}_n)|) ,$$

then let $r_n = \bar{r}_n$ (see (2.2.11)).

(B) Otherwise, let $r_n = \hat{r}_n$ (see (2.2.3) and (2.2.10)).

Observe that

$$r_0 > 2^{11} \tag{2.4.3}$$

by (2.2.4), (2.2.5), (A) and (B). Moreover, let

$$\left.\begin{aligned} \Delta_n = \boldsymbol{\Delta}_n(r_0, \dots, r_n) \ , \quad \Lambda_n = \boldsymbol{\Lambda}_n(r_0, \dots, r_n) \ , \\ \Gamma_n^\pm = \boldsymbol{\Gamma}_n^\pm(r_0, \dots, r_n) \ , \quad \overline{K}_n = \overline{\boldsymbol{K}}_n(r_0, \dots, r_n) \ , \\ K_n = \boldsymbol{K}_n(r_0, \dots, r_n) \ , \quad R_n^\pm = \boldsymbol{R}^\pm(r_n) \ , \quad \lambda_n = \boldsymbol{\lambda}_n(r_n) \ . \end{aligned}\right\} \tag{2.4.4}$$

All functions in (2.4.4) are monotone increasing on every coordinate. So, if $\hat{\boldsymbol{r}}_{\boldsymbol{n}}$ denotes the $(n+1)$-tuple $(\hat{r}_0, \dots, \hat{r}_n)$, we get

$$\boldsymbol{\Delta}_n(\hat{\boldsymbol{r}}_n) \le \Delta_n \le \boldsymbol{\Delta}_n(\bar{\boldsymbol{r}}_n) \ , \quad \boldsymbol{R}_n^\pm(\hat{r}_n) \le R_n^\pm \le \boldsymbol{R}_n^\pm(\bar{r}_n) \ , \tag{2.4.5}$$

and so on. From (2.2.14), (3.2.1) and (2.4.4), it follows that

$$\mathfrak{r}_n > \Gamma_n^\pm \ge R_m^\pm \tag{2.4.6}$$

for $m = 0, \dots, n$. Finally, let

$$r_n^- = r_n \ , \quad r_n^+ = r_n s_n \ . \tag{2.4.7}$$

By (2.2.7), (2.2.13), (2.4.4) and (2.4.7), we have

$$r_n^\pm \le R_n^\pm \ . \tag{2.4.8}$$

Proposition 2.4.1. *For $n \in \mathbb{N}$, there are disjoint subsets $X_n^+, X_n^- \subset X$ and a graph structure E_n on $X_n := X_n^- \sqcup X_n^+$ such that the following properties are satisfied:*

(i) $\mathfrak{X}_n \subset X_n \subset X_{n-1}$.

(ii) For all $(m, x) \in \mathfrak{P}_{n-1}$, we have

$$h_{j,x}\left(X_n^\pm \cap D_{-1}(p, \mathfrak{r}_m - \overline{K}_n)\right) = X_n^\pm \cap D_{-1}(x, \mathfrak{r}_m - \overline{K}_n) \ .$$

(iii) For all $x \in X_n^\pm$, we have

$$\eta_n(|D_{n-1}(x, r_n^\pm)|) \ge (6 + |D_{n-1}(x, r_n^\pm s_n)|)^2 \ .$$

(iv) X_n is $(2r_n^+ + 1)$-separated and R_n^+-relatively dense in (X_{n-1}, d_{n-1}).

(v) (X_n, E_n) is a connected graph. Let d_n denote the induced metric.

(vi) We have $d_n \le d_{n-1} \le \lambda_n d_n$ and $d_n \le d_{-1} \le \Lambda_n d_n$.

(vii) We have

$$\deg X_n \le \ \Delta_n, \ 4(\deg X_{n-1} - 1)^{2R_n^+} \ .$$

(viii) For any $(m, x) \in \mathfrak{P}_{n-1}$, the restriction of $h_{j,x}$ to $X_n \cap D_{-1}(p, \mathfrak{r}_m - K_n)$ is an $(s_{n+1}R_{n+1}^+ + \Gamma_n^+)$-short scale isometry with respect to d_n.

Remark 3. Note that $K_n < \mathfrak{t}_n, \mathfrak{r}_n$ by (3.2.1), (3.2.3) and the fact that $\bar{r}_n > r_n$. This and the inequality $K_n > \overline{K}_n$ yield $\mathfrak{r}_n - \overline{K}_n, \mathfrak{r}_n - K_n > 0$ in (ii) and (viii).

Remark 4. In accordance with the discussion at the beginning of the section, to prove Theorem 2.1.26, if X does not satisfy the hypothesis of Theorem 2.1.26 (ii), items (ii) and (viii) must be omitted, and only the inclusion "$X_n \subset X_{n-1}$" must be considered in (i).

The rest of this section is devoted to the proof of the above proposition. We proceed by induction on n. The following lemma follows from Proposition 2.3.3, (2.4.1) and (2.3.5). The items are irregularly numbered so that there is an obvious correspondence with those of Proposition 2.4.1.

Lemma 2.4.2. *The following properties hold:*

(i') $\mathfrak{X}_0 \subset X_{-1}$.

(ii') For all $(m, x) \in \mathfrak{P}_{-1}$, *we have*

$$h_{j,x}\left(X_{-1} \cap D_{-1}(p, \mathfrak{r}_m)\right) = X_{-1} \cap D_{-1}(x, \mathfrak{r}_m)\ .$$

(iv') X_{-1} *is* $(2r_{-1}s_{-1} + 1)$*-separated and* R_{-1}^{+}*-relatively dense in* X.

(v') (X_{-1}, E_{-1}) *is a connected graph.*

(vi') We have $d_{-1} = d = \lambda_{-1}d_{-1} = \Lambda_{-1}d_{-1}$.

(vii') We have $\deg X_{-1} = \Delta_{-1}$.

(viii') For any $(m, x) \in \mathfrak{P}_{-1}$, *the restriction of* $h_{j,x}$ *to* $X_n \cap D_{-1}(p, \mathfrak{r}_m)$ *is an* $(s_0R_0^{+} + \Gamma_0^{+})$*-short scale isometry with respect to* d_{-1}.

This lemma can be considered the extension to $n = -1$ of properties (i), (ii) and (iv)–(viii) of Proposition 2.4.1. In this way, we include the case $n = 0$ in the induction step. Thus suppose that, for $n \geq 0$, we have already defined X_m, E_m, d_m and r_m for $m < n$, satisfying all required properties. When we invoke the induction hypothesis with some item, e.g. (i), it will refer to Lemma 2.4.2 (i') if $n = 0$, and to Proposition 2.4.1 (i) if $n > 0$.

By (2.4.5), we have $\Delta_{n-1} \leq \boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})$. From this inequality, and the definitions of η_n and $\overline{\eta}_n$ in (2.2.9) and (2.4.2), we obtain, for $a \in \mathbb{N}$,

$$\eta_n(a) \geq \overline{\eta}_n(a)\ . \tag{2.4.9}$$

Let $\hat{\mathfrak{c}}_n \colon X_{n-1} \to \{n, n+1, \dots\}$ be defined by

$$\begin{aligned}\hat{\mathfrak{c}}_n(x) = \min\{\, l \in \mathbb{N} \mid l \geq n,\ \exists y \in \mathfrak{X}_l \text{ so that } (l, y) \in \mathfrak{P}_{n-1} \\ \text{and } x \in D_{-1}(y, \mathfrak{r}_l - K_{n-1})\,\}.\end{aligned} \tag{2.4.10}$$

This map is well-defined because $\mathfrak{r}_l \to \infty$ as $l \to \infty$ by (3.2.1) and (3.2.3). By Proposition 2.3.3 (i), for each $x \in X_{n-1}$, there is a unique point $\hat{\mathfrak{p}}_n(x) \in \mathfrak{X}_{\hat{\mathfrak{c}}_n(x)}$ such that $x \in D_{n-1}(\hat{\mathfrak{p}}_n(x), \mathfrak{r}_{\hat{\mathfrak{c}}_n(x)} - K_{n-1})$. This defines a map $\hat{\mathfrak{p}}_n \colon \hat{\mathfrak{c}}_n^{-1}(\{n, n+1, \dots\}) \to \mathfrak{X}_n$.

Lemma 2.4.3. *For $m \geq n$, there are ordered sets $(Y_n^m, \leq_n^m)$ such that the following properties hold:*

(a) Y_n^m is a maximal $2r_n$-separated subset of $(D_{-1}(p, \mathfrak{r}_m - K_{n-1}) \cap X_{n-1}, d_{n-1})$ containing p.

(b) If $m > n$, then $Y_n^{m-1} \subset Y_n^m$, and the map $(Y_n^{m-1}, \leq_n^{m-1}) \hookrightarrow (Y_n^m, \leq_n^m)$ is order-preserving.

(c) For any $(l, z) \in \mathfrak{P}_{n-1}^m$, we have $\mathfrak{h}_{l,z}^m(Y_n^l) = Y_n^m \cap D_{-1}(z, \mathfrak{r}_l - K_{n-1})$, and the map

$$\mathfrak{h}_{l,z}^m \colon \left(Y_n^l, \leq_n^l\right) \to (Y_n^m \cap D_{-1}(z, \mathfrak{r}_l - K_{n-1}), \leq_n^m)$$

is order-preserving.

(d) For all $x, y \in Y_n^m$, we have $x <_n^m y$ if one of the following conditions holds:

(1) $\hat{\mathfrak{c}}_n(x) < \hat{\mathfrak{c}}_n(y)$;

(2) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$ and $d_{-1}(\hat{\mathfrak{p}}_n(x), p) < d_{-1}(\hat{\mathfrak{p}}_n(y), p)$; or

(3) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$, $\hat{\mathfrak{p}}_n(x) = \hat{\mathfrak{p}}_n(y)$ and $d_{-1}(x, \hat{\mathfrak{p}}_n(x)) < d_{-1}(y, \hat{\mathfrak{p}}_n(x))$.

Proof. We proceed by induction on m. Let Y_n^n be any maximal $2r_n$-separated subset of $(D_{-1}(p, \mathfrak{r}_n - K_{n-1}) \cap X_{n-1}, d_{n-1})$ containing p. Let $\leq_n^n$ be any order relation on Y_n^n such that, if $d_{-1}(x, p) < d_{-1}(y, p)$, then $x <_n^n y$. Since $\hat{\mathfrak{c}}_n(x) = n$ and $\hat{\mathfrak{p}}_n(x) = p$ for all $x \in Y_n^n$, this relation satisfies the properties of the statement for $m = n$.

Suppose that we have defined Y_n^l and $\leq_n^l$ for $n \leq l < m$, satisfying the stated properties. Let

$$\widetilde{Y}_n^m = \bigsqcup_{(l,z) \in \overline{\mathfrak{P}}_{n-1}^m} \mathfrak{h}_{l,z}^m(Y_n^l) \ .$$

By the induction hypothesis with (viii), for every $(l, z) \in \overline{\mathfrak{P}}_{n-1}^m$, the set $\mathfrak{h}_{l,z}(Y_n^l) = \mathfrak{h}_{l,z}^m(Y_n^l)$ is contained in X_{n-1} and is $2r_n$-separated with respect to d_{n-1}. Arguing like in the proof of Proposition 2.3.1 (i), we get that $\widetilde{Y}_n^m$ is a maximal $2r_n$-separated subset of

$$\bigsqcup_{(l,z) \in \overline{\mathfrak{P}}_{n-1}^m} D_{-1}(z, \mathfrak{r}_l - K_{n-1}) \ ,$$

with respect to d_{n-1}, containing p. Now, let Y_n^m be any maximal $2r_n$-separated subset of $(D_{-1}(p, \mathfrak{r}_n - K_{n-1}) \cap X_{n-1}, d_{n-1})$ containing $\widetilde{Y}_n^m$; in particular, Y_n^m safisfies (a).

Let $\widetilde{\leq}_n^m$ be any ordering of $\widetilde{Y}_n^m$ satisfying the analogues of (b), (1) and (2) with $\widetilde{Y}_n^m$ instead of Y_n^m. Then, by the induction hypothesis with (3) and the definition of $\widetilde{Y}_n^m$, the order $\widetilde{\leq}_n^m$ also satisfies the analogue of (3). Let $\widehat{\leq}_n^m$ be any ordering of $\widehat{Y}_n^m := Y_n^m \setminus \widetilde{Y}_n^m$ satisfying the analogue of (3) with $\widehat{Y}_n^m$ instead of Y_n^m. Let $\leq_n^m$ be the order relation on Y_n^m defined by $\widetilde{\leq}_n^m$ and $\widehat{\leq}_n^m$ on $\widetilde{Y}_n^m$ and $\widehat{Y}_n^m$, respectively, and satisfying $x \leq_n^m y$ for all $x \in \widetilde{Y}_n^m$ and $y \in \widehat{Y}_n^m$. It is easy to check that $\leq_n^m$ satisfies the stated properties. □

Let $Y_n = \bigcup_{m \geq n} Y_n^m$. Like in the case of the relations $\preceq_n^m$ (Section 2.3), the order relations $\leq_n^m$ define an order relation $\leq_n$ on Y_n.

Lemma 2.4.4. *The ordered sets $(Y_n, \leq_n)$ satisfy the following properties:*

(a) Y_n is a maximal $2r_n$-separated subset of (X_{n-1}, d_{n-1}) containing p, and therefore it is $2r_n$-relatively dense in (X_{n-1}, d_{n-1}).

(b) For any $(l, z) \in \mathfrak{P}_{n-1}$, we have $\mathfrak{h}_{l,z}(Y_n^l) = Y_n \cap D_{-1}(z, \mathfrak{r}_l - K_{n-1})$, and the map

$$\mathfrak{h}_{l,z} \colon (Y_n^l, \leq_n^l) \to (Y_n \cap D_{-1}(z, \mathfrak{r}_l - K_{n-1}), \leq_n)$$

is order-preserving.

(c) For all $x, y \in Y_n$, we have $x <_n y$ if one of the following conditions holds:

(1) $\hat{\mathfrak{c}}_n(x) < \hat{\mathfrak{c}}_n(y)$;

(2) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$ and $d_{-1}(\hat{\mathfrak{p}}_n(x), p) < d_{-1}(\hat{\mathfrak{p}}_n(y), p)$; or

(3) $\hat{\mathfrak{c}}_n(x) = \hat{\mathfrak{c}}_n(y)$, $\hat{\mathfrak{p}}_n(x) = \hat{\mathfrak{p}}_n(y)$ and $d_{-1}(x, \hat{\mathfrak{p}}_n(x)) < d_{-1}(y, \hat{\mathfrak{p}}_n(x))$.

(d) $(Y_n, \leq_n)$ is well-ordered.

Proof. Properties (a)–(c) follow from Lemma 2.4.3 (a)–(c) and the definition of $(Y_n, \leq_n)$. So let us prove (d). By (1), it is enough to prove that, for each $m \geq n$, the ordered subset $(Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m), \leq_n)$ is well-ordered. By (2), the subsets $\{ y \in Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m) \mid d_{-1}(\hat{\mathfrak{p}}(y), p) \leq l \}$, with $l \in \mathbb{N}$, form an increasing sequence of finite initial segments of $(Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m), \leq_n)$ covering $Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m)$. Since

$$\begin{aligned}\{ y \in Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m) \mid d_{-1}(\hat{\mathfrak{p}}(y), p) \leq l \} &\subset \bigcup_{y \in Y_n,\ d_{-1}(y,p) \leq l} D_{-1}(y, \mathfrak{r}_m - K_{n-1}) \\ &\subset D_{-1}(p, l + \mathfrak{r}_m - K_{n-1}) \ ,\end{aligned}$$

all sets $\{ y \in Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m) \mid d_{-1}(\hat{\mathfrak{p}}(y), p) \leq l \}$ are finite, and therefore well-ordered with $\leq_n$. Then it easily follows that $Y_n \cap \hat{\mathfrak{c}}_n^{-1}(m)$ is well-ordered, completing the proof of (d). □

Remark 5. Note that $\{n\} \times \mathfrak{X}_n \subset \mathfrak{P}_{n-1}$ by definition. By Lemma 3.2.15 (a),(b), for any $x \in \mathfrak{X}_n$, we have $x = \mathfrak{h}_{n,x}(p) \subset Y_n$, yielding $\mathfrak{X}_n \subset Y_n$.

Remark 6. For any $x \in D_{-1}(p, \mathfrak{r}_n - K_{n-1})$, we have $\hat{\mathfrak{c}}_n(x) = n$ and $\hat{\mathfrak{p}}_n(x) = p$ by definition. So, by (2), $D_{-1}(p, \mathfrak{r}_n - K_{n-1})$ is an initial segment of Y_n. Therefore p is the least element of Y_n by (3).

Let now

$$\begin{aligned}Y_n^- &= \{ y \in Y_n \mid \eta_n(|D_{n-1}(y, r_n^+)|) < (6 + |D_{n-1}(y, r_n^+ s_n)|)^2 \} \ , \\ Y_n^+ &= \{ y \in Y_n \mid \eta_n(|D_{n-1}(y, r_n^+)|) \geq (6 + |D_{n-1}(y, r_n^+ s_n)|)^2 \} \ .\end{aligned}$$

Lemma 2.4.5. *We have*

$$y \in D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2) \Longrightarrow D_{n-1}(y, r_n^+ s_n) \subset D_{-1}(p, \mathfrak{r}_l - K_{n-1}) \ .$$

Proof. By the induction hypothesis with Proposition 2.4.1 (vi), we have

$$\begin{aligned}d_{-1}(x, p) &\leq d_{-1}(x, y) + d_{-1}(y, p) \leq \Lambda_{n-1} d_{n-1}(x, y) + d_{n-1}(y, p) \\ &\leq \Lambda_{n-1} r_n s_n^2 + \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2 = \mathfrak{r}_l - K_{n-1} \ .\end{aligned}$$

□

Lemma 2.4.6. *For any $(l,z) \in \mathfrak{P}_{n-1}$ and $y \in Y_n \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, we have that $y \in Y_n^{\pm}$ if and only if $\mathfrak{h}_{l,z}(y) \in Y_n^{\pm}$.*

Proof. By Lemma 2.4.5, we have $D_{n-1}(y, r_n s_n^2) \subset D_{-1}(p, \mathfrak{r}_l - K_{n-1}) \subset \operatorname{dom} \mathfrak{h}_{l,z}$. Then $|D_{n-1}(y, r_n s_n^i)| = |D_{n-1}(\mathfrak{h}_{l,z}(y), r_n s_n^i)|$ for $i = 1, 2$ because $\mathfrak{h}_{l,z}$ is a $s_n R_n^+$-short scale isometry on $(D_{-1}(p, \mathfrak{r}_l - K_{n-1}), d_{n-1})$. □

Using that $(Y_n, \leq_n)$ is a well-ordered set (Lemma 3.2.15 (d)), let $X_n^+ \subset Y_n^+$ be the subset inductively defined as follows:

- If y_0 is the least element of $(Y_n^+, \leq_n)$, then $y_0 \in X_n^+$.
- For all $y \in Y_n^+$ such that $y >_n y_0$, we have $y \in X_n^+$ if and only if $d_{n-1}(y, y') > 2 r_n s_n$ for all $y' \in X_n^+$ with $y' <_n y$.

Remark 7. Observe that X_n^+ is $(2 r_n s_n + 1)$-separated and $2 r_n s_n$-relatively dense in (Y_n^+, d_{n-1}).

Remark 8. Note that Lemma 3.2.15 (b) yields $Y_n^l = Y_n \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1})$ because $\mathfrak{h}_{l,p} = \mathrm{id}$ by Proposition 2.3.3 (vi).

Lemma 2.4.7. *For all $z \in \mathfrak{X}_n$ and $y \in Y_n \cap D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, we have $y \in X_n^+$ if and only if $\mathfrak{h}_{n,z}(y) \in X_n^+$.*

Proof. By Lemma 2.4.6, it is enough to prove the statement for points $y \in Y_n^+$. We proceed by induction on the elements of $Y_n^+ \cap D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ using $\leq_n$. Let y_1 be the least element of $Y_n^+ \cap D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. We first show that $y_1, \mathfrak{h}_{n,z}(y_1) \in X_n^+$, establishing the desired property for y_1.

By absurdity, suppose that $y_1 \notin X_n^+$. This means that $y_1 >_n y_0$ and there is some $u \in X_n^+$ such that $u <_n y_1$ and $d_{n-1}(y_1, u) \leq 2 r_n s_n$. Since $s_n > 2$ by (2.2.1) and (2.2.8), it follows from Lemma 2.4.5 that $u \in D_{-1}(p, \mathfrak{r}_n - K_{n-1})$. Then $\hat{\mathfrak{c}}_n(y_1) = \hat{\mathfrak{c}}_n(u) = n$ and $\hat{\mathfrak{p}}_n(y_1) = \hat{\mathfrak{p}}_n(u) = p$. Lemma 3.2.15 (3) and the assumption that $u <_n y_1$ yield $d_{-1}(p, u) \leq d_{-1}(p, y_1)$. So, in fact, $u \in D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, contradicting the hypothesis that y_1 is the least element of $D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. This shows that $y_1 \in X_n^+$.

By Lemma 3.2.15 (b) and Remark 8, the map $\mathfrak{h}_{n,z}$ preserves $\leq_n$ over $D_{-1}(p, \mathfrak{r}_n - K_{n-1})$. So, using the same argument, we get $\mathfrak{h}_{n,z}(y_1) \in X_n^+$.

Now, given $y \in Y_n^+ \cap D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ so that $y_1 <_n y$, suppose that the result is true for all $y' \in Y_n^+ \cap D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ with $y' <_n y$. By definition, we have $y \notin X_n^+$ if and only if there is some $u \in X_n^+$ such that $u <_n y$ and $d_{n-1}(u, p) \leq 2 r_n s_n$. Using the same argument as before, we obtain that, necessarily, $u \in D_{-1}(p, \mathfrak{r}_n - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. By the induction hypothesis, we have $\mathfrak{h}_{n,z}(u) \in X_n^+$. Then $y \notin X_n^+$ if and only if there is some $u \in D_{-1}(\mathfrak{r}_n - K_{n-1})$ with $\mathfrak{h}_{n,z}(u) \in X_n^+$ and $d_{n-1}(\mathfrak{h}_{n,z}(u), \mathfrak{h}_{n,z}(y)) \leq 2 r_n s_n$. But, by the induction hypothesis with (viii), we have $d_{n-1}(\mathfrak{h}_{n,z}(u), \mathfrak{h}_{n,z}(y)) = d_{n-1}(u, y) \leq 2 r_n s_n$. So $y \in X_n^+$ if and only if $\mathfrak{h}_{n,z}(y) \in X_n^+$, as desired. □

Proposition 2.4.8. *For all $(l,z) \in \mathfrak{P}_{n-1}$ and $y \in Y_n \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, we have $y \in X_n^+$ if and only if $\mathfrak{h}_{l,z}(y) \in X_n^+$.*

Proof. We proceed by induction on $l \geq n$. The case $l = n$ is precisely the statement of Lemma 2.4.7. Therefore take any $l > n$ and suppose that the result is true for $n \leq l' < l$.

By Lemma 2.4.6, it is enough to prove the statement for points $y \in Y_n^+$. We proceed by induction on the elements of $Y_n^+ \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ using $\leq_n$. Let y_1 be the least element of $Y_n^+ \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. We will prove that $y_1 \notin X_n^+$ if and only if $\mathfrak{h}_{l,z}(y_1) \notin X_n^+$, establishing the desired property for y_1.

The condition $y_1 \notin X_n^+$ means that $y_1 >_n y_0$ and there is some $u \in X_n^+$ such that $u <_n y_1$ and $d_{n-1}(y_1, u) \leq 2 r_n s_n$. Since $s_n > 2$ by (2.2.1) and (2.2.8), it follows from Lemma 2.4.5 that $u \in D_{-1}(p, \mathfrak{r}_l - K_{n-1})$, and therefore $\hat{\mathfrak{c}}(y_1), \hat{\mathfrak{c}}(u) \leq l$. We will consider several cases about u.

Suppose that $\hat{\mathfrak{c}}_n(u) > \hat{\mathfrak{c}}_n(y_1)$. Then $y_1 <_n u$ by Lemma 3.2.15 (1), contradicting the assumption that $u <_n y_1$.

Suppose then that $\hat{\mathfrak{c}}(y_1) = \hat{\mathfrak{c}}(u) = l$. Thus $\hat{\mathfrak{p}}(y_1) = \hat{\mathfrak{p}}(u) = p$. Lemma 3.2.15 (3) and the assumption that $u <_n y_1$ yield $d_{-1}(p, u) \leq d_{-1}(p, y_1)$. Therefore $u \in Y_n^+ \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, contradicting the hypothesis that y_1 is the least element in $Y_n^+ \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$.

Suppose finally that $\hat{\mathfrak{c}}(u) < l$. Then $\mathfrak{h}_{\hat{\mathfrak{c}}(u), \hat{\mathfrak{p}}(u)}(u) \in X_n^+$ by the induction hypothesis with l. But, by the induction hypothesis with (viii), we have $d_{n-1}(\mathfrak{h}_{l,z}(u), \mathfrak{h}_{l,z}(y_1)) = d_{n-1}(u, y_1) \leq 2 r_n s_n$. So $\mathfrak{h}_{l,z}(y_1) \notin X_n^+$.

Thus far, we have proved that $y_1 \notin X_\subset D_{-1}(n^+$ implies $\mathfrak{h}_{l,z}(y_1) \notin X_n^+$. The proof of the converse implication is similar

Now, given $y \in Y_n^+ \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ so that $y_1 <_n y$, suppose that the result is true for all $y' \in Y_n^+ \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$ with $y' <_n y$. By definition, $y \notin X_n^+$ if and only if there is some $u \in X_n^+$ such that $u <_n y$ and $d_{n-1}(u, p) \leq 2 r_n s_n$. Using the same argument as before, we obtain that, either $\hat{\mathfrak{c}}_n(u) < l$, or $u \in D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. If $\hat{\mathfrak{c}}_n(u) < l$, we get $\mathfrak{h}_{l,z}(y) \notin X_n^+$ arguing as before. If $u \in D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$, then $\mathfrak{h}_{l,z}(u) \in X_n^+$ by the induction hypothesis in $Y_n^+ \cap D_{-1}(p, \mathfrak{r}_l - K_{n-1} - \Lambda_{n-1} r_n s_n^2)$. Thus $y \notin X_n^+$ if and only if there is some $u \in D_{-1}(\mathfrak{r}_l - K_{n-1})$ with $\mathfrak{h}_{l,z}(u) \in X_n^+$ and $d_{n-1}(\mathfrak{h}_{l,z}(u), \mathfrak{h}_{l,z}(y)) \leq 2 r_n s_n$. But $d_{n-1}(\mathfrak{h}_{l,z}(u), \mathfrak{h}_{l,z}(y)) = d_{n-1}(u, y) \leq 2 r_n s_n$ by the induction hypothesis with (viii). So $y \in X_n^+$ if and only if $\mathfrak{h}_{l,z}(y) \in X_n^+$, as desired. □

Let

$$X_n^- = \{\, y \in Y_n^- \mid d_{n-1}(y, X_n^+) > r_n(2s_n + 1) \,\} . \tag{2.4.11}$$

Lemma 2.4.9. *We have $p \in X_n$.*

Proof. Suppose first that condition (A) is satisfied in the definition of r_n, and consequently $r_n = \bar{r}_n$. Then there is some point $x \in D_{n-1}(p, \hat{r}_n(2s_n + 1))$ such that

$$(|D_{n-1}(x, \hat{r}_n s_n)| + 6)^2 \geq \eta_n(|D_{n-1}(x, \hat{r}_n)|) . \tag{2.4.12}$$

So $D_{n-1}(x, \hat{r}_n s_n) \subset D_{n-1}(p, \hat{r}_n(3s_n + 1))$, and therefore

$$|D_{n-1}(p, r_n)| = |D_{n-1}(p, \hat{r}_n(3s_n + 1))| \geq |D_{n-1}(x, \hat{r}_n s_n)| . \tag{2.4.13}$$

Using (2.2.11), (2.2.12), (2.4.9), (2.4.12) and (2.4.13), we get

$$\begin{aligned}\eta_n(|D_{n-1}(p,r_ns_n)|) &\geq \eta_n(|D_{n-1}(x,\hat{r}_ns_n)|) \geq \eta_n\big(\sqrt{\eta_n(|D_{n-1}(x,\hat{r}_n)|)}-6\big)\\ &\geq \overline{\eta}_n\big(\sqrt{\overline{\eta}_n(|D_{n-1}(x,\hat{r}_n)|)}-6\big) > \overline{\eta}_n\big(\sqrt{\overline{\eta}_n(\hat{r}_n)}-6\big)\\ &\geq \Big(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})-1)^{\bar{r}_ns_n^2}+6\Big)^2 .\end{aligned}$$

The assumption $r_n=\bar{r}_n$ implies $\bar{\boldsymbol{r}}_{n-1}=(r_0,\dots,r_{n-1})$ and $\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})=\Delta_{n-1}$ according to (2.4.4). Hence, by Corollary 2.1.19,

$$\begin{aligned}\eta_n(|D_{n-1}(p,r_ns_n)|) &\geq \Big(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1})-1)^{\bar{r}_ns_n^2}+6\Big)^2\\ &= \Big(4(\Delta_{n-1}-1)^{r_ns_n^2}+6\Big)^2 \geq (|D_{n-1}(p,r_ns_n^2)|+6)^2 ,\end{aligned}$$

and therefore $p\in Y_n^+$. Thus the statement follows in this case from Remark 6 and the definition of X_n^+.

Suppose now that condition (B) holds. Then $p\in Y_n^-$ and $Y_n^+\cap D_{n-1}(p,r_n(2s_n+1))=\emptyset$, and the statement also follows in this second case. □

By (2.2.15), (2.2.16) and (2.4.4), we have

$$\overline{K}_n = K_{n-1}+\Lambda_n(r_ns_n^2+r_n(2s_n+1)) , \tag{2.4.14}$$

$$K_n = \overline{K}_n+\Lambda_n(s_{n+1}R_{n+1}^+ +\Gamma_n^+ +2R_n^+) . \tag{2.4.15}$$

Lemma 2.4.10. *For all $(l,z)\in\mathfrak{P}_{n-1}$ and $y\in Y_n\cap D_{-1}(p,\mathfrak{r}_l-\overline{K}_n)$, we have $y\in X_n^-$ if and only if $\mathfrak{h}_{l,z}(y)\in X_n^-$.*

Proof. Let $y\in Y_n\cap D_{-1}(p,\mathfrak{r}_l-\overline{K}_n)$. Then, by (2.4.14),

$$y\in Y_n\cap D_{-1}(p,\mathfrak{r}_l-K_{n-1}-\Lambda_{n-1}(r_ns_n^2+r_n(2s_n+1))) .$$

By Lemma 2.4.6, we can assume $y,\mathfrak{h}_{l,z}(y)\in Y_n^-$. Hence, by definition, $y\notin X_n^-$ if and only if there is some $x\in X_n^+$ with $d_{n-1}(y,x)\leq r_n(2s_n+1)$. In this case, by the induction hypothesis with (vi), we have $d_{-1}(x,y)\leq\Lambda_{n-1}r_n(2s_n+1)$. Therefore, by the triangle inequality, $x\in D_{-1}(p,\mathfrak{r}_l-K_{n-1}-\Lambda_{n-1}r_ns_n^2)\subset D_{-1}(p,\mathfrak{r}_m-K_n)$. Applying now Proposition 2.4.8, we get $\mathfrak{h}_{l,z}(x)\in X_n^+$. Also, by the induction hypothesis with (viii), $\mathfrak{h}_{l,z}$ is a $s_nR_n^+$-short scale isometry on $(X_{n-1}\cap D_{-1}(p,\mathfrak{r}_m-K_n),d_{n-1})$. Therefore $\mathfrak{h}_{l,z}(x)\in X_n^+$ and $d_{n-1}(\mathfrak{h}_{l,z}(x),\mathfrak{h}_{l,z}(y))\leq r_n(2s_n+1)$, obtaining $\mathfrak{h}_{l,z}(y)\notin X_n^-$.

The proof of the converse implication is similar. □

Let us prove (i). By Lemma 2.4.9, we have $p\in X_n$ and $(n,x)\in\mathfrak{P}_{n-1}$ for each $x\in\mathfrak{X}_{n,}$. Proposition 2.4.8 and Lemma 2.4.10 then imply $x=\mathfrak{h}_{n,x}(p)\in X_n$ for all $x\in\mathfrak{X}_n$, obtaining $\mathfrak{X}_n\subset X_n$. The inclusion $X_n\subset X_{n-1}$ follows from Lemma 3.2.15 (a) and the fact that $X_n\subset Y_n$. This completes the proof of (i).

For all $(m,x)\in\mathfrak{P}_{n-1}$, the map $\mathfrak{h}_{m,x}\colon(D_{-1}(p,\mathfrak{r}_m),p)\to(D_{-1}(x,\mathfrak{r}_m),x)$ is a pointed isometry by definition. Therefore $\mathfrak{h}_{m,x}(D_{-1}(p,\mathfrak{r}_m-\overline{K}_n))=D_{-1}(x,\mathfrak{r}_m-\overline{K}_n)$. Then property (ii) follows from Proposition 2.4.8 and Lemma 2.4.10.

Let us prove (iii). For $x \in X_n^+$, the result is an immediate consequence of the definition of Y_n^+ and the fact that $X_n^+ \subset Y_n^+$. So assume $x \in X_n^-$. By absurdity, suppose that

$$(|D_{n-1}(x, r_n s_n)| + 6)^2 > \eta_n(|D_{n-1}(x, r_n)|) .$$

Since η_n is an increasing function, and using (2.4.9), (2.2.12), (2.4.4) and Corollary 2.1.19, we get

$$\begin{aligned} \eta_n(|D_{n-1}(x, r_n s_n)|) &\geq \eta_n\big(\sqrt{\eta_n(|D_{n-1}(x, r_n)|)} - 6\big) \geq \overline{\eta}_n\big(\sqrt{\overline{\eta}_n(|D_{n-1}(x, r_n)|)} - 6\big) \\ &> \overline{\eta}_n\big(\sqrt{\overline{\eta}_n(r_n)} - 6\big) > \Big(4(\boldsymbol{\Delta}_{n-1}(\bar{\boldsymbol{r}}_{n-1}) - 1)^{\bar{r}_n s_n^2} + 6\Big)^2 \\ &= \Big(4(\Delta_{n-1} - 1)^{\bar{r}_n s_n^2} + 6\Big)^2 \geq (|D_{n-1}(x, r_n s_n^2)| + 6)^2 . \end{aligned}$$

Therefore $x \notin Y_n^-$ by definition, contradicting the assumption that $x \in X_n^-$, which completes the proof of (iii).

Let us prove (iv). First, define

$$\mathcal{Z}_{n-1}^- = \{ z \in X_{n-1} \mid d_{n-1}(z, X_n^+) - 2r_n s_n > d_{n-1}(z, X_n^-) - r_n \} , \tag{2.4.16}$$

$$Z_{n-1}^+ = \{ z \in X_{n-1} \mid d_{n-1}(z, X_n^+) - 2r_n s_n \leq d_{n-1}(z, X_n^-) - r_n \} . \tag{2.4.17}$$

Thus $X_{n-1} = \mathcal{Z}_{n-1}^- \cup Z_{n-1}^+$. On the other hand, using (2.2.7), (2.2.13), (2.4.1) and (2.4.4), we get

$$R_n^- = 4r_n - 1 , \quad R_n^+ = r_n(2s_n + 3) .$$

Lemma 2.4.11. *X_n^+ is $(2r_n s_n + 1)$-separated and R_n^+-relatively dense in (Z_{n-1}^+, d_{n-1}).*

Proof. By Remark 7, we only need to show that X_n^+ is R_n^+-relatively dense in (Z_{n-1}^+, d_{n-1}). Take an arbitrary point $z \in Z_{n-1}^+$. Since Y_n is $2r_n$-relatively dense in (X_{n-1}, d_{n-1}) by Lemma 3.2.15 (a), there is some $y \in Y_n$ with $d_{n-1}(x, z) \leq 2r_n$.

If $y \in Y_n^+$, then, by Remark 7, there is some $x \in X_n^+$ with $d_{n-1}(y, x) \leq 2r_n s_n$. Using the triangle inequality, we get

$$d(z, x) \leq d(z, y) + d(y, x) \leq 2r_n + 2r_n s_n < r_n(2s_n + 3) = R_n^+ .$$

If $y \in X_n^-$, we have $d_{n-1}(z, X_n^-) \leq 2r_n$. Then (2.4.17) implies $d_{n-1}(z, X_n^+) - 2r_n s_n \leq r_n$, obtaining $d_{n-1}(z, X_n^+) \leq r_n(2s_n + 1) < R_n^+$.

Finally, suppose that $y \in Y_n^- \setminus X_n^-$. By (2.4.11), there is some point $x \in X_n^+$ with $d_{n-1}(x, y) \leq r_n(2s_n + 1)$, and the lemma follows applying the triangle inequality:

$$d(z, x) \leq d(z, y) + d(y, x) \leq 2r_n + r_n(2s_n + 1) = r_n(2s_n + 3) = R_n^+ . \qquad \square$$

Lemma 2.4.12. *X_n^- is $(2r_n s_n + 1)$-separated and R_n^--relatively dense in $(\mathcal{Z}_{n-1}^-, d_{n-1})$.*

Proof. Let $z \in \mathcal{Z}_{n-1}^-$. Like in Lemma 2.4.11, there is some $y \in Y_n$ with $d_{n-1}(z, y) \leq 2r_n$.

In the case where $y \in X_n^-$, the lemma is trivial.

If $y \in X_n^+$, then $d_{n-1}(z, X_n^+) \leq 2r_n$, yielding $d_{n-1}(z, X_n^+) - 2r_n s_n \leq 2r_n(1 - s_n)$. Using (2.4.16), we get $d_{n-1}(y, X_n^-) - r_n < 2r_n(1 - s_n)$, and therefore $d_{n-1}(y, X_n^-) < 2r_n(2 - s_n)$. However, by (2.2.1) and (2.2.8), we have $s_n > 2$, reaching a contradiction. Therefore $y \notin X_n^+$.

Now, suppose $y \in Y_n^+ \setminus X_n^+$. By Remark 7, there is some $x \in X_n^+$ with $d_{n-1}(x,y) \leq 2r_n s_n$, and we get $d_{n-1}(z,x) \leq 2r_n(s_n+1)$ using the triangle inequality. Then (2.4.16) yields

$$\begin{aligned} d_{n-1}(z,X_n^-) &< d_{n-1}(z,X_n^+) - 2r_n s_n + r_n \leq d_{n-1}(z,x) - 2r_n s_n + r_n \\ &\leq 2r_n(s_n+1) - 2r_n s_n + r_n = 3r_n \leq R_n^- \ . \end{aligned}$$

Finally, suppose $y \in Y_n^- \setminus X_n^-$. By (2.4.11), there is some point $x \in X_n^+$ with $d_{n-1}(x,y) \leq r_n(2s_n+1)$, obtaining $d_{n-1}(z,X_n^+) \leq r_n(2s_n+3)$ by the triangle inequality. Therefore $d_{n-1}(z,X_n^+) - 2r_n s_n \leq 3r_n$, obtaining $d_{n-1}(z,X_n^-) < 4r_n$ by (2.4.16); i.e., $d_{n-1}(z,X_n^-) \leq 4r_n - 1 = R_n^-$. □

To finish the proof of Proposition 2.4.1 (iv), it remains to show that $d_{n-1}(X_n^-,X_n^+) \geq 2r_n s_n + 1$, which follows from (2.4.11).

To prove the next items of Proposition 2.4.1, we need some more preliminary results.

Lemma 2.4.13. *For all $z \in X_{n-1}$, we have $z \in Z_{n-1}^+$ if and only if*

$$d_{n-1}(z, X_n^+ \cap D_{n-1}(z,R_n^+)) - 2r_n s_n \leq d_{n-1}(z, X_n^- \cap D_{n-1}(z,R_n^+)) - r_n \ . \tag{2.4.18}$$

Proof. Suppose first that $z \in Z_{n-1}^+$. Lemma 2.4.11 implies $X_n^+ \cap D_{n-1}(z,R_n^+) \neq \emptyset$, and therefore

$$d_{n-1}(z, X_n^+ \cap D_{n-1}(z,R_n^+)) = d_{n-1}(z,X_n^+) \ .$$

Then (2.4.16) implies (2.4.18).

Suppose now that (2.4.18) holds for some $z \in X_{n-1}$. Property (iv) implies that at least one of the inequalities $d_{n-1}(z,X_n^-) \leq R_n^+$ or $d_{n-1}(z,X_n^+) \leq R_n^+$ is satisfied. So at least the left-hand side of (2.4.18) is finite. Therefore (2.4.18) yields (2.4.16). □

Corollary 2.4.14. *For all $u \in X_{n-1} \cap D_{-1}(p, \mathfrak{r}_l - \overline{K}_n - \Lambda_{n-1}R_n^+)$ and $(l,z) \in \mathfrak{P}_{n-1}$, we have $u \in \mathcal{Z}_{n-1}^\pm$ if and only if $\mathfrak{h}_{l,z}(u) \in \mathcal{Z}_{n-1}^\pm$.*

Proof. Let $u \in X_{n-1} \cap D_{-1}(p, \mathfrak{r}_l - \overline{K}_n - \Lambda_{n-1}R_n^+)$ and $(l,z) \in \mathfrak{P}_{n-1}$. Since $X_{n-1} = Z_{n-1}^- \cup Z_{n-1}^+$, it is enough to prove that $u \in Z_{n-1}^+$ if and only if $\mathfrak{h}_{l,z}(u) \in Z_{n-1}^+$.

