Ixchel Dzohara Gutiérrez Rodríguez

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DEPARTAMENTO DE MATEMATICAS

Bach-flat manifolds and conformally Einstein structures

by

IXCHEL DZOHARA GUTIÉRREZ RODRÍGUEZ

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Introduction

A central problem in pseudo-Riemannian geometry is the existence of “optimal metrics”, meaning those whose curvature has the property of being most evenly distributed about the manifold. The approach to determine such metrics usually focuses on finding critical metrics for some natural curvature functionals.

Let M be compact and τ_g denote the scalar curvature of a pseudo-Riemannian metric g on M. The simplest and most natural curvature functional defined on the space of metrics is the one given by the integral of the scalar curvature: $\mathcal{S} : g \mapsto \mathcal{S}(g) = \int_M \tau_g \, dvol_g$, where $dvol_g$ is the volume element determined by the metric g. A metric g is $\mathcal{S}$-critical if its Einstein tensor $\rho_g - \frac{1}{2}\tau_g g$ vanishes, where ρ_g denotes the Ricci tensor of (M, g). Since the curvature functional $\mathcal{S}$ is sensitive to scalings of the metric, one restricts its action to metrics within constant volume. The corresponding critical metrics are the Einstein ones. Hence one could argue that *Einstein metrics*, i.e., those whose Ricci tensor is proportional to the metric, are the most natural optimal metrics on a pseudo-Riemannian manifold.

Einstein metrics are somehow meaningless in dimension two. The Gauss-Bonnet Theorem shows that $\mathcal{S}(g) = 4\pi\chi[M]$, where $\chi[M]$ denotes the Euler characteristic of M, and thus all metrics are $\mathcal{S}$-critical in dimension two. The three-dimensional case is very rigid and Einstein metrics are just those of constant sectional curvature. Hence they are locally isometric to a pseudo-sphere, to a pseudo-Euclidean space or to a pseudo-hyperbolic space. The first non-trivial situation occurs in dimension four, where non-trivial examples exist. The classification of four-dimensional Einstein metrics is a widely open problem and a central question is the existence of such metrics.

There are several strategies to construct Einstein metrics. A classical one consists on deforming a given metric by a conformal factor so that the metric becomes Einstein after a suitable conformal rescaling. In this case (M, g) is said to be *conformally Einstein*, i.e., if there is an Einstein representative of the conformal class $[g]$. A second more recent strategy makes use of the Ricci flow which under suitable conditions converges to an Einstein metric. There are however metrics which remain invariant (up to scaling and diffeomorphisms) by the Ricci flow: the *Ricci solitons*.

Brinkmann showed in [14] that an n-dimensional manifold (M, g) is conformally Einstein if and only if the equation

$$(n-2)\operatorname{Hes}_\varphi + \varphi\,\rho - \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\,\tau\}g = 0 \qquad (1)$$

has a positive solution. Even though in dimension 2 the equation is trivial, in higher dimensions the integration is surprisingly difficult and the equation above is overdetermined in most cases.

Furthermore the conformally Einstein metric, if exists, it is unique up to homotheties in the Riemannian setting [14, 106]. An important issue is, therefore, to characterize conformally Einstein spaces by some more manageable tensorial equations.

Let (M, g) be conformally Einstein and assume $\overline{g} = e^{2\sigma} g$ to be Einstein. Since Einstein metrics have harmonic Weyl tensor one trivially has $\overline{\operatorname{div}}\, \overline{W} = 0$, where W denotes the Weyl conformal curvature tensor of (M, g). The fact that the Weyl tensor rescales under conformal transformations gives that

$$(\operatorname{div}_4 W)(X, Y, Z) + W(X, Y, Z, \nabla\sigma) = 0$$

is a necessary condition for (M, g) to be conformally Einstein. A second necessary condition is obtained as follows. Let $\mathcal{W} : g \mapsto \mathcal{W}(g) = \int_M \|W\|^2 \, dvol_g$ be the curvature functional determined by the L^2-norm of the Weyl conformal curvature tensor. $\mathcal{W}$-critical metrics where characterized by Bach in [6], showing that a four-dimensional metric is $\mathcal{W}$-critical if and only if the Bach tensor $\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \frac{1}{2} W[\rho]$ vanishes identically. Clearly any Einstein metric is *Bach-flat* ($\mathfrak{B} = 0$). Moreover, a specific feature of dimension four is that $\mathcal{W}$ is conformally invariant and thus conformally Einstein metrics are Bach-flat in dimension four.

Kozameh, Newman and Tod showed in [72] that the two necessary conditions:

$$(i) \quad \mathfrak{B} = 0, \qquad (ii) \quad (\operatorname{div}_4 W)(X, Y, Z) + W(X, Y, Z, \nabla\sigma) = 0, \tag{2}$$

are also sufficient to be conformally Einstein if (M, g) is *weakly-generic*, i.e., the Weyl tensor viewed as a map $TM \to \bigotimes^3 TM$ is injective. In the Kähler case the situation is simpler, since any Bach-flat Riemannian Kähler metric is conformally Einstein [48]. Despite all these results, the classification of conformally Einstein manifolds is an open question nowadays, with only partial results available. See for example [75] for a recent classification of conformally Einstein product manifolds.

Our purpose on Part I of this thesis is to address the classification of four-dimensional conformally Einstein metrics in the homogeneous case. The homogeneity assumption allows a simplification of the conformally Einstein equation, reducing Equation (1) to a system of algebraic equations by using the conditions in Equation (2). Four-dimensional homogeneous Einstein metrics were described by Jensen [70], who showed that they are symmetric in the Riemannian case. Hence they are locally a real or complex space form or locally a product of two surfaces of constant equal Gauss curvature. The conformally Einstein situation is richer and Chapter 2 is devoted to prove the following classification result.

Theorem 2.1. *Let (M, g) be a four-dimensional complete and simply connected conformally Einstein homogeneous Riemannian manifold. Then (M, g) is locally symmetric or otherwise it is homothetic to one of the Lie groups determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \tfrac{1}{4} e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \tfrac{1}{4} e_3 \,.$$

(ii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$[e_1, e_2] = e_3, \quad [e_4, e_1] = e_1 - \alpha e_2, \quad [e_4, e_2] = \alpha e_1 + e_2, \quad [e_4, e_3] = 2e_3 \,.$$

(iii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ *given by*

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha+1)^2\, e_2, \quad [e_4, e_3] = \alpha^2\, e_3, \qquad \alpha > 1\,.$$

Here $\{e_1, \dots, e_4\}$ *is an orthonormal basis. Moreover, the Lie groups* $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle)$ *in Assertion (ii) are half conformally flat.*

In addition to four-dimensional conformally Einstein metrics, the Hirzebruch signature formula shows that *self-dual* and *anti-self-dual* metrics are also Bach-flat. As a consequence of the analysis in Chapter 2 we obtain a classification of homogeneous metrics which are strictly Bach-flat, i.e., those which are neither half conformally flat nor conformally Einstein, as follows:

Theorem 2.4. *Let* (M, g) *be a four-dimensional complete and simply connected strictly Bach-flat homogeneous Riemannian manifold. Then* (M, g) *is homothetic to one of the Lie groups determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{e}(1,1)$ *given by*

$$[e_2, e_3] = e_1, \qquad [e_1, e_3] = (2+\sqrt{3})\, e_2,$$
$$[e_4, e_1] = \sqrt{6+3\sqrt{3}}\, e_1, \qquad [e_4, e_2] = \sqrt{6+3\sqrt{3}}\, e_2\,.$$

(ii) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ *given by*

$$[e_1, e_2] = e_3, \qquad [e_4, e_1] = \tfrac{1}{4}\sqrt{7-3\sqrt{5}}\, e_1,$$
$$[e_2, e_4] = \tfrac{1}{4}\sqrt{7+3\sqrt{5}}\, e_2, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}}\, e_3\,.$$

Here $\{e_1, \dots, e_4\}$ *is an orthonormal basis.*

It is worth emphasizing that the two examples in Theorem 2.4 were previously constructed by Abbena, Garbiero and Salamon [1].

A crucial step in the proof of Theorem 2.1 and Theorem 2.4 is the description of four-dimensional homogeneous Riemannian manifolds by Bérard-Bergery [9]: they are either symmetric or a Lie group with a left-invariant Riemannian metric. An analogous statement clearly fails in the Lorentzian and neutral signature cases, since pseudo-Riemannian homogeneous spaces are not necessarily reductive.

Non-reductive four-dimensional homogeneous spaces were classified by Fels and Renner [54], and we explicitly use their classification to determine all non-reductive conformally Einstein metrics in Chapter 3 as follows:

Theorem 3.1. *Let* (M, g) *be a conformally Einstein four-dimensional non-reductive homogeneous space. Then* (M, g) *is Einstein, locally conformally flat, or locally isometric to:*

(i) $(\mathbb{R}^4, g)$ *with metric given by*

$$\begin{aligned} g = {} & (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ & - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ & + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4, \end{aligned}$$

where a*,* b *and* c *are arbitrary constants with* $ab \neq 0$*.*

(ii) $(\mathbb{R}^4, g)$ *with metric given by*

$$\begin{aligned} g = &(4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ &- (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ &+ b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 - \tfrac{3a}{4}\, dx^3 \circ dx^3, \end{aligned}$$

where a, b and c are arbitrary constants with $ab \neq 0$.

(iii) $(\mathbb{R}^4, g)$ *with metric given by*

$$\begin{aligned} g = &-2ae^{2x^4}\, dx^1 \circ dx^3 + ae^{2x^4} dx^2 \circ dx^2 \\ &+ b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where a, b, c and q are arbitrary constants with $abq \neq 0$.

(iv) $(\mathfrak{U} \subset \mathbb{R}^4, g_+)$ *with metric given by*

$$\begin{aligned} g_+ = &\, 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 \\ &+ b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where $\mathfrak{U} = \{(x^1, \dots, x^4) \in \mathbb{R}^4 \,/\, \cos(x^4) \neq 0\}$, and a, b, c and q are arbitrary constants with $ab \neq 0$ and $b \neq -q$, or

$(\mathbb{R}^4, g_-)$ *with metric given by*

$$\begin{aligned} g_- = &\, 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cosh(x^4)^2 dx^2 \circ dx^2 \\ &+ b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where a, b, c and q are arbitrary constants with $ab \neq 0$ and $b \neq q$.

Moreover, all the cases (i)–(iv) are in the conformal class of a Ricci-flat metric which is unique (up to an homothety) only in Case (i). Otherwise the space of conformally Ricci-flat metrics is either two or three-dimensional.

A second more recent approach to the construction of Einstein metrics is given by the *Ricci flow*, i.e., a one-parameter family of metrics $g(t)$ on a manifold M which satisfies the equation $\frac{\partial}{\partial t} g(t) = -2\rho_{g(t)}$. The Ricci flow is well-posed in the Riemannian context in the sense that for any closed manifold M and any initial metric $g(0)$, there is a unique solution $g(t)$ for sufficiently small t. Hamilton [64] showed that the Ricci flow converges to an Einstein metric under suitable conditions thus showing the existence of Einstein metrics. It is an important observation that, if the initial metric $g(0)$ is Einstein, then it remains invariant under the flow (up to homothetical scaling). Furthermore a solution of the flow is said to be *self-similar* if it remains invariant up to scalings and diffeomorphisms. Such solutions –usually referred to as Ricci solitons– are characterized by the existence of a vector field X on M so that

$$\mathcal{L}_X g + \rho = \lambda g, \tag{3}$$

where $\mathcal{L}$ denotes the Lie derivative and λ is a real constant. Ricci solitons are therefore generalizations of Einstein metrics and their classification is an important issue in understanding the Ricci flow. If X is a gradient, then Equation (3) becomes

$$\mathrm{Hes}_f + \rho = \lambda g, \tag{4}$$

for some potential function f and (M, g, f) is called a *gradient Ricci soliton*. The geometry of the Ricci tensor strongly depends on the sign of the Ricci curvatures. While positive Ricci curvature is a strong condition with topological consequences, Lohkamp [80] showed that any manifold admits complete metrics with negative Ricci curvature. Correspondingly, the study of Ricci solitons depends on the sign of the soliton constant λ; a Ricci soliton (M, g, X) is called *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

While there exist several classification results for gradient Ricci solitons, the generic case (3) is still pretty unknown. Even in the homogeneous case a complete classification is not yet available in dimension four. Since all Bach-flat left-invariant Riemannian metrics are realized on solvable Lie groups (cf. Theorem 2.1 and Theorem 2.4) one has the following description of homogeneous Bach-flat Ricci solitons.

Theorem 2.16. *Let (M, g) be a four-dimensional complete and simply connected Bach-flat Riemannian homogeneous Ricci soliton. Then (M, g) is Einstein, a locally conformally flat gradient Ricci soliton $N^3(c) \times \mathbb{R}$, where $N^3(c)$ is a space form, or homothetic to one of the algebraic Ricci solitons determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \tfrac{1}{4}e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \tfrac{1}{4}e_3 \,.$$

(ii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha+1)^2 \, e_2, \quad [e_4, e_3] = \alpha^2 \, e_3, \qquad \alpha > 1 \,.$$

(iii) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$[e_1, e_2] = e_3, \qquad [e_4, e_1] = \tfrac{1}{4}\sqrt{7 - 3\sqrt{5}}\, e_1,$$
$$[e_2, e_4] = \tfrac{1}{4}\sqrt{7 + 3\sqrt{5}}\, e_2, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}}\, e_3 \,.$$

The gradient Ricci soliton Equation (4) encodes geometric information of the manifold in terms of the Ricci curvature and the second fundamental form of the level sets of the potential function f. Since the Ricci tensor completely determines the curvature in the locally conformally flat case, substantial progress have been made towards a classification of gradient Ricci solitons under some assumptions on the Weyl curvature. Locally conformally flat gradient Ricci solitons are locally warped products with one-dimensional base in the Riemannian case [55] and a complete description is available in the complete shrinking and steady cases [35, 94]. The Lorentzian situation allows another family of examples whose underlying structure is that of a plane wave [17].

Weaker assumptions on the Weyl conformal tensor have been investigated, half conformal flatness being an important example. While (anti-)self-dual gradient Ricci solitons are locally conformally flat in the Riemannian setting [39], the neutral signature case allows non-trivial examples [16] given by Riemannian extensions of affine gradient Ricci solitons. Generalizing the half conformally flat situation, Bach-flat gradient Ricci solitons have been investigated in [34]. Complete Bach-flat shrinking gradient Ricci solitons, as well as steady gradient Ricci solitons of positive Ricci curvature whose scalar curvature attains a maximum at some interior point, are locally conformally flat in the Riemannian category.

Our purpose in Part II of this thesis is to show the existence of strictly Bach-flat gradient Ricci solitons in the neutral signature case. This question is motivated by the existence of self-dual gradient Ricci solitons which are not locally conformally flat [16]. The desired metrics are constructed by a perturbation of the classical Riemannian extensions introduced by Patterson and Walker [92]. Let (Σ, D) be an affine surface and let T and Φ be a parallel $(1,1)$-tensor field and an arbitrary symmetric $(0,2)$-tensor field on Σ, respectively. The data (Σ, D, T, Φ) determines a neutral signature metric on the cotangent bundle $T^*\Sigma$ given by

$$g_{D,\Phi,T} = \iota T \circ \iota T + g_D + \pi^*\Phi, \tag{5}$$

where ι denotes the evaluation map on the cotangent bundle, $\pi : T^*\Sigma \to \Sigma$ is the canonical projection and g_D is the *Patterson-Walker's Riemannian extension*.

In Chapter 4, we show that the metrics in Equation (5) provide a large family of strictly Bach-flat manifolds. Indeed:

Theorem 4.1. *Let (Σ, D, T) be a torsion free affine surface equipped with a parallel $(1,1)$-tensor field T. Let Φ be an arbitrary symmetric $(0,2)$-tensor field on Σ. Then the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$ vanishes if and only if T is either a multiple of the identity or nilpotent.*

If T is a multiple of the identity, then the metrics $g_{D,\Phi,T}$ are self-dual and thus we are specially interested in the nilpotent case ($T^2 = 0$, $T \neq 0$). Moreover, since the deformation tensor field Φ does not play any role in Theorem 4.1 it may be used to construct an infinite family of non-isometric Bach-flat metrics for any given data (D, T) on Σ. A suitable choice of Φ enables the construction of the desired new examples of Bach-flat gradient steady Ricci solitons, where as a matter of notation, $\widehat{\Phi}(X,Y) = \Phi(TX, TY)$ in Equation (6).

Theorem 4.6. *Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1,1)$-tensor field T and let Φ be a symmetric $(0,2)$-tensor field on Σ. Let $h \in \mathcal{C}^\infty(\Sigma)$ be a smooth function. Then $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ is a Bach-flat gradient Ricci soliton if and only if $dh(\ker(T)) = 0$ and*

$$\widehat{\Phi} = -\operatorname{Hes}_h^D - 2\rho_s^D. \tag{6}$$

Moreover the soliton is steady and isotropic.

We emphasize that the corresponding potential function has degenerate level set hypersurfaces and their underlying structure is never locally conformally flat, in sharp contrast with the Riemannian situation. The pseudo-Riemannian metrics in Theorem 4.1 are never self-dual, but they can be anti-self-dual in some cases. This fact allows the construction of anti-self-dual gradient Ricci solitons which are not locally conformally flat, just requiring that both T and Φ are parallel.

Theorem 4.12. *Let (Σ, D, T, Φ) be an affine surface with symmetric Ricci tensor equipped with a parallel nilpotent $(1,1)$-tensor field T and a parallel symmetric $(0,2)$-tensor field Φ.*

(i) The (Σ, D, h) is an affine gradient Ricci soliton with $dh(\ker(T)) = 0$ if and only if $(T^*\Sigma, g_{D,\widehat{\Phi},T}, f = h \circ \pi)$ is an anti-self-dual steady gradient Ricci soliton which is not locally conformally flat.

(ii) The (Σ, D, h) is an affine gradient Ricci soliton with $dh(\ker(T)) = 0$ if and only if there exist local coordinates (u^1, u^2) on Σ so that the only non-zero Christoffel symbol is given by ${}^u\Gamma_{11}{}^2 = P(u^1) + u^2 Q(u^1)$ and the potential function $h(u^1)$ is determined by $h''(u^1) = -2Q(u^1)$, for any $P, Q \in \mathcal{C}^\infty(\Sigma)$.

The constructions in Chapter 4 require the existence of affine surfaces admitting a parallel nilpotent tensor field, which is a rather restrictive condition. We therefore investigate in Chapter 5 the existence of parallel $(1,1)$-tensor fields on affine surfaces. One says that a tensor field T is a *Kähler* (resp. *para-Kähler*) structure if T is parallel and $T^2 = -\operatorname{Id}$ (resp. $T^2 = \operatorname{Id}$). T is *nilpotent Kähler* if $T^2 = 0$ and $DT = 0$. Since the trace of any parallel tensor is constant, one may express $T = \frac{1}{2}\operatorname{tr}(T)\operatorname{Id} + (T - \frac{1}{2}\operatorname{tr}(T)\operatorname{Id})$ so that it decomposes into a scalar multiple of the identity and a trace free tensor field.

If (Σ, D) is an affine surface with skew-symmetric Ricci tensor $\rho_{sk}^D \neq 0$, then ρ_{sk}^D defines a volume element. Moreover, ρ_{sk}^D is said to be recurrent, i.e., $D\rho_{sk}^D = \omega \otimes \rho_{sk}^D$ for some one-form ω. Parallel trace free $(1,1)$-tensor fields can be rescaled to be either Kähler, para-Kähler or nilpotent Kähler with a recurrent condition as follows:

Theorem 5.1. *Let (Σ, D) be a simply connected affine surface with $\rho_s^D \neq 0$.*

(i) (Σ, D) admits a Kähler structure if and only if $\det(\rho_s^D) > 0$ and ρ_s^D is recurrent.

(ii) (Σ, D) admits a para-Kähler structure if and only if $\det(\rho_s^D) < 0$ and ρ_s^D is recurrent.

(iii) (Σ, D) admits a nilpotent Kähler structure if and only if ρ_s^D is of rank one and recurrent.

Surfaces with skew-symmetric Ricci tensor (equivalently, $\rho_s^D = 0$) admit Kähler, para-Kähler and nilpotent Kähler structures simultaneously (see Lemma 5.6). We use homogeneous affine surfaces to illustrate Theorem 5.1, showing that all the different possibilities are realizable. The results in Section 5.3 give explicit expressions of all parallel nilpotent Kähler structures on homogeneous surfaces.

Finally in Chapter 6, we consider some generalizations of Theorem 4.1 to construct Riemannian extensions with non-parallel tensor field T which are Bach-flat. Theorem 6.1 extends the construction in Theorem 4.1, showing that the modified Riemannian extension $(T^*\Sigma, g_{D,\Phi,T})$ determined by a non-parallel nilpotent tensor field T remains Bach-flat under some conditions on the affine connection. The underlying question relies on determining the conditions on the connection once the nilpotent endomorphism is given. Conversely, one may consider the reverse problem of constructing nilpotent endomorphisms on Σ such that the modified Riemannian extension (5) is Bach-flat once the connection D is given. We use the Cauchy-Kovalevski Theorem to show that any Patterson-Walker Riemannian extension may be locally deformed by a suitable nilpotent endomorphism field to be Bach-flat in the real analytic category.

Theorem 6.7. *Let (Σ, D) be a real analytic affine surface. Then there exist locally defined nilpotent $(1,1)$-tensor fields T such that the modified Riemannian extension $(T^*\Sigma, g_{D,\Phi,T})$ is Bach-flat.*

It is a remarkable fact that modified Riemannian extensions (5) have vanishing scalar curvature invariants if and only if T is nilpotent (cf. Theorem 6.8). Hence we introduce some new invariants in Section 6.3 which are not of Weyl type. These invariants, which strongly depend on the Ricci curvature of (Σ, D), allow one to distinguish some isometry classes of Bach-flat metrics.

Chapter 1
Preliminaries

Throughout this chapter we will introduce some concepts and notation that will be necessary in the development of this thesis. We shall omit most of the proofs and instead provide references for more details.

1.1 Pseudo-Riemannian manifolds

A *pseudo-Riemannian* manifold (M, g) is a smooth manifold M of dimension n equipped with a metric tensor, i.e., with a symmetric and non-degenerate $(0, 2)$-tensor field. A non-zero vector $v \in T_pM$ is called *timelike* if $g(v, v) < 0$, *spacelike* if $g(v, v) > 0$ or *null* if $g(v, v) = 0$. We denote by $S_p^-(M)$, $S_p^+(M)$, $S_p^0(M)$ the set of timelike unit vectors, spacelike unit vectors and null vectors, respectively, at a point $p \in M$.

Recall that the signature of the metric g is the pair $(n - \nu, \nu)$ such that $n - \nu$ is the number of negative eigenvalues and ν is the number of positive eigenvalues in the associated matrix. For example, an n-dimensional pseudo-Riemannian manifold (M, g) is Riemannian if the signature is $(0, n)$ and Lorentzian if the signature is $(1, n - 1)$. Moreover, if n is even and the signature of g is $(\frac{n}{2}, \frac{n}{2})$ then the manifold has neutral signature. We denote by TM and T^*M the tangent and the cotangent fiber bundles of the corresponding manifold. Let $\mathfrak{X}(M)$ be the space of tangent vector fields to M. We represent vector fields by $X, Y, Z, \dots$ and tangent vectors at a given point by $x, y, z, \dots$.

For any pseudo-Riemannian manifold (M, g) there exists a unique adapted linear connection ∇ which is torsion free and parallel, i.e.,

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \quad \text{and} \quad \nabla g = 0\,.$$

Such connection is called the *Levi-Civita connection*. The Koszul formula gives the following expression of the Levi-Civita connection:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $[\cdot, \cdot]$ represents the Lie bracket. The connection can be characterized by means of the *Christoffel symbols*. Let $(x^1, \dots, x^n)$ be local coordinates. We define the Christoffel symbols of the *first kind* by

$$\Gamma_{ij\ell} = \frac{1}{2}\left(\frac{\partial g_{\ell j}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\right)$$

and the Christoffel symbols of the *second kind* by

$$\Gamma_{ij}{}^{k} = g^{k\ell}\Gamma_{ij\ell},$$

where $(g^{\alpha\beta})$ denote the inverse matrix of $(g_{\alpha\beta})$. Therefore, we obtain

$$\nabla_{\partial_{x^i}}\partial_{x^j} = \Gamma_{ij}{}^{k}\partial_{x^k},$$

where we use the notation $\partial_{x^i} := \frac{\partial}{\partial x^i}$ to represent the locally defined coordinate vector fields.

1.1.1 Differentiable operators

Let (M, g) be a pseudo-Riemannian manifold and let $f\colon M \to \mathbb{R}$ be a differentiable function. We define the *gradient operator* $\nabla\colon C^\infty(M) \to \mathfrak{X}(M)$ on M as follows:

$$g(\nabla f, X) = X(f), \quad \text{for all } X \in \mathfrak{X}(M)\,.$$

In a system of local coordinates $(x^1, \dots, x^n)$, the gradient of the function f is given by:

$$\nabla f = \sum_{i,k=1}^{n} g^{ik}\frac{\partial f}{\partial x^k}\partial_{x^i}.$$

The *Hessian operator* of f is defined by the endomorphism $\mathrm{h}_f\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by the second covariant derivative

$$\mathrm{h}_f(X) = \nabla_X\nabla f\,.$$

Now, we can define a new symmetric tensor field of type $(0, 2)$, the *Hessian tensor* Hes_f, given by

$$\mathrm{Hes}_f(X, Y) = g(\mathrm{h}_f(X), Y) = g(\nabla_X\nabla f, Y) = XYf - (\nabla_X Y)f\,.$$

In terms of a local coordinate system the Hessian tensor is given by:

$$\begin{aligned}\mathrm{Hes}_f(\partial_{x^i}, \partial_{x^j}) &= \frac{\partial^2 f}{\partial x^i\partial x^j} + \frac{1}{2}g^{k\ell}\left(\frac{\partial g_{ij}}{\partial x^\ell} - \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{\ell i}}{\partial x^j}\right)\frac{\partial f}{\partial x^k} \\ &= \frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma_{ij}{}^{k}\frac{\partial f}{\partial x^k}\,.\end{aligned}$$

We define the *divergence* of a vector field X by the function $\operatorname{div} X = \operatorname{tr}(\nabla X)$. Considering an orthonormal frame $\{E_1, \dots, E_n\}$ we have

$$\operatorname{div} X = \sum \varepsilon_i g(\nabla_{E_i}X, E_i),$$

where $\varepsilon_i = g(E_i, E_i)$. In general, if T is a tensor field of type $(0, s)$, we define the divergence on the r-th argument as the $(0, s-1)$-tensor field given by

$$(\operatorname{div}_r T)(X_1, \dots, X_{s-1}) = \sum_{i=1}^{n} \varepsilon_i(\nabla_{E_i}T)(X_1, \dots, X_{r-1}, E_i, X_r, \dots, X_{s-1}),$$

for all $X_1, \dots, X_{s-1} \in \mathfrak{X}(M)$. Since r-divergence of T is given by the r-th trace of ∇T the definition above does not depend on the choice of the local frame.

1.1.2 The curvature tensor

The Levi-Civita connection having been defined, we introduce the *curvature operator*, denoted by R, or curvature tensor of type (1,3) by setting

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z\,.$$

In local coordinates $(x^1, \dots, x^n)$ the components of the curvature tensor are given by

$$R(\partial_{x^i}, \partial_{x^j})\partial_{x^k} = R_{ijk}{}^{\ell}\partial_{x^\ell}.$$

The *curvature tensor* of type (0,4) is given by

$$R(X,Y,Z,V) = g(R(X,Y)Z,V)\,.$$

Hence its components are given by $R_{ijk\ell} = g_{\ell r}R_{ijk}{}^r$.

The curvature tensor has the following algebraic symmetries:

$$\begin{aligned} &a)\quad R(X,Y,Z,V) = -R(Y,X,Z,V) = -R(X,Y,V,Z),\\ &b)\quad R(X,Y,Z,V) + R(Y,Z,X,V) + R(Z,X,Y,V) = 0,\\ &c)\quad R(X,Y,Z,V) = R(Z,V,X,Y),\end{aligned} \tag{1.1}$$

and the differential identity

$$d)\quad (\nabla_X R)(Y,Z,U,V) + (\nabla_Y R)(Z,X,U,V) + (\nabla_Z R)(X,Y,U,V) = 0\,.$$

Identities $b)$ and $d)$ are known as the *first Bianchi identity* and the *second Bianchi identity*, respectively. A tensor of type $(0,4)$, $A\colon V \times V \times V \times V \to \mathbb{R}$, on a vector space V is called an *algebraic curvature tensor* if it satisfies the identities (1.1).

The *sectional curvature* of a given Riemannian manifold (M,g) is the real function κ defined on the Grassmannian of 2-planes by

$$\kappa(\pi) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where $\pi = \langle\{X,Y\}\rangle$ is a two-dimensional subspace of T_pM.

In the pseudo-Riemannian setting one must consider the restriction to the Grassmannian of non-degenerate planes, i.e., those where

$$g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0\,.$$

If $\kappa(\pi)$ is independent of $\pi \subset T_pM$, then the curvature tensor is given by

$$R(X,Y,Z,V) = \kappa R^0(X,Y,Z,V),$$

where R^0 is the *standard algebraic curvature tensor* given by

$$R^0(X,Y,Z,V) = g(X,Z)g(Y,V) - g(X,V)g(Y,Z)\,.$$

If M is connected and $\dim(M) \geq 3$, then the second Bianchi identity guarantees that κ is necessarily a global constant if it is pointwise constant. A pseudo-Riemannian manifold of constant sectional curvature is locally isometric to a pseudo-sphere $\mathbb{S}^n_\nu$, to a pseudo-Euclidean space $\mathbb{E}^n_\nu$ or to a pseudo-hyperbolic space $\mathbb{H}^n_\nu$ depending on the sign $\kappa > 0$, $\kappa = 0$ or $\kappa < 0$ and the signature ν (see [88]).

We denote by ρ the *Ricci tensor* defined by

$$\rho(X,Y) = \mathrm{tr}(Z \mapsto R(X,Z)Y)$$

and the *Ricci operator*, Ric, which is the associated $(1,1)$-tensor field defined by $g(\mathrm{Ric}(X),Y) = \rho(X,Y)$. The curvature identities (1.1) show that the Ricci tensor is symmetric, or equivalently, the Ricci operator is self-adjoint. Moreover the *scalar curvature* τ is given by

$$\tau = \mathrm{tr}(\mathrm{Ric})\,.$$

The Ricci tensor and the scalar curvature can be expressed in coordinates by

$$\rho_{ij} = g^{r\ell} R_{irj\ell}, \quad \tau = g^{ij}\rho_{ij}\,.$$

Any two-dimensional pseudo-Riemannian manifold satisfies $\rho = \frac{\tau}{2} g$. A pseudo-Riemannian manifold of dimension $n \geq 3$ is called an *Einstein space* if its Ricci tensor is a constant multiple of the metric, $\rho = \lambda g$. Tracing on the previous expression one gets

$$\rho = \frac{\tau}{n} g, \tag{1.2}$$

and using the second Bianchi identity we obtain that τ is constant, when M connected and $\dim(M) \geq 3$.

In dimension 3, the Einstein condition is equivalent to constant sectional curvature. In dimension $n \geq 4$, there exist Einstein metrics which are not of constant sectional curvature. For instance $\mathbb{S}^2 \times \mathbb{S}^2$ is Einstein but the sectional curvature is clearly not constant. Dimension four appears therefore as the first non-trivial case for consideration.

1.1.3 The Weyl tensor

Let D and B be two symmetric bilinear forms on a vector space V. The *Kulkarni-Nomizu* product $D \odot B$ is the $(0,4)$-tensor on V defined as follows:

$$\begin{aligned}(D \odot B)(x,y,z,v) &= D(x,z)B(y,v) + D(y,v)B(x,z) \\ &\quad -D(x,v)B(y,z) - D(y,z)B(x,v),\end{aligned}$$

where $x, y, z, v \in V$. An easy calculation shows that $D \odot B$ is an algebraic curvature tensor on $(V, \langle \cdot, \cdot \rangle)$, i.e., a (0,4)-tensor on V satisfying the algebraic identities (1.1) of the curvature tensor. As a basic example, the standard algebraic curvature tensor R^0 is given by $R^0 = \frac{1}{2}\langle \cdot, \cdot \rangle \odot \langle \cdot, \cdot \rangle$.

The *Schouten tensor*, $\mathfrak{S}$, of an algebraic curvature tensor A on an n-dimensional inner product vector space $(V, \langle \cdot, \cdot \rangle)$ is the symmetric tensor field of type $(0, 2)$ defined by

$$\mathfrak{S}_A = \frac{1}{n-2}\left(\rho_A - \frac{\tau_A}{2(n-1)}\langle \cdot, \cdot \rangle\right),$$

where ρ_A and τ_A are the Ricci tensor and the scalar curvature associated to A.

The Weyl tensor arises from the Kulkarni-Nomizu product of the Schouten tensor and the metric tensor; $W_A = A - \mathfrak{S}_A \odot \langle \cdot, \cdot \rangle$. Hence the *Weyl tensor*, W, of a pseudo-Riemannian manifold (M, g) is defined by:

$$W = R - \mathfrak{S} \odot g,$$

or equivalently at each point $p \in M$

$$\begin{aligned} W(x, y, z, v) &= R(x, y, z, v) + \tfrac{\tau}{(n-1)(n-2)}\Big\{g(x, z)g(y, v) - g(x, v)g(y, z)\Big\} \\ &- \tfrac{1}{(n-2)}\Big\{\rho(x, z)g(y, v) - \rho(x, v)g(y, z) + \rho(y, v)g(x, z) - \rho(y, z)g(x, v)\Big\}, \end{aligned}$$

for all $x, y, z, v \in T_pM$.

An important property of the Weyl tensor to be used in this work is that it is trace free. Indeed, one has:

Lemma 1.1. *The Ricci curvature of the Weyl conformal tensor vanishes identically.*

Proof. Let $\{E_1, E_2, \ldots, E_n\}$ be a pseudo-orthonormal frame, where $g(E_i, E_j) = \varepsilon_i\delta_{ij}$ and $\varepsilon_i \in \{\pm 1\}$. We denote by $\rho_W(X, Y) = \operatorname{tr}(Z \to W(X, Z)Y)$ the Ricci tensor of the Weyl conformal tensor. Then,

$$\begin{aligned} \rho_W &= \sum_i \varepsilon_i W(X, E_i, Y, E_i) \\ &= \rho(X, Y) + \tfrac{\tau}{(n-1)(n-2)}\Big\{n\, g(X, Y) - g(X, Y)\Big\} \\ &\quad -\tfrac{1}{n-2}\Big\{n\,\rho(X, Y) - \rho(X, Y) + \tau\, g(X, Y) - \rho(X, Y)\Big\} \\ &= \rho(X, Y) + \tfrac{\tau}{n-2}g(X, Y) - \rho(X, Y) - \tfrac{\tau}{n-2}g(X, Y) = 0\,. \end{aligned}$$

□

A pseudo-Riemannian manifold (M, g) is called *locally conformally flat* if for each point $p \in M$ there exists an open neighborhood $\mathcal{U}$ of p and a smooth function $\sigma\colon \mathcal{U} \to \mathbb{R}$ so that the metric $\bar{g} = e^{2\sigma}g$ is flat.

The vanishing of the Weyl tensor characterizes locally conformally flat spaces in dimension $n \geq 4$. Observe that $W = 0$ in dimension $n = 3$. In fact, 3-dimensional locally conformally flat manifolds are characterized by the total symmetry of the covariant derivative of the Schouten tensor $(\nabla_X \mathfrak{S})(Y,Z) = (\nabla_Y \mathfrak{S})(X,Z)$ [73]. Explicitly one has:

$$\begin{aligned}(\nabla_X \mathfrak{S})(Y,Z) &- (\nabla_Y \mathfrak{S})(X,Z)\\ &= \tfrac{1}{(n-2)}\Big\{(\nabla_X \rho)(Y,Z) - (\nabla_Y \rho)(X,Z)\\ &\qquad - \tfrac{1}{2(n-1)}(X(\tau)g(Y,Z) - Y(\tau)g(X,Z))\Big\}.\end{aligned}$$

At the level of local differential geometry, the most important invariant of a conformal structure is given by the conformal Weyl tensor, which satisfies $\bar{W} = e^{2\sigma}W$ for any two conformally related metrics $\bar{g} = e^{2\sigma}g$.

1.2 Curvature decomposition

Given an n-dimensional real vector space V with basis $\{e_1,\dots,e_n\}$, a *bivector* of V is an element of the form

$$\sum_{i,j=1}^{n} a_{ij} e_i \wedge e_j,$$

where $a_{ij} \in \mathbb{R}$. The set of all elements of this form is called the bivector space $\Lambda^2 V$. It has the following properties:

- $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0 \quad \forall i,j \in \{1,\dots,n\}$.
- The set $e_1 \wedge e_2, \dots, e_1 \wedge e_n, e_2 \wedge e_3, \dots, e_{n-1} \wedge e_n$ is a basis of $\Lambda^2 V$.

In consequence, $\Lambda^2 V$ has a vector space structure of dimension $\frac{n(n-1)}{2}$. We define the *wedge product* of two elements $x, y \in V$, with $x = x^i e_i$ and $y = y^j e_j$, by:

$$x \wedge y = \left(\sum_{i=1}^{n} x^i e_i\right) \wedge \left(\sum_{j=1}^{n} y^j e_j\right) = \sum_{i<j} (x^i y^j - x^j y^i) e_i \wedge e_j \in \Lambda^2 V\,.$$

Let $\langle\cdot,\cdot\rangle$ be an inner product on V. Then, it naturally extends to an inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\Lambda^2 V$ as follows (see for example [73]):

$$\langle\!\langle x \wedge y, z \wedge t\rangle\!\rangle = \langle x,z\rangle\langle y,t\rangle - \langle x,t\rangle\langle y,z\rangle\,. \tag{1.3}$$

Moreover, if $\{e_1,\dots,e_n\}$ is a $\langle\cdot,\cdot\rangle$-orthonormal basis of V, then $e_i \wedge e_j$ $(i<j)$ is a $\langle\!\langle\cdot,\cdot\rangle\!\rangle$-orthonormal basis of $\Lambda^2 V$.

Each algebraic curvature tensor in $(V, \langle\cdot,\cdot\rangle)$ induces a unique self-adjoint endomorphism in $(\Lambda^2 V, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$ as follows. Given a curvature tensor A, we define the endomorphism $\widetilde{A}\colon \Lambda^2 V \to \Lambda^2 V$ by

$$\langle\!\langle \widetilde{A}(x \wedge y), z \wedge w\rangle\!\rangle = A(x,y,z,w) \quad \text{for all} \quad x,y,z,w \in V\,.$$

The converse is not true in general since a given self-adjoint endomorphism of $\Lambda^2 V$ may fail to satisfy the first Bianchi identity. There exists a bijective correspondence between the set of algebraic curvature tensors $\mathcal{A}$ and the set $\widetilde{\mathcal{A}}$ of self-adjoint endomorphisms of $\Lambda^2 V$ satisfying

$$\langle\!\langle \widetilde{A}(X \wedge Y), Z \wedge T \rangle\!\rangle + \langle\!\langle \widetilde{A}(Y \wedge Z), X \wedge T \rangle\!\rangle + \langle\!\langle \widetilde{A}(Z \wedge X), Y \wedge T \rangle\!\rangle = 0\,.$$

In particular, the standard curvature tensor R^0 corresponds to the endomorphism $\tilde{R}^0 = \mathrm{Id}_{\Lambda^2}$.

The following result provides a decomposition of any algebraic curvature tensor. It is also a motivation for the previously described tensors previously.

Theorem 1.2. [73] *Let A be an algebraic curvature tensor in an n-dimensional inner product vector space $(V, \langle \cdot, \cdot \rangle)$. Then it decomposes as:*

$$A = \mathfrak{U}_A + \mathfrak{Z}_A + W_A,$$

where

$$\mathfrak{U}_A = \frac{\tau_A}{2n(n-1)} \langle \cdot, \cdot \rangle \odot \langle \cdot, \cdot \rangle, \qquad \mathfrak{Z}_A = \frac{1}{n-2} \left(\rho_A - \frac{\tau_A}{n} \langle \cdot, \cdot \rangle \right) \odot \langle \cdot, \cdot \rangle$$

and $W_A = A - \mathfrak{U}_A - \mathfrak{Z}_A = A - \mathfrak{S}_A \odot \langle \cdot, \cdot \rangle$ is the Weyl tensor associated to the algebraic curvature tensor A.

The components $\mathfrak{U}_A, \mathfrak{Z}_A, W_A$ in Theorem 1.2 correspond to the following:

- $\mathfrak{U}_A$ is the orthogonal projection on the space of algebraic curvature tensors of constant sectional curvature.
- The vanishing of the component $\mathfrak{Z}_A$ corresponds with Einstein algebraic curvature tensors.
- In dimension ≥ 4, the vanishing of the component W_A represents locally conformally flat algebraic curvature tensors.

1.3 Self-duality and anti-self-duality

We work at the purely algebraic setting and assume $\dim(V) = 4$. Let $\{e_1, \dots, e_4\}$ be an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$. Then it follows from Equation (1.3) that

$$\begin{aligned}
\langle\!\langle e_1 \wedge e_2, e_1 \wedge e_2 \rangle\!\rangle &= \langle e_1, e_1 \rangle \langle e_2, e_2 \rangle - \langle e_1, e_2 \rangle \langle e_2, e_1 \rangle = \varepsilon_1 \varepsilon_2, \\
\langle\!\langle e_1 \wedge e_3, e_1 \wedge e_3 \rangle\!\rangle &= \langle e_1, e_1 \rangle \langle e_3, e_3 \rangle - \langle e_1, e_3 \rangle \langle e_3, e_1 \rangle = \varepsilon_1 \varepsilon_3, \\
\langle\!\langle e_1 \wedge e_4, e_1 \wedge e_4 \rangle\!\rangle &= \langle e_1, e_1 \rangle \langle e_4, e_4 \rangle - \langle e_1, e_4 \rangle \langle e_4, e_1 \rangle = \varepsilon_1 \varepsilon_4, \\
\langle\!\langle e_2 \wedge e_3, e_2 \wedge e_3 \rangle\!\rangle &= \langle e_2, e_2 \rangle \langle e_3, e_3 \rangle - \langle e_2, e_3 \rangle \langle e_3, e_2 \rangle = \varepsilon_2 \varepsilon_3, \\
\langle\!\langle e_2 \wedge e_4, e_2 \wedge e_4 \rangle\!\rangle &= \langle e_2, e_2 \rangle \langle e_4, e_4 \rangle - \langle e_2, e_4 \rangle \langle e_4, e_2 \rangle = \varepsilon_2 \varepsilon_4, \\
\langle\!\langle e_3 \wedge e_4, e_3 \wedge e_4 \rangle\!\rangle &= \langle e_3, e_3 \rangle \langle e_4, e_4 \rangle - \langle e_3, e_4 \rangle \langle e_4, e_3 \rangle = \varepsilon_3 \varepsilon_4,
\end{aligned}$$

where $\varepsilon_i = \langle e_i, e_i \rangle$. If $\langle \cdot, \cdot \rangle$ is positive definite then so is $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, while if $\langle \cdot, \cdot \rangle$ has neutral signature then $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is an inner product of signature $(4, 2)$ on $\Lambda^2 V$.

Now, let $vol := e_1 \wedge e_2 \wedge e_3 \wedge e_4$ be a volume element on V and define the *Hodge-star* operator $\star\colon \Lambda^2 V \to \Lambda^2 V$ by $\alpha \wedge \star\beta = \langle\!\langle \alpha, \beta \rangle\!\rangle \cdot vol$ for all $\alpha, \beta \in \Lambda^2 V$. This operator satisfies the following properties in Riemannian or neutral signature:

(i) $\star^2 = \mathrm{Id}_{\Lambda^2 V}$,

(ii) $\star$ is a self-adjoint operator.

The case when $(V, \langle \cdot, \cdot \rangle)$ is of Lorentzian signature is essentially different since in this setting one has $\star^2 = -\mathrm{Id}_{\Lambda^2 V}$, thus defining a complex structure on $\Lambda^2 V$ and the induced inner product is of neutral signature $(3, 3)$.

The action of the Hodge-star operator on the basis $\{e_i \wedge e_j\}$ is determined by:

$$\star(e_1 \wedge e_2) = \varepsilon_3\varepsilon_4 e_3 \wedge e_4, \quad \star(e_1 \wedge e_3) = -\varepsilon_2\varepsilon_4 e_2 \wedge e_4, \quad \star(e_1 \wedge e_4) = \varepsilon_2\varepsilon_3 e_2 \wedge e_3,$$

where the remaining elements are obtained using that $\star^2 = \pm\,\mathrm{Id}_{\Lambda^2 V}$, depending on the signature of $(V, \langle \cdot, \cdot \rangle)$.

In the Riemannian and neutral signature cases, since $\star^2 = \mathrm{Id}_{\Lambda^2 V}$, the eigenspaces corresponding to the eigenvalues ± 1 of $\star$ decompose $\Lambda^2 V$ as $\Lambda^2 V = \Lambda^2_+ \oplus \Lambda^2_-$, where

$$\Lambda^2_+ V = \{\alpha \in \Lambda^2 V \mid \star\alpha = \alpha\}, \qquad \Lambda^2_- V = \{\alpha \in \Lambda^2 V \mid \star\alpha = -\alpha\}\,.$$

The space $\Lambda^2_+ V$ is called the space of self-dual 2-forms and $\Lambda^2_- V$ is called the space of anti-self-dual 2-forms. Furthermore, for any algebraic curvature tensor A on $(V, \langle \cdot, \cdot \rangle)$ the associated Weyl tensor satisfies $\star\tilde{W}_A = \tilde{W}_A \star$ and thus the endomorphism $\tilde{W}_A$ decomposes accordingly.

Hence an algebraic curvature tensor A on $(V, \langle \cdot, \cdot \rangle)$ is said to be *self-dual* (resp. *anti-self-dual*) if $\tilde{W}_A(\Lambda^2_- V) \equiv 0$ (resp. $\tilde{W}_A(\Lambda^2_+ V) \equiv 0$). Further, A is said to be *locally conformally flat* if $W_A = 0$. Whenever the orientation is not specified, we will say that A is *half conformally flat* if A is either self-dual or anti-self-dual.

The half conformally flat condition can now be stated in terms of the components of the Weyl tensor in an orthonormal basis as follows. The existence of a field of 2-planes on a four-dimensional manifold determines a natural orientation on itself, for that reason self-dual and anti-self-dual conditions have a clear meaning.

Lemma 1.3. [21] *$(V, \langle \cdot, \cdot \rangle, A)$ is half conformally flat if and only if*

$$W_A(e_1, e_i, x, y) = \sigma_{ijk}\varepsilon_j\varepsilon_k W_A(e_j, e_k, x, y),$$

for each $x, y \in V$, where $\{i, j, k\} = \{2, 3, 4\}$, σ_{ijk} is the signature of the corresponding permutation and $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$.

Since we are interested in pseudo-Riemannian manifolds, we can reformulate the half conformally flat condition in the previous lemma for a *pseudo-orthonormal* basis $\{t, u, v, w\}$, i.e., a basis of $(V, \langle\cdot,\cdot\rangle)$ so that the inner product expresses as

$$\langle\cdot,\cdot\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{1.4}$$

i.e., the only non-zero products are given by $\langle t, v\rangle = \langle v, t\rangle = \langle u, w\rangle = \langle w, u\rangle = 1$.

Lemma 1.4. [21] $(V, \langle\cdot,\cdot\rangle, A)$ *is half conformally flat if and only if*

$$W_A(t, v, x, y) = W_A(u, w, x, y), \quad W_A(t, w, x, y) = 0, \quad W_A(u, v, x, y) = 0,$$

for all $x, y \in V$*, where* $\{t, u, v, w\}$ *is a pseudo-orthonormal basis of* $(V, \langle\cdot,\cdot\rangle)$.

Let (M, g) be an oriented four-dimensional pseudo-Riemannian manifold of Riemannian or neutral signature. Then, M is called *self-dual* (resp. *anti-self-dual*) if (T_pM, g_p, R_p) is self-dual (resp. anti-self-dual) for all $p \in M$. If any of the previous cases occur then (M, g) is called *half conformally flat*.

1.4 Conformal transformations and Einstein manifolds

In this section we consider conformal deformations of pseudo-Riemannian metrics with special attention to their influence on the curvature.

1.4.1 Conformal transformations

A *conformal map* between two pseudo-Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$ is a smooth map $F\colon (M, g) \to (\overline{M}, \overline{g})$ such that $F^*\overline{g} = \varphi^{-2}g$, for a non-zero smooth function $\varphi\colon M \to \mathbb{R}$, i.e.,

$$\overline{g}_{F(p)}(F_*(p)X, F_*(p)Y) = \varphi^{-2}(p)g_p(X, Y) \quad \text{for all} \quad p \in M,$$

and any $X, Y \in \mathfrak{X}(M)$. Moreover, two pseudo-Riemannian manifolds are *conformal* if there is a conformal map between them. Conformallity defines an equivalence relation in the space of metrics and we denote by $[g]$ the conformal class of a pseudo-Riemannian metric g.

Weyl showed in [103] that, although the definition of W evidently depends on the metric g, the Weyl tensor W actually depends on the conformal class of the metric. If two metrics $\overline{g} = \varphi^{-2}g$ are conformally equivalent, then the Weyl conformal tensors W and $\overline{W}$ of type (1,3) are equal to each other. However, the corresponding Weyl conformal tensors of type (0,4) rescale as $\overline{W} = \varphi^{-2}W$. The converse is true if $W = 0$ in which case both metrics are locally conformally flat, but not in general. Hall [63] showed that in dimension four the following partial converse holds.

Theorem 1.5. [63] *Let* (M, g) *be a four-dimensional Riemannian manifold. Let* $\overline{g}$ *be a Riemannian metric on* M *so that the Weyl conformal curvature tensors of type* $(1, 3)$ *satisfy* $W = \overline{W}$ *on some open set* $\mathcal{U} \subset M$ *where* $W \neq 0$. *Then* g *and* $\overline{g}$ *are conformally related on* $\mathcal{U}$.

Since we are interested in Einstein metrics one may wonder how the Einstein condition behaves under conformal transformations. The following lemma follows after some standard calculations.

Lemma 1.6. [73] *Let (M, g) be a pseudo-Riemannian manifold of dimension n and let $\varphi\colon M \to \mathbb{R}$ be a non-zero smooth function. If we consider in M the metric given by $\overline{g} = \varphi^{-2}g$ then one has:*

(i) If ∇ and $\overline{\nabla}$ denote the Levi-Civita connections of g and $\overline{g}$, respectively, then

$$\overline{\nabla}_X Y - \nabla_X Y = -X(\log\varphi)Y - Y(\log\varphi)X + g(X,Y)\nabla(\log\varphi)\,. \tag{1.5}$$

(ii) If R and $\overline{R}$ denote the curvature tensors of type $(0,4)$ of g and $\overline{g}$, respectively, then

$$\begin{aligned}\overline{R}(X,Y)Z - R(X,Y)Z =& \langle\nabla_X\nabla(\log\varphi), Z\rangle Y + \langle\nabla_Y\nabla(\log\varphi), Z\rangle X \\ &- \langle X, Z\rangle\nabla_Y\nabla(\log\varphi) + \langle Y, Z\rangle\nabla_X\nabla(\log\varphi) \\ &+ (Y\log\varphi)(Z\log\varphi)X - (X\log\varphi)(Z\log\varphi)Y \\ &- \langle\nabla(\log\varphi), \nabla(\log\varphi)\rangle\cdot R^0(X,Y)Z \\ &+ ((X\log\varphi)\langle Y,Z\rangle - (Y\log\varphi)\langle X,Z\rangle)\nabla\log\varphi\,.\end{aligned}$$

(iii) If ρ and $\overline{\rho}$ denote the Ricci tensors of g and $\overline{g}$, respectively, then

$$\overline{\rho} - \rho = \varphi^{-2}((n-2)\cdot\varphi\,\mathrm{Hes}_\varphi + (\varphi\Delta\varphi - (n-1)\|\nabla\varphi\|^2)g),$$

where $\Delta\varphi = \mathrm{tr}_g(\mathrm{Hes}_\varphi)$ is the Laplacian.

Assertion (iii) in Lemma 1.6 shows that the Einstein condition is not necessarily preserved by a conformal transformation. The following result was originally proven by Brinkmann [15] (see also [74]).

Theorem 1.7. *Let (M, g) be an Einstein manifold of $\dim(M) = n \geq 3$. A conformal metric $\overline{g} = \varphi^{-2}g$ is Einstein if and only if $\mathrm{Hes}_\varphi = \dfrac{\Delta\varphi}{n}g$.*

It follows from the work of Brinkmann that a Riemannian four-dimensional Einstein metric admits a conformally related Einstein deformation if and only if it is of constant sectional curvature. On the other hand, the indefinite setting allows the existence of conformally-related Einstein metrics which are of non-constant sectional curvature. Examples of this situation will appear in Chapter 3.

1.4.2 Conformally Einstein manifolds

Einstein metrics are among the most privileged ones, since they are considered optimal metrics, i.e., those whose curvature has the property of being most evenly distributed on the manifold. For that reason, Einstein metrics are central in geometry. One strategy to construct an Einstein metric consists in deforming an initial metric by a conformal factor so that the resulting metric becomes Einstein. We make this more precise as follows.

Definition 1.8. A pseudo-Riemannian manifold (M, g) is called *locally conformally Einstein* if for any point $p \in M$ there exists a neighborhood $\mathcal{U}$ of $p \in M$ and a smooth function $\varphi\colon \mathcal{U} \to \mathbb{R}$ so that $\overline{g} = \varphi^{-2} g$ is a locally defined Einstein metric.

An application of Lemma 1.6–(iii) gives the following:

Theorem 1.9. [14] *A pseudo-Riemannian manifold (M, g) is conformally Einstein if and only if the following equation has a positive solution*

$$(n-2)\operatorname{Hes}_\varphi + \varphi\rho = \frac{1}{n}((n-2)\Delta\varphi + \varphi\tau)g, \tag{1.6}$$

where $n = \dim(M)$.

Equation (1.6) will be called the *conformally Einstein equation*. Observe that the conformally Einstein equation is generically overdetermined. Moreover solutions (if exist) are unique as shown by Brinkmann and Yau in the Riemannian setting.

Theorem 1.10. [15, 106] *Let M and N be two connected Riemannian Einstein manifolds of dimension ≥ 3 and let $F\colon M \to N$ be a conformal diffeomorphism. Then either F is a homothety or both M and N have constant curvature.*

It is relevant to emphasize that uniqueness of Einstein metrics in the conformal class is not true in higher signatures. In Chapter 3 we will show the existence of non-reductive homogeneous conformally Einstein pseudo-Riemannian manifolds (of neutral or Lorentzian signature) where the space of conformally Einstein metrics has dimension 2 or 3.

Equation (1.6) is trivial in dimension two, but its integration is surprisingly difficult in higher dimensions. Three-dimensional manifolds are locally conformally Einstein if and only if they are locally conformally flat. Hence this dimension is exceptional and there is a tensorial characterization of the conformally Einstein property. However, in dimension ≥ 4, there are examples which are conformally Einstein but not locally conformally flat. The conformally Einstein equation implies that the eigenspaces of the Hessian operator h_φ must coincide with the eigenspaces of the Ricci operator. Moreover, the eigenvalues of h_φ are determined by the eigenvalues of Ric and conversely.

In what follows of this section, we will show some consequences of the conformally Einstein Equation (1.6) aimed to obtain a tensorial characterization of the conformally Einstein property. The following is an important observation.

Lemma 1.11. *Let (M,g) be a pseudo-Riemannian manifold. Then*

$$(\operatorname{div} W)(X,Y,Z) = (n-3)\Big\{(\nabla_X \mathfrak{S})(Y,Z) - (\nabla_Y \mathfrak{S})(X,Z)\Big\},$$

where $\mathfrak{S} = \frac{1}{n-2}\Big(\rho - \frac{\tau}{2(n-1)}g\Big)$ is the Schouten tensor. In particular $\operatorname{div} W = 0$ if (M,g) is Einstein.

Proof. Fix a point $p \in M$ and specialize a local orthonormal frame $\{E_1,\dots,E_n\}$ so that $\nabla_{E_i}E_j\mid_p = 0$. Further let X, Y, Z be vector fields on M and assume that $\nabla_{E_i}X\mid_p = \nabla_{E_i}Y\mid_p = \nabla_{E_i}Z\mid_p = 0$. Then the divergence of the Weyl tensor is given by:

$$(\operatorname{div} W)(X,Y,Z) = \sum_i \varepsilon_i (\nabla_{E_i} W)(X,Y,Z,E_i)\,.$$

Recall that the expression of the Weyl tensor $W = R - \mathfrak{S} \odot g$ is given by

$$W(X,Y,Z,T) = \Big\{R + \tfrac{\tau}{(n-1)(n-2)}R^0 - \tfrac{1}{n-2}(\rho \odot g)\Big\}(X,Y,Z,T)\,. \tag{1.7}$$

Now, we compute the covariant derivative of each term in Equation (1.7). First of all we apply the second Bianchi identity to compute

$$\begin{aligned}\textstyle\sum_i \varepsilon_i E_i R(X,Y,Z,E_i) &= -\textstyle\sum_i \varepsilon_i X R(Y,E_i,Z,E_i) - \sum_i \varepsilon_i Y R(E_i,X,Z,E_i)\\ &= Y\rho(X,Z) - X\rho(Y,Z)\,.\end{aligned}$$

Since the standard algebraic curvature tensor R^0 is parallel, the derivative of the second term in Equation (1.7) becomes

$$\begin{aligned}&\tfrac{1}{(n-1)(n-2)}\textstyle\sum_i \varepsilon_i E_i\left(\tau R^0(X,Y,Z,E_i)\right)\\ &\qquad= \tfrac{1}{(n-1)(n-2)}R^0(X,Y,Z,E_i)\varepsilon_i E_i(\tau)\\ &\qquad= \tfrac{1}{(n-1)(n-2)}\Big\{g(X,Z)Y(\tau) - g(Y,Z)X(\tau)\Big\}\,.\end{aligned}$$

Proceeding in an analogous way with the derivative of the third term in Equation (1.7) one has

$$\begin{aligned}&\tfrac{1}{n-2}\textstyle\sum_i \varepsilon_i E_i(\rho \odot g)(X,Y,Z,E_i)\\ &\quad= \tfrac{1}{n-2}\textstyle\sum_i \varepsilon_i\Big\{g(Y,E_i)E_i\rho(X,Z) - g(Y,Z)E_i\rho(X,E_i)\\ &\qquad\qquad + g(X,Z)E_i\rho(Y,E_i) - g(X,E_i)E_i\rho(Y,Z)\Big\}\\ &\quad= \tfrac{1}{n-2}\{Y\rho(X,Z) - g(Y,Z)\operatorname{div}\rho(X) + g(X,Z)\operatorname{div}\rho(Y) - X\rho(Y,Z)\}\\ &\quad= \tfrac{1}{2(n-2)}\Big\{2Y\rho(X,Z) - g(Y,Z)X(\tau) + g(X,Z)Y(\tau) - 2X\rho(Y,Z)\Big\}\,.\end{aligned}$$

Now, adding all the expressions above one gets:

$$\begin{aligned}
&(\operatorname{div} W)(X,Y,Z)\\
&= Y\rho(X,Z) - X\rho(Y,Z) + \tfrac{1}{(n-2)(n-1)}\Big\{g(X,Z)Y(\tau) - g(Y,Z)X(\tau)\Big\}\\
&\quad - \tfrac{1}{2(n-2)}\Big\{2Y\rho(X,Z) - g(Y,Z)X(\tau) + g(X,Z)Y(\tau) - 2X\rho(Y,Z)\Big\}\\
&= \tfrac{n-3}{n-2}\Big\{Y\rho(X,Z) - X\rho(Y,Z) - \tfrac{1}{2(n-1)}\left(g(X,Z)Y(\tau) - g(Y,Z)X(\tau)\right)\Big\}\\
&= (n-3)\Big\{(\nabla_X\mathfrak{S})(Y,Z) - (\nabla_Y\mathfrak{S})(X,Z)\Big\}.
\end{aligned}$$

Finally observe that if (M,g) is Einstein, then $\rho = \frac{\tau}{n}g$ with $\tau\in\mathbb{R}$, and thus the Schouten tensor $\mathfrak{S} = \dfrac{1}{n-2}\left(\rho - \dfrac{\tau}{2(n-1)}g\right)$ is parallel. Hence $\operatorname{div} W = 0$. □

Let $\overline{g}$ and g be conformally related so that $\overline{g} = \varphi^{-2}g$ and, for sake of simplicity, we set $\sigma = -\log\varphi$. Let $\overline{W}$ and $\overline{\operatorname{div}_4}\,\overline{W}$ denote the Weyl tensor and its divergence with respect to the metric $\overline{g}$. Then one has:

Lemma 1.12. *Let $\overline{g} = e^{2\sigma}g$ be two conformally related metrics. Then*

$$(\overline{\operatorname{div}_4}\,\overline{W})(X,Y,Z) = (\operatorname{div}_4 W)(X,Y,Z) + (n-3)W(X,Y,Z,\nabla\sigma) \tag{1.8}$$

for all vector fields X,Y,Z on M.

Proof. Let $\{E_1,E_2,\dots,E_n\}$ be a local g-orthonormal frame and set $\overline{E}_i = \frac{1}{e^\sigma}E_i$ such that $\{\overline{E}_1,\overline{E}_2,\dots,\overline{E}_n\}$ is a local $\bar{g}$-orthonormal frame, where $g(E_i,E_j) = \varepsilon_i\delta_{ij}$. Then

$$\begin{aligned}
&(\overline{\operatorname{div}_4}\,\overline{W})(X,Y,Z)\\
&= \sum_i \varepsilon_i(\overline{\nabla}_{\overline{E}_i}\overline{W})(X,Y,Z,\overline{E}_i)\\
&= \sum_i \varepsilon_i\frac{1}{e^{2\sigma}}\Big\{\overline{\nabla}_{E_i}\overline{W}(X,Y,Z,E_i) - \overline{W}(\overline{\nabla}_{E_i}X,Y,Z,E_i) - \overline{W}(X,\overline{\nabla}_{E_i}Y,Z,E_i)\\
&\qquad\qquad - \overline{W}(X,Y,\overline{\nabla}_{E_i}Z,E_i) - \overline{W}(X,Y,Z,\overline{\nabla}_{E_i}E_i)\Big\}.
\end{aligned}$$

Next, we expand separately each one of the five terms above to obtain:

$$\begin{aligned}
\overline{\nabla}_{E_i}\overline{W}(X,Y,Z,E_i) &= E_i(e^{2\sigma}W(X,Y,Z,E_i))\\
&= 2e^{2\sigma}E_i(\sigma)W(X,Y,Z,E_i) + e^{2\sigma}\nabla_{E_i}W(X,Y,Z,E_i),
\end{aligned}$$

$$
\begin{aligned}
&\overline{W}(\overline{\nabla}_{E_i}X, Y, Z, E_i)\\
&\quad = e^{2\sigma}W(\nabla_{E_i}X + E_i(\sigma)X + X(\sigma)E_i - g(E_i, X)\nabla\sigma, Y, Z, E_i)\\
&\quad = e^{2\sigma}\Big\{W(\nabla_{E_i}X, Y, Z, E_i) + W(E_i(\sigma)X, Y, Z, E_i) + W(X(\sigma)E_i, Y, Z, E_i)\\
&\qquad\quad - g(E_i, X)W(\nabla\sigma, Y, Z, E_i)\Big\}\\
&\quad = e^{2\sigma}\Big\{W(\nabla_{E_i}X, Y, Z, E_i) + W(X, Y, Z, g(\nabla\sigma, E_i)E_i)\\
&\qquad\quad + X(\sigma)W(E_i, Y, Z, E_i) - W(\nabla\sigma, Y, Z, g(E_i, X)E_i)\Big\},\\
&\overline{W}(X, \overline{\nabla}_{E_i}Y, Z, E_i)\\
&\quad = e^{2\sigma}W(X, \nabla_{E_i}Y + E_i(\sigma)Y + Y(\sigma)E_i - g(E_i, Y)\nabla\sigma, Z, E_i)\\
&\quad = e^{2\sigma}\Big\{W(X, \nabla_{E_i}Y, Z, E_i) + W(X, E_i(\sigma)Y, Z, E_i) + W(X, Y(\sigma)E_i, Z, E_i)\\
&\qquad\quad - g(E_i, Y)W(X, \nabla\sigma, Z, E_i)\Big\}\\
&\quad = e^{2\sigma}\Big\{W(X, \nabla_{E_i}Y, Z, E_i) + W(X, Y, Z, g(\nabla\sigma, E_i)E_i)\\
&\qquad\quad + Y(\sigma)W(X, E_i, Z, E_i) - W(X, \nabla\sigma, Z, g(E_i, Y)E_i)\Big\},\\
&\overline{W}(X, Y, \overline{\nabla}_{E_i}Z, E_i)\\
&\quad = e^{2\sigma}W(X, Y, \nabla_{E_i}Z + E_i(\sigma)Z + Z(\sigma)E_i - g(E_i, Z)\nabla\sigma, E_i)\\
&\quad = e^{2\sigma}\Big\{W(X, Y, \nabla_{E_i}Z, E_i) + W(X, Y, E_i(\sigma)Z, E_i) + W(X, Y, Z(\sigma)E_i, E_i)\\
&\qquad\quad - g(E_i, Z)W(X, Y, \nabla\sigma, E_i)\Big\}\\
&\quad = e^{2\sigma}\Big\{W(X, Y, \nabla_{E_i}Z, E_i) + W(X, Y, Z, \langle\nabla\sigma, E_i\rangle E_i)\\
&\qquad\quad + Z(\sigma)W(X, Y, E_i, E_i) - W(X, Y, \nabla\sigma, g(E_i, Z)E_i)\Big\},\\
&\overline{W}(X, Y, Z, \overline{\nabla}_{E_i}E_i)\\
&\quad = e^{2\sigma}W(X, Y, Z, \nabla_{E_i}E_i + E_i(\sigma)E_i + E_i(\sigma)E_i - g(E_i, E_i)\nabla\sigma)\\
&\quad = e^{2\sigma}\Big\{W(X, Y, Z, \nabla_{E_i}E_i) + 2W(X, Y, Z, E_i(\sigma)E_i)\\
&\qquad\quad - g(E_i, E_i)W(X, Y, Z, \nabla\sigma)\Big\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(\overline{\operatorname{div}_4}\,\overline{W})(X,Y,Z)\\
&\quad=\sum_i \varepsilon_i e^{2\sigma}\Big\{\nabla_{E_i}W(X,Y,Z,E_i)-W(\nabla_{E_i}X,Y,Z,E_i)-W(X,\nabla_{E_i}Y,Z,E_i)\\
&\qquad\qquad -W(X,Y,\nabla_{E_i}Z,E_i)-W(X,Y,Z,\nabla_{E_i}E_i)\Big\}\\
&\qquad+(n-3)W(X,Y,Z,\nabla\sigma)\\
&\qquad+W(\nabla\sigma,Y,Z,X)+W(X,\nabla\sigma,Z,Y)+W(X,Y,\nabla\sigma,Z)\\
&\quad=(\operatorname{div}_4 W)(X,Y,Z)+(n-3)W(X,Y,Z,\nabla\sigma),
\end{aligned}
$$

which finishes the proof. □

An immediate consequence of Lemma 1.11 and Lemma 1.12 is that, if $\overline{g}=\varphi^{-2}g$ is Einstein, then

$$(\operatorname{div}_4 W)(X,Y,Z)-(n-3)W(X,Y,Z,\nabla\sigma)=0 \tag{1.9}$$

for all vector fields X,Y,Z on M where $\varphi=e^{-\sigma}$ [74, 79]. Observe that the tensorial condition involves $\nabla\sigma$, which makes (1.9) in a certain way unmanageable since the conformal deformation σ is unknown.

Remark 1.13. The identity in Equation (1.9) is satisfied for any divergence, i.e.,

$$(\operatorname{div}_3 W)(X,Y,Z)-(n-3)W(X,Y,\nabla\sigma,X)=0.$$

Next we compute the divergence in Equation (1.9). As a matter of notation, let $W[\Phi]$ denote the action of the Weyl conformal curvature tensor on the space of symmetric $(0,2)$-tensor fields by (see [10])

$$W[\Phi](X,Y)=\sum_{i,j}\varepsilon_i\varepsilon_j W(E_i,X,Y,E_j)\Phi(E_j,E_i)\,.$$

Lemma 1.1 shows that, for any function $f\in C^\infty(M)$, one has $W[f\,g]=0$ since W is trace free and $W[g]=\rho_W=0$.

Now, fix a point $p\in M$ and let $\{E_1,E_2,\dots,E_n\}$ be a local g-orthonormal frame around $p\in M$ such that $\nabla_{E_i}E_j|_p=0$ for any i,j and let X, Y, and Z be vector fields such that$\nabla_{E_i}X|_p=\nabla_{E_i}Y|_p=\nabla_{E_i}Z|_p=0$ for all i. We set $T(X,Y,Z)=W(X,Y,Z,\nabla\sigma)$ and compute $\operatorname{div}_2 T$ in Equation (1.9) to get:

$$0=\operatorname{div}_2\operatorname{div}_4 W(X,Y)+(n-3)\operatorname{div}_2 T(X,Y)\,. \tag{1.10}$$

Furthermore

$$
\begin{aligned}
&\operatorname{div}_2 T(X,Y)\\
&\quad = \textstyle\sum_i (\nabla_{E_i} T)(X,E_i,Y) = \displaystyle\sum_i \nabla_{E_i} W(X,E_i,Y,\nabla\sigma)\\
&\quad = \sum_{i,j} \nabla_{E_i}\left(\langle\nabla\sigma,E_j\rangle W(X,E_i,Y,E_j)\right) = \sum_{i,j}\Big\{\langle\nabla_{E_i}\nabla\sigma,E_j\rangle W(X,E_i,Y,E_j)\Big\}\\
&\quad = \sum_{i,j}\operatorname{Hes}_\sigma(E_i,E_j)W(X,E_i,Y,E_j) + \sum_j\langle\nabla_\sigma,E_j\rangle\sum_i \nabla_{E_i}W(X,E_i,Y,E_j)\\
&\quad = W[\operatorname{Hes}_\sigma] + \sum_j\langle\nabla\sigma,E_j\rangle \operatorname{div}_2 W(X,Y,E_j)\,.
\end{aligned}
$$

Now, by Remark 1.13 one gets

$$
\begin{aligned}
&\operatorname{div}_2 T(X,Y)\\
&\quad = W[\operatorname{Hes}_\sigma] - (n-3)\sum_j\langle\nabla\sigma,E_j\rangle W(X,\nabla\sigma,Y,E_j)\\
&\quad = W[\operatorname{Hes}_\sigma] - (n-3)\sum_{i,j}\langle\nabla\sigma,E_i\rangle\langle\nabla\sigma,E_j\rangle W(X,E_i,Y,E_j)\\
&\quad = W[\operatorname{Hes}_\sigma] - (n-3)W[d\sigma\otimes d\sigma],
\end{aligned}
$$

and thus

$$\operatorname{div}_2\operatorname{div}_4 W(X,Y) + (n-3)W[\operatorname{Hes}_\sigma] - (n-3)^2 W[d\sigma\otimes d\sigma] = 0\,. \tag{1.11}$$

Since $\varphi = e^{-\sigma}$, one has $d\varphi = -e^{-\sigma}d\sigma$ and $\operatorname{Hes}_\varphi = e^{-\sigma}\left(-\operatorname{Hes}_\sigma + d\sigma\otimes d\sigma\right)$. Hence $(n-2)\operatorname{Hes}_\varphi + \varphi\rho = (n-2)e^{-\sigma}\left(-\operatorname{Hes}_\sigma + d\sigma\otimes d\sigma\right) + e^{-\sigma}\rho$, and the conformally Einstein Equation (1.6) becomes

$$\operatorname{Hes}_\sigma = \frac{1}{n-2}\rho + d\sigma\otimes d\sigma - e^\sigma\cdot\xi\cdot g,$$

where $\xi = \frac{1}{n(n-2)}\{(n-2)\Delta\varphi + \varphi\tau\}$. Finally, substituting in Equation (1.11) and using Lemma 1.1, one gets

$$
\begin{aligned}
0 &= \operatorname{div}_2\operatorname{div}_4 W(X,Y) + (n-3)W\left[\tfrac{1}{n-2}\rho + d\sigma\otimes d\sigma - e^\sigma\,\xi\, g\right] - (n-3)^2 W[d\sigma\otimes d\sigma]\\
&= \operatorname{div}_2\operatorname{div}_4 W + \tfrac{n-3}{n-2}W[\rho] + (n-3)\Big\{W[d\sigma\otimes d\sigma] - (n-3)W[d\sigma\otimes d\sigma]\Big\}\\
&= \operatorname{div}_2\operatorname{div}_4 W + \tfrac{n-3}{n-2}W[\rho] - (n-3)(n-4)W[d\sigma\otimes d\sigma]\,.
\end{aligned}
$$

In the special case of $\dim(M) = 4$, one obtains the necessary conditions (i) and (ii) in Theorem 1.14 to be conformally Einstein. Moreover, these conditions are also sufficient in some special cases as the following shows.

Theorem 1.14. [72] *Let (M,g) be a four-dimensional manifold such that the conformal metric $\bar{g}=\varphi^{-2}g$ is Einstein. Then*

(i) $\operatorname{div}_2\operatorname{div}_4 W+\frac{1}{2}W[\rho]=0$,

(ii) $(\operatorname{div}_4 W)(X,Y,Z)-W(X,Y,Z,\nabla\sigma)=0$,

where $\sigma=-\log\varphi$, for some function $\varphi\in\mathcal{C}^\infty(M)$.

Conversely, conditions (i) and (ii) above are also sufficient if (M,g) is assumed to be weakly-generic, i.e., the Weyl curvature operator (viewed as a map $W:TM\to\bigotimes^3 TM$) is injective.

Observe that condition (i) in Theorem 1.14 is a tensorial equation on (M,g) which is independent of the conformal factor.

1.5 Additional structures on manifolds

In this section we briefly review some basic notation on Kähler and para-Kähler structures that will appear in subsequent chapters.

Kähler structures

A complex manifold is a differentiable manifold with a holomorphic atlas. If a real manifold M of dimension $n=2m$ admits a globally defined tensor field J of type $(1,1)$ such that

$$J^2=-\operatorname{Id}, \tag{1.12}$$

then (M,J) is called an *almost complex manifold* and J is an *almost complex structure* on M. As the word indicates, almost complex means that it is "not quite" complex. If the almost complex structure corresponds to the underlying structure of a complex manifold, then it is said to be *integrable* and a fundamental result of Newlander and Nirenberg [85] shows that an almost complex structure J on M is integrable if and only if the Nijenhuis tensor N_J vanishes, where

$$N_J(X,Y)=[JX,JY]-J[JX,Y]-J[X,JY]+J^2[X,Y]\,.$$

A pseudo-Riemannian metric g on M is called an *almost Hermitian metric* if the almost complex structure J is an isometry in each tangent space, i.e.,

$$g(JX,JY)=g(X,Y)\quad\text{for all}\quad X,Y\in\mathfrak{X}(M)\,. \tag{1.13}$$

The triple (M,g,J) is called *almost Hermitian manifold*. An almost Hermitian manifold (M,g,J) is said to be *Hermitian* if the almost complex structure is integrable.

Associated to any almost Hermitian structure (g,J) there exists a non-degenerate 2-form, called the *Kähler form* and given by:

$$\Omega(X,Y)=g(JX,Y)\,.$$

The covariant derivative of the almost complex structure, the Nijenhuis tensor and the differential of the Kähler 2-form Ω are related by (see [105]).

$$2g((\nabla_X J)Y, Z) + 3d\Omega(X, Y, Z) - 3d\Omega(X, JY, JZ) - g(JX, N_J(Y, Z)) = 0\,.$$

A *symplectic manifold* (M, Ω) is a manifold equipped with a closed and non-degenerate two form (i.e., $d\Omega = 0$ and $\Omega^m = \Omega \wedge \overset{m}{\cdots} \wedge \Omega \neq 0$). If (M, g, J) is an almost Hermitian manifold with closed Kähler form, then M is said to be *almost Kähler*. If M is a complex manifold with an Hermitian metric and Ω is closed then M is called a *Kähler manifold* with Kähler metric g. In other words, Kähler manifolds are characterized by the parallelizability of their complex structure, $\nabla J = 0$, and their curvature tensor satisfies

$$R(X, Y, Z, W) = R(JX, JY, Z, W)\,.$$

A consequence of the previous identity is that any Kähler manifold of constant sectional curvature is necessarily flat. We define the *holomorphic sectional curvature* as the restriction of the sectional curvature to non-degenerate holomorphic planes π (i.e., non-degenerate planes invariant by the complex structure; $J(\pi) \subset \pi$) and it is given by

$$H(\pi) = \frac{R(X, JX, X, JX)}{g(X, X)^2}\,.$$

It is important to emphasize that the holomorphic sectional curvature determines the curvature tensor in the Kähler case. Moreover, a Kähler manifold has constant holomorphic sectional curvature c if and only if the curvature tensor is given by

$$R = \frac{c}{4}(R^0 + R^J),$$

where R^0 is the standard algebraic curvature tensor and

$$R^J(X, Y)Z = g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ\,.$$

A Kähler manifold of constant holomorphic sectional curvature is locally isometric to the complex space $\mathbb{C}^m_\nu$ (if $c = 0$), to the complex projective space $\mathbb{CP}^m_\nu$ (if $c > 0$) or to the complex hyperbolic space $\mathbb{CH}^m_\nu$ (if $c < 0$) [8].

An almost Hermitian manifold (M, g, J) is said to be *locally conformally Kähler* (resp. *locally conformally symplectic*) if there is a local conformal deformation $\bar{g} = e^{2\sigma} g$ so that $(M, \bar{g}, J)$ becomes Kähler (resp. symplectic). One has the following characterizations (see [52, 101] and references therein).

