VÍCTOR SANMARTÍN LÓPEZ

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2020

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del
Departamento
de Geometría y Topología

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CENTRO INTERNACIONAL DE ESTUDOS
DE DOUTORAMENTO E AVANZADOS
DA USC (CIEDUS)

TESE DE DOUTORAMENTO

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ESCOLA DE DOUTORAMENTO INTERNACIONAL
PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

SANTIAGO DE COMPOSTELA
ANO 2019

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Introduction

"By symmetry we mean the existence of different viewpoints from which the system appears the same. It is only slightly overstating the case to say that physics is the study of symmetry."

This comment is due to the Nobel Prize Winner in Physics 1977, Philip Anderson, and reveals that *symmetry* lies at the very core of science. Whatever the challenge in our society, its solution requires the development of novel ideas that often originate from mathematical models. Then, using the symmetries of the mathematical model one may reduce the degrees of freedom of the problem, making it simpler. The main aim of this thesis is precisely to investigate geometric models or structures by examining their symmetries.

In this sense, Felix Klein described geometry as the study of those properties in a space that are invariant under a given transformation (symmetry) group. In Riemannian geometry, this group is the *isometry group*, that is, the group of distance-preserving transformations of a given Riemannian manifold. The action of a subgroup of the isometry group of a given manifold is called an *isometric action*. Its cohomogeneity is the lowest codimension of the orbits. An orbit whose codimension is greater than the cohomogeneity of the action is called a *singular orbit*. The orbits of maximal dimension are called *regular*. A submanifold is said to be *(extrinsically) homogeneous* if it is an orbit of the action of a subgroup of the isometry group on the ambient manifold.

The problem of classifying homogeneous hypersurfaces (equivalently cohomogeneity one actions up to orbit equivalence) in Euclidean spaces stems from Geometrical Optics and traces back to the work by Somigliana [96] at the beginning of the 20th century. Incidentally, his result initiated the study of one of the geometric objects of interest in this thesis: *isoparametric hypersurfaces*. A hypersurface of a Riemannian manifold is called isoparametric if it and its nearby equidistant hypersurfaces have constant mean curvature. Homogeneous hypersurfaces are always examples of isoparametric hypersurfaces. In the 1930s, Levi-Civita [75], Segre [93] and Cartan [25, 27, 26] restarted the study of these objects from a geometric viewpoint. Cartan [25] proved that, in space forms, a hypersurface is isoparametric if and only if its principal curvatures are constant. Segre [93] and Cartan [25] classified these objects in Euclidean and real hyperbolic spaces, respectively. All examples known to Cartan had a common property: they were homogeneous. However, unlike the Euclidean and hyperbolic cases, spheres do admit non-homogeneous examples [53]. In fact, the problem in spheres turned out to be much more involved and rich, and it was included by the Fields Medallist Yau in his influential list of problems in geometry [111].

Cohomogeneity one actions have been usually investigated from a Lie-theoretic point of view or, as mentioned above, from the viewpoint of their regular orbits (homogeneous hypersurfaces) and related concepts (such as isoparametric hypersurfaces). However, it is also interesting to approach cohomogeneity one actions with regard to their singular orbits. Indeed, if one considers a cohomogeneity one action with a singular orbit on a connected complete Riemannian manifold, then the principal curvatures of this singular orbit, counted with multiplicities, do not depend on the normal directions. It is really interesting to investigate the classification of submanifolds having this geometric property of singular orbits of cohomogeneity one actions. In this thesis, we call these objects CPC submanifolds. Note that CPC submanifolds have constant principal curvatures in the sense introduced by Heintze, Olmos and Thorbergsson [58] in the context of isoparametric submanifolds.

The relation between cohomogeneity one actions and CPC submanifolds mentioned above was generalized in the following result [54]: if M is a submanifold of a Riemannian manifold with codimension greater than or equal to two for which the tubes around it (for sufficiently small radii) are isoparametric hypersurfaces with constant principal curvatures, then M is a CPC submanifold. Therefore, CPC submanifolds play a crucial role in the study of cohomogeneity one actions and isoparametric hypersurfaces. In particular, in standard real space forms one can show by using Jacobi field theory that a submanifold M is CPC if and only if the tubes (of sufficiently small radii) around M have constant principal curvatures. Thus, in real space forms, classifying CPC submanifolds is equivalent to classifying isoparametric hypersurfaces. Note that totally geodesic submanifolds are always CPC, and CPC submanifolds are clearly minimal.

Another central concept in this thesis, which is also related to the aforementioned notions, is that of austere submanifold. A submanifold M is said to be *austere* if, at every point, the principal curvatures (counted with multiplicities) with respect to any unit normal vector are invariant under change of sign. One of the main sources of interest in this notion stems from its relation with other concepts, such as isoparametric hypersurfaces, homogeneous hypersurfaces, minimal submanifolds and CPC submanifolds. Indeed, austere submanifolds constitute an intermediate class between CPC submanifolds and minimal submanifolds, and as mentioned above, the focal sets of isoparametric families of hypersurfaces with constant principal curvatures are CPC, and hence austere. Moreover, homogeneous austere hypersurfaces are CPC.

Austere submanifolds were introduced by Harvey and Lawson [57] in the context of calibrated geometries. They proved for instance that the normal bundle of an austere submanifold of the Euclidean space $\mathbb{R}^n$ is a special Lagrangian submanifold of the cotangent bundle $T^*\mathbb{R}^n$. Since then, austere submanifolds have been investigated because of their own geometric interest. In fact, the austerity condition imposes a highly overdetermined, non-linear second order PDE on the submanifold which clearly implies the vanishing of the mean curvature. However, in dimension higher than two it turns out to be much stronger than minimality. Moreover, as a particular class of minimal submanifolds, their investigation is interesting on their own from the viewpoint of Riemannian submanifold geometry. Thus, Bryant [22] initiated a systematic study of austere submanifolds of the

Euclidean space, and derived the classification of austere submanifolds of dimension 3 in $\mathbb{R}^n$. This result was generalized by Dajczer and Florit [38] to austere submanifolds of arbitrary dimension in $\mathbb{R}^n$, under the assumption that their Gauss map has rank two, and by Ionel and Ivey [64] to 4-dimensional submanifolds without the assumption on the Gauss map. Austere submanifolds have also been studied in spheres [36], [62] and complex projective spaces [65], for example. Some other related notions, such as weakly reflective submanifolds [62] or arid submanifolds [101], have also been recently investigated.

In this thesis, we have mostly focused on the study of isoparametric hypersurfaces, CPC submanifolds and austere submanifolds in the setting of symmetric spaces of non-compact type.

According to the original definition given by Cartan [24], a Riemannian symmetric space is a Riemannian manifold characterized by the property that curvature is invariant under parallel translation. This geometric definition has the surprising effect of bringing the theory of Lie groups into the picture, and it turns out that Riemannian symmetric spaces are intimately related to semisimple Lie groups. To a large extent, many geometric problems in symmetric spaces can be reduced to the study of properties of semisimple Lie algebras, thus transforming difficult geometric questions into linear algebra problems that one might be able to solve.

For this reason, the family of Riemannian symmetric spaces has been a setting where many geometric properties can be tackled and tested. They are often a source of examples and counterexamples. The set of symmetric spaces is a large family encompassing many interesting examples of Riemannian manifolds such as spaces of constant curvature, projective and hyperbolic spaces, Grassmannians, compact Lie groups and more. Apart from Differential Geometry, symmetric spaces have also been studied from the point of view of Global Analysis and Harmonic Analysis, being non-compact symmetric spaces of particular relevance (see for example [60]). They are also an outstanding family in the theory of holonomy, constituting a class of their own in Berger's classification of holonomy groups.

Roughly speaking, there are three types of symmetric spaces: Euclidean spaces, symmetric spaces of compact type (in case the group of isometries is compact semisimple) and symmetric spaces of non-compact type (if the group of isometries is non-compact semisimple). Symmetric spaces of compact and non-compact type are in some way dual to each other, but they usually have many different properties. Symmetric spaces of non-compact type are diffeomorphic to Euclidean spaces, and thus their topology is trivial, whereas in compact symmetric spaces topology does play a relevant role.

Every symmetric space of non-compact type is isometric to a solvable Lie group endowed with a left-invariant metric. Indeed, this Lie group is the solvable part AN of the Iwasawa decomposition of the isometry group of the symmetric space (see Section 1.5 for details). In our experience, this provides a wealth of examples of many interesting concepts, compared to their compact counterparts. In fact, from the viewpoint of submanifold geometry, one can consider interesting types of submanifolds by looking at orbits of the action of subgroups of the solvable Lie group AN or, equivalently, by looking at subalgebras of the Lie algebra of such Lie group. For this reason, a good understanding of the restricted root space decomposition associated with the symmetric space is crucial. Of course, not every submanifold (even homogeneous submanifold) can be regarded as an orbit of the action of a Lie subgroup of AN, but very important types of examples arise in this way, sometimes combined with some additional constructions, as we will see throughout this thesis.

In the following lines we present the main contributions and goals of this thesis.

Isoparametric hypersurfaces in complex hyperbolic spaces

One of the main contributions of this thesis is *the classification of isoparametric hypersurfaces in complex hyperbolic spaces*. Firstly, Chapter 2 is devoted to an exposition of the origin of isoparametric hypersurfaces as well as some well-known results concerning them, and also to describing the construction method and some geometric data of the known examples of isoparametric hypersurfaces in complex hyperbolic spaces. Then, in Chapter 3 we prove the classification result of isoparametric hypersurfaces in complex hyperbolic spaces. Indeed, we see that each isoparametric hypersurface in the complex hyperbolic space corresponds to one of the examples described in Chapter 2. It is important to emphasize the existence of non-homogeneous isoparametric families of hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$ [41]. The classification result proved in Chapter 3 constituted the first complete classification of isoparametric hypersurfaces in a whole family of symmetric spaces after Cartan's work for real hyperbolic spaces [25] in 1938. This classification result obtained in Chapter 3 has given rise to the articles [43] and [44].

The first step of the proof is to check the good behaviour of isoparametric hypersurfaces with respect to the Hopf fibration associated with $\mathbb{C}H^n$. In other words, we start by showing that a hypersurface is isoparametric in $\mathbb{C}H^n$ if and only if its pullback with respect to the Hopf map is a Lorentzian isoparametric hypersurface in the anti-De Sitter space H_1^{2n+1}. This allows us to study isoparametric hypersurfaces in $\mathbb{C}H^n$ by analyzing their lifts to Lorentzian isoparametric hypersurfaces in H_1^{2n+1}. There are two main reasons to start our work by inspecting these Lorentzian isoparametric hypersurfaces: since H_1^{2n+1} has constant sectional curvature, it is easier to solve the Jacobi equation in order to examine parallel translation of hypersurfaces; and we have a generalization of the Cartan formula, which, roughly speaking, is a formula that allows us to obtain bounds on the number of distinct principal curvatures of a Lorentzian isoparametric hypersurface. Hence, working in the anti-De Sitter space we are able to extract the fundamental geometric information about the Lorentzian isoparametric hypersurface, and deduce its implications to the initial hypersurface in $\mathbb{C}H^n$. Finally, we prove a rigidity result that reveals several interesting aspects of the geometry of the examples.

This classification result of isoparametric hypersurfaces in complex hyperbolic spaces has very interesting consequences. On the one hand, we deduce that isoparametric hypersurfaces of $\mathbb{C}H^n$ with constant principal curvatures are homogeneous hypersurfaces. Moreover, we obtain that each isoparametric hypersurface in a complex hyperbolic space has (pointwise) the same principal curvatures as a homogeneous one. On the other hand, we also obtain that the focal submanifold of an isoparametric hypersurface in the complex hyperbolic space is locally homogeneous.

Isoparametric hypersurfaces in the anti-De Sitter space

The concept of isoparametric hypersurface has been also generalized to the context of semi-Riemannian geometry, just by additionally requiring the hypersurface to have non-degenerate induced metric. Moreover, as follows from Hanh's work [56], a hypersurface in a semi-Riemannian space form is isoparametric if and only if it has constant principal curvatures with constant algebraic multiplicities.

Isoparametric hypersurfaces have been investigated in the Lorentzian setting, where the breadth of examples seems to be much richer than in the Riemannian case. In fact, these objects are supposed to be classified in the Minkowski space by Magid [80], although Burth [23] pointed out some gaps in Magid's arguments. Some partial classifications have been achieved in De Sitter spaces. Indeed, Nomizu [83] proved, using the fact that the number of principal curvatures is bounded from above by two, that spacelike hypersurfaces with constant principal curvatures in De Sitter spaces are tubes around totally geodesic submanifolds. He also conjectured in the same paper [83] that examples of spacelike isoparametric hypersurfaces with more than two principal curvatures would appear in the anti-De Sitter space.

The main aim of Chapter 4 is to give a negative answer to this conjecture proposed by Nomizu. Indeed, we will show that two is an upper bound for the number of principal curvatures of a spacelike isoparametric hypersurface in the anti-De Sitter space. In order to prove this bound, we generalize Ferus' work [52] to study isoparametric hypersurfaces in semi-Riemannian space forms focusing, in this particular case, on anti-De Sitter spaces. The bound achieved on the number of principal curvatures leads to a *classification of spacelike isoparametric hypersurfaces in anti-De Sitter spaces*: non-totally umbilical spacelike isoparametric hypersurfaces are tubes around totally geodesic submanifolds. This classification has been published in the article [92].

CPC submanifolds

Chapter 5 is devoted to the study of CPC submanifolds, that is, submanifolds whose principal curvatures, counted with multiplicities, do not depend on the normal direction. It is evident from the discussion above that CPC submanifolds arise in various geometric contexts. However, there seems to be no systematic study in a more general setting. This is somewhat surprising, given that the condition on the principal curvatures is remarkably simple and natural.

The main purpose of Chapter 5 is to present a *systematic approach to the construction, description and classification of homogeneous CPC submanifolds in irreducible Riemannian symmetric spaces of non-compact type* and rank greater or equal than 2. Recall that totally geodesic submanifolds and singular orbits of cohomogeneity one actions are always examples of CPC submanifolds. The main contribution of Chapter 5 is *to provide a large number of new examples of non-totally geodesic CPC submanifolds that are not orbits of cohomogeneity one actions.* To our knowledge, only one example under such conditions was previously known: a particular 11-dimensional submanifold of the Cayley hyperbolic plane [41]. The contents of Chapter 5 have given rise to the article [14], and jointly with other results, to the survey [45].

In order to construct and describe the geometry of this new family of non-totally geodesic CPC submanifolds that are not orbits of cohomogeneity one actions, we restrict our attention to submanifolds that arise as orbits of certain subgroups of the solvable part AN of the Iwasawa decomposition associated with the symmetric space of non-compact type. Moreover, we have developed an original and promising technique based on the examination of the information codified in the root system of each symmetric space.

More precisely, the Levi-Civita connection is the main tool for studying many geometric properties in the context of submanifold theory. In symmetric spaces one can use Lie algebraic tools to describe this connection. In principle, it is quite hard to handle this in full generality because it involves many calculations that relate root spaces of positive roots in a complicated way. In order to tackle this difficulty, we introduce in this thesis a generalization of the classical concept of α-string of λ [69, p. 152], where α, λ are roots. For each given solvable submanifold, using this more general notion of string, we describe a partition of the set of positive roots explicitly. Each string constitutes one of the sets of the partition. This can be interpreted as a decomposition of the tangent space of the submanifold. Roughly speaking, the Levi-Civita connection is determined if we calculate it when restricted to each subspace of the decomposition induced by the strings. The main goal of this approach is that strings can adopt just a few different configurations that we perfectly control. This means that we just need to calculate the Levi-Civita connection when restricted to a very reduced number of different kinds of subspaces.

To sum up, we are able to describe the Levi-Civita connection of a symmetric space of non-compact type (and thus the shape operator of each solvable submanifold) with very simple and short calculations using the information codified in the root system, independently of the rank of the space under consideration. In Chapter 5 we have used this tool to study CPC submanifolds, but we believe that it might be applied to study totally geodesic, austere and minimal submanifolds.

Austere submanifolds in classical and exceptional symmetric spaces

An important tool for the study of symmetric spaces of non-compact type and rank higher than one stems from their so-called horospherical decomposition, which is intimately related to the theory of parabolic subalgebras of real semisimple Lie algebras [50, Section 2.17]. These subalgebras are parametrized (up to conjugacy) by the subsets Φ of a set Π of simple roots for the restricted root space decomposition of a real semisimple Lie algebra $\mathfrak{g}$. Thus, given a symmetric space of non-compact type $M \cong G/K$, the horospherical decomposition associated with the choice $\Phi \subset \Pi$ states that M is diffeomorphic to the Cartesian product of certain totally geodesic submanifold B_Φ of M, an abelian subgroup A_Φ of G and a nilpotent subgroup N_Φ of G. Even more than that, the connected solvable subgroup $S_\Phi = A_\Phi N_\Phi$ of G acts freely and isometrically on M, and all the orbits of such action are mutually congruent. Tamaru [102] proved that these orbits are Einstein solvmanifolds and, from the viewpoint of extrinsic geometry, minimal submanifolds of M.

In Chapters 6 and 7 we deepen into the investigation of the extrinsic geometry of such orbits by *classifying which S_Φ-orbits are austere submanifolds*. The austerity condition on the S_Φ-orbits turns out to be reflected on certain algebraic and combinatorial properties of the pair (Π, Φ). Analyzing these properties requires a profound understanding of the restricted root system associated with the symmetric space M. In order to address this problem we introduce the notion of Φ-*string*, which generalizes the classical concept of string in the theory of root systems. Moreover, to each Φ-string we associate certain diagram which will help us to understand its structure. Roughly speaking, the austerity of the S_Φ-orbits is codified in certain symmetry conditions of the diagrams of the Φ-strings. After proving several basic results for Φ-strings and their diagrams, we develop a rather exhaustive case-by-case study of these objects for each possible root system.

Due to the length of this analysis, we divide the exposition into two chapters. Chapter 6 is devoted to the setup of the problem, the introduction and general properties of Φ-strings and their diagrams, and the classification in symmetric spaces of classical type. The investigation of exceptional symmetric spaces, together with specific tools for their study, constitutes the content of Chapter 7.

Structure of the thesis

This thesis is organized as follows.

Chapter 1 is devoted to the introduction of the basic notions, concepts and terminology to be used in this work. More precisely, we introduce the notion of semi-Riemannian manifold (Section 1.1), the main general tools in order to study submanifold theory (Section 1.2), some basic facts about isometric actions (Section 1.3) and we finally introduce and describe symmetric spaces (Section 1.4), with special focus on the algebraic description of those of non-compact type (Section 1.5). Finally, we briefly construct anti-De Sitter and complex hyperbolic spaces (Section 1.6).

In Chapter 2, we start with an exposition of the origin of isoparametric hypersurfaces (Section 2.1) together with some well-known results and classifications concerning these geometrical objects (Section 2.2). Furthermore, in this chapter we also construct and describe the examples of isoparametric hypersurfaces in complex hyperbolic spaces (Section 2.5), using the identification of the complex hyperbolic space with a solvable Lie group with a left-invariant metric (Section 2.3) and some facts related to real subspaces of complex vector spaces (Section 2.4).

The original contributions of this work are located from Chapter 3 to 7.

In Chapter 3 we classify isoparametric hypersurfaces in complex hyperbolic spaces. This means that we show that an isoparametric hypersurface in the complex hyperbolic space is one of the examples introduced and described in Chapter 2. We start by checking the good behaviour of the Hopf map (Section 3.1) in order to analyze the lift of each isoparametric hypersurface of the complex hyperbolic space to the anti-De Sitter space (Section 3.2). Then, we focus on the possibilities for the shape operator of this lift, with special attention to one non-diagonalizable case (Section 3.3). Finally, with a rigidity argument (Section 3.4) we conclude the classification result (Section 3.5).

In Chapter 4 we generalize Ferus' work [52] to the semi-Riemannian setting (Section 4.1) and obtain a classification result of spacelike isoparametric hypersurfaces in anti-De Sitter spaces (Section 4.2).

Chapter 5 is devoted to the development of a systematic approach to the construction (Sections 5.2 and 5.3), classification (Section 5.4) and description (Sections 5.5 and 5.6) of homogeneous CPC submanifolds in irreducible symmetric spaces of non-compact type and rank greater or equal than 2, based on the inspection of the geometric information codified in root systems. In particular, we provide a large number of new examples of non-totally geodesic CPC submanifolds that are not orbits of cohomogeneity one actions.

The remaining chapters are devoted to studying the austerity of certain orbits that arise from the theory of parabolic subgroups of real semisimple Lie groups. Due to the length of this work, we have divided this last part into two different chapters. Firstly, in Chapter 6 we explain the general setting (Section 6.1), introduce the main tools to be used and their properties (Section 6.2) and conclude the classification in classical symmetric spaces (Section 6.3). Finally, in Chapter 7 we finish the classification by analyzing exceptional symmetric spaces with an exhaustive case-by-case analysis.

Chapter 1
Preliminaries

This first chapter is completely devoted to the introduction of the basic notions, concepts and terminology to be used in this thesis.

In Section 1.1 we introduce the notion of semi-Riemannian manifold. Moreover, we also fix our sign convention for the curvature tensor. Section 1.2 focuses on the main tools and ingredients in order to study submanifold geometry. In Section 1.3 we introduce the concept of isometric action and some of the main notions related to it, such as homogeneous submanifold or principal, exceptional and singular orbit. In Section 1.4 we introduce the concept and basic ideas concerning Riemannian symmetric spaces. Moreover, in Section 1.5 we describe symmetric spaces of non-compact type algebraically and see that they can be regarded as solvable Lie groups with a left-invariant metric. Finally, in Section 1.6 we briefly construct anti-De Sitter and complex hyperbolic spaces.

1.1 Semi-Riemannian manifolds

Let M be a smooth differentiable manifold of dimension n. Indeed, in this thesis we will always assume that manifolds are smooth and second countable. If $p \in M$, then T_pM denotes the tangent space of M at p, TM is the tangent bundle of M, and $\Gamma(TM)$ is the module of smooth vector fields on M. In general, if $\mathcal{D}$ is a distribution along M, we denote by $\Gamma(\mathcal{D})$ the module of sections of $\mathcal{D}$, that is, the vector fields $X \in \Gamma(TM)$ such that $X_p \in \mathcal{D}_p$ for each $p \in M$.

Let T be a symmetric bilinear tensor in a vector space V. We will say that T is non-degenerate if $T(x, y) = 0$ for all $y \in V$ implies that $x = 0$. Any non-degenerate symmetric bilinear tensor in a vector space is linearly congruent to a diagonal matrix of the form $\operatorname{diag}(1, \overset{r}{\cdots}, 1, -1, \overset{s}{\cdots}, -1)$. The signature of the tensor T is by definition the pair (r, s).

A *semi-Riemannian manifold* is a pair $(M, \langle\cdot,\cdot\rangle)$, where M is a manifold and $\langle\cdot,\cdot\rangle$ is a non-degenerate symmetric bilinear tensor field of type $(0, 2)$ and constant signature. In other words, for each point $p \in M$, the tangent space T_pM is endowed with a non-degenerate symmetric bilinear tensor $\langle\cdot,\cdot\rangle_p$. We define the signature of the manifold M as the signature of its non-degenerate symmetric bilinear tensor field $\langle\cdot,\cdot\rangle$. In particular, if the signature is $(n, 0)$, then M is said to be a *Riemannian manifold*. If the signature of the manifold M is $(n-1, 1)$, then M is said to be a *Lorentzian manifold*.

Let V be a vector space with non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. Recall that $v \in V$ is spacelike, timelike, or null if $\langle v, v\rangle$ is positive, negative, or zero, respectively. We also write $\|v\| = \sqrt{|\langle v, v\rangle|}$ for $v \in V$.

Moreover, if U and W are subspaces of V, we will denote $U \ominus W = \{u \in U : \langle u, w\rangle = 0, \text{ for all } w \in W\}$. We do not require $W \subset U$. This will be convenient when dealing with non-definite scalar products, especially if there are null vectors in W. If $\langle\cdot,\cdot\rangle$ is positive definite, this notation stands for the orthogonal complement of W in U.

One of the central concepts in geometry is the curvature. Its study is addressed by means of the curvature tensor R, which is defined according to the convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where ∇ denotes the Levi-Civita connection of M, that is, the unique symmetric torsion-free connection of M. We say that a manifold is *flat* if the curvature tensor vanishes identically. Moreover, a semi-Riemmanian manifold M is said to have *constant curvature* c if its curvature tensor can be written as $R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$ for all vector fields X, Y and Z in M.

1.2 Geometry of submanifolds

Let $(\bar{M}, \langle\cdot,\cdot\rangle)$ be a semi-Riemannian manifold and M an embedded submanifold of $\bar{M}$ such that the restriction of $\langle\cdot,\cdot\rangle$ to M is non-degenerate (this is automatically true if $\bar{M}$ is Riemannian). The normal bundle of M is denoted by νM. Thus, $\Gamma(\nu M)$ denotes the module of all normal vector fields to M. A canonical orthogonal decomposition holds at each point $p \in M$, namely, $T_p\bar{M} = T_pM \oplus \nu_pM$. In this thesis, the symbol $\oplus$ will always denote direct sum (not necessarily orthogonal direct sum).

Let us denote by $\bar{\nabla}$ and $\bar{R}$ the Levi-Civita connection and the curvature tensor of $\bar{M}$, respectively, and by ∇ and R the corresponding objects for M. The second fundamental form $I\!I$ of M is defined by the Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + I\!I(X,Y)$$

for any X, $Y \in \Gamma(TM)$. Let $\xi \in \Gamma(\nu M)$ be a normal vector field. The shape operator $\mathcal{S}_\xi$ of M with respect to ξ is the operator on M defined by $\langle \mathcal{S}_\xi X, Y\rangle = \langle I\!I(X,Y), \xi\rangle$, where X, $Y \in \Gamma(TM)$. Furthermore, denote by $\nabla^\perp$ the normal connection of M, that is, $\nabla^\perp_X \xi = (\bar{\nabla}_X \xi)^\perp$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Then we have the Weingarten formula

$$\bar{\nabla}_X \xi = -\mathcal{S}_\xi X + \nabla^\perp_X \xi.$$

The extrinsic geometry of M is controlled by Gauss, Codazzi and Ricci equations

$$\begin{aligned}
\langle \bar{R}(X,Y)Z, W\rangle &= \langle R(X,Y)Z, W\rangle - \langle I\!I(Y,Z), I\!I(X,W)\rangle + \langle I\!I(X,Z), I\!I(Y,W)\rangle, \\
\langle \bar{R}(X,Y)Z, \xi\rangle &= \langle (\nabla^\perp_X I\!I)(Y,Z) - (\nabla^\perp_Y I\!I)(X,Z), \xi\rangle, \\
\langle R^\perp(X,Y)\xi, \eta\rangle &= \langle \bar{R}(X,Y)\xi, \eta\rangle + \langle [\mathcal{S}_\xi, \mathcal{S}_\eta] X, Y\rangle,
\end{aligned}$$

where X, Y, Z, $W \in \Gamma(TM)$, ξ, $\eta \in \Gamma(\nu M)$, $(\nabla^{\perp}_X II)(Y,Z) = \nabla^{\perp}_X II(Y,Z) - II(\nabla_X Y, Z) - II(Y, \nabla_X Z)$, and $R^{\perp}$ is the curvature tensor of the normal bundle of M, which is defined by $R^{\perp}(X,Y)\xi = [\nabla^{\perp}_X, \nabla^{\perp}_Y]\xi - \nabla^{\perp}_{[X,Y]}\xi$.

The *mean curvature vector* H of a semi-Riemannian submanifold M is defined as the trace of the second fundamental form. In this sense, let $\{X_i\}_{i=1}^n$ be a local orthonormal basis of TM. Then, we have that

$$H = \sum_{i=1}^{n} \langle X_i, X_i \rangle II(X_i, X_i).$$

In particular, let ξ be an element in $\Gamma(\nu M)$. The mean curvature vector of M with respect to the normal vector ξ is the trace of the shape operator $\mathcal{S}_\xi$. A submanifold is said to be *minimal* if its mean curvature vector vanishes.

A submanifold is said to be *totally umbilical* if there exists a function λ such that $II = \lambda\langle\cdot,\cdot\rangle H$. In particular, when $\lambda = 0$ or, equivalently, when the second fundamental form II vanishes identically we say that M is a *totally geodesic* submanifold. This is equivalent to saying that every geodesic in M is also a geodesic in $\bar{M}$.

Let ξ be a unit normal vector field defined on an open subset U of the submanifold M. We say that $\lambda\colon U \subset M \to \mathbb{R}$ is a *principal curvature* of M with respect to ξ at $p \in M$ if there exists a vector field $X \in \Gamma(TU)$ such that $\mathcal{S}_\xi X_p = \lambda(p) X_p$, for each $p \in U$.

The vector X_p is then called a principal curvature vector at $p \in M$. By $T_\lambda(p)$ we denote the eigenspace of $\lambda(p)$ at p, and we call it the principal curvature space of $\lambda(p)$. Under certain assumptions, T_λ defines a smooth distribution along M. If $\bar{M}$ is a Riemannian manifold, then the shape operator $\mathcal{S}$ is diagonalizable at every point, since it is a self-adjoint map and the metric is positive definite. However, if M is not Riemannian, this is not necessarily true, and the Jordan canonical form of $\mathcal{S}$ might have a non-diagonal structure. In such situations it is important to distinguish between the *geometric multiplicity* of a principal curvature λ, that is, $\dim\ker(\mathcal{S} - \lambda)$, and its *algebraic multiplicity* m_λ, that is, the multiplicity of λ as a zero of the characteristic polynomial of $\mathcal{S}$. Obviously, the geometric multiplicity is always less or equal than the algebraic multiplicity. In the Riemannian setting both quantities are the same and we simply talk about the multiplicity of λ. In any case, the number of distinct principal curvatures at p is denoted by $g(p)$. In principle, g does not need to be a constant function.

The concepts of totally umbilical, totally geodesic and minimal submanifolds can be rewritten in terms of principal curvatures. Indeed, the submanifold M is totally umbilical if and only if for each normal vector ξ at each $p \in M$ all the eigenvalues of $\mathcal{S}_\xi$ coincide, for each $p \in M$; the submanifold M is totally geodesic if and only if for each unit normal vector ξ the shape operator $\mathcal{S}_\xi$ vanishes; and the submanifold M is minimal if and only if for each normal vector ξ the trace of the shape operator $\mathcal{S}_\xi$ is zero.

Another class of submanifolds that we will study in this thesis is the class of austere submanifolds (Chapter 6 and Chapter 7). A submanifold M of $\bar{M}$ is said to be *austere* if, for any unit normal vector, the set of principal curvatures with respect to such normal vector, counted with multiplicities, is invariant under multiplication by -1. Equivalently,

M is austere in $\bar{M}$ if all odd degree symmetric polynomials in the principal curvatures of M vanish. Clearly, totally geodesic submanifolds are austere, and austere submanifolds are always minimal.

A submanifold M of a Riemannian manifold $\bar{M}$ has *constant principal curvatures* if the principal curvatures of M are constant for any parallel normal vector field of M along any piecewise differentiable curve in M. Submanifolds with constant principal curvatures were introduced and studied by Heintze, Olmos and Thorbergsson [58] in the context of isoparametric submanifolds.

Assume now that M is a hypersurface of $\bar{M}$, that is, a submanifold of codimension one. Then, locally and up to sign, there exists a unique smooth normal vector field $\xi \in \Gamma(\nu M)$, with $\langle \xi, \xi \rangle = \epsilon \in \{-1, 1\}$. Put $\langle \xi, \xi \rangle = \epsilon$. In this case we write $\mathcal{S} = \mathcal{S}_\xi$ for the shape operator with respect to ξ. The Gauss and Weingarten formulas can now be written as

$$\bar{\nabla}_X Y = \nabla_X Y + \epsilon \langle \mathcal{S}X, Y \rangle \xi, \qquad \bar{\nabla}_X \xi = -\mathcal{S}X.$$

Then, the Gauss and Codazzi equations reduce to

$$\begin{aligned} \langle \bar{R}(X,Y)Z, W \rangle &= \langle R(X,Y)Z, W \rangle - \epsilon \langle \mathcal{S}Y, Z \rangle \langle \mathcal{S}X, W \rangle + \epsilon \langle \mathcal{S}X, Z \rangle \langle \mathcal{S}Y, W \rangle, \\ \langle \bar{R}(X,Y)Z, \xi \rangle &= \langle (\nabla_X \mathcal{S})Y - (\nabla_Y \mathcal{S})X, Z \rangle, \end{aligned}$$

whereas the Ricci equation does not give further information for hypersurfaces.

In the context of hypersurfaces, the mean curvature vector H is proportional to the vector ξ. Hence, we will usually refer to the *mean curvature* of the hypersurface, which is defined as the trace of its shape operator $\mathcal{S}$. Recall that locally and up to sign, there exists a unique unit normal vector field $\xi \in \Gamma(\nu M)$. A hypersurface is said to have *constant principal curvatures* if the eigenvalues of the shape operator $\mathcal{S} = \mathcal{S}_\xi$ are the same at every point. In this case, we will denote by T_λ the distribution on M formed by the principal curvature spaces of λ and by $\Gamma(T_\lambda)$ the set of all sections of T_λ, that is, the vector fields $X \in \Gamma(TM)$ such that $\mathcal{S}X = \lambda X$.

1.3 Isometric actions

In this section, we briefly review the basic concepts, notations and terminology related to the study of isometric actions on Riemannian manifolds. In particular, we introduce the concept of *(extrinsically) homogeneous submanifold*, that will be of great interest in this thesis. We refer to [6, Chapter 3] for a more detailed exposition on the topic.

Let $\bar{M}$ be a Riemannian manifold and let G be a Lie group. An *isometric action* φ of the Lie group G on the Riemannian manifold $\bar{M}$ is a smooth map

$$\varphi \colon G \times \bar{M} \to \bar{M}, \quad (g, p) \mapsto gp$$

satisfying that:

(i) $\varphi(g, \varphi(g', p)) = \varphi(gg', p)$ for all g, $g' \in G$ and all $p \in \bar{M}$,

(ii) $\varphi(e,p) = p$ for all $p \in \bar{M}$, where e is the identity element of G, and

(iii) the map $\varphi_g \colon \bar{M} \to \bar{M}$ given by $p \mapsto \varphi_g(p) = \varphi(g,p)$ is an isometry of $\bar{M}$, for each $g \in G$.

From now on, we will write gp instead of $\varphi(g,p)$ for the sake of simplicity. Let us introduce two crucial concepts concerning (isometric) actions. For each point $p \in \bar{M}$, the *orbit* of the action of G through p is defined as

$$G \cdot p = \{gp : g \in G\}$$

and the *isotropy group* or *stabilizer* at p is defined as

$$G_p = \{g \in G : gp = p\}.$$

If there is a point $p \in \bar{M}$ such that $G \cdot p = \bar{M}$, then the action φ is said to be *transitive* or that G acts *transitively* on $\bar{M}$. Moreover, when the isometric action is transitive, then $\bar{M}$ is said to be a *(Riemannian) homogeneous manifold*.

Furthermore, each orbit $G \cdot p$ of the isometric action of G on $\bar{M}$ is a submanifold (generally immersed) of $\bar{M}$. One may study the intrinsic geometry of this orbit with the induced metric. However, we will be interested in the geometry of the orbit $G \cdot p$ in relation to the geometry of $\bar{M}$, that is, the extrinsic geometry of $G \cdot p$. In this sense, an *(extrinsically) homogeneous submanifold* of $\bar{M}$ is an orbit of an isometric action on $\bar{M}$. Moreover, G acts transitively by isometries on each orbit $G \cdot p$ (with the induced metric). Hence, each orbit $G \cdot p = G/G_p$ is a Riemannian homogeneous manifold.

The group of isometries of $\bar{M}$, which we denote by $\mathrm{Isom}(\bar{M})$, turns out to be a Lie group [82]. Hence, we can consider a Lie group homomorphism $\rho \colon G \to I(\bar{M})$ defined as $\rho(g) = \varphi_g$. If this associated map ρ is injective, then the action is called *effective*. This means that the Lie group G is isomorphic to a subgroup of $\mathrm{Isom}(\bar{M})$. The action is said to be *free* if for every $p \in \bar{M}$ and every g, $h \in G$, the equality $gp = hp$ implies $g = h$. Finally, we say that G acts *simply transitively* on $\bar{M}$ when the action is free and transitive.

Consider two isometric actions $G \times \bar{M} \to \bar{M}$ and $G' \times \bar{M}' \to \bar{M}'$. We say that these isometric actions are *orbit equivalent* if there exists an isometry $f \colon \bar{M} \to \bar{M}'$ that maps the orbits of the G-action on $\bar{M}$ to the orbits of the G'-action on $\bar{M}'$. Furthermore, both isometric actions are said to be *conjugate* or *equivalent* if there is an isometry $f \colon \bar{M} \to \bar{M}'$ and a Lie group isomorphism $\psi \colon G \to G'$ such that $f(gp) = \psi(g)f(p)$ for all $p \in \bar{M}$ and all $g \in G$. It easily follows that two conjugate actions are in particular orbit equivalent.

Given an isometric action, we can derive certain orthogonal representations in a natural way. Recall that, roughly speaking, a representation of a Lie group G on a vector space V is a Lie group homomorphism $\rho \colon G \to GL(V)$. This representation ρ is said to be orthogonal if $\rho(g)$ is an orthogonal transformation of V for each $g \in G$.

As usual in this section, let $\varphi \colon G \times \bar{M} \to \bar{M}$ be an isometric action on a Riemannian manifold $\bar{M}$, and let $p \in \bar{M}$. Note that the isotropy group G_p fixes p and leaves the orbit $G \cdot p$ invariant. Hence, the differential of each isometry φ_g, for $g \in G_p$, leaves the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$ invariant. On the one hand, we call the action

$$G_p \times T_p(G \cdot p) \to T_p(G \cdot p), \qquad (g, X) \mapsto (\varphi_g)_{*p} X,$$

the *isotropy representation* of the action φ at p. On the other hand, the action

$$G_p \times \nu_p(G \cdot p) \to \nu_p(G \cdot p), \qquad (g, \xi) \mapsto (\varphi_g)_{*p} \xi,$$

is usually called the *slice representation* of the action φ at p.