The induction hypothesis with (vi) and the triangle inequality yield $D_{n-1}(u,R_n^+) \subset D_{-1}(p,\mathfrak{r}_l - \overline{K}_n) \subset \operatorname{dom} \mathfrak{h}_{l,z}$. Proposition 2.4.8, Lemma 2.4.10 and the induction hypothesis with (viii) imply that the restriction of $\mathfrak{h}_{l,z}$ to $D_{-1}(p,\mathfrak{r}_l - \overline{K}_n)$ preserves $X_n^\pm$ and is an R_n^+-partial isometry with respect to d_{n-1}. Then the result follows from Lemma 2.4.13. □

Remark 9. Note that (2.4.14) yields $K_n \geq \overline{K}_n + \Lambda_{n-1}R_n^+$. Then $\mathfrak{r}_l - \overline{K}_n - \Lambda_{n-1}R_n^+ > 0$ in Corollary 2.4.14 by (3.2.1).

Recall the definition of $r_n^\pm$ given in (2.4.7).

Lemma 2.4.15. *If $x \in X_n^\pm$, then $D_{n-1}(x,r_n^\pm) \subset Z_{n-1}^\pm$.*

Proof. For $x \in X_n^-$, suppose on the contrary that there is some $z \in D_{n-1}(x,r_n)$ such that

$$d_{n-1}(z,X_n^+) - 2r_n s_n \leq d_{n-1}(z,X_n^-) - r_n \ .$$

In particular, $d_{n-1}(z, X_n^+) \leq 2r_ns_n$ because $d_{n-1}(z, X_n^-) \leq d_{n-1}(z, x) \leq r_n$. By the triangle inequality, it follows that

$$d_{n-1}(x, X_n^+) \leq d_{n-1}(x, z) + d_{n-1}(z, X_n^+) \leq r_n + 2r_ns_n = r_n(2s_n + 1) ,$$

contradicting the definition of X_n^- in (2.4.11).

The proof when $x \in X_n^+$ is similar. □

For every $x \in X_n^\pm$, let

$$\overline{C}_{n,n-1}(x) = \{ z \in Z_{n-1}^\pm \mid d_{n-1}(z, x) = d_{n-1}(z, X_n^\pm) \} . \tag{2.4.19}$$

Remark 10. Observe that the sets $\overline{C}_{n,n-1}(x)$, for $x \in X_n$, cover X_{n-1}.

Lemma 2.4.16. *For $x \in X_n^\pm$, we have $\overline{C}_{n,n-1}(x) \subset D_{n-1}(x, R_n^\pm)$.*

Proof. This is a direct consequence of Lemmas 2.4.11 and 2.4.12. □

Define a graph structure E_n on X_n by declaring that $x, y \in X_n$ are joined by an edge if

$$d_{n-1}(\overline{C}_{n,n-1}(x), \overline{C}_{n,n-1}(y)) \leq 1 . \tag{2.4.20}$$

To prove (v), let $x, y \in X_n$. By the induction hypothesis with (v), X_{n-1} is connected, and, by construction, $X_n \subset X_{n-1}$. So there is some path in (X_{n-1}, E_{n-1}) of the form $(u_0 = x, u_1, \dots, u_a = y)$. By Remark 10, for each $i = 0, \dots, a$, there is some $z_i \in X_n$ such that $u_i \in \overline{C}_{n,n-1}(z_i)$, $z_0 = x$ and $z_a = y$. Clearly, $d_{n-1}(\overline{C}_{n,n-1}(z_{i-1}), \overline{C}_{n,n-1}(z_i)) \leq 1$ for $i = 1, \dots, a$. Thus $(z_0, \dots, z_a)$ is a path in X_n connecting x to y.

Let us prove (vi). For $x, y \in X_n$ with $d_n(x, y) = a$, there is a finite sequence $(x_0 = x, x_1, \dots, x_a = y)$ in X_n such that $d_n(\overline{C}_{n,n-1}(x_{i-1}), \overline{C}_{n,n-1}(x_i)) \leq 1$ for $i = 1, \dots, a$. By Lemma 2.4.16, (2.2.7) and (2.4.4), we have $d_{n-1}(x_{i-1}, x_i) \leq 2R_n^+ + 1 = \lambda_n$. Then (vi) follows from the triangle inequality, using (2.2.13), (2.4.1) and (2.4.4).

Let us prove (vii). For $x, y \in X_n$, if xE_ny, then $d_{n-1}(x, y) \leq 2R_n^+ + 1$ by (2.4.20) and Lemma 2.4.16. So

$$|S_n(x, 1)| \leq |D_{n-1}(x, 2R_n^+ + 1)| \leq 4(\deg X_{n-1} - 1)^{2R_n^+}$$

by Corollary 2.1.19. Then the bound $\deg X_n \leq \Delta_n$ follows by induction with (vii), using (2.2.7), (2.2.13) and (2.4.4).

Let us prove (viii). Let $(m, z) \in \mathfrak{P}_{n-1}$ and $x \in X_n \cap D_{-1}(p, \mathfrak{r}_m - \overline{K}_n - 2\Lambda_nR_n^+)$. Then

$$\overline{C}_{n,n-1}(x) \subset D_{-1}(p, \mathfrak{r}_m - \overline{K}_n - \Lambda_nR_n^+) \subset \operatorname{dom} \mathfrak{h}_{m,z} \tag{2.4.21}$$

by Lemma 2.4.16, Proposition 2.3.3 (v), and the induction hypothesis with (vi) and (viii). Recall that $\mathfrak{P}_{n-1} \subset \mathfrak{P}_{n-2}$ by (2.3.4) and (2.3.5). Furthermore, from the induction hypothesis with (viii), Proposition 2.3.3 (v), Corollary 2.4.14, (2.4.19) and (2.4.21), it follows that

$$\mathfrak{h}_{m,z}\left(\overline{C}_{n,n-1}(x)\right) = \overline{C}_{n,n-1}\left(\mathfrak{h}_{m,z}(x)\right) . \tag{2.4.22}$$

So, for $x, y \in X_n \cap D_{-1}(p, \mathfrak{r}_m - \overline{K}_n - 2\Lambda_nR_n^+)$, (2.4.20) holds if and only if

$$d_{n-1}(\overline{C}_{n,n-1}(\mathfrak{h}_{m,z}(x)), \overline{C}_{n,n-1}(\mathfrak{h}_{m,z}(y))) \leq 1 .$$

Therefore xE_ny if and only if $\mathfrak{h}_{m,z}(x)E_n\mathfrak{h}_{m,z}(y)$. Then (viii) is a consequence of Corollary 2.1.12, (2.4.15) and the induction hypothesis with (vi). This completes the proof of Proposition 2.4.1.

2.5 Clusters

In order to define the colorings satisfying the conditions of Theorem (2.1.26), we will divide the sets X_{n-1} into "clusters", denoted by $C_{n,n-1}(x)$ and indexed by $x \in X_n$. These will be used in Section 2.6 to construct the suitable colorings locally on this family of sets.

In Section 2.4, we have defined well-ordered sets $(Y_n, \leq_n)$ for $n \in \mathbb{N}$, whose restrictions to the subsets X_n determine a family of well-orders $\leq_n$. For $n \in \mathbb{N}$, let $\pi_n^\pm \colon \mathcal{Z}_{n-1}^\pm \to X_n^\pm$ be defined by

$$\pi_{n-1}^\pm(u) = \inf\{\, x \in X_n^\pm \mid d_{n-1}(u,x) = d_{n-1}(u, X_n^\pm) \,\} \;, \tag{2.5.1}$$

with respect to $\leq_n$. For each $n \in \mathbb{N}$ and $x \in X_n^\pm$, let $C_{n,n-1}(x) = (\pi_n^\pm)^{-1}(x)$. These sets form a partition of X_{n-1}, and satisfy

$$C_{n,n-1}(x) = \overline{C}_{n,n-1}(x) \setminus \bigcup_{x' \in X_n^\pm,\ x' <_n x} \overline{C}_{n,n-1}(x') \tag{2.5.2}$$

for $x \in X_n^\pm$, by (2.4.19) and (2.5.1). For $-1 \leq m < n-1$, we continue defining sets $C_{n,m}(x)$ and $\overline{C}_{n,m}(x)$ by reverse induction on m, taking

$$C_{n,m}(x) = \bigcup_{u \in C_{n,m+1}(x)} C_{m+1,m}(u) \;, \quad \overline{C}_{n,m}(x) = \bigcup_{u \in \overline{C}_{n,m+1}(x)} \overline{C}_{m+1,m}(u) \;.$$

It is straightforward to check that, for $-1 \leq l_1 < l_2 < l_3 \leq n$,

$$C_{l_3,l_1}(x) = \bigcup_{u \in C_{l_3,l_2}(x)} C_{l_2,l_1}(u) \;, \quad \overline{C}_{l_3,l_1}(x) = \bigcup_{u \in \overline{C}_{l_3,l_2}(x)} \overline{C}_{l_2,l_1}(u) \;. \tag{2.5.3}$$

By (2.2.13) and (2.4.4), we have

$$\Gamma_0^\pm = R_0^\pm \;, \quad \Gamma_n^\pm = R_n^\pm \Lambda_{n-1} + \Gamma_{n-1}^\pm \;. \tag{2.5.4}$$

Lemma 2.5.1. $C_{n,-1}(x) \subset \overline{C}_{n,-1}(x) \subset D_{-1}(x, \Gamma_n^\pm)$.

Proof. We proceed by induction on n. For $n = 0$ and $x \in X_0^\pm$, we have $C_{0,-1}(x) \subset D_{-1}(x, R_0^\pm)$ by Lemma 2.4.16 and (2.5.2). Now take any $n > 0$ and suppose that $C_{m,-1}(y) \subset \overline{C}_{m,-1}(y) \subset D_{-1}(y, \Gamma_m^\pm)$ for $0 \leq m < n$ and $y \in X_m^\pm$. By (2.5.3),

$$C_{n,-1}(x) = \bigcup_{u \in C_{n,n-1}(x)} C_{n,n-1}(u) \;, \quad \overline{C}_{n,-1}(x) = \bigcup_{u \in \overline{C}_{n,n-1}(x)} \overline{C}_{n,n-1}(u) \;.$$

We get $d_{n-1}(x,u) \leq R_n^+$ for all $u \in \overline{C}_{n,n-1}(x)$ by Lemma 2.4.16 and (2.5.2). So $d_{-1}(x,u) \leq \Lambda_{n-1} R_n^+$ by Proposition 2.4.1 (vii). Then the result follows easily from the induction hypothesis using the triangle inequality. □

Lemma 2.5.2. *For every $n \in \mathbb{N}$ and $x \in X_n^\pm$, we have $D_{n-1}(x, r_n^\pm) \subset C_{n,n-1}(x)$.*

Proof. For $u \in D_{n-1}(x, r_n^\pm)$, we have $u \in Z_{n-1}^\pm$ by Lemma 2.4.15, and $d_{n-1}(u, X_n) \leq r_n^\pm$ by definition. Then the result follows from (2.5.1) and the fact that $X_n^\pm$ is $(2r_n^+ + 1)$-separated by Proposition 2.4.1 (iv). □

The following result follows from Lemma 2.5.2 by induction.

Corollary 2.5.3. *For every $n \in \mathbb{N}$ and $x \in X_n$, we have $D_{-1}(x, \sum_{i=0}^{n} r_i) \subset C_{n,n-1}(x)$.*

The following lemma states that every $C_{n,n-1}(x)$ is a star-shaped subset of (X_{n-1}, E_{n-1}) with center x.

Lemma 2.5.4. *For $x \in X_n^{\pm}$ and $u \in C_{n,n-1}(x)$, any geodesic segment in (X_{n-1}, E_{n-1}) of the form $\tau = (x = \tau_0, \ldots, \tau_l = u)$ is a path in $C_{n,n-1}(x)$.*

Proof. We prove that $\tau_k \in C_{n,n-1}(x)$ by reverse induction on $k = 0, \ldots, l$. We have $\tau_l = u \in C_{n,n-1}(x)$ by hypothesis. Now, suppose that $\tau_{k+1} \in C_{n,n-1}(x)$ for some $k = 0, \ldots, l-1$. Assume by absurdity that $\tau_k \notin C_{n,n-1}(x)$. Since τ is a geodesic segment,

$$\begin{aligned} d_{n-1}(\tau_k, X_n^{\pm}) &\le d_{n-1}(\tau_k, x) = d_{n-1}(\tau_{k+1}, x) - 1 \\ &= d_{n-1}(\tau_{k+1}, X_n^{\pm}) - 1 \le d_{n-1}(\tau_k, X_n^{\pm}) \ , \end{aligned}$$

and therefore $\tau_k \in \overline{C}_{n,n-1}(x)$. So, according to (2.5.1), there must be some $y \in X_n^{\pm}$ such that $d_{n-1}(\tau_k, y) = d_{n-1}(\tau_k, x) = k$ and $y <_n x$. But then $d_{n-1}(\tau_{k+1}, y) \le k+1 = d_{n-1}(\tau_{k+1}, x)$, yielding $\tau_{k+1} \notin C_{n,n-1}(x)$ by (2.5.1), a contradiction. □

Lemma 2.5.5. *Let $x \in X_n \cap D_{-1}(p, \mathfrak{r}_m - K_{n-1} - 2\Lambda_{n-1} R_n^{+})$ and $(m, z) \in \mathfrak{P}_{n-1}$. Then $C_{n,n-1}(x) \subset \operatorname{dom} \mathfrak{h}_{m,z}$ and $\mathfrak{h}_{m,z}(C_{n,n-1}(x)) = C_{n,n-1}(\mathfrak{h}_{m,z}(x))$.*

Proof. It is an immediate consequence of (2.4.21), (2.4.22), (2.5.2) and Lemma 3.2.15 (b). □

2.6 Colorings

2.6.1 Colorings χ_n

Given $a \in \mathbb{N}$, let $[a] = \{0, \ldots, a-1\}$. For $n \in \mathbb{N}$ and $x \in X_n^{\pm}$, let

$$H_{n,x} = \left[\eta_n \left(\left|D_{n-1}\left(x, r_n^{\pm}\right)\right|\right)\right] \ , \quad I_{n,x} = \left[5 + \left|D_{n-1}\left(x, r_n^{\pm} s_n\right)\right|\right] \ . \tag{2.6.1}$$

The standard ordering of $\mathbb{N}$ and the calligraphic ordering of $I_{n,x}^2$ can be used to realize $I_{n,x}^2$ as an initial segment of $\mathbb{N}$. Since $|I_{n,x}|^2 \le |H_{n,x}|$ by Proposition 2.4.1 (iii), the sets $I_{n,x}$ and $I_{n,x}^2$ become initial segments of $H_{n,x}$. For $n \in \mathbb{N}$, let

$$\mathcal{H}_n = \bigcup_{x \in X_n} H_{n,x} \ , \quad \mathcal{I}_n = \bigcup_{x \in X_n} I_{n,x} \ . \tag{2.6.2}$$

From now on, when referring to a coloring $\phi \colon X_n \to \mathcal{H}_n$ (respectively, $\phi \colon X_n \to \mathcal{I}_n$), we assume $\phi(x) \in H_{n,x}$ (respectively, $\phi(x) \in I_{n,x}$) for all $x \in X_n$.

Proposition 2.6.1. *For every $n \in \mathbb{N}$, there is a coloring $\chi_n \colon X_n \to \mathcal{I}_n$ satisfying the following conditions:*

(i) We have $\chi_n(x) = 0$ if and only if $x \in \mathfrak{X}_n$.

(ii) *For all* $x, y \in X_n^\pm$ *with* $d_{n-1}(x,y) \leq r_n^\pm s_n$*, we have* $\chi_n(x) < \chi_n(y)$ *if and only if* $x <_n y$*. In particular, if* $0 < d_{n-1}(x,y) \leq r_n^\pm s_n$*, then* $\chi_n(x) \neq \chi_n(y)$*.*

(iii) *For every* $(m,z) \in \mathfrak{P}_{n-1}$*, the map* $\mathfrak{h}_{m,z} \colon (D_n(p, \Gamma_m^+), \chi_n) \to (D_n(z, \Gamma_m^+), \chi_n)$ *is color-preserving.*

Proof. First, set $\chi_n(x) = 0$ for all $x \in \mathfrak{X}_n$. Then we define $\chi_n(x)$ for $x \in X_n^\pm \setminus \mathfrak{X}_n$ by induction using $\leq_n$. Let $A_x^\pm := \{\, y \in X_n^\pm \mid y <_n x \,\}$, and let

$$\chi_n(x) = \min \left(I_{n,x} \setminus (\{0\} \cup \chi_n(A_x^\pm \cap D_{n-1}(x, r_n^\pm s_n))) \right) . \tag{2.6.3}$$

Note that this is well defined since

$$|A_x^\pm \cap D_{n-1}(x, r_n^\pm s_n)| \leq |D_{n-1}(x, r_n^\pm s_n)| - 1 \leq |I_{n,x}| - 1 .$$

With this definition, it is obvious that χ_n satisfies (i) and (ii).

To prove (iii), we show by induction on $(X_n \setminus \mathfrak{X}_n, \leq_n)$ that, if $x \subset D_n(z, \Gamma_m^+)$ for $(m,z) \in \mathfrak{P}_{n-1}$, then $\chi_n(x) = \chi_n(\mathfrak{h}_{m,z}^{-1}(x))$. By Remark 6, the set $X_n \cap D_{-1}(p, \mathfrak{r}_n - K_{n-1})$ is an initial segment of $(X_n, \leq_n)$. For $x \in X_n \cap D_{-1}(p, \mathfrak{r}_n - K_{n-1})$, the result is trivial since $\mathfrak{h}_{m,p}$ is the identity. Suppose $x \in X_n \cap D_n(z, \Gamma_m^+)$ for some $(m,z) \in \mathfrak{P}_{n-1}$ with $z \neq p$. By (3.2.1) and (2.4.4), we have $D_{n-1}(x, r_n^\pm s_n) \subset D_{-1}(z, \mathfrak{r}_n - K_{n-1})$. Thus

$$\mathfrak{h}_{m,z} \colon (D_{n-1}(\mathfrak{h}_{m,z}^{-1}(x), r_n^\pm s_n), \leq_n) \to (D_{n-1}(x, r_n^\pm s_n), \leq_n) \tag{2.6.4}$$

is order-preserving and an $r_n^\pm s_n$-short scale isometry with respect to d_{n-1} by Proposition 2.4.1 (viii) and Lemma 3.2.15 (b). Therefore

$$A_x^\pm \cap D_{n-1}(x, r_n^\pm s_n) = \mathfrak{h}_{m,z}(A_{\mathfrak{h}_{m,z}^{-1}(x)}^\pm \cap D_{n-1}(\mathfrak{h}_{m,z}^{-1}(x), r_n^\pm s_n)) .$$

Then, by the induction hypothesis, we have

$$\chi_n(A_x^\pm \cap D_{n-1}(x, r_n^\pm s_n)) = \chi_n(A_{\mathfrak{h}_{m,z}^{-1}(x)}^\pm \cap D_{n-1}(\mathfrak{h}_{m,z}^{-1}(x), r_n^\pm s_n)) .$$

Moreover $I_{n,x} = I_{n,\mathfrak{h}_{m,y}^{-1}(x)}$ because (2.6.4) is order-preserving and an $r_n^\pm s_n$-short scale isometry with respect to d_{n-1}. Then the result follows from (2.6.3). □

2.6.2 Equivalences

We will define, by induction on $n \in \mathbb{N}$, the notion of n-equivalence between points $x, y \in X_n$. In addition, an explicit family of n-equivalences will be constructed, together with an induced equivalence relation.

Consider the restriction of the graph structure E_{n-1} to $C_{n,n-1}(x)$, for every $n \in \mathbb{N}$ and $x \in X_n$.

Definition 2.6.2. For $x, y \in X_0$, a 0-*equivalence* from x to y, denoted by $f \colon x \to y$, is a pointed graph isomorphism

$$f \colon (\overline{C}_{0,-1}(x), x) \to (\overline{C}_{0,-1}(y), y)$$

such that $f(C_{0,-1}(x)) = C_{0,-1}(f(x))$.

Let $\sim_0^\pm$ be the equivalence relation on $X_0^\pm$ defined by declaring $x \sim_0^\pm y$ for $x, y \in X_0^\pm$ if there is some 0-equivalence $(\overline{C}_{0,-1}(x), x) \to (\overline{C}_{0,-1}(y), y)$. Let Φ_0 be the map defined on $X_0 = X_0^+ \cup X_0^-$ that sends each point $x \in X_0^\pm$ to its equivalence class with respect to $\sim_0^\pm$. The range of this map is obviously finite.

Lemma 2.6.3. *For $n \in \mathbb{N}$, there are disjoint subsets $X_0^{-,\Phi}, X_0^{+,\Phi} \subset X_0$ satisfying the following properties:*

(a) The sets $X_0^{\pm,\Phi}$ are maximal among the subsets of $X_0^\pm$ where Φ_0 is injective.

(b) For $u \in X_0^{\pm,\Phi}$ and $v \in X_0^\pm$, if $\Phi_0(u) = \Phi_0(v)$, then $d_0(u,p) \le d_0(v,p)$.

Proof. Take in each $\sim_0^\pm$-equivalence class a representative that minimizes the d_0-distance to p. □

By Lemma 2.6.3, for every point $x \in X_0^\pm$, there is a unique element $u \in X_0^{\pm,\Phi}$ satisfying $\Phi_0(x) = \Phi_0(u)$. Let $\mathrm{rep}_0^\pm \colon X_0^\pm \to X_0^{\pm,\Phi}$ be the maps determined by this correspondence, and let $\mathrm{rep}_0 \colon X_0 \to X_0^\Phi := X_0^{+,\Phi} \cup X_0^{-,\Phi}$ be their union.

Lemma 2.6.4. *For all $(m,y) \in \mathfrak{P}_{-1}$ and $x \in X_0^\pm \cap D_0(p, \Gamma_0^+)$, the following properties hold:*

(a) $\overline{C}_{0,-1}(x) \subset \operatorname{dom} \mathfrak{h}_{m,y}$.

(b) The restriction

$$\mathfrak{h}_{m,y} \colon (\overline{C}_{0,-1}(x), x) \to (\overline{C}_{0,-1}(\mathfrak{h}_{m,y}(x)), \mathfrak{h}_{m,y}(x))$$

is a 0-equivalence; in particular, $x \sim_0 \mathfrak{h}_{m,y}(x)$ and $p \sim_0 y$.

Proof. By Lemma 2.5.1 and the triangle inequality,

$$\overline{C}_{0,-1}(x) \subset D_{-1}(x, \Gamma_0^+) \subset D_{-1}\left(p, \Lambda_0 \Gamma_m^+ + \Gamma_0^+\right) . \tag{2.6.5}$$

By (3.2.1) and (2.4.4),

$$\mathfrak{r}_m > 4\Lambda_m \Gamma_m^+ + K_m .$$

The assumption $(m,y) \in \mathfrak{P}_{-1}$ implies $m \ge 0$ according to (2.3.5). So $\Lambda_m \ge \Lambda_0 \ge \Lambda_{-1} = 1$ by (2.2.13) and (2.4.4), $K_m \ge K_0 > K_{-1} = 0$ by (2.2.15), (2.2.16) and (2.4.4), and $\Gamma_m^+ \ge R_0^+$ by (2.4.6). Therefore

$$\mathfrak{r}_m - K_{-1} - 2\Lambda_{-1} R_0^+ > 4\Lambda_m \Gamma_m^+ + K_m - 2R_0^+ > \Lambda_0 \Gamma_m^+ + R_0^+ .$$

Then (2.6.5) yields

$$\overline{C}_{0,-1}(x) \subset D_{-1}\left(p, \mathfrak{r}_m - K_{-1} - 2\Lambda_{-1} R_0^+\right) , \tag{2.6.6}$$

completing the proof of (a) because $\operatorname{dom} \mathfrak{h}_{m,y} = D_{-1}(p, \mathfrak{r}_m)$.

Property (b) follows from (2.4.22) and Proposition 2.4.1 (viii). □

Proposition 2.6.5. *For $x \in X_0^\pm$, there is a 0-equivalence*

$$h_{0,x} \colon \left(\overline{C}_{0,-1}\left(\mathrm{rep}_0(x)\right), \mathrm{rep}_0(x)\right) \to \left(\overline{C}_{0,-1}(x), x\right)$$

satisfying the following properties:

(i) If $x \in X_0^{\pm,\Phi}$, then $h_{0,x}$ is the identity on $\overline{C}_{0,-1}(x)$.

(ii) For $(m,y) \in \mathfrak{P}_{-1}$ and $x \in X_0 \cap D_0(y,\Gamma_0^+)$, we have $h_{0,x} = \mathfrak{h}_{m,y} \circ h_{0,\mathfrak{h}_{m,y}^{-1}(x)}$.

(iii) If $x \in \mathfrak{X}_0$, then $h_{0,x} = \mathfrak{h}_{0,x}|_{\overline{C}_{0,-1}(x)}$.

Proof. First, set $h_{0,x} = \mathrm{id}_{C_{0,-1}(x)}$ for every $x \in X_0^{\pm,\Phi}$, so that (i) is satisfied. Now, we define $h_{0,x}$ independently for $x \in A_m \setminus A_{m-1}$, where

$$A_m = \dot{\bigcup_{y \in \mathfrak{X}_m}} D_n(y,\Gamma_m^+) \cap X_0 \setminus X_0^\Phi$$

for $m \geq n$, and $A_{-1} = \emptyset$. Note that A_m is a union of disjoint subsets by Proposition 2.3.3 (i), since $\mathfrak{s}_m \geq \Gamma_m^+$ by (3.2.2) and (2.4.4). This completes the definition of $h_{0,x}$ for all $x \in X_0$ because $X_0 = \bigcup_{m \geq 0} A_m$ since $p \in \mathfrak{X}_m$ (Proposition 2.3.3 (i) and $\Gamma_m^+ \uparrow \infty$. Moreover (iii) is a direct consequence of (i) and (ii), and therefore we only have to check (ii).

Let $x \in A_m \setminus A_{m-1}$ for $m \geq 0$. On the one hand, if

$$x \in \left(D_0(p,\Gamma_m^+) \setminus X_0^\Phi\right) \setminus A_{m-1}\ ,$$

then let $h_{0,x}$ be any 0-equivalence $(\overline{C}_{0,-1}(\mathrm{rep}_0(x)), \mathrm{rep}_0(x)) \to (\overline{C}_{0,-1}(x), x)$. On the other hand, if

$$x \in \left(D_0(y,\Gamma_m^+) \setminus X_0^\Phi\right) \setminus A_{m-1}$$

for some $y \in \mathfrak{X}_m \setminus \{p\}$, then $\mathrm{rep}_0(x) \in D_0(p,\Gamma_m^+)$ by Lemma 2.6.3 (b), and let $h_{0,x} = \mathfrak{h}_{m,y} \circ h_{0,\mathfrak{h}_{m,y}^{-1}(x)}$. Note that this composite is well defined because

$$\mathrm{im}\, h_{0,\mathfrak{h}_{m,y}^{-1}(x)} = D_{-1}(x, r_0^\pm) \subset D_{-1}(x, R_0^\pm) \subset \mathrm{dom}\, \mathfrak{h}_{m,y}$$

by Lemma 2.6.4 (a) and (2.4.8). Property (ii) is obvious with this definition of $h_{0,x}$. □

Now, given any integer $n > 0$, suppose that we have already defined the equivalence relations $\sim_m$, the sets X_m^Φ, and maps rep_m and $h_{m,x}$ for $0 \leq m < n$. Let

$$\mathcal{C}_{n,-1}(x) = \bigcup_{v \in D_n(x,n)} \overline{C}_{n,-1}(v)\ , \quad \mathcal{C}_{n,n-1}(x) = \bigcup_{v \in D_n(x,n)} \overline{C}_{n,n-1}(v)\ .$$

Definition 2.6.6. For $n \in \mathbb{N}$ and $x, y \in X_n^\pm$, a pointed graph isomorphism

$$f\colon\ (\mathcal{C}_{n,-1}(x), x) \to (\mathcal{C}_{n,-1}(y), y)$$

is called an *n-equivalence* from x to y, denoted by $f\colon x \to y$, if it satisfies the following properties for $0 \leq m < n$ and $v \in D_n(x,n)$:

(i) We have $f(D_n(x,n)) = D_n(f(x),n)$.

(ii) We have $f(\overline{C}_{n,n-1}(v)) = \overline{C}_{n,n-1}(f(v))$ and $f(C_{n,n-1}(v)) = C_{n,n-1}(f(v))$.

(iii) We have

$$f\left(X^{\pm}_{n-1}\cap\mathcal{C}_{n,n-1}(x)\right)=X^{\pm}_{n-1}\cap\mathcal{C}_{n,n-1}(y)\;,$$

and

$$f\colon\ (\mathcal{C}_{n,n-1}(x),\chi_{n-1})\to(\mathcal{C}_{n,n-1}(y),\chi_{n-1})$$

is a color-preserving graph isomorphism with respect to E_{n-1}.

(iv) We have

$$f\left(\mathfrak{X}_{n-1}\cap\mathcal{C}_{n,n-1}(x)\right)=\mathfrak{X}_{n-1}\cap\mathcal{C}_{n,n-1}(y)\;.$$

(v) For all $u\in\operatorname{Pen}_{n-1}(C_{n,n-1}(x),1)$, the restriction $f\colon\mathcal{C}_{n-1,-1}(u)\to\mathcal{C}_{n-1,-1}(f(u))$ equals $h_{n-1,f(u)}\circ h^{-1}_{n-1,u}$; in particular, it is an $(n-1)$-equivalence.

Remark 11. Note that $X^{\pm}_{n-1}\cap\overline{C}_{n,n-1}(x),\overline{C}_{n-1,-1}(u)\subset\overline{C}_{n,-1}(x)$ by (2.5.3).

Remark 12. For $u\in\operatorname{Pen}_{n-1}(C_{n,n-1}(x),1)$ and $v\in D_{n-1}(u,n-1)$, $d_n(x,\pi_n(v))\leq n$ by Proposition 2.4.1 (vi) and the definition of E_n. So $\overline{C}_{n-1,-1}(v)\subset\operatorname{dom}f$ in Definition 2.6.6 (v).

The following lemma is an immediate consequence of Definitions 2.6.6 and 2.6.16.

Lemma 2.6.7. *For $n\in\mathbb{N}$, the family of n-equivalences between points of $X^{\pm}_n$ is closed by the operations of composition and inversion of maps.*

According to Lemma 2.6.7, for $n\in\mathbb{N}$, an equivalence relation $\sim^{\pm}_n$ on $X^{\pm}_n$ is defined by declaring $x\sim^{\pm}_n y$ if there is some n-equivalence between x and y. Let Φ_n be the map defined on $X_n=X^+_n\sqcup X^-_n$ that sends each point $x\in X^{\pm}_n$ to its equivalence class with respect to $\sim^{\pm}_n$. The range of each of these maps is obviously finite.

Lemma 2.6.8. *For $n\in\mathbb{N}$, there are disjoint subsets $X^{-,\Phi}_n,X^{+,\Phi}_n\subset X_n$ satisfying the following properties:*

(a) The sets $X^{\pm,\Phi}_n$ are maximal among the subsets of $X^{\pm}_n$ where Φ_n is injective.

(b) For $u\in X^{\pm,\Phi}_n$ and $v\in X^{\pm}_n$, if $\Phi_n(u)=\Phi_n(v)$, then $d_n(u,p)\leq d_n(v,p)$.

Proof. This follows by taking in each $\sim^{\pm}_n$-equivalence class a representative that minimizes the d_n-distance to p. □

By Lemma 2.6.8, for every point $x\in X^{\pm}_n$, there is a unique element $u\in X^{\pm,\Phi}_n$ satisfying $\Phi_n(x)=\Phi_n(u)$. Let $\operatorname{rep}^{\pm}_n\colon X^{\pm}_n\to X^{\pm,\Phi}_n$ be the maps determined by this correspondence, and let $\operatorname{rep}_n\colon X_n\to X^{\Phi}_n:=X^{+,\Phi}_n\sqcup X^{-,\Phi}_n$ be their union.

Lemma 2.6.9. *For all $(m,y)\in\mathfrak{P}_{n-1}$ and $x\in X^{\pm}_n\cap D_n(p,\Gamma^+_m)$, the following properties hold:*

(a) $\mathcal{C}_{n,-1}(v)\subset\operatorname{dom}\mathfrak{h}_{m,y}$.

(b) The restriction

$$\mathfrak{h}_{m,y}\colon\ (\mathcal{C}_{n,-1}(x),x)\to(\mathcal{C}_{n,-1}(\mathfrak{h}_{m,y}(x)),\mathfrak{h}_{m,y}(x))$$

is an n-equivalence; in particular, $x\sim_n\mathfrak{h}_{m,y}(x)$ and $p\sim_n y$.

Proof. By Lemma 2.5.1, $\overline{C}_{n,-1}(v) \subset D_{-1}(v, \Gamma_n^+)$ for every $v \in D_n(x,n)$. Using the triangle inequality, we get

$$\mathfrak{C}_{n,-1}(v) \subset D_{-1}(x, \Gamma_n^+ + n\Lambda_n) \subset D_{-1}\left(p, \Lambda_n(\Gamma_m^+ + n) + \Gamma_n^+\right) . \tag{2.6.7}$$

By (3.2.1) and (2.4.4), we have

$$\mathfrak{r}_m > 4\Lambda_m(\Gamma_m^+ + m) + K_m .$$

The assumption $(m,y) \in \mathfrak{P}_{n-1}$ implies $m \geq n$ according to (3.2.20). So $\Lambda_m \geq \Lambda_n > \Lambda_{n-1}$ by (2.2.13) and (2.4.4), $K_m \geq K_n > K_{n-1}$ by (2.2.15), (2.2.16) and (2.4.4), and $\Gamma_m^+ \geq R_n^+$ by (2.4.6). Therefore

$$\mathfrak{r}_m - K_n > 4\Lambda_m(\Gamma_m^+ + m) + K_m - K_n > \Lambda_n(\Gamma_m^+ + n) + \Gamma_n^+ .$$

Then (2.6.7) yields

$$\mathfrak{C}_{n,-1}(v) \subset D_{-1}\left(p, \mathfrak{r}_m - K_n\right) , \tag{2.6.8}$$

completing the proof of (a) because $\operatorname{dom} \mathfrak{h}_{m,y} = D_{-1}(p, \mathfrak{r}_m)$.

Let us prove (b). We proceed by induction on n. For $n = 0$, the result follows from Lemma 2.6.4 (b). So suppose that, given some $n > 0$, the result is true for $0 \leq m < n$. Definition 2.6.6 (i) follows from Proposition 2.4.1 (viii) and (2.6.8). By Lemma 2.5.5, (2.4.22) and (2.6.8), we get $\mathfrak{h}_{m,y}(\overline{C}_{n,n-1}(u)) = \overline{C}_{n,n-1}(\mathfrak{h}_{m,y}(u))$ and $\mathfrak{h}_{m,y}(C_{n,n-1}(u)) = C_{n,n-1}(\mathfrak{h}_{m,y}(u))$ for every $v \in D_n(x,n)$ and $u \in \overline{C}_{n,l}(v)$. Thus Definition 2.6.6 (ii) is satisfied. The map $\mathfrak{h}_{m,y} \colon \mathfrak{C}_{n,n-1}(v) \to \mathfrak{C}_{n,n-1}(w)$ is a graph isomorphism that preserves χ_{n-1} by Propositions 2.4.1 (viii) and 2.6.1 (iii). Therefore

$$\mathfrak{h}_{m,y}(X_m^\pm \cap \mathfrak{C}_{n,n-1}(x)) = X_m^\pm \cap \mathfrak{C}_{n,n-1}(y)$$

by Proposition 2.4.1 (ii),(viii). Hence $\mathfrak{h}_{m,y}$ satisfies Definition 2.6.6 (iii). Definition 2.6.6 (v) follows by the induction hypothesis. By Proposition 3.2.9 (iii), we have $\mathfrak{X}_{n-1} \cap D(y, \mathfrak{r}_l) = \mathfrak{h}_{m,y}(\mathfrak{X}_{n-1}^m)$ for each $(m,y) \in \mathfrak{P}_{n-1}$. In particular, for $(m,y) = (m,p)$, we obtain $\mathfrak{X}_{n-1}^m = \mathfrak{X}_{n-1} \cap D(p, \mathfrak{r}_m)$. So

$$\mathfrak{X}_{n-1} \cap D(y, \mathfrak{r}_l) = \mathfrak{h}_{m,y}(\mathfrak{X}_{n-1} \cap D(p, \mathfrak{r}_l)) ,$$

and Definition 2.6.6 (iv) follows using (2.6.7) and (a), since $\mathfrak{r}_m \geq R_n^+ \geq r_n^\pm$ according to (2.4.6)–(2.4.8). Therefore $\mathfrak{h}_{m,y}$ satisfies Definition 2.6.6 (iv). This completes the proof of (b). □

Proposition 2.6.10. *For $n \in \mathbb{N}$ and $x \in X_n$, there is an n-equivalence $h_{n,x} \colon \operatorname{rep}_n(x) \to x$satisfying the following properties:*

(i) If $x \in X_n^\Phi$, then $h_{n,x}$ is the identity on $\mathfrak{C}_{n,-1}(x)$.

(ii) For $(m,y) \in \mathfrak{P}_{n-1}$ and $x \in X_n \cap D_n(y, \Gamma_m^+)$, we have $h_{n,x} = \mathfrak{h}_{m,y} \circ h_{n,\mathfrak{h}_{m,y}^{-1}(x)}$.

(iii) If $x \in \mathfrak{X}_n$, then $h_{n,x} = \mathfrak{h}_{n,x}$ on $\mathfrak{C}_{n,-1}(x)$.

Proof. First, define $h_{n,x}$ as the identity on $\mathcal{C}_{n,-1}(x)$ for every $x \in X_n^\Phi$, so that (i) is satisfied. Now, we define $h_{n,x}$ independently for $x \in A_m \setminus A_{m-1}$, where

$$A_m = \bigcup_{y \in \mathfrak{X}_m}^{\cdot} D_n(y, \Gamma_m^+) \cap X_n \setminus X_n^\Phi$$

for $m \geq n$, and $A_{n-1} = \emptyset$. Note that A_m is a union of disjoint subsets by Proposition 2.3.3 (i), since $\mathfrak{s}_m \geq \Gamma_m^+$ by (3.2.2) and (2.4.4). This completes the definition of $h_{n,x}$ for all $x \in X_n$ because $X_n = \bigcup_{m \geq n} A_m$ since $p \in \mathfrak{X}_m$ (Proposition 2.3.3 (i) and $\Gamma_m^+ \uparrow \infty$. Moreover (iii) is a direct consequence of (i) and (ii), and therefore we only have to check (ii).

Let $x \in A_m \setminus A_{m-1}$ for $m \geq n$. On the one hand, if

$$x \in \left(D_n(p, \Gamma_m^+) \cap X_n \setminus X_n^\Phi\right) \setminus A_{m-1} ,$$

then let $h_{n,x} \colon \operatorname{rep}_n(x) \to x$ be any n-equivalence, whose existence is guaranteed by the definition of rep_n. On the other hand, if

$$x \in \left(D_n(y, \Gamma_m^+) \cap X_n \setminus X_n^\Phi\right) \setminus A_{m-1}$$

for some $y \in \mathfrak{X}_m \setminus \{p\}$, then $\operatorname{rep}_n(x) \in D_n(p, \Gamma_m^+)$ by Lemmas 2.6.3 (b) and 2.6.8 (b), and let $h_{n,x} = \mathfrak{h}_{m,y} \circ h_{n,\mathfrak{h}_{m,y}^{-1}(x)}$. Note that this composite is well defined because, for $x \in X_n^\pm$,

$$\operatorname{im} h_{n,\mathfrak{h}_{m,y}^{-1}(x)} = D_{n-1}(x, r_n^\pm) \subset D_{n-1}(x, R_n^\pm) \subset \operatorname{dom} \mathfrak{h}_{m,y}$$

by Lemma 2.6.9 (a) and (2.4.8). Property (ii) is obvious with this definition of $h_{n,x}$. □

Remark 13. In accordance with the discussion at the beginning of Section 2.4, only Proposition 2.6.10 (i) is needed to prove Theorem 2.1.26 (i), whereas the whole Proposition 2.6.10 is needed to prove Theorem 2.1.26 (ii).