- (M, g, J) is *locally conformally Kähler* if and only if J is integrable and $d\Omega = \theta \wedge \Omega,\ d\theta = 0$, where θ is a closed 1-form.

- (M, g, J) is *locally conformally symplectic* if and only if $d\Omega = \theta \wedge \Omega,\ d\theta = 0$, where θ is a closed 1-form.

Let (M, g, J) be a four-dimensional Kähler manifold and orient it so that the Kähler 2-form is self-dual ($\Omega \in \Lambda^2_+$). Then the self-dual Weyl curvature operator satisfies

$$W^+ = \frac{\tau}{12}\,\mathrm{diag}[2, -1, -1]\,.$$

Hence the self-dual part of the Weyl tensor of any locally conformally Kähler metric has two-equal eigenvalues. The following converse, proven by Derdziński, is important for our purposes in Chapter 2.

Theorem 1.15. [48] *Let (M, g) be an oriented four-dimensional Riemannian Einstein manifold such that W^+ has at most two different eigenvalues at each point. Then $\overline{g} = (24\|W^+\|^2)^{\frac{1}{3}} g$ is Kähler on the open set where $W^+ \neq 0$.*

Para-Kähler structures

A (1,1)-tensor field $\mathfrak{J}$ on a $2m$-dimensional manifold M is said to be an *almost product structure* if $\mathfrak{J}^2 = \mathrm{Id}$. In this case the pair $(M, \mathfrak{J})$ is called an *almost product manifold*. An *almost para-complex manifold* is an almost product manifold such that the bundles T^+M and T^-M associated with the two eigenvalues ± 1 of $\mathfrak{J}$ have the same rank.

An *almost para-Hermitian manifold* $(M, g, \mathfrak{J})$ is a manifold M endowed with an almost para-complex structure $\mathfrak{J}$ and a metric tensor g such that $g(\mathfrak{J}X, \mathfrak{J}Y) = -g(X, Y)$. We define the non-degenerate 2-form of the almost para-Hermitian manifold by

$$\Omega(X, Y) = g(\mathfrak{J}X, Y),$$

for any vector fields X, Y on M.

Let Ω be a 2-form on M. Ω is called an *almost symplectic form* if it is non-degenerate, i.e., $\Omega^m \neq 0$ and the pair (M, Ω) is said to be an *almost symplectic manifold*. Let $L \subset M$ be an m-dimensional submanifold of an almost symplectic manifold. If $\Omega|_L = 0$ then L is a *Lagrangian submanifold*. An almost symplectic manifold is an almost para-Hermitian manifold if its tangent bundle decomposes as a Whitney sum of Lagrangian subbundles. Observe that $TM = L_1 \oplus L_2$ and the $(1,1)$-tensor field defined by $\mathfrak{J} = \pi_{L_1} - \pi_{L_2}$ (where π_{L_1} and π_{L_2} are the projections of TM on L_1 and L_2, respectively) determines an almost para-complex structure on M. Furthermore the metric tensor is determined by the para-complex structure and the 2-form Ω as $g(X, Y) = \Omega(\mathfrak{J}X, Y)$.

A *para-Kähler manifold* is a symplectic manifold which is diffeomorphic to a product of Lagrangian submanifolds. One has the following relationship between Ω, the integrability of $\mathfrak{J}$ and the covariant derivative of $\mathfrak{J}$:

$$2g((\nabla_X \mathfrak{J})Y, Z) + 3d\Omega(X, Y, Z) + 3d\Omega(X, \mathfrak{J}Y, \mathfrak{J}Z) + g(\mathfrak{J}X, N_{\mathfrak{J}}(Y, Z)) = 0,$$

where $N_{\mathfrak{J}}(X, Y) = [\mathfrak{J}X, \mathfrak{J}Y] - \mathfrak{J}[\mathfrak{J}X, Y] - \mathfrak{J}[X, \mathfrak{J}Y] + \mathfrak{J}^2[X, Y]$ is the Nijenhuis tensor of the almost para-complex structure. This relationship allows to characterize para-Kähler manifolds through the parallelism of $\mathfrak{J}$. Hence one has that $(M, g, \mathfrak{J})$ is a *para-Kähler manifold* if and only if

$$\mathfrak{J}^2 = \mathrm{Id}, \quad g(\mathfrak{J}X, \mathfrak{J}Y) = -g(X, Y) \quad \text{and} \quad \nabla\mathfrak{J} = 0\,.$$

We refer to [44, 46] for more details and references. In this case, the curvature tensor satisfies

$$R(X, Y, Z, W) = -R(\mathfrak{J}X, \mathfrak{J}Y, Z, W)\,.$$

We define the *para-holomorphic sectional curvature* by the restriction of the sectional curvature to non-degenerate para-holomorphic planes, i.e.,
non-degenerate planes π such that $\mathfrak{J}(\pi) \subset \pi$:

$$H(\pi) = -\frac{R(X, \mathfrak{J}X, X, \mathfrak{J}X)}{g(X, X)^2}\,.$$

As in the Kähler case, the para-holomorphic sectional curvature determines the curvature of a para-Kähler manifold and it is constant if and only if

$$R = -\frac{c}{4}(R^0 + R^{\mathfrak{J}}),$$

where R^0 is the standard algebraic curvature tensor and

$$R^{\mathfrak{J}}(X, Y)Z = g(\mathfrak{J}X, Z)\mathfrak{J}Y - g(\mathfrak{J}Y, Z)\mathfrak{J}X + 2g(\mathfrak{J}X, Y)\mathfrak{J}Z\,.$$

A para-Kähler manifold of constant para-holomorphic sectional curvature is locally isometric (or anti-isometric) to $\mathbb{R}^{2m}$ if $c = 0$ or to the para-complex projective space $\mathbb{P}^m(\mathbb{B})$ if $c \neq 0$ [58]. Further, observe that the para-complex projective space is locally isometric to the cotangent bundle of a flat affine manifold equipped with a suitable Riemannian extension [29].

1.6 The Bach tensor

The Bach tensor arises as the gradient of the quadratic curvature functional given by the L^2-norm of the Weyl curvature tensor. The purpose of this section is to introduce the Bach tensor in dimension four and give some examples of Bach-flat metrics (we refer to [6] for more details). We have already encountered the Bach tensor in Theorem 1.14–(i).

Definition 1.16. Let (M, g) be a four-dimensional pseudo-Riemannian manifold. The *Bach tensor* is the symmetric $(0, 2)$-tensor field defined by

$$\mathfrak{B} = \mathrm{div}_2\,\mathrm{div}_4\, W + \frac{1}{2}W[\rho]\,.$$

The Bach tensor in dimension four is symmetric, trace free, divergence free and conformally invariant [95, 96]. Clearly locally conformally flat metrics are Bach-flat. Moreover, a straightforward calculation shows that the Bach tensor of any Einstein metric vanishes identically and Theorem 1.14–(i) shows that conformally Einstein metrics are Bach-flat as well.

It is important to emphasize that Bach flatness is a necessary but not sufficient condition for a manifold to be conformally Einstein. For instance the left-invariant Bach-flat metrics constructed by Abbena, Garbiero and Salamon in [1] fail to satisfy Equation (1.9) and thus they are not locally conformally Einstein. However, Bach flatness is also a sufficient condition to conformally Einstein in some special cases as shown by Derdziński [48] (see also [77]).

Theorem 1.17. *Let (M, g, J) be a four-dimensional positive definite Kähler manifold. Then it is conformally Einstein if and only the Bach tensor vanishes.*

An additional motivation for studying the Bach tensor with a different geometrical flavor is as follows. Let $\mathcal{W}$ be the quadratic curvature functional given by the L^2-norm of the conformal Weyl tensor

$$\mathcal{W} : g \mapsto \mathcal{W}(g) = \int_M \|W_g\|^2 \, dvol_g \,. \tag{1.14}$$

It quantifies the deflection of a Riemannian metric g from being locally conformally flat. A remarkable property is that $\mathcal{W}$ is conformally invariant in dimension four. Indeed, if $\overline{g} = e^{2\sigma} g$ and $n = 4$, then

$$\begin{aligned} \|\overline{W}\|^2 \, dvol_{\overline{g}} &= \overline{W}_{ijk\ell}\overline{W}^{ijk\ell} \, dvol_{\overline{g}} \\ &= e^{2\sigma} W_{ijk\ell} e^{2\sigma} e^{-8\sigma} W^{ijk\ell} e^{4\sigma} \, dvol_g = \|W\|^2 \, dvol_g \,. \end{aligned}$$

The Euler-Lagrange equations for $\mathcal{W}$-critical metrics were obtained by Bach [6], who showed that a metric is $\mathcal{W}$-critical if and only if $\mathfrak{B} = 0$.

Remark 1.18. In addition to conformally Einstein manifolds, half conformally flat metrics are also Bach-flat. Let M be an oriented four-dimensional manifold. Recall from the Hirzebruch signature formula that (see [48] and [5])

$$\tau[M] = \frac{1}{12\pi^2} \int_M (\|W^+\|^2 - \|W^-\|^2) dV, \tag{1.15}$$

where $\tau[M]$ denotes the Hirzebruch signature of M. Hence

$$\begin{aligned} \mathcal{W}(g) = \int_M \|W\|^2 \, dvol_g &= \int_M \left(\|W^+\|^2 + \|W^-\|^2\right) \, dvol_g \\ &= \pm 12\pi^2 \tau[M] + 2 \int_M \|W^\mp\|^2 \, dvol_g, \end{aligned}$$

which shows that half conformally flat metrics are extremal for the functional $\mathcal{W}$, and thus Bach-flat.

1.7 Affine geometry

An *affine manifold* is a pair (M, D) of a manifold M and an affine torsion free connection D. The Ricci tensor ρ^D is defined by setting $\rho^D(X, Y) := \mathrm{tr}(Z \to R^D(X, Z)Y)$. Since the Ricci tensor need not be symmetric in general, we introduce the symmetrization ρ_s^D and the skew-symmetrization ρ_{sk}^D by setting:

$$\begin{aligned}\rho_s^D(X,Y) &:= \tfrac{1}{2}\{\rho^D(X,Y)+\rho^D(Y,X)\},\\ \rho_{sk}^D(X,Y) &:= \tfrac{1}{2}\{\rho^D(X,Y)-\rho^D(Y,X)\}\,.\end{aligned} \tag{1.16}$$

An affine manifold (M,D) is *flat* if the associated curvature tensor R^D vanishes. In this case, there exists local coordinates where the Christoffel symbols are zero. Two connections D and $\overline{D}$ are said to be *projectively equivalent* if there is a 1-form ω such that $\overline{D}_X Y = D_X Y + \omega(X)Y + \omega(Y)X$ for all vector fields X,Y on M. One says that (M,D) is *projectively flat* if the connection D is projectively equivalent to a flat affine connection. Two-dimensional projectively flat affine structures are characterized as follows

Theorem 1.19. [87] *Let (M,D) be an affine surface. Then (M,D) is projectively flat if and only if ρ^D, $D\rho^D$ are totally symmetric.*

An affine manifold is *curvature recurrent* (resp. *Ricci recurrent*) if $DR^D = \omega\otimes R^D$ (resp. $D\rho^D = \omega\otimes\rho^D$) for some 1-form ω, and (M,D) is said to be *locally symmetric* if $DR^D = 0$. Since the curvature tensor of any affine surface is determined by the Ricci tensor as $R^D(X,Y)Z = \rho^D(X,Z)Y - \rho^D(Y,Z)X$, one has that curvature recurrent and Ricci recurrent conditions are equivalent in the two-dimensional case.

Curvature recurrent surfaces appear in a natural way in the study of affine connections with skew-symmetric Ricci tensor since any affine surface with skew-symmetric Ricci tensor is curvature recurrent around any point where the curvature is non-zero. We refer to Wong [104] for a classification of curvature recurrent surfaces. The following results will be used in this memory:

(i) Let (Σ,D) be a curvature recurrent affine surface with symmetric Ricci tensor of rank one. Then there exist local coordinates (x^1,x^2) where the unique non-zero component of D is given by
$$D_{\partial_{x^1}}\partial_{x^1} = a(x^1,x^2)\partial_{x^2}$$
for some smooth function $a(x^1,x^2)$ [104].

(ii) Let (Σ,D) be a curvature recurrent affine surface with non-degenerate symmetric Ricci tensor. Then, there is a pseudo-Riemannian metric g on M such that D is the Levi-Civita connection of g [104].

(iii) Let (Σ,D) be a curvature recurrent affine surface with skew-symmetric Ricci tensor. Then there exist local coordinates (x^1,x^2) where the unique non-zero components of D are given by
$$D_{\partial_{x^1}}\partial_{x^1} = -\partial_{x^1}\theta(x^1,x^2)\partial_{x^1}, \qquad D_{\partial_{x^2}}\partial_{x^2} = \partial_{x^2}\theta(x^1,x^2)\partial_{x^2},$$
for some smooth function $\theta(x^1,x^2)$ [49, 104].

1.7.1 Riemannian extensions

The existence of a parallel distribution on a Riemannian manifold (M, g), i.e., a distribution $\mathfrak{V}$ such that $\nabla\mathfrak{V} \subset \mathfrak{V}$, leads to a local de Rham decomposition. This local decomposition extends to the pseudo-Riemannian setting whenever the parallel distribution $\mathfrak{V}$ is non-degenerate. We say that a pseudo-Riemannian manifold is a *Walker manifold* if it admits a parallel and degenerate distribution $\mathfrak{V}$. Walker showed in [102] the existence of local coordinates where the metric takes a simple form as follows (see [19] for more information on Walker manifolds).

Theorem 1.20. [102] *Let M be an n-dimensional Walker manifold and let $\mathfrak{V}$ be an r-dimensional parallel and degenerate distribution. Then, there exist adapted coordinates on the Walker manifold M, $(x^1, \ldots, x^{n-r}, x^{n-r+1}, \ldots, x^n)$, such that the metric is given by*

$$(g_{ij}) = \begin{pmatrix} B & H & \mathrm{Id}_r \\ H^t & A & 0 \\ \mathrm{Id}_r & 0 & 0 \end{pmatrix},$$

where Id_r is the identity matrix of order r and A, B, H are matrices whose coefficients are functions of the coordinates verifying:

(i) A and B are symmetric matrices of order $(n-2r) \times (n-2r)$ and $r \times r$, respectively. H is a matrix of order $r \times (n-2r)$ matrix and H^t denotes its transposed.

(ii) A and H do not depend on the coordinates $(x^{n-r+1}, \ldots, x^n)$.

Moreover, the null parallel distribution $\mathfrak{V}$ is locally generated by the coordinate vector fields $\{\partial_{x^{n-r+1}}, \ldots, \partial_{x^n}\}$.

The canonical form in the previous theorem simplifies if the parallel distribution has full dimension and the manifold has even dimension $n = 2m$. In this case, there exist Walker coordinates $(x^1, \ldots, x^m, x_{1'}, \ldots, x_{m'})$ such that the metric is given by the matrix (see [19]):

$$(g_{ij}) = \begin{pmatrix} B & \mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix}, \tag{1.17}$$

where B is an $m \times m$ symmetric matrix whose entries are functions of the coordinates. When the metric is in the form (1.17) the Christoffel symbols and the curvature operator are given as follows:

Lemma 1.21. [29] *Let $(M, g, \mathfrak{V})$ be a Walker manifold of dimension $n = 2m$, where $\dim(\mathfrak{V}) = m$. Then, the non-zero Christoffel symbols are given by*

$$\Gamma_{ij}{}^{k} = -\tfrac{1}{2}\partial_{x_{k'}} g_{ij}, \quad \Gamma_{i'j}{}^{k'} = \tfrac{1}{2}\partial_{x_{i'}} g_{jk},$$

$$\Gamma_{ij}{}^{k'} = -\tfrac{1}{2}(\partial_{x^k} g_{ij} + \partial_{x^j} g_{ik} + \partial_{x^i} g_{jk} + \textstyle\sum_s g_{ks}\partial_{x_{s'}} g_{ij}),$$

where the sum is taken for all $s = 1, \ldots, m$.

For any Walker manifold the curvature tensor satisfies the following conditions (see [50]):

$$R(\mathfrak{V},\mathfrak{V}^\perp,\cdot,\cdot)=0,\quad R(\mathfrak{V},\mathfrak{V},\cdot,\cdot)=0,\quad R(\mathfrak{V}^\perp,\mathfrak{V}^\perp,\mathfrak{V},\cdot)=0\,.$$

Moreover, the non-zero components of the curvature tensor are as follows.

Lemma 1.22. [29] *Let $(M,g,\mathfrak{V})$ be a Walker manifold of dimension $n=2m$, where $\dim(\mathfrak{V})=m$. Then, the non-zero components of the curvature tensor of type (1,3) are given by (up to symmetries):*

$$R^h_{jik}=-\tfrac{1}{2}(\partial_{x^i}\partial_{x_{h'}}g_{jk}-\partial_{x^j}\partial_{x_{h'}}g_{ik})-\tfrac{1}{4}(\partial_{x_{s'}}g_{ik}\partial_{x_{h'}}g_{js}-\partial_{x_{s'}}g_{jk}\partial_{x_{h'}}g_{is}),$$

$$\begin{aligned}R^{h'}_{jik}=&-\tfrac{1}{2}(\partial_{x^j}\partial_{x^k}g_{ih}-\partial_{x^j}\partial_{x^h}g_{ik}+\partial_{x^i}\partial_{x^h}g_{jk}-\partial_{x^i}\partial_{x^k}g_{jh})\\ &-\tfrac{1}{4}\Big\{\partial_{x_{s'}}g_{ik}(\partial_{x^h}g_{js}-\partial_{x^s}g_{jh}-\partial_{x^j}g_{sh}-g_{ht}\partial_{x_{t'}}g_{js})\\ &\quad-\partial_{x_{s'}}g_{jk}(\partial_{x^h}g_{is}-\partial_{x^s}g_{ih}-\partial_{x^i}g_{sh}-g_{ht}\partial_{x_{t'}}g_{is})\\ &\quad-\partial_{x_{s'}}g_{jh}(\partial_{x^s}g_{ik}-\partial_{x^k}g_{is}-\partial_{x^i}g_{ks}-g_{st}\partial_{x_{t'}}g_{ik})\\ &\quad+\partial_{x_{s'}}g_{ih}(\partial_{x^s}g_{jk}-\partial_{x^k}g_{js}-\partial_{x^j}g_{ks}-g_{st}\partial_{x_{t'}}g_{jk})\\ &\quad+2\partial_{x^j}(g_{hs}\partial_{x_{s'}}g_{ik})-2\partial_{x^i}(g_{hs}\partial_{x_{s'}}g_{jk})\Big\},\end{aligned}$$

$$R^h_{ji'k}=-\tfrac{1}{2}\partial_{x_{i'}}\partial_{x_{h'}}g_{jk},$$

$$\begin{aligned}R^{h'}_{jik}=&-\tfrac{1}{2}(\partial_{x^h}\partial_{x_{i'}}g_{jk}-\partial_k\partial_{x_{i'}}g_{jh})\\ &-\tfrac{1}{4}(\partial_{x_{s'}}g_{jk}\partial_{x_{i'}}g_{sh}+\partial_{x_{s'}}g_{jh}\partial_{x_{i'}}g_{sk}-2\partial_{x_{i'}}(g_{hs}\partial_{x_{s'}}g_{jk})),\end{aligned}$$

$$R^{h'}_{jik'}=-\tfrac{1}{2}(\partial_{x^j}\partial_{x_{k'}}g_{ih}-\partial_{x^i}\partial_{x_{k'}}g_{jh})-\tfrac{1}{4}(\partial_{x_{k'}}g_{is}\partial_{x_{s'}}g_{jh}-\partial_{x_{k'}}g_{js}\partial_{x_{s'}}g_{ih}),$$

$$R^{h'}_{ji'k'}=\tfrac{1}{2}\partial_{x_{i'}}\partial_{x_{k'}}g_{jh},$$

where $1\le s\le m$.

The study of the geometry in dimension four is central in this thesis. In this case, we take coordinates $(x^1,x^2,x_{1'},x_{2'})$ such that the metric (1.17) takes the form:

$$g=\begin{pmatrix}a&c&1&0\\c&b&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

where a, b and c are functions in the coordinates $(x^1,x^2,x_{1'},x_{2'})$.

A special class of Walker metrics is given by the Riemannian extensions and their modifications. A feature of these metrics is that they provide a link between affine and pseudo-Riemannian geometry. Hence one may use pseudo-Riemannian techniques to investigate affine problems and vice versa.

The Riemannian extensions are a family of distinguished metrics on the cotangent bundle of an affine manifold. Let T^*M be the cotangent bundle of an m-dimensional manifold M and let $\pi\colon T^*M \to M$ be the projection. Let $\tilde{p} = (p, \omega)$ denote a point of T^*M, where $p \in M$ and $\omega \in T_p^*M$. Local coordinates $(x^1, \dots, x^m)$ in an open set $\mathcal{U}$ of M induce local coordinates $(x^1, \dots, x^m, x_{i'}, \dots, x_{m'})$ in $\pi^{-1}(\mathcal{U})$, where one sets for any 1-form

$$\omega = \sum x_{i'} dx^i .$$

For each vector field X on M, *the evaluation* of X is the real valued function $\iota X\colon T^*M \to \mathbb{R}$ given by $\iota X(p, \omega) = \omega(X_p)$. Setting $X = X^i \partial_{x^i}$ one has

$$\iota X(x^i, x_{i'}) = \sum x_{i'} X^i .$$

Vector fields on T^*M are characterized by their action on evaluations ιX and one defines *the complete lift* to T^*M of a vector field X on M by $X^c(\iota Z) = \iota[X, Z]$ for all vector fields $Z \in \mathfrak{X}(M)$. Moreover, $(0, s)$-tensor fields on T^*M are characterized by their action on complete lifts of vector fields on M. Hence, for any $(1, 1)$-tensor field T on M, its evaluation is the 1-form ιT on T^*M characterized by $\iota T(X^c) = \iota(TX)$. In induced local coordinates one has the expression $\iota T = x_{k'} T^k{}_i \, dx^i$.

Considering a torsion free connection D on M, the cotangent bundle T^*M can be equipped with a pseudo-Riemannian metric g_D of signature (m, m), which is called the *Riemannian extension* of D [92], characterized by

$$g_D(X^c, Y^c) = -\iota(D_X Y + D_Y X),$$

where X^c, Y^c denote the complete lifts to T^*M of vector fields X, Y on M. In induced local coordinates $(x^1, \dots, x^m, x_{i'}, \dots, x_{m'})$ on T^*M, the Riemannian extension has the expression

$$g_D = 2\, dx^i \circ dx_{i'} - 2x_{k'}\, {}^D\Gamma_{ij}{}^k dx^i \circ dx^j, \tag{1.18}$$

where ${}^D\Gamma_{ij}{}^k$ are the Christoffel symbols of D with respect to $(x^1, \dots, x^m)$ on M and "$\circ$" denotes the symmetric product $\omega_1 \circ \omega_2 := \frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)$. In matrix form:

$$g_D = \begin{pmatrix} -2x_{k'}\, {}^D\Gamma_{ij}{}^k & \mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix} .$$

Riemannian extensions are a particular class of Walker metrics with parallel degenerate distribution $\mathfrak{V} = \ker(\pi_*) = \operatorname{span}\{\partial_{x_{1'}}, \dots, \partial_{x_{m'}}\}$.

Modified Riemannian extensions

A generalization of Riemannian extensions can be constructed as follows. Consider (M, D) an n-dimensional affine manifold where D is a torsion free connection on M. Let Φ be a symmetric

$(0,2)$-tensor field on M. Then the *deformed Riemannian extension*, $g_{D,\Phi} = g_D + \pi^*\Phi$, is a first perturbation of the Riemannian extension and is characterized by

$$g_{D,\Phi}(X^c, Y^c) = -\iota(D_X Y + D_Y X) + \Phi(X,Y) \circ \pi,$$

where X^c and Y^c denote the complete lifts to T^*M of vector fields X, Y on M. In local coordinates one has

$$g_{D,\Phi} = 2\, dx^i \circ dx_{i'} - \{2x_{k'}{}^D\Gamma_{ij}{}^k - \Phi_{ij}\} dx^i \circ dx^j\,.$$

A second perturbation is as follows. Let $T = T^k{}_i\, dx^i \otimes \partial_{x^k}$ and $S = S^k{}_i\, dx^i \otimes \partial_{x^k}$ be $(1,1)$-tensor fields on M. The evaluations ιT and ιS define 1-forms on T^*M. The *modified Riemannian extension*, $g_{D,\Phi,T,S}$ is the neutral signature metric on T^*M defined by (see [29])

$$g_{D,\Phi,T,S} = \iota T \circ \iota S + g_D + \pi^*\Phi, \tag{1.19}$$

where Φ is a symmetric $(0,2)$-tensor field on M. In local coordinates one has

$$\begin{aligned} g_{D,\Phi,T,S} &= 2\, dx^i \circ dx_{i'} \\ &\quad + \{\tfrac{1}{2} x_{r'} x_{s'} (T^r{}_i S^s{}_j + T^r{}_j S^s{}_i) - 2x_{k'}{}^D\Gamma_{ij}{}^k + \Phi_{ij}\} dx^i \circ dx^j\,. \end{aligned}$$

Modified Riemannian extensions are characterized among Walker metrics by their curvature as follows (see [2]):

(i) A Walker manifold satisfies $R(\mathfrak{V}, \cdot)\mathfrak{V} = 0$ if and only if it is locally a deformed Riemannian extension.

(ii) A Walker manifold satisfies $(\nabla_{\mathfrak{V}} R)(\mathfrak{V}, \cdot)\mathfrak{V} = 0$ if and only if it is locally a modified Riemannian extension. Hence, locally symmetric Walker metrics are modified Riemannian extensions.

The case when T is a multiple of the identity ($T = c\,\mathrm{Id}$, $c \neq 0$) and $S = \mathrm{Id}$ is of special interest. It was shown in [29] that for any affine manifold (M, D), the modified Riemannian extension $g_{D,\Phi,c\,\mathrm{Id},\mathrm{Id}}$ is an Einstein metric on T^*M if and only if the deformation tensor Φ is the symmetric part of the Ricci tensor of (M, D).

Theorem 1.23. *The modified Riemannian extension $g_{D,\Phi,c\,\mathrm{Id},\mathrm{Id}}$ on the cotangent bundle T^*M of an m-dimensional affine manifold (M, D) is Einstein if and only if $\Phi = \frac{4}{c(m-1)}\rho_s^D$, with $c \neq 0$.*

Proof. Let $g_{D,\Phi,c\,\mathrm{Id},\mathrm{Id}} = c\iota\,\mathrm{Id} \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi$ be a modified Riemannian extension and let τ be its scalar curvature. The trace free Ricci tensor is given by $\rho_0 = \rho - \frac{\tau}{2m} g_{D,\Phi,c,\mathrm{Id}}$ and determined by

$$\rho_0 = 2\pi^*\rho_s^D - \frac{1}{2}c(m-1)\pi^*\Phi,$$

from where the result follows. □

A slight generalization of the modified Riemannian extension allowed a complete description of self-dual Walker metrics as follows.

Theorem 1.24. [29,51] *A four-dimensional Walker metric is self-dual if and only if it is locally isometric to the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D), with metric tensor*

$$g = \iota X(\iota\,\mathrm{Id} \circ \iota\,\mathrm{Id}) + \iota\,\mathrm{Id} \circ \iota T + g_D + \pi^*\Phi,$$

where X, T, D and Φ are a vector field, a $(1,1)$-tensor field, a torsion free affine connection and a symmetric $(0,2)$-tensor field on Σ, respectively.

1.8 The Ricci flow: Ricci solitons

The Ricci flow was introduced by Hamilton in [64] aimed to solve the Poincaré conjecture: *any three-dimensional closed and simply connected manifold is homeomorphic to $\mathbb{S}^3$*. The Ricci flow is defined by the evolution equation

$$\frac{\partial}{\partial t} g(t) = -2\rho_{g(t)}, \tag{1.20}$$

where $g(t)$ is a 1-parameter family of Riemannian metrics on M. For any C^∞ metric g_0 on a closed manifold M, there is a unique solution $g(t)$, $t \in [0, \varepsilon)$, to the Ricci flow equation for some $\varepsilon > 0$, with $g(0) = g_0$. The idea of the Ricci flow is to deform the original metric $g(0)$ into a distinguished one by its Ricci curvature (see for example [42]). For example, if M is two-dimensional, the Ricci flow deforms a metric conformally to one of constant curvature and thus gives a proof of the two-dimensional uniformization theorem [38].

The first example of solution to the Ricci flow equation is given by Einstein metrics, where the solution is

$$g(t) = (1 - 2\lambda t) g_0, \quad \text{where} \quad \begin{cases} t \in (-\infty, \frac{1}{2\lambda}) & \text{if} \quad \lambda > 0, \\ t \in (\frac{1}{2\lambda}, \infty) & \text{if} \quad \lambda < 0, \\ t \in (-\infty, \infty) & \text{if} \quad \lambda = 0, \end{cases}$$

for an Einstein initial metric $g(0)$ such that $\rho_{g(0)} = \lambda g(0)$. Moreover, in any of the cases $g(0)$ remains invariant modulo homotheties.

Generalizing the behaviour of Einstein metrics, and allowing the initial metric to change not only by homotheties but also by diffeomorphisms, a solution $g(t)$ of the Ricci flow is said to be *self-similar* if there exists a positive function $\sigma(t)$ and a one-parameter group of diffeomorphisms $\psi(t) : M \to M$ such that

$$g(t) = \sigma(t)\psi(t)^* g(0)\,. \tag{1.21}$$

Remark 1.25. If Equation (1.21) defines a solution of the Ricci flow, then differentiating (1.21) yields

$$-\,2\,\rho(g(t)) = \sigma'(t)\psi(t)^* g_0 + \sigma(t)\psi(t)^*(\mathcal{L}_X g_0), \tag{1.22}$$

where $g_0 = g(0)$, X is the time-dependent vector field such that $X(\psi(t)(p)) = \frac{d}{dt}(\psi(t)(p))$ for any $p \in M$, and $\sigma' = \frac{d\sigma}{dt}$.

Since $\rho(g(t)) = \psi(t)^*\rho(g_0)$, one can drop the pull-backs in Equation (1.22) and get:

$$-\,2\,\rho(g_0) = \sigma'(t)g_0 + \mathcal{L}_{\tilde{X}(t)}g_0, \tag{1.23}$$

where $\tilde{X}(t) = \sigma(t)X(t)$. Put $\lambda = -\frac{1}{2}\,\dot{\sigma}(0)$ and $X_0 = \frac{1}{2}\,\tilde{X}(0)$, so that Equation (1.23) becomes

$$-2\,\rho(g_0) = -2\lambda\,g_0 + 2\,\mathcal{L}_{X_0}g_0 \quad \text{at} \quad t = 0\,.$$

This shows that for any self-similar solution of the Ricci flow there exists a vector field on M satisfying

$$\mathcal{L}_X g + \rho = \lambda\,g\,.$$

Conversely, let X be a complete vector field on a pseudo-Riemannian manifold (M, g) and denote by $\psi(t) : M \to M$ with $\psi(0) = \mathrm{Id}_M$ the family of diffeomorphisms generated by X according to

$$\frac{\partial}{\partial t}\psi(t)(p) = \frac{1}{1-2\lambda t}X(\psi(t)(p)),$$

which is defined for all $t < \frac{1}{2\lambda}$ if $\lambda > 0$ and for all $t > \frac{1}{2\lambda}$ if $\lambda < 0$. Considering now the one-parameter family of metrics

$$g(t) = (1-2\lambda t)\psi(t)^* g,$$

one has

$$\begin{aligned} \tfrac{\partial}{\partial t}\,g(t) &= -2\lambda\,\psi(t)^* g + (1-2\lambda)\psi(t)^*\left(\mathcal{L}_{\frac{1}{1-2\lambda t}X}g\right) \\ &= \psi(t)^*\left(-2\lambda\,g + \mathcal{L}_{X(\psi(t)(p))}g\right)\,. \end{aligned}$$

Now, if $\mathcal{L}_X g + \rho = \lambda g$, then

$$\frac{\partial}{\partial t}\,g(t) = \psi(t)^*(-2\,\rho) = -2\,\psi(t)^*\rho = -2\,\rho(\psi(t)^* g) = -2\,\rho(g(t)),$$

which shows that $g(t)$ is a solution of the Ricci flow.

The above motivates the following definition.

Definition 1.26. A triple (M, g, X) where (M, g) is a pseudo-Riemannian manifold and X is a vector field on M satisfying

$$\mathcal{L}_X g + \rho = \lambda\,g \tag{1.24}$$

is called a *Ricci soliton*. A Ricci soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

A Ricci soliton whose vector field can be written as the gradient of some function $f\colon M \to \mathbb{R}$ is called a *gradient Ricci soliton*. In this case, we may compute $\mathcal{L}_X g_0 = 2\,\mathrm{Hes}_{g_0}(f)$ and we have

$$\mathrm{Hes}_{g_0}(f) + \rho(g_0) = \lambda g_0\,. \tag{1.25}$$

We call the function f the *potential function*. If the potential function is constant, then the gradient Ricci soliton is trivial since Equation (1.25) reduces to the Einstein equation. In consequence, gradient Ricci solitons are natural extensions of Einstein manifolds.

Gradient Ricci solitons codify geometric information of the manifold in terms of the Ricci curvature and the second fundamental form of the level sets of the potential function f. Moreover, they appear as singularities of the Ricci flow [41], so it is important to understand the geometry and topology of gradient Ricci solitons and their classification.

As a result of several works, the classification of complete locally conformally flat gradient shrinking Ricci solitons has been finally achieved [83, 94]. Since the Ricci tensor determines the curvature tensor in the locally conformally flat case, it follows that a locally conformally flat gradient Ricci soliton, not necessarily complete, is locally a warped product in the Riemannian case [55]. Four-dimensional half conformally flat (i.e., self-dual or anti-self-dual) gradient Ricci solitons have been investigated in the Riemannian and neutral signature cases [16,39]. While they are locally conformally flat in the Riemannian situation, neutral signature allows other examples given by Riemannian extensions of affine gradient Ricci solitons.

On the other hand, since Bach-flat metrics contain half conformally flat and conformally Einstein metrics as special cases, a natural problem is to classify Bach-flat gradient Ricci solitons. The Riemannian case was investigated in the shrinking and steady cases in [34, 36]. In all situations the Bach-flat condition reduces to the locally conformally flat one under some natural assumptions. In Chapter 4 we construct new examples of Bach-flat gradient Ricci solitons in the neutral signature case where the corresponding potential functions have degenerate level set hypersurfaces and their underlying structure is never locally conformally flat, in sharp contrast with the Riemannian situation. These metrics are realized as modified Riemannian extensions on the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D).

Self-dual gradient Ricci solitons

Let (M, g, f) be a gradient Ricci soliton. The level set hypersurfaces of the potential function play a distinguished role in analyzing the geometry of gradient Ricci solitons. Hence we say that the soliton is *non-isotropic* if ∇f is a nowhere null vector (i.e., $\|\nabla f\|^2 \neq 0$), and that the soliton is *isotropic* if $\|\nabla f\|^2 = 0$, but $\nabla f \neq 0$.

Non-isotropic gradient Ricci solitons lead to local warped product decompositions in the locally conformally flat and half conformally flat cases, and their geometry resembles the Riemannian situation [16, 17]. The isotropic case is, however, in sharp contrast with the positive definite setting since ∇f gives rise to a Walker structure. Self-dual gradient Ricci solitons which are not locally conformally flat are isotropic and steady. Moreover, they are described in terms of Riemannian extensions as follows.

Theorem 1.27. [16] *Let (M, g, f) be a four-dimensional self-dual gradient Ricci soliton of neutral signature which is not locally conformally flat. Then (M, g) is locally isometric to the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) equipped with a deformed Riemannian extension $g_{D,\Phi} = g_D + \pi^*\Phi$.*

Moreover any such gradient Ricci soliton is steady and the potential function is given by

$f = h \circ \pi$ for some $h \in \mathcal{C}^\infty(\Sigma)$ satisfying the affine gradient Ricci soliton equation

$$\mathrm{Hes}_h^D + 2\rho_s^D = 0, \tag{1.26}$$

for any symmetric $(0,2)$-tensor field Φ on Σ.

An affine surface (Σ, D) is an *affine gradient Ricci soliton* if there is a function $h \in \mathcal{C}^\infty(\Sigma)$ satisfying Equation (1.26).

The previous result relates affine geometry of (Σ, D) and pseudo-Riemannian geometry of $(T^*\Sigma, g_{D,\Phi})$, allowing the construction of an infinite family of steady gradient Ricci solitons on $T^*\Sigma$ for any initial data (Σ, D, h) satisfying Equation (1.26). It is important to remark here that the existence of affine gradient Ricci solitons imposes some restrictions on (Σ, D), as shown in [18] in the locally homogeneous case. Moreover, note that Equation (1.26) does not depend on the deformation tensor Φ. In consequence any affine gradient Ricci soliton gives rise to an infinite family of self-dual gradient Ricci solitons just varying the deformation tensor Φ.

1.9 Homogeneous spaces

Homogeneity is central in differential geometry. In pseudo-Riemannian geometry, roughly speaking, homogeneity means that for any two points there exists an isometry sending one point to another. Thus geometry is the same at each point. In affine geometry, the notion of homogeneity means that for any two points there exists an affine transformation sending one point into the other. It is important to emphasize that a pseudo-Riemannian manifold may be affine homogeneous for the Levi-Civita connection but not homogeneous. In this section, we treat homogeneity from the point of view of pseudo-Riemannian and affine geometry.

Riemannian homogeneous spaces

A connected Riemannian manifold (M, g) is said to be *homogeneous* if the group of isometries acts transitively on M. This means that if $p, q \in M$ are any two points then there exists an isometry φ of (M, g) such that $\varphi(p) = q$. Note that, in this case, the connected component of the identity of the isometry group acts transitively on M as well. This definition of homogeneity is equivalent to the existence of a connected Lie group G and a smooth map

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ (q,p) & \mapsto & q\,p = L_q(p) \end{array}$$

such that for all $q_1, q_2 \in G$ it satisfies:

(i) L_{q_1} is an isometry of (M, g).

(ii) $L_{q_1} L_{q_2} = L_{q_1 q_2}$.

(iii) For $p_1, p_2 \in M$ there exists an element $q_1 \in G$ such that $L_{q_1}(p_1) = p_2$.

Now, we suppose that G acts effectively on M, i.e., L_q is the identity transformation of M if and only if q is the identity element $e \in G$. Note that we can always replace G by the quotient group G/K, where K is the kernel of the map $q \mapsto L_q$ of G in the isometry group. Thus, if G is a connected Lie group which acts on (M, g) as a transitive and effective group of isometries, then G can be identified with a Lie subgroup of the isometry group.

Let $p \in M$ and let $H = \{q \in G |\ qp = p\}$ be the isotropy subgroup of p. Then M is diffeomorphic to the quotient G/H and we have the canonical projection

$$\pi \colon G \longrightarrow G/H\,.$$

It is a principal fiber bundle over M with structure group H. The subgroup H is closed but not necessarily connected. A Riemannian metric g on G/H is called *G-invariant* if the action $t_q \colon G/H \to G/H$ with $t_q(sH) = qsH$ is an isometry, for all $q \in G$. In this case $(G/H, g)$ is called a homogeneous Riemannian space. One says that (M, g) is *locally homogeneous* if for each $p, q \in M$, there exist neighborhoods $\mathcal{U}$ of p and $\mathcal{V}$ of q, and a local isometry $\varphi \colon \mathcal{U} \to \mathcal{V}$ such that $\varphi(p) = q$.

Simply connected homogeneous Riemannian manifolds of dimension 2 are symmetric. Three-dimensional complete and simply connected homogeneous Riemannian manifolds are either symmetric spaces or Lie groups with a left-invariant Riemannian metric [97] (see [81] for a modern exposition and [23] for an extension to the three-dimensional Lorentzian setting). The same result holds true in the four-dimensional case, as shown by Bérard-Bergery:

Theorem 1.28. [9] *Let (M, g) be a four-dimensional complete and simply connected Riemannian homogeneous manifold. Then either (M, g) is symmetric or it is isometric to a Lie group with a left-invariant metric.*

In particular, either M is one of the groups $\widetilde{SL(2, \mathbb{R})} \times \mathbb{R}$, $SU(2) \times \mathbb{R}$ or it is a solvable Lie group. Four-dimensional solvable Lie algebras are obtained as extensions of the three-dimensional unimodular Lie algebras: the abelian Lie algebra $\mathfrak{r}^3$, the Heisenberg algebra $\mathfrak{h}^3$, the Poincaré algebra $\mathfrak{e}(1, 1)$ of the group of rigid motions of the Minkowski 2-space and the Euclidean algebra $\mathfrak{e}(2)$ of the group of rigid motions of the Euclidean 2-space. Moreover, the solvable and simply connected four-dimensional Lie groups are the following:

(i) The non-trivial semi-direct products $\mathbb{R} \ltimes E(2)$ and $\mathbb{R} \ltimes E(1, 1)$.

(ii) The semi-direct products $\mathbb{R} \ltimes \mathbb{R}^3$.

(iii) The non-nilpotent semi-direct products $\mathbb{R} \ltimes H^3$, where H^3 is the Heisenberg group.

Let (M, g) be a connected n-dimensional Riemannian manifold. Further let $M = G/H$, where G is a group of isometries of M acting transitively and effectively on M. We denote by H the isotropy group at a point $p \in M$. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{h}$ the Lie algebra of H.

Definition 1.29. $M = G/H$ is called *reductive* if there exists a vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{1.27}$$

where $\mathfrak{m}$ is the $\mathrm{Ad}(H)$-invariant subspace on $\mathfrak{g}$, i.e., $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ (see [71, 100]).

Note that when H is connected $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and that H is always connected if M is simply connected. Moreover, if H is compact, the decomposition always exists since we can take $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{g}$.

If G/H is a homogeneous reductive space which admits a pseudo-Riemannian metric with G acting by isometries, then the curvature tensor R takes a simpler form, which facilitates the study of the geometry of these spaces. It is important to emphasize that reductivity is not an intrinsic property of (M, g) but of the description of M as a coset space G/H. For example, in neutral or Lorentzian signature reductive decompositions may not exist. Fels and Renner [54] classified four-dimensional non-reductive homogeneous spaces, and their work will be essential in the development of Chapter 3.

Half conformally flat homogeneous manifolds

De Smedt and Salamon [47] classified half conformally flat left-invariant Riemannian metrics on Lie groups, showing the following.

Theorem 1.30. [47] *A four-dimensional homogeneous manifold is strictly anti-self-dual if and only if it is a complex space form or a simply connected Lie group G_α corresponding to the solvable Lie algebra $\mathfrak{g}_\alpha$ given by*

$$[e_1, e_2] = e_2 - \alpha e_3,\ [e_1, e_3] = \alpha e_2 + e_3,\ [e_1, e_4] = 2e_4,\ [e_2, e_3] = -e_4, \tag{1.28}$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

Note that the choice of orientation has no role at all in Theorem 1.30 so that one may replace anti-self-duality by self-duality.

Homogeneous affine surfaces

We say that an affine surface (Σ, D) is locally homogeneous if given any two points p and q of Σ, there exists a local diffeomorphism Ψ intertwining p and q such that $\Psi^* D = D$. The following result was proved by Opozda [90] (see [22] for a different proof). It is fundamental in the subject.

Theorem 1.31. [90] *Let (Σ, D) be a locally homogeneous affine surface which is not flat. Then at least one of the following three possibilities holds which describe the local geometry:*

($\mathcal{A}$) There exists a coordinate atlas such that the Christoffel symbols ${}^D\Gamma_{ij}{}^k$ are constant.

($\mathcal{B}$) There exists a coordinate atlas such that the Christoffel symbols have the form

$${}^D\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k,$$

for $C_{ij}{}^k$ constant and $x^1 > 0$.

($\mathcal{C}$) D is the Levi-Civita connection of a metric of constant Gauss curvature.

Surfaces of Type $\mathcal{A}$ and Type $\mathcal{B}$ have different geometric properties. For example, the Ricci tensor of any Type $\mathcal{A}$ surface is symmetric and this may fail for a Type $\mathcal{B}$ surface. Indeed, the Ricci tensor of a Type $\mathcal{B}$ surface may even be skew-symmetric; this is closely related to the existence of non-flat affine Osserman structures [60]. The geometry of Type $\mathcal{B}$ surfaces is not so rigid as that of the Type $\mathcal{A}$ surfaces. On the other hand, any Type $\mathcal{A}$ surface is projectively flat and again this may fail for a Type $\mathcal{B}$ surface.

Remark 1.32. The different types $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are not exclusive (see [18]).

(i) There are no non-flat surfaces which are both of Type $\mathcal{A}$ and Type $\mathcal{C}$.

(ii) The only non-flat surfaces which are of both Type $\mathcal{B}$ and Type $\mathcal{C}$ are the hyperbolic plane and the Lorentzian analogue realized as the half plane models with metrics $ds^2 = \frac{1}{(x^1)^2}\{(dx^1)^2 + (dx^2)^2\}$ and $ds^2 = \frac{1}{(x^1)^2}\{(dx^1)^2 - (dx^2)^2\}$, respectively.

(iii) A Type $\mathcal{B}$ affine surface is also of Type $\mathcal{A}$ if and only if it is flat or the Christoffel symbols satisfy ${}^D\Gamma_{12}{}^1 = {}^D\Gamma_{22}{}^1 = {}^D\Gamma_{22}{}^2 = 0$.

Part I

Conformally Einstein homogeneous manifolds

Chapter 2
Conformally Einstein homogeneous Riemannian manifolds

The existence of conformally Einstein metrics amounts to understand a rather complicated PDE as Brinkmann showed in [14]. Homogeneity allows a reduction of the problem to a system of algebraic equations and our purpose in this chapter is to provide a complete description of homogeneous conformally Einstein metrics in dimension four. Previous work of Jensen [70] showed that four-dimensional homogeneous Einstein metrics are symmetric and thus locally a product of two surfaces of constant sectional curvature or a real or a complex space form. Our main result provides a classification of conformally Einstein and Bach-flat homogeneous four-manifolds. In this chapter we report on work investigated in [28].

Theorem 2.1. *Let (M, g) be a four-dimensional complete and simply connected conformally Einstein homogeneous Riemannian manifold. Then (M, g) is locally symmetric or otherwise it is homothetic to one of the Lie groups determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \tfrac{1}{4}e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \tfrac{1}{4}e_3 .$$

(ii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$[e_1, e_2] = e_3, \quad [e_4, e_1] = e_1 - \alpha e_2, \quad [e_4, e_2] = \alpha e_1 + e_2, \quad [e_4, e_3] = 2e_3 .$$

(iii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha + 1)^2 e_2, \quad [e_4, e_3] = \alpha^2 e_3, \qquad \alpha > 1 .$$

Here $\{e_1, \dots, e_4\}$ is an orthonormal basis. Moreover, the Lie groups $(G_\alpha, \langle \cdot , \cdot \rangle)$ in Assertion (ii) are half conformally flat.

Remark 2.2. Following the notation in [4], the underlying Lie algebras in Theorem 2.1 are $\mathfrak{r}'_{4,1,\frac{1}{4\alpha}}$ if $\alpha \neq 0$ or $\mathfrak{r}_{4,\frac{1}{4},\frac{1}{4}}$ if $\alpha = 0$ in Assertion (i), $\mathfrak{d}'_{4,\frac{1}{\alpha}}$ if $\alpha \neq 0$ or $\mathfrak{d}_{4,\frac{1}{2}}$ if $\alpha = 0$ in Assertion (ii) and $\mathfrak{r}_{4,(\alpha+1)^2,\alpha^2}$ in Assertion (iii).

Remark 2.3. Recall that if two Riemannian metrics are conformally equivalent, $\widetilde{g} = e^{2\sigma} g$, then their Weyl tensors of type $(1, 3)$ coincide and thus $\widetilde{W} = e^{2\sigma} W$ for the Weyl tensors of type $(0, 4)$. The converse does not hold in general, but it is true in dimension four on any open set where $W \neq 0$ (see [63]). Furthermore, if the conformal manifolds (M, g) and $(M, \widetilde{g})$ are both homogeneous, then $\|W\|^2$ and $\|\widetilde{W}\|^2$ are constant and, since $\|\widetilde{W}\|^2 = e^{-4\sigma}\|W\|^2$, either g and $\widetilde{g}$ are homothetic or otherwise both metrics are locally conformally flat. We will make extensively use of these facts to obtain the different homothety classes in Theorem 2.1.

Theorem 2.4. *Let (M, g) be a four-dimensional complete and simply connected strictly Bach-flat homogeneous Riemannian manifold. Then (M, g) is homothetic to one of the Lie groups determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{e}(1,1)$ given by

$$[e_2, e_3] = e_1, \qquad [e_1, e_3] = (2+\sqrt{3})\, e_2,$$
$$[e_4, e_1] = \sqrt{6+3\sqrt{3}}\, e_1, \qquad [e_4, e_2] = \sqrt{6+3\sqrt{3}}\, e_2 \,.$$

(ii) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$[e_1, e_2] = e_3, \qquad [e_4, e_1] = \tfrac{1}{4}\sqrt{7-3\sqrt{5}}\, e_1,$$
$$[e_2, e_4] = \tfrac{1}{4}\sqrt{7+3\sqrt{5}}\, e_2, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}}\, e_3 \,.$$

Here $\{e_1, \dots, e_4\}$ is an orthonormal basis.

Remark 2.5. The underlying Lie algebras in Theorem 2.4 are $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ in Case (i) and $\mathfrak{d}_{4,\mu}$ with $\mu = \frac{1}{10}(5 - 3\sqrt{5})$ in Case (ii), following again the notation in [4].

The chapter is organized as follows. In Section 2.1 we give the coordinate expressions of the metrics as well as the underlying structure of conformally Einstein and strictly Bach-flat manifolds. In Section 2.2 locally symmetric Bach-flat four-manifolds are shown to be either Einstein or locally conformally flat (cf. Lemma 2.8). Hence the analysis of the Bach-flat condition is considered separately for the different four-dimensional Lie groups through Sections 2.4–2.7. The components of the Bach tensor give polynomials in the corresponding structure constants. Therefore, determining the Bach-flat Lie groups equals to solve some rather complicated polynomial systems. We make use of Gröbner bases theory previously introduced in Section 2.3. The proofs of Theorems 2.1 and 2.4 are completed in Section 2.8 and in Section 2.9. Finally in Section 2.10, as an application of the previous results, we determine the four-dimensional homogeneous Bach-flat Ricci solitons.

2.1 Coordinate expressions

As a matter of notation, for a given orthonormal basis $\{e_1, \dots, e_4\}$ on a Lie algebra $\mathfrak{g}$, we denote by $\{E_i^\pm\}$ the corresponding orthonormal basis of self-dual and anti-self-dual two-forms in $\Lambda^2_\pm(\mathfrak{g})$ given by:

$$E_1^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^2 \pm e^3 \wedge e^4\right),$$
$$E_2^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^3 \mp e^2 \wedge e^4\right),$$
$$E_3^\pm = \frac{1}{\sqrt{2}}\left(e^1 \wedge e^4 \pm e^2 \wedge e^3\right),$$

where $\{e^i\}$ is the dual basis of $\{e_i\}$.

Conformally Einstein homogeneous metrics in Theorem 2.1–(i)

The structure equations of $\mathfrak{g}_\alpha$ corresponding to Theorem 2.1–(i):

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \frac{1}{4}e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \frac{1}{4}e_3$$

are given in the dual basis $\{e^k\}$ by

$$\begin{aligned} de^4 &= 0, & de^1 &= e^1 \wedge e^4, \\ de^2 &= \tfrac{1}{4}e^2 \wedge e^4 - \alpha e^3 \wedge e^4, & de^3 &= \alpha e^2 \wedge e^4 + \tfrac{1}{4}e^3 \wedge e^4 . \end{aligned} \tag{2.1}$$

Integrating the expressions above gives coordinates (x, y, z, t) on $\mathbb{R}^4$ where

$$e^1 = e^{-t}dx, \quad e^2 = e^{-\frac{1}{4}t}(dy - \alpha z dt), \quad e^3 = e^{-\frac{1}{4}t}(dz + \alpha y dt), \quad e^4 = dt,$$

so that the metric expresses as

$$g_\alpha = e^{-2t}dx^2 + e^{-\frac{1}{2}t}(dy - \alpha z dt)^2 + e^{-\frac{1}{2}t}(dz + \alpha y dt)^2 + dt^2 . \tag{2.2}$$

Now, a straightforward calculation shows that the conformal metric $\widetilde{g}_\alpha = e^{\frac{3}{2}t} g_\alpha$ is Ricci-flat.

Observe that $W^\pm = \frac{1}{8}\operatorname{diag}[1, 1, -2]$. Therefore, the self-dual and anti-self-dual Weyl curvature operators have a distinguished eigenvalue with one-dimensional corresponding eigenspace, which define an almost Hermitian structure and an opposite one. The structure Equations (2.1) show that the underlying almost complex structures ($J^\pm e_1 = e_4$, $J^\pm e_2 = \pm e_3$) are integrable and moreover the corresponding Kähler forms satisfy $d\Omega_\pm = \theta \wedge \Omega_\pm$ with $\theta = -\frac{1}{4}e^4$. Hence $(G_\alpha, \langle\cdot\,,\cdot\rangle, J^\pm)$ is conformally Kähler and opposite-Kähler since both $J^\pm$ are integrable. Alternatively, results in [48] show that, since $\widetilde{g}_\alpha$ is Einstein and $\widetilde{W}^+ = \widetilde{W}^-$, the conformal metric $g^c_\alpha = (24\|\widetilde{W}^+\|^2)^{\frac{1}{3}}\widetilde{g}_\alpha$ is Kähler with respect to both orientations, where $\|\widetilde{W}^+\|^2 = \frac{3}{32}e^{-3t}$ in the coordinates (x, y, z, t) of Equation (2.2). Finally, observe that the Kähler metric g^c_α is locally a product $N \times \mathbb{R}^2$, where N is a warped product.

Conformally Einstein homogeneous metrics in Theorem 2.1–(ii)

A direct calculation shows that the Weyl tensor of $(G_\alpha, \langle\cdot\,,\cdot\rangle)$ corresponding to Theorem 2.1–(ii) satisfies $W^+ = 0$ and $W^- = \operatorname{diag}[-2, 1, 1]$. Hence, the distinguished eigenvalue of W^- with corresponding one-dimensional eigenspace defines a two-form E_1^- on G_α. The structure equations

$$\begin{aligned} de^4 &= 0, & de^1 &= e^1 \wedge e^4 + \alpha e^2 \wedge e^4, \\ de^2 &= -\alpha e^1 \wedge e^4 + e^2 \wedge e^4, & de^3 &= 2e^3 \wedge e^4 - e^1 \wedge e^2, \end{aligned} \tag{2.3}$$

show that the underlying almost complex structure ($J^- e_1 = e_2$, $J^- e_3 = -e_4$) is integrable and moreover $dE_1^- = \theta \wedge E_1^-$ with $\theta = e^4$. Hence $(G_\alpha, \langle\cdot\,,\cdot\rangle, J^-)$ is conformally opposite-Kähler, since J^- induces an opposite orientation on G_α. Alternatively, results in [48] show that, since

$\widetilde{g}_\alpha$ is Einstein, the conformal metric $g^c_\alpha = (24\|\widetilde{W^-}\|^2)^{\frac{1}{3}}\widetilde{g}_\alpha$ is Kähler with respect to the opposite orientation, where $\|\widetilde{W}\|^2 = 6e^{-12t}$ in the coordinates (x,y,z,t) where the metric expresses as:

$$\begin{aligned} g_\alpha &= e^{-2t}(dx+\alpha y dt)^2 + e^{-2t}(dy - \alpha x dt)^2 \\ &\quad + e^{-4t}(dz + \tfrac{1}{2}(xdy - ydx) - \tfrac{1}{2}\alpha(x^2+y^2)dt)^2 + dt^2\,. \end{aligned} \tag{2.4}$$

Let V be a vector space equipped with an inner product $\langle\cdot,\cdot\rangle$ and let A be an algebraic curvature tensor on V. Fix $z \in V$. The associated *Jacobi operator* is defined by the linear map

$$\mathcal{J}_A(z)\colon V \to V, \qquad \mathcal{J}_A(z)(x) \mapsto (A(z,\cdot)z)x = A(z,x)z\,.$$

It is possible to restrict the domain of this operator to $z^\perp$ by the curvature identities (1.1). Observe that this operator is self-adjoint. Indeed:

$$\langle \mathcal{J}_A(z)(x), y\rangle = \langle A(z,x)z, y\rangle = A(z,x,z,y) = \langle A(z,y)z, x\rangle = \langle x, \mathcal{J}_A(z)(y)\rangle\,.$$

Let $z \in V$ be a unit vector and let $\mathcal{J}_A(z)$ be the associated Jacobi operator. If $\{x_1,\dots,x_{n-1}\}$ is an orthonormal basis for $z^\perp$, then

$$\mathrm{tr}(\mathcal{J}_A(z)) = \sum_{i=1}^{n-1}\varepsilon_i\langle \mathcal{J}_A(z)x_i, x_i\rangle = \sum_{i=1}^{n-1}\varepsilon_i\langle A(z,x_i)z, x_i\rangle = \rho_A(z,z)\,.$$

If $x \in z^\perp$ is a unit non-zero vector, then $\pi = \langle\{x,z\}\rangle$ is a non-degenerate plane of V, i.e., the restriction of $\langle\cdot,\cdot\rangle$ to π is non-degenerate. In consequence, the sectional curvature of π is given by:

$$\kappa_A(\pi) = \frac{\langle A(z,x)z,x\rangle}{\langle x,x\rangle\langle z,z\rangle - \langle x,z\rangle^2} = \frac{\langle \mathcal{J}_A(z)x,x\rangle}{\langle x,x\rangle\langle z,z\rangle}\,.$$

In particular, if we restrict to the definite positive case, the eigenvalues of the Jacobi operator $\mathcal{J}_A(z)$ represent the extremal values of the sectional curvature of all planes containing z.

Let A be an algebraic curvature tensor in a vector space equipped with an inner product $(V,\langle\,\cdot\,,\,\cdot\,\rangle)$ of signature $(\nu, n-\nu)$. We say that $(V,\langle\,\cdot\,,\,\cdot\,\rangle, A)$ is *spacelike Osserman* (resp. *timelike Osserman*) if the (possibly complex) eigenvalues of the associated Jacobi operator $\mathcal{J}_A$ are constant in the spacelike pseudo-sphere $S^+(V)$ (resp. in the timelike pseudo-sphere $S^-(V)$). Assuming $\nu > 0$ and $n-\nu > 0$, both conditions are equivalent [59] and we will say that $(V,\langle\,\cdot\,,\,\cdot\,\rangle, A)$ is ***Osserman***.

In a purely geometric context, we must differentiate between pointwise Osserman and global Osserman conditions. A pseudo-Riemannian manifold (M,g) is called *pointwise Osserman* if the eigenvalues of the Jacobi operators $\mathcal{J}(x) = R(x,\cdot)x$ do not depend on the unit vector $x \in S^\pm_p(M)$ but they can change from point to point. If the eigenvalues of the Jacobi operators do not vary from point to point then (M,g) is called *globally Osserman*. Observe that any isotropic pseudo-Riemannian manifold is globally Osserman. Therefore, real, complex and para-complex space forms are examples of globally Osserman manifolds.

Since the Ricci tensor of a pseudo-Riemannian manifold is obtained from the trace of the Jacobi operators, $\rho(x,x) = \mathrm{tr}(\mathcal{J}(x))$, any pointwise Osserman manifold is necessarily Einstein and then it has constant sectional curvature in dimension 3. In this case, pointwise and globally Osserman conditions are equivalent. Moreover, Osserman condition is non-trivial for dimension ≥ 4. Although the description of Osserman manifolds is still an open question, in certain situations it is known a complete classification (see [11, 12, 40] and [59] for more information).

Remark 2.6. Let (M, g) be a four-dimensional pseudo-Riemannian globally Osserman manifold. It was shown in [12] that, if the Jacobi operators are diagonalizable, then (M, g) is locally isometric to a real, complex or para-complex space form. Note that there are many pointwise Osserman manifolds in dimension four which do not correspond to the situation above [61].

Observe that in Theorem 2.1–(ii) the conformal metric $(\mathbb{R}^4, \widetilde{g}_\alpha = e^{3t} g_\alpha)$ is Ricci-flat and anti-self-dual. Hence we obtain a pointwise Osserman manifold [61]. Furthermore, for any unit vector field X, the Jacobi operator $\mathcal{J}(X) = R(X, \cdot)X$ has eigenvalues $\mu = 0$, $\mu = -e^{-3t}$ and $\mu = \frac{1}{2} e^{-3t}$, the latter with multiplicity two. Since the non-zero eigenvalues are in a ratio $-1 : \frac{1}{2}$ they do not correspond to the eigenvalue structure of any globally Osserman manifold.

Conformally Einstein homogeneous metrics in Theorem 2.1–(iii)

The eigenvalue structure of the self-dual and anti-self-dual Weyl curvature tensors corresponding to Theorem 2.1–(iii) is given by:

$$W_\alpha^+ = \alpha(\alpha+1)\,\mathrm{diag}[\alpha, -(\alpha+1), 1] = W_\alpha^- \,. \tag{2.5}$$

This shows that $\{E_i^+, E_i^-\}$, $i = 1, 2, 3$, define pairs of two-forms on G_α so that $E_i^+ \wedge E_i^- = 0$ and $E_i^+ \wedge E_i^+ = -E_i^- \wedge E_i^-$ for all $i = 1, 2, 3$. Furthermore, writing the structure equations of the Lie algebra $(\mathfrak{g}_\alpha, \langle \cdot, \cdot \rangle_\alpha)$ as

$$de^4 = 0, \quad de^1 = e^1 \wedge e^4, \quad de^2 = (\alpha+1)^2 e^2 \wedge e^4, \quad de^3 = \alpha^2 e^3 \wedge e^4, \tag{2.6}$$

one has $dE_i^\pm = \theta_i \wedge E_i^\pm$ with $\theta_1 = -(\alpha^2 + 2\alpha + 2)e^4$, $\theta_2 = -(\alpha^2+1)e^4$ and $\theta_3 = -(2\alpha^2 + 2\alpha + 1)e^4$. Therefore $\{E_i^+, E_i^-\}$ is a conformal symplectic pair on G_α for all $i = 1, 2, 3$ (see [7] for more information about symplectic pairs). In particular the six two-forms $E_i^\pm$ are conformally symplectic. Furthermore, integrating the expressions in Equation (2.6) gives coordinates (x, y, z, t) on $\mathbb{R}^4$ where

$$e^1 = e^{-t} dx, \quad e^2 = e^{-(\alpha+1)^2 t} dy, \quad e^3 = e^{-\alpha^2 t} dz, \quad e^4 = dt,$$

so that the metric expresses as

$$g_\alpha = e^{-2t} dx^2 + e^{-2(\alpha+1)^2 t} dy^2 + e^{-2\alpha^2 t} dz^2 + dt^2 \,. \tag{2.7}$$

As a consequence, $(\mathbb{R}^4, g_\alpha)$ has the structure of a multiply warped space of the form $\mathbb{R} \times_{f_1} \mathbb{R} \times_{f_2} \mathbb{R} \times_{f_3} \mathbb{R}$. Finally, a straightforward calculation shows that the conformal metric $\widetilde{g}_\alpha = e^{2(\alpha^2+\alpha+1)t} g_\alpha$ is Ricci-flat.

Remark 2.7. Bach-flat Kähler metrics are conformally Einstein [48]. Due to the conformal invariance of the Bach tensor, any Bach-flat conformally Kähler manifold is also conformally Einstein. The converse result is certainly not true. For instance, the eigenvalue structure of $W^\pm$ shows that the homogeneous spaces corresponding to Theorem 2.1–(iii) cannot be Kähler with respect to any conformal metric.

2.2 Conformally Einstein symmetric spaces

Four-dimensional homogeneous Einstein manifolds are locally symmetric [70]. Furthermore, any locally conformally flat homogeneous manifold is locally symmetric [99].

Lemma 2.8. *A four-dimensional locally symmetric Bach-flat manifold is Einstein or locally conformally flat.*

Proof. Let (M^4, g) be locally symmetric. Then it is an Einstein manifold or it is locally a product of the form $\mathbb{R} \times N^3(c)$, $\mathbb{R}^2 \times N^2(c)$ or $N_1^2(c_1) \times N_2^2(c_2)$, where $N^k(c)$ is a k-dimensional manifold of constant curvature c. In the case $\mathbb{R} \times N^3(c)$, (M, g) is locally conformally flat since $N^3(c)$ is of constant curvature. An explicit calculation of the Bach tensor shows that $\mathbb{R}^2 \times N^2(c)$, where $N^2(c)$ is a surface of constant curvature, is Bach-flat if and only if $c = 0$, thus (M, g) being flat. Finally, the Bach tensor of $N_1^2(c_1) \times N_2^2(c_2)$ vanishes if and only if $c_1^2 - c_2^2 = 0$, thus leading to locally conformal flatness ($c_1 = -c_2$) or to an Einstein manifold ($c_1 = c_2$). □

The above lemma shows that four-dimensional locally symmetric Bach-flat metrics are either Einstein or locally conformally flat. The existence of left-invariant Riemannian metrics with zero Bach tensor which are neither conformally Einstein nor half conformally flat was established in [1]. We will show that the examples constructed by Abbena, Garbiero and Salamon are the only possible ones within the framework of four-dimensional homogeneous manifolds.

2.3 Gröbner bases

Gröbner bases were introduced by Bruno Buchberger around the 1960's. Ever since, dozens of applications have been found for Gröbner bases. Nonetheless, to the best of our knowledge, this topic had never been applied in Riemannian geometry. This section contains a short introduction to the theory of Gröbner bases. In the rest of the chapter, Gröbner bases will play an important role.

2.3.1 Monomial order and ideals

The notion of order of terms in polynomials is the principal ingredient in the division algorithm and Gaussian elimination, where the success of both algorithms depends on working systematically with the leading terms of polynomials. Furthermore, we might intuit that when we work with arbitrary polynomials in several variables, where there is no standard order, the order we choose is fundamental. Based on this fact, what properties should this order have?

Given a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the exponents $\alpha = (\alpha_1, \dots, \alpha_n)$ are elements of $\mathbb{Z}_{\geq 0}^n$ and this observation establishes a one-to-one correspondence between the monomials in $\mathbb{R}[x_1, \dots, x_n]$ and $\mathbb{Z}_{\geq 0}^n$. A *monomial order* $>$ on $\mathbb{R}[x_1, \dots, x_n]$ is a relation $>$ on $\mathbb{Z}_{\geq 0}^n$, or equivalently, a relation on the set of monomials x^α where $\alpha \in \mathbb{Z}_{\geq 0}^n$, satisfying:

(i) $>$ is a total order on $\mathbb{Z}_{\geq 0}^n$.

(ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.

(iii) $>$ is a well-order on $\mathbb{Z}^n_{\geq 0}$.

We are specially interested in the following monomial orderings:

- *Lexicographical Order*: We say that $\alpha >_{lex} \beta$ if in the vector $\alpha - \beta \in \mathbb{Z}^n$, the leftmost non-zero entry is positive.
- *Graded Lexicographical Order*: We say that $\alpha >_{grlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and $\alpha >_{lex} \beta$, where $|\alpha| = \sum_i \alpha_i$.
- *Graded Reverse Lexicographical Order*: We say that $\alpha >_{grevlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost non-zero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.

The lexicographical order is analogous to the order of words used in dictionaries: $a > b > \cdots > y > z$ or $x_1 > x_2 > \cdots > x_n$. Observe that a variable dominates any monomial involving only smaller variables, regardless of its total degree. Hence, we could take the total degrees of the monomials into account and order monomials of bigger degree first and, after that, one may use the graded lexicographical order.

Let $\mathfrak{P} = \sum_\alpha a_\alpha x^\alpha$ be a non-zero polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ and let $>$ be a monomial order. The *multidegree* of $\mathfrak{P}$ is the maximum $\alpha \in \mathbb{Z}^n_{\geq 0}$ so that $a_\alpha \neq 0$, where the maximum is taken with respect to the given monomial order. The corresponding monomial is called the *leading term* $LT(\mathfrak{P}) = a_\alpha x^\alpha$. A *monomial ideal* is a polynomial ideal that can be generated by monomials. Therefore, a polynomial $\mathfrak{P}$ belongs to a monomial ideal $\mathcal{I}$ if and only if every term of $\mathfrak{P}$ lies in $\mathcal{I}$. Let $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a non-zero ideal and fix a monomial order on $\mathbb{R}[x_1, \ldots, x_n]$. We denote by $LT(\mathcal{I})$ the set of leading terms of non-zero elements of $\mathcal{I}$, i.e.,

$$LT(\mathcal{I}) = \{cx^\alpha : \text{ there exists } \mathfrak{P} \in \mathcal{I} \setminus \{0\} \text{ with } LT(\mathfrak{P}) = cx^\alpha\},$$

and we denote by $\langle LT(\mathcal{I}) \rangle$ the ideal generated by the elements of $LT(\mathcal{I})$. Observe that $LT(\mathfrak{P}_i) \in LT(\mathcal{I}) \subset \langle LT(\mathcal{I}) \rangle$ which implies $\langle LT(\mathfrak{P}_1), \ldots, LT(\mathfrak{P}_k) \rangle \subset \langle LT(\mathcal{I}) \rangle$. However, it is important to emphasize that if $\mathcal{I} = \langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k \rangle$, then $\langle LT(\mathcal{I}) \rangle$ may be strictly larger than the ideal $\langle LT(\mathfrak{P}_1), \ldots, LT(\mathfrak{P}_k) \rangle$.

For example, consider $\mathcal{I} = \langle \mathfrak{P}_1, \mathfrak{P}_2 \rangle$ the ideal generated by $\mathfrak{P}_1 = x^3 - 2xy$ and $\mathfrak{P}_2 = x^2y - 2y^2 + x$, where we fix the graded lexicographical order on monomials. Then $x \cdot \mathfrak{P}_2 - y \cdot \mathfrak{P}_1 = x^2$, so $x^2 \in \mathcal{I}$. Therefore, $x^2 = LT(x^2) \in \langle LT(\mathcal{I}) \rangle$ but $x^2 \notin \langle LT(\mathfrak{P}_1), LT(\mathfrak{P}_2) \rangle$. Hence, $\langle LT(\mathcal{I}) \rangle \neq \langle LT(\mathfrak{P}_1), LT(\mathfrak{P}_2) \rangle$.

The next result is crucial and it is known as the Hilbert Basis Theorem:

Theorem 2.9. [66] *Every ideal $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ has a finite generating set.*

For monomial ideals this result is called Dickson's Lemma.

The importance of the above result is not only that every ideal has a finite basis, but also that its proof is based on $\langle LT(\mathbf{g}_1), \ldots, LT(\mathbf{g}_\nu) \rangle = \langle LT(\mathcal{I}) \rangle$ (see for example [45]).

Definition 2.10. Fix a monomial order on the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$. A finite subset $\mathcal{G} = \{\mathrm{g}_1, \ldots, \mathrm{g}_\nu\}$ of an ideal $\mathcal{I}$ is said to be a *Gröbner basis* (or Gröbner-Shirshov basis) with respect to some monomial order if

$$\langle LT(\mathbf{g}_1), \ldots, LT(\mathbf{g}_\nu)\rangle = \langle LT(\mathcal{I})\rangle \,.$$

The Hilbert Basis Theorem guarantees that any non-zero ideal $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal $\mathcal{I}$ is a basis of $\mathcal{I}$. However, how can we know that a given basis of an ideal is a Gröbner basis? Buchberger's algorithm (among others) provides a constructive algorithm to find one such basis. This rather simple notion allows us to have simple algorithmic solutions to different problems.

- The remainder of the division algorithm applied to a polynomial $\mathfrak{P}$ divided by a Gröbner basis $\mathcal{G}$ of an ideal $\mathcal{I}$ is zero if and only if $\mathfrak{P}$ belongs to $\mathcal{I}$, a property that does not necessarily hold if $\mathcal{G}$ is not a Gröbner basis. Therefore, this fact provides an algorithm to check the Ideal Membership Problem.

- As another example, when the set of solutions of a polynomial system is not too large, the calculation of a Gröbner basis with respect to the lexicographical order gives rise to elimination theory, simplifying the problem of finding all common roots, thus generalizing the classical Gaussian method of the linear case.

Just as a matter of curiosity, let us mention that Gröbner bases even generalize the simplex method used in mathematical optimization. We refer the interested reader to [45] for more information on the theory of Gröbner bases.

2.3.2 Gröbner basis in homogeneous manifolds

One of the most important applications of Gröbner bases is to eliminate variables. We pleasantly found out that these methods can be very useful to classify homogeneous geometric structures such as Einstein metrics, Bach-flat structures or Ricci solitons.

The components of the Bach tensor for a left-invariant metric on a Lie group give polynomials on the structure constants. Hence, to obtain a full classification of Bach-flat Lie groups, one needs to solve the corresponding polynomial system of equations. When the system under consideration is simple, it is an elementary problem to get all common roots, but if the number of equations and their degrees increase, it may become a quite unmanageable assignment. Gröbner bases theory provides very powerful tools to solve large polynomial systems of equations. The basic idea is to use elimination theory. But, how does elimination work? Consider the following polynomial system:

$$x^2 + y + z = 1, \quad x + y^2 + z = 1, \quad x + y + z^2 = 1, \tag{2.8}$$

and let $\mathcal{I} = \langle x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\rangle \subset \mathbb{R}[x, y, z]$ be the ideal. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and we obtain:

$$\begin{aligned} g_1 &= x + y + z^2 - 1, \\ g_2 &= y^2 - y - z^2 + z, \\ g_3 &= z^2(2y + z^2 - 1), \\ g_4 &= z^2(z^4 - 4z^2 + 4z - 1)\,. \end{aligned} \tag{2.9}$$

The equations in (2.8) and $g_1 = g_2 = g_3 = g_4 = 0$ given by (2.9) have the same solutions and g_4 involves only z, then the possible z's are 0, 1 and $-1 \pm \sqrt{2}$. Now, substituting these values into g_2 and g_3 one can determine all possible solutions for y. Finally, g_1 gives the corresponding x's.

Given $\mathcal{I} = \langle \mathfrak{P}_1, \dots, \mathfrak{P}_k \rangle \subset \mathbb{R}[x_1, \dots, x_n]$, the ν-th elimination ideal $\mathcal{I}_\nu$ is the ideal of $\mathbb{R}[x_{\nu+1}, \dots, x_n]$ defined by $\mathcal{I}_\nu = \mathcal{I} \cap \mathbb{R}[x_{\nu+1}, \dots, x_n]$. Therefore, $\mathcal{I}_\nu$ consists of all $\mathfrak{P}_1 = \cdots = \mathfrak{P}_k = 0$. In other words, eliminating $x_1, \dots, x_\nu$ means finding non-zero polynomials in the ν-th elimination ideal $\mathcal{I}_\nu$. It is important to emphasize that different order of the variables leads to different elimination ideals. Note that if two sets of polynomials generate the same ideal, the corresponding zero sets must be identical.

Proposition 2.11. [45] *Let $\mathcal{I} \subset \mathbb{R}[x_1, \dots, x_n]$ be an ideal and let $\mathcal{G}$ be a Gröbner basis of $\mathcal{I}$ with respect to the lexicographical order. Then, for every $0 \leq \nu \leq n$, the set $\mathcal{G}_\nu = \mathcal{G} \cap \mathbb{R}[x_1, \dots, x_n]$ is a Gröbner basis of the ν-th elimination ideal $\mathcal{I}_\nu$.*

The above result shows that a Gröbner basis for the lexicographical order eliminates not only the first variable, but also the first two variables, the first three variables, and so on. Therefore, our strategy for solving the rather large polynomial systems consists of obtaining "better" polynomials that belong to the ideals generated by the corresponding polynomial systems.