An isometric action $\varphi\colon G \times \bar{M} \to \bar{M}$ is said to be *proper* if, for any two points p, $q \in \bar{M}$, there exist open neighbourhoods U_p and U_q of p and q in $\bar{M}$, respectively, such that the set $\{g \in G : gU_p \cap U_q \neq \emptyset\}$ is relatively compact in G. Another equivalent definition is that the map

$$G \times \bar{M} \to \bar{M} \times \bar{M}, \qquad (g, p) \mapsto (p, gp)$$

is a proper map, that is, the inverse image of each compact set in $\bar{M} \times \bar{M}$ is also compact in $G \times \bar{M}$. This kind of isometric actions comes motivated for the following reason. If one considers the space of orbits of the action of G on $\bar{M}$, namely $\bar{M}/G$ with the quotient topology, it is not necessarily Hausdorff. However, if the Lie group G acts properly on $\bar{M}$, then $\bar{M}/G$ is a Hausdorff space. Moreover, each isotropy group G_p is compact, and each orbit $G \cdot p$ is closed in $\bar{M}$ and therefore an embedded submanifold [39].

In order to finish this section, we will focus on the three different types of orbits that can appear when one considers a proper isometric action. Let us consider an orbit $G \cdot p$, for some $p \in \bar{M}$. If for each $q \in \bar{M}$ the isotropy group G_p at p is conjugate in G to some subgroup of G_q, then $G \cdot p$ is said to be a *principal orbit*. Equivalently, any orbit $G \cdot p$ of a proper action is principal if and only if the slice representation at p is trivial. It is interesting to point out that the union of all principal orbits is a dense and open subset of $\bar{M}$. Principal orbits are orbits of maximal dimension. The codimension of any principal orbit is the *cohomogeneity* of the action. An *exceptional orbit* is any non-principal orbit of maximal dimension. Finally, a *singular orbit* is an orbit whose dimension is less than the dimension of a principal orbit. In other words, an orbit whose codimension is greater than the cohomogeneity of the action is called a singular orbit.

1.3.1 Cohomogeneity one actions

A *cohomogeneity one action* of a Lie group G on a Riemannian manifold $\bar{M}$ is an isometric action of G on $\bar{M}$ whose principal orbits have codimension one. In such a case, $\bar{M}$ is called a *cohomogeneity one manifold*.

In this thesis we will not study cohomogeneity one actions directly. However, on the one hand, in Chapter 2 and in Chapter 3 we deal with isoparametric hypersurfaces, which can be understood as generalizations of principal orbits of cohomogeneity one actions. On the other hand, in Chapter 5 we introduce the notion of CPC submanifold, which is intimately related to the notion of cohomogeneity one action.

The classification of cohomogeneity one actions (up to orbit equivalence) is an important problem in differential geometry. This is probably due to the fact that they allow to reduce certain partial differential equations on $\bar{M}$ to ordinary differential equations. Indeed, cohomogeneity one actions have been successfully utilized, for example, in the construction of Einstein, Einstein-Kähler and Einstein-Weyl structures [1, 21], in the inspection of Yang-Mills equations [105], and also to construct hyper-Kähler Calabi metrics [37] and special Lagrangian submanifolds [68].

Cohomogeneity one actions have been classified in space forms. In non-positive curvature, this traces back to the works of Somigliana [96], Levi-Civita [75], Segre [93] and Cartan [25]. We include these results in Section 2.1. The classification in spheres follows from the work of Hsiang and Lawson [61] decades later.

1.4 Symmetric spaces

In this thesis, we will be particularly interested in submanifold theory of symmetric spaces. Therefore, this section is completely devoted to a brief introduction to them. Indeed, we start by presenting the notion of symmetric space and the first properties one can derive from it. We will also distinguish among the different types of symmetric spaces and we finally introduce the main algebraic tools to be used in this thesis.

There are several references that the reader may like to consult to obtain further information on this topic. Probably, the most well-known and complete references are Helgason's book [59] and Loos' books [78, 79]. Eschenburg's survey [51] and Ziller's notes [112] are great references. The books by Besse [20], Kobayashi and Nomizu [73], O'Neill [86] and Wolf [109] also include nice chapters on symmetric spaces. In this section we mainly follow [59] and [112].

Firstly, let us fix some notations concerning Lie groups and Lie algebras. In general, for each Lie group G, we denote its Lie algebra by the corresponding gothic letter $\mathfrak{g}$. We denote by Exp the Lie exponential map. Consider the conjugation map $I_g\colon G \to G$, $h \mapsto ghg^{-1}$, for each $g \in G$. Let $\operatorname{Aut}(\mathfrak{g})$ be the group of automorphisms of the Lie algebra $\mathfrak{g}$, that is, the linear bijective transformations $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ such that $\varphi[X,Y] = [\varphi X, \varphi Y]$ for all X, $Y \in \mathfrak{g}$. Then, the Lie group adjoint map $\operatorname{Ad}\colon G \to \operatorname{Aut}(\mathfrak{g})$, $g \to (I_g)_{*e}$, is defined as the differential of I_g at the identity element e of G. Moreover, the differential of Ad at the identity element of G leads to the Lie algebra adjoint map $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$, $X \mapsto \operatorname{ad}(X) = [X, \cdot\,]$.

Let M be a connected Riemannian manifold and take a point $o \in M$. Take $r > 0$ sufficiently small such that normal coordinates are defined on the open geodesic ball $B_o(r) = \{p \in M : d(o,p) < r\}$. We can always consider a smooth map $\sigma_o\colon B_o(r) \to B_o(r)$ that sends each $p = \exp_o(v)$ to $\sigma_o(p) = \exp_o(-v)$, for $v \in T_oM$, $\|v\| < r$. We call this map a local geodesic symmetry. A Riemannian manifold M is said to be *locally symmetric* if at each point there is a ball such that the corresponding local geodesic symmetry is a local isometry. Moreover, a locally symmetric space is characterized by the fact that $\nabla R = 0$. A connected Riemannian manifold M is called a *(Riemannian) symmetric space* if each local geodesic symmetry can be extended to a global isometry.

We can easily deduce from the definition that symmetric spaces are complete, since geodesics can be extended by using geodesic reflections. Moreover, symmetric spaces are examples of homogeneous spaces, that is, for any p_1, $p_2 \in M$ there is an isometry φ of M mapping p_1 to p_2. Actually, it suffices to take $\varphi = \sigma_q$, where q is the midpoint of a geodesic joining p_1 and p_2.

Let M be a symmetric space. In the following lines we will focus on a more algebraic description of symmetric spaces. Let $G = \mathrm{Isom}^0(M)$ be the connected component of the identity of the isometry group of M. Fix a point $o \in M$, consider the geodesic symmetry s_o at the point o and let $K = G_o$ be the isotropy group of G at o. Note that K is compact. Then, M is diffeomorphic to the coset space G/K by means of the map $\Phi\colon G/K \to M$ defined by $gK \to g(o)$. With a pullback of the metric of M, the map Φ becomes an isometry. Since for each $h \in G$ the map defined by $gK \to hgK$ is an isometry, the induced metric $\langle \cdot, \cdot \rangle$ in G/K is G-invariant. The *isotropy representation* of the symmetric space $M \cong G/K$ at the point o is the orthogonal representation defined by $K \times T_oM \to T_oM$, $(k, v) \to k_{*o}v$.

Recall that o is a fixed point in M and that $K = G_o$. Let us define the involutive Lie group automorphism $s\colon G \to G$, $g \mapsto \sigma_o g \sigma_o$. It satisfies $G_s^0 \subset K \subset G_s$, where $G_s = \{g \in G : s(g) = g\}$ and G_s^0 is the connected component of the identity of G_s. The differential $\theta = s_*\colon \mathfrak{g} \to \mathfrak{g}$ of s is a Lie algebra automorphism called the *Cartan involution* of the symmetric space (at the Lie algebra level). The isotropy Lie algebra $\mathfrak{k}$ is the eigenspace of θ with eigenvalue 1. Let $\mathfrak{p}$ be the (-1)-eigenspace of θ. The eigenspace decomposition of θ then reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which is called the *Cartan decomposition*. Moreover, it easily follows that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Let B be the Killing form of the Lie algebra $\mathfrak{g}$, that is, $B(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ for X, $Y \in \mathfrak{g}$. Using the bracket relations and the definition of the Killing form it follows that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal subspaces with respect to B.

Since the vector space $\mathfrak{p}$ is a complementary subspace to $\mathfrak{k}$ in $\mathfrak{g}$, it can be identified with T_oM by means of the map Φ. Thus, $\mathfrak{p}$ can be endowed with an inner product that turns out to be $\mathrm{Ad}(K)$-invariant. Indeed, the isotropy representation of G/K explained above is equivalent to the adjoint representation of K in $\mathfrak{p}$, $K \times \mathfrak{p} \to \mathfrak{p}$, given by $(k, X) \to \mathrm{Ad}(k)X$.

If M is a connected, complete, locally symmetric Riemannian manifold, then its Riemannian universal covering is a symmetric space. In particular, every locally symmetric space is locally isometric to a symmetric space.

Let $M = G/K$ be a symmetric space and let us write $\widetilde{M}$ for its universal covering. The isotropy representation allows to distinguish different types of symmetric spaces. Indeed, if the restriction of the isotropy representation of $M \cong G/K$ to the connected component K^0 of the identity of K is irreducible, then we say that the symmetric space M is *irreducible*. This turns out to be equivalent to the property that the universal cover $\widetilde{M}$ of M (which is always a symmetric space) cannot be written as a non-trivial product of symmetric spaces, unless $\widetilde{M}$ is some Euclidean space $\mathbb{R}^n$. Furthermore, according to De Rham Theorem, we can decompose the universal covering as $\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k$, where $\widetilde{M}_0$ is locally isometric to a Euclidean space and $\widetilde{M}_i$ is a simply connected irreducible symmetric space, with $i \in \{1, \dots, k\}$. A symmetric space G/K is said to be semisimple if its universal covering does not have a Euclidean factor.

A symmetric space $M \cong G/K$ is said to be of *compact type*, of *non-compact type* or of *Euclidean type* if $B|_{\mathfrak{p}\times\mathfrak{p}}$, the restriction to $\mathfrak{p}$ of the Killing form B of $\mathfrak{g}$, is negative definite, positive definite or identically zero, respectively. If M is irreducible, then Schur's lemma implies that $B|_{\mathfrak{p}\times\mathfrak{p}}$ is a scalar multiple of the induced metric on $\mathfrak{p} \cong T_oM$ and, according to the sign of such scalar, M falls into exactly one of the three possible types. It turns out that if M is of compact type, then G is a compact semisimple Lie group, and M is compact and of non-negative sectional curvature; if M is of non-compact type, then G is a non-compact real semisimple Lie group, and M is non-compact (indeed, diffeomorphic to a Euclidean space) and with non-positive sectional curvature; and if M is of Euclidean type, its Riemannian universal covering is a Euclidean space $\mathbb{R}^n$. Moreover, in general, the universal cover of a symmetric space M splits as a product $\widetilde{M} = M_0 \times M_+ \times M_-$, where $M_0 = \mathbb{R}^n$ is of Euclidean type, M_+ is of compact type, and M_- is of non-compact type.

Symmetric spaces of compact and non-compact type are related via the notion of duality. Being more specific, there is a one-to-one correspondence between simply symmetric spaces of compact type and (necessarily simply connected) symmetric spaces of non-compact type. Without entering into details, the trick at the Lie algebra level to obtain the dual symmetric space is to change $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ by the new Lie algebra $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$, where $i = \sqrt{-1}$ is the imaginary unit. Let G^* be the simply connected real Lie group whose Lie algebra is $\mathfrak{g}^*$. Then, we obtain that G^*/K is a symmetric space that we call the dual symmetric space of G/K. If G/K is of compact type, then G^*/K is of non-compact type, and if G/K is of non-compact type, then G^*/K is of compact type. In spite of the simplicity of this procedure, dual symmetric spaces have, of course, very different geometric and even topological properties. However, dual symmetric spaces have equivalent isotropy representations and, therefore, irreducibility is preserved by duality.

Among different kinds of Riemannian submanifolds, the totally geodesic ones typically play an important role. This is particularly true in the case of symmetric spaces. Indeed, although the classification problem of totally geodesic submanifolds in symmetric spaces is still outstanding, these submanifolds are, intrinsically, also symmetric, and admit a neat algebraic characterization. The *rank* of a symmetric space M is defined as the maximal dimension of a totally geodesic and flat submanifold of M or, equivalently, the dimension of a maximal abelian subspace of $\mathfrak{p}$. The rank is an invariant that is preserved under duality.

Another interesting problem in symmetric spaces is the classification of cohomogeneity one actions. This classification was achieved in irreducible symmetric spaces of compact type. In the rank one case, Hsiang and Lawson [61] obtained the classification in spheres, Takagi [98] on complex projective spaces and Iwata for the quaternionic [66] and the Cayley [67] cases. Many years later, Kollross classified cohomogeneity one actions on irreducible compact symmetric spaces of rank greater than one [74].

The techniques utilized to classify cohomogeneity one actions in compact symmetric spaces do not hold for non-compact symmetric spaces, where the problem remains open. However, cohomogeneity one actions in symmetric spaces of non-compact type have been thoroughly investigated and classifications have been achieved by Berndt and Tamaru under the following extra assumptions: cohomogeneity one actions that produce regular foliations [15]; cohomogeneity one actions with a totally geodesic singular orbit [16]; and cohomogeneity one actions in rank one symmetric spaces of non-compact type [17], except for the quaternionic hyperbolic spaces $\mathbb{H}H^n$, $n \geq 3$. Also Berndt and Tamaru have made remarkable progress in [16] for higher rank (see also [13]). However, a complete classification is still open.

Among the different kinds of symmetric spaces, the Hermitian ones constitute a remarkable subclass. Indeed, one of them, namely the complex hyperbolic space will be of great interest in this thesis (see Chapter 2 and Chapter 3). We recall some definitions concerning complex, Hermitian and Kähler manifolds following [73].

On the one hand, an endomorphism J of a vector space V is said to be *a complex structure* if $J^2 = -\operatorname{id}$. Note that V is a complex vector space if and only it has a complex structure. Moreover, if V has an inner product $\langle \cdot, \cdot \rangle$ such that $\langle u, v \rangle = \langle Ju, Jv \rangle$ for all u, $v \in V$, then $\langle \cdot, \cdot \rangle$ is said to be a *Hermitian inner product.* An *almost complex structure* on a manifold M is a tensor field that defines a complex structure in each tangent vector space T_pM, with $p \in M$.

A *complex manifold* is a manifold that admits charts with image onto open subsets of $\mathbb{C}^n$ such that the coordinate transitions are holomorphic. This induces an almost complex structure J on M, which is as an endomorphism of the tangent bundle TM of M such that $J^2 = -\operatorname{id}$. If M is a complex Riemannian manifold, and the metric is Hermitian in each tangent space, then M is called a *Hermitian manifold.* A *Kähler manifold* is a Hermitian manifold M satisfying $\nabla J = 0$, where ∇ is the Levi-Civita connection of M. The endomorphism J is known as the *Kähler structure* or the *complex structure* of M.

Thus, a symmetric space M is *Hermitian* if it is a Hermitian manifold and the geodesic symmetries s_p, $p \in M$, are holomorphic transformations. It occurs that every Hermitian symmetric space is Kähler. Moreover, a symmetric space M is Hermitian if and only if its dual is Hermitian, and every Hermitian symmetric space is simply connected.

Let M be a Kähler manifold and denote by J and R its complex structure and its curvature tensor, respectively. The *holomorphic sectional curvature* K_h of M is defined as the restriction of the sectional curvature K to J-invariant 2-dimensional subspaces of the form $\{v, Jv\}$, with $v \in T_pM$, for $p \in M$. Note that K_h can be thought as a function that maps each unit vector $v \in T_pM$ to the real number $K(v, Jv) = \langle R(v, Jv)Jv, v \rangle$.

A Kähler manifold is said to have *constant holomorphic curvature* if K_h is constant for any unit tangent vector of M. If M has constant holomorphic curvature c then its curvature tensor reads [111]

$$R(X,Y)Z = \frac{c}{4}\left(\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY - 2\langle JX, Y\rangle JZ\right).$$

1.5 Symmetric spaces of non-compact type

In this thesis we will focus on symmetric spaces of non-compact type and this section is devoted to describe the tools and structures that will be used throughout this work in order to study them. The main purpose will be to explain the fact that any symmetric space of non-compact type is isometric to a solvable Lie group endowed with a left-invariant metric.

For more information or detailed proofs one can consult, for instance, Eberlein's [50, Chapter 2], Helgason's [59, Chapter VI] or Knapp's books [69, Chapter VI, Section 4-5]. A nice survey that includes a detailed description of the space $SL_n(\mathbb{R})/SO_n$ can be found in [4]. We refer to the survey [45] for an exposition of the study of submanifold geometry of symmetric spaces. In the following lines, we mainly follow [4], [69] and [45].

We will start by describing some important decompositions of the Lie algebra of the isometry group (Subsection 1.5.1), and then we present the Lie group model of a symmetric space of non-compact type (Subsection 1.5.2).

1.5.1 Root space and Iwasawa decompositions

Let $M \cong G/K$ be an arbitrary symmetric space of non-compact type. Then $\mathfrak{g}$ is a real semisimple Lie algebra, which implies that its Killing form B is non-degenerate. Indeed, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is B-orthogonal, $B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite (due to the compactness of K), and $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite (since M is of non-compact type). Hence, by reverting the sign on $\mathfrak{k} \times \mathfrak{k}$ or, equivalently, by defining

$$\langle X, Y\rangle_{B_\theta} = -B(\theta X, Y),$$

for $X, Y \in \mathfrak{g}$, we have that $\langle \cdot, \cdot\rangle_{B_\theta}$ defines a positive definite inner product on $\mathfrak{g}$. It is easy to check that this inner product satisfies

$$\langle \mathrm{ad}(X)Y, Z\rangle_{B_\theta} = -\langle Y, \mathrm{ad}(\theta X)Z\rangle_{B_\theta}, \tag{1.1}$$

for all $X, Y, Z \in \mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. One can show that any two choices for $\mathfrak{a}$ are conjugate under the adjoint action of K (similar to the fact that any two maximal abelian subalgebras of a compact Lie algebra are conjugate to each other). Moreover, recall that the rank of $M \cong G/K$ is the dimension of $\mathfrak{a}$. For each $H \in \mathfrak{a}$, $X, Y \in \mathfrak{g}$, we have that

$$\langle \mathrm{ad}(H)X, Y\rangle_{B_\theta} = -\langle X, \mathrm{ad}(\theta H)Y\rangle_{B_\theta} = \langle X, \mathrm{ad}(H)Y\rangle_{B_\theta},$$

which means that each operator $\mathrm{ad}(H) \in \mathrm{End}(\mathfrak{g})$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot\rangle_{B_\theta}$. Moreover, if $H_1, H_2 \in \mathfrak{a}$, then $[\mathrm{ad}(H_1), \mathrm{ad}(H_2)] = \mathrm{ad}[H_1, H_2] = 0$, since $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a Lie algebra homomorphism and $\mathfrak{a}$ is abelian. Thus, $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ constitutes a commuting family of self-adjoint endomorphisms of $\mathfrak{g}$. Therefore, they diagonalize simultaneously. Their common eigenspaces are called the *restricted root spaces*, whereas their non-zero eigenvalues (which depend linearly on $H \in \mathfrak{a}$) are called the *restricted roots* of $\mathfrak{g}$. In other words, if for each covector $\lambda \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\},$$

then any $\mathfrak{g}_\lambda \neq 0$ is a restricted root space, and any $\lambda \neq 0$ such that $\mathfrak{g}_\lambda \neq 0$ is a restricted root. Note that $\mathfrak{g}_0$ is always non-zero, since $\mathfrak{a} \subset \mathfrak{g}_0$. If $\Sigma = \{\lambda \in \mathfrak{a}^* : \lambda \neq 0,\ \mathfrak{g}_\lambda \neq 0\}$ denotes the set of restricted roots, then we have the following $\langle \cdot, \cdot\rangle_{B_\theta}$-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda\right), \tag{1.2}$$

which is called the *restricted root space decomposition* of $\mathfrak{g}$.

Observe that these definitions depend on the choice of $o \in M$ (or, equivalently, of a Cartan involution θ of $\mathfrak{g}$) and on the choice of the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. However, different choices of o and $\mathfrak{a}$ give rise to decompositions that are conjugate under the adjoint action of G. For simplicity, in this thesis we will not specify this dependence and we will also omit the adjective "restricted". It is easy to check that we have the bracket relation

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu} \tag{1.3}$$

for any λ, $\mu \in \mathfrak{a}^*$. Moreover, we have the following properties:

(i) $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ and, hence, $\lambda \in \Sigma$ if and only if $-\lambda \in \Sigma$.

(ii) $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$.

For each $\lambda \in \Sigma$, define $H_\lambda \in \mathfrak{a}$ by the relation $B(H_\lambda, H) = \lambda(H)$, for all $H \in \mathfrak{a}$. Then we can introduce an inner product on $\mathfrak{a}^*$ defined by $\langle \lambda, \mu \rangle := B(H_\lambda, H_\mu)$. We will write $|\lambda|^2 = \langle \lambda, \lambda \rangle$ for the induced norm on $\mathfrak{a}^*$. Thus, with a bit more work one can show that Σ is an abstract root system in $\mathfrak{a}^*$, that is, it satisfies (see [69, p. 149]):

(a) $\mathfrak{a}^* = \operatorname{span} \Sigma$,

(b) for $\alpha, \beta \in \Sigma$, the number $A_{\alpha,\beta} = 2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$ is an integer,

(c) for $\alpha, \beta \in \Sigma$, we have $\beta - A_{\alpha,\beta}\, \alpha \in \Sigma$.

This system may be non-reduced, that is, there may exist $\lambda \in \Sigma$ such that $2\lambda \in \Sigma$.

Now we can define a positivity criterion on Σ by declaring those roots that lie at one of the two half-spaces determined by a hyperplane in $\mathfrak{a}^*$ not containing any root to be positive. If Σ^+ denotes the set of positive roots, then $\Sigma = \Sigma^+ \cup (-\Sigma^+)$.

We define here the concept of string [69, p. 152], since it will play a crucial role in this thesis. Let $\alpha \in \Sigma$ and $\lambda \in \Sigma \cup \{0\}$. The *$\alpha$-string* containing λ is defined as the set of all elements in $\Sigma \cup \{0\}$ of the form $\lambda + n\alpha$ with $n \in \mathbb{Z}$.

We state now a result concerning the algebraic structure of the root system Σ. In particular, it provides really useful information about the Cartan integers, that is, the integers of the form

$$A_{\alpha,\lambda} = \frac{2\langle \alpha, \lambda \rangle}{|\alpha|^2},$$

where α, $\lambda \in \Sigma$. The calculation of Cartan integers allows to control how roots are constructed, and in particular, they allow to determine strings explicitly.

Proposition 1.5.1. [69, Proposition 2.48] *Let Σ be the restricted root system of a Riemannian symmetric space of non-compact type.*

(i) *If $\alpha \in \Sigma$, then $-\alpha \in \Sigma$.*

(ii) *If* $\alpha \in \Sigma$ *and* $\lambda \in \Sigma \cup \{0\}$*, then*

$$A_{\alpha,\lambda} = \frac{2\langle \lambda, \alpha \rangle}{|\alpha|^2} \in \{0, \pm 1, \pm 2, \pm 3 \pm 4\},$$

and ± 4 *occurs only when* Σ *is non-reduced and* $\lambda = \pm 2\alpha$*.*

(iii) *If* $\alpha, \lambda \in \Sigma$ *are non-proportional and* $|\lambda| \leq |\alpha|$*, then* $A_{\alpha,\lambda} \in \{0, \pm 1\}$*.*

(iv) *If* $\alpha, \lambda \in \Sigma$ *with* $\langle \alpha, \lambda \rangle > 0$*, then* $\alpha - \lambda \in \Sigma \cup \{0\}$*. If* $\alpha, \lambda \in \Sigma$ *with* $\langle \alpha, \lambda \rangle < 0$*, then* $\alpha + \lambda \in \Sigma \cup \{0\}$*.*

(v) *If* $\alpha \in \Sigma$ *and* $\lambda \in \Sigma \cup \{0\}$*, then the* α*-string containing* λ *has the form* $\lambda + n\alpha$ *for* $-p \leq n \leq q$ *with* $p, q \geq 0$*. There are no gaps. Furthermore* $p - q = A_{\alpha,\lambda}$*. The* α*-string containing* λ *contains at most four roots.*

As it is usual in the theory of root systems, one can consider a subset $\Pi \subset \Sigma^+$ of simple roots. A positive root is *simple* if it cannot be written as the sum of two positive roots. The set of simple roots Π is a basis of $\mathfrak{a}^*$ made of positive roots such that any $\lambda \in \Sigma$ is a linear combination of the roots in Π where all coefficients are either non-negative integers or non-positive integers. More precisely, each root $\lambda \in \Sigma$ can be written as $\lambda = \sum_{\alpha \in \Pi} n_\alpha \alpha$, where the coefficients n_α are either all non-negative or all non-positive integers depending on whether λ is a positive root or a negative root, respectively. For each root $\lambda = \sum_{\alpha \in \Pi} n_\alpha \alpha \in \Sigma$, the sum

$$l(\lambda) = \sum_{\alpha \in \Pi} n_\alpha \tag{1.4}$$

is called the *level* of the root λ. Note that positive roots have positive level and negative roots have negative level. Of course, the cardinality of Π agrees with the dimension of $\mathfrak{a}$, that is, with the rank of G/K. The set Π of simple roots allows to construct the Dynkin diagram associated with the root system Σ, which is a graph whose nodes correspond to the simple roots. The nodes corresponding to the simple roots α, $\beta \in \Pi$ are joined by $A_{\alpha,\beta} \cdot A_{\beta,\alpha}$ edges. Moreover, if the system is non-reduced, two collinear positive roots are drawn as two concentric nodes. Due to the properties of the root space decomposition, the subspace

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$$

of $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$ such that $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$. Any two choices of positivity criteria on Σ give rise to isomorphic Dynkin diagrams and to nilpotent subalgebras $\mathfrak{n}$ that are conjugate by an element of $N_K(\mathfrak{a})$.

A fundamental result in what follows is the *Iwasawa decomposition theorem.* At the Lie algebra level, it states that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is a vector space direct sum (but neither orthogonal direct sum nor semidirect product). Let us denote by A and N the connected Lie subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Since $\mathfrak{a}$ normalizes $\mathfrak{n}$, the semidirect product AN is the connected Lie subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$. Then the Iwasawa decomposition theorem at the Lie group level states that the multiplication map

$$K \times A \times N \to G, \qquad (k, a, n) \mapsto kan$$

is an analytic diffeomorphism, and the Lie groups A and N are simply connected. Indeed, as A is abelian and N is nilpotent, they are both diffeomorphic to Euclidean spaces [69, Theorem 1.127]. Hence, the semidirect product AN is also diffeomorphic to a Euclidean space.

1.5.2 The solvable Lie group model

As above, let $M \cong G/K$ be a symmetric space of non-compact type, where K is the isotropy group at some point $o \in M$. Consider the smooth map $\phi\colon G \to M$, $h \mapsto h(o)$. The restriction $\phi|_{AN}\colon AN \to M$ is injective; indeed, if $\phi(h) = \phi(h')$ with $h, h' \in AN$, then $h^{-1}h'(o) = o$, and hence $h^{-1}h' \in K \cap AN$, which, by the Iwasawa decomposition, implies that $h^{-1}h' = e$. It is also onto: if $p \in M$, then by the transitivity of G there exists $h \in G$ such that $h(p) = o$, but using the Iwasawa decomposition we can write $h = kan$, with $k \in K$, $a \in A$, $n \in N$, and then $p = h^{-1}(o) = n^{-1}a^{-1}k^{-1}(o) = (an)^{-1}(o)$. Finally, $\phi|_{AN}$ is a local diffeomorphism, since $\ker \phi_{*e} = \mathfrak{k}$, hence $(\phi|_{AN})_{*e}\colon \mathfrak{a} \oplus \mathfrak{n} \to T_oM$ is an isomorphism, and by homogeneity any other differential $(\phi|_{AN})_{*h}$ is also bijective.

Therefore, $\phi|_{AN}\colon AN \to M$ is a diffeomorphism. If we denote by g the Riemannian metric on M, we can pull it back to obtain a Riemannian metric $(\phi|_{AN})^*g$ on AN. Hence, we trivially have that (M, g) and $(AN, (\phi|_{AN})^*g)$ are isometric Riemannian manifolds.

Let now $h, h' \in AN \subset G$, and denote by L_h the left multiplication by h in G. Then

$$(h^{-1} \circ \phi|_{AN} \circ L_h)(h') = h^{-1}(hh'(o)) = h'(o) = \phi|_{AN}(h'),$$

from where we get $h^{-1} \circ \phi|_{AN} \circ L_h = \phi|_{AN}$ as maps from AN to M. Since h^{-1} is an isometry of (M, g), and using the previous equality, we have

$$L_h^*(\phi|_{AN})^*g = L_h^*(\phi|_{AN})^*(h^{-1})^*g = (h^{-1} \circ \phi|_{AN} \circ L_h)^*g = (\phi|_{AN})^*g.$$

This shows that $(\phi|_{AN})^*g$ is a left-invariant metric on the Lie group AN.

Altogether, we have seen that *any symmetric space $M \cong G/K$ of non-compact type is isometric to a solvable Lie group AN endowed with a left-invariant metric*. In particular, any symmetric space of non-compact type is diffeomorphic to a Euclidean space and, since it is non-positively curved, it is a Hadamard manifold. This allows us to regard any of these spaces as an open Euclidean ball endowed with certain metric, as happens with the well-known ball model of the real hyperbolic space.

Moreover, it is sometimes useful to view a symmetric space of non-compact type M as a dense and open subset of a bigger compact topological space $M \cup M(\infty)$ which, in this case, would be homeomorphic to a closed Euclidean ball. In order to do so, one defines an equivalence relation on the family of complete, unit-speed geodesics in M: if γ and σ are two of them, we declare them equivalent if they are asymptotic, that is, if $d(\gamma(t), \sigma(t)) \leq C$, for certain constant C and for all $t \geq 0$. Each equivalence class of asymptotic geodesics is called a *point at infinity*, and the set $M(\infty)$ of all of them is the *ideal boundary* of M. By endowing $M \cup M(\infty)$ with the so-called cone topology, $M \cup M(\infty)$ becomes homeomorphic to a closed Euclidean ball whose interior corresponds to M and its boundary to $M(\infty)$. Two geodesics are asymptotic precisely when they converge to the same point in $M(\infty)$. We refer to [50, §1.7] for more details.

The Lie group model turns out to be a powerful tool for the study of submanifolds of symmetric spaces of non-compact type. The reason is that one can *consider interesting types of submanifolds by looking at subgroups of AN* or, equivalently, at subalgebras of $\mathfrak{a} \oplus \mathfrak{n}$. A good understanding of the root space decomposition is crucial for that. Of course, not every submanifold (even extrinsically homogeneous submanifold) of M can be regarded as a Lie subgroup of AN, but very important types of examples arise in this way, sometimes combined with some additional constructions. In any case, if one wants to study submanifolds of AN with particular geometric properties, one needs to have manageable expressions for the left-invariant metric on AN and its Levi-Civita connection. We obtain the appropriate formulas below.

Let us denote by $\langle \cdot, \cdot \rangle_{AN}$ the inner product on $\mathfrak{a} \oplus \mathfrak{n}$ given by the left-invariant metric $(\phi|_{AN})^* g$ on AN in order to avoid confusions with the inner product $\langle \cdot, \cdot \rangle_{B_\theta}$. Assume for the moment that M is irreducible. Then, recall that the inner product $\phi^* g_o$ on $T_o M$ induced by the metric g on M is a scalar multiple of modified Killing form $\langle \cdot, \cdot \rangle_{B_\theta}$. Thus, after a rescaling of the metric we can and will assume that $\phi^* g_o = \langle \cdot, \cdot \rangle_{B_\theta}$. Now, we will find the relation between $\langle \cdot, \cdot \rangle_{AN}$ and $\langle \cdot, \cdot \rangle_{B_\theta}$. Thus, if $X, Y \in \mathfrak{a} \oplus \mathfrak{n}$, and denoting orthogonal projections (with respect to $\langle \cdot, \cdot \rangle_{B_\theta}$) with subscripts, we have

$$\begin{aligned}\langle X, Y\rangle_{AN} &= (\phi|_{AN})^* g_o(X_{\mathfrak{k}} + X_{\mathfrak{p}}, Y_{\mathfrak{k}} + Y_{\mathfrak{p}}) = g_o(\phi_* X_{\mathfrak{p}}, \phi_* Y_{\mathfrak{p}}) = \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}}\rangle_{B_\theta} \\ &= \langle \frac{1-\theta}{2} X, \frac{1-\theta}{2} Y\rangle_{B_\theta} = \frac{1}{4}\langle 2X_{\mathfrak{a}} + X_{\mathfrak{n}} - \theta X_{\mathfrak{n}}, 2Y_{\mathfrak{a}} + Y_{\mathfrak{n}} - \theta Y_{\mathfrak{n}}\rangle_{B_\theta} \\ &= \frac{1}{4}\left(4\langle X_{\mathfrak{a}}, Y_{\mathfrak{a}}\rangle_{B_\theta} + \langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle_{B_\theta} + \langle \theta X_{\mathfrak{n}}, \theta Y_{\mathfrak{n}}\rangle_{B_\theta}\right) \\ &= \langle X_{\mathfrak{a}}, Y_{\mathfrak{a}}\rangle_{B_\theta} + \frac{1}{2}\langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle_{B_\theta}. \qquad (1.5)\end{aligned}$$

If M is reducible, one can adapt the argument (by defining $\langle \cdot, \cdot \rangle_{B_\theta}$ as a suitable multiple of B_θ on each factor) to prove the same formula. Using Koszul formula and relations (1.5) and (1.1), one can obtain an important formula for the Levi-Civita connection ∇ of the Lie group AN. Indeed, if $X, Y, Z \in \mathfrak{a} \oplus \mathfrak{n}$, and taking into account that $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] \subset \mathfrak{n}$, we have

$$\begin{aligned}\langle \nabla_X Y, Z\rangle_{AN} &= \frac{1}{2}\big(\langle [X,Y],Z\rangle_{AN} - \langle [Y,Z],X\rangle_{AN} - \langle [X,Z],Y\rangle_{AN}\big)\\ &= \frac{1}{4}\big(\langle [X,Y],Z\rangle_{B_\theta} - \langle [Y,Z],X\rangle_{B_\theta} - \langle [X,Z],Y\rangle_{B_\theta}\big)\\ &= \frac{1}{4}\langle [X,Y] + [\theta X, Y] - [X,\theta Y], Z\rangle_{B_\theta}. \end{aligned} \tag{1.6}$$

Note that we have started and finished with different inner products. Thus, in order to obtain an explicit formula for $\nabla_X Y$, one has to impose some restrictions on X and Y. For example, if X and Y do not belong to the same root space, then $[\theta X, Y]$ and $[X, \theta Y]$ are orthogonal to $\mathfrak{a}$, whence in this case $2\nabla_X Y = \big([X,Y] + [\theta X, Y] - [X, \theta Y]\big)_{\mathfrak{a}\oplus\mathfrak{n}}$.

1.6 Anti-De Sitter and complex hyperbolic spaces

Let $\mathbb{R}^{n+1}_2$, $n \geq 3$, denote the $(n+1)$-dimensional real vector space endowed with the semi-Riemannian metric $\langle x, y\rangle = -x_1y_1 - x_2y_2 + \sum_{i=3}^{n+1} x_iy_i$. This metric has signature $(n-1, 2)$. We define the anti-De Sitter space of radius r, $H^n_1(r)$, as

$$H^n_1(r) = \{x \in \mathbb{R}^{n+1}_2 \mid \langle x, x\rangle = -r^2\}.$$

Let D denote the Levi-Civita connection of $\mathbb{R}^{n+1}_2$ and $\mathcal{S}$ the shape operator of $H^n_1(r)$ as a submanifold of $\mathbb{R}^{n+1}_2$. Consider the normal vector field to $H^n_1(r)$ given by $\xi_z = z/r$, for each $z \in H^n_1(r)$, and let X be a tangent vector to the anti-De Sitter space. Note that $\langle \xi, \xi\rangle = -1$. Then, we have

$$\mathcal{S}X = -\frac{1}{r}X.$$

for each tangent vector X to $H^n_1(r)$. Therefore, we deduce that

$$I\!I(X,Y) = \langle I\!I(X,Y), \xi\rangle\langle \xi,\xi\rangle\xi = -\langle \mathcal{S}X, Y\rangle\xi = \frac{1}{r}\langle X, Y\rangle\xi,$$

for all vector fields X, Y and Z tangent to the anti-De Sitter space $H^n_1(r)$. Let $\tilde{\nabla}$ be the Levi-Civita connection of H^n_1. Then, the Gauss formula reads as

$$D_X Y = \tilde{\nabla}_X Y + \frac{\langle X, Y\rangle}{r}\xi,$$

and Gauss equation can be written as

$$\tilde{R}(X,Y)Z = -\frac{1}{r^2}(\langle Y, Z\rangle X - \langle X, Z\rangle Y).$$

Therefore, the anti-De Sitter space is a Lorentzian manifold with negative constant (sectional) curvature $-1/r^2$. It can be thought as the Lorentzian analogue of the real hyperbolic space.

In the following lines we briefly construct the complex hyperbolic space. In order to do so, consider the anti-De Sitter spaces of the form $H_1^{2n+1}(r) \subset \mathbb{R}_2^{2n+2}$. Let J be a complex structure $\mathbb{R}_2^{2n+2}$, satisfying $\langle Jx, Jy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}_2^{2n+2}$. This allows us to identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, where the multiplication by the imaginary unit i is induced by J. We will consider the following equivalence relation on $H_1^{2n+1}(r)$: two elements z, $z' \in H_1^{2n+1}(r)$ are related if and only if there exists an element $\lambda \in S^1 \subset \mathbb{C}$ such that $z' = \lambda z$. The complex hyperbolic space is the smooth quotient manifold $\mathbb{C}H^n = H_1^{2n+1}(r)/\sim$. The canonical projection $\pi\colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ is the so-called *Hopf map*.