Remark 14. Note that the definitions of $\sim_n^\pm$, Φ_n e $\operatorname{rep}_n^\pm$, and the properties of $X_n^{\pm,\Phi}$ already guarantee the existence of n-equivalences $h_{n,x}$. Moreover there is no problem to assume (i) and (iii). So the really new contribution of Proposition 2.6.10 is (ii).

2.6.3 Weak equivalences

Next we introduce another notion of equivalence very similar to that of n-equivalence. We need both concepts due to how we prove the crucial Lemma 2.6.42. In this result we first prove that a certain map is an n-weak equivalence, and use that to conclude that it is in fact an n-equivalence over a smaller domain.

Definition 2.6.11. For $x, y \in X_0$, a 0-*weak equivalence* from x to y, denoted $f \colon x \to y$, is a pointed graph isomorphism $(D_{-1}(x, r_n^\pm), x) \to (D_{-1}(y, r_n^\pm), y)$

Let $\widehat{\sim}_0^\pm$ be the equivalence relation on $X_0^\pm$ defined by declaring $x \widehat{\sim}_0^\pm y$ for $x, y \in X_0^\pm$ if there is some 0-weak equivalence $(D_{-1}(x, r_0^\pm), x) \to (D_{-1}(y, r_0^\pm), y)$. Let $\widehat{\Phi}_0$ be the map defined on $X_0 = X_0^+ \cup X_0^-$ that sends each point $x \in X_0^\pm$ to its equivalence class with respect to $\widehat{\sim}_0^\pm$. The range of this map is obviously finite.

Lemma 2.6.12. *Let $f\colon x \to y$ be a 0-equivalence. Then the restriction of f to $D_{-1}(x, r_0^{\pm})$ is a 0-weak equivalence; in particular, $x \sim_0 y$ implies $x \widehat{\sim}_0 y$.*

Lemma 2.6.13. *For $n \in \mathbb{N}$, there are disjoint subsets $X_0^{-,\widehat{\Phi}}, X_0^{+,\widehat{\Phi}} \subset X_0$ satisfying the following properties:*

(a) The sets $X_0^{\pm,\widehat{\Phi}}$ are maximal among the subsets of $X_0^{\pm}$ where $\widehat{\Phi}_0$ is injective.

(b) For $u \in X_0^{\pm,\widehat{\Phi}}$ and $v \in X_0^{\pm}$, if $\widehat{\Phi}_0(u) = \widehat{\Phi}_0(v)$, then $d_0(u,p) \le d_0(v,p)$.

(c) We have $X_0^{\pm,\widehat{\Phi}} \subset X_0^{\pm,\Phi}$.

Proof. Take in each $\widehat{\sim}_0^{\pm}$-equivalence class a representative that minimizes the d_0-distance to p. □

By Lemma 2.6.3, for each point $x \in X_0^{\pm}$, there is a unique element $u \in X_0^{\pm,\widehat{\Phi}}$ satisfying $\widehat{\Phi}_0(x) = \widehat{\Phi}_0(u)$. Let $\widehat{\mathrm{rep}}_0^{\pm}\colon X_0^{\pm} \to X_0^{\pm,\widehat{\Phi}}$ be the maps determined by this correspondence, and let $\widehat{\mathrm{rep}}_0\colon X_0 \to X_0^{\widehat{\Phi}} := X_0^{+,\widehat{\Phi}} \cup X_0^{-,\widehat{\Phi}}$ be their union.

The following lemma follows from Lemmas 2.6.4 and 2.6.12.

Lemma 2.6.14. *For all $(m,y) \in \mathfrak{P}_{-1}$ and $x \in X_0^{\pm} \cap D_0(p, \Gamma_0^{+})$, the following properties hold:*

(a) $D_{-1}(x, r_0^{\pm})(x) \subset \operatorname{dom} \mathfrak{h}_{m,y}$.

(b) The restriction

$$\mathfrak{h}_{m,y}\colon (D_{-1}(x, r_0^{\pm})(x), x) \to (D_{-1}(x, r_0^{\pm})(x), \mathfrak{h}_{m,y}(x))$$

is a 0-weak equivalence; in particular, $x \widehat{\sim}_0 \mathfrak{h}_{m,y}(x)$ and $p \widehat{\sim}_0 y$.

Proposition 2.6.15. *For $x \in X_0^{\pm}$, there is a 0-weak equivalence*

$$\hat{h}_{0,x}\colon \big(D_{-1}(\widehat{\mathrm{rep}}_0(x), r_0^{\pm}), \widehat{\mathrm{rep}}_0(x)\big) \to \big(D_{-1}(x, r_0^{\pm}), x\big)$$

satisfying the following properties:

(i) If $x \in X_0^{\pm,\widehat{\Phi}}$, then $\hat{h}_{0,x}$ is the identity on $D_{-1}(x, r_0^{\pm})(x)$.

(ii) For all $x \in X_0^{\pm}$, $\hat{h}_{0,x} = h_{0,x} \circ \hat{h}_{0,\mathrm{rep}_0(x)}$.

Proof. First, for every $x \in X_0^{\pm,\widehat{\Phi}}$, let $\hat{h}_{0,x}$ be the identity on $D_{-1}(x, r_0^{\pm})$. Then, for points $x \in X_0^{\pm,\Phi} \setminus X_0^{\pm,\widehat{\Phi}}$, let $\hat{h}_{0,x}\colon \widehat{\mathrm{rep}}_0(x) \to x$ be any 0-weak equivalence. Finally, for every $x \in X_0 \setminus X_0^{\pm,\Phi}$, let $\hat{h}_{0,x} = h_{0,x} \circ \hat{h}_{0,\mathrm{rep}_0(x)}$. □

Now, given any integer $n > 0$, suppose that we have already defined the equivalence relations $\widehat{\sim}_m$, the sets $X_m^{\widehat{\Phi}}$, and maps $\widehat{\mathrm{rep}}_m$ and $\hat{h}_{m,x}$ for $0 \le m < n$. Let

$$\mathscr{C}_n(x) = \bigcup_{u \in D_{n-1}(x, r_n^{\pm})} \overline{C}_{n-1,-1}(u) \ .$$

Definition 2.6.16. For $n \in \mathbb{N}$ and $x, y \in X_n^\pm$, a pointed graph isomorphism

$$f\colon (\mathscr{C}_n(x), x) \to (\mathscr{C}_n(y), y)$$

is called an *n-weak equivalence* from x to y, denoted $f : x \to y$, if it satisfies the following properties for $0 \leq m < n$ and $v \in D_n(x, n)$:

(i) We have $f(D_{n-1}(x, r_n^\pm)) = D_{n-1}(y, r_n^\pm)$.

(ii) We have
$$f\left(X_{n-1}^\pm \cap D_{n-1}(x, r_n^\pm)\right) = X_{n-1}^\pm \cap D_{n-1}(y, r_n^\pm) ,$$
and
$$f\colon \left(D_{n-1}(x, r_n^\pm), \chi_{n-1}\right) \to \left(D_{n-1}(y, r_n^\pm), \chi_{n-1}\right)$$
is a color-preserving graph isomorphism with respect to E_{n-1}.

(iii) We have
$$f\left(\mathfrak{X}_{n-1} \cap D_{n-1}(x, r_n^\pm)\right) = \mathfrak{X}_{n-1} \cap D_{n-1}(x, r_n^\pm) .$$

(iv) For every $u \in D_{n-1}(x, r_n^\pm - 1)$, the restriction $f\colon \mathcal{C}_{n-1}(u) \to \mathcal{C}_{n-1}(f(u))$ equals $h_{n-1,f(u)} \circ h_{n-1,u}^{-1}$; in particular, it is an $(n-1)$-equivalence.

Remark 15. Note that, for $n > 0$, $x \in X_n$ and $u \in D_{n-1}(x, r_n^\pm - 1)$, we have $\mathcal{C}_{n-1}(u) \subset \mathscr{C}_n(x)$ because $D_{n-1}(u, 1) \subset D_{n-1}(x, r_n^\pm)$.

The following lemma is an immediate consequence of Definitions 2.6.6 and 2.6.16.

Lemma 2.6.17. *The family of n-weak equivalences between points of $X_n^\pm$ is closed by the operations of composition and inversion of maps. Moreover the composition of an n-weak equivalence and an n-equivalence is an n-weak equivalence; in particular, every n-equivalence is an n-weak equivalence.*

According to Lemma 2.6.17, for $n \in \mathbb{N}$, an equivalence relation $\widehat{\sim}_n^\pm$ on $X_n^\pm$ is defined by declaring $x \widehat{\sim}_n^\pm y$ if there is some n-equivalence between x and y. Let $\widehat{\Phi}_n$ be the map defined on $X_n = X_n^+ \cup X_n^-$ that sends each point $x \in X_n^\pm$ to its equivalence class with respect to $\widehat{\sim}_n^\pm$. The range of each of these maps is obviously finite.

Lemma 2.6.18. *For $n \in \mathbb{N}$, there are disjoint subsets $X_n^{-,\widehat{\Phi}}, X_n^{+,\widehat{\Phi}} \subset X_n$ satisfying the following properties:*

(a) We have $X_n^{\pm,\widehat{\Phi}} \subset X_n^{\pm,\Phi}$.

(b) The sets $X_n^{\pm,\widehat{\Phi}}$ are maximal among the subsets of $X_n^\pm$ where $\widehat{\Phi}_n$ is injective.

(c) For $u \in X_n^{\pm,\widehat{\Phi}}$ and $v \in X_n^\pm$, if $\widehat{\Phi}_n(u) = \widehat{\Phi}_n(v)$, then $d_n(u, p) \leq d_n(v, p)$.

Proof. Take in each $\widehat{\sim}_n^\pm$-equivalence class a representative that minimizes the d_n-distance to p. □

By Lemma 2.6.8, for each point $x \in X_n^\pm$, there is a unique element $u \in X_n^{\pm,\widehat{\Phi}}$ satisfying $\widehat{\Phi}_n(x) = \widehat{\Phi}_n(u)$. Let $\widehat{\mathrm{rep}}_n^\pm\colon X_n^\pm \to X_n^{\pm,\widehat{\Phi}}$ be the maps determined by this correspondence, and let $\widehat{\mathrm{rep}}_n\colon X_n \to X_n^{\widehat{\Phi}} := X_n^{+,\widehat{\Phi}} \cup X_n^{-,\widehat{\Phi}}$ be their union.

The following result follows from Lemmas 2.6.14 and 2.6.17.

Lemma 2.6.19. *For all* $(m, y) \in \mathfrak{P}_{-1}$ *and* $x \in X_n^{\pm} \cap D_n(p, \Gamma_0^{+})$*, the following properties hold.*

(a) $\mathscr{C}_n(x) \subset \operatorname{dom} \mathfrak{h}_{m,y}$*.*

(b) The restriction

$$\mathfrak{h}_{m,y} \colon (\mathscr{C}_n(x), x) \to (\mathscr{C}_{n-1,-1}(\mathfrak{h}_{m,y}(x)), \mathfrak{h}_{m,y}(x))$$

is a n-weak equivalence; in particular, $x \mathrel{\widehat{\sim}_n} \mathfrak{h}_{m,y}(x)$ *and* $p \mathrel{\widehat{\sim}_n} y$*.*

Proposition 2.6.20. *For* $x \in X_n^{\pm}$*, there is a n-weak equivalence* $\hat{h}_{n,x} \colon \widehat{\operatorname{rep}}_n(x) \to x$ *satisfying the following properties:*

(i) If $x \in X_n^{\pm,\widehat{\Phi}}$*, then* $\hat{h}_{n,x}$ *is the identity on* $D_{n-1}(x, r_n^{\pm})$*.*

(ii) For all $x \in X_n^{\pm}$*,* $\hat{h}_{n,x} = h_{n,x} \circ \hat{h}_{n,\operatorname{rep}_n(x)}$*.*

Proof. The proof is identical to that of Proposition 2.6.15. □

2.6.4 BFS-orderings

We introduce a special kind of orderings on graphs that are used to produce aperiodic colorings. They are essentially a reformulation of the *breadth-first search spanning trees* in [27].

Definition 2.6.21. Let (A, x) be a pointed connected graph with finite vertex degrees endowed with an order relation $\leq$. Define the *parent map*, $\operatorname{Pa} \colon A \setminus \{x\} \to A$, by

$$\operatorname{Pa}(u) = \min S(u, 1) \; . \tag{2.6.9}$$

For $v \in A$, its *children set*, denoted by $\operatorname{Ch}(v)$, is

$$\operatorname{Ch}(v) = \operatorname{Pa}^{-1}(v) = S(v, 1) \setminus \left(\bigcup_{w<v} S(w, 1) \cup \{x\} \right) . \tag{2.6.10}$$

Definition 2.6.22. A *BFS-ordering* on a pointed connected graph (A, x) is an order $\trianglelefteq$ on A satisfying the following conditions for all $u, v \in A$:

(i) If $d(x, u) < d(x, v)$, then $u \lhd v$.

(ii) If $u, v \neq x$ and $\operatorname{Pa}(u) \lhd \operatorname{Pa}(v)$, then $u \lhd v$.

There exists a BFS-ordering $\trianglelefteq$ on any pointed connected graph (A, x) with finite vertex degrees. It can be defined on $D(x, n)$ by induction on $n \in \mathbb{N}$ as follows. First, declare x to be the least element in A. Then the restriction of $\trianglelefteq$ to $S(x, 1)$ is any order, and declare the points in $D(x, 1)$ to be an initial segment of $\trianglelefteq$. Next, the restriction of $\trianglelefteq$ to $S(x, 2)$ is any order such that $u \lhd v$ if

$$\min(S(1, u) \cap D(1, x)) \lhd \min(S(1, v) \cap D(1, x)) \; ,$$

and so on. This argument gives the following result.

Lemma 2.6.23. *Let $a \in \mathbb{N}$, let (A, x) be a pointed connected graph with finite vertex degrees. Then there is a BFS-ordering $\trianglelefteq$ on (A, x).*

Given an isomorphism of graphs, $f\colon A \to B$, and an order relation $\leq_A$ on A ($\leq_A \subset A \times A$), the corresponding push-forward order relation on B is $(f \times f)(\leq_A) \subset B \times B$, simply denoted by $f(\leq_A)$.

Recall that $C_{n,n-1}(x)$ is a connected subgraph of (X_{n-1}, E_{n-1}) by Lemma 2.5.4. Consider the n-equivalences $h_{n,x}$, for $n \in \mathbb{N}$ and $x \in X_n$, given by Proposition 2.6.10.

Proposition 2.6.24. *For $n \in \mathbb{N}$ and $x \in X_n$, there is a BFS-ordering $\trianglelefteq_{n,x}$ on the pointed connected graph $(C_{n,n-1}(x), x)$ satisfying $\trianglelefteq_{n,x} = h_{n,\mathrm{rep}_n(x)}(\trianglelefteq_{n,\mathrm{rep}_n(x)})$.*

Proof. Take any BFS-ordering $\trianglelefteq_{n,x}$ on $(C_{n,n-1}(x), x)$ for $x \in X_n^\Phi$ (Lemma 2.6.23). Then define $\trianglelefteq_{n,x} = h_{n,\mathrm{rep}_n(x)}(\trianglelefteq_{n,\mathrm{rep}_n(x)})$ for $x \in X_n \setminus X_n^\Phi$. □

From now on, for every $n \in \mathbb{N}$ and $x \in X_n$, the notation $\mathrm{Pa}_{n,x}$ and $\mathrm{Ch}_{n,x}$ is used for the parent map and children sets on the pointed connected graph $(C_{n,n-1}(x), x)$, with the BFS-ordering $\trianglelefteq_{n,x}$ given by Proposition 2.6.24.

Lemma 2.6.25. *Let $n \in \mathbb{N}$ and $x \in X_n$. The following properties hold for every $u \in C_{n,n-1}(x)$:*

(a) If $u \neq x$, then $d_{n-1}(x, \mathrm{Pa}_{n,x}(u)) = d_{n-1}(x, u) - 1$.

(b) We have

$$\dot{\bigcup_{v \in C_{n,n-1}(x)}} \mathrm{Ch}_{n,x}(v) = C_{n,n-1}(x) \setminus \{x\}\ .$$

(c) If $u \neq x$, then $|\mathrm{Ch}_{n,x}(u)| \leq \Delta_{n-1} - 1$.

Proof. Property (a) is an easy consequence of Definitions 2.6.21 and 2.6.22 (i). Property (c) follows from (a) and Definition 2.6.21, whereas (b) is obvious. □

2.6.5 Adapted colorings for $n = 0$

When we sketched the outline of the proof in Section 1.1.5, it was said that we needed to construct many colorings $\psi_{n,x}^i$ on the clusters $C_{n,n-1}(x)$ that break the symmetries of the cluster. These are the building blocks that we will use to construct the colorings in the statement of Theorem 2.1.26.

Definition 2.6.26. For $x \in X_0$, a coloring $\psi\colon C_{0,-1}(x) \to [\Delta]$ is said to be *adapted* if it satisfies the following two conditions:

(i) There is a geodesic segment in (X_{-1}, E_{-1}) of the form $\tau = (x = \tau_0, \dots, \tau_5)$ such that

$$\psi^{-1}(0) \cap D_{-1}(x, 7) = \begin{cases} \{\tau_0, \tau_1, \tau_2, \tau_5\} & \text{if } x \in X_0^- \\ \{\tau_0, \tau_1, \tau_2, \tau_4, \tau_5\} & \text{if } x \in X_0^+\ . \end{cases}$$

(ii) For all $u \in C_{0,-1}(x)$, the coloring ψ is injective on $\mathrm{Ch}_{0,x}(u)$.

It is said that ψ is *strongly adapted* if it is adapted and moreover the following property holds:

(iii) We have $\psi^{-1}(0) \setminus D_{-1}(x,7) = \emptyset$.

Lemma 2.6.27. *For every $x \in X_0^{\pm}$, there is a strongly adapted coloring $\psi_x\colon C_{0,-1}(x) \to [\Delta]$.*

Proof. First, choose a geodesic segment in (X_{-1}, E_{-1}) of the form $\tau = (x = \tau_0, \dots, \tau_5)$, which is contained in $C_{0,-1}(x)$ because $D_{-1}(x, r_0^{\pm}) \subset C_{0,-1}(x)$ (Lemma 2.5.2), and $r_0^{\pm} > 2^{11}$ by (2.4.3) and (2.4.7). Consider the set

$$\begin{aligned} T_0^- &= \{\tau_0, \tau_1, \tau_2, \tau_5\} && \text{if } x \in X_0^-, \text{ or} \\ T_0^+ &= \{\tau_0, \tau_1, \tau_2, \tau_4, \tau_5\} && \text{if } x \in X_0^+. \end{aligned}$$

Color the corresponding set $T_0^{\pm}$ with the color 0, depending on whether $x \in X_0^-$ or $x \in X_0^+$. In both cases $\psi_x(x) = 0$. The sets $\mathrm{Ch}_{0,x}(u)$, for $u \in C_{0,-1}(x)$, form a partition of $C_{0,-1}(x) \setminus \{x\}$ by Lemma 2.6.25 (b). Moreover $|\mathrm{Ch}_{0,x}(u) \setminus T_n^{\pm}| \leq \Delta - 1$ by Lemma 2.6.25 (c). So, for each $u \in C_{0,-1}(x)$, we can color the points in $\mathrm{Ch}_{0,x}(u) \setminus T_n^{\pm}$ with different colors from $\{1, \dots, \Delta - 1\}$. This procedure defines a coloring $\psi_x\colon C_{0,-1}(x) \to [\Delta]$ satisfying all conditions of Definition 2.6.26. □

For a colored graph, (X, ϕ), and a graph isomorphism, $h : X \to Y$, the notation $h(\phi)$ is used for the corresponding pushforward coloring of Y.

Proposition 2.6.28. *There is a family of strongly adapted colorings, $\psi_{0,x}^0\colon C_{0,-1}(x) \to [\Delta]$, for $x \in X_0$, satisfying $\psi_{0,x}^0 = h_{0,x}(\psi_{0,\mathrm{rep}_0(x)}^0)$.*

Proof. If $x \in X_0^{\Phi}$, take any strongly adapted coloring (Lemma 2.6.27). If $x \in X_0 \setminus X_0^{\Phi}$, let $\psi_{0,x}^0 = h_{0,x}(\psi_{0,\mathrm{rep}_0(x)}^0)$. It is trivial to check that $h_{0,x}(\psi_{0,\mathrm{rep}_0(x)}^0)$ satisfies the properties (i) and (iii) of Definition 2.6.26, whereas its property (ii) follows from Proposition 2.6.24. □

Proposition 2.6.29. *There is a family of colorings, $\psi_{0,x}^i\colon C_{0,-1}(x) \to [\Delta]$, for $x \in X_0$ and $i \in H_{0,x}$, satisfying the following properties:*

(i) The coloring $\psi_{0,x}^0$ is strongly adapted.

(ii) We have $\psi_{0,x}^i = h_{0,x}(\psi_{0,\mathrm{rep}_0(x)}^i)$.

(iii) For $i \in H_{0,x}$, the coloring $\psi_{0,x}^i$ is adapted.

(iv) For $x \in X_0$ and $i, j \in H_{0,x}$, let $A = C_{0,-1}(x)$ (respectively, $A = D_{-1}(x, r_n^{\pm})$), and let $f\colon (A, x, \psi_{0,x}^i) \to (A, x, \psi_{0,x}^j)$ be a color-preserving 0-equivalence (respectively, 0-weak equivalence). Then f is the identity map on A, and $i = j$.

Proof. First, for $i = 0$, we take the strongly adapted colorings $\psi_{0,x}^0$ constructed in Proposition 2.6.28. So (i) is satisfied.

For every $x \in X_0^{\pm,\Phi}$, choose a maximal 3-separated subset $N_{0,x}$ of $C_{-1}(x, 10, r_0^{\pm})$, together with an enumeration of its powerset,

$$\mathcal{P}(N_{0,x}) = \{\, \mathcal{N}_{0,x}^0 = \emptyset,\ \mathcal{N}_{0,x}^1,\ \dots \} \,.$$

We have $|D_{-1}(x,10)| \le \Delta^{11}$ by Corollary 2.1.19. Thus $|C_{-1}(x,10,r_0^\pm)| \ge |D_{-1}(x,r_0^\pm)| - \Delta^{11}$ (recall that $r_0^\pm > 2^{11}$). By Lemma 2.1.7, $N_{0,x}$ is 2-relatively dense in $C_{-1}(x,10,r_0^\pm)$. So

$$|N_{0,x}| \ge \left\lfloor \left(|D_{-1}(x,r_n^\pm)| - \Delta^{11}\right)/\Delta^3 \right\rfloor \ge \left\lfloor \left(|D_{-1}(x,r_n^\pm)| - \Delta^{11} - 1\right)/\Delta^3 \right\rfloor \tag{2.6.11}$$

by Lemma 2.1.20. Therefore

$$|\mathcal{P}(N_{0,x})| \ge \exp_2\left(\left\lfloor \left(|D_{-1}(x,r_n^\pm)| - \Delta^{11} - 1\right)/\Delta^3 \right\rfloor\right) = \eta_0(|D_{-1}(x,r_n^\pm)|) .$$

Thus an injective map $H_{0,x} \to \mathcal{P}(N_{0,x})$ is well defined by $i \mapsto \mathcal{N}_{0,x}^i$.

If $x \notin X_0^\Phi$, let $N_{0,x} = h_{0,x}(N_{0,\mathrm{rep}_0(x)})$ and $\mathcal{N}_{0,x}^i = h_{0,x}(\mathcal{N}_{0,\mathrm{rep}_0(x)}^i)$, so that $N_{0,x}$ satisfies (2.6.11). Then define

$$\psi_{0,x}^i(u) = \begin{cases} \psi_{0,x}^0(u) & \text{if } u \notin \mathcal{N}_{0,x}^i \\ 0 & \text{if } u \in \mathcal{N}_{0,x}^i . \end{cases}$$

Note that this definition agrees with the previous one in the case $i = 0$. Property (ii) follows immediately from Proposition 2.6.28 and the fact that $\mathcal{N}_{0,x}^i = h_{0,x}(\mathcal{N}_{0,\mathrm{rep}_0(x)}^i)$.

To prove (iii), note that $\psi_{0,x}^i = \psi_{0,x}^0$ on $D_{-1}(x,10)$ by construction. So Definition 2.6.26 (i) is trivially satisfied by $\psi_{0,x}^i$. For every $u \in C_{0,-1}(x)$, we have $\mathrm{Ch}_{0,x}(u) \subset D_{-1}(u,1)$, which yields $d(v,w) \le 2$ for all $v,w \in \mathrm{Ch}_{0,x}(u)$. Hence $N_{0,x} \cap \mathrm{Ch}_{0,x}(u)$ has at most one point because $N_{0,x}$ is 3-separated, and therefore $\mathcal{N}_{0,x}^i \cap \mathrm{Ch}_{0,x}(u)$ has at most one point. The coloring $\phi_{0,x}^0$ assigns different colors to all points in $\mathrm{Ch}_{0,x}(u)$ (Definition 2.6.26 (ii). If $u \in D_{-1}(x,9)$, then $\mathrm{Ch}_{0,x}(u) \subset D_{-1}(x,10)$, and therefore $\psi_{0,x}^i$ also assigs different colors to all points in $\mathrm{Ch}_{0,x}(u)$ since $\psi_{0,x}^i = \psi_{0,x}^0$ on $D_{-1}(x,10)$. If $u \in C_{0,-1}(x) \setminus D_{-1}(x,9)$, then $\psi_{0,x}^0$ assigns different colors to all points in $\mathrm{Ch}_{0,x}(u)$, all of them different from 0, and it follows from the definition that $\psi_{0,x}^i$ assigns different colors to those points too. Thus Definition 2.6.26 (ii) is satisfied by $\psi_{0,x}^i$, and the coloring $\psi_{0,x}^i$ is adapted.

To prove (iv), suppose first that $A = C_{0,-1}(x)$ and f is a 0-equivalence. For all $u \in C_{0,-1}(x)$, we show that f is the identity map on $\mathrm{Ch}_{n,x}(u)$, and that $\mathcal{N}_{0,x}^i \cap \mathrm{Ch}_{n,x}(u) = \mathcal{N}_{0,x}^j \cap \mathrm{Ch}_{n,x}(u)$, using induction on u with $\trianglelefteq_{0,x}$. This will complete the proof because it follows that f is the identity map and $\mathcal{N}_{0,x}^i = \mathcal{N}_{0,x}^j$, yielding $i = j$.

First, we have $f(x) = x$ by Definition 2.6.26 (i), since x is the unique point having the correct coloring pattern on some geodesic segment of the form $\tau = (x = \tau_0, \dots, \tau_5)$. Also, we have

$$\mathcal{N}_{0,x}^i \cap \mathrm{Ch}_{n,x}(x) = \mathcal{N}_{0,x}^j \cap \mathrm{Ch}_{n,x}(x) = \emptyset$$

since $N_{0,x} \cap D(x,10) = \emptyset$.

Suppose now that, for some $u \in C_{0,-1}(x)$ with $d_{-1}(u,x) > 0$, f is the identity map on $\mathrm{Ch}_{0,x}(v)$ and $\mathcal{N}_{0,x}^i \cap \mathrm{Ch}_{n,x}(v) = \mathcal{N}_{0,x}^j \cap \mathrm{Ch}_{n,x}(v)$ for all $v \lhd_{0,x} u$. In particular, f is the identity map on $\mathrm{Ch}_{n,x}(\mathrm{Pa}_{n,x}(u))$, and therefore $f(u) = u$. Furthermore this implies $f(\mathrm{Ch}_{0,x}(u)) = \mathrm{Ch}_{0,x}(u)$ by (2.6.10). By definition, for $l = i,j$, we have $\psi_{0,x}^l = \psi_{0,x}^0$ on $\mathrm{Ch}_{0,x}(u) \setminus N_{0,x}$, and $\psi_{0,x}^l(u) = 0$ if $u \in \mathcal{N}_{0,x}^l$. Recall that $N_{0,x} \cap \mathrm{Ch}_{0,x}(u)$ has at most one point, which is denoted by w. By (iii) and Definition 2.6.26 (ii), $\psi_{0,x}^0$ is injective on $\mathrm{Ch}_{0,x}(u) \setminus \{w\}$. Thus $\psi_{0,x}^i$ and $\psi_{0,x}^j$ agree and are injective on $\mathrm{Ch}_{0,x}(u) \setminus \{w\}$, and therefore

f is the identity on $\mathrm{Ch}_{0,x}(u) \setminus \{w\}$. But this yields $f(w) = w$, and f is color preserving only if $\mathrm{Ch}_{0,x}(u) \cap \mathcal{N}^i_{0,x} = \mathrm{Ch}_{0,x}(u) \cap \mathcal{N}^j_{0,x}$.

The proof of (iv) when $A = D_{-1}(x, r_0^{\pm})$ and f is a 0-weak equivalence is similar. □

Corollary 2.6.30. *Let $x, y \in X_0$, $i \in H_{0,x}$ and $j \in H_{0,y}$, let $A = C_{0,-1}(x)$ (respectively, $A = D_{-1}(x, r_n^{\pm})$), and let $f\colon (A, x, \psi^i_{0,x}) \to (A, x, \psi^j_{0,x})$ be a color-preserving 0-equivalence (respectively, 0-weak equivalence). Then $i = j$ and $f = h_{n,y} \circ h_{n,x}^{-1}$ on A.*

Proof. Suppose that $A = C_{0,-1}(x)$. Since there is a 0-equivalence between x and y, we have $\Phi_0(x) = \Phi_0(y)$ and $\mathrm{rep}_0(x) = \mathrm{rep}_0(y) =: z$. So $h^*_{0,x}\psi^l_{0,x} = \psi^l_{0,z}$ for $l = i, j$ by Proposition 2.6.29 (ii). Then

$$h_{0,y}^{-1} \circ f \circ h_{0,x}\colon (C_{0,-1}(z), z, \psi^i_{0,z}) \to (C_{0,-1}(z), z, \psi^j_{0,z})$$

is a color-preserving 0-equivalence. The result follows from Proposition 2.6.29 (iv).

The case where $A = D_{-1}(x, r_n^{\pm})$ is similar. □

2.6.6 Adapted colorings for $n > 0$

Definition 2.6.31. Let $x \in X_n$. A coloring $\psi\colon C_{n,n-1}(x) \to \mathcal{I}_{n-1}$ is said to be *adapted* if the following conditions are satisfied:

(i) We have $\psi^{-1}(0) = \mathfrak{X}_{n-1} \cap C_{n,n-1}(x)$.

(ii) We have

$$\psi^{-1}(1) = \begin{cases} \{x\} & \text{if } x \in X_n^- \setminus \mathfrak{X}_{n-1} \\ \emptyset & \text{otherwise .} \end{cases}$$

(iii) We have

$$\psi^{-1}(2) = \begin{cases} \{x\} & \text{if } x \in X_n^+ \setminus \mathfrak{X}_{n-1} \\ \emptyset & \text{otherwise .} \end{cases}$$

(iv) If $x \in \mathfrak{X}_{n-1} \cap X_n^+$, then $\psi^{-1}(3) = \{y\}$ for some $y \in S_{n-1}(x, 1)$, otherwise $\psi^{-1}(3) = \emptyset$.

(v) If $x \in \mathfrak{X}_{n-1} \cap X_n^-$, then $\psi^{-1}(4) = \{y\}$ for some $y \in S_{n-1}(x, 1)$, otherwise $\psi^{-1}(4) = \emptyset$.

The coloring ψ is *strongly adapted* if it is adapted and, additionally, it satisfies the following condition:

(vi) $\psi^{-1}(5) = \emptyset$.

Recall that the sets $C_{n,n-1}(x)$, for $x \in X_n$, form a partition of X_{n-1} by definition.

Lemma 2.6.32. *Consider a family of adapted colorings, $\psi_x\colon C_{n,n-1}(x) \to \mathcal{I}_{n-1}$, for $x \in X_n$, whose combination is denoted by ψ. For every $u \in X_{n-1}$, we have $u \in X_n$ if and only if, either $\psi(u) \in \{1, 2\}$, or $\psi(u) = 0$ and there is some $v \in S_{n-1}(u, 1)$ such that $\psi(v) \in \{3, 4\}$.*

By Proposition 2.4.1 (vi), and Lemmas 2.4.16 and 2.5.1 , we have $d_{-1}(u,v) \leq 2\Lambda_{n-1}R_n^+$ for any $u,v \in C_{n,n-1}(x)$. On the other hand, if $u,v \in \mathfrak{X}_{n-1}$, then $d_{-1}(u,v) \geq \mathfrak{s}_{n-1}$ by Proposition 2.3.3 (i). Since $\mathfrak{s}_{n-1} > 3\Lambda_{n-1}\Gamma_n^+ \geq 3\Lambda_{n-1}R_n^+$ by (3.2.2), (2.4.4) and (2.4.6), it follows that

$$|C_{n,n-1}(x) \cap \mathfrak{X}_{n-1}| \leq 1 \ . \tag{2.6.12}$$

Lemma 2.6.33. *For every $x \in X_n$, there is a strongly adapted coloring $\psi_x \colon C_{n,n-1}(x) \to \mathcal{I}_{n-1}$.*

Proof. First, note that $[7] \subset I_{n-1,u}$ for all $u \in C_{n,n-1}(x)$ by (2.6.1). Define $\psi_x(u) = 0$ for every $u \in C_{n,n-1}(x) \cap \mathfrak{X}_{n-1}$. In the case where $x \in \mathfrak{X}_{n-1}$, choose some $y \in S_{n-1}(x,1)$ and define

$$\psi_x(y) = \begin{cases} 3 & \text{if } x \in \mathfrak{X}_n^- \\ 4 & \text{if } x \notin \mathfrak{X}_n^+ \ . \end{cases}$$

If $x \notin \mathfrak{X}_{n-1}$, set

$$\psi_x(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{X}_n^- \\ 2 & \text{if } x \notin \mathfrak{X}_n^+ \ . \end{cases}$$

Let A be the set of points in $C_{n,n-1}(x)$ that have been already colored at this point. For $u \in C_{n,n-1}(x) \setminus A$, let $\phi_x(u)$ be any color in $I_{n-1,u} \setminus [6]$. □

Proposition 2.6.34. *There is a family of strongly adapted colorings, $\psi_{n,x}^0 \colon C_{n,n-1}(x) \to \mathcal{I}_{n-1}$, for $x \in X_n$, satisfying $\psi_{n,x}^0 = h_{n,x}(\psi_{n,\operatorname{rep}_n(x)}^0)$.*

Proof. This follows from Lemma 2.6.33 like Proposition 2.6.28. □

Proposition 2.6.35. *There is a family of colorings, $\psi_{n,x}^i \colon C_{n,n-1}(x) \to \mathcal{I}_{n-1}$, for $x \in X_n$ and $i \in H_{n,x}$, satisfying the following properties:*

(i) The coloring $\psi_{n,x}^0$ is strongly adapted.

(ii) We have $\psi_{n,x}^i = h_{n,x}(\psi_{n,\operatorname{rep}_n(x)}^i)$.

(iii) Each coloring $\psi_{n,x}^i$ is adapted.

(iv) There are sets $\mathcal{N}_{n,x}^i \subset C_{n-1}(x,10,r_n^\pm - 1)$, for $x \in X_n$ and $i \in H_{n,x}$, satisfying:

(a) $\mathcal{N}_{n,x}^i = \hat{h}_{n,x}(\mathcal{N}_{n,\widehat{\operatorname{rep}}_n(x)}^i)$;
(b) $(\psi_{n-1,x}^i)^{-1}(4) = \mathcal{N}_{n,x}^i$; and
(c) $\mathcal{N}_{n,x}^i \neq \mathcal{N}_{n,x}^j$ if $i \neq j$.

Proof. First, for $i = 0$, we take the strongly adapted colorings $\phi_{0,x}^0$ constructed in Proposition 2.6.28, so that (i) is satisfied.

For every $x \in X_n^{\pm,\widehat{\Phi}}$, let $N_{n,x}$ be a maximal subset of $C_{n-1}(x,10,r_n^\pm) \setminus \mathfrak{X}_{n-1}$ that is $r_{n-1}^2 s_{n-1}$-separated with respect to d_{n-2}. Choose an enumeration of the powerset $\mathcal{P}(N_{n,x})$,

$$\mathcal{P}(N_{n,x}) = \{\, \mathcal{N}_{n,x}^0 := \emptyset,\, \mathcal{N}_{n,x}^1,\, \dots \,\} \ .$$

We have $|D_{n-1}(x,10)| \leq (\deg X_{n-1})^{11}$ and $|C_{n,n-1}(x) \cap \mathfrak{X}_{n-1}| \leq 1$ by Corollary 2.1.19 and (2.6.12). Therefore

$$|C_{n-1}(x,10,r_n^{\pm}) \setminus \mathfrak{X}_{n-1}| \geq |D_{n-1}(x,r_n^{\pm})| - (\deg X_{n-1})^{11} - 1 \ .$$

By Lemma 2.1.7, $N_{n,x}$ is $(r_{n-1}^2 s_{n-1} - 1)$-relatively dense in $|C_{n-1}(x,10,r_n^{\pm})|$ with respect to d_{n-2}, so

$$|N_{n,x}| \geq \left\lfloor \big(|D_{n-1}(x,r_n^{\pm})| - (\deg X_{n-1})^{11} - 1\big)/(\deg X_{n-2})^{r_{n-1}^2 s_{n-1}} \right\rfloor \tag{2.6.13}$$

by Lemma 2.1.20. Therefore, by (2.4.2),

$$\begin{aligned} |\mathcal{P}(N_{n,x})| &\geq \exp_2\left(\left\lfloor \big(|D_{n-1}(x,r_n^{\pm})| - (\deg X_{n-1})^{11} - 1\big)/(\deg X_{n-2})^{r_{n-1}^2 s_{n-1}} \right\rfloor\right) \\ &= \eta_n(|D_{n-1}(x,r_n^{\pm})|) \ . \end{aligned}$$

Thus an injective map $H_{n,x} \to \mathcal{P}(N_{n,x})$ is well defined by $i \mapsto \mathcal{N}_{n,x}^i$.

If $x \notin X_0^{\widehat{\Phi}}$, let $N_{n,x} = \hat{h}_{n,x}(N_{n,\mathrm{rep}_n(x)})$ and $\mathcal{N}_{n,x}^i = \hat{h}_{n,x}(\mathcal{N}_{n,\mathrm{rep}_n(x)}^i)$, so that $N_{n,x}$ satisfies (2.6.13). Then define

$$\psi_{n,x}^i(u) = \begin{cases} \psi_{n,x}^0(u) & \text{if } u \notin \mathcal{N}_{n,x}^i \\ 4 & \text{if } u \in \mathcal{N}_{n,x}^i \ . \end{cases}$$

With this definition, (i) is obvious because $\mathcal{N}_{n,x}^0 = \emptyset$. Property (ii) follows immediately from Proposition 2.6.34 and the fact that $\mathcal{N}_{0,x}^i = h_{0,x}(\mathcal{N}_{0,\mathrm{rep}_0(x)}^i)$ if $x \notin X_0^{\Phi}$. Finally, (iv) follows since $\mathcal{N}_{n,x}^i \neq \mathcal{N}_{n,x}^j$ for $i \neq j$. □

Remark 16. In Section 2.6.1, it was said that $I_{n,x}^2$ is considered as an initial segment of $H_{n,x}$ for every $x \in X_n$. Let $\iota_{n,x}$ denote the inclusion $I_{n,x}^2 \hookrightarrow H_{n,x}$. From now on, the notation $\psi_{n,x}^{i,j}$ will refer to the coloring $\psi_{n,x}^{\iota_{n,x}(i,j)}$.

2.6.7 Colorings ϕ_n^N

In this subsection we proceed to define the colorings ϕ_n^N, which will give us the colorings ϕ^N in the statement of Theorem 2.1.27. First we define the notion of a *rigid* coloring, which are those obtained by combining different colorings $\psi_{0,x}^i$ over clusters $C_{0,-1}(x)$.

Definition 2.6.36. Let $n \in \mathbb{N}$ and $x \in X_n$. A coloring $\phi\colon C_{n,-1}(x) \to [\Delta]$ is called *rigid* if, for all $u \in C_{n,0}(x)$, there is some $i \in H_{n,x}$ such that the restriction of ϕ to $C_{0,-1}(u)$ equals $\psi_{0,x}^i$.