2.4 Left-invariant metrics on $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{g}_3$ be a semi-direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{e}(1,1)$ or $\mathfrak{g}_3 = \mathfrak{e}(2)$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle_3$ its restriction to $\mathfrak{g}_3$. Following the work of Milnor [82], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \qquad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \qquad [\mathbf{v}_1, \mathbf{v}_2] = 0, \tag{2.10}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \lambda_2 \neq 0$. Moreover, the associated Lie group corresponds to $E(2)$ (resp. $E(1,1)$) if $\lambda_1 \lambda_2 > 0$ (resp. $\lambda_1 \lambda_2 < 0$). The algebra of derivations of $\mathfrak{g}_3$ is given by

$$\operatorname{der}(\mathfrak{g}_3) = \left\{ \begin{pmatrix} b & a & c \\ -\frac{\lambda_2}{\lambda_1} a & b & d \\ 0 & 0 & 0 \end{pmatrix} ;\, a, b, c, d \in \mathbb{R} \right\}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$, with $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ given by Equation (2.10), and $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \mathfrak{g}_3$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{g}_3$, set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 -$

$\sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{g}_3$ so that

$$\begin{aligned}
[e_2, e_3] &= \lambda_1 e_1, \\
[e_3, e_1] &= \lambda_2 e_2, \\
[e_4, e_1] &= \tfrac{1}{R}\{b e_1 - \lambda_2(\tfrac{a}{\lambda_1} + k_3) e_2\}, \\
[e_4, e_2] &= \tfrac{1}{R}\{(a + k_3\lambda_1) e_1 + b e_2\}, \\
[e_4, e_3] &= \tfrac{1}{R}\{(c - k_2\lambda_1) e_1 + (d + k_1\lambda_2) e_2\}, \quad R > 0\,.
\end{aligned} \tag{2.11}$$

Lemma 2.12. *The Lie group* $\mathbb{R}e_4 \ltimes E(1,1)$ *admits a non-symmetric Bach-flat left-invariant metric if and only if it is isomorphically homothetic to a Lie group determined by the solvable Lie algebra given by*

$$\begin{aligned}
&[e_2, e_3] = e_1, && [e_1, e_3] = (2 + \sqrt{3}) e_2, \\
&[e_4, e_1] = \sqrt{6 + 3\sqrt{3}}\, e_1, && [e_4, e_2] = \sqrt{6 + 3\sqrt{3}}\, e_2\,.
\end{aligned}$$

Moreover, the Lie group $\mathbb{R}e_4 \ltimes E(2)$ *does not admit any non-symmetric Bach-flat left-invariant metric.*

Proof. We start analyzing the Bach tensor of $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$. In order to simplify the expressions we use the notation $A = \frac{a}{\lambda_1} + k_3$, $C = c - k_2\lambda_1$ and $D = d + k_1\lambda_2$. Moreover, since the structure constants of $\mathfrak{g}_3$ satisfy $\lambda_1\lambda_2 \neq 0$, one may work with a homothetic basis $\widetilde{e}_k = \frac{1}{\lambda_1} e_k$ so that we may assume $\lambda_1 = 1$. A long but straightforward calculation shows that the components of the Bach tensor, with the structure constants in Equation (2.11), are given by

$$\begin{aligned}
&\mathfrak{B}_{11} = \tfrac{1}{24R^4}\mathfrak{P}_{11},\ \mathfrak{B}_{12} = \tfrac{1}{12R^4}\mathfrak{P}_{12},\ \mathfrak{B}_{13} = \tfrac{1}{12R^4}\mathfrak{P}_{13},\ \mathfrak{B}_{14} = \tfrac{\lambda_2}{12R^3}\mathfrak{P}_{14}, \\
&\mathfrak{B}_{22} = \tfrac{1}{24R^4}\mathfrak{P}_{22},\ \mathfrak{B}_{23} = \tfrac{1}{12R^4}\mathfrak{P}_{23},\ \mathfrak{B}_{24} = \tfrac{1}{12R^3}\mathfrak{P}_{24},\ \mathfrak{B}_{33} = \tfrac{1}{24R^4}\mathfrak{P}_{33}, \\
&\mathfrak{B}_{34} = \tfrac{1}{12R^3}\mathfrak{P}_{34},\ \mathfrak{B}_{44} = \tfrac{1}{24R^4}\mathfrak{P}_{44},
\end{aligned} \tag{2.12}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$\begin{aligned}
\mathfrak{P}_{11} = {}& 12(A^2 + R^2)^2\lambda_2^4 - 4(A^2 + R^2)^2\lambda_2^3 - (20b^2 - C^2 - 8D^2)(A^2 + R^2)\lambda_2^2 \\
&+ (12A^4 - 4(2b^2 - 3C^2 - D^2 - 6R^2)A^2 - 42bCDA - 4R^2(2b^2 - 3C^2 - D^2 - 3R^2))\lambda_2 \\
&- 20A^4 + (28b^2 - 40C^2 + 3D^2 - 40R^2)A^2 - 42bCDA - 20R^4 + (3D^2 - 40C^2)R^2 \\
&- 4(C^2 + D^2)(5C^2 + D^2) + b^2(43C^2 + D^2 + 28R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{12} = {}& -16b(A^2 + R^2)A\lambda_2^3 - 8CD(A^2 + R^2)\lambda_2^2 - (5CDA^2 - b(5C^2 - 16D^2)A + 5CDR^2)\lambda_2 \\
&+ 16bA^3 - 8CDA^2 + b(16C^2 - 5D^2 + 16R^2)A + CD(21b^2 - 8(C^2 + D^2 + R^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{13} = {}& -8AD(A^2 + R^2)\lambda_2^3 + (4AD(A^2 + R^2) - 3bCR^2)\lambda_2^2 \\
&+ (DA^3 - 9bCA^2 + D(12b^2 + R^2 - 8(C^2 + D^2))A - 12bCR^2)\lambda_2 \\
&+ 3b(8CA^2 + 3bDA - 3b^2C + 8C(C^2 + D^2 + R^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{14} &= -8D(A^2+R^2)\lambda_2^2 + (4DA^2+3bCA+4DR^2)\lambda_2 \\
&\quad + DA^2 + 3bCA + D(3b^2+R^2-8(C^2+D^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{22} &= -20(A^2+R^2)^2\lambda_2^4 + 12(A^2+R^2)^2\lambda_2^3 + (28b^2+3C^2-40D^2)(A^2+R^2)\lambda_2^2 \\
&\quad - (4A^4+4(2b^2-C^2-3D^2+2R^2)A^2 - 42bCDA + 4R^2(2b^2-C^2-3D^2+R^2))\lambda_2 \\
&\quad + 12A^4 - (20b^2-8C^2-D^2-24R^2)A^2 + 42bCDA \\
&\quad + 12R^4 + (8C^2+D^2)R^2 - 4(C^2+D^2)(C^2+5D^2) + b^2(C^2+43D^2-20R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{23} &= -(AC-24bD)(A^2+R^2)\lambda_2^2 - (4(AC+3bD)R^2 + A(4CA^2+9bDA+9b^2C))\lambda_2 \\
&\quad + 8CA^3 + 4C(2(C^2+D^2+R^2)-3b^2)A - 3bD(3b^2+R^2-8(C^2+D^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{24} &= -C(A^2+R^2)\lambda_2^2 - (4CA^2-3bDA+4CR^2)\lambda_2 \\
&\quad + 8CA^2 + 3bDA - 3b^2C + 8C(C^2+D^2+R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{33} &= -4(A^2-3R^2)(A^2+R^2)\lambda_2^4 + 4(A^2-3R^2)(A^2+R^2)\lambda_2^3 \\
&\quad - ((12b^2+C^2-8D^2)A^2 + 3(4b^2+C^2-8D^2)R^2)\lambda_2^2 \\
&\quad + 2(2A^4+2(6b^2-C^2-D^2-2R^2)A^2 + 9bCDA + 6R^2(2b^2-C^2-D^2-R^2))\lambda_2 \\
&\quad - 4A^4 - (12b^2-8C^2+D^2-8R^2)A^2 - 18bCDA \\
&\quad + 12R^4 - 3(4b^2-8C^2+D^2)R^2 + 3(C^2+D^2)(4(C^2+D^2)-19b^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{34} &= -8A(A^2+R^2)\lambda_2^4 + 8A(A^2+R^2)\lambda_2^3 + A(C^2-8D^2)\lambda_2^2 \\
&\quad + (8A^3+4(C^2+D^2+2R^2)A + 9bCD)\lambda_2 \\
&\quad - 8A^3 - 9bCD - A(8C^2-D^2+8R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{44} &= 4(3A^2-R^2)(A^2+R^2)\lambda_2^4 - 4(3A^4+2R^2A^2-R^4)\lambda_2^3 \\
&\quad + ((4b^2-3C^2+24D^2)A^2 + (4b^2-C^2+8D^2)R^2)\lambda_2^2 \\
&\quad + (4R^4-4(2A^2+2b^2+C^2+D^2)R^2 - 2A(6A^3+4b^2A+6(C^2+D^2)A+9bCD))\lambda_2 \\
&\quad + 12A^4 + (4b^2+24C^2-3D^2+8R^2)A^2 + 18bCDA \\
&\quad - 4R^4 + (4b^2+8C^2-D^2)R^2 + (C^2+D^2)(13b^2+12(C^2+D^2)).
\end{aligned}$$

Hence, $\mathbb{R}e_4 \ltimes E(1,1)$ or $\mathbb{R}e_4 \ltimes E(2)$ admit a Bach-flat left-invariant metric if and only if the structure constants in Equation (2.11) satisfy the equations $\{\mathfrak{P}_{ij} = 0\}$.

Let $\mathcal{I} \subset \mathbb{R}[A, b, \lambda_2, C, D, R]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and a detailed analysis of that basis shows that the polynomial

$$\begin{aligned}
\mathfrak{g}_0 \;=\; & D^6(C^2+D^2)(2D^2+R^2)(25D^2+4R^2)(16D^2+5R^2) \\
& \times(9D^2+16R^2)(25D^2+24R^2)(80D^4+R^4-16D^2R^2)
\end{aligned} \tag{2.13}$$

belongs to $\mathcal{G}$. Since the zero sets of $\{\mathfrak{P}_{ij} = 0\}$ and $\mathcal{I} = \langle \mathfrak{P}_{ij} \rangle = \langle \mathcal{G} \rangle$ coincide, then necessarily $D = 0$.

Next, we compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{D\}$ with respect to the lexicographical order and we get that the polynomial

$$\mathbf{g}_1 = C^4(9C^2 + 4R^2)(25C^2 + 16R^2)(49C^2 + 24R^2)\lambda_2^3$$

belongs to $\mathcal{G}_1$. Thus, since $\lambda_2 \neq 0$, we get $C = 0$.

Now, for $C = D = 0$, Equation (2.12) implies that

$$\mathfrak{P}_{34} = -8(\lambda_2 - 1)^2 A(A^2 + R^2)(\lambda_2^2 + \lambda_2 + 1)$$

and therefore we are led to the following possibilities:

$$(1)\, \lambda_2 = 1, \qquad (2)\, A = 0\,.$$

Case (1):

$C = 0$, $D = 0$, $\lambda_2 = 1$. In this case, a direct calculation shows that the corresponding Lie group given by Equation (2.11) is locally conformally flat and therefore a symmetric manifold [99].

Case (2):

$C = 0$, $D = 0$, $A = 0$. Excluding $\lambda_2 = 1$ solved in the previous case, Equation (2.12) implies that the Bach-flat condition is equivalent to

$$b^2 - R^2(\lambda_2^2 + \lambda_2 + 1) = 0, \qquad 3R^2 - b^2(\lambda_2 + 4) = 0,$$

from where it easily follows that

$$b = \pm R, \qquad \lambda_2 = -1,$$

in which case a straightforward calculation shows that the manifold is Einstein and thus locally symmetric [70], or otherwise

$$b = \pm R\sqrt{6 + 3\sqrt{3}}, \qquad \lambda_2 = -2 - \sqrt{3}, \quad \text{or} \tag{2.14}$$

$$b = \pm R\sqrt{6 - 3\sqrt{3}}, \qquad \lambda_2 = -2 + \sqrt{3}\,. \tag{2.15}$$

Now, considering the isometry $e_4 \mapsto -e_4$ one has $b > 0$ in both cases. Setting $\bar{e}_1 = (2 + \sqrt{3})e_2$, $\bar{e}_2 = (2 + \sqrt{3})e_1$, $\bar{e}_3 = (2 + \sqrt{3})e_3$, $\bar{e}_4 = (2 + \sqrt{3})e_4$ one interchanges the brackets given by Equations (2.15) and (2.14). Moreover since this isomorphism transforms the original metric $\langle \cdot, \cdot \rangle$ into a homothetic one $\langle \cdot, \cdot \rangle^* = (2 + \sqrt{3})^2 \langle \cdot, \cdot \rangle$ and we work modulo homotheties, we change the metric so that $\bar{e}_i$ remains an orthonormal basis. Hence we reduce this case to the homothetically isomorphic Lie algebra given by $b = R\sqrt{6 + 3\sqrt{3}}$ with $\lambda_2 = -2 - \sqrt{3}$.

Note that $\lambda_1\lambda_2 = \lambda_2 < 0$; hence the group is $\mathbb{R}e_4 \ltimes E(1,1)$ and a straightforward calculation shows that this case is not locally symmetric. This finishes the proof. $\square$

2.5 Left-invariant metrics on $\mathbb{R}e_4 \ltimes H^3$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{h}^3$ be a semi-direct extension of the Heisenberg algebra $\mathfrak{h}^3$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle_3$ its restriction to $\mathfrak{h}^3$. Then, there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{h}^3$ such that (see [82])

$$[\mathbf{v}_3, \mathbf{v}_2] = 0, \qquad [\mathbf{v}_3, \mathbf{v}_1] = 0, \qquad [\mathbf{v}_1, \mathbf{v}_2] = \lambda_3 \mathbf{v}_3, \tag{2.16}$$

where $\lambda_3 \neq 0$ is a real number. The algebra of all derivations of $\mathfrak{h}^3$ is given with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ by

$$\operatorname{der}(\mathfrak{h}^3) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ \hat{h} & \hat{f} & \alpha_{11} + \alpha_{22} \end{pmatrix} ; \alpha_{ij}, \hat{f}, \hat{h} \in \mathbb{R} \right\} .$$

We rotate the basis elements $\{\mathbf{v}_1, \mathbf{v}_2\}$ so that the matrix $A = (\alpha_{ij})$ decomposes as the sum of a diagonal matrix and a skew-symmetric matrix. Hence

$$\operatorname{der}(\mathfrak{h}^3) = \left\{ \begin{pmatrix} a & c & 0 \\ -c & d & 0 \\ h & f & a+d \end{pmatrix} ; a, c, d, f, h \in \mathbb{R} \right\} ,$$

and consider the Lie algebra $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \mathfrak{h}^3$ given by

$$\begin{aligned} &[\mathbf{v}_3, \mathbf{v}_2] = 0, && [\mathbf{v}_3, \mathbf{v}_1] = 0, \\ &[\mathbf{v}_1, \mathbf{v}_2] = \gamma \mathbf{v}_3, && [\mathbf{v}_4, \mathbf{v}_1] = a\mathbf{v}_1 - c\mathbf{v}_2 + h\mathbf{v}_3, \\ &[\mathbf{v}_4, \mathbf{v}_2] = c\mathbf{v}_1 + d\mathbf{v}_2 + f\mathbf{v}_3, && [\mathbf{v}_4, \mathbf{v}_3] = (a+d)\mathbf{v}_3 . \end{aligned}$$

Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{h}^3$, set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{h}^3$ so that

$$\begin{aligned} &[e_1, e_2] = \gamma e_3, \\ &[e_4, e_1] = \tfrac{1}{R}\{a e_1 - c e_2 + (h + k_2\gamma) e_3\}, \\ &[e_4, e_3] = \tfrac{1}{R}(a+d) e_3, \\ &[e_4, e_2] = \tfrac{1}{R}\{c e_1 + d e_2 + (f - k_1\gamma) e_3\}, \quad R > 0 . \end{aligned} \tag{2.17}$$

Lemma 2.13. *The group $\mathbb{R}e_4 \ltimes H^3$ admits a non-symmetric Bach-flat left-invariant metric if and only if it is isomorphically homothetic to a Lie group determined by one of the following solvable Lie algebras:*

(i) The Lie algebra given by

$$\begin{aligned} &[e_1, e_2] = e_3, && [e_4, e_1] = e_1 - \alpha\, e_2, \\ &[e_4, e_2] = \alpha\, e_1 + e_2, && [e_4, e_3] = 2\, e_3 . \end{aligned}$$

In this case $\mathbb{R}e_4 \ltimes H^3$ is half conformally flat.

(ii) The Lie algebra given by

$$[e_1,e_2]=e_3, \qquad [e_3,e_4]=\tfrac{\sqrt{5}}{2\sqrt{2}}\,e_3,$$
$$[e_4,e_1]=\tfrac14\sqrt{7-3\sqrt{5}}\,e_1, \quad [e_2,e_4]=\tfrac14\sqrt{7+3\sqrt{5}}\,e_2\,.$$

Proof. First we obtain the Bach tensor of $\mathbb{R}e_4\ltimes H^3$. In order to simplify the expressions we use the notation $F=f-k_1\gamma$ and $H=h+k_2\gamma$. Moreover, since the structure constant of $\mathfrak{h}^3$ satisfies $\gamma\neq 0$, one may work with a homothetic basis $\widetilde{e}_k=\frac{1}{\gamma}e_k$ so that we may assume $\gamma=1$. A long but straightforward calculation shows that the components of the Bach tensor, with the structure constants in Equation (2.17), are given by

$$\begin{aligned}
&\mathfrak{B}_{11}=\tfrac{1}{24R^4}\mathfrak{P}_{11},\ \mathfrak{B}_{12}=\tfrac{1}{12R^4}\mathfrak{P}_{12},\ \mathfrak{B}_{13}=\tfrac{1}{12R^4}\mathfrak{P}_{13},\ \mathfrak{B}_{14}=\tfrac{1}{12R^3}\mathfrak{P}_{14},\\
&\mathfrak{B}_{22}=\tfrac{1}{24R^4}\mathfrak{P}_{22},\ \mathfrak{B}_{23}=\tfrac{1}{12R^4}\mathfrak{P}_{23},\ \mathfrak{B}_{24}=\tfrac{1}{12R^3}\mathfrak{P}_{24},\ \mathfrak{B}_{33}=\tfrac{1}{24R^4}\mathfrak{P}_{33},\\
&\mathfrak{B}_{34}=0, \qquad \mathfrak{B}_{44}=\tfrac{1}{24R^4}\mathfrak{P}_{44},
\end{aligned}\tag{2.18}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$\begin{aligned}
\mathfrak{P}_{11}&=24ac^2d-16a^3d+48ad^3+84a^2c^2+16a^2d^2-108c^2d^2\\
&+(F^2-20(H^2+R^2))a^2-21(F^2-H^2)c^2-3(4F^2+19H^2+4R^2)d^2\\
&+78FHac-4(22H^2+7R^2)ad+78FHcd-4(F^2+H^2+R^2)(F^2-3(H^2+R^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{12}&=-58a^2cd+58acd^2-18a^3c+24ac^3-24c^3d+18cd^3-12FHa^2+21FHc^2-12FHd^2\\
&+(31F^2-2(4H^2+R^2))ac-53FHad+(8F^2-31H^2+2R^2)cd+8FH(F^2+H^2+R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{13}&=53Facd-3Fc^3-9Hd^3+33Fa^2c-28Ha^2d+3Hac^2-48Had^2+24Hc^2d-9Fcd^2\\
&+16H(F^2+H^2+R^2)a-8F(F^2+H^2+R^2)c+24H(F^2+H^2+R^2)d,
\end{aligned}$$

$$\mathfrak{P}_{14}=-3Fa^2+3Fc^2+3Hac-14Fad-15Hcd+8F(F^2+H^2+R^2),$$

$$\begin{aligned}
\mathfrak{P}_{22}&=24ac^2d+48a^3d-16ad^3-108a^2c^2+16a^2d^2+84c^2d^2\\
&-3(19F^2+4(H^2+R^2))a^2+21(F^2-H^2)c^2-(20F^2-H^2+20R^2)d^2\\
&-78FHac-4(22F^2+7R^2)ad-78FHcd+4(F^2+H^2+R^2)(3F^2-H^2+3R^2),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{23}&=-53Hacd-9Fa^3+3Hc^3+9Ha^2c-48Fa^2d+24Fac^2-28Fad^2+3Fc^2d-33Hcd^2\\
&+24F(F^2+H^2+R^2)a+8H(F^2+H^2+R^2)c+16F(F^2+H^2+R^2)d,
\end{aligned}$$

$$\mathfrak{P}_{24}=-3Hc^2+3Hd^2-15Fac+14Had+3Fcd-8H(F^2+H^2+R^2),$$

$$\begin{aligned}
\mathfrak{P}_{33}&=24ac^2d-16a^3d-16ad^3-12a^2c^2-48a^2d^2-12c^2d^2\\
&+(43F^2+28(H^2+R^2))a^2-9(F^2+H^2)c^2+(28F^2+43H^2+28R^2)d^2\\
&-54FHac+(104(F^2+H^2)+44R^2)ad+54FHcd-20(F^2+H^2+R^2)^2,
\end{aligned}$$

$$\mathfrak{P}_{44}=-72ac^2d-16a^3d-16ad^3+36a^2c^2+16a^2d^2+36c^2d^2$$

$$+ (13F^2 + 4(H^2 + R^2))a^2 + 9(F^2 + H^2)c^2 + (4F^2 + 13H^2 + 4R^2)d^2 + 54FHac$$
$$- (16(F^2 + H^2) - 12R^2)ad - 54FHcd + 4(3(F^2 + H^2) - R^2)(F^2 + H^2 + R^2)\,.$$

Therefore, $\mathbb{R}e_4 \ltimes H^3$ admits a Bach-flat left-invariant metric if and only if the structure constants given in Equation (2.17) satisfy the equations $\{\mathfrak{P}_{ij} = 0\}$. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, H, F, R]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and a detailed analysis of the Gröbner basis shows that the polynomial

$$\begin{aligned} \mathfrak{g}_0 = {} & FR^4(2F^2+R^2)^4(4F^2+R^2)(F^2+H^2+R^2)^2(4F^2+9R^2)(9F^2+11R^2) \\ & \times ((F^2 - H^2)^2 + F^2R^2 + H^2R^2)(10000F^4 + 10200F^2R^2 + 3087R^4) \\ & \times (606208F^4 + 861952F^2R^2 + 144669R^4) \end{aligned}$$

belongs to $\mathcal{G}$. Since the zero sets of $\{\mathfrak{P}_{ij} = 0\}$ and $\mathcal{I} = \langle \mathfrak{P}_{ij} \rangle = \langle \mathcal{G} \rangle$ coincide and $R > 0$, then necessarily $F = 0$.

Next, we compute a Gröbner basis $\mathcal{G}'$ of the ideal generated by $\mathcal{G} \cup \{F\}$ with respect to the lexicographical order and we get that the polynomial

$$\mathfrak{g}_0' = H(H^2 + R^2)(4H^2 + R^2)(4H^2 + 9R^2)(9H^2 + 11R^2)$$

belongs to $\mathcal{G}'$. Thus, we get $H = 0$.

Now, computing a Gröbner basis $\mathcal{G}''$ of the ideal generated by $\mathcal{G}' \cup \{H\}$ with respect to the graded reverse lexicographical order we find that the polynomial

$$\mathfrak{g}_0'' = (a - d)(24c^2 - 8ad - R^2)R^4$$

belongs to $\mathcal{G}''$ and therefore we are led to the following possibilities:

$$(1)\, a = d, \qquad (2)\, 24c^2 - 8ad - R^2 = 0\,.$$

Case (1):

$F = 0$, $H = 0$, $a = d$. In this case, Equation (2.18) implies that the Bach-flat condition is equivalent to

$$4d^4 + R^4 - 5d^2R^2 = 0,$$

from where we easily get

$$d = \pm R \quad \text{or} \quad d = \pm\frac{R}{2}\,.$$

If $d = \pm\frac{R}{2}$, the manifold is half conformally flat and Einstein, thus locally symmetric and homothetic to the complex hyperbolic plane. For $d = \pm R$, the isometry $e_4 \mapsto -e_4$ lets us to take $d = R$. Now, a direct calculation shows that the manifold is half conformally flat and non-symmetric, hence obtaining Assertion (i) in Lemma 2.13. Furthermore, it follows from [47] that all the Lie groups in Lemma 2.13–(i) are isometric.

Case (2):

$F=0$, $H=0$, $24c^2-8ad-R^2=0$. Equation (2.18) implies that

$$\mathfrak{P}_{12}=-(a-d)c(18a^2+18d^2+68ad+R^2)\,.$$

Since $a=d$ was already solved in the previous case, we compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by $\mathcal{G}''\cup\{c(18a^2+18d^2+68ad+R^2)\}\subset\mathbb{R}[R,a,c,d,H,F]$ with respect to the lexicographical order and we get that the polynomial

$$\mathbf{g}_2=cd^4(25c^4+18c^2d^2+d^4)(961c^4+1298c^2d^2+121d^4)$$

belongs to $\mathcal{G}_2$. Thus, we have two possibilities:

$$(2.i)\; d=0, \qquad (2.ii)\; c=0\,.$$

Case (2.i):

$F=0$, $H=0$, $24c^2-8ad-R^2=0$, $d=0$. In this case, from Equation (2.18) we get that the Bach-flat condition is equivalent to

$$33a^2c^2-R^4=0, \qquad ac(3a^2+4c^2)=0,$$

which does not hold since $R>0$.

Case (2.ii):

$F=0$, $H=0$, $24c^2-8ad-R^2=0$, $c=0$. Since $d=0$ was solved in the previous case, we have $a=-\frac{R^2}{8d}$ and Equation (2.18) implies that the Bach-flat condition is equivalent to

$$64d^4-56d^2R^2+R^4=0\,.$$

Thus, it follows that

$$d=\pm\frac{1}{4}R\sqrt{7-3\sqrt{5}} \quad \text{or} \tag{2.19}$$

$$d=\pm\frac{1}{4}R\sqrt{7+3\sqrt{5}}, \tag{2.20}$$

and a straightforward calculation shows that none of these cases is locally symmetric. Note that if we take $e_4\mapsto -e_4$ one may assume that $d>0$ in both cases. Moreover, taking $\bar{e}_1=-e_2$, $\bar{e}_2=-e_1$, $\bar{e}_3=-e_3$, $\bar{e}_4=-e_4$, one reduces this case to an only homothetically isomorphic Lie algebra. For $d=-\frac{1}{4}R\sqrt{7+3\sqrt{5}}$ we get Assertion (ii) in Lemma 2.13, finishing the proof. □

2.6 Left-invariant metrics on $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{r}^3$ be a semi-direct extension of the abelian Lie algebra $\mathfrak{r}^3$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_3$ its restriction to $\mathfrak{r}^3$. The algebra of all derivations $\mathfrak{D}$ of $\mathfrak{r}^3$ is $\mathfrak{gl}(3,\mathbb{R})$. If we fix $\mathfrak{D} \in \mathfrak{gl}(3,\mathbb{R})$, there exists a $\langle\cdot,\cdot\rangle_3$-orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{r}^3$ where $\mathfrak{D}$ decomposes as a sum of a diagonal matrix and a skew-symmetric matrix. Hence

$$\operatorname{der}(\mathfrak{r}^3) = \left\{ \begin{pmatrix} a & -b & -c \\ b & f & -h \\ c & h & p \end{pmatrix} ;\ a,b,c,f,h,p \in \mathbb{R} \right\}.$$

Now, the corresponding semi-direct product $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{r}^3$, is given by

$$\begin{aligned} &[\mathbf{v}_1, \mathbf{v}_2] = 0, && [\mathbf{v}_1, \mathbf{v}_3] = 0, \\ &[\mathbf{v}_2, \mathbf{v}_3] = 0, && [\mathbf{v}_4, \mathbf{v}_1] = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3, \\ &[\mathbf{v}_4, \mathbf{v}_2] = -b\mathbf{v}_1 + f\mathbf{v}_2 + h\mathbf{v}_3, && [\mathbf{v}_4, \mathbf{v}_3] = -c\mathbf{v}_1 - h\mathbf{v}_2 + p\mathbf{v}_3, \end{aligned}$$

with respect to some basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ so that $\mathfrak{g} = \mathbb{R}\mathbf{v}_4 \oplus \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{r}^3$, set $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1,2,3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, \ldots, e_4\}$ of $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{r}^3$ so that

$$\begin{aligned} &[e_4, e_1] = \tfrac{1}{R}\{ae_1 + be_2 + ce_3\}, \qquad [e_4, e_2] = \tfrac{1}{R}\{-be_1 + fe_2 + he_3\}, \\ &[e_4, e_3] = \tfrac{1}{R}\{-ce_1 - he_2 + pe_3\}, \quad R > 0\,. \end{aligned} \tag{2.21}$$

Lemma 2.14. *The group $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ admits a non-symmetric Bach-flat left-invariant metric if and only if it is isomorphically homothetic to a Lie group determined by one of the following solvable Lie algebras:*

(i) $[e_4, e_1] = e_1, \quad [e_4, e_2] = \frac{1}{4}e_2 + \alpha\, e_3, \quad [e_4, e_3] = \frac{1}{4}e_3 - \alpha\, e_2\,.$

(ii) $[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha+1)^2 e_2, \quad [e_4, e_3] = \alpha^2\, e_3, \qquad \alpha > 0\,.$

Proof. A long but straightforward calculation shows that the components of the Bach tensor of $\mathbb{R}e_4 \ltimes \mathbb{R}^3$, with the structure constants in Equation (2.21), are given by

$$\begin{aligned} &\mathfrak{B}_{11} = \tfrac{1}{6R^4}\mathfrak{P}_{11}, \ \mathfrak{B}_{12} = \tfrac{1}{6R^4}\mathfrak{P}_{12}, \ \mathfrak{B}_{13} = \tfrac{1}{6R^4}\mathfrak{P}_{13}, \ \mathfrak{B}_{14} = 0, \\ &\mathfrak{B}_{22} = \tfrac{1}{6R^4}\mathfrak{P}_{22}, \ \mathfrak{B}_{23} = \tfrac{1}{6R^4}\mathfrak{P}_{23}, \ \mathfrak{B}_{24} = 0, \qquad \mathfrak{B}_{33} = \tfrac{1}{6R^4}\mathfrak{P}_{33}, \\ &\mathfrak{B}_{34} = 0, \qquad \mathfrak{B}_{44} = \tfrac{1}{6R^4}\mathfrak{P}_{44}, \end{aligned} \tag{2.22}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$\begin{aligned} \mathfrak{P}_{11} = {} & a^4 + 9a^2b^2 + 9a^2c^2 - (f+p)a^3 + 6(f+2p)ab^2 + 6(2f+p)ac^2 - (2f^2+2p^2+7fp)a^2 \\ & - 3f(5f+4p)b^2 - 3p(4f+5p)c^2 + 18h(f-p)bc + 3(f+p)(f^2+p^2)a \end{aligned}$$

$$- (f-p)^2(f^2+3h^2+p^2+fp),$$

$$\begin{aligned}\mathfrak{P}_{12} &= -12abc^2 - 2a^3b - 12ab^3 + 12fb^3 + 2(9f+5p)a^2b + 6ha^2c + 3(f+3p)bc^2 \\ &- (18f^2+3h^2-p^2)ab + 6h(2f-p)ac + (2f^3+12fh^2-10f^2p-fp^2-9h^2p)b \\ &+ 6h(f+p)(f-2p)c,\end{aligned}$$

$$\begin{aligned}\mathfrak{P}_{13} &= -12ab^2c - 2a^3c - 12ac^3 + 12pc^3 - 6ha^2b + 2(5f+9p)a^2c + 3(3f+p)b^2c \\ &+ 6h(f-2p)ab + (f^2-3h^2-18p^2)ac + 6h(f+p)(2f-p)b \\ &+ (2p^3-9fh^2-f^2p-10fp^2+12h^2p)c,\end{aligned}$$

$$\begin{aligned}\mathfrak{P}_{22} &= -a^4 - 15a^2b^2 - 3a^2c^2 - 18habc + (3f+p)a^3 + 6(f-2p)ab^2 + 6pac^2 \\ &- f(2f-3p)a^2 + 3f(3f+4p)b^2 - 3p^2c^2 + 18hpbc \\ &- (f^3-p^3-12fh^2+7f^2p-3fp^2+12h^2p)a + (f-p)(f^3+p^3+9fh^2-2fp^2+15h^2p),\end{aligned}$$

$$\begin{aligned}\mathfrak{P}_{23} &= -12a^2bc - 6(f+p)abc + 9hab^2 - 9hac^2 + h(f-p)a^2 - 3h(4f-p)b^2 \\ &- 3h(f-4p)c^2 + 6(f+p)^2bc + 10h(f^2-p^2)a - 2h(f-p)(f^2-8pf+6h^2+p^2),\end{aligned}$$

$$\begin{aligned}\mathfrak{P}_{33} &= -a^4 - 3a^2b^2 - 15a^2c^2 + 18habc + (f+3p)a^3 + 6fab^2 - 6(2f-p)ac^2 \\ &+ p(3f-2p)a^2 - 3f^2b^2 + 3p(4f+3p)c^2 - 18fhbc \\ &+ (f^3-p^3+3f^2p-7fp^2-12fh^2+12h^2p)a - (f-p)(f^3+p^3-2f^2p+15fh^2+9h^2p),\end{aligned}$$

$$\begin{aligned}\mathfrak{P}_{44} &= a^4 + 9a^2b^2 + 9a^2c^2 - 3(f+p)a^3 - 18fab^2 - 18pac^2 + (4f^2+4p^2+fp)a^2 \\ &+ 9f^2b^2 + 9p^2c^2 - (f+p)(3f^2+3p^2-4fp)a + (f-p)^2(f^2+9h^2+p^2-fp)\,.\end{aligned}$$

Hence, $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ admits a Bach-flat left-invariant metric if and only if the structure constants in Equation (2.21) satisfy the equations $\{\mathfrak{P}_{ij} = 0\}$. We consider separately the cases $a = 0$ and $a \neq 0$.

Case $a = 0$

Let $\mathcal{I}_0 \subset \mathbb{R}[b, f, c, h, p]$ be the ideal generated by the seven polynomials $\mathfrak{P}_{ij}$ in Equation (2.22). We compute a Gröbner basis $\mathcal{G}_0$ of $\mathcal{I}_0$ with respect to the graded reverse lexicographical order and get that it contains the polynomial

$$\mathbf{g}_0 = p^8(f-p)^2\,.$$

Since the zero sets of $\{\mathfrak{P}_{ij} = 0\}$ and $\mathcal{I}_0 = \langle \mathfrak{P}_{ij} \rangle = \langle \mathcal{G}_0 \rangle$ coincide, we are led to the following cases:

$$(1)\, p = 0, \qquad (2)\, f = p\,.$$

Case (1):

$a = 0$, $p = 0$. In this case, one checks using Equation (2.22) that

$$\mathfrak{P}_{44} = f^2(9b^2 + f^2 + 9h^2)$$

and therefore necessarily $f = 0$. Now, a direct calculation shows that, in such a case, the manifold is Einstein and therefore symmetric [70].

Case (2):

$a = 0$, $f = p$. Equation (2.22) implies that

$$\mathfrak{P}_{44} = 9(b^2 + c^2)p^2 \,.$$

Since $p = 0$ corresponds to Case (1), we have $b = c = 0$ and a direct calculation shows that the manifold is locally conformally flat and thus symmetric [99].

Case $a \neq 0$

Taking $a \neq 0$ in Equation (2.21), we may work with a homothetic basis $\widetilde{e}_k = \frac{1}{a} e_k$ so that we may assume, without loss of generality, $a = 1$.

Let $\mathcal{I} \subset \mathbb{R}[p, f, b, c, h]$ be the ideal generated by the seven polynomials $\mathfrak{P}_{ij}$ in Equation (2.22). Computing a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order we find that the following polynomial is in the basis:

$$\begin{aligned}\mathbf{g} = {} & (f-1)ch^2(24h^6 + 69h^4 - 12h^2 - 8)(128h^6 + 63h^4 - 324h^2 - 216)(16h^2 + 9) \\ & \times (16h^2+81)(80h^4+95h^2+32)(464h^4+2175h^2+1824)(2116h^4+4884h^2+1089) \\ & \times (49532953600h^{12} + 100931329200h^{10} + 67210421265h^8 + 16039857600h^6 \\ & \quad + 1904414976h^4 + 177168384h^2 + 11943936) \\ & \times (14705175456768h^{12} - 11441136851376h^{10} + 3165906982755h^8 \\ & \quad + 580502490560h^6 + 263837594880h^4 + 2944180224h^2 + 127844352) \,.\end{aligned}$$

Note that only the first five factors provide real roots, so we consider the following cases:

$$\begin{aligned}& (1)\, f = 1, \qquad (2)\, c = 0, \qquad (3)\, h = 0, \\ & (4)\, (24h^6 + 69h^4 - 12h^2 - 8)(128h^6 + 63h^4 - 324h^2 - 216) = 0 \,.\end{aligned}$$

Case (1):

$a = 1$, $f = 1$. We compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{f - 1\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order, and we get that

$$\mathbf{g}_1 = (p-1)c^2 \quad \text{and} \quad \mathbf{g}_1' = (p-1)h^2$$

belong to $\mathcal{G}_1$. Thus, we have two possibilities:

$$(1.i)\, p = 1, \qquad (1.ii)\, c = h = 0\,.$$

Case (1.i): $a = 1$, $f = 1$, $p = 1$. In this case, a direct calculation shows that the manifold is Einstein and therefore symmetric [70].

Case (1.ii): $a = 1$, $f = 1$, $c = h = 0$. Equation (2.22) implies that

$$\mathfrak{P}_{44} = (p-1)^2 p(p-4)\,.$$

Note that $p = 1$ corresponds to the previous case and for $p = 0$ a direct calculation shows that the manifold is locally conformally flat and thus symmetric [99]. Now, if $p = 4$, Equation (2.22) shows that the manifold is Bach-flat and, moreover, one easily checks that it is non-symmetric. This is a particular case of Assertion (ii) in Lemma 2.14 if $b = 0$ (taking $\alpha = 1$ and using the homothetic isomorphism $\bar{e}_1 = e_1$, $\bar{e}_2 = e_3$, $\bar{e}_3 = e_2$, $\bar{e}_4 = e_4$). If $b \neq 0$, it corresponds to Assertion (i) just considering the homothetic isomorphism $\bar{e}_1 = \frac{1}{4}\, e_3$, $\bar{e}_2 = \frac{1}{4}\, e_2$, $\bar{e}_3 = \frac{1}{4}\, e_1$, $\bar{e}_4 = \frac{1}{4}\, e_4$.

Case (2):

$a = 1$, $c = 0$. We consider the ideal generated by $\mathcal{G} \cup \{c\} \subset \mathbb{R}[p, h, f, b, c]$ and compute a Gröbner basis $\mathcal{G}_2$ for it with respect to the lexicographical order, obtaining that the polynomial

$$\begin{aligned} \mathrm{g}_2 \;=\;& (f-1)b^2(b^4+90b^2+81)(5b^4+5b^2+2)(25b^4+2b^2+1) \\ & \times(49b^4+138b^2+9)(725b^4+8613b^2+2850)(2116b^4+4884b^2+1089) \end{aligned}$$

belongs to $\mathcal{G}_2$. Excluding $f = 1$ solved in Case (1), the only real root for g_2 corresponds to the factor b^2, so necessarily $b = 0$.

Next, we compute a new Gröbner basis $\mathcal{G}_2'$ for the ideal generated by $\mathcal{G}_2 \cup \{b\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order and we find that the polynomials

$$\begin{aligned} \mathrm{g}_2' &= (f-1)(4f-1)h^2(8h^2-1)(8h^2+3)(8h^2+9), \\ \mathrm{g}_2'' &= (f-1)(4f-1)h^2(320fh^4+128h^4+320fh^2+40f^2+152h^2-5f+4) \end{aligned}$$

belong to $\mathcal{G}_2'$. As a consequence, and since $f = 1$ was solved in Case (1), one easily checks that we have two possibilities:

$$(2.i)\, f = \tfrac{1}{4}, \qquad (2.ii)\, h = 0\,.$$

Case (2.i): $a = 1$, $c = 0$, $b = 0$, $f = \frac{1}{4}$. Computing a Gröbner basis $\mathcal{G}_{21}$ for the ideal generated by $\mathcal{G}_2' \cup \{4f - 1\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order we get that the polynomial

$$\mathrm{g}_{21} = (4p-1)(4p-9)$$

belongs to $\mathcal{G}_{21}$. Now, we have:

- If $p = \frac{1}{4}$, Equation (2.22) implies that the manifold is Bach-flat and, moreover, one easily checks that it is non-symmetric, corresponding to Assertion (i) in Lemma 2.14.
- If $p = \frac{9}{4}$, then we use again Equation (2.22) to get that the Bach-flat condition is equivalent to $h = 0$ and, in such a case, a direct calculation shows that the manifold is non-symmetric. This is a particular case of Assertion (ii) in Lemma 2.14, just taking $\alpha = \frac{1}{2}$ and considering the homothetic isomorphism $\bar{e}_1 = e_1$, $\bar{e}_2 = e_3$, $\bar{e}_3 = e_2$, $\bar{e}_4 = e_4$.

Case (2.ii): $a = 1$, $c = 0$, $b = 0$, $h = 0$. We compute a Gröbner basis $\mathcal{G}_{22}$ for the ideal generated by $\mathcal{G}_2' \cup \{h\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order and we find that the polynomial

$$\mathbf{g}_{22} = (f-1)((f-p)^2 - 2f - 2p + 1)(f^2 + f + 1)$$

belongs to $\mathcal{G}_{22}$. Excluding $f = 1$ solved in Case (1), it follows that necessarily

$$f = (1 + \sqrt{p})^2 \quad \text{or} \tag{2.23}$$

$$f = (-1 + \sqrt{p})^2 \tag{2.24}$$

and Equation (2.22) shows that the manifold is Bach-flat in both cases. Now, we set $p = \alpha^2$ with $f = (1 + \alpha)^2$ in the first possibility and $p = \beta^2$ with $f = (-1 + \beta)^2$ in the last one. Taking $\alpha = -1 + \beta$ and considering $\bar{e}_1 = e_1$, $\bar{e}_2 = e_3$, $\bar{e}_3 = e_2$, $\bar{e}_4 = e_4$, we get that the two possibilities are homothetic so we identify both cases. Moreover, a straightforward calculation shows that if $f = 0$ or $p = 0$ the manifold is locally conformally flat and thus symmetric [99], while it is non-symmetric if $f \cdot p \neq 0$. This last case corresponds to Assertion (ii) in Lemma 2.14.

Case (3):

$a = 1$, $h = 0$. We consider the ideal generated by $\mathcal{G} \cup \{h\} \subset \mathbb{R}[p, f, b, c, h]$ and compute a Gröbner basis $\mathcal{G}_3$ for it with respect to the lexicographical order, obtaining that the polynomial

$$\mathbf{g}_3 = (f-1)cb(14c^2 + 33)(5c^4 - 25c^2 + 32)(1421c^4 + 28623c^2 + 45600)$$

belongs to $\mathcal{G}_3$. Since $f = 1$ and $c = 0$ were solved in the previous cases, we get that necessarily $b = 0$.

Next, we compute a Gröbner basis $\mathcal{G}_3'$ for the ideal generated by $\mathcal{G}_3 \cup \{b\} \subset \mathbb{R}[p, f, b, c, h]$ with respect to the lexicographical order and the polynomial

$$\mathbf{g}_3' = (f-1)c^2(f-4)f(c^4 + 90c^2 + 81)(25c^4 + 2c^2 + 1)(49c^4 + 138c^2 + 9)$$

belongs to $\mathcal{G}_3'$. As a consequence, we must consider the following two possibilities:

$$(3.i)\, f = 0, \qquad (3.ii)\, f = 4\,.$$

Case (3.i): $a = 1$, $h = 0$, $b = 0$, $f = 0$. Equation (2.22) implies that the Bach-flat condition is equivalent to $p = 1$ and, in that case, a direct calculation shows that the manifold is locally conformally flat, and thus symmetric [99].

Case (3.ii): $a = 1$, $h = 0$, $b = 0$, $f = 4$. Assuming $c \neq 0$, since it was solved in Case (2), a straightforward calculation using Equation (2.22) shows that the Bach-flat condition is equivalent to $p = 1$. Moreover, a direct calculation shows that, in such a case, the manifold is not symmetric. If one takes $\bar{e}_1 = \frac{1}{4}\, e_2$, $\bar{e}_2 = \frac{1}{4}\, e_1$, $\bar{e}_3 = \frac{1}{4}\, e_3$, $\bar{e}_4 = \frac{1}{4}\, e_4$, then it corresponds to a homothetic case of Assertion (i) in Lemma 2.14.

Case (4):

$a = 1$, $(24h^6 + 69h^4 - 12h^2 - 8)(128h^6 + 63h^4 - 324h^2 - 216) = 0$. In this last case, it is hard to get a good Gröbner basis if we use $\mathcal{G}$ as the starting point as in the previous cases. Instead, we analyze in detail the polynomials in $\mathcal{G}$ (39 specifically) and we find that excluding the factors previously solved (i.e., factors involving $f - 1$, c and h), just one of those polynomials depends only on c and h and has the form

$$\mathbf{g}_4 = (f-1)ch^2 Q(c,h)$$

where $Q(c,h) = \delta c^4 + S(h)c^2 + T(h)$, with $\delta > 0$ and where $S(h)$, $T(h)$ are polynomials with only even powers of h.

In the last step, we use the polynomial $Q(c,h)$ to compute a Gröbner basis $\mathcal{G}_4$ of the ideal generated by

$$Q(c,h) \cup \{(24h^6 + 69h^4 - 12h^2 - 8)(128h^6 + 63h^4 - 324h^2 - 216)\} \subset \mathbb{R}[c,h]$$

with respect to the graded reverse lexicographical order and we find that

$$\begin{aligned}\mathbf{g}_4' = {} & 9408954328h^8 + 3490462417c^4h^4 + 8504049964c^2h^6 + 631105440c^6 \\ & + 48352913472h^6 + 4976629248c^4h^2 + 38523345312c^2h^4 + 5583229368c^4 \\ & + 72029134968h^4 + 37011199020c^2h^2 + 10563992784c^2 + 38487215664h^2 \\ & + 5890415904\end{aligned}$$

belongs to $\mathcal{G}_4$. Therefore we conclude that there is no solution in this case, finishing the proof. □

2.7 Left-invariant metrics on $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

Let $\mathfrak{g} = \mathfrak{g}_3 \times \mathbb{R}$ be a direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle_3$ denote its restriction to $\mathfrak{g}_3$. Following the work of Milnor [82], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \qquad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \qquad [\mathbf{v}_1, \mathbf{v}_2] = \lambda_3 \mathbf{v}_3, \tag{2.25}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_1\lambda_2\lambda_3 \neq 0$. Moreover, the associated Lie group corresponds to $SU(2)$ (resp. $SL(2,\mathbb{R})$) if $\lambda_1, \lambda_2, \lambda_3$ are all positive (resp. if any of $\lambda_1, \lambda_2, \lambda_3$ is negative).

Let $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ are given by Equation (2.25) and $\mathfrak{g}=\mathfrak{g}_3\oplus\mathbb{R}\mathbf{v}_4$. Since $\mathbb{R}\mathbf{v}_4$ needs not to be orthogonal to $\mathfrak{g}_3$, set $k_i=\langle\mathbf{v}_i,\mathbf{v}_4\rangle$, for $i=1,2,3$. Let $\hat{e}_4=\mathbf{v}_4-\sum_i k_i\mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1,\dots,e_4\}$ of $\mathfrak{g}=\mathfrak{g}_3\oplus\mathbb{R}$ so that

$$\begin{aligned}
&[e_1,e_2]=\lambda_3e_3, && [e_2,e_3]=\lambda_1e_1,\\
&[e_3,e_1]=\lambda_2e_2, && [e_1,e_4]=\tfrac{1}{R}(k_3\lambda_2e_2-k_2\lambda_3e_3),\\
&[e_2,e_4]=\tfrac{1}{R}(k_1\lambda_3e_3-k_3\lambda_1e_1), && [e_3,e_4]=\tfrac{1}{R}(k_2\lambda_1e_1-k_1\lambda_2e_2),
\end{aligned}\tag{2.26}$$

where $R>0$.

Lemma 2.15. *The Lie groups $\widetilde{SL(2,\mathbb{R})}\times\mathbb{R}$ and $SU(2)\times\mathbb{R}$ do not admit any non-symmetric Bach-flat left-invariant metric.*

Proof. Since the structure constants of $\mathfrak{g}_3$ satisfy $\lambda_1\lambda_2\lambda_3\neq0$, one may work with a homothetic basis $\widetilde{e}_k=\frac{1}{\lambda_1}e_k$ so that we may assume $\lambda_1=1$. A long but straightforward calculation shows that the components of the Bach tensor of $\widetilde{SL(2,\mathbb{R})}\times\mathbb{R}$ or $SU(2)\times\mathbb{R}$, with the structure constants in Equation (2.26), are given by

$$\begin{aligned}
&\mathfrak{B}_{11}=\tfrac{1}{24R^4}\mathfrak{P}_{11},\ \mathfrak{B}_{12}=\tfrac{1}{12R^4}\mathfrak{P}_{12},\ \mathfrak{B}_{13}=\tfrac{1}{12R^4}\mathfrak{P}_{13},\ \mathfrak{B}_{14}=\tfrac{1}{12R^3}\mathfrak{P}_{14},\\
&\mathfrak{B}_{22}=\tfrac{1}{24R^4}\mathfrak{P}_{22},\ \mathfrak{B}_{23}=\tfrac{1}{12R^4}\mathfrak{P}_{23},\ \mathfrak{B}_{24}=\tfrac{1}{12R^3}\mathfrak{P}_{24},\ \mathfrak{B}_{33}=\tfrac{1}{24R^4}\mathfrak{P}_{33},\\
&\mathfrak{B}_{34}=\tfrac{1}{12R^3}\mathfrak{P}_{34},\ \mathfrak{B}_{44}=\tfrac{1}{24R^4}\mathfrak{P}_{44},
\end{aligned}\tag{2.27}$$

where the polynomials $\mathfrak{P}_{ij}$'s correspond to:

$$\begin{aligned}
\mathfrak{P}_{11}&=-4(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3)k_1^4\\
&+4(3\lambda_3^4-\lambda_3^3+3\lambda_3-5)k_2^4\\
&+4(3\lambda_2^4-\lambda_2^3+3\lambda_2-5)k_3^4\\
&-(((\lambda_3-4)\lambda_3+24)\lambda_2^2-(8\lambda_3^2+4\lambda_3+3)\lambda_3^2+2(2\lambda_3^2+\lambda_3-6)\lambda_2\lambda_3)k_1^2k_2^2\\
&+(8\lambda_2^4-4(\lambda_3-1)\lambda_2^3-(\lambda_3-1)(\lambda_3+3)\lambda_2^2-24\lambda_3^2+4(\lambda_3+3)\lambda_2\lambda_3)k_1^2k_3^2\\
&+((4(6\lambda_3-1)\lambda_3+1)\lambda_2^2-2(2\lambda_3^2+\lambda_3-6)\lambda_2+(\lambda_3+12)\lambda_3-40)k_2^2k_3^2\\
&+R^2(\lambda_2-\lambda_3)^2(8\lambda_2^2+8\lambda_3^2+(8\lambda_3+4)\lambda_2+4\lambda_3+3)k_1^2\\
&-R^2(\lambda_3-1)((3\lambda_3+1)\lambda_2^2+4((3\lambda_3+2)\lambda_3+3)\lambda_2-8(((3\lambda_3+2)\lambda_3+2)\lambda_3+5))k_2^2\\
&-R^2(\lambda_2-1)((3\lambda_2+1)\lambda_3^2+4((3\lambda_2+2)\lambda_2+3)\lambda_3-8(((3\lambda_2+2)\lambda_2+2)\lambda_2+5))k_3^2\\
&+4R^4(3\lambda_2^4-(3\lambda_3+1)\lambda_2^3+(3\lambda_3-1)\lambda_3^3+\lambda_2^2\lambda_3+((-3\lambda_3^2+\lambda_3-1)\lambda_3+3)\lambda_2+3\lambda_3-5),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{12}&=(\lambda_2-\lambda_3^2)((8\lambda_2+5)\lambda_2+8)k_1k_2k_3^2\\
&-(8\lambda_3^4-8\lambda_2^3-(\lambda_3-4)\lambda_2^2\lambda_3-(4\lambda_3-1)\lambda_2\lambda_3^2)k_1^3k_2\\
&-(8\lambda_3^4-4\lambda_3^3-\lambda_3^2+(\lambda_3+4)\lambda_2\lambda_3-8\lambda_2)k_1k_2^3\\
&-R^2(8\lambda_3^4-4(\lambda_2+1)\lambda_3^3-((\lambda_2-3)\lambda_2+1)\lambda_3^2+10(\lambda_2+1)\lambda_2\lambda_3-((8\lambda_2+5)\lambda_2+8)\lambda_2)k_1k_2,
\end{aligned}$$

$$\mathfrak{P}_{13}=-(\lambda_2^2-\lambda_3)((8\lambda_3+5)\lambda_3+8)k_1k_2^2k_3$$

$$
\begin{aligned}
&- (8\lambda_2^4 - 8\lambda_3^3 - (4\lambda_2 - 1)\lambda_2^2\lambda_3 - (\lambda_2 - 4)\lambda_2\lambda_3^2)k_1^3k_3\\
&+ (8\lambda_3 - (8\lambda_3^2 \ - 4\lambda_2^2 + \lambda_2\lambda_3 - \lambda_2 + 4\lambda_3)\lambda_2)k_1k_3^3\\
&- R^2(8\lambda_2^4 - 4(\lambda_3 + 1)\lambda_2^3 - ((\lambda_3 - 3)\lambda_3 + 1)\lambda_2^2 + 10(\lambda_3 + 1)\lambda_2\lambda_3 - ((8\lambda_3 + 5)\lambda_3 + 8)\lambda_3)k_1k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{14} &= -8(\lambda_2 - \lambda_3)^2(\lambda_2^2 + \lambda_3^2 + \lambda_3\lambda_2)k_1^3\\
&- (8\lambda_3^4 - 4(\lambda_2 + 1)\lambda_3^3 - (\lambda_2 - 1)^2\lambda_3^2 + 8\lambda_2^2 - 4(\lambda_2 + 1)\lambda_2\lambda_3)k_1k_2^2\\
&- (8\lambda_2^4 - 4(\lambda_3 + 1)\lambda_2^3 - (\lambda_3 - 1)^2\lambda_2^2 + 8\lambda_3^2 - 4(\lambda_3 + 1)\lambda_2\lambda_3)k_1k_3^2\\
&- R^2(\lambda_2 - \lambda_3)^2(8\lambda_2^2 + 8\lambda_3^2 + (8\lambda_3 - 4)\lambda_2 - 4\lambda_3 - 1)k_1,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{22} &= -4(5\lambda_2^4 - 3\lambda_3^4 - 3\lambda_2^3\lambda_3 + \lambda_2\lambda_3^3)k_1^4\\
&- 4(\lambda_3^4 - \lambda_3^3 - \lambda_3 + 1)k_2^4\\
&- 4(5\lambda_2^4 - 3\lambda_2^3 + \lambda_2 - 3)k_3^4\\
&+ (3((\lambda_3 + 4)\lambda_3 - 8)\lambda_2^2 + (8\lambda_3^2 - 4\lambda_3 - 1)\lambda_3^2 + 2((2\lambda_3 - 1)\lambda_3 + 2)\lambda_2\lambda_3)k_1^2k_2^2\\
&- (40\lambda_2^4 - 12(\lambda_3 + 1)\lambda_2^3 - (\lambda_3 - 1)^2\lambda_2^2 - 24\lambda_3^2 + 4(\lambda_3 + 1)\lambda_2\lambda_3)k_1^2k_3^2\\
&- (3(8\lambda_3^2 - 4\lambda_3 - 1)\lambda_2^2 - (4\lambda_3^2 - 2\lambda_3 + 4)\lambda_2 + (\lambda_3 + 4)\lambda_3 - 8)k_2^2k_3^2\\
&- R^2(40\lambda_2^4 - 12(2\lambda_3 + 1)\lambda_2^3 + (4\lambda_3 - 1)\lambda_2^2 - 3(8\lambda_3^2 - 4\lambda_3 - 1)\lambda_3^2 - 2(-4\lambda_3^2 + 2\lambda_3 + 1)\lambda_2\lambda_3)k_1^2\\
&+ R^2(\lambda_3 - 1)^2(3\lambda_2^2 + 4(\lambda_3 + 1)\lambda_2 + 8(\lambda_3^2 + \lambda_3 + 1))k_2^2\\
&+ R^2(\lambda_2 - 1)((\lambda_2 + 3)\lambda_3^2 + 4((3\lambda_2 + 2)\lambda_2 + 3)\lambda_3 - 8((\lambda_2(5\lambda_2 + 2) + 2)\lambda_2 + 3))k_3^2\\
&- 4R^4(5\lambda_2^4 - 3(\lambda_3 + 1)\lambda_2^3 + \lambda_2^2\lambda_3 + (\lambda_3 - 1)^2(\lambda_3 + 1)\lambda_2 - 3(\lambda_3^4 - \lambda_3^3 - \lambda_3 + 1)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{23} &= (\lambda_2\lambda_3 - 1)(8\lambda_2^2 + 8\lambda_3^2 + 5\lambda_2\lambda_3)k_1^2k_2k_3\\
&+ ((\lambda_3 + (8\lambda_3^2 - 4\lambda_3 - 1)\lambda_2 + 4)\lambda_3 - 8)k_2^3k_3\\
&+ (((8\lambda_2 - 4)\lambda_2\lambda_3 + \lambda_2 - \lambda_3 + 4)\lambda_2 - 8)k_2k_3^3\\
&+ R^2(8\lambda_2^3\lambda_3 + (5(\lambda_3 - 2)\lambda_3 + 1)\lambda_2^2 + ((2\lambda_3 - 3)(4\lambda_3 + 1)\lambda_3 + 4)\lambda_2 + (\lambda_3 + 4)\lambda_3 - 8)k_2k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{24} &= -8(\lambda_3^4 - \lambda_3^3 - \lambda_3 + 1)k_2^3\\
&- (8\lambda_3^4 - 4(\lambda_2 + 1)\lambda_3^3 + 8\lambda_2^2 - (\lambda_2 - 1)^2\lambda_3^2 - 4(\lambda_2 + 1)\lambda_2\lambda_3)k_1^2k_2\\
&- ((8\lambda_3^2 - 4\lambda_3 - 1)\lambda_2^2 - (4\lambda_3^2 - 2\lambda_3 + 4)\lambda_2 - (\lambda_3 + 4)\lambda_3 + 8)k_2k_3^2\\
&+ R^2(\lambda_3 - 1)^2(\lambda_2^2 + 4(\lambda_3 + 1)\lambda_2 - 8(\lambda_3^2 + \lambda_3 + 1))k_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{33} &= 4(3\lambda_2^4 - 5\lambda_3^4 - \lambda_2^3\lambda_3 + 3\lambda_2\lambda_3^3)k_1^4\\
&- 4(5\lambda_3^4 - 3\lambda_3^3 + \lambda_3 - 3)k_2^4\\
&- 4(\lambda_2^4 - \lambda_2^3 - \lambda_2 + 1)k_3^4\\
&- (40\lambda_3^4 - 12(\lambda_2 + 1)\lambda_3^3 - 24\lambda_2^2 - (\lambda_2 - 1)^2\lambda_3^2 + 4(\lambda_2 + 1)\lambda_2\lambda_3)k_1^2k_2^2\\
&+ (8\lambda_2^4 + 4(\lambda_3 - 1)\lambda_2^3 + (\lambda_3 - 1)(3\lambda_3 + 1)\lambda_2^2 - 24\lambda_3^2 + 4(3\lambda_3 + 1)\lambda_2\lambda_3)k_1^2k_3^2\\
&- ((4(6\lambda_3 - 1)\lambda_3 + 1)\lambda_2^2 - 2(2\lambda_3 + 1)(3\lambda_3 - 2)\lambda_2 - (3\lambda_3 + 4)\lambda_3 - 8)k_2^2k_3^2\\
&+ R^2(24\lambda_2^4 - 40\lambda_3^4 - 4(2\lambda_3 + 3)\lambda_2^3 + 12\lambda_3^3 + (4\lambda_3 - 3)\lambda_2^2 + \lambda_3^2 + 2(2(6\lambda_3 - 1)\lambda_3 + 1)\lambda_2\lambda_3)k_1^2\\
&+ R^2(\lambda_3 - 1)((\lambda_3 + 3)\lambda_2^2 + 4((3\lambda_3 + 2)\lambda_3 + 3)\lambda_2 - 8(((5\lambda_3 + 2)\lambda_3 + 2)\lambda_3 + 3))k_2^2\\
&+ R^2(\lambda_2 - 1)^2(8\lambda_2^2 + 4(\lambda_3 + 2)\lambda_2 + (3\lambda_3 + 4)\lambda_3 + 8)k_3^2
\end{aligned}
$$

$$
\begin{aligned}
&- 4R^4(5\lambda_3^4 - 3(\lambda_2+1)\lambda_3^3 + \lambda_2\lambda_3^2 + (\lambda_2-1)^2(\lambda_2+1)\lambda_3 - 3(\lambda_2-1)^2(\lambda_2^2+\lambda_2+1)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{34} &= -8(\lambda_2^4 - \lambda_2^3 - \lambda_2 + 1)k_3^3\\
&- (8\lambda_2^4 - 4(\lambda_3+1)\lambda_2^3 - (\lambda_3-1)^2\lambda_2^2 + 8\lambda_3^2 - 4(\lambda_3+1)\lambda_2\lambda_3)k_1^2k_3\\
&- ((8\lambda_3^2 - 4\lambda_3 - 1)\lambda_2^2 - (4\lambda_3^2 - 2\lambda_3 + 4)\lambda_2 - (\lambda_3+4)\lambda_3 + 8)k_2^2k_3\\
&- R^2(\lambda_2-1)^2(8\lambda_2^2 - 4(\lambda_3-2)\lambda_2 - (\lambda_3+4)\lambda_3 + 8)k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{44} &= 12(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_3\lambda_2)k_1^4\\
&+ 12(\lambda_3-1)^2(\lambda_3^2+\lambda_3+1)k_2^4\\
&+ 12(\lambda_2-1)^2(\lambda_2^2+\lambda_2+1)k_3^4\\
&+ 3(8\lambda_3^4 - 4(\lambda_2+1)\lambda_3^3 + 8\lambda_2^2 - (\lambda_2-1)^2\lambda_3^2 - 4(\lambda_2+1)\lambda_2\lambda_3)k_1^2k_2^2\\
&+ 3(8\lambda_2^4 - 4(\lambda_3+1)\lambda_2^3 - (\lambda_3-1)^2\lambda_2^2 + 8\lambda_3^2 - 4(\lambda_3+1)\lambda_2\lambda_3)k_1^2k_3^2\\
&+ 3((8\lambda_3^2 - 4\lambda_3 - 1)\lambda_2^2 - 2(2\lambda_3^2 - \lambda_3 + 2)\lambda_2 - (\lambda_3+4)\lambda_3 + 8)k_2^2k_3^2\\
&+ R^2(\lambda_2-\lambda_3)^2(8\lambda_2^2 + 8\lambda_3^2 + (8\lambda_3-4)\lambda_2 - 4\lambda_3 - 1)k_1^2\\
&- R^2(\lambda_3-1)^2(\lambda_2^2 + 4(\lambda_3+1)\lambda_2 - 8(\lambda_3^2+\lambda_3+1))k_2^2\\
&+ R^2(\lambda_2-1)^2(8\lambda_2^2 - 4(\lambda_3-2)\lambda_2 - (\lambda_3+4)\lambda_3 + 8)k_3^2\\
&- 4R^4(\lambda_2^4 - (\lambda_3+1)\lambda_2^3 + \lambda_2^2\lambda_3 - (\lambda_3-1)^2(\lambda_3+1)\lambda_2 + (\lambda_3-1)^2(\lambda_3^2+\lambda_3+1))\,.
\end{aligned}
$$

Therefore, $\widetilde{SL(2,\mathbb{R})}\times\mathbb{R}$ or $SU(2)\times\mathbb{R}$ admit a Bach-flat left-invariant metric if and only if the structure constants in Equation (2.26) satisfy the equations $\{\mathfrak{P}_{ij}=0\}$.

Let $\mathcal{I}\subset\mathbb{R}[\lambda_2,\lambda_3,k_1,k_2,k_3,R]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded reverse lexicographical order. A detailed analysis of the Gröbner basis shows that the polynomial

$$
\mathbf{g}_0 = (\lambda_2-\lambda_3)k_1k_2^2k_3^2(k_2^2+k_3^2+R^2)(k_1^2+k_2^2+k_3^2+R^2) \tag{2.28}
$$

belongs to the basis. Since the zero sets of $\{\mathfrak{P}_{ij}=0\}$ and $\mathcal{I}=\langle\mathfrak{P}_{ij}\rangle=\langle\mathcal{G}\rangle$ coincide, we are led to the following cases:

$$
(1)\,\lambda_2=\lambda_3, \qquad (2)\,k_1=0, \qquad (3)\,k_2=0, \qquad (4)\,k_3=0\,.
$$

Case (1):

$\lambda_2=\lambda_3$. A direct calculation using Equation (2.27) implies that

$$
\mathfrak{P}_{14} = -3(\lambda_3-1)^2k_1(k_2^2+k_3^2)\lambda_3^2
$$

and therefore we have the following possibilities:

$$
(1.i)\,\lambda_3=1, \qquad (1.ii)\,k_1=0, \qquad (1.iii)\,k_2=k_3=0\,.
$$

Case (1.i):

$\lambda_2 = \lambda_3$, $\lambda_3 = 1$. In this case we have $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and a direct calculation shows that the corresponding Lie group given by Equation (2.26) is locally conformally flat, and thus a symmetric manifold [99].

Case (1.ii):

$\lambda_2 = \lambda_3$, $k_1 = 0$. Computing a Gröbner basis of the ideal generated by $\mathcal{G} \cup \{\lambda_2 - \lambda_3, k_1\}$ with respect to the graded reverse lexicographical order, we find that the polynomial

$$\mathbf{g}_{12} = (\lambda_3 - 1)(k_2^2 + k_3^2 + R^2)^3 R^2$$

belongs to the ideal, leading to the solution $\lambda_3 = 1$ in Case (1.i).

Case (1.iii):

$\lambda_2 = \lambda_3$, $k_2 = k_3 = 0$. A direct calculation using Equation (2.27) shows that

$$\mathfrak{P}_{44} = -4(\lambda_3 - 1)^2 R^4$$

and thus $\lambda_3 = 1$, which corresponds to Case (1.i).

Case (2):

$k_1 = 0$. Computing a Gröbner basis $\mathcal{G}_2$ of the ideal generated by $\mathcal{G} \cup \{k_1\}$ with respect to the graded reverse lexicographical order, one has that the polynomial

$$\mathbf{g}_2 = k_2 k_3 (\lambda_2 - \lambda_3)(k_3^2 + 2R^2)(k_2^2 + k_3^2 + R^2)^2$$

belongs to the basis. Since $\lambda_2 = \lambda_3$ was solved in Case (1), we have the following possibilities:

$$(2.i)\; k_2 = 0, \qquad (2.ii)\; k_3 = 0\,.$$

Case (2.i):

$k_1 = 0$, $k_2 = 0$. We compute a Gröbner basis $\mathcal{G}_{21}$ of the ideal generated by $\mathcal{G}_2 \cup \{k_2\}$ with respect to the lexicographical order and we obtain that the polynomial $(\lambda_3 - 1)^2 \lambda_3^4 R^6$ belongs to the basis. Since $\lambda_3 \neq 0$ the only possible solution is $\lambda_3 = 1$. Now, computing a new Gröbner basis $\mathcal{G}'_{21}$ of the ideal generated by $\mathcal{G}_{21} \cup \{\lambda_3 - 1\}$ with respect to the graded reverse lexicographical order we find that the polynomial $(\lambda_2 - 1)\lambda_2^2 R^4$ belongs to the basis. Thus we get the solution $\lambda_2 = \lambda_3 = 1$ which corresponds to Case (1.i).

Case (2.ii):

$k_1 = 0$, $k_3 = 0$. Considering the ideal $\mathcal{G}_2 \cup \{k_3\}$ and computing a Gröbner basis with respect to the lexicographical order, we find that the polynomial $k_2(\lambda_3 - 1)^2(k_2^2 + R^2)^3$ belongs to the basis. Since $k_2 = 0$ was treated in the previous case, we have $\lambda_3 = 1$, which together with $k_1 = k_3 = 0$ let us to get $\mathfrak{P}_{44} = -4(\lambda_2 - 1)^2\lambda_2^2 R^4$ from Equation (2.27). Hence, necessarily $\lambda_2 = \lambda_3 = 1$ and we are again in Case (1.i).

Case (3):

$k_2 = 0$. Computing a Gröbner basis $\mathcal{G}_3$ of the ideal generated by $\mathcal{G} \cup \{k_2\}$ with respect to the graded reverse lexicographical order, we find that the polynomial

$$\mathbf{g}_3 = (\lambda_2 - 1)(\lambda_3 - 1)k_3(k_3^2 + R^2)^2(k_1^2 + k_3^2 + R^2)R^2$$

belongs to the basis. Therefore we consider the following possibilities:

$$(3.i)\ \lambda_2 = 1, \qquad (3.ii)\ \lambda_3 = 1, \qquad (3.iii)\ k_3 = 0\,.$$

Case (3.i):

$k_2 = 0$, $\lambda_2 = 1$. Adding the polynomial $\lambda_2 - 1$ to $\mathcal{G}_3$ and computing a Gröbner basis with respect to the lexicographical order, we find that the polynomial $(\lambda_3 - 1)k_1^2(3k_3^2 + R^2)R^2$ belongs to the basis. Therefore, we are led to the previously considered Case (1) or Case (2).

Case (3.ii):

$k_2 = 0$, $\lambda_3 = 1$. Adding the polynomial $\lambda_3 - 1$ to $\mathcal{G}_3$ and computing a Gröbner basis with respect to the lexicographical order, we find that the polynomial $(\lambda_2 - 1)k_1 R^4$ belongs to the basis. Therefore, we are led to the previously considered Case (1) or Case (2).

Case (3.iii):

$k_2 = 0$, $k_3 = 0$. Adding the polynomial k_3 to $\mathcal{G}_3$ and computing a Gröbner basis with respect to the lexicographical order, we find in this case that the polynomial $(\lambda_3 - 1)^2\lambda_3^2 R^6$ belongs to the basis. This leads to Case (3.ii).

Case (4):

$k_3 = 0$. Computing a Gröbner basis $\mathcal{G}_4$ of the ideal generated by $\mathcal{G} \cup \{k_3\}$ with respect to the graded reverse lexicographical order, we find that the polynomial

$$\mathbf{g}_4 = k_1 k_2(\lambda_3 - 1)(k_2^2 + R^2)^2(k_1^2 + k_2^2 + R^2)(k_1^2 + k_2^2 + 4R^2)R^2$$

belongs to the basis. Since the cases $k_1 = 0$ and $k_2 = 0$ were already considered, one necessarily has $\lambda_3 = 1$. Using Equation (2.27), since $k_3 = 0$ and $\lambda_3 = 1$ we get $\mathfrak{P}_{24} = -3k_1^2 k_2(\lambda_2 - 1)^2$. Therefore, $\lambda_2 = 1 = \lambda_3$ and this leads again to Case (1), finishing the proof. $\square$

2.8 Conformally Einstein four-dimensional Lie groups

The purpose of this section is to complete the proof of Theorem 2.1 based on the analysis in sections 2.4–2.7.

Proof of Assertion (i) in Theorem 2.1. Consider the different Lie groups given by Lemma 2.14. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the left-invariant metric determined by Lemma 2.14–(i). Considering the homothetic metric $\langle\,\cdot\,,\,\cdot\,\rangle^* = \frac{27}{8}\langle\,\cdot\,,\,\cdot\,\rangle$, we obtain that the Ricci operator of $\langle\,\cdot\,,\,\cdot\,\rangle^*$ takes the form $\mathrm{Ric} = -\frac{1}{9}\,\mathrm{diag}[4,1,1,3]$ in the basis $\{e_1,\dots,e_4\}$. Moreover, the self-dual and anti-self-dual Weyl curvature operators become $W^\pm = \frac{1}{27}\,\mathrm{diag}[1,1,-2]$ in the induced basis of self-dual and anti-self-dual two-forms. The expressions of $W^\pm$ show that the Weyl curvature operator has maximal rank. Hence, the necessary condition in Theorem 1.14–(ii) to be conformally Einstein is also sufficient. Let $\mathbb{T} \in \mathfrak{g}$ be an arbitrary vector and set $\mathbb{T} = \sum_k \mathbb{T}^k e_k$. A straightforward calculation shows that $(\mathrm{div}_4\, W)(e_i,e_j,e_k) - W(e_i,e_j,e_k,\mathbb{T}) = 0$ if and only if $\mathbb{T} = -\frac{2}{9}\,e_4$. This shows that left-invariant metrics given by Lemma 2.14–(i) are conformally Einstein.

Now, denoting by W_{ij} the Weyl endomorphism given by $W(e_i,e_j)$, the non-zero components of the Weyl tensor of type (1,3) are given by:

$$W_{12} = \begin{pmatrix} 0 & \frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_{13} = \begin{pmatrix} 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{14} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{pmatrix}, \quad W_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{8} & 0 & 0 \end{pmatrix}, \quad W_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & -\frac{1}{8} & 0 \end{pmatrix}.$$

Since the Weyl tensor of type (1,3) does not depend on α, then it now follows from the work of Hall [63] that all the left-invariant metrics in Lemma 2.14–(i) are homothetic (but not necessarily isomorphic). This completes the proof of Assertion (i) in Theorem 2.1. □

Proof of Assertion (ii) in Theorem 2.1. Let $(G_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a half conformally flat Lie group given by Lemma 2.13–(i) (see also Theorem 1.30). Following [47], let $\{e^k\}$ denote the dual basis of $\{e_k\}$ so that the structure equations are given by

$$\begin{aligned} &de^4 = 0, && de^1 = e^1\wedge e^4 + \alpha e^2 \wedge e^4, \\ &de^2 = -\alpha e^1 \wedge e^4 + e^2\wedge e^4, && de^3 = 2e^3\wedge e^4 - e^1\wedge e^2\,. \end{aligned} \tag{2.29}$$

Integrating the expressions above gives coordinates (x, y, z, t) on $\mathbb{R}^4$ where (see [47])

$$e^1 = e^{-t}(dx + \alpha y dt), \quad e^2 = e^{-t}(dy - \alpha x dt),$$
$$e^3 = -e^{-2t}\left(dz + \tfrac{1}{2}(xdy - ydx) - \tfrac{1}{2}\alpha(x^2+y^2)dt\right), \quad e^4 = dt,$$

so that the metric expresses as

$$\begin{aligned} g_\alpha = {} & e^{-2t}(dx + \alpha y dt)^2 + e^{-2t}(dy - \alpha x dt)^2 \\ & + e^{-4t}(dz + \tfrac{1}{2}(xdy - ydx) - \tfrac{1}{2}\alpha(x^2+y^2)dt)^2 + dt^2 . \end{aligned} \tag{2.30}$$

Now, a straightforward calculation shows that the conformal metric $\widetilde{g}_\alpha = e^{3t} g_\alpha$ is Ricci-flat, and thus $(G_\alpha, \langle \cdot\,, \cdot \rangle)$ is conformally Einstein. This proves Assertion (ii) in Theorem 2.1. □

Proof of Assertion (iii) in Theorem 2.1. Let $(\mathfrak{g}_\alpha, \langle \cdot\,, \cdot \rangle_\alpha)$ be a Lie algebra given by Lemma 2.14–(ii), and set

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha+1)^2 e_2, \quad [e_4, e_3] = \alpha^2 e_3, \qquad \alpha > 0,$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

Considering the homothetic metric $\langle \cdot\,, \cdot \rangle_\alpha^* = 6(\alpha^2+\alpha+1)^2 \langle \cdot\,, \cdot \rangle_\alpha$, the Ricci operator of $\langle \cdot\,, \cdot \rangle^*$ and the self-dual and anti-self-dual Weyl curvature operators take the forms

$$\begin{aligned} \mathrm{Ric}_\alpha &= -\tfrac{1}{3(\alpha^2+\alpha+1)} \operatorname{diag}[1, (\alpha+1)^2, \alpha^2, \alpha^2+\alpha+1], \\ W_\alpha^+ &= \tfrac{\alpha(\alpha+1)}{6(\alpha^2+\alpha+1)^2} \operatorname{diag}[\alpha, -(\alpha+1), 1] = W_\alpha^-, \end{aligned} \tag{2.31}$$

when expressed in the $\langle \cdot\,, \cdot \rangle_\alpha^*$-orthogonal basis $\{e_1, \dots, e_4\}$ and the induced basis of two-forms. Therefore $W_\alpha^\pm$ has three-distinct eigenvalues unless $\alpha = 1$.

The necessary condition in Theorem 1.14–(ii) to be conformally Einstein is also sufficient in this case since by Equation (2.31) the Weyl tensor has maximal rank. Let $\mathbb{T} \in \mathfrak{g}_\alpha$ be an arbitrary vector and set $\mathbb{T} = \sum_k \mathbb{T}^k e_k$. A straightforward calculation shows that $(\operatorname{div}_4 W)(e_i, e_j, e_k) - W(e_i, e_j, e_k, \mathbb{T}) = 0$ if and only if $\mathbb{T} = -\frac{1}{6(\alpha^2+\alpha+1)} e_4$. This shows that left-invariant metrics given by Lemma 2.14–(ii) are conformally Einstein.

In the special case $\alpha = 1$, one has that the Ricci is given by $\mathrm{Ric} = -\frac{1}{9}\operatorname{diag}[1, 4, 1, 3]$, $W^+ = W^- = \frac{1}{27}\operatorname{diag}[1, -2, 1]$ and considering a new basis $\bar{e}_1 = \frac{1}{4}e_2$, $\bar{e}_2 = \frac{1}{4}e_1$, $\bar{e}_3 = \frac{1}{4}e_3$, $\bar{e}_4 = \frac{1}{4}e_4$, the non-zero components of the Weyl tensor of type (1,3) are given by:

$$W_{12} = \begin{pmatrix} 0 & \frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_{13} = \begin{pmatrix} 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{14} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{pmatrix}, \quad W_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{8} & 0 & 0 \end{pmatrix}, \quad W_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & -\frac{1}{8} & 0 \end{pmatrix}.$$

Now, it follows from the work of Hall [63] that the left-invariant metric for $\alpha = 1$ is homothetic to left-invariant metrics in Lemma 2.14–(i). Hence we assume $\alpha \neq 1$.

Furthermore, replacing α by α^{-1} in Equation (2.31) one has that $e_1 \mapsto e_3$ defines an orientation reversing homothety between the left-invariant metrics $\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\alpha^{-1}}$. We therefore may assume $\alpha > 1$. Considering the homothetic metric $\langle\,\cdot\,,\,\cdot\,\rangle^*_\alpha$, a straightforward calculation shows that $\tau_\alpha = -1$ and $\|\rho_\alpha\|^2 = \frac{1}{3}$. Moreover, the norm of the Weyl tensor satisfies $\|W_\alpha\|^2 = \frac{4\alpha^2(\alpha+1)^2}{9(\alpha^2+\alpha+1)^3}$. Hence two metrics $\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$ and $\langle\,\cdot\,,\,\cdot\,\rangle_\beta$ with $\alpha, \beta \in (1, +\infty)$ are homothetic if and only if $\alpha^2(\alpha+1)^2(\beta^2+\beta+1)^3 = \beta^2(\beta+1)^2(\alpha^2+\alpha+1)^3$, and thus $\alpha = \beta$.

□

2.9 Strictly Bach-flat four-dimensional Lie groups

The purpose of this section is to complete the proof of Theorem 2.4 based on the analysis in sections 2.4–2.7.

Proof of Theorem 2.4. Consider the left-invariant metric on $\mathbb{R}e_4 \ltimes E(1,1)$ given in Lemma 2.12. The Lie brackets are given, with respect to an orthonormal basis $\{e_1, \ldots, e_4\}$, by

$$[e_1, e_3] = (2+\sqrt{3})e_2, \quad [e_2, e_3] = e_1,$$
$$[e_4, e_1] = \sqrt{6+3\sqrt{3}}e_1, \quad [e_4, e_2] = \sqrt{6+3\sqrt{3}}e_2\,.$$

Now, an explicit calculation shows that the Ricci operator, in the basis $\{e_1, \ldots, e_4\}$, takes the form $\mathrm{Ric} = -(2+\sqrt{3})\,\mathrm{diag}[6+\sqrt{3}, 6-\sqrt{3}, 3, 6]$.