Now, we will equip the complex hyperbolic space with a metric that is induced by the metric of the anti-De Sitter space. In order to do so, let ξ be a unit vector normal to $H_1^{2n+1}(r)$, and define $V = J\xi$. Since J is a complex structure, we have that V is a vector field tangent to $H_1^{2n+1}(r)$. Moreover, it satisfies $\langle V, V\rangle = -1$. Consider the orthogonal decomposition

$$T_z H_1^{2n+1}(r) = \mathbb{R}V_z \oplus V_z^{\perp}$$

of the tangent space of $H_1^{2n+1}(r)$ at z, where $V_z^{\perp}$ denotes the orthogonal complement of V_z in $T_z H_1^{2n+1}(r)$ with respect to the Lorentzian metric of the anti-De Sitter space $H_1^{2n+1}(r)$. Moreover, $\ker \pi_{*z} = \mathbb{R}V_z$. Thus, arguing by dimensions, it follows that $\pi_{*z}|_{V_z^{\perp}}$ is an isomorphism between the vector spaces $V_z^{\perp}$ and $T_{\pi(z)}\mathbb{C}H^n$, for each $z \in H_1^{2n+1}(r)$. Therefore, for each $X_{\pi(z)} \in T_{\pi(z)}\mathbb{C}H^n$ we define the horizontal lift X_z^L of $X_{\pi(z)}$ to z as the unique tangent vector in $V_z^{\perp}$ such that $\pi_* X_z^L = X$. This allows to define a metric in the complex hyperbolic space given by

$$\langle X, Y\rangle = \left\langle X^L, Y^L\right\rangle$$

for the vectors X, $Y \in T_{\pi(z)}\mathbb{C}H^n$, which is independent of the base point $z \in H_1^{2n+1}(r)$ of the lifts. Moreover, the (well-defined) map J given by

$$JX = \pi_{*z} J X_z^L$$

defines a complex structure for the complex hyperbolic space (for the sake of simplicity we use J for both complex structures). An important point here is the fact that the metric of $H_1^{2n+1}(r)$ is positive definite on $V_z^{\perp}$ and, hence, the metric on $\mathbb{C}H^n$ is positive definite. This means that $\mathbb{C}H^n$ becomes a Riemannian manifold. This metric, called the *Bergman metric* of $\mathbb{C}H^n$, makes $\pi\colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ a semi-Riemannian submersion. Moreover, the Bergman metric satisfies that $\langle JX, JY\rangle = \langle X, Y\rangle$ for any tangent vectors X and Y. From the formulas for semi-Riemannian submersions (see [85] or [86, p. 213]), the Levi-Civita connection of $\mathbb{C}H^n$ is given by

$$\bar{\nabla}_X Y = \pi_* \left(\tilde{\nabla}_{X^L} Y^L\right),$$

for tangent vector fields X, Y on $\mathbb{C}H^n$. Using this formula one can show that J is Kähler. Again, the theory of semi-Riemannian submersions allows to calculate the holomorphic sectional curvature of $\mathbb{C}H^n$, which turns out to be $-4/r^2$ for every $X \in T\mathbb{C}H^n$. Therefore,

$\mathbb{C}H^n$ is a space of constant holomorphic curvature $c = -4/r^2$. Now, the curvature tensor $\bar{R}$ of $\mathbb{C}H^n$ reads

$$\bar{R}(X,Y)Z = \frac{c}{4}\Big(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ\Big),$$

for $X, Y, Z \in T\mathbb{C}H^n$.

Chapter 2

Isoparametric hypersurfaces in the complex hyperbolic space: the examples

This chapter is devoted, on the one hand, to an exposition of the origin of isoparametric hypersurfaces, as well as some well-known results concerning them in the context of Riemannian geometry. On the other hand, we will describe the construction method and some geometric data of the known examples of isoparametric hypersurfaces in complex hyperbolic spaces. In Chapter 3 we will prove that these examples constitute a classification of isoparametric hypersurfaces in complex hyperbolic spaces.

We organize this chapter in the following way. In Section 2.1 we explain the origin of the concepts of isoparametric function and isoparametric hypersurface. In Section 2.2 we state the classification results of isoparametric hypersurfaces in Euclidean spaces and real hyperbolic spaces. We also provide some information on the problem in spheres, which is much more involved. In Section 2.3 we include a description of the complex hyperbolic space as a symmetric space. Section 2.4 is devoted to the study of the structure of a real subspace of a complex vector space by means of the notion of Kähler angle, which will allow to distinguish among the examples of isoparametric hypersurfaces in complex hyperbolic spaces that we introduce in Section 2.5.

2.1 Origin of the problem

The origin of the study of isoparametric hypersurfaces traces us back to the work of Somigliana in 1919 [96], where he addressed the following problem in the context of Geometric Optics. Let $\varphi\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be a solution to the wave equation

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial t^2},$$

where Δ is the Laplace operator of $\mathbb{R}^3$, that is, with respect to the space variables. We think of t as the time variable. A wavefront of φ is defined as the set of points that have a common oscillating state at a given instant $t = t_0$. Mathematically, for each instant t_0, they are the level surfaces of the function $f_{t_0}(x) = \varphi(x, t_0)$.

Somigliana was interested in waves satisfying two particular conditions. Firstly, let us assume that φ is a stationary wave, that is, its wavefronts do not depend on the time. Then, we can write f instead of f_{t_0}. Moreover, we have that the map $c(t) = \varphi(x_0, t)$ does not depend on the chosen point x_0, but on the wavefront containing the point x_0. Thus, for each $x \in f^{-1}(c(t_0))$ we have that

$$\Delta f(x) = \Delta\varphi(x, t_0) = \frac{\partial^2 \varphi}{\partial t^2}(x, t_0) = c''(t_0).$$

Hence, in mathematical terms, the stationary condition means that the Laplacian Δf is constant along the level sets of φ. Secondly, let us assume that the wavefronts of φ are parallel, that is, equidistant to each other. Somigliana refers to this condition as Huygens principle. Mathematically, this means that $||\operatorname{grad} f||$ is constant along the level sets of f, where $\operatorname{grad} f$ denotes the gradient of f.

In summary, an stationary wave φ with equidistant wavefronts leads to a function f whose Laplacian and norm of its gradient are constant along the level sets of f. The generalization of this idea is behind the origin of isoparametric hypersurfaces.

Indeed, the term *isoparametric hypersurface* was probably introduced by the mathematician Levi-Civita [75] in the year 1937, and it is intimately related to what we have explained above. Let $f\colon \bar{M} \to \mathbb{R}$ be a smooth function, where $\bar{M}$ is a Riemannian manifold. The *first* and the *second differential parameters* of f are, respectively,

$$\Delta_1 f = \|\operatorname{grad} f\|^2 \quad \text{and} \quad \Delta_2 f = \Delta f,$$

where Δ is the Laplace-Beltrami operator of $\bar{M}$ and $\operatorname{grad} f$ denotes the gradient of f. When the first and the second differential parameters of a non-constant function f are constant along the level sets of f, we say that f is an *isoparametric function*. A hypersurface is said to be an *isoparametric hypersurface* if it is a regular level set of an isoparametric function. In particular, note that f is isoparametric if and only if there exist real functions F_1 and F_2 of real variable such that

$$\Delta_1 f = F_1(f) \quad \text{and} \quad \Delta_2 f = F_2(f).$$

Usually, it is required that the function F_1 is smooth and the function F_2 is continuous in order to avoid dealing with complicated examples. We refer to [107] for more details.

Cartan found out an equivalent more geometric definition for isoparametric hypersurfaces. Thus, a hypersurface is isoparametric if and only if it and its *sufficiently close parallel hypersurfaces have constant mean curvature* [27]. Let us be more precise, and state this equivalent definition in the more general setting of semi-Riemannian geometry. Given a non-degenerate hypersurface M of a semi-Riemannian manifold $\bar{M}$, for $r \in \mathbb{R}$ close enough to zero we define the map $\Phi^r\colon M \to \bar{M}$ by $\Phi^r(p) = \exp_p(r\xi_p)$, where exp is the semi-Riemannian exponential map of $\bar{M}$ and ξ is a unit normal vector field on M. For a fixed r, $\Phi^r(M)$ is not necessarily a submanifold of $\bar{M}$, but at least locally and for r small enough, it is a hypersurface of $\bar{M}$. A parallel hypersurface at a distance r to a given hypersurface M is precisely a hypersurface of the form $\Phi^r(M)$. Thus M is isoparametric if and only if $\Phi^r(M)$ is a hypersurface with constant mean curvature, for all $r \in (-\epsilon, \epsilon)$ and some $\epsilon > 0$. In this thesis, we will use this second definition.

Isoparametric hypersurfaces have been studied thoroughly and their study has revealed many connections with different areas of mathematics such as Riemannian geometry, but also Lie group theory, algebraic geometry, algebraic topology, differential equations and Hilbert spaces. Even some applications in Physics have been found. For instance, see [89] and [94] for the appearance of isoparametric hypersurfaces in some problems of fluid mechanics, or [55] for certain relation between isoparametric families and Dirac operators.

2.2 Some classification results

In the following lines we will present some important classification results concerning isoparametric hypersurfaces. However, for a complete and more detailed approach to this topic, we refer to the surveys [103], [28], [104] and [32], and to the books [88] and [6].

As explained above, the study of isoparametric hypersurfaces traces back to the work of Somigliana [96], who studied isoparametric surfaces of the 3-dimensional Euclidean space, motivated by a problem in the context of Geometric Optics. This study was generalized by Segre [93], who classified isoparametric hypersurfaces in any Euclidean space. Indeed, Segre proved that one can extend the results of [96] and [75] to Euclidean spaces of arbitrary dimension.

Theorem 2.2.1. [93] *Let M be an isoparametric hypersurface in a Euclidean space $\mathbb{R}^n$. Then M has one or two constant principal curvatures and it is an open part of one of the following hypersurfaces:*

(i) *An affine hyperplane $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$.*

(ii) *A sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.*

(iii) *A generalized cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k-1}$, with $k \in \{1, \dots, n-2\}$.*

It is interesting to point out that these examples are all homogeneous. Hence, the classification of isoparametric hypersurfaces is equivalent to the classification of cohomogeneity one actions in the context of Euclidean spaces. The observation about the constancy of the principal curvatures of these examples can be extended. Indeed, Cartan characterized isoparametric hypersurfaces in real space forms as hypersurfaces with *constant principal curvatures* [25]. Furthermore, he derived a fundamental formula relating principal curvatures and their multiplicities in hypersurfaces with constant principal curvatures in spaces with constant sectional curvature. Indeed, consider a manifold with constant sectional curvature c and let g be the number of distinct constant principal curvatures of one of its isoparametric hypersurfaces. For each principal curvature λ_i, we write m_i for its multiplicity. Then, Cartan proved that

$$\sum_{j=1,\, \lambda_j \neq \lambda_i}^{g} m_j \frac{c + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0, \quad \text{for each } i = 1, \dots, g.$$

Using this formula, it is possible to see that if $c < 0$ then $g \in \{1, 2\}$. With this information, Cartan derived the classification of isoparametric hypersurfaces in real hyperbolic spaces $\mathbb{R}H^n$.

Theorem 2.2.2. [25] *Let M be an isoparametric hypersurface in a real hyperbolic space $\mathbb{R}H^n$. Then, M has one or two constant principal curvatures and it is an open part of one of the following hypersurfaces:*

(i) *A totally geodesic real hyperbolic hyperspace $\mathbb{R}H^{n-1}$ in $\mathbb{R}H^n$ or one of its equidistant hypersurfaces.*

(ii) *A tube around a totally geodesic real hyperbolic space $\mathbb{R}H^k$ in $\mathbb{R}H^n$, for some $k \in \{1, \dots, n-2\}$.*

(iii) *A geodesic sphere in $\mathbb{R}H^n$.*

(iv) *A horosphere in $\mathbb{R}H^n$.*

Again, all these examples are homogeneous hypersurfaces. Cartan also made progress in spheres [26], and succeeded in classifying isoparametric hypersurfaces with one, two or three distinct principal curvatures. However, it turns out that the classification of isoparametric hypersurfaces in spheres is very involved. In fact, its complete classification was considered one of the most outstanding problems in Differential Geometry [111]. It was a surprise at that moment to find inhomogeneous examples: the first such examples were constructed by Ozeki and Takeuchi [87], and these were generalized by Ferus, Karcher and Münzner [53] by using Clifford modules. As of this writing, it is not clear if the classification problem remains still open or not. Many mathematicians have worked in this problem and we include some of the main references. Some important progress has been made by Stolz [97], Cecil, Chi and Jensen [29], Immervoll [63] and Chi [33, 34, 35] for four distinct principal curvatures, and by Dorfmeister and Neher [49], Miyaoka [81] and Siffert [95] for six distinct principal curvatures. See the surveys [28] and [103] for a more detailed story of the problem in spheres and related topics.

Recall that, in real space forms, a hypersurface is isoparametric if and only if it has constant principal curvatures. This is not true in a general Riemannian manifold. Thus, it makes sense to study both isoparametric hypersurfaces or hypersurfaces with constant principal curvatures in non-flat complex space forms, that is, complex projective and hyperbolic spaces. The classification of real hypersurfaces with constant principal curvatures in complex projective spaces is known for Hopf hypersurfaces [70], and for two or three distinct principal curvatures [99, 100]; all known examples are open parts of homogeneous hypersurfaces. Using the classification results in spheres, Domínguez-Vázquez [46] derived the classification of isoparametric hypersurfaces in $\mathbb{C}P^n$, $n \neq 15$. A consequence of this classification is that inhomogeneous isoparametric hypersurfaces in $\mathbb{C}P^n$ are relatively common. See also [48] for a recent classification of isoparametric hypersurfaces in quaternionic projective spaces $\mathbb{H}P^n$, $n \neq 7$.

Moreover, real hypersurfaces with constant principal curvatures in complex hyperbolic spaces have been classified under the assumption that the hypersurface is Hopf [2], or if the number of distinct constant principal curvatures is two [84] or three [8], [9]. All of these examples are again homogeneous. See Section 2.5 for more information.

In Chapter 2 and Chapter 3, we deal with isoparametric hypersurfaces in complex hyperbolic spaces. Apart from the homogenous examples classified by Berndt and Tamaru in [17], there are also some inhomogeneous examples that were built by Díaz-Ramos and Domínguez-Vázquez [40]. We will explain all these examples in Subsection 2.5.2.

2.3 The complex hyperbolic space described as a symmetric space

The complex hyperbolic space $\mathbb{C}H^n$ is a rank one symmetric space of non-compact type. Hence, this section is devoted to describing $\mathbb{C}H^n$ according to the algebraic information provided in Section 1.5. Firstly, note that $\mathbb{C}H^n = G/K$, where $G = \mathrm{SU}(1,n)$ is the connected component of the identity of the isometry group of $\mathbb{C}H^n$, and $K = \mathrm{S}(\mathrm{U}(1)\mathrm{U}(n))$ is (up to a finite kernel) the isotropy group at a point $o \in \mathbb{C}H^n$. Let $\mathfrak{g} = \mathfrak{su}(1,n)$ and $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$ be the Lie algebras of G and K, respectively. Recall that ad and Ad denote the adjoint maps of $\mathfrak{g}$ and G, respectively. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the *Cartan decomposition* of $\mathfrak{g}$ with respect to $o \in \mathbb{C}H^n$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. Recall also that the Killing form B allows to define a positive definite inner product $\langle X, Y\rangle_{B_\theta} = -B(\theta X, Y)$ on the Lie algebra $\mathfrak{g}$ satisfying the relation $\langle \mathrm{ad}(X)Y, Z\rangle_{B_\theta} = -\langle Y, \mathrm{ad}(\theta X)Y\rangle_{B_\theta}$ for all X, Y, $Z \in \mathfrak{g}$.

Take now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. It can be proved that the dimension of $\mathfrak{a}$ is one, which is the rank of the symmetric space $G/K = \mathbb{C}H^n$. Recall that the set $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ is a family of commuting self-adjoint (with respect to B_θ) endomorphisms of $\mathfrak{g}$, and hence simultaneously diagonalizable. In this particular case, the (restricted) root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ reads

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},$$

for certain covector $\alpha \in \mathfrak{a}^*$. Therefore, the restricted roots are -2α, $-\alpha$, α and 2α. Furthermore, it can be seen that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are isomorphic to $\mathbb{C}^{n-1}$ and that $\mathfrak{g}_{2\alpha}$, $\mathfrak{g}_{-2\alpha}$ and $\mathfrak{a}$ are isomorphic to $\mathbb{R}$. In particular, this means that $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 2n-2$ and $\dim \mathfrak{g}_{2\alpha} = \dim \mathfrak{g}_{-2\alpha} = \dim \mathfrak{a} = 1$.

At this point, we will fix a positivity criterion in the set of roots. Let us say that α is a positive root. Define $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ as the sum of the root spaces corresponding to all positive roots. These choices (the point o, the maximal abelian subspace $\mathfrak{a}$ and the notion of positivity) determine a point at infinity x in the ideal boundary $\mathbb{C}H^n(\infty)$ of $\mathbb{C}H^n$, that is, an equivalence class of geodesics that are asymptotic to the geodesic starting at $o \in \mathbb{C}H^n$, with direction $\mathfrak{a} \subset \mathfrak{p} \cong T_o\mathbb{C}H^n$ and the orientation determined by the fact that α is positive.

Due to the properties of the root space decomposition, $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ with center $\mathfrak{g}_{2\alpha}$; in fact $\mathfrak{n}$ is isomorphic to the $(2n-1)$-dimensional Heisenberg algebra (see [19, Chapter 3] for a description of generalized Heisenberg algebras). Then $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, since $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$ is nilpotent. Now $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the so-called Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ with respect $o \in \mathbb{C}H^n$ and $x \in \mathbb{C}H^n(\infty)$.

2.4 Real subspaces of a complex vector space

In this section we compile some information on the structure of a real subspace of a complex vector space V. This will be needed to present the examples of isoparametric hypersurfaces appearing in Theorem 3.0.4, and it will be also an important tool in the proof of such classification result. We follow [42].

Let W be a real subspace of V, that is, a subspace of V with the underlying structure of real vector space (as opposed to a complex subspace of V). We denote by J the complex structure of V, and assume that V, as a real vector space, carries an inner product $\langle \cdot, \cdot \rangle$ for which J is an isometry.

Let $\xi \in W$ be a non-zero vector. The *Kähler angle* of ξ with respect to W is the angle $\varphi_\xi \in [0, \pi/2]$ between $J\xi$ and W. For each $\xi \in W$, we write $J\xi = F\xi + P\xi$, where $F\xi$ is the orthogonal projection of $J\xi$ onto W, and $P\xi$ is the orthogonal projection of $J\xi$ onto $V \ominus W$, the orthogonal complement of W in V. Then, the Kähler angle of W with respect to ξ is determined by $\langle F\xi, F\xi \rangle = \cos^2(\varphi_\xi)\langle \xi, \xi \rangle$. Hence, if ξ has unit length, φ_ξ is determined by the fact that $\cos(\varphi_\xi)$ is the length of the orthogonal projection of $J\xi$ onto W. Furthermore, it readily follows from $J^2 = -\operatorname{id}$ that $\langle P\xi, P\xi \rangle = \sin^2(\varphi_\xi)\langle \xi, \xi \rangle$.

A subspace W of a complex vector space is said to have *constant Kähler angle* $\varphi \in [0, \pi/2]$ if all non-zero vectors of W have the same Kähler angle φ. In particular, a totally real subspace is a subspace with constant Kähler angle $\pi/2$, and a subspace is complex if and only if it has constant Kähler angle 0. It is also known that a subspace W with constant Kähler angle has even dimension unless $\varphi = \pi/2$.

Following the ideas in [42, Theorem 2.6], we consider the skew-adjoint linear map $F\colon W \to W$, that is, $\langle F\xi, \eta \rangle = -\langle \xi, F\eta \rangle$ for any ξ, $\eta \in W$, and the symmetric bilinear form $(\xi, \eta) \mapsto \langle F\xi, F\eta \rangle$. Hence, it follows that there is an orthonormal basis $\{\xi_1, \dots, \xi_k\}$ of W and Kähler angles $\varphi_1, \dots, \varphi_k$ such that $\langle F\xi_i, F\xi_j \rangle = \cos^2(\varphi_i)\delta_{ij}$, for all i, $j \in \{1, \dots, k\}$, and where δ_{ij} is the Kronecker delta. We call $\varphi_1, \dots, \varphi_k$ the *principal Kähler angles* of W, and $\xi_1, \dots, \xi_k$ are called *principal Kähler vectors*. Moreover, as it is proved in [42, Section 2.3], the subspace W can be written as $W = \oplus_{\varphi \in \Phi} W_\varphi$, where $\Phi \subset [0, \pi/2]$ is a finite subset, $W_\varphi \neq 0$ for each $\varphi \in \Phi$, and each W_φ has constant Kähler angle φ. Furthermore, if φ, $\psi \in \Phi$ and $\varphi \neq \psi$, then W_φ and W_ψ are complex-orthogonal, i.e. $\mathbb{C}W_\varphi \perp \mathbb{C}W_\psi$. The elements of Φ are precisely the principal Kähler angles, the subspaces W_φ are called the *principal Kähler subspaces*, and their dimension is called their multiplicity.

Denote by $W^\perp = V \ominus W$ the orthogonal complement of W in V. Then, we can also take the decomposition of $W^\perp$ in subspaces of constant Kähler angle $W^\perp = \oplus_{\varphi\in\Psi} W^\perp_\varphi$. It is known that $\Phi \setminus \{0\} = \Psi \setminus \{0\}$ and $\dim W_\varphi = \dim W^\perp_\varphi$ for each $\varphi \in \Phi \setminus \{0\}$, that is, except possibly for complex subspaces in W or $W^\perp$, the Kähler angles of W and $W^\perp$ and their multiplicities are the same. We have $\mathbb{C}W_\varphi = W_\varphi \oplus W^\perp_\varphi$ for $\varphi \in \Phi \setminus \{0\}$, and moreover, $F^2\xi = -\cos^2(\varphi)\xi$ for each $\xi \in W_\varphi$ and each $\varphi \in \Phi$. Conversely, if $\xi \in W$ satisfies $F^2\xi = -\cos^2(\varphi)\xi$, then it follows from the decomposition of W in subspaces of constant Kähler angle that $\xi \in W_\varphi$.

Finally, two subspaces W and $\hat{W}$ of $V \cong \mathbb{C}^n$ are congruent by an element of $U(n)$ if and only if they have the same principal Kähler angles with the same multiplicities, that is, if $W = \oplus_{\varphi\in\Phi} W_\varphi$ and $\hat{W} = \oplus_{\varphi\in\Psi} \hat{W}_\varphi$ are as above, then they are congruent by an element of $U(n)$ if and only if $\Phi = \Psi$ and $\dim W_\varphi = \dim \hat{W}_\psi$ whenever $\varphi = \psi$.

2.5 The examples

The main purpose of this section is to present the examples of isoparametric hypersurfaces in complex hyperbolic spaces. Let M be a hypersurface in the complex hyperbolic space and let ξ be a unit normal vector field. The tangent vector field $J\xi$ to M is called *the Reeb or Hopf vector field* of M. A real hypersurface M in a complex hyperbolic space $\mathbb{C}H^n$ is *Hopf* at a point $p \in M$ if $J\xi_p$ is a principal curvature vector of the shape operator. We say that M is *Hopf* if it is Hopf at all points.

2.5.1 The standard examples

The standard set of homogeneous examples of real hypersurfaces in complex hyperbolic spaces is known as Montiel's list [84]. Berndt [2] classified these examples in the following sense:

Theorem 2.5.1. *Let M be a connected Hopf real hypersurface with constant principal curvatures of the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 2$. Then, M is holomorphically congruent to an open part of:*

(i) *a tube around a totally geodesic $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$, or*

(ii) *a tube around a totally geodesic $\mathbb{R}H^n$, or*

(iii) *a horosphere.*

Remark 2.5.2. In order to use Theorem 2.5.1 efficiently (see for example Corollary 3.2.13 and Proposition 3.2.14), we need to know the principal curvatures and their multiplicities for a Hopf real hypersurface with constant principal curvatures. These can be found for example in [2] or [10].

A tube of radius $r > 0$ around a totally geodesic $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$, has the following principal curvatures:

$$\lambda_1 = \frac{\sqrt{-c}}{2} \tanh\Big(\frac{r\sqrt{-c}}{2}\Big), \qquad \lambda_2 = \frac{\sqrt{-c}}{2} \coth\Big(\frac{r\sqrt{-c}}{2}\Big), \qquad \lambda_3 = \sqrt{-c}\coth\Big(r\sqrt{-c}\Big),$$

with multiplicities $2k$, $2(n-k-1)$, and 1. Thus, the number of principal curvatures is $g = 2$ if $k = 0$ or $k = n-1$, and $g = 3$ otherwise. The Hopf vector is associated with λ_3.

A tube of radius $r > 0$ around a totally geodesic $\mathbb{R}H^n$ has three principal curvatures

$$\lambda_1 = \frac{\sqrt{-c}}{2} \tanh\Big(\frac{r\sqrt{-c}}{2}\Big), \qquad \lambda_2 = \frac{\sqrt{-c}}{2} \coth\Big(\frac{r\sqrt{-c}}{2}\Big), \qquad \lambda_3 = \sqrt{-c}\tanh\Big(r\sqrt{-c}\Big),$$

with multiplicities $n-1$, $n-1$, and 1, except when $r = \frac{1}{\sqrt{-c}} \log\big(2+\sqrt{3}\big)$, in which case $\lambda_1 = \lambda_3$. The Hopf vector is associated with λ_3.

Finally, a horosphere has two distinct principal curvatures

$$\lambda_1 = \frac{\sqrt{-c}}{2}, \qquad\qquad \lambda_2 = \sqrt{-c},$$

with multiplicities $2(n-1)$ and 1. The Hopf vector is associated with λ_2.

It was believed for some time that, as it is the case for complex projective spaces, the Hopf hypersurfaces with constant principal curvatures (Theorem 2.5.1) should give the list of homogeneous hypersurfaces in complex hyperbolic spaces. However, Lohnherr and Reckziegel found in [77] an example of a homogeneous hypersurface that is not Hopf, namely, case (iv) in Theorem 3.0.4. Later, new examples of non-Hopf homogeneous hypersurfaces in complex hyperbolic spaces were found in [5], and Berndt and Tamaru classified all homogeneous hypersurfaces in [17]. The construction method of these non-Hopf examples was generalized by Díaz-Ramos and Domínguez-Vázquez in [40] for the complex hyperbolic space, and in [41] for Damek-Ricci spaces. These examples are in general not homogeneous, but they are isoparametric, and the rest of this section is devoted to presenting their definition and main properties.

2.5.2 Tubes around the submanifolds $W_{\mathfrak{w}}$

Before starting with the description of the examples themselves, we need to introduce some facts about the Lie group and Riemannian structures of the solvable part of the Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$. With the notations introduced in Section 2.3, throughout this section B will be the unit left-invariant vector field of $\mathfrak{a}$ determined by the point at infinity x. That is, the geodesic through o whose initial speed is B converges to x. We also set $Z = JB \in \mathfrak{g}_{2\alpha}$, and thus, $\mathfrak{a} = \mathbb{R}B$ and $\mathfrak{g}_{2\alpha} = \mathbb{R}Z$. Moreover, $\mathfrak{g}_\alpha$ is J-invariant, so it is isomorphic to $\mathbb{C}^{n-1}$. The Lie algebra structure on $\mathfrak{a} \oplus \mathfrak{n}$ is given by the formulas

$$[B,Z] = \sqrt{-c}\,Z, \quad 2\,[B,U] = \sqrt{-c}\,U, \quad [U,V] = \sqrt{-c}\,\langle JU,V\rangle Z, \quad [Z,U] = 0, \tag{2.1}$$

where U, $V \in \mathfrak{g}_\alpha$.

In Section 3.4 we will also need the group structure of the semidirect product AN. A standard reference for this is [19]. The product structure is given by

$$\begin{aligned}
&\mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(aB + U + xZ) \cdot \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(bB + V + yZ) \\
&\quad = \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}\Bigg((a+b)B + \rho\Big(\frac{a+b}{2}\Big)^{-1}\Big(\rho(a/2)U + e^{a/2}\rho(b/2)V\Big) \\
&\qquad\qquad + \rho(a+b)^{-1}\Big(\rho(a)x + e^{a}\rho(b)y + \frac{1}{2}e^{a/2}\sqrt{-c}\rho\left(\frac{a}{2}\right)\rho\left(\frac{b}{2}\right)\langle JU, V\rangle\Big)Z\Bigg)
\end{aligned} \tag{2.2}$$

for all a, b, x, $y \in \mathbb{R}$ and U, $V \in \mathfrak{g}_\alpha$. Here, $\mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}} \colon \mathfrak{a}\oplus\mathfrak{n} \to AN$ denotes the Lie exponential map of AN, and $\rho\colon \mathbb{R} \to \mathbb{R}$ is the analytic function defined by

$$\rho(s) = \begin{cases} \frac{e^s-1}{s} & \text{if } s \neq 0, \\ 1 & \text{if } s = 0. \end{cases}$$

The Levi-Civita connection of AN is given by

$$\begin{aligned}
\nabla_{aB+U+xZ}(bB + V + yZ) = \sqrt{-c}\Big\{&\Big(\frac{1}{2}\langle U, V\rangle + xy\Big)B \\
&- \frac{1}{2}\Big(bU + yJU + xJV\Big) + \Big(\frac{1}{2}\langle JU, V\rangle - bx\Big)Z\Big\},
\end{aligned} \tag{2.3}$$

where a, b, x, $y \in \mathbb{R}$, U, $V \in \mathfrak{g}_\alpha$, and all vector fields are considered to be left-invariant.

In order to construct the examples corresponding to cases (iv) to (vi) of Theorem 3.0.4, let $\mathfrak{w}$ be a proper real subspace of $\mathfrak{g}_\alpha$, that is, a subspace of $\mathfrak{g}_\alpha$, $\mathfrak{w} \neq \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is regarded as a real vector space. We define $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$, the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, and write $k = \dim \mathfrak{w}^\perp$. It follows from the bracket relations above that $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ is a solvable Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. We define

$$W_{\mathfrak{w}} = S_{\mathfrak{w}} \cdot o, \text{ where } \mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha},$$

the orbit of the group $S_{\mathfrak{w}}$ through the point o, where $S_{\mathfrak{w}}$ is the connected subgroup of AN whose Lie algebra is $\mathfrak{s}_{\mathfrak{w}}$. Hence, $W_{\mathfrak{w}}$ is a homogeneous submanifold of $\mathbb{C}H^n$; it was proved in [40] that $W_{\mathfrak{w}}$ is minimal and tubes around $W_{\mathfrak{w}}$ are isoparametric hypersurfaces of $\mathbb{C}H^n$.

We give some more information on $W_{\mathfrak{w}}$ and its tubes. As we have seen in Section 2.4, we can decompose $\mathfrak{w}^\perp = \oplus_{\varphi\in\Phi}\mathfrak{w}^\perp_\varphi$ as a direct sum of complex-orthogonal subspaces of constant Kähler angle. The elements of Φ are the principal Kähler angles of $\mathfrak{w}^\perp$. Recall that $F\colon \mathfrak{w}^\perp \to \mathfrak{w}^\perp$ and $P\colon \mathfrak{w}^\perp \to \mathfrak{w}$ map any $\xi \in \mathfrak{w}^\perp$ to the orthogonal projections of $J\xi$ onto $\mathfrak{w}^\perp$ and $\mathfrak{w}$ respectively. Let $\mathfrak{c}$ be the maximal complex subspace of $\mathfrak{s}_{\mathfrak{w}}$, that is, $\mathfrak{c} = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{w}^\perp) \oplus \mathfrak{g}_{2\alpha}$. Then, $\mathfrak{s}_{\mathfrak{w}} = \mathfrak{c} \oplus P\mathfrak{w}^\perp$ and $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{c} \oplus P\mathfrak{w}^\perp \oplus \mathfrak{w}^\perp$. Denoting by $\mathfrak{C}$, $P\mathfrak{W}^\perp$, and $\mathfrak{W}^\perp$ the corresponding left-invariant distributions on AN, then the tangent bundle of $W_{\mathfrak{w}}$ is $TW_{\mathfrak{w}} = \mathfrak{C} \oplus P\mathfrak{W}^\perp$ and the normal bundle is $\nu W_{\mathfrak{w}} = \mathfrak{W}^\perp$. It follows from [40, p. 1039] that the second fundamental form of $W_{\mathfrak{w}}$ is determined by the trivial symmetric bilinear extension of

$$2I\!I(Z, P\xi) = -\sqrt{-c}\,(JP\xi)^{\perp}, \quad \xi \in \nu W_{\mathfrak{w}},$$

where $(\cdot)^{\perp}$ denotes orthogonal projection onto $\nu W_{\mathfrak{w}}$. It can be shown that this expression for the second fundamental form implies that the complex distribution $\mathfrak{C}$ on $W_{\mathfrak{w}}$ is autoparallel, and hence $W_{\mathfrak{w}}$ is ruled by totally geodesic complex hyperbolic subspaces (see Lemma 3.4.6).

If $k = 1$, that is, if $\mathfrak{w}$ is a real hyperplane in $\mathfrak{g}_{\alpha}$, then the corresponding $W_{\mathfrak{w}}$ is denoted by W^{2n-1} and is called the *Lohnherr hypersurface* [77]. It follows that W^{2n-1} and its equidistant hypersurfaces are homogeneous hypersurfaces of $\mathbb{C}H^n$. These were also studied by Berndt in [3], and correspond to case (iv) of Theorem 3.0.4. The corresponding foliation on $\mathbb{C}H^n$ is sometimes called the solvable foliation.

Thus, we assume from now on $k > 1$. If $\mathfrak{w}^{\perp}$ has constant Kähler angle $\varphi = 0$, then $W_{\mathfrak{w}}$ is congruent to a totally geodesic complex hyperbolic space. If $\mathfrak{w}^{\perp}$ has constant Kähler angle $\varphi \in (0, \pi/2]$, then $W_{\mathfrak{w}}$ is denoted by W_{φ}^{2n-k}. These are the so-called *Berndt-Brück submanifolds*, and it is proved in [5] that the tubes around W_{φ}^{2n-k} are homogeneous. Moreover, it follows from [17] that a real hypersurface in $\mathbb{C}H^n$ is homogeneous if and only if it is congruent to one of the Hopf examples in Theorem 2.5.1, to W^{2n-1} or one of its equidistant hypersurfaces, or to a tube around a W_{φ}^{2n-k}.

In general, however, a tube around a submanifold $W_{\mathfrak{w}}$ is not necessarily homogeneous. For an arbitrary $\mathfrak{w}$, the mean curvature H^r of the tube M^r of radius r around the submanifold $W_{\mathfrak{w}}$ is [40]

$$H^r = \frac{\sqrt{-c}}{2\sinh\frac{r\sqrt{-c}}{2}\cosh\frac{r\sqrt{-c}}{2}}\left(k - 1 + 2n\sinh^2\frac{r\sqrt{-c}}{2}\right).$$

Therefore, for every $r > 0$, the tube M^r of radius r around $W_{\mathfrak{w}}$ is a hypersurface with constant mean curvature, and hence, tubes around the submanifold $W_{\mathfrak{w}}$ constitute an isoparametric family of hypersurfaces in $\mathbb{C}H^n$.

Remark 2.5.3. With the notation as above, if γ_{ξ} denotes the geodesic through a point $o \in W_{\mathfrak{w}}$ with $\dot\gamma_{\xi}(0) = \xi \in \nu_o W_{\mathfrak{w}}$, then the characteristic polynomial of the shape operator of M^r at $\gamma_{\xi}(r)$ with respect to $-\gamma_{\xi}'(r)$ is

$$p_{r,\xi}(x) = (\lambda - x)^{2n-k-2}\left(-\frac{c}{4\lambda} - x\right)^{k-2} f_{\lambda,\varphi_{\xi}}(x),$$

where $\lambda = \frac{\sqrt{-c}}{2}\tanh\frac{r\sqrt{-c}}{2}$, φ_{ξ} is the Kähler angle of ξ with respect to $\nu_o W_{\mathfrak{w}}$, and

$$\begin{aligned} f_{\lambda,\varphi}(x) = &-x^3 + \left(-\frac{c}{4\lambda} + 3\lambda\right)x^2 + \frac{1}{2}\left(c - 6\lambda^2\right)x \\ &+ \frac{16\lambda^4 - 16c\lambda^2 - c^2 + (c + 4\lambda^2)^2\cos(2\varphi)}{32\lambda}. \end{aligned}$$

As was pointed out in [40], at $\gamma_\xi(r)$, M^r has the same principal curvatures, with the same multiplicities, as a tube of radius r around the $W^{2n-k}_{\varphi_\xi}$, $\varphi_\xi \in [0, \pi/2]$. However, in general, the principal curvatures and the number g of principal curvatures vary from point to point in M^r.

Finally, we summarize the examples of isoparametric hypersurfaces in complex hyperbolic spaces that we have presented. On the one hand, in Theorem 2.5.1 we have introduced the examples of the Montiel's list. This list constitutes the classification of Hopf hypersurfaces with constant principal curvatures in complex hyperbolic spaces. On the other hand, the rest of the examples are constructed following the same procedure.

Indeed, let $\mathfrak{w}$ be a real subspace of $\mathfrak{g}_\alpha$, that is, a subspace of $\mathfrak{g}_\alpha$ with the underlying structure of real vector space. We define the Lie subalgebra $\mathfrak{s}_{\mathfrak{w}}$ of $\mathfrak{g}$ by $\mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, and denote by $S_{\mathfrak{w}}$ the connected closed subgroup of $SU(1,n)$ whose Lie algebra is $\mathfrak{s}_{\mathfrak{w}}$. Then, we define $W_{\mathfrak{w}}$ as the orbit through o of the subgroup $S_{\mathfrak{w}}$. It was shown in [40] that $W_{\mathfrak{w}}$ is a homogeneous minimal submanifold of $\mathbb{C}H^n$, and that the tubes around it are isoparametric hypersurfaces of $\mathbb{C}H^n$. We denote by $\mathfrak{w}^\perp$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$. These procedure gives rise to three kinds of examples.

If $\mathfrak{w}$ is a hyperplane of $\mathfrak{g}_\alpha$, then $W_{\mathfrak{w}}$ is a real hypersurface of $\mathbb{C}H^n$ denoted by W^{2n-1}, and it was shown in [3] that the equidistant hypersurfaces to W^{2n-1} are homogeneous.

If $\mathfrak{w}^\perp$ has constant Kähler angle φ, then the corresponding $W_{\mathfrak{w}}$ is denoted by W^{2n-k}_φ. Here k is the codimension of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, and it can be proved [5] that k is even if $\varphi \neq \pi/2$. Moreover, it follows from [5] that the tubes around W^{2n-k}_φ are homogeneous. In particular, if $\varphi = 0$, the submanifold W^{2n-k}_0 is a totally geodesic complex hyperbolic space and we recover the examples in Theorem 2.5.1 (i).

If $\mathfrak{w}^\perp$ does not have constant Kähler angle, then the tubes around $W_{\mathfrak{w}}$ are not homogeneous (indeed, they have non-constant principal curvatures) but are still isoparametric [40].

Chapter 3
Isoparametric hypersurfaces in the complex hyperbolic space: the classification

Recall that an isoparametric hypersurface of a Riemannian manifold is a hypersurface such that all its sufficiently close parallel hypersurfaces have constant mean curvature. The aim of this chapter is to prove the following classification result of isoparametric hypersurfaces in complex hyperbolic spaces. To our knowledge, this is the first complete classification in a whole family of Riemannian manifolds since Cartan's classification of isoparametric hypersurfaces in real hyperbolic spaces [25]. The results of this chapter have been published in the article [43]; see also [44] for an alternative proof of the fact that isoparametric hypersurfaces in $\mathbb{C}H^n$ have the same principal curvatures as the homogeneous examples.