Lemma 2.6.37. *For all $x_1, x_2 \in X_n^+$, if $d_n(x_1,x_2) \leq 2$, then $d_{n-1}(x_1,x_2) < r_n^+ s_n$.*

Proof. By the definition of E_n, there is a point $x_3 \in X_n$ and points, $u_1 \in \overline{C}_{n,n-1}(x_1)$, $u_2 \in \overline{C}_{n,n-1}(x_2)$ and $u_3, u_3' \in \overline{C}_{n,n-1}(x_3)$, such that $u_1 E_{n-1} u_3$ and $u_3' E_{n-1} u_2$. By Lemma 2.4.16, the triangle inequality, (2.2.13) and (2.4.4), we get

$$d_{n-1}(x_1,x_2) \leq 4R_n^+ + 2 = 4(r_n(2s_n+3)) + 2 \leq 20 r_n s_n < r_n s_n^2 \ ,$$

since $s_n > 20$ by (2.2.1) and (2.2.8). □

Lemma 2.6.38. *For all $x_1, x_2, x_3 \in X_n^-$, if $x_1 E_n x_2 E_n x_3$, then $d_{n-1}(x_1, x_3) < r_n^- s_n$.*

Proof. By the definition of E_n, there are points, $u_1 \in \overline{C}_{n,n-1}(x_1)$, $u_2, u_2' \in \overline{C}_{n,n-1}(x_2)$ and $u_3 \in \overline{C}_{n,n-1}(x_3)$, such that $u_1 E_{n-1} u_2$ and $u_2' E_{n-1} u_3$. By Lemma 2.4.16, the triangle inequality, (2.2.13) and (2.4.4), we get

$$d_{n-1}(x_1, x_2) \leq 4R_n^- + 2 = 4(4r_n + 2) + 2 \leq 26r_n < r_n s_n \ ,$$

since $s_n > 26$ by (2.2.1) and (2.2.8). □

Proposition 2.6.39. *For $n \in \mathbb{N}$ and $x \in X_n^\pm$, let $A = C_{n,-1}(x)$ (respectively, $A = D_{n-1}(x, r_n^\pm - 1)$), let $\zeta\colon \bigcup_{a \in A} C_{n-1,-1}(a) \to [\Delta]$ be a rigid coloring, and let $f\colon x \to x$ be an n-equivalence (respectively, an n-weak equivalence) preserving ζ. Then f is the identity map on A.*

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 0$, then the result follows from Proposition 2.6.29 (iv). Therefore suppose that $n > 0$ and the result is true for $0 \leq m < n$. By hypothesis, f is an n-(weak) equivalence and $f(x) = x$. Thus, $f(C_{n-1,n-2}(x)) = C_{n-1,n-2}(x)$ and $f : x \to x$ is an $(n-1)$-equivalence by Definitions 2.6.6 (v) and 2.6.16 (iv). Hence f is the identity on $C_{n-1,n-2}(x)$ by the induction hypothesis.

Let us prove that f is the identity on $C_{n-1,n-2}(u)$ by induction on $u \in A$, using $\trianglelefteq_{n,x}$. The case $u = x$ was proved in the previous paragraph. Thus suppose $u \neq x$. By the induction hypothesis, we have $f(\mathrm{Pa}_{n,x}(u)) = \mathrm{Pa}_{n,x}(u)$. Then we have $(f(u), f(\mathrm{Pa}_{n,x}(u))) \in E_{n-1}$ by Definitions 2.6.6 (iii) and 2.6.16 (ii), and therefore $(f(u), \mathrm{Pa}_{n,x}(u)) \in E_{n-1}$. We consider the following cases.

If $u, f(u) \in X_{n-1}^+$, then $d_{n-2}(u, f(u)) < r_{n-1}^+ s_{n-1}$ by Lemma 2.6.37. If $u, \mathrm{Pa}_{n,x}(u) \in X_{n-1}^-$, then $f(u) \in X_{n-1}^-$ by Definitions 2.6.6 (iii) and 2.6.16 (ii), obtaining $d_{n-2}(u, f(u)) < r_{n-1}^- s_{n-1}$ by Lemma 2.6.38. By Definitions 2.6.6 (iii) and 2.6.16 (ii), we have $\chi_{n-1}(u) = \chi_{n-1}(f(u))$. Thus Proposition 2.6.1 (ii) yields $f(u) = u$ in these two cases.

Finally, suppose that $u, f(u) \in X_{n-1}^-$ and $\mathrm{Pa}_{n,x}(u) \in X_{n-1}^+$. By the definition of E_{n-1}, there is some $u' \in X_{n-1}^+ \cap D_{n-1}(\mathrm{Pa}_{n,x}(u), 1)$ such that there are $v \in \overline{C}_{n-1,n-2}(u)$ and $v' \in C_{n-1,n-2}(u')$ with $v E_{n-2} v'$. Note that this implies $d_{n-1}(x, u') \leq d_{n-1}(x, u)$. If f is an n-equivalence, then this implies $u' \in \mathcal{C}_{n,n-1}(x)$, whereas if f is an n-weak equivalence, we obtain $u' \in D_{n-1}(x, r_n^\pm - 1)$. In any case, using Definitions 2.6.6 and 2.6.16 we get that f restricts to a $(n-1)$-equivalence from u' to $f(u')$. Since $(u', \mathrm{Pa}_{n,x}(u)) \in E_{n-1}$ and $f(\mathrm{Pa}_{n,x}(u)) = \mathrm{Pa}_{n,x}(u)$, we obtain $d_{n-2}(u, f(u)) < r_{n-1}^+ s_{n-1}$, and the same argument of the previous paragraph gives us $f(u') = u'$. Then the induction hypothesis (on n) yields $f(v') = v'$. Therefore $d_{n-2}(v, f(v)) \leq 2$, and we obtain $d_{n-2}(u, f(u)) \leq 2R_n^- + 2$. Then $f(u) = u$ as before, and we get that f is the identity on $C_{n-1,-1}(u)$ by the induction hypothesis. □

Corollary 2.6.40. *For $n \in \mathbb{N}$ and $x, y \in X_n^\pm$, let $A = C_{n,-1}(x)$ (respectively, $A = D_{n-1}(x, r_n^\pm - 1)$), let $\zeta\colon \bigcup_{a \in A} C_{n-1,-1}(a) \to [\Delta]$ and $\hat{\zeta}\colon \bigcup_{b \in f(A)} C_{n-1,-1}(b) \to [\Delta]$ be rigid colorings, and let $f\colon x \to y$ be an n-equivalence (respectively, n-weak equivalence) satisfying $f^*\hat{\zeta} = \zeta$. Then $f = h_{n,y} \circ h_{n,x}^{-1}$ (respectively, $f = \hat{h}_{n,y} \circ \hat{h}_{n,x}^{-1}$).*

Definition 2.6.41. For $N \in \mathbb{N}$, let $\phi_n^N\colon X_n \to \mathcal{I}_n^2$ and $\psi_{-1}^N\colon X_{-1} \to [\Delta]$ be defined by reverse induction on $n = -1, \ldots, N$ as follows:

- For $n = N$, let $\phi_N^N = (\chi_N, 0)$.
- For $0 \le n < N$, define ϕ_n^N so that, for every $x \in X_{n+1}$,

$$\phi_n^N|_{C_{n+1,n}(x)} = \left(\psi_{n,x}^{\phi_{n+1}^N(x)}, \chi_n(x)\right) . \tag{2.6.14}$$

- Finally, define ϕ_{-1}^N so that, for every $x \in X_0$,

$$\phi_{-1}^N|_{C_{0,-1}(x)} = \psi_{-1,x}^{\phi_0^N(x)} . \tag{2.6.15}$$

Remark 17. It follows from Proposition 2.6.1 (ii) that $\phi_n^N(x) \neq \phi_n^N(y)$ for $x, y \in X_n^\pm$ if $0 < d_{n-1}(x, y) < r_n^\pm s_n$.

Remark 18. By Definitions 2.6.1 (i) and 2.6.31 (i), for all $0 \le m \le N$ and $x \in X_m$, the value $\phi_m^N(x)$ determines whether $x \in \mathfrak{X}_m$.

We now prove the crucial lemma from which we will derive Theorem 2.1.27. Let $W_0 = 10$ and $W_i = 2$ for $i > 0$, and let Υ_n be recursively defined by

$$\Upsilon_{-1} = 0 \,, \quad \Upsilon_n = \Upsilon_{n-1} + \Lambda_{n-1}(W_n + 3R_n^+ + 1) + \Gamma_n^+ + \Lambda_n \,. \tag{2.6.16}$$

Lemma 2.6.42. *Fix $0 \le n \le N$ and $R > \Upsilon_n$. Let $A \subset X$ and $x \in A$ be such that $D_{-1}(x, R) \subset A$, and let $f\colon (A, x, \phi_{-1}^N) \to (f(A), f(x), \phi_{-1}^N)$ be a pointed colored graph isomorphism with respect to the restriction of E_{-1}. Then the following properties hold for $0 \le m \le n$ and $0 \le l \le n + 1$:*

(a) The restriction

$$f\colon \left(X_{l-1} \cap D_{-1}(x, R - \Upsilon_{l-1}), x, \phi_{l-1}^N\right) \to \left(X_{l-1} \cap D_{-1}(f(x), R - \Upsilon_{l-1}), f(x), \phi_{l-1}^N\right)$$

is a pointed colored graph isomorphism with respect to E_{l-1}.

(b) For any $z \in X_{m-1} \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}W_m)$, we have $z \in X_m^\pm$ if and only if $f(z) \in X_m^\pm$.

(c) For all $z \in X_m \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + r_m^+))$, the restriction of f is an m-weak equivalence.

(d) For any $z \in X_m \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + r_m^+))$, we have $\phi_m^N(z) = \phi_m^N(f(z))$.

(e) For any $z \in X_m \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + r_m^+ + 1))$, we have $z \in \mathfrak{X}_m$ if and only if $f(z) \in \mathfrak{X}_m$.

(f) For all $z \in X_{m-1} \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 2R_m^+))$, we have $z \in Z_{m-1}^\pm$ if and only if $f(z) \in Z_{m-1}^\pm$.

(g) For any $z \in X_m \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+))$, we have $f(\overline{C}_{m,m-1}(z)) = \overline{C}_{m,m-1}(f(z))$.

(h) For any $z \in X_m \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+) - \Lambda_m)$, we have $f(C_{m,m-1}(z)) = C_{m,m-1}(f(z))$.

(i) *For all* $z, z' \in X_m \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+ + 1) - \Gamma_m^+)$, *we have* $zE_m z'$ *if and only if* $f(z)E_m f(z')$.

(j) *For all* $z \in X_m \cap D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+ + 1) - \Gamma_m^+ - \Lambda_m)$, *the restriction of* f *to* $\bigcup_{u \in D_m(z,1)} \overline{C}_{m,-1}(u)$ *is an* m*-equivalence.*

Proof. We proceed by induction on m and l. For $l = 0$, (a) is true by hypothesis. When $l > 0$, (a) follows from (2.6.16) and the induction hypothesis for $m = l - 1$ with (d) and (i). For $m = 0, \dots, n$, we are going to derive (b)–(i) from (a), completing the proof of the lemma.

Let us prove (b). The coloring ϕ_{m-1}^N is adapted by Remark 17. For every $z \in X_{m-1}$, we have $z \in X_m^\pm$ if and only if the colored set $(D_{m-1}(z, W_m/2), \phi_{m-1}^N)$ has one of the patterns described in Definition 2.6.26 (i) and Lemma 2.6.32. By Proposition 2.4.1 (vi) and the triangle inequality, we get

$$D_{m-1}(z, W_m) \subset D_{-1}(z, \Lambda_{m-1} W_m) \subset D_{-1}(x, R - \Upsilon_m).$$

Therefore the restriction $f\colon D_{m-1}(z, W_m/2) \to D_{m-1}(f(z), W_m/2)$ is an isometry by Corollary 2.1.13. The induction hypothesis with (a) implies that the set $D_{m-1}(z, W_m/2)$ has one of the patterns of Definition 2.6.26 (i) and Lemma 2.6.32 if and only if the set $D_{m-1}(f(z), W_m/2)$ does. Then (b) follows from (a).

To prove (c), let $z \in X_m^\pm$. If $m = 0$, (c) is obvious. Thus suppose $m > 0$. We have $f(z) \in X_m^\pm$ by (b). By Proposition 2.4.1 (vi),

$$D_{m-1}(z, r_m^\pm) \subset D_{-1}(z, \Lambda_{m-1} r_m^+) \subset D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1} W_m) .$$

Now, in Definition 2.6.16, the properties (i) and (ii) follow from (a), the property (iii) holds by the induction hypothesis with (e), and the property (iv) follows from the induction hypothesis with (j).

Let us prove (d). By Definition 2.6.41, the restriction of ϕ_{m-1}^N to $C_{m,m-1}(z)$ equals $(\psi_{m-1,z}^i, \chi_{m-1}(z))$ for some $i \in H_{m,z}$. Then $\phi_m^N(z) = \phi_m^N(f(z))$ if and only if the restrictions of ϕ_{m-1}^N to $C_{m,m-1}(z)$ and $C_{m,m-1}(f(z))$ are equal to $(\psi_{m-1,z}^i, \chi_{m-1}(z))$ and $(\psi_{m-1,f(z)}^i, \chi_{m-1}(f(z)))$, respectively. Furthermore i is determined by $(\phi_{m-1}^N)^{-1}(4) \cap D_{m-1}(z, r_m^\pm - 1) = \mathcal{N}_{m,x}^i$ if $m > 0$, or by $(\phi_{-1}^N)^{-1}(0) \cap C_{-1}(z, 10, r_0^\pm - 1) = \mathcal{N}_{0,x}^i$ if $m = 0$. By (a),

$$f((\phi_{m-1}^N)^{-1}(4) \cap D_{m-1}(z, r_m^\pm - 1)) = (\phi_{m-1}^N)^{-1}(4) \cap D_{m-1}(f(z), r_m^\pm - 1)$$

if $m > 0$, and

$$f((\phi_{-1}^N)^{-1}(0) \cap C_{-1}(z, 10, r_0^\pm - 1)) = (\phi_{-1}^N)^{-1}(0) \cap C_{-1}(f(z), 10, r_0^\pm - 1)$$

if $m = 0$. Since $\chi_{m-1}(z) = \chi_{m-1}(f(z))$ by (c) and Definition 2.6.16 (ii), property (d) follows from Proposition 2.6.35 (a).

Property (e) follows from (d) and Remark 18.

Let us prove (f). Let $z \in D_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 2R_m^+))$. By (a), Proposition 2.4.1 (vi) and Corollary 2.1.12, the restriction of f to $D_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + R_m^+))$ preserves $X_n^\pm$ and is an R_m^+-short scale isometry with respect to E_{m-1}. Then z satisfies (2.4.18) if and only if $f(z)$ does, and (f) follows.

To prove (g), let $z \in X_m \cap D_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+))$. By Lemma 2.4.16, we have $\overline{C}_{m,m-1}(z) \subset D_{m-1}(z, R_n^+)$. Using Proposition 2.4.1 (vi) and the triangle inequality, we get

$$\overline{C}_{m,m-1}(z) \subset D_{m-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 2R_m^+)) .$$

Therefore, for all $u \in \overline{C}_{m,m-1}(z)$, we have $u \in Z_{m-1}^\pm$ if and only if $f(u) \in Z_{m-1}^\pm$ by (f). Let $y \in X_m$ such that $d_{m-1}(u, X_m) = d_{m-1}(u, y)$. Thus $d_{m-1}(u, y) \leq R_m^+$ by Proposition 2.4.1 (iv), yielding $d_{-1}(u, y) \leq \Lambda_{m-1} R_m^+$ by Proposition 2.4.1 (vi). By (a), (b) and Corollary 2.1.12, we get $f(y) \in X_m^\pm$ if and only if $y \in X_m^\pm$ and $d_{m-1}(u, y) = d_{m-1}(f(u), f(y))$. Then (g) follows by (2.4.19).

Let us prove (h). By Proposition 2.4.1 (vi) and the triangle inequality, we get

$$D_m(z, 1) \subset D_{-1}(z, \Lambda_m) \subset D_{-1}(x, R - \Upsilon_{m-1} - \Lambda_{m-1}(W_m + 3R_m^+))) .$$

Therefore $f(\overline{C}_{n,n-1}(u)) = \overline{C}_{n,n-1}(f(u))$ for all $u \in D_m(z, 1)$ by (g). Moreover $\phi_m^N(u) = \phi_m^N(f(u))$ for all $u \in D_m(z, 1)$ by (d). In particular, this yields $\chi_m(u) = \chi_m(f(u))$. Then the result follows from Proposition 2.6.1 (ii) and (2.5.2).

Property (i) follows easily from (g), Corollary 2.1.12 and the definition of E_m.

Finally, (j) follows from (a), (b), (d), (e) and the induction hypothesis with (j). □

We are now in position to complete the proof of Theorem 2.1.27. Consider the increasing sequence of positive integers ε_n of the statement of Theorems 2.1.26 and 2.1.27, used in Section 2.2. Let $\delta_n = 4\Gamma_n^+ + \Upsilon_n + 2\Lambda_n$.

Proposition 2.6.43. *For $0 \leq n \leq N$ and $u \in X$, let*

$$f\colon \left(D_{-1}(u, \delta_n), u, \phi_{-1}^N\right) \to \left(D_{-1}(f(u), \delta_n), f(u), \phi_{-1}^N\right)$$

be a color-preserving pointed graph isomorphism with respect to E_{-1}. Then, either $f(u) = u$, or $d_{-1}(u, f(u)) > \varepsilon_n$.

Proof. Let $x \in X_n^\pm$ such that $u \in C_{n,-1}(x)$. We have $d_{-1}(u, x) \leq \Gamma_n^+$ by Lemma 2.5.1, and $D_{-1}(x, 3\Gamma_n^+ + \Upsilon_n + 2\Lambda_n) \subset \operatorname{dom} f$ by the triangle inequality. By Lemma 2.6.42 (b),(d), we obtain $f(x) \in X_n^\pm$ and $\phi_n^N(x) = \phi_n^N(f(x))$. In particular, $\chi_n(x) = \chi_n(f(x))$. Therefore, either $f(x) = x$, or $d_{n-1}(x, f(x)) \geq r_n^\pm s_n$ by Proposition 2.6.1 (ii).

If $f(x) = x$, then $f(u) = u$ by Proposition 2.6.39 and the result follows. So suppose $d_{n-1}(x, f(x)) \geq 2r_n^\pm s_n$. By Lemma 2.5.1, $d_{-1}(u, x) = d_{-1}(f(u), f(x)) \leq \Gamma_n^\pm$. Then, by the triangle inequality, $d(u, f(u)) \geq r_n^\pm s_n - 2\Gamma_n^\pm$. Applying now Lemma 2.2.1, we get $d(u, f(u)) \geq \varepsilon_n$. □

This completes the proof of Theorem 2.1.27 (i) taking $\phi^N = \phi_{-1}^N$.

Proposition 2.6.44. *For $n = 0, \dots, N$, $x \in \mathfrak{X}_n$ and $u \in C_{n,m}(p)$, we have $\phi_m^N(u) = \phi_m^N(\mathfrak{h}_{n,x}(u))$ for $-1 \leq m \leq n$.*

Proof. We proceed by inverse induction on m. For $m = N$, we have $\phi_N^N = (\chi_N, 0)$. So $\phi_N^N(u) = \phi_N^N(\mathfrak{h}_{n,x}(u))$ by Proposition 2.6.1 (iii).

Suppose that, for $0 \leq m < N - 1$, the result is true for $m + 1$. Let $u \in C_{n,m}(p)$ and $z \in C_{n,m+1}(p)$ such that $u \in C_{m+1,m}(z)$. By the induction hypothesis, $\phi_{m+1}^N(z) = \phi_{m+1}^N(\mathfrak{h}_{n,x}(z))$. By the definition of ϕ_{m+1}^N, Lemmas 2.6.4 and 2.6.9, and Corollary 2.6.40, this means that the restrictions of ϕ^N_{m+1} to $C_{m+1,m}(z)$ and $C_{m+1,m}(\mathfrak{h}_{n,x}(z))$ equal $\psi^{i,j}_{m,x}$ and $\psi^{i,j}_{m,\mathfrak{h}_{n,x}(z)}$ for some $(i,j) \in I^2_{m,x} \subset H_{m,x}$ (see Remark 16). But $\psi^{i,j}_{m,\mathfrak{h}_{n,x}(z)} = \mathfrak{h}_{n,x}(\psi^{i,j}_{m,x})$ by Proposition 2.6.35 (ii). □

Propositions 2.4.1 and 2.6.44 (i), together with Corollary 2.5.3, yield $\mathfrak{X}_n \subset \widehat{\Omega}_n$ for $n \leq N$ by taking $\phi^N = \phi^N_{-1}$, with the set $\widehat{\Omega}_n$ defined in Theorem 2.1.27 (ii). Then Theorem 2.1.27 (ii) follows from Propositions 2.3.4 and 2.4.1 (i) taking $\alpha_n = 2\mathfrak{s}_n + \mathfrak{t}_l + 3\omega_n$.

Chapter 3

Realization of Riemannian manifolds as leaves

3.1 Preliminaries

3.1.1 Riemannian manifolds

Let M be a connected complete Riemannian n-manifold, g its metric tensor, d its distance function, ∇ its Levi-Civita connection, R its curvature tensor, $\operatorname{inj}(x)$ its injectivity radius at $x \in M$, and $\operatorname{inj} = \inf_{x\in M} \operatorname{inj}(x)$ (its injectivity radius). If necessary, we may add "M" as a subindex or superindex to this notation, or the subindex or superindex "i" when a family of Riemannian manifolds M_i is considered. Since M is complete, it is proper as metric space.

Let $T^{(0)}M = M$, and $T^{(m)}M = TT^{(m-1)}M$ for $m \in \mathbb{Z}^+$. If $l < j$, then $T^{(l)}M$ is sometimes identified with a regular submanifold of $T^{(m)}M$ via zero sections. Any C^m map between Riemannian manifolds, $h : M \to M'$, induces a map $h_*^{(m)} : T^{(m)}M \to T^{(m)}M'$ defined by $h_*^{(0)} = h$ and $h_*^{(m)} = (h_*^{(m-1)})_*$ for $m \in \mathbb{Z}^+$.

The Levi-Civita connection determines a decomposition $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$, as direct sum of the horizontal and vertical subbundles. Consider the *Sasaki metric* $g^{(1)}$ on TM, which is the unique Riemannian metric such that $\mathcal{H} \perp \mathcal{V}$ and the canonical identities $\mathcal{H}_\xi \equiv T_\xi M \equiv \mathcal{V}_\xi$ are isometries for every $\xi \in TM$. For $m \geq 2$, consider the *Sasaki metric* $g^{(m)} = (g^{(m-1)})^{(1)}$ on $T^{(m)}M$. The notation $d^{(m)}$ is used for the corresponding distance function, and the corresponding open and closed balls of center $v \in T^{(m)}M$ and radius $r > 0$ are denoted by $B^{(m)}(v,r)$ and $D^{(m)}(v,r)$. For $l < j$, $T^{(l)}M$ is totally geodesic in $T^{(m)}M$ and $g^{(m)}|_{T^{(l)}M} = g^{(l)}$.

Let $D \subset M$ be a compact domain[1] and $m \in \mathbb{N}$. The C^m tensors on D of a fixed type form a Banach space with the norm $\| \ \|_{C^m,D,g}$ defined by

$$\|A\|_{C^m,D,g} = \max_{0\leq l\leq m,\ x\in D} |\nabla^l A(x)| \,.$$

By taking the projective limit as $m \to \infty$, we get the Fréchet space of C^∞ tensors on D of that type equipped with the C^∞ *topology* (see e.g. [39]). Similar definitions apply to the

[1] A regular submanifold of the same dimension as M, possibly with boundary.

space of C^m or C^∞ functions on M with values in an Euclidean space or in a separable Hilbert space $\mathfrak{H}$.

Recall that a C^1 map between Riemannian manifolds, $h\colon M \to M'$, is called a (λ-) *quasi-isometry* if there is some $\lambda \geq 1$ such that $\lambda^{-1}|v| \leq |h_*(v)| \leq \lambda |v|$ for all $v \in TM$.

For $m \in \mathbb{N}$, a partial map $h : M \rightarrowtail M'$ is called a C^m *local diffeomorphism* if $\operatorname{dom} h$ and $\operatorname{im} h$ are open in M and M', respectively, and $h : \operatorname{dom} h \to \operatorname{im} h$ is a C^m diffeomorphism. If moreover $h(x) = x'$ for distinguished points, $x \in \operatorname{dom} h$ and $x' \in \operatorname{im} h$, then h is said to be *pointed*, and the notation $h : (M, x) \rightarrowtail (M', x')$ is used. The term (*pointed*) *local homeomorphism* is used in the C^0 case.

For $m \in \mathbb{N}$, $R > 0$ and $\lambda \geq 1$, an (m, R, λ)-*pointed partial quasi-isometry*[2] (or simply an (m, R, λ)-*p.p.q.i.*) is a pointed partial map $h\colon (M, x) \rightarrowtail (M', x')$, with $\operatorname{dom} h = D(x, R)$, which can be extended to a C^{m+1}-diffeomorphism $\tilde h$ between open subsets such that $D^{(m)}_M(x, R) \subset \operatorname{dom} \tilde h^{(m)}_*$ and $\tilde h^{(m)}_*$ is a λ-quasi-isometry of some neighborhood of $D^{(m)}_M(x, R)$ in $T^{(m)}M$ to $T^{(m)}M'$. The following result has an elementary proof.

Proposition 3.1.1. *Let $h\colon (M, x) \rightarrowtail (M, y)$ be an (m, R, λ)-p.p.q.i. and $h'\colon (M, x) \rightarrowtail (M, y')$ an (m', R', λ')-p.p.q.i. Then $h^{-1}\colon (M, y) \rightarrowtail (M, x)$ is an $(m, \lambda^{-1}R, \lambda)$-p.p.q.i. If $m' \geq m$ and $R\lambda + d(x, y) \leq R'$, then $h' \circ h\colon (M, x) \rightarrowtail (M, h'(y))$ is an $(m, R, \lambda\lambda')$-p.p.q.i.*

In the following two results, E is a (real) Hilbert bundle over M, equipped with an orthogonal connection ∇. Let $C^m(M; E)$ denote the space of its C^m sections ($m \in \mathbb{N} \cup \{\infty\}$), and E_x its fiber over any $x \in M$.

Proposition 3.1.2 (Cf. [8, Proposition 3.11])**.** *Let $\mathcal{S} \subset C^{m+1}(M; E)$ for $m \in \mathbb{N}$, and let $x_0 \in M$. Then $\mathcal{S}$ is precompact in $C^m(M; E)$ if*

(i) $\sup_{s\in\mathcal{S}} \sup_D |\nabla^k s| < \infty$ *for every compact subset $D \subset M$ and $1 \leq k \leq m + 1$; and*

(ii) $\{\, (\nabla^k s)(x_0) \mid s \in \mathcal{S} \,\}$ *is precompact in $E_{x_0} \otimes \bigotimes_k T^*_{x_0}M$ for all $0 \leq k \leq m$.*

Proof. We proceed by induction on m. Consider the case $m = 0$. From (i) for $k = 1$, it follows that $\mathcal{S}$ is equicontinuous on the interior of D, and therefore on M because D is an arbitrary compact subset. Moreover (ii) for $k = 0$ states that $\{\, s(x_0) \mid s \in \mathcal{S} \,\}$ is precompact in E_{x_0}. So $\mathcal{S}$ is precompact in $C(M; E)$ by the Arzelà-Ascoli theorem.

Now assume that $m \geq 1$ and the result is true for $m - 1$. Given $x \in M$, $0 \leq t, u \leq 1$ and a piecewise smooth path $c : [0, 1] \to M$ from x_0 to x, let $P^u_{c,t} : E_{c(t)} \to E_{c(u)}$ be the ∇-parallel transport along c from u to v. For any $e \in E_{x_0}$ and $\alpha \in C^{m-1}(M; E \otimes T^*M)$, let

$$Q_c(e, \alpha) = P^1_{c,0}(e) + \int_0^1 P^1_{c,t}\alpha(c'(t))\, dt \in E_x\, .$$

This expression defines a continuous map $Q_c : E_{x_0} \times C^{m-1}(M; E \otimes T^*M) \to E_x$. In particular, for any $s \in C^m(M; E)$, we have

$$Q_c(s(x), \nabla s) = s(y) \tag{3.1.1}$$

[2]The extension $\tilde h$ is an (m, R, λ)-pointed local quasi-isometry, as defined in [6]. On the other hand, any (m, R, λ)-pointed local quasi-isometry defines an (m, R, λ)-pointed partial quasi-isometry by restriction. Thus both notions are equivalent.

because

$$P^1_{c,t}\nabla_{c'(t)}s = \frac{d}{du}P^t_{c,u}sc(u)|_{u=t}$$

if c is smooth at t.

Let

$$T : C^m(M;E) \to E_{x_0} \times C^{m-1}(M;E\otimes T^*M)$$

be defined by $T(s) = (s(x_0), \nabla s)$, and let $\Omega(M, x_0)$ denote the set of piecewise smooth loops $d : [0,1] \to M$ based at x_0.

Claim 1. The following properties hold:

(a) We have

$$\operatorname{im} T = \{\, (e,\alpha) \in E_{x_0} \times C^{m-1}(M;E\otimes T^*M) \mid Q_d(e,\alpha) = e \ \forall d \in \Omega(M,x_0) \,\}\ .$$

(b) T is a closed embedding, and $T^{-1} : \operatorname{im} T \to C^m(M;E)$ is given by $T^{-1}(e,\alpha)(x) = Q_c(e,\alpha)$, where $c : [0,1] \to M$ is any piecewise smooth path from x_0 to x.

If $(e,\alpha) \in \operatorname{im} T$, then $Q_d(e,\alpha) = e$ for all $d \in \Omega(M,x_0)$ by (3.1.1).

Now suppose that $Q_d(e,\alpha) = e$ for all $d \in \Omega(M,x_0)$. Then a section $s \in C^m(M;E)$ is well defined by $s(x) = Q_c(e,\alpha)$, where $c : [0,1] \to M$ is any piecewise smooth path from x_0 to x. By choosing the constant path at x_0, it follows that $s(x_0) = e$. On the other hand, given $x \in M$ and $X \in T_xM$, there is a piecewise smooth path $c : [0,1] \to M$ from x_0 to x with $c'(1) = X$. Hence

$$\begin{aligned}\nabla_X s &= \frac{d}{du}P^1_{c,u}sc(u)|_{u=1} = \frac{d}{du}P^1_{c,u}\Big(P^u_{c,0}(e) + \int_0^u P^u_{c,t}\alpha(c'(t))\,dt\Big)\Big|_{u=1}\\ &= \frac{d}{du}\Big(P^1_{c,0}(e) + \int_0^u P^1_{c,t}\alpha(c'(t))\,dt\Big)\Big|_{u=1} = \alpha(X)\ .\end{aligned}$$

So $\nabla s = \alpha$, and therefore $Ts = (e,\alpha)$. Thus $(e,\alpha) \in \operatorname{im} T$, completing the proof of (a).

The above argument also shows that T is injective, and $T^{-1} : \operatorname{im} T \to C^m(M;E)$ is given by $T^{-1}(e,\alpha)(x) = Q_c(e,\alpha)$, where $c : [0,1] \to M$ is any piecewise smooth path from x_0 to x. Thus $T^{-1} : \operatorname{im} T \to C^m(M;E)$ is continuous, showing that T is an embedding.

Finally, $\operatorname{im} T$ is closed by (a) and the continuity of $Q_d : E_{x_0} \times C^{m-1}(M;E\otimes T^*M) \to E_{x_0}$ for every $d \in \Omega(M,x_0)$. Thus T is also a closed map, and the proof of Claim 1 is finished.

By Claim 1, it is enough to prove that $T(\mathcal{S})$ is precompact in $E_{x_0} \times C^{m-1}(M;E\otimes T^*M)$. But

$$T(\mathcal{S}) \subset \{\, s(x_0) \mid s \in \mathcal{S} \,\} \times \nabla(\mathcal{S})\ ,$$

where the first factor is already known to be precompact in E_{x_0}. On the other hand, we have $\nabla(\mathcal{S}) \subset C^m(M;E\otimes T^*M)$, and this subspace satisfies (i) for $1 \le k \le m$ and (ii) for $0 \le k \le m-1$. So $\nabla(\mathcal{S})$ is precompact in $C^{m-1}(M;E\otimes T^*M)$ by the induction hypothesis. Thus $T(\mathcal{S})$ is precompact in $E_{x_0} \times C^{m-1}(M;E\otimes T^*M)$ because it is contained in a precompact subspace. □

Corollary 3.1.3. *Let $\mathcal{S} \subset C^\infty(M;E)$ and $x_0 \in M$. Then $\mathcal{S}$ is precompact in $C^\infty(M;E)$ if and only if conditions (i) and (ii) in Proposition 3.1.2 are satisfied for all $k \in \mathbb{N}$.*

Proof. The "only if" part follows from the continuity of

$$\nabla^k : C^\infty(M;E) \to C^\infty\Big(M; E \otimes \bigotimes_k T^*M\Big)$$

for all $k \in \mathbb{N}$. The "if" part is true by Proposition 3.1.2 since $C^\infty(M;E) = \bigcap_m C^m(M;E)$ with the inverse limit topology. □

Recall that M is said to be of *bounded geometry* if $\operatorname{inj}_M > 0$ and $\sup_M |\nabla^m R_M| < \infty$ for all $m \in \mathbb{N}$. For a given manifold M of bounded geometry, the optimal bounds of the previous inequalities will be referred to as the *geometric bounds* of M. Let $B_r = B_{\mathbb{R}^n}(0,r)$ ($r > 0$).

Proposition 3.1.4 (See [57, Theorem A.1], [58, Theorem 2.5], [55, Proposition 2.4], [29])**.** *M is of bounded geometry if and only if there is some $0 < r_0 < \operatorname{inj}_M$ such that, for normal parametrizations $\kappa_x : B_{r_0} \to B_M(x, r_0)$ ($x \in M$), the corresponding metric coefficients, g_{ij} and g^{ij}, as a family of C^∞ functions on B_{r_0} parametrized by x, i and j, lie in a bounded subset of the Fréchet space $C^\infty(B_{r_0})$.*

Proposition 3.1.5 (See the proof of [58, Proposition 3.2], [63, A1.2 and A1.3])**.** *Suppose that M is of bounded geometry. For every $\tau > 0$, there is some map $c\colon \mathbb{R}^+ \to \mathbb{N}$, depending only on τ and the geometric bounds of M, such that, for any τ-separated subset $X \subset M$, and all $x \in M$ and $\delta > 0$, we have $|D(x,\delta) \cap X| \le c(\delta)$.*

Proposition 3.1.6. *Let X be a τ-separated η-relatively dense subset of a manifold of bounded geometry M for some $0 < \tau < \eta$. Given $0 < \varepsilon < \tau/2$ and $\sigma > 0$, let $\tau' = \tau - 2\varepsilon$ and $\eta' = \eta + \varepsilon$. Then there is some $0 < P = P(\varepsilon) < \sigma$, depending only on τ, ε, σ and the geometric bounds of M, with $P(\varepsilon) \to 0$ as $\varepsilon \to 0$, and such that, for every $0 < \rho < P$ and $A \subset X$ satisfying $d(a,b) \notin (\sigma - \rho, \sigma + \rho)$ for all $a, b \in A$, there is an ε-perturbation $X' \subset M$ of X such that $A \subset X'$ and $d(x',y') \notin (\sigma - \rho, \sigma + \rho)$ for all $x', y' \in X'$. In particular, X' is τ'-separated and η'-relatively dense by Lemma 2.1.6.*

Proof. By Propositions 3.1.4 and 3.1.5, the following properties hold:

(a) There are $C, P_0 > 0$ such that every τ'-separated subset $Y \subset M$ satisfies $|Y \cap D(y, \sigma + \rho + \tau/2)| \le C$ for all $y \in Y$ and $0 < \rho < P_0$.

(b) There is some $K = K(\varepsilon) > 0$, with $K(\varepsilon) \to 0$ as $\varepsilon \to 0$, and such that $\operatorname{vol} B(x, \varepsilon) \ge K$ for all $x \in M$.

(c) Given $0 < L < K/C$, there is some $0 < P = P(\varepsilon) \le P_0$, with $P(\varepsilon) \to 0$ as $\varepsilon \to 0$, and such that $\operatorname{vol} C(x, \sigma - \rho, \sigma + \rho) \le L$ for $x \in M$ and $0 < \rho < P$.

Take any $0 < \rho < P$.

Claim 2. Let $X' \subset M$ be a τ'-separated subset, and let

$$A = \{\, x \in X' \mid d(x,y) \notin (\sigma - \rho, \sigma + \rho)\ \forall y \in X' \,\} .$$

Then, for all $x \in X' \setminus A$, there is some $\hat{x} \in M$ such that $d(x, \hat{x}) < \varepsilon$ and

$$((X' \setminus \{x\}) \cup \{\hat{x}\}) \cap C(\hat{x}, \sigma - \rho, \sigma + \rho) = \emptyset .$$

By (a), the subset

$$Z = \{ z \in X \mid B(x,\varepsilon) \cap C(z,\sigma-\rho,\sigma+\rho) \neq \emptyset \} \subset X \cap C(z,\sigma-\rho,\sigma+\rho+\tau/2)$$

has cardinality at most C. Thus, by (c) and (b), for all $x \in X' \setminus A$,

$$\begin{aligned}\operatorname{vol}\Big(B(x,\varepsilon) \cap \bigcup_{z\in Z} C(z,\sigma-\rho,\sigma+\rho)\Big) &\leq \sum_{z\in Z} \operatorname{vol} C(z,\sigma-\rho-\varepsilon,\sigma+\rho+\varepsilon) \\ &\leq CL < K \leq \operatorname{vol} B(x,\varepsilon)\ .\end{aligned}$$

So there is some $\hat{x} \in B(x,\varepsilon)$ such that $\hat{x} \notin C(y,\sigma-\rho,\sigma+\rho)$ for every $y \in Z$. Therefore $\hat{x} \notin C(y,\sigma-\rho,\sigma+\rho)$ for all $y \in X'$, and Claim 2 follows.

Let $x_1, x_2, \dots$ be a (finite or infinite) sequence enumerating the elements of $X \setminus A$. Then X' is defined as the union of A and a set of elements x'_i defined by induction on i as follows. Equivalently, taking $X_0 = X$, we construct the sets $X_i = (X_{i-1} \setminus \{x_i\}) \cup \{x'_i\}$ by induction on i. Assuming that X_{i-1} is defined for some i, by Claim 2, there is some x'_i such that $d(x_i, x'_i) < \varepsilon$ and $X_i \cap C(x'_i,\sigma-\rho,\sigma+\rho) = \emptyset$. These conditions yield the desired properties of X'; in particular, it is a τ'-separated η'-relatively dense subset of M by the triangle inequality. □

Proposition 3.1.7. *Let X be an ε-relatively dense subset of M for some $\varepsilon > 0$, and let h be an isometry of M. If ε is small enough and $h = \mathrm{id}$ on X, then $h = \mathrm{id}$ on M.*

Proof. Fix any $x_0 \in M$ and $0 < r_0 < \operatorname{inj}_M(x_0)$. For $0 < r \leq r_0$, let $\check{B}(r)$ denote the open ball $B(0,r)$ in $T_{x_0}M$. Moreover let $\check{X} = \exp_{x_0}^{-1}(X) \subset T_{x_0}M$. There is some $\lambda \geq 1$ such that $\exp_{x_0} : \check{B}(r_0) \to B_M(x_0,r_0)$ is a λ-bi-Lipschitz diffeomorphism. Since X is an ε-relatively dense subset of M, for all $x \in B_M(x_0, r_0-\varepsilon)$, there is some $y \in X \cap B_M(x_0,r_0)$ with $d_M(x,y) < \varepsilon$. Hence, for all $v \in \check{B}(r_0-\varepsilon)$, there is some $w \in \check{X} \cap \check{B}(r_0)$ with $|v-w| < \lambda\varepsilon$. If ε is small enough, it follows that $\check{X} \cap \check{B}(r_0)$ generates $T_{x_0}M$. Since $h_* = \mathrm{id}$ on $\check{X} \cap \check{B}(r_0)$ because $h = \mathrm{id}$ on X, we get $h_* = \mathrm{id}$ on $T_{x_0}M$, yielding $h = \mathrm{id}$ on M. □

3.1.2 Foliated spaces

A *foliated space* $\mathfrak{X} \equiv (\mathfrak{X}, \mathcal{F})$ of *dimension* n is a Polish space $\mathfrak{X}$ equipped with a partition $\mathcal{F}$ (*foliated structure* or *lamination*) into injectively immersed manifolds (*leaves*) so that $\mathfrak{X}$ has an open cover $\{U_i\}$ with homeomorphisms $\phi_i : U_i \to B_i \times \mathfrak{T}_i$, for some open balls $B_i \subset \mathbb{R}^n$ and Polish spaces $\mathfrak{T}_i$, such that the slices $B_i \times \{*\}$ correspond to open sets in the leaves (*plaques*); every (U_i, ϕ_i) is called a *foliated chart* and $\mathcal{U} = \{U_i, \phi_i\}$ a *foliated atlas*. The corresponding changes of foliated coordinates are locally of the form $\phi_i \circ \phi_j^{-1}(y,z) \mapsto (f_{ij}(y,z), h_{ij}(z))$. Let $p_i : U_i \to \mathfrak{T}_i$ denote the projection defined by every ϕ_i, whose fibers are the plaques. The subspaces transverse to the leaves are called *transversals*; for instance, the subspaces $\phi_i^{-1}(\{*\} \times \mathfrak{T}_i) \equiv \mathfrak{T}_i$ are local transversals. A transversal is said to be *complete* if it meets all leaves. $\mathfrak{X}$ is called a *matchbox manifold* if it is compact and connected, and its local transversals are totally disconnected.