Let $\{E_i^\pm\}$ be the corresponding orthonormal basis of self-dual and anti-self-dual two-forms: $E_1^\pm = \frac{1}{\sqrt{2}}(e^1 \wedge e^2 \pm e^3 \wedge e^4)$, $E_2^\pm = \frac{1}{\sqrt{2}}(e^1 \wedge e^3 \mp e^2 \wedge e^4)$, and $E_3^\pm = \frac{1}{\sqrt{2}}(e^1 \wedge e^4 \pm e^2 \wedge e^3)$. Then, the self-dual and anti-self-dual Weyl curvature operators are given by

$$W^+ = \tfrac{2+\sqrt{3}}{2}\,\mathrm{diag}[2, -1-3\sqrt{2}-\sqrt{3}, -1+3\sqrt{2}+\sqrt{3}],$$
$$W^- = \tfrac{2+\sqrt{3}}{2}\,\mathrm{diag}[2, -1+3\sqrt{2}-\sqrt{3}, -1-3\sqrt{2}+\sqrt{3}]\,.$$

Finally, observe that the metric in Lemma 2.12 is not conformally Einstein. Indeed, considering the corresponding left-invariant metric $\langle\cdot,\cdot\rangle$, a straightforward calculation shows that, for any vector $\mathbb{T} \in \mathfrak{g}$, the necessary condition in Theorem 1.14–(ii) gives

$$(\mathrm{div}_4 W)(e_1, e_2, e_3) - W(e_1, e_2, e_3, \mathbb{T}) = \frac{3}{2}(5+3\sqrt{3}) \neq 0,$$

and thus $(G, \langle\,\cdot\,,\,\cdot\,\rangle)$ is strictly Bach-flat.

Consider the left-invariant metrics on $\mathbb{R}e_4 \ltimes H^3$ at Lemma 2.13–(ii). The Lie brackets are given, with respect to an orthonormal basis $\{e_1, \dots, e_4\}$, by

$$[e_1, e_2] = e_3, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}} e_3,$$
$$[e_4, e_1] = \tfrac{1}{4}\sqrt{7 - 3\sqrt{5}}\, e_1, \quad [e_2, e_4] = \tfrac{1}{4}\sqrt{7 + 3\sqrt{5}}\, e_2 .$$

Now, a explicit calculation shows that the Ricci operator, in the basis $\{e_1, \dots, e_4\}$, takes the form $\mathrm{Ric} = -\frac{3}{8}\,\mathrm{diag}[3 - \sqrt{5}, 3 + \sqrt{5}, 2, 4]$, and the self-dual and anti-self-dual Weyl curvature operators are given by

$$W^+ = -\tfrac{1}{8}\,\mathrm{diag}[2 + \sqrt{10}, -1 - \sqrt{7 + 3\sqrt{5}}, -1 + \sqrt{7 - 3\sqrt{5}}],$$
$$W^- = -\tfrac{1}{8}\,\mathrm{diag}[2 - \sqrt{10}, -1 + \sqrt{7 + 3\sqrt{5}}, -1 - \sqrt{7 - 3\sqrt{5}}] .$$

In order to show that the left-invariant metrics in Lemma 2.13–(ii) are strictly Bach-flat, we consider the necessary condition in Theorem 1.14–(ii) to be conformally Einstein. Let $\mathbb{T} \in \mathfrak{g}$ be an arbitrary vector and set $\mathbb{T} = \sum_k \mathbb{T}^k e_k$. Then one has

$$(\mathrm{div}_4 W)(e_1, e_2, e_3) - W(e_1, e_2, e_3, \mathbb{T}) = \tfrac{1}{16}(3 + 2\sqrt{10}\mathbb{T}^4),$$
$$(\mathrm{div}_4 W)(e_1, e_4, e_1) - W(e_1, e_4, e_1, \mathbb{T}) = -\tfrac{1}{32}(3\sqrt{3 - \sqrt{5}} + 4\mathbb{T}^4),$$

which are not compatible and thus the Lie group is strictly Bach-flat. □

2.10 Bach-flat homogeneous Ricci solitons

Recall from Section 1.8 that Ricci solitons are self-similar solutions of the Ricci flow $\frac{\partial}{\partial t} g(t) = -2\rho_{g(t)}$, i.e., they are fixed points of the flow up to diffeomorphisms and rescaling. On a Lie group one may consider a stronger condition and search for fixed points of the flow up to automorphisms of the Lie group instead of diffeomorphisms. This observation led Lauret [76] to introduce algebraic Ricci solitons as follows. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A left-invariant metric $\langle \cdot, \cdot \rangle$ on G is called an *algebraic Ricci soliton* if

$$\mathfrak{D} = \mathrm{Ric} - \lambda\, \mathrm{Id} \tag{2.32}$$

is a derivation of the Lie algebra, i.e., $\mathfrak{D}[X, Y] = [\mathfrak{D}X, Y] + [X, \mathfrak{D}Y]$ for all $X, Y \in \mathfrak{g}$, where Ric denotes the Ricci operator $\langle \mathrm{Ric}(X), Y \rangle = \rho(X, Y)$ and $\lambda \in \mathbb{R}$. Let $\mathfrak{D}$ be a derivation given by Equation (2.32) and let φ_t denote the one-parameter family of automorphisms determined by $d\varphi_t|_e = \exp \frac{t}{2}\mathfrak{D}$. Then the vector field X given by $X(p) = \frac{d}{dt}\varphi_t(p)|_{t=0}$ satisfies Equation (1.24), thus defining a Ricci soliton on G. It is important to recognize that both Equations (1.24) and (2.32) are invariant by homotheties. Hence, aimed to characterize Bach-flat homogeneous Ricci solitons we shall work modulo homotheties.

Let $\Delta_X u = \Delta u - g(X, \nabla u)$ be the X-Laplacian on a Ricci soliton structure (M, g, X) (see for example [37]). Then $\frac{1}{2}\Delta_X \tau = \lambda\tau - \|\mathrm{Ric}\|^2$, which shows that a steady Ricci soliton ($\lambda = 0$)

with constant scalar curvature is Ricci-flat, and hence flat in the homogeneous setting (see [3] and [98] for an extension to the locally homogeneous setting). Furthermore, four-dimensional homogeneous shrinking Ricci solitons have bounded curvature and thus they are gradient [84]. Hence, (M, g) is *rigid*, i.e., it splits as a product $N \times \mathbb{R}^k$ where N is Einstein and the potential function is given by the projection into the Euclidean factor [93]. Every homogeneous expanding Ricci soliton is necessarily non-compact, and all known non-gradient examples are algebraic Ricci solitons on manifolds isometric to solvable Lie groups with left-invariant metrics [67].

The following result describes all homogeneous Bach-flat Ricci solitons.

Theorem 2.16. *Let (M, g) be a four-dimensional complete and simply connected Bach-flat Riemannian homogeneous Ricci soliton. Then (M, g) is Einstein, a locally conformally flat gradient Ricci soliton $N^3(c) \times \mathbb{R}$, where $N^3(c)$ is a space form, or homothetic to one of the algebraic Ricci solitons determined by the following solvable Lie algebras:*

(i) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \tfrac{1}{4}e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \tfrac{1}{4}e_3 \,.$$

(ii) The Lie algebra $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ given by

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha+1)^2\, e_2, \quad [e_4, e_3] = \alpha^2\, e_3, \qquad \alpha > 1 \,.$$

(iii) The Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ given by

$$[e_1, e_2] = e_3, \qquad [e_4, e_1] = \tfrac{1}{4}\sqrt{7 - 3\sqrt{5}}\, e_1,$$
$$[e_2, e_4] = \tfrac{1}{4}\sqrt{7 + 3\sqrt{5}}\, e_2, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}}\, e_3 \,.$$

Proof. Let (M, g) be a homogeneous Ricci soliton, i.e., $\mathcal{L}_X g + \rho = \lambda g$. If $\lambda = 0$, then (M, g) is flat. Moreover, if $\lambda > 0$, then X is the gradient of a potential function and one has $\operatorname{Hes}_f + \rho = \lambda g$. Homogeneity now means that either f is constant or otherwise ∇f is a parallel vector field [93]. Hence (M, g) is Einstein or it splits as a product $N \times \mathbb{R}^k$ where N is Einstein. Since $\dim(N) \leq 3$, it is of constant sectional curvature and thus $N \times \mathbb{R}^k$ is locally symmetric. Now Lemma 2.8 shows that (M, g) is Bach-flat if and only if it is either Einstein or $M = N^3(c) \times \mathbb{R}$. Next we consider the expanding case ($\lambda < 0$).

First of all observe that all the homogeneous Bach-flat metrics in Theorem 2.1 and Theorem 2.4 are realized on solvable Lie groups. It was shown by Jablonski [68] that if a solvmanifold is a Ricci soliton, then it is isometric to a solvsoliton. Hence in what follows we examine the existence of solutions to Equation (2.32) within the Lie algebras in Theorem 2.1 and Theorem 2.4.

A straightforward calculation shows that half conformally flat Lie groups in Theorem 2.1–(ii) are not algebraic Ricci solitons. Indeed, if $\mathfrak{D} = \operatorname{Ric} - \lambda \operatorname{Id}$ is a derivation then λ should satisfy the equations $\lambda + 6 = 0$ and $\lambda + \frac{3}{2} = 0$ which are incompatible.

Let $\mathfrak{g}_\alpha$ be a Lie algebra as in Theorem 2.1–(i). Then a straightforward calculation shows that $\mathfrak{D} = \operatorname{Ric} + \frac{9}{8} \operatorname{Id}$ is a derivation and thus it defines an algebraic Ricci soliton. Analogously, Lie algebras $\mathfrak{g}_\alpha$ in Theorem 2.1–(iii) are algebraic Ricci solitons, just considering the derivation $\mathfrak{D} = \operatorname{Ric} + 2(\alpha^2 + \alpha + 1)^2 \operatorname{Id}$.

The Lie algebra corresponding to Theorem 2.4–(i) is not an algebraic Ricci soliton since λ should satisfy the incompatible equations $\lambda - \sqrt{3} = 0$ and $\lambda + 12 + 7\sqrt{3} = 0$. On the contrary the Lie algebra $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}_3$ given at Theorem 2.4–(ii) is an algebraic Ricci soliton, with $\mathfrak{D} = \operatorname{Ric} + \frac{3}{2}\operatorname{Id}$. □

Chapter 3
Conformally Einstein non-reductive homogeneous manifolds

The purpose of this chapter is to analyze the conformally Einstein equation for a class of strictly pseudo-Riemannian four-dimensional homogeneous spaces, namely the non-reductive ones. We determine explicitly which non-reductive homogeneous four-manifolds are conformally Einstein and give all the possible conformally Einstein metrics in each case. It is worth remarking that all Einstein metrics inside each conformal class are Ricci-flat and, moreover, they are not unique depending on the cases, allowing the existence of two-parameter and three-parameter families of Ricci-flat conformal metrics in some cases.

It is important to emphasize that although any locally conformally Einstein metric is Bach-flat, there are examples of strictly Bach-flat manifolds, i.e., which are neither half conformally flat nor locally conformally Einstein (see for example [1, 33, 78] and the references therein). In this chapter we report on work investigated in [32]. Now, our main result can be stated as follows.

Theorem 3.1. *Let (M, g) be a conformally Einstein four-dimensional non-reductive homogeneous space. Then (M, g) is Einstein, locally conformally flat, or locally isometric to:*

(i) $(\mathbb{R}^4, g)$ *with metric given by*

$$\begin{aligned} g = {} & (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ & - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ & + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4, \end{aligned}$$

where a, b and c are arbitrary constants with $ab \neq 0$.

(ii) $(\mathbb{R}^4, g)$ *with metric given by*

$$\begin{aligned} g = {} & (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ & - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ & + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 - \tfrac{3a}{4}\, dx^3 \circ dx^3, \end{aligned}$$

where a, b and c are arbitrary constants with $ab \neq 0$.

(iii) $(\mathbb{R}^4, g)$ *with metric given by*

$$\begin{aligned} g = {} & -2ae^{2x^4}\, dx^1 \circ dx^3 + ae^{2x^4} dx^2 \circ dx^2 \\ & + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where a, b, c and q are arbitrary constants with $abq \neq 0$.

(iv) $(\mathfrak{U} \subset \mathbb{R}^4, g_+)$ *with metric given by*

$$\begin{aligned} g_+ &= 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 \\ &\quad + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where $\mathfrak{U} = \{(x^1, \ldots, x^4) \in \mathbb{R}^4 \,/\, \cos(x^4) \neq 0\}$*, and* a*,* b*,* c *and* q *are arbitrary constants with* $ab \neq 0$ *and* $b \neq -q$*, or*

$(\mathbb{R}^4, g_-)$ *with metric given by*

$$\begin{aligned} g_- &= 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cosh(x^4)^2 dx^2 \circ dx^2 \\ &\quad + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where a*,* b*,* c *and* q *are arbitrary constants with* $ab \neq 0$ *and* $b \neq q$*.*

Moreover, all the cases (i)–(iv) are in the conformal class of a Ricci-flat metric which is unique (up to an homothety) only in Case (i). Otherwise the space of conformally Ricci-flat metrics is either two or three-dimensional.

This chapter is organized as follows. The classification of the non-reductive four-dimensional homogeneous spaces given in [54] and the local form of the metrics corresponding to the different classes obtained in [25] are briefly reviewed in Section 3.1. The classification of all Bach-flat non-reductive four-dimensional homogeneous spaces is given in Theorem 3.9. The conformally Einstein equation is treated in Section 3.3 where Theorem 3.1 is stated, classifying the conformally Einstein non-reductive four-dimensional homogeneous spaces. All the curvature calculations are carried out in Section 3.2, while the proof of Theorem 3.1 is given in Section 3.3.

3.1 Classification of four-dimensional non-reductive homogeneous manifolds

We recall that a pseudo-Riemannian manifold is homogeneous if there is a group of isometries which acts transitively on M. Let G be such a group of isometries and let H denote the isotropy group at some fixed point. Then (M, g) can be identified with the quotient space $(G/H, \tilde{g})$, where $\tilde{g}$ is an invariant metric on G. A homogeneous space G/H is said to be *reductive* if the associated Lie algebra admits a decomposition of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is an $Ad(H)$-invariant complement of $\mathfrak{h}$. While every Riemannian homogeneous space is reductive, there are pseudo-Riemannian homogeneous spaces without any reductive decomposition. The geometry of reductive pseudo-Riemannian manifolds presents some similarities with the Riemannian case (see for example [57]), but little is known about the non-reductive case. The geometry of non-reductive homogeneous spaces is therefore an important aspect towards a good understanding of pseudo-Riemannian homogeneous manifolds.

Recall that any homogeneous pseudo-Riemannian manifold is reductive in dimension two and three. So the first non-trivial examples appear in dimension four. Both Lorentzian and neutral signature examples may occur. In dimension four, a complete classification of non-reductive homogeneous spaces was obtained in [54] (see Section 3.1.1). Later on a coordinate description was given in [25] which we recall in order to state our results.

3.1.1 Classification of Fels and Renner

We consider $M = G/H$ and denote by $(\mathfrak{g}, \mathfrak{h})$ the pair of Lie algebras corresponding to G and H, respectively. The Lie algebras in dimension ≤ 4 were classified in [91]. Following the same notation we introduce the relevant Lie algebras for our purpose.

- $A^1_{4,9}$ is the solvable Lie algebra determined by:
  $$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3\,.$$
- $A_{5,30}$ is the solvable Lie algebra determined by:
  $$[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2, \quad [e_1, e_5] = (\alpha + 1)e_1,$$
  $$[e_2, e_5] = \alpha e_2, \quad [e_3, e_5] = (\alpha - 1)e_3, \quad [e_4, e_5] = e_4,$$
  with $\alpha \in \mathbb{R}$.
- $A_{5,36}$ is the solvable Lie algebra determined by:
  $$[e_2, e_3] = e_1, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2,$$
  $$[e_2, e_5] = -e_2, \quad [e_3, e_5] = e_3\,.$$
- $A_{5,37}$ is the solvable Lie algebra determined by:
  $$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2,$$
  $$[e_3, e_4] = e_3, \quad [e_2, e_5] = -e_3, \quad [e_3, e_5] = e_2\,.$$

Now, we provide a classification when the signature is Lorentzian.

Theorem 3.2. [54] *Let $(M = G/H, g)$ be a four-dimensional homogeneous Lorentzian manifold, where H is connected. If M is non-reductive then the pair $(\mathfrak{g}, \mathfrak{h})$ is isomorphic to one of the following:*

(A.1) *The Lie algebra $\mathfrak{g}$ is the 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$, where $\mathfrak{s}(2)$ is the 2-dimensional solvable Lie algebra. There is a basis $\{e_1, \dots, e_5\}$ such that $\mathfrak{g}$ is determined by:*
$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_4, e_5] = e_4\,.$$
The subalgebras are given by $\mathfrak{h} = \operatorname{span}\{h_1 = e_3 + e_4\}$ and $\mathfrak{m} = \operatorname{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_5, u_4 = e_3 - e_4\}$. With respect to the basis $\{\theta^1, \dots, \theta^4\}$, dual to $\{u_1, \dots, u_4\}$, we have the description of the left-invariant metric:
$$g = a(\theta^1 \circ \theta^1 - \theta^1 \circ \theta^3 + 2\,\theta^2 \circ \theta^4) + b\,\theta^2 \circ \theta^2 + 2c\,\theta^2 \circ \theta^3 + q\,\theta^3 \circ \theta^3, \tag{3.1}$$
where $a(a - 4q) \neq 0$.

(A.2) *The Lie algebra* $\mathfrak{g}$ *is the* 1*-parameter family of* 5*-dimensional solvable Lie algebras* $A_{5,30}$. *There is a basis* $\{e_1, \dots, e_5\}$ *such that* $\mathfrak{g}$ *is determined by:*

$$[e_1, e_5] = (\alpha + 1)e_1, \quad [e_2, e_4] = e_1, \quad [e_2, e_5] = \alpha e_2,$$
$$[e_3, e_4] = e_2, \quad [e_3, e_5] = (\alpha - 1)e_3, \quad [e_4, e_5] = e_4,$$

where $\alpha \in \mathbb{R}$. *The subalgebras are given by* $\mathfrak{h} = \mathrm{span}\{h_1 = e_4\}$ *and* $\mathfrak{m} = \mathrm{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = e_5\}$. *With respect to the basis* $\{\theta^1, \dots, \theta^4\}$, *dual to* $\{u_1, \dots, u_4\}$, *we have the description of the left-invariant metric:*

$$g = a(-2\,\theta^1 \circ \theta^3 + \theta^2 \circ \theta^2) + b\,\theta^3 \circ \theta^3 + 2c\,\theta^3 \circ \theta^4 + q\,\theta^4 \circ \theta^4, \tag{3.2}$$

where $aq \neq 0$.

(A.3) *The Lie algebra* $\mathfrak{g}$ *is one of the 5-dimensional Lie algebras* $A_{5,37}$, $A_{5,36}$. *There is a basis* $\{e_1, \dots, e_5\}$ *such that* $\mathfrak{g}$ *is determined by:*

$$[e_1, e_4] = 2e_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2,$$
$$[e_2, e_5] = -\varepsilon e_3, \quad [e_3, e_4] = e_3, \quad [e_3, e_5] = e_2,$$

with $\varepsilon = 1$ *for* $A_{5,37}$ *and* $\varepsilon = -1$ *for* $A_{5,36}$. *The subalgebras are given by* $\mathfrak{h} = \mathrm{span}\{h_1 = e_3\}$ *and* $\mathfrak{m} = \mathrm{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\}$. *With respect to the basis* $\{\theta^1, \dots, \theta^4\}$, *dual to* $\{u_1, \dots, u_4\}$, *we have the description of the left-invariant metric:*

$$g = a(2\,\theta^1 \circ \theta^4 + \theta^2 \circ \theta^2) + b\,\theta^3 \circ \theta^3 + 2c\,\theta^3 \circ \theta^4 + q\,\theta^4 \circ \theta^4, \tag{3.3}$$

where $ab \neq 0$.

(A.4) *The Lie algebra* $\mathfrak{g}$ *is the 6-dimensional Lie algebra of Schrödinger* $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{n}(3)$, *where* $\mathfrak{n}(3)$ *is the 3-dimensional Lie algebra of Heisenberg. There is a basis* $\{e_1, \dots, e_6\}$ *such that* $\mathfrak{g}$ *is determined by:*

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$
$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5, \quad [e_4, e_5] = e_6\,.$$

The subalgebras are given by $\mathfrak{h} = \mathrm{span}\{h_1 = e_3 + e_6, h_2 = e_5\}$ *and* $\mathfrak{m} = \mathrm{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 - e_6, u_4 = e_4\}$. *With respect to the basis* $\{\theta^1, \dots, \theta^4\}$, *dual to* $\{u_1, \dots, u_4\}$, *we have the description of the left-invariant metric:*

$$g = a(\theta^1 \circ \theta^1 + 2\theta^2 \circ \theta^3 + \frac{1}{2}\,\theta^4 \circ \theta^4) + b\,\theta^2 \circ \theta^2, \tag{3.4}$$

where $a \neq 0$.

(A.5) *The Lie algebra* $\mathfrak{g}$ *is the 7-dimensional Lie algebra* $\mathfrak{sl}(2, \mathbb{R}) \ltimes A^1_{4,9}$. *There is a basis* $\{e_1, \dots, e_7\}$ *such that* $\mathfrak{g}$ *is determined by:*

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_1, e_5] = -e_5, \quad [e_1, e_6] = e_6,$$
$$[e_2, e_3] = e_1, \quad [e_2, e_5] = e_6, \quad [e_3, e_6] = e_5, \quad [e_4, e_7] = 2e_4,$$
$$[e_5, e_6] = e_4, \quad [e_5, e_7] = e_5, \quad [e_6, e_7] = e_6\,.$$

The subalgebras are given by $\mathfrak{h} = \mathrm{span}\{h_1 = e_1 + e_7, h_2 = e_3 - e_4, h_3 = e_5\}$ *and* $\mathfrak{m} = \mathrm{span}\{u_1 = e_1 - e_7, u_2 = e_2, u_3 = e_3 + e_4, u_4 = e_6\}$. *With respect to the basis* $\{\theta^1, \ldots, \theta^4\}$, *dual to* $\{u_1, \ldots, u_4\}$, *we have the description of the left-invariant metric:*

$$g = a(\theta^1 \circ \theta^1 + \frac{1}{2}\,\theta^2 \circ \theta^3 + \frac{1}{8}\,\theta^4 \circ \theta^4), \tag{3.5}$$

where $a \neq 0$.

The following theorem gives a list when the signature of the manifold is $(2,2)$.

Theorem 3.3. [54] *Let* $(M = G/H, g)$ *be a homogeneous pseudo-Riemannian manifold of dimension four and signature* $(2,2)$, *where* H *is connected. If* M *is non-reductive then the pair* $(\mathfrak{g}, \mathfrak{h})$ *is isomorphic to one of the following:*

(A.1) – (A.3) *The corresponding pairs of Lie algebras in Theorem 3.2.*

(B.1) *The Lie algebra* $\mathfrak{g}$ *is the 5-dimensional Lie algebra* $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. *There is a basis* $\{e_1, \ldots, e_5\}$ *such that* $\mathfrak{g}$ *is determined by:*

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$
$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5\,.$$

The subalgebras are given by $\mathfrak{h} = \mathrm{span}\{h_1 = e_3\}$ *and* $\mathfrak{m} = \mathrm{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\}$. *With respect to the basis* $\{\theta^1, \ldots, \theta^4\}$, *dual to* $\{u_1, \ldots, u_4\}$, *we have the description of the left-invariant metric:*

$$g = 2a(\theta^1 \circ \theta^3 + \theta^2 \circ \theta^4) + b\,\theta^2 \circ \theta^2 + 2c\,\theta^2 \circ \theta^3 + q\,\theta^3 \circ \theta^3, \tag{3.6}$$

where $a \neq 0$.

(B.2) *The Lie algebra* $\mathfrak{g}$ *is the 6-dimensional Lie algebra of Schrödinger* $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$ *as in* (A.4) *of Theorem 3.2, with the subalgebras* $\mathfrak{h} = \mathrm{span}\{h_1 = e_3 - e_6, h_2 = e_5\}$ *and* $\mathfrak{m} = \mathrm{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 + e_6, u_4 = e_4\}$. *With respect to the basis* $\{\theta^1, \ldots, \theta^4\}$, *dual to* $\{u_1, \ldots, u_4\}$, *we have the description of the left-invariant metric:*

$$g = a(-\theta^1 \circ \theta^1 + 2\theta^2 \circ \theta^3 + \frac{1}{2}\,\theta^4 \circ \theta^4) + b\,\theta^2 \circ \theta^2, \tag{3.7}$$

where $a \neq 0$.

(B.3) *The Lie algebra* $\mathfrak{g}$ *is the 7-dimensional Lie algebra* $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2 \oplus \mathbb{R}$. *There is a basis* $\{e_1, \ldots, e_7\}$ *such that* $\mathfrak{g}$ *is determined by:*

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$
$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5\,.$$

The subalgebras are given by $\mathfrak{h} = \operatorname{span}\{h_1 = e_3, h_2 = e_5 + e_6\}$ *and* $\mathfrak{m} = \operatorname{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = e_4\}$*. With respect to the basis* $\{\theta^1, \ldots, \theta^4\}$*, dual to* $\{u_1, \ldots, u_4\}$*, we have the description of the left-invariant metric:*

$$g = 2a(\theta^1 \circ \theta^3 + \theta^2 \circ \theta^4) + b\,\theta^3 \circ \theta^3, \tag{3.8}$$

where $a \neq 0$.

The following theorem gives a complete classification when the space is simply connected.

Theorem 3.4. [54] *If* $(M = G/H, g)$ *is a four-dimensional homogeneous simply connected and non-reductive pseudo-Riemannian manifold, then:*

(i) M *is diffeomorphic to* $\mathbb{R}^4$.

(ii) If G *is the complete group of isometries then the pair of Lie algebras for* G/H *is equivalent to one of the cases in Theorem 3.2 excluding the case* (A.5)*, or to one of the cases in Theorem 3.3.*

Conversely, for any pair of Lie algebras in Theorem 3.2 excluding the case (A.5)*, or for any pair of Lie algebras in Theorem 3.3, there is a pseudo-Riemannian metric in* $\mathbb{R}^4$ *(subject to the signature conditions), where the group of isometries acts transitively on* $\mathbb{R}^4$*. The Lie algebra of the symmetry group is given by the Lie algebra* $\mathfrak{g}$ *and the Lie algebra of the isotropy group at a point is given by* $\mathfrak{h}$.

3.1.2 Description in coordinates

Calvaruso, Fino and Zaeim established the following coordinate description, which will be used in what follows:

Theorem 3.5. [25] *Let* (M, g) *be a non-reductive homogeneous pseudo-Riemannian manifold of dimension four. Then it is locally isometric to one of the following:*

(A.1) $\mathbb{R}^4$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$\begin{aligned} g = {} & (4b(x^2)^2 + a)\,dx^1 \circ dx^1 + 4bx^2\,dx^1 \circ dx^2 \\ & - (4ax^2x^4 - 4cx^2 + a)\,dx^1 \circ dx^3 + 4ax^2\,dx^1 \circ dx^4 \\ & + b\,dx^2 \circ dx^2 - 2(ax^4 - c)\,dx^2 \circ dx^3 + 2a\,dx^2 \circ dx^4 + q\,dx^3 \circ dx^3, \end{aligned}$$

where a*,* b*,* c *and* q *are arbitrary constants with* $a(a - 4q) \neq 0$.

(A.2) $\mathbb{R}^4$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$\begin{aligned} g = {} & -2ae^{2\alpha x^4}\,dx^1 \circ dx^3 + ae^{2\alpha x^4} dx^2 \circ dx^2 + b\,e^{2(\alpha-1)x^4} dx^3 \circ dx^3 \\ & + 2ce^{(\alpha-1)x^4} dx^3 \circ dx^4 + q\,dx^4 \circ dx^4, \end{aligned}$$

where a*,* b*,* c*,* q *and* α *are arbitrary constants with* $aq \neq 0$.

(A.3) *An open subset* $\mathfrak{U} \subset \mathbb{R}^4$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$\begin{aligned} g_+ &= 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 \\ &\quad + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where a*,* b*,* c *and* q *are arbitrary constants with* $ab \neq 0$ *and* $\mathfrak{U} = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4; \cos(x^4) \neq 0\}$*, or*

$$\begin{aligned} g_- &= 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cosh(x^4)^2 dx^2 \circ dx^2 \\ &\quad + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where a*,* b*,* c *and* q *are arbitrary constants with* $ab \neq 0$ *and* $\mathfrak{U} = \mathbb{R}^4$*.*

(A.4) $\mathbb{R}^4$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$\begin{aligned} g &= \left(\tfrac{a}{2}(x^4)^2 + 4b(x^2)^2 + a\right) dx^1 \circ dx^1 + 4bx^2 dx^1 \circ dx^2 \\ &\quad + ax^2(4 + (x^4)^2) dx^1 \circ dx^3 + a(1 + 2x^2x^3)x^4 dx^1 \circ dx^4 + b\, dx^2 \circ dx^2 \\ &\quad + \tfrac{a}{2}(4 + (x^4)^2) dx^2 \circ dx^3 + ax^3x^4 dx^2 \circ dx^4 + \tfrac{a}{2}\, dx^4 \circ dx^4, \end{aligned}$$

where a *and* b *are arbitrary constants with* $a \neq 0$*.*

(A.5) $(\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}^2$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$\begin{aligned} g &= -\tfrac{ax^4}{4x^2} dx^1 \circ dx^2 + \tfrac{a}{4}\, dx^1 \circ dx^4 + \tfrac{a(2+2x^1x^4+(x^3)^2)}{8(x^2)^2} dx^2 \circ dx^2 \\ &\quad - \tfrac{ax^3}{4x^2} dx^2 \circ dx^3 - \tfrac{ax^1}{4x^2} dx^2 \circ dx^4 + \tfrac{a}{8}\, dx^3 \circ dx^3, \end{aligned}$$

where $a \neq 0$ *is an arbitrary constant.*

(B.1) $\mathbb{R}^4$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$\begin{aligned} g &= (q((x^3)^2 + 4x^2x^3x^4 + 4(x^2)^2(x^4)^2) \\ &\quad + 4cx^2x^3 + 8c(x^2)^2x^4 + 2ax^3 + 4b(x^2)^2)\, dx^1 \circ dx^1 \\ &\quad + 2(q(x^3x^4 + 2x^2(x^4)^2) + 4cx^2x^4 + cx^3 + 2bx^2) dx^1 \circ dx^2 \\ &\quad + 2(q(x^3 + 2x^2x^4) + 2cx^2 + a) dx^1 \circ dx^3 + 4ax^2 dx^1 \circ dx^4 \\ &\quad + (q(x^4)^2 + 2cx^4 + b) dx^2 \circ dx^2 + 2(qx^4 + c) dx^2 \circ dx^3 \\ &\quad + 2a\, dx^2 \circ dx^4 + q\, dx^3 \circ dx^3, \end{aligned}$$

where a*,* b*,* c *and* q *are arbitrary constants with* $a \neq 0$*.*

(B.2) $\mathfrak{U} = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4; x^4 \neq \pm 2\}$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$g = \left(a - \tfrac{a(x^4)^2}{2} + 4b(x^2)^2\right) dx^1 \circ dx^1 + 4bx^2 dx^1 \circ dx^2$$
$$- ax^2((x^4)^2 - 4)dx^1 \circ dx^3 - a(1 + 2x^2x^3)x^4 dx^1 \circ dx^4 + b\, dx^2 \circ dx^2$$
$$- \tfrac{1}{2}a((x^4)^2 - 4)dx^2 \circ dx^3 - ax^3x^4 dx^2 \circ dx^4 - \tfrac{1}{2}a\, dx^4 \circ dx^4,$$

where a and b are arbitrary constants with $a \neq 0$.

(B.3) $\mathbb{R}^4$ *with coordinates* (x^1, x^2, x^3, x^4) *and metric tensor*

$$g = -2ae^{-x^2}x^3 dx^1 \circ dx^2 + 2ae^{-x^2} dx^1 \circ dx^3$$
$$+ 2(2b(x^3)^2 - ax^4)dx^2 \circ dx^2 - 4bx^3 dx^2 \circ dx^3$$
$$+ 2a\, dx^2 \circ dx^4 + b\, dx^3 \circ dx^3,$$

where a and b are arbitrary constants with $a \neq 0$.

It is worth emphasizing that the spaces of Types (A.1)–(A.3) admit metrics both of Lorentzian and neutral signature depending on the values of the constants defining the corresponding metrics. Metrics of Type $(A.4)$ and Type $(A.5)$ are always Lorentzian, while metrics of Types (B.1)–(B.3) are of neutral signature $(2, 2)$.

3.1.3 Ricci and Weyl tensors of non-reductive four-dimensional homogeneous manifolds

We describe the curvature of non-reductive four-dimensional homogeneous manifolds analyzing the Ricci tensor and the Weyl curvature tensor case by case. As a consequence we obtain Theorem 3.7 and Theorem 3.8. We consider separately all the possibilities in Theorem 3.5.

Type (A.1)

Consider the metric tensor

$$\begin{aligned} g = {} & (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ & - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ & + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 + q\, dx^3 \circ dx^3 . \end{aligned} \tag{3.9}$$

It immediately follows from the above expression that $\det(g) = \frac{1}{4}a^3(a - 4q)$, which shows that the metric (3.9) is Lorentzian if $a(a-4q) < 0$ and of neutral signature otherwise. Further observe that the restriction $a(a - 4q) \neq 0$ in Theorem 3.5–(A.1) ensures that g is non-degenerate.

The Ricci operator is given by

$$\mathrm{Ric} = \frac{1}{a} \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & -2x^2 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{8b(a+4q)x^2}{a(a-4q)} & \frac{4b(a+4q)}{a(a-4q)} & \frac{2(ax^4-c)}{a} & -2 \end{pmatrix}, \tag{3.10}$$

showing that the manifold is never Einstein for any $a, b, c, q \in \mathbb{R}$. The non-zero components of the Weyl tensor are (up to the usual symmetries):

$$\begin{aligned} &W_{1212} = \tfrac{8bq-6ab}{a-4q}, && W_{1213} = -\tfrac{16bqx^2}{a-4q}, && W_{1223} = -\tfrac{8bq}{a-4q}, \\ &W_{1313} = -\tfrac{8bq(x^2)^2(a+4q)}{a(a-4q)}, && W_{1323} = -\tfrac{4bqx^2(a+4q)}{a(a-4q)}, && W_{2323} = -\tfrac{2bq(a+4q)}{a(a-4q)}\,. \end{aligned} \tag{3.11}$$

Note that if $b = 0$ then the manifold is locally conformally flat.

Type (A.2)

Consider the metric tensor

$$\begin{aligned} g = &-2ae^{2\alpha x^4}\, dx^1 \circ dx^3 + ae^{2\alpha x^4} dx^2 \circ dx^2 + be^{2(\alpha-1)x^4} dx^3 \circ dx^3 \\ &+ 2ce^{(\alpha-1)x^4} dx^3 \circ dx^4 + q\, dx^4 \circ dx^4\,. \end{aligned} \tag{3.12}$$

It immediately follows from the above expression that $\det(g) = -a^3 q\, e^{6\alpha x^4}$, which shows that the metric (3.12) is Lorentzian if $aq > 0$ and of neutral signature otherwise. Further observe that the restriction $aq \neq 0$ in Theorem 3.5–(A.2) ensures that g is non-degenerate.

The Ricci operator is given by

$$\mathrm{Ric} = -\frac{3\alpha^2}{q} \begin{pmatrix} 1 & 0 & \frac{b(3\alpha-2)}{3a\alpha^2} e^{-2x^4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.13}$$

Hence (M, g) is Einstein if and only if $b = 0$ (with scalar curvature $\tau = -12\frac{\alpha^2}{q}$), or $\alpha = \frac{2}{3}$ (with scalar curvature $\tau = -\frac{16}{3q}$), or Ricci-flat if $\alpha = 0$. The non-zero components of the Weyl tensor are given by

$$W_{2323} = -\frac{(\alpha-2)a\, be^{2(2\alpha-1)x^4}}{2q}, \qquad W_{3434} = \frac{1}{2}(\alpha-2)be^{2(\alpha-1)x^4}\,. \tag{3.14}$$

Note that if $\alpha = 2$ or $b = 0$ then the manifold is locally conformally flat.

Type (A.3)

Two distinct cases have to be considered for Type (A.3) metrics. Let $\mathfrak{U}$ be the open set in $\mathbb{R}^4$ determined by $\mathfrak{U} = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4; \cos(x^4) \neq 0\}$ and the metric tensor

$$\begin{aligned} g_+ = &\,2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 \\ &+ b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4\,. \end{aligned} \tag{3.15}$$

Now $\det(g_+) = -a^3 b \cos(x^4)^2\, e^{6x^3}$ shows that the metric (3.15) is Lorentzian if $ab > 0$ and of neutral signature otherwise. Further observe that the restriction $ab \neq 0$ in Theorem 3.5–(A.3) ensures that g_+ is non-degenerate.

The Ricci operator is given by

$$\mathrm{Ric} = -\frac{3}{b}\begin{pmatrix} 1 & 0 & 0 & -\frac{(b+q)e^{-2x^3}}{3a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.16}$$

and thus (M, g) is Einstein if and only if $b = -q$. The non-zero components of the Weyl tensor are:

$$W_{2424} = \frac{ae^{2x^3}(b+q)\cos(x^4)^2}{2b}, \qquad W_{3434} = -\frac{b+q}{2}\,. \tag{3.17}$$

Now we consider the second case for Type (A.3) metrics. Let M be $\mathbb{R}^4$ with metric tensor

$$\begin{aligned} g_- = {} & 2ae^{2x^3}\,dx^1 \circ dx^4 + ae^{2x^3}\cosh(x^4)^2 dx^2 \circ dx^2 \\ & + b\,dx^3 \circ dx^3 + 2c\,dx^3 \circ dx^4 + q\,dx^4 \circ dx^4\,. \end{aligned} \tag{3.18}$$

Next $\det(g_-) = -a^3 b\cosh(x^4)^2\,e^{6x^3}$ shows that the metric (3.18) is Lorentzian if $ab > 0$ and of neutral signature otherwise. Further observe that the restriction $ab \neq 0$ in Theorem 3.5–(A.3) ensures that g_- is non-degenerate.

The Ricci operator is given by

$$\mathrm{Ric} = -\frac{3}{b}\begin{pmatrix} 1 & 0 & 0 & \frac{(b-q)e^{-2x^3}}{3a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.19}$$

and thus (M, g) is Einstein if and only if $b = q$ in which case the manifold is locally conformally flat. The non-vanishing components of the Weyl tensor are:

$$W_{2424} = -\frac{ae^{2x^3}(b-q)\cosh(x^4)^2}{2b}, \qquad W_{3434} = \frac{b-q}{2}\,. \tag{3.20}$$

Type (A.4)

Consider the metric tensor

$$\begin{aligned} g = {} & \left(\tfrac{a}{2}(x^4)^2 + 4b(x^2)^2 + a\right) dx^1 \circ dx^1 + 4bx^2 dx^1 \circ dx^2 \\ & + ax^2(4 + (x^4)^2)dx^1 \circ dx^3 + a(1 + 2x^2x^3)x^4 dx^1 \circ dx^4 \\ & + b\,dx^2 \circ dx^2 + \tfrac{a}{2}(4 + (x^4)^2)dx^2 \circ dx^3 \\ & + ax^3x^4 dx^2 \circ dx^4 + \tfrac{a}{2}\,dx^4 \circ dx^4\,. \end{aligned} \tag{3.21}$$

It follows from the above expression that $\det(g) = -\frac{1}{32}a^4(4 + (x^4)^2)^2$, which shows that the metric (3.21) is Lorentzian. Further observe that the restriction $a \neq 0$ in Theorem 3.5–(A.4) ensures that g is non-degenerate.

The Ricci operator is given by

$$\mathrm{Ric} = -\frac{3}{a}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{40bx^2}{3a((x^4)^2+4)} & \frac{20b}{3a((x^4)^2+4)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.22}$$

which shows that (M,g) is Einstein if and only if $b=0$ in which case the manifold is of constant sectional curvature taking into account that the non-zero components of the Weyl tensor are given by

$$\begin{array}{lll} W_{1212} = \frac{3}{4}b\left((x^4)^2-2\right), & W_{1214} = -\frac{3}{2}bx^2x^4, & W_{1224} = -\frac{3bx^4}{4}, \\ W_{1414} = 3b(x^2)^2, & W_{1424} = \frac{3bx^2}{2}, & W_{2424} = \frac{3b}{4}. \end{array} \tag{3.23}$$

Type (A.5)

Let $M = (\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}^2$ and let (x^1,x^2,x^3,x^4) be the coordinates. Consider the metric tensor

$$\begin{aligned} g = & -\tfrac{ax^4}{4x^2}dx^1\circ dx^2 + \tfrac{a}{4}\,dx^1\circ dx^4 + \tfrac{a(2+2x^1x^4+(x^3)^2)}{8(x^2)^2}dx^2\circ dx^2 \\ & - \tfrac{ax^3}{4x^2}dx^2\circ dx^3 - \tfrac{ax^1}{4x^2}dx^2\circ dx^4 + \tfrac{a}{8}\,dx^3\circ dx^3\,. \end{aligned} \tag{3.24}$$

Since $\det(g) = -\frac{a^4}{2048(x^2)^2}$, the metric (3.24) is Lorentzian and the restriction $a \neq 0$ in Theorem 3.5–(A.5) ensures that g is non-degenerate.

The Ricci tensor is given by

$$\rho = \begin{pmatrix} 0 & \frac{3x^4}{2x^2} & 0 & -\frac{3}{2} \\ \frac{3x^4}{2x^2} & -\frac{3\left((x^3)^2+2x^1x^4+2\right)}{2(x^2)^2} & \frac{3x^3}{2x^2} & \frac{3x^1}{2x^2} \\ 0 & \frac{3x^3}{2x^2} & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{3x^1}{2x^2} & 0 & 0 \end{pmatrix}, \tag{3.25}$$

from where it follows that the corresponding Ricci operator is a multiple of the identity, $\mathrm{Ric} = -\frac{12}{a}\,\mathrm{Id}$, and thus Einstein. Moreover, the Weyl tensor vanishes identically. Therefore any Type (A.5) manifold has constant sectional curvature.

Type (B.1)

Let $M = \mathbb{R}^4$ with coordinates (x^1,x^2,x^3,x^4) and metric tensor

$$\begin{aligned} g = & (q((x^3)^2 + 4x^2x^3x^4 + 4(x^2)^2(x^4)^2) \\ & +4cx^2x^3 + 8c(x^2)^2x^4 + 2ax^3 + 4b(x^2)^2)\,dx^1\circ dx^1 \\ & + 2(q(x^3x^4 + 2x^2(x^4)^2) + 4cx^2x^4 + cx^3 + 2bx^2)dx^1\circ dx^2 \\ & + 2(q(x^3 + 2x^2x^4) + 2cx^2 + a)dx^1\circ dx^3 + 4ax^2dx^1\circ dx^4 \\ & + (q(x^4)^2 + 2cx^4 + b)dx^2\circ dx^2 + 2(qx^4 + c)dx^2\circ dx^3 \\ & + 2a\,dx^2\circ dx^4 + q\,dx^3\circ dx^3\,. \end{aligned} \tag{3.26}$$

Since $\det(g) = a^4$ and the component $g_{44} = 0$, the metric (3.26) is of neutral signature and the restriction $a \neq 0$ in Theorem 3.5–(B.1) ensures that g is non-degenerate.

The Ricci operator is given by

$$\mathrm{Ric} = \begin{pmatrix} \frac{3q}{2a^2} & 0 & 0 & 0 \\ 0 & \frac{3q}{2a^2} & 0 & 0 \\ 0 & 0 & \frac{3q}{2a^2} & 0 \\ \frac{15}{a^3}x^2(bq-c^2) & \frac{15}{2a^3}(bq-c^2) & 0 & \frac{3q}{2a^2} \end{pmatrix}, \tag{3.27}$$

from where it follows that (M,g) is Einstein if and only if $c^2 - bq = 0$. The non-zero components of the Weyl tensor are given by

$$\begin{aligned}
W_{1212} &= \tfrac{-6a^2\left(b+2cx^4+q(x^4)^2\right)+ax^3\left(-7bq+6c^2-qx^4\left(2c+qx^4\right)\right)+5q(x^3)^2\left(c^2-bq\right)}{2a^2},\\
W_{1213} &= \tfrac{2x^2\left(a\left(7bq-6c^2+qx^4\left(2c+qx^4\right)\right)+10qx^3\left(bq-c^2\right)\right)-a\left(6a+qx^3\right)\left(c+qx^4\right)}{4a^2},\\
W_{1214} &= -\tfrac{2x^2\left(6a+qx^3\right)\left(c+qx^4\right)+qx^3\left(2a+qx^3\right)}{4a},\\
W_{1223} &= \tfrac{a\left(7bq-6c^2+qx^4\left(2c+qx^4\right)\right)+10qx^3\left(bq-c^2\right)}{4a^2},\\
W_{1224} &= -\tfrac{\left(6a+qx^3\right)\left(c+qx^4\right)}{4a}, \quad W_{1234} = -\tfrac{q\left(2a+qx^3\right)}{4a},\\
W_{1313} &= \tfrac{q\left(-a^2+2ax^2\left(c+qx^4\right)+20(x^2)^2\left(c^2-bq\right)\right)}{2a^2},\\
W_{1314} &= \tfrac{qx^2\left(-2a+2x^2\left(c+qx^4\right)+qx^3\right)}{2a},\\
W_{1323} &= \tfrac{q\left(a\left(c+qx^4\right)+20x^2\left(c^2-bq\right)\right)}{4a^2}, \quad W_{1324} = \tfrac{q\left(x^2\left(c+qx^4\right)-a\right)}{2a},\\
W_{1334} &= \tfrac{q^2x^2}{2a}, \quad W_{1423} = \tfrac{q\left(2x^2\left(c+qx^4\right)+qx^3\right)}{4a}, \quad W_{1424} = -qx^2,\\
W_{1414} &= -2q(x^2)^2, \quad W_{2334} = \tfrac{q^2}{4a}, \quad W_{2424} = -\tfrac{q}{2},\\
W_{2323} &= \tfrac{5q\left(c^2-bq\right)}{2a^2}, \quad W_{2324} = \tfrac{q\left(c+qx^4\right)}{4a}\,.
\end{aligned} \tag{3.28}$$

Note that if $b = c = q = 0$ then $W = 0$.

Type (B.2)

Let $\mathfrak{U} = \{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4; x^4 \neq \pm 2\}$ with coordinates (x^1,x^2,x^3,x^4) and metric tensor

$$\begin{aligned}
g = &\left(a - \tfrac{a(x^4)^2}{2} + 4b(x^2)^2\right) dx^1 \circ dx^1 + 4bx^2 dx^1 \circ dx^2 \\
&- ax^2((x^4)^2-4)dx^1 \circ dx^3 - a(1+2x^2x^3)x^4 dx^1 \circ dx^4 \\
&+ b\, dx^2 \circ dx^2 - \tfrac{1}{2}a((x^4)^2-4)dx^2 \circ dx^3 \\
&- ax^3x^4 dx^2 \circ dx^4 - \tfrac{1}{2}a\, dx^4 \circ dx^4\,.
\end{aligned} \tag{3.29}$$

Since $\det(g) = \frac{1}{32}a^4((x^4)^2 - 4)^2$ and the component $g_{33} = 0$, the metric (3.29) is of neutral signature and the restriction $a \neq 0$, $x^4 \neq \pm 2$ in Theorem 3.5–(B.2) ensures that g is non-degenerate.

The Ricci operator is given by

$$\operatorname{Ric} = -\frac{3}{a}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{40bx^2}{3a((x^4)^2-4)} & -\frac{20b}{3a((x^4)^2-4)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.30}$$

which shows that (M, g) is Einstein if and only if $b = 0$. In this case, the manifold is locally conformally flat since the non-zero components of the Weyl tensor are determined by

$$\begin{aligned} &W_{1212} = -\tfrac{3}{4}b\left((x^4)^2 + 2\right), \quad W_{1214} = \tfrac{3}{2}bx^2x^4, \quad W_{1224} = \tfrac{3bx^4}{4}, \\ &W_{1414} = -3b(x^2)^2, \qquad\qquad W_{1424} = -\tfrac{3bx^2}{2}, \quad W_{2424} = -\tfrac{3b}{4}. \end{aligned} \tag{3.31}$$

Note that $W = 0$ if and only if the manifold has constant sectional curvature.

Type (B.3)

Let $M = \mathbb{R}^4$ with coordinates (x^1, x^2, x^3, x^4) and metric tensor

$$\begin{aligned} g = &-2ae^{-x^2}x^3dx^1 \circ dx^2 + 2ae^{-x^2}dx^1 \circ dx^3 \\ &+ 2(2b(x^3)^2 - ax^4)dx^2 \circ dx^2 - 4bx^3dx^2 \circ dx^3 \\ &+ 2a\,dx^2 \circ dx^4 + b\,dx^3 \circ dx^3. \end{aligned} \tag{3.32}$$

Since $\det(g) = a^4e^{-2x^2}$ and the component $g_{44} = 0$, the metric (3.32) is of neutral signature and the restriction $a \neq 0$ in Theorem 3.5–(B.3) ensures that g is non-degenerate.

A straightforward calculation shows that *the Ricci operator of any Type* (B.3) *metric vanishes identically and hence they are all Ricci-flat*. The Weyl tensor is not necessarily zero and the only non-zero component of the Weyl tensor is given by

$$W_{2323} = -3b, \tag{3.33}$$

which shows that (M, g) is flat if and only if $b = 0$.

Remark 3.6. As a consequence of the expressions of the Ricci and the Weyl tensor in this section, a metric given by Theorem 3.5 is of constant sectional curvature κ if and only if it corresponds to one of the following (see also [24, 26, 54]):

Type (A.2) with $b = 0$, in which case $\kappa = -\frac{\alpha^2}{q}$.

Type (A.3) with $b = -\varepsilon q$, in which case $\kappa = \varepsilon\frac{1}{q}$.

Type (A.4) with $b = 0$, in which case $\kappa = -\frac{1}{a}$.

Type (A.5), in which case $\kappa = -\frac{4}{a}$.

Type (B.1) with $q = c = b = 0$, in which case is flat.

Type (B.2) with $b = 0$, in which case $\kappa = -\frac{1}{a}$.

Type (B.3) with $b = 0$, in which case is flat.

Fels and Renner [54] classified the Einstein non-reductive four-dimensional homogeneous spaces, showing that they must be of Type (A.2) or (B.3) (see also [24, 25, 27]). The following theorem summarizes all the previous results.

Theorem 3.7. *Let (M, g) be a manifold given by Theorem 3.5. Then (M, g) is Einstein if and only if it has constant sectional curvature or it corresponds to one of the following:*

(i) Type (A.2) with $\alpha = \frac{2}{3}$ and $aq \neq 0$:

$$g = -2ae^{\frac{4}{3}x^4}\, dx^1 \circ dx^3 + ae^{\frac{4}{3}x^4} dx^2 \circ dx^2 + be^{-\frac{2}{3}x^4} dx^3 \circ dx^3 + 2ce^{-\frac{1}{3}x^4} dx^3 \circ dx^4 + q\, dx^4 \circ dx^4 \,.$$

(ii) Type (B.1) with $q = c = 0 \neq ba$:

$$g = (2ax^3 + 4b(x^2)^2)\, dx^1 \circ dx^1 + 2(2bx^2) dx^1 \circ dx^2 + 2a\, dx^1 \circ dx^3 + 4ax^2 dx^1 \circ dx^4 + b\, dx^2 \circ dx^2 + 2a\, dx^2 \circ dx^4.$$

(iii) Type (B.1) with $q \neq 0$, $b = \frac{c^2}{q}$ and $a \neq 0$:

$$\begin{aligned} g = {} & \Big(q((x^3)^2 + 4x^2x^3x^4 + 4(x^2)^2(x^4)^2) \\ & +4cx^2x^3 + 8c(x^2)^2x^4 + 2ax^3 + \tfrac{4c^2}{q}(x^2)^2\Big)\, dx^1 \circ dx^1 \\ & + 2(q(x^3x^4 + 2x^2(x^4)^2) + 4cx^2x^4 + cx^3 + \tfrac{2c^2}{q}x^2) dx^1 \circ dx^2 \\ & + 2(q(x^3 + 2x^2x^4) + 2cx^2 + a) dx^1 \circ dx^3 + 4ax^2 dx^1 \circ dx^4 \\ & + (q(x^4)^2 + 2cx^4 + \tfrac{c^2}{q}) dx^2 \circ dx^2 + 2(qx^4 + c) dx^2 \circ dx^3 \\ & + 2a\, dx^2 \circ dx^4 + q\, dx^3 \circ dx^3. \end{aligned}$$

(iv) Type (B.3) with $ab \neq 0$:

$$g = -2ae^{-x^2}x^3 dx^1 \circ dx^2 + 2ae^{-x^2} dx^1 \circ dx^3 + 2(2b(x^3)^2 - ax^4) dx^2 \circ dx^2 - 4bx^3 dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 + b\, dx^3 \circ dx^3 \,.$$

In all the cases, the manifold is of neutral signature.

Some generalizations of the Einstein condition were studied in [24] and [26] showing which of these manifolds admit Ricci solitons.

The main goal of this chapter is to study the conformal geometry of these spaces aimed to describe all the conformally Einstein non-reductive homogeneous spaces. Clearly the Einstein cases mentioned above as well as the locally conformally flat cases already described in [27] should be discarded, since they all are conformally Einstein.

Theorem 3.8. *Let (M, g) be a manifold given by Theorem 3.5. Then (M, g) is locally conformally flat if and only if it is of constant curvature or it corresponds to one of the following cases:*

(i) Type (A.1) with $b = 0$ and $a(a - 4q) \neq 0$:

$$\begin{aligned} g = {} & a\, dx^1 \circ dx^1 - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 \\ & + 4ax^2\, dx^1 \circ dx^4 - 2(ax^4 - c)\, dx^2 \circ dx^3 \\ & + 2a\, dx^2 \circ dx^4 + q\, dx^3 \circ dx^3 \,. \end{aligned}$$

(ii) Type (A.2) with $\alpha = 2$ and $abq \neq 0$:

$$\begin{aligned} g = {} & -2ae^{4x^4}\, dx^1 \circ dx^3 + ae^{4x^4} dx^2 \circ dx^2 + be^{2x^4} dx^3 \circ dx^3 \\ & + 2ce^{x^4} dx^3 \circ dx^4 + q\, dx^4 \circ dx^4 \,. \end{aligned}$$

3.2 Bach-flat non-reductive homogeneous manifolds

In this section we briefly schedule some basic facts about the curvature of non-reductive homogeneous spaces. All the curvature expressions are obtained after some straightforward calculations that we omit. We consider separately all the possibilities in Theorem 3.5 and analyze the Bach tensor case by case. As a consequence, one obtains the proof of Theorem 3.9.

3.2.1 Non-reductive spaces admitting Lorentzian and neutral signature metrics

With the notation of Theorem 3.5 at hand, the non-reductive four-dimensional homogeneous manifolds admitting both Lorentzian and neutral signature metrics are those corresponding to Types (A.1), (A.2) and (A.3).

Type (A.1)

Consider the metric tensor

$$\begin{aligned} g = {} & (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ & - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ & + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 + q\, dx^3 \circ dx^3, \end{aligned} \tag{3.34}$$

where $a(a-4q)\neq 0$. The non-zero components of the divergence of the Weyl tensor are given by (up to symmetries):

$$\begin{aligned}
\operatorname{div}_4 W_{121} &= \tfrac{12bx^2(a+4q)}{a(a-4q)},\\
\operatorname{div}_4 W_{122} &= \tfrac{6b(a+4q)}{a(a-4q)},\\
\operatorname{div}_4 W_{131} &= -\tfrac{32bq(x^2)^2}{a^2-4aq},\\
\operatorname{div}_4 W_{132} &= \operatorname{div}_4 W_{231} = -\tfrac{16bqx^2}{a^2-4aq},\\
\operatorname{div}_4 W_{232} &= -\tfrac{8bq}{a^2-4aq}\,.
\end{aligned} \tag{3.35}$$

The Bach tensor is given by

$$\mathfrak{B} = \begin{pmatrix} -\frac{256b\,q\,(3a+4q)\,(x^2)^2}{a^2(a-4q)^2} & -\frac{128b\,q\,(3a+4q)x^2}{a^2(a-4q)^2} & 0 & 0 \\ -\frac{128b\,q\,(3a+4q)\,x^2}{a^2(a-4q)^2} & -\frac{64b\,q\,(3a+4q)}{a^2(a-4q)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.36}$$

An immediate consequence of previous expression is that a *Type (A.1) non-reductive homogeneous space is Bach-flat if and only if one of the following holds:* $b=0$, $q=0$ *or* $q=-\frac{3a}{4}$. *Moreover:*

(1) If $b=0$, *then Equation* (3.11) *shows that* (M,g) *is locally conformally flat.*

(2) If $b\neq 0$, *then* (M,g) *is neither locally conformally flat nor Einstein.*

Type (A.2)

Consider the metric tensor

$$\begin{aligned} g = &-2ae^{2\alpha x^4}\,dx^1\circ dx^3 + ae^{2\alpha x^4}dx^2\circ dx^2 + be^{2(\alpha-1)x^4}dx^3\circ dx^3 \\ &+ 2ce^{(\alpha-1)x^4}dx^3\circ dx^4 + q\,dx^4\circ dx^4, \end{aligned} \tag{3.37}$$

where $aq\neq 0$. The only non-zero component of the divergence of the Weyl tensor is given by

$$\operatorname{div}_4 W_{343} = \frac{(\alpha-2)(3\alpha-2)\,be^{2(\alpha-1)x^4}}{2q}\,. \tag{3.38}$$

In this case the Bach tensor is expressed with respect to the coordinate basis as

$$\mathfrak{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(\alpha-2)(\alpha-1)(3\alpha-2)be^{2(\alpha-1)x^4}}{q^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.39}$$

Hence *a non-reductive homogeneous space of Type (A.2) is Bach-flat if and only if* $b=0$, $\alpha=\frac{2}{3}$, $\alpha=1$ *or* $\alpha=2$. *Moreover:*

(1) If $b = 0$ then Equations (3.13) *and* (3.14) *show that the manifold is of constant sectional curvature $\kappa = -\frac{\alpha^2}{q}$.*

(2) If $\alpha = \frac{2}{3}$, then $W_{3434} = -\frac{2}{3}be^{-\frac{2x^4}{3}}$ and hence the manifold is not locally conformally flat, unless $b = 0$.

(3) If $\alpha = 1$ then $W_{3434} = -\frac{b}{2}$, which shows that (M, g) is not locally conformally flat unless $b = 0$.

(4) If $\alpha = 2$, then Equation (3.14) *shows that (M, g) is locally conformally flat but not Einstein unless $b = 0$.*

Type (A.3)

Two distinct cases have to be considered for Type (A.3) metrics. Let $\mathfrak{U}$ be the open set in $\mathbb{R}^4$ determined by $\mathfrak{U} = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4; \cos(x^4) \neq 0\}$ and the metric tensor

$$\begin{aligned} g_+ = {} & 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 \\ & + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned} \tag{3.40}$$

where $ab \neq 0$. The only non-zero component of the divergence of the Weyl tensor is

$$\mathrm{div}_4\, W_{344} = \frac{b+q}{2b}\,. \tag{3.41}$$

Now, a long but straightforward computation shows that (M, g_+) *is always Bach-flat. Moreover, (M, g_+) is locally conformally flat if and only if $b = -q$ by Equation* (3.17)*, in which case it is Einstein and thus of constant sectional curvature $\kappa = \frac{1}{q}$.*

Now we consider the second case for Type (A.3) metrics. Let M be $\mathbb{R}^4$ with metric tensor

$$\begin{aligned} g_- = {} & 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cosh(x^4)^2 dx^2 \circ dx^2 \\ & + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned} \tag{3.42}$$

where $ab \neq 0$. The only non-zero component of the divergence of the Weyl tensor is

$$\mathrm{div}_4\, W_{344} = \frac{1}{2}\left(\frac{q}{b} - 1\right)\,. \tag{3.43}$$

Furthermore, a long but straightforward computation shows that (M, g_-) *is always Bach-flat. Moreover, (M, g_-) is locally conformally flat if and only if $b = q$ by Equation* (3.20)*, in which case it is Einstein and thus of constant sectional curvature $\kappa = -\frac{1}{q}$.*

Hence any Einstein Type (A.3) manifold is necessarily of constant sectional curvature.

3.2.2 Non-reductive spaces admitting only Lorentzian metrics

Type (A.4)

Consider the metric tensor

$$\begin{aligned} g = &\left(\tfrac{a}{2}(x^4)^2 + 4b(x^2)^2 + a\right) dx^1 \circ dx^1 + 4bx^2 dx^1 \circ dx^2 \\ &+ ax^2(4+(x^4)^2) dx^1 \circ dx^3 + a(1+2x^2x^3)x^4 dx^1 \circ dx^4 + b dx^2 \circ dx^2 \\ &+ \tfrac{a}{2}(4+(x^4)^2) dx^2 \circ dx^3 + ax^3x^4 dx^2 \circ dx^4 + \tfrac{a}{2}\, dx^4 \circ dx^4, \end{aligned} \tag{3.44}$$

where $a \neq 0$. The only non-zero components of the divergence of the Weyl tensor are given by

$$\operatorname{div}_4 W_{121} = -\frac{15bx^2}{a}, \qquad \operatorname{div}_4 W_{122} = -\frac{15b}{2a}\,. \tag{3.45}$$

The Bach tensor is given by

$$\mathfrak{B} = \begin{pmatrix} -\frac{120b(x^2)^2}{a^2} & -\frac{60bx^2}{a^2} & 0 & 0 \\ -\frac{60bx^2}{a^2} & -\frac{30b}{a^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.46}$$

Hence, *a Type (A.4) metric is Bach-flat if and only if* $b = 0$*, in which case* (M, g) *is locally conformally flat by Equation* (3.23)*, and thus of constant sectional curvature* $\kappa = -\frac{1}{a}$, as the Ricci operator shows.

Type (A.5)

Let $M = (\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}^2$ and let (x^1, x^2, x^3, x^4) be the coordinates. Consider the metric tensor

$$\begin{aligned} g = &-\tfrac{ax^4}{4x^2} dx^1 \circ dx^2 + \tfrac{a}{4}\, dx^1 \circ dx^4 + \tfrac{a(2+2x^1x^4+(x^3)^2)}{8(x^2)^2} dx^2 \circ dx^2 \\ &- \tfrac{ax^3}{4x^2} dx^2 \circ dx^3 - \tfrac{ax^1}{4x^2} dx^2 \circ dx^4 + \tfrac{a}{8}\, dx^3 \circ dx^3, \end{aligned} \tag{3.47}$$

where $a \neq 0$. In this case, the manifold is Einstein and the Weyl tensor vanishes identically. Therefore, any Type (A.5) metric is always of constant sectional curvature $\kappa = -\frac{4}{a}$.

3.2.3 Non-reductive spaces admitting only neutral signature metrics.

There exist three different families of non-reductive homogeneous four-manifolds which admit exclusively neutral signature metrics.

Type (B.1)

Let $M = \mathbb{R}^4$ with coordinates (x^1, x^2, x^3, x^4) and metric tensor

$$\begin{aligned} g = {} & (q((x^3)^2 + 4x^2x^3x^4 + 4(x^2)^2(x^4)^2) \\ & +4cx^2x^3 + 8c(x^2)^2x^4 + 2ax^3 + 4b(x^2)^2)\, dx^1 \circ dx^1 \\ & + 2(q(x^3x^4 + 2x^2(x^4)^2) + 4cx^2x^4 + cx^3 + 2bx^2) dx^1 \circ dx^2 \\ & + 2(q(x^3 + 2x^2x^4) + 2cx^2 + a) dx^1 \circ dx^3 + 4ax^2 dx^1 \circ dx^4 \\ & + (q(x^4)^2 + 2cx^4 + b) dx^2 \circ dx^2 + 2(qx^4 + c) dx^2 \circ dx^3 \\ & + 2a\, dx^2 \circ dx^4 + q\, dx^3 \circ dx^3, \end{aligned} \tag{3.48}$$

where $a \neq 0$. In this case, the non-zero components of the divergence of the Weyl tensor are given by

$$\begin{aligned} \operatorname{div}_4 W_{121} &= -\frac{15x^2\left(6a - qx^3\right)\left(c^2 - bq\right)}{4a^3}, \\ \operatorname{div}_4 W_{122} &= -\frac{15\left(6a - qx^3\right)\left(c^2 - bq\right)}{8a^3}, \\ \operatorname{div}_4 W_{232} &= -\frac{15q\left(c^2 - bq\right)}{8a^3}, \\ \operatorname{div}_4 W_{131} &= 4(x^2)^2 \operatorname{div}_4 W_{232}, \\ \operatorname{div}_4 W_{132} &= \operatorname{div}_4 W_{231} = 2x^2 \operatorname{div}_4 W_{232}\,. \end{aligned} \tag{3.49}$$

The Bach tensor is given by

$$\mathfrak{B} = \begin{pmatrix} \frac{240q\left(c^2-bq\right)(x^2)^2}{a^4} & \frac{120q\left(c^2-bq\right)x^2}{a^4} & 0 & 0 \\ \frac{120q\left(c^2-bq\right)x^2}{a^4} & \frac{60q\left(c^2-bq\right)}{a^4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.50}$$

Thus, *a Type (B.1) metric is Bach-flat if and only if $q = 0$ or $c^2 - bq = 0$, in the latter case being Einstein. Moreover,*

(1) *If $q = 0$, then the Ricci operator in Equation* (3.27) *is either zero or two-step nilpotent and Equation* (3.28) *gives $W_{1224} = -\frac{3}{2}c$, thus distinguishing the following two cases:*

   (a) *If $q = 0$ and $c = 0$, then (M, g) is Ricci-flat and the only non-zero component of the Weyl tensor is $W_{1212} = -3b$. Therefore (M, g) is flat if $q = c = b = 0$.*
   Otherwise, if $q = c = 0 \neq b$, then the Jacobi operators $\mathcal{J}(x)(\,\cdot\,) = R(x, \cdot\,)x$ are two-step nilpotent. Hence (M, g) is Osserman and thus half conformally flat. (See [59] and the references therein for further information about Osserman manifolds).

   (b) *If $q = 0$ and $c \neq 0$, then (M, g) is not locally conformally flat. Moreover the conformal Jacobi operators $\mathcal{J}_W(x)(\,\cdot\,) = W(x, \cdot\,)x$ are nilpotent and (M, g) is half conformally flat. (See [86] and the references therein for further information about conformally Osserman manifolds).*

(2) *If* $q \neq 0$ *and* $b = \frac{c^2}{q}$, *then Equation* (3.28) *shows that* $W_{1334} = \frac{q^2 x^2}{2a}$ *and hence* (M,g) *is not locally conformally flat. Equation* (3.27) *shows that* (M,g) *is Einstein and moreover the Jacobi operator* $\mathcal{J}(x)(\,\cdot\,) = R(x,\cdot\,)x$ *associated to any unit vector* x *has constant eigenvalues* $\{0, \varepsilon_x \frac{q}{a^2}, \varepsilon_x \frac{q}{4a^2}, \varepsilon_x \frac{q}{4a^2}\}$, *where* $g(x,x) = \varepsilon_x = \pm 1$.

Moreover (M,g) *is locally isometric to a para-complex space form of constant para-holomorphic sectional curvature* $H = -\frac{q}{a^2}$, *and thus a modified Riemannian extension as in [29].*

Type (B.2)

Let $\mathfrak{U} = \{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4; x^4 \neq \pm 2\}$ with coordinates (x^1,x^2,x^3,x^4) and metric tensor

$$\begin{aligned} g = &\left(a - \tfrac{a(x^4)^2}{2} + 4b(x^2)^2\right) dx^1 \circ dx^1 + 4bx^2 dx^1 \circ dx^2 \\ &- ax^2((x^4)^2 - 4)dx^1 \circ dx^3 - a(1 + 2x^2x^3)x^4 dx^1 \circ dx^4 + bdx^2 \circ dx^2 \\ &- \tfrac{1}{2}a((x^4)^2 - 4)dx^2 \circ dx^3 - ax^3x^4 dx^2 \circ dx^4 - \tfrac{1}{2}a\, dx^4 \circ dx^4, \end{aligned} \tag{3.51}$$

where $a \neq 0$. The non-zero components of the divergence of the Weyl tensor are given by

$$\operatorname{div}_4 W_{121} = -\frac{15bx^2}{a}, \qquad \operatorname{div}_4 W_{122} = -\frac{15b}{2a}\,. \tag{3.52}$$

The Bach tensor is given by

$$\mathfrak{B} = \begin{pmatrix} -\frac{120b(x^2)^2}{a^2} & -\frac{60bx^2}{a^2} & 0 & 0 \\ -\frac{60bx^2}{a^2} & -\frac{30b}{a^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.53}$$

Now it follows from the previous expressions that *a metric* (3.51) *is Bach-flat if and only if* $b = 0$, *in which case it is Einstein and locally conformally flat, and thus of constant sectional curvature* $\kappa = -\frac{1}{a}$.

Type (B.3)

Let $M = \mathbb{R}^4$ with coordinates (x^1,x^2,x^3,x^4) and metric tensor

$$\begin{aligned} g = &-2ae^{-x^2}x^3 dx^1 \circ dx^2 + 2ae^{-x^2} dx^1 \circ dx^3 \\ &+ 2(2b(x^3)^2 - ax^4)dx^2 \circ dx^2 - 4bx^3 dx^2 \circ dx^3 \\ &+ 2a\, dx^2 \circ dx^4 + b\, dx^3 \circ dx^3, \end{aligned} \tag{3.54}$$

where $a \neq 0$. The divergence of the Weyl tensor and the Bach tensor are both zero. *Any non-reductive metric of Type (B.3) with* $b \neq 0$ *has two-step nilpotent Jacobi operators and thus it is Osserman.* Therefore, it is Einstein and half conformally flat.

Bach-flat metrics are critical points for the functional

$$\mathcal{W} : g \mapsto \mathcal{W}(g) = \int_M \|W\|^2 \, dvol_g$$

and one has that locally conformally Einstein metrics are Bach-flat. Hence, aimed to describe all the non-reductive four-dimensional homogeneous conformally Einstein metrics, one has the following result.

Theorem 3.9. *Let (M, g) be a manifold given by Theorem 3.5. Then (M, g) is Bach-flat if and only if it is locally conformally flat, Einstein or one of the following:*

(i) Type (A.1) with $q = 0$ and $ab \neq 0$:

$$\begin{aligned} g = &(4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ &- (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ &+ b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 \,. \end{aligned}$$

(ii) Type (A.1) with $q = -\frac{3a}{4}$ and $ab \neq 0$:

$$\begin{aligned} g = &(4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 \\ &- (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 \\ &+ b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 - \tfrac{3a}{4}\, dx^3 \circ dx^3 \,. \end{aligned}$$

(iii) Type (A.2) with $\alpha = 1$ and $abq \neq 0$:

$$\begin{aligned} g = &-2ae^{2x^4}\, dx^1 \circ dx^3 + ae^{2x^4} dx^2 \circ dx^2 \\ &+ b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4 \,. \end{aligned}$$

(iv) Type (A.3) with metric

$$\begin{aligned} g_+ = &\,2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 \\ &+ b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where $ab \neq 0$ and $b \neq -q$, or with metric

$$\begin{aligned} g_- = &\,2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cosh(x^4)^2 dx^2 \circ dx^2 \\ &+ b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4, \end{aligned}$$

where $ab \neq 0$ and $b \neq q$.

(v) Type (B.1) with $q = 0$ and $ac \neq 0$:

$$\begin{aligned} g = &(4cx^2x^3 + 8c(x^2)^2x^4 + 2ax^3 + 4b(x^2)^2)\, dx^1 \circ dx^1 \\ &+ 2(4cx^2x^4 + cx^3 + 2bx^2) dx^1 \circ dx^2 + 2(2cx^2 + a) dx^1 \circ dx^3 \\ &+ 4ax^2 dx^1 \circ dx^4 + (2cx^4 + b) dx^2 \circ dx^2 \\ &+ 2c\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 \,. \end{aligned}$$

Half conformally flat non-reductive homogeneous spaces

A special class of Bach-flat spaces is that of half conformally flat manifolds. While half conformally flat Lorentzian metrics are locally conformally flat, there are many strictly half conformally flat examples in the Riemannian and neutral signature settings. Recall that a four-dimensional manifold is half conformally flat if and only if is *conformally Osserman* [13], i.e., the spectrum of the conformal Jacobi operators $\mathcal{J}_W(x)(\,\cdot\,) = W(x,\,\cdot\,)x$ is constant on the unit pseudo-spheres $S^{\pm}(T_pM)$ at each point $p \in M$ (see [86] and the references therein).

An explicit calculation of the conformal Jacobi operators shows that a metric given by Theorem 3.5 is half conformally flat and not locally conformally flat if and only if it corresponds to one of the following cases:

Type (A.1) with $q = 0$ and $ab \neq 0$.

Type (A.1) with $q = -\frac{3}{4}a$ and $ab \neq 0$.

Type (B.1) with $q = c = 0$ and $ab \neq 0$.

Type (B.1) with $q = 0$ and $ac \neq 0$.

Type (B.1) with $aq \neq 0$ and $b = \frac{c^2}{q}$.

Type (B.3) with $ab \neq 0$.

Note that this agrees with the description of (anti-)self-dual non-reductive homogeneous spaces in [27]. Moreover, the conformal Jacobi operators are two-step nilpotent in all cases but the one corresponding to Type (B.1) with $aq \neq 0$ and $b = \frac{c^2}{q}$ where they diagonalize.

It is worth mentioning that in some of the cases above the manifold is also Einstein and thus *pointwise Osserman*, i.e., the spectrum of the Jacobi operators $\mathcal{J}(x)(\,\cdot\,) = R(x,\,\cdot\,)x$ is constant on the unit pseudo-spheres $S^{\pm}(T_pM)$ at each point $p \in M$ (see [59] for further information about Osserman manifolds).

More precisely, a metric given by Theorem 3.5 is Osserman if and only if it is of constant sectional curvature (cf. Remark 3.6) or otherwise:

(i) (M, g) is of Type (B.1) with $q = c = 0$ and $ab \neq 0$, in which case the Jacobi operators are two-step nilpotent, or

(ii) (M, g) is of Type (B.1) with $aq \neq 0$ and $b = \frac{c^2}{q}$. In this case, for any unit spacelike vector the corresponding Jacobi operator $\mathcal{J}(x)(\,\cdot\,) = R(x,\,\cdot\,)x$ is diagonalizable with eigenvalues $\{0, \varepsilon_x\frac{q}{a^2}, \frac{1}{4}\varepsilon_x\frac{q}{a^2}, \frac{1}{4}\varepsilon_x\frac{q}{a^2}\}$, where $\varepsilon_x = g(x,x)$; thus the manifold is locally isometric to a complex or para-complex space form [59]. A long but straightforward calculation shows that for any non-null vector x, the vector space $\operatorname{span}\{x\} \oplus \ker(\mathcal{J}(x) - \varepsilon_x\frac{q}{a^2}\,\mathrm{Id})$ is of Lorentzian signature. Hence (M, g) is a para-complex space form.

(iii) (M, g) is of Type (B.3) with $ab \neq 0$, in which case the Jacobi operators are two-step nilpotent.

Moreover, it is worth emphasizing that in all the cases above the manifold is locally symmetric.

Remark 3.10. A long but straightforward calculation shows that a metric given by Theorem 3.5 of non-constant sectional curvature is locally symmetric if and only if it is

> Type (A.1) with $b = 0$ and $a(a - 4q) \neq 0$, in which case (M, g) is locally conformally flat with diagonalizable Ricci operator. Hence locally isometric to a product $\mathbb{R} \times N$, where N is of constant sectional curvature $\kappa_N = -\frac{1}{a}$,

or it corresponds to one of the following cases:

> Type (B.1) with $q = c = 0$ and $ab \neq 0$, in which case (M, g) is Osserman with two-step nilpotent Jacobi operators.
>
> Type (B.1) with $aq \neq 0$ and $b = \frac{c^2}{q}$, in which case (M, g) is a para-complex space form.
>
> Type (B.3) with $ab \neq 0$, in which case (M, g) is Osserman with two-step nilpotent Jacobi operators.

See [59] for a classification of locally symmetric four-dimensional Osserman manifolds and [26] for a description of gradient Ricci solitons on non-reductive homogeneous spaces, where metrics of Type (A.1) with $b = 0$ play a distinguished role.

3.3 Non-reductive conformally Einstein homogeneous manifolds

The purpose of this section is to prove Theorem 3.1, determining which non-reductive homogeneous four-manifolds contain an Einstein metric in their conformal class. We will exclude from our analysis the trivial cases of Einstein and locally conformally flat manifolds. Moreover, we will obtain the explicit form of the conformal Einstein metric. Since any conformally Einstein manifold is necessarily Bach-flat, Theorem 3.9 shows that the analysis of the conformally Einstein equation

$$2\,\mathrm{Hes}_\varphi + \varphi\,\rho = \frac{1}{4}\{2\Delta\varphi + \varphi\,\tau\}g \tag{3.55}$$

must be carried out only for the following cases:

(i) Type (A.1) with $q = 0$ and $ab \neq 0$:

$$\begin{aligned} g = {} & (4b(x^2)^2 + a)\,dx^1 \circ dx^1 + 4bx^2\,dx^1 \circ dx^2 \\ & - (4ax^2x^4 - 4cx^2 + a)\,dx^1 \circ dx^3 + 4ax^2\,dx^1 \circ dx^4 \\ & + b\,dx^2 \circ dx^2 - 2(ax^4 - c)\,dx^2 \circ dx^3 + 2a\,dx^2 \circ dx^4\,. \end{aligned}$$

(ii) Type (A.1) with $q = -\frac{3a}{4}$ and $ab \neq 0$:

$$g = (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 - \tfrac{3a}{4}\, dx^3 \circ dx^3 .$$

(iii) Type (A.2) with $\alpha = 1$ and $abq \neq 0$:

$$g = -2ae^{2x^4}\, dx^1 \circ dx^3 + ae^{2x^4} dx^2 \circ dx^2 + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4 .$$

(iv) Type (A.3) with metric

$$g_+ = 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4,$$

where $ab \neq 0$ and $b \neq -q$, or with metric

$$g_- = 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cosh(x^4)^2 dx^2 \circ dx^2 + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4,$$

where $ab \neq 0$ and $b \neq q$.