Theorem 3.0.4. *Let M be a connected real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 2$. Then, M is isoparametric if and only if M is congruent to an open part of:*

(i) *a tube around a totally geodesic complex hyperbolic space $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$, or*

(ii) *a tube around a totally geodesic real hyperbolic space $\mathbb{R}H^n$, or*

(iii) *a horosphere, or*

(iv) *a ruled homogeneous minimal Lohnherr hypersurface W^{2n-1}, or some of its equidistant hypersurfaces, or*

(v) *a tube around a ruled homogeneous minimal Berndt-Brück submanifold W^{2n-k}_φ, for $k \in \{2, \dots, n-1\}$, $\varphi \in (0, \pi/2]$, where k is even if $\varphi \neq \pi/2$, or*

(vi) *a tube around a ruled homogeneous minimal submanifold $W_\mathfrak{w}$, for some proper real subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ such that $\mathfrak{w}^\perp$, the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, has non-constant Kähler angle.*

Note that all the examples in the above classification result have been presented in Section 2.5. We state some of the direct consequences of Theorem 3.0.4.

Corollary 3.0.5. [17] *A real hypersurface of $\mathbb{C}H^n$, $n \geq 2$, is homogeneous if and only if it is congruent to one of the examples (i) through (v) in Theorem 3.0.4.*

For $n = 2$, $\mathfrak{g}_\alpha$ is a complex line and thus the examples (v) and (vi) are not possible. Compare also with the classification of real hypersurfaces in $\mathbb{C}H^2$ with constant principal curvatures [9].

Corollary 3.0.6. *An isoparametric hypersurface in* $\mathbb{C}H^2$ *is as open part of a homogeneous hypersurface.*

Nevertheless, for $n \geq 3$ there are inhomogeneous examples: one family up to congruence for $\mathbb{C}H^3$, and infinitely many for $\mathbb{C}H^n$, $n \geq 4$.

Since the examples in (vi) of Theorem 3.0.4 are the only ones that do not have constant principal curvatures we also get:

Corollary 3.0.7. *An isoparametric hypersurface of* $\mathbb{C}H^n$ *has constant principal curvatures if and only if it is an open part of a homogeneous hypersurface of* $\mathbb{C}H^n$.

Moreover, since the examples in Theorem 3.0.4 have the same pointwise principal curvatures as the homogeneous hypersurfaces in $\mathbb{C}H^n$, we have the following result. An alternative shorter proof can be found in [44].

Corollary 3.0.8. *Let* M *be an isoparametric hypersurface in* $\mathbb{C}H^n$. *Then, the principal curvatures of* M *are pointwise the same as the principal curvatures of a homogeneous hypersurface of* $\mathbb{C}H^n$.

Another important consequence of the classification is that each isoparametric hypersurface of $\mathbb{C}H^n$ is an open part of a complete, topologically closed, isoparametric hypersurface which, in turn, is a regular leaf of a singular Riemannian foliation on $\mathbb{C}H^n$ whose leaves of maximal dimension are all isoparametric. Thus, an isoparametric hypersurface in $\mathbb{C}H^n$ determines an isoparametric family of hypersurfaces that fills the whole ambient space and that admits at most one singular leaf.

We can determine the congruence classes of isoparametric families of hypersurfaces in $\mathbb{C}H^n$. Note that, apart from the horosphere foliation $\mathcal{F}_H$, the family $\mathcal{F}_{\mathbb{R}H^n}$ of tubes around a totally geodesic $\mathbb{R}H^n$, and the family $\mathcal{F}_o$ of geodesic spheres around any point $o \in \mathbb{C}H^n$, any other family is given by the collection of tubes around a submanifold $W_{\mathfrak{w}}$ (see Subsection 2.5.2), where $\mathfrak{w}$ is any real subspace of codimension at least one in $\mathfrak{g}_\alpha$. Thus, we have

Theorem 3.0.9. *The moduli space of congruence classes of isoparametric families of hypersurfaces of* $\mathbb{C}H^n$ *is isomorphic to the disjoint union*

$$\{\mathcal{F}_H, \mathcal{F}_{\mathbb{R}H^n}, \mathcal{F}_o\} \amalg \left(\coprod_{k=0}^{2n-3} G_k(\mathbb{R}^{2n-2})/U(n-1) \right),$$

where $G_k(\mathbb{R}^{2n-2})/U(n-1)$ *stands for the orbit space of the standard action of the unitary group* $U(n-1)$ *on the Grassmannian of real vector subspaces of dimension* k *of* $\mathbb{C}^{n-1}$.

As we will see in Section 3.2, the classification of isoparametric hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$ is intimately related to the study of Lorentzian isoparametric hypersurfaces in the anti-De Sitter space H_1^{2n+1}. Following the ideas of Magid in [80], Xiao gave parametrizations of Lorentzian isoparametric hypersurfaces in H_1^{2n+1} [106]. Burth [23] pointed out some crucial gaps in Magid's arguments, which Xiao's proof depends on. However, the classification of isoparametric hypersurfaces in $\mathbb{C}H^n$ does not follow right away from an eventual classification of Lorentzian isoparametric hypersurfaces in the anti-De Sitter space H_1^{2n+1}, as the projection via the Hopf map $\pi\colon H_1^{2n+1} \to \mathbb{C}H^n$ depends in a very essential way on the complex structure of the semi-Euclidean space $\mathbb{R}_2^{2n+2}$ where the anti-De Sitter space lies. This is precisely the main difficulty of this approach in the classification of isoparametric submanifolds of complex projective spaces [46] using the Hopf map from an odd-dimensional sphere.

Therefore, although the starting point of our arguments is the fact that isoparametric hypersurfaces in $\mathbb{C}H^n$ lift to Lorentzian isoparametric hypersurfaces in H_1^{2n+1}, our approach is independent of [80] and [106]. The shape operator of a Lorentzian isoparametric hypersurface does not need to be diagonalizable and, indeed, it can adopt four distinct Jordan canonical forms. Using the Lorentzian version of Cartan's fundamental formula, some algebraic arguments, and Gauss and Codazzi equations, we determine the hypersurfaces in $\mathbb{C}H^n$ that lift to Lorentzian hypersurfaces of three of the four types. The remaining case is much more involved. Working in the anti-De Sitter space, we start using Jacobi field theory in order to extract information about the shape operator of the focal submanifold (Proposition 3.3.6). The key step is to justify the existence of a common eigenvector to all shape operators of the focal submanifold (Proposition 3.3.7). This allows us to define a smooth vector field which is crucial to show that the second fundamental form of the focal set in the complex hyperbolic space coincides with that of one of the submanifolds $W_{\mathfrak{w}}$. After a study of the normal bundle of this focal set, the obtention of a reduction of codimension result, together with a more geometric construction of the submanifolds $W_{\mathfrak{w}}$ (Proposition 3.4.2), we prove a rigidity result for these submanifolds (Theorem 3.4.1); although the proof of this result is convoluted, it reveals several interesting aspects of the geometry of the ruled minimal submanifolds $W_{\mathfrak{w}}$ in relation to the geometry of the ambient complex hyperbolic space. Altogether, this will allow us to conclude the proof of Theorem 3.0.4.

This chapter is organized as follows. In Section 3.1 we describe the relation of the complex hyperbolic space $\mathbb{C}H^n$ with the anti-De Sitter space by means of the Hopf map. Section 3.2 is basically devoted to presenting Cartan's fundamental formula for Lorentzian space forms and some of its algebraic consequences. It turns out that cases (ii) and (iii) in Theorem 3.0.4 can be handled at this point. For the remaining cases, a more thorough study of the focal set is needed, and this is carried out in Section 3.3. The ingredients utilized here are the Gauss and Codazzi equations of a hypersurface (Subsection 3.3.1), Jacobi field theory (Subsection 3.3.2), and a detailed study of the geometry of the focal submanifold (Subsection 3.3.3). In Section 3.4 we give a characterization of the submanifolds $W_{\mathfrak{w}}$ in terms of their second fundamental form. We need a reduction of codimension argument in Subsection 3.4.1, and the proof is concluded in Subsection 3.4.2. We finish the proofs of Theorem 3.0.4 and Theorem 3.0.9 in Section 3.5.

3.1 Complex hyperbolic space and the Hopf map

In this section we will focus on the Hopf map, which was introduced in Section 1.6, and the role that it plays in order to study isoparametric hypersurfaces in complex hyperbolic spaces. Indeed, recall from Section 1.6 the definition of the vector field V on H_1^{2n+1} by means of $V_q = J\sqrt{-c}\, q/2$ for each $q \in H_1^{2n+1}$. This vector field is tangent to the S^1-flow and $\langle V, V\rangle = -1$. We have the linear isometry $T_q H_1^{2n+1} \cong T_{\pi(q)}\mathbb{C}H^n \oplus \mathbb{R}V_q$, and the following relations between the Levi-Civita connections $\tilde{\nabla}$ and $\bar{\nabla}$ of H_1^{2n+1} and $\mathbb{C}H^n$, respectively:

$$\tilde{\nabla}_{X^L} Y^L = (\bar{\nabla}_X Y)^L + \frac{\sqrt{-c}}{2}\langle JX^L, Y^L\rangle V, \tag{3.1}$$

$$\tilde{\nabla}_V X^L = \tilde{\nabla}_{X^L} V = \frac{\sqrt{-c}}{2}(JX)^L = \frac{\sqrt{-c}}{2} JX^L, \tag{3.2}$$

for all X, $Y \in \Gamma(T\mathbb{C}H^n)$, and where X^L denotes the horizontal lift of X and J denotes the complex structure on $\mathbb{C}^{n+1}$ as well. These formulas follow from the fundamental equations of semi-Riemannian submersions [85].

Let now M be a real hypersurface in $\mathbb{C}H^n$. Sometimes we say 'real' to emphasize that M has real codimension one, as opposed to 'complex' codimension one. Then $\tilde{M} = \pi^{-1}(M)$ is a hypersurface in H_1^{2n+1} which is invariant under the S^1-action. Thus $\pi|_{\tilde{M}} \colon \tilde{M} \to M$ is a semi-Riemannian submersion with timelike totally geodesic S^1-fibers. Conversely, if $\tilde{M}$ is a Lorentzian hypersurface in H_1^{2n+1} which is invariant under the S^1-action, then $M = \pi(\tilde{M})$ is a real hypersurface in $\mathbb{C}H^n$, and $\pi|_{\tilde{M}} \colon \tilde{M} \to M$ is a semi-Riemannian submersion with timelike totally geodesic fibers. If ξ is a (local) unit normal vector field to M, then ξ^L is a (local) spacelike unit normal vector field to $\tilde{M}$. In order to simplify the notation, we will denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and of $\tilde{M}$. Denote by $\mathcal{S}$ and $\tilde{\mathcal{S}}$ the shape operators of M and $\tilde{M}$, respectively.

The Gauss and Weingarten formulas for the hypersurface $\tilde{M}$ in H_1^{2n+1} are, as we have seen, $\tilde{\nabla}_X Y = \nabla_X Y + \langle \tilde{\mathcal{S}}X, Y\rangle \xi^L$, and $\tilde{\nabla}_X \xi^L = -\tilde{\mathcal{S}}X$. Using (3.1) and (3.2), for any $X \in \Gamma(TM)$, we have

$$\tilde{\mathcal{S}}X^L = (\mathcal{S}X)^L + \frac{\sqrt{-c}}{2}\langle J\xi^L, X^L\rangle V, \qquad \tilde{\mathcal{S}}V = -\frac{\sqrt{-c}}{2}J\xi^L. \tag{3.3}$$

In particular, $\mathcal{S}X = \pi_* \tilde{\mathcal{S}}X^L$.

Let $X_1, \dots, X_{2n-1}$ be a local frame on M consisting of principal directions with corresponding principal curvatures $\lambda_1, \dots, \lambda_{2n-1}$ (obviously, some can be repeated). Then $X_1^L, \dots, X_{2n-1}^L, V$ is a local frame on $\tilde{M}$ with respect to which $\tilde{\mathcal{S}}$ is represented by the matrix

$$\begin{pmatrix} \lambda_1 & & 0 & -\frac{b_1\sqrt{-c}}{2} \\ & \ddots & & \vdots \\ 0 & & \lambda_{2n-1} & -\frac{b_{2n-1}\sqrt{-c}}{2} \\ \frac{b_1\sqrt{-c}}{2} & \dots & \frac{b_{2n-1}\sqrt{-c}}{2} & 0 \end{pmatrix}, \tag{3.4}$$

where $b_i = \langle J\xi, X_i \rangle$, $i = 1, \dots, 2n-1$, are S^1-invariant functions on (an open set of) $\tilde{M}$.

As a consequence of (3.4), M and $\tilde{M}$ have the same mean curvatures. Since horizontal geodesics in H_1^{2n+1} are mapped via π to geodesics in $\mathbb{C}H^n$, it follows that π maps equidistant hypersurfaces to $\tilde{M}$ to equidistant hypersurfaces to M. Therefore, M is isoparametric if and only if $\tilde{M}$ is isoparametric. This allows us to study isoparametric hypersurfaces in $\mathbb{C}H^n$ by analyzing which Lorentzian isoparametric hypersurfaces in H_1^{2n+1} can result by lifting isoparametric hypersurfaces in $\mathbb{C}H^n$ to the anti-De Sitter space. It is instructive to note that, whereas the isoparametric condition behaves well with respect to the Hopf map, this is not so for the constancy of the principal curvatures of a hypersurface, since the functions b_i might be non-constant.

3.2 Lorentzian isoparametric hypersurfaces

In this section we present the possible eigenvalue structures of the shape operator of a Lorentzian isoparametric hypersurface in the anti-De Sitter space H_1^{2n+1} and use this information to deduce some algebraic properties of an isoparametric hypersurface in the complex hyperbolic space $\mathbb{C}H^n$.

Let $\tilde{M}$ be a Lorentzian isoparametric hypersurface in H_1^{2n+1}. Then we know by [56, Proposition 2.1] that it has constant principal curvatures with constant algebraic multiplicities. The shape operator $\tilde{\mathcal{S}}_q$ at a point q is a self-adjoint endomorphism of $T_q\tilde{M}$. It is known (see for example [86, Chapter 9]) that there exists a basis of $T_q\tilde{M}$ where $\tilde{\mathcal{S}}_q$ assumes one of the following Jordan canonical forms:

$$\text{I.}\ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2n} \end{pmatrix} \qquad \text{II.}\ \begin{pmatrix} \lambda_1 & 0 & & & \\ \varepsilon & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{2n-1} \end{pmatrix},\ \varepsilon = \pm 1$$

$$\text{III.}\ \begin{pmatrix} \lambda_1 & 0 & 1 & & & \\ 0 & \lambda_1 & 0 & & & \\ 0 & 1 & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{2n-2} \end{pmatrix} \qquad \text{IV.}\ \begin{pmatrix} a & -b & & & \\ b & a & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_{2n} \end{pmatrix}$$

Here, the $\lambda_i \in \mathbb{R}$ can be repeated and, in case IV, $\lambda_1 = a + ib, \lambda_2 = a - ib$ $(b \neq 0)$ are the complex eigenvalues of $\tilde{\mathcal{S}}_q$. In cases I and IV the basis with respect to which $\tilde{\mathcal{S}}_q$ is represented is orthonormal (with the first vector being timelike), while in cases II and III the basis is semi-null. A semi-null basis is a basis $\{u, v, e_1, \dots, e_{m-2}\}$ for which all inner products are zero except $\langle u, v \rangle = \langle e_i, e_i \rangle = 1$, for all $i = 1, \dots, m-2$. We will say that a point $q \in \tilde{M}$ is of type I, II, III or IV if the canonical form of $\tilde{\mathcal{S}}_q$ is of type I, II, III or IV, respectively.

Remark 3.2.1. It can be seen by direct calculation that all points of the lift of a tube around a totally geodesic $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$, are of type I. Similarly, all points of the lift of a horosphere are of type II, and all points of the lift of a tube around a totally geodesic $\mathbb{R}H^n$ are of type IV. For the Lohnherr hypersurface W^{2n-1} and its equidistant hypersurfaces, or for the tubes around the Berndt-Brück submanifolds W^{2n-k}_{φ}, all points of their lifts are of type III. Nevertheless, it is important to point out that, in general, the lift of a tube around a submanifold $W_{\mathfrak{w}}$ does not have constant type: there might be points of type I (if $\varphi_\xi = 0$ in the notation of Subsection 2.5.2) and of type III (otherwise).

Cartan's fundamental formula can be generalized to semi-Riemannian space forms. See [56], or [23, Satz 2.3.6] for a proof:

Proposition 3.2.2. *Let $\tilde{M}$ be a Lorentzian isoparametric hypersurface in the anti-De Sitter space H_1^{2n+1} of curvature $c/4$. If its (possibly complex) principal curvatures are $\lambda_1, \dots, \lambda_{\tilde{g}}$ with algebraic multiplicities $m_1, \dots, m_{\tilde{g}}$, respectively, and if for some $i \in \{1, \dots, \tilde{g}\}$ the principal curvature λ_i is real and its algebraic and geometric multiplicities coincide, then:*

$$\sum_{j=1,\, j\neq i}^{\tilde{g}} m_j \frac{c + 4\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0.$$

Now let M be an isoparametric real hypersurface in $\mathbb{C}H^n$ and $\tilde{M} = \pi^{-1}(M)$ its lift to H_1^{2n+1}. Then, $\tilde{M}$ is a Lorentzian isoparametric hypersurface in the anti-De Sitter space. We use Cartan's fundamental formula to analyze the eigenvalue structure of M. Our approach here will be mostly based on elementary algebraic arguments.

We denote by ξ a (local) unit normal vector field of M. For a point $q \in \tilde{M}$, the shape operator $\tilde{\mathcal{S}}$ of $\tilde{M}$ at q with respect to ξ^L_q can adopt one of the four possible types described above. We will analyze the possible principal curvatures of M at the point $p = \pi(q)$ going through the four cases.

The following is an elementary result that we state without proof.

Lemma 3.2.3. *Let $c < 0$, $p > 0$, and define $\phi \colon \mathbb{R} \setminus \{p\} \to \mathbb{R}$ by $\phi(x) = \frac{c+4px}{p-x}$. Then $\phi(x) > 0$ if and only if $x > 0$ and $|x + \frac{c}{4x}| < |p + \frac{c}{4p}|$.*

We begin with a consequence of Cartan's fundamental formula that will be used in Subsections 3.2.1, 3.2.2 and 3.2.3. See [23, §2.4] and [106, Lemma 2.3].

Lemma 3.2.4. *Let $q \in \tilde{M}$ be a point of type I, II or III. Then the number $\tilde{g}(q)$ of constant principal curvatures at q satisfies $\tilde{g}(q) \in \{1, 2\}$. Moreover, if $\tilde{g}(q) = 2$ and the principal curvatures are λ and μ, then $c + 4\lambda\mu = 0$.*

Proof. Let Λ be the set of principal curvatures of $\tilde{M}$ at q. The algebraic multiplicity of $\lambda \in \Lambda$ is denoted by m_λ. If q is of type II or III, then the algebraic and geometric multiplicities of only one principal curvature $\mu_0 \in \Lambda$ of $\tilde{M}$ at q do not coincide.

By Proposition 3.2.2, we have

$$m_{\mu_0} \sum_{\mu \in \Lambda \setminus \{\mu_0\}} m_\mu \frac{c + 4\mu_0 \mu}{\mu_0 - \mu} = \sum_{\lambda \in \Lambda} m_\lambda \left(\sum_{\mu \in \Lambda \setminus \{\lambda\}} m_\mu \frac{c + 4\lambda\mu}{\lambda - \mu} \right)$$
$$= \sum_{\lambda < \mu} m_\lambda m_\mu (c + 4\lambda\mu) \left(\frac{1}{\lambda - \mu} + \frac{1}{\mu - \lambda} \right) = 0.$$

Since $m_{\mu_0} \neq 0$, we have that the fundamental formula of Cartan is also satisfied for μ_0.

Now let q be a point of type I, II or III. Then we have

$$\sum_{\mu \in \Lambda \setminus \{\lambda\}} m_\mu \frac{c + 4\lambda\mu}{\lambda - \mu} = 0, \text{ for each } \lambda \in \Lambda. \tag{3.5}$$

By a suitable choice of the normal vector field, we can assume that Λ^+, the set of positive principal curvatures, is non-empty; otherwise, there would be only one principal curvature $\lambda = 0$, and hence $\tilde{g} = 1$. Let $\lambda_0 \in \Lambda$ be a positive principal curvature that minimizes $\lambda \in \Lambda^+ \mapsto |\lambda + c/(4\lambda)|$. By Lemma 3.2.3 (with $p = \lambda_0$) we have $(c + 4\lambda_0\mu)/(\lambda_0 - \mu) \leq 0$ for all $\mu \in \Lambda \setminus \{\lambda_0\}$. Therefore, (3.5) implies $\tilde{g} \in \{1, 2\}$, and if $\tilde{g} = 2$, then $\Lambda = \{\lambda_0, \mu\}$ and $c + 4\lambda_0\mu = 0$. □

We will make extensive use of the relations, see (3.3),

$$\tilde{\mathcal{S}}V = -\frac{\sqrt{-c}}{2} J\xi^L \qquad \text{and} \qquad \langle \tilde{\mathcal{S}}V, V \rangle = 0,$$

where V is a timelike unit vector field on H_1^{2n+1} tangent to the fibers of the Hopf map π. In order to simplify the notation, we will put $v = V_q$, $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_q$, $\mathcal{S} = \mathcal{S}_p$, and remove the base point of a vector field from the notation whenever it does not lead to confusion.

3.2.1 Type I points

We start our study with the diagonal setting.

Proposition 3.2.5. *If $q \in \tilde{M}$ is of type I and $p = \pi(q)$, then M is Hopf at p, and $g(p) \in \{2, 3\}$. The principal curvatures of M at p are:*

$$\lambda \in \left(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2} \right), \lambda \neq 0, \quad \mu = -\frac{c}{4\lambda} \in \left(-\infty, -\frac{\sqrt{-c}}{2} \right) \cup \left(\frac{\sqrt{-c}}{2}, \infty \right), \quad \textit{and} \;\; \lambda + \mu.$$

The first two principal curvatures coincide with those of $\tilde{M}$ (one of them might not exist as a principal curvature of M at p) and the last one is of multiplicity one and corresponds to the Hopf vector.

Proof. According to Lemma 3.2.4, let λ and $\mu = -c/(4\lambda)$ be the eigenvalues of $\tilde{\mathcal{S}}$ (μ might not exist). We assume that the principal curvature space $T_\lambda(q)$ has Lorentzian signature.

First, assume that there exist two distinct principal curvatures λ and μ. Since $c+4\lambda\mu = 0$, we have $\lambda, \mu \neq 0$. We can write $v = u + w$, where $u \in T_\lambda(q)$, and $w \in T_\mu(q)$. Since $-1 = \langle v, v\rangle = \langle u, u\rangle + \langle w, w\rangle$, we have that u is timelike, and

$$0 = \langle \tilde{\mathcal{S}}v, v\rangle = \lambda\langle u, u\rangle + \mu\langle w, w\rangle = (\lambda - \mu)\langle u, u\rangle - \mu,$$

whence $\langle u, u\rangle = \frac{\mu}{\lambda-\mu} < 0$ and $\langle w, w\rangle = \frac{\lambda}{\mu-\lambda} > 0$. In addition:

$$J\xi^L = -\frac{2}{\sqrt{-c}}\tilde{\mathcal{S}}v = -\frac{2}{\sqrt{-c}}(\lambda u + \mu w).$$

Both $T_\lambda(q) \ominus \mathbb{R}u$ and $T_\mu(q) \ominus \mathbb{R}w$ are orthogonal to v and $J\xi^L$, and so, by (3.3), they descend via π_{*q} to eigenvectors of $\mathcal{S}$ (which are orthogonal to $J\xi$) corresponding to the eigenspaces of λ and μ, respectively. For dimension reasons, $J\xi$ belongs to one eigenspace of $\mathcal{S}$. Since $\pi_* v = 0$, we have $\pi_* w = -\pi_* u$, and thus, by (3.3),

$$\begin{aligned}\mathcal{S}J\xi &= \frac{-2}{\sqrt{-c}}(\lambda^2\pi_* u + \mu^2\pi_* w) = \frac{-2}{\sqrt{-c}}(\lambda^2 - \mu^2)\pi_* u \\ &= \frac{-2}{\sqrt{-c}}(\lambda + \mu)(\lambda\pi_* u + \mu\pi_* w) = (\lambda + \mu)J\xi.\end{aligned}$$

Therefore M has $g(p) \in \{2, 3\}$ principal curvatures at p: λ, μ and $\lambda + \mu$, where one of the first two might not exist (depending on whether $T_\lambda(q) \ominus \mathbb{R}u$ or $T_\mu(q) \ominus \mathbb{R}w$ might be zero) and where the last one is of multiplicity one and corresponds to the Hopf vector. Since $4\lambda\mu + c = 0$ and $\frac{\lambda}{\mu-\lambda}, \frac{\mu}{\mu-\lambda} > 0$ it readily follows that $|\mu| > |\lambda|$, and thus $|\lambda| < \sqrt{-c}/2$.

Now assume that there is just one principal curvature λ. Then $\tilde{\mathcal{S}}v = \lambda v$ and $0 = \langle \tilde{\mathcal{S}}v, v\rangle = -\lambda$, but then $J\xi^L = -\frac{2}{\sqrt{-c}}\tilde{\mathcal{S}}v = 0$, which makes no sense. So this case is impossible. □

Remark 3.2.6. Note that for a certain $r \in \mathbb{R}$, one can write

$$\lambda = \frac{\sqrt{-c}}{2}\tanh\left(\frac{r\sqrt{-c}}{2}\right), \ \mu = \frac{\sqrt{-c}}{2}\coth\left(\frac{r\sqrt{-c}}{2}\right), \quad \text{and } \lambda + \mu = \sqrt{-c}\coth(r\sqrt{-c}).$$

Therefore, if M is an isoparametric hypersurface that lifts to a type I hypersurface, then M is a Hopf real hypersurface with constant principal curvatures and, according to the classification of Hopf real hypersurfaces with constant principal curvatures in the complex hyperbolic space (Theorem 2.5.1) and to the principal curvatures of M, it is an open part of a tube around a totally geodesic $\mathbb{C}H^k$, $k \in \{0, \ldots, n-1\}$. However, as we have mentioned in Remark 3.2.1, it is possible for an isoparametric hypersurface of $\mathbb{C}H^n$ to have points of type I and III in the same connected component. We will have to address this difficulty later in this chapter.

3.2.2 Type II points

Now we tackle the second possibility for the Jordan canonical form of the shape operator.

Proposition 3.2.7. *If $q \in \tilde{M}$ is of type II and $p = \pi(q)$, then M is Hopf at p, and $g(p) = 2$. Moreover, $\tilde{M}$ has one principal curvature $\lambda = \pm\sqrt{-c}/2$, and the principal curvatures of M at p are λ and 2λ. The second one has multiplicity one and corresponds to the Hopf vector.*

Proof. Let λ and $\mu = -c/(4\lambda)$ be the eigenvalues of $\tilde{\mathcal{S}}$ (μ might not exist). Assume $\tilde{\mathcal{S}}$ has a type II matrix expression with respect to a semi-null basis $\{e_1, e_2, \dots, e_{2n}\}$, where $\tilde{\mathcal{S}}e_1 = \lambda e_1 + \varepsilon e_2$, with $\varepsilon \in \{-1, 1\}$, $T_\lambda(q) = \operatorname{span}\{e_2, \dots, e_k\}$ and $T_\mu(q) = \operatorname{span}\{e_{k+1}, \dots, e_{2n}\}$. As a precaution, for the calculations that follow we observe that $e_1 \notin T_\lambda(q)$, but it still makes sense to write, for example, $T_\lambda(q) \ominus \mathbb{R}e_1 = \operatorname{span}\{e_3, \dots, e_k\}$.

First, assume that $\tilde{M}$ has two distinct principal curvatures $\lambda, \mu \neq 0$ at q with $c+4\lambda\mu = 0$. We can assume that $v = r_1 e_1 + r_2 e_2 + u + w$, where $u \in T_\lambda(q)$, $\langle e_1, u\rangle = \langle e_2, u\rangle = 0$, $w \in T_\mu(q)$ and $r_1, r_2 \in \mathbb{R}$. We have $-1 = \langle v, v\rangle = 2r_1r_2 + \langle u, u\rangle + \langle w, w\rangle$, so $r_1, r_2 \neq 0$. If $u \neq 0$, we define

$$e_1' = e_1 - \frac{\langle u, u\rangle}{2r_1^2} e_2 + \frac{1}{r_1} u, \qquad\qquad e_2' = e_2,$$

and then we have $\langle e_i', e_j'\rangle = \langle e_i, e_j\rangle$, $\tilde{\mathcal{S}}e_1' = \lambda e_1' + \varepsilon e_2'$, $\tilde{\mathcal{S}}e_2' = \lambda e_2'$, and $v = r_1 e_1' + (r_2 + \langle u, u\rangle/(2r_1))e_2' + w$. This means that we could have assumed from the very beginning $u = 0$.

Thus, we have $-1 = \langle v, v\rangle = 2r_1r_2 + \langle w, w\rangle$ and $\tilde{\mathcal{S}}v = r_1\lambda e_1 + r_1\varepsilon e_2 + r_2\lambda e_2 + \mu w$, and hence $J\xi^L = -2(r_1\lambda e_1 + (r_1\varepsilon + r_2\lambda)e_2 + \mu w)/\sqrt{-c}$. Taking into account that $2r_1r_2 = -1 - \langle w, w\rangle$, we have

$$\begin{aligned}
1 = \langle J\xi^L, J\xi^L\rangle &= -\frac{4}{c}\left(2r_1^2\lambda\varepsilon + 2r_1r_2\lambda^2 + \langle w, w\rangle\mu^2\right)\\
&= -\frac{4}{c}\left(2r_1^2\lambda\varepsilon - \lambda^2 + \langle w, w\rangle(\mu^2 - \lambda^2)\right),\\
0 = \langle \tilde{\mathcal{S}}v, v\rangle &= 2r_1r_2\lambda + r_1^2\varepsilon + \langle w, w\rangle\mu = r_1^2\varepsilon - \lambda + \langle w, w\rangle(\mu - \lambda).
\end{aligned}$$

These two equations give a linear system in the unknowns r_1^2 and $\langle w, w\rangle$. As $\lambda \neq \mu$ and $c + 4\lambda\mu = 0$, it is immediate to prove that this system is compatible and determined, and $r_1^2 = -(c + 4\lambda\mu)/(4\varepsilon(\lambda - \mu)) = 0$, which gives a contradiction. Therefore, there cannot be two distinct eigenvalues of $\tilde{\mathcal{S}}$.

If $\tilde{\mathcal{S}}$ has just one eigenvalue λ, similar calculations as above (or just setting $w = 0$ everywhere) yield $2\lambda\varepsilon r_1^2 = -\frac{c}{4} + \lambda^2$, and $\varepsilon r_1^2 = \lambda$, which is only possible if $\lambda = \pm\sqrt{-c}/2$ and $r_1^2 = \sqrt{-c}/2$.

Now, $T_\lambda(q) \ominus \mathbb{R}e_1$ is orthogonal to v and $J\xi^L$. Thus, when we apply π_{*q}, the vectors in $T_\lambda(q) \ominus \mathbb{R}e_1$ descend to eigenvectors of $\mathcal{S}$ associated with the eigenvalue λ, which are also orthogonal to $J\xi$. For dimension reasons, $J\xi$ must also be an eigenvector of $\mathcal{S}$. Furthermore, by (3.3), and since $0 = \pi_* v = r_1 \pi_* e_1 + r_2 \pi_* e_2$, we get

$$\mathcal{S}J\xi = -\frac{2}{\sqrt{-c}}(r_1\lambda^2\pi_*e_1 + (2r_1\varepsilon\lambda + r_2\lambda^2)\pi_*e_2) = -\frac{4\lambda r_1\varepsilon}{\sqrt{-c}}\pi_*e_2 = 2\lambda J\xi\,.$$

In conclusion, M has $g(p) = 2$ principal curvatures at p. One is $\lambda = \pm\sqrt{-c}/2$, which coincides with the unique principal curvature of $\tilde{M}$, and the other one is $2\lambda = \pm\sqrt{-c}$, which has multiplicity one and corresponds to the Hopf vector. □

3.2.3 Type III points

Now we will assume that the minimal polynomial of the shape operator $\tilde{\mathcal{S}}$ has a triple root. This case is much more involved than the others, and indeed, Section 3.3 will be mainly devoted to dealing with this possibility. For type III points we will always take vectors $\{e_1, e_2, e_3\}$ such that

$$\begin{gathered}\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0, \qquad \langle e_1, e_2\rangle = \langle e_3, e_3\rangle = 1,\\ \tilde{\mathcal{S}}e_1 = \lambda e_1, \qquad \tilde{\mathcal{S}}e_2 = \lambda e_2 + e_3, \qquad \tilde{\mathcal{S}}e_3 = e_1 + \lambda e_3.\end{gathered} \tag{3.6}$$

Proposition 3.2.8. *Let $q \in \tilde{M}$ be a point of type III and let λ be the principal curvature of $\tilde{M}$ at q whose algebraic and geometric multiplicities do not coincide. Then, $\tilde{g}(q) \in \{1, 2\}$, $\lambda \in \left(-\sqrt{-c}/2, \sqrt{-c}/2\right)$; if there are two principal curvatures at q and we denote the other one by μ, then $c + 4\lambda\mu = 0$.*

Proof. Let λ and $\mu = -c/(4\lambda)$ be the eigenvalues of $\tilde{\mathcal{S}}$ (μ might not exist). Recall that $c + 4\lambda\mu = 0$ from Proposition 3.2.4. Assume that $\tilde{\mathcal{S}}$ has a type III matrix expression, and take $\{e_1, e_2, e_3\}$ as in (3.6). The spaces $T_\lambda(q) \ominus \mathbb{R}e_2$ (recall that $e_2 \notin T_\lambda(q)$) and $T_\mu(q)$ are spacelike. By changing the sign of the normal vector we can further assume $\lambda \geq 0$.

First, assume that there exist two distinct principal curvatures $\lambda, \mu \neq 0$ with $c + 4\lambda\mu = 0$. We can write $v = r_1e_1 + r_2e_2 + r_3e_3 + u + w$, where $u \in T_\lambda(q) \ominus \mathbb{R}e_2$, $w \in T_\mu(q)$. Taking an appropriate orientation of $\{e_1, e_2, e_3\}$ we can further assume $r_2 \geq 0$. We have $-1 = \langle v, v\rangle = 2r_1r_2 + r_3^2 + \langle u, u\rangle + \langle w, w\rangle$. In particular, $r_2 > 0$ and $r_1 < 0$. If $u \neq 0$, we define

$$e_1' = e_1, \quad e_2' = -\frac{\langle u, u\rangle}{2r_2^2}e_1 + e_2 + \frac{1}{r_2}u, \quad e_3' = e_3. \tag{3.7}$$

Then, the e_i''s satisfy (3.6), and also $v = (r_1 + \langle u, u\rangle/(2r_2))e_1' + r_2e_2' + r_3e_3' + w$. This shows that we could have assumed from the very beginning $u = 0$.

Thus we have $-1 = \langle v, v\rangle = 2r_1r_2 + r_3^2 + \langle w, w\rangle$, and $\tilde{\mathcal{S}}v = (r_1\lambda + r_3)e_1 + r_2\lambda e_2 + (r_2 + r_3\lambda)e_3 + \mu w$, and hence $J\xi^L = -2\left((r_1\lambda + r_3)e_1 + r_2\lambda e_2 + (r_2 + r_3\lambda)e_3 + \mu w\right)/\sqrt{-c}$.

Taking into account that $2r_1r_2 = -1 - r_3^2 - \langle w, w\rangle$ we have

$$\begin{aligned} 1 = \langle J\xi^L, J\xi^L\rangle &= -\frac{4}{c}\left(2r_1r_2\lambda^2 + 4r_2r_3\lambda + r_2^2 + r_3^2\lambda^2 + \langle w, w\rangle\mu^2\right) \\ &= -\frac{4}{c}\left(4r_2r_3\lambda + r_2^2 - \lambda^2 + (\mu^2 - \lambda^2)\langle w, w\rangle\right), \\ 0 = \langle \tilde{\mathcal{S}}v, v\rangle &= 2r_1r_2\lambda + 2r_2r_3 + r_3^2\lambda + \mu\langle w, w\rangle = 2r_2r_3 - \lambda + (\mu - \lambda)\langle w, w\rangle. \end{aligned}$$

Canceling the r_2r_3 addend, we get

$$r_2^2 + (\mu - \lambda)^2\langle w, w\rangle = -\frac{c}{4} - \lambda^2.$$

Since $r_2 > 0$, we deduce $\lambda \in \left(-\sqrt{-c}/2, \sqrt{-c}/2\right)$, $\lambda \neq 0$.

If $\tilde{M}$ has just one principal curvature $\lambda \geq 0$ at q, calculations are very similar to what we did above, just putting $w = 0$. We also get $\lambda \in (-\sqrt{-c}/2, \sqrt{-c}/2)$, although in this case $\lambda = 0$ is possible. □

3.2.4 Type IV points

The final possibility for the Jordan canonical form of a self-adjoint operator of a Lorentzian vector space concerns the existence of a complex eigenvalue. Since an isoparametric hypersurface in the anti-De Sitter space has constant principal curvatures, if there is a complex eigenvalue at a point, then there is a complex eigenvalue at all points. Since type IV matrices are the only ones with a non-real eigenvalue we conclude

Lemma 3.2.9. *If $\tilde{M}$ is a connected isoparametric hypersurface of the anti-De Sitter space, and $q \in \tilde{M}$ is a point of type IV, then all the points of $\tilde{M}$ are of type IV.*

As a consequence of Cartan's fundamental formula (Proposition 3.2.2) we have (cf. [23, Satz 2.4.3] or [106, Lemma 2.4]):

Lemma 3.2.10. *Let $q \in \tilde{M}$ be a point of type IV and let $a \pm ib$ ($b \neq 0$) be the non-real complex conjugate principal curvatures at q. We denote by Λ the set of real principal curvatures at q. Then $\tilde{g}(q) \in \{3, 4\}$ and*

$$a(4\lambda^2 - c) - \lambda(4a^2 + 4b^2 - c) = 0, \text{ for each } \lambda \in \Lambda.$$

If $\tilde{g}(q) = 4$, the real principal curvatures λ and μ satisfy $c + 4\lambda\mu = 0$.