We can assume that $\mathcal{U}$ is *regular* in the sense that it is locally finite, every ϕ_i can be extended to a foliated chart whose domain contains $\overline{U_i}$, and every plaque of U_i meets at most one plaque of U_j. In this case, the the maps h_{ij} define unique homeomorphisms

$h_{ij} : p_j(U_i \cap U_j) \to p_i(U_i \cap U_j)$ (*elementary holonomy transformations*), which generate a pseudogroup $\mathcal{H}$ on $\mathfrak{T} := \bigsqcup_i \mathfrak{T}_i$. This $\mathcal{H}$ is unique up to Haefliger's equivalences [37,38], and its equivalence class is called the *holonomy pseudogroup*. The $\mathcal{H}$-orbits are equipped with a connected graph structure so that a pair of points is joined by an edge if they correspond by some h_{ij}. The projections p_i define an identity between the leaf space $\mathfrak{X}/\mathcal{F}$ and the orbit space $\mathfrak{T}/\mathcal{H}$. Moreover we can choose points $y_i \in B_i$ so that the corresponding local transversals $\phi_i^{-1}(\{y_i\} \times \mathfrak{T}_i)$ are disjoint. Then their union is a complete transversal homeomorphic to $\mathfrak{T}$, and the $\mathcal{H}$-orbits are given by the intersection of the complete transversal with the leaves. If $\mathfrak{X}$ is compact, then $\mathcal{U}$ is finite, and therefore the vertex degrees of the $\mathcal{H}$-orbits is bounded by the finite number of maps h_{ij}. Moreover the coarse quasi-isometry class of the $\mathcal{H}$-orbits is independent of $\mathcal{U}$ in this case.

If the functions $y \mapsto f_{ij}(y, z)$ are C^∞ with partial derivatives of arbitrary order depending continuously on z, then $\mathcal{U}$ defines a C^∞ *structure* on $\mathfrak{X}$, which is called a C^∞ *foliated space* when equipped with such structure. In this case, C^∞ bundles and sections also make sense on $\mathfrak{X}$, defined by requiring that their local descriptions are C^∞ in a similar sense. For instance, the tangent bundle $T\mathfrak{X}$ (or $T\mathcal{F}$) is the C^∞ vector bundle on $\mathfrak{X}$ that consists of the vectors tangent to the leaves, and a *Riemannian metric* on $\mathfrak{X}$ consists of Riemannian metrics on the leaves that define a C^∞ section on $\mathfrak{X}$. This gives rise to the concept of *Riemannian foliated space*. If $\mathfrak{X}$ is a compact C^∞ foliated space, then the differentiable quasi-isometry type of every leaf is independent of the choice of the Riemannian metric on $\mathfrak{X}$, and is coarsely quasi-isometric to the corresponding $\mathcal{H}$-orbits (see e.g. [10, Section 10.3]).

Many of the concepts and properties of foliated spaces are direct generalizations from foliations. Several results about foliations have obvious versions for foliated spaces, like the holonomy group and holonomy cover of the leaves, and the Reeb's local stability theorem. This can be seen in the following standard references about foliated spaces: [51], [17, Chapter 11], [18, Part 1] and [33].

3.1.3 The spaces $\mathcal{M}_*^n$ and $\widehat{\mathcal{M}}_*^n$

For any $n \in \mathbb{N}$, consider triples (M, x, f), where (M, x) is a pointed complete connected Riemannian n-manifold and $f : M \to \mathfrak{H}$ is a C^∞ function to a separable (real) Hilbert space (of finite or infinite dimension). Two such triples, (M, x, f) and (M', x', f'), are said to be *equivalent* if there is a pointed isometry $h : (M, x) \to (M', x')$ such that $h^* f' = f$. Let[3] $\widehat{\mathcal{M}}_*^n = \widehat{\mathcal{M}}_*^n(\mathfrak{H})$ be the set[4] of equivalence classes $[M, x, f]$ of the above triples (M, x, f). A sequence $[M_i, x_i, f_i] \in \widehat{\mathcal{M}}_*^n$ is said to be C^∞ *convergent* to $[M, x, f] \in \widehat{\mathcal{M}}_*^n$ if, for any compact domain $D \subset M$ containing x, there are pointed C^∞ embeddings $h_i : (D, x) \to (M_i, x_i)$, for large enough i, such that $h_i^* g_i \to g_M|_D$ and $h_i^* f_i \to f|_D$ as $i \to \infty$ in the C^∞ topology[5]. In other words, for all $m \in \mathbb{N}$, $R, \varepsilon > 0$ and $\lambda > 1$, there is an (m, R, λ)-

[3] In [6,8,10], the notation $\mathcal{M}_*(n)$ and $\widehat{\mathcal{M}}_*(n)$ was used instead of $\mathcal{M}_*^n$ and $\widehat{\mathcal{M}}_*^n$, adding the superindex "∞" when equipped with the topology defined by the C^∞ convergence.

[4] The cardinality of each complete connected Riemannian n-manifold is less than or equal to the cardinality of the continuum, and therefore it may be assumed that its underlying set is contained in $\mathbb{R}$. With this assumption, $\widehat{\mathcal{M}}_*^n$ is a well defined set.

[5] The C^{m+1} embeddings and C^m convergence of [8, Definition 1.1] and [6, Definition 1.2], for arbitrary order m, can be assumed to be C^∞ embeddings and C^∞ convergence [39, Theorem 2.2.7].

p.p.q.i. $h_i : (M, x) \rightarrowtail (M_i, x_i)$, for i large enough, with $|\nabla^l(f - h_i^* f_i)| < \varepsilon$ on $D_M(x, R)$ for $0 \le l \le m$ [8, Propositions 6.4 and 6.5]. The C^∞ convergence describes a Polish topology on $\widehat{\mathcal{M}}_*^n$ [6, Theorem 1.3]. The evaluation map $\mathrm{ev} : \widehat{\mathcal{M}}_*^n \to \mathfrak{H}$, $\mathrm{ev}([M, x, f]) = f(x)$, is continuous.

For any connected complete Riemannian n-manifold M and any C^∞ function $f : M \to \mathfrak{H}$, there is a canonical continuous map $\hat\iota_{M,f} : M \to \widehat{\mathcal{M}}_*^n$ defined by $\hat\iota_{M,f}(x) = [M, x, f]$, whose image is denoted by $[M, f]$. Note that $[M, f] = [N, y]$ if (M, f) and (N, y) are pointed isometric, and $[M, f] \cap [N, y] = \emptyset$ otherwise. We have $[M, f] \equiv \mathrm{Iso}(M, f)\backslash M$, where $\mathrm{Iso}(M, f)$ denotes the group of isometries of M preserving f. The images of these maps form a canonical partition of $\widehat{\mathcal{M}}_*^n$, which is considered when using saturations or minimal sets in $\widehat{\mathcal{M}}_*^n$. Any bounded linear map between Hilbert spaces, $\Phi : \mathfrak{H} \to \mathfrak{H}'$, induces a relation-preserving continuous map $\Phi_* : \widehat{\mathcal{M}}_*^n(\mathfrak{H}) \to \widehat{\mathcal{M}}_*^n(\mathfrak{H}')$, given by $\Phi_*([M, x, f]) = [M, x, \Phi \circ f]$, which defines a functor.

Lemma 3.1.8. *The saturation of any open subset of $\widehat{\mathcal{M}}_*^n$ is open, and therefore the closure of any saturated subset of $\widehat{\mathcal{M}}_*^n$ is saturated.*

Proof. Let $\mathcal{V}$ be the saturation of some open $\mathcal{U} \subset \widehat{\mathcal{M}}_*^n$, and let $[M, x, f] \in \mathcal{V}$. Then there is some $y \in M$ such that $[M, y, f] \in \mathcal{U}$. Since $\mathcal{U}$ is open, there are $m \in \mathbb{N}$, $R, \varepsilon > 0$ and $\lambda > 1$ so that, for all $[M', y', f'] \in \widehat{\mathcal{M}}_*^n$, if there is an (m, R, λ)-p.p.q.i. $h : (M, y) \rightarrowtail (M', y')$ with $|\nabla^l(f - h^* f')| < \varepsilon$ on $D_M(y, R)$ for $0 \le l \le m$, then $[M', y', f'] \in \mathcal{U}$. We can assume that $R > d_M(x, y)$. Take any convergent sequence $[M_i, x_i, f_i] \to [M, x, f]$ in $\widehat{\mathcal{M}}_*^n$. For i large enough, there is some $(m, 2R, \lambda)$-p.p.q.i. $h_i : (M, x) \rightarrowtail (M_i, x_i)$ with $|\nabla^l(f - h_i^* f_i)| < \varepsilon$ on $D_M(x, 2R)$ for $0 \le l \le m$. Since $D_M(y, R) \subset D_M(x, 2R)$, it follows that $[M_i, h_i(y), f_i] \in \mathcal{U}$ for i large enough. Therefore $[M_i, x_i, f_i] \in \mathcal{V}$ for i large enough, showing that $\mathcal{V}$ is open.

The last part of the statement follows from the first part and Lemma 2.1.1. □

Let $\hat d : (\widehat{\mathcal{M}}_*^n)^2 \to [0, \infty]$ be the metric with possible infinite values induced by d_M on every equivalence class $[M, f] \equiv \mathrm{Iso}(M, f)\backslash M$, and equal to ∞ on non-related pairs.

Lemma 3.1.9. *For every open $\mathcal{U} \subset \widehat{\mathcal{M}}_*^n$, the map $\hat d(\cdot, \mathcal{U}) : \widehat{\mathcal{M}}_*^n \to [0, \infty]$ is upper semi-continuous.*

Proof. To prove the upper semicontinuity of $\hat d(\cdot, \mathcal{U})$ at any point $[M, x, f]$, we can assume that $\hat d([M, x, f], \mathcal{U}) < \infty$, and therefore there is some $y \in M$ such that $[M, y, f] \in \mathcal{U}$. Take a convergent sequence $[M_i, x_i, f_i] \to [M, x, f]$ in $\widehat{\mathcal{M}}_*^n$, and let $\varepsilon > 0$. We can also suppose that

$$\hat d([M, x, f], [M, y, f]) < \hat d([M, x, f], \mathcal{U}) + \varepsilon/3 ,$$
$$d_M(x, y) < \hat d([M, x, f], [M, y, f]) + \varepsilon/3 .$$

Since $\mathcal{U}$ is open, there are $m \in \mathbb{N}$, $R > d_M(x, y) + \varepsilon$, $1 < \lambda < (d_M(x, y) + \varepsilon/3)/d_M(x, y)$ and $0 < \delta < \varepsilon$ so that, for all $[M', y', f'] \in \widehat{\mathcal{M}}_*^n$, if there is an (m, R, λ)-p.p.q.i. $h : (M, y) \rightarrowtail (M', y')$ with $|\nabla^l(f - h^* f')| < \delta$ on $D_M(y, R)$ for $0 \le l \le m$, then $[M', y', f'] \in \mathcal{U}$. By the convergence $[M_i, x_i, f_i] \to [M, x, f]$, for i large enough, there is some $(m, 2R, \lambda)$-p.p.q.i.

$h_i : (M, x) \rightarrowtail (M_i, x_i)$ with $|\nabla^l(f - h_i^* f_i)| < \delta$ on $D_M(x, 2R)$ for $0 \le l \le m$. Since $D_M(y, R) \subset D_M(x, 2R)$, it follows that $[M_i, y_i, f_i] \in \mathfrak{U}$ for $y_i = h_i(y)$, and

$$\begin{aligned}\hat{d}([M_i, x_i, f_i], [M_i, y_i, f_i]) &\le d_i(x_i, y_i) \le \lambda d_M(x, y) < d_M(x, y) + \varepsilon/3 \\ &< \hat{d}([M, x, f], \mathfrak{U}) + \varepsilon\ .\end{aligned}$$

Hence $\hat{d}([M_i, x_i, f_i], \mathfrak{U}) < \hat{d}([M, x, f], \mathfrak{U}) + \varepsilon$ for i large enough. □

It is said that (M, f) (or f) is (*locally*) *non-periodic* (or (*locally*) *aperiodic*) if $\hat{\iota}_{M,f}$ is (locally) injective; i.e., aperiodicity means $\mathrm{Iso}(M, f) = \{\mathrm{id}_M\}$, and local aperiodicity means that the canonical projection $M \to \mathrm{Iso}(M, f)\backslash M$ is a covering map. More strongly, (M, f) (or f) is said to be *limit aperiodic* if (M', f') is aperiodic for all $[M', x', f'] \in \overline{[M, f]}$. On the other hand, (M, f) (or f) is said to be *repetitive* if, given any $p \in M$, for all $m \in \mathbb{N}$, $R, \varepsilon > 0$ and $\lambda > 1$, the set

$$\begin{aligned}\{\, x \in M \mid \exists \text{ an } (m, R, \lambda)\text{-p.p.q.i. } h\colon (M, p) \rightarrowtail (M, x)& \\ \text{with } |\nabla^l(f - h^* f)| < \varepsilon \text{ on } D_M(p, R)\ \forall l \le m \,\}&\end{aligned} \tag{3.1.2}$$

is relatively dense in M. Clearly, this property is independent of the choice of p.

Proposition 3.1.10. *The following holds for any connected complete Riemannian n-manifold M:*

(i) If (M, f) is repetitive, then $\overline{[M, f]}$ is minimal.

(ii) If $\overline{[M, f]}$ is compact and minimal, then (M, f) is repetitive.

Proof. By Lemma 3.1.8, $\overline{[M, f]}$ is saturated, and therefore its minimality can be considered.

Item (i) follows by showing that $[M, f] \subset \overline{[M', f']}$ for every equivalence class $[M', f'] \subset \overline{[M, f]}$. In fact, it is enough to prove that $[M, f] \cap \overline{[M', f']} \ne \emptyset$ because $\overline{[M', f']}$ is saturated. Fix any $p \in M$, and let $m \in \mathbb{N}$, $R, \varepsilon > 0$ and $\lambda > 1$. By the repetitiveness of (M, f), for some $c > 0$, there is a c-relatively dense subset $X \subset M$ such that, for all $x \in X$, there is an $(m, R, \lambda^{1/2})$-p.p.q.i. $h_x : (M, p) \rightarrowtail (M, x)$ with $|\nabla^l(f - h_x^* f)| < \varepsilon/2$ and $|\nabla^l h_x^* \phi| < \frac{3}{2} h_x^* |\nabla^l \phi|$ on $D_M(x, R)$ for $0 \le l \le m$ and $\phi \in C^\infty(M)$. On the other hand, since $[M', f'] \subset \overline{[M, f]}$, given any $y' \in M'$, there are some $y \in M$ and an $(m, \lambda^{1/2} c + \lambda R, \lambda^{1/2})$-p.p.q.i. $h : (M', y') \rightarrowtail (M, y)$ so that $|\nabla^l(f - (h^{-1})^* f')| < \varepsilon/3$ on $h(D_{M'}(x, \lambda^{1/2} c + \lambda R))$ for $0 \le l \le m$. Take some $x \in X$ with $d_M(x, y) \le c$. We have $D_M(y, c) \subset h(D_{M'}(y', \lambda^{1/2} c))$, and therefore there is some $x' \in D_{M'}(y', \lambda^{1/2} c)$ with $h(x') = x$. By Proposition 3.1.1, the composite $h^{-1} h_x$ defines an (m, R, λ)-p.p.q.i. $(M, p) \rightarrowtail (M', x')$. Moreover

$$\begin{aligned}|\nabla^l(f - (h^{-1} h_x)^* f')| &\le |\nabla^l(f - h_x^* f)| + |\nabla^l(h_x^* f - (h^{-1} h_x)^* f')| \\ &\le |\nabla^l(f - h_x^* f)| + \frac{3}{2} h_x^* |\nabla^l(f - (h^{-1})^* f')| < \frac{\varepsilon}{2} + \frac{3}{2}\frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

on $D_M(p, R)$ for $0 \le l \le m$. Since m, R, ε and λ are arbitrary, we get $[M, p, f] \in [M, f] \cap \overline{[M', f']}$.

To prove (ii), fix any $p \in M$, and take $m \in \mathbb{N}$, $R, \varepsilon > 0$ and $\lambda > 1$. The set

$$\mathcal{U} = \{ [M', x', f'] \in \widehat{\mathcal{M}}^n_* \mid \exists \text{ an } (m, R, \lambda)\text{-p.p.q.i. } h\colon (M, p) \rightarrowtail (M', x') \\ \text{with } |\nabla^l (f - h^* f')| < \varepsilon \text{ on } D_M(p, R)\ \forall l \leq m \}$$

is an open neighborhood of $[M, p, f]$ in $\widehat{\mathcal{M}}^n_*$. By Lemma 3.1.9, and the compactness and minimality of $\overline{[M, f]}$, we have $\hat{d}(\cdot, \mathcal{U}) \leq c$ on $\widehat{\mathcal{M}}^n_*$ for some $c > 0$. So (3.1.2) is a c-relatively dense subset of M. Since m, R, ε and λ are arbitrary, we get that (M, f) is repetitive. □

The non-periodic and locally non-periodic pairs (M, f) define saturated subspaces $\widehat{\mathcal{M}}^n_{*,\mathrm{np}} \subset \widehat{\mathcal{M}}^n_{*,\mathrm{lnp}} \subset \widehat{\mathcal{M}}^n_*$. The pairs (M, f) where f is an immersion define a saturated Polish subspace $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}} \subset \widehat{\mathcal{M}}^n_{*,\mathrm{lnp}}$. The following properties hold [6, Theorem 1.4]:

- $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$ is open and dense in $\widehat{\mathcal{M}}^n_*$.
- $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$ is a foliated space with the restriction of the canonical partition.
- The foliated space $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$ has unique C^∞ structure such that $\mathrm{ev} : \widehat{\mathcal{M}}^n_* \to \mathfrak{H}$ is C^∞. Furthermore $\hat{\iota}_{M,f} : M \to \widehat{\mathcal{M}}^n_*$ is also C^∞ for all pairs (M, f) where f is an immersion.
- Every map $\hat{\iota}_{M,f} : M \to [M, f] \equiv \mathrm{Iso}(M, f)\backslash M$ is the holonomy covering of the leaf $[M, f]$. Thus $\widehat{\mathcal{M}}^n_{*,\mathrm{np}} \cap \widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$ is the union of leaves without holonomy.
- The C^∞ foliated space $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$ has a Riemannian metric so that every map $\hat{\iota}_{M,f} : M \to [M, f] \equiv \mathrm{Iso}(M, f)\backslash M$ is a local isometry.

By forgetting the functions f, we get a Polish space $\mathcal{M}^n_*$ [8, Theorem 1.2]. We have $\mathcal{M}^n_* \equiv \widehat{\mathcal{M}}^n_*(0)$, using the zero Hilbert space. The forgetful or underlying map $\mathfrak{u} : \widehat{\mathcal{M}}^n_* \to \mathcal{M}^n_*$, $\mathfrak{u}([M, x, f]) = [M, x]$, is continuous. We also have a canonical partition defined by the images $[M]$ of canonical continuous maps $\iota_M : M \to \mathcal{M}^n_*$, $\iota_M(x) = [M, x]$, giving rise to the conditions on M of being (*locally*) *non-periodic* (or (*locally*) *aperiodic*), and the subspaces $\mathcal{M}^n_{*,\mathrm{np}} \subset \mathcal{M}^n_{*,\mathrm{lnp}} \subset \mathcal{M}^n_*$. The condition on M to be *repetitive* is also defined by forgetting about the functions, and the obvious version without functions of Proposition 3.1.10 is true. Then the following properties hold for $n \geq 2$ [8, Theorem 1.3]:

- $\mathcal{M}^n_{*,\mathrm{lnp}}$ is open and dense in $\mathcal{M}^n_*$.
- $\mathcal{M}^n_{*,\mathrm{lnp}}$ is a foliated space with the restriction of the canonical partition.
- The foliated space $\mathcal{M}^n_{*,\mathrm{lnp}}$ has a unique C^∞ and Riemannian structures such that every map $\iota_M : M \to [M] \equiv \mathrm{Iso}(M)\backslash M$ is a local isometry. Furthermore $\mathfrak{u} : \widehat{\mathcal{M}}^n_{*,\mathrm{imm}} \to \mathcal{M}^n_{*,\mathrm{lnp}}$ is a C^∞ foliated map.
- Every map $\iota_M : M \to [M] \equiv \mathrm{Iso}(M)\backslash M$ is the holonomy covering of the leaf $[M]$. Thus $\mathcal{M}^n_{*,\mathrm{np}}$ is the union of leaves without holonomy.

Moreover $\overline{[M]}$ is compact if and only if M is of bounded geometry [8, Theorem 12.3] (see also [25], [53, Chapter 10, Sections 3 and 4]).

For $m \in \mathbb{N}$, consider quadruples (M, x, f, ξ), where (M, x, f) is like in the definition of $\widehat{\mathcal{M}}^n_*$ and $\xi \in \bigotimes_m T^*_x M$ with $|\xi| = 1$. An equivalence between such quadruples, $(M, x, f, \xi) \sim (M', x', f', \xi')$, means that there is an isometry $h : M \to M'$ defining an equivalence $(M, x, f) \sim (M', x', f')$ with $h^*\xi' = \xi$. The corresponding equivalence classes, denoted by $[M, x, f, \xi]$, define a set $\mathcal{S}^m\widehat{\mathcal{M}}^n_* = \mathcal{S}^m\widehat{\mathcal{M}}^n_*(\mathfrak{H})$, like in the case of $\widehat{\mathcal{M}}^n_*$. Moreover the C^∞ *convergence* $[M_i, x_i, f_i, \xi_i] \to [M, x, f, \xi]$ in $\mathcal{S}^m\widehat{\mathcal{M}}^n_*$ means that, for all $m \in \mathbb{N}$, $R, \varepsilon > 0$ and $\lambda > 1$, there is an (m, R, λ)-p.p.q.i. $h_i : (M, x) \rightarrowtail (M_i, x_i)$, for i large enough, such $|\nabla^l(f - h_i^* f_i)| < \varepsilon$ on $D_M(x, R)$ for $0 \le l \le m$ and $h_i^*\xi_i \to \xi$. Like in the case of $\widehat{\mathcal{M}}^n_*$, it can be proved that this convergence defines a Polish topology on $\mathcal{S}^m\widehat{\mathcal{M}}^n_*$. Moreover there are continuous maps $\hat{\iota}_{M,f,\xi} : M \to \mathcal{S}^m\widehat{\mathcal{M}}^n_*$, defined by $\hat{\iota}_{M,f,\xi}(x) = [M, x, f, \xi]$, whose images $[M, f, \xi]$ form a canonical partition of $\mathcal{S}^m\widehat{\mathcal{M}}^n_*$ satisfying the same basic properties as the canonical partition of $\widehat{\mathcal{M}}^n_*$. We also have a continuous forgetful or underlying map $\mathfrak{u} : \mathcal{S}^m\widehat{\mathcal{M}}^n_* \to \widehat{\mathcal{M}}^n_*$ given by $\mathfrak{u}([M, x, f, \xi]) = [M, x, f]$.

Proposition 3.1.11. *The map* $\mathfrak{u} : \mathcal{S}^m\widehat{\mathcal{M}}^n_* \to \widehat{\mathcal{M}}^n_*$ *is proper.*

Proof. For any compact subset $\mathcal{K} \subset \widehat{\mathcal{M}}^n_*$, take a sequence $[M_i, x_i, f_i, \xi_i]$ in $\mathfrak{u}^{-1}(\mathcal{K})$. Since $\mathcal{K}$ is compact, after taking a subsequence if necessary, we can assume that $[M_i, x_i, f_i]$ converges to some element $[M, x, f]$ in $\mathcal{K}$. Thus there are sequences, $m_i \uparrow \infty$ in $\mathbb{N}$, $0 < R_i \uparrow \infty$, $0 < \varepsilon_i \downarrow 0$ and $1 < \lambda_i \downarrow 1$, such that, for every i, there is some an (m_i, R_i, λ_i)-p.p.q.i. $h_i : (M, x) \rightarrowtail (M_i, x_i)$ with $|\nabla^l(f - h_i^* f_i)| < \varepsilon_i$ on $D_M(x, R_i)$ for $0 \le l \le m_i$. Since $\lambda_i^{-m} \le |h_i^*\xi_i| \le \lambda_i^m$ for all i, some subsequence $h_{i_k}^*\xi_{i_k}$ is convergent in $\bigotimes_m T^*_x M$ to some ξ with $|\xi| = 1$. Using h_{i_k}, it follows that the subsequence $[M_{i_k}, x_{i_k}, f_{i_k}, \xi_{i_k}]$ converges to $[M, x, f, \xi]$ in $\mathfrak{u}^{-1}(\mathcal{K})$, showing that $\mathfrak{u}^{-1}(\mathcal{K})$ is compact. □

For every $m \in \mathbb{N}$, a well-defined continuous map $\nabla^m : \mathcal{S}^m\widehat{\mathcal{M}}^n_* \to \mathfrak{H}$ is given by $\nabla^m([M, x, f, \xi]) = (\nabla^m f)(x, \xi)$.

Proposition 3.1.12. *Let M be a complete connected Riemannian n-manifold, and let $f \in C^\infty(M, \mathfrak{H})$ and $x_0 \in M$. Then $\overline{[M, f]}$ is compact if and only if M is of bounded geometry and $\nabla^m(\mathfrak{u}^{-1}([M, f]))$ is precompact in $\mathfrak{H}$ for all $m \in \mathbb{N}$.*

Proof. Assume that $\overline{[M, f]}$ is compact to prove the "only if" part. The map $\mathfrak{u} : \widehat{\mathcal{M}}^n_{*,\mathrm{imm}} \to \mathcal{M}^n_*$ defines a map $\mathfrak{u} : \overline{[M, f]} \to \overline{[M]}$ with dense image because $\iota_M = \mathfrak{u} \circ \hat{\iota}_{M,f}$. By the compactness of $\overline{[M, f]}$, it follows that this map is surjective, and therefore $\overline{[M]}$ is compact. So M is of bounded geometry. Furthermore $\nabla^m(\mathfrak{u}^{-1}(\overline{[M, f]}))$ is compact in $\mathfrak{H}$ by Proposition 3.1.11.

The "if" part follows by showing that any sequence $[M, f, x_p]$ in $[M, f]$ has a subsequence that is convergent in $\widehat{\mathcal{M}}^n_*$. Since $\overline{[M]}$ is compact and $\mathfrak{u} : \widehat{\mathcal{M}}^n_* \to \mathcal{M}^n_*$ continuous, we can suppose that $[M, x_p]$ converges to some point $[M', x']$ in $\mathcal{M}^n_*$. Take a sequence of compact domains D_q in M' such that $B_{M'}(x', q+1) \subset D_q$. For every q, there are pointed C^∞ embeddings $h_{q,p} : (D_q, x') \to (M, x_p)$, for p large enough, such that $h_{q,p}^* g_M \to g_N$ on D_q with respect to the C^∞ topology. Let $f'_{q,p} = h_{q,p}^* f$ on D_q. From the compactness of $\nabla^m(\mathfrak{u}^{-1}(\overline{[M, f]}))$, it easily follows that, for all q, m, we have $\sup_p \sup_{D_q} |\nabla^m f'_{q,p}| < \infty$, and the elements $(\nabla^m f'_{q,p})(x')$ form a precompact subset of $\mathfrak{H} \otimes \bigotimes_m T_{x'}M^*$. Hence the functions $f'_{q,p}$ form a precompact subset of $C^\infty(D_q, \mathbb{E})$ with the C^∞ topology by Corollary 3.1.3. So some subsequence $f'_{q,p(q,\ell)}$ is convergent to some $f'_q \in C^\infty(\widetilde{\Omega}_q, \mathbb{E})$ with respect to the C^∞ topology. In fact, arguing inductively on q, it is easy to see that we can assume that each $f'_{q+1,p(q+1,\ell)}$ is a subsequence of $f'_{q,p(q,\ell)}$, and therefore f'_{q+1} extends f'_q. Thus the functions f'_q can be combined to define a function $f' \in C^\infty(M', \mathbb{E})$. Take sequences $\ell_q, m_q \uparrow \infty$ in $\mathbb{N}$ so that

$$\|f' - h^*_{q,p(q,\ell_q)} f\|_{C^{m_q},D_q,g_N} = \|f'_q - f'_{q,p(q,\ell_q)}\|_{C^{m_q},\widetilde{\Omega}_q,g_N} \to 0 \ .$$

Considering f' as an $\mathfrak{H}$-valued function, we get that $[M, f, x_{p(q,\ell_q)}] \to [M', f', x']$ in $\widehat{\mathcal{M}}^n_*$ as $q \to \infty$. □

The following is an elementary consequence of Proposition 3.1.12.

Corollary 3.1.13. *Let M be a complete connected Riemannian n-manifold, and let $f \in C^\infty(M, \mathfrak{H})$. Suppose that* $\dim \mathfrak{H} < \infty$. *Then $\overline{[M, f]}$ is compact if and only if M is of bounded geometry and* $\sup_M |\nabla^m f| < \infty$ *for all $m \in \mathbb{N}$.*

Corollary 3.1.14. *Let M be a complete connected Riemannian n-manifold, let $fH = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be a direct sum decomposition of Hilbert spaces, and let*

$$f \equiv (f_1, f_2) \in C^\infty(M, \mathfrak{H}) \equiv C^\infty(M, \mathfrak{H}_1) \oplus C^\infty(M, \mathfrak{H}_2) \ .$$

Then $\overline{[M, f]}$ is compact if and only if $\overline{[M, f_1]}$ and $\overline{[M, f_2]}$ are compact.

Proof. Assume that $\overline{[M, f]}$ is compact to prove the "only if" part. Let $\Pi_a : \mathfrak{H} \to \mathfrak{H}_2$ ($a = 1, 2$) denote the factor projections. The induced maps $\Pi_{a*} : \widehat{\mathcal{M}}^n_*(\mathfrak{H}) \to \widehat{\mathcal{M}}^n_*(\mathfrak{H}_a)$ define continuous maps $\Pi_{a*} : \overline{[M, f]} \to \overline{[M, f_a]}$, whose images are dense because $\hat\iota_{M,f_a} = \Pi_{a*} \circ \hat\iota_{M,f}$. By the compactness of $\overline{[M, f]}$, it follows that these maps are surjective and the spaces $\overline{[M, f_a]}$ are compact.

Now assume that every space $\overline{[M, f_a]}$ ($a = 1, 2$) is compact to prove the "if" part. By Proposition 3.1.12, this means that M is of bounded geometry and, for all $m \in \mathbb{N}$, each set $\nabla^m(\mathfrak{u}^{-1}([M, f_a]))$ is precompact in $\mathfrak{H}_a$. Since

$$\nabla^m(\mathfrak{u}^{-1}([M, f])) \subset \nabla^m(\mathfrak{u}^{-1}([M, f_1])) \times \nabla^m(\mathfrak{u}^{-1}([M, f_2]))$$

for every m because $(\nabla^m f)(x, \xi) = ((\nabla^m f_1)(x, \xi), (\nabla^m f_2)(x, \xi))$ for all $x \in M$ and $\xi \in \bigotimes_m T^*_x M$, we get that $\nabla^m(\mathfrak{u}^{-1}([M, f]))$ is precompact in $\mathfrak{H}$ for all m. Hence $\overline{[M, f]}$ is compact by Proposition 3.1.12. □

Proposition 3.1.15. *Let M be a complete connected Riemannian n-manifold, and let $f \in C^\infty(M, \mathfrak{H})$. Then the following properties hold:*

(i) If $\overline{[M, f]}$ is a compact subspace of $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$, then $\inf_M |\nabla f| > 0$.

(ii) If $\inf_M |\nabla f| > 0$, *then $\overline{[M, f]} \subset \widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$.*

Proof. This holds because the mapping $[M', x', f'] \mapsto |(\nabla f')(x')|$ is well defined and continuous on $\widehat{\mathcal{M}}^n_*$. □

Proposition 3.1.16. *In any minimal compact Riemannian foliated space, all leaves without holonomy are repetitive.*

Proof. This is a direct consequence of the Reeb's local stability theorem and the fact that $L \cap U$ is relatively dense in L for all leaf L and open $U \neq \emptyset$ in a minimal compact foliated space [10, Second proof of Theorem 1.13, p. 123]. □

Example 3.1.17. For any compact C^∞ foliated space $\mathfrak{X}$, there is a C^∞ embedding into some separable Hilbert space, $h : \mathfrak{X} \to \mathfrak{H}$ [17, Theorem 11.4.4]. Suppose that $\mathfrak{X}$ is transitive and without holonomy, and endowed with a Riemannian metric. Let M be a dense leaf of $\mathfrak{X}$, which is of bounded geometry, and let $f = h|_M \in C^\infty(M, \mathfrak{H})$. We have $\inf_M |\nabla f| = \min_{\mathfrak{X}} |\nabla h| > 0$. So $\mathfrak{X}' := \overline{[M, f]}$ is a Riemannian foliated subspace of $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}$ (Proposition 3.1.15 (ii)). Since $\mathfrak{X}$ is compact and without holonomy, and M is dense in $\mathfrak{X}$, it follows from the Reeb's local stability theorem that the leaves of $\mathfrak{X}'$ are the subspaces $[L, h|_L]$, for leaves L of $\mathfrak{X}$, and the combination of the corresponding maps maps $\hat\iota_{L,h|_L}$ is an isometric foliated surjective map $\hat\iota_{\mathfrak{X},h} : \mathfrak{X} \to \mathfrak{X}'$. Using that $\mathrm{ev} \circ \hat\iota_{\mathfrak{X},h} = h$, we get that $\hat\iota_{\mathfrak{X},h} : \mathfrak{X} \to \mathfrak{X}'$ is an isometric foliated diffeomorphism, and $\mathrm{ev} : \mathfrak{X}' \to \mathfrak{H}$ is a C^∞ embedding whose image is $h(\mathfrak{X})$. Thus $\mathfrak{X}'$ is compact and without holonomy, and (M, f) is limit aperiodic. If moreover $\mathfrak{X}$ is minimal, then (M, f) is repetitive by Proposition 3.1.16.

3.1.4 The spaces $\mathcal{CM}^n_*$ and $\widehat{\mathcal{CM}}^n_*$

Like in Section 3.1.3, using distinguished closed subsets $C \subset M$ instead of C^∞ functions $f : M \to \mathfrak{H}$, we get set $\mathcal{CM}^n_*$ of equivalence classes $[M, x, C]$ of triples (M, x, C), where the equivalence $(M, x, C) \sim (M', x', C')$ means that there is a pointed isometry $h : (M, x) \to (M', x')$ with $h(C) = C'$. A sequence $[M_i, x_i, C_i] \in \mathcal{CM}^n_*$ is said to be C^∞-*Chabauty convergent* to $[M, x, C] \in \mathcal{CM}^n_*$ if, for any compact domain $D \subset M$ containing x, there are pointed C^∞ embeddings $h_i : (D, x) \to (M_i, x_i)$, for large enough i, such that $h_i^* g_i \to g_M|_D$ in the C^∞ topology and $h_i^{-1}(C_i) \to C \cap D$ in the Chabauty (or Fell) topology [1, Section A.4]. In other words, this convergence also means that, for all $m \in \mathbb{N}$, $R > \varepsilon > 0$ and $\lambda > 1$, there is some (m, R, λ)-p.p.q.i. $h_i : (M, x) \rightarrowtail (M_i, x_i)$, for i large enough, such that:

(a) for all $y \in D_M(x, R-\varepsilon) \cap C$, there is some $y_i \in h_i^{-1}(C_i) \subset D_M(x, R)$ with $d_M(y, y_i) < \varepsilon$; and,

(b) for all $y_i \in D_M(x, R-\varepsilon) \cap h_i^{-1}(C_i)$, there is some $y \in D_M(x, R)$ with $d_M(y, y_i) < \varepsilon$.

The C^∞-Chabauty convergence describes a Polish topology on $\mathcal{CM}^n_*$ [1, Theorem A.17], and the forgetful or underlying map $\mathfrak{u} : \mathcal{CM}^n_* \to \mathcal{M}^n_*$, $\mathfrak{u}([M, x, C]) = [M, x]$, is continuous. There are also canonical continuous maps $\iota_{M,C} : M \to \mathcal{CM}^n_*$, $\iota_{M,C}(x) = [M, x, C]$, whose images, denoted by $[M, C]$, form a canonical partition of $\mathcal{CM}^n_*$. We have $[M, C] \equiv \mathrm{Iso}(M, C)\backslash M$, where $\mathrm{Iso}(M, C)$ denotes the group of isometries of M preserving C. There are obvious versions of Lemmas 3.1.8 and 3.1.9 in this setting, as well as obvious versions of (*limit*) *aperiodicity* for (M, C). Similarly, the *repetitivity* of (M, C) can be defined like in the case of (M, f) in Section 3.1.3, using (a) and (b) instead of the condition on f in (3.1.2). The obvious version of Proposition 3.1.10 holds in this setting.

Now fix some countable set F like in Section 2.1.4. A set $\widehat{\mathcal{CM}}^n_* = \widehat{\mathcal{CM}}^n_*(F)$ can be defined like $\mathcal{CM}^n_*$, using equivalence classes $[M, x, C, \phi]$ of quadruples (M, x, C, ϕ), for closed subsets $C \subset M$ with locally constant colorings $\phi : C \to F$, where the equivalence $(M, x, C, \phi) \sim (M', x', C', \psi')$ means that there is a pointed isometry $h : (M, x) \to (M', x')$ with $h(C) = C'$ and $h^*\phi' = \phi$. The convergence $[M_i, x_i, C_i, \phi_i] \to [M, x, C, \phi]$ in $\widehat{\mathcal{CM}}^n_*$ can be defined like in the case of $\mathcal{CM}^n_*$, adding the condition $\phi(y) = \phi_i h_i(y_i)$ in (a) and (b). Like in [1, Theorem A.17], it can be probed that this convergence defines a Polish topology on $\widehat{\mathcal{CM}}^n_*$, and the forgetful or underlying map $\mathfrak{u} : \widehat{\mathcal{CM}}^n_* \to \mathcal{CM}^n_*$, $\mathfrak{u}([M, x, C, \phi]) = [M, x, C]$, is continuous. There are also canonical continuous maps $\hat\iota_{M,C,\phi} : M \to \widehat{\mathcal{CM}}^n_*$, $\hat\iota_{M,C,\phi}(x) = [M, x, C, \phi]$, whose images, denoted by $[M, C, \phi]$, form a canonical partition of $\mathcal{CM}^n_*$ satisfying the obvious versions of Lemmas 3.1.8 and 3.1.9. Similarly, the concepts of (*limit*) *aperiodicity* and *repetitivity* have obvious versions for (M, C, ϕ), satisfying the obvious version of Proposition 3.1.10.

3.2 Repetitive Riemannian manifolds

Let M be a complete connected Riemannian manifold and fix a distinguished point $p \in M$. For $i \in \mathbb{N}$, $R > 0$, and $\lambda \geq 1$, let

$$\Omega(i, R, \lambda) = \{\, x \in M \mid \exists \text{ an } (i, R, \lambda)\text{-p.p.q.i. } f\colon (M, p) \rightarrowtail (M, x) \,\} .$$

Suppose that M is repetitive; i.e., the sets $\Omega(i, R, \lambda)$ are relatively dense in M. We will hereafter consider sequences $0 < r_i, s_i, t_i \uparrow \infty$ and $\lambda_i \downarrow 1$ satisfying a list of conditions that can be achieved by assuming that these divergences and convergence are fast enough. For integers $i, j \geq 0$, we will use the notation

$$\Lambda_{i,j} = \prod_{k=i}^{j} \lambda_k , \quad \Lambda_i = \prod_{k \geq i} \lambda_k ;$$

in particular,[6] $\Lambda_{i,j} = 1$ if $j < i$. Let ω_i denote the smallest positive real such that the set $\Omega_i = \Omega(i, r_i, \lambda_i)$ is ω_i-relatively dense in M. For notational convenience, let also $r_{-1} = s_{-1} = t_{-1} = \omega_{-1} = 0$, and fix any $\lambda_{-1} > 1$. For $i \geq 0$, we can assume

$$r_i > \frac{\lambda_0^5}{\lambda_0 - 1}(r_{i-1} + s_{i-1} + t_{i-1} + 2\omega_{i-1} + 1) , \tag{3.2.1}$$

$$s_i > 2\lambda_0^5(r_i + s_{i-1} + \omega_i) , \tag{3.2.2}$$

$$t_i > \lambda_0^3(5t_{i-1} + r_i + s_{i-1} + 2\omega_{i-1} + 1) , \tag{3.2.3}$$

$$t_i > 4\frac{\lambda_i^4 + \lambda_i^2 - 1}{\lambda_i^2} r_i + t_{i-1} + \Lambda_i(s_{i-1} + 2\omega_{i-1} + \omega_i) , \tag{3.2.4}$$

$$\lambda_i^2 < \lambda_{i-1} , \tag{3.2.5}$$

$$2^{2^{-i}} > \frac{r_i(\lambda_i^5 - 1)\lambda_{i-1}^2}{r_{i-1}(\lambda_{i-1}^5 - 1)\lambda_i^2}, \frac{r_i(\lambda_i^6 - 1)\lambda_{i-1}^2}{r_{i-1}(\lambda_{i-1}^6 - 1)\lambda_i^2} . \tag{3.2.6}$$

[6] An empty product is assumed to be 1.