(v) Type (B.1) with $q = 0$ and $ac \neq 0$:

$$g = (4cx^2x^3 + 8c(x^2)^2x^4 + 2ax^3 + 4b(x^2)^2)\, dx^1 \circ dx^1 + 2(4cx^2x^4 + cx^3 + 2bx^2)dx^1 \circ dx^2 + 2(2cx^2 + a)dx^1 \circ dx^3 + 4ax^2 dx^1 \circ dx^4 + (2cx^4 + b)dx^2 \circ dx^2 + 2c\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 .$$

Theorem 3.1. *Let (M, g) be a conformally Einstein four-dimensional non-reductive homogeneous space. Then (M, g) is Einstein, locally conformally flat, or locally isometric to:*

(i) $(\mathbb{R}^4, g)$ *with metric given by*

$$g = (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4,$$

where a, b and c are arbitrary constants with $ab \neq 0$.

(ii) $(\mathbb{R}^4, g)$ *with metric given by*

$$g = (4b(x^2)^2 + a)\, dx^1 \circ dx^1 + 4bx^2\, dx^1 \circ dx^2 - (4ax^2x^4 - 4cx^2 + a)\, dx^1 \circ dx^3 + 4ax^2\, dx^1 \circ dx^4 + b\, dx^2 \circ dx^2 - 2(ax^4 - c)\, dx^2 \circ dx^3 + 2a\, dx^2 \circ dx^4 - \tfrac{3a}{4}\, dx^3 \circ dx^3,$$

where a, b and c are arbitrary constants with $ab \neq 0$.

(iii) $(\mathbb{R}^4, g)$ *with metric given by*

$$g = -2ae^{2x^4}\, dx^1 \circ dx^3 + ae^{2x^4} dx^2 \circ dx^2 + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4,$$

where a, b, c *and* q *are arbitrary constants with* $abq \neq 0$.

(iv) $(\mathfrak{U} \subset \mathbb{R}^4, g_+)$ *with metric given by*

$$g_+ = 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cos(x^4)^2 dx^2 \circ dx^2 + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4,$$

where $\mathfrak{U} = \{(x^1, \dots, x^4) \in \mathbb{R}^4 \,/\, \cos(x^4) \neq 0\}$, *and* a, b, c *and* q *are arbitrary constants with* $ab \neq 0$ *and* $b \neq -q$, *or*

$(\mathbb{R}^4, g_-)$ *with metric given by*

$$g_- = 2ae^{2x^3}\, dx^1 \circ dx^4 + ae^{2x^3} \cosh(x^4)^2 dx^2 \circ dx^2 + b\, dx^3 \circ dx^3 + 2c\, dx^3 \circ dx^4 + q\, dx^4 \circ dx^4,$$

where a, b, c *and* q *are arbitrary constants with* $ab \neq 0$ *and* $b \neq q$.

Moreover, all the cases (i)–(iv) are in the conformal class of a Ricci-flat metric which is unique (up to an homothety) only in Case (i). Otherwise the space of conformally Ricci-flat metrics is either two or three-dimensional.

In what follows we will use the necessary conditions obtained by Kozameh, Newman and Tod [72] stated in Theorem 1.14. Relating the solutions φ of the conformally Einstein Equation (1.6) and the conformal deformation σ by $\sigma = -\log(\varphi)$, as a matter of notation, we define a $(0, 3)$-tensor field $\mathcal{C}$ by $\mathcal{C} = (\mathrm{div}_4 W)(X, Y, Z) + W(X, Y, Z, \nabla\sigma)$. Obviously, $\mathcal{C}_{ijk} = -\mathcal{C}_{jik}$ for all $i, j, k \in \{1, \dots, 4\}$ and $\mathcal{C} = 0$ is equivalent to Theorem 1.14–(ii).

Recall that conditions (i)–(ii) in Theorem 1.14 are also sufficient to be conformally Einstein if (M, g) is *weakly-generic* (i.e., the Weyl tensor, viewed as a map $TM \to \bigotimes^3 TM$ is injective). Note that cases (i)–(iii) in Theorem 3.9 are not weakly-generic and thus we must study the existence of solutions of Equation (3.55) case by case. In opposition, metrics corresponding to Theorem 3.9–(iv) are weakly-generic.

3.3.1 Type (A.1) with $q = 0$ and $ab \neq 0$ or $q = -\frac{3a}{4}$ and $ab \neq 0$

We consider the two possibilities separately.

Type (A.1) with $q = 0$ and $ab \neq 0$

In this case by Equation (3.35) the non-zero components of the divergence of the Weyl tensor are given by

$$\mathrm{div}_4 W_{121} = \frac{12\, bx^2}{a}, \qquad \mathrm{div}_4 W_{122} = \frac{6\, b}{a}, \tag{3.56}$$

and, by Equation (3.11), the only non-zero component of the Weyl tensor is given by

$$W_{1212} = -6b, \tag{3.57}$$

which shows that (M, g) is not weakly-generic. For an arbitrary positive function $\varphi(x^1, x^2, x^3, x^4)$ on M, let $\sigma = -\log(\varphi)$. Then a straightforward calculation shows that the gradient of σ is given in the coordinate basis by

$$\begin{aligned}\nabla\sigma = & \tfrac{4}{a^2\varphi}\left\{\left(ax^4-c\right)\varphi_4 + a\varphi_3\right\}\partial_{x^1}\\ & -\tfrac{2}{a^2\varphi}\left\{\varphi_4\left(a+4x^2\left(ax^4-c\right)\right)+4ax^2\varphi_3\right\}\partial_{x^2}\\ & +\tfrac{4}{a^2\varphi}\left\{a\left(2\varphi_3-2x^2\varphi_2+\varphi_1\right)+2\varphi_4\left(ax^4-c\right)\right\}\partial_{x^3}\\ & +\tfrac{2}{a^3\varphi}\left\{\varphi_4\left(ab+4ax^4\left(ax^4-2c\right)+4c^2\right)+2a\varphi_1\left(ax^4-c\right)\right.\\ & \qquad \left. +a\left(4\varphi_3\left(ax^4-c\right)+\varphi_2\left(4x^2\left(c-ax^4\right)-a\right)\right)\right\}\partial_{x^4},\end{aligned}$$

where $\varphi_i = \frac{\partial}{\partial x^i}\varphi$ denote the corresponding partial derivatives.

Thus, the only non-zero components of the tensor $\mathcal{C} = \operatorname{div}_4 W + W(\cdot,\cdot,\cdot,\nabla\sigma)$ are those given by

$$\begin{aligned} a^2\varphi\,\mathcal{C}_{121} &= -12b\left(\varphi_4\left(a-4x^2\left(c-ax^4\right)\right)+4ax^2\varphi_3+2ax^2\varphi\right),\\ a^2\varphi\,\mathcal{C}_{122} &= -12b\left(-2\varphi_4\left(c-ax^4\right)+2a\varphi_3+a\varphi\right). \end{aligned} \tag{3.58}$$

Since $\mathcal{C} = 0$ is a necessary condition for (M, g) to be conformally Einstein, $a\varphi(\mathcal{C}_{121} - 2x^2\mathcal{C}_{122}) = -12b\varphi_4$ must be zero and, since $b \neq 0$, in this case φ does not depend on the coordinate x^4. Then

$$\mathcal{C}_{122} = \frac{-12b(\varphi+2\varphi_3)}{a\varphi}, \qquad \mathcal{C}_{121} = 2x^2\mathcal{C}_{122}\,.$$

Hence, $\mathcal{C} = 0$ shows that

$$\varphi(x^1, x^2, x^3) = e^{-\frac{x^3}{2}}\phi(x^1, x^2), \tag{3.59}$$

for some smooth function $\phi(x^1, x^2)$.

Now, we analyze the existence of solutions of Equation (3.55) for some φ as above. In order to simplify the notation, set

$$\mathcal{E} = 2\operatorname{Hes}_\varphi + \varphi\,\rho - \frac{1}{4}\{2\Delta\varphi + \varphi\,\tau\}g,$$

and determine the conditions for $\mathcal{E} = 0$.

Since $\mathcal{E}(\partial_{x^1}, \partial_{x^1}) = 2e^{-\frac{x^3}{2}}\phi_{11}(x^1, x^2)$, any solution of Equation (3.55) must be of the form given in Equation (3.59) with $\phi(x^1, x^2) = \alpha_1(x^2) + x^1\alpha_2(x^2)$ for some smooth functions α_1, α_2 on M. Taking into account that

$$\mathcal{E}(\partial_{x^1}, \partial_{x^2}) = -2e^{-\frac{x^3}{2}}(\alpha_1'(x^2) + (x^1-1)\alpha_2'(x^2)),$$

we can show that $\alpha_1(x^2) = \mu_1$ and $\alpha_2(x^2) = \mu_2$ for some constants μ_1, μ_2. Further, the component $\mathcal{E}(\partial_{x^2}, \partial_{x^4}) = -2\mu_2 e^{-\frac{x^3}{2}}$ shows that $\mu_2 = 0$ and hence Equation (3.59) reduces to

$$\varphi = \mu_1 e^{-\frac{x^3}{2}}\,.$$

Now, a straightforward calculation shows that $\mathcal{E} = 0$ holds and the conformal metric $\bar{g} = \varphi^{-2} g$ is Ricci-flat.

Remark 3.11. Since any non-reductive homogeneous manifold of Type (A.1) with $q = 0$ and $ab \neq 0$ is conformally Osserman with two-step nilpotent conformal Jacobi operators, and this property is conformally invariant, the metric $\bar{g}$ is Osserman with two-step nilpotent Jacobi operators.

Type (A.1) with $q = -\frac{3a}{4}$ and $ab \neq 0$.

Proceeding as in the previous case, for an arbitrary positive function $\varphi(x^1, x^2, x^3, x^4)$ on M we consider $\sigma = -\log(\varphi)$. Then

$$\begin{aligned}
\nabla\sigma = &\tfrac{1}{a^2\varphi}\left\{a\left(\varphi_3 + 3x^2\varphi_2 - \tfrac{3}{2}\varphi_1\right) + \varphi_4\left(ax^4 - c\right)\right\}\partial_{x^1}\\
&- \tfrac{1}{a^2\varphi}\left\{2\varphi_4\left(x^2\left(ax^4 - c\right) + a\right) + ax^2\left(2\varphi_3 + 6x^2\varphi_2 - 3\varphi_1\right)\right\}\partial_{x^2}\\
&+ \tfrac{1}{a^2\varphi}\left\{a\left(2\varphi_3 - 2x^2\varphi_2 + \varphi_1\right) + 2\varphi_4\left(ax^4 - c\right)\right\}\partial_{x^3}\\
&+ \tfrac{1}{a^3\varphi}\left\{2\varphi_4\left(a^2(x^4)^2 + ab - 2acx^4 + c^2\right)\right.\\
&\qquad \left. + a\left(\varphi_1\left(ax^4 - c\right) - 2\left(\varphi_2\left(x^2\left(ax^4 - c\right) + a\right) + \varphi_3\left(c - ax^4\right)\right)\right)\right\}\partial_{x^4}\,.
\end{aligned}$$

Recall from Equation (3.35) that the non-zero components of the divergence of the Weyl tensor are given by

$$\begin{aligned}
&\operatorname{div}_4 W_{121} = -\tfrac{6bx^2}{a}, \qquad \operatorname{div}_4 W_{122} = -\tfrac{3b}{a}, \qquad \operatorname{div}_4 W_{131} = \tfrac{6b(x^2)^2}{a},\\
&\operatorname{div}_4 W_{232} = \tfrac{3b}{2a}, \qquad \operatorname{div}_4 W_{132} = \operatorname{div}_4 W_{231} = \tfrac{3bx^2}{a}\,.
\end{aligned}$$

Equation (3.11) shows that the non-zero components of the Weyl tensor are

$$\begin{aligned}
&W_{1212} = -3b, \qquad W_{1213} = 3bx^2, \qquad W_{1223} = \tfrac{3b}{2},\\
&W_{1313} = -3b(x^2)^2, \quad W_{1323} = -\tfrac{3bx^2}{2}, \quad W_{2323} = -\tfrac{3b}{4}\,.
\end{aligned}$$

Then, the non-zero components of the tensor field $\mathcal{C}$ are given by

$$\begin{aligned}
&\mathcal{C}_{131} = -x^2\mathcal{C}_{121}, \quad \mathcal{C}_{231} = -\tfrac{1}{2}\mathcal{C}_{121},\\
&\mathcal{C}_{132} = -x^2\mathcal{C}_{122}, \quad \mathcal{C}_{232} = -\tfrac{1}{2}\mathcal{C}_{122},\\
&\mathcal{C}_{133} = -x^2\mathcal{C}_{123}, \quad \mathcal{C}_{233} = -\tfrac{1}{2}\mathcal{C}_{123},
\end{aligned}$$

where

$$\begin{aligned} a^2\varphi\,\mathcal{C}_{121} &= 6b\,(x^2\,(2a\varphi + a\varphi_1 - 2a\varphi_3 - 2a\varphi_4 x^4 + 2c\varphi_4) \\ &\qquad -a\varphi_4 - 2a\varphi_2 (x^2)^2)\,, \\ a^2\varphi\,\mathcal{C}_{122} &= 3b\,(2a\varphi + a\varphi_1 - 2a\varphi_3 - 2a\varphi_2 x^2 - 2a\varphi_4 x^4 + 2c\varphi_4)\,, \\ a^2\varphi\,\mathcal{C}_{123} &= -3ab\varphi_4\,. \end{aligned}$$

Since $ab \neq 0$ and $\mathcal{C}_{123} = 0$ the function $\varphi(x^1, x^2, x^3, x^4)$ does not depend on the coordinate x^4 and the tensor field $\mathcal{C}_{121}$ reduces to

$$\mathcal{C}_{122} = \frac{3b}{a\varphi}\left\{2\varphi + \varphi_1 - 2\varphi_3 - 2\varphi_2 x^2\right\}, \qquad \mathcal{C}_{121} = 2x^2\mathcal{C}_{122}\,.$$

A solution of the differential equation $2\varphi = 2\varphi_3 + 2x^2\varphi_2 - \varphi_1$ is necessarily of the form

$$\varphi(x^1, x^2, x^3) = e^{-2x^1}\phi(e^{2x^1}x^2, 2x^1 + x^3) = e^{-2x^1}(\phi\circ\psi)(x^1, x^2, x^3), \tag{3.60}$$

where $\psi(x^1, x^2, x^3) = (e^{2x^1}x^2, 2x^1 + x^3)$ and $\phi(z, \omega)$ is an arbitrary function for $z = e^{2x^1}x^2$ and $\omega = 2x^1 + x^3$.

Now, we analyze the existence of solutions of Equation (3.55) for some φ as in Equation (3.60). Setting

$$\mathcal{E} = 2\,\mathrm{Hes}_\varphi + \varphi\,\rho - \frac{1}{4}\{2\Delta\varphi + \varphi\,\tau\}g,$$

one has $\mathcal{E}(\partial_{x^2}, \partial_{x^2}) = 2e^{2x^1}\partial^2_{z^2}\phi = 0$, and hence

$$\varphi(x^1, x^2, x^3) = e^{-2x^1}\left(e^{2x^1}x^2\hat{\phi}(2x^1 + x^3) + \bar{\phi}(2x^1 + x^3)\right)$$

for some smooth functions $\hat{\phi}(\omega)$, $\bar{\phi}(\omega)$. Considering the component $\mathcal{E}(\partial_{x^2}, \partial_{x^3}) = 2\hat{\phi}' - \hat{\phi} = 0$, one has that $\hat{\phi}(2x^1 + x^3) = \mu e^{\frac{1}{2}(2x^1+x^3)}$ for some constant μ. Now the only non-zero components of the tensor field $\mathcal{E}$ are given by

$$\mathcal{E}(\partial_{x^1}, \partial_{x^1}) = 2\mathcal{E}(\partial_{x^1}, \partial_{x^3}) = 2\mathcal{E}(\partial_{x^3}, \partial_{x^3}) = 4e^{-2x^1}\left(\bar{\phi} - 3\bar{\phi}' + 2\bar{\phi}''\right)\,.$$

Hence $\mathcal{E} = 0$ gives $\bar{\phi}(2x^1 + x^3) = \mu_1 e^{\frac{1}{2}(2x^1+x^3)} + \mu_2 e^{2x^1+x^3}$ and thus any solution of the conformally Einstein equation is of the form

$$\varphi(x^1, x^2, x^3, x^4) = \mu_1 e^{x^3} + \mu_2 e^{\frac{1}{2}x^3 - x^1} + \mu_3\,x^2\,e^{\frac{1}{2}x^3 + x^1}\,.$$

Moreover, any of the conformal metrics $\bar{g} = \varphi^{-2}g$ is Ricci-flat.

Remark 3.12. Since any non-reductive homogeneous manifold of Type (A.1) with $q = -\frac{3}{4}a$ and $ab \neq 0$ is conformally Osserman with two-step nilpotent conformal Jacobi operators, any conformal Einstein metric $\bar{g}$ is Osserman with two-step nilpotent Jacobi operators. Moreover, there is a 3-parameter family of conformally equivalent Osserman metrics. This shows that the cases $q = 0$ and $q = -\frac{3}{4}a$ are essentially different since the space of conformally Einstein metrics is one-dimensional in the first case and three-dimensional in the second one.

3.3.2 Type (A.2) with $\alpha = 1$ and $abq \neq 0$

Let $\varphi(x^1, x^2, x^3, x^4)$ be a positive function on M and $\sigma = -2\log(\varphi)$. Then

$$\nabla\sigma = \tfrac{2}{a^2 q\varphi}\left\{ae^{-2x^4}(q\varphi_3 - c\varphi_4) - e^{-4x^4}\varphi_1(c^2 - bq)\right\}\partial_{x^1} \\ - \tfrac{2}{a\varphi}\varphi_2 e^{-2x^4}\partial_{x^2} + \tfrac{2}{a\varphi}\varphi_1 e^{-2x^4}\partial_{x^3} - \tfrac{1}{aq\varphi}\left\{2a\varphi_4 + 2c\varphi_1 e^{-2x^4}\right\}\partial_{x^4}\,.$$

It follows from Equations (3.14) and (3.38) that the non-zero components of the Weyl tensor and its divergence are given by

$$W_{2323} = \frac{ab}{2q}e^{2x^4}, \quad W_{3434} = -\frac{b}{2} \qquad \text{and} \qquad \operatorname{div}_4 W_{343} = -\frac{b}{2q},$$

respectively, from where it follows that (M, g) is not weakly-generic. Therefore, the only non-zero components of the tensor field $\mathcal{C} = \operatorname{div}_4 W + W(\cdot,\cdot,\cdot,\nabla\sigma)$ are those given by

$$\begin{aligned} &\mathcal{C}_{232} = -\tfrac{b}{q}\tfrac{\varphi_1}{\varphi}, \qquad \mathcal{C}_{233} = -\tfrac{b}{q}\tfrac{\varphi_2}{\varphi}, \qquad \mathcal{C}_{344} = -\tfrac{b}{a}\tfrac{\varphi_1}{\varphi}e^{-2x^4}, \\ &\mathcal{C}_{343} = \tfrac{b}{a\,q\,\varphi}\left(a(\varphi - \varphi_4) - c\varphi_1 e^{-2x^4}\right). \end{aligned} \tag{3.61}$$

Since $a\,b \neq 0$, the first two expressions in Equation (3.61) show that $\varphi(x^1, x^2, x^3, x^4)$ does not depend on the coordinates x^1 and x^2. Hence, the tensor field $\mathcal{C}$ reduces to

$$\mathcal{C}_{343} = \frac{b(\varphi - \varphi_4)}{q\varphi},$$

where φ is a smooth function on the coordinates (x^3, x^4) and it follows from $\mathcal{C}_{343} = 0$ that $\varphi(x^3, x^4) = \phi(x^3)e^{x^4}$, for some smooth function $\phi(x^3)$.

Considering now the conformally Einstein Equation (3.55), and setting

$$\mathcal{E} = 2\operatorname{Hes}_\varphi + \varphi\,\rho - \frac{1}{4}\{2\Delta\varphi + \varphi\,\tau\}g,$$

the only non-zero component of the tensor $\mathcal{E}$ is

$$\mathcal{E}(\partial_{x^3}, \partial_{x^3}) = \frac{e^{x^4}(2q\phi'' - b\phi)}{q}.$$

Integrating $\mathcal{E}(\partial_{x^3}, \partial_{x^3}) = 0$ we obtain that

$$\begin{cases} \varphi = e^{x^4 - x^3\sqrt{\frac{b}{2q}}}\left(\kappa_1 e^{x^3\sqrt{\frac{2b}{q}}} + \kappa_2\right), & \text{if} \quad b\,q > 0, \\ \varphi = e^{x^4}\left(\kappa_1\cos\left(x^3\sqrt{-\frac{b}{2q}}\right) + \kappa_2\sin\left(x^3\sqrt{-\frac{b}{2q}}\right)\right), & \text{if} \quad b\,q < 0\,. \end{cases} \tag{3.62}$$

Moreover, a long but straightforward computation shows that the metric $\bar{g} = \varphi^{-2}g$ for any function φ given by Equation (3.62) is Ricci-flat.

Remark 3.13. For each of the possibilities in Equation (3.62) there are at least two conformal metrics which are Einstein (indeed, Ricci-flat). Moreover, for any of the conformal Einstein metrics, there are some conformal deformation of the metric which remains Einsteinian.

Further observe that no metric (A.2) with $\alpha = 1$ and $abq \neq 0$ is half conformally flat, and hence they are not in the conformal class of any Osserman metric.

3.3.3 Type (A.3) with metrics $g_\pm$ and $b \neq \mp q$, $ab \neq 0$

We will briefly schedule the proof of the case corresponding to g_+. The analysis of g_- is completely analogous. Hence assume $b \neq -q$ and $ab \neq 0$. As in the previous cases, let $\varphi(x^1, x^2, x^3, x^4)$ be a positive function and set $\sigma = -\log(\varphi)$. Then

$$\begin{aligned}\nabla\sigma = &\tfrac{1}{a^2 b\varphi}\left\{2e^{-4x^3}\left(ae^{2x^3}\left(c\varphi_3 - b\varphi_4\right) + \varphi_1\left(bq - c^2\right)\right)\right\}\partial_{x^1}\\ &- \tfrac{2\varphi_2}{a\varphi}\left\{e^{-2x^3}\sec\left(x^4\right)^2\right\}\partial_{x^2} + \tfrac{2}{a\,b\varphi}\left\{c\varphi_1 e^{-2x^3} - a\varphi_3\right\}\partial_{x^3}\\ &- \tfrac{2}{a\varphi}\varphi_1 e^{-2x^3}\partial_{x^4}\,.\end{aligned}$$

It follows from Equations (3.17) and (3.41) that the non-zero components of the tensor $\mathcal{C} = \operatorname{div}_4 W + W(\cdot,\cdot,\cdot,\nabla\sigma)$ are given by

$$\begin{aligned}&b\,\varphi\,\mathcal{C}_{242} = (b+q)\cos\left(x^4\right)^2\varphi_1, \quad b\,\varphi\,\mathcal{C}_{244} = -(b+q)\varphi_2,\\ &a\varphi\,\mathcal{C}_{343} = -(b+q)e^{-2x^3}\varphi_1,\\ &a\,b\,\varphi\,\mathcal{C}_{344} = -(b+q)e^{-2x^3}\left(a\left(\varphi - \varphi_3\right)e^{2x^3} + c\varphi_1\right)\,.\end{aligned} \tag{3.63}$$

Since $ab \neq 0$ and $b \neq -q$ the first two equations show that φ does not depend on the coordinates x^1 and x^2 and the tensor field $\mathcal{C}$ reduces to

$$b\,\varphi\,\mathcal{C}_{344} = -(b+q)\left(\varphi - \varphi_3\right),$$

where φ is a smooth function on the coordinates (x^3, x^4). Now $\mathcal{C}_{344} = 0$ gives $\varphi(x^3, x^4) = \phi(x^4)e^{x^3}$, for some smooth function $\phi(x^4)$.

Consider now the conformally Einstein equation and set, as in the previous cases, $\mathcal{E} = 2\operatorname{Hes}_\varphi + \varphi\,\rho - \frac{1}{4}\{2\Delta\varphi + \varphi\,\tau\}g$. A straightforward calculation shows that the only non-zero component of the tensor field $\mathcal{E}$ is given by

$$\mathcal{E}(\partial_{x^4}, \partial_{x^4}) = \frac{1}{b}e^{x^3}\left((b-q)\phi + 2b\phi''\right),$$

which shows that $\phi(x^4)$ is determined by the equation $\phi'' = -\frac{b-q}{2b}\phi$. Hence the conformal deformation $\varphi(x^3, x^4)$ is given by

$$\begin{cases}\varphi_+ = (\mu_1 x^4 + \mu_2)e^{x^3}, & \text{if } b - q = 0,\\ \varphi_+ = e^{x^3 - x^4\sqrt{\frac{q-b}{2b}}}\left(\mu_1 e^{x^4\sqrt{\frac{2(q-b)}{b}}} + \mu_2\right), & \text{if } b(q-b) > 0,\\ \varphi_+ = e^{x^3}\left(\mu_1\cos\left(x^4\sqrt{\frac{b-q}{2b}}\right) + \mu_2\sin\left(x^4\sqrt{\frac{b-q}{2b}}\right)\right), & \text{if } b(q-b) < 0\,.\end{cases} \tag{3.64}$$

Moreover in all the cases above the conformal metric $\bar{g}_+ = \varphi_+^{-2} g_+$ is Ricci-flat.

The case of g_- is obtained in a completely analogous way. For any metric g_- given by Equation (3.18), the conformal metric $\bar{g}_- = \varphi_-^{-2} g_-$ is Ricci-flat, where

$$\begin{cases} \varphi_- = (\mu_1 x^4 + \mu_2) e^{x^3}, & \text{if } b + q = 0, \\ \varphi_- = e^{x^3 - x^4 \sqrt{\frac{q+b}{2b}}} \left(\mu_1 e^{x^4 \sqrt{\frac{2(q+b)}{b}}} + \mu_2 \right), & \text{if } b(q+b) > 0, \\ \varphi_- = e^{x^3} \left(\mu_1 \cos\left(x^4 \sqrt{-\frac{b+q}{2b}} \right) + \mu_2 \sin\left(x^4 \sqrt{-\frac{b+q}{2b}} \right) \right), & \text{if } b(q+b) < 0 . \end{cases} \tag{3.65}$$

Remark 3.14. For each of the possibilities in Equations (3.64) and (3.65) there are at least two conformal metrics which are Einstein. Equivalently, for any conformally Einstein metric, there are some conformal deformation of the metric which remains to be Einstein.

Further observe that no metric of Type (A.3) with $\varepsilon = \pm 1$, $b \neq \mp q$ and $ab \neq 0$ is half conformally flat, and hence (M, g) is not in the conformal class of an Osserman metric.

3.3.4 Type (B.1) with $q = 0$ and $ac \neq 0$

Setting $q = 0$ in Equations (3.28) and (3.49), the non-zero components of the Weyl tensor and its divergence are given by

$$\operatorname{div}_4 W_{121} = -\tfrac{45c^2x^2}{2a^2}, \qquad \operatorname{div}_4 W_{122} = -\tfrac{45c^2}{4a^2}, \qquad \text{and}$$

$$\begin{aligned} W_{1212} &= \tfrac{3c^2x^3}{a} - 3\left(b + 2cx^4\right), & W_{1213} &= -\tfrac{3c\left(a+2cx^2\right)}{2a}, & & \\ W_{1214} &= -3cx^2, & W_{1223} &= -\tfrac{3c^2}{2a}, & W_{1224} &= -\tfrac{3c}{2}, \end{aligned}$$

respectively. This shows that, in opposition to the previous cases, (M, g) is weakly-generic and thus $\mathcal{C} = \operatorname{div}_4 W + W(\cdot,\cdot,\cdot,\nabla\sigma) = 0$ is a necessary and sufficient condition to be conformally Einstein.

As in the previous cases, consider $\varphi(x^1, x^2, x^3, x^4)$ a positive function and set $\sigma = -\log(\varphi)$. Express the gradient of σ as

$$\begin{aligned} \nabla\sigma = & \tfrac{2}{a^2\varphi}\left\{c\varphi_4 - a\varphi_3\right\}\partial_{x^1} + \tfrac{2}{a^2\varphi}\left\{2x^2\left(a\varphi_3 - c\varphi_4\right) - a\varphi_4\right\}\partial_{x^2} \\ & + \tfrac{2}{a^2\varphi}\left\{x^3\left(2a\varphi_3 - c\varphi_4\right) - a\varphi_1 + 2a\varphi_2 x^2\right\}\partial_{x^3} \\ & + \tfrac{2}{a^2\varphi}\left\{b\varphi_4 - \varphi_2\left(a + 2cx^2\right) + c\varphi_1 - c\varphi_3 x^3 + 2c\varphi_4 x^4\right\}\partial_{x^4} . \end{aligned}$$

Now, the components $\mathcal{C}_{123}$ and $\mathcal{C}_{124}$ of the tensor field $\mathcal{C} = \operatorname{div}_4 W + W(\cdot,\cdot,\cdot,\nabla\sigma)$ are given by

$$a\,\varphi\,\mathcal{C}_{123} = 3c\varphi_3, \qquad a\,\varphi\,\mathcal{C}_{124} = 3c\varphi_4,$$

and, since $c \neq 0$ and $\mathcal{C}_{123} = \mathcal{C}_{124} = 0$, the function φ is independent of the coordinates x^3 and x^4. Assuming φ to be a smooth function on the coordinates (x^1, x^2), the non-zero components of $\mathcal{C}$ reduce to

$$\mathcal{C}_{121} = -\frac{3c\,(a\varphi_1 - 15c\varphi x^2)}{a^2\varphi}, \qquad \mathcal{C}_{122} = \frac{3c\,(15c\varphi - 2a\varphi_2)}{2a^2\varphi}\,. \tag{3.66}$$

A straightforward computation shows that $\mathcal{C}_{122} = 0$ if and only if

$$\varphi = \phi(x^1)e^{\frac{15c}{2a}x^2}, \tag{3.67}$$

for some smooth function $\phi(x^1)$. Then $\mathcal{C}_{121}$ becomes

$$\mathcal{C}_{121} = -\frac{3c\,(a\phi'(x^1) - 15cx^2\phi(x^1))}{a^2\phi(x^1)}, \tag{3.68}$$

from where it follows that ϕ vanishes identically, and hence $\varphi \equiv 0$, which is a contradiction. Hence this manifold is not conformally Einstein.

Remark 3.15. Observe that the conformally Einstein metrics in Theorem 3.1–(i) are always of neutral signature, while metrics corresponding to cases (ii) and (iii) may be either Lorentzian or of neutral signature $(2,2)$, depending on the choice of the parameters defining the metrics (3.12), (3.15) and (3.18).

Remark 3.16. Let (M,g) be a non-reductive and not locally symmetric homogeneous pseudo-Riemannian manifold of dimension four with $\mathfrak{g}$ the isometry algebra and $\mathfrak{h}$ its isotropy subalgebra. Then, the pair of Lie algebras $(\mathfrak{g},\mathfrak{h})$ is isomorphic to one of the following Types: (A.1), (A.2), (A.3), (A.4), (B.1) or (B.2). Conversely, for every pair of Lie algebras $(\mathfrak{g},\mathfrak{h})$ in this list there exits a non-reductive homogeneous pseudo-Riemannian four-dimensional manifold with isometry algebra $\mathfrak{g}$.

Moreover, the $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$ (excluding (A.3) with $\varepsilon = 1$) is a subalgebra of $\mathfrak{g}$ which implies that each case in the list is locally isometric to a Lie group G with a left-invariant metric as follows:

(i) The Lie group $\mathbb{R} \times \widetilde{SL(2,\mathbb{R})}$ is locally isometric to Type (A.1) with Lie algebra $\mathfrak{m}$ given by

$$[u_1,u_2] = 2u_2, \quad [u_1,u_4] = -2u_4, \quad [u_2,u_4] = 2u_1\,.$$

(ii) The Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ is locally isometric to Type (A.2) with Lie algebra $\mathfrak{m}$ given by

$$[u_1,u_4] = (\alpha+1)u_1, \quad [u_2,u_4] = \alpha\,u_2, \quad [u_3,u_4] = (\alpha-1)\,u_3, \quad \text{or}$$

to Types (A.4) (B.2) with Lie algebra $\mathfrak{m}$ given by

$$[u_1,u_2] = 2u_2, \quad [u_1,u_4] = u_4\,.$$

(iii) The Lie group $\mathbb{R} \ltimes E(1,1)$ is locally isometric to Type (A.3) for $\varepsilon = -1$ with Lie algebra $\mathfrak{m}$ given by

$$[u_1,u_3] = 2u_1, \quad [u_2,u_3] = u_2, \quad [u_2,u_4] = u_2\,.$$

(iv) The Lie group $\mathbb{R} \ltimes H^3$ is locally isometric to Type (B.1) with Lie algebra $\mathfrak{m}$ given by

$$[u_1,u_2] = 2u_2, \quad [u_1,u_3] = u_3, \quad [u_1,u_4] = -u_4, \quad [u_2,u_4] = u_3\,.$$

Part II

New examples of Bach-flat metrics and Ricci solitons

Chapter 4
Bach-flat isotropic gradient Ricci solitons

Bach-flat structures and Ricci solitons are natural generalizations of Einstein metrics. The Riemannian situation is quite rigid, since Bach-flat four-dimensional complete gradient Ricci solitons are locally conformally flat in the shrinking case [36]. The steady case is more involved and triviality of complete Bach-flat gradient Ricci solitons is proved in [34] under the assumptions of positive Ricci tensor and scalar curvature attaining a maximum at some interior point.

The purpose of this chapter is to show the existence of non-trivial examples of Bach-flat gradient Ricci solitons in neutral signature. For that, we construct a family of Bach-flat metrics (see Theorem 4.1) analyzing the existence of gradient Ricci solitons. Examples of self-dual gradient Ricci solitons which are not locally conformally flat were already known in signature $(2,2)$. Hence, our description can be considered as a generalization of Theorem 1.27 allowing to obtain non-trivial anti-self-dual examples at the same time. In this chapter we report on work investigated in [33].

4.1 Bach-flat Riemannian extensions determined by a parallel tensor field

Our first main result concerns the construction of Bach-flat metrics:

Theorem 4.1. *Let (Σ, D, T) be a torsion free affine surface equipped with a parallel $(1,1)$-tensor field T. Let Φ be an arbitrary symmetric $(0,2)$-tensor field on Σ. Then the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$ vanishes if and only if T is either a multiple of the identity or nilpotent.*

Proof. In order to compute the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$, first of all observe that being T parallel imposes some restrictions on the components $T^j{}_i$ as well as on the Christoffel symbols of the connection D:

$$DT = 0 \Rightarrow \begin{cases} T^1{}_2\,{}^D\Gamma_{11}{}^2 - T^2{}_1\,{}^D\Gamma_{12}{}^1 = 0, \\ T^1{}_2\,{}^D\Gamma_{12}{}^2 - T^2{}_1\,{}^D\Gamma_{22}{}^1 = 0, \\ T^2{}_1\,{}^D\Gamma_{11}{}^1 + (T^2{}_2 - T^1{}_1)\,{}^D\Gamma_{11}{}^2 - T^2{}_1\,{}^D\Gamma_{12}{}^2 = 0, \\ T^1{}_2\,{}^D\Gamma_{11}{}^1 + (T^2{}_2 - T^1{}_1)\,{}^D\Gamma_{12}{}^1 - T^1{}_2\,{}^D\Gamma_{12}{}^2 = 0, \\ T^2{}_1\,{}^D\Gamma_{12}{}^1 + (T^2{}_2 - T^1{}_1)\,{}^D\Gamma_{12}{}^2 - T^2{}_1\,{}^D\Gamma_{22}{}^2 = 0, \\ T^1{}_2\,{}^D\Gamma_{12}{}^1 + (T^2{}_2 - T^1{}_1)\,{}^D\Gamma_{22}{}^1 - T^1{}_2\,{}^D\Gamma_{22}{}^2 = 0\,. \end{cases} \tag{4.1}$$

Then, expressing the Bach tensor $\mathfrak{B}_{ij} = \mathfrak{B}(\partial_{x^i}, \partial_{x^j})$ in induced coordinates $(x^i, x_{i'})$, a long but straightforward calculation shows that

$$(\mathfrak{B}_{ij}) = \left(\begin{array}{cc|c} \mathfrak{B}_{11} & \mathfrak{B}_{12} & \multirow{2}{*}{$\tilde{\mathfrak{B}}$} \\ \mathfrak{B}_{12} & \mathfrak{B}_{22} & \\ \hline \multicolumn{2}{c|}{\tilde{\mathfrak{B}}} & 0 \end{array} \right), \tag{4.2}$$

where

$$\tilde{\mathfrak{B}} = \frac{1}{6}((T^1{}_1 - T^2{}_2)^2 + 4T^1{}_2T^2{}_1) \cdot (T^1{}_1 + T^2{}_2) \cdot \begin{pmatrix} T^1{}_1 - T^2{}_2 & 2T^2{}_1 \\ 2T^1{}_2 & T^2{}_2 - T^1{}_1 \end{pmatrix}$$

and where the coefficients $\mathfrak{B}_{11}$, $\mathfrak{B}_{12}$ and $\mathfrak{B}_{22}$ can be written in terms of $\mathfrak{d} = \det(T)$ and $\mathfrak{t} = \operatorname{tr}(T)$ as follows:

$$\begin{aligned}
\mathfrak{B}_{11} = & -\tfrac{1}{6}\left\{10\mathfrak{d}^3 - 2(\mathfrak{t}^2 + 13T^2{}_2\mathfrak{t} - 15(T^2{}_2)^2)\mathfrak{d}^2\right. \\
& \qquad \left. +(5\mathfrak{t} - T^2{}_2)(\mathfrak{t} - T^2{}_2)\mathfrak{t}^2\mathfrak{d} - (\mathfrak{t} - T^2{}_2)^2\mathfrak{t}^4\right\} x_{1'}^2 \\
& - \tfrac{1}{6}\left\{(T^2{}_1)^2(30\mathfrak{d}^2 + \mathfrak{t}^2\mathfrak{d} - \mathfrak{t}^4)\right\} x_{2'}^2 \\
& - \tfrac{1}{3}\left\{(13\mathfrak{t} - 30T^2{}_2)\mathfrak{d}^2 + (3\mathfrak{t} - T^2{}_2)\mathfrak{t}^2\mathfrak{d} - (\mathfrak{t} - T^2{}_2)\mathfrak{t}^4\right\} T^2{}_1\, x_{1'}x_{2'} \\
& - \tfrac{1}{3}\left\{({}^D\Gamma_{11}{}^1 + 2{}^D\Gamma_{12}{}^2)(\mathfrak{t} - 2T^2{}_2) + 2T^2{}_1{}^D\Gamma_{22}{}^2\right\} (\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}\, x_{1'} \\
& - \tfrac{1}{3}\left\{{}^D\Gamma_{11}{}^2(\mathfrak{t} - 2T^2{}_2) + 2T^2{}_1{}^D\Gamma_{12}{}^2\right\} (\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}\, x_{2'} \\
& - \tfrac{1}{6}\left\{10\mathfrak{d}^2 + (3\mathfrak{t}^2 - 22T^2{}_2\mathfrak{t} + 14(T^2{}_2)^2)\mathfrak{d}\right. \\
& \qquad \left. -(\mathfrak{t}^2 - 4T^2{}_2\mathfrak{t} + 2(T^2{}_2)^2)\mathfrak{t}^2\right\} \Phi_{11} \\
& - \tfrac{1}{3}\left\{(11\mathfrak{t} - 14T^2{}_2)\mathfrak{d} - 2(\mathfrak{t} - T^2{}_2)\mathfrak{t}^2\right\} T^2{}_1\Phi_{12} \\
& + \tfrac{1}{3}\left\{\mathfrak{t}^2 - 7\mathfrak{d}\right\} (T^2{}_1)^2\Phi_{22} \\
& - \tfrac{2}{3}(\partial_{x^2}{}^D\Gamma_{11}{}^2 - \partial_{x^1}{}^D\Gamma_{12}{}^2)(4\mathfrak{d} - \mathfrak{t}^2), \\
\mathfrak{B}_{12} = & -\tfrac{1}{6}\left\{(13\mathfrak{t} - 30T^2{}_2)\mathfrak{d}^2 + (3\mathfrak{t} - T^2{}_2)\mathfrak{t}^2\mathfrak{d} - (\mathfrak{t} - T^2{}_2)\mathfrak{t}^4\right\} T^1{}_2 x_{1'}^2 \\
& + \tfrac{1}{6}\left\{(17\mathfrak{t} - 30T^2{}_2)\mathfrak{d}^2 - (2\mathfrak{t} + T^2{}_2)\mathfrak{t}^2\mathfrak{d} + T^2{}_2\mathfrak{t}^4\right\} T^2{}_1 x_{2'}^2 \\
& + \tfrac{1}{6}\left\{20\mathfrak{d}^3 + 4(4\mathfrak{t}^2 - 15T^2{}_2\mathfrak{t} + 15(T^2{}_2)^2)\mathfrak{d}^2\right. \\
& \qquad \left. -(3\mathfrak{t}^2 + 2T^2{}_2\mathfrak{t} - 2(T^2{}_2)^2)\mathfrak{t}^2\mathfrak{d} + 2(\mathfrak{t} - T^2{}_2)T^2{}_2\mathfrak{t}^4\right\} x_{1'}x_{2'} \\
& - \tfrac{1}{3}\left\{{}^D\Gamma_{12}{}^1(\mathfrak{t} - 2T^2{}_2) + 2T^2{}_1{}^D\Gamma_{22}{}^1\right\} (\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_{1'} \\
& - \tfrac{1}{3}\left\{{}^D\Gamma_{12}{}^2(\mathfrak{t} - 2T^2{}_2) + 2T^2{}_1{}^D\Gamma_{22}{}^2\right\} (\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_{2'} \\
& - \tfrac{1}{6}\left\{(11\mathfrak{t} - 14T^2{}_2)\mathfrak{d} - 2(\mathfrak{t} - T^2{}_2)\mathfrak{t}^2\right\} T^1{}_2\Phi_{11} \\
& + \tfrac{1}{6}\left\{4\mathfrak{d}^2 + (6\mathfrak{t}^2 - 28T^2{}_2\mathfrak{t} + 28(T^2{}_2)^2)\mathfrak{d} - (\mathfrak{t} - 2T^2{}_2)^2\mathfrak{t}^2\right\} \Phi_{12} \\
& + \tfrac{1}{6}\left\{(3\mathfrak{t} - 14T^2{}_2)\mathfrak{d} + 2T^2{}_2\mathfrak{t}^2\right\} T^2{}_1\Phi_{22} \\
& - \tfrac{1}{3}\left\{(\partial_{x^2}{}^D\Gamma_{11}{}^1 - \partial_{x^1}{}^D\Gamma_{12}{}^1 - \partial_{x^2}{}^D\Gamma_{12}{}^2 + \partial_{x^1}{}^D\Gamma_{22}{}^2)(\mathfrak{t}^2 - 4\mathfrak{d})\right\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{B}_{22} = &-\tfrac{1}{6}\left\{30\mathfrak{d}^2 - \mathfrak{t}^4 + \mathfrak{t}^2\mathfrak{d}\right\}(T^1{}_2)^2 x_{1'}^2 \\
&-\tfrac{1}{6}\left\{10\mathfrak{d}^3 + 2(\mathfrak{t}^2 - 17T^2{}_2\mathfrak{t} + 15(T^2{}_2)^2)\mathfrak{d}^2 \right.\\
&\quad \left. +(4\mathfrak{t} + T^2{}_2)T^2{}_2\mathfrak{t}^2\mathfrak{d} - (T^2{}_2)^2\mathfrak{t}^4\right\} x_{2'}^2 \\
&+\tfrac{1}{3}\left\{(17\mathfrak{t} - 30T^2{}_2)\mathfrak{d}^2 - (2\mathfrak{t} + T^2{}_2)\mathfrak{t}^2\mathfrak{d} + T^2{}_2\mathfrak{t}^4\right\} T^1{}_2 x_{1'} x_{2'} \\
&-\tfrac{1}{3}\left\{{}^D\Gamma_{22}{}^1(\mathfrak{t} - 2T^2{}_2) + 2T^1{}_2\,{}^D\Gamma_{22}{}^2\right\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t} x_{1'} \\
&+\tfrac{1}{3}\left\{{}^D\Gamma_{22}{}^2(\mathfrak{t} - 2T^2{}_2) - 2T^2{}_1\,{}^D\Gamma_{22}{}^1\right\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t} x_{2'} \\
&-\tfrac{1}{3}(7\mathfrak{d} - \mathfrak{t}^2)(T^1{}_2)^2\Phi_{11} \\
&+\tfrac{1}{3}\left\{(3\mathfrak{t} - 14T^2{}_2)T^1{}_2\mathfrak{d} + 2T^1{}_2T^2{}_2\mathfrak{t}^2\right\}\Phi_{12} \\
&-\tfrac{1}{6}\left\{10\mathfrak{d}^2 - (5\mathfrak{t}^2 + 6T^2{}_2\mathfrak{t} - 14(T^2{}_2)^2)\mathfrak{d} + \mathfrak{t}^4 - 2(T^2{}_2)^2\mathfrak{t}^2\right\}\Phi_{22} \\
&-\tfrac{2}{3}(\partial_{x^2}{}^D\Gamma_{12}{}^1 - \partial_{x^1}{}^D\Gamma_{22}{}^1)(\mathfrak{t}^2 - 4\mathfrak{d})\,.
\end{aligned}$$

Suppose first that the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$ vanishes. We start analyzing the case $T^1{}_2 = 0$. In this case, the expression of $\tilde{\mathfrak{B}}$ in Equation (4.2) reduces to

$$\tilde{\mathfrak{B}} = \frac{1}{6}(T^1{}_1 - T^2{}_2)^2 \cdot (T^1{}_1 + T^2{}_2) \cdot \begin{pmatrix} T^1{}_1 - T^2{}_2 & 2T^2{}_1 \\ 0 & T^2{}_2 - T^1{}_1 \end{pmatrix}. \tag{4.3}$$

If $T^2{}_2 = T^1{}_1$, we differentiate the component $\mathfrak{B}_{11}$ in Equation (4.2) twice with respect to $x_{2'}$ to obtain $T^2{}_1 T^1{}_1 = 0$. Thus, either $T^2{}_1 = 0$ and T is a multiple of the identity, or $T^1{}_1 = 0$ and, in such a case, T is determined by $T\partial_{x^1} = T^2{}_1\partial_{x^2}$ and therefore it is nilpotent. If $T^2{}_2 \neq T^1{}_1$, then Equation (4.3) implies that $T^2{}_2 = -T^1{}_1$. In this case, we differentiate the component $\mathfrak{B}_{22}$ in Equation (4.2) twice with respect to $x_{2'}$ and obtain $T^1{}_1 = 0$. Thus, as before, T is nilpotent.

Next we analyze the case $T^1{}_2 \neq 0$. We use Equation (4.1) to express

$$\begin{aligned}
{}^D\Gamma_{11}{}^1 &= \tfrac{T^1{}_1 - T^2{}_2}{T^1{}_2}\,{}^D\Gamma_{12}{}^1 + \tfrac{T^2{}_1}{T^1{}_2}\,{}^D\Gamma_{22}{}^1, & {}^D\Gamma_{11}{}^2 &= \tfrac{T^2{}_1}{T^1{}_2}\,{}^D\Gamma_{12}{}^1, \\
{}^D\Gamma_{12}{}^2 &= \tfrac{T^2{}_1}{T^1{}_2}\,{}^D\Gamma_{22}{}^1, & {}^D\Gamma_{22}{}^2 &= {}^D\Gamma_{12}{}^1 - \tfrac{T^1{}_1 - T^2{}_2}{T^1{}_2}\,{}^D\Gamma_{22}{}^1\,.
\end{aligned}$$

Considering the component $\tilde{\mathfrak{B}}_{11}$ in Equation (4.2),

$$\tilde{\mathfrak{B}}_{11} = \frac{1}{6}(T^1{}_1 - T^2{}_2) \cdot (T^1{}_1 + T^2{}_2) \cdot ((T^1{}_1 - T^2{}_2)^2 + 4T^1{}_2T^2{}_1),$$

we analyze separately the vanishing of each one of the three factors in $\tilde{\mathfrak{B}}_{11}$.

Assume that $T^2{}_2 = T^1{}_1$. In this case, component $\tilde{\mathfrak{B}}_{12}$ in Equation (4.2) reduces to $\tilde{\mathfrak{B}}_{12} = \frac{8}{3}T^1{}_2(T^2{}_1)^2T^1{}_1$; since we are assuming that $T^1{}_2 \neq 0$, then either $T^2{}_1 = 0$ or $T^2{}_1 \neq 0$ and $T^1{}_1 = 0$. If $T^2{}_1 = 0$, the only non-zero component of the Bach tensor is given by $\mathfrak{B}_{22} = -(T^1{}_2)^2(T^1{}_1)^2(3(T^1{}_1)^2x_{1'}^2 + \Phi_{11})$, from where it follows that $T^1{}_1 = 0$ and hence T is determined by $T\partial_{x^2} = T^1{}_2\partial_{x^1}$ and is nilpotent. If $T^2{}_1 \neq 0$ and $T^1{}_1 = 0$, then we differentiate the component $\mathfrak{B}_{12}$ in Equation (4.2) with respect to $x_{1'}$ and $x_{2'}$ to get $T^1{}_2T^2{}_1 = 0$, which is not possible since both $T^1{}_2$ and $T^2{}_1$ are non-null.

Suppose now that $T^2{}_2 = -T^1{}_1$. In this case, we differentiate the component $\mathfrak{B}_{22}$ in Equation (4.2) twice with respect to $x_{1'}$ and as a consequence we obtain $T^1{}_2(T^1{}_2T^2{}_1 + (T^1{}_1)^2) = 0$; since we are assuming $T^1{}_2 \neq 0$, it follows that $T^2{}_1 = -\frac{(T^1{}_1)^2}{T^1{}_2}$. Thus, the (1,1)-tensor field T is given by $T\partial_{x^1} = T^1{}_1\partial_{x^1} - \frac{(T^1{}_1)^2}{T^1{}_2}\partial_{x^2}$ and $T\partial_{x^2} = T^1{}_2\partial_{x^1} - T^1{}_1\partial_{x^2}$, and therefore it is nilpotent as well.

Finally, suppose that $(T^1{}_1 - T^2{}_2)^2 + 4T^1{}_2T^2{}_1 = 0$; since $T^1{}_2 \neq 0$, this is equivalent to $T^2{}_1 = -\frac{(T^1{}_1-T^2{}_2)^2}{4T^1{}_2}$. Now, we differentiate the component $\mathfrak{B}_{22}$ in Equation (4.2) twice with respect to $x_{1'}$ to obtain $T^1{}_2(T^1{}_1 + T^2{}_2) = 0$. Thus, we have that $T^2{}_2 = -T^1{}_1$ and T is given by $T\partial_{x^1} = T^1{}_1\partial_{x^1} - \frac{(T^1{}_1)^2}{T^1{}_2}\partial_{x^2}$ and $T\partial_{x^2} = T^1{}_2\partial_{x^1} - T^1{}_1\partial_{x^2}$, which again implies that T is nilpotent.

To conclude the proof we show the "only if" part. If T is a multiple of the identity, then $(T^*\Sigma, g_{D,\Phi,T})$ is self-dual by Theorem 1.24 and therefore it has vanishing Bach tensor. Thus, we suppose T is parallel and nilpotent and, in this case, we can choose a system of coordinates (x^1, x^2) such that T is determined by $T\partial_{x^1} = \partial_{x^2}$ and $T\partial_{x^2} = 0$. Hence, examining Equation (4.2), clearly $\tilde{\mathfrak{B}} = 0$ and, since $\mathfrak{d} = \mathfrak{t} = 0$, one easily checks that $\mathfrak{B}_{11} = \mathfrak{B}_{12} = \mathfrak{B}_{22} = 0$, showing that the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$ vanishes. □

Remark 4.2. We emphasize that even though the Bach tensor of the metrics $g_{D,\Phi,T}$ depends on the choice of Φ (as shown in the proof of Theorem 4.1), the existence of Bach-flat metrics in Theorem 4.1 is independent of the symmetric $(0,2)$-tensor field Φ, thus providing an infinite family of examples for each initial data (Σ, D, T). Moreover, note that the metrics $g_{D,\Phi,T}$ are generically non-isometric for different deformation tensor fields Φ.

The Bach-flat modified Riemannian extensions in Theorem 4.1 obtained from a $(1,1)$-tensor field of the form $T = c\,\mathrm{Id}$ are not of interest for our purposes since they all are half conformally flat (cf. Theorem 1.24). Hence, in what follows we focus on the case when T is a parallel nilpotent $(1,1)$-tensor field and refer to $g_{D,\Phi,T}$ as a *nilpotent Riemannian extension*.

Remark 4.3. The nilpotent Riemannian extensions to be considered in what remains of this chapter are those induced by a parallel nilpotent (1,1)-tensor field T on an affine surface (Σ, D). In this case, there exist suitable coordinates (x^1, x^2) where $T\partial_{x^1} = \partial_{x^2}$ and $T\partial_{x^2} = 0$, and it follows from Equation (4.1) that the Christoffel symbols of D satisfy

$$^D\Gamma_{12}{}^1 = 0, \quad ^D\Gamma_{12}{}^2 = {}^D\Gamma_{11}{}^1, \quad ^D\Gamma_{22}{}^1 = 0, \quad ^D\Gamma_{22}{}^2 = 0\,.$$

A straightforward calculation shows that the Ricci tensor satisfies

$$\rho^D = \begin{pmatrix} \partial_{x^2}{}^D\Gamma_{11}{}^2 - \partial_{x^1}{}^D\Gamma_{11}{}^1 & \partial_{x^2}{}^D\Gamma_{11}{}^1 \\ -\partial_{x^2}{}^D\Gamma_{11}{}^1 & 0 \end{pmatrix}.$$

Hence, ρ_s^D is either zero or of rank one and one easily gets that ρ_s^D is recurrent, i.e., $D\rho_s^D = \eta \otimes \rho_s^D$, with recurrence one-form

$$\eta = \{\partial_{x^1} \ln \rho_s^D(\partial_{x^1}, \partial_{x^1}) - 2{}^D\Gamma_{11}{}^1\}dx^1 + \partial_{x^2} \ln \rho_s^D(\partial_{x^1}, \partial_{x^1})dx^2 \tag{4.4}$$

(see also Theorem 5.1–(iii)).

Moreover, in the special case that the Ricci tensor is symmetric ($\rho^D_{sk} = 0$ or, equivalently, $\partial_{x^2}{}^D\Gamma_{11}{}^1 = 0$), work of Wong [104] shows that ρ^D is recurrent and of rank one if and only if there exist local coordinates where the only non-zero Christoffel symbol is ${}^D\Gamma_{11}{}^2(x^1, x^2)$. Furthermore, in this case one has

$$\rho^D = \partial_{x^2}{}^D\Gamma_{11}{}^2(x^1, x^2)dx^1 \otimes dx^1$$

and $D\rho^D = \omega \otimes \rho^D$, where the recurrence one-form is given by

$$\begin{aligned} \omega &= (\partial_{x^1} \ln \rho^D_{11})dx^1 + (\partial_{x^2} \ln \rho^D_{11})dx^2 \\ &= \frac{\partial_{x^1}\partial_{x^2}{}^D\Gamma_{11}{}^2}{\partial_{x^2}{}^D\Gamma_{11}{}^2}dx^1 + \frac{\partial_{x^2}\partial_{x^2}{}^D\Gamma_{11}{}^2}{\partial_{x^2}{}^D\Gamma_{11}{}^2}dx^2 \,. \end{aligned} \tag{4.5}$$

4.2 Bach-flat gradient Ricci solitons

Let Φ be a symmetric $(0,2)$-tensor field on (Σ, D, T). One uses the nilpotent structure T to construct an associated symmetric $(0,2)$-tensor field $\widehat{\Phi}$ given by $\widehat{\Phi}(X,Y) = \Phi(TX, TY)$, for all vector fields X, Y on Σ.

Further, proceeding as in Lemma 5.4, let (x^1, x^2) be local coordinates where $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$ (just interchanging the order of the coordinates in Assertion (i) of Lemma 5.4). Setting $\Phi = \Phi_{ij}dx^i \otimes dx^j$ one has that $\widehat{\Phi}$ expresses as $\widehat{\Phi} = \widehat{\Phi}_{ij}dx^i \otimes dx^j = \Phi_{22}dx^1 \otimes dx^1$.

4.2.1 Einstein nilpotent Riemannian extensions

Riemannian extensions $g_{D,\Phi,T}$ with $T = c\,\mathrm{Id}$ are Einstein if and only if the deformation tensor Φ is given by the symmetric part of the Ricci tensor (cf. Theorem 1.23). In the nilpotent case ($T^2 = 0$) one has:

Theorem 4.4. *Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1,1)$-tensor field T and let Φ be a symmetric $(0,2)$-tensor field on Σ. Then $(T^*\Sigma, g_{D,\Phi,T})$ is Einstein (indeed, Ricci-flat) if and only if $\widehat{\Phi} = -2\rho^D_s$.*

Proof. Let (x^1, x^2) be local coordinates on Σ so that $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. A straightforward calculation shows that the Ricci tensor of any nilpotent Riemannian extension $g_{D,\Phi,T}$ is determined by

$$\rho(\partial_{x^1}, \partial_{x^1}) = \Phi(\partial_{x^2}, \partial_{x^2}) + 2\rho^D_s(\partial_{x^1}, \partial_{x^1}),$$

the other components being zero. Hence the Ricci operator is nilpotent and $g_{D,\Phi,T}$ has zero scalar curvature.

Moreover, the Ricci tensor vanishes if and only if $\Phi(\partial_{x^2}, \partial_{x^2}) + 2\rho^D_s(\partial_{x^1}, \partial_{x^1}) = 0$. The result now follows. □

Remark 4.5. The Weyl tensor of a pseudo-Riemannian manifold is harmonic if and only if $\operatorname{div}_4 W$ vanishes. Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1,1)$-tensor field T and let Φ be a symmetric $(0,2)$-tensor field on Σ. Let (x^1, x^2) be local coordinates on Σ so that $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. A straightforward calculation shows that the divergence of the Weyl tensor of $(T^*\Sigma, g_{D,\Phi,T})$ is given by

$$2(\operatorname{div}_4 W)(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}) = \{\partial_{x^2}\Phi(\partial_{x^2}, \partial_{x^2}) + 2\partial_{x^2}\rho_s^D(\partial_{x^1}, \partial_{x^1})\},$$

the other components being zero. Hence $(T^*\Sigma, g_{D,\Phi,T})$ *has harmonic Weyl tensor if and only if* $\widehat{D\Phi} = -2\,\widehat{\eta}\otimes\rho_s^D$, *where* $\widehat{\eta}(X) = \eta(TX)$, η *being the recurrence one-form given in Equation* (4.4) *and* $\widehat{D\Phi}(X, Y; Z) = D\Phi(TX, TY; TZ)$.

4.2.2 Gradient Ricci solitons on nilpotent Riemannian extensions

Recall from Theorem 1.27 that the affine gradient Ricci soliton equation determines the potential function of any self-dual gradient Ricci soliton which is not locally conformally flat, independently of the deformation tensor Φ. The next theorem shows that, in contrast with the previous situation, for any $h \in \mathcal{C}^\infty(\Sigma)$ with $dh(\ker(T)) = 0$, one may use the symmetric $(0,2)$-tensor field $\operatorname{Hes}_h^D + 2\rho_s^D$ to determine a deformation tensor field Φ so that the resulting nilpotent Riemannian extension is a Bach-flat steady gradient Ricci soliton with potential function $f = h\circ\pi$.

Theorem 4.6. *Let* (Σ, D, T) *be an affine surface equipped with a parallel nilpotent* $(1,1)$*-tensor field* T *and let* Φ *be a symmetric* $(0,2)$*-tensor field on* Σ*. Let* $h \in \mathcal{C}^\infty(\Sigma)$ *be a smooth function. Then* $(T^*\Sigma, g_{D,\Phi,T}, f = h\circ\pi)$ *is a Bach-flat gradient Ricci soliton if and only if* $dh(\ker(T)) = 0$ *and*

$$\widehat{\Phi} = -\operatorname{Hes}_h^D - 2\rho_s^D\,. \tag{4.6}$$

Moreover the soliton is steady and isotropic.

Proof. Let (x^1, x^2) be local coordinates on Σ so that $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. Setting $f = h\circ\pi$, one has that $\operatorname{Hes}_f(\partial_{x^1}, \partial_{x_{1'}}) + \rho(\partial_{x^1}, \partial_{x_{1'}}) = \lambda g(\partial_{x^1}, \partial_{x_{1'}})$ leads to $\lambda = 0$, which shows that the soliton is steady. A straightforward calculation shows that the remaining non-zero terms in the gradient Ricci soliton equation are given by

$$\begin{aligned}
\operatorname{Hes}_f(\partial_{x^2}, \partial_{x^2}) + \rho(\partial_{x^2}, \partial_{x^2}) &= \partial_{x^2}\partial_{x^2}h,\\
\operatorname{Hes}_f(\partial_{x^1}, \partial_{x^2}) + \rho(\partial_{x^1}, \partial_{x^2}) &= \partial_{x^1}\partial_{x^2}h - {}^D\Gamma_{11}{}^1\partial_{x^2}h,\\
\operatorname{Hes}_f(\partial_{x^1}, \partial_{x^1}) + \rho(\partial_{x^1}, \partial_{x^1}) &= x_{2'}\,\partial_{x^2}h - {}^D\Gamma_{11}{}^2\,\partial_{x^2}h + \partial_{x^1}\partial_{x^1}h - {}^D\Gamma_{11}{}^1\partial_{x^1}h\\
&\quad + \Phi_{22} + 2\partial_{x^2}{}^D\Gamma_{11}{}^2 - 2\partial_{x^1}{}^D\Gamma_{11}{}^1\,.
\end{aligned}$$

It immediately follows from the equation $(\operatorname{Hes}_f + \rho)(\partial_{x^1}, \partial_{x^1}) = 0$ that $\partial_{x^2}h = 0$, which shows that $dh(\ker(T)) = 0$. The only remaining equation now becomes

$$\begin{aligned}\mathrm{Hes}_f(\partial_{x^1}, \partial_{x^1}) + \rho(\partial_{x^1}, \partial_{x^1}) \\ &= \partial_{x^1}\partial_{x^1}h - {}^D\Gamma_{11}{}^1\partial_{x^1}h + \Phi_{22} + 2\partial_{x^2}{}^D\Gamma_{11}{}^2 - 2\partial_{x^1}{}^D\Gamma_{11}{}^1 \\ &= \Phi(\partial_{x^2}, \partial_{x^2}) + \mathrm{Hes}_h^D(\partial_{x^1}, \partial_{x^1}) + 2\rho_s^D(\partial_{x^1}, \partial_{x^1}),\end{aligned}$$

from which Equation (4.6) follows. Moreover, it also follows from the form of the potential function that $\nabla f = h'(x^1)\partial_{x_{1'}}$, and thus $\|\nabla f\|^2 = 0$ (equivalently, the level hypersurfaces of the potential function are degenerate submanifolds of $T^*\Sigma$), which shows that the soliton is isotropic. □

Remark 4.7. The potential functions of the gradient Ricci solitons in Theorem 4.6 are of the form $f = h \circ \pi$ for some $h \in \mathcal{C}^\infty(\Sigma)$. Next we show that this is indeed the case if the Ricci tensor of (Σ, D) is non-symmetric.

We consider $(T^*\Sigma, g_{D,\Phi,T}, f)$ a gradient Ricci soliton with potential function $f \in \mathcal{C}^\infty(T^*\Sigma)$. Take local coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$ as in the proof of Theorem 4.6. Observe that, since $\mathrm{Hes}_f(\partial_{x_{i'}}, \partial_{x_{j'}}) = \partial_{x_{i'}}\partial_{x_{j'}} f(x^1, x^2, x_{1'}, x_{2'})$, it follows from the expression of the Ricci tensor in Theorem 4.4 and the metric tensor (1.19), that the potential function is determined by $f = \iota X + h \circ \pi$, for some $h \in \mathcal{C}^\infty(\Sigma)$ and some vector field X on Σ, where ιX is the evaluation map acting on X.

Further set $X = A(x^1, x^2)\partial_{x^1} + B(x^1, x^2)\partial_{x^2}$ in the local coordinates (x^1, x^2) above, for some $A, B \in \mathcal{C}^\infty(\Sigma)$. Then $\mathrm{Hes}_f(\partial_{x^2}, \partial_{x_{1'}}) = \partial_{x^2}A(x^1, x^2)$, from where it follows that $X = A(x^1)\partial_{x^1} + B(x^1, x^2)\partial_{x^2}$. Considering $\mathrm{Hes}_f(\partial_{x^2}, \partial_{x_{2'}}) = -A''(x^1) + \partial_{x^2}B(x^1, x^2)$, one has that $X = A(x^1)\partial_{x^1} + (P(x^1) + x^2A'(x^1))\partial_{x^2}$ for some smooth function $P(x^1)$. Next the component

$$\begin{aligned}\mathrm{Hes}_f(\partial_{x^1}, \partial_{x_{2'}}) &= A(x^1){}^D\Gamma_{11}{}^2 - x_{2'}A(x^1) \\ &\quad + {}^D\Gamma_{11}{}^1(P(x^1) + x^2A'(x^1)) + P'(x^1) + x^2A''(x^1)\end{aligned}$$

shows that $A = 0$ and it reduces to $\mathrm{Hes}_f(\partial_{x^1}, \partial_{x_{2'}}) = P'(x^1) + P(x^1){}^D\Gamma_{11}{}^1$. A solution $P(x^1)$ of the equation $P'(x^1) + P(x^1){}^D\Gamma_{11}{}^1 = 0$ either vanishes identically (and hence $X = 0$) or it is nowhere zero, in which case $\partial_{x^2}{}^D\Gamma_{11}{}^1 = 0$ (see the proof of Theorem 4.13). In the latter case Remark 4.3 shows that the Ricci tensor of (Σ, D) is symmetric and thus recurrent of rank one. Theorem 4.6 describes all possible gradient Ricci solitons on $(T^*\Sigma, g_{D,\Phi,T})$ whenever ρ_{sk}^D is non-zero.

Remark 4.8. The tensor field $\mathbb{D}_{ijk} = -2\,\mathrm{div}_4 W_{ijk} + W_{ijk\ell}\nabla_\ell f$ introduced in [36] plays an essential role in analyzing the geometry of Bach-flat gradient Ricci solitons. Local conformal flatness in [34, 36] follows from $\mathbb{D} = 0$, which is obtained under some natural assumptions.

Gradient Ricci solitons in Theorem 4.6 satisfy $\nabla f = h'(x^1)\partial_{x_{1'}}$. Therefore, a straightforward calculation shows that $\mathbb{D}$ is completely determined by

$$\mathbb{D}_{121} = -2h'(x^1)\partial_{x^2}\,{}^D\Gamma_{11}{}^1(x^1, x^2),$$

the other components being zero. Hence it follows from Remark 4.3 that the tensor field $\mathbb{D}$ vanishes if and only if the Ricci tensor ρ^D is symmetric. However Theorem 4.9 shows that $(T^*\Sigma, g_{D,\Phi,T})$ is never locally conformally flat.

4.3 Half conformally flat nilpotent Riemannian extensions

The existence of a null distribution $\mathfrak{V}$ on a four-dimensional manifold (M, g) of neutral signature defines a natural orientation on M: the one which, for any basis $\{u, v\}$ of $\mathfrak{V}$, makes the bivector $u \wedge v$ self-dual (see [49]). We consider on T^*M the orientation which agrees with $\mathfrak{V} = \ker(\pi_*)$, and thus self-duality and anti-self-duality are not interchangeable. The following result shows that they are essentially different for nilpotent Riemannian extensions.

Theorem 4.9. *Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1,1)$-tensor field T. Then*

(i) *$(T^*\Sigma, g_{D,\Phi,T})$ is never self-dual for any deformation tensor field Φ.*

(ii) *If $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual, then D is either a flat connection or (Σ, D) is recurrent with symmetric Ricci tensor of rank one.*

In the latter case there exist local coordinates (u^1, u^2) where the only non-zero Christoffel symbol is ${}^u\Gamma_{11}{}^2$ and the tensor field T is given by $T\partial_{u^1} = \partial_{u^2}$, $T\partial_{u^2} = 0$. Moreover, $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual if and only if the symmetric $(0,2)$-tensor field Φ satisfies the equations:

$$\begin{aligned} \widehat{D\Phi} &= -2\widehat{\omega} \otimes \rho^D, \\ 0 &= \tfrac{1}{2}\widehat{\Phi} \otimes \widehat{\Phi}(\partial_{x^1}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) + 2(\widehat{\Phi} \otimes \rho^D)(\partial_{x^1}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) \\ &\quad + D^2\Phi(\partial_{x^1}, \partial_{x^1}; T\partial_{x^1}, T\partial_{x^1}) + D^2\Phi(T\partial_{x^1}, T\partial_{x^1}; \partial_{x^1}, \partial_{x^1}) \\ &\quad - 2\, D^2\Phi(\partial_{x^1}, T\partial_{x^1}; T\partial_{x^1}, \partial_{x^1}), \end{aligned} \tag{4.7}$$

where $\widehat{D\Phi}(X,Y,Z) = D\Phi(TX, TY; TZ)$, ω is the recurrence one-form given by $D\rho^D = \omega \otimes \rho^D$, and $\widehat{\omega}(X) = \omega(TX)$.

Proof. A direct computation using the expression of the anti-self-dual curvature operator of any four-dimensional Walker metric obtained in [51] shows that, for any nilpotent Riemannian extension $g_{D,\Phi,T}$, W^- takes the form

$$W^- = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \tag{4.8}$$

thus showing that the anti-self-dual Weyl curvature operator W^- is nilpotent and so $(T^*\Sigma, g_{D,\Phi,T})$ is never self-dual, which proves Assertion (i).

Next we show Assertion (ii). As a matter of notation we write $\partial_{x^s} f = f_s$, $\partial_{x^r}\partial_{x^s} f = f_{rs}$. Let (M, g) be a four-dimensional Walker metric and set the metric components $g_{11} = a$, $g_{12} = c$ and $g_{22} = b$, where g_{ij} are functions of the Walker coordinates $(x^1, x^2, x_{1'}, x_{2'})$. Then the self-dual Weyl curvature operator takes the form (see [51])

$$W^+ = \begin{pmatrix} W_{11}^+ & W_{12}^+ & W_{11}^+ + \frac{\tau}{12} \\ -W_{12}^+ & \frac{\tau}{6} & -W_{12}^+ \\ -W_{11}^+ - \frac{\tau}{12} & -W_{12}^+ & -W_{11}^+ - \frac{\tau}{6} \end{pmatrix}, \tag{4.9}$$

where

$$
\begin{aligned}
W_{11}^{+} = \tfrac{1}{12}(&6ca_1b_2 - 6a_1b_{1'} - 6ba_1c_2 + 12a_1c_{2'} - 6ca_2b_1 + 6a_2b_{2'} \\
&+ 6ba_2c_1 + 6a_{1'}b_1 - 6a_{2'}b_2 - 12a_{2'}c_1 + 6ab_1c_2 - 6ab_2c_1 \\
&+ 12b_2c_{1'} - 12b_{1'}c_2 - a_{11} - 12c^2a_{11} - 12bca_{12} + 24ca_{12'} \\
&- 3b^2a_{22} + 12ba_{22'} - 12a_{2'2'} - 3a^2b_{11} + 12ab_{11'} - b_{22} \\
&- 12b_{1'1'} + 12acc_{11} - 2c_{12} + 6abc_{12} - 24cc_{11'} - 12ac_{12'} \\
&- 12bc_{21'} + 24c_{1'2'}),
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
W_{12}^{+} = \tfrac{1}{4}(&-2ca_{11} - ba_{12} + 2a_{12'} + ab_{12} - 2b_{21'} + ac_{11} - 2cc_{12} \\
&- 2c_{11'} - bc_{22} + 2c_{22'})\,.
\end{aligned}
\tag{4.11}
$$

Since any anti-self-dual metric is Bach-flat, we proceed as in the proof of Theorem 4.1 considering local coordinates (x^1, x^2) on the surface Σ such that T is determined by $T\partial_{x^1} = \partial_{x^2}$ and $T\partial_{x^2} = 0$. Since T is parallel, the Christoffel symbols must satisfy:

$$
{}^D\Gamma_{12}{}^1 = 0, \quad {}^D\Gamma_{12}{}^2 = {}^D\Gamma_{11}{}^1, \quad {}^D\Gamma_{22}{}^1 = 0, \quad {}^D\Gamma_{22}{}^2 = 0\,.
$$

Next, we analyze the self-dual Weyl curvature operator, which is completely determined by the scalar curvature and its components W_{11}^+ and W_{12}^+ already described in Equations (4.10) and (4.11). The scalar curvature is zero by Theorem 4.4, and $W_{12}^+ = -2\partial_{x^2}{}^D\Gamma_{11}{}^1$, from where it follows that the Ricci tensor ρ^D is symmetric of rank one and recurrent (see Remark 4.3). Take local coordinates (u^1, u^2), as in Remark 4.3, so that the only non-zero Christoffel symbol is ${}^u\Gamma_{11}{}^2$ and $T\partial_{u^1} = \partial_{u^2}$, $T\partial_{u^2} = 0$. Finally, we compute the component W_{11}^+ given by Equation (4.10) in the coordinates $(u^1, u^2, u_{1'}, u_{2'})$ of $T^*\Sigma$, obtaining

$$
\begin{aligned}
W_{11}^{+} = &\left(\partial_{x^2}\Phi_{22} + 2\partial_{x^2}\partial_{x^2}{}^u\Gamma_{11}{}^2\right) u_{2'} - \tfrac{1}{2}(\Phi_{22})^2 - 2\Phi_{22}\partial_{x^2}{}^u\Gamma_{11}{}^2 \\
&- \partial_{x^2}\Phi_{22}{}^u\Gamma_{11}{}^2 + 2\partial_{x^1}\partial_{x^2}\Phi_{12} - \partial_{x^2}\partial_{x^2}\Phi_{11} - \partial_{x^1}\partial_{x^1}\Phi_{22}\,.
\end{aligned}
$$

Thus $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual if and only if

$$
\begin{aligned}
&\partial_{x^2}\Phi_{22} + 2\partial_{x^2}\partial_{x^2}{}^u\Gamma_{11}{}^2 = 0, \\
&\tfrac{1}{2}(\Phi_{22})^2 + 2\Phi_{22}\partial_{x^2}{}^u\Gamma_{11}{}^2 + \partial_{x^2}\Phi_{22}{}^u\Gamma_{11}{}^2 \\
&\qquad\qquad = 2\partial_{x^1}\partial_{x^2}\Phi_{12} - \partial_{x^2}\partial_{x^2}\Phi_{11} - \partial_{x^1}\partial_{x^1}\Phi_{22},
\end{aligned}
$$

from where Equation (4.7) follows. □

4.3.1 Anti-self-dual gradient Ricci solitons

Self-dual gradient Ricci solitons which are not locally conformally flat are described in Theorem 1.27. In contrast, no explicit examples of strictly anti-self-dual gradient Ricci solitons were previously reported. In this section we use nilpotent Riemannian extensions to construct anti-self-dual isotropic gradient Ricci solitons. In this case, Theorem 4.9 shows that (Σ, D) must have symmetric Ricci tensor.

Proposition 4.10. *Let* (Σ, D, T, Φ) *be an affine surface with symmetric Ricci tensor equipped with a parallel nilpotent* $(1,1)$*-tensor field* T *and a parallel symmetric* $(0,2)$*-tensor field* Φ*. Then* $(T^*\Sigma, g_{D,\Phi,T})$ *is anti-self-dual if and only if* $\widehat{\omega} = 0$ *and* $\widehat{\Phi} = 0$*, where* ω *is the recurrence one-form given in Equation* (4.5)*.*

Proof. If the Ricci tensor ρ^D is symmetric of rank one and Φ is parallel, then the equations in Theorem 4.9 reduce to $\widehat{\omega} = 0$ and $\widehat{\Phi} = 0$, which proves the result. If (Σ, D) is a flat surface then a straightforward calculation shows that anti-self-duality is equivalent to $\widehat{\Phi} = 0$, being Φ a parallel tensor. □

Since the deformation tensor Φ of any gradient Ricci soliton in Theorem 4.6 must satisfy $\widehat{\Phi} = -\operatorname{Hes}_h^D - 2\rho_s^D$, the condition $\widehat{\Phi} = 0$ in the previous proposition restricts the consideration of Ricci solitons on $(T^*\Sigma, g_{D,\Phi,T})$ to those originated by affine gradient Ricci solitons on (Σ, D).