Proof. Let $a + ib$, $a - ib$ ($b \neq 0$) be the two complex principal curvatures, both with multiplicity one, and as usual we denote by m_λ the multiplicity of $\lambda \in \Lambda$. Since $n \geq 2$, we have $\Lambda \neq \emptyset$. By Proposition 3.2.2, for each $\lambda \in \Lambda$ we have

$$2\frac{a(4\lambda^2 - c) - \lambda(4a^2 + 4b^2 - c)}{(\lambda - a)^2 + b^2} + \sum_{\mu \in \Lambda \setminus \{\lambda\}} m_\mu \frac{c + 4\lambda\mu}{\lambda - \mu} = 0. \tag{3.8}$$

We denote by Λ^+ the set of positive principal curvatures at q. We define the map $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = a(4x^2 - c) - x(4a^2 + 4b^2 - c)$.

Assume $a \leq 0$ and $\Lambda^+ \neq \emptyset$. We define λ_0 to be a positive principal curvature that minimizes $\lambda \in \Lambda^+ \mapsto |\lambda + c/(4\lambda)|$. Then, by Lemma 3.2.3 we get $(c + 4\lambda_0\mu)/(\lambda_0 - \mu) \leq 0$ for all $\mu \in \Lambda \setminus \{\lambda_0\}$. Since $f(\lambda_0) < 0$, this gives a contradiction with (3.8). Thus, there cannot be positive principal curvatures if $a \leq 0$. Similarly, we get that all real principal curvatures are non-negative if $a \geq 0$. In particular, if $a = 0$ then $\Lambda = \{0\}$ and hence $\tilde{g} = 3$.

From now on we will assume, without losing generality, that $a > 0$. Then, all real principal curvatures are non-negative. But from (3.8) one sees that in fact $\lambda > 0$ for all $\lambda \in \Lambda$, that is, $\Lambda = \Lambda^+$.

The function f is a quadratic function with discriminant $(c + 4a^2 - 4b^2)^2 + 64a^2b^2 > 0$, so f has exactly two zeroes, say x_1 and x_2. We have $x_1x_2 = -c/4 > 0$ and $x_1 + x_2 = (a^2 + b^2 - c/4)/a > 0$, so we can assume $0 < x_1 < x_2 = -c/(4x_1)$.

If $\lambda > 0$, note that $\lambda \in (x_1, x_2)$ if and only if $|\lambda + c/(4\lambda)| < |x_1 + c/(4x_1)|$. If $\Lambda \cap (x_1, x_2) \neq \emptyset$, we define λ_0 to be a principal curvature that minimizes $\lambda \in \Lambda \mapsto |\lambda + c/(4\lambda)|$. Then $f(\lambda_0) < 0$ and $(c + 4\lambda_0\mu)/(\lambda_0 - \mu) \leq 0$ for all $\mu \in \Lambda \setminus \{\lambda_0\}$ by Lemma 3.2.3 (with $p = \lambda_0$), contradiction with (3.8). Thus, let λ_0 be a principal curvature that maximizes $\lambda \in \Lambda \mapsto |\lambda + c/(4\lambda)|$. In this case, $f(\lambda_0) \geq 0$ and $(c + 4\lambda_0\mu)/(\lambda_0 - \mu) \geq 0$ for all $\mu \in \Lambda \setminus \{\lambda_0\}$ by Lemma 3.2.3 (with $p = \mu$). Hence, by (3.8) we get $f(\lambda_0) = 0$, $\Lambda \subset \{x_1, x_2\}$, and the assertion follows. □

Before starting an algebraic analysis of the shape operator we need to prove the following inequality, which requires obtaining information from the Codazzi and Gauss equations.

Lemma 3.2.11. *With the notation as above we have* $4a^2 + 4b^2 + c \geq 0$.

Proof. First, recall that by Lemma 3.2.9, $\tilde{M}$ is of type IV everywhere with the same principal curvatures. We denote by λ and μ the real principal curvatures (μ might not exist), and by T_λ and T_μ the corresponding smooth principal curvature distributions. We also consider smooth vector fields E_1 and E_2 such that $\tilde{\mathcal{S}}E_1 = aE_1 + bE_2$, $\tilde{\mathcal{S}}E_2 = -bE_1 + aE_2$, $\langle E_1, E_1\rangle = -1$, $\langle E_2, E_2\rangle = 1$, $\langle E_1, E_2\rangle = 0$.

First of all we claim

$$\nabla_{E_i} E_j \in \Gamma(T_\lambda \oplus T_\mu), \qquad \text{for } i, j \in \{1, 2\}. \tag{3.9}$$

In order to prove this, note that $\langle E_i, E_j\rangle$ is constant, so in particular $\langle \nabla_{E_i} E_j, E_j\rangle = 0$. On the other hand, by the Codazzi equation,

$$\begin{aligned} 0 &= \langle \tilde{R}(E_1, E_2)E_2, \xi^L\rangle = \langle (\nabla_{E_1}\tilde{\mathcal{S}})E_2, E_2\rangle - \langle (\nabla_{E_2}\tilde{\mathcal{S}})E_1, E_2\rangle \\ &= \langle \nabla_{E_1}\tilde{\mathcal{S}}E_2, E_2\rangle - \langle \tilde{\mathcal{S}}\nabla_{E_1}E_2, E_2\rangle - \langle \nabla_{E_2}\tilde{\mathcal{S}}E_1, E_2\rangle + \langle \tilde{\mathcal{S}}\nabla_{E_2}E_1, E_2\rangle \\ &= -2b\langle \nabla_{E_1}E_1, E_2\rangle, \end{aligned}$$

so $\langle \nabla_{E_1}E_1, E_2\rangle = 0$. Similarly, writing the Codazzi equation with (E_1, E_2, E_1) gives $\langle \nabla_{E_2}E_2, E_1\rangle = 0$. Altogether this proves (3.9).

Now let $X \in \Gamma(T_\nu)$, with $\nu \in \{\lambda, \mu\}$. By applying the Codazzi equation to (E_1, X, E_2), (E_2, X, E_1), (E_1, X, E_1), and (E_2, X, E_2), we obtain

$$\begin{aligned}(\nu - a)\langle\nabla_{E_1}E_2, X\rangle + b\langle\nabla_{E_1}E_1, X\rangle &= (\nu - a)\langle\nabla_{E_2}E_1, X\rangle - b\langle\nabla_{E_2}E_2, X\rangle = 0\\ (\nu - a)\langle\nabla_{E_1}E_1, X\rangle - b\langle\nabla_{E_1}E_2, X\rangle &= (\nu - a)\langle\nabla_{E_2}E_2, X\rangle + b\langle\nabla_{E_2}E_1, X\rangle\\ &= 2b\langle\nabla_X E_1, E_2\rangle.\end{aligned}$$

From this we get the following relations:

$$\begin{aligned}\langle\nabla_{E_1}E_1, X\rangle = \langle\nabla_{E_2}E_2, X\rangle &= \frac{2b(\nu - a)}{(\nu - a)^2 + b^2}\langle\nabla_X E_1, E_2\rangle,\\ \langle\nabla_{E_1}E_2, X\rangle = -\langle\nabla_{E_2}E_1, X\rangle &= -\frac{2b^2}{(\nu - a)^2 + b^2}\langle\nabla_X E_1, E_2\rangle.\end{aligned} \tag{3.10}$$

Now we use the Gauss equation and (3.9) to get

$$\begin{aligned}-\frac{c}{4} &= \langle\tilde{R}(E_1, E_2)E_2, E_1\rangle\\ &= \langle R(E_1, E_2)E_2, E_1\rangle - \langle\tilde{\mathcal{S}}E_2, E_2\rangle\langle\tilde{\mathcal{S}}E_1, E_1\rangle + \langle\tilde{\mathcal{S}}E_2, E_1\rangle\langle\tilde{\mathcal{S}}E_2, E_1\rangle\\ &= \langle\nabla_{E_1}E_2, \nabla_{E_2}E_1\rangle - \langle\nabla_{E_1}E_1, \nabla_{E_2}E_2\rangle - \langle\nabla_{\nabla_{E_1}E_2}E_2, E_1\rangle\\ &\quad + \langle\nabla_{\nabla_{E_2}E_1}E_2, E_1\rangle + a^2 + b^2.\end{aligned}$$

Finally, let $\{X_1, \dots, X_k\}$ be an orthonormal basis of $\Gamma(T_\lambda \oplus T_\mu)$ such that $\tilde{\mathcal{S}}X_i = \nu_i X_i$, with $\nu_i \in \{\lambda, \mu\}$. Taking into account (3.9), and writing the previous covariant derivatives with respect to the previous basis, (3.10) implies

$$\begin{aligned}-\frac{c}{4} - a^2 - b^2 &= \langle\nabla_{E_1}E_2, \nabla_{E_2}E_1\rangle - \langle\nabla_{E_1}E_1, \nabla_{E_2}E_2\rangle\\ &\quad - \langle\nabla_{\nabla_{E_1}E_2}E_2, E_1\rangle + \langle\nabla_{\nabla_{E_2}E_1}E_2, E_1\rangle\\ &= \sum_{i=1}^{k}\langle\nabla_{E_1}E_2, X_i\rangle\langle\nabla_{E_2}E_1, X_i\rangle - \sum_{i=1}^{k}\langle\nabla_{E_1}E_1, X_i\rangle\langle\nabla_{E_2}E_2, X_i\rangle\\ &\quad - \sum_{i=1}^{k}\langle\nabla_{E_1}E_2, X_i\rangle\langle\nabla_{X_i}E_2, E_1\rangle + \sum_{i=1}^{k}\langle\nabla_{E_2}E_1, X_i\rangle\langle\nabla_{X_i}E_2, E_1\rangle\\ &= -\sum_{i=1}^{k}\frac{8b^2}{(\nu_i - a)^2 + b^2}\langle\nabla_{X_i}E_1, E_2\rangle^2 \leq 0,\end{aligned}$$

from where the result follows. □

Proposition 3.2.12. *If $q \in \tilde{M}$ is of type IV and $p = \pi(q)$, then M is Hopf at p. Let λ and μ be the real principal curvatures of $\tilde{M}$ at q (μ might not exist). Then the principal curvatures of M at p are*

$$\lambda,\ \mu,\quad \text{and}\quad 2a = \frac{4c\lambda}{c - 4\lambda^2} \in \left(-\sqrt{-c}, \sqrt{-c}\right),$$

where $2a$ is the principal curvature associated with the Hopf vector.

Proof. Let $a \pm ib$ be the non-real complex eigenvalues of $\tilde{\mathcal{S}}$ ($b \neq 0$). Let λ and $\mu = -c/4\lambda$ be the real eigenvalues of $\tilde{\mathcal{S}}$ (μ might not exist). Assume that $\tilde{\mathcal{S}}$ has a type IV matrix expression and let e_1, $e_2 \in T_q\tilde{M}$ such that $\tilde{\mathcal{S}}e_1 = ae_1 + be_2$, $\tilde{\mathcal{S}}e_2 = -be_1 + ae_2$, $\langle e_1, e_1\rangle = -1$, $\langle e_2, e_2\rangle = 1$, $\langle e_1, e_2\rangle = 0$.

We can assume that $v = r_1e_1 + r_2e_2 + u + w$, where $u \in T_\lambda(q)$, $w \in T_\mu(q)$, and r_1, $r_2 \in \mathbb{R}$. If there is only one principal curvature λ, then μ and $T_\mu(q)$ do not exist and it suffices to put $w = 0$ throughout. We have $-1 = \langle v, v\rangle = -r_1^2 + r_2^2 + \langle u, u\rangle + \langle w, w\rangle$ and $\tilde{\mathcal{S}}v = (r_1a - r_2b)e_1 + (r_2a + r_1b)e_2 + \lambda u + \mu w$, and hence

$$J\xi^L = -2((r_1a - r_2b)e_1 + (r_2a + r_1b)e_2 + \lambda u + \mu w)/\sqrt{-c}.$$

Taking into account that $\langle u, u\rangle = -1 + r_1^2 - r_2^2 - \langle w, w\rangle$ we have

$$\begin{aligned} 1 &= \langle J\xi^L, J\xi^L\rangle = -\frac{4}{c}\big((-a^2 + b^2 + \lambda^2)(r_1^2 - r_2^2) + 4abr_1r_2 + (\mu^2 - \lambda^2)\langle w, w\rangle - \lambda^2\big),\\ 0 &= \langle \tilde{\mathcal{S}}v, v\rangle = (\lambda - a)(r_1^2 - r_2^2) + 2br_1r_2 + (\mu - \lambda)\langle w, w\rangle - \lambda. \end{aligned}$$

We can view the previous two equations as a linear system in the variables $r_1^2 - r_2^2$ and r_1r_2. The matrix of this system has determinant $-8b((a-\lambda)^2 + b^2)/c \neq 0$, and thus has a unique solution. In fact,

$$r_1^2 - r_2^2 = \frac{-c - 8a\lambda + 4\lambda^2 + 4(\lambda + \mu - 2a)(\lambda - \mu)\langle w, w\rangle}{4((a - \lambda)^2 + b^2)}.$$

Then we have

$$0 \leq \langle u, u\rangle = -1 + r_1^2 - r_2^2 - \langle w, w\rangle = -\frac{4a^2 + 4b^2 + c + 4((a - \mu)^2 + b^2)\langle w, w\rangle}{4((a - \lambda)^2 + b^2)}.$$

Hence, as we knew that $4a^2 + 4b^2 + c \geq 0$ by Lemma 3.2.11, we must have $4a^2 + 4b^2 + c = 0$, and thus $u = w = 0$.

This implies that $T_\lambda(q)$ and $T_\mu(q)$ are orthogonal to v and $J\xi^L$, and therefore, they descend to the λ and μ eigenspaces of $\mathcal{S}$ respectively, and they are orthogonal to $J\xi$. Again, for dimension reasons, $J\xi$ must be an eigenvector of $\mathcal{S}$ and thus M is Hopf at p. We also have, taking into account $0 = \pi_{*q}v = r_1\pi_*e_1 + r_2\pi_*e_2$ and $b^2 = -a^2 - c/4$,

$$\begin{aligned} \mathcal{S}J\xi &= -\frac{2}{\sqrt{-c}}((r_1a^2 - 2r_2ab - r_1b^2)\pi_*e_1 + (2r_1ab - r_2b^2 + r_2a^2)\pi_*e_2)\\ &= -\frac{2}{\sqrt{-c}}(2a(ar_1 - br_2)\pi_*e_1 + 2a(br_1 + ar_2)\pi_*e_2 + \frac{c}{4}\pi_*v) = 2aJ\xi. \end{aligned}$$

Lemma 3.2.10 and $4a^2 + 4b^2 + c = 0$ yield $a = 2c\lambda/(c - 4\lambda^2)$. If $|2a| \geq \sqrt{-c}$, then $0 = 4a^2 + 4b^2 + c \geq 4b^2$, which is impossible because $b \neq 0$. Therefore, $|2a| < \sqrt{-c}$, that is, the principal curvature associated with the Hopf vector in M is in $(-\sqrt{-c}, \sqrt{-c})$. □

Corollary 3.2.13. *Let M be a connected isoparametric hypersurface in $\mathbb{C}H^n$ which lifts to a type IV hypersurface in H_1^{2n+1} at some point. Then M is an open part of a tube around a totally geodesic $\mathbb{R}H^n$.*

Proof. By Lemma 3.2.9, every point of $\tilde{M}$ is of type IV. From Proposition 3.2.12 and the fact that $\tilde{M}$ has constant principal curvatures, we deduce that M is Hopf and has constant principal curvatures. From the classification of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ (Theorem 2.5.1), it follows that the unique such hypersurface whose Hopf principal curvature is less than $\sqrt{-c}$ in absolute value (see Remark 2.5.2) is a tube around a totally geodesic $\mathbb{R}H^n$. □

3.2.5 Variation of the Jordan canonical form

As was pointed out in Remark 3.2.1, there are examples of isoparametric hypersurfaces in $\mathbb{C}H^n$ whose lift to the anti-De Sitter space might have varying Jordan canonical form. We clarify this a bit more in the following

Proposition 3.2.14. *Let M be a connected isoparametric hypersurface in $\mathbb{C}H^n$, $n \geq 2$, and denote by $\tilde{M} = \pi^{-1}(M)$ its lift to the anti-De Sitter space. Then,*

(i) *If a point $q \in \tilde{M}$ is of type IV, then all the points of $\tilde{M}$ are of type IV, and M is an open part of a tube around a totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$.*

(ii) *If a point $q \in \tilde{M}$ is of type II, then all the points of $\tilde{M}$ are of type II, and M is an open part of a horosphere in $\mathbb{C}H^n$.*

(iii) *If there is a point $q \in \tilde{M}$ of type III, then there is a neighborhood of q where all points are of type III.*

Proof. The first statement is consequence of Lemma 3.2.9 and Corollary 3.2.13.

Assume now that $q \in \tilde{M}$ is of type II, and recall that $\tilde{M}$ has constant principal curvatures. Then, according to Proposition 3.2.7, $\tilde{M}$ has exactly one principal curvature at q that is $\pm\sqrt{-c}/2$. If $q_0 \in \tilde{M}$ is another point of type I or III, then propositions 3.2.5 and 3.2.8 say that $\pm\sqrt{-c}/2$ cannot be a principal curvature of $\tilde{M}$ at q_0. Since $\tilde{M}$ is connected we conclude that all the points of $\tilde{M}$ are of type II. But now the classification of Hopf real hypersurfaces with constant principal curvatures in complex hyperbolic spaces (Theorem 2.5.1 together with Remark 2.5.2) implies that M is an open part of a horosphere.

Finally, assume that $q \in \tilde{M}$ is of type III. By definition, the difference between the algebraic and geometric multiplicities of λ is a lower semi-continuous function on $\tilde{M}$. In our case, this function can only take the values 0 (at points of type I) and 2 (at points of type III). Hence we conclude. □

3.3 Type III hypersurfaces

The aim of this section is to study isoparametric hypersurfaces of the anti-De Sitter space all of whose points are of type III, and determine the extrinsic geometry of their focal submanifolds.

Let M be a connected isoparametric real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 2$. We denote by $\tilde{M} = \pi^{-1}(M)$ its lift to the anti-De Sitter space. Assume that there are points in $\tilde{M}$ of type III. According to Proposition 3.2.14, if $q \in \tilde{M}$ is a point of type III, then there is a neighborhood of q where all points are also of type III.

Thus, we assume that we are working on a connected open subset $\tilde{\mathcal{W}}$ of $\tilde{M} = \pi^{-1}(M)$ where all points are of type III. We denote by ξ a unit (spacelike) normal vector field along $\tilde{\mathcal{W}}$. We know that $\tilde{M}$ has at most two distinct constant principal curvatures (see Proposition 3.2.8). We call λ the principal curvature whose algebraic and geometric multiplicities do not coincide, and μ the other one, if it exists. Note that if there are two distinct principal curvatures, then $c+4\lambda\mu = 0$. We denote by T_λ and T_μ the corresponding principal curvature distributions, and choose smooth vector fields E_1, E_2, $E_3 \in \Gamma(T\tilde{\mathcal{W}})$ satisfying (3.6) at each point. Recall that $T_\lambda = \mathbb{R}E_1 \oplus (T_\lambda \ominus \mathbb{R}E_2)$.

We also denote $m_\lambda = \dim T_\lambda + 2$ and $m_\mu = \dim T_\mu$, the algebraic multiplicities of λ and μ. Since $\tilde{\mathcal{W}}$ is isoparametric and all points are of type III, m_λ and m_μ are constant functions, and in principle $m_\lambda \geq 3$, $m_\mu \geq 0$. In fact, μ might not exist, and in this case, $m_\mu = 0$.

3.3.1 Covariant derivatives of an isoparametric hypersurface

Recall that ξ^L denotes a unit normal vector field along $\tilde{\mathcal{W}}$. By ∇ and R we denote the Levi-Civita connection and curvature tensor of $\tilde{\mathcal{W}}$, and by $\tilde{\nabla}$ and $\tilde{R}$ the Levi-Civita connection of the anti-De Sitter spacetime, respectively. The aim of this subsection is to prove the following result:

Proposition 3.3.1. *For any $W \in \Gamma(T_\mu)$ we have $\tilde{\nabla}_{E_1} W \in \Gamma(T_\mu)$.*

We may assume $m_\mu > 0$; otherwise, if $m_\mu = 0$, this is trivial. We will carry out the proof in several steps. The first step almost finishes the argument except for an E_1-component.

Lemma 3.3.2. *For any $W \in \Gamma(T_\mu)$ we have $\tilde{\nabla}_{E_1} W \in \Gamma(\mathbb{R}E_1 \oplus T_\mu)$.*

Proof. First, recall that $\tilde{\nabla}_{E_1} W = \nabla_{E_1} W + \langle \mathcal{S}E_1, W\rangle \xi^L = \nabla_{E_1} W$, so it suffices to work with ∇. Let $X \in \Gamma(\mathbb{R}E_3 \oplus T_\lambda)$. The result follows if we show $\langle \nabla_{E_1} W, X\rangle = 0$. First of all, the Codazzi equation and the fact that $\mathcal{S}$ is self-adjoint imply:

$$\begin{aligned} 0 &= \langle \tilde{R}(E_1, W)X, \xi^L\rangle = \langle (\nabla_{E_1}\mathcal{S})W, X\rangle - \langle (\nabla_W \mathcal{S})E_1, X\rangle \\ &= \mu\langle \nabla_{E_1} W, X\rangle - \langle \nabla_{E_1} W, \mathcal{S}X\rangle - \lambda\langle \nabla_W E_1, X\rangle + \langle \nabla_W E_1, \mathcal{S}X\rangle. \end{aligned}$$

Taking $X \in \Gamma(T_\lambda)$ in this formula gives $0 = (\mu - \lambda)\langle \nabla_{E_1} W, X\rangle$. In particular, $\langle \nabla_{E_1} W, E_1\rangle = 0$. Using this, $\langle \nabla_W E_1, E_1\rangle = 0$ (because $\langle E_1, E_1\rangle = 0$), and putting $X = E_3$ in the previous equation yields

$$\begin{aligned}0 &= \mu\langle \nabla_{E_1} W, E_3\rangle - \langle \nabla_{E_1} W, E_1 + \lambda E_3\rangle - \lambda\langle \nabla_W E_1, E_3\rangle + \langle \nabla_W E_1, E_1 + \lambda E_3\rangle\\ &= (\mu - \lambda)\langle \nabla_{E_1} W, E_3\rangle,\end{aligned}$$

from where the assertion follows. □

Thus, in order to conclude the proof of Proposition 3.3.1 it just remains to show that $\langle \nabla_{E_1} W, E_2\rangle = 0$. This will take most of the effort of this subsection. The next lemma is known (see for example [56, Propostion 2.6]), but we include its proof here for the sake of completeness.

Lemma 3.3.3. *T_μ is an autoparallel distribution: if W_1, $W_2 \in \Gamma(T_\mu)$, then $\nabla_{W_1} W_2 \in \Gamma(T_\mu)$.*

Proof. Let $X \in \Gamma(\mathbb{R}E_2 \oplus \mathbb{R}E_3 \oplus T_\lambda)$. It suffices to prove that $\langle \nabla_{W_1} W_2, X\rangle = 0$. Since $\mathcal{S}$ is self-adjoint and $\mathcal{S}X$ is orthogonal to T_μ, the Codazzi equation implies

$$\begin{aligned}0 &= \langle \tilde{R}(X, W_1)W_2, \xi^L\rangle = \langle (\nabla_X \mathcal{S})W_1, W_2\rangle - \langle (\nabla_{W_1}\mathcal{S})X, W_2\rangle\\ &= -\langle \nabla_{W_1}\mathcal{S}X, W_2\rangle + \langle \mathcal{S}\nabla_{W_1} X, W_2\rangle = \langle \nabla_{W_1} W_2, \mathcal{S}X - \mu X\rangle.\end{aligned}$$

Taking $X \in \Gamma(T_\lambda)$ in this formula yields $0 = (\lambda - \mu)\langle \nabla_{W_1} W_2, X\rangle = 0$. In particular, $\langle \nabla_{W_1} W_2, E_1\rangle = 0$. This, and setting $X = E_3$ above yields

$$0 = \langle \nabla_{W_1} W_2, E_1 + \lambda E_3 - \mu E_3\rangle = (\lambda - \mu)\langle \nabla_{W_1} W_2, E_3\rangle.$$

This equation, and setting $X = E_2$ in the previous equation yields

$$0 = \langle \nabla_{W_1} W_2, \lambda E_2 + E_3 - \mu E_2\rangle = (\lambda - \mu)\langle \nabla_{W_1} W_2, E_2\rangle,$$

as we wanted to show. □

In order to finish the proof of Proposition 3.3.1 we use the Gauss equation to get

$$\begin{aligned}0 &= \langle \tilde{R}(W, E_1)W, E_3\rangle = \langle R(W, E_1)W, E_3\rangle\\ &\quad + \langle \mathcal{S}W, W\rangle\langle \mathcal{S}E_1, E_3\rangle - \langle \mathcal{S}E_1, W\rangle\langle \mathcal{S}W, E_3\rangle\\ &= \langle \nabla_W \nabla_{E_1} W, E_3\rangle - \langle \nabla_{E_1}\nabla_W W, E_3\rangle - \langle \nabla_{\nabla_W E_1} W, E_3\rangle + \langle \nabla_{\nabla_{E_1} W} W, E_3\rangle.\end{aligned} \tag{3.11}$$

Lemma 3.3.2 yields $\nabla_{E_1} W \in \Gamma(\mathbb{R}E_1 \oplus T_\mu)$. Write $\nabla_{E_1} W = \langle \nabla_{E_1} W, E_2\rangle E_1 + (\nabla_{E_1} W)_{T_\mu}$ accordingly. By Lemma 3.3.3, we have that $\nabla_W (\nabla_{E_1} W)_{T_\mu} \in \Gamma(T_\mu)$, and thus we get $\langle \nabla_W (\nabla_{E_1} W)_{T_\mu}, E_3\rangle = 0$. Since $\langle E_1, E_3\rangle = 0$, then we deduce the equality

$$\langle \nabla_W \nabla_{E_1} W, E_3\rangle = \langle \nabla_{E_1} W, E_2\rangle\langle \nabla_W E_1, E_3\rangle.$$

From Lemma 3.3.3 we have $\nabla_W W \in \Gamma(T_\mu)$, and thus Lemma 3.3.2 implies that $\nabla_{E_1}\nabla_W W \in \Gamma(\mathbb{R}E_1 \oplus T_\mu)$. Hence, $\langle \nabla_{E_1}\nabla_W W, E_3\rangle = 0$.

Lemma 3.3.2 yields $\nabla_{E_1} W \in \Gamma(\mathbb{R}E_1 \oplus T_\mu)$, which together with lemmas 3.3.2 and 3.3.3 gives $\langle \nabla_{\nabla_{E_1} W} W, E_3\rangle = 0$.

Hence, (3.11) now reads

$$0 = \langle \nabla_{E_1} W, E_2\rangle\langle \nabla_W E_1, E_3\rangle - \langle \nabla_{\nabla_W E_1} W, E_3\rangle. \tag{3.12}$$

Lemma 3.3.4. *Let $U \in \Gamma(T_\lambda \ominus \mathbb{R}E_2)$ and $W \in \Gamma(T_\mu)$. Then,*

$$\langle \nabla_W E_1, E_3\rangle = (\lambda - \mu)\langle \nabla_{E_1} W, E_2\rangle, \tag{3.13}$$

$$\langle \nabla_{E_3} W, E_3\rangle = -2\langle \nabla_{E_1} W, E_2\rangle, \tag{3.14}$$

$$\langle \nabla_W E_1, U\rangle = -(\lambda - \mu)\langle \nabla_U W, E_3\rangle. \tag{3.15}$$

Proof. The Codazzi equation and Lemma 3.3.2 imply

$$\begin{aligned} 0 &= \langle \tilde{R}(E_1, W)E_2, \xi^L\rangle = \langle (\nabla_{E_1}\mathcal{S})W, E_2\rangle - \langle (\nabla_W \mathcal{S})E_1, E_2\rangle \\ &= \mu\langle \nabla_{E_1} W, E_2\rangle - \langle \nabla_{E_1} W, \mathcal{S}E_2\rangle - \lambda\langle \nabla_W E_1, E_2\rangle + \langle \nabla_W E_1, \mathcal{S}E_2\rangle \\ &= (\mu - \lambda)\langle \nabla_{E_1} W, E_2\rangle + \langle \nabla_W E_1, E_3\rangle, \end{aligned}$$

from where we get (3.13).

We also have

$$0 = \langle \tilde{R}(E_3, W)E_1, \xi^L\rangle = \langle (\nabla_{E_3}\mathcal{S})W, E_1\rangle - \langle (\nabla_W \mathcal{S})E_3, E_1\rangle = (\mu - \lambda)\langle \nabla_{E_3} W, E_1\rangle.$$

Thus, $\langle \nabla_{E_3} W, E_1\rangle = 0$. This, the Codazzi equation, and (3.13) yield

$$\begin{aligned} 0 &= \langle \tilde{R}(E_3, W)E_3, \xi^L\rangle = \langle (\nabla_{E_3}\mathcal{S})W, E_3\rangle - \langle (\nabla_W \mathcal{S})E_3, E_3\rangle \\ &= (\mu - \lambda)\langle \nabla_{E_3} W, E_3\rangle - \langle \nabla_{E_3} W, E_1\rangle - 2\langle \nabla_W E_1, E_3\rangle \\ &= -(\lambda - \mu)\langle \nabla_{E_3} W, E_3\rangle - 2(\lambda - \mu)\langle \nabla_{E_1} W, E_2\rangle, \end{aligned}$$

which gives (3.14).

Now, the Codazzi equation and Lemma 3.3.2 imply

$$\begin{aligned} 0 &= \langle \tilde{R}(E_1, U)W, \xi^L\rangle = \langle (\nabla_{E_1}\mathcal{S})U, W\rangle - \langle (\nabla_U \mathcal{S})E_1, W\rangle \\ &= (\lambda - \mu)\langle \nabla_{E_1} U, W\rangle - (\lambda - \mu)\langle \nabla_U E_1, W\rangle = -(\lambda - \mu)\langle \nabla_U E_1, W\rangle, \end{aligned}$$

and thus we get $\langle \nabla_U E_1, W\rangle = 0$. This implies

$$\begin{aligned} 0 &= \langle \tilde{R}(E_3, U)W, \xi^L\rangle = \langle (\nabla_{E_3}\mathcal{S})U, W\rangle - \langle (\nabla_U \mathcal{S})E_3, W\rangle \\ &= (\lambda - \mu)\langle \nabla_{E_3} U, W\rangle - \langle \nabla_U E_1, W\rangle - (\lambda - \mu)\langle \nabla_U E_3, W\rangle \\ &= (\lambda - \mu)(\langle \nabla_{E_3} U, W\rangle - \langle \nabla_U E_3, W\rangle), \end{aligned}$$

from where we obtain $\langle \nabla_{E_3} W, U\rangle = \langle \nabla_U W, E_3\rangle$. Finally, this equation gives

$$\begin{aligned} 0 &= \langle \tilde{R}(E_3, W)U, \xi^L\rangle = \langle (\nabla_{E_3}\mathcal{S})W, U\rangle - \langle (\nabla_W \mathcal{S})E_3, U\rangle \\ &= (\mu - \lambda)\langle \nabla_{E_3} W, U\rangle - \langle \nabla_W E_1, U\rangle = -(\lambda - \mu)\langle \nabla_U W, E_3\rangle - \langle \nabla_W E_1, U\rangle, \end{aligned}$$

which concludes the proof of the lemma. □

Now we come back to (3.12) and finish the proof of Proposition 3.3.1.

Using Lemma 3.3.3 we see that $\nabla_W E_1 \in \Gamma(\mathbb{R}E_1 \oplus \mathbb{R}E_3 \oplus T_\lambda)$. Take $\{U_1, \dots, U_k\}$ an orthonormal basis of vector fields of the distribution $T_\lambda \ominus \mathbb{R}E_2$. Thus, we can write, taking into account (3.13) and (3.15),

$$\begin{aligned}\nabla_W E_1 &= \langle \nabla_W E_1, E_2\rangle E_1 + \langle \nabla_W E_1, E_3\rangle E_3 + \sum_{i=1}^{k} \langle \nabla_W E_1, U_i\rangle U_i \\ &= \langle \nabla_W E_1, E_2\rangle E_1 + (\lambda - \mu)\langle \nabla_{E_1} W, E_2\rangle E_3 - (\lambda-\mu) \sum_{i=1}^{k} \langle \nabla_{U_i} W, E_3\rangle U_i.\end{aligned} \tag{3.16}$$

Hence, using (3.16), Lemma 3.3.2, (3.13) and (3.14), Equation (3.12) becomes

$$\begin{aligned}0 &= (\lambda-\mu)\langle \nabla_{E_1} W, E_2\rangle^2 - \langle \nabla_W E_1, E_2\rangle\langle \nabla_{E_1} W, E_3\rangle \\ &\quad - (\lambda-\mu)\langle \nabla_{E_1} W, E_2\rangle\langle \nabla_{E_3} W, E_3\rangle + (\lambda-\mu)\sum_{i=1}^{k}\langle \nabla_{U_i} W, E_3\rangle\langle \nabla_{U_i} W, E_3\rangle \\ &= (\lambda-\mu)\Big(3\langle \nabla_{E_1} W, E_2\rangle^2 + \sum_{i=1}^{k}\langle \nabla_{U_i} W, E_3\rangle^2\Big).\end{aligned}$$

Since the addends are all non-negative, we must have

$$\langle \nabla_{E_1} W, E_2\rangle = 0, \text{ and } \langle \nabla_U W, E_3\rangle = 0 \text{ for any } U \in \Gamma(T_\lambda \ominus \mathbb{R}E_2) \text{ and } W \in \Gamma(T_\mu),$$

which is what was left to finish the proof of Proposition 3.3.1.

3.3.2 Parallel hypersurfaces and the focal manifold

We continue to denote by $\tilde{\mathcal{W}}$ a connected open subset of the Lorentzian isoparametric hypersurface $\tilde{M} = \pi^{-1}(M)$ of the anti-De Sitter space H_1^{2n+1} where all points are of type III, and let $\mathcal{W} = \pi(\tilde{\mathcal{W}}) \subset M$. If ξ denotes a unit normal vector field along $\mathcal{W}$, then ξ^L is a unit vector field along $\tilde{\mathcal{W}}$. As a matter of notation, $\tilde{\gamma}_q$ will be the geodesic in H_1^{2n+1} such that $\tilde{\gamma}_q(0) = q \in \tilde{\mathcal{W}}$ and $\tilde{\gamma}_q'(0) = \xi_q^L$. Accordingly, we write $\gamma_p = \pi \circ \tilde{\gamma}_q$ for the geodesic in $\mathbb{C}H^n$ with initial conditions $\gamma_p(0) = p = \pi(q)$ and $\gamma_p'(0) = \xi_p$.

Recall from Section 2.1 the definition of the map $\tilde{\Phi}^t \colon \tilde{\mathcal{W}} \to H_1^{2n+1}$, given by $\tilde{\Phi}^t(q) = \exp_q(t\xi^L) = \tilde{\gamma}_q(t)$, where exp is the semi-Riemannian exponential map. For a fixed r, $\Phi^r(M)$ is not necessarily a submanifold of H_1^{2n+1}, but at least locally and for r small enough, it is a hypersurface of H_1^{2n+1}. We also consider the vector field η^t along $\tilde{\Phi}^t$ defined by $\eta^t(q) = \tilde{\gamma}_q'(t)$.

The differential of $\tilde{\Phi}^t$ is given by $\tilde{\Phi}^t_{*q}(X) = \zeta_X(t)$, where ζ_X is a Jacobi vector field along $\tilde{\gamma}_q$ with initial conditions $\zeta_X(0) = X \in T_q\tilde{\mathcal{W}}$, and $\zeta_X'(0) = -\tilde{\mathcal{S}}X$, where $(\cdot)'$ stands for covariant differentiation along $\tilde{\gamma}_q$ (see [6, §8.2]). Since H_1^{2n+1} is a space of constant sectional curvature $c/4$ and $\tilde{\gamma}'$ is spacelike, it follows that the Jacobi equation is written as $4\zeta_X'' + c\zeta_X = 0$.

Let $\mathcal{P}_X(t)$ denote the parallel translation of $X \in T_q\tilde{\mathcal{W}}$ along $\tilde{\gamma}_q$. For $\nu \in \mathbb{R}$, we also define

$$g_\nu(t) = \cosh\Big(\frac{t\sqrt{-c}}{2}\Big) - \frac{2\nu}{\sqrt{-c}}\sinh\Big(\frac{t\sqrt{-c}}{2}\Big) \quad \text{and} \quad h(t) = -\frac{2}{\sqrt{-c}}\sinh\Big(\frac{t\sqrt{-c}}{2}\Big).$$

Solving the Jacobi equation we get

$$\begin{aligned} \zeta_X(t) &= g_\lambda(t)\mathcal{P}_X(t), \quad \text{if } X \in T_\lambda(q), \\ \zeta_X(t) &= g_\mu(t)\mathcal{P}_X(t), \quad \text{if } X \in T_\mu(q), \\ \zeta_{E_2(q)}(t) &= g_\lambda(t)\mathcal{P}_{E_2(q)}(t) + h(t)\mathcal{P}_{E_3(q)}(t), \\ \zeta_{E_3(q)}(t) &= h(t)\mathcal{P}_{E_1(q)}(t) + g_\lambda(t)\mathcal{P}_{E_3(q)}(t). \end{aligned} \tag{3.17}$$

Since we are denoting by λ the principal curvature whose geometric and algebraic multiplicities do not coincide, it follows from Proposition 3.2.8 that $|\lambda| < \sqrt{-c}/2$. We assume, changing the orientation if necessary, that $\lambda \geq 0$. Recall that, if a second distinct principal curvature μ exists, then $c + 4\lambda\mu = 0$, which implies $\lambda, \mu \neq 0$. We may choose $r \geq 0$ such that

$$\lambda = \frac{\sqrt{-c}}{2}\tanh\Big(\frac{r\sqrt{-c}}{2}\Big) \quad \text{and} \quad \mu = \frac{\sqrt{-c}}{2}\coth\Big(\frac{r\sqrt{-c}}{2}\Big). \tag{3.18}$$

Coming back to the differential of $\tilde{\Phi}^t$, it now follows from $\tilde{\Phi}^t_*(X) = \zeta_X(t)$ and (3.17) that, if $t \in [0, r)$, then $\tilde{\Phi}^t_*$ is an isomorphism for each $q \in \tilde{\mathcal{W}}$. This is simply because $g_\lambda, g_\mu > 0$ in $[0, r)$. Therefore, by making $\tilde{\mathcal{W}}$ smaller if necessary, we conclude that $\tilde{\mathcal{W}}^t = \tilde{\Phi}^t(\tilde{\mathcal{W}})$ is an equidistant hypersurface to $\tilde{\mathcal{W}}$ for each $t \in [0, r)$, and η^t can be seen as a unit normal vector field along $\tilde{\mathcal{W}}^t$.