When $i < j$, (3.2.5) yields

$$\Lambda_{i,j} < \Lambda_i < \prod_{k\geq i} \lambda_i^{2^{i-k}} = \lambda_i^2 \; . \tag{3.2.7}$$

Finally, let $\widetilde{\Omega}_i = \Omega(i, r_i, \Lambda_i)$ and $\widetilde{\Omega}_{i,j} = \Omega(i, r_i, \Lambda_{i,j})$.

Lemma 3.2.1. *For $i < j$,*

$$r_j \frac{\lambda_j \Lambda_j^2 - 1}{\Lambda_j} < 4 \frac{\lambda_i^5 - 1}{\lambda_i^2} r_i \; , \quad r_j \frac{\lambda_j^2 \Lambda_j^2 - 1}{\Lambda_j} < 4 \frac{\lambda_i^6 - 1}{\lambda_i^2} r_i \; . \tag{3.2.8}$$

Proof. We will prove the first inequality, the proof of the second one being similar. For $i \leq k \leq j$, let

$$f(k) = \frac{\lambda_k^5 - 1}{\lambda_k^2} r_k \; .$$

We have to show that

$$r_j \frac{\lambda_j \Lambda_j^2 - 1}{\Lambda_j} \leq 4 f(i) \; . \tag{3.2.9}$$

By (3.2.7),

$$\lambda_j^3 \Lambda_j^2 - \lambda_j^2 \leq \lambda_j^5 \Lambda_j - \Lambda_j \; ,$$

and therefore

$$r_j \frac{\lambda_j \Lambda_j^2 - 1}{\Lambda_j} \leq r_j \frac{\lambda_j^5 - 1}{\lambda_j^2} = f(l) \; . \tag{3.2.10}$$

On the other hand, (3.2.6) yields

$$f(l) = \frac{f(l)}{f(l-1)} \frac{f(l-1)}{f(l-2)} \cdots \frac{f(i+1)}{f(i)} f(i) < 2^{2^{-l}} 2^{2^{-l+1}} \cdots 2^{2^{-i+1}} f(i) < 4 f(i) \; , \tag{3.2.11}$$

and (3.2.9) follows from (3.2.10) and (3.2.11). □

For $i \in \mathbb{N}$, let $M_i^i = \{p\}$ and let $h_{i,p}^i = \mathrm{id}_{D(p,r_i)}$. In Proposition 3.2.2, for integers $0 \leq i < j$, we will continue defining subsets $M_i^j \subset M$ and an $(i, r_i, \Lambda_{i,j-1})$-p.p.q.i. $h_{i,z}^j \colon (M,p) \to (M,z)$ for every $z \in M_i^j$. Using this notation, let

$$P_i^j = \{\, (l,z) \in \mathbb{N} \times M \mid i < l < j,\ z \in M_l^j \,\} \; . \tag{3.2.12}$$

Note that $P_k^j \subset P_i^j$ if $i \leq k < j$. Moreover, let $<$ be the binary relation on P_i^j defined by declaring $(l,z) < (l',z')$ if $l < l'$ and $z \in h_{l',z'}^j(M_l^{l'})$, and let $\leq$ denote its reflexive closure. We will prove that $\leq$ is in fact a partial order relation (Lemma 3.2.3 (b)). Let $\overline{P}_i^j$ denote the set of maximal elements of $(P_i^j, \leq)$, which is nonempty because all chains in P_i^j are finite.

Proposition 3.2.2. *For all integers $0 \leq i < j$, there is a set*[7] *$M_i^j = \widehat{M}_i^j \,\dot\cup\, \widetilde{M}_i^j \subset M$ and, for every $x \in M_i^j$, there is an $(i, r_i, \Lambda_{i,j-1})$-p.p.q.i. $h_{i,x}^j \colon (M,p) \rightarrowtail (M,x)$ satisfying the following properties:*

[7] The dotted union symbol denotes a union of disjoint subsets.

(i) $\widehat{M}_i^j$ *is a maximal* s_i*-separated subset of*

$$\Omega_i \cap D(p, r_j - t_i) \setminus \bigcup_{(l,z)\in\overline{P}_i^j}^{\cdot} D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i)) \ .$$

(ii) M_i^j *is an* $s_i/\Lambda_{i+1,j-1}$*-separated subset of* $\widetilde{\Omega}_{i,j-1} \cap D(p, r_j - t_i)$.

(iii) *For every* $(l, z) \in P_i^j$ *and* $x \in M_i^j \cap h_{l,z}^j(D(p, r_l))$, *we have* $h_{i,x}^j = h_{l,z}^j \circ h_{i,x'}^l$, *where* $x' = (h_{l,z}^j)^{-1}(x)$.

(iv) *For any* $(l, z) \in P_i^j$, *we have* $M_i^j \cap h_{l,z}^j(D(p, r_l)) = h_{l,z}^j(M_i^l)$.

(v) *For any* $x \in M_i^j$ *and* $(l, z) \in P_i^j$, *either* $d(x, z) \geq \lambda_l \Lambda_j(r_l + s_i)$ *or* $x \in h_{l,z}^j(M_i^l)$.

(vi) *For all integers* $0 \leq k \leq l$ *such that either* $l < j$ *and* $k \geq i$, *or* $l = j$ *and* $k > i$, *we have* $M_k^l \subset M_i^j$ *and* $h_{i,z}^j = h_{k,z}^l|_{D(p,r_i)}$ *for any* $z \in M_k^l$.

(vii) *We have* $p \in M_i^j$ *and* $h_{i,p}^j = \mathrm{id}_{D(p,r_i)}$.

Remark 19. In Proposition 3.2.2 (iii), the equality $h_{i,x}^j = h_{l,z}^j \circ h_{i,x'}^l$ holds on $D(p, r_i)$ because

$$h_{i,x'}^l(D(p, r_i)) \subset D(x', \Lambda_{i,j-1} r_i) \subset D(p, r_l) \ . \tag{3.2.13}$$

Here, the last inclusion is true since, for all $y \in D(x', \Lambda_{i,j-1} r_i)$,

$$d(y, p) \leq d(y, x') + d(x', p) \leq \Lambda_{i,j-1} r_i + r_l - t_i < r_l$$

because $x' \in M_i^l \subset D(p, r_l - t_i)$ by (ii) and (iv), and $t_i > \Lambda_{i,j-1} r_i$ by (3.2.3) and (3.2.7).

The proof of Proposition 3.2.2 is long and has several intermediate steps. By Remark 19, for integers $0 \leq i < j$, Items (i) to (vii) refer only to points $z \in M_k^l$ or pointed quasi-isometries $h_{k,z}^l$ where either $l < j$, or $l = j$ and $k \geq i$. This allows us to proceed inductively in the following way: First, for $i \geq 0$, we define M_i^{i+1} and $h_{i,z}^{i+1}$ for $z \in M_i^{i+1}$. Then, for $0 \leq i < j-1$, we construct M_i^j and $h_{i,z}^j$ for $z \in M_i^j$ under the assumption that we have already defined M_k^l and $h_{k,z}^l$ when either $l < j$, or $l = j$ and $k > i$.

For $i \geq 0$, let $\widehat{M}_i^{i+1} = M_i^{i+1}$ be any maximal s_i-separated subset of $\Omega_i \cap D(p, r_j - t_i)$ containing p, and let $\widetilde{M}_i^{i+1} = \emptyset$. Let $h_{i,p}^{i+1} = \mathrm{id}_{B(p,r_i)}$ and, for each $x \in M_i^{i+1} \setminus \{p\}$, let $h_{i,x}^{i+1} \colon (M, p) \rightarrowtail (M, z)$ be any pointed (i, r_i, λ_i)-p.p.q.i. These definitions satisfy Items (i) to (vii) in Proposition 3.2.2 because $P_i^{i+1} = \emptyset$.

Now, given $0 \leq i < j-1$, suppose that M_k^l and $h_{k,z}^l$ are defined if either $l < j$, or $l = j$ and $k > i$.

Lemma 3.2.3. *We have the following:*

(a) *For* $(l, z), (l, z') \in P_i^j$, *any of the following properties yields* $z = z'$:

(1) $d(z, z') \leq 2r_l + 2s_i$,

(2) $d(z, z') < s_l/\Lambda_{l+1,j-1}$, *or*

(3) $(l,z) \leq (l,z')$.

(b) $(P_i^j, \leq)$ *is a partially ordered set.*

Proof. Let us prove (a). It is obvious that (3) yields $z = z'$ since $\leq$ is the reflexive closure of $<$. Item (1) implies (2) because, since $i < l$, we get $2r_l + 2s_i < s_l/\lambda_0^5 < s_l/\Lambda_{l+1,j-1}$ by (3.2.2) and (3.2.7). According to (3.2.12), we have $l > i$ and $z, z' \in M_l^j$, so (2) yields $z = z'$ because M_l^j is $s_l/\Lambda_{l+1,j-1}$-separated by the induction hypothesis.

Let us prove (b). First, let us show that the reflexive relation $\leq$ is also transitive. Suppose $(l,z) < (l',z') < (l'',z'')$, which means $l < l' < l''$, $z \in h^j_{l',z'}(M_l^{l'})$, and $z' \in h^j_{l'',z''}(M_{l'}^{l''})$. By the induction hypothesis with (iv), it is enough to show $z \in h^j_{l'',z''}(D(p,r_{l''}))$ in order to obtain $z \in h^j_{l'',z''}(M_l^{l''})$ and thus $(l,z) < (l'',z'')$, so let us prove it.

By hypothesis, we have $z = h^j_{l',z'}(y)$ for some $y \in M_l^{l'}$, which is contained in $D(p,r_{l'})$ by the induction hypothesis with (ii). We also have $z'' \in P^j_{l'}$ by (3.2.12), so the induction hypothesis with (iii) yields $h^j_{l',z'} = h^j_{l'',z''} \circ h^{l''}_{l',y'}$ on $D(p,r_{l'})$, where $y' = (h^j_{l'',z''})^{-1}(z')$. By Remark 19,

$$y'' := h^{l''}_{l',y'}(y) \in h^{l''}_{l',y'}(D(p,r_{l'})) \subset D(p,r_{l''}) \; .$$

Thus $z = h^j_{l'',z''}(y'') \in h^j_{l'',z''}(D(p,r_{l''}))$, proving the transitivity of $\leq$.

Finally, let us prove that $\leq$ is antisymmetric. Let $(l,z),(l',z') \in P_i^j$ be such that $(l,z) \leq (l',z')$ and $(l',z') \leq (l,z)$. By the definition of $\leq$, we get $l = l'$. Thus $z = z'$ by (a), and therefore $(l,z) = (l',z')$. □

Lemma 3.2.4. *The following properties hold:*

(a) For $(l,z),(l',z') \in P_i^j$, *if* $l < l'$ *and* $d(z,z') < \lambda_l\Lambda_j(r_{l'} + s_l)$, *then* $(l,z) \leq (l',z')$.

(b) For every

$$x \in \bigcup_{(l,z)\in P_i^j} h^j_{l,z}(D(p,r_l))$$

there is a unique $(l,z) \in \overline{P}_i^j$ *such that* $x \in h^j_{l,z}(D(p,r_l))$. *In particular, for all* $(k,y) \in P_i^j$, *there is a unique* $(l,z) \in \overline{P}_i^j$ *satisfying* $(k,y) \leq (l,z)$.

(c) For $(l,z),(l',z') \in P_i^j$, *we have*

$$D(z, \lambda_l\Lambda_{l,j-1}(r_l + s_i)) \subset D(z', \lambda_l\Lambda_{l',j-1}(r_{l'} + s_i))$$

if $(l,z) < (l',z')$.

Proof. Item (a) follows from a simple application of the induction hypothesis with (v).

Let us prove (b). Suppose by absurdity that there are $(l,z) \neq (l',z')$ in $\overline{P}_i^j$ such that $h^j_{l,z}(D(p,r_l))$ and $h^j_{l',z'}(D(p,r_{l'}))$ intersect at some point $x \in M$. By the induction hypothesis, $h^j_{l,z}$ and $h^j_{l',z'}$ are $(l,r_l,\Lambda_{l,j-1})$ and $(l,r_l,\Lambda_{l,j-1})$-p.p.q.i., respectively. In the case where $l < l'$, then (3.2.2) and (3.2.7) yield

$$d(z,z') \leq d(z,x) + d(x,z') \leq \Lambda_{l,j-1}r_l + \Lambda_{l',j-1}r_{l'} < \Lambda_l(r_{l'} + s_l) < \lambda_0^2(r_{l'} + s_l) \; .$$

Thus $(l,z) < (l',z')$ by (a), contradicting the maximality of (l,z). If, on the other hand, $l = l'$, then the induction hypothesis with (ii) yields

$$s_l/\Lambda_{l+1,j-1} \le d(z,z') \le d(z,x) + d(x,z') \le 2\Lambda_{l,j-1} r_l \ .$$

In particular, $s_l \le 2\lambda_0^2 r_l$ by (3.2.7), contradicting (3.2.2). The second assertion of (b) follows from the first one because $(k,y) \le (l,z)$ yields $y \in h^j_{l,z}(D(p,r_l)) \cap h^j_{l',z'}(D(p,r_{l'}))$.

Let us prove (c). We are assuming that $(l,z) < (l',z')$, so $z \in h^j_{l',z'}(M^{l'}_l)$. Since $h^j_{l',z'} \colon (M,p) \rightarrowtail (M,z')$ is an $(l', r_{l'}, \Lambda_{l',j-1})$-p.p.q.i., we have $d(z',z) \le \Lambda_{l',j-1} r_{l'}$ by the induction hypothesis with (ii), so

$$D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i)) \subset D(z', \Lambda_{l',j-1} r_{l'} + \lambda_l \Lambda_{l,j-1}(r_l + s_i)) \ .$$

But now (3.2.1) yields

$$\lambda_l \Lambda_{l',j-1}(r_{l'} + s_i) > \Lambda_{l',j-1} r_{l'} + (\lambda_l - 1)\Lambda_{l',j-1} r_{l'} \ge \Lambda_{l',j-1} r_{l'} + \lambda_l \Lambda_{l,j-1}(r_l + s_i) \ . \qquad \square$$

Let us define the disjoint sets $\widetilde{M}^j_i$ and $\widehat{M}^j_i$, whose union is the definition of M^j_i. First, let

$$\widetilde{M}^j_i = \bigcup_{(l,z) \in \overline{P}^j_i} h^j_{l,z}(M^l_i) \ . \tag{3.2.14}$$

Note that this set is well-defined since $M^l_i \subset D(p, r_l) = \operatorname{dom} h^j_{l,z}$ by the induction hypothesis with (ii). Second, take any maximal s_i-separated subset

$$\widehat{M}^j_i \subset \Omega_i \cap D(p, r_j - t_i) \setminus \dot{\bigcup_{(l,z) \in \overline{P}^j_i}} D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i)) \ . \tag{3.2.15}$$

We have $\widetilde{M}^j_i \cap \widehat{M}^j_i = \emptyset$ since, for all $(l,z) \in \overline{P}^j_i$,

$$h^j_{l,z}(M^l_i) \subset h^j_{l,z}(D(p,r_l)) \subset D(z, \Lambda_{l,j-1} r_l) \subset D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i))$$

because $h^j_{l,z} \colon (M,p) \rightarrowtail (M,z)$ is an $(l, r_l, \Lambda_{l,j-1})$-p.p.q.i. by the induction hypothesis.

The definition of the partial maps $h^j_{i,x}$ depends on whether $x \in \widehat{M}^j_i$ or $x \in \widetilde{M}^j_i$. If $x \in \widehat{M}^j_i$, let $h^j_{i,x}$ be any (i, r_i, λ_i)-p.p.q.i. $(M,p) \rightarrowtail (M,x)$, which exists because $x \in \Omega_i$. If $x \in \widetilde{M}^j_i$, then the induction hypothesis with (ii) yields

$$x \in \bigcup_{(k,y) \in \overline{P}^j_i} h^j_{k,y}(M^k_i) \subset \bigcup_{(k,y) \in \overline{P}^j_i} B(y, \Lambda_{k,j-1} r_k) \ .$$

By Lemma 3.2.4 (b), there is a unique $(l,z) \in \overline{P}^j_i$ such that $x \in h^j_{l,z}(D(p,r_l))$. Then define $h^j_{i,x} = h^j_{l,z} \circ h^l_{i,x'}$, where $x' = (h^j_{l,z})^{-1}(x)$. Note that $\operatorname{im}(h^l_{i,x'}) \subset \operatorname{dom}(h^j_{l,z})$, as explained in Remark 19.

Lemma 3.2.5. *If $(l,z) \in \overline{P}^j_i$, then $z \in \widehat{M}^j_l$.*

Proof. The statement is true for $l = j-1$ because $\widehat{M}^j_{j-1} = M^j_{j-1}$ by definition. Suppose by absurdity that $l < j-1$ and $z \in \widetilde{M}^j_l$. Then, by (3.2.14), there is some $(l',z') \in \overline{P}^j_i$ with $l' > l$ and $z \subset h^j_{l',z'}(M^{l'}_l)$. Thus $(l,z) < (l',z')$, a contradiction. $\square$

Lemma 3.2.6. *The following properties hold for every* $x \in M_i^j$*:*

(a) If $x \in \widehat{M}_i^j$*, then the partial map* $h_{i,x}^j$ *is an* (i, r_i, λ_i)*-p.p.q.i.* $(M, p) \rightarrowtail (M, x)$*.*

(b) The partial map $h_{i,x}^j$ *can be expressed as a product* $h_{i_L,x_L}^j \cdots h_{i,x_1}^{j_L}$ $(1 \le L \le j - i)$*, where* $i_1 > \cdots > i_L = i$*,* $j = j_1 > \cdots > j_L$ *and* $x_l \in \widehat{M}_{i_l}^{j_l}$ $(1 \le l \le L)$*.*

(c) The partial map $h_{i,x}^j$ *is an* $(i, r_i, \Lambda_{i,j-1})$*-p.p.q.i.* $(M, p) \rightarrowtail (M, x)$*.*

Proof. Item (a) holds by the definition of $h_{i,x}^j$ when $x \in \widehat{M}_i^j$, so let us prove Items (b) and (c) by induction. When $j = i + 1$, we have $M_i^j = \widehat{M}_i^j$ and so Items (b) and (c) hold trivially. Suppose the result is true when either either $l < j$, or $l = j$ and $k > i$. We only have to consider the case where $x \in \widetilde{M}_i^j$. Let $(l, z) \in \overline{P}_i^j$ be the unique pair satisfying $x \in B(z, r_l)$ (Lemma 3.2.4 (b)), and let $x' = (h_{l,z}^j)^{-1}(x)$. By the induction hypothesis, $h_{i,x'}^l : (M, p) \rightarrowtail (M, x')$ is an $(i, r_i, \Lambda_{i,l-1})$-p.p.q.i. and can be written as a product $h_{i_K,x_K}^{j_K} \cdots h_{i_1,x_1}^j$ $(1 \le L \le l - i)$, where $i_1 < \cdots < i_K = i$, $j = j_1 > \cdots > j_K = l$ and $x_k \in \widehat{M}_{i_k}^{j_k}$ $(1 \le k \le K)$. By the definition of $h_{i,x}^j$ when $x \in \widetilde{M}_i^j$, we have

$$h_{i,x}^j = h_{l,z}^j h_{i,x'}^l = h_{l,z}^j h_{i_K,x_K}^l \cdots h_{i,x_1}^{j_1} ,$$

and Item (b) follows from Lemma 3.2.5. Finally, Item (c) follows from the equality $h_{i,x}^j = h_{l,z}^j h_{i,x'}^l$, the induction hypothesis and Proposition 3.1.1. □

Once we have made the relevant definitions, let us show that they satisfy the properties listed in Proposition 3.2.2. Item (i) is guaranteed by the definition of $\widehat{M}_i^j$, so we really start by proving (ii).

The inclusion $M_i^j \subset \widetilde{\Omega}_{i,j-1}$ is obvious by Lemma 3.2.6 (c). Let us prove that $M_i^j \subset D(p, r_j - t_i)$. We have $\widehat{M}_i^j \subset D(p, r_j - t_i)$ by construction, so let us show that $\widetilde{M}_i^j \subset D(p, r_j - t_i)$. By the induction hypothesis with (ii), we have $z \in D(p, r_j - t_l)$ for all $(l, z) \in \overline{P}_i^j$. Then $D(z, \lambda_l r_l) \subset D(p, r_j - t_i)$ because, for any $y \in D(z, \lambda_l r_l)$,

$$d(y, p) \le d(y, z) + d(z, p) < \lambda_l r_l + r_j - t_l < r_j - t_i$$

by (3.2.3). Thus $\widetilde{M}_i^j \subset D(p, r_j - t_i)$ according to (3.2.14), since $h_{l,z}^j : (M, p) \rightarrowtail (M, z)$ is an (l, r_l, λ_l)-p.p.q.i. for all $(l, z) \in \overline{P}_i^j$ by Lemmas 3.2.5 and 3.2.6, and $M_i^l \subset D(p, r_l)$ by the induction hypothesis with (ii).

The proof of (ii) is concluded by showing that M_i^j is $s_i/\Lambda_{i,j-1}$-separated. To begin with, we prove that $\widetilde{M}_i^j$ is $s_i/\Lambda_{i,j-1}$-separated. Let $(l, z) \in \overline{P}_i^j$. By the induction hypothesis, M_i^l is $s_i/\Lambda_{i,l-1}$-separated and $h_{l,z}^j : (M, p) \rightarrowtail (M, z)$ is an $(l, r_l, \Lambda_{l,j-1})$-quasi-isometry. Thus $h_{l,z}^j(M_i^l)$ is $s_i/\Lambda_{i,j-1}$-separated. Moreover $h_{l,z}^j(M_i^l) \subset D(z, \Lambda_{l,j-1} r_l)$ by the induction hypothesis with (ii). By (3.2.14), it is enough to show that

$$d(z, z') \ge \Lambda_{l,j-1} r_l + \Lambda_{l',j-1} r_{l'} + s_i/\Lambda_{i,j-1} \tag{3.2.16}$$

for $(l, z) \ne (l', z')$ in $\overline{P}_i^j$. If $l = l'$, then, by (3.2.2) and (3.2.7),

$$s_l/\Lambda_{l+1,j-1} > s_l/\lambda_0^2 > \lambda_0(2r_l + s_i) > 2\Lambda_{i,l-1} r_l + s_i/\Lambda_{i,j-1} . \tag{3.2.17}$$

Thus (3.2.16) follows from the induction hypothesis with (ii) applied to M_l^j. If $l < l'$, then (3.2.2) yields

$$s_l \geq \lambda_0(r_l + s_i) \geq \Lambda_{l,j-1} r_l + s_i/\Lambda_{i,j-1} \;.$$

So, applying Lemma 3.2.4 (a) and (3.2.7), we get

$$d(z,z') \geq \lambda_l \Lambda_j (r_{l'} + s_l) \geq \Lambda_{l',j-1} r_{l'} + \Lambda_{l,j-1} r_l + s_i/\Lambda_{i,j-1} \;.$$

The set $\widehat{M}_i^j$ is s_i-separated by construction. Thus, to prove that $M_i^j = \widehat{M}_i^j \cup \widetilde{M}_i^j$ is $s_i/\Lambda_{i,j-1}$-separated, it suffices to show that $d(\widetilde{M}_i^j, \widehat{M}_i^j) \geq s_i/\Lambda_{i,j-1}$. Let $\tilde{x} \in \widetilde{M}_i^j$ and $\hat{x} \in \widehat{M}_i^j$. By (3.2.13), (3.2.14) and (3.2.15), there is some $(l,z) \in \overline{P}_i^j$ such that $\tilde{x} \in D(z, \Lambda_{l,j-1} r_l)$ and $\hat{x} \notin D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i))$. By the triangle inequality, we get $d(\tilde{x}, \hat{x}) \geq s_i$, which concludes the proof of (ii).

Let us prove (iii). Let $(l,z) \in P_i^j$ and $x \in M_i^j \cap h_{l,z}^j(D(p, r_l))$. We have

$$\widehat{M}_i^j \cap D(z, \Lambda_l(r_l + s_i)) = \emptyset \tag{3.2.18}$$

by (3.2.15) and Lemma 3.2.4 (b),(c), so $x \in \widetilde{M}_i^j$. Consider first the case where $(l,z) \in \overline{P}_i^j$. Then the equality $h_{i,x}^j = h_{l,z}^j \circ h_{i,x'}^l$, for $x' = (h_{l,z}^j)^{-1}(x)$, is precisely the definition of $h_{i,z}^j$. Therefore we can suppose that $(l,z) \in P_i^j \setminus \overline{P}_i^j$. According to Lemma 3.2.4 (b), there is a unique $(l',z') \in \overline{P}_i^j$ such that $(l,z) < (l',z')$ and $x \in \operatorname{im}(h_{l',z'}^j)$. We have already proved that $h_{i,x}^j = h_{l',z'}^j \circ h_{i,x'}^{l'}$ for $x' = (h_{l',z'}^j)^{-1}(x)$. Moreover, by the induction hypothesis with (iii), if $y = (h_{l',z'}^j)^{-1}(z)$ and $x'' = (h_{l,z}^j)^{-1}(x)$, we have $(h_{l,y}^{l'})^{-1}(x') = x''$, $h_{l,z}^j = h_{l',z'}^j \circ h_{l,y}^{l'}$ and $h_{i,x'}^{l'} = h_{l,y}^{l'} \circ h_{i,x''}^l$. Therefore

$$h_{i,x}^j = h_{l',z'}^j \circ h_{i,x'}^{l'} = h_{l',z'}^j \circ h_{l,y}^{l'} \circ h_{i,x''}^l = h_{l,z}^j \circ h_{i,x''}^l \;.$$

Let us prove (iv). Let $(l,z) \in P_i^j$. By (3.2.18), we only have to show that

$$\widetilde{M}_i^j \cap h_{l,z}^j(D(p, r_l)) = h_{l,z}^j(M_i^l) \;. \tag{3.2.19}$$

Consider first the case where $(l,z) \in \overline{P}_i^j$. For $(l',z') \in \overline{P}_i^j \setminus \{(l,z)\}$, by (ii) and (3.2.16),

$$h_{l,z}^j(D(p,r_l)) \cap h_{l',z'}^j\big(M_i^{l'}\big) \subset D(z, \Lambda_{l,j-1} r_l) \cap D(z', \Lambda_{l',j-1} r_{l'}) = \emptyset$$

and $M_i^l \subset D(p, r_l)$, yielding (3.2.19), as desired.

Suppose now that $(l,z) \in P_i^j \setminus \overline{P}_i^j$. Then, according to Lemma 3.2.4 (b), there is a unique $(l',z') \in \overline{P}_i^j$ such that $(l,z) < (l',z')$. We have already proved that

$$M_i^j \cap h_{l',z'}^j(D(p, r_{l'})) = h_{l',z'}^j(M_i^{l'}) \;.$$

Let $y = (h_{l',z'}^j)^{-1}(z)$. By the induction hypothesis with (iv), we know that $M_i^{l'} \cap h_{l,y}^{l'}(D(p,r_l)) = h_{l,y}^{l'}(M_i^l)$. Thus (3.2.19) follows using (iii):

$$\begin{aligned} M_i^j \cap h_{l,z}^j(D(p,r_l)) &= M_i^j \cap h_{l',z'}^j \circ h_{l,y}^{l'}(D(p,r_l)) = h_{l',z'}^j\big(M_i^{l'}\big) \cap h_{l',z'}^j \circ h_{l,y}^{l'}(D(p,r_l)) \\ &= h_{l',z'}^j\big(M_i^{l'} \cap h_{l,y}^{l'}(D(p,r_l))\big) = h_{l',z'}^j\big(h_{l,y}^{l'}(M_n^l)\big) = h_{l,z}^j(M_i^l) \;. \end{aligned}$$

Let us prove (v). If $(l,z)\in\overline{P}_i^j$, then the result follows from (3.2.14) and (3.2.15). So suppose $(l,z)\notin\overline{P}_i^j$. Consider first the case where $x\in\widehat{M}_i^j$. By Lemma 3.2.4 (b), there is a unique $(l',z')\in\overline{P}_i^j$ such that $(l,z)<(l',z')$, and so Lemmas 3.2.5 and 3.2.6, and (ii) give

$$z\in h^j_{l',z'}(M_l^{l'})\subset h^j_{l',z'}(D(z',r_{l'}-t_l))\subset D(z',\lambda_{l'}(r_{l'}-t_l))\ .$$

Then (3.2.15), (3.2.3) and (3.2.7) yield

$$d(x,z)\ge d(x,z')-d(z',z)\ge\lambda_l\Lambda_{l',j-1}(r_{l'}+s_i)-\lambda_{l'}(r_{l'}-t_l)>t_l>\lambda_l\Lambda_j(r_l+s_i)\ .$$

Consider now the case where $x\in\widetilde{M}_i^j$. Then there is a unique $(l',z')\in\overline{P}_i^j$ such that $x\in h^j_{l',z'}(M_i^{l'})$. If $(l,z)=(l',z')$, then $x\in h^j_{l,z}(M_i^l)$. If $(l,z)\ne(l',z')$ and $l=l'$, then $d(z,z')\ge s_l/\Lambda_{l+1,j-1}$ by (ii). Thus (3.2.2) and (3.2.7) yield

$$d(x,z)\ge d(z,z')-d(x,z')\ge s_l/\Lambda_{l+1,j-1}-\Lambda_{l,j-1}r_l\ge\lambda_l\Lambda_j(r_l+s_i)\ .$$

If $l<l'$ and $(l,z)\nleq(l',z')$, then Lemma 3.2.4 (a), (3.2.2) and (3.2.7) yield

$$d(x,z)\ge d(z,z')-d(x,z')\ge\lambda_l\Lambda_j(r_{l'}+s_l)-\Lambda_{l',j-1}r_{l'}>\mathfrak{s}_l>\lambda_l\Lambda_j(r_l+s_i)\ .$$

This holds since $\lambda_l\Lambda_j-\Lambda_{l',j-1}>0$ by (3.2.7). If $l>l'$, then Lemma 3.2.4 (a), (3.2.2) and (3.2.7) yield

$$\begin{aligned}d(x,z)&\ge d(z,z')-d(x,z')\ge\lambda_{l'}\Lambda_j(r_l+s_{l'})-\lambda_{l'}r_{l'}\\&>\lambda_{l'}\Lambda_j r_l+s_{l'}-\lambda_{l'}r_{l'}>\lambda_l\Lambda_j(r_l+s_i)\ .\end{aligned}$$

At this point, only the case $(l,z)<(l',z')$ remains to be considered; i.e., $l<l'$ and $z\in h^j_{l',z'}(M_l^{l'})$. Let $x'=(h^j_{l',z'})^{-1}(x)\in M_i^{l'}$ and $y=(h^j_{l',z'})^{-1}(z)\in M_l^{l'}$. By the induction hypothesis with (v), either $x'\in h^{l'}_{l,y}(M_i^l)$, or $d(x',y)\ge\lambda_l\Lambda_{l'}(r_l+s_i)$. In the first case, we have $x\in h^j_{l',z'}\circ h^{l'}_{l,y}(M_i^l)=h^j_{l,z}(M_i^l)$ by (iii). In the second case, the fact that $h^j_{l',z'}$ is an $(l',r_{l'},\Lambda_{l',j-1})$-p.p.q.i. $(M,p)\rightarrowtail(M,z')$ gives

$$d(x,z)\ge\frac{d(x',y)}{\Lambda_{l',j-1}}\ge\lambda_l\frac{\Lambda_{l'}}{\Lambda_{l',j-1}}(r_l+s_i)=\lambda_l\Lambda_j(r_l+s_i)\ .$$

Lemma 3.2.7. *$M_i^{j-1}\subset M_i^j$, and $h^j_{i,z}=h^{j-1}_{i,z}$ for all $z\in M_i^{j-1}$.*

Proof. Let $z\in M_i^{j-1}$. By (ii) and the induction hypothesis with (vii), we have $z\in B(p,r_{j-1})$, $p\in M^j_{j-1}$ and $h^j_{j-1,p}=\mathrm{id}_{B(p,r_{j-1})}$. By the definitions of P_i^j in (3.2.12) and $<$, it is immediate that $(j-1,p)\in\overline{P}_i^j$. Then $z\in M_i^{j-1}=h^j_{j-1,p}(M_i^{j-1})\subset\widetilde{M}_i^j$. Using (iii), we see that

$$h^j_{i,z}=h^j_{j-1,p}\circ h^{j-1}_{i,z}=\mathrm{id}_{B(p,r_{j-1})}\circ h^{j-1}_{i,z}=h^{j-1}_{i,z}\ .$$ □

Lemma 3.2.8. *$M^j_{i+1}\subset M_i^j$, and $h^j_{i,z}=h^j_{i+1,z}|_{B(p,r_i)}$ for every $z\in M^j_{i+1}$.*

Proof. Let $z\in M^j_{i+1}$. Then $(i+1,z)\in P_i^j$. Moreover $p\in M_i^{i+1}$ and $h^{i+1}_{i,p}=\mathrm{id}_{B(p,r_i)}$ by definition, and

$$z=h^j_{i+1,z}(p)\subset h^j_{i+1,z}(M_i^{i+1})\subset M_i^j$$

by (iv). Therefore, by (iii),

$$h^j_{i,z}=h^j_{i+1,z}\circ h^{i+1}_{i,p}=h^j_{i+1,z}\circ\mathrm{id}_{B(p,r_i)}=h^j_{i+1,z}|_{B(p,r_i)}\ .$$ □

Now (vi) follows from Lemmas 3.2.7 and 3.2.8 by induction.

Finally, (vii) follows from (vi) and the definitions of M_i^i and $h_{i,p}^i$, completing the proof of Proposition 3.2.2.

Remark 20. Refining Proposition 3.2.2 (vii), note that $p \in \widehat{M}_i^{i+1}$ by definition, and $p \in \widetilde{M}_i^j$ for $j > i+1$ by the argument of Lemma 3.2.7.

Remark 21. Note that, in the course of the proof of Proposition 3.2.2, the only properties needed from the sets Ω_i are $\Omega_i \subset \Omega(i, r_i, \lambda_i)$ and the fact that Ω_i is relatively dense in M. Therefore Proposition 3.2.2 also holds by substituting the sets Ω_i with a prescribed family of subsets of M satisfying the above conditions, after possibly changing the value of ω_i. Similarly, the choice of (i, r_i, λ_i)-p.p.q.i. $h_{i,x}^j$ for $x \in \widehat{M}_i^j$ is arbitrary. So, if we have for every x in Ω_i a prescribed (i, r_i, λ_i)-p.p.q.i. $f_x \colon (M, p) \rightarrowtail (M, x)$, then we can also assume that $h_{i,x}^j = f_x$ for every $x \in \widehat{M}_i^j$. Thus every map $h_{i,x}^j$ is a composition of the form $f_I \cdots f_1$ by Lemma 3.2.6.

For $i \in \mathbb{N}$, let

$$M_i = \bigcup_{j \geq i} M_i^j \ , \quad P_i = \bigcup_{j > i} P_i^j = \{ (j, x) \in \mathbb{N} \times M \mid j > i,\ x \in M_i^j \} \ . \tag{3.2.20}$$

For every $x \in M_i$, there is some $j \geq i$ such that $x \in M_i^j$. Then let $h_{i,x} = h_{i,x}^j$, which is independent of j by Proposition 3.2.2 (vi). Thus the order relations $\leq$ on the sets P_i^j $(j \geq i)$ fit well to define an order relation $\leq$ on P_i; more precisely, $\leq$ is the reflexive closure of the relation $<$ on P_i defined by setting $(j, x) < (j', x')$ if $j < j'$ and $x \in h_{j',x'}(M_j^{j'})$. The following result is a direct consequence of Proposition 3.2.2.

Proposition 3.2.9. *The following properties hold:*

(i) M_i *is an* s_i/Λ_{i+1}*-separated subset of* $\widetilde{\Omega}_i$.

(ii) For every $x \in M_i$, $h_{i,x}$ *is an* (i, r_i, Λ_i)*-p.p.q.i.* $(M, p) \rightarrowtail (M, x)$.

(iii) For any $(l, z) \in P_i$, *we have* $M_i \cap h_{l,z}(D(p, r_l)) = h_{l,z}(M_i^l)$.

(iv) For every $(j, y) \in P_i$ *and* $x \in M_i \cap h_{j,y}(D(p, r_j))$, *we have* $h_{i,x} = h_{j,y} \circ h_{i,x'}$, *where* $x' = h_{j,y}^{-1}(x)$.

(v) For any $x \in M_i$ *and* $(j, y) \in P_i$, *either* $d(x, z) \geq \lambda_l(r_l + s_i)$, *or* $x \in h_{l,z}(M_i^l)$.

(vi) For $i \leq j$, *we have* $M_j \subset M_i$, *and* $h_{i,x} = h_{j,x}|_{D(p,r_i)}$ *for* $x \in M_j$.

(vii) We have $p \in M_i$ *and* $h_{i,p} = \mathrm{id}_{D(p,r_i)}$.

For integers $0 \leq i < j$, let

$$I_i^j = D(p, r_j - t_i - \omega_n) \setminus \bigcupdot_{(l,z) \in \overline{P}_i^j} D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i) + \omega_n); . \tag{3.2.21}$$

Lemma 3.2.10. *The set* $S(p, r_j - t_i - \omega_i)$ *is contained in* I_i^j.

Proof. Since

$$S(p, r_j - t_i - \omega_i) \subset D(p, r_j - t_i - \omega_i) ,$$

the lemma follows by proving that, for $(l, z) \in \overline{P}_i^j$,

$$S(p, r_j - t_i - \omega_i) \cap D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i) + \omega_i) = \emptyset . \tag{3.2.22}$$

On the one hand, $d(p, z) \leq r_j - t_l$ by Proposition 3.2.2 (ii). On the other hand, by (3.2.3) and (3.2.7),

$$t_l > t_i + \lambda_l \Lambda_{l,j-1}(r_l + s_i) + 2\omega_i ,$$

and therefore

$$r_j - t_l + \lambda_l \Lambda_{l,j-1}(r_l + s_i) + \omega_i < r_j - t_i - \omega_i .$$

Thus

$$D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i) + \omega_i) \subset B(p, r_j - t_i - \omega_i) ,$$

and (3.2.22) follows. □

Lemma 3.2.11. *For all $z \in I_i^j$, we have $d(z, M_i^j) \leq \omega_i + s_i$.*

Proof. Let $z \in I_i^j$. Since Ω_i is ω_i-relatively dense in M, there is some $y \in \Omega_i$ with $d(y, z) \leq \omega_i$. Thus, by (3.2.21),

$$y \in D(p, r_j - t_i) \setminus \dot{\bigcup_{(l,u) \in \overline{P}_i^j}} D(u, \lambda_l \Lambda_{l,j-1}(r_l + s_i)) .$$

Then, by Proposition 3.2.2 (i), the set $\widehat{M}_i^j \cup \{y\}$ cannot be s_i-separated and properly contain $\widehat{M}_i^j$. So, either $y \in \widehat{M}_i^j$, or there is some $x \in \widehat{M}_i^j \setminus \{y\}$ with $d(x, y) < s_i$. In the former case $d(z, M_i^j) \leq d(z, y) \leq \omega_i$, whereas in the latter $d(z, M_i^j) \leq d(z, x) \leq d(z, y) + d(y, x) \leq \omega_i + s_i$. □

For $i \in \mathbb{N}$, let

$$I_i = \bigcup_{(j,z) \in P_i} h_{j,z}(I_i^j) .$$

Lemma 3.2.12. *I_i is relatively dense in M, where the implied constant depends only on $r_i, s_i, t_i, \omega_i, \lambda_i$, and λ_0.*

Proof. Let $x \in M$. We have $D(x, \omega_i) \subset D(p, r_j - t_i)$ for j large enough. If $x \notin I_i$, then $x \notin h_{j,p}(I_i^j) = I_i^j$. So, according to (3.2.21), there is some $(l, z) \in P_i^j$ such that

$$x \in D(z, \lambda_l \Lambda_{l,j-1}(r_l + s_i) + \omega_i) . \tag{3.2.23}$$

We can suppose that the pair (l, z) minimizes $d(x, z)$ among the elements in P_i^j satisfying (3.2.23). Moreover we can assume that l is the least value such that (l, z) is in P_i^j and satisfies the above properties.