Proposition 4.11. *Let* (Σ, D, T) *be an affine surface equipped with a parallel nilpotent* $(1,1)$*-tensor field* T *and let* $h \in \mathcal{C}^\infty(\Sigma)$*. Then*

(i) The (Σ, D, T, h) *is an affine gradient Ricci soliton with* $dh(\ker(T)) = 0$ *if and only if* $(T^*\Sigma, g_{D,\widehat{\Phi},T}, f = h \circ \pi)$ *is a Bach-flat steady gradient Ricci soliton for any symmetric* $(0,2)$*-tensor field* Φ*.*

(ii) The (Σ, D, T, h) *is a non-flat affine gradient Ricci soliton with* $dh(\ker(T)) = 0$ *if and only if the recurrence one-form* η *given in Equation* (4.4) *satisfies* $\widehat{\eta} = 0$*.*

Proof. Since T is nilpotent, $\widehat{\Phi}(TX, TY) = 0$ for any $(0,2)$-tensor field Φ. Hence Equation (4.6) shows that $(T^*\Sigma, g_{D,\widehat{\Phi},T}, f = h \circ \pi)$ is a gradient Ricci soliton if and only if (Σ, D, T, h) is an affine gradient Ricci soliton with $dh(\ker(T)) = 0$, which shows Assertion (i). Next take local coordinates (x^1, x^2) on Σ so that $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$. Since $\rho_s^D = (\partial_{x^2}{}^D\Gamma_{11}{}^2 - \partial_{x^1}{}^D\Gamma_{11}{}^1)dx^1 \otimes dx^1$ (see Remark 4.3), one has

$$(\operatorname{Hes}_h^D + 2\rho_s^D)(\partial_{x^2}, \partial_{x^2}) = \partial_{x^2}\partial_{x^2} h\,.$$

Thus $h(x^1, x^2) = x^2 P(x^1) + Q(x^1)$ for some $P, Q \in \mathcal{C}^\infty(\Sigma)$. Hence $dh(\ker(T)) = 0$ holds if and only if $P = 0$. Since $h(x^1, x^2) = Q(x^1)$ one has that

$$(\operatorname{Hes}_h^D + 2\rho_s^D)(\partial_{x^1}, \partial_{x^2}) = 0,$$

and the only remaining equation is

$$\begin{aligned} 0 = (\operatorname{Hes}_h^D + 2\rho_s^D)(\partial_{x^1}, \partial_{x^1}) &= Q'' + 2(\partial_{x^2}{}^D\Gamma_{11}{}^2 - \partial_{x^1}{}^D\Gamma_{11}{}^1) \\ &= Q'' + 2\rho^D(\partial_{x^1}, \partial_{x^1})\,. \end{aligned}$$

Therefore, the integrability condition becomes $\partial_{x^2}\rho^D(\partial_{x^1}, \partial_{x^1}) = 0$.

Hence, it follows from Equation (4.4) that (Σ, D, T, h) is an affine gradient Ricci soliton with $dh(\ker(T)) = 0$ if and only if the symmetric part of the Ricci tensor ρ_s^D is recurrent with recurrence one-form η satisfying $\eta(\ker(T)) = 0$. Assertion (ii) now follows. □

A direct application of previous propositions gives the desired examples.

Theorem 4.12. *Let (Σ, D, T, Φ) be an affine surface with symmetric Ricci tensor equipped with a parallel nilpotent $(1,1)$-tensor field T and a parallel symmetric $(0,2)$-tensor field Φ.*

(i) The (Σ, D, h) is an affine gradient Ricci soliton with $dh(\ker(T)) = 0$ if and only if $(T^*\Sigma, g_{D,\widehat{\Phi},T}, f = h \circ \pi)$ is an anti-self-dual steady gradient Ricci soliton which is not locally conformally flat.

(ii) The (Σ, D, h) is an affine gradient Ricci soliton with $dh(\ker(T)) = 0$ if and only if there exist local coordinates (u^1, u^2) on Σ so that the only non-zero Christoffel symbol is given by ${}^u\Gamma_{11}{}^2 = P(u^1) + u^2 Q(u^1)$ and the potential function $h(u^1)$ is determined by $h''(u^1) = -2Q(u^1)$, for any $P, Q \in \mathcal{C}^\infty(\Sigma)$.

Proof. $(T^*\Sigma, g_{D,\widehat{\Phi},T}, f = h \circ \pi)$ is a gradient Ricci soliton by Proposition 4.11–(i). Anti-self-duality now follows from Proposition 4.10 and Proposition 4.11–(ii), showing Assertion (i).

Assertion (ii) follows from Proposition 4.11–(ii) and the expression of the recurrence form ω in Equation (4.5). Take local coordinates (u^1, u^2) on Σ as in the proof of Proposition 4.11–(ii). Then it follows from Equation (4.5) that $\widehat{\omega} = 0$ if and only if $\partial_{x^2}\partial_{x^2}{}^u\Gamma_{11}{}^2 = 0$. Thus

$$ {}^u\Gamma_{11}{}^2(u^1, u^2) = P(u^1) + u^2 Q(u^1) $$

for some $P, Q \in \mathcal{C}^\infty(\Sigma)$ and $h''(u^1) = -2Q(u^1)$. □

4.4 Conformally Einstein nilpotent Riemannian extensions

Since nilpotent Riemannian extensions are not weakly-generic (see the expression of W^- in the proof of Theorem 4.9), we will analyze the conformally Einstein Equation (1.6):

$$ (n-2)\operatorname{Hes}_\varphi + \varphi\,\rho - \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\,\tau\} g = 0, $$

seeking for solutions on nilpotent Riemannian extensions $(T^*\Sigma, g_{D,\Phi,T})$.

Theorem 4.13. *Let (Σ, D, T) be a torsion free affine surface equipped with a parallel nilpotent $(1,1)$-tensor field T. Then any solution of Equation (1.6) is of the form $\varphi = \iota X + \phi \circ \pi$ for some vector field X on Σ such that $X \in \ker(T)$ and $\operatorname{tr}(DX) = 0$.*

Moreover $(T^*\Sigma, g_{D,\Phi,T})$ is conformally Einstein if and only if one of the following holds:

(i) The conformally Einstein Equation (1.6) admits a solution $\varphi = \phi \circ \pi$ for some $\phi \in \mathcal{C}^\infty(\Sigma)$ with $d\phi(\ker(T)) = 0$, and the deformation tensor Φ is determined by $\phi\,\widehat{\Phi} + 2(\operatorname{Hes}^D_\phi + \phi\,\rho^D_s) = 0$.

(ii) The conformally Einstein Equation (1.6) admits a solution $\varphi = \iota X + \phi \circ \pi$ for some $\phi \in \mathcal{C}^\infty(\Sigma)$ and some non-zero vector field X on Σ such that $X \in \ker(T)$ and $\operatorname{tr}(DX) = 0$. In this case, the Ricci tensor ρ^D is symmetric of rank one and recurrent.

Moreover there are local coordinates (u^1, u^2) *on* Σ *so that*

$$\varphi(u^1, u^2, u_{1'}, u_{2'}) = \kappa u_{2'} + \phi(u^1, u^2)$$

is a solution of Equation (1.6) *if and only if*

$$\begin{aligned} &d\phi(T\partial_{x^1}) = \tfrac{\mu}{2}\Phi(T\partial_{x^1}, T\partial_{x^1}),\\ &\mathrm{Hes}^D_\phi(\partial_{x^1}, \partial_{x^1}) + \phi\,\rho^D(\partial_{x^1}, \partial_{x^1})\\ &\qquad = -\tfrac{1}{2}(\phi + 2\mu\,{}^u\Gamma_{11}{}^2)\Phi(T\partial_{x^1}, T\partial_{x^1})\\ &\qquad\quad + \tfrac{\mu}{2}\left\{2(D_{\partial_{x^1}}\Phi)(T\partial_{x^1}, \partial_{x^1}) - (D_{T\partial_{x^1}}\Phi)(\partial_{x^1}, \partial_{x^1})\right\}. \end{aligned}$$

Proof. Let (x^1, x^2) be local coordinates on Σ so that $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. Since T is parallel we obtain directly from Equation (4.1) that

$${}^D\Gamma_{12}{}^1 = 0, \quad {}^D\Gamma_{12}{}^2 = {}^D\Gamma_{11}{}^1, \quad {}^D\Gamma_{22}{}^1 = 0, \quad {}^D\Gamma_{22}{}^2 = 0\,.$$

In order to analyze the conformally Einstein Equation (1.6) consider the symmetric $(0,2)$-tensor field $\mathcal{E} = 2\,\mathrm{Hes}_\varphi + \varphi\,\rho - \frac{1}{4}\{2\Delta\varphi + \varphi\,\tau\}g$ and set $\mathcal{E} = 0$. Let $\mathcal{E}_{ij} = \mathcal{E}(\partial_{x^i}, \partial_{x^j})$ and let $\varphi \in \mathcal{C}^\infty(T^*\Sigma)$ be a solution of Equation (1.6). Then one computes

$$\mathcal{E}_{33} = 2\partial_{x_{1'}}\partial_{x_{1'}}\varphi, \quad \mathcal{E}_{34} = 2\partial_{x_{1'}}\partial_{x_{2'}}\varphi, \quad \mathcal{E}_{44} = 2\partial_{x_{2'}}\partial_{x_{2'}}\varphi,$$

to show that any solution of Equation (1.6) must be of the form

$$\varphi(x^1, x^2, x_{1'}, x_{2'}) = A(x^1, x^2)x_{1'} + B(x^1, x^2)x_{2'} + \psi(x^1, x^2), \tag{4.12}$$

for some smooth functions A, B and ψ depending only on the coordinates (x^1, x^2). This shows that any solution of the conformally Einstein equation on $(T^*\Sigma, g_{D,\Phi,T})$ is of the form $\varphi = \iota X + \psi \circ \pi$, where ιX is the evaluation of a vector field $X = A\partial_{x^1} + B\partial_{x^2}$ on Σ, $\psi \in \mathcal{C}^\infty(\Sigma)$ and $\pi : T^*\Sigma \to \Sigma$ is the projection.

Now, the conformally Einstein condition given in Equation (1.6) can be expressed in matrix form as follows:

$$(\mathcal{E}_{ij}) = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \partial_{x^1}A - \partial_{x^2}B & 2({}^D\Gamma_{11}{}^2 A + {}^D\Gamma_{11}{}^1 B + \partial_{x^1}B - Ax_{2'}) \\ * & \mathcal{E}_{22} & 2\partial_{x^2}A & -\partial_{x^1}A + \partial_{x^2}B \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \tag{4.13}$$

where positions with $*$ are not written since the matrix is symmetric, and where

$$
\begin{aligned}
\mathcal{E}_{11} \;=\;& -(\partial_{x^1}A - \partial_{x^2}B - 4{}^D\Gamma_{11}{}^1 A)x_{2'}^2 \\
& +\{A\Phi_{22} + 2(\partial_{x^1}\partial_{x^1}A - {}^D\Gamma_{11}{}^2\partial_{x^2}A \\
& \quad + {}^D\Gamma_{11}{}^1\partial_{x^2}B + A\partial_{x^2}{}^D\Gamma_{11}{}^2 - B\partial_{x^2}{}^D\Gamma_{11}{}^1)\}x_{1'} \\
& -\{B\Phi_{22} + 2A\Phi_{12} - 2(\partial_{x^1}\partial_{x^1}B + {}^D\Gamma_{11}{}^2\partial_{x^1}A \\
& \quad - {}^D\Gamma_{11}{}^1\partial_{x^1}B + (\partial_{x^1}{}^D\Gamma_{11}{}^2 - 2{}^D\Gamma_{11}{}^1{}^D\Gamma_{11}{}^2)A \\
& \quad + (\partial_{x^1}{}^D\Gamma_{11}{}^1 - 2({}^D\Gamma_{11}{}^1)^2)B + \partial_{x^2}\psi)\}x_{2'} \\
& +2\partial_{x^2}Ax_{1'}x_{2'} \\
& -(\partial_{x^1}A + \partial_{x^2}B)\Phi_{11} + 2({}^D\Gamma_{11}{}^2A + {}^D\Gamma_{11}{}^1B)\Phi_{12} \\
& +(2{}^D\Gamma_{11}{}^2B + \psi)\Phi_{22} - A\partial_{x^1}\Phi_{11} + B\partial_{x^2}\Phi_{11} - 2B\partial_{x^1}\Phi_{12} + 2\partial_{x^1}\partial_{x^1}\psi \\
& -2{}^D\Gamma_{11}{}^1\partial_{x^1}\psi - 2{}^D\Gamma_{11}{}^2\partial_{x^2}\psi - 2(\partial_{x^1}{}^D\Gamma_{11}{}^1 - \partial_{x^2}{}^D\Gamma_{11}{}^2)\psi,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{12} \;=\;& 2(\partial_{x^1}\partial_{x^2}A - {}^D\Gamma_{11}{}^1\partial_{x^2}A + A\partial_{x^2}{}^D\Gamma_{11}{}^1)x_{1'} \\
& +2(\partial_{x^1}\partial_{x^2}B + {}^D\Gamma_{11}{}^1\partial_{x^1}A + A\partial_{x^2}{}^D\Gamma_{11}{}^2)x_{2'} \\
& -(\partial_{x^1}A + \partial_{x^2}B)\Phi_{12} + 2{}^D\Gamma_{11}{}^1B\Phi_{22} - A\partial_{x^2}\Phi_{11} - B\partial_{x^1}\Phi_{22} \\
& +2\partial_{x^1}\partial_{x^2}\psi - 2{}^D\Gamma_{11}{}^1\partial_{x^2}\psi,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{22} \;=\;& 2\partial_{x^2}\partial_{x^2}Ax_{1'} + 2(\partial_{x^2}\partial_{x^2}B + 2A\partial_{x^2}{}^D\Gamma_{11}{}^1)x_{2'} \\
& -(\partial_{x^1}A + \partial_{x^2}B + 2{}^D\Gamma_{11}{}^1A)\Phi_{22} - 2A\partial_{x^2}\Phi_{12} \\
& +A\partial_{x^1}\Phi_{22} - B\partial_{x^2}\Phi_{22} + 2\partial_{x^2}\partial_{x^2}\psi\,.
\end{aligned}
$$

First, we use the component $\mathcal{E}_{14} = 2({}^D\Gamma_{11}{}^2A + {}^D\Gamma_{11}{}^1B + \partial_{x^1}B - Ax_{2'})$ in Equation (4.13); note that $\partial_{x_{2'}}\mathcal{E}_{14} = -2A$, and therefore $A(x^1, x^2) = 0$, which shows that $X \in \ker(T)$. Now component $\mathcal{E}_{13}$ in Equation (4.13) gives $\partial_{x^2}B = 0$, which implies $B(x^1, x^2) = P(x^1)$ for some smooth function P depending only on the coordinate x^1, i.e., the vector field $X = B\partial_{x^2}$ satisfies $\operatorname{tr}(DX) = 0$.

At this point, the conformal function φ has the coordinate expression

$$\varphi(x^1, x^2, x_{1'}, x_{2'}) = P(x^1)x_{2'} + \psi(x^1, x^2)$$

and the possible non-zero components in Equation (4.13) are $\mathcal{E}_{11}$, $\mathcal{E}_{12}$, $\mathcal{E}_{22}$ and $\mathcal{E}_{14}$. Considering the component $\mathcal{E}_{14} = 2(P'(x^1) + {}^D\Gamma_{11}{}^1(x^1, x^2)P(x^1))$, we distinguish two cases depending on whether the function P vanishes identically or not. Indeed, if $P(x^1)$ is a solution of the equation $\mathcal{E}_{14} = 0$, then

$$
\begin{aligned}
&\partial_{x^1}\left(P(x^1)e^{\int {}^D\Gamma_{11}{}^1(x^1,x^2)dx^1}\right) \\
&\qquad = e^{\int {}^D\Gamma_{11}{}^1(x^1,x^2)dx^1}\left\{P'(x^1) + P(x^1){}^D\Gamma_{11}{}^1(x^1, x^2)\right\} = 0,
\end{aligned}
$$

which shows that

$$P(x^1)e^{\int {}^D\Gamma_{11}{}^1(x^1,x^2)dx^1} = \mathcal{Q}(x^2),$$

for some smooth function $\mathcal{Q}(x^2)$. Now, if the function $\mathcal{Q}(x^2)$ vanishes at some point, then $P(x^1) = 0$ at each point. Otherwise, if $\mathcal{Q}(x^2) \neq 0$ at each point, so is $P(x^1)$.

First, suppose that $P(x^1) \equiv 0$, and hence $\varphi = \psi \circ \pi$. In this case, component $\mathcal{E}_{22}$ in Equation (4.13) yields $\partial_{x^2}\partial_{x^2}\psi = 0$, which implies $\psi(x^1, x^2) = Q(x^1)x^2 + \phi(x^1)$ for some smooth functions Q and ϕ depending only on the coordinate x^1. Now, the only components in Equation (4.13) which could be non-null are

$$\begin{aligned} \mathcal{E}_{11} &= 2Qx_{2'} + (Q\Phi_{22} + 2Q'' - 2\,{}^D\Gamma_{11}{}^1 Q' - 2(\partial_{x^1}{}^D\Gamma_{11}{}^1 - \partial_{x^2}{}^D\Gamma_{11}{}^2)Q)x^2 \\ &\quad +\phi\Phi_{22} + 2\phi'' - 2\,{}^D\Gamma_{11}{}^1\phi' - 2(\partial_{x^1}{}^D\Gamma_{11}{}^1 - \partial_{x^2}{}^D\Gamma_{11}{}^2)\phi - 2{}^D\Gamma_{11}{}^2 Q, \\ \mathcal{E}_{12} &= 2(Q' - {}^D\Gamma_{11}{}^1 Q)\,. \end{aligned}$$

Now, $\partial_{x_{2'}}\mathcal{E}_{11} = 2Q$, implies $Q = 0$, thus showing that $d\varphi(\ker(T)) = 0$. Then $\mathcal{E}_{12} = 0$ and the component $\mathcal{E}_{11}$ reduces to

$$\mathcal{E}_{11} = \phi\Phi_{22} + 2\phi'' - 2\,{}^D\Gamma_{11}{}^1\phi' - 2(\partial_{x^1}{}^D\Gamma_{11}{}^1 - \partial_{x^2}{}^D\Gamma_{11}{}^2)\phi\,.$$

Since $\varphi(x^1, x^2, x_{1'}, x_{2'}) = \phi(x^1)$, ϕ must be non-null and we obtain that $\mathcal{E}_{11} = 0$ is equivalent to

$$\begin{aligned} \Phi_{22} &= -\tfrac{2}{\phi}\left\{\phi'' - {}^D\Gamma_{11}{}^1\phi' - (\partial_{x^1}{}^D\Gamma_{11}{}^1 - \partial_{x^2}{}^D\Gamma_{11}{}^2)\,\phi\right\} \\ &= -\tfrac{2}{\phi}\left\{\mathrm{Hes}^D_\phi(\partial_{x^1}, \partial_{x^1}) + \phi\,\rho^D_s(\partial_{x^1}, \partial_{x^1})\right\}, \end{aligned}$$

from where Assertion (i) is obtained.

Finally, we analyze the case in which the function $P(x^1)$ does not vanish identically. Since $\mathcal{E}_{14} = 2(P'(x^1) + {}^D\Gamma_{11}{}^1(x^1, x^2)P(x^1))$, we have $\partial_{x^2}{}^D\Gamma_{11}{}^1 = 0$. Now it follows that the Ricci tensor ρ^D is symmetric of rank one and recurrent (see Remark 4.3). Specialize the local coordinates (u^1, u^2) on Σ so that the only non-zero Christoffel symbol of D is ${}^u\Gamma_{11}{}^2(u^1, u^2)$ and $T\partial_{u^1} = \partial_{u^2}$, $T\partial_{u^2} = 0$. Then any solution of the conformally Einstein equation takes the form

$$\varphi(u^1, u^2, u_{1'}, u_{2'}) = \mathcal{A}(u^1)u_{2'} + \phi(u^1, u^2)\,.$$

Now, considering the component $\mathcal{E}_{41}$ of the conformally Einstein equation in the new coordinates (u^1, u^2), one has $\mathcal{E}_{41} = 2\mathcal{A}'(u^1)$, which shows that $\varphi(u^1, u^2, u_{1'}, u_{2'}) = \mu u_{2'} + \phi(u^1, u^2)$ for some $\mu \neq 0$. Considering now the component

$$\begin{aligned} \mathcal{E}_{11} &= (2\partial_{x^2}\phi - \mu\Phi_{22})u_{2'} + 2\partial_{x^1}\partial_{x^1}\phi - 2\partial_{x^2}\phi{}^u\Gamma_{11}{}^2 \\ &\quad + 2\phi\partial_{x^2}{}^u\Gamma_{11}{}^2 + \phi\Phi_{22} + 2\mu\Phi_{22}{}^u\Gamma_{11}{}^2 + \mu\partial_{x^2}\Phi_{11} - 2\mu\partial_{x^1}\Phi_{12}, \end{aligned}$$

it follows that the conformally Einstein equation reduces to

$$\begin{aligned} \mu\Phi_{22} &= 2\partial_{x^2}\phi, \\ (\phi + 2\mu\,{}^u\Gamma_{11}{}^2)\Phi_{22} &= -2(\mathrm{Hes}^D_\phi(\partial_{u^1}, \partial_{u^1}) + \phi\rho^D(\partial_{u^1}, \partial_{u^1})) \\ &\quad +\mu(2\partial_{x^1}\Phi_{12} - \partial_{x^2}\Phi_{11}), \end{aligned}$$

from where Assertion (ii) is obtained. □

4.5 Examples

4.5.1 Nilpotent Riemannian extensions with flat base

Let (Σ, D) be a flat torsion free affine surface. Take local coordinates on Σ so that all Christoffel symbols vanish. Let T be a parallel nilpotent $(1,1)$-tensor field. Since T is parallel, its components $T^j{}_i$ are necessarily constant on the given coordinates. Hence one may further specialize the local coordinates (x^1, x^2), by using a linear transformation, so that $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$ and all the Christoffel symbols ${}^D\Gamma_{ij}{}^k$ remain identically zero. Now Theorem 4.1 shows that $(T^*\Sigma, g_{D,\Phi,T})$ *is Bach-flat for any symmetric* $(0,2)$*-tensor field* Φ *on* Σ. Moreover it follows from Theorem 4.6 that $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ *is a steady gradient Ricci soliton for any* $h \in \mathcal{C}^\infty(\Sigma)$ *with* $dh \circ T = 0$ *and any symmetric* $(0,2)$*-tensor field* Φ *such that* $\Phi_{22}(x^1, x^2) = -h''(x^1)$. Further note from Remark 4.8 that the steady gradient Ricci soliton $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ satisfies $\mathbb{D} = 0$. Moreover, since $\Phi_{22} = -h''(x^1)$, one has that $(T^*\Sigma, g_{D,\Phi,T})$ is in the conformal class of an Einstein metric (just considering the conformal metric $\bar{g} = \phi^{-2} g_{D,\Phi,T}$ determined by the equation $\phi''(x^1) - \frac{1}{2}\phi(x^1)h''(x^1) = 0$).

Remark 4.14. Set $\Sigma = \mathbb{R}^2$ with usual coordinates (x^1, x^2) and put $T\partial_{x^1} = \partial_{x^2}$, $T\partial_{x^2} = 0$. For any smooth function $h(x^1)$ consider the deformation tensor Φ given by $\Phi_{22}(x^1, x^2) = -h''(x^1)$ (the other components being zero). Then, the non-zero Christoffel symbols of $g_{D,\Phi,T}$ are given by

$$\Gamma_{11}{}^2 = -x_{2'} = -\Gamma_{12'}{}^{1'},\ \Gamma_{11}{}^{2'} = -h''(x^1)x_{2'},\ \Gamma_{12}{}^{2'} = -\frac{1}{2}h^{(3)}(x^1) = -\Gamma_{22}{}^{1'}\,.$$

Hence a curve $\gamma(t) = (x^1(t), x^2(t), x_{1'}(t), x_{2'}(t))$ is a geodesic if and only if

$$\begin{aligned}
&\ddot{x}^1(t) = 0, \qquad\qquad \ddot{x}^2(t) - x_{2'}(t)\,\dot{x}^1(t)^2 = 0,\\
&\ddot{x}_{1'}(t) + 2\,x_{2'}(t)\,\dot{x}^1(t)\dot{x}_{2'}(t) + \tfrac{1}{2}\,h^{(3)}(x^1(t))\,\dot{x}^2(t)^2 = 0,\\
&\ddot{x}_{2'}(t) - h''(x^1(t))\,x_{2'}(t)\,\dot{x}^1(t)^2 - h^{(3)}(x^1(t))\,\dot{x}^1(t)\,\dot{x}^2(t) = 0\,.
\end{aligned}$$

Thus $x^1(t) = at + b$ for some $a, b \in \mathbb{R}$ and

$$\begin{aligned}
&\ddot{x}^2(t) - a^2\,x_{2'}(t) = 0,\\
&\ddot{x}_{2'}(t) - h''(at+b)\,a^2\,x_{2'}(t) - h^{(3)}(at+b)\,a\,\dot{x}^2(t) = 0,\\
&\ddot{x}_{1'}(t) + 2a\,x_{2'}(t)\,\dot{x}_{2'}(t) + \tfrac{1}{2}\,h^{(3)}(at+b)\,\dot{x}^2(t)^2 = 0\,.
\end{aligned}$$

Now the first two equations above are linear and thus $x^2(t)$ and $x_{2'}(t)$ are globally defined. Finally, since $\ddot{x}_{1'}(t) + 2a\,x_{2'}(t)\,\dot{x}_{2'}(t) + \frac{1}{2}\,h^{(3)}(at+b)\,\dot{x}^2(t)^2 = 0$ is also linear on $x_{1'}(t)$, one has that geodesics are globally defined.

Then it follows from Theorem 4.6 that $(T^*\mathbb{R}^2, g_{D,\Phi,T}, f = h \circ \pi)$ *is a geodesically complete steady gradient Ricci soliton, which is conformally Einstein* by Theorem 4.13.

4.5.2 Nilpotent Riemannian extensions with non-recurrent base

Let $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ be a non-trivial Bach-flat steady gradient Ricci soliton as in Theorem 4.6. Further assume that the Ricci tensor ρ^D is non-symmetric, i.e., $\rho^D_{sk} \neq 0$ (equivalently, $\partial_{x^2}{}^D\Gamma_{11}{}^1 \neq 0$ as shown in Remark 4.3). Then it follows from Theorem 4.9 that $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat.

Theorem 4.13 shows that $(T^*\Sigma, g_{D,\Phi,T})$ is conformally Einstein if and only if there exists a positive $\phi \in \mathcal{C}^\infty(\Sigma)$ with $d\phi \circ T = 0$ such that

$$\phi\,\widehat{\Phi} + 2(\mathrm{Hes}^D_\phi + \phi\,\rho^D_s) = 0\,.$$

Therefore, it follows from Theorem 4.6 that $\mathrm{Hes}^D_h = \frac{2}{\phi}\,\mathrm{Hes}^D_\phi$, which means $(2\frac{\phi'}{\phi} - h')^D\Gamma_{11}{}^1 = 2\frac{\phi''}{\phi} - h''$. Taking derivatives with respect to x^2 and, since $\partial_{x^2}{}^D\Gamma_{11}{}^1 \neq 0$, the equation above splits into

$$\frac{2\phi'}{\phi} - h' = 0 \quad \text{and} \quad \frac{2\phi''}{\phi} - h'' = 0,$$

which only admits constant solutions. Summarizing the above one has the following: *Let (Σ, D, T) be an affine surface with non-symmetric Ricci tensor (i.e., $\rho^D_{sk} \neq 0$). Then any Bach-flat gradient Ricci soliton $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ is neither half conformally flat nor conformally Einstein.*

Chapter 5
Parallel tensors on affine surfaces

Motivated by the results in Chapter 4, one is interested in the existence of affine surfaces admitting a parallel tensor of type (1,1) which is nilpotent and their explicit description. It is important to emphasize that any parallel tensor decomposes as an scalar multiple of the identity plus a trace free part. In consequence, one can reduce the problem and restrict the study to trace free parallel tensor fields.

We say that (Σ, D, T) is a *Kähler surface* if T is a complex structure ($T^2 = -\operatorname{Id}$) and $DT = 0$. If the parallel tensor field is a para-complex structure ($T^2 = \operatorname{Id}$), then (Σ, D, T) is called *para-Kähler*. Finally (Σ, D, T) is said to be *nilpotent Kähler* if T is a nilpotent parallel tensor field of type (1,1). Let (Σ, D) be an affine surface with the non-zero skew-symmetric Ricci tensor $\rho_{sk}^D \neq 0$. Then ρ_{sk}^D defines a volume element. Furthermore, ρ_{sk}^D is recurrent, i.e., $D\rho_{sk}^D = \omega \otimes \rho_{sk}^D$. The symmetric Ricci tensor is not recurrent in general. We will prove the following result in Section 5.2.

Theorem 5.1. *Let (Σ, D) be a simply connected affine surface with $\rho_s^D \neq 0$.*

(i) (Σ, D) *admits a Kähler structure if and only if* $\det(\rho_s^D) > 0$ *and* ρ_s^D *is recurrent.*

(ii) (Σ, D) *admits a para-Kähler structure if and only if* $\det(\rho_s^D) < 0$ *and* ρ_s^D *is recurrent.*

(iii) (Σ, D) *admits a nilpotent Kähler structure if and only if* ρ_s^D *is of rank one and recurrent.*

Affine surfaces admitting a trace free parallel (1,1)-tensor field have appeared in the literature in several contexts.

(1) Affine surfaces with parallel shape operator have been investigated in [69], where it is shown that any such surface is either an equiaffine sphere or the shape operator is two-step nilpotent, thus corresponding to Case (iii) above.

(2) Let (Σ, D) be an affine surface equipped with a parallel volume form Ω. Since $d\Omega = 0$ and $D\Omega = 0$, there is a notion of symplectic sectional curvature (see [56]).

A symplectic surface (Σ, D, Ω) has zero symplectic sectional curvature if and only if the Ω-Ricci operator $\Omega(\operatorname{Ric}^\Omega(X), Y) = \rho^D(X, Y)$ is a nilpotent Kähler structure. Furthermore the symplectic sectional curvature is positive definite (resp. negative definite) if and only if $\operatorname{Ric}^\Omega$ is a Kähler (resp. para-Kähler) structure [56].

This chapter is organized as follows. In section 5.1 we study the relation between parallel tensors and the Ricci tensor as well as the dimension of the spaces of parallel tensors. The proof of the Theorem 5.1 is given in Section 5.2. In Section 5.3 we analyze the existence of parallel (1,1)-tensor fields on the Type $\mathcal{A}$ and Type $\mathcal{B}$ homogeneous surfaces. In this chapter we report on work investigated in [30].

5.1 The space of parallel tensor fields on a surface

Let (Σ, D) be an affine surface and let (x^1, x^2) be a system of local coordinates on Σ. Let T be a tensor field of type (1,1). Expand $T = T^i{}_j \partial_{x^i} \otimes dx^j$. We say that T is *parallel* if $DT = 0$. Let $\mathcal{P}(\Sigma, D)$ be the set of parallel tensors of type $(1,1)$ on (Σ, D):

$$\mathcal{P}(\Sigma, D) = \{T^i{}_j : \partial_{x^k} T^i{}_j + {}^D\Gamma_{k\ell}{}^i\, T^\ell{}_j - {}^D\Gamma_{kj}{}^\ell\, T^i{}_\ell = 0, \quad \text{for all } \ i, j, k\}\,.$$

Let $\operatorname{tr}(T) := T^i{}_i$ be the trace of the endomorphism. Let

$$\mathcal{P}^0(\Sigma, D) := \{T \in \mathcal{P}(\Sigma, D) : \operatorname{tr}(T) = 0\}$$

be the space of trace free parallel tensors of type (1,1). If $T \in \mathcal{P}(\Sigma, D)$, $\operatorname{tr}(T)$ is constant and expressing $T = \frac{1}{2}\operatorname{tr}(T)\operatorname{Id} + (T - \frac{1}{2}\operatorname{tr}(T)\operatorname{Id})$ decomposes

$$\mathcal{P}(\Sigma, D) = \operatorname{Id}\cdot\mathbb{R} \oplus \mathcal{P}^0(\Sigma, D)\,.$$

If $0 \neq T \in \mathcal{P}^0(\Sigma, D)$, then the eigenvalues of T are $\{\pm\lambda\}$ so $\operatorname{tr}(T^2) = 2\lambda^2$. If $2\lambda^2 < 0$ (resp. $2\lambda^2 > 0$), we can rescale T so $T^2 = -\operatorname{Id}$ (resp. $T^2 = \operatorname{Id}$) and T defines a Kähler (resp. para-Kähler) structure on Σ; the almost complex (resp. almost para-complex) structure being integrable as Σ is a surface [44, 85]. Finally, if $\lambda = 0$, then T is nilpotent and defines what we will call a *nilpotent Kähler structure*.

Lemma 5.2. *If (Σ, D) is a connected affine surface, then $\mathcal{P}(\Sigma, D)$ is a unital algebra with $\dim(\mathcal{P}(\Sigma, D)) \leq 4$. Let $T \in \mathcal{P}(\Sigma, D)$. The eigenvalues of T are constant on Σ. If T vanishes at any point of Σ, then T vanishes identically.*

Proof. Let $M_2(\mathbb{F})$ be the unital algebra of 2×2 matrices with entries in a field $\mathbb{F}$ and let $M_2^0(\mathbb{F}) \subset M_2(\mathbb{F})$ be the linear subspace of trace free matrices. The sum and product of parallel tensors of type (1,1) is again parallel. Since $\operatorname{Id} = (\delta^i{}_j)$ is parallel, $\mathcal{P}(\Sigma, D)$ is a unital algebra. Fix a point $p \in \Sigma$. Since Σ is connected, a parallel tensor is defined by its value at a single point. Thus the map $T \to T(p)$ is a unital algebra homomorphism which embeds $\mathcal{P}(\Sigma, D)$ into $M_2(\mathbb{R})$ relative to the coordinate basis. Thus $\mathcal{P}(\Sigma, D)$ has dimension at most 4. Let $T \in \mathcal{P}(\Sigma, D)$. Since $d\{\operatorname{tr}(T)\} = \operatorname{tr}(DT) = 0$, $\operatorname{tr}(T)$ is constant. By replacing T by $T - \frac{1}{2}\operatorname{tr}(T)\operatorname{Id}$, we may assume that $T \in \mathcal{P}^0(\Sigma, D)$ is trace free. The eigenvalues of T are then $\{\lambda(p), -\lambda(p)\}$ so $\operatorname{tr}(T^2) = 2\lambda^2(p)$. Since T^2 is parallel, this implies $\lambda^2(\cdot)$ is constant and hence the eigenvalues themselves are constant. □

The symmetric Ricci tensor plays a crucial role. The proof of the following theorem will be obtained in this section after a case by case analysis.

Theorem 5.3. *Let (Σ, D) be a simply connected affine surface.*

(i) If $\dim(\mathcal{P}^0(\Sigma, D)) = 1$, then exactly one of the following possibilities holds:

(a) (Σ, D) admits a Kähler structure and $\operatorname{Rank}(\rho_s^D) = 2$.

(b) (Σ, D) *admits a para-Kähler structure and* $\mathrm{Rank}(\rho_s^D) = 2$.

(c) (Σ, D) *admits a nilpotent Kähler structure and* $\mathrm{Rank}(\rho_s^D) = 1$.

(ii) $\dim(\mathcal{P}^0(\Sigma, D)) \neq 2$.

(iii) $\dim(\mathcal{P}^0(\Sigma, D)) = 3$ *if and only if* $\rho_s^D = 0$. *This implies* (Σ, D) *admits Kähler, para-Kähler, and nilpotent Kähler structures.*

Generically, of course, $\dim(\mathcal{P}^0(\Sigma, D)) = 0$. Thus, there exist examples with $\mathrm{Rank}(\rho_s^D) = 1$ (resp. $\mathrm{Rank}(\rho_s^D) = 2$) where $\dim(\mathcal{P}^0(\Sigma, D)) = 0$ as we shall show in Remark 5.17 (resp. Remark 5.13). What is somewhat surprising is that the existence of parallel $(1,1)$-tensor fields is completely characterized by the geometry of the symmetric part of the Ricci tensor ρ_s^D.

Let T be a tensor of type (1,1) on a smooth surface Σ such that the eigenvalues of T are constant; this is equivalent, of course, to assuming either that $\mathrm{tr}(T)$ and $\mathrm{tr}(T^2)$ are constant on Σ or that $\mathrm{tr}(T)$ and $\det(T)$ are constant on Σ. By subtracting a suitable multiple of the identity from T, we can assume T is trace free. We have the following useful observation.

Lemma 5.4. *Let* $0 \neq T$ *be a trace free tensor of type (1,1) on an affine surface* Σ *with* $\det(T) \in \{0, \pm 1\}$.

(i) If $\det(T) = 0$, *we can choose local coordinates so* $T = \partial_{x^1} \otimes dx^2$.

(ii) If $\det(T) = 1$, *we can choose local coordinates so* $T = \partial_{x^2} \otimes dx^1 - \partial_{x^1} \otimes dx^2$.

(iii) If $\det(T) = -1$, *we can choose local coordinates so* $T = \partial_{x^1} \otimes dx^1 - \partial_{x^2} \otimes dx^2$.

Proof. Let $0 \neq T$ be nilpotent. Let Y_1 be a non-zero vector field which is defined locally so that $TY_1 \neq 0$. Then $Y_2 := TY_1$ spans $\ker(T)$. Choose local coordinates (y^1, y^2) so that $Y_2 = \partial_{y^2}$. Then $T\partial_{y^1}$ is a non-zero multiple of ∂_{y^2}, i.e., $T\partial_{y^1} = f\partial_{y^2}$. Let $X_1 = \partial_{y^1} + g\partial_{y^2}$ and $X_2 = f\partial_{y^2}$ where g remains to be determined. Then $TX_1 = X_2$. We have $[X_1, X_2] = (\partial_{y^1} f + g\partial_{y^2} f - f\partial_{y^2} g)\partial_{y^2}$. Solve the ODE

$$\partial_{y^2} g(y^1, y^2) = f^{-1}\{\partial_{y^1} f + g\partial_{y^2} f\} \quad \text{with} \quad g(y^1, 0) = 0\,.$$

This ensures $[X_1, X_2] = 0$. Since $\{X_1, X_2\}$ are linearly independent, we can choose local coordinates (x^1, x^2) so $\partial_{x^1} = X_1$ and $\partial_{x^2} = X_2$. We then have $T\partial_{x^1} = \partial_{x^2}$ and $T\partial_{x^2} = 0$; Assertion (i) follows after interchanging the roles of x^1 and x^2.

If $\det(T) = 1$, then $T^2 = -\mathrm{Id}$ and T defines an almost complex structure. Since Σ is a surface, the Nirenberg-Newlander Theorem [85] shows that we can choose local coordinates so $T\partial_{x^1} = \partial_{x^2}$ and $T\partial_{x^2} = -\partial_{x^1}$. Assertion (ii) now follows.

Let $\det(T) = -1$. Then $T^2 = \mathrm{Id}$ and T defines an almost para-complex structure. Since we are in dimension 2, the para-complex structure is integrable and we can choose local coordinates so $T\partial_{x^1} = \partial_{x^1}$ and $T\partial_{x^2} = -\partial_{x^2}$ (see for example [44]). Assertion (iii) follows. □

The proof of Theorem 5.3 follows after a case by case analysis of the different local forms in Lemma 5.4. First, we consider the existence of parallel tensor fields on affine surfaces with skew-symmetric Ricci tensor. After that, we will analyze nilpotent Kähler structures, Kähler structures and para-Kähler structures.

The case of skew-symmetric Ricci tensor

Lemma 5.5. *Let (Σ, D) be an affine surface which is not flat.*

(i) $\rho_s^D = 0$ *if and only if there is a coordinate atlas with locally defined* φ *so:*

$$\begin{array}{lll} {}^D\Gamma_{11}{}^1 = 0, & {}^D\Gamma_{11}{}^2 = 0, & {}^D\Gamma_{12}{}^1 = \partial_{x^1}\varphi, \\ {}^D\Gamma_{12}{}^2 = 0, & {}^D\Gamma_{22}{}^1 = \partial_{x^2}\varphi, & {}^D\Gamma_{22}{}^2 = \partial_{x^1}\varphi\,. \end{array} \tag{5.1}$$

(ii) If Equation (5.1) *holds, then* $\rho^D = -\partial_{x^1}\partial_{x^1}\varphi\, dx^1 \wedge dx^2$*, and*

$$\mathcal{P}^0(\Sigma, D) = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2\varphi \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\varphi & -\varphi^2 \\ 1 & \varphi \end{pmatrix} \right\}.$$

Proof. Suppose $\rho_s^D = 0$. Fix a local basis $\{e_1, e_2\}$ for $T_p\Sigma$. Let $\sigma(t) := \exp_p(te_2)$. Extend e_1 along σ to be parallel and let $\Psi(s,t) := \exp_{\sigma(t)}(se_1(t))$. This gives a system of local coordinates where $D_{\partial_t}\partial_t|_{s=0} = 0$, $D_{\partial_t}\partial_s|_{s=0} = 0$, $D_{\partial_s}\partial_s = 0$, i.e.,

$$\begin{array}{lll} {}^D\Gamma_{22}{}^1(0,x^2) = 0, & {}^D\Gamma_{22}{}^2(0,x^2) = 0, & {}^D\Gamma_{12}{}^1(0,x^2) = 0, \\ {}^D\Gamma_{12}{}^2(0,x^2) = 0, & {}^D\Gamma_{11}{}^1(x^1,x^2) = 0, & {}^D\Gamma_{11}{}^2(x^1,x^2) = 0\,. \end{array}$$

We have $0 = \rho_{s,11}^D = -({}^D\Gamma_{12}{}^2)^2 - \partial_{x^1}{}^D\Gamma_{12}{}^2 = 0$. Since ${}^D\Gamma_{12}{}^2(0,x^2) = 0$, this ODE implies ${}^D\Gamma_{12}{}^2 = 0$. Setting $\rho_{s,12}^D = 0$ then yields $\partial_{x^1}\{{}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2\} = 0$. Since ${}^D\Gamma_{12}{}^1(0,x^2) = 0$ and ${}^D\Gamma_{22}{}^2(0,x^2) = 0$, we conclude ${}^D\Gamma_{12}{}^1 = {}^D\Gamma_{22}{}^2$. Setting $\rho_{s,22}^D = 0$ yields $-\partial_{x^2}{}^D\Gamma_{22}{}^2 + \partial_{x^1}{}^D\Gamma_{22}{}^1 = 0$. Consequently, ${}^D\Gamma_{22}{}^2 = \partial_{x^1}\varphi$ and ${}^D\Gamma_{22}{}^1 = \partial_{x^2}\varphi$ for some smooth function φ. This yields the relations of Equation (5.1). Conversely, if Equation (5.1) holds, then a direct computation shows that $\rho_s^D = 0$ and that the three endomorphisms of Assertion (ii) are parallel. Since these endomorphisms are linearly independent and $\dim(\mathcal{P}^0(\Sigma, D)) \leq 3$, Assertion (ii) holds. □

The case of nilpotent Kähler structures

Lemma 5.6. *Let (Σ, D) be an affine surface which is not flat.*

(i) If (Σ, D) admits a nilpotent Kähler structure, there is a coordinate atlas so

$${}^D\Gamma_{11}{}^1 = 0, \quad {}^D\Gamma_{11}{}^2 = 0, \quad {}^D\Gamma_{12}{}^2 = 0, \quad {}^D\Gamma_{22}{}^2 = {}^D\Gamma_{12}{}^1\,. \tag{5.2}$$

(ii) If Equation (5.2) *holds, then* $\rho_s^D = (\partial_{x^1}{}^D\Gamma_{22}{}^1 - \partial_{x^2}{}^D\Gamma_{12}{}^1)\, dx^2 \otimes dx^2$ *and* $T = \partial_{x^1} \otimes dx^2 \in \mathcal{P}^0(\Sigma, D)$.

(iii) If Equation (5.2) *holds and if* $\dim(\mathcal{P}^0(\Sigma, D)) \geq 2$*, then*

$${}^D\Gamma_{12}{}^1 = -\partial_{x^1}\psi \quad \text{and} \quad {}^D\Gamma_{22}{}^1 = -\partial_{x^2}\psi \tag{5.3}$$

for some smooth function ψ.

(iv) If Equations (5.2) *and* (5.3) *hold, then* $\rho^D = \partial_{x^1}\partial_{x^1}\psi\, dx^1 \wedge dx^2$ *and*

$$\mathcal{P}^0(\Sigma, D) = \operatorname{span}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \psi & -\psi^2 \\ 1 & -\psi \end{pmatrix}, \begin{pmatrix} 1 & -2\psi \\ 0 & -1 \end{pmatrix}\right\}.$$

Proof. Let $0 \neq T \in \mathcal{P}^0(\Sigma, D)$ be nilpotent. By Lemma 5.4, we may choose coordinates so $T = \partial_{x^1} \otimes dx^2$. Setting $DT = 0$ yields the following relations from which Equation (5.2) follows (see also [33]):

$$D_{\partial_{x^1}} T = 0: \quad \begin{pmatrix} -{}^D\Gamma_{11}{}^2 & {}^D\Gamma_{11}{}^1 - {}^D\Gamma_{12}{}^2 \\ 0 & {}^D\Gamma_{11}{}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_{\partial_{x^2}} T = 0: \quad \begin{pmatrix} -{}^D\Gamma_{12}{}^2 & {}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2 \\ 0 & {}^D\Gamma_{12}{}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume Equation (5.2) holds. A direct computation establishes Assertion (ii). To prove Assertion (iii), assume in addition that $\dim(\mathcal{P}^0(\Sigma, D)) \geq 2$ and choose $S \in \mathcal{P}^0(\Sigma, D)$ so S and T are linearly independent. We must establish the relations of Equation (5.3).

Case 1:

Suppose that S is nilpotent. Express

$$S = \begin{pmatrix} S^1{}_1 & S^1{}_2 \\ S^2{}_1 & -S^1{}_1 \end{pmatrix}; \quad ST = \begin{pmatrix} 0 & S^1{}_1 \\ 0 & S^2{}_1 \end{pmatrix}.$$

Since $ST \in \mathcal{P}(\Sigma, D)$, $\operatorname{tr}(ST) = S^2{}_1$ is constant. Thus $S^2{}_1 = c$ for $c \in \mathbb{R}$ and

$$S = \begin{pmatrix} S^1{}_1 & S^1{}_2 \\ c & -S^1{}_1 \end{pmatrix}.$$

If $c = 0$, then $\det(S) = -(S^1{}_1)^2 = 0$ implies $S^1{}_1 = 0$ so $S = S^1{}_2\, T$. Since S and T are parallel, $dS^1{}_2 = 0$ so $S^1{}_2 \in \mathbb{R}$ and S and T are linearly dependent contrary to our assumption. Thus $c \neq 0$ and we may rescale S to assume $c = 1$. Setting $\det(S) = 0$ yields $S^1{}_2 = -(S^1{}_1)^2$ so

$$S = \begin{pmatrix} S^1{}_1 & -(S^1{}_1)^2 \\ 1 & -S^1{}_1 \end{pmatrix}.$$

We compute the covariant derivative $DS = S^i{}_{j;k}\partial_{x^i} \otimes dx^j \otimes \partial_{x^k}$, where the components $S^i{}_{j;k} = \partial_{x^k} S^i{}_j + {}^D\Gamma_{k\ell}{}^i S^\ell{}_j - {}^D\Gamma_{kj}{}^\ell S^i{}_\ell$ to get $0 = S^2{}_{2;1} = -{}^D\Gamma_{12}{}^1 - \partial_{x^1} S^1{}_1$ and $0 = S^2{}_{2;2} = -{}^D\Gamma_{22}{}^1 - \partial_{x^2} S^1{}_1$. This yields the additional relations given in Equation (5.3).

Case 2:

Suppose that S is not nilpotent. The map $S \to S(p)$ is an algebra morphism which embeds $\mathcal{P}(\Sigma, D)$ in $M_2(\mathbb{R})$. Consequently, if $\dim(\mathcal{P}^0(\Sigma, D)) = 3$, then $\dim(\mathcal{P}(\Sigma, D)) = 4$ and $\mathcal{P}(\Sigma, D)$ contains a linearly independent nilpotent element $S \in \mathcal{P}(\Sigma, D)$ and the argument given in Case 1 pertains. We therefore assume $\dim(\mathcal{P}(\Sigma,D))=3$ and that any nilpotent element of $\mathcal{P}(\Sigma,D)$ is a constant multiple of T. Express

$$S=\left(\begin{array}{cc} S^1{}_1 & S^1{}_2 \\ S^2{}_1 & -S^1{}_1 \end{array}\right) \quad \text{and} \quad T=\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right).$$

We compute

$$ST=\left(\begin{array}{cc} 0 & S^1{}_1 \\ 0 & S^2{}_1 \end{array}\right).$$

As ST is parallel, $\operatorname{tr}(ST)=S^2{}_1$ is constant so $S^2{}_1=c$ for some constant c and

$$ST-\tfrac{c}{2}\operatorname{Id}=\left(\begin{array}{cc} -\frac{c}{2} & S^1{}_1 \\ 0 & \frac{c}{2} \end{array}\right), \quad STS=\left(\begin{array}{cc} cS^1{}_1 & -(S^1{}_1)^2 \\ c^2 & -cS^1{}_1 \end{array}\right).$$

Since $\dim(\mathcal{P}^0(\Sigma,D))=2$, there must exist a non-trivial real dependence relation of the form $0=a_1T+a_2(ST-\frac{c}{2}\operatorname{Id})+a_3STS$, i.e.,

$$\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)=\left(\begin{array}{cc} -\frac{1}{2}a_2c+a_3cS^1{}_1 & a_1+a_2S^1{}_1-a_3(S^1{}_1)^2 \\ a_3c^2 & \frac{1}{2}a_2c-a_3cS^1{}_1 \end{array}\right).$$

If $c\neq 0$, the relation $a_3c^2=0$ implies $a_3=0$. The relation $\frac{1}{2}a_2c-a_3cS^1{}_1=0$ then implies $a_2=0$. And then finally the relation $a_1+a_2S^1{}_1-a_3(S^1{}_1)^2=0$ implies $a_1=0$. Thus $c=0$ so we have

$$S=\left(\begin{array}{cc} S^1{}_1 & S^1{}_2 \\ 0 & -S^1{}_1 \end{array}\right), \quad T=\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right).$$

Since the eigenvalues of S are constant, $S^1{}_1$ is constant as well. If $S^1{}_1=0$, then $DS=0$ implies $S^1{}_2\in\mathbb{R}$ and hence S and T are not linearly independent. Thus we may assume $S^1{}_1=1$. We set $S^1{}_2=-2\psi$. Setting $DS=0$ then shows that ${}^D\Gamma_{12}{}^1=-\partial_{x^1}\psi$ and ${}^D\Gamma_{22}{}^1=-\partial_{x^2}\psi$ which yields, as desired, Equation (5.3).

Assertion (iv) follows by a direct computation. □

The case of Kähler structures

Lemma 5.7. *Let (Σ,D) be an affine surface which is not flat.*

(i) If (Σ,D) admits a Kähler structure, then there is a coordinate atlas so

$${}^D\Gamma_{11}{}^1={}^D\Gamma_{12}{}^2=-{}^D\Gamma_{22}{}^1, \quad {}^D\Gamma_{11}{}^2=-{}^D\Gamma_{12}{}^1=-{}^D\Gamma_{22}{}^2. \tag{5.4}$$

(ii) If Equation (5.4) *holds, then*

$$\rho_s^D=(\partial_{x^2}{}^D\Gamma_{11}{}^2-\partial_{x^1}{}^D\Gamma_{11}{}^1)\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$$

and

$$T=\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)\in\mathcal{P}^0(\Sigma,D).$$

(iii) If Equation (5.4) *holds and if* $\dim(\mathcal{P}^0(\Sigma, D)) \geq 2$, *there exists smooth* ψ *so*

$$ {}^D\Gamma_{11}{}^1 = \frac{1}{2}\,\partial_{x^2}\psi \quad \textit{and} \quad {}^D\Gamma_{11}{}^2 = \frac{1}{2}\,\partial_{x^1}\psi\,. \tag{5.5} $$

(iv) If Equations (5.4) *and* (5.5) *hold, then*

$$ \rho^D = \frac{1}{2}(\partial_{x^1}\partial_{x^1} + \partial_{x^2}\partial_{x^2})\psi\, dx^1 \wedge dx^2 $$

and

$$ \mathcal{P}^0(\Sigma, D) = \operatorname{span}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{pmatrix}, \begin{pmatrix} \sin\psi & \cos\psi \\ \cos\psi & -\sin\psi \end{pmatrix} \right\}. $$

Proof. Suppose $T \in \mathcal{P}^0(\Sigma, D)$ satisfies $T^2 = -\operatorname{Id}$. By Lemma 5.4, we can choose local coordinates so $T = \partial_{x^2} \otimes dx^1 - \partial_{x^1} \otimes dx^2$. Setting $DT = 0$ yields the relations:

$$ D_{\partial_{x^1}} T = 0 : \quad \begin{pmatrix} {}^D\Gamma_{11}{}^2 + {}^D\Gamma_{12}{}^1 & -{}^D\Gamma_{11}{}^1 + {}^D\Gamma_{12}{}^2 \\ -{}^D\Gamma_{11}{}^1 + {}^D\Gamma_{12}{}^2 & -{}^D\Gamma_{11}{}^2 - {}^D\Gamma_{12}{}^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, $$

$$ D_{\partial_{x^2}} T = 0 : \quad \begin{pmatrix} {}^D\Gamma_{12}{}^2 + {}^D\Gamma_{22}{}^1 & -{}^D\Gamma_{12}{}^1 + {}^D\Gamma_{22}{}^2 \\ -{}^D\Gamma_{12}{}^1 + {}^D\Gamma_{22}{}^2 & -{}^D\Gamma_{12}{}^2 - {}^D\Gamma_{22}{}^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. $$

These relations establish Equation (5.4). A direct computation establishes Assertion (ii). Suppose $\dim(\mathcal{P}(\Sigma, D)) \geq 3$. Choose $S \in \mathcal{P}^0(\Sigma, D)$ to be linearly independent of T. Express

$$ S = \begin{pmatrix} S^1{}_1 & S^1{}_2 \\ S^2{}_1 & -S^1{}_1 \end{pmatrix}, \; T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; S + \varepsilon T = \begin{pmatrix} S^1{}_1 & S^1{}_2 - \varepsilon \\ S^2{}_1 + \varepsilon & -S^1{}_1 \end{pmatrix}. $$

We have $\det(S + \varepsilon T) = \varepsilon^2 + \varepsilon(S^2{}_1 - S^1{}_2) - (S^1{}_1)^2 - S^2{}_1 S^1{}_2$. We use the quadratic formula to solve the equation $\det(S + \varepsilon T) = 0$ setting:

$$ \varepsilon = \tfrac{1}{2}\left\{ (S^1{}_2 - S^2{}_1) \pm \sqrt{(S^1{}_2 + S^2{}_1)^2 + 4(S^1{}_1)^2} \right\}. $$

Since S and T are assumed linearly independent, $S + \varepsilon T$ is a non-trivial nilpotent element. We can then apply Lemma 5.6 and Assertion (ii) to see $\rho_s^D = 0$ and derive the relations of Equation (5.5). This proves Assertion (iii); Assertion (iv) follows by a direct computation. □

The case of para-Kähler structures

Lemma 5.8. *Let* (Σ, D) *be an affine surface which is not flat.*

(i) If (Σ, D) *admits a para-Kähler structure, then there is a coordinate atlas so*

$$ {}^D\Gamma_{11}{}^2 = 0, \quad {}^D\Gamma_{12}{}^1 = 0, \quad {}^D\Gamma_{12}{}^2 = 0, \quad {}^D\Gamma_{22}{}^1 = 0\,. \tag{5.6} $$

(ii) If Equation (5.6) *holds, then*

$$T=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\in\mathcal{P}^0(\Sigma,D)$$

and

$$\rho_s^D=-\frac{1}{2}(\partial_{x^2}{}^D\Gamma_{11}{}^1+\partial_{x^1}{}^D\Gamma_{22}{}^2)\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

(iii) If Equation (5.6) *holds and if* $\dim(\mathcal{P}^0(\Sigma,D))\geq 2$*, then there exists a locally defined smooth function* θ *such that*

$${}^D\Gamma_{11}{}^1=\partial_{x^1}\theta\quad\textit{and}\quad {}^D\Gamma_{22}{}^2=-\partial_{x^2}\theta\,. \tag{5.7}$$

(iv) If Equations (5.6) *and* (5.7) *hold, then* $\rho^D=\partial_{x^1}\partial_{x^2}\theta\,dx^1\wedge dx^2$ *and*

$$\mathcal{P}^0(\Sigma,D)=\operatorname{span}\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix},e^{-\theta}\begin{pmatrix}0&1\\0&0\end{pmatrix},e^{\theta}\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}.$$

Proof. Let $T\in\mathcal{P}^0(\Sigma,D)$ satisfy $T^2=\mathrm{Id}$. We apply Lemma 5.4 to see we may choose local coordinates so $T=\partial_{x^1}\otimes dx^1-\partial_{x^2}\otimes dx^2$. Setting $DT=0$ yields the relations

$$D_{\partial_{x^1}}T=0:\quad\begin{pmatrix}0&-2{}^D\Gamma_{12}{}^1\\2{}^D\Gamma_{11}{}^2&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$D_{\partial_{x^2}}T=0:\quad\begin{pmatrix}0&-2{}^D\Gamma_{22}{}^1\\2{}^D\Gamma_{12}{}^2&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

This yields Equation (5.6). Suppose $\dim(\mathcal{P}^0(\Sigma,D))\geq 2$. If $\dim(\mathcal{P}^0(\Sigma,D))=3$, then $\mathcal{P}^0(\Sigma,D)$ contains a nilpotent element and we may apply Lemma 5.6 to conclude $\rho_s^D=0$ and Assertion (ii) gives the relations of Equation (5.7) for suitably chosen θ. We therefore suppose $\dim(\mathcal{P}^0(\Sigma,D))=2$. Let $\{S,T\}$ be linearly independent elements of $\mathcal{P}^0(\Sigma,D)$. Expand

$$S=\begin{pmatrix}S^1{}_1&S^1{}_2\\S^2{}_1&-S^1{}_1\end{pmatrix},\quad T=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad ST=\begin{pmatrix}S^1{}_1&-S^1{}_2\\S^2{}_1&S^1{}_1\end{pmatrix}.$$

Since $\operatorname{tr}(ST)=2S^1{}_1$ is constant, we obtain $S^1{}_1$ is constant. Define $\widehat{S}=S-S^1{}_1T$. Then $\widehat{S}$ is parallel and $\widehat{S}\neq 0$ since S and T are linearly independent. We then have

$$\widehat{S}=\begin{pmatrix}0&S^1{}_2\\S^2{}_1&0\end{pmatrix},\quad T=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \widehat{S}T=\begin{pmatrix}0&-S^1{}_2\\S^2{}_1&0\end{pmatrix}.$$

Since $\widehat{S}\pm\widehat{S}T$ are nilpotent and not both are zero, $\mathcal{P}(\Sigma,D)$ contains a non-trivial nilpotent element and we can use Lemma 5.6 to conclude $\rho_s^D=0$ and Assertion (ii) establishes Assertion (iii). Assertion (iv) follows by a direct computation. □

5.2 Characterization of affine surfaces admitting parallel tensor fields

The purpose of this section is to prove Theorem 5.1, which characterizes the existence of parallel tensor fields by the recurrence of the symmetric part of the Ricci tensor. We recall the result for the convenience of the reader.

Theorem 5.1. *Let (Σ, D) be a simply connected affine surface with $\rho_s^D \neq 0$.*

(i) (Σ, D) admits a Kähler structure if and only if $\det(\rho_s^D) > 0$ and ρ_s^D is recurrent.

(ii) (Σ, D) admits a para-Kähler structure if and only if $\det(\rho_s^D) < 0$ and ρ_s^D is recurrent.

(iii) (Σ, D) admits a nilpotent Kähler structure if and only if ρ_s^D is of rank one and recurrent.

Proof.
Assertion (i). Let (Σ, D) be an affine surface with $\rho_s^D \neq 0$ admitting a Kähler structure. Take local coordinates as in Lemma 5.7. Then the relations in Equation (5.4) show that $\det(\rho_s^D) > 0$ and ρ_s^D is recurrent, i.e., $D\rho_s^D = \omega \otimes \rho_s^D$ with

$$\omega = -(2{}^D\Gamma_{11}{}^1 - \partial_{x^1} \log \rho_{s,11}^D)\, dx^1 - (2{}^D\Gamma_{11}{}^2 - \partial_{x^2} \log \rho_{s,22}^D)\, dx^2 .$$

Conversely, if ρ_s^D is recurrent and $\det(\rho_s^D) > 0$, there exist local coordinates (x^1, x^2) so that $\rho_s^D = \psi(x^1, x^2)\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$ (see for example Theorem 3.2 in [104]). Now a straightforward calculation using $D\rho_s^D = \omega \otimes \rho_s^D$ gives the relations of Equation (5.4) and thus Assertion (ii) in Lemma 5.7 shows that (Σ, D) is Kähler.

Assertion (ii). Let (Σ, D) be an affine surface with $\rho_s^D \neq 0$ admitting a para-Kähler structure. Take local coordinates as in Lemma 5.8. Then the relations in Equation (5.6) show that $\det(\rho_s^D) < 0$ and ρ_s^D is recurrent, i.e., $D\rho_s^D = \omega \otimes \rho_s^D$ with

$$\omega = -({}^D\Gamma_{11}{}^1 - \partial_{x^1} \log \rho_{s,12}^D)\, dx^1 - ({}^D\Gamma_{22}{}^2 - \partial_{x^2} \log \rho_{s,12}^D)\, dx^2 .$$

Conversely, if ρ_s^D is recurrent and $\det(\rho_s^D) < 0$, there exist local coordinates (x^1, x^2) so that $\rho_s^D = \psi(x^1, x^2)\,(dx^1 \otimes dx^2 + dx^2 \otimes dx^1)$ (see for example Theorem 3.2 in [104]). Now a straightforward calculation using $D\rho_s^D = \omega \otimes \rho_s^D$ gives the relations of Equation (5.6) and thus Assertion (ii) in Lemma 5.8 shows that (Σ, D) admits a para-Kähler structure.

Assertion (iii). Let (Σ, D) be an affine surface with $\rho_s^D \neq 0$. Assume (Σ, D) admits a nilpotent Kähler structure. Take adapted coordinates as in Lemma 5.6 so that the Christoffel symbols are given by the relations in Equation (5.2). Then ρ_s^D is recurrent of rank one with recurrence 1-form given by

$$\omega = \partial_{x^1} \log \rho_{s,22}^D\, dx^1 - (2{}^D\Gamma_{12}{}^1 - \partial_{x^2} \log \rho_{s,22}^D)\, dx^2 .$$

Conversely, let (Σ, D) be a recurrent affine surface with $\mathrm{Rank}(\rho_s^D) = 1$. Take local coordinates (x^1, x^2) so that $\ker(\rho_s^D) = \mathrm{span}\{\partial_{x^1}\}$ (see Theorem 4.1 in [104]). If $\rho_s^D = \rho_{s,22}^D dx^2 \otimes dx^2$, a straightforward calculation shows that $D\rho_s^D = \omega\otimes\rho_s^D$ for some 1-form ω if and only if ${}^D\Gamma_{11}{}^2 = 0$ and ${}^D\Gamma_{12}{}^2 = 0$. Furthermore, one has

$$\begin{aligned}
\rho_{s,12}^D &= \tfrac{1}{2}\left(\partial_{x^1}({}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2) - \partial_{x^2}{}^D\Gamma_{11}{}^1\right), \qquad \rho_{s,11}^D = 0,\\
\rho_{s,22}^D &= {}^D\Gamma_{11}{}^{1D}\Gamma_{22}{}^1 + {}^D\Gamma_{12}{}^1\left({}^D\Gamma_{22}{}^2 - {}^D\Gamma_{12}{}^1\right) + \partial_{x^1}{}^D\Gamma_{22}{}^1 - \partial_{x^2}{}^D\Gamma_{12}{}^1\,.
\end{aligned}$$

Since $\rho_{s,12}^D = 0$ one has the additional relation

$$ {}^D\Gamma_{11}{}^1 = \mu(x^1) + \int \partial_{x^1}\left({}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2\right)dx^2\,. $$

Change the coordinates as $(u^1, u^2) = (x^1 + a(x^1), x^2)$ so that

$$\begin{aligned}
du^1 &= (1+a')dx^1, \quad du^2 = dx^2,\\
\partial_{u^1} &= (1+a')^{-1}\partial_{x^1}, \quad \partial_{u^2} = \partial_{x^2}\,.
\end{aligned}$$

Now, one has that

$$ {}^{uD}\Gamma_{11}{}^2 = 0, \quad {}^{uD}\Gamma_{12}{}^2 = 0, \quad {}^{uD}\Gamma_{12}{}^1 = {}^{xD}\Gamma_{12}{}^1, \quad {}^{uD}\Gamma_{22}{}^2 = {}^{xD}\Gamma_{22}{}^2 $$

and

$$\begin{aligned}
{}^{uD}\Gamma_{11}{}^1 &= \frac{1}{1+a'(x^1)}\left({}^{xD}\Gamma_{11}{}^1 - \frac{a''(x^1)}{1+a'(x^1)}\right)\\
&= \frac{1}{1+a'(x^1)}\left(\mu(x^1) - \frac{a''(x^1)}{1+a'(x^1)} + \int \partial_{x^1}\left({}^{uD}\Gamma_{12}{}^1 - {}^{uD}\Gamma_{22}{}^2\right)dx^2\right)\\
&= \frac{1}{1+a'(x^1)}\left(\mu(x^1) - \frac{a''(x^1)}{1+a'(x^1)}\right)\\
&\quad + \frac{1}{1+a'(x^1)}\int \partial_{x^1}\left({}^{uD}\Gamma_{12}{}^1 - {}^{uD}\Gamma_{22}{}^2\right)dx^2\\
&= \frac{1}{1+a'(x^1)}\left(\mu(x^1) - \frac{a''(x^1)}{1+a'(x^1)}\right) + \int \partial_{u^1}\left({}^{uD}\Gamma_{12}{}^1 - {}^{uD}\Gamma_{22}{}^2\right)du^2\,.
\end{aligned}$$

Hence choosing $a(x^1)$ to be a solution of $a'' - \mu a' - \mu = 0$ one may assume that

$$ {}^D\Gamma_{11}{}^1 = \int \partial_{x^1}\left({}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2\right)dx^2\,. $$

Let $T = T^1{}_2\,\partial_{x^1}\otimes dx^2$ be a nilpotent tensor field on (Σ, D). Then T is parallel if and only if

$$\begin{aligned}
T^1{}_{2;2} &= \partial_{x^2}T^1{}_2 + ({}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2)T^1{}_2 = 0 \quad \text{and}\\
T^1{}_{2;1} &= \partial_{x^1}T^1{}_2 + T^1{}_2{}^D\Gamma_{11}{}^1 = 0\,.
\end{aligned}$$

Use the equation $T^1{}_{2;2} = 0$ and set $T^1{}_2 = e^{-\int ({}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2) dx^2}$. Then

$$\begin{aligned} T^1{}_{2;1} &= \partial_{x^1} T^1{}_2 + T^1{}_2 {}^D\Gamma_{11}{}^1 \\ &= e^{-\int ({}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2) dx^2} \left(-\partial_{x^1} \int ({}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2) dx^2 + {}^D\Gamma_{11}{}^1 \right) \\ &= 0, \end{aligned}$$

thus showing that T is a nilpotent Kähler structure. □

Remark 5.9. Let (Σ, D) be a simply connected affine surface with $\operatorname{Rank}(\rho_s^D) = 1$. The following conditions are equivalent:

(i) The symmetric part of the Ricci tensor is recurrent: $D\rho_s^D = \omega \otimes \rho_s^D$.

(ii) The kernel of the symmetric part of the Ricci tensor is a parallel distribution: $D \ker(\rho_s^D) \subset \ker(\rho_s^D)$.

(iii) The kernel of ρ_s^D is spanned by a recurrent vector field: $\ker(\rho_s^D) = \operatorname{span}\{X\}$ and $DX = \eta \otimes X$.

Consequently, if ρ_s^D has rank one and if $\ker(\rho_s^D)$ is parallel, then the affine surface admits a nilpotent Kähler structure (see for example [89]).

Indeed, assume that $\operatorname{Rank}(\rho_s^D) = 1$. Choose local coordinates so that the symmetric Ricci tensor has the form $\rho_s^D = \rho_{s,22}^D dx^2 \otimes dx^2$. A straightforward calculation shows that any of the conditions of the observation is equivalent to the condition ${}^D\Gamma_{11}{}^2 = {}^D\Gamma_{12}{}^2 = 0$.

5.3 Parallel tensor fields on homogeneous surfaces

Homogeneous surfaces were discussed in Chapter 1. For the convenience of the reader, we recall the following result of Opozda [90]. It is fundamental in the subject.

Theorem 1.31. *Let (Σ, D) be a locally homogeneous affine surface which is not flat. Then at least one of the following three possibilities holds which describe the local geometry:*

($\mathcal{A}$) There exists a coordinate atlas such that the Christoffel symbols ${}^D\Gamma_{ij}{}^k$ are constant.

($\mathcal{B}$) There exists a coordinate atlas such that the Christoffel symbols have the form

$${}^D\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$$

for $C_{ij}{}^k$ constant and $x^1 > 0$.

($\mathcal{C}$) D is the Levi-Civita connection of a metric of constant Gauss curvature.

Homogeneous Type $\mathcal{C}$ surfaces have symmetric and parallel Ricci tensor, which is a multiple of the metric. Hence any such surface admits either a Kähler or a para-Kähler structure, depending on the signature of the metric.

In what remains of this chapter we analyze the existence of parallel $(1,1)$-tensor fields on the other two types of homogeneous surfaces.

5.3.1 Parallel tensor fields on Type $\mathcal{A}$ homogeneous surfaces

The Ricci tensor of any Type $\mathcal{A}$ homogeneous model is symmetric. Furthermore, the Ricci tensor is recurrent if and only if it is of rank one (see Lemma 2.3 in [18]). Therefore Theorem 5.1 (iii) shows that a Type $\mathcal{A}$ homogeneous surface admits a parallel tensor field if and only if the Ricci tensor is of rank one, in which case it is a nilpotent Kähler surface. The construction in Theorem 4.6 make an explicit use of the nilpotent Kähler structure. Therefore, it is important to have concrete expressions. We begin with a useful algebraic fact that we will use to explicitly determine all nilpotent Kähler structures on Type $\mathcal{A}$ homogeneous models. As a matter of notation, let $\mathfrak{K}(\Sigma, D)$ be the Lie algebra of affine Killing vector fields.

Lemma 5.10. *Let D be a Type $\mathcal{A}$ connection on $\Sigma = \mathbb{R}^2$ which is not flat and which satisfies $\mathcal{P}^0(\Sigma, D) \neq \{0\}$. There exists $(a_1, a_2) \in \mathbb{R}^2$ and $0 \neq \mathfrak{t} \in M_2^0(\mathbb{R})$ so that $\mathcal{P}^0(\Sigma, D) = e^{a_1x^1+a_2x^2}\mathfrak{t} \cdot \mathbb{R}$.*

Proof. It is convenient to complexify and set $\mathcal{P}^0_{\mathbb{C}}(\Sigma, D) := \mathcal{P}^0(\Sigma, D) \otimes_{\mathbb{R}} \mathbb{C}$. If $K \in \mathfrak{K}(\Sigma, D)$ and if $T \in \mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$, then the Lie derivative $\mathcal{L}_K T$ belongs to $\mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$. Thus $\mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$ is a finite dimensional complex $\mathfrak{K}(\Sigma, D)$ module. If D defines a Type $\mathcal{A}$ structure on $\mathbb{R}^2$, the Christoffel symbols are constant and ∂_{x^1} and ∂_{x^2} are affine Killing vector fields. If X and Y are vector fields, then we have $\mathcal{L}_X Y = [X, Y]$ is the Lie bracket. Thus $\mathcal{L}_{\partial_{x^i}}\partial_{x^j} = 0$ and dually $\mathcal{L}_{\partial_{x^i}} dx^j = 0$; if $T = T^i{}_j \partial_{x^i} \otimes dx^j$, then $\{\mathcal{L}_{\partial_{x^k}} T\}^i{}_j = \partial_{x^k}\{T^i{}_j\}$; the components of T do not interact. The operators ∂_{x^1} and ∂_{x^2} commute and act on the finite-dimensional vector space $\mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$. Consequently, there is a non-trivial joint eigenvector so $\partial_{x^1} T^i{}_j = a_1 T^i{}_j$ and $\partial_{x^2} T^i{}_j = a_2 T^i{}_j$; this implies $T = e^{a_1x^1+a_2x^2}\mathfrak{t}$ for $0 \neq \mathfrak{t} \in M_2^0(\mathbb{C})$. Since (Σ, D) is not flat, the Ricci tensor is non-zero. Since the Ricci tensor is symmetric for a Type $\mathcal{A}$ geometry, $\rho_s^D \neq 0$. Theorem 5.3 then implies $\dim(\mathcal{P}^0_{\mathbb{C}}(\Sigma, D)) = 1$. Thus the real and imaginary parts of T are linearly dependent and we can assume T is real. The desired result now follows. □

Lemma 5.11. *Let $(\Sigma, D) = (\mathbb{R}^2, D)$ be a Type $\mathcal{A}$ structure which is not flat. Then $\mathcal{P}^0(\Sigma, D) \neq \{0\}$ if and only if (Σ, D) is linearly equivalent to a Type $\mathcal{A}$ structure with ${}^D\Gamma_{11}{}^2 = {}^D\Gamma_{12}{}^2 = 0$. In this setting,*

$$\rho^D = (-{}^D\Gamma_{12}{}^1\, {}^D\Gamma_{12}{}^1 + {}^D\Gamma_{11}{}^1\, {}^D\Gamma_{22}{}^1 + {}^D\Gamma_{12}{}^1\, {}^D\Gamma_{22}{}^2) dx^2 \otimes dx^2 \,.$$

Let $a_1 := -{}^D\Gamma_{11}{}^1$, let $a_2 := {}^D\Gamma_{22}{}^2 - {}^D\Gamma_{12}{}^1$, and let $T = e^{a_1x^1+a_2x^2}\partial_{x^1} \otimes dx^2$. Then $\mathcal{P}^0(\Sigma, D) = T \cdot \mathbb{R}$ is 1-dimensional and nilpotent.

Proof. Let D define a Type $\mathcal{A}$ structure on $\mathbb{R}^2$ with $\mathcal{P}^0(\Sigma, D) \neq \{0\}$ which is not flat. We apply Lemma 5.10 to choose (a_1, a_2) so that $0 \neq T = e^{a_1x^1+a_2x^2}\mathfrak{t} \in \mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$ for some $0 \neq \mathfrak{t} \in M_2^0(\mathbb{C})$. By Lemma 5.2, the eigenvalues of T are constant. Assume the eigenvalues are non-zero. This implies $e^{a_1x^1+a_2x^2}$ is constant and hence $a_1 = a_2 = 0$. By rescaling T, we may assume the eigenvalues are ± 1 and hence, after making a complex linear change of coordinates, we may assume $T^1{}_1 = 1$, $T^2{}_2 = -1$, and $T^1{}_2 = T^2{}_1 = 0$. Setting $DT = 0$ then yields the relations

$${}^D\Gamma_{12}{}^1 = {}^D\Gamma_{11}{}^2 = {}^D\Gamma_{22}{}^1 = {}^D\Gamma_{12}{}^2 = 0 \,.$$

This forces the Ricci tensor to be zero which is false. Thus no Type $\mathcal{A}$ geometry which is not flat admits a Kähler or a para-Kähler structure.

We may therefore assume the eigenvalues of T are constant and zero. After making a linear change of coordinates, we can assume $T = e^{a_1x^1+a_2x^2}\partial_{x^1}\otimes dx^2$. We compute $DT = 0$ if and only if

$$
\begin{aligned}
{}^D\Gamma_{11}{}^2 = 0, \quad a_1 + {}^D\Gamma_{11}{}^1 - {}^D\Gamma_{12}{}^2 = 0\\
{}^D\Gamma_{12}{}^2 = 0, \quad a_2 + {}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2 = 0\,.
\end{aligned}
$$

Thus (Σ, D) admits a non-trivial parallel nilpotent tensor of type $(1,1)$ if and only if ${}^D\Gamma_{11}{}^2 = {}^D\Gamma_{12}{}^2 = 0$. We make a direct computation to determine ρ^D. Since the Ricci tensor is symmetric, we use Theorem 5.3 to see $\dim(\mathcal{P}^0(\Sigma, D)) = 1$. □

We say that two Type $\mathcal{A}$ structures on $\mathbb{R}^2$ are *linearly equivalent* if there exists an element $\Theta \in \mathrm{GL}(2,\mathbb{R})$ which intertwines the two structures. As a consequence of Lemma 5.11 we have:

Theorem 5.12. *Let $(\Sigma, D) = (\mathbb{R}^2, D)$ be a Type $\mathcal{A}$ structure which is not flat. Then $\mathcal{P}^0(\Sigma, D) \neq \{0\}$ if and only if the Ricci tensor is of rank one. Furthermore, (Σ, D) is linearly equivalent to a structure where ${}^D\Gamma_{11}{}^2 = 0$ and ${}^D\Gamma_{12}{}^2 = 0$, and $\mathcal{P}^0(\Sigma, D) = T\cdot\mathbb{R}$, where $T = e^{-{}^D\Gamma_{11}{}^1x^1+({}^D\Gamma_{22}{}^2-{}^D\Gamma_{12}{}^1)x^2}\partial_{x^1}\otimes dx^2$.*

Remark 5.13. If (Σ, D) is a Type $\mathcal{A}$ geometry which is not flat, then (Σ, D) is neither Kähler nor para-Kähler. Furthermore, any Type $\mathcal{A}$ surface with $\mathrm{Rank}(\rho_s^D) = 2$ satisfies $\dim(\mathcal{P}^0(\Sigma, D)) = 0$.