We now determine the extrinsic geometry of the hypersurface $\tilde{\mathcal{W}}^t$. For each $t \in [0, r)$ it is known that the shape operator $\tilde{\mathcal{S}}^t$ of $\tilde{\mathcal{W}}^t$ at $\tilde{\Phi}^t(q)$ with respect to $\eta^t(q)$ is determined by the formula $\tilde{\mathcal{S}}^t\tilde{\Phi}^t_{*q}X = -\zeta'_X(t)$ for each $X \in T_q\tilde{\mathcal{W}}$ (again, see [6, §8.2]). Before using the explicit expressions of the Jacobi vector fields in terms of the parallel translation obtained above, we define the functions

$$\begin{aligned} \lambda(t) &= \frac{\sqrt{-c}}{2}\tanh\Big(\frac{\sqrt{-c}}{2}(r-t)\Big), \qquad \mu(t) = \frac{\sqrt{-c}}{2}\coth\Big(\frac{\sqrt{-c}}{2}(r-t)\Big), \\ \alpha(t) &= \frac{2}{\sqrt{-c}}\cosh^3\Big(\frac{r\sqrt{-c}}{2}\Big)\operatorname{sech}^3\Big(\frac{\sqrt{-c}}{2}(r-t)\Big)\sinh\Big(\frac{t\sqrt{-c}}{2}\Big), \\ \beta(t) &= \cosh^2\Big(\frac{r\sqrt{-c}}{2}\Big)\operatorname{sech}^2\Big(\frac{\sqrt{-c}}{2}(r-t)\Big), \end{aligned} \tag{3.19}$$

which are positive for each $t \in [0, r)$, and the vector fields along $\tilde{\Phi}^t$

$$\begin{aligned} E_1^t(q) &= \beta(t)\mathcal{P}_{E_1(q)}(t), \\ E_2^t(q) &= -\frac{\alpha(t)^2}{8\beta(t)^3}\mathcal{P}_{E_1(q)}(t) + \frac{1}{\beta(t)}\mathcal{P}_{E_2(q)}(t) - \frac{\alpha(t)}{2\beta(t)^2}\mathcal{P}_{E_3(q)}(t), \\ E_3^t(q) &= \frac{\alpha(t)}{2\beta(t)}\mathcal{P}_{E_1(q)}(t) + \mathcal{P}_{E_3(q)}(t). \end{aligned} \tag{3.20}$$

Now, using (3.17) and $\tilde{\mathcal{S}}^t\tilde{\Phi}^t_{*q}X = -\zeta'_X(t)$, it follows after some calculations that $\tilde{\mathcal{W}}^t$ has principal curvatures $\lambda(t)$ and $\mu(t)$ with algebraic multiplicities m_λ and m_μ, and the tangent vectors E_1^t, E_2^t, E_3^t satisfy (3.6) at each point (with $\lambda(t)$ instead of λ). Moreover, the principal curvature spaces of $\tilde{\mathcal{W}}^t$ are obtained by parallel translation of T_λ and T_μ along the geodesics $\tilde{\gamma}_q$, that is, $T_{\lambda(t)} = \mathcal{P}_{T_\lambda}(t)$ and $T_{\mu(t)} = \mathcal{P}_{T_\mu}(t)$. In particular, $\tilde{\mathcal{W}}^t$ is isoparametric for all $t \in [0, r)$, and all points of $\tilde{\mathcal{W}}^t$ are of type III.

Finally, we show that the S^1-fiber of π is tangent to $\tilde{\mathcal{W}}^t$ for each $t \in [0, r)$. This follows from the fact that the vertical vector field V satisfies

$$\langle \tilde{\gamma}'_q(0), V_{\tilde{\gamma}_q(0)}\rangle = 0 \quad \text{and} \quad \frac{d}{dt}\langle \tilde{\gamma}'_q, V\rangle = \langle \tilde{\gamma}'_q, \tilde{\nabla}_{\tilde{\gamma}'_q(t)}V\rangle = 0,$$

for all t, because V is a Killing vector field (and thus $\tilde{\nabla}V$ is skew-symmetric with respect to the metric).

We can summarize the information obtained about $\tilde{\mathcal{W}}^t$ so far as follows

Proposition 3.3.5. *If $t \in [0, r)$, then the S^1-fibers of π are tangent to the parallel hypersurface $\tilde{\mathcal{W}}^t$, which has constant principal curvatures $\lambda(t)$ and $\mu(t)$ with algebraic multiplicities m_λ and m_μ. All points of $\tilde{\mathcal{W}}^t$ are of type III, $\{E_1^t, E_2^t, E_3^t\}$ are three tangent vector fields satisfying (3.6) at each point (with $\lambda(t)$ instead of λ), and the spaces $T_{\lambda(t)} \ominus \mathbb{R}E_2^t$ and $T_{\mu(t)}$ are obtained by parallel translation of $T_\lambda \ominus \mathbb{R}E_2$ and T_μ along normal geodesics.*

Now we focus our attention on $t = r$. Recall from Proposition 3.2.8 that if $\lambda = 0$, then μ does not exist and $m_\mu = 0$. In general, it follows from (3.17) that $\ker \tilde{\Phi}^r_* = T_\mu$, and thus, $\tilde{\Phi}^r$ has constant rank $2n - m_\mu$. Hence, making $\tilde{\mathcal{W}}$ smaller if necessary, we deduce that $\tilde{\mathcal{W}}^r$ is an embedded submanifold of H_1^{2n+1} of codimension $m_\mu + 1$.

Let $q_r \in \tilde{\mathcal{W}}^r$. The map $\eta^r\colon (\tilde{\Phi}^r)^{-1}(q_r) \to \nu^1_{q_r}\tilde{\mathcal{W}}^r$, $q \mapsto \eta^r(q)$, from $(\tilde{\Phi}^r)^{-1}(q_r) \subset \tilde{\mathcal{W}}$ to the unit normal space $\nu^1_{q_r}\tilde{\mathcal{W}}^r$ of $\tilde{\mathcal{W}}^r$ at q_r is differentiable. By (3.17),

$$\tilde{\nabla}_X\eta^r = \zeta'_X(r) = -\frac{\sqrt{-c}}{2}\operatorname{csch}\Big(\frac{r\sqrt{-c}}{2}\Big)\mathcal{P}_X(t),$$

for each $X \in T_\mu(q)$ with $q \in (\tilde{\Phi}^r)^{-1}(q_r)$. Since $T_\mu(q)$ is the tangent space of $(\tilde{\Phi}^r)^{-1}(q_r)$ at q, it follows that $\eta^r((\tilde{\Phi}^r)^{-1}(q_r))$ is open in $\nu^1_{q_r}\tilde{\mathcal{W}}^r$.

As we have seen above, $\langle \tilde{\gamma}'_q(t), V\rangle = 0$ for all t and all $q \in \tilde{\mathcal{W}}$. Setting $t = r$ we get $\langle \eta^r, V\rangle = 0$ for all $q_r \in \tilde{\mathcal{W}}^r$, and since η^r maps $\tilde{\mathcal{W}}$ to an open subset of the unit normal bundle of $\tilde{\mathcal{W}}^r$ we get that V is orthogonal to $\nu\tilde{\mathcal{W}}^r$, and thus tangent to $\tilde{\mathcal{W}}^r$. This implies that $\tilde{\mathcal{W}}^r$ contains locally the S^1-fiber of the submersion $\pi\colon H_1^{2n+1} \to \mathbb{C}H^n$.

On the other hand, the tangent space $T_{q_r}\tilde{\mathcal{W}}^r = \tilde{\Phi}^r_{*q}(T_\lambda(q) \oplus \mathbb{R}E_2(q) \oplus \mathbb{R}E_3(q))$ is, according to (3.17), precisely the parallel translation of $T_\lambda(q) \oplus \mathbb{R}E_2(q) \oplus \mathbb{R}E_3(q)$ along the geodesic $\tilde{\gamma}_q$ for $q \in \tilde{\mathcal{W}}$. Again by (3.17), $(\nu_{q_r}\tilde{\mathcal{W}}^r) \ominus \mathbb{R}\eta^r(q)$ is obtained by parallel translation of $T_\mu(q)$ along $\tilde{\gamma}_q$.

In order to determine the geometry of the submanifold $\tilde{\mathcal{W}}^r$, we take $q \in \tilde{\mathcal{W}}$ and calculate the shape operator $\tilde{\mathcal{S}}^r_{\eta^r(q)}$ of $\tilde{\mathcal{W}}^r$ at $q_r = \tilde{\Phi}^r(q)$ with respect to $\eta^r(q)$. It is known that

$\tilde{\mathcal{S}}^r_{\eta^r(q)}\tilde{\Phi}^r_{*q}X = -(\zeta'_X(t))^\top$ for each $X \in T_q\tilde{\mathcal{W}}$, where $(\cdot)^\top$ denotes orthogonal projection onto the tangent space $T\tilde{\mathcal{W}}^r$.

Taking this into account, and using (3.19) and (3.20) for $t = r$, one can see that $\tilde{\mathcal{S}}^r_{\eta^r(q)}$ has exactly one principal curvature $\lambda(r) = 0$, and $\{E^r_1(q), E^r_2(q), E^r_3(q)\}$ are vectors satisfying the same relations as in (3.6) for $\tilde{\mathcal{S}}^r_{\eta^r(q)}$ at q_r (with $\lambda = 0$ in (3.6)). The parallel translation of $T_\lambda(q) \ominus \mathbb{R}E_2(q)$ along the normal geodesic $\tilde{\gamma}_q$ is in the kernel of $\tilde{\mathcal{S}}^r_{\eta^r(q)}$.

In particular it follows that $(\tilde{\mathcal{S}}^r_{\eta^r(q)})^2 \neq 0$ and $(\tilde{\mathcal{S}}^r_{\eta^r(q)})^3 = 0$ for each $q \in \tilde{\mathcal{W}}$. Since $\eta^r(\tilde{\mathcal{W}})$ is open in the unit normal bundle of $\tilde{\mathcal{W}}^r$, the analiticity of $(\tilde{\mathcal{S}}^r_\eta)^3$ with respect to η implies that $(\tilde{\mathcal{S}}^r_\eta)^3 = 0$ for any $\eta \in \nu\tilde{\mathcal{W}}^r$.

We summarize these results in the following

Proposition 3.3.6. *The submanifold $\tilde{\mathcal{W}}^r$ has codimension $m_\mu + 1$ in H^{2n+1}_1 and the S^1-fibers of π are tangent to it. Moreover, if $q_r = \tilde{\Phi}^r(q)$, with $q \in \tilde{\mathcal{W}}$, then $(\nu_{q_r}\tilde{\mathcal{W}}^r) \ominus \mathbb{R}\eta^r(q)$ is obtained by parallel translation of $T_\mu(q)$ along a geodesic normal to $\tilde{\mathcal{W}}$ through q. For any $\eta \in \nu^1_{q_r}\tilde{\mathcal{W}}^r$, the shape operator $\tilde{\mathcal{S}}^r_\eta$ is 3-step nilpotent, and its kernel is obtained by parallel translation of $T_\lambda(q)$ along a geodesic normal to $\tilde{\mathcal{W}}$ through q.*

It is worthwhile to emphasize that, although $E^r_1(q)$, $E^r_2(q)$, $E^r_3(q)$ are tangent vectors at $q_r \in \tilde{\mathcal{W}}^r$, these depend on $q \in \tilde{\mathcal{W}}$. The next subsection is devoted to a more thorough study of the geometry of the focal submanifold $\tilde{\mathcal{W}}^r$.

3.3.3 Algebraic study of the focal submanifold

Let $q_r \in \tilde{\mathcal{W}}^r$. The main idea in what follows is to prove Proposition 3.3.7, which implies that a certain vector does not depend on the choice of $q \in (\tilde{\Phi}^r)^{-1}(q_r)$. This vector will be fundamental to determine the geometry of $\pi(\tilde{\mathcal{W}}^r)$, which is the aim of this subsection. We continue using the notation introduced in Section 3.3.2.

Proposition 3.3.7. *Let $q_r \in \tilde{\mathcal{W}}^r$. Then, the map*

$$(\tilde{\Phi}^r)^{-1}(q_r) \to T_{q_r}\tilde{\mathcal{W}}^r, \quad q \mapsto -\frac{1}{\langle V_{q_r}, E^r_1(q)\rangle}E^r_1(q) - V_{q_r},$$

is constant in $(\tilde{\Phi}^r)^{-1}(q_r)$.

Proof. Let $q \in (\tilde{\Phi}^r)^{-1}(q_r)$ and let $\zeta_{q_r} \in \nu_{q_r}\tilde{\mathcal{W}}^r \ominus \mathbb{R}\eta^r(q)$ be a unit vector. We calculate $\tilde{\mathcal{S}}^r_{\zeta_{q_r}}E^r_1(q)$. Let σ be an integral curve of E_1 in $\tilde{\mathcal{W}}$ and extend ζ_{q_r} to a smooth vector field ζ along $s \mapsto \tilde{\Phi}^r(\sigma(s))$ in such a way that $\langle \zeta_{\tilde{\Phi}^r(\sigma(s))}, \eta^r_{\tilde{\Phi}^r(\sigma(s))}\rangle = 0$. Then, there exists a unique vector field $Y \in \Gamma(\sigma^*T_\mu)$ along σ tangent to T_μ such that $\mathcal{P}_{Y_{\sigma(s)}}(r) = \zeta_{\tilde{\Phi}^r(\sigma(s))}$ for all s by Proposition 3.3.6. We define the geodesic variation $F(s,t) = \exp_{\sigma(s)}(t\xi^L_{\sigma(s)})$, where ξ^L is the unit normal vector of $\tilde{\mathcal{W}}$ that was fixed at the beginning of Subsection 3.3.2. We use Proposition 3.3.5 twice, and Proposition 3.3.1 applied to $\tilde{\mathcal{W}}^t$, $t \in [0, r)$, to conclude that

$\mathcal{P}_{Y_{\sigma(s)}}(t) \in T_{\mu(t)}(F(s,t))$ and $\tilde{\nabla}_{E_1^t(\sigma(s))}\mathcal{P}_{Y_{\sigma(s)}}(t) \in T_{\mu(t)}(F(s,t))$. By Proposition 3.3.5 we have that the principal curvature distribution of $\tilde{\mathcal{W}}^t$ associated with $\mu(t)$ at $F(s,t)$ is the parallel translation of $T_\mu(\sigma(s))$ along a normal geodesic, that is, $T_{\mu(t)}(F(s,t)) = \mathcal{P}_{T_\mu(\sigma(s))}(t)$. By continuity we get $\tilde{\nabla}_{E_1^r(\sigma(s))}\mathcal{P}_{Y_{\sigma(s)}}(r) \in \mathcal{P}_{T_\mu(\sigma(s))}(r)$. Combining this with $\zeta_{\tilde{\Phi}^r(\sigma(s))} = \mathcal{P}_{Y_{\sigma(s)}}(r)$ and Proposition 3.3.6 yields $\tilde{\nabla}_{E_1^r(\sigma(s))}\zeta \in (\nu_{\tilde{\Phi}^r(\sigma(s))}\tilde{\mathcal{W}}^r) \ominus \mathbb{R}\eta^r_{\sigma(s)}$. Therefore,

$$\tilde{\mathcal{S}}^r_{\zeta_{q_r}} E_1^r(q) = -(\tilde{\nabla}_{E_1^r(q)}\zeta)^\top = 0,$$

as we wanted to calculate.

Since $\zeta_{q_r} \in \nu_{q_r}\tilde{\mathcal{W}}^r \ominus \mathbb{R}\eta^r(q)$ was arbitrary and we already had $\tilde{\mathcal{S}}_{\eta^r(q)}E_1^r(q) = 0$ by Proposition 3.3.6 and (3.20), we conclude that $\tilde{\mathcal{S}}^r_\eta E_1^r(q) = 0$, for any $\eta \in \nu_{q_r}\tilde{\mathcal{W}}^r$. Since q is also arbitrary, we get

$$\tilde{\mathcal{S}}^r_\eta E_1^r(q) = 0, \text{ for any } \eta \in \nu_{q_r}\tilde{\mathcal{W}}^r, \text{ and any } q \in (\tilde{\Phi}^r)^{-1}(q_r). \tag{3.21}$$

Now take another point $\hat{q} \in (\tilde{\Phi}^r)^{-1}(q_r)$. According to Proposition 3.3.6, we can write $E_1^r(\hat{q}) = a_1E_1^r(q) + a_2E_2^r(q) + a_3E_3^r(q) + u$, with $a_i \in \mathbb{R}$, and $u \in (\ker \tilde{\mathcal{S}}^r_{\eta^r(q)}) \ominus \mathbb{R}E_2^r(q)$. By (3.21) we have

$$0 = \tilde{\mathcal{S}}^r_{\eta^r(q)}E_1^r(\hat{q}) = a_2E_3^r(q) + a_3E_1^r(q).$$

Thus, $a_2 = a_3 = 0$. On the other hand, since $E_1^r(\hat{q})$ is a null vector, we also obtain $0 = \langle E_1^r(\hat{q}), E_1^r(\hat{q})\rangle = \langle u, u\rangle$, and as u is spacelike, we get $u = 0$. Thus, $E_1^r(\hat{q}) = a_1E_1^r(q)$, which easily implies the result. □

The submanifold $\tilde{\mathcal{W}}^t$ contains locally the S^1-fiber of the semi-Riemannian submersion $\pi\colon H_1^{2n+1} \to \mathbb{C}H^n$ as we have seen in propositions 3.3.5 and 3.3.6. If we denote $\mathcal{W}^t = \pi(\tilde{\mathcal{W}}^t)$, $t \in [0,r]$, and consider the map $\Phi^t\colon \mathcal{W} \to \mathbb{C}H^n$, $p \mapsto \Phi^t(p) = \exp_p(t\xi_p)$, then it follows that $\Phi^t(\pi(\tilde{\mathcal{W}})) = \pi(\tilde{\Phi}^t(\tilde{\mathcal{W}}))$, that is, $\Phi^t(\mathcal{W}) = \pi(\tilde{\mathcal{W}}^t)$, or in other words, the Hopf map commutes with the parallel displacement map.

Coming back to the study of the geometry of the submanifold $\mathcal{W}^r$, we write $V_{q_r} = s_1(q)E_1^r(q) + s_2(q)E_2^r(q) + s_3(q)E_3^r(q) + u_q$, for $s_i(q) \in \mathbb{R}$ and $u_q \in T_{q_r}\tilde{\mathcal{W}}^r \ominus (\mathbb{R}E_1^r(q) \oplus \mathbb{R}E_2^r(q) \oplus \mathbb{R}E_3^r(q)) = (\ker \tilde{\mathcal{S}}^r_{\eta^r(q)}) \ominus \mathbb{R}E_2^r(q)$. Arguing as in (3.7), we can assume $u_q = 0$. Note that the procedure at the beginning of the proof of Proposition 3.2.8 which leads to (3.7) does not change the vector $E_1^r(q)$. Thus, $-1 = \langle V, V\rangle = 2s_1s_2 + s_3^2$, which immediately implies $s_1, s_2 \neq 0$. We can assume, changing the signs of $E_1(q)$, $E_2(q)$ and $E_3(q)$, that $s_2 > 0$.

If ξ is now a unit normal vector field of $\mathcal{W}^r$, we write $J\xi = P\xi + F\xi$, where $P\xi$ is the orthogonal projection of $J\xi$ onto $T\mathcal{W}^r$ and $F\xi$ is the orthogonal projection of $J\xi$ onto $\nu\mathcal{W}^r$. We also write $J\xi^L = P\xi^L + F\xi^L$ for $\tilde{\mathcal{W}}^r$, accordingly. Notice that $P(\xi^L) = (P\xi)^L$ and $F(\xi^L) = (F\xi)^L$. From (3.2) we get $\tilde{\mathcal{S}}^r_{\xi^L}V = -(\tilde{\nabla}_V\xi^L)^\top = -(\sqrt{-c}/2)P\xi^L$. Hence, taking $\xi \in \Gamma(\nu\mathcal{W}^r)$ such that $\xi^L_{q_r} = \eta^r(q)$ we get

$$0 = -\frac{\sqrt{-c}}{2}\langle P\xi^L, V\rangle = \langle \tilde{\mathcal{S}}^r_{\eta^r}V, V\rangle = \langle s_2E_3^r(q) + s_3E_1^r(q), V\rangle = 2s_2s_3,$$

which implies $s_3 = 0$. We may also write

$$J\eta^r = -\frac{2}{\sqrt{-c}}\tilde{\mathcal{S}}^r_{\eta^r}V + F\eta^r = -\frac{2s_2}{\sqrt{-c}}\,E^r_3(q) + F\eta^r. \tag{3.22}$$

Thus, $1 = \langle J\eta^r, J\eta^r\rangle = -(4/c)s_2^2 + \langle F\eta^r, F\eta^r\rangle$, and consequently we can choose a real number $\varphi(q) \in (0, \pi/2]$, such that

$$s_2(q) = \frac{\sqrt{-c}}{2}\sin(\varphi(q)), \qquad \langle F\eta^r(q), F\eta^r(q)\rangle = \cos^2(\varphi(q)).$$

If $\mathcal{S}^r_\xi$ denotes the shape operator of $\mathcal{W}^r$ with respect to $\xi \in \Gamma(\nu\mathcal{W}^r)$, then (3.1) implies

$$\tilde{\mathcal{S}}^r_{\xi^L}X^L = (\mathcal{S}^r_\xi X)^L + \frac{\sqrt{-c}}{2}\langle J\xi^L, X^L\rangle V \quad\text{and}\quad \mathcal{S}^r_\xi X = \pi_*\tilde{\mathcal{S}}^r_{\xi^L}X^L, \quad\text{for each } X \in T\mathcal{W}^r.$$

The vectors in $(\ker\tilde{\mathcal{S}}^r_{\eta^r(q)}) \ominus \mathbb{R}E^r_2(q)$ are orthogonal to $J\eta^r(q)$ and V_{q_r} by (3.22), and by the previous equation, project bijectively onto $\ker\mathcal{S}^r_{\pi_*\eta^r(q)}$. For dimension reasons, there are only two eigenvectors left to determine $\mathcal{S}^r_{\pi_*\eta^r(q)}$ completely.

In view of Proposition 3.3.7 we can define

$$Z_{\pi(q_r)} = \pi_*\Big(-\frac{1}{\langle V_{q_r}, E^r_1(q)\rangle}E^r_1(q) - V_{q_r}\Big) = -\frac{1}{\langle V_{q_r}, E^r_1(q)\rangle}\pi_*E^r_1(q), \text{ for } q \in (\tilde{\Phi}^r)^{-1}(q_r).$$

Note that this vector field is smooth because E^t_1 is smooth along the map $\tilde{\Phi}^t$ by the smooth dependence on the initial conditions of solutions to an ordinary differential equation. For the subsequent calculations, we consider $\xi \in \nu_{\pi(q_r)}\mathcal{W}^r$ such that its lift to $\nu_{q_r}\tilde{\mathcal{W}}^r$ satisfies $\xi^L = \eta^r(q)$. Thus we can write $P\xi^L = P\eta^r$. We have

$$Z^L_{q_r} = -\frac{1}{\langle V, E^r_1(q)\rangle}E^r_1(q) - V_{q_r}, \qquad P\xi^L = -\sin(\varphi(q))E^r_3(q).$$

These two vectors are tangent to $\tilde{\mathcal{W}}^r$ and orthogonal to V. Thus they are mapped isometrically to Z and $P\xi$ respectively; in particular, $\|P\xi\| = \sin(\varphi(q))$. Furthermore, by (3.22) we also have $\langle Z^L_{q_r}, J\eta^r(q)\rangle = 0$ for any $q \in (\tilde{\Phi}^r)^{-1}(q_r)$. Since $\eta^r((\tilde{\Phi}^r)^{-1}(q_r))$ is open in $\nu^1_{q_r}\tilde{\mathcal{W}}^r$, we deduce that Z^L is orthogonal to $J\nu\tilde{\mathcal{W}}^r$, and hence, Z is orthogonal to $J\nu\mathcal{W}^r$. Thus, we have that $T\mathcal{W}^r \ominus P\nu\mathcal{W}^r$ is the maximal complex distribution of $T\mathcal{W}^r$ and Z is tangent to it.

Using the above formulas we obtain

$$\mathcal{S}^r_\xi Z = \pi_{*q_r}\tilde{\mathcal{S}}^r_{\xi^L}Z^L = -\pi_{*q_r}\tilde{\mathcal{S}}^r_{\xi^L}V = \frac{\sqrt{-c}}{2}\pi_{*q_r}P\xi^L = \frac{\sqrt{-c}}{2}P\xi,$$

$$\mathcal{S}^r_\xi P\xi = \pi_{*q_r}\tilde{\mathcal{S}}^r_{\xi^L}P\xi^L = -\sin(\varphi(q))\pi_{*q_r}E^r_1(q) = \sin(\varphi(q))s_2(q)Z = \frac{\sqrt{-c}}{2}\sin^2(\varphi(q))Z.$$

Therefore, by analiticity of $\mathcal{S}^r_\xi$ with respect to ξ,

$$\langle I\!I(Z, P\xi), \eta\rangle = \langle\mathcal{S}^r_\eta Z, P\xi\rangle = \frac{\sqrt{-c}}{2}\langle P\eta, P\xi\rangle = -\frac{\sqrt{-c}}{2}\langle\eta, JP\xi\rangle,$$

for all ξ, $\eta \in \nu\mathcal{W}^r$. We can summarize the results obtained so far in

Proposition 3.3.8. *The vector field Z is tangent to the maximal complex distribution of $T\mathcal{W}^r$. The second fundamental form of $\mathcal{W}^r$ is determined by the trivial symmetric bilinear extension of*

$$2\,I\!I(Z, P\xi) = -\sqrt{-c}\,(JP\xi)^{\perp},$$

for any $\xi \in \nu\mathcal{W}^r$.

3.4 Rigidity of the focal submanifold

In this section we prove that a submanifold of $\mathbb{C}H^n$ under the conditions of Proposition 3.3.8 is congruent to an open part of a submanifold $W_{\mathfrak{w}}$ defined in Subsection 2.5.2. The precise statement is as follows.

Theorem 3.4.1. *Let M be a connected $(2n-k)$-dimensional submanifold of $\mathbb{C}H^n$, $n \geq 2$. Assume that there exists a smooth unit vector field Z tangent to the maximal complex distribution of M such that the second fundamental form $I\!I$ of M is given by the trivial symmetric bilinear extension of*

$$2\,I\!I(Z, P\xi) = -\sqrt{-c}\,(JP\xi)^{\perp}, \tag{3.23}$$

for $\xi \in \nu M$, where $P\xi$ is the tangential component of $J\xi$, and $(\cdot)^{\perp}$ denotes orthogonal projection onto the normal space νM. Then, a point $o \in M$ and $B_o = -JZ_o$ determine an Iwasawa decomposition $\mathfrak{su}(1,n) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}$ of the Lie algebra of the isometry group of $\mathbb{C}H^n$, such that M is congruent to an open part of the minimal submanifold $W_{\mathfrak{w}}$, where $\mathfrak{w} = T_oM \ominus (\mathbb{R}B_o \oplus \mathbb{R}Z_o) \subset \mathfrak{g}_{\alpha}$.

Before beginning the proof, we start with a more geometric construction of the submanifolds $W_{\mathfrak{w}}$. This will make use of several Lie theoretic concepts that were introduced in Subsection 2.5.2. See [19] for further details.

Proposition 3.4.2. *Let $k \in \{1, \dots, n-1\}$, fix a totally geodesic $\mathbb{C}H^{n-k}$ in $\mathbb{C}H^n$ and points $o \in \mathbb{C}H^{n-k}$ and $x \in \mathbb{C}H^{n-k}(\infty)$. Let KAN be the Iwasawa decomposition of $SU(1,n)$ with respect to o and x, and let $\hat{H}$ be the subgroup of AN that acts simply transitively on $\mathbb{C}H^{n-k}$. Now, let $\mathfrak{v}$ be a proper subspace of $\nu_o\mathbb{C}H^{n-k}$ such that $\mathfrak{v} \cap J\mathfrak{v} = 0$. Left translation of $\mathfrak{v}$ by $\hat{H}$ to all points of $\mathbb{C}H^{n-k}$ determines a subbundle $\mathfrak{V}$ of the normal bundle $\nu\mathbb{C}H^{n-k}$. At each point $p \in \mathbb{C}H^{n-k}$ attach the horocycles determined by x and the linear lines in $\mathfrak{V}_p$. The resulting subset M of $\mathbb{C}H^n$ is congruent to the submanifold $W_{\mathfrak{w}}$, where $\mathfrak{w} = (\hat{\mathfrak{h}} \ominus (\mathfrak{a} \oplus \mathfrak{g}_{2\alpha})) \oplus \mathfrak{v} \subset \mathfrak{g}_{\alpha}$.*

Proof. Let $W_{\mathfrak{w}}$ be the minimal submanifold of $\mathbb{C}H^n$ constructed from the Iwasawa decomposition KAN associated with o and x and from $\mathfrak{w} = (\hat{\mathfrak{h}} \ominus (\mathfrak{a} \oplus \mathfrak{g}_{2\alpha})) \oplus \mathfrak{v}$, as described in Subsection 2.5.2. We recall that $T_o\mathbb{C}H^n$ is now identified with $\mathfrak{a} \oplus \mathfrak{n}$ and we denote by $\mathfrak{w}^{\perp} = \mathfrak{g}_{\alpha} \ominus \mathfrak{w}$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_{\alpha}$. We have that the Lie algebra of $\hat{H}$ is $\hat{\mathfrak{h}} = \mathfrak{s}_{\mathfrak{w}} \ominus P\mathfrak{w}^{\perp}$, with $\mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, and where, as usual, $P\xi$ denotes the orthogonal projection of $J\xi$ on $\mathfrak{w}$ for each $\xi \in \mathfrak{w}^\perp$. Since $\mathfrak{v} \cap J\mathfrak{v} = 0$, we have that $\hat{\mathfrak{h}}$ is the maximal complex subspace of $\mathfrak{s}_\mathfrak{w}$.

Let $p \in W_\mathfrak{w}$. By definition, there exists an isometry $s \in S_\mathfrak{w}$ with $p = s(o)$. There is a unique vector X in the Lie algebra $\mathfrak{s}_\mathfrak{w}$ of $S_\mathfrak{w}$ such that $s = \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(X)$. We can write $X = aB + U + W + xZ$ with a, $x \in \mathbb{R}$, $U \in \hat{\mathfrak{h}} \ominus (\mathfrak{a} \oplus \mathfrak{g}_{2\alpha})$, and $W \in \mathfrak{v}$. Since U and W are complex-orthogonal, we get $[U, W] = 0$ by (2.1) from Section 2.5. Using this notation we can define the elements $g = \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(\rho(a/2)W)$ and $h = \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(aB + U + xZ) \in \hat{H}$. Using (2.2) we obtain,

$$gh = \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}\Big(\rho\Big(\frac{a}{2}\Big)W\Big) \cdot \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(aB + U + xZ) = \mathrm{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(aB + U + W + xZ) = s.$$

By construction, $h(o) \in \mathbb{C}H^{n-k}$, and $s(o) = g(h(o))$ is in the horocycle through $h(o)$, tangent to $\mathbb{R}W$, and with center x at infinity. Hence, $p = s(o) \in M$ and we conclude that $W_\mathfrak{w} \subset M$.

Now we prove the converse. Let σ be a horocycle such that $\sigma(0) = o$, $\sigma'(0) = U \in \mathfrak{v}$, $\|U\| = 1$, and $2(\bar{\nabla}_{\sigma'}\sigma')(0) = \sqrt{-c}\,B$. We show that σ is contained in $W_\mathfrak{w}$. First, using (2.3), we get $\bar{\nabla}_B B = \bar{\nabla}_B U = 0$, $2\bar{\nabla}_U B = -\sqrt{-c}\,U$ and $2\bar{\nabla}_U U = \sqrt{-c}\,B$. Hence, it follows that the distribution generated by B and U is autoparallel and its integral submanifolds are totally geodesic real hyperbolic spaces $\mathbb{R}H^2$ of curvature $c/4$. Now, we denote by τ an integral curve of the left-invariant vector field U such that $\tau(0) = o$. Using (2.3) we get $\bar{\nabla}_U\bar{\nabla}_U U + \langle\bar{\nabla}_U U, \bar{\nabla}_U U\rangle U = 0$. Thus, τ is a cycle in a totally geodesic $\mathbb{R}H^2$ of curvature $c/4$, and since $2(\bar{\nabla}_{\tau'}\tau')(0) = \sqrt{-c}B$, it follows that τ is a horocycle determined by o, U and the point at infinity x. By uniqueness of solutions to ordinary differential equations we get $\tau = \sigma$, and thus σ is contained in $W_\mathfrak{w}$.

If σ is an arbitrary horocycle determined by initial conditions $p \in \mathbb{C}H^{n-k}$, $U_p \in \mathfrak{V}_p$ and $\sqrt{-c}\,B_p/2$, then there is a unique $h \in \hat{H}$ such that $p = h(o)$. Since h is an isometry of $\mathbb{C}H^n$, it is easy to see that $h^{-1} \circ \sigma$ satisfies the conditions of horocycle in the previous paragraph. Hence, $h^{-1} \circ \sigma$ is contained in $W_\mathfrak{w}$, from where it follows that σ is contained in $W_\mathfrak{w}$ because $h \in \hat{H} \subset S_\mathfrak{w}$. This shows that $M \subset W_\mathfrak{w}$ and finishes the proof of the proposition. □

The rest of this section is devoted to the proof of the rigidity result given by Theorem 3.4.1. In what follows, M will denote a submanifold of $\mathbb{C}H^n$ under the assumptions of Theorem 3.4.1.

3.4.1 The structure of the normal bundle

For $\xi \in \nu M$ recall that $J\xi = P\xi + F\xi$, where $P\xi$ and $F\xi$ denote the orthogonal projections of $J\xi$ onto TM and νM respectively. The maps $P\colon \nu M \to TM$ and $F\colon \nu M \to \nu M$ are vector bundle homomorphisms. We will use some of their properties in the rest of this chapter. We start with

Lemma 3.4.3. *The endomorphism F of νM is parallel with respect to the normal connection of M, that is, $\nabla^\perp F = 0$.*

Proof. Let ξ, $\eta \in \Gamma(\nu M)$ and $X \in \Gamma(TM)$. Using (3.23) we get

$$\langle I\!I(Z, P\xi), \eta\rangle = -\frac{\sqrt{-c}}{2}\langle JP\xi, \eta\rangle = \frac{\sqrt{-c}}{2}\langle P\xi, P\eta\rangle = -\frac{\sqrt{-c}}{2}\langle \xi, JP\eta\rangle = \langle I\!I(Z, P\eta), \xi\rangle.$$

This relation yields $\langle I\!I(X, P\xi), \eta\rangle = \langle I\!I(X, P\eta), \xi\rangle$ using the fact that $I\!I$ is obtained by the trivial symmetric bilinear extension of (3.23). Since $\mathbb{C}H^n$ is Kähler,

$$\begin{aligned}\langle \nabla^\perp_X F\xi, \eta\rangle &= \langle \bar\nabla_X J\xi, \eta\rangle - \langle \bar\nabla_X P\xi, \eta\rangle = -\langle \bar\nabla_X \xi, P\eta + F\eta\rangle - \langle I\!I(X, P\xi), \eta\rangle \\ &= \langle I\!I(X, P\eta), \xi\rangle - \langle \nabla^\perp_X \xi, F\eta\rangle - \langle I\!I(X, P\xi), \eta\rangle = -\langle \nabla^\perp_X \xi, J\eta\rangle \\ &= \langle F\nabla^\perp_X \xi, \eta\rangle.\end{aligned}$$

Hence, $(\nabla^\perp_X F)\xi = \nabla^\perp_X F\xi - F\nabla^\perp_X \xi = 0$, as we wanted to show. □

For each $p \in M$, the normal space $\nu_p M$ is a real vector subspace of the complex vector space $T_p\mathbb{C}H^n$. According to Section 2.4, $\nu_p M$ has a decomposition as a sum of subspaces of constant Kähler angle. These angles are called the principal Kähler angles of $\nu_p M$. We show that they do not depend on $p \in M$.

Proposition 3.4.4. *The principal Kähler angles of νM and their multiplicities are constant along M.*

Proof. Let p, $q \in M$ be two arbitrary points, and let $\sigma\colon [0, 1] \to M$ be a smooth curve in M such that $\sigma(0) = p$ and $\sigma(1) = q$. We take a basis $\{\xi_1, \dots, \xi_k\}$ of principal Kähler vectors, that is, an orthonormal basis of $\nu_p M$ such that $\langle F\xi_i(p), F\xi_j(p)\rangle = \cos^2(\varphi_i(p))\delta_{ij}$, for i, $j \in \{1, \dots, k\}$ (see Section 2.4). We extend this basis to a $\nabla^\perp$-parallel orthonormal basis $\{\xi_1(t), \dots, \xi_k(t)\}$ of smooth vector fields along σ. Since F is parallel by Lemma 3.4.3, it follows that $\langle F\xi_i, F\xi_j\rangle$ is constant along σ. Therefore, $\{\xi_1(1), \dots, \xi_k(1)\}$ is also a basis of principal Kähler vectors of $\nu_q M$, and it follows that the principal Kähler angles and their multiplicities of νM at p and q coincide. □

Let Φ be the set of constant principal Kähler angles of νM. As in Section 2.4 we write $\nu_p M = \oplus_{\varphi\in\Phi}\mathfrak{W}^\perp_\varphi(p)$, where each $\mathfrak{W}^\perp_\varphi(p)$ has constant Kähler angle φ. Since the principal Kähler angles are constant, $\mathfrak{W}^\perp_\varphi$ is a smooth vector subbundle of νM. If $\mathfrak{W}^\perp_0$ is non-zero we can simplify matters because there is a reduction of codimension.