Consider first the case where $x \notin h_{l,z}(B(p, r_l - t_i - \omega_i))$. Let $\tau\colon [0, 1] \to M$ be a minimizing geodesic segment with $\tau(0) = x$ and $\tau(1) = z$. There is some $a \in [0, 1)$ such that

$$\tau(a) \in h_{l,z}(S(p, r_l - t_i - \omega_i)) \subset C(z, (r_l - t_i - \omega_i)/\Lambda_l, \Lambda_l(r_j - t_i - \omega_i)) ,$$

where the last inclusion holds because $h_{l,z}\colon (M,p) \rightarrowtail (M,z)$ is an (l,r_l,Λ_l)-p.p.q.i. Then, by (3.2.23), (3.2.5) and (3.2.7),

$$\begin{aligned} d(x,\tau(a)) &= d(x,z) - d(\tau(a),z) \leq \lambda_l\Lambda_l(r_l+s_i)+\omega_i-(r_l-t_i-\omega_i)/\Lambda_l \\ &< r_l(\lambda_l\Lambda_l^2-1)/\Lambda_l+\lambda_0^2 s_i+t_i+2\omega_i\ . \end{aligned}$$

Using Lemma 3.2.1, we get

$$d(x,h_{l,z}(S(p,r_l-t_i-\omega_i))) < 4r_i\frac{\lambda_i^5-1}{\lambda_i^2}+\lambda_0^3 s_i+t_i+2\omega_i\ ,$$

and then the result follows from Lemma 3.2.10.

Suppose now that $x \in h_{l,z}(B(p,r_l-t_i-\omega_i))$. Then $h_{l,z}^{-1}(x) \notin I_i^l$ because $h_{l,z}(I_i^l) \subset I_i$. Therefore

$$h_{l,z}^{-1}(x) \in D(z',\lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)+\omega_i) \tag{3.2.24}$$

for some $(l',z') \in P_i^l$, according to (3.2.21). Assume first $z' \neq p$, and let us prove that $z \neq p$. Suppose by absurdity that $z = p$. We have $h_{l,z}^{-1}(x) = x$ by Proposition 3.2.9 (vii). So now (3.2.24) gives

$$x \in D(z',\lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)+\omega_i) \subset D(z',\lambda_{l'}\Lambda_{l',j-1}(r_{l'}+s_i)+\omega_i)\ .$$

Since $d(x,z) \leq d(x,z')$, we get $x \in D(z,\lambda_{l'}\Lambda_{l',j-1}(r_{l'}+s_i)+\omega_i)$, contradicting our choice of (l,z) because $l' < l$.

Since $p \in M_{l'}$ by Proposition 3.2.9 (vii), we have $d(p,z') \geq s_{l'}/\Lambda_{l'+1}$ by Proposition 3.2.9 (i). So, by (3.2.24),

$$d(p,h_{l,z}^{-1}(x)) \geq d(p,z') - d(z',h_{l,z}^{-1}(x)) \geq s_{l'}/\Lambda_{l'+1} - \lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)-\omega_i\ . \tag{3.2.25}$$

Note that $z' \in M_{l'}^l \subset D(p,r_l-t_{l'}) \subset \operatorname{dom} h_{l,z}$ by Proposition 3.2.2 (ii). Moreover $(l',h_{l,z}(z')) \in P_i^j$ according to (3.2.12) because $h_{l,z}(z') = h_{l,z}^j(z') \in h_{l,z}^j(M_{l'}^l) \subset M_{l'}^j$ by Proposition 3.2.2 (iv), using that $(l,z) \in P_{l'}^m$. Since $h_{l,z}\colon (M,p) \rightarrowtail (M,z)$ is an (l,r_l,Λ_l)-p.p.q.i., and using (3.2.25), (3.2.2), (3.2.5), (3.2.7) and (3.2.24), it follows that

$$\begin{aligned} d(z,x) &\geq \Lambda_l^{-1}(s_{l'}/\Lambda_{l'+1}-\lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)-\omega_i) \\ &> 2\lambda_0^3(r_{l'}+s_i+\omega_i)-\lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)-\omega_i \\ &> \lambda_0^3(r_{l'}+s_i)+\tilde{\omega}_i > \Lambda_l\lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)+\omega_i > d(h_{l,z}(z'),x)\ . \end{aligned}$$

This contradicts the assumption that (l,z) minimizes $d(z,x)$ because $(l',h_{l,z}(z')) \in P_i^j$.

At this point, only the case $z' = p$ remains to be considered. Then, since $h_{l,z}\colon (M,p) \rightarrowtail (M,z)$ is an (l,r_l,Λ_l)-p.p.q.i., and using (3.2.24), (3.2.5) and (3.2.7), we get

$$d(x,z) \leq \Lambda_l d(h_{l,z}^{-1}(x),p) \leq \Lambda_l\big(\lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)+\omega_i\big) < \lambda_{l'}^2\Lambda_{l'}(r_{l'}+s_i)+\lambda_0\omega_i\ .$$

Note that Proposition 3.2.2 (vi) yields $(l',z) \in P_i^j$ since $i < l' < l$. Thus the minimality of l gives

$$d(x,z) \geq \lambda_{l'}\Lambda_{l',l-1}(r_{l'}+s_i)+\tilde{\omega}_i > r_{l'}-t_i-\omega_i\ ,$$

using (3.2.23). Then, arguing like in the second paragraph of the proof, we construct a minimizing geodesic segment τ from x to z that meets $h_{l',z}(S(z, r_{l'} - t_i - \omega_i))$ at a point $\tau(a)$ satisfying

$$\begin{aligned} d(x,\tau(a)) = d(x,z) - d(\tau(a),z) &\le \lambda_{l'}^2\Lambda_{l'}(r_{l'} + s_i) + \lambda_0\omega_i - (r_{l'} - t_i - \omega_i)/\Lambda_l \\ &< r_{l'}\frac{\lambda_{l'}^2\Lambda_{l'}^2 - 1}{\Lambda_{l'}} + \lambda_0^2 s_i + t_i + (1+\lambda_0)\omega_i \ . \end{aligned}$$

Using Lemma 3.2.1, we get

$$d(x, h_{l',z}(S(z, r_{l'} - t_i - \omega_i))) \le 4r_i\frac{\lambda_i^6 - 1}{\lambda_i^2} + \lambda_0^2 s_i + t_i + (1+\lambda_0)\omega_i \ ,$$

and then the result follows from Lemma 3.2.10. □

Proposition 3.2.13. *M_i is relatively dense in M, where the implied constant depends only on $r_i, s_i, t_i, \omega_i, \lambda_i$, and λ_0.*

Proof. Note that $M_i \subset I_i$ since, for all $x \in M_i$, we have $x \in h_{i+1,x}(D(p, r_{i+1} - t_i)) = h_{i+1,x}(I_i^{i+1})$. By Lemma 3.2.12, it is enough to show that M_i is relatively dense in I_i. Let $y \in I_i$. By definition of I_i, there is some $(l,z) \in P_i$ such that $y \in h_{l,z}(I_i^l)$. By Lemma 3.2.11, there is some $x \in M_i^l \subset \operatorname{dom} h_{l,z}$ such that $d(h_{l,z}^{-1}(y), x) \le \omega_i + s_i$. By Proposition 3.2.9 (iii), we have $h_{l,z}(x) \in M_i$. Then the fact that $h_{l,z}\colon (M,p) \rightarrowtail (M,z)$ is an (l, r_l, Λ_l)-p.p.q.i. gives

$$d(y, M_i) \le d(y, h_{l,z}(x)) \le \Lambda_l(\omega_i + s_i) \le \Lambda_i(\omega_i + s_i) \ . \qquad \square$$

Proposition 3.2.14. *For every $\eta > 0$, there is a separated η-relatively dense subset $X \subset M$ such that $p \in X$ and*

$$X \cap h_{l,z}(D(p, r_l)) = h_{l,z}(X \cap D(p, r_l)) \tag{3.2.26}$$

for all $(l,z) \in P_0$.

We will derive this result from the following auxiliary lemma.

Lemma 3.2.15. *For any $\eta > 0$ and $0 < \delta < \eta/\Lambda_1$, there are sets $X_1 \subset X_2 \subset \cdots \subset M$ containing p such that:*

(a) every X_i is $\delta/\Lambda_{1,i-1}$-separated and $\delta\Lambda_{1,i-1}$-relatively dense in $D(p, r_i)$; and,

(b) for all $(l,z) \in P_0^i$,

$$X_i \cap h_{l,z}(D(p, r_l)) = h_{l,z}(X_l) \ .$$

Proof. We proceed by induction on $i \in \mathbb{Z}^+$. Let X_1 be a maximal δ-separated subset of $D(p, r_1)$ containing p, given by Zorn's lemma. By Lemma 2.1.3, it is also δ-relatively dense in $D(p, r_1)$.

Now, given any $i > 1$, suppose that we have already defined X_k for $1 \le k < i$ satisfying (a) and (b). Let

$$\widetilde{X}_i = \bigcup_{(l,z)\in\overline{P}_0^i}^{\cdot} h_{l,z}^i(X_l) \ .$$

Note that $X_{i-1} \subset \widetilde{X}_i$ by Proposition 3.2.2 (vii). The following assertion follows from the induction hypothesis with (a) and Proposition 3.2.9 (ii).

Claim 3. $\widetilde{X}_i$ is $\delta/\Lambda_{1,i-1}$-separated $\delta\Lambda_{1,i-1}$-relatively dense in

$$\dot{\bigcup_{(l,z)\in\overline{P}_0^i}} h_{l,z}^i(D(p,r_l)) \; .$$

Let X_i be a maximal $\delta/\Lambda_{1,i-1}$-separated subset of $D(p,r_i)$ satisfying

$$X_i \cap \dot{\bigcup_{(l,z)\in\overline{P}_0^i}} h_{l,z}^i(D(p,r_l)) = \widetilde{X}_i \; , \tag{3.2.27}$$

whose existence is guaranteed by Zorn's lemma and Claim 3. To establish (a), we still have to prove that $d(x,X_i) \le \delta\Lambda_{1,i-1}$ for every $x \in D(p,r_i)$. If

$$x \in \dot{\bigcup_{(l,z)\in\overline{P}_0^i}} h_{l,z}^i(D(p,r_l)) \; ,$$

then this inequality follows from Claim 3 and (3.2.27), so assume the opposite. Suppose by absurdity that $d(x,X_i) > \delta\Lambda_{1,i-1}$. Then $\{x\}\cup X_i$ is a $\delta\Lambda_{1,i-1}$-separated subset of $D(p,r_i)$ that still satisfies (3.2.27) and properly contains X_i, contradicting the maximality of X_i.

Let us prove (b). If $(l,z) \in \overline{P}_0^i$, then the result follows from (3.2.27). If $(l,z) \notin \overline{P}_0^i$, then Lemma 3.2.4 (b) states that there is a unique $(l',z') \in \overline{P}_0^i$ such that $(l,z) < (l',z')$. Proposition 3.2.2 (iii) yields $h_{l,z} = h_{l,z}^i = h_{l',z'}^i \circ h_{l,z''}^{l'}$, where $z'' = (h_{l',z'}^i)^{-1}(z)$. By the induction hypothesis, we have $X_{l'} \cap h_{l,z''}(D(p,r_l)) = h_{l,z''}(X_l)$, and therefore

$$h_{l',z'}(X_{l'}) \cap h_{l,z}(D(p,r_l)) = h_{l',z'}(X_{l'} \cap h_{l,z''}(D(p,r_l))) = h_{l',z'}(h_{l,z''}(X_l)) = h_{l,z}(X_l) \; .$$

Thus the result follows by showing that

$$h_{l',z'}(X_{l'}) \cap h_{l,z}(D(p,r_l)) = X_i \cap h_{l,z}(D(p,r_l)) \; . \tag{3.2.28}$$

First, note that $X_i \cap h_{l,z}(D(p,r_l)) = \widetilde{X}_i \cap h_{l,z}(D(p,r_l))$ by (3.2.27). Then, by the definition of $\widetilde{X}_i$, (3.2.28) follows if we prove that $h_{l',z'}(D(p,r_{l'})) \cap h_{j,y}(D(p,r_j)) = \emptyset$ for all $(j,y) \in \overline{P}_0^i \setminus \{(l',z')\}$. Recall that $h_{l',z'}\colon (M,p) \rightarrowtail (M,z')$ is an $(l',r_{l'},\lambda_{l'})$-p.p.q.i. and $h_{j,y}\colon (M,p) \rightarrowtail (M,y)$ a (j,r_j,λ_j)-p.p.q.i. by Claims 3.2.5 and 3.2.6; in particular, $h_{l',z'}(D(p,r_{l'})) \subset D(z',\lambda_{l'}r_{l'})$ and $h_{j,y}(D(p,r_j)) \subset D(y,\lambda_j r_j)$. But $D(z,\lambda_{l'}r_{l'}) \cap D(y,\lambda_j r_j) = \emptyset$, as follows with the following argument. If $l' = j$, Proposition 3.2.2 (ii) and (3.2.17) give

$$d(y,z) \ge s_j/\Lambda_{j,i-1} > 2\Lambda_{0,j-1}r_j + s_0/\Lambda_{0,i-1} > 2\lambda_j r_j \; .$$

In the case $l' < j$, we have $(j,y) \in P_{l'}^i$ and $z' \notin h_{j,y}^i(M_{l'}^j)$ since (l',z') is maximal. Therefore Proposition 3.2.2 (v) and (3.2.2) give

$$d(y,z) \ge \lambda_j\Lambda_i(r_j + s_{l'}) > \lambda_j r_j + \lambda_{l'}r_{l'} \; .$$

The case $m < l'$ is similar, completing the proof of (3.2.28). □

Proof of Proposition 3.2.14. For any $\delta < \eta/\Lambda_1$, let X be the union of the sets X_i given by Lemma 3.2.15. By Lemma 3.2.15 (a) and since $r_i \uparrow \infty$, this set is δ/Λ_1-separated and $\delta\Lambda_1$-relatively dense in M; in particular, it is η-relatively dense in M since $\delta\Lambda_1 < \eta$. Finally, (3.2.26) follows easily from (3.2.20), Lemma 3.2.15 (b) and Proposition 3.2.9 (vii). □

Remark 22. According to the proofs of Proposition 3.2.14 and Lemma 3.2.15, we can assume the separating constant of X to be any $\tau < \eta/\Lambda_1^2$. Therefore we can take s_1 as large as desired and Λ_1 as close to 1 as desired, and still assume that X is η-relatively dense and τ-separated. This follows because, according to (3.2.1)–(3.2.7), enlarging s_i only forces Λ_1 to be smaller.

Proposition 3.2.16. *In Proposition 3.2.14, given any $\sigma > 0$, we can assume that there is some $0 < \rho < \sigma$ such that, for all $l \in \mathbb{Z}^+$ and $x, y \in X$,*

$$\{x, y\} \subset D(p, r_l) \Rightarrow d(x, y) \notin [(\sigma - \rho)/\Lambda_l, \Lambda_l(\sigma + \rho)] \ . \qquad (3.2.29)$$

In particular, $d(x, y) \notin (\sigma - \rho, \sigma + \rho)$ for all $x, y \in X$.

Proof. Given $\eta > \eta' > 0$, take some $X' \subset M$ satisfying the statement of Proposition 3.2.14 with η'. For $i \in \mathbb{Z}^+$, let X'_i, $\widetilde{X}'_i$ and δ be like in the statement and proof of Lemma 3.2.15 with η'.

Claim 4. There are subsets X_i ($i \in \mathbb{Z}^+$) satisfying (3.2.29), and there are bijections $f_i \colon X'_i \to X_i$ such that:

(a) $d(y, f_i(y)) \le 3\Lambda_{1,i-1}\varepsilon/2$ for all $x, y \in X'_i$;

(b) X_i is $(\delta - 3\varepsilon)/\Lambda_{1,i-1}$-separated and $(\delta + 3\varepsilon/2)\Lambda_{1,i-1}$-relatively dense in $B(p, r_i)$;

(c) $X_i \subset X_l$ and $f_i = f_l|_{X'_i}$ for all $1 \le l \le i$; and,

(d) for all $(l, z) \in P_0^i$,
$$X_i \cap h_{l,z}(D(p, r_l)) = h_{l,z}(X_l) \ .$$

We proceed by induction on $i \in \mathbb{Z}^+$. First, for $\varepsilon > 0$ small enough and since $\delta < \eta'/\Lambda_1$, we have

$$3\varepsilon\Lambda_1/2 < \eta/\Lambda_1 - \delta < (\eta - \eta')/\Lambda_1 \ . \qquad (3.2.30)$$

There is also an assignment $\varepsilon \mapsto P(\varepsilon) > 0$ given by Proposition 3.1.6 such that $\sigma > P(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Choose $\rho, \rho_1 > 0$ satisfying $\rho < \rho_1 < P(\varepsilon/2)$. Once r_1 is fixed, we can choose λ_1 close enough to 1 so that

$$\rho_1 > (1 - 1/\Lambda_1)\sigma + \rho/\Lambda_1, (\Lambda_1 - 1)\sigma + \Lambda_1\rho \ . \qquad (3.2.31)$$

Let Z_1 be any ε-perturbation of X'_1 such that $Z_1 \subset B(p, r_1 - \varepsilon/2)$. Then, using Proposition 3.1.6, let X_1 be an $\varepsilon/2$-perturbation of Z_1 such that, for all $x, y \in X_1$,

$$d(x, y) \notin [\sigma - \rho_1, \sigma + \rho_1] \ . \qquad (3.2.32)$$

Let $f_1 \colon X'_1 \to X_1$ be the induced bijection, so that (a) is satisfied for $i = 1$. This can be done since we chose $\rho\Lambda_1 < \rho_1 < P(\varepsilon/2)$. Then (3.2.31) implies (3.2.29) for $x, y \in X_1$, whereas (b) follows from Proposition 3.1.6. Items (c) and (d) are vacuous for $i = 1$. Note that we also have $X_1 \subset B(p, r_1)$.

Now, given any integer $i > 1$, assume that we have sets X_j and bijections $f_j \colon X'_i \to X_i$ for $1 \le j < i$ satisfying the properties of Claim 4. Let

$$\widetilde{X}_i = \bigsqcup_{(l,z)\in\overline{P}_0^i} h_{l,z}^i(X_l) ,$$

like in the proof of Lemma 3.2.15. We get

$$d(x,y) \notin [(\sigma-\rho)/\Lambda_i, \Lambda_i(\sigma+\rho)]$$

for $(l,z) \in \overline{P}_0^i$ and $x, y \in \widetilde{h}_{l,z}^i(X_l)$ by Proposition 3.2.9 (ii). By Remark 22, we may assume that $s_0\lambda_0 > \sigma + \rho_1 > \sigma + \rho_i$. By (3.2.16), we have

$$d(z,z') \ge \Lambda_{l,i-1} r_l + \Lambda_{l',i-1} r_{l'} + s_0/\lambda_0$$

for all $(l,z), (l',z') \in \overline{P}_0^i$, $(l,z) \ne (l',z')$. So, by the triangle inequality, we have

$$d(x,y) > s_1/\Lambda_{1,i-1} > \Lambda_i(\sigma+\rho)$$

for $x \in h_{l,z}^i(X_l)$ and $y \in h_{l',z}^i(X'_l)$. This shows that (3.2.29) is satisfied for every $x, y \in \widetilde{X}_i$.

Note that

$$X'_i \cap \bigsqcup_{(l,z)\in P_0^i} h_{l,z}(D(p,r_l)) = \widetilde{X}'_i .$$

Since X'_i is finite, it follows that there is some $0 < \varepsilon_i < \varepsilon$ such that

$$\mathrm{CPen}(X'_i \setminus \widetilde{X}'_i, \varepsilon_i) \subset D(p,r_i) \setminus \bigsqcup_{(l,z)\in P_0^i} h_{l,z}(D(p,r_l)) .$$

Choose ρ_i such that $\rho < \rho_i < \rho_1 < P(\varepsilon_i/2) < P(\varepsilon/2)$. Once we have fixed r_i, we can choose λ_i close enough to 1 that

$$\rho_i > (1 - 1/\Lambda_i)\sigma + \rho/\Lambda_i, (\Lambda_i - 1)\sigma + \Lambda_i\rho . \tag{3.2.33}$$

Let $\widehat{Z}_i$ be an ε_i-perturbation of $X'_i \setminus \widetilde{X}'_i$ such that

$$\widehat{Z}_i \subset B(p, r_i - \varepsilon_i/2) , \quad \mathrm{CPen}(\widehat{Z}_i, \varepsilon_i/2) \cap \bigsqcup_{(l,z)\in P_0^i} h_{l,z}(D(p,r_l)) = \emptyset .$$

Now, by Proposition 3.1.6, there is an $\varepsilon_i/2$-perturbation X_i of $\widehat{Z}_i \cup \widetilde{X}_i$ satisfying

$$d(x,y) \notin [\sigma - \rho_l, \sigma + \rho_l]$$

for all $x, y \in X_i$ and $\widetilde{X}_i \subset X_i$. Let $\hat{h}_i \colon X'_i \setminus \widetilde{X}'_i \to \widehat{X}_i$ denote the induced bijection. Note that

$$\widehat{X}_i \subset B(p,r_i) \setminus \bigsqcup_{(l,z)\in P_0^i} h_{l,z}(D(p,r_l)) . \tag{3.2.34}$$

Now (3.2.33) implies (3.2.29) for all $x, y \in X_i$.

Let $\tilde{f}_i\colon \widetilde{X}'_i \to \widetilde{X}_i$ be given by $\tilde{h}_i(y) = h_{l,z} f_l(h_{l,z}^{-1}(y))$, where (l, z) is the only element in $\overline{P}_0^i$ such that $y \in h_{l,z}(D(p, r_l))$. By Proposition 3.2.9 (ii) and the induction hypothesis with (a), this map satisfies $d(y, \tilde{f}_i(y)) \le 3\Lambda_{1,i-1}\varepsilon/2$ for all $y \in X'_i \setminus \widetilde{X}'_i$. The combination of $\hat{f}_i$ and $\tilde{f}_i$ into a map $f_i\colon X'_i \to X_i$ is the desired bijection, and trivially satisfies both (a) and (c). Item (b) follows from (a) and Proposition 3.1.6, whereas (d) follows from the definition of $\widetilde{X}_i$ and (3.2.34), completing the proof of Claim 4.

By Claim 4 (b), the set $X = \bigcup_i X_i$ is $(\delta - 3\varepsilon)/\Lambda_1$-separated and $(\delta + 3\varepsilon/2)\Lambda_1$-relatively dense in X. Therefore it is also η-relatively dense by (3.2.30). According to Claim 4 (d), X satisfies all the requirements of Proposition 3.2.14. Moreover X satisfies (3.2.29) because every X_i does. □

3.3 Repetitive colored graphs

The results of Section 3.2 have obvious versions for (colored) connected graphs with finite vertex degrees, using (colored) graph repetitivity with respect to pointed partial quasi-isometries and graph-theoretic geodesic segments (Section 2.1.3). By Corollary 2.1.25, the proofs are essentially the same, omitting the use of m and taking $\Lambda_0 < 2$. Moreover Corollary 2.1.25 shows that $\widetilde{\Omega}_i = \Omega_i$ and $\widetilde{\Omega}_{i,j} = \Omega_{i,j}$ in this version by taking λ_0 close enough to 1. Note that the version for (colored) graphs of Proposition 3.2.14 is trivial. The versions for colored graphs of Propositions 3.2.2, 3.2.9 and 3.2.13, and other observations, are explicitly stated here because they will be used in the proof of Theorem 1.2.1. Their versions without colorings can be considered as the particular case of colorings by one color.

Let (X, ϕ) be a colored connected graph with finite vertex degrees. Fix any $p \in X$. For $i \in \mathbb{N}$ and $R > 0$, let $\Omega(i, R)$ be the set of elements $x \in X$ such that there exists a pointed color-preserving graph isomorphism $(D_X(p, R), p, \phi) \to (D_X(x, R), x, \phi)$. Suppose that (X, ϕ) is repetitive; i.e., the sets $\Omega(i, R)$ are relatively dense in X. Take sequences $0 < r_i, s_i, t_i \uparrow \infty$, and let ω_i denote the smallest positive real such that the set $\widetilde{\Omega}_i := \Omega(i, r_i)$ is ω_i-relatively dense in X. Let also $r_{-1} = s_{-1} = t_{-1} = \omega_{-1} = 0$. Suppose that r_i, s_i and t_i satisfy Eqs. (3.2.1) to (3.2.4). For $i \in \mathbb{N}$, let $X_i^i = \{p\}$ and $h_{i,p}^i = \mathrm{id}_{D(p,r_i)}$. In Proposition 3.3.1, we will continue defining a subset $X_i^j \subset X$ for every $0 \le i < j$, and a pointed color-preserving graph isomorphism $h_{i,z}^j : (D(p, r_i), p, \phi) \to (D(z, r_i), z, \phi)$ for every $z \in M_i^j$. Using this notation, let

$$P_i^j = \{\, (l, z) \in \mathbb{N} \times M \mid n < l < m,\ z \in M_l^m \,\} .$$

Note that $P_k^j \subset P_i^j$ if $i \le k < j$. Moreover, let $<$ be the binary relation on P_i^j defined by declaring $(l, z) < (l', z')$ if $l < l'$ and $z \in h_{l',z'}^j(M_l^{l'})$, and let $\le$ denote its reflexive closure, which is a partial order relation (the analogue of Lemma 3.2.3 (b)). Let $\overline{P}_i^j$ denote the set of maximal elements of $(P_i^j, \le)$, which is nonempty since all chains in P_i^j are finite. For every $(k, y) \in P_i^j$, there is a unique $(l, z) \in \overline{P}_i^j$ so that $(k, y) \le (l, z)$ (the analogue of Lemma 3.2.4 (b)).

Proposition 3.3.1. *For all integers $0 \le i < j$, there is a set $X_i^j = \widehat{X}_i^j \dot\cup \widetilde{X}_i^j \subset X$ and, for every $z \in X_i^j$, there is a pointed color-preserving graph isomorphism $h_{i,z}^j : (D(p,r_i),p,\phi) \to (D(z,r_i),z,\phi)$ satisfying the following properties:*

(i) $\widehat{X}_i^j$ is a maximal s_i-separated subset of

$$\Omega_i \cap D(p, r_j - t_i) \setminus \dot{\bigcup_{(l,z)\in \overline{P}_i^j}} D(z, r_l + s_i) .$$

(ii) X_i^j is an s_i-separated subset of $\Omega_i \cap D(p, r_j - t_i)$.

(iii) For every $(l,z) \in P_i^j$ and $x \in X_i^j \cap D(z,r_l)$, we have $h_{i,x}^j = h_{l,z}^j \circ h_{i,x'}^l$, where $x' = (h_{l,z}^j)^{-1}(x)$.

(iv) For any $(l,z) \in P_i^j$, we have $X_i^j \cap D(z,r_l) = h_{l,z}^j(X_i^l)$.

(v) For any $x \in X_i^j$ and $(l,z) \in P_i^j$, either $d(x,z) \ge r_l + s_i$, or $x \in h_{l,z}^j(X_i^l)$.

(vi) For all integers $0 \le k \le l$ such that either $l < j$ and $k \ge i$, or $l = j$ and $k > i$, we have $X_k^l \subset X_i^j$ and $h_{i,z}^j = h_{k,z}^l|_{D(p,r_i)}$ for any $z \in X_k^l$.

(vii) We have $p \in X_i^j$ and $h_{i,p}^j = \mathrm{id}_{D(p,r_i)}$.

For $i \in \mathbb{N}$, let

$$X_i = \bigcup_{j\ge i} X_i^j \ , \quad P_i = \bigcup_{j>i} P_i^j = \{ (j,x) \in \mathbb{N} \times X \mid j > i, \, x \in X_i^j \} .$$

For all $x \in X_i$, there is some $j \ge i$ such that $x \in X_i^j$. Thus let $h_{i,x} = h_{i,x}^j$, which is independent of j by Proposition 3.3.1 (vi). Hence the order relations $\le$ on the sets P_i^j ($j \ge i$) define an order relation $\le$ on P_i, which is the reflexive closure of the relation $<$ on P_i given by setting $(j,x) < (j',x')$ if $j < j'$ and $x \in h_{j',x'}(X_j^{j'})$.

Proposition 3.3.2. *The following properties hold:*

(i) X_i is an s_i-separated subset of Ω_i.

(ii) For all $x \in X_i$, $h_{i,x} : (D(p,r_i),p,\phi) \to (D(x,r_i),x,\phi)$ is a pointed color-preserving graph isomorphism.

(iii) For any $(l,z) \in P_i$, we have $X_i \cap D(z,r_l) = h_{l,z}(X_i^l)$.

(iv) For every $(j,y) \in P_i$ and $x \in X_i \cap D(y,r_j)$, we have $h_{i,x} = h_{j,y} \circ h_{i,x'}$, where $x' = h_{j,y}^{-1}(x)$.

(v) For any $x \in X_i$ and $(j,y) \in P_i$, either $d(x,z) \ge r_l + s_i$, or $x \in h_{l,z}(X_i^l)$.

(vi) For $i \le j$, we have $X_j \subset X_i$, and $h_{i,x} = h_{j,x}|_{D(p,r_i)}$ for $x \in X_j$.

(vii) We have $p \in X_i$ and $h_{i,p} = \mathrm{id}_{D(p,r_i)}$.

Remark 23. Using the same argument as in Remark 21, we can assume that Ω_i ($i \in \mathbb{N}$) is any family of relatively dense subsets of $\Omega(i, r_i)$, so that $\widehat{X}_i^j \subset \Omega_i$. If, for every $x \in \Omega_i$, we have a prescribed (i, r_i)-p.p.q.i. $f_{i,x} \colon (X, p) \rightarrowtail (X, x)$, then we may assume that $h_{i,x}^j = f_{i,x}$ for every $x \in \widehat{X}_i^j$. Finally we have that, for every $x \in X_i$, $h_{i,x}^j$ is a composition of the form $f_{i_I, x_I} \cdots f_{i_1, x_1}$ by the analogue of Lemma 3.2.6.

The following result is the analogue for colored graphs of Lemma 3.2.12.

Proposition 3.3.3. *X_i is relatively dense in X, where the implied constant depends only on r_i, s_i, t_i, ω_i.*

3.4 Realization of manifolds as leaves

3.4.1 Realization in compact foliated spaces without holonomy

Theorem 3.4.1. *For any (repetitive) connected Riemannian manifold M of bounded geometry, there is a (minimal) compact Riemannian foliated space $\mathfrak{X}$ without holonomy with a leaf isometric to M.*

To prove this theorem, the construction of $\mathfrak{X}$ begins with the following result.

Proposition 3.4.2. *Let M be a (repetitive) connected Riemannian manifold of bounded geometry. For every $\eta > 0$, there is some separated η-relatively dense subset $X \subset M$, and some coloring ϕ of X by finitely many colors such that (M, X, ϕ) is (repetitive and) limit aperiodic.*

Proof. Let $0 < \tau < \eta$. When M is not assumed to be repetitive, choose $0 < \varepsilon < \eta - \tau$ and take any $(\tau + 2\varepsilon)$-separated $(\eta - \varepsilon)$-relatively dense subset $\widehat{X} \subset M$ (Corollary 2.1.4). By Proposition 3.1.6, there are $\rho > 0$, $\sigma \geq 3\eta$ and a τ-separated η-relatively dense subset X such that

$$d_M(x, y) \notin (\sigma - \rho, \sigma + \rho) \quad \forall x, y \in X . \tag{3.4.1}$$

The set X becomes a graph by declaring that there is an edge connecting points x and y if $0 < d_M(x, y) \leq \sigma$.

Claim 5. The graph X is connected, and $X \cap D_M(x, r) \subset D_X(x, \lfloor r/\eta \rfloor + 1)$ for all $x \in X$ and $r > 0$.

Let $x, y \in X$ and $k = \lfloor d(x, y)/\eta \rfloor + 1$. Since M is connected, there is a finite sequence $x = u_0, u_1, \ldots, u_k = y$ such that $d_M(u_{i-1}, u_i) < \eta$ ($i = 1, \ldots, k$). Using that X is η-relatively dense in M, we get another finite sequence $x = z_0, z_1, \ldots, z_k = y$ in X so that $d_M(u_i, z_i) < \eta$ for all i. Then

$$d_M(z_{i-1}, z_i) \leq d_M(z_{i-1}, u_{i-1}) + d_M(u_{i-1}, u_i) + d_M(u_i, z_i) < 3\eta \leq \sigma .$$

So, either $z_{i-1} = z_i$, or there is an edge between z_{i-1} and z_i. Thus, omitting consecutive repetitions, $z_0, z_1, \ldots, z_k$ gives rise to a graph-theoretic path between x and y in X. This shows that X is a connected graph and $d_X(x, y) \leq k$, as desired.

By Proposition 3.1.5, there is some $c \in \mathbb{N}$ such that, for all $x \in M$, the disk $D_M(x, \sigma) \cap X$ has at most c points, obtaining that $\deg X \leq c$. Now Theorem 1.1.4 ensures that there exists a limit aperiodic coloring $\phi : X \to \{1, \dots, c\}$. By the definition of the graph structure of X, we also get

$$D_X(x, r) \subset D_M(x, r\sigma) \tag{3.4.2}$$

for all $x \in X$ and $r \in \mathbb{N}$.

For $n = \dim M$, take a class $[M', X', \phi'] \subset \overline{[M, X, \phi]}$ in $\widehat{\mathcal{CM}}^n_*(\{1, \dots, c\})$ (Section 3.1.4). Consider the graph structure on X' defined by declaring that there is an edge connecting points x' and y' if $0 < d_{M'}(x', y') \le \sigma$.

Claim 6. We have that:

(a) X' is τ-separated and η-relatively dense in M',

(b) X' is a connected graph and $X' \cap D_{M'}(x', r) \subset D_{X'}(x', \lfloor r/\eta \rfloor + 1)$ for all $x' \in X'$ and $r > 0$,

(c) $\deg X' \le c$, and

(d) $[X', \phi'] \subset \overline{[X, \phi]}$ in $\widehat{\mathcal{G}}_*(\{1, \dots, c\})$.

Given $x' \in X'$, $m \in \mathbb{Z}^+$, $R > \delta > 0$ and $\lambda \ge 1$, there are some $x \in X$ and an (m, R, λ)-p.p.q.i. $h : (M', x') \rightarrowtail (M, x)$ such that:

- for all $u \in D(x', R - \delta) \cap X'$, there is some $v \in h^{-1}(X) \subset D(x', R)$ with $d(u, v) < \delta$ and $\phi'(u) = \phi h(v)$; and,
- for all $v \in D(x', R - \delta) \cap h^{-1}(X)$, there is some $u \in X' \cap D(x', R)$ with $d(u, v) < \delta$ and $\phi'(u) = \phi h(v)$.

For the sake of simplicity, let $\bar{y} = h^{-1}(y)$ for every $y \in \operatorname{im} h$. Since $X \cap h(D_{M'}(x', R))$, $X' \cap D_{M'}(x', R)$ and $h(D_{M'}(x', R))$ are compact, given any $0 < \tau' < \tau$, we can assume that $\lambda - 1$ and δ are so small that

$$2\lambda\delta < \tau\ . \tag{3.4.3}$$

For any $y' \in X' \cap D_{M'}(x', R - \delta)$, there is some $y \in X \cap h(D_{M'}(x', R))$ such that $d_{M'}(y', \bar{y}) < \delta$ and $\phi'(y') = \phi(y)$. If $z \in X \cap h(D_{M'}(x', R))$ also satisfies $d_{M'}(y', \bar{z}) < \delta$, then, by (3.4.3),

$$d_M(y, z) \le \lambda d_{M'}(\bar{y}, \bar{z}) \le \lambda(d_{M'}(y', \bar{z}) + d_{M'}(y', \bar{y})) < 2\lambda\delta < \tau\ ,$$

yielding $y = z$ because X is τ-separated. So y is uniquely associated to y', and therefore the assignment $y' \mapsto y$ defines a color-preserving map

$$\tilde{h} : X' \cap D_{M'}(x', R - \delta) \to X \cap h(D_{M'}(x', R))\ ;$$

in particular, $\tilde{h}(x') = h(x') = x$. Since h is an (m, R, λ)-p.p.q.i., for all $y', z' \in X' \cap D_{M'}(x', R - \delta)$,

$$(d_{M'}(y', z') - 2\delta)/\lambda < d_M(\tilde{h}(y'), \tilde{h}(z')) < \lambda(d_{M'}(y', z') + 2\delta)\ . \tag{3.4.4}$$

Furthermore, either $d_M(\tilde{h}(y'), \tilde{h}(z')) = 0$, or $d_M(\tilde{h}(y'), \tilde{h}(z')) \ge \tau$ because X is τ-separated. So, either $d_{M'}(y', z') < 2\delta$, or $d_{M'}(y', z') > \tau/\lambda - 2\delta$ by (3.4.4). Since the choice of δ, λ

and R was arbitrary, we infer that X' is a τ-separated subset of M'. In particular, $\tilde{h}$ is injective by (3.4.3) and (3.4.4).

By taking δ and $\lambda - 1$ small enough, we can also assume that

$$\lambda(\sigma - \rho + 2\delta) < \sigma < (\sigma + \rho - 2\delta)/\lambda \,. \tag{3.4.5}$$

Given $y', z' \in X' \cap D_{M'}(x', R - \delta)$, let $y = \tilde{h}(y')$ and $z = \tilde{h}(z')$ in $X \cap h(D_{M'}(x', R))$. If $d_{M'}(y', z') < \sigma - \rho$, then, by (3.4.5),

$$d_M(y, z) \leq \lambda d_{M'}(\bar{y}, \bar{z}) < \lambda(d_{M'}(y', z') + 2\delta) < \sigma \,.$$

If $d_{M'}(y', z') \geq \sigma + \rho$, then, by (3.4.5),

$$d_M(y, z) \geq d_{M'}(\bar{y}, \bar{z})/\lambda > (d_{M'}(y', z') - 2\delta)/\lambda > \sigma \,.$$

These inequalities, (3.4.4), and the injectivity of $\tilde{h}$ show that

$$\tilde{h} : X' \cap D_{M'}(x', R - \delta) \to \tilde{h}(X' \cap D_{M'}(x', R - \delta)) \tag{3.4.6}$$

is a color-preserving graph isomorphism.

Like in (3.4.4), for all $y' \in X' \cap D_{M'}(x', R - \delta)$,

$$(d_{M'}(x', y') - \delta)/\lambda < d_M(x, \tilde{h}(y')) < \lambda(d_{M'}(x', y') + \delta) \,. \tag{3.4.7}$$

We use these inequalities to show that

$$X \cap D_M(x, (R - 2\delta)/\lambda) \subset \tilde{h}(X' \cap D_{M'}(x', R - \delta)) \subset X \cap D_M(x, \lambda R) \,. \tag{3.4.8}$$

Here, the second inclusion is a direct consequence of (3.4.4). To show the first inclusion, observe that $D_M(x, (R - 2\delta)/\lambda) \subset h(D_{M'}(x, R - 2\delta))$ because $h : (M', x') \rightarrowtail (M, x)$ is an (m, R, λ)-p.p.q.i. Thus, for any $y \in X \cap D_M(x, (R - 2\delta)/\lambda)$, we have $\bar{y} \in D_{M'}(x', R - 2\delta)$ with $h(\bar{y}) = y$. Moreover there is some $y' \in X'$ such that $d_{M'}(y', \bar{y}) \leq \delta$. Then $d_{M'}(x', y') \leq d_{M'}(x', \bar{y}) + \delta \leq R - \delta$, and $\tilde{h}(y') = y$ by the definition of $\tilde{h}$. So $y \in \tilde{h}(X' \cap D_{M'}(x', R - \delta))$, completing the proof of (3.4.8).

Now, for any $y' \in D_{M'}(x', (R - 2\delta)/(\lambda - \eta)\lambda)$, we get $h(y') \in D_M(x, (R - 2\delta)/\lambda - \eta)$ because $h : (M', x') \rightarrowtail (M, x)$ is an (m, R, λ)-p.p.q.i. Since X is η-relatively dense, there is some $y \in M$ such that $d(h(y'), y) \leq \eta$. We have $y \in D_M(x, (R - 2\delta)/\lambda)$ by the triangle inequality. Moreover $y \in \operatorname{im} \tilde{h}$ by (3.4.8). So $\tilde{h}^{-1}(y) \in X'$ and

$$d(y', \tilde{h}^{-1}(y)) < d(y', \bar{y}) + \delta \leq \lambda d(h(y'), y) + \delta \leq \lambda\eta + \delta \,.$$

Since R is arbitrarily large, and δ and $\lambda - 1$ are arbitrarily small, it follows that X' is η-relatively dense in M', completing the proof of (a).

Item (b) follows from (a) with the same argument as in Claim 5. Finally, (c) and (d) follow using (3.4.8) and the color-preserving graph isomorphisms (3.4.6). This completes the proof of Claim 6.

Claim 7. If η is small enough, then (M, X, ϕ) is limit aperiodic.

Consider any class $[M',X',\phi']\subset\overline{[M,X,\phi]}$ in $\widehat{\mathcal{C}\mathcal{M}}{}^n_*(\{1,\dots,c\})$, and let h be an isometry of M' preserving X' and ϕ'. Then h defines a color-preserving graph automorphism (X',ϕ') with the above graph structure. By Claim 6 and since (X,ϕ) is limit aperiodic, we get that $h=\mathrm{id}$ on X'. By Proposition 3.1.7, it follows that $h=\mathrm{id}$ on M' if η is small enough. So (M',X',ϕ') is aperiodic, completing the proof of Claim 7.