Remark 5.14. Let (Σ, D) be a Type $\mathcal{A}$ surface with Ricci tensor of rank one and let $T = e^{a_1x^1+a_2x^2}\partial_{x^1}\otimes dx^2$ be a nilpotent Kähler structure as in Theorem 5.12. A straightforward calculation shows that the corresponding modified Riemannian extension $(T^*\Sigma, g_{D,\Phi,T})$ with deformation tensor field $\Phi \equiv 0$ is anti-self-dual. This is due to the fact that any Type $\mathcal{A}$ homogeneous geometry is projectively flat. Moreover it has been shown in [18] that any Type $\mathcal{A}$ surface with Ricci tensor of rank one admits affine gradient Ricci solitons (i.e., smooth functions $f \in \mathcal{C}^\infty(\Sigma)$ satisfying $\mathrm{Hes}_f + 2\rho_s^D = 0$) so that $df(\ker(\rho^D)) = 0$. Hence $(T^*\Sigma, g_{D,0,T}, h = \pi^*f)$ is an anti-self-dual gradient Ricci soliton which is never locally conformally flat. In this setting, the soliton is steady (i.e., $\lambda = 0$) and isotropic (i.e., $\|d\pi^*f\|^2 = 0$).

In a more general setting, results in [20] show that any Type $\mathcal{A}$ surface with Ricci tensor of rank one admits solutions of the affine quasi-Einstein equation (i.e., smooth functions $f \in \mathcal{C}^\infty(\Sigma)$ satisfying $\mathrm{Hes}_f + 2\rho_s^D - \mu\, df\otimes df = 0$) so that $df(\ker(\rho^D)) = 0$. Hence $(T^*\Sigma, g_{D,0,T}, h = \pi^*f)$ is an anti-self-dual quasi-Einstein manifold which is never locally conformally flat.

Results of [18] show that if (Σ, D) is a Type $\mathcal{A}$ geometry which is not flat, then either $\dim(\mathfrak{K}(\Sigma, D)) = 2$ or $\dim(\mathfrak{K}(\Sigma, D)) = 4$.

Theorem 5.15. *Let $(\Sigma, D) = (\mathbb{R}^2, D)$ be a Type $\mathcal{A}$ structure. The following assertions are equivalent:*

(i) $\mathrm{Rank}(\rho^D) = 1$.

(ii) $\mathcal{P}^0(\Sigma, D) \neq \{0\}$.

(iii) $\dim(\mathcal{P}^0(\Sigma, D)) = 1$.

(iv) $\dim(\mathfrak{K}(\Sigma, D)) = 4$.

Proof. Results of [18] (see Lemma 2.3) show that ρ_s^D has rank one if and only if (Σ, D) is linearly equivalent to a structure where ${}^D\Gamma_{11}{}^2 = 0$ and ${}^D\Gamma_{12}{}^2 = 0$. The equivalence of Assertion (i), Assertion (ii), and Assertion (iii) then follows from Theorem 5.12. The equivalence of Assertion (i) and Assertion (iv) follows from Theorem 3.4 of [18]. □

5.3.2 Parallel tensor fields on Type $\mathcal{B}$ homogeneous surfaces

The situation is more complicated in the Type $\mathcal{B}$ setting. For instance, Remark 5.17 shows the existence of simply connected affine surfaces with $\operatorname{Rank}(\rho_s^D) = 1$ but with non-recurrent ρ_s^D and $\dim(\mathcal{P}^0(\Sigma, D)) = 0$. Also, in contrast with Type $\mathcal{A}$ surfaces, there are non-flat Type $\mathcal{B}$ surfaces with $\rho_s^D = 0$. This situation is discussed in Lemma 5.19.

Let $(\Sigma, D) = (\mathbb{R}^+ \times \mathbb{R}, D)$ where ${}^D\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$ and $C_{ij}{}^k \in \mathbb{R}$ be a Type $\mathcal{B}$ surface which is not flat such that $\mathcal{P}^0(\Sigma, D)$ is non-trivial. In Lemma 5.16, we give an algebraic criteria for determining when $\mathcal{P}^0(\Sigma, D)$ is non-trivial. In Lemmas 5.21–5.29, we use this criteria to divide the analysis into five different cases and to determine when $\dim(\mathcal{P}^0(\Sigma, D)) = 1$ or $\dim(\mathcal{P}^0(\Sigma, D)) = 3$. We first prove an analogue of Lemma 5.10 in this setting.

Lemma 5.16. *If D is a Type $\mathcal{B}$ connection on $\Sigma = \mathbb{R}^+ \times \mathbb{R}$ and if $\mathcal{P}^0(\Sigma, D) \neq \{0\}$, then there exists $\alpha \in \mathbb{C}$ and $0 \neq \mathfrak{t} \in M_2^0(\mathbb{C})$ so that $(x^1)^\alpha \mathfrak{t} \in \mathcal{P}_{\mathbb{C}}^0(\Sigma, D)$.*

Proof. Let D define a Type $\mathcal{B}$ structure on $\mathbb{R}^+ \times \mathbb{R}$. The vector fields ∂_{x^2} and $X := x^1\partial_{x^1} + x^2\partial_{x^2}$ are affine Killing vector fields (see [18]). We have:

$$\begin{array}{lll} \mathcal{L}_X(\partial_{x^i}) = [X, \partial_{x^i}] = -\partial_{x^i} & \mathcal{L}_X(dx^j) = dx^j, & \mathcal{L}_X(\partial_{x^i} \otimes dx^j) = 0, \\ \mathcal{L}_{\partial_{x^2}}(\partial_{x^i}) = 0, & \mathcal{L}_{\partial_{x^2}}(dx^j) = 0, & \mathcal{L}_{\partial_{x^2}}(\partial_{x^i} \otimes dx^j) = 0\,. \end{array}$$

Therefore the components do not interact and we have:

$$\{\mathcal{L}_X T\}^i{}_j = X T^i{}_j \quad \text{and} \quad \{\mathcal{L}_{\partial_{x^2}} T\}^i{}_j = \partial_{x^2} T^i{}_j\,.$$

Because $\mathcal{P}_{\mathbb{C}}^0(\Sigma, D)$ is a finite-dimensional ∂_{x^2} module, we can find a non-trivial complex eigenvector, i.e., $0 \neq T \in \mathcal{P}_{\mathbb{C}}^0(\Sigma, D)$ so $\partial_{x^2} T^i{}_j = a_2 T^i{}_j$. This implies that $T^i{}_j = e^{a_2 x^2} t^i{}_j(x^1)$. Applying X^k yields

$$X^k(T^i{}_j) = e^{a_2 x^2}\{a_2^k (x^2)^k t^i{}_j(x^1) + O((x^2)^{k-1})\}\,.$$

Thus if $a_2 \neq 0$, the elements $\{T, \mathcal{L}_X T, \dots, \mathcal{L}_X T^k\}$ are linearly independent for any k. This is false since $\dim(\mathcal{P}_{\mathbb{C}}^0(\Sigma, D)) \leq 3$. Therefore, $T = t^i{}_j(x^1)$. We let $\mathcal{V} \neq \{0\}$ be the subspace of all elements of $\mathcal{P}_{\mathbb{C}}^0(\Sigma, D)$ where $T = T(x^1)$. Choose a non-trivial eigenvector of $\mathcal{L}_X$. Then $x^1\partial_{x^1} T = \alpha T$ implies $T(x^1) = (x^1)^\alpha \mathfrak{t}$ for some $\mathfrak{t} \in M_2^0(\mathbb{C})$. □

Remark 5.17. In the Type $\mathcal{A}$ setting, the condition $\operatorname{Rank}(\rho_s^D) = 1$ implies $\mathcal{P}^0(\Sigma, D)$ is non-trivial. This fails in the Type $\mathcal{B}$ setting. Let (Σ, D) be the Type $\mathcal{B}$ surface defined by setting $C_{22}{}^2 = (3+2\sqrt{3})/3$ and $C_{ij}{}^k = 1$ otherwise. We compute that

$$\rho_s^D = \frac{1}{(x^1)^2}\begin{pmatrix} 1+\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}}-1 \end{pmatrix}$$

and, consequently, ρ_s^D has rank one. Assume $\dim(\mathcal{P}^0(\Sigma, D)) \geq 1$. It follows from Lemma 5.16 that there exists an element in $\mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$ of the form $T = (x^1)^\alpha(\mathfrak{t}^i{}_j)$ where $0 \neq (\mathfrak{t}^i{}_j) \in M_2^0(\mathbb{C})$. Setting $T^i{}_{j;2} = 0$ yields the relations:

$$(x^1)^{\alpha-1}\begin{pmatrix} \mathfrak{t}^2{}_1 - \mathfrak{t}^1{}_2 & -2\mathfrak{t}^1{}_1 - \frac{2}{\sqrt{3}}\mathfrak{t}^1{}_2 \\ 2\mathfrak{t}^1{}_1 + \frac{2}{\sqrt{3}}\mathfrak{t}^2{}_1 & \mathfrak{t}^1{}_2 - \mathfrak{t}^2{}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We solve this relation to see $\mathfrak{t}^2{}_1 = \mathfrak{t}^1{}_2$ and $\mathfrak{t}^1{}_1 = -\frac{1}{\sqrt{3}}\mathfrak{t}^1{}_2$. Substituting these relations and setting $T^i{}_{j;1} = 0$ then yields:

$$(x^1)^{\alpha-1}\begin{pmatrix} -\frac{\alpha}{\sqrt{3}}\mathfrak{t}^1{}_2 & \left(\alpha+\frac{2}{\sqrt{3}}\right)\mathfrak{t}^1{}_2 \\ \left(\alpha-\frac{2}{\sqrt{3}}\right)\mathfrak{t}^1{}_2 & \frac{\alpha}{\sqrt{3}}\mathfrak{t}^1{}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This shows $\mathfrak{t}^1{}_2 = 0$ and hence $T = 0$. This shows $\mathcal{P}^0(\Sigma, D)$ is trivial. The result also follows from Theorem 5.1 just observing that the symmetric Ricci tensor ρ_s^D is not recurrent.

Definition 5.18. We follow the discussion of [18] and introduce the following surfaces of Type $\mathcal{B}$.

(1) For $c \in \mathbb{R}$, let $\mathcal{Q}_c$ be the affine manifold of Type $\mathcal{B}$ defined by

$$C_{11}{}^1 = 0,\ C_{11}{}^2 = c,\ C_{12}{}^1 = 1,\ C_{12}{}^2 = 0,\ C_{22}{}^1 = 0,\ C_{22}{}^2 = 1\,.$$

Since $\rho^D = (x^1)^{-2}dx^1 \wedge dx^2$, $\rho_s^D = 0$.

(2) For $0 \neq c \in \mathbb{R}$, let $\mathcal{P}^{\pm}_{0,c}$ be the affine manifold of Type $\mathcal{B}$ defined by

$$\begin{aligned} &C_{11}{}^1 = \mp c^2 + 1, && C_{11}{}^2 = c, && C_{12}{}^1 = 0, \\ &C_{12}{}^2 = \mp c^2, && C_{22}{}^1 = \pm 1, && C_{22}{}^2 = \pm 2c\,. \end{aligned}$$

Since $\rho^D = \pm(x^1)^{-2}c\,dx^1 \wedge dx^2$, $\rho_s^D = 0$.

By Theorem 5.3, $\rho_s^D = 0$ if and only if $\dim(\mathcal{P}^0(\Sigma, D)) = 3$. We give a complete description of Type $\mathcal{B}$ manifolds which are not flat where $\rho_s^D = 0$ as follows.

Lemma 5.19.

(i) If (Σ, D) *is a Type* $\mathcal{B}$ *manifold which is not flat but which has* $\rho_s^D = 0$*, then* (Σ, D) *is linearly equivalent either to* $\mathcal{Q}_c$ *or to* $\mathcal{P}_{0,c}^{\pm}$*.*

(ii) If $(\Sigma, D) = \mathcal{Q}_c$ *for* $c \neq 0$*, then*

$$\mathcal{P}^0_{\mathbb{C}}(\mathcal{Q}_c) = \operatorname{span}\left\{\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}, \quad (x^1)^{2\sqrt{c}}\begin{pmatrix} \sqrt{c} & 1 \\ -c & -\sqrt{c} \end{pmatrix}, \right.$$
$$\left. (x^1)^{-2\sqrt{c}}\begin{pmatrix} -\sqrt{c} & 1 \\ -c & \sqrt{c} \end{pmatrix}\right\}.$$

(iii) If $(\Sigma, D) = \mathcal{Q}_c$ *for* $c = 0$*, then*

$$\mathcal{P}^0(\mathcal{Q}_0) = \operatorname{span}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -\log(x^1) & 1 - \log(x^1)^2 \\ 1 & -\log(x^1) \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -\log(x^1) & -1 - \log(x^1)^2 \\ 1 & -\log(x^1) \end{pmatrix}\right\}.$$

(iv) If $(\Sigma, D) = \mathcal{P}_{0,c}^{\pm}$*, then*

$$\mathcal{P}^0(\mathcal{P}_{0,c}^{\pm}) = \operatorname{span}\left\{(x^1)^{-1}\begin{pmatrix} -c & 1 \\ -c^2 & c \end{pmatrix}, \right.$$
$$(x^1)^{-1}\begin{pmatrix} \pm\frac{1}{2}(x^1 \mp 2cx^2) & x^2 \\ \pm c(x^1 \mp cx^2) & \mp\frac{1}{2}(x^1 \mp 2cx^2) \end{pmatrix},$$
$$\left. (x^1)^{-1}\begin{pmatrix} \pm x^2(x^1 \mp cx^2) & (x^2)^2 \\ -(x^1 \mp cx^2)^2 & \mp x^2(x^1 \mp cx^2) \end{pmatrix}\right\}.$$

Proof. Assertion (i) follows from Lemma 4.6 in [18]; the remaining assertions follow from a direct computation. □

Remark 5.20. Suppose that (Σ, D) is a Type $\mathcal{B}$ surface with $\mathcal{P}^0(\Sigma, D)$ non-trivial. And by Lemma 5.16, there exists $\alpha \in \mathbb{C}$ and $0 \neq \mathfrak{t} \in M_2^0(\mathbb{C})$ so that $T := (x^1)^\alpha \mathfrak{t} \in \mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$. If α is complex, then the real and imaginary parts of T are linearly dependent and both belong to $\mathcal{P}^0(\Sigma, D)$. This implies $\dim(\mathcal{P}^0(\Sigma, D)) \geq 2$ and hence $\rho_s^D = 0$. Lemma 5.19 then yields $(\Sigma, D) = \mathcal{Q}_c$ for $c < 0$ and α is purely imaginary.

In view of Lemma 5.19, we will assume $\rho_s^D \neq 0$ henceforth. Let (Σ, D) be a Type $\mathcal{B}$ geometry with $\mathcal{P}^0(\Sigma, D)$ non-trivial and, since $\rho_s^D \neq 0$, $\dim(\mathcal{P}^0(\Sigma, D)) = 1$. By Lemma 5.16, there exists $\alpha \in \mathbb{C}$ and $0 \neq \mathfrak{t} \in M_2^0(\mathbb{C})$ so that $(x^1)^\alpha \mathfrak{t} \in \mathcal{P}^0_{\mathbb{C}}(\Sigma, D)$. By Remark 5.20, $\alpha \in \mathbb{R}$ and thus, by taking real and imaginary parts, we may assume that $0 \neq \mathfrak{t} \in M_2^0(\mathbb{R})$. Suppose $\alpha = 0$. We deal with the case $\mathfrak{t}^1{}_2 \neq 0$ in Lemma 5.21, the case $\mathfrak{t}^1{}_2 = 0$ and $\mathfrak{t}^2{}_1 \neq 0$ in Lemma 5.23, and the case $\mathfrak{t}^1{}_2 = \mathfrak{t}^2{}_1 = 0$ and $\mathfrak{t}^1{}_1 \neq 0$ in Lemma 5.25. We then turn to the situation where $\alpha \neq 0$. Since $\det(T) = (x^1)^{2\alpha}\det(\mathfrak{t})$ is constant and since $\alpha \neq 0$ is real, we conclude that $\mathfrak{t}$ is nilpotent. In Lemma 5.27, we assume $\mathfrak{t}^1{}_2 \neq 0$ and in Lemma 5.29, we assume $\mathfrak{t}^1{}_2 = 0$ to complete our analysis.

Lemma 5.21. *Let D define a Type $\mathcal{B}$ structure on $\mathbb{R}^+ \times \mathbb{R}$ with $\rho_s^D \neq 0$. Suppose that there exists $0 \neq \mathfrak{t} \in \mathcal{P}^0(\Sigma, D) \cap M_2(\mathbb{R})$ with $\mathfrak{t}^1{}_2 \neq 0$. Rescale $\mathfrak{t}$ to assume that $\mathfrak{t}^1{}_2 = 1$. Then:*

(i) The Christoffel symbols are determined by

$$\begin{aligned} C_{11}{}^1 &= C_{22}{}^1\,\mathfrak{t}^2{}_1 + 2(C_{22}{}^2 + 2C_{22}{}^1\,\mathfrak{t}^1{}_1)\mathfrak{t}^1{}_1, & C_{12}{}^1 &= C_{22}{}^2 + 2C_{22}{}^1\,\mathfrak{t}^1{}_1, \\ C_{11}{}^2 &= (C_{22}{}^2 + 2C_{22}{}^1\,\mathfrak{t}^1{}_1)\mathfrak{t}^2{}_1, & C_{12}{}^2 &= C_{22}{}^1\,\mathfrak{t}^2{}_1\,. \end{aligned}$$

(ii) The symmetric part of the Ricci tensor is given by

$$\rho_s^D = (x^1)^{-2} C_{22}{}^1 \begin{pmatrix} \mathfrak{t}^2{}_1 & -\mathfrak{t}^1{}_1 \\ -\mathfrak{t}^1{}_1 & -1 \end{pmatrix}, \qquad C_{22}{}^1 \neq 0\,.$$

(iii) The space of trace free parallel tensor fields is generated by

$$\mathcal{P}^0(\Sigma, D) = \begin{pmatrix} \mathfrak{t}^1{}_1 & 1 \\ \mathfrak{t}^2{}_1 & -\mathfrak{t}^1{}_1 \end{pmatrix} \cdot \mathbb{R}\,.$$

Proof. The equations $D_{\partial_{x^i}}\mathfrak{t} = 0$, $i = 1, 2$ become:

$$\begin{pmatrix} C_{12}{}^1\mathfrak{t}^2{}_1 - C_{11}{}^2 & C_{11}{}^1 - C_{12}{}^2 - 2C_{12}{}^1\mathfrak{t}^1{}_1 \\ -C_{11}{}^1\mathfrak{t}^2{}_1 + C_{12}{}^2\mathfrak{t}^2{}_1 + 2C_{11}{}^2\mathfrak{t}^1{}_1 & C_{11}{}^2 - C_{12}{}^1\mathfrak{t}^2{}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} C_{22}{}^1\mathfrak{t}^2{}_1 - C_{12}{}^2 & C_{12}{}^1 - C_{22}{}^2 - 2C_{22}{}^1\mathfrak{t}^1{}_1 \\ -C_{12}{}^1\mathfrak{t}^2{}_1 + C_{22}{}^2\mathfrak{t}^2{}_1 + 2C_{12}{}^2\mathfrak{t}^1{}_1 & C_{12}{}^2 - C_{22}{}^1\mathfrak{t}^2{}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

These equations yield the relations amongst the $C_{ij}{}^k$; a direct computation then yields ρ_s^D; we obtain $C_{22}{}^1 \neq 0$ since $\rho_s^D \neq 0$. Furthermore, since $\rho_s^D \neq 0$, we have $\dim(\mathcal{P}^0(\Sigma, D)) = 1$ and the element given spans $\mathcal{P}^0(\Sigma, D)$. □

Remark 5.22. Let $\mathfrak{t}$ be a nilpotent Kähler tensor field as in Lemma 5.21. Then, in contrast with Remark 5.14, the modified Riemannian extension $(T^*\Sigma, g_{D,0,\mathfrak{t}})$ is never anti-self-dual. Indeed, the affine structures in Lemma 5.21 are never projectively flat unless $\rho_s^D = 0$.

Lemma 5.23. *Let D define a Type $\mathcal{B}$ structure on $\mathbb{R}^+ \times \mathbb{R}$ with $\rho_s^D \neq 0$. Suppose that there exists $0 \neq \mathfrak{t} \in \mathcal{P}^0(\Sigma, D) \cap M_2(\mathbb{R})$ with $\mathfrak{t}^1{}_2 = 0$ and $\mathfrak{t}^2{}_1 \neq 0$. Rescale $\mathfrak{t}$ to assume $\mathfrak{t}^2{}_1 = 1$. Then:*

(i) The Christoffel symbols are determined by

$$C_{11}{}^1 = C_{12}{}^2 + 2C_{11}{}^2\mathfrak{t}^1{}_1,\ C_{12}{}^1 = 0,\ C_{22}{}^1 = 0,\ C_{22}{}^2 = -2C_{12}{}^2\mathfrak{t}^1{}_1\,.$$

(ii) The Ricci tensor is given by

$$\rho^D = (x^1)^{-2}C_{12}{}^2\begin{pmatrix} 1 & -2\mathfrak{t}^1{}_1 \\ 0 & 0 \end{pmatrix}, \qquad C_{12}{}^2 \neq 0\,.$$

(iii) The space of trace free parallel tensor fields is generated by

$$\mathcal{P}^0(\Sigma, D) = \begin{pmatrix} \mathfrak{t}^1{}_1 & 0 \\ 1 & -\mathfrak{t}^1{}_1 \end{pmatrix} \cdot \mathbb{R}\,.$$

Proof. Setting $D\mathfrak{t} = 0$ yields the relations

$$\begin{pmatrix} C_{12}{}^1 & -2C_{12}{}^1\mathfrak{t}^1{}_1 \\ -C_{11}{}^1 + C_{12}{}^2 + 2C_{11}{}^2\mathfrak{t}^1{}_1 & -C_{12}{}^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} C_{22}{}^1 & -2C_{22}{}^1\mathfrak{t}^1{}_1 \\ -C_{12}{}^1 + C_{22}{}^2 + 2C_{12}{}^2\mathfrak{t}^1{}_1 & -C_{22}{}^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We solve these relations to obtain the relations amongst the $C_{ij}{}^k$. We then compute ρ^D. Since $\rho_s^D \neq 0$, $C_{12}{}^2 \neq 0$. Furthermore, since $\rho_s^D \neq 0$, $\dim(\mathcal{P}^0(\Sigma, D)) = 1$ and the element given spans $\mathcal{P}^0(\Sigma, D)$. □

Remark 5.24. The modified Riemannian extensions of nilpotent tensor fields in Lemma 5.23 corresponding to $\mathfrak{t}^1{}_1 = 0$ are anti-self-dual whenever the deformation tensor field $\Phi \equiv 0$. In this case Lemma 5.23 gives $C_{12}{}^1 = 0,\ C_{22}{}^1 = 0,\ C_{22}{}^2 = 0$, and thus (Σ, D) is also of type $\mathcal{A}$ (see Remark 1.32). In this case, Remark 5.14 applies.

Lemma 5.25. *Let D define a Type $\mathcal{B}$ structure on $\mathbb{R}^+ \times \mathbb{R}$ with $\rho_s^D \neq 0$. Suppose that there exists $0 \neq \mathfrak{t} \in \mathcal{P}^0(\Sigma, D) \cap M_2(\mathbb{R})$ with $\mathfrak{t}^1{}_2 = \mathfrak{t}^2{}_1 = 0$. Rescale $\mathfrak{t}$ to assume $\mathfrak{t}^1{}_1 = 1$. Then:*

(i) The Christoffel symbols are determined by

$$C_{11}{}^2 = 0, \quad C_{12}{}^1 = 0, \quad C_{12}{}^2 = 0, \quad C_{22}{}^1 = 0\,.$$

(ii) The Ricci tensor is given by

$$\rho^D = (x^1)^{-2}C_{22}{}^2 dx^1 \otimes dx^2, \qquad C_{22}{}^2 \neq 0\,.$$

(iii) The space of trace free parallel tensor fields is generated by

$$\mathcal{P}^0(\Sigma, D) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \mathbb{R}\,.$$

Proof. Let $\mathfrak{t} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Setting $D\mathfrak{t} = 0$ yields the relations

$$\begin{pmatrix} 0 & -2C_{12}{}^1 \\ 2C_{11}{}^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2C_{22}{}^1 \\ 2C_{12}{}^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The relations in Lemma 5.25 concerning the $C_{ij}{}^k$ now follow. We determine ρ^D by a direct computation; since $\rho_s^D \neq 0$, $C_{22}{}^2 \neq 0$. Furthermore, since $\rho_s^D \neq 0$, $\dim(\mathcal{P}^0(\Sigma, D)) = 1$ and the element given spans $\mathcal{P}^0(\Sigma, D)$. □

Remark 5.26. Theorem 5.12 shows that type $\mathcal{A}$ surfaces with $\dim(\mathcal{P}^0(\Sigma, D)) \geq 1$ have dimension 1 in the non-flat case and $\mathcal{P}^0(\Sigma, D)$ is generated by a nilpotent Kähler structure. In opposition, the Type $\mathcal{B}$ geometries in Lemma 5.21 with $\dim(\mathcal{P}^0(\Sigma, D)) = 1$ contain Kähler, para-Kähler and nilpotent Kähler examples. On the other hand, the Type $\mathcal{B}$ geometries treated in Lemma 5.23 and Lemma 5.25 only admit para-Kähler structures.

Lemma 5.27. *Let D define a Type $\mathcal{B}$ structure on $\mathbb{R}^+ \times \mathbb{R}$ with $\rho_s^D \neq 0$. Suppose that there exists $0 \neq \mathfrak{t} \in M_2(\mathbb{R})$ with $\mathfrak{t}^1{}_2 \neq 0$ and that there exists $\alpha \neq 0$ so that $(x^1)^\alpha \mathfrak{t} \in \mathcal{P}^0(\Sigma, D)$. Rescale $\mathfrak{t}$ so that $\mathfrak{t}^1{}_2 = 1$. Then:*

(i) The Christoffel symbols are determined by

$$\begin{aligned} &C_{12}{}^1 = C_{22}{}^2 + 2C_{22}{}^1\mathfrak{t}^1{}_1, && C_{11}{}^2 = \mathfrak{t}^1{}_1(-C_{11}{}^1 + \mathfrak{t}^1{}_1(C_{22}{}^2 + C_{22}{}^1\mathfrak{t}^1{}_1)), \\ &C_{12}{}^2 = -C_{22}{}^1(\mathfrak{t}^1{}_1)^2, && \alpha = -C_{11}{}^1 + \mathfrak{t}^1{}_1(2C_{22}{}^2 + 3C_{22}{}^1\mathfrak{t}^1{}_1) \neq -1. \end{aligned}$$

(ii) The symmetric part of the Ricci tensor is given by

$$\rho_s^D = -(x^1)^{-2}C_{22}{}^1(1+\alpha)\begin{pmatrix} (\mathfrak{t}^1{}_1)^2 & \mathfrak{t}^1{}_1 \\ \mathfrak{t}^1{}_1 & 1 \end{pmatrix}, \qquad C_{22}{}^1 \neq 0.$$

(iii) The space of trace free parallel tensor fields is generated by

$$\mathcal{P}^0(\Sigma, D) = (x^1)^\alpha \begin{pmatrix} \mathfrak{t}^1{}_1 & 1 \\ -(\mathfrak{t}^1{}_1)^2 & -\mathfrak{t}^1{}_1 \end{pmatrix} \cdot \mathbb{R}.$$

Proof. As noted previously, $\alpha \neq 0$ implies $\mathfrak{t}$ is nilpotent. Since we assumed $\mathfrak{t}^1{}_2 = 1$,

$$T = (x^1)^\alpha \begin{pmatrix} \mathfrak{t}^1{}_1 & 1 \\ -(\mathfrak{t}^1{}_1)^2 & -\mathfrak{t}^1{}_1 \end{pmatrix}.$$

The conditions $D_{\partial_{x^i}} T = 0$ $(i = 1, 2)$ imply the vanishing of the matrices

$$\begin{pmatrix} -C_{11}{}^2 - (C_{12}{}^1\mathfrak{t}^1{}_1 - \alpha)\mathfrak{t}^1{}_1 & C_{11}{}^1 - C_{12}{}^2 + \alpha - 2C_{12}{}^1\mathfrak{t}^1{}_1 \\ \mathfrak{t}^1{}_1(2C_{11}{}^2 + (C_{11}{}^1 - C_{12}{}^2 - \alpha)\mathfrak{t}^1{}_1) & C_{11}{}^2 + (C_{12}{}^1\mathfrak{t}^1{}_1 - \alpha)\mathfrak{t}^1{}_1 \end{pmatrix}$$

and

$$\begin{pmatrix} -C_{12}{}^2 - C_{22}{}^1(\mathfrak{t}^1{}_1)^2 & C_{12}{}^1 - C_{22}{}^2 - 2C_{22}{}^1\mathfrak{t}^1{}_1 \\ \mathfrak{t}^1{}_1(2C_{12}{}^2 + (C_{12}{}^1 - C_{22}{}^2)\mathfrak{t}^1{}_1) & C_{12}{}^2 + C_{22}{}^1(\mathfrak{t}^1{}_1)^2 \end{pmatrix}.$$

We solve these relations to obtain the relations amongst the $C_{ij}{}^k$. The expression of α and ρ_s^D then follows by a direct computation. Since $\rho_s^D \neq 0$, we obtain $C_{22}{}^1 \neq 0$, $\alpha \neq 0$, and $\alpha \neq -1$. Furthermore, since $\rho_s^D \neq 0$, $\dim(\mathcal{P}^0(\Sigma, D)) = 1$ and the element given spans $\mathcal{P}^0(\Sigma, D)$. □

Remark 5.28. Let T be a nilpotent Kähler tensor field as in Lemma 5.27. The modified Riemannian extension $(T^*\Sigma, g_{D,0,T})$ is not anti-self-dual.

Lemma 5.29. *Let D define a Type $\mathcal{B}$ structure on $\mathbb{R}^+ \times \mathbb{R}$ with $\rho_s^D \neq 0$. Suppose that there exists $0 \neq \mathfrak{t} \in M_2(\mathbb{R})$ with $\mathfrak{t}^1{}_2 = 0$ and that there exists $\alpha \neq 0$ so that $(x^1)^\alpha \mathfrak{t} \in \mathcal{P}^0(\Sigma, D)$. Since $\mathfrak{t}$ is nilpotent, $\mathfrak{t}^1{}_1 = 0$ and $\mathfrak{t}^2{}_1 \neq 0$. Rescale $\mathfrak{t}$ so that $\mathfrak{t}^2{}_1 = 1$. Then:*

(i) The Christoffel symbols are determined by

$$C_{12}{}^1 = 0, \quad C_{22}{}^1 = 0, \quad C_{22}{}^2 = 0, \quad \alpha = C_{11}{}^1 - C_{12}{}^2 \notin \{0, -1\}.$$

(ii) The Ricci tensor is given by

$$\rho^D = (x^1)^{-2}(1+\alpha)C_{12}{}^2 dx^1 \otimes dx^1.$$

(iii) The space of trace free parallel tensor fields is generated by

$$\mathcal{P}^0(\Sigma, D) = (x^1)^{C_{11}{}^1 - C_{12}{}^2}\partial_{x^2} \otimes dx^1 \cdot \mathbb{R}.$$

Proof. Setting $DT = 0$ yields the vanishing of the matrices

$$\begin{pmatrix} C_{12}{}^1 & 0 \\ -C_{11}{}^1 + C_{12}{}^2 + \alpha & -C_{12}{}^1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_{22}{}^1 & 0 \\ -C_{12}{}^1 + C_{22}{}^2 & -C_{22}{}^1 \end{pmatrix}.$$

The relations amongst the $C_{ij}{}^k$ follows and α is determined. A direct computation yields the Ricci tensor. Since $\rho^D = \rho_s^D \neq 0$, $\dim(\mathcal{P}^0(\Sigma, D)) = 1$ and the element given spans $\mathcal{P}^0(\Sigma, D)$. □

Chapter 6
General examples of Bach-flat manifolds in neutral signature

In this chapter we generalize the construction in Chapter 4 to characterize Bach-flat Riemannian extensions of affine surfaces admitting a nilpotent structure without assuming any parallelizability condition.

This chapter is organized as follows. In Sections 6.1 and 6.2 we make use of the Cauchy-Kovalevski Theorem to show that any nilpotent Riemannian extension can be locally deformed to be Bach-flat in the real analytic case (see Theorem 6.1). In Section 6.3 we show that all these metrics have vanishing scalar curvature invariants (see Theorem 6.8). For that reason, in Section 6.3.2 we shall introduce suitable invariants which are not of Weyl type to distinguish different classes. Finally, in Section 6.4, we shall exhibit some specific examples of Bach-flat manifolds. In this chapter we report on work investigated in [31].

6.1 Bach-flat Riemannian extensions

Let (Σ, D) be an affine surface. If (x^1, x^2) are local coordinates on Σ, let $(x_{1'}, x_{2'})$ be the associated dual coordinates on the cotangent bundle. Let $T = T^r{}_i\, \partial_{x^r} \otimes dx^i$ be a tensor field of type $(1,1)$ on Σ and let Φ_{ij} be a symmetric $(0,2)$-tensor on Σ. The associated modified Riemannian extension

$$\begin{aligned} g_{D,\Phi,T} &= 2\, dx^i \circ dx_{i'} \\ &\quad + \left\{ \tfrac{1}{2} x_{r'} x_{s'} (T^r{}_i T^s{}_j + T^r{}_j T^s{}_i) - 2 x_{k'}\, {}^D\Gamma_{ij}{}^k + \Phi_{ij} \right\} dx^i \circ dx^j \end{aligned} \tag{6.1}$$

is invariantly defined and independent of the particular system of local coordinates (see for example the discussion in Chapter 1 and [29]). Let

$$\mathcal{S}_T := \{p \in \Sigma : T(p) = \lambda(p)\,\mathrm{Id}\} \quad \text{and} \quad \mathcal{O}_T := \Sigma - \mathcal{S}_T\,.$$

The space $\mathcal{S}_T$ is the set of points where T is a scalar multiple of the identity; $\mathcal{O}_T$ is the complementary space.

Theorem 6.1. *Let (Σ, D) be an affine surface, let T be a tensor field of type $(1,1)$, and let Φ be a symmetric $(0,2)$-tensor.*

(i) If $\Sigma = \mathcal{S}_T$, then $(T^*\Sigma, g_{D,\Phi,T})$ is half conformally flat and hence Bach-flat.

(ii) $\mathcal{O}_T$ is an open subset of Σ. If $p \in \mathcal{O}_T$ and if $\mathfrak{B}(p) = 0$, then $T(p)^2 = 0$.

(iii) If T is nilpotent on Σ and if $T(p) \neq 0$, then there exist local coordinates near p so that $T = \partial_{x^1} \otimes dx^2$. The following assertions are equivalent in such a coordinate system:

(a) $(T^*\Sigma, g_{D,\Phi,T})$ *is Bach-flat.*

(b) ${}^D\Gamma_{11}{}^2 = 0$ *and* $({}^D\Gamma_{11}{}^1)^2 - {}^D\Gamma_{11}{}^1\, {}^D\Gamma_{12}{}^2 + \partial_{x^1}({}^D\Gamma_{11}{}^1 - {}^D\Gamma_{12}{}^2) = 0$.

Proof. A direct computation shows that if $T = f\,\mathrm{Id}$ for $f \in C^\infty(\Sigma)$, then $(T^*\Sigma, g_{D,\Phi,T})$ is self-dual [29], and thus $\mathfrak{B} = 0$; this establishes Assertion (i). Consequently, we assume henceforth that there exists a point p of Σ where $T(p) \neq f(p)\,\mathrm{Id}$.

Let $\mathrm{Coeff}[\mathfrak{B}_{k\ell}; x_{i'}x_{j'}]$ be the coefficient of $x_{i'}x_{j'}$ in $\mathfrak{B}_{k\ell}$. A straightforward calculation shows that the components of the Bach tensor are quadratic polynomials in the fiber coordinates $(x_{1'}, x_{2'})$, and moreover one has:

$$\begin{aligned}\mathrm{Coeff}[\mathfrak{B}_{11}; x_{1'}x_{1'}] = \tfrac{1}{6}\big\{&\big(-30(\det T)^2 - \det T(\operatorname{tr} T)^2 + (\operatorname{tr} T)^4\big)\, T^1{}_1(x^1,x^2)^2\\ &+2\det T \operatorname{tr} T\,\big(17\det T - 2(\operatorname{tr} T)^2\big)\, T^1{}_1(x^1,x^2)\\ &-2(\det T)^2\,\big(5\det T + (\operatorname{tr} T)^2\big)\big\}\,,\end{aligned}$$

$$\begin{aligned}\mathrm{Coeff}[\mathfrak{B}_{11}; x_{1'}x_{2'}] = \tfrac{1}{6}T^1{}_2(x^1,x^2)\big\{&\big(-30(\det T)^2 - \det T(\operatorname{tr} T)^2 + (\operatorname{tr} T)^4\big)\, T^1{}_1(x^1,x^2)\\ &+\det T \operatorname{tr} T\,\big(17\det T - 2(\operatorname{tr} T)^2\big)\big\}\,,\end{aligned}$$

$$\mathrm{Coeff}[\mathfrak{B}_{11}; x_{2'}x_{2'}] = \tfrac{1}{6}T^1{}_2(x^1,x^2)^2\big\{-30(\det T)^2 - \det T(\operatorname{tr} T)^2 + (\operatorname{tr} T)^4\big\}\,,$$

$$\begin{aligned}\mathrm{Coeff}[\mathfrak{B}_{12}; x_{1'}x_{1'}] = \tfrac{1}{3}T^2{}_1(x^1,x^2)\big\{&\big(-30(\det T)^2 - \det T(\operatorname{tr} T)^2 + (\operatorname{tr} T)^4\big)\, T^1{}_1(x^1,x^2)\\ &+\det T \operatorname{tr} T\,\big(17\det T - 2(\operatorname{tr} T)^2\big)\big\}\,,\end{aligned}$$

$$\begin{aligned}\mathrm{Coeff}[\mathfrak{B}_{12}; x_{1'}x_{2'}] = \tfrac{1}{6}\big\{&-2\,\big(30(\det T)^2 \operatorname{tr} T + \det T(\operatorname{tr} T)^3 - (\operatorname{tr} T)^5\big)\, T^1{}_1(x^1,x^2)\\ &+2\,\big(30(\det T)^2 + \det T(\operatorname{tr} T)^2 - (\operatorname{tr} T)^4\big)\, T^1{}_1(x^1,x^2)^2\\ &+\det T\,\big(20(\det T)^2 + 16\det T(\operatorname{tr} T)^2 - 3(\operatorname{tr} T)^4\big)\big\}\,,\end{aligned}$$

$$\begin{aligned}\mathrm{Coeff}[\mathfrak{B}_{12}; x_{2'}x_{2'}] = \tfrac{1}{3}T^1{}_2(x^1,x^2)\big\{&\big(30(\det T)^2 + \det T(\operatorname{tr} T)^2 - (\operatorname{tr} T)^4\big)\, T^1{}_1(x^1,x^2)\\ &-13(\det T)^2 \operatorname{tr} T - 3\det T(\operatorname{tr} T)^3 + (\operatorname{tr} T)^5\big\}\,,\end{aligned}$$

$$\mathrm{Coeff}[\mathfrak{B}_{22}; x_{1'}x_{1'}] = \tfrac{1}{6}T^2{}_1(x^1,x^2)^2\big\{-30(\det T)^2 - \det T(\operatorname{tr} T)^2 + (\operatorname{tr} T)^4\big\}\,,$$

$$\begin{aligned}\mathrm{Coeff}[\mathfrak{B}_{22}; x_{1'}x_{2'}] = \tfrac{1}{6}T^2{}_1(x^1,x^2)\big\{&\big(30(\det T)^2 + \det T(\operatorname{tr} T)^2 - (\operatorname{tr} T)^4\big)\, T^1{}_1(x^1,x^2)\\ &-13(\det T)^2 \operatorname{tr} T - 3\det T(\operatorname{tr} T)^3 + (\operatorname{tr} T)^5\big\}\,,\end{aligned}$$

$$\begin{aligned}\mathrm{Coeff}[\mathfrak{B}_{22}; x_{2'}x_{2'}] = \tfrac{1}{6}\big\{&\big(26(\det T)^2 \operatorname{tr} T + 6\det T(\operatorname{tr} T)^3 - 2(\operatorname{tr} T)^5\big)\, T^1{}_1(x^1,x^2)\\ &+\big(-30(\det T)^2 - \det T(\operatorname{tr} T)^2 + (\operatorname{tr} T)^4\big)\, T^1{}_1(x^1,x^2)^2\\ &-10(\det T)^3 + 2(\det T)^2(\operatorname{tr} T)^2 - 5\det T(\operatorname{tr} T)^4 + (\operatorname{tr} T)^6\big\}\,.\end{aligned}$$

Next we analyze the different possibilities for the eigenvalues of $T(p)$, showing firstly that they cannot be complex. Assume that $T^1{}_1 = T^2{}_2$ and $T^1{}_2 = -T^2{}_1$. Then $\det(T) = (T^1{}_1)^2 +$

$(T^1{}_2)^2$ and $\operatorname{tr}(T) = 2T^1{}_1$, so we get

$$\text{Coeff}[\mathfrak{B}_{11}; x_{1'}x_{1'}] = -\frac{1}{3}(T^1{}_2)^2 \left\{3(T^1{}_1)^4 + 5(T^1{}_2)^4\right\},$$

from where it follows that $T^1{}_2 = 0$.

Next assume that $T(p)$ has two distinct real eigenvalues and set $T^1{}_2 = T^2{}_1 = 0$. Then $\det(T) = T^1{}_1 T^2{}_2$ and $\operatorname{tr}(T) = T^1{}_1 + T^2{}_2$, so we have

$$\begin{aligned}
\text{Coeff}[\mathfrak{B}_{11}; x_{1'}x_{1'}] &= \tfrac{1}{6}(T^1{}_1)^2(T^1{}_1 - T^2{}_2)^2\left((T^1{}_1)^2 + T^1{}_1T^2{}_2 - 5(T^2{}_2)^2\right),\\
\text{Coeff}[\mathfrak{B}_{22}; x_{2'}x_{2'}] &= \tfrac{1}{6}(T^2{}_2)^2(T^1{}_1 - T^2{}_2)^2\left((T^2{}_2)^2 + T^1{}_1T^2{}_2 - 5(T^1{}_1)^2\right),
\end{aligned}$$

and thus

$$\begin{aligned}
&T^1{}_1\left((T^1{}_1)^2 + T^1{}_1T^2{}_2 - 5(T^2{}_2)^2\right) = 0 \quad \text{and}\\
&T^2{}_2\left((T^2{}_2)^2 + T^1{}_1T^2{}_2 - 5(T^1{}_1)^2\right) = 0\,.
\end{aligned}$$

If $T^1{}_1 = 0$, then $T^2{}_2 \neq 0$ and the second identity fails. Similarly, if $T^2{}_2 = 0$, then $T^1{}_1 \neq 0$ and the first identity fails. Thus $T^1{}_1 \neq 0$ and $T^2{}_2 \neq 0$ and we obtain

$$\begin{aligned}
&(T^1{}_1)^2 + T^1{}_1T^2{}_2 - 5(T^2{}_2)^2 = 0 \quad \text{and}\\
&(T^2{}_2)^2 + T^1{}_1T^2{}_2 - 5(T^1{}_1)^2 = 0\,.
\end{aligned}$$

Adding the two identities yields $4(T^1{}_1)^2 - 2T^1{}_1T^2{}_2 + 4(T^2{}_2)^2 = 0$. The only real solution to this is $(T^1{}_1, T^2{}_2) = (0,0)$ which is false since we assumed the eigenvalues to be distinct.

Thus the eigenvalues of $T(p)$ must be real and equal. Since $T(p)$ is not a scalar multiple of the identity, we must have non-trivial Jordan normal form at p. If we choose coordinates so

$$T(p) = T^1{}_1(\partial_{x^1} \otimes dx^1 + \partial_{x^2} \otimes dx^2) + \partial_{x^1} \otimes dx^2,$$

we obtain that $\text{Coeff}[\mathfrak{B}_{11}; x_{2'}x_{2'}] = -3(T^1{}_1)^4$. Thus $T^1{}_1 = 0$ and T is nilpotent. This completes the proof of Assertion (ii).

Now assume that T is nilpotent. By Lemma 5.4, we may choose coordinates so $T = \partial_{x^1} \otimes dx^2$. Examining $\mathfrak{B}_{11}$ yields ${}^D\Gamma_{11}{}^2 = 0$. Examining $\mathfrak{B}_{22}$ yields the remaining relation of Assertion (iii-b). A direct computation shows that if the relations of Assertion (iii-b) are satisfied, then the Riemannian extension is Bach-flat. □

By Theorem 6.1 we may decompose $\Sigma = \mathcal{S}_T \dot\cup \mathcal{O}_T$ as the disjoint union of the set of points where T is a scalar multiple of the identity and the set of points where T is nilpotent and has non-trivial Jordan normal form. In the real analytic setting, if $\mathcal{O}_T$ is non-empty and if Σ is connected, then $\mathcal{O}_T$ is dense in Σ and T is always nilpotent. Next we provide an example in the smooth category where this observation fails.

Example 6.2. Let $\Sigma = \mathbb{R}^2$ and let $\alpha(x^2)$ be a smooth real valued function which vanishes to infinite order at $x^2 = 0$ and which is positive for $x^2 \neq 0$. Impose the conditions of Theorem 6.1–(iii-b) and assume that ${}^D\Gamma_{11}{}^2 = 0$ and $({}^D\Gamma_{11}{}^1)^2 - {}^D\Gamma_{11}{}^1\, {}^D\Gamma_{12}{}^2 + \partial_{x^1}({}^D\Gamma_{11}{}^1 - {}^D\Gamma_{12}{}^2) = 0$.

Let

$$T(x^1,x^2)=\begin{cases}\begin{pmatrix}\alpha(x^2) & 0\\ 0 & \alpha(x^2)\end{pmatrix} & \text{if} \quad x^2\leq 0,\\ \begin{pmatrix}0 & \alpha(x^2)\\ 0 & 0\end{pmatrix} & \text{if} \quad x^2\geq 0\,.\end{cases}$$

One may then compute that $\mathfrak{B}=0$ so this yields a Bach-flat manifold where the Jordan normal form of T changes at $x^2=0$. Furthermore, if we only assume that α is C^k for $k\geq 2$, we still obtain a solution; thus there is no hypo-ellipticity present when considering the solutions to the equations $\mathfrak{B}=0$.

Remark 6.3. We note that the auxiliary tensor Φ plays no role in the analysis. We also note that we can express the conditions of Theorem 6.1-(iii-b) in the form

$${}^D\Gamma_{11}{}^2=0,\quad {}^D\Gamma_{11}{}^1=-\partial_{x^1}\beta,\quad {}^D\Gamma_{12}{}^2={}^D\Gamma_{11}{}^1+c\cdot e^{\beta}$$

for smooth functions $c=c(x^2)$ and $\beta=\beta(x^1,x^2)$.

6.2 Deformation of nilpotent Riemannian extensions

Theorem 6.1 permits us to construct connections so the Riemannian extension is Bach-flat once the nilpotent endomorphism is given. Next, we focus on the reverse problem of constructing nilpotent endomorphisms so the Riemannian extension is Bach-flat once the connection is given; this is, in certain sense, a more natural question.

For sake of completeness we include the following statement of the Cauchy-Kovalevski Theorem, which is an essential argument in proving Theorem 6.7. We refer to [53] for a discussion of Theorem 6.4.

We assume the following boundary-value problem:

$$\begin{aligned}\mathbf{u}_{x^n}=\sum_{j=1}^{n-1}\mathbf{B}_j(\mathbf{u},x')\mathbf{u}_{x^j}+\mathbf{c}(\mathbf{u},x') \qquad &\text{for} \quad |x|<r,\\ \mathbf{u}=0 \qquad &\text{for} \quad |x'|<r, \quad x^n=0,\end{aligned} \tag{6.2}$$

where $\mathbf{B}_j=(b_j^{k\ell})$ with $j=1,\dots,n-1$, $\mathbf{c}=(c^1,\dots,c^m)$ and u_{x^k} denotes partial derivative.

Theorem 6.4 (Cauchy-Kovalevski). *Assume $\{\mathbf{B}_j\}_{j=1}^{n-1}$ and $\mathbf{c}$ are real analytic functions. Then, there exist $r>0$ and a real analytic function*

$$\mathbf{u}=\sum_{\alpha}\mathbf{u}_\alpha x^\alpha$$

solving the boundary-value problem (6.2).

Roughly speaking, one compute all the derivatives at the origin of a possible solution and uses them to construct the formal Taylor's series of an anticipated solution. The proof of the theorem reduces to show that the series converges about the origin. The convergence of the series could be establish, indirectly, by the method of the majorant.

If T is a scalar multiple of the identity, then $(T^*\Sigma, g_{D,\Phi,T})$ is half conformally flat. We focus, therefore, on the case T which is nilpotent henceforth and assume, unless otherwise noted, that $\Sigma = \mathcal{O}_T$. We work locally. Fix $p \in \Sigma$ and a local system of coordinates defined near p. We wish to find $0 \neq T$ nilpotent so that $(T^*\Sigma, g_{D,\Phi,T})$ is Bach-flat. Since either $T^1{}_2(p) \neq 0$ or $T^2{}_1(p) \neq 0$, we assume for the sake of definiteness that $T^1{}_2(p) \neq 0$. This implies that we may expand T near p in the form

$$T = \alpha(x^1, x^2) \begin{pmatrix} \xi(x^1, x^2) & 1 \\ -\xi^2(x^1, x^2) & -\xi(x^1, x^2) \end{pmatrix}. \tag{6.3}$$

For sake of simplicity we introduce the following notation to be used in Definition 6.5 and the proof of Lemma 6.6 and Theorem 6.7. Let $\phi(x^1, x^2)$ be a smooth function. Then $\phi^{(1,0)} = \partial_{x^1}\phi$, $\phi^{(2,0)} = \partial_{x^1}\partial_{x^1}\phi$ and so forth.

Definition 6.5. We introduce the following operators:

$$\begin{aligned}
\mathcal{P}_1(\xi) :=& -\xi^{(1,0)} + \xi\,\xi^{(0,1)} + {}^D\Gamma_{22}{}^1\xi^3 - (2{}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2)\xi^2 \\
&+ ({}^D\Gamma_{11}{}^1 - 2{}^D\Gamma_{12}{}^2)\xi + {}^D\Gamma_{11}{}^2, \\
\mathcal{P}_2(\xi,\alpha) :=& \,\alpha\alpha^{(2,0)} + \xi^2\alpha\alpha^{(0,2)} - 2\xi\alpha\alpha^{(1,1)} + (\alpha^{(1,0)})^2 + \xi^2(\alpha^{(0,1)})^2 - 2\xi\alpha^{(1,0)}\alpha^{(0,1)} \\
&- \alpha\alpha^{(1,0)}\left(2\xi^{(0,1)} - 5{}^D\Gamma_{22}{}^1\xi^2 + 2(4{}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2)\xi - 3{}^D\Gamma_{11}{}^1 + 2{}^D\Gamma_{12}{}^2\right) \\
&+ \alpha\alpha^{(0,1)}\left(2\xi\xi^{(0,1)} - 6{}^D\Gamma_{22}{}^1\xi^3 + (10{}^D\Gamma_{12}{}^1 - 3{}^D\Gamma_{22}{}^2)\xi^2\right. \\
&\left.-4({}^D\Gamma_{11}{}^1 - {}^D\Gamma_{12}{}^2)\xi - {}^D\Gamma_{11}{}^2\right) + 6\xi^4\alpha^2({}^D\Gamma_{22}{}^1)^2 \\
&- 2\xi^3\alpha^2\left(({}^D\Gamma_{22}{}^1)^{(0,1)} + 9{}^D\Gamma_{12}{}^1{}^D\Gamma_{22}{}^1 - 3{}^D\Gamma_{22}{}^1{}^D\Gamma_{22}{}^2\right) \\
&- \xi^2\alpha^2\left(4{}^D\Gamma_{22}{}^1\xi^{(0,1)} - 3({}^D\Gamma_{12}{}^1)^{(0,1)} - 2({}^D\Gamma_{22}{}^1)^{(1,0)} + ({}^D\Gamma_{22}{}^2)^{(0,1)}\right. \\
&\left.- 12({}^D\Gamma_{12}{}^1)^2 - ({}^D\Gamma_{22}{}^2)^2 - 7{}^D\Gamma_{11}{}^1{}^D\Gamma_{22}{}^1 + 7{}^D\Gamma_{12}{}^1{}^D\Gamma_{22}{}^2 + 9{}^D\Gamma_{12}{}^2{}^D\Gamma_{22}{}^1\right) \\
&+ \xi\alpha^2\left(2(3{}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2)\xi^{(0,1)} - ({}^D\Gamma_{11}{}^1)^{(0,1)} - 3({}^D\Gamma_{12}{}^1)^{(1,0)}\right. \\
&+({}^D\Gamma_{12}{}^2)^{(0,1)} + ({}^D\Gamma_{22}{}^2)^{(1,0)} - 2({}^D\Gamma_{11}{}^1 - {}^D\Gamma_{12}{}^2)(4{}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2) \\
&\left.+4{}^D\Gamma_{11}{}^2{}^D\Gamma_{22}{}^1\right) - \alpha^2\left(2({}^D\Gamma_{11}{}^1 - {}^D\Gamma_{12}{}^2)\xi^{(0,1)} - ({}^D\Gamma_{11}{}^1)^{(1,0)}\right. \\
&\left.+({}^D\Gamma_{12}{}^2)^{(1,0)} - ({}^D\Gamma_{11}{}^1)^2 + {}^D\Gamma_{11}{}^1{}^D\Gamma_{12}{}^2 + 3{}^D\Gamma_{11}{}^2{}^D\Gamma_{12}{}^1 - {}^D\Gamma_{11}{}^2{}^D\Gamma_{22}{}^2\right).
\end{aligned}$$

Lemma 6.6. *Let (Σ, D) be an affine surface. Let T have the form given in Equation* (6.3) *and let Φ be arbitrary. The modified Riemannian extension $(T^*\Sigma, g_{D,\Phi,T})$ of Equation* (6.1) *is Bach-flat if and only if α and ξ are solutions to the partial differential equations $\mathcal{P}_1(\xi) = 0$ and $\mathcal{P}_2(\xi,\alpha) = 0$.*

Proof. We suppose T is a nilpotent tensor field of type $(1,1)$. Then $\mathrm{tr}(T) = 0$ and $\det(T) = 0$. If we assume that $T^1{}_2(p) \neq 0$, then T has the form given in Equation (6.3). A direct computation shows

$$\mathfrak{B} = \begin{pmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} & 0 & 0 \\ \mathfrak{B}_{12} & \mathfrak{B}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and thus only $\mathfrak{B}_{11}$, $\mathfrak{B}_{12}$ and $\mathfrak{B}_{22}$ are relevant. We observe that

$$\begin{aligned} \text{Coeff}[\mathfrak{B}_{11}, \alpha^{(2,0)}] &= -4\alpha\xi^2, \\ \text{Coeff}[\mathfrak{B}_{12}, \alpha^{(2,0)}] &= -4\alpha\xi, \\ \text{Coeff}[\mathfrak{B}_{22}, \alpha^{(2,0)}] &= -4\alpha\,. \end{aligned}$$

We therefore define $\mathfrak{Q}_1 := \mathfrak{B}_{11} - \mathfrak{B}_{12}\xi$, $\mathfrak{Q}_2 := \mathfrak{B}_{11} - \mathfrak{B}_{22}\xi^2$ and $\mathfrak{Q}_3 := 2\mathfrak{Q}_1 - \mathfrak{Q}_2$. We may then express $\mathfrak{Q}_3 = -4\alpha^2(\mathcal{P}_1)^2$ and thus the vanishing of $\mathfrak{Q}_3$ is equivalent to the vanishing of $\mathcal{P}_1$. We set $\mathcal{P}_1 = 0$ and express $\xi^{(1,0)} = F_{(1,0)}(\xi, {}^D\Gamma, \xi^{(0,1)})$. Differentiating this relation permits us to express $\xi^{(1,1)} = F_{(1,1)}(\xi, {}^D\Gamma, d^D\Gamma, \xi^{(0,1)}, \xi^{(0,2)})$ and $\xi^{(2,0)} = F_{(2,0)}(\xi, {}^D\Gamma, d^D\Gamma, \xi^{(0,1)}, \xi^{(0,2)})$. Substituting these relations then yields $\mathfrak{Q}_1 = 0$ and $\mathfrak{Q}_2 = 0$. Thus only $\mathfrak{B}_{11}$ plays a role. Substituting these relations permits us to express $\mathfrak{B}_{11} = -4\xi^2\mathcal{P}_2$. The desired result now follows. □

Theorem 6.7. *Let (Σ, D) be a real analytic affine surface. Then there exist locally defined nilpotent $(1,1)$-tensor fields T such that the modified Riemannian extension $(T^*\Sigma, g_{D,\Phi,T})$ is Bach-flat.*

Proof. Suppose (Σ, D) is real analytic. The operator $\mathcal{P}_1(\xi)$ of Definition 6.5 takes the form:

$$\mathcal{P}_1(\xi) = -\xi^{(1,0)} + \xi\xi^{(0,1)} + f(\xi, {}^D\Gamma)\,.$$

Given a real analytic function $\xi_0(x^2)$, the Cauchy-Kovalevski Theorem shows that there is a unique real solution to the equation $\mathcal{P}_1(\xi) = 0$ with $\xi(0, x^2) = \xi_0(x^2)$. Once ξ is determined, the operator $\mathcal{P}_2(\xi, \alpha)$ of Definition 6.5 takes the form

$$\mathcal{P}_2(\xi, \alpha) = \alpha\alpha^{(2,0)} - 2\xi\alpha\alpha^{(1,1)} + \xi^2\alpha\alpha^{(0,2)} + F(\alpha, d\alpha; {}^D\Gamma, d^D\Gamma; \xi, d\xi)\,.$$

Given real analytic functions $\alpha_0(x^2)$ and $\alpha_1(x^2)$, there exists a unique local solution to the equation $\mathcal{P}_2(\xi, \alpha) = 0$ with $\alpha(0, x^2) = \alpha_0(x^2)$ and $\alpha^{(1,0)}(0, x^2) = \alpha_1(x^2)$. Thus given D, there are many nilpotent T so that $(T^*\Sigma, g_{D,\Phi,T})$ is Bach-flat in this setting; the auxiliary tensor Φ plays no role in the analysis. □

6.3 Invariants of nilpotent Riemannian extensions

6.3.1 VSI manifolds

A pseudo-Riemannian manifold is said to be VSI (vanishing scalar curvature invariants) if all the scalar Weyl invariants (i.e., invariants formed by a complete contraction of indices in the Riemann curvature tensor $R_{ijk\ell}$ and its covariant derivatives) vanish.

Theorem 6.8. *Let $(T^*\Sigma, g_{D,\Phi,T})$ with $T \neq 0$. The following assertions are equivalent:*

(i) $(T^*\Sigma, g_{D,\Phi,T})$ *is VSI.*

(ii) $\|R\|^2 = \|\rho\|^2 = 0$.

(iii) $\|\rho\|^2 = \tau = 0$.

(iv) T *is nilpotent.*

We will show the different implications in Theorem 6.8 in the rest of this subsection. Clearly if $(T^*\Sigma, g_{D,\Phi,T})$ is VSI, then $\|R\|^2 = \|\rho\|^2 = \tau = 0$. Thus Assertion (i) of Theorem 6.8 implies Assertions (ii) and (iii).

Assertions (ii) or (iii) imply Assertion (iv) in Theorem 6.8

A direct computation shows that τ is a quadratic polynomial in the components of T and that $\|R\|^2$ and $\|\rho\|^2$ are fourth order polynomials in the components of T; the other variables do not enter. Moreover

$$\begin{aligned}
\tau &= -2(\det T - (\operatorname{tr} T)^2), \\
\|\rho\|^2 &= (\det T)^2 - 6 \det T (\operatorname{tr} T)^2 + 2(\operatorname{tr} T)^4, \\
\|R\|^2 &= 4(3(\det T)^2 - 4 \det T (\operatorname{tr} T)^2 + (\operatorname{tr} T)^4) .
\end{aligned}$$

Hence $\|\rho\|^2 = \tau = 0$ or $\|\rho\|^2 = \|R\|^2 = 0$ if and only if $\det(T) = \operatorname{tr}(T) = 0$.

Example 6.9 (The vanishing of just one invariant)**.** The next example shows that the conditions $\|R\|^2 = \tau = 0$ do not suffice to get that T is nilpotent nor does the condition $\|\rho\|^2 = 0$ suffice to get that T is nilpotent. Indeed, $\|R\|^2 = \tau = 0$ if and only if $\det(T) = (\operatorname{tr}(T))^2$.

Let $r(x^1, x^2) > 0$ be an arbitrary smooth function and let θ be constant. Set

$$T = r(x^1, x^2) \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} .$$

This example is not nilpotent and we have

$$\begin{aligned}
\tau &= 2r(x^1, x^2)^2 \{1 + 2\cos(2\theta)\}, \\
\|\rho\|^2 &= r(x^1, x^2)^4 \{1 + 4\cos(2\theta) + 4\cos(4\theta)\}, \\
\|R\|^2 &= 4r(x^1, x^2)^4 \{1 + 2\cos(4\theta)\} .
\end{aligned}$$

Choosing $\theta = \frac{\pi}{3}$ one has that $\tau = \|R\|^2 = 0$ but $\|\rho\|^2 = -3r(x^1, x^2)^4 \neq 0$. Moreover, setting $\theta = \frac{1}{2} \arctan\left(\frac{1+\sqrt{7}}{1-\sqrt{7}}\right)$ one has $\|\rho\|^2 = 0$ but $\tau = (1 + \sqrt{7}) r(x^1, x^2)^2$ and $\|R\|^2 = 2(2 - \sqrt{7}) r(x^1, x^2)^4$.

Assertion (iv) implies Assertion (i) in Theorem 6.8

In the final step we will show that T is nilpotent implies that $(T^*\Sigma, g_{D,\Phi,T})$ is VSI. Although this fact already follows from the results in [43, 65], we include a direct proof for sake of completeness. Before establishing this implication, we must derive an additional technical result.

Assume that T is nilpotent. By Lemma 5.4, we may choose coordinates so $T = \partial_{x^1} \otimes dx^2$. Let $g = g_{D,\Phi,T}$. Then$\{g^{ij}, \Gamma_{ij}{}^k, R_{abcd;e_1\dots e_k}\}$ are polynomial expressions in the fiber coordinates $x_{1'}$ and $x_{2'}$ whose coefficients depend on the variables $\{{}^D\Gamma_{ij}{}^k, \Phi_{ij}\}$ and their derivatives with respect to x^1 and x^2. In such a coordinate system, one computes that the possibly non-zero components of the tensor g^{ij}, of the Christoffel symbols, and of the curvature R are, up to the usual $\mathbb{Z}_2$ symmetries,

$$\begin{array}{llllllll}
g^{11'}, & g^{22'}, & g^{1'1'}, & g^{1'2'}, & g^{2'2'}, & \Gamma_{11}{}^1, & \Gamma_{11}{}^2, & \Gamma_{11}{}^{1'}, \\
\Gamma_{11}{}^{2'}, & \Gamma_{12}{}^1, & \Gamma_{12}{}^2, & \Gamma_{12}{}^{1'}, & \Gamma_{12}{}^{2'}, & \Gamma_{11'}{}^{1'}, & \Gamma_{11'}{}^{2'}, & \Gamma_{12'}{}^{1'}, \\
\Gamma_{12'}{}^{2'}, & \Gamma_{22}{}^1, & \Gamma_{22}{}^2, & \Gamma_{22}{}^{1'}, & \Gamma_{22}{}^{2'}, & \Gamma_{21'}{}^{1'}, & \Gamma_{21'}{}^{2'}, & \Gamma_{22'}{}^{1'}, \\
\Gamma_{22'}{}^{2'}, & R_{1212}, & R_{1211'}, & R_{1212'}, & R_{1221'}, & R_{1222'}, & R_{21'21'}\,. &
\end{array} \tag{6.4}$$

Of particular interest is the fact that $R_{21'21'} = -1$. Let $\mathfrak{o}(\cdot)$ be the maximal order of an expression in the dual variables $\{x_{1'}, x_{2'}\}$. Thus if $\mathfrak{o}(\cdot) = 0$, these variables do not occur, if $\mathfrak{o}(\cdot) = 1$, the expression is linear in the variables $\{x_{1'}, x_{2'}\}$, and so forth. In other words, we define $\mathfrak{o}(x_{1'}) = \mathfrak{o}(x_{2'}) = 1$ and extend $\mathfrak{o}$ to a derivation. If $\mathfrak{o}(R_{ijk\ell}) = 2$, then $R_{ijk\ell}$ is at most quadratic in $\{x_{1'}, x_{2'}\}$; if $\mathfrak{o}(R_{ijk\ell}) = 1$, then $R_{ijk\ell}$ is at most linear in $\{x_{1'}, x_{2'}\}$; and if $\mathfrak{o}(R_{ijk\ell}) = 0$, then $R_{ijk\ell}$ does not involve $\{x_{1'}, x_{2'}\}$. We have:

$$\begin{array}{llll}
\mathfrak{o}(\Gamma_{11}{}^1) = 0, & \mathfrak{o}(\Gamma_{11}{}^2) = 0, & \mathfrak{o}(\Gamma_{11}{}^{1'}) = 1, & \mathfrak{o}(\Gamma_{11}{}^{2'}) = 2, \\
\mathfrak{o}(\Gamma_{12}{}^1) = 0, & \mathfrak{o}(\Gamma_{12}{}^2) = 0, & \mathfrak{o}(\Gamma_{12}{}^{1'}) = 1, & \mathfrak{o}(\Gamma_{12}{}^{2'}) = 2, \\
\mathfrak{o}(\Gamma_{11'}{}^{1'}) = 0, & \mathfrak{o}(\Gamma_{11'}{}^{2'}) = 0, & \mathfrak{o}(\Gamma_{12'}{}^{1'}) = 0, & \mathfrak{o}(\Gamma_{12'}{}^{2'}) = 0, \\
\mathfrak{o}(\Gamma_{22}{}^1) = 1, & \mathfrak{o}(\Gamma_{22}{}^2) = 0, & \mathfrak{o}(\Gamma_{22}{}^{1'}) = 2, & \mathfrak{o}(\Gamma_{22}{}^{2'}) = 2, \\
\mathfrak{o}(\Gamma_{21'}{}^{1'}) = 0, & \mathfrak{o}(\Gamma_{21'}{}^{2'}) = 1, & \mathfrak{o}(\Gamma_{22'}{}^{1'}) = 0, & \mathfrak{o}(\Gamma_{22'}{}^{2'}) = 0,
\end{array}$$

$$\begin{array}{llll}
\mathfrak{o}(R_{1212}) = 2, & \mathfrak{o}(R_{1211'}) = 1, & \mathfrak{o}(R_{1212'}) = 0, & \mathfrak{o}(R_{1221'}) = 1, \\
\mathfrak{o}(R_{1222'}) = 1, & \mathfrak{o}(R_{21'21'}) = 0\,. & &
\end{array}$$

We define the *defect* by setting

$$\mathfrak{d}(\Gamma_{ij}{}^k) := -\sum_{n=1}^{2}\{\delta_{i,n} + \delta_{j,n} - \delta_{k,n}\} + \sum_{n=1}^{2}\{\delta_{i,n'} + \delta_{j,n'} - \delta_{k,n'}\},$$

$$\mathfrak{d}(R_{i_1 i_2 i_3 i_4; i_5 \dots i_\nu}) := \sum_{n=1}^{\nu}\{\delta_{i_n,1'} + \delta_{i_n,2'} - \delta_{i_n,1} - \delta_{i_n,2}\}\,.$$

In brief, we count, with multiplicity, each lower index '1' or '2' with a -1 and '1′' or '2′' with a $+1$ and reverse the sign for upper indices. This will play an important role in contracting indices subsequently. We then set $\mathfrak{r} = \mathfrak{o} + \mathfrak{d}$ and compute:

$$\begin{array}{llll}
\mathfrak{r}(\Gamma_{11}{}^1) = -1, & \mathfrak{r}(\Gamma_{11}{}^2) = -1, & \mathfrak{r}(\Gamma_{11}{}^{1'}) = -2, & \mathfrak{r}(\Gamma_{11}{}^{2'}) = -1, \\
\mathfrak{r}(\Gamma_{12}{}^1) = -1, & \mathfrak{r}(\Gamma_{12}{}^2) = -1, & \mathfrak{r}(\Gamma_{12}{}^{1'}) = -2, & \mathfrak{r}(\Gamma_{12}{}^{2'}) = -1, \\
\mathfrak{r}(\Gamma_{11'}{}^{1'}) = -1, & \mathfrak{r}(\Gamma_{11'}{}^{2'}) = -1, & \mathfrak{r}(\Gamma_{12'}{}^{1'}) = -1, & \mathfrak{r}(\Gamma_{12'}{}^{2'}) = -1, \\
\mathfrak{r}(\Gamma_{22}{}^1) = 0, & \mathfrak{r}(\Gamma_{22}{}^2) = -1, & \mathfrak{r}(\Gamma_{22}{}^{1'}) = -1, & \mathfrak{r}(\Gamma_{22}{}^{2'}) = -1, \\
\mathfrak{r}(\Gamma_{21'}{}^{1'}) = -1, & \mathfrak{r}(\Gamma_{21'}{}^{2'}) = 0, & \mathfrak{r}(\Gamma_{22'}{}^{1'}) = -1, & \mathfrak{r}(\Gamma_{22'}{}^{2'}) = -1, \\
\mathfrak{r}(R_{1212}) = -2, & \mathfrak{r}(R_{1211'}) = -1, & \mathfrak{r}(R_{1212'}) = -2, & \mathfrak{r}(R_{1221'}) = -1, \\
\mathfrak{r}(R_{1222'}) = -1, & \mathfrak{r}(R_{21'21'}) = 0\,. & &
\end{array} \tag{6.5}$$

Lemma 6.10. *Suppose that* $R_{i_1i_2i_3i_4;j_1\ldots j_\nu} \neq 0$*. Then* $\mathfrak{r}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) \leq 0$*. Furthermore,* $\mathfrak{r}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) = 0$ *if and only if* $\nu = 0$ *and* $R_{i_1i_2i_3i_4} = \pm R_{21'21'}$*.*

Proof. As a matter of notation, throughout the proof of this lemma ∂_j denotes ∂_{x^j} if $j \in \{1,2\}$ and ∂_{x_j} if $j \in \{1',2'\}$. Let $R_{i_1i_2i_3i_4} \neq 0$. By Equation (6.5), $\mathfrak{r}(R_{i_1i_2i_3i_4}) \leq 0$ with equality if and only if $R_{i_1i_2i_3i_4} = \pm R_{21'21'}$. This establishes the result if $\nu = 0$. Next we suppose $\nu = 1$ and examine ∇R. We expand

$$\begin{aligned}
R_{i_1i_2i_3i_4;j} &= \partial_j R_{i_1i_2i_3i_4} - \sum_a \Gamma_{ji_1}{}^a\, R_{ai_2i_3i_4} - \sum_a \Gamma_{ji_2}{}^a\, R_{i_1ai_3i_4} \\
&\quad - \sum_a \Gamma_{ji_3}{}^a\, R_{i_1i_2ai_4} - \sum_a \Gamma_{ji_4}{}^a\, R_{i_1i_2i_3a}\,.
\end{aligned}$$

We examine different cases separately, depending on the kind of addend which gives the order in the above expression for $R_{i_1i_2i_3i_4;j}$.

Case 1. $\mathfrak{o}(R_{i_1i_2i_3i_4;j}) = \mathfrak{o}(\Gamma_{ji_1}{}^a\, R_{ai_2i_3i_4})$. (For any other addend of this type the argument is similar).

In this case, since $\mathfrak{d}(R_{i_1i_2i_3i_4;j}) = \mathfrak{d}(\Gamma_{ji_1}{}^a\, R_{ai_2i_3i_4})$, we have

$$\mathfrak{r}(R_{i_1i_2i_3i_4;j}) = \mathfrak{r}(\Gamma_{ji_1}{}^a\, R_{ai_2i_3i_4}) = \mathfrak{r}(\Gamma_{ji_1}{}^a) + \mathfrak{r}(R_{ai_2i_3i_4})\,.$$

Suppose $r(R_{i_1i_2i_3i_4;j}) \geq 0$. Since Equation (6.5) implies that $\mathfrak{r}(\Gamma_{ji_1}{}^a) \leq 0$ and $\mathfrak{r}(R_{ai_2i_3i_4}) \leq 0$, we conclude that

$$\mathfrak{r}(\Gamma_{ji_1}{}^a) = \mathfrak{r}(R_{ai_2i_3i_4}) = 0,$$

which is a contradiction since, again by Equation (6.5), $\mathfrak{r}(\Gamma_{ji_1}{}^a) = 0$ implies that $a \in \{1,2'\}$, while $\mathfrak{r}(R_{ai_2i_3i_4}) = 0$ implies that $a \in \{2,1'\}$. So, we conclude that necessarily $\mathfrak{r}(R_{i_1i_2i_3i_4;j}) < 0$.

Case 2. $\mathfrak{o}(R_{i_1i_2i_3i_4;j}) = \mathfrak{o}(\partial_j R_{i_1i_2i_3i_4})$, with $j \in \{1,2\}$.

Note that $\mathfrak{o}(\partial_j R_{i_1i_2i_3i_4}) \leq \mathfrak{o}(R_{i_1i_2i_3i_4})$ and $\mathfrak{d}(R_{i_1i_2i_3i_4;j}) = \mathfrak{d}(R_{i_1i_2i_3i_4}) - 1$. Thus,

$$\begin{aligned}\mathfrak{r}(R_{i_1i_2i_3i_4;j}) &= \mathfrak{o}(\partial_j R_{i_1i_2i_3i_4}) + \mathfrak{d}(R_{i_1i_2i_3i_4}) - 1 \\ &\leq \mathfrak{o}(R_{i_1i_2i_3i_4}) + \mathfrak{d}(R_{i_1i_2i_3i_4}) - 1 = \mathfrak{r}(R_{i_1i_2i_3i_4}) - 1\,.\end{aligned}$$

Now, since $\mathfrak{r}(R_{i_1i_2i_3i_4}) \leq 0$, we conclude that $\mathfrak{r}(R_{i_1i_2i_3i_4;j}) < 0$.

Case 3. $\mathfrak{o}(R_{i_1i_2i_3i_4;j}) = \mathfrak{o}(\partial_j R_{i_1i_2i_3i_4})$, with $j \in \{1',2'\}$.

In this case, a key observation is that $\mathfrak{o}(\partial_j R_{i_1i_2i_3i_4}) < \mathfrak{o}(R_{i_1i_2i_3i_4})$. Indeed, analyzing the components with $\partial_j R_{i_1i_2i_3i_4} \neq 0$ we distinguish two cases: the components $R_{1211'}$, $R_{1221'}$ and $R_{1222'}$ are of the form $x_{1'}F_1(x^1,x^2) + F_2(x^1,x^2)$ and hence $\mathfrak{o}(\partial_j R_{i_1i_2i_3i_4}) = 0 < 1 = \mathfrak{o}(R_{i_1i_2i_3i_4})$. On the other hand, R_{1212} is of the form $(x_{1'})^2F_1(x^1,x^2) + x_{1'}F_2(x^1,x^2) + x_{2'}F_3(x^1,x^2) + x_{1'}x_{2'}F_4(x^1,x^2) + F_5(x^1,x^2)$ and therefore $\mathfrak{o}(\partial_j R_{1212}) = 1 < 2 = \mathfrak{o}(R_{1212})$.

Hence $\mathfrak{o}(\partial_j R_{i_1i_2i_3i_4}) < \mathfrak{o}(R_{i_1i_2i_3i_4})$ and since $\mathfrak{d}(R_{i_1i_2i_3i_4;j}) = \mathfrak{d}(R_{i_1i_2i_3i_4}) + 1$ we have:

$$\begin{aligned}\mathfrak{r}(R_{i_1i_2i_3i_4;j}) &= \mathfrak{o}(\partial_j R_{i_1i_2i_3i_4}) + \mathfrak{d}(R_{i_1i_2i_3i_4}) + 1 \\ &< \mathfrak{o}(R_{i_1i_2i_3i_4}) + \mathfrak{d}(R_{i_1i_2i_3i_4}) + 1 = \mathfrak{r}(R_{i_1i_2i_3i_4}) + 1\,.\end{aligned}$$

Now, note that $R_{i_1i_2i_3i_4} \neq \pm R_{21'21'}$ in this case and therefore $\mathfrak{r}(R_{i_1i_2i_3i_4}) < 0$, so we conclude that $\mathfrak{r}(R_{i_1i_2i_3i_4;j}) < 0$.

In the second part of the proof, for $\nu \geq 2$, we proceed by induction in two conditions. In particular, we suppose that $\mathfrak{o}(\partial_{j_\nu} R_{i_1i_2i_3i_4;j_1\dots j_{\nu-1}}) < \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\dots j_{\nu-1}})$ whenever $j_\nu \in \{1',2'\}$ and $\partial_{j_\nu} R_{i_1i_2i_3i_4;j_1\dots j_{\nu-1}} \neq 0$, and we also suppose that if $R_{i_1i_2i_3i_4;j_1\dots j_\nu} \neq 0$ then $\mathfrak{r}(R_{i_1i_2i_3i_4;j_1\dots j_\nu}) < 0$. Next we show that both conditions hold for $\nu+1$. We expand

$$\begin{aligned}R_{i_1i_2i_3i_4;j_1\dots j_{\nu+1}} &= \partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\dots j_\nu} - \sum_a \Gamma_{j_{\nu+1}i_1}{}^a\, R_{ai_2i_3i_4;j_1\dots j_\nu} \\ &\quad - \sum_a \Gamma_{j_{\nu+1}i_2}{}^a\, R_{i_1ai_3i_4;j_1\dots j_\nu} - \sum_a \Gamma_{j_{\nu+1}i_3}{}^a\, R_{i_1i_2ai_4;j_1\dots j_\nu} \\ &\quad - \sum_a \Gamma_{j_{\nu+1}i_4}{}^a\, R_{i_1i_2i_3a;j_1\dots j_\nu}\,.\end{aligned}$$

As in the case $\nu = 1$ we analyze separately the different cases depending on the kind of addend which gives the order in the expression for $R_{i_1i_2i_3i_4;j_1\dots j_{\nu+1}}$.

Case 1. $\mathfrak{o}(R_{i_1i_2i_3i_4;j_1\dots j_{\nu+1}}) = \mathfrak{o}(\Gamma_{j_{\nu+1}i_1}{}^a\, R_{ai_2i_3i_4;j_1\dots j_\nu})$. (For any other addend of this type the argument is similar).