Proposition 3.4.5. *If $\mathfrak{W}^\perp_0 \neq 0$ there exists a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ containing M where 0 is no longer a principal Kähler angle of M in $\mathbb{C}H^k$, the normal bundle of M is obtained by inclusion, and the second fundamental form is obtained by restriction.*

Proof. We first show that each distribution $\mathfrak{W}^\perp_\varphi$ is parallel with respect to the normal connection. Let $\varphi \in \Phi$, $\xi \in \Gamma(\mathfrak{W}^\perp_\varphi)$ and $X \in \Gamma(TM)$. As we argued in Section 2.4, we have $F^2\xi = -\cos^2(\varphi)\xi$. Since $\nabla^\perp F = 0$ by Lemma 3.4.3, we get

$$F^2\nabla^\perp_X \xi = \nabla^\perp_X F^2\xi = \nabla^\perp_X(-\cos^2(\varphi)\xi) = -\cos^2(\varphi)\nabla^\perp_X \xi,$$

and again from the results in Section 2.4 it follows that $\nabla^{\perp}_X \xi \in \Gamma(\mathfrak{W}^{\perp}_{\varphi})$. Therefore

$$\nabla^{\perp}\mathfrak{W}^{\perp}_{\varphi} \subset \mathfrak{W}^{\perp}_{\varphi} \quad \text{for each } \varphi \in \Phi. \tag{3.24}$$

Recall from Section 2.4 that we can decompose $TM = \mathfrak{W}_0 \oplus (\oplus_{\varphi\in\Phi\setminus\{0\}}\mathfrak{W}_{\varphi})$ with $\mathbb{C}\mathfrak{W}^{\perp}_{\varphi} = \mathfrak{W}^{\perp}_{\varphi} \oplus \mathfrak{W}_{\varphi}$ and $\dim \mathfrak{W}^{\perp}_{\varphi} = \dim \mathfrak{W}_{\varphi}$ for all $\varphi \in \Phi \setminus \{0\}$. Now we consider the bundle

$$\mathcal{F} = TM \oplus \left(\bigoplus_{\varphi\in\Phi\setminus\{0\}} \mathfrak{W}^{\perp}_{\varphi} \right) = \mathfrak{W}_0 \oplus \left(\bigoplus_{\varphi\in\setminus\{0\}} \mathbb{C}\mathfrak{W}^{\perp}_{\varphi} \right)$$

along M. Then, $\mathcal{F}$ is a complex vector bundle and, at a point $p \in M$, $\mathcal{F}_p$ is the tangent space of a totally geodesic complex hyperbolic space $\mathbb{C}H^{n-m_0^{\perp}}$, $m_0^{\perp} = \dim_{\mathbb{C}} \mathfrak{W}^{\perp}_0$, in $\mathbb{C}H^n$. Using (3.23) and (3.24) we get $\bar{\nabla}_X \phi = \nabla^{\mathcal{F}}_X \phi$ for each $\phi \in \Gamma(\mathcal{F})$ and where $\nabla^{\mathcal{F}}$ denotes the connection on $\mathcal{F}$ induced from $\bar{\nabla}$. Hence, by [90, Theorem 1 (with $h = 0$ in the notation of this paper)] we conclude that M is contained in the totally geodesic $\mathbb{C}H^{n-m_0^{\perp}}$ mentioned above. □

In other words, what Proposition 3.4.5 states is that we can, and we will, assume from now on that $\mathfrak{W}^{\perp}_0 = 0$. Otherwise, we just take a smaller complex hyperbolic space where this condition is fulfilled.

3.4.2 Proof of Theorem 3.4.1

In order to prove Theorem 3.4.1 we use the construction of $W_{\mathfrak{w}}$ as described in Proposition 3.4.2. Part of the proof goes along the lines of the rigidity result in [10], although the argument here is more involved.

As we have just seen in Subsection 3.4.1, we may assume that the normal bundle νM does not contain a non-zero complex subbundle. We decompose the tangent bundle TM of M orthogonally into $TM = \mathfrak{C} \oplus \mathfrak{D}$, where $\mathfrak{C}$ is the maximal complex subbundle of TM. Thus, $\mathfrak{D} \cap J\mathfrak{D} = 0$. For each $\xi \in \Gamma(\nu M)$ we have $J\xi = P\xi + F\xi$, where $P\xi \in \Gamma(\mathfrak{D})$ and $F\xi \in \Gamma(\nu M)$. Since $\mathfrak{D} = P\nu M$, then we argued in Section 2.4 that $\mathfrak{D}$ has the same Kähler angles, with the same multiplicities as νM (note that 0 is not a Kähler angle of νM by the assumption we have made after Subsection 3.4.1). Since the principal Kähler angles are never 0, it follows that $P\colon \nu M \to \mathfrak{D}$ is an isomorphism of vector bundles.

Lemma 3.4.6. *The distribution $\mathfrak{C}$ is autoparallel and each integral submanifold is an open part of a totally geodesic complex hyperbolic space $\mathbb{C}H^{n-k}$ in $\mathbb{C}H^n$.*

Proof. For all $U, V \in \Gamma(\mathfrak{C})$ and $\xi \in \Gamma(\nu M)$ we have, using (3.23) and $\bar{\nabla} J = 0$,

$$\langle \bar{\nabla}_U V, \xi \rangle = \langle I\!I(U, V), \xi \rangle = 0, \quad \text{and} \quad \langle \bar{\nabla}_U V, J\xi \rangle = -\langle J\bar{\nabla}_U V, \xi \rangle = -\langle I\!I(U, JV), \xi \rangle = 0.$$

Thus $\mathfrak{C}$ is autoparallel and as $\mathfrak{C}$ is a complex subbundle of complex rank $n - k$, each of its integral manifolds is an open part of a totally geodesic $\mathbb{C}H^{n-k}$ in $\mathbb{C}H^n$. □

From now on we fix $o \in M$ and let $\mathcal{L}_o$ be the leaf of $\mathfrak{C}$ through o, which is an open part of a totally geodesic $\mathbb{C}H^{n-k}$ in $\mathbb{C}H^n$ by Lemma 3.4.6. We have

Lemma 3.4.7. *If $\gamma\colon I \to \mathcal{L}_o$ is a curve with $\gamma(0) = o$ then the normal spaces of M along γ are uniquely determined by the differential equation*

$$2\bar{\nabla}_{\gamma'}\eta + \sqrt{-c}\langle\gamma', Z\rangle J\eta = 0 \tag{3.25}$$

for $\eta \in \Gamma(\gamma^*\nu\mathcal{L}_o)$, where $\gamma^*\nu\mathcal{L}_o$ is the bundle of vectors along γ that are orthogonal to $\mathcal{L}_o$.

Proof. Let $X \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Using (3.23) we get

$$-\langle\bar{\nabla}_{\gamma'}\xi, X\rangle = \langle I\!I(\gamma', X), \xi\rangle = \langle\gamma', Z\rangle\frac{\langle X, P\xi\rangle}{\langle P\xi, P\xi\rangle}\langle I\!I(Z, P\xi), \xi\rangle = \frac{\sqrt{-c}}{2}\langle\gamma', Z\rangle\langle P\xi, X\rangle,$$

which implies

$$\bar{\nabla}_{\gamma'}\xi = -\frac{\sqrt{-c}}{2}\langle\gamma', Z\rangle P\xi + \nabla^{\perp}_{\gamma'}\xi, \tag{3.26}$$

where $\nabla^{\perp}$ is the normal connection of M. Now, we take a vector field X along γ with $X_0 \in \nu_o M$ and satisfying (3.25). We write $X = U + J\eta + \xi$, where we have $U \in \Gamma(\gamma^*\mathfrak{C})$, ξ, $\eta \in \Gamma(\gamma^*\nu M)$ and $U_0 = \eta_0 = 0$. Using (3.26) and taking into account that $\bar{\nabla}J = 0$, we obtain

$$\begin{aligned} 0 &= 2\bar{\nabla}_{\gamma'}X + \sqrt{-c}\langle\gamma', Z\rangle JX \\ &= 2\bar{\nabla}_{\gamma'}U + 2J\bar{\nabla}_{\gamma'}\eta + 2\bar{\nabla}_{\gamma'}\xi + \sqrt{-c}\langle\gamma', Z\rangle JU + \sqrt{-c}\langle\gamma', Z\rangle J^2\eta + \sqrt{-c}\langle\gamma', Z\rangle J\xi \\ &= 2\bar{\nabla}_{\gamma'}U + \sqrt{-c}\langle\gamma', Z\rangle JU + P\left(2\nabla^{\perp}_{\gamma'}\eta + \sqrt{-c}\langle\gamma', Z\rangle F\eta\right) \\ &\quad + 2\nabla^{\perp}_{\gamma'}\xi + \sqrt{-c}\langle\gamma', Z\rangle F\xi + F\left(2\nabla^{\perp}_{\gamma'}\eta + \sqrt{-c}\langle\gamma', Z\rangle F\eta\right). \end{aligned}$$

We have that $2\bar{\nabla}_{\gamma'}U + \sqrt{-c}\langle\gamma', Z\rangle JU$ is tangent to $\mathfrak{C}$ since $\mathfrak{C}$ is a complex autoparallel distribution. Thus, it follows that $2\bar{\nabla}_{\gamma'}U + \sqrt{-c}\langle\gamma', Z\rangle JU = 0$. Since $U_0 = 0$, the uniqueness of solutions to ordinary differential equations implies $U_t = 0$ for all t, and thus $X \in \Gamma(\gamma^*\nu\mathcal{L}_o)$. Similarly, the component tangent to $P\nu M$ in the previous equation yields $2\nabla^{\perp}_{\gamma'}\eta + \sqrt{-c}\langle\gamma', Z\rangle F\eta = 0$ and since $\eta_0 = 0$ we have $\eta_t = 0$ for any t by uniqueness of solution. Hence, $X_t \in \nu_{\gamma(t)}M$ for all t, which proves our assertion. □

We define $B = -JZ$.

The point $o \in M$ and the tangent vector B_o uniquely determine a point at infinity $x \in \mathbb{C}H^n(\infty)$ and thus, a corresponding Iwasawa decomposition $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ of the isometry group of $\mathbb{C}H^n$, where $\mathfrak{a} = \mathbb{R}B_o$ and $\mathfrak{g}_{2\alpha} = \mathbb{R}Z_o$. We define the subspace $\mathfrak{w} = T_oM \ominus (\mathbb{R}B_o \oplus \mathbb{R}Z_o) \subset \mathfrak{g}_\alpha$ and consider the submanifold $W_\mathfrak{w}$ defined by this Iwasawa decomposition and $\mathfrak{w}$. As we have already seen, the integral submanifold $\mathcal{L}_o$ is an open part of a totally geodesic $\mathbb{C}H^{n-k}$ contained in $\mathbb{C}H^n$ that is tangent to the maximal complex distribution of $W_\mathfrak{w}$ at o. Since by Lemma 3.4.7 the normal bundle is uniquely determined by the ordinary differential equation (3.25), and both M and $W_\mathfrak{w}$ satisfy the hypotheses of Theorem 3.4.1, it follows that $\nu_p M = \nu_p W_{\mathfrak{w}}$ for each $p \in \mathcal{L}_o$. As a consequence, $\nu_p M$ is obtained by left translation of $\nu_o M$ by the subgroup of AN that acts simply transitively on $\mathcal{L}_o$. In view of Proposition 3.4.2 it only remains to prove that for each $p \in \mathcal{L}_o$ the horocycles determined by the point at infinity x and the lines of $P\nu_p M$ are locally contained in M.

Before continuing our argument we need to calculate certain covariant derivatives of some vector fields.

Lemma 3.4.8. *Let $X \in \Gamma(TM \ominus \mathbb{R}B)$ and $\xi \in \Gamma(\nu M)$. Then*

$$\bar\nabla_X B = -\frac{\sqrt{-c}}{2}X - \frac{\sqrt{-c}}{2}\langle X, Z\rangle Z, \tag{3.27}$$

$$\bar\nabla_B P\xi = P\nabla^\perp_B \xi, \tag{3.28}$$

$$\bar\nabla_{P\xi} P\xi = \frac{\sqrt{-c}}{2}\langle P\xi, P\xi\rangle B + P\nabla^\perp_{P\xi}\xi. \tag{3.29}$$

Proof. Let $\eta \in \Gamma(\nu M)$ be a local unit vector field. Using (3.23) we obtain $\langle \bar\nabla_X B, \eta\rangle = \langle I\!I(X,B), \eta\rangle = 0$. Moreover, $\langle \bar\nabla_X B, B\rangle = 0$. Next, (3.23) yields

$$\begin{aligned} 2\langle \bar\nabla_X B, P\eta\rangle &= -2\langle \bar\nabla_X JZ, J\eta - F\eta\rangle = -2\langle I\!I(X,Z), \eta\rangle - 2\langle I\!I(X,B), F\eta\rangle \\ &= -2\langle X, P\eta\rangle\langle I\!I(P\eta, Z), \eta\rangle / \langle P\eta, P\eta\rangle = -\sqrt{-c}\langle X, P\eta\rangle. \end{aligned} \tag{3.30}$$

Now, let $Y \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$ and assume that $X \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$. For any $\xi \in \Gamma(\nu M)$ we have $\langle \nabla_{P\eta} JY, P\xi\rangle = \langle I\!I(P\eta, Y), \xi\rangle - \langle I\!I(P\eta, JY), F\xi\rangle = \sqrt{-c}\langle Y, Z\rangle\langle P\eta, P\xi\rangle/2$. This, the explicit expression for the curvature tensor $\bar R$ of $\mathbb{C}H^n$, the Codazzi equation, (3.23) and $\bar\nabla J = 0$ imply

$$\begin{aligned} c\langle P\eta, P\eta\rangle\langle X, Y\rangle &= 4\langle \bar R(X, P\eta)JY, \eta\rangle = 4\langle (\nabla^\perp_X I\!I)(P\eta, JY) - (\nabla^\perp_{P\eta} I\!I)(X, JY), \eta\rangle \\ &= -4\langle I\!I(P\eta, \nabla_X JY), \eta\rangle + 4\langle I\!I(X, \nabla_{P\eta} JY), \eta\rangle \\ &= -4\langle \nabla_X JY, Z\rangle\langle I\!I(P\eta, Z), \eta\rangle + 4\langle X, Z\rangle\langle I\!I(Z, \nabla_{P\eta} JY), \eta\rangle \\ &= 2\sqrt{-c}\langle P\eta, P\eta\rangle\langle \bar\nabla_X B, Y\rangle - c\langle P\eta, P\eta\rangle\langle X, Z\rangle\langle Z, Y\rangle. \end{aligned}$$

Thus, if $X \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$ we have, taking into account $\bar\nabla_X B \in \Gamma(\mathfrak{C})$, that $2\bar\nabla_X B = -\sqrt{-c}\,(X + \langle X, Z\rangle Z)$.

Next we assume that $X \in \Gamma(P\nu M)$ and we put $X = P\xi$ with $\xi \in \Gamma(\nu M)$. Then, we have $\langle \nabla_{JY} P\xi, Z\rangle = -\langle \bar\nabla_{JY} Z, J\xi - F\xi\rangle = -\langle I\!I(JY, B), \xi\rangle + \langle I\!I(JY, Z), F\xi\rangle = 0$. This, together with the expression for $\bar R$, the Codazzi equation, (3.23) and $\bar\nabla J = 0$ yields

$$\begin{aligned} 0 &= 2\langle \bar R(P\xi, JY)P\xi, \xi\rangle = 2\langle (\nabla^\perp_{P\xi} I\!I)(JY, P\xi) - (\nabla^\perp_{JY} I\!I)(P\xi, P\xi), \xi\rangle \\ &= -2\langle I\!I(\nabla_{P\xi} JY, P\xi), \xi\rangle + 4\langle I\!I(\nabla_{JY} P\xi, P\xi), \xi\rangle \\ &= -2\langle \nabla_{P\xi} JY, Z\rangle\langle I\!I(Z, P\xi), \xi\rangle + 4\langle \nabla_{JY} P\xi, Z\rangle\langle I\!I(Z, P\xi), \xi\rangle \\ &= -\sqrt{-c}\langle P\xi, P\xi\rangle\langle \bar\nabla_{P\xi} JY, Z\rangle = \sqrt{-c}\langle P\xi, P\xi\rangle\langle \bar\nabla_{P\xi} B, Y\rangle. \end{aligned}$$

Hence $\langle \bar\nabla_{P\xi} B, Y\rangle = 0$, and using (3.30) we get $2\bar\nabla_{P\xi} B = -\sqrt{-c}\, P\xi$. Altogether we get (3.27).

Now we prove (3.28). Let $\xi, \zeta \in \Gamma(\nu M)$ and $Y \in \Gamma(\mathfrak{C})$. As $\mathfrak{C}$ is autoparallel, we have $\langle \bar\nabla_B P\xi, Y\rangle = 0$. Using (3.23) we get $\langle \bar\nabla_B P\xi, \zeta\rangle = \langle I\!I(B, P\xi), \zeta\rangle = 0$. Moreover, using (3.23), we obtain $\mathcal{S}_\xi B = 0$ and thus

$$\begin{aligned}\langle \bar\nabla_B P\xi, P\zeta\rangle &= \langle \bar\nabla_B (J - F)\xi, P\zeta\rangle = -\langle \bar\nabla_B \xi, JP\zeta\rangle + \langle F\xi, \bar\nabla_B P\zeta\rangle \\ &= \langle \mathcal{S}_\xi B, JP\zeta\rangle - \langle \nabla^\perp_B \xi, JP\zeta\rangle = \langle P\nabla^\perp_B \xi, P\zeta\rangle.\end{aligned}$$

This implies (3.28).

Finally, if $Y \in \Gamma(\mathfrak{C})$, using again (3.23) we have

$$\begin{aligned}2\langle \bar\nabla_{P\xi} P\xi, Y\rangle &= -2\langle \bar\nabla_{P\xi} Y, J\xi - F\xi\rangle \\ &= 2\langle JY, Z\rangle\langle I\!I(P\xi, Z), \xi\rangle + 2\langle Y, Z\rangle\langle I\!I(P\xi, Z), F\xi\rangle \\ &= -\sqrt{-c}\langle P\xi, P\xi\rangle\langle JZ, Y\rangle - \sqrt{-c}\langle Y, Z\rangle\langle JP\xi, F\xi\rangle \\ &= \sqrt{-c}\langle P\xi, P\xi\rangle\langle B, Y\rangle,\end{aligned}$$

where we have used $\langle JP\xi, F\xi\rangle = \langle JP\xi, J\xi - P\xi\rangle = \langle P\xi, \xi\rangle - \langle JP\xi, P\xi\rangle = 0$. Obviously, (3.23) implies $\langle \bar\nabla_{P\xi} P\xi, \zeta\rangle = \langle I\!I(P\xi, P\xi), \zeta\rangle = 0$. Using (3.23) we obtain

$$\begin{aligned}\langle \bar\nabla_{P\xi} P\xi, P\zeta\rangle &= \langle \bar\nabla_{P\xi} (J - F)\xi, P\zeta\rangle = -\langle \bar\nabla_{P\xi} \xi, JP\zeta\rangle + \langle \bar\nabla_{P\xi} P\zeta, F\xi\rangle \\ &= \langle \mathcal{S}_\xi P\xi, JP\zeta\rangle - \langle \nabla^\perp_{P\xi} \xi, JP\zeta\rangle = \langle P\nabla^\perp_{P\xi} \xi, P\zeta\rangle.\end{aligned}$$

Altogether this yields (3.29). □

The next lemma basically says that the point at infinity determined by B does not depend on the point $o \in M$ that was chosen.

Lemma 3.4.9. *The vector field B is a geodesic vector field and all its integral curves are pieces of geodesics in $\mathbb{C}H^n$ converging to the point $x \in \mathbb{C}H^n(\infty)$.*

Proof. Since $B \in \Gamma(\mathfrak{C})$ we have $\bar\nabla_B B \in \Gamma(\mathfrak{C})$. Clearly, $\langle \bar\nabla_B B, B\rangle = 0$. Let $X \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$ and $\eta \in \Gamma(\nu M)$ be a local unit normal vector field. Using the expression for $\bar R$, the Codazzi equation, (3.23) and $\bar\nabla J = 0$ we obtain

$$\begin{aligned}0 &= 2\langle \bar R(B, P\eta)JX, \eta\rangle = 2\langle (\nabla^\perp_B I\!I)(P\eta, JX) - (\nabla^\perp_{P\eta} I\!I)(B, JX), \eta\rangle \\ &= -2\langle I\!I(P\eta, \nabla_B JX), \eta\rangle = -2\langle \nabla_B JX, Z\rangle\langle I\!I(P\eta, Z), \eta\rangle \\ &= \sqrt{-c}\langle JP\eta, \eta\rangle\langle \bar\nabla_B JX, Z\rangle = \sqrt{-c}\langle P\eta, P\eta\rangle\langle \bar\nabla_B B, X\rangle.\end{aligned}$$

This yields $\langle \bar\nabla_B B, X\rangle = 0$ and hence $\bar\nabla_B B = 0$. This implies that the integral curves of B are geodesics in $\mathbb{C}H^n$.

Now let $X \in \Gamma(TM \ominus \mathbb{R}B)$ be a unit vector field, and γ an integral curve of X. We define the geodesic variation $F(s,t) = \exp_{\gamma(s)}(tB_{\gamma(s)})$, where $F_s(t) = F(s,t)$ are integral curves of B. We prove that $d(F(s_1,t),F(s_2,t))$ tends to 0 as t goes to infinity, where d stands for the Riemannian distance function of $\mathbb{C}H^n$.

The transversal vector field of F, $\zeta(s,t) = (\partial F/\partial s)(s,t)$, is a Jacobi field along each F_s satisfying

$$4\frac{\partial^2\zeta}{\partial t^2} + c\zeta + 3c\langle\zeta, Z\rangle Z = 0, \qquad \zeta(s,0) = X_{\gamma(s)}, \qquad \frac{\partial\zeta}{\partial t}(s,0) = \bar{\nabla}_{X_{\gamma(s)}}B.$$

If $\mathcal{P}_X$ denotes $\bar{\nabla}$-parallel translation of X along F_s, one can directly show that

$$\zeta(s,t) = e^{-t\sqrt{-c}/2}\mathcal{P}_X(s,t) + (e^{-t\sqrt{-c}} - e^{-t\sqrt{-c}/2})\langle X_{F_s(0)}, Z_{F_s(0)}\rangle Z_{F_s(t)},$$

where we have used (3.27) and the fact that Z is a parallel vector field along F_s since $\bar{\nabla}_{B_{F(s,t)}}Z = J\bar{\nabla}_{B_{F(s,t)}}B = 0$. It is easy to see that $\lim_{t\to\infty}\|\zeta(s,t)\| = 0$. Using the mean value theorem of integral calculus we get

$$d(F(s_1,t),F(s_2,t)) \leq \int_{s_1}^{s_2}\|\frac{\partial F}{\partial s}(s,t)\|\,ds = \int_{s_1}^{s_2}\|\zeta(s,t)\|\,ds = (s_2-s_1)\|\zeta(s_*,t)\| \to 0,$$

for some $s_* \in (s_1,s_2)$. Therefore the integral curves of B are geodesics converging to the point $x \in \mathbb{C}H^n(\infty)$ at infinity. □

Now take $p \in \mathcal{L}_o$ and let $\xi_p \in \nu_p M$ be a unit vector. As we argued before, the theorem will follow if we prove that the horocycle determined by $P\xi_p/\|P\xi_p\|$ and the point $x \in \mathbb{C}H^n(\infty)$ is locally contained in M. To this end we will construct a local unit vector field $\xi \in \Gamma(\nu M)$ such that the aforementioned horocycle is an integral curve of $P\xi/\|P\xi\|$.

Let $\gamma\colon I \to M$ be a curve satisfying the initial value problem

$$\nabla_{\gamma'}\gamma' = \frac{\sqrt{-c}}{2}\langle\gamma',\gamma'\rangle B, \qquad \gamma'(0) = P\xi_p/\|P\xi_p\|. \tag{3.31}$$

Lemma 3.4.10. *A curve γ satisfying* (3.31) *is parametrized by arc length and remains tangent to $P\nu M$.*

Proof. Write $\gamma' = aB + xZ + X + P\eta$ for certain differentiable functions $a, x\colon I \to \mathbb{R}$, and vector fields $X \in \Gamma(\gamma^*(\mathfrak{C} \ominus (\mathbb{R}B \oplus \mathbb{R}Z)))$ and $\eta \in \Gamma(\gamma^*\nu M)$. As $Z = JB$, the definition of γ and (3.27) show

$$\frac{dx}{dt} = \frac{d}{dt}\langle\gamma', Z\rangle = \langle\nabla_{\gamma'}\gamma', Z\rangle + \langle\nabla_{\gamma'}Z,\gamma'\rangle = \sqrt{-c}\,\langle xB - \frac{1}{2}JX - \frac{1}{2}JP\eta,\gamma'\rangle = \sqrt{-c}\,ax.$$

Since $x(0) = 0$, the uniqueness of solutions to ordinary differential equations gives $x(t) = 0$ for all t.

Let $Y \in \Gamma(\mathbb{R}B \oplus P\nu M)$ and $\zeta \in \Gamma(\nu M)$. Then, (3.23) yields $\langle\bar{\nabla}_Y X, \zeta\rangle = \langle I\!I(Y,X),\zeta\rangle = 0$ and $\langle\bar{\nabla}_Y X, J\zeta\rangle = -\langle I\!I(Y,JX),\zeta\rangle = 0$. Since $\bar{\nabla}_Y B \in \Gamma(P\nu M)$ by (3.27), we have $\langle\bar{\nabla}_Y X, B\rangle = -\langle\bar{\nabla}_Y B, X\rangle = 0$. Moreover, we have also the equality $2\langle\bar{\nabla}_X X, B\rangle =$

$-2\langle\bar\nabla_X B, X\rangle = \sqrt{-c}\langle X, X\rangle$ and $\langle\bar\nabla_X X, P\eta\rangle = -\langle II(X, JX), \eta\rangle - \langle II(X, X), F\eta\rangle = 0$. Hence, using also the definition of the curve γ,

$$\begin{aligned}\frac{d}{dt}\langle X, X\rangle &= \frac{d}{dt}\langle\gamma', X\rangle = \langle\nabla_{\gamma'}\gamma', X\rangle + \langle\nabla_{\gamma'}X, \gamma'\rangle\\ &= a\langle\bar\nabla_{\gamma'}X, B\rangle + \langle\bar\nabla_{\gamma'}X, X\rangle + \langle\bar\nabla_{\gamma'}X, P\eta\rangle = \langle\bar\nabla_{\gamma'}X, X\rangle + a\langle\bar\nabla_X X, B\rangle\\ &= \langle\nabla_{\gamma'}X, X\rangle + \frac{a\sqrt{-c}}{2}\langle X, X\rangle = \frac{1}{2}\frac{d}{dt}\langle X, X\rangle + \frac{a\sqrt{-c}}{2}\langle X, X\rangle.\end{aligned}$$

This gives $(d/dt)\langle X, X\rangle = a\sqrt{-c}\langle X, X\rangle$. Since $\langle X(0), X(0)\rangle = 0$ we get $\langle X(t), X(t)\rangle = 0$ for all t, and thus $X = 0$.

The definition of γ gives

$$\frac{d}{dt}\langle\gamma', \gamma'\rangle = 2\langle\nabla_{\gamma'}\gamma', \gamma'\rangle = a\sqrt{-c}\langle\gamma', \gamma'\rangle.$$

Using again the definition of γ, the fact that B is geodesic and (3.27), we get

$$\begin{aligned}\frac{da}{dt} &= \frac{d}{dt}\langle\gamma', B\rangle = \langle\nabla_{\gamma'}\gamma', B\rangle + \langle\nabla_{\gamma'}B, \gamma'\rangle\\ &= \frac{\sqrt{-c}}{2}\big(\langle\gamma', \gamma'\rangle - \langle P\eta, \gamma'\rangle\big) = \frac{\sqrt{-c}}{2}\big(\langle\gamma', \gamma'\rangle - \langle P\eta, P\eta\rangle\big).\end{aligned}$$

Finally, from (3.28) and (3.29) we obtain

$$\begin{aligned}\frac{d}{dt}\langle P\eta, P\eta\rangle &= \frac{d}{dt}\langle\gamma', P\eta\rangle = \langle\nabla_{\gamma'}\gamma', P\eta\rangle + \langle\nabla_{\gamma'}P\eta, \gamma'\rangle\\ &= \frac{a\sqrt{-c}}{2}\langle P\eta, P\eta\rangle + a\langle P\nabla^\perp_B\eta, P\eta\rangle + \langle P\nabla^\perp_{P\eta}\eta, P\eta\rangle\\ &= \frac{a\sqrt{-c}}{2}\langle P\eta, P\eta\rangle + \langle\bar\nabla_{\gamma'}P\eta, P\eta\rangle = \frac{a\sqrt{-c}}{2}\langle P\eta, P\eta\rangle + \frac{1}{2}\frac{d}{dt}\langle P\eta, P\eta\rangle,\end{aligned}$$

and thus

$$\frac{d}{dt}\langle P\eta, P\eta\rangle = a\sqrt{-c}\langle P\eta, P\eta\rangle.$$

If we define $b = \langle\gamma', \gamma'\rangle$ and $h = \langle P\eta, P\eta\rangle$, we get the initial value problem:

$$a' = \frac{\sqrt{-c}}{2}(b - h), \quad b' = \sqrt{-c}\,ab, \quad h' = \sqrt{-c}\,ah, \quad a(0) = 0, \quad b(0) = h(0) = 1.$$

Again, by uniqueness of solution we deduce $a(t) = 0$, $b(t) = h(t) = 1$ for all t. Hence, $\langle\gamma'(t), \gamma'(t)\rangle = 1$ and $\gamma'(t) \in P\nu M$ for all t as we wanted to show. □

Let us assume then that $\gamma\colon I \to M$ is a curve satisfying equation (3.31). Since the map $P\colon \nu M \to \mathfrak{D} = P\nu M$ is an isomorphism of vector bundles, there exists a smooth unit normal vector field η of M in a neighborhood of p such that $\gamma'(t) = P\eta_{\gamma(t)}/\|P\eta_{\gamma(t)}\|$ for all sufficiently small t. Since B is a unit vector field and γ is orthogonal to B, we can find a hypersurface $\mathcal{N}$ in M containing γ and transversal to B in a small neighborhood of p. The restriction of η to this hypersurface $\mathcal{N}$ is a smooth unit normal vector field along $\mathcal{N}$. We define ξ to be the unit normal vector field on a neighborhood of p such that $\xi = \eta$ on $\mathcal{N}$, and such that ξ is obtained by $\nabla^\perp$-parallel translation along the integral curves of B. It follows that ξ is smooth by the smooth dependence on initial conditions of ordinary differential equations, and by definition $\nabla^\perp_B \xi = 0$.

The definition of ξ and equations (3.27) and (3.28) imply $2[B, P\xi] = 2\bar{\nabla}_B P\xi - 2\bar{\nabla}_{P\xi} B = \sqrt{-c}\, P\xi$. Thus, the distribution generated by B and $P\xi$ is integrable. We denote by $\mathcal{U}$ the integral submanifold through p.

Lemma 3.4.11. *We have:*

(i) *The norm of $P\xi$ is constant along the integral curves of $P\xi$, that is, $P\xi(\|P\xi\|) = 0$.*

(ii) $\bar{\nabla}_{P\xi} P\xi = \sqrt{-c}\langle P\xi, P\xi\rangle B$.

(iii) *The submanifold $\mathcal{U}$ is an open part of a totally geodesic $\mathbb{R}H^2$ in $\mathbb{C}H^n$.*

Proof. We calculate $\bar{\nabla}_{P\xi} P\xi$. Equation (3.23) implies that $\mathcal{S}_\eta B = 0$ for all $\eta \in \nu M$. Then, for any $\eta, \zeta \in \nu M$ the Ricci equation of M yields

$$\langle R^\perp(B, P\xi)\eta, \zeta\rangle = \langle \bar{R}(B, P\xi)\eta, \zeta\rangle + \langle [\mathcal{S}_\eta, \mathcal{S}_\zeta]B, P\xi\rangle = 0,$$

where $R^\perp$ denotes the curvature tensor of the normal connection $\nabla^\perp$. This, $2[B, P\xi] = \sqrt{-c}\, P\xi$, and the definition of ξ give

$$0 = R^\perp(B, P\xi)\xi = \nabla^\perp_B \nabla^\perp_{P\xi}\xi - \nabla^\perp_{P\xi}\nabla^\perp_B \xi - \nabla^\perp_{[B,P\xi]}\xi = \nabla^\perp_B \nabla^\perp_{P\xi}\xi - \frac{\sqrt{-c}}{2}\nabla^\perp_{P\xi}\xi,$$

and therefore,

$$2\nabla^\perp_B \nabla^\perp_{P\xi}\xi = \sqrt{-c}\, \nabla^\perp_{P\xi}\xi. \tag{3.32}$$

By definition of ξ, along γ we have $\gamma'(t) = P\xi_{\gamma(t)}/\|P\xi_{\gamma(t)}\|$, and thus, along γ we get

$$\begin{aligned}\bar{\nabla}_{P\xi} P\xi &= \bar{\nabla}_{\|P\xi\|\gamma'}\big(\|P\xi\|\gamma'\big) = \|P\xi\|\Big(\gamma'(\|P\xi\|)\gamma' + \|P\xi\|\bar{\nabla}_{\gamma'}\gamma'\Big)\\ &= \gamma'(\|P\xi\|)P\xi + \frac{\sqrt{-c}}{2}\langle P\xi, P\xi\rangle B.\end{aligned}$$

Comparing this equation above with (3.29) yields $P\nabla^\perp_{P\xi}\xi = \gamma'(\|P\xi\|)P\xi$, and since we have that $P\colon \nu M \to \mathfrak{D} = P\nu M$ is an isomorphism of vector bundles we get $\nabla^\perp_{P\xi}\xi = \gamma'(\|P\xi\|)\xi$. Finally, $\langle \xi, \xi\rangle = 1$ implies $\langle \nabla^\perp_{P\xi}\xi, \xi\rangle = 0$. Thus, $\gamma'(\|P\xi\|) = 0$, which is our first assertion, and hence $\nabla^\perp_{P\xi}\xi = 0$ along γ.

Now, let α be an integral curve of B such that $\alpha(0) = \gamma(s)$. We have just shown that $\nabla^\perp_{P\xi}\xi\,|_{\alpha(0)} = \nabla^\perp_{P\xi}\xi\,|_{\gamma(s)} = 0$. Next, from (3.32) and since $\mathcal{S}_\eta B = 0$ for each $\eta \in \nu M$, we get

$$2\bar{\nabla}_{\alpha'}\nabla^\perp_{P\xi}\xi\,|_t = 2\nabla^\perp_B \nabla^\perp_{P\xi}\xi\,|_{\alpha(t)} - 2\mathcal{S}_{\nabla^\perp_{P\xi}\xi}B\,|_{\alpha(t)} = \sqrt{-c}\,\nabla^\perp_{P\xi}\xi\,|_{\alpha(t)}.$$

Hence, by the uniqueness of solutions to differential equations we get $\nabla^{\perp}_{\bar{P}\xi}\xi\,|_{\alpha(t)} = 0$ for all t, and consequently $2\bar{\nabla}_{P\xi}P\xi = \sqrt{-c}\,\langle P\xi, P\xi\rangle B$ along the integral submanifold $\mathcal{U}$. This is our second assertion.

Since B is a geodesic vector field we have $\bar{\nabla}_B B = 0$. By (3.27) we have $2\bar{\nabla}_{P\xi}B = -\sqrt{-c}\,P\xi$, and by definition of ξ and (3.28) we get $\bar{\nabla}_B P\xi = P\bar{\nabla}^{\perp}_B\xi = 0$. Together with (ii) we deduce that $\mathcal{U}$ is an open part of a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$. □

We define $\bar{P}\xi = P\xi/\|P\xi\|$ along $\mathcal{U}$. From Lemma 3.4.11 we obtain $2\bar{\nabla}_{\bar{P}\xi}\bar{P}\xi = \sqrt{-c}\,B$. Using this and (3.27) we obtain

$$\bar{\nabla}_{\bar{P}\xi}\bar{\nabla}_{\bar{P}\xi}\bar{P}\xi + \langle\bar{\nabla}_{\bar{P}\xi}\bar{P}\xi, \bar{\nabla}_{\bar{P}\xi}\bar{P}\xi\rangle\bar{P}\xi = \frac{\sqrt{-c}}{2}\bar{\nabla}_{\bar{P}\xi}B - \frac{c}{4}\langle B, B\rangle\bar{P}\xi = 0.$$

Therefore, the integral curves of $\bar{P}\xi$ are horocycles contained in $\mathcal{U}$ with center $x \in \mathbb{C}H^n(\infty)$, where $\mathcal{U}$ is an open part of a totally geodesic real hyperbolic plane in $\mathbb{C}H^n$. The rigidity of totally geodesic submanifolds of Riemannian manifolds (see e.g. [6, p. 230]), and of horocycles in real hyperbolic planes (see e.g. [6, pp. 24–26]), together with the construction method described in Proposition 3.4.2, imply that a neighborhood of any o in M is congruent to an open part of a submanifold $W_{\mathfrak{w}}$ determined by the point $o \in \mathbb{C}H^n$, $x \in \mathbb{C}H^n(\infty)$ and $\mathfrak{w} = T_oM \ominus (\mathbb{R}B_o \oplus \mathbb{R}Z_o)$.

The argument above was local, so we still need to prove that the connected submanifold M is contained in the $W_{\mathfrak{w}}$ stated above. Since $W_{\mathfrak{w}}$ is an orbit of a Lie group action on an analytic manifold, it follows that $W_{\mathfrak{w}}$ is analytic and complete. Since M is a smooth minimal submanifold in an analytic Riemannian manifold, it is well known that M is also an analytic submanifold of $\mathbb{C}H^n$. As an open neighborhood of M is contained in $W_{\mathfrak{w}}$ it follows that M is an open part of the submanifold $W_{\mathfrak{w}}$.

3.5 Proofs of Theorem 3.0.4 and Theorem 3.0.9

We are now ready to summarize our arguments and conclude the proofs of Theorem 3.0.4 and Theorem 3.0.9 of this chapter.

Proof of Theorem 3.0.4. Assume that M is a connected isoparametric hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 2$. Then, its lift to the anti-De Sitter space $\tilde{M} = \pi^{-1}(M)$ is also an isoparametric hypersurface. If at some point the shape operator of $\tilde{M}$ is of type II or of type IV, then by Proposition 3.2.14 we have that M is an open part of a horosphere or a tube around a totally geodesic real hyperbolic space $\mathbb{R}H^n$ in $\mathbb{C}H^n$, respectively. This corresponds to cases (iii) and (ii) of Theorem 3.0.4. If all points of $\tilde{M}$ are of type I, then Remark 3.2.6 implies that M is an open part of a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ (Theorem 3.0.4 (i)).