Now assume that M is repetitive, and take the separated η-relatively dense subset $X\subset M$ given by Proposition 3.2.14. Moreover assume that X satisfies the additional conditions of Proposition 3.2.16 for any given $\sigma\ge 3\eta$ and with some $0<\rho<\sigma$. Define a graph structure on X using σ and ρ like in the previous case. According to Proposition 3.2.14, for every $(l,z)\in P_0$, we have a pointed bijection

$$h_{l,z}:(X\cap D_M(p,r_l),p)\to(X\cap h_{l,z}(D_M(p,r_l)),z) \tag{3.4.9}$$

for every $(l,z)\in P_0$, which are pointed graph isomorphisms by (3.2.29) in Proposition 3.2.16. As before, the graph X is connected, there is some $c\in\mathbb{N}$ such that $\deg X\le c$, there is a repetitive limit aperiodic coloring $\phi:X\to\{1,\dots,c\}$, and (M,X,ϕ) is limit aperiodic if η is small enough.

Let us prove that we can assume that (M,X,ϕ) is repetitive in this case. To construct ϕ and prove its limit aperiodicity and repetitivity, the argument of Theorem 1.1.4 uses the versions without colorings of Propositions 3.3.1 to 3.3.3. Given other sequences $0<r'_i,s'_i,t'_i\uparrow\infty$ satisfying Eqs. (3.2.1) to (3.2.4), we can also suppose in Section 3.2 that $r_i\ge\Lambda\sigma r'_i$, yielding $D_X(x,r'_i)\subset X\cap D_M(x,r_i)$ for all $x\in X$ and $i\in\mathbb{N}$. So, according to Remark 2, the versions without colorings of Propositions 3.3.1 and 3.3.2 hold with the maps

$$h_{l,z}:(D_X(p,r'_l),p)\to(D_X(z,r'_l)),z)\;. \tag{3.4.10}$$

induced by the pointed graph isomorphisms (3.4.9). Then the proof of Theorem 1.1.4 describes the repetitivity of the colored graph (X,ϕ) using the pointed graph isomorphisms (3.4.10). By Claim 5, any sequence $0<r''_l\to\infty$ with $\lfloor r'_l\rfloor\ge\lfloor r''_l/\eta\rfloor+1$ if $r'_l\ge 1$ satisfies $X\cap D_M(p,r''_l)\subset D_X(p,r'_l)$. Thus the (l,r''_l,Λ_l)-p.p.q.i. $(M,p)\rightarrowtail(M,z)$ defined by $h_{l,z}$ can be used to describe the repetitivity of (M,X,ϕ) □

As explained in Section 3.1.3, Theorem 3.4.1 holds with the Riemannian foliated subspace $\mathfrak{X}=\overline{[M,f]}\subset\widehat{\mathcal{M}}{}^n_{*,\mathrm{imm}}$ ($n=\dim M$), where $f\in C^\infty(M,\mathfrak{H})$ is given by the following result.

Proposition 3.4.3 (Cf. [6, Proposition 7.1])**.** *Let M be a (repetitive) connected Riemannian manifold. There is some (repetitive) limit aperiodic $f\in C^\infty(M,\mathfrak{H})$, where $\mathfrak{H}$ is a finite-dimensional Hilbert space, so that* $\sup_M|\nabla^m f|<\infty$ *for all $m\in\mathbb{N}$ and* $\inf_M|\nabla f|>0$.

Proof. Take $r_0>0$ and normal parametrizations $\kappa_x:B_{r_0}\to B_M(x,r_0)$ ($x\in M$) like in Proposition 3.1.4. For any $0<r<r_0$, take X, c and ϕ like in Proposition 3.4.2 with $\eta=2r/3$. Write $X=\{\,x_i\mid i\in I\,\}$ for some index set I, and let $\kappa_i=\kappa_{x_i}:B_r\to B_M(x_i,r)$ and $\phi_i=\phi(x_i)$ ($i\in I$). Consider the graph structure on X defined in the proof of Proposition 3.4.2, using $\sigma=3\eta=2r$. Since $\deg X\le c$, there is a coloring $\alpha:X\to\{1,\dots,c+1\}$ such that adjacent vertices have different colors. Let $X_k=\alpha^{-1}(k)$ and $I_k=\{\,i\in I\mid x_i\in X_k\,\}$ ($k=1,\dots,c+1$).

For $n = \dim M$, let S be an isometric copy in $\mathbb{R}^{n+1}$ of the standard n-dimensional sphere so that $0 \in S$. Choose some function $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $\rho(x)$ depends only on $|x|$, $0 \le \rho \le 1$, $\rho(x) = 1$ if $|x| \le r/2$, and $\rho(x) = 0$ if $|x| \ge r$. Take also some C^∞ map $\tau : \mathbb{R}^n \to \mathbb{R}^{n+1}$ that restricts to a diffeomorphism $B_r \to S \setminus \{0\}$ and maps $\mathbb{R}^n \setminus B_r$ to 0. Let $V = \tau(B_{r/2}) \subset S$ and $y_0 = \tau(0) \in V$. Let $\rho_i = \rho \circ \kappa_i^{-1}$ and $\tau_i = \tau \circ \kappa_i^{-1}$. For $k = 1, \dots, c+1$, let $f^k = (f_1^k, f_2^k) : M \to \mathbb{R}^{n+2} = \mathbb{R} \times \mathbb{R}^{n+1}$ be the extension by zero of the combination of the compactly supported functions $(\rho_i \cdot \phi_i, \rho_i \cdot \tau_i)$ on the disjoint balls $B_M(x_i, r)$, for $i \in I_k$. Let $f = (f^1, \dots, f^{c+1}) : M \to (\mathbb{R}^{n+2})^{c+1} \equiv \mathbb{R}^{(c+1)(n+2)} =: \mathfrak{H}$. Note that $\sup_M |\nabla^m f| < \infty$ for all $m \in \mathbb{N}$ and $\inf_M |\nabla f| > 0$. We can write $f = (f_1, f_2) : M \to \mathfrak{H} \equiv \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $f_1 = (f_1^1, \dots, f_1^{c+1}) : M \to \mathbb{R}^{c+1} =: \mathfrak{H}_1$ and $f_2 = (f_2^1, \dots, f_2^{c+1}) : M \to (\mathbb{R}^{n+1})^{c+1} \equiv \mathbb{R}^{(c+1)(n+1)} =: \mathfrak{H}_2$.

Claim 8. If r is small enough, then f is limit aperiodic.

Take any class $[M', f'] \in \overline{[M, f]}$. Then $[M'] \in \overline{[M]}$, obtaining that $\mathrm{inj}_{M'} \ge \mathrm{inj}_M > r_0$ and M' satisfies the property stated in Proposition 3.1.4. We can consider $f' = (f'^1, \dots, f'^{c+1}) : M' \to (\mathbb{R}^{n+2})^{c+1} \equiv \mathbb{R}^{(c+1)(n+2)} = \mathfrak{H}$ with $f'^k = (f_1'^k, f_2'^k) : M' \to \mathbb{R} \times \mathbb{R}^{n+1} \equiv \mathbb{R}^{n+2}$. Given $x' \in M'$, there are sequences, $0 < R_p \uparrow \infty$, $0 < \eta_p \downarrow 0$, $m_p \uparrow \infty$ in $\mathbb{N}$, of smooth compact domains $D_p \subset M'$ with $B_{M'}(x', R_p) \subset D_p \subset B_{M'}(x', R_{p+1})$, and of C^∞ embeddings $h_p : D_p \to M$, such that

$$q \ge p \Longrightarrow \|h_q^* g_M - g_{M'}\|_{C^{m_q}, D_p, g_{M'}}, \|h_q^* f - f'\|_{C^{m_q}, D_p, g_{M'}} < \eta_q \ .$$

Let $X'_k = (f_2'^k)^{-1}(y_0) \subset M'$ and $X' = X'_1 \cup \cdots \cup X'_{c+1}$. Write $X' = \{\, x'_a \mid a \in A \,\}$ for some index set A, and let $A_k = \{\, a \in A \mid x'_a \in X'_k \,\}$. For any $a \in A_k$, we have $D_{M'}(x'_a, r) \subset D_p$ for p large enough. Let $\bar{x}_{a,q} = h_q(x'_a)$ for $q \ge p$. Then $f_2^k(\bar{x}_{a,q}) \to f_2'^k(x'_a) = y_0$ as $q \to \infty$. By the definition of f_2^k, it follows that there is a sequence $i_{a,q} \in I_k$ such that $d_M(x_{i_{a,q}}, \bar{x}_{a,q}) \to 0$. Given $0 < \theta < r/2$, we get $h_q(D_{M'}(x'_a, \theta)) \subset B_M(x_{i_{a,q}}, r/2)$ for $q \ge p$ large enough, and $\kappa_{i_{a,q}}^{-1} h_q = \tau^{-1} f_2^k h_q \to \tau^{-1} f_2'^k$ with respect to the C^∞ topology on $D_{M'}(x'_a, \theta)$. Thus there is some normal parametrization $\kappa'_a : B_r \to B_{M'}(x'_a, r)$ such that $\tau^{-1} f_2'^k = \kappa_a'^{-1}$ on $D_{M'}(x'_a, \theta)$. Since θ is arbitrary, we get $f_2'^k = \tau \kappa_a'^{-1}$ on $B_{M'}(x'_a, r/2)$; in particular, $f_2'^k : B_{M'}(x'_a, r/2) \to V$ is a diffeomorphism.

Now, using the properties of X and the convergence $d_M(x_{i_{a,q}}, \bar{x}_{a,q}) \to 0$, it easily follows that X' is also separated and η-relatively dense in M', and, for all $x' \in M'$, the ball $B_{M'}(x', \sigma) \cap X'$ has at most c points. Hence, like in the case of X, the set X' becomes a connected graph with $\deg X' \le c$ by attaching an edge between x'_a and x'_b $(a, b \in A)$ if $0 < d_{M'}(x'_a, x'_b) < \sigma$. Let $\widetilde{D}_p$ denote the set of points x'_a in X' such that $D_{M'}(x'_a, r) \subset D_p$. From the convergence $d_M(x_{i_{a,q}}, \bar{x}_{a,q}) \to 0$, we also get that, if p and q are large enough with $q \ge p$, then, for all $a, b \in A$ with $x'_a, x'_b \in \widetilde{D}_p$, there is an edge in X between $x_{i_{a,q}}$ and $x_{i_{b,q}}$ if and only if there is an edge in X' between x'_a and x'_b. Thus an injection $\tilde{h}_{p,q} : \widetilde{D}_p \to X$ is defined by $\tilde{h}_{p,q}(x'_a) = x_{i_{a,q}}$, and $\tilde{h}_{p,q} : \widetilde{D}_p \to \tilde{h}_{p,q}(\widetilde{D}_p)$ is a graph isomorphism. Moreover, for any $N \in \mathbb{Z}^+$ and $a \in A$, we have $D_{X'}(x'_a, N) \subset \widetilde{D}_p$ if $D_{M'}(x'_a, 2Nr) \subset D_p$, which holds for p large enough. Then there is a pointed isomorphism $(B_{X'}(x'_a, N), x'_a) \to (B_X(x_{i_{a,q}}, N), x_{i_{a,q}})$ if p and q are large enough with $q \ge p$, yielding $[X', x'_a] \in \overline{[X]}$, and therefore $\overline{[X']} \subset \overline{[X]}$. Furthermore, $f_1^k(\bar{x}_{a,q}) = f_1^k(x_{i_{a,q}}) = \phi_{i_{a,q}} = (\widetilde{F}^*_{p,q} \phi)(x'_a)$ if $d_M(x_{i_{a,q}}, \bar{x}_{a,q}) < r/2$ and $i_{a,q} \in I_k$, and $f_1^k(\bar{x}_{a,q}) = (h_q^* f_1^k)(x'_a) \to f_1'^k(x'_a)$ as $q \to \infty$. So a coloring $\phi' : X' \to \{1, \dots, c\}$ is defined by taking $\phi' = f_1'^k$ on every

X'_k, and we have $\tilde{h}_{p,q}\phi = \phi'$ on $D_{X'}(x'_a, N)$. Hence $[X', x'_a, \phi'] \in \overline{[X, \phi]}$, and therefore $\overline{[X', \phi']} \subset \overline{[X, \phi]}$. Moreover (X', ϕ') is aperiodic because (X, ϕ) is limit aperiodic.

Let us prove that (M', f') is aperiodic. Let h be an isometry of M' such that $h^* f' = f'$. Then $h^* f'^k_j = f'^k_j$ for all $k = 1, \dots, c+1$ and $j = 1, 2$. So $h(X') = X'$ and $h : X' \to X'$ is a graph isomorphism preserving ϕ'. Since (X', ϕ') is aperiodic, it follows that h is the identity on X'. So $h = \mathrm{id}$ on M' if r is small enough by Proposition 3.1.7. This completes the proof of Claim 8.

When M is repetitive, the repetitivity of f is a direct consequence of the repetitivity of (M, X, ϕ). □

3.4.2 Replacing compact foliated spaces with matchbox manifolds

Theorem 3.4.4. *For any (minimal) transitive compact C^∞ foliated space $\mathfrak{X}$ without holonomy, there is a C^∞ (minimal) matchbox manifold $\mathfrak{M}$ without holonomy, and there is a C^∞ surjective foliated map $\pi : \mathfrak{M} \to \mathfrak{X}$ that restricts to diffeomorphisms between the leaves of $\mathfrak{M}$ and $\mathfrak{X}$.*

Proof. Fix any dense leaf M of $\mathfrak{X}$, an auxiliary Riemannian metric on $\mathfrak{X}$, and a C^∞ embedding $h : \mathfrak{X} \to \mathfrak{H}_1$ into some separable Hilbert space. Let $f_1 = h|_M$ and $\mathfrak{M}_1 = \overline{[M, f_1]}$ in $\widehat{\mathcal{M}}^n_*(\mathfrak{H}_1)$ ($n = \dim M$). Then (M, f_1) is limit aperiodic, $\mathfrak{M}_1$ is compact, and we have an induced isometric diffeomorphism between Riemannian foliated spaces $\hat{\iota}_{\mathfrak{X},h} : \mathfrak{X} \to \mathfrak{M}_1$ (Example 3.1.17).

There are regular foliated atlases $\mathfrak{U} = \{U_i, \phi_i\}$ and $\widetilde{\mathfrak{U}} = \{\widetilde{U}_i, \tilde{\phi}_i\}$ of $\mathfrak{X}$ ($i = 1, \dots, c$), with foliated charts $\phi_i : U_i \to B_i \times \mathfrak{T}_i$ and $\tilde{\phi}_i : \widetilde{U}_i \to \widetilde{B}_i \times \widetilde{\mathfrak{T}}_i$, such that $\overline{U_i} \subset \widetilde{U}_i$ and $\phi_i = \tilde{\phi}_i|_{U_i}$. Thus $\overline{B}_i \subset \widetilde{B}_i$ in $\mathbb{R}^n$ ($n = \dim \mathfrak{X}$), and every $\mathfrak{T}_i$ is a relatively compact subspace of $\widetilde{\mathfrak{T}}_i$. Moreover the projections $\tilde{p}_i = \mathrm{pr}_2 \tilde{\phi}_i : \widetilde{U}_i \to \widetilde{\mathfrak{T}}_i$ extend the projections $p_i = \mathrm{pr}_2 \phi_i : U_i \to \mathfrak{T}_i$, and the elementary holonomy transformations $\tilde{h}_{ij} : \tilde{p}_i(\widetilde{U}_i \cap \widetilde{U}_j) \to \tilde{p}_j(\widetilde{U}_i \cap \widetilde{U}_j)$ defined by $\widetilde{\mathfrak{U}}$ extend the elementary holonomy transformations $h_{ij} : p_i(U_i \cap U_j) \to \tilde{p}_j(U_i \cap U_j)$ defined by $\mathfrak{U}$. Let $\mathfrak{I}$ denote the set of all finite sequences of indices in $\{1, \dots, c\}$. For every $I = (i_0, i_1, \dots, i_k) \in \mathfrak{I}$, let $\tilde{h}_I = \tilde{h}_{i_{k-1}i_k} \cdots \tilde{h}_{i_1 i_0}$ and $h_I = h_{i_{k-1}i_k} \cdots h_{i_1 i_0}$, which may be empty maps. There are points $y_i \in B_i$ such that the local transversals $\tilde{\phi}_i^{-1}(\{y_i\} \times \widetilde{\mathfrak{T}}_i) \equiv \widetilde{\mathfrak{T}}_i$ have disjoint closures in $\mathfrak{X}$, and therefore we can realize $\widetilde{\mathfrak{T}} := \bigsqcup_i \widetilde{\mathfrak{T}}_i$ as a complete transversal in $\mathfrak{X}$ (Section 3.1.2). Hence $\phi_i^{-1}(\{y_i\} \times \mathfrak{T}_i) \equiv \mathfrak{T}_i$ and $\mathfrak{T} := \bigsqcup_i \mathfrak{T}_i$ also have these properties.

Since $\mathfrak{X}$ is Polish and compact, it is locally compact and second countable, and therefore $\widetilde{\mathfrak{T}}$ is also locally compact and second countable. Then there is a countable base of relatively compact open subsets V_k ($k \in \mathbb{N}$) of $\widetilde{\mathfrak{T}}$. Fix any relatively compact open subset $\mathfrak{S}_i$ of every $\widetilde{\mathfrak{T}}_i$ containing $\overline{\mathfrak{T}_i}$, and let $\mathfrak{S} = \bigsqcup_i \mathfrak{S}_i$. Given a metric on $\widetilde{\mathfrak{T}}$ inducing its topology, we can suppose that there is a sequence $0 = k_0 < k_1 < \cdots$ in $\mathbb{N}$ such that the sets $V_{k_m}, \dots, V_{k_{m+1}-1}$ cover $\overline{\mathfrak{S}}$ and have diameter $< 1/(m+1)$ for all $m \in \mathbb{N}$. Using $K = \{0,1\}^{\mathbb{N}}$ as a model of the Cantor space, let $\psi : \widetilde{\mathfrak{T}} \to K$ be defined by

$$\psi(x)(k) = \begin{cases} 0 & \text{if } x \notin V_k \\ 1 & \text{if } x \in V_k \,. \end{cases}$$

Since $\mathfrak{I}$ is countable, $K^{\mathfrak{I}}$ is homeomorphic to K. Let $\Psi : \widetilde{\mathfrak{T}} \to K^{\mathfrak{I}}$ be the map defined by

$$\Psi(x)(I) = \begin{cases} \psi\tilde{h}_I(x) & \text{if } x \in \operatorname{dom}\tilde{h}_I \\ 0 & \text{if } x \notin \operatorname{dom}\tilde{h}_I\ , \end{cases}$$

where $0 \equiv (0, 0, \dots) \in K$. Observe that $\Psi(x)$ determines $\Psi\tilde{h}_I(x)$ for all $x \in \widetilde{\mathfrak{T}}$ and $I \in \mathfrak{I}$ with $x \in \operatorname{dom}\tilde{h}_I$.

Claim 9. For any sequence x_a in $\mathfrak{S}$, if $\psi(x_a)$ is convergent in K, then x_a is convergent in $\widetilde{\mathfrak{T}}$, and $\lim_a x_a$ depends only on $\lim_a \psi(x_a)$.

The convergence of $\psi(x_a)$ in K means that, for every $m \in \mathbb{N}$, there is some $a_m \in \mathbb{N}$ such that $\psi(x_a)(k) = \psi(x_b)(k)$ for all $k < k_{m+1}$ and $a, b \geq a_m$. Since the sets $V_{k_m}, \dots, V_{k_{m+1}-1}$ cover $\mathfrak{S}$, it follows that there is a sequence $l_m \in \mathbb{N}$ such that $k_m \leq l_m < k_{m+1}$ and $x_a \in V_{l_m}$ for all $a \geq a_m$. Thus the limit set $\bigcap_k \overline{\{\, x_a \mid a \geq a_m \,\}}$ is a nonempty subset of $\bigcap_m \overline{V_{l_m}}$, which consists of a unique point of $\mathfrak{S}$ because every $\overline{V_{l_m}}$ is compact with diameter $< 1/(m+1)$. Thus x_a is convergent in $\widetilde{\mathfrak{T}}$.

Now let y_a be another sequence in $\mathfrak{S}$ such that $\psi(y_a)$ is convergent in K and $\lim_a \psi(y_a) = \lim_a \psi(x_a)$. We have already proved that y_a is convergent in $\widetilde{\mathfrak{T}}$. Moreover, taking a_m large enough in the above argument, we also get $\psi(y_a)(k) = \psi(x_a)(k)$ for all $k < k_{m+1}$ and $a \geq a_m$. This yields $y_a \in V_{l_m}$ for all $a \geq a_m$, and therefore $\lim_a y_a = \lim_a x_a$. This completes the proof of Claim 9.

According to Claim 9, a continuous map $\varpi : \overline{\psi(\mathfrak{S})} \to \overline{\mathfrak{S}}$ is defined by $\varpi(\xi) = x$ if $\{x\} = \bigcap_{k \in \xi^{-1}(1)} \overline{V_k}$, and we have $\varpi\psi = \operatorname{id}$ on $\mathfrak{S}$. Let $X_i = \mathfrak{T}_i \cap M$ and $X = \bigcup_i X_i = \mathfrak{T} \cap M$, which is a Delone set in M (see e.g. [10, Proposition 10.5]).

For every i, let $\lambda_i : \mathfrak{X} \to [0,1]$ be a C^∞ function with $\lambda_i = 1$ on $\mathfrak{T}_i$ and $\lambda_i = 0$ on $\widetilde{\mathfrak{T}} \setminus \overline{\mathfrak{S}_i}$. Fix an embedding $\sigma : K^{\mathfrak{I}} \to \mathbb{R}$, and let $f_2 = (f_2^1, \dots, f_2^c) : M \to \mathbb{R}^c =: \mathfrak{H}_2$, where $f_2^i(x) = \lambda_i(x) \cdot \sigma\Psi\tilde{p}_i(x)$. We have $\sup_M |\nabla^m f_2| = \max_i \sup_{\mathfrak{X}} |\nabla^m \lambda_i| < \infty$ for all $m \in \mathbb{N}$. So $\mathfrak{M}_2 := \overline{[M, f_2]}$ is compact by Corollary 3.1.13.

Consider the C^∞ function $f = (f_1, f_2) : M \to \mathfrak{H} := \mathfrak{H}_1 \oplus \mathfrak{H}_2$, and $\mathfrak{M} = \overline{[M, f]}$ in $\widehat{\mathcal{M}}^n_*(\mathfrak{H})$. Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are compact, we get that $\mathfrak{M}$ is also compact by Corollary 3.1.14. We have $\inf_M |\nabla f| \geq \inf_M |\nabla f_1| = \inf_{\mathfrak{X}} |\nabla\tilde{h}| > 0$, and therefore $\mathfrak{M} \subset \widehat{\mathcal{M}}^n_{*,\mathrm{imm}}(\mathfrak{H})$ by Proposition 3.1.15 (ii). The function (M, f) is limit aperiodic because (M, f_1) is limit aperiodic, and therefore $\mathfrak{M}$ has no holonomy (Section 3.1.3).

For $a = 1, 2$, let $\Pi_a : \mathfrak{H} \to \mathfrak{H}_a$ denote the corresponding factor projection. Then $\Pi_{1*} : \mathfrak{M} \to \mathfrak{M}_1$ is a surjective C^∞ foliated map restricting to isometries between the leaves, and therefore $\pi := (\hat{\iota}_{\mathfrak{X},h_1})^{-1} \circ \Pi_{1*} : \mathfrak{M} \to \mathfrak{X}$ is also a surjective C^∞ foliated map restricting to isometries between the leaves. Thus every leaf of $\mathfrak{M}$ is of the form $[M', f']$, where M' is a leaf of $\mathfrak{X}$ and $f' = (f_1', f_2') : M' \to \mathfrak{H}$, where $f_1' = h|_{M'}$ and $[M', f_2'] \subset \mathfrak{M}_2$.

Let $p_i' : U_i' := \pi^{-1}(U_i) \to \mathfrak{T}_i' := \pi^{-1}(\mathfrak{T}_i)$ be defined by $p_i'([M', x', f']) = [M', p_i(x'), f']$, for leaves M' of $\mathfrak{X}$, and let $\phi_i' = (\operatorname{pr}_1 \phi_i\pi, p_i') : U_i' \to B_i \times \mathfrak{T}_i'$, where $\operatorname{pr}_1 : B_i \times \mathfrak{T}_i \to B_i$ is the first factor projection. Using the description of the C^∞ foliated structure of $\widehat{\mathcal{M}}^n_{*,\mathrm{imm}}(\mathfrak{H})$ given in [6, Section 5], it is easy to check that $\{U_i', \phi_i'\}$ is a C^∞ foliated atlas of $\mathfrak{M}$. Thus $\mathfrak{T}' = \bigcup_i \mathfrak{T}_i' \equiv \bigsqcup_i \mathfrak{T}_i'$ is a complete transversal of $\mathfrak{M}$.

Claim 10. The map $\operatorname{ev} : \overline{\mathfrak{T}'} \to \mathfrak{H}$ is an embedding whose image if $\overline{f(X)}$.

Since $\operatorname{ev} : \overline{\mathfrak{T}'} \to \mathfrak{H}$ is a continuous map defined on a compact space, and $\{\, [M, x, f] \mid x \in X \,\}$ is dense in $\overline{\mathfrak{T}'}$, it is enough to prove that $\operatorname{ev} : \overline{\mathfrak{T}'} \to \mathfrak{H}$ is injective. Let

$[M', x', f'], [M'', x'', f''] \in \overline{\mathfrak{T}'}$ with $f'(x') = f''(x'')$. We can assume that M' and M'' are leaves of $\mathfrak{X}$, $x' \in M' \cap \overline{\mathfrak{T}}$, $x'' \in M'' \cap \overline{\mathfrak{T}}$, $f' = (f'_1, f'_2)$ with $f'_1 = h|_{M'}$, and $f'' = (f''_1, f''_2)$ with $f''_1 = h|_{M''}$. Then $h(x') = h(x'')$, yielding $x' = x''$ and $M' = M''$. On the other hand, there are sequences x'_m and x''_m in $M \cap \mathfrak{T}$ converging to x' in $\overline{\mathfrak{T}}$ such that (M, x'_m, f_2) and (M, x''_m, f_2) are C^∞-convergent to (M', x', f'_2) and (M', x', f''_2), respectively. If $x' \in \overline{\mathfrak{T}_i}$, we can assume that $x'_m, x''_m \in M \cap \mathfrak{T}_i$ for all m. Writing $f'_2 = (f'^1_2, \dots, f'^c_2)$ and $f''_2 = (f''^1_2, \dots, f''^c_2)$, we get

$$\lim_m \sigma\Psi(x'_m) = f'^i(x') = f''^i(x') = \lim_m \sigma\Psi(x''_m) \; .$$

So $\lim_m \Psi(x'_m) = \lim_m \Psi(x''_m)$, yielding $\lim_m \Psi h_I(x'_m) = \lim_m \Psi h_I(x''_m)$ for all $I \in \mathfrak{I}$. Since $h_I(x'_m)$ and $h_I(x''_m)$ converge to $\tilde{h}_I(x')$ in $\overline{\mathfrak{T}}$, using the Reeb's local stability theorem and the definition of f_2, it follows that both (M, x'_k, f_2) and (M, x''_k, f_2) are C^∞-convergent to the same triple with first components (M', x'). Therefore $f'_2 = f''_2$, yielding $[M', x', f'] = [M'', x'', f'']$, as desired.

According to Claim 10, $\overline{\mathfrak{T}'}$ is homeomorphic to the subspace

$$\overline{f(X)} = \overline{\{\, (f_1(x), f_2(x)) \mid x \in X \,\}} \subset f_1(\overline{\mathfrak{T}}) \times (\sigma(K^{\mathfrak{I}}))^c \; .$$

By the conditions on the functions λ_i, this subspace is homeomorphic to the subspace

$$\begin{aligned} \bigsqcup_i \overline{\{\, (x, \Psi(x)) \mid x \in X_i \,\}} &= \bigsqcup_i \overline{\{\, (\varpi(\xi), \xi) \mid \xi \in \Psi(X_i) \,\}} \\ &= \bigsqcup_i \{\, (\varpi(\xi), \xi) \mid \xi \in \overline{\Psi(X_i)} \,\} \subset \bigsqcup_i \overline{\mathfrak{T}_i} \times K^{\mathfrak{I}} \equiv \overline{\mathfrak{T}} \times K^{\mathfrak{I}} \; , \end{aligned}$$

which in turn is homeomorphic to the subspace $\bigcup_i \overline{\Psi(X_i)} \subset K^{\mathfrak{I}}$ because ϖ is continuous. So $\overline{\mathfrak{T}'}$ and $\mathfrak{T}'$ are zero-dimensional, obtaining that $\mathfrak{M}$ is a matchbox manifold.

Now suppose that $\mathfrak{X}$ is minimal. Then (M, f_1) is repetitive (Example 3.1.17). A simple refinement of the proof of Proposition 3.1.16 also shows that (M, f_2) is repetitive. In both cases, this property can be described with the same partial pointed quasi-isometries given by the Reeb's local stability theorem. So (M, f) is also repetitive, and therefore $\mathfrak{M}$ is minimal by Proposition 3.1.10 (i). □

As explained in Section 1.2.4, Theorem 1.2.1 is a direct consequence of Theorems 3.4.1 and 3.4.4.

3.4.3 Attaching flat bundles to foliated spaces

Let $\mathfrak{X} \equiv (\mathfrak{X}, \mathcal{F})$ be a compact C^∞ foliated space of dimension n, and let M be a leaf of $\mathfrak{X}$. On the other hand, let $\rho : E \to M$ be a locally compact flat bundle with typical fiber F and horizontal foliated structure $\mathcal{H}$. It can be described as the suspension of its holonomy homomorphism $h : \pi_1 M \to \mathrm{Homeo}(F)$, whose image is its holonomy group G; they are well defined up to conjugation in $\mathrm{Homeo}(F)$. Any foliated concept of E refers to $\mathcal{H}$. The C^∞ differentiable structure of M induces a C^∞ differentiable structure of $\mathcal{H}$. Assume that F is a non-compact locally compact Polish space; then E also has these properties. The notation $E_x = \rho^{-1}(x)$ and $E_X = \rho^{-1}(X)$ will be used for $x \in M$ and $X \subset M$.

The one-point compactifications $E_x^+ = \{x\} \sqcup E_x$ of the fibers E_x ($x \in M$) are the fibers of another C^∞ flat bundle $\rho^+ : E^+ \to M$; thus $E^+ \equiv M \sqcup E$ as sets. Its typical fiber is the one-point compactification $F^+ = \{\infty\} \cup F$ of F, the leaves of its horizontal foliation $\mathcal{H}^+$ are M and the leaves of $\mathcal{H}$, its holonomy homomorphism $h^+ : \pi_1 M \to \operatorname{Homeo}(F^+)$ is induced by h, and its holonomy group is denoted by G^+. The more specific notation $h_x^+ : \pi_1(M, x) \to \operatorname{Homeo}(F^+)$ and G_x^+ will be used to indicate the base point $x \in M$.

Let $\mathfrak{X}' = \mathfrak{X} \sqcup E$, equipped with the following topology. Take any foliated chart $U \equiv B \times \mathfrak{T}$ of $\mathfrak{X}$, for some ball $B \subset \mathbb{R}^n$ and some local transversal $\mathfrak{T}$. We have $M \cap U \equiv B \times D$ for some countable subset $D \subset \mathfrak{T}$. Since the plaques of U are contractible, ρ has a local trivialization $E_{M \cap U} \equiv (M \cap U) \times F$ of flat bundle. Let $\mathfrak{T}' = \mathfrak{T} \sqcup (D \times F)$, endowed with the topology with basic open sets of the form

$$\mathfrak{V} = \emptyset \sqcup \Big(\bigcup_z (\{z\} \times R_z) \Big) \equiv \bigcup_z (\{z\} \times R_z) \;, \quad \mathfrak{W} = \mathfrak{T} \sqcup \Big(\bigcup_z (\{z\} \times S_z) \Big) \;,$$

where z runs in D, R_z and S_z are open in F, $\overline{R_z}$ is compact for all z, $R_z = \emptyset$ for all but finitely many z, $F \setminus S_z$ is compact for all z, and $S_z = F$ for all but finitely many z. Then $\mathfrak{X}$ has a topology with basic open sets of the form

$$V \equiv \emptyset \sqcup \Big(B \times \bigcup_z (\{z\} \times R_z) \Big) \equiv B \times \mathfrak{V} \;, \quad W \equiv U \sqcup \Big(B \times \bigcup_z (\{z\} \times S_z) \Big) \equiv B \times \mathfrak{W} \;,$$

for all possible foliated charts $U \equiv B \times \mathfrak{T}$ of $\mathfrak{X}$. Using these basic open sets, it is easy to check that $\mathfrak{X}'$ is Hausdorff, second countable and compact. So $\mathfrak{X}'$ is metrizable [45, Proposition 4.6], hence Polish. In particular, the sets

$$U' = U \sqcup E_{M \cap U} \equiv (B \times \mathfrak{T}) \sqcup (B \times D \times F) = B \times \mathfrak{T}'$$

are open in $\mathfrak{X}'$, and the fibers $B \times \{*\}$ correspond to open subsets of leaves of $\mathcal{F}$ or $\mathcal{H}$. Thus these identities are foliated charts of a foliated structure $\mathcal{F}'$ on $\mathfrak{X}'$, and its leaves are the leaves of $\mathcal{F}$ and $\mathcal{H}$. As sets, we can write $\mathfrak{X}' \equiv \mathfrak{X} \cup_{\mathrm{id}_M} E^+$ and $\mathfrak{T}' \equiv \mathfrak{T} \cup_{\mathrm{id}_D} (D \times F^+)$, where we consider $D \equiv D \times \{\infty\} \subset D \times F^+$; we can also write $\mathfrak{T}' = \mathfrak{T} \sqcup E_D \equiv \mathfrak{T} \cup_{\mathrm{id}_D} E_D^+$.

Consider a regular foliated atlas of $\mathfrak{X}$ consisting of charts $U_i \equiv B_i \times \mathfrak{T}_i$, for balls $B_i \subset \mathbb{R}^n$ and local transversal $\mathfrak{T}_i$. As before, take local trivializations $E_{M \cap U_i} \equiv (M \cap U_i) \times F$ of the flat bundle ρ, write $M \cap U_i \equiv B_i \times D_i$ for countable subsets $D_i \subset \mathfrak{T}_i$, and consider the induced foliated charts $U_i' \equiv B_i \times \mathfrak{T}_i'$ of $\mathcal{F}'$, where $U_i' = U_i \sqcup E_{M \cap U_i}$ and $\mathfrak{T}_i' = \mathfrak{T}_i \sqcup (D_i \times F)$, endowed with Polish topologies. The changes of coordinates of the foliated charts $U_i \equiv B_i \times \mathfrak{T}_i$ are of the form $(y, z) \mapsto (f_{ij}(y, z), h_{ij}(z))$, where every mapping $y \mapsto f_{ij}(y, z)$ is C^∞ with all of its partial derivatives of arbitrary order depending continuously on z. On the other hand, the changes of the local trivializations $E_{M \cap U_i} \equiv (M \cap U_i) \times F \equiv B_i \times D_i \times F$ are of the form $(y, z, u) \mapsto (y, g_{ij}(z, u))$, where the maps g_{ij} are independent of y by the compatibility with $\mathcal{H}$. Then the changes of coordinates of the foliated charts $U_i' \equiv B_i \times \mathfrak{T}_i'$ are of the form

$$(y, z') \mapsto \begin{cases} (f_{ij}(x, z'), h_{ij}(z')) \in B_j \times \mathfrak{T}_j & \text{if } z' \in \mathfrak{T}_i \\ (f_{ij}(x, z), (h_{ij}(z), g_{ij}(z, u))) \in B_j \times (D_j \times F) & \text{if } z' = (z, u) \in D_i \times F \;. \end{cases}$$

Thus the charts $U_i' \equiv B_i \times \mathfrak{T}_i'$ define a C^∞ structure on $\mathfrak{X}' \equiv (\mathfrak{X}', \mathcal{F}')$. The corresponding elementary holonomy transformations h_{ij}' are combinations of maps h_{ij} and g_{ij}. Using these foliated charts, it also follows that $\mathfrak{X}$ and E are embedded C^∞ foliated subspaces of $\mathfrak{X}'$, E^+ is an injectively immersed C^∞ foliated subspace of $\mathfrak{X}'$, and the combination $\pi : \mathfrak{X}' \to \mathfrak{X}$ of $\mathrm{id}_{\mathfrak{X}}$ and ρ (or ρ^+) is a C^∞ foliated retraction. The fibers of π are

$$\pi^{-1}(x) = \begin{cases} \{x\} \sqcup \emptyset \equiv \{x\} & \text{if } x \in \mathfrak{X} \setminus M \\ \{x\} \sqcup E_x = E_x^+ & \text{if } x \in M\ . \end{cases}$$

Lemma 3.4.5. *Suppose that the restrictions of ρ to the leaves of $\mathcal{H}$ are regular coverings of the leaves of $\mathcal{F}$, and that the leaf M of $\mathcal{F}$ has no holonomy. Then the holonomy group of the leaf M of $\mathcal{F}'$ is isomorphic to the group of germs at ∞ of the elements of the subgroup $G^+ \subset \mathrm{Homeo}(F^+)$.*

Proof. With the above notation, fix an index i_0 and some point $x_0 \in D_{i_0} \equiv \mathfrak{T}_{i_0} \cap M \equiv \mathfrak{T}'_{i_0} \cap M$, considering $\mathfrak{T}_{i_0} \subset \mathfrak{X}$ and $\mathfrak{T}'_{i_0} \subset \mathfrak{X}'$. Let $c : [0,1] \to M$ be a loop based at x_0. Since the holonomy group of M in $\mathfrak{X}$ is trivial, there is a family of leafwise loops $c_x : [0,1] \to \mathfrak{X}$, depending continuously on x in some open neighborhood $\mathfrak{T}_0$ of x_0 in $\mathfrak{T}_{i_0}$, such that $c_{x_0} = c$. Let $D_0 = D_{i_0} \cap \mathfrak{T}_0$. From the above description of the elementary holonomy transformations h'_{ij}, it follows that the holonomy defined by $[c] \in \pi_1(M, x_0)$ is the germ at $x_0 \equiv (x_0, \infty)$ of the homeomorphism g_c of $\mathfrak{T}'_{i_0} = \mathfrak{T}_{i_0} \sqcup (D_{i_0} \times F)$ given by

$$g_c(z') = \begin{cases} z' & \text{if } z' \in \mathfrak{T}_0 \\ (x, h_x([c_x])(u)) & \text{if } z' = (x,u) \in D_0 \times F\ , \end{cases}$$

using $[c_x] \in \pi_1(M, x)$. Since the restrictions of ρ to the leaves of $\mathcal{H}$ are regular coverings of M, we easily get that $h_x^+([c_x])(u) = u$ for some $x \in D_0$ and $u \in F^+$ close enough to ∞ if and only if $h_{x_0}^+([c])(u) = u$ for $u \in F^+$ close enough to ∞. So, by restricting every g_c to $\{x_0\} \times F^+ \equiv F^+$, we get an isomorphism from the holonomy group of the leaf M of $\mathcal{F}'$ at x_0 to the group of germs of the elements of $G_{x_0}^+$ at ∞. □

Proofs of Corollaries 1.2.2 and 1.2.3. Let M be non-compact connected Riemannian manifold of bounded geometry. By Theorem 1.2.1, M is isometric to a leaf in some Riemannian matchbox manifold $\mathfrak{M}$ without holonomy. Now Corollaries 1.2.2 and 1.2.3 follow by considering the foliated space $\mathfrak{M}'$ constructed as above with $\mathfrak{M}$ and an appropriate flat bundle E over M, and lifting the Riemannian metric of $\mathfrak{M}$ to $\mathfrak{M}'$.

In the case of Corollary 1.2.2, we can use the trivial flat bundle $E = M \times K$ over M, where K is the Cantor space. By the density of M in $\mathfrak{M}$, it follows that $\mathfrak{M}'$ has a compact zero-dimensional complete transversal $\mathfrak{T}'$ without isolated points, and therefore $\mathfrak{T}'$ is homeomorphic to the Cantor space.

In the case of Corollary 1.2.3, let Γ denote the group of deck transformations of the given regular covering $\widetilde{M}$ of M, equipped with the discrete topology. If Γ is infinite, we can take $E = \widetilde{M}$, whose typical fiber is $F = \Gamma$. If Γ is finite, we can take $E = \widetilde{M} \times \mathbb{Z}$, whose typical fiber is $F = \Gamma \times \mathbb{Z}$. In any case, F is non-compact, and the action of Γ on itself by left translations induces a canonical action of Γ on F, which in turn induces an action on F^+. By Lemma 3.4.5 and the regularity of the covering $\widetilde{M}$ of M, the holonomy group of M in $\mathfrak{M}'$ is isomorphic to the group of germs at ∞ of the action of the elements of Γ on F^+, which is itself isomorphic to Γ. □

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