In this case, $\mathfrak{d}(R_{i_1i_2i_3i_4;j_1\dots j_{\nu+1}}) = \mathfrak{d}(\Gamma_{j_{\nu+1}i_1}{}^a\, R_{ai_2i_3i_4;j_1\dots j_\nu})$, so we have

$$\mathfrak{r}(R_{i_1i_2i_3i_4;j_1\dots j_{\nu+1}}) = \mathfrak{r}(\Gamma_{j_{\nu+1}i_1}{}^a\, R_{ai_2i_3i_4;j_1\dots j_\nu}) = \mathfrak{r}(\Gamma_{j_{\nu+1}i_1}{}^a) + \mathfrak{r}(R_{ai_2i_3i_4;j_1\dots j_\nu})\,.$$

Since $\mathfrak{r}(\Gamma_{j_{\nu+1}i_1}{}^a) \leq 0$ by Equation (6.5) and we are assuming $\mathfrak{r}(R_{ai_2i_3i_4;j_1\dots j_\nu}) < 0$, we conclude that $\mathfrak{r}(R_{i_1i_2i_3i_4;j_1\dots j_{\nu+1}}) < 0$.

Case 2. $\mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) = \mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu})$, with $j_{\nu+1} \in \{1,2\}$.

Using $\mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) \leq \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu})$ and that, in this case, $\mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) = \mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) - 1$, we have

$$\begin{aligned}\mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) &= \mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + \mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) - 1 \\ &\leq \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + \mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) - 1 \\ &= \mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) - 1 .\end{aligned}$$

Since we are assuming $\mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) < 0$, we get $\mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) < 0$.

Case 3. $\mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) = \mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu})$, with $j_{\nu+1} \in \{1',2'\}$.

Suppose that $\mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) < \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu})$. Note that, $\mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) = \mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + 1$, then

$$\begin{aligned}\mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) &= \mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + \mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + 1 \\ &< \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + \mathfrak{d}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + 1 \\ &= \mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) + 1 .\end{aligned}$$

Since we are assuming $\mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) < 0$, we conclude $\mathfrak{x}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu+1}}) < 0$.

We finish the proof showing that if $\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu} \neq 0$ then

$$\mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu}) < \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_\nu}),$$

where $j_{\nu+1} \in \{1',2'\}$. We analyze the three different kind of addends in the expression of $\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_\nu}$ showing that the order of each addend is always smaller than the order of the addend from which it derives. This, in particular, implies the above inequality.

• For $\partial_{j_{\nu+1}}\partial_{j_\nu} R_{i_1i_2i_3i_4;j_1\ldots j_{\nu-1}}$, since we are assuming

$$\mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_{\nu-1}}) < \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu-1}}),$$

we have

$$\begin{aligned}\mathfrak{o}(\partial_{j_{\nu+1}}\partial_{j_\nu} R_{i_1i_2i_3i_4;j_1\ldots j_{\nu-1}}) &= \mathfrak{o}(\partial_{j_\nu}\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_{\nu-1}}) \\ &\leq \mathfrak{o}(\partial_{j_{\nu+1}} R_{i_1i_2i_3i_4;j_1\ldots j_{\nu-1}}) \\ &< \mathfrak{o}(R_{i_1i_2i_3i_4;j_1\ldots j_{\nu-1}}) .\end{aligned}$$

• For $\partial_{j_{\nu+1}} ({\Gamma_{j_\nu i_1}}^a) R_{ai_2i_3i_4;j_1\ldots j_{\nu-1}}$, a straightforward calculation show that $\mathfrak{o}(\partial_\ell {\Gamma_{ij}}^k)) < \mathfrak{o}({\Gamma_{ij}}^k)$ whenever $\mathfrak{o}({\Gamma_{ij}}^k) > 0$ and $\ell \in \{1',2'\}$, so we get

$$\begin{aligned}\mathfrak{o}\left(\partial_{j_{\nu+1}} ({\Gamma_{j_\nu i_1}}^a) R_{ai_2i_3i_4;j_1\ldots j_{\nu-1}}\right) &= \mathfrak{o}(\partial_{j_{\nu+1}} {\Gamma_{j_\nu i_1}}^a) + \mathfrak{o}(R_{ai_2i_3i_4;j_1\ldots j_{\nu-1}}) \\ &< \mathfrak{o}({\Gamma_{j_\nu i_1}}^a) + \mathfrak{o}(R_{ai_2i_3i_4;j_1\ldots j_{\nu-1}}) \\ &= \mathfrak{o}({\Gamma_{j_\nu i_1}}^a\, R_{ai_2i_3i_4;j_1\ldots j_{\nu-1}}) .\end{aligned}$$

• Finally, for an addend ot the type $\Gamma_{j_\nu i_1}{}^{a}\,\partial_{j_{\nu+1}}\left(R_{ai_2i_3i_4;j_1\dots j_{\nu-1}}\right)$, and since we are assuming $\mathfrak{o}\left(\partial_{j_{\nu+1}}R_{ai_2i_3i_4;j_1\dots j_{\nu-1}}\right) < \mathfrak{o}(R_{ai_2i_3i_4;j_1\dots j_{\nu-1}})$ we have

$$\begin{aligned}\mathfrak{o}\left(\Gamma_{j_\nu i_1}{}^{a}\,\partial_{j_{\nu+1}}\left(R_{ai_2i_3i_4;j_1\dots j_{\nu-1}}\right)\right) &= \mathfrak{o}(\Gamma_{j_\nu i_1}{}^{a}) + \mathfrak{o}(\partial_{j_{\nu+1}}R_{ai_2i_3i_4;j_1\dots j_{\nu-1}})\\ &< \mathfrak{o}(\Gamma_{j_\nu i_1}{}^{a}) + \mathfrak{o}(R_{ai_2i_3i_4;j_1\dots j_{\nu-1}})\\ &= \mathfrak{o}(\Gamma_{j_\nu i_1}{}^{a}\,R_{ai_2i_3i_4;j_1\dots j_{\nu-1}})\,.\end{aligned}$$

□

Now we are ready to show that Assertion (iv) implies Assertion (i) in Theorem 6.8. Suppose T is nilpotent. Let $\mathcal{W}$ be a Weyl scalar invariant formed from the curvature tensor and its covariant derivatives. By Equation (6.4), we can contract an index '1' against an index '1′' and an index '2' against an index '2′'. We can also contract indices $\{1', 2'\}$ against $\{1', 2'\}$. Consequently, if $A = R_{i_1i_2i_3i_4;j_1\dots j_\nu}\dots$ is a monomial, then

$$\deg_1(A) \le \deg_{1'}(A) \quad \text{and} \quad \deg_2(A) \le \deg_{2'}(A),$$

where $\deg_\ell(A)$ denotes the number of times that the index 'ℓ' appears in the monomial A. The inequality can, of course, be strict as we can also contract an index '1′' or '2′' against an index '1′' or '2′'. This implies that $\mathfrak{d}(A) \ge 0$. Since $\mathfrak{o}(A) \ge 0$, this implies $\mathfrak{x}(A) \ge 0$. By Lemma 6.10, $\mathfrak{x}(A) \le 0$. Thus we have $\mathfrak{x}(A) = 0$. This implies A is a power of $R_{21'21'}$. Since we cannot contract an index '2' against an index '1′', we see that $\mathcal{W} = 0$. This shows Assertion (iv) implies Assertion (i) in Theorem 6.8.

6.3.2 Invariants which are not of Weyl Type

Let (Σ, D) be an affine surface and let $(T^*\Sigma, g_{D,\Phi,T})$ be the associated Riemannian extension where T is nilpotent. We begin by decomposing the curvature and the Ricci tensor of the Riemannian extension $(T^*\Sigma, g_{D,\Phi,T})$. Choose coordinates so $T = \partial_{x^1} \otimes dx^2$. Let $\{R, \rho\}$ be the curvature operator and Ricci tensor of $(T^*\Sigma, g_{D,\Phi,T})$ and let $\{R^D, \rho^D, \rho^D_s, \rho^D_{sk}\}$ be the curvature operator, Ricci tensor, and the symmetric and skew-symmetric Ricci tensors of (Σ, D). Let $\mathfrak{V} := \operatorname{span}\{\partial_{x_{1'}}, \partial_{x_{2'}}\}$ be the "vertical" space and let $\mathfrak{H} := \operatorname{span}\{\partial_{x^1}, \partial_{x^2}\}$ be the "horizontal" space. These are, of course, not invariantly defined. We may then decompose

$$R(X,Y) = \begin{pmatrix} R^{\mathfrak{H}}_{\mathfrak{H}} = \begin{pmatrix} R_{XY1}{}^{1} & R_{XY2}{}^{1} \\ R_{XY1}{}^{2} & R_{XY2}{}^{2}\end{pmatrix} & R^{\mathfrak{H}}_{\mathfrak{V}} = \begin{pmatrix} R_{XY1'}{}^{1} & R_{XY2'}{}^{1} \\ R_{XY1'}{}^{2} & R_{XY2'}{}^{2}\end{pmatrix} \\ R^{\mathfrak{V}}_{\mathfrak{H}} = \begin{pmatrix} R_{XY1}{}^{1'} & R_{XY2}{}^{1'} \\ R_{XY1}{}^{2'} & R_{XY2}{}^{2'}\end{pmatrix} & R^{\mathfrak{V}}_{\mathfrak{V}} = \begin{pmatrix} R_{XY1'}{}^{1'} & R_{XY2'}{}^{1'} \\ R_{XY1'}{}^{2'} & R_{XY2'}{}^{2'}\end{pmatrix}\end{pmatrix}.$$

The next result follows by a direct computation.

Lemma 6.11. *Take $(T^*\Sigma, g_{D,\Phi,T})$ where $T = \partial_{x^1} \otimes dx^2$, as above.*

(i) $R^{\mathfrak{H}}_{\mathfrak{V}}(X,Y)=0$ *for all* X,Y *vector fields on* $T^*\Sigma$*, i.e.,*

$$R_{abi'}{}^j=0 \quad \textit{for} \quad 1\le i\le 2, 1\le j\le 2\,.$$

(ii) $\{R^{\mathfrak{H}}_{\mathfrak{H}}+(R^{\mathfrak{V}}_{\mathfrak{V}})^t\}(X,Y)=0$ *for all* X,Y *vector fields on* $T^*\Sigma$*, i.e.,*
$R_{ab1}{}^1+R_{ab1'}{}^{1'}=0$, $R_{ab2}{}^2+R_{ab2'}{}^{2'}=0$,
$R_{ab1}{}^2+R_{ab2'}{}^{1'}=0$, $R_{ab2}{}^1+R_{ab1'}{}^{2'}=0\,.$

$R^{\mathfrak{H}}_{\mathfrak{H}}(\partial_{x^i},\partial_{x^j})=0$ *for* $i<j$ *and* $(i,j)\notin\{(1,2),(2,1')\}\,.$

$$\begin{pmatrix} R_{21'1}{}^1 & R_{21'2}{}^1 \\ R_{21'1}{}^2 & R_{21'2}{}^2\end{pmatrix}=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}.$$

$$\begin{pmatrix} R_{121}{}^1 & R_{122}{}^1 \\ R_{121}{}^2 & R_{122}{}^2\end{pmatrix}=\begin{pmatrix} R^D_{121}{}^1 & R^D_{122}{}^1 \\ R^D_{121}{}^2 & R^D_{122}{}^2\end{pmatrix} + x_{1'}\begin{pmatrix} -{}^D\Gamma_{11}{}^2 & {}^D\Gamma_{11}{}^1-{}^D\Gamma_{12}{}^2 \\ 0 & {}^D\Gamma_{11}{}^2\end{pmatrix}.$$

$\operatorname{tr}(R^{\mathfrak{H}}_{\mathfrak{H}}(X,Y))=2(\pi^*\rho^D_{sk})(X,Y)$ *for all* X,Y *vector fields on* $T^*\Sigma\,.$

(iii) $\rho_{i'j}=\rho_{i'j'}=0$ *for all* $i,j=1,2\,.$

$$\begin{pmatrix}\rho_{11} & \rho_{21}\\ \rho_{12} & \rho_{22}\end{pmatrix}=2\rho^D_s + \begin{pmatrix} 0 & 2x_{1'}{}^D\Gamma_{11}{}^2 \\ 2x_{1'}{}^D\Gamma_{11}{}^2 & 2x_{1'}({}^D\Gamma_{12}{}^2-2{}^D\Gamma_{11}{}^1)-2x_{2'}{}^D\Gamma_{11}{}^2+\Phi_{11}\end{pmatrix}.$$

(iv) $\nabla R(i,j,1,1;k)+\nabla R(i,j,2,2;k)=0$ *unless* $\{i,j,k\}\in\{1,2\}\,.$

The manifold $(T^*\Sigma, g_{D,\Phi,T})$ is a Walker manifold; $\mathfrak{V}:=\operatorname{span}\{\partial_{x_{1'}},\partial_{x_{2'}}\}$ is a null parallel distribution of rank two by Equation (6.1) and Equation (6.4). Generically, this is the only such distribution and $\mathfrak{V}$ is invariantly defined. We use $\mathfrak{V}$ as an additional piece of structure and redefine $\mathfrak{H}:=T(T^*\Sigma)/\mathfrak{V}$ and let $\pi:T(T^*\Sigma)\to\mathfrak{H}$ be the natural projection. By Lemma 6.11, $\pi R(X,Y)v=0$ for $v\in\mathfrak{V}$ and thus $\pi R(X,Y)$ descends to a well-defined map that, via an abuse of notation, we continue to denote by $R^{\mathfrak{H}}_{\mathfrak{H}}(X,Y)$ of $\mathfrak{H}$. Let $\{X_{1'},X_{2'}\}$ be a local frame for $\mathfrak{V}$. Choose $\{X_1,X_2\}$ so that

$$g(X_1,X_{1'})=g(X_2,X_{2'})=1 \quad\text{and}\quad g(X_1,X_{2'})=g(X_2,X_{1'})=0\,. \tag{6.6}$$

We note that $\{X_1,X_2\}$ is not uniquely defined by these relations as we can add an arbitrary element of $\mathfrak{V}$ to either X_1 or X_2 and preserve Equation (6.6). However $\{\pi X_1,\pi X_2\}$ is uniquely defined by Equation (6.6). And, in particular, if we take $X_{1'}=\partial_{x_{1'}}$ and $X_{2'}=\partial_{x_{2'}}$, then we may take $X_1=\partial_{x^1}$ and $X_2=\partial_{x^2}$.

We use Lemma 6.11 to introduce some additional quantities.

(1) Since $\rho(X,Y)=0$ if either X or Y belongs to $\mathfrak{V}$, ρ descends to a map from $\mathfrak{H}\oplus\mathfrak{H}$ to $\mathbb{R}$ that we shall denote by $\rho^{\mathfrak{H}}\in S^2(\mathfrak{H}^*)$. Let $\pi: T^*\Sigma\to\Sigma$. Since $\pi_*(\mathfrak{V})=0$, $\pi_*:\mathfrak{H}\to T\Sigma$. If ${}^D\Gamma_{11}{}^2=0$, if $2{}^D\Gamma_{11}{}^2={}^D\Gamma_{12}{}^2$, and if $\Phi_{11}=0$, then $\rho^{\mathfrak{H}}=2\pi^*\rho_s^D$.

(2) Let $\Omega(X,Y)=\operatorname{tr}(R_{\mathfrak{H}}^{\mathfrak{H}}(X,Y))$. Then $\Omega(X,Y)=0$ if either X or Y belongs to $\mathfrak{V}$ so Ω descends to an alternating bilinear map from $\mathfrak{H}\oplus\mathfrak{H}$ to $\mathbb{R}$ that we shall denote by $\Omega^{\mathfrak{H}}\in\Lambda^2(\mathfrak{H}^*)$. We have $\Omega^{\mathfrak{H}}=2\pi^*\rho_{sk}^D$.

(3) As $\mathfrak{V}$ is parallel, $\nabla R(X,Y;Z)$ maps $\mathfrak{V}$ to $\mathfrak{V}$. Consequently, $\nabla R(X,Y;Z)$ extends to an endomorphism $(\nabla R)^{\mathfrak{H}}(X,Y;Z)$ of $\mathfrak{H}$. A direct computation shows that

$$\operatorname{tr}((\nabla R)^{\mathfrak{H}}(X,Y;Z))=0,$$

if X, Y, or Z belongs to $\mathfrak{V}$. We may therefore regard $\operatorname{tr}((\nabla R)^{\mathfrak{H}}(X,Y;Z))\in\Lambda^2(\mathfrak{H})\otimes\mathfrak{H}^*$. Assuming that $\Omega^{\mathfrak{H}}\neq 0$, we may decompose $\operatorname{tr}((\nabla R)^{\mathfrak{H}})=\omega^{\mathfrak{H}}\otimes\Omega^{\mathfrak{H}}$ for $\omega^{\mathfrak{H}}\in\mathfrak{H}^*$. Moreover, one has $d\omega^{\mathfrak{H}}=\Omega^{\mathfrak{H}}$.

Definition 6.12. Suppose that we are at a point of $(T^*\Sigma, g_{D,\Phi,T})$ where $\rho^{\mathfrak{H}}$ defines a non-degenerate symmetric bilinear form on $\mathfrak{H}$. We may then define

$$\beta_1 := \|\Omega^{\mathfrak{H}}\|^2_{\rho^{\mathfrak{H}}} = \frac{(R_{121}{}^1+R_{122}{}^2)^2}{\rho_{11}\rho_{22}-\rho_{12}\rho_{12}}\,.$$

If we also assume that $\Omega^{\mathfrak{H}}\neq 0$ (i.e., $\rho_{sk}^D\neq 0$) or, equivalently, that $\beta_1\neq 0$, then $\omega^{\mathfrak{H}}$ is well-defined and we may set

$$\beta_2 := \|\omega^{\mathfrak{H}}\|^2_{\rho^{\mathfrak{H}}}\,.$$

We have

$$\omega_1^{\mathfrak{H}}=\frac{R_{121}{}^1{}_{;1}+R_{122}{}^2{}_{;1}}{R_{121}{}^1+R_{122}{}^2},\quad \omega_2^{\mathfrak{H}}=\frac{R_{121}{}^1{}_{;2}+R_{122}{}^2{}_{;2}}{R_{121}{}^1+R_{122}{}^2},$$

$$\beta_2 := \frac{\rho_{22}^{\mathfrak{H}}\omega_1^{\mathfrak{H}}\omega_1^{\mathfrak{H}}+\rho_{11}^{\mathfrak{H}}\omega_2^{\mathfrak{H}}\omega_2^{\mathfrak{H}}-2\rho_{12}^{\mathfrak{H}}\omega_1^{\mathfrak{H}}\omega_2^{\mathfrak{H}}}{\rho_{11}^{\mathfrak{H}}\rho_{22}^{\mathfrak{H}}-\rho_{12}^{\mathfrak{H}}\rho_{12}^{\mathfrak{H}}}\,.$$

It is obvious from the discussion given above that β_1 and β_2 are isometry invariants of $(T^*\Sigma, g_{D,\Phi,T})$ where defined.

Generically, β_1 and β_2 are very complicated expressions which involve non-trivial dependence on the fiber variables and which involve the endomorphism Φ. It is interesting to note that if we consider a nilpotent (1,1)-tensor field T given by $T=\partial_{x^2}\otimes dx^1$, then proceeding in a completely analogous way as in Lemma 6.11 one can construct the invariants β_1 and β_2. In the next section we will present examples of Bach-flat manifolds where both invariants are calculated.

Remark 6.13. The facts that $(R_{\mathfrak{H}}^{\mathfrak{H}})\in\Lambda^2(\mathfrak{H}^*)$, $\omega^{\mathfrak{H}}=\operatorname{tr}(\nabla R)^{\mathfrak{H}}/\Omega^{\mathfrak{H}}\in\mathfrak{H}^*$ and $d\omega^{\mathfrak{H}}=\Omega^{\mathfrak{H}}$ is, of course, not true for a general Walker manifold. This observation perhaps can be useful in studying when a general Walker manifold is one of our special examples. All of these are pull-backs of similar identities on the base.

6.4 Examples of Bach-flat manifolds

The existence of a null distribution $\mathfrak{V}$ on a four-dimensional manifold (N, g) of neutral signature defines a natural orientation on N: the one which, for any basis $\{u, v\}$ of $\mathfrak{V}$, makes the bivector $u \wedge v$ self-dual (see Chapter 1 and [49]). We consider on $T^*\Sigma$ the orientation which agrees with $\mathfrak{V} = \ker(\pi_*)$, and thus self-duality and anti-self-duality are not interchangeable. Let

$$\begin{aligned} e_1 &= \partial_{x^1} + \tfrac{1}{2}(1 - (g_{D,\Phi,T})_{11})\partial_{x_{1'}}, \\ e_2 &= \partial_{x^2} - (g_{D,\Phi,T})_{12}\partial_{x_{1'}} + \tfrac{1}{2}(1 - (g_{D,\Phi,T})_{22})\partial_{x_{2'}}, \\ e_3 &= \partial_{x^1} - \tfrac{1}{2}(1 + (g_{D,\Phi,T})_{11})\partial_{x_{1'}}, \\ e_4 &= \partial_{x^2} - (g_{D,\Phi,T})_{12}\partial_{x_{1'}} - \tfrac{1}{2}(1 + (g_{D,\Phi,T})_{22})\partial_{x_{2'}} \end{aligned}$$

be an orthonormal basis of $(T^*\Sigma, g_{D,\Phi,T})$ with $\varepsilon_1 = \varepsilon_2 = 1 = -\varepsilon_3 = -\varepsilon_4$ (where $\varepsilon_i = g(e_i, e_i)$). Note that $e^1 \wedge \cdots \wedge e^4$ agrees with the orientation determined by $\mathfrak{V} = \ker(\pi_*)$. Then, the spaces of self-dual and anti-self-dual 2-forms, $\Lambda^2_\pm = \langle \{E_1^\pm, E_2^\pm, E_3^\pm\} \rangle$, have induced basis

$$E_1^\pm = \frac{e^1 \wedge e^2 \pm e^3 \wedge e^4}{\sqrt{2}}, \quad E_2^\pm = \frac{e^1 \wedge e^3 \pm e^2 \wedge e^4}{\sqrt{2}}, \quad E_3^\pm = \frac{e^1 \wedge e^4 \mp e^2 \wedge e^3}{\sqrt{2}}.$$

Here observe that the Hodge star operator satisfies

$$e^i \wedge e^j \wedge \star(e^k \wedge e^\ell) = (\delta^i_k \delta^j_\ell - \delta^i_\ell \delta^j_k)\, \varepsilon_i \varepsilon_j\, e^1 \wedge e^2 \wedge e^3 \wedge e^4 .$$

Further note that $\langle E_1^\pm, E_1^\pm \rangle = 1$, $\langle E_2^\pm, E_2^\pm \rangle = -1$, $\langle E_3^\pm, E_3^\pm \rangle = -1$, where $\langle \cdot, \cdot \rangle$ is the inner product defined in Chapter 1, and let $W^\pm_{ij} = W^\pm(E_i^\pm, E_j^\pm)$ denote the components of the self-dual and anti-self-dual parts of the Weyl curvature tensor.

Let $0 \neq T = T^j{}_i(x^1, x^2)$ be a nilpotent tensor field of type $(1,1)$ as in Equation (6.3). A straightforward calculation shows that

$$W^-_{11} = -\frac{1}{2}\alpha(x^1, x^2)^2(\xi(x^1, x^2)^2 + 1)^2, \qquad W^+_{12} = -2\rho^D_{sk}(\partial_{x^1}, \partial_{x^2}) .$$

Therefore, W^- is always non-null and the non-symmetry of ρ^D guarantees that $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat.

We recall that a manifold is conformally Einstein if and only if the equation

$$(n-2)\,\mathrm{Hes}_\varphi + \varphi\, \rho - \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\, \tau\} g = 0 \tag{6.7}$$

has a positive solution, where the conformal metric is given by $\bar{g} = \varphi^{-2} g$. It was shown in [62,72] that any four-dimensional conformally Einstein manifold satisfies

$$(i) \quad \mathrm{div}_4 W - W(\cdot, \cdot, \cdot, \nabla \log \varphi) = 0, \qquad (ii) \quad \mathfrak{B} = 0 . \tag{6.8}$$

Conditions (i)–(ii) above are also sufficient to be conformally Einstein if (N, g) is *weakly-generic*. In our case, it is easy to check that Riemannian extensions $g_{D,\Phi,T}$ for T nilpotent are not weakly-generic (see Chapter 1).

From now on we introduce the notation:

$$\mathcal{E} = 2\,\mathrm{Hes}_\varphi + \varphi\,\rho - \frac{1}{4}\{2\Delta\varphi + \varphi\,\tau\}g \quad \text{and} \quad \widetilde{\mathcal{E}} = \mathrm{div}_4\,W - W(\cdot,\cdot,\cdot,\nabla\log\varphi)\,.$$

For the Riemannian extension $(T^*\Sigma, g_{D,\Phi,T})$, we compute

$$\mathcal{E}_{1'1'} = 2\partial_{x_{1'}}\partial_{x_{1'}}\varphi, \quad \mathcal{E}_{1'2'} = 2\partial_{x_{1'}}\partial_{x_{2'}}\varphi, \quad \mathcal{E}_{2'2'} = 2\partial_{x_{2'}}\partial_{x_{2'}}\varphi,$$

to show that any solution of Equation (6.7) must be of the form

$$\varphi(x^1, x^2, x_{1'}, x_{2'}) = A(x^1, x^2)x_{1'} + B(x^1, x^2)x_{2'} + \psi(x^1, x^2), \tag{6.9}$$

for some smooth functions A, B and ψ depending only on the coordinates (x^1, x^2). This shows that any solution of the conformally Einstein equation on $(T^*\Sigma, g_{D,\Phi,T})$ is of the form $\varphi = \iota X + \psi \circ \pi$, where ιX is the evaluation of a vector field $X = A\partial_{x^1} + B\partial_{x^2}$ on Σ, $\psi \in \mathcal{C}^\infty(\Sigma)$ and $\pi : T^*\Sigma \to \Sigma$ is the projection.

Although in some cases we shall discuss some interesting families of anti-self-dual and conformally Einstein manifolds, our main purpose is to construct strictly Bach-flat examples with $DT \neq 0$ (examples with T parallel were previously constructed in Chapter 5).

6.4.1 Locally homogeneous setting

For a Type $\mathcal{A}$ or Type $\mathcal{B}$ homogeneous affine surface we investigate the existence of nilpotent tensor fields so that the corresponding nilpotent Riemannian extension is Bach-flat. We begin with a simple case.

Example 6.14. Setting

$${}^D\Gamma_{11}{}^1 = 0, \quad {}^D\Gamma_{11}{}^2 = 0, \quad {}^D\Gamma_{12}{}^1 = 1, \quad {}^D\Gamma_{12}{}^2 = 1, \quad {}^D\Gamma_{22}{}^1 = 0, \quad {}^D\Gamma_{22}{}^2 = 0,$$

we have a Type $\mathcal{A}$ affine surface which is not flat since the Ricci tensor of D is $\rho^D = -(dx^1 - dx^2)^2$. Despite of the Bach-flat condition is quite complicated, there exist many examples of nilpotent tensor fields of type $(1,1)$ which give rise to Bach-flat manifolds. For instance, if α_i's are smooth functions of one variable, then the following nilpotent endomorphisms lead to Bach-flat manifolds:

$$T := \alpha_2(x^2)\sqrt{e^{2x^1} + \alpha_1(x^2)}\,\left\{\partial_{x^1} \otimes dx^2\right\},$$

$$\widetilde{T} := \alpha_2(x^1)\sqrt{e^{2x^2} + \alpha_1(x^1)}\,\left\{\partial_{x^2} \otimes dx^1\right\}.$$

In the rest of this subsection we consider nilpotent tensor fields $T \in M_2(\mathbb{R})$, i.e., T has constant entries in the given coordinates and analyze Type $\mathcal{A}$ and Type $\mathcal{B}$ affine surfaces by separate.

Example 6.15. Let D be a Type $\mathcal{A}$ structure on $\mathbb{R}^2$, i.e., the Christoffel symbols of D are constant. The Ricci tensor is symmetric in this setting. Let $0 \neq T \in M_2(\mathbb{R})$ be nilpotent. Make a linear change of coordinates to ensure $T = \partial_{x^1} \otimes dx^2$. A direct computation shows $\mathfrak{B} = 0$ if and only if ${}^D\Gamma_{11}{}^2 = 0$ and $({}^D\Gamma_{11}{}^1)^2 - {}^D\Gamma_{11}{}^1\, {}^D\Gamma_{12}{}^2 = 0$.

If $(T^*\Sigma, g_{D,\Phi,T})$ is Bach-flat and $\Phi = 0$, then $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual and conformally Einstein. Indeed, if ${}^D\Gamma_{11}{}^1 = {}^D\Gamma_{11}{}^2 = 0$ and ${}^D\Gamma_{12}{}^2 = 0$, then the conformal metric $\varphi^{-2} g_{D,\Phi,T}$ is Einstein just taking $\varphi = x_{1'}\, \mathrm{e}^{-{}^D\Gamma_{12}{}^1 x^2}$. If ${}^D\Gamma_{11}{}^1 = {}^D\Gamma_{11}{}^2 = 0$ and ${}^D\Gamma_{12}{}^2 \neq 0$ then $\varphi^{-2} g_{D,\Phi,T}$ is an Einstein metric with conformal factor $\varphi = \mathrm{e}^{-{}^D\Gamma_{12}{}^2 x^1 + {}^D\Gamma_{12}{}^1 x^2}$. If ${}^D\Gamma_{11}{}^2 = 0$ and ${}^D\Gamma_{11}{}^1 = {}^D\Gamma_{12}{}^2$ then again $\varphi = x_{1'}\, \mathrm{e}^{-{}^D\Gamma_{12}{}^1 x^2}$ defines an Einstein conformal metric.

Next we construct strictly Bach-flat Riemannian extensions considering the case ${}^D\Gamma_{11}{}^1 = {}^D\Gamma_{11}{}^2 = 0$ and assuming ${}^D\Gamma_{12}{}^2 \neq 0$. In this case,

$$\partial_{x_{1'}} W_{11}^+ = \partial_{x^1} \Phi_{11}(x^1, x^2) - 2\,{}^D\Gamma_{12}{}^2 \Phi_{11}(x^1, x^2)\,.$$

A straightforward calculation shows that the possible conformal functions take the form $\varphi = \mu\, \mathrm{e}^{-{}^D\Gamma_{12}{}^2 x^1 + {}^D\Gamma_{12}{}^1 x^2}$ (with $\mu \in \mathbb{R}$) and, in such a case,

$$\mathcal{E}_{22} = \varphi(x^1, x^2, x_{1'}, x_{2'}) \Phi_{11}(x^1, x^2)\,.$$

Hence, we conclude that if

$$\Phi_{11}(x^1, x^2) \neq 0 \quad \text{and} \quad \partial_{x^1} \Phi_{11}(x^1, x^2) - 2\,{}^D\Gamma_{12}{}^2 \Phi_{11}(x^1, x^2) \neq 0,$$

i.e., $\Phi_{11}(x^1, x^2) \neq e^{2\,{}^D\Gamma_{12}{}^2 x^1} P(x^2)$, where P is a smooth function depending only on the coordinate x^2, then $(T^*\Sigma, g_{D,\Phi,T})$ is strictly Bach-flat.

Moreover, since $(D_{\partial_{x^1}} T)\partial_{x^2} = -{}^D\Gamma_{12}{}^2 \partial_{x^1}$, we have $DT \neq 0$ in this case.

Remark 6.16. Let (Σ, D) be a Type $\mathcal{A}$ surface. Since any Type $\mathcal{A}$ surface has $\rho^D_{sk} = 0$, the invariant $\beta_1 = 0$ whenever it is defined. Hence the invariant β_2 is not defined in this setting.

Example 6.17. Let D be a Type $\mathcal{B}$ structure on $\mathbb{R}^+ \times \mathbb{R}$, i.e., the Christoffel symbols of D take the form ${}^D\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$. Let $0 \neq T \in M_2(\mathbb{R})$ be nilpotent. The map $(x^1, x^2) \to (x^1, ax^2 + bx^1)$ defines an action of the "$ax + b$" group on such structures and modulo such an action, we may assume T takes one of the following two forms:

(1) $T = \partial_{x^1} \otimes dx^2$. A direct computation shows that $(T^*\Sigma, g_{D,\Phi,T})$ is Bach-flat if and only if $C_{11}{}^2 = 0$ and $(C_{11}{}^1 - 1)(C_{11}{}^1 - C_{12}{}^2) = 0$.

(2) $T = \partial_{x^2} \otimes dx^1$. A direct computation shows that $(T^*\Sigma, g_{D,\Phi,T})$ is Bach-flat if and only if $C_{22}{}^1 = 0$ and $C_{22}{}^2 (C_{12}{}^1 - C_{22}{}^2) = 0$.

Case (1). In this case, ρ^D is symmetric if and only if $C_{12}{}^1 + C_{22}{}^2 = 0$.

Case (1.1) $C_{11}{}^2 = 0$, $C_{11}{}^1 = 1$

First of all, note that $DT \neq 0$ in this case since $(D_{\partial_{x^2}} T)\partial_{x^1} = -\frac{C_{12}{}^2}{x^1} \partial_{x^1}$ and $(D_{\partial_{x^1}} T)\partial_{x^2} = \frac{1 - C_{12}{}^2}{x^1} \partial_{x^1}$.

Assume that ρ^D is non-symmetric, i.e., ${C_{12}}^1 + {C_{22}}^2 \neq 0$. Then $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat. Moreover, a straightforward calculation shows that the possible conformal functions take the form $\varphi = (x^1)^{2-{C_{12}}^2} P(x^2)$, where P is a smooth function depending only on the coordinate x^2 and, in such a case,

$$-2(x^1)^3\, \widetilde{\mathcal{E}}_{121} = {C_{12}}^1(5 - 4{C_{12}}^2) - {C_{22}}^2\,.$$

Hence, ρ^D non-symmetric and ${C_{22}}^2 \neq {C_{12}}^1(5-4{C_{12}}^2)$ imply $(T^*\Sigma, g_{D,\Phi,T})$ is strictly Bach-flat.

If ρ^D is non-symmetric and ${C_{22}}^2 = {C_{12}}^1(5 - 4{C_{12}}^2)$, we distinguish two cases depending on ${C_{12}}^2$ equals 1 or not. If ${C_{12}}^2 = 1$ then a straightforward calculation shows that $(T^*\Sigma, g_{D,\Phi,T})$ is conformally Einstein if and only if $\Phi_{11}(x^1, x^2) = A(x^2) - \frac{B(x^2)}{x^1} + \frac{4{C_{22}}^1}{(x^1)^2}$ and the possible conformal functions take the form $\varphi = x^1 P(x^2)$, where A, B and P are smooth functions depending only on the coordinate x^2 satisfying

$$2P''(x^2) + A(x^2)P(x^2) = 0, \qquad 2{C_{12}}^1 P'(x^2) + B(x^2)P(x^2) = 0\,.$$

If ${C_{12}}^2 \neq 1$, then it is easy to check that $(T^*\Sigma, g_{D,\Phi,T})$ is conformally Einstein if and only if $\Phi_{11}(x^1, x^2) = \frac{4({C_{22}}^1 + 2({C_{12}}^1)^2({C_{12}}^2-1))}{(x^1)^2}$ and the possible conformal functions take the form $\varphi = \mu(x^1)^{2-{C_{12}}^2}$, where $\mu \in \mathbb{R}$.

Remark 6.18. In a more general setting, without imposing the non-symmetric condition on ρ^D, assume ${C_{12}}^2 \neq 1$ and $\Phi_{11}(x^1, x^2) = P(x^2) \neq 0$, where P is a smooth function depending only on the coordinate x^2 with $P' \neq 0$. In this case, we compute $\partial_{x_{1'}}\partial_{x^1}\partial_{x^1}\left((x^1)^3 W_{11}^+\right) = -4({C_{12}}^2 - 1)P(x^2)$ so $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat. Moreover, a straightforward calculation shows that the possible conformal functions take the form $\varphi = (x^1)^{2-{C_{12}}^2}\mu$ (with $\mu \in \mathbb{R}$) and, in such a case,

$$\partial_{x^2}(\mathcal{E}_{22}) = \varphi(x^1, x^2, x_{1'}, x_{2'})\, \partial_{x^2}\Phi_{11}(x^1, x^2)\,.$$

Therefore, we conclude that if ${C_{12}}^2 \neq 1$ and $\Phi_{11}(x^1, x^2) = P(x^2) \neq 0$ with $P'(x^2) \neq 0$, then $(T^*\Sigma, g_{D,\Phi,T})$ is strictly Bach-flat.

Invariants β_1 and β_2 in Case (1.1) **(${C_{11}}^2 = 0$, ${C_{11}}^1 = 1$)**

In this case, assuming that $\rho^{\mathfrak{H}}$ is non-degenerate, one has

$$\beta_1 = ({C_{12}}^1 + {C_{22}}^2)^2 \Delta^{-1}$$

where

$$\begin{aligned}\Delta = {} & 2(2 - {C_{12}}^2){C_{12}}^2(x^1)^2\Phi_{11} - 4(2 - {C_{12}}^2)^2{C_{12}}^2 x^1 x_{1'} \\ & - (4{C_{12}}^2 + 1)({C_{12}}^1)^2 + 4({C_{12}}^2 - 2){C_{22}}^1({C_{12}}^2)^2 \\ & - ({C_{22}}^2)^2 + 2(1 - 2({C_{12}}^2 - 1){C_{12}}^2){C_{12}}^1{C_{22}}^2\,.\end{aligned}$$

It now follows that $\beta_1 = 0$ if and only if the Ricci tensor ρ^D of (Σ, D) is symmetric. Moreover β_1 is a non-zero constant if and only if either $C_{12}{}^2 = 0$, in which case $\beta_1 = -\frac{(C_{12}{}^1 + C_{22}{}^2)^2}{(C_{12}{}^1 - C_{22}{}^2)^2}$, or $C_{12}{}^2 = 2$, and then $\beta_1 = -\frac{(C_{12}{}^1 + C_{22}{}^2)^2}{(3C_{12}{}^1 + C_{22}{}^2)^2}$. Further, if β_1 is non-zero then one has

$$\begin{aligned}\beta_2 = \{&(C_{12}{}^2+3)^2(x^1)^2\Phi_{11} + 2(C_{12}{}^2-2)(C_{12}{}^2+3)^2x^1x_{1'}\\&-2(C_{12}{}^2+3)^2C_{12}{}^2C_{22}{}^1 - 2((C_{12}{}^2-1)C_{12}{}^2+3)(C_{22}{}^2)^2\\&-2((4C_{12}{}^2+9)C_{12}{}^2+6)(C_{12}{}^1)^2\\&-2((3C_{12}{}^2-4)C_{12}{}^2-9)C_{12}{}^1C_{22}{}^2\}\Delta^{-1},\end{aligned}$$

which is generically non-constant.

Case (1.2) $C_{11}{}^2 = 0$, $C_{12}{}^2 = C_{11}{}^1$

In this case DT is determined by

$$(D_{\partial_{x^2}}T)\partial_{x^1} = -\frac{C_{11}{}^1}{x^1}\partial_{x^1}, \qquad (D_{\partial_{x^2}}T)\partial_{x^2} = \frac{C_{12}{}^1 - C_{22}{}^2}{x^1}\partial_{x^1} + \frac{C_{11}{}^1}{x^1}\partial_{x^2}\,.$$

If ρ^D is non-symmetric, i.e $C_{12}{}^1 + C_{22}{}^2 \neq 0$, then $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat and, moreover, a straightforward calculation shows that the possible conformal functions take the form $\varphi = (x^1)^{C_{11}{}^1}P(x^2)$, where P is a smooth function depending only on the coordinate x^2. In such a case,

$$\mathcal{E}_{12} = (x^1)^{-2}(C_{22}{}^2 - C_{12}{}^1)\varphi(x^1, x^2, x_{1'}, x_{2'})\,.$$

Hence, if ρ^D is non-symmetric and $C_{22}{}^2 \neq C_{12}{}^1$ then $(T^*\Sigma, g_{D,\Phi,T})$ is strictly Bach-flat.

Now, if ρ^D is non-symmetric and $C_{22}{}^2 = C_{12}{}^1$ then a straightforward calculation shows that $(T^*\Sigma, g_{D,\Phi,T})$ is conformally Einstein if and only if $\Phi_{11}(x^1, x^2) = A(x^2) - \frac{B(x^2)}{x^1} + \frac{2C_{22}{}^1(C_{11}{}^1+1)}{(x^1)^2}$ and the possible conformal functions take the form $\varphi = (x^1)^{C_{11}{}^1}P(x^2)$, where A, B and P are smooth functions depending only on the coordinate x^2 satisfying

$$2P''(x^2) + A(x^2)P(x^2) = 0, \qquad 2C_{12}{}^1P'(x^2) + B(x^2)P(x^2) = 0\,.$$

Invariants β_1 and β_2 in Case (1.2) ($C_{11}{}^2 = 0$, $C_{12}{}^2 = C_{11}{}^1$)

Assume that $\rho^{\mathfrak{H}}$ is non-degenerate. Then

$$\beta_1 = (C_{12}{}^1 + C_{22}{}^2)^2\Delta^{-1}$$

where

$$\begin{aligned}\Delta = {}&2C_{11}{}^1(x^1)^2\Phi_{11} - 4(C_{11}{}^1)^2x^1x_{1'} - (4(C_{11}{}^1)^2+1)(C_{12}{}^1)^2 - (C_{22}{}^2)^2\\&-4C_{11}{}^1C_{22}{}^1 + 2C_{12}{}^1C_{22}{}^2\,.\end{aligned}$$

Therefore $\beta_1 = 0$ if and only if the Ricci tensor of (Σ, D) is symmetric. Moreover, one has that β_1 is a non-zero constant if and only if $C_{11}{}^1 = 0$, in which case $\beta_1 = -\frac{(C_{12}{}^1+C_{22}{}^2)^2}{(C_{12}{}^1-C_{22}{}^2)^2}$. Furthermore, if $\rho^{\mathfrak{H}}$ is non-degenerate and $\beta_1 \neq 0$, then

$$\begin{aligned}\beta_2 = \{&4(C_{11}{}^1+1)^2(x^1)^2\Phi_{11} - 8(C_{11}{}^1+1)^2C_{11}{}^1(x^1)x_{1'}\\ &- 2(C_{11}{}^1+2)(C_{22}{}^2)^2 - 8(C_{11}{}^1+1)^2C_{22}{}^1\\ &- 2(C_{11}{}^1(8C_{11}{}^1+9)+2)(C_{12}{}^1)^2 + 4(3C_{11}{}^1+2)C_{12}{}^1C_{22}{}^2\}\Delta^{-1}\,.\end{aligned}$$

Case (2). In this case, ρ^D is symmetric if and only if $C_{12}{}^1 = 0$.

Case (2.1) $C_{22}{}^1 = 0$, $C_{22}{}^2 = 0$

If ρ^D is not symmetric, i.e., $C_{12}{}^1 \neq 0$, then $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat. Moreover, a straightforward calculation shows that the possible conformal functions take the form $\varphi = \mathrm{e}^{-\frac{C_{12}{}^1x^2}{x^1}}P(x^1)$, where P is a smooth function depending only on the coordinate x^1 and, in such a case,

$$\partial_{x^2}\left((x^1)^3\,\mathrm{e}^{\frac{C_{12}{}^1x^2}{x^1}}\,\mathcal{E}_{12}\right) = -4(C_{12}{}^1)^2P(x^1)\,.$$

Hence, we conclude that if ρ^D is non-symmetric then $(T^*\Sigma, g_{D,\Phi,T})$ is strictly Bach-flat. Moreover, $DT \neq 0$ since $(D_{\partial_{x^1}}T)\partial_{x^2} = -\frac{C_{12}{}^1}{x^1}\partial_{x^2}$.

Invariants β_1 and β_2 in Case (2.1) ($C_{22}{}^1 = 0$, $C_{22}{}^2 = 0$)

Assuming that $\rho^{\mathfrak{H}}$ is non-degenerate, one has

$$\beta_1 = (C_{12}{}^1)^2\Delta^{-1}$$

where

$$\Delta = (C_{12}{}^1)^2\{-2(x^1)^2\Phi_{22} - 4C_{12}{}^1x^1x_{2'} - 4C_{11}{}^1C_{12}{}^2 + 4C_{11}{}^2C_{12}{}^1 - 1\}\,.$$

One now checks that β_1 is never constant in this case. Moreover, if $\rho^{\mathfrak{H}}$ is non-degenerate and $\beta_1 \neq 0$, then

$$\begin{aligned}\beta_2 = (C_{12}{}^1)^2\{&(x^1)^2\Phi_{22} + 2C_{12}{}^1x^1x_{2'} - 12C_{12}{}^2 - 2C_{11}{}^2C_{12}{}^1 - 4\\ &- 2(C_{11}{}^1)^2 - 8(C_{12}{}^2)^2 - 6(C_{12}{}^2+1)C_{11}{}^1\}\Delta^{-1}\,.\end{aligned}$$

It follows that β_2 is constant if and only if $2C_{11}{}^1 + 4C_{12}{}^2 + 3 = 0$, in which case $\beta_2 = -\frac{1}{2}$.

Case (2.2) $C_{22}{}^1 = 0$, $C_{22}{}^2 = C_{12}{}^1$

If ρ^D is non-symmetric, i.e., $C_{12}{}^1 \neq 0$, then $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat. A straightforward calculation shows that the possible conformal functions take the form $\varphi = \mathrm{e}^{\frac{C_{12}{}^1x^2}{x^1}}P(x^1)$, where P is a smooth function depending only on the coordinate x^1 and, in such a case,

$$(x^1)^2\mathcal{E}_{12} = -2C_{12}{}^1\varphi(x^1, x^2, x_{1'}, x_{2'})\,.$$

Hence, we conclude that if ρ^D is non-symmetric then $(T^*\Sigma, g_{D,\Phi,T})$ is strictly Bach-flat. Moreover, $(D_{\partial_{x^1}}T)\partial_{x^2} = -\frac{C_{12}{}^1}{x^1}\partial_{x^2}$ and therefore $DT \neq 0$.

Invariants β_1 and β_2 in Case (2.2) ($C_{22}{}^1 = 0$, $C_{22}{}^2 = C_{12}{}^1$)

In this case, $\rho^{\mathfrak{H}}$ is non-degenerate if and only if $C_{12}{}^1 C_{12}{}^2 \neq 0$. We get

$$\begin{aligned} \beta_1 &= -(C_{12}{}^2)^{-2}, \\ \beta_2 &= -\left((x^1)^2\Phi_{22} - 2C_{12}{}^1 x^1 x_{2'} - 4(C_{12}{}^2)^2 - 2C_{12}{}^2\right)(C_{12}{}^2)^{-2}. \end{aligned}$$

In contrast with the previous cases, β_1 is constant while β_2 is never constant.

6.4.2 Non-locally homogeneous setting

Example 6.19. Impose the relations of Remark 6.3 and set

$$^D\Gamma_{11}{}^2 = 0, \quad {}^D\Gamma_{11}{}^1 = -\partial_{x^1}\beta, \quad {}^D\Gamma_{12}{}^2 = -\partial_{x^1}\beta + ce^{\beta},$$

for smooth functions $c = c(x^2)$ and $\beta = \beta(x^1, x^2)$. We consider the nilpotent endomorphism given by $T^1{}_1 = 0$, $T^2{}_2 = 0$, $T^2{}_1 = 0$ and $T^1{}_2 = e^f$, where f is a function $f(x^1, x^2)$. This yields Bach-flat manifold if and only if

$$0 = \partial_{x^1} f\left(2ce^{\beta} + \partial_{x^1}\beta\right) - 2(\partial_{x^1} f)^2 - \partial_{x^1}\partial_{x^1} f\,.$$

In particular, any function $f = f(x^2)$ will work in this instance. Further, assume that $c = 0$ and $\partial_{x^1}{}^D\Gamma_{12}{}^1 = 0$. In this case, ρ^D is symmetric if and only if $\partial_{x^1}{}^D\Gamma_{22}{}^2 + 2\partial_{x^1}\partial_{x^2}\beta = 0$.

If $\partial_{x^1}{}^D\Gamma_{22}{}^2 + 2\partial_{x^1}\partial_{x^2}\beta \neq 0$ then $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat since $W_{21}^+ = -\partial_{x^1}{}^D\Gamma_{22}{}^2 - 2\partial_{x^1}\partial_{x^2}\beta$. Moreover, a straightforward calculation shows that the possible conformal functions take the form $\varphi = \mathrm{e}^{-\beta(x^1,x^2)}\,P(x^2)$, where P is a smooth function depending only on the coordinate x^2 and, in such a case,

$$\mathcal{E}_{12} = -\varphi(x^1, x^2, x_{1'}, x_{2'})(\partial_{x^1}{}^D\Gamma_{22}{}^2 + 2\partial_{x^1}\partial_{x^2}\beta)\,.$$

Hence, we conclude that if ρ^D is non-symmetric then $(T^*\Sigma, g_{D,\Phi,T})$ is strictly Bach-flat. Moreover, in this case, $DT = 0$ if and only if $\partial_{x^1}\beta = 0$ and ${}^D\Gamma_{12}{}^1 - {}^D\Gamma_{22}{}^2 + f' = 0$.

Let us impose further relations interchanging the roles of the indices to specialize the remaining three Christoffel symbols:

$$\begin{aligned} &^D\Gamma_{11}{}^2 = 0, \quad {}^D\Gamma_{11}{}^1 = -\partial_{x^1}\beta, \quad {}^D\Gamma_{12}{}^2 = -\partial_{x^1}\beta + ce^{\beta}, \quad \text{for} \quad c = c(x^2), \\ &^D\Gamma_{22}{}^1 = 0, \quad {}^D\Gamma_{22}{}^2 = -\partial_{x^2}\tilde\beta, \quad {}^D\Gamma_{12}{}^1 = -\partial_{x^2}\tilde\beta + \tilde c e^{\tilde\beta}, \quad \text{for} \quad \tilde c = \tilde c(x^1)\,. \end{aligned}$$

Then, in addition, we have the solutions $T = e^{\tilde f}\partial_{x^2} \otimes dx^1$ where $\tilde f$ is a function $\tilde f(x^1, x^2)$ satisfying

$$0 = \partial_{x^2}\tilde f(2\tilde c e^{\tilde\beta} + \partial_{x^2}\tilde\beta) - 2(\partial_{x^2}\tilde f)^2 - \partial_{x^2}\partial_{x^2}\tilde f\,.$$

Resumo

Un problema central na xeometría de Riemann é a existencia de métricas “óptimas”, é dicir, aquelas cuxa curvatura ten a propiedade de ser a mellor distribuída uniformemente sobre unha variedade. O enfoque para determinar tales métricas xeralmente céntrase en atopar métricas críticas para algúns funcionais naturais de curvatura.

Sexa M unha variedade compacta e τ_g a curvatura escalar dunha métrica pseudo-Riemanniana g en M. O funcional de curvatura máis simple e máis natural definido sobre o espazo de métricas vén dado pola integral da curvatura escalar: $\mathcal{S}\colon g \mapsto \mathcal{S}(g) = \int_M \tau_g \, d\,vol_g$, onde $d\,vol_g$ é o elemento de volume determinado pola métrica g. Unha métrica g dise $\mathcal{S}$-crítica cando o seu tensor de Ricci $\rho_g - \frac{1}{2}\tau_g g$ se anula, onde ρ_g denota o tensor de Ricci de (M, g). Como a curvatura do funcional $\mathcal{S}$ é sensíbel a reescalamentos da métrica, a súa acción restrínxese a métricas de volume constante. As métricas críticas correspondentes son as métricas de Einstein. Polo tanto, poderíase argumentar que as métricas de Einstein, é dicir, aquelas cuxo tensor de Ricci é proporcional á métrica, son as métricas óptimas máis naturais sobre unha variedade pseudo-Riemanniana.

As métricas de Einstein son dalgunha maneira insignificantes en dimensión dous. O teorema de Gauss-Bonnet mostra que $\mathcal{S}(g) = 4\pi\chi[M]$, onde $\chi[M]$ denota a característica de Euler de M, e por conseguinte tódalas métricas son $\mathcal{S}$-críticas en dimensión dous. O caso en dimensión tres é moi ríxido e as métricas de Einstein son xusto aquelas de curvatura seccional constante. De feito, son localmente isométricas á pseudo-esfera, ao espazo pseudo-Euclidiano ou ao espazo pseudo-hiperbólico. A primeira situación non trivial dáse en dimensión catro. A clasificación de métricas de Einstein en dimensión catro é un problema amplamente aberto e unha pregunta central é a existencia de tales métricas.

Existen diversas estratexias para construír métricas de Einstein. Unha construción clásica consiste en deformar unha métrica dada por un factor conforme tal que a métrica se convirta nunha métrica de Einstein tras un axeitado reescalamento conforme. Unha variedade de Riemann (M, g) dise conforme Einstein se este enfoque ten éxito, é dicir, se existe un representante Einstein na clase conforme $[g]$. Unha segunda estratexia máis recente fai uso do fluxo de Ricci, o cal baixo condicións apropiadas converxe a unha métrica de Einstein. Non obstante, existen métricas que permanecen invariantes (salvo reescalamentos e difeomorfismos) baixo o fluxo de Ricci: os *solitóns de Ricci*.

Brinkmann mostrou en [14] que unha variedade (M, g) de dimensión n é conforme Einstein se, e só se, a ecuación

$$(n-2)\operatorname{Hes}_\varphi + \varphi\,\rho - \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\,\tau\}g = 0 \tag{1}$$

ten solución positiva. A pesar de que en dimensión dous a ecuación é trivial, en dimensións superiores a súa integración é sorprendetemente difícil e ademais esta é sobredeterminada na maioría dos casos. Ademais unha métrica conforme Einstein no caso Riemanniano, se existe, é única salvo homotecias [14, 106]. Un problema importante é caracterizar espazos conforme Einstein por certas ecuacións tensoriais máis manexábeis.

Sexa (M, g) unha variedade conforme Einstein e supoñamos que $\overline{g} = e^{2\sigma} g$ é Einstein. Como as métricas de Einstein teñen tensor de Weyl harmónico tense trivialmente que $\overline{\operatorname{div}}\, \overline{W} = 0$, onde W denota o tensor curvatura de Weyl da variedade (M, g).O feito de que o tensor de Weyl se reescale baixo transformacións conformes implica que $(\operatorname{div}_4 W)(X, Y, Z) + W(X, Y, Z, \nabla\sigma) = 0$ é unha condición necesaria para que (M, g) sexa conforme Einstein. Unha segunda condición necesaria obtense da seguinte maneira: sexa $\mathcal{W}\colon g \mapsto \mathcal{W}(g) = \int_M \|W\|^2\, dvol_g$ o funcional curvatura determinado pola norma L^2 do tensor curvatura de Weyl conforme. As métricas $\mathcal{W}$-críticas foron caracterizadas por Bach en [6], onde mostra que unha métrica de dimensión catro é $\mathcal{W}$-crítica se, e só se, o tensor de Bach $\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \dfrac{1}{2} W[\rho]$ é identicamente nulo. Claramente toda métrica de Einstein resulta Bach-chá ($\mathfrak{B} = 0$). Máis aínda, unha característica específica en dimensión catro é que $\mathcal{W}$ é un invariante conforme e polo tanto as métricas conforme Einstein son métricas Bach-chás en dimensión catro.

Kozameh, Newman e Tod mostraron en [72] que ás dúas condicións necesarias:

$$(i) \quad \mathfrak{B} = 0, \qquad (ii) \quad (\operatorname{div}_4 W)(X, Y, Z) + W(X, Y, Z, \nabla\sigma) = 0, \tag{2}$$

son suficientes para ser conforme Einstein se (M, g) é debilmente-xenérica, é dicir, o tensor de Weyl visto como unha aplicación $TM \to \bigotimes^3 TM$ é inxectiva. No caso Kähler a situación é simple, pois toda métrica Kähler Bach-chá é conforme Einstein [48]. A pesar de todos estes resultados, a clasificación de variedades conforme Einstein é, a día de hoxe, un problema aberto, con tan só resultados parciais. Ver por exemplo [75] para unha clasificación de variedades produto conforme Einstein.

O noso propósito na Primeira parte desta monografía é abordar a clasificación de métricas conforme Einstein en dimensión catro para o caso homoxéneo. A homoxeneidade permite unha simplificación da ecuación conforme Einstein, reducindo a Ecuación (1) a un sistema de ecuacións alxébricas mediante o uso das condicións na Ecuación (2). As métricas de Einstein homoxéneas en dimensión catro foron descritas por Jensen [70], quen mostrou que no caso Riemannianno estas son simétricas. Polo tanto, son localmente un “space form” real ou complexo, ou son localmente un produto de duás superficies de igual curvatura de Gauss constante. A situación conforme Einstein é máis rica, por iso o Capítulo 2 destínase a probar o seguinte resultado:

Theorem 2.1. *Sexa (M, g) unha variedade de Riemann de dimensión catro homoxénea, conforme Einstein, completa e simplemente conexa. Entón (M, g) é localmente simétrica ou é homotética a un dos seguintes grupos de Lie determinados polas seguintes álxebras de Lie solubles:*

(i) A álxebra de Lie $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ dada por

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \tfrac{1}{4}e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \tfrac{1}{4}e_3\,.$$

(ii) *A álxebra de Lie* $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ *dada por*

$$[e_1, e_2] = e_3, \quad [e_4, e_1] = e_1 - \alpha e_2, \quad [e_4, e_2] = \alpha e_1 + e_2, \quad [e_4, e_3] = 2e_3 .$$

(iii) *A álxebra de Lie* $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ *dada por*

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha + 1)^2 e_2, \quad [e_4, e_3] = \alpha^2 e_3, \qquad \alpha > 1 .$$

Aquí $\{e_1, \dots, e_4\}$ *é unha base ortonormal. Ademais, os grupos de Lie* $(G_\alpha, \langle \cdot , \cdot \rangle)$ *na afirmación (ii) son semi-conformemente chans.*

En resumo, para métricas conforme Einstein en dimensión catro, a fórmula da sinatura de Hirzebruch mostra que as métricas auto-duais e anti-auto-duais son tamén Bach-chás. Como consecuencia directa da análise no Capítulo 2, obtemos unha clasificación de métricas homoxéneas que son estritamente Bach-chás, é dicir, aquelas métricas que non son conforme Einstein, nin semi-conformemente chás:

Theorem 2.4. *Sexa* (M, g) *unha variedade de Riemann de dimensión catro homoxénea, Bach-chá estrita, completa e simplemente conexa. Entón* (M, g) *é homotética a un dos grupos de Lie determinados polas seguintes álxebras de Lie solubles:*

(i) *A álxebra de Lie* $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{e}(1, 1)$ *dada por*

$$[e_2, e_3] = e_1, \qquad [e_1, e_3] = (2 + \sqrt{3})\, e_2,$$
$$[e_4, e_1] = \sqrt{6 + 3\sqrt{3}}\, e_1, \qquad [e_4, e_2] = \sqrt{6 + 3\sqrt{3}}\, e_2 .$$

(ii) *A álxebra de Lie* $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ *dada por*

$$[e_1, e_2] = e_3, \qquad [e_4, e_1] = \tfrac{1}{4}\sqrt{7 - 3\sqrt{5}}\, e_1,$$
$$[e_2, e_4] = \tfrac{1}{4}\sqrt{7 + 3\sqrt{5}}\, e_2, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}}\, e_3 .$$

Aquí $\{e_1, \dots, e_4\}$ *é unha base ortonormal.*

É un feito notable que os dous exemplos do Teorema 2.4 foron previamente construídos por Abbena, Garbiero e Salamon [1].

Un paso crucial na proba do Teorema 2.1 e na proba do Teorema 2.4 é a descrición que fai Bérard-Bergery [9] das variedades homoxéneas de Riemann en dimensión catro: estas son, ou simétricas, ou un grupo de Lie cunha métrica Riemanniana invariante pola esquerda. Claramente, unha afirmación análoga non funciona nos casos Lorentziano e de sinatura neutra, pois os espazos homoxéneos pseudo-Riemannianos non son necesariamente redutivos. Os espazos homoxéneos non redutivos en dimensión 4 foron clasificados por Fels e Renner [54]. Neste traballo, empregamos explicitamente a súa clasificación para determinar todalas métricas conforme Einstein non redutivas. No Capítulo 3 o teorema principal é o seguinte:

Theorem 3.1. *Sexa* (M, g) *un espazo homoxéneo de dimensión catro conforme Einstein e non redutivo. Entón* (M, g) *é Einstein, localmente conformemente chan ou localmente isométrico a:*

(i) $(\mathbb{R}^4, g)$ *coa métrica determinada por*

$$g = (4b(x^2)^2 + a)\,dx^1 \circ dx^1 + 4bx^2\,dx^1 \circ dx^2 - (4ax^2x^4 - 4cx^2 + a)\,dx^1 \circ dx^3 + 4ax^2\,dx^1 \circ dx^4 + b\,dx^2 \circ dx^2 - 2(ax^4 - c)\,dx^2 \circ dx^3 + 2a\,dx^2 \circ dx^4,$$

onde a, b *e* c *son constantes arbitrarias tales que* $ab \neq 0$.

(ii) $(\mathbb{R}^4, g)$ *coa métrica determinada por*

$$g = (4b(x^2)^2 + a)\,dx^1 \circ dx^1 + 4bx^2\,dx^1 \circ dx^2 - (4ax^2x^4 - 4cx^2 + a)\,dx^1 \circ dx^3 + 4ax^2\,dx^1 \circ dx^4 + b\,dx^2 \circ dx^2 - 2(ax^4 - c)\,dx^2 \circ dx^3 + 2a\,dx^2 \circ dx^4 - \tfrac{3a}{4}\,dx^3 \circ dx^3,$$

onde a, b *e* c *son constantes arbitrarias tales que* $ab \neq 0$.

(iii) $(\mathbb{R}^4, g)$ *coa métrica determinada por*

$$g = -2ae^{2x^4}\,dx^1 \circ dx^3 + ae^{2x^4} dx^2 \circ dx^2 + b\,dx^3 \circ dx^3 + 2c\,dx^3 \circ dx^4 + q\,dx^4 \circ dx^4,$$

onde a, b, c *e* q *son constantes arbitrarias tales que* $abq \neq 0$.

(iv) $(\mathfrak{U} \subset \mathbb{R}^4, g_+)$ *coa métrica determinada por*

$$g_+ = 2ae^{2x^3}\,dx^1 \circ dx^4 + ae^{2x^3}\cos(x^4)^2 dx^2 \circ dx^2 + b\,dx^3 \circ dx^3 + 2c\,dx^3 \circ dx^4 + q\,dx^4 \circ dx^4,$$

onde $\mathfrak{U} = \{(x^1, \ldots, x^4) \in \mathbb{R}^4 \,/\, \cos(x^4) \neq 0\}$*, e* a, b, c *e* q *son constantes arbitrarias tales que* $ab \neq 0$ *e* $b \neq -q$*, ou*

$(\mathbb{R}^4, g_-)$ *coa métrica determinada por*

$$g_- = 2ae^{2x^3}\,dx^1 \circ dx^4 + ae^{2x^3}\cosh(x^4)^2 dx^2 \circ dx^2 + b\,dx^3 \circ dx^3 + 2c\,dx^3 \circ dx^4 + q\,dx^4 \circ dx^4,$$

onde a, b, c *e* q *son constantes arbitrarias tales que* $ab \neq 0$ *e* $b \neq q$.

Máis aínda, tódolos casos (i)–(iv) están na clase conforme dunha métrica Ricci-chá, a cal é única (salvo homotecias) só no Caso (i). Noutro caso, o espazo das métricas conformes Ricci-chás é de dimensión dous o tres.

Unha segunda aproximación máis recente para levar a cabo a construción de métricas de Einstein vén dada polo fluxo de Ricci, é dicir, unha familia 1-paramétrica de métricas $g(t)$ sobre unha variedade M que satisfai a ecuación $\frac{\partial}{\partial t} g(t) = -2\rho_{g(t)}$. O fluxo de Ricci está ben formulado no contexto Riemanniano no sentido de que para toda variedade pechada M e toda métrica inicial $g(0)$, existe unha única solución $g(t)$ para t suficientemente pequeno. Hamilton en [64] mostrou que o fluxo de Ricci converxe a unha métrica de Einstein baixo condicións axeitadas, mostrando así a existencia de métricas de Einstein. Unha observación importante é que, se a métrica inicial $g(0)$ é Einstein, entón permanece invariante baixo o fluxo (salvo reescalamento homotético). Polo tanto, unha solución do fluxo dise que é auto-similar se permanece invariante baixo reescalamentos e difeomorfismos. Tales solucións, usualmente referidas como solitóns de Ricci, están caracterizadas pola existencia dun campo de vectores X en M tal que

$$\mathcal{L}_X g + \rho = \lambda g, \tag{3}$$

onde $\mathcal{L}$ denota a derivada de Lie e λ é unha constante real. Os solitóns de Ricci son polo tanto xeneralizacións das métricas de Einstein e súa clasificación é un problema importante para entender o fluxo de Ricci. Se X é un gradiente, entón a Ecuación (3) convértese en

$$\operatorname{Hes}_f + \rho = \lambda g, \tag{4}$$

para algunha función potencial f, e (M, g, f) dise un *soliton de Ricci gradiente*.

A xeometría do tensor de Ricci depende fortemente do signo das curvaturas de Ricci. Mentres a curvatura de Ricci positiva é unha condición moi forte con consecuencias topolóxicas, Lohkamp [80] mostrou que toda variedade admite métricas completas con curvatura de Ricci negativa. Correspondentemente, o estudo dos solitóns de Ricci depende do signo da constante λ do solitón; un solitón de Ricci (M, g, X) dise *contractivo*, *estábel* ou *expansivo* se $\lambda > 0$, $\lambda = 0$ ou $\lambda < 0$, respectivamente.

Mentres que se coñecen certos resultados de clasificación para solitóns de Ricci gradientes, o caso xenérico (3) é aínda bastante descoñecido. Incluso no caso homoxéneo, non existe aínda unha clasificación completa en dimensión catro. Tendo en conta que todalas métricas invariantes á esquerda Bach-chás se realizan sobre grupos de Lie resolubles (cf. Theorem 2.1 and Theorem 2.4), tense a seguinte descrición dos solitóns Bach-chans homoxéneos.

Theorem 2.16. *Sexa (M, g) un solitón de Ricci Riemanniano de dimensión catro homoxéneo, Bach-chan, completo e simplemente conexo. Entón (M, g) é Einstein, un solitón de Ricci gradiente localmente conformemente chan da forma $N^3(c) \times \mathbb{R}$, onde $N^3(c)$ é un espazo de curvatura constante, ou homotético a un dos seguintes solitóns de Ricci alxébricos determinados polas seguintes álxebras de Lie solubles:*

(i) A álxebra de Lie $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ determinada por

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = \tfrac{1}{4}e_2 + \alpha e_3, \quad [e_4, e_3] = -\alpha e_2 + \tfrac{1}{4}e_3 \,.$$

(ii) A álxebra de Lie $\mathfrak{g}_\alpha = \mathbb{R}e_4 \ltimes \mathfrak{r}^3$ determinada por

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = (\alpha+1)^2\, e_2, \quad [e_4, e_3] = \alpha^2\, e_3, \qquad \alpha > 1\,.$$

(iii) A álxebra de Lie $\mathfrak{g} = \mathbb{R}e_4 \ltimes \mathfrak{h}^3$ determinada por

$$[e_1, e_2] = e_3, \qquad [e_4, e_1] = \tfrac{1}{4}\sqrt{7 - 3\sqrt{5}}\, e_1,$$
$$[e_2, e_4] = \tfrac{1}{4}\sqrt{7 + 3\sqrt{5}}\, e_2, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}}\, e_3 \,.$$

A ecuación do solitón de Ricci gradiente (4) codifica a información da variedade en términos da curvatura de Ricci e da segunda forma fundamental dos conxuntos de nivel da función potencial f. Como o tensor de Ricci determina completamente a curvatura no caso localmente conformemente chan, fixéronse moitos esforzos para obter unha clasificación de solitóns de Ricci gradientes baixo algunhas condicións do tensor curvatura de Weyl. No caso Riemanniano os solitóns de Ricci gradientes localmente conformemente chans son localmente produtos warped con base de dimensión 1 [55] e tense unha descrición completa para o caso completo contractivo e estábel [35, 94]. A situación Lorentziana permite outras familias de exemplos cuxa estrutura subxacente é a dunha onda plana [17].

Foron investigadas suposicións máis débiles sobre o tensor de Weyl conforme. A propiedade de ser semi-conformemente chá é un exemplo importante. Mentres os solitóns de Ricci (anti)-auto-duais son localmente conformemente chans no caso Riemanniano [39], o caso de sinatura neutra permite exemplos non triviais [16] dados por *extensións de Riemann* de solitóns de Ricci gradientes. Xeneralizando a situación semi-conformemente chá, solitóns de Ricci gradientes Bach-chans foron investigados en [34]. Os solitóns de Ricci gradientes expansivos completos Bach-chans, así como os solitóns de Ricci gradientes estábeis con curvatura de Ricci positiva cuxa curvatura escalar alcanza un máximo nalgún punto interior, son localmente conformemente chans na categoría Riemanniana.

O noso obxectivo na segunda parte desta monografía é mostrar a existencia de solitóns de Ricci gradientes Bach-chans estritos no caso de sinatura neutra. Ista pregunta está motivada pola existencia de solitóns de Ricci gradientes auto-duais que non son localmente conformemente chans [16]. As métricas desexadas constrúense por unha perturbación das extensións de Riemann clásicas introducidas por Patterson e Walker [92]. Sexa (Σ, D) unha superficie afín, sexan T e Φ un campo de tensores paralelo de tipo $(1, 1)$ e un campo de tensores simétrico arbitrario de tipo $(0, 2)$ sobre Σ respectivamente. Os datos (Σ, D, T, Φ) determinan unha métrica de signatura neutra sobre o fibrado cotanxente $T^*\Sigma$ dada por

$$g_{D,\Phi,T} = \iota T \circ \iota T + g_D + \pi^*\Phi, \tag{5}$$

onde ι denota a aplicación avaliación sobre o fibrado cotanxente, $\pi : T^*\Sigma \to \Sigma$ é a proxección canónica e g_D denota a extensión de Riemann de Patterson-Walker.

No Capítulo 4, mostramos que as métricas (5) constituen unha grande familia de variedades Bach-chás estritas. En efecto,

Theorem 4.1. *Sexa* (Σ, D, T) *unha superficie afín libre de torsión equipada cun campo de tensores paralelo* T *de tipo* $(1, 1)$*. Sexa* Φ *un campo de tensores simétrico arbitrario de tipo* $(0, 2)$ *sobre* Σ*. Entón o tensor de Bach de* $(T^*\Sigma, g_{D,\Phi,T})$ *anúlase se, e só se,* T *é un múltiplo da identidade ou é nilpotente.*

Se T é un múltiplo da identidade, entón as métricas $g_{D,\Phi,T}$ son auto-duais e entón son de especial interese no caso nilpotente ($T^2 = 0$, $T \neq 0$). Ademais, como o campo de tensores deformación Φ non xoga ningún papel no Teorema 4.1, este pódese usar para construír unha familia infinita de variedades Bach-chás non isométricas para calquera (D, T) sobre Σ. Unha elección adecuada de Φ permite a construción de novos exemplos de solitóns de Ricci gradientes Bach-chans estábeis, onde por notación, $\widehat{\Phi}(X, Y) = \Phi(TX, TY)$ na Ecuación (6).

Theorem 4.6. *Sexa* (Σ, D, T) *unha superficie afín equipada cun campo de tensores paralelo nilpotente* T *de tipo* $(1,1)$ *e sexa* Φ *un campo de tensores simétrico de tipo* $(0,2)$ *sobre* Σ. *Sexa* $h \in \mathcal{C}^\infty(\Sigma)$ *unha función diferenciable. Entón* $(T^*\Sigma,\ g_{D,\Phi,T},\ f = h \circ \pi)$ *é un solitón de Ricci gradiente Bach-chan se, e só se,* $dh(\ker(T)) = 0$ *e*

$$\widehat{\Phi} = -\operatorname{Hes}_h^D - 2\rho_s^D\,. \tag{6}$$

Ademais o solitón é estábel e isotrópico.

Destacamos que a función potencial correspondente ten hipersuperficies de nivel dexeneradas e a súa estrutura subxacente nunca é localmente conformemente chá, en contraste coa situación no caso Riemanniano. No Teorema 4.1 as métricas pseudo-Riemannianas nunca son auto-duais, pero poden ser anti-auto-duais nalgúns casos. Este feito permite a construción de solitóns de Ricci gradientes anti-auto-duais que non son localmente conformemente chans, simplemente requírese que ambos T e Φ sexan paralelos.

Theorem 4.12. *Sexa* (Σ, D, T, Φ) *unha superficie afín con tensor de Ricci simétrico equipada cun campo de tensores paralelo nilpotente* T *de tipo* $(1,1)$ *e un campo de tensores paralelo simétrico* Φ *de tipo* $(0,2)$.

(i) (Σ, D, h) *é un solitón de Ricci gradiente afín con* $dh(\ker(T)) = 0$ *se, e só se,* $(T^*\Sigma, g_{D,\widehat{\Phi},T}, f = h \circ \pi)$ *é un solitón de Ricci gradiente estábel e anti-auto-dual que non é localmente conformemente chan.*

(ii) (Σ, D, h) *é un solitón de Ricci gradiente afín con* $dh(\ker(T)) = 0$ *se, e só se, existen coordenadas locais* (u^1, u^2) *en* Σ *tal que o único símbolo de Christoffel distinto de cero está determinado por* ${}^u\Gamma_{11}{}^2 = P(u^1) + u^2 Q(u^1)$ *e a función potencial* $h(u^1)$ *está determinada por* $h''(u^1) = -2Q(u^1)$, *para todos* $P, Q \in \mathcal{C}^\infty(\Sigma)$.

A construción no Capítulo 4 require da existencia de superficies afíns que admitan un campo de tensores paralelo e nilpotente, o cal é bastante restritivo. Polo tanto, no Capítulo 5 investigamos a existencia de campos de tensores paralelos de tipo $(1,1)$ sobre superficies afíns. Dise que un campo de tensores T é unha estrutura Kähler (resp. para-Kähler), se T é paralelo e $T^2 = -\operatorname{Id}$ (resp. $T^2 = \operatorname{Id}$). Ademais T é Kähler nilpotente se $T^2 = 0$ e $DT = 0$. Como a traza de todo tensor paralelo é constante, podemos expresar $T = \frac{1}{2}\operatorname{tr}(T)\operatorname{Id} + (T - \frac{1}{2}\operatorname{tr}(T)\operatorname{Id})$ de tal xeito que se descompoña nun múltiplo escalar da identidade e un campo de tensores sen traza.

Se (Σ, D) é unha superficie afín con tensor de Ricci anti-simétrico $\rho_{sk}^D \neq 0$, entón ρ_{sk}^D define un elemento de volume. Máis aínda, ρ_{sk}^D dise recorrente, é dicir, $D\rho_{sk}^D = \omega \otimes \rho_{sk}^D$ para algunha 1-forma ω. Os campos de tensores paralelos de tipo $(1,1)$ sen traza poden reescalarse para ser Kähler, para-Kähler ou Kähler nilpotente cunha condición de recorrencia:

Theorem 5.1. *Sexa* (Σ, D) *unha superficie afín simplemente conexa con* $\rho_s^D \neq 0$.

(i) (Σ, D) *admite unha estrutura Kähler se, e só se,* $\det(\rho_s^D) > 0$ *e* ρ_s^D *é recorrente.*

(ii) (Σ, D) *admite unha estrutura para-Kähler se, e só se,* $\det(\rho_s^D) < 0$ *e* ρ_s^D *é recorrente.*

(iii) (Σ, D) *admite unha estrutura Kähler nilpotente se, e só se,* ρ_s^D *é de rango un e recorrente.*

As superficies con tensor de Ricci anti-simétrico (equivalentemente $\rho_s^D = 0$) admiten simultaneamente estruturas Kähler, para-Kähler e Kähler nilpotente (ver Lemma 5.6). Usamos superficies homoxéneas afíns para ilustrar o Teorema 5.1, mostrando que todas as posibilidades distintas poden realizarse. Os resultados na Sección 5.3 dan expresións explícitas de estruturas paralelas Kähler nilpotentes sobre superficies homoxéneas.

Finalmente, dentro do Capítulo 6 consideramos algunhas xeneralizacións do Teorema 4.1 para a construcións de extensións de Riemann (5) Bach-chás cun campo de tensores T non paralelo. O Teorema 6.1 estende a construción do Teorema 4.1, mostrando que a extensión de Riemann $(T^*\Sigma, g_{D,\Phi,T})$ determinada por un campo de tensores nilpotente T non paralelo segue sendo Bach-chá baixo algunhas condicións na conexión afín. A pregunta subxacente está baseada en determinar as condicións na conexión unha vez que se proporciona o endomorfismo nilpotente. Reciprocamente, poderiamos considerar o problema inverso de construír endomorfismos nilpotentes en Σ tal que a extensión de Riemann (5) é Bach-chá unha vez que D está determinada. Usamos o Teorema de Cauchy-Kovalevski para mostrar que toda extensión de Riemann de Patterson-Walker pódese deformar localmente por un campo de endomorfismos nilpotente adecuado para ser Bach-chá na categoría real analítica.

Theorem 6.7. *Sexa* (Σ, D) *unha superficie afín real e analítica. Entón existen campos de tensores nilpotentes* T *de tipo* $(1, 1)$ *definidos localmente tales que a extensión de Riemann modificada* $(T^*\Sigma, g_{D,\Phi,T})$ *é Bach-chá.*

É importante destacar o feito de que os invariantes escalares da curvatura das extensións de Riemann modificadas (5) son nulos se, e só se, T é nilpotente (Teorema 6.8). Polo cal na Sección 6.3 introducimos algúns invariantes novos que non son de tipo Weyl. Estes invariantes, que dependen fortemente da curvatura de Ricci de (Σ, D), permiten distinguir algunhas clases de isometrías de métricas Bach-chás.

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