Finally, if there is a point $q \in \tilde{M}$ of type III, then there is a neighborhood $\tilde{\mathcal{W}}$ of q where all points are of type III by Proposition 3.2.14. Then, by the results of Section 3.3, there is $r \geq 0$ such that the parallel displacement at distance r, that is, $\mathcal{W}^r = \Phi^r(\pi(\tilde{\mathcal{W}}))$, is a submanifold of $\mathbb{C}H^n$ such that its second fundamental form is given by the trivial symmetric bilinear extension of $2II(Z, P\xi) = -\sqrt{-c}\,(JP\xi)^\perp$, $\xi \in \nu\mathcal{W}^r$, where Z is a vector field tangent to the maximal complex distribution of $\mathcal{W}^r$, and $(\cdot)^\perp$ denotes orthogonal projection on $\nu\mathcal{W}^r$. Using Theorem 3.4.1 we conclude that there exists an Iwasawa decomposition $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ of the Lie algebra of the isometry group of $\mathbb{C}H^n$ and a subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$, such that $\mathcal{W}^r$ is an open part of $W_\mathfrak{w}$.

Therefore, we have proved that there is an open subset of M that is an open part of a tube of radius r around the submanifold $W_\mathfrak{w}$. Since both M and the tubes around $W_\mathfrak{w}$ are smooth hypersurfaces with constant mean curvature, they are real analytic hypersurfaces of $\mathbb{C}H^n$. Thus, we conclude that M is an open part of a tube of radius r around $W_\mathfrak{w}$. Note that $W_\mathfrak{w}$ is minimal, as shown in [40], and ruled by totally geodesic complex hyperbolic subspaces, as follows from Lemma 3.4.6.

If $\mathfrak{w}$ is a hyperplane, $W_\mathfrak{w}$ is denoted by W^{2n-1}, and we get one of the examples in Theorem 3.0.4 (iv). In this case we can have $r = 0$ and we get exactly W^{2n-1}. Both W^{2n-1} and its equidistant hypersurfaces are homogeneous (see for example [3]).

If $\mathfrak{w}^\perp$ has constant Kähler angle $\varphi \in (0, \pi/2]$, then $W_\mathfrak{w}$ is denoted by W_φ^{2n-k}, where k is the codimension. If $\varphi \neq \pi/2$, then k is even [5]. In any case the tubes around W_φ^{2n-k} are homogeneous as was shown in [5]. These correspond to case (v) of Theorem 3.0.4.

If $\mathfrak{w}^\perp$ does not have constant Kähler angle, then the tubes around $W_\mathfrak{w}$ are isoparametric but not homogeneous [40]. These remaining examples correspond to case (vi) of Theorem 3.0.4. □

Proof of Theorem 3.0.9. An isoparametric family corresponding to cases (iii) or (iv) in Theorem 3.0.4 cannot be congruent to a family in one of the other four cases, since the former are regular Riemannian foliations, whereas the latter families always have a singular leaf. Foliations in cases (iii) and (iv) give rise to exactly two congruence classes. Indeed, the family in (iv) has a minimal leaf W^{2n-1} whereas the family in (iv) does not (see Remark 2.5.2). Furthermore, all horosphere foliations are mutually congruent, as well as all solvable foliations. Now, any family in (i) and (ii) has a totally geodesic singular leaf, whereas the singular leaf $W_\mathfrak{w}$ in (v) and (vi) is not totally geodesic. Moreover, the classification of totally geodesic submanifolds of $\mathbb{C}H^n$ allows to distinguish between cases (i) and (ii).

In order to finish the proof, it is convenient to consider the families (i), (iv), (v) and (vi) as tubes around a submanifold $W_\mathfrak{w}$ as described in Subsection 2.5.2. Thus, a totally geodesic $\mathbb{C}H^k$, $k \in \{1, \dots, n-1\}$, corresponds to a submanifold $W_\mathfrak{w}$, where $\mathfrak{w} \subset \mathfrak{g}_\alpha$ is complex, a Lohnherr submanifold W^{2n-1} corresponds to a hyperplane $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, and a Berndt-Brück submanifold W_φ^{2n-k} corresponds to a subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ whose orthogonal complement in $\mathfrak{g}_\alpha$ has constant Kähler angle. Thus, the congruence classes of isoparametric families of hypersurfaces in $\mathbb{C}H^n$ are parametrized by the disjoint union of the singular foliation by geodesic spheres $\mathcal{F}_o$, the horosphere foliation $\mathcal{F}_H$, the singular foliation $\mathcal{F}_{\mathbb{R}H^n}$ of tubes around a totally geodesic $\mathbb{R}H^n$, and the congruence classes of isoparametric families of tubes around the submanifolds $W_\mathfrak{w}$, which we still have to determine.

The submanifold $W_\mathfrak{w}$ depends on the choice of a root space decomposition. Since any two such decompositions are conjugate by an element of $SU(1,n)$, it suffices to take a fixed root space decomposition $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}$, real subspaces $\mathfrak{w}_1$, $\mathfrak{w}_2 \subset \mathfrak{g}_\alpha$ and determine when the family of tubes around $W_{\mathfrak{w}_1}$ and $W_{\mathfrak{w}_2}$ are congruent. By dimension reasons, and by the minimality of W^{2n-1} if both $\mathfrak{w}_1$, $\mathfrak{w}_2$ are hyperplanes, such families are congruent if and only if the two submanifolds $W_{\mathfrak{w}_1} = S_1 \cdot o$ and $W_{\mathfrak{w}_2} = S_2 \cdot o$ are congruent, where S_i is the connected Lie subgroup of $SU(1,n)$ with Lie algebra $\mathfrak{s}_i = \mathfrak{a} \oplus \mathfrak{w}_i \oplus \mathfrak{g}_{2\alpha}$, $i = 1, 2$.

Let ϕ be an isometry of $\mathbb{C}H^n$ such that $\phi(W_{\mathfrak{w}_1}) = W_{\mathfrak{w}_2}$, and assume, without loss of generality, that $\phi(o) = o$. The identification $T_o\mathbb{C}H^n \cong \mathfrak{a} \oplus \mathfrak{n}$ thus allows us to deduce that $\phi_*(\mathfrak{a} \oplus \mathfrak{w}_1 \oplus \mathfrak{g}_{2\alpha}) = \mathfrak{a} \oplus \mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha}$. We consider the Kähler angle decompositions $\mathfrak{w}_i = \oplus_{\varphi \in \Phi_i} \mathfrak{w}_{i,\varphi}$ as described in Section 2.4. Since ϕ is an isometry of $\mathbb{C}H^n$ fixing o, it follows that ϕ_* is a unitary or anti-unitary transformation of $T_o\mathbb{C}H^n \cong \mathfrak{a} \oplus \mathfrak{n} \cong \mathbb{C}^n$. Hence, it maps subspaces of constant Kähler angle to subspaces of the same constant Kähler angle, and thus we have $\Phi := \Phi_1 = \Phi_2$ and

$$\phi_*(\mathfrak{a} \oplus \mathfrak{w}_{1,0} \oplus \mathfrak{g}_{2\alpha}) = (\mathfrak{a} \oplus \mathfrak{w}_{2,0} \oplus \mathfrak{g}_{2\alpha}), \ \phi_*(\mathfrak{w}_{1,\varphi}) = \mathfrak{w}_{2,\varphi}, \ \text{for all } \varphi \in \Phi \setminus \{0\}.$$

Therefore, $\mathfrak{w}_1$ and $\mathfrak{w}_2$ have the same Kähler angles with the same multiplicities. Now set $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K, the isotropy group at o. It is known (see e.g. [42]) that $\mathfrak{k}_0$ is a Lie subalgebra of $\mathfrak{g}$ and that the connected subgroup K_0 of $G = SU(1,n)$ whose Lie algebra is $\mathfrak{k}_0$ acts on $\mathfrak{g}_\alpha$, and its action is equivalent to the standard action of $U(n-1)$ on $\mathbb{C}^{n-1}$. The action of K_0 on $\mathfrak{a}$ and on $\mathfrak{g}_{2\alpha}$ is trivial. Since $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are subspaces of $\mathfrak{g}_\alpha$ with the same Kähler angles and the same multiplicities, it follows that there exists $k \in K_0$ such that $\mathrm{Ad}(k)\mathfrak{w}_1 = \mathfrak{w}_2$ (see the end of Section 2.4 or [42, Remark 2.10] for further details), and thus, $k(W_{\mathfrak{w}_1}) = W_{\mathfrak{w}_2}$.

As a consequence, we have proved that the congruence classes of the submanifolds of type $W_{\mathfrak{w}}$ are in one-to-one correspondence with proper real subspaces of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ modulo the action of $K_0 = U(n-1)$. Altogether this implies Theorem 3.0.9. □

Chapter 4

Isoparametric hypersurfaces in the anti-De Sitter space

In Chapter 2 we have presented a quick review of the origin of isoparametric hypersurfaces. Indeed, we have mentioned some Riemannian spaces where a classification of these kind of hypersurfaces is known. However, it also makes sense to study isoparametric hypersurfaces in the semi-Riemannian case, and more precisely, in Lorentzian space forms, where the breadth of examples is much richer than in the Riemannian case. In fact, in the previous chapter Lorentzian isoparametric hypersurfaces in anti-De Sitter spaces played a crucial role in the classification of isoparametric hypersurfaces in complex hyperbolic spaces.

In this chapter we restrict our attention to spacelike isoparametric hypersurfaces, and we prove the following result, which has been published in the article [92].

Theorem 4.0.1. *Spacelike isoparametric hypersurfaces with more than one principal curvature in the anti-De Sitter space H_1^n, $n \geq 3$, are tubes around totally geodesic submanifolds of H_1^n.*

As we have already used in the previous chapter, a hypersurface in a Lorentzian space form is isoparametric if and only if it has constant principal curvatures with constant algebraic multiplicities. In this context, some remarkable progress has been made as well. For instance, these objects are supposed to be classified in the Minkowski space by Magid [80], although in [23] Burth pointed out some gaps in Magid's arguments. There are also partial classifications in the De Sitter space. In this space, Nomizu [83] proved, using the fact that the number of principal curvatures is bounded from above by two, that spacelike hypersurfaces with constant principal curvatures are tubes around totally geodesic submanifolds. He also conjectured in the same paper [83] that examples of spacelike isoparametric hypersurfaces with more than two principal curvatures would appear in the anti-De Sitter space H_1^n. In this chapter we answer this question negatively, proving that the number of principal curvatures of a spacelike isoparametric hypersurface in H_1^n is less or equal than two, using a different technique than that in [76], where the same question is addressed.

This chapter is organized as follows. In Section 4.1 we recall some notations and conventions and we start with a general procedure following [52] in order to study the geometry of spacelike isoparametric hypersurfaces in anti-De Sitter spaces. This process will allow us to finish the proof of Theorem 4.0.1 in Section 4.2.

4.1 General procedure

Recall from Section 1.6 the construction and geometry of the anti-De Sitter space. We have considered the vector space $\mathbb{R}_2^{n+1}$, $n \geq 3$, provided with the semi-Riemannian metric $\langle x, y\rangle = -x_1y_1 - x_2y_2 + \sum_{i=3}^{n+1} x_iy_i$. We have defined the anti-De Sitter space of radius r, $H_1^n(r)$, as

$$H_1^n(r) = \{x \in \mathbb{R}_2^{n+1} \mid \langle x, x\rangle = -r^2\}.$$

Recall that the anti-De Sitter space is a Lorentzian manifold with constant negative sectional curvature $-1/r^2$, whose curvature tensor $\bar{R}$ then reads $\bar{R}(X,Y)Z = -\frac{1}{r^2}(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$, for $X, Y, Z \in \Gamma(H_1^n(r))$. We will write $\bar{\nabla}$ for the Levi-Civita connection of the anti-De Sitter space.

Let $M \subset H_1^n(r)$ be an isoparametric hypersurface, that is, a hypersurface with constant principal curvatures, $\lambda_1, \dots, \lambda_{n-1}$, and whose corresponding algebraic multiplicities, $m_{\lambda_1}, \dots, m_{\lambda_{n-1}}$, are constant along M [56]. Note that, implicitly, we are assuming that M is a non-degenerate hypersurface of $H_1^n(r)$. Let us denote by ∇ and R the Levi-Civita connection and the curvature tensor of M, respectively. Locally and up to sign, we can take a unique unit normal vector field $\xi \in \Gamma(\nu M)$. We write $\varepsilon = \langle \xi, \xi\rangle \in \{-1, 1\}$.

Now, let λ be a real principal curvature of M with constant geometric multiplicity. Under these assumptions, it is easy to check that $T_\lambda = \ker(\mathcal{S} - \lambda I)$ constitutes a distribution on M. In fact, T_λ defines an autoparallel and, consequently, integrable distribution. In order to prove this last claim, let $X, Y \in \Gamma(T_\lambda)$. Then, for each vector field $Z \in \Gamma(TM)$, we obtain

$$\begin{aligned}
\langle (\mathcal{S} - \lambda I)(\nabla_X Y), Z\rangle &= \langle \nabla_X(\mathcal{S} - \lambda I)Y - (\nabla_X(\mathcal{S} - \lambda I))Y, Z\rangle \\
&= -\langle (\nabla_X(\mathcal{S} - \lambda I))Y, Z\rangle = -\langle Y, (\nabla_X(\mathcal{S} - \lambda I))Z\rangle \\
&= -\langle Y, (\nabla_Z(\mathcal{S} - \lambda I))X\rangle \\
&= -\langle Y, \nabla_Z(\mathcal{S} - \lambda I)X\rangle + \langle Y, (\mathcal{S} - \lambda I)\nabla_Z X\rangle = 0,
\end{aligned}$$

where in the third equality we have used the symmetry of $\nabla_X(\mathcal{S} - \lambda I)$ and in the fourth one the Codazzi equation (see Section 1.2). Therefore, we can construct L_λ, the integral submanifolds of the distribution T_λ through a point $p \in M$. Assume, in what follows, that the geometric and algebraic multiplicities of λ coincide. In this case, for each $p \in M$ and each $\mu \in \operatorname{Spec}(\mathcal{S})\backslash\{\lambda\}$, it is possible to select $r(\mu, p)$ big enough in such a way that, if $T_\mu(p) = \ker(\mathcal{S} - \mu I)_p^{r(\mu,p)}$, we obtain the orthogonal decomposition

$$T_pM = T_\lambda(p) \oplus \bigoplus_{\mu \neq \lambda} T_\mu(p).$$

Take now an element X in $T_\lambda(p)$ orthogonal to all the elements of $T_\lambda(p)$. Since $T_\lambda(p)$ is orthogonal to all the generalized eigenspaces $T_\mu(p)$ with $\mu \neq \lambda$ [85], we deduce that X is orthogonal to all the elements in T_pM. But taking into account that M is non-degenerate, we can conclude that $X = 0$. Therefore, L_λ is a non-degenerate submanifold of M. In fact, L_λ is totally geodesic as a submanifold of M and totally umbilical as a submanifold of $H_1^n(r)$.

The next step is trying to understand the behaviour of the generalized eigenspaces T_μ, with $\mu \neq \lambda$, with respect to T_λ. In this sense, following Ferus' ideas [52], we examine the behaviour of $(\mathcal{S} - \lambda I)$ along a geodesic curve in L_λ. In order to do that, we introduce, as in [52], a tensor field $\mathcal{C}$ defined by

$$\mathcal{C}_X(Y) = -\mathcal{V}\nabla_Y \mathcal{H}X,$$

where, for each $p \in M$, $\mathcal{H}_p$ and $\mathcal{V}_p$ denote the orthogonal projections onto $\ker(\mathcal{S} - \lambda I)_p$ and $\operatorname{Im}(\mathcal{S} - \lambda I)_p$, respectively. It is easy to check that both $\mathcal{H}$ and $\mathcal{V}$ are parallel along L_λ.

Ferus' work focused on Riemannian geometry and consequently some of his results must be adapted to the more general semi-Riemannian setting. The next lemma constitutes a generalization of Lemmas 1 and 2 in [52]. Note that, even though the final claim is exactly the same, the arguments utilised in the proof should be modified slightly.

Lemma 4.1.1. *Let X be a vector $T_\lambda(p)$ and $Y \in T_pM$. Then:*

(i) $(\nabla_X(\mathcal{S} - \lambda I))Y = (\mathcal{S} - \lambda I) \circ \mathcal{C}_X(Y)$.

(ii) $(\nabla_X \mathcal{C})_X Y = \mathcal{C}_X^2(Y) + R_X Y$*, where* $R_X Y = \mathcal{V}R(\mathcal{V}Y, X)X$.

Let $\gamma\colon I \subset \mathbb{R} \to L_\lambda$ be a unit speed geodesic in L_λ, with $\eta = \langle \dot\gamma, \dot\gamma\rangle \in \{1, -1\}$. Using the Gauss equation and taking into account that $\dot\gamma \in \Gamma(\gamma^* T_\lambda)$, we obtain the Jacobi operator

$$\begin{aligned} R_{\dot\gamma}(X) &= \mathcal{V}\bar{R}(\mathcal{V}X, \dot\gamma)\dot\gamma + \varepsilon\langle \mathcal{S}\dot\gamma, \dot\gamma\rangle \mathcal{V}\mathcal{S}\mathcal{V}X - \varepsilon\langle \mathcal{S}\mathcal{V}X, \dot\gamma\rangle \mathcal{V}\mathcal{S}\dot\gamma \\ &= \mathcal{V}\bar{R}(\mathcal{V}X, \dot\gamma)\dot\gamma + \varepsilon\lambda\eta\mathcal{S}\mathcal{V}X - \varepsilon\lambda^2\langle \mathcal{V}X, \dot\gamma\rangle \mathcal{V}\dot\gamma = \mathcal{V}\bar{R}(\mathcal{V}X, \dot\gamma)\dot\gamma + \varepsilon\lambda\eta\mathcal{S}\mathcal{V}X \end{aligned}$$

and recalling that $H_1^n(r)$ has constant curvature, we substitute $\bar{R}$ by its value in the above equation to obtain

$$R_{\dot\gamma}X = \eta(c + \varepsilon\lambda\mathcal{S})\mathcal{V}X. \tag{4.1}$$

For each $t \in I$, we construct an endomorphism $\mathcal{A}(t)$ of the real vector space $T_{\gamma(t)}M = \ker(\mathcal{S} - \lambda I)_{\gamma(t)} \oplus \operatorname{Im}(\mathcal{S} - \lambda I)_{\gamma(t)}$, defined as the inverse of $(\mathcal{S} - \lambda I)_{\gamma(t)}|_{\operatorname{Im}(\mathcal{S}-\lambda I)_{\gamma(t)}}$ when restricted to $\operatorname{Im}(\mathcal{S} - \lambda I)_{\gamma(t)}$, and defined as zero for the elements in $\ker(\mathcal{S} - \lambda I)_{\gamma(t)}$. Thus, $\mathcal{A}(t)$ is a tensor field along the curve γ. It is convenient to remark here that along γ the equality $\mathcal{V} = \mathcal{A}(\mathcal{S} - \lambda I) = (\mathcal{S} - \lambda I)\mathcal{A}$ holds. Taking derivatives along γ in this last equality, and using Lemma 4.1.1 we may write

$$\begin{aligned} 0 &= (\nabla_{\dot\gamma}\mathcal{V})\mathcal{A} = (\nabla_{\dot\gamma}\mathcal{A}(\mathcal{S} - \lambda I))\mathcal{A} = \{(\nabla_{\dot\gamma}\mathcal{A})(\mathcal{S} - \lambda I) + \mathcal{A}(\nabla_{\dot\gamma}(\mathcal{S} - \lambda I))\}\mathcal{A} \\ &= \{(\nabla_{\dot\gamma}\mathcal{A})(\mathcal{S} - \lambda I) + \mathcal{A}(\mathcal{S} - \lambda I)\mathcal{C}_{\dot\gamma}\}\mathcal{A} = (\nabla_{\dot\gamma}\mathcal{A})(\mathcal{S} - \lambda I)\mathcal{A} + \mathcal{V}\mathcal{C}_{\dot\gamma}\mathcal{A} = \nabla_{\dot\gamma}\mathcal{A} + \mathcal{C}_{\dot\gamma}\mathcal{A}. \end{aligned}$$

Taking derivatives again and using the expression above together with (4.1) and Lemma 4.1.1 we obtain

$$\begin{aligned} 0 &= (\nabla^2_{\dot\gamma}\mathcal{A}) + (\nabla_{\dot\gamma}\mathcal{C}_{\dot\gamma}\mathcal{A}) = (\nabla^2_{\dot\gamma}\mathcal{A}) + (\nabla_{\dot\gamma}\mathcal{C}_{\dot\gamma})\mathcal{A} + \mathcal{C}_{\dot\gamma}(\nabla_{\dot\gamma}\mathcal{A}) \\ &= (\nabla^2_{\dot\gamma}\mathcal{A}) + \mathcal{C}^2_{\dot\gamma}\mathcal{A} + R_{\dot\gamma}\mathcal{A} - \mathcal{C}^2_{\dot\gamma}\mathcal{A} = (\nabla^2_{\dot\gamma}\mathcal{A}) + \eta(c+\varepsilon\lambda\mathcal{S})\mathcal{V}\mathcal{A} \\ &= (\nabla^2_{\dot\gamma}\mathcal{A}) + \eta(c+\varepsilon\lambda\mathcal{S} - \varepsilon\lambda^2 I + \varepsilon\lambda^2 I)\mathcal{A} \\ &= (\nabla^2_{\dot\gamma}\mathcal{A}) + \eta(c+\varepsilon\lambda^2)\mathcal{A} + \eta\varepsilon\lambda\mathcal{V}. \end{aligned}$$

We can also rewrite this differential equation in the following way:

$$\nabla^2_{\dot\gamma}\{(c+\varepsilon\lambda^2)\mathcal{A} + \varepsilon\lambda\mathcal{V}\} + \eta(c+\varepsilon\lambda^2)\{(c+\varepsilon\lambda^2)\mathcal{A} + \varepsilon\lambda\mathcal{V}\} = 0. \tag{4.2}$$

The next step is to solve this equation with the purpose of understanding and extracting all the relevant information codified in it. In fact, in the Riemannian case, it seems that all the information can be summarized in the Cartan formula. However, although it is possible to rewrite a semi-Riemannian version of the Cartan formula using (4.2) as well, there is some more information which would not remain summarized in it. Actually, this geometric information will lead us to conclude a bound on the number of principal curvatures of a spacelike isoparametric hypersurfaces in the anti-De Sitter space. In other settings, like Lorentzian isoparametric hypersurfaces in De Sitter spaces, this procedure presented so far is still valid and it might be utilized to obtain some results concerning the number of principal curvatures and the relations between them.

4.2 Spacelike isoparametric hypersurfaces in the anti-De Sitter space

We will focus now our attention on a spacelike hypersurface with constant principal curvatures in the anti-De Sitter space $H^n_1(r)$. Therefore, this isoparametric hypersurface M has diagonalizable shape operator at each point p in M. Assume that we have more than one constant principal curvature and select, without loss of generality, $\lambda = \lambda_1$. In this particular situation, we could develop the process we have just explained and, moreover, we have that the constant $\eta(c+\varepsilon\lambda^2)$ is strictly less than zero ($c < 0$, $\varepsilon = -1$, $\eta = 1$). Under all these conditions, we can easily integrate equation (4.2), and writing $F(t) = (c+\varepsilon\lambda^2)\mathcal{A}(t)+\varepsilon\lambda\mathcal{V}(t)$ and $k = \eta(c+\varepsilon\lambda^2)$ for the sake of simplicity, its solution may be written as

$$F(t) = \cosh\left(\sqrt{-k}t\right)\mathcal{P}_{F(0)}(t) + \frac{1}{\sqrt{-k}}\sinh\left(\sqrt{-k}t\right)\mathcal{P}_{(\nabla_{\dot\gamma}F)(0)}(t), \tag{4.3}$$

where $\mathcal{P}_{F(0)}(t)$ and $\mathcal{P}_{(\nabla_{\dot\gamma}F)(0)}(t)$ denote the parallel transport of the endomorphisms $F(0)$ and $(\nabla_{\dot\gamma}F)(0)$ of $T_{\gamma(0)}M$ along the curve γ from the point $\gamma(0) = p$ to the point $\gamma(t)$. We will show that $(c+\varepsilon\lambda^2)\mathcal{A}(t) + \varepsilon\lambda\mathcal{V}(t)$ is a self-adjoint endomorphism for each $t \in I$ by checking that both $\mathcal{A}(t)$ and $\mathcal{V}(t)$ are self-adjoint endomorphisms for all $t \in I$. This is clear for $\mathcal{A}$ because it is the inverse of a self-adjoint operator. For $\mathcal{V}$, we can compute

$$\langle \mathcal{V}X, Y\rangle = \langle \mathcal{V}X, \mathcal{H}Y + \mathcal{V}Y\rangle = \langle \mathcal{V}X, \mathcal{V}Y\rangle = \langle \mathcal{H}X + \mathcal{V}X, \mathcal{V}Y\rangle = \langle X, \mathcal{V}Y\rangle.$$

At this point, we can determine the eigenvalue structure of the endomorphism $F(t)$ of the real vector space $T_{\gamma(t)}M$, for each $t \in I$. Firstly, by hypothesis, we know that the principal curvatures of M and their algebraic and geometric multiplicities are constant along M. But, taking into account that for each $t \in I$ the tensor field $\mathcal{A}(t)$ is zero when restricted to $\ker(\mathcal{S} - \lambda I)_{\gamma(t)}$ and the inverse of $(\mathcal{S} - \lambda I)_{\gamma(t)}|_{\mathrm{Im}(\mathcal{S}-\lambda I)_{\gamma(t)}}$ when restricted to $\mathrm{Im}(\mathcal{S} - \lambda I)_{\gamma(t)}$, we can deduce that the eigenvalues of $\mathcal{A}(t)$ are: zero with algebraic and geometric multiplicity m_λ, and $\frac{1}{\lambda_i-\lambda}$ with algebraic and geometric multiplicity m_{λ_i}, for $i = 2, 3, \ldots, n-1$. On the other hand, the spectrum of the endomorphism $\mathcal{V}$ is 0 with multiplicity m_λ, and 1 with multiplicity $n - 1 - m_\lambda$. Note that these eigenvalues together with their algebraic and geometric multiplicities are constant along M precisely because they only depend on the dimension of the subspaces involved in the orthogonal decomposition $\ker(\mathcal{S}-\lambda I)_q \oplus \mathrm{Im}(\mathcal{S}-\lambda I)_q$. The dimensions of these subspaces are constant because M is isoparametric and, thus, the eigenvalues of $\mathcal{V}$ are constant along the curve γ.

Therefore, the tensor field $F(t) = (c+\varepsilon\lambda^2)\mathcal{A}(t) + \varepsilon\lambda\mathcal{V}(t)$ has constant eigenvalues with constant algebraic and geometric multiplicities. These eigenvalues are: 0 with geometric and algebraic multiplicity m_λ, and $\frac{c+\varepsilon\lambda\lambda_i}{\lambda_i-\lambda}$ with algebraic and geometric multiplicities m_{λ_i}, for $i = 2, \ldots, n-1$.

Note at this point that the parallel transport $\mathcal{P}_{F(0)}(t)$ of the endomorphism $F(0)$ along γ has the same eingenvalues as those of $F(0)$ for all $t \in I$. Furthermore, the eigenvectors are exactly the parallel translation of these of $F(0)$. So parallel translation of endomorphisms also preserves algebraic and geometric multiplicities. In fact, let $\{X_1, \ldots, X_{n-1}\}$ be an orthonormal basis of $T_{\gamma(0)}M$. Then, writing F instead of $F(0)$ for the sake of simplicity, we may deduce

$$\begin{aligned}\nabla_{\dot\gamma}\langle \mathcal{P}_F(t)\mathcal{P}_{X_i}(t), \mathcal{P}_F(t)\mathcal{P}_{X_j}(t)\rangle = {}& \langle (\nabla_{\dot\gamma}\mathcal{P}_F\mathcal{P}_{X_i})(t), \mathcal{P}_{X_j}(t)\rangle \\ & + \langle \mathcal{P}_F(t)\mathcal{P}_{X_i}(t), (\nabla_{\dot\gamma}\mathcal{P}_{X_j})(t)\rangle \\ =& \langle (\nabla_{\dot\gamma}\mathcal{P}_F)(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{X_j}(t)\rangle \\ & + \langle F(\nabla_{\dot\gamma}\mathcal{P}_{X_i})(t), \mathcal{P}_{X_j}(t)\rangle = 0.\end{aligned}$$

Therefore, the function $t \in I \to \langle \mathcal{P}_{F(0)}(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{F(0)}(t)\mathcal{P}_{X_j}(t)\rangle$ is constant and takes the value $\delta_{ij}\lambda_i = \delta_{ij}\lambda_j$ at zero. Thus, our claim is proved. This might be thought as a particularization of a more general result which claims that the parallel transport of an endomorphism along a curve preserves eigenvalues together with their algebraic and geometric multiplicities. Moreover, the eigenvectors of the parallel transport of an endomorphism are exactly the parallel transport of the eigenvectors of the initial endomorphism. Thus $\mathcal{P}_{(\nabla_{\dot\gamma}F)(0)}(t)$ also has constant eigenvalues with constant algebraic multiplicities for all $t \in I$.

It is important to remark that $(\nabla_{\dot\gamma}F)(t)$ is a self-adjoint endomorphism for each $t \in I$. Since $F(t)$ is self-adjoint, then we have that $\langle F(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{X_j}(t)\rangle = \langle \mathcal{P}_{X_i}(t), F(t)\mathcal{P}_{X_j}(t)\rangle$, where $\{X_1, \ldots, X_{n-1}\}$ is again, as above, an orthonormal basis of $T_{\gamma(0)}M$. Taking derivatives in the left hand side we get $\langle (\nabla_{\dot\gamma}F)(t)\mathcal{P}_{X_i}(t), \mathcal{P}_{X_j}(t)\rangle$. By symmetry, in the right hand side we obtain $\langle \mathcal{P}_{X_i}(t), (\nabla_{\dot\gamma}F)(t)\mathcal{P}_{X_j}(t)\rangle$. Thus, $(\nabla_{\dot\gamma}F)(t)$ is a self-adjoint endomorphism of the real vector space $T_{\gamma(t)}M$ for all $t \in I$.

This means, in particular, that each one of the addends of $F(t)$ in (4.3) diagonalizes with real eigenvalues. Furthermore, taking into account that F has constant eigenvalues along the geodesic curve γ, one may argue that the map $t \in I \to \operatorname{tr}(F(t))$ is a constant function. But it is then clear that $F^2(t)$ diagonalizes with real eigenvalues, the square of the eigenvalues of F, for all $t \in I$. Therefore, the function $\operatorname{tr}(F^2(t))$ is again constant and we may write

$$0 = \left.\frac{d^2}{dt^2}\right|_{t=0} \operatorname{tr}(F^2(t)) = \operatorname{tr}((\nabla^2_{\dot\gamma}F^2)(0)) = 2|k|\operatorname{tr}(F^2(0)) + 2\operatorname{tr}((\nabla_{\dot\gamma}F)^2(0)). \tag{4.4}$$

But this last equalitiy clearly implies that both $F(0)$ and $(\nabla_{\dot\gamma}F)(0)$ are the zero endomorphims. Consequently, $F(t) = 0$ for all $t \in I$ by (4.3) and recalling the definition of F we have just shown that $(c + \varepsilon\lambda^2)\mathcal{A}(t) = -\varepsilon\lambda\mathcal{V}(t)$. If we now decompose $T_{\gamma(t)}M$ into $\ker(\mathcal{S} - \lambda I)_{\gamma(t)}$ and $\operatorname{Im}(\mathcal{S} - \lambda I)_{\gamma(t)}$ as usual, and we express both families of endomorphisms in their matrix form with respect to that decomposition,

$$(c + \varepsilon\lambda^2)\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{1}{\lambda_i - \lambda}\,\mathrm{id} \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\varepsilon\lambda\,\mathrm{id} \end{array}\right),$$

one can easily deduce that M only has two principal curvatures: λ, the curvature we had assumed at the very beginning to have the same algebraic and geometric multiplicity, and $\frac{-c}{\varepsilon\lambda} = \frac{c}{\lambda}$. According to the bound achieved in Lemma 3.2.4 and Lemma 3.2.10 on the number of principal curvatures for a Lorentzian isoparametric hypersurface in the anti-De Sitter space, this allows to state the following

Proposition 4.2.1. *Let $M \subset H^n_1$ be an isoparametric hypersurface. Then, the number of principal curvatures is less or equal than two.*

Finally, coming back to spacelike isoparametric hypersurfaces in anti-De Sitter spaces, it is easy to check, using Jacobi vector field theory, that the focal submanifold of the spacelike isoparametric hypersurface M considered above is a totally geodesic submanifold. Hence Theorem 4.0.1 follows.

Chapter 5
CPC submanifolds

In the previous chapters we have investigated isoparametric hypersurfaces in different contexts. An important subclass of isoparametric hypersurfaces is that of homogeneous hypersurfaces. In this chapter we focus on a property of focal sets of homogeneous hypersurfaces (see Section 5.1 for more details).

In this sense, we will say that a connected complete submanifold P of a Riemannian manifold M is CPC (constant principal curvatures) if its principal curvatures, counted with multiplicities, are independent of the normal direction (parametrized by the unit normal vectors of P). Note that our notion of constant principal curvatures is more restrictive than the one studied in [58]: every CPC submanifold is a submanifold with constant principal curvatures in the sense of [58].

The main purpose of this chapter is to present a systematic approach to the construction, classification and description of homogeneous CPC submanifolds in irreducible Riemannian symmetric spaces of non-compact type and rank ≥ 2. The contents of this chapter have given rise to the paper [14], and jointly with other results, to the survey [45].

It is evident that totally geodesic submanifolds are CPC submanifolds. Since totally geodesic submanifolds of irreducible Riemannian symmetric spaces are not yet classified, unless the rank of M is ≤ 2 [109, 30, 31, 72, 71], we cannot expect to achieve full classification results of CPC submanifolds.

Thus, we will restrict our attention to CPC submanifolds arising from orbits of certain subgroups of the solvable part of the Iwasawa decomposition associated with a symmetric space of non-compact type. More precisely, let $M = G/K$ be an irreducible Riemannian symmetric space of non-compact type, where $G = I^o(M)$ is the identity component of the isometry group of M and K is the isotropy group of G at a point $o \in M$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha\right)$ be the induced restricted root space decomposition of $\mathfrak{g}$, where Σ denotes the set of restricted roots. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the corresponding Iwasawa decomposition of $\mathfrak{g}$. Denote by AN the solvable closed connected subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$. Then M is isometric to AN endowed with a suitable left-invariant Riemannian metric (see Section 1.5). Let Π be a set of simple roots for Σ and denote by Π' the set of simple roots $\alpha \in \Pi$ with $2\alpha \notin \Delta$. Note that there is at most one simple root in Π that does not belong to Π', and this happens precisely when the restricted root system of G/K is of type BC_r. Denote by $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$ the principal isotropy subalgebra of $\mathfrak{k}$. This chapter is completely devoted to prove the following result.

Theorem 5.0.2. *Let $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$ be a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ with $V \subseteq \bigoplus_{\alpha\in\Pi'} \mathfrak{g}_\alpha$. Let S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. Then the orbit $S \cdot o$ is a CPC submanifold of $M = G/K$ if and only if one of the following statements holds:*

(I) *There exists a simple root* $\lambda \in \Pi'$ *with* $V \subset \mathfrak{g}_\lambda$.

(II) *There exist two non-orthogonal simple roots* $\alpha_0, \alpha_1 \in \Pi'$ *with* $|\alpha_0| = |\alpha_1|$ *and subspaces* $V_0 \subseteq \mathfrak{g}_{\alpha_0}$ *and* $V_1 \subseteq \mathfrak{g}_{\alpha_1}$ *such that* $V = V_0 \oplus V_1$ *and one of the following conditions holds:*

(i) $V_0 \oplus V_1 = \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$;

(ii) $V_0 \oplus V_1$ *is a proper subset of* $\mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$ *and*

(a) V_0 *and* V_1 *are isomorphic to* $\mathbb{R}$; *or*

(b) V_0 *and* V_1 *are isomorphic to* $\mathbb{C}$ *and there exists* $T \in \mathfrak{k}_0$ *such that* $\mathrm{ad}(T)$ *defines complex structures on* V_0 *and* V_1 *and vanishes on* $[V_0, V_1]$; *or*

(c) V_0 *and* V_1 *are isomorphic to* $\mathbb{H}$ *and there exists a subset* $\mathfrak{l} \subseteq \mathfrak{k}_0$ *such that* $\mathrm{ad}(\mathfrak{l})$ *defines quaternionic structures on* V_0 *and* V_1 *and vanishes on* $[V_0, V_1]$.

Moreover, only the submanifolds given by (I) *and* (II)(i) *can appear as singular orbits of cohomogeneity one actions.*

Note that this result has three different aspects: a construction part, a classification part and a description part. We will first construct the submanifolds introduced in Theorem 5.0.2 (in particular we see that all the cases occur) and prove that their principal curvatures are independent of the normal direction. We will then prove that there are no other such submanifolds under the hypotheses of Theorem 5.0.2. Finally, some of them can be described as singular orbits of cohomogeneity one actions, but one of the main goals of this work is that most of those examples do not come from cohomogeneity one actions.

The submanifolds in (I) can be thought of as canonical extensions of submanifolds in real hyperbolic spaces. According to [25], all these examples are singular orbits of cohomogeneity one actions. Thus, using a result due to Ge and Tang [54] we will obtain directly that their principal curvatures are independent of the normal direction. Therefore, in this chapter we will focus mainly on the submanifolds presented in (II).

We will construct the submanifolds of Theorem 5.0.2 explicitly and compute their shape operator. For this purpose, we first generalize the concept of strings generated by a single root [69, p. 152] to strings generated by two roots. This more general concept will then induce a natural decomposition of the tangent space of the submanifold into subspaces that are invariant under the shape operator. The root space structure will then allow us to calculate explicitly the shape operator when restricted to each of these invariant subspaces. This technique is original and we hope that it can be applied also in other situations. We will also construct explicitly the complex and quaternionic structures mentioned in Theorem 5.0.2.

This chapter is organized as follows. In Section 5.1 we expose the main the motivations for studying CPC submanifolds and their connections with geometrical objects such as cohomogeneity one actions or isoparametric hypersurfaces. We also state some results that we will need for our investigations. In Section 5.2, we start by introducing the general setting for constructing the new examples. We show that in order to understand the principal curvatures of those examples it suffices to determine a decomposition of the tangent space by invariant subspaces with respect to the shape operator. We also determine one of these invariant subspaces. Calculating the shape operator when restricted to such a subspace turns out to be equivalent to studying Theorem 5.0.2 for a symmetric space of non-compact type whose Dynkin diagram is of type A_2. Thus, in the final part of the section, we prove the construction and classification part of Theorem 5.0.2 for the symmetric spaces $SL_3(\mathbb{R})/SO_3$, $SL_3(\mathbb{C})/SU_3$, $SL_3(\mathbb{H})/Sp_3$ and E_6^{-26}/F_4. In Section 5.3 we will show that all the examples of Theorem 5.0.2 are indeed CPC submanifolds. Thus, in Section 5.3 we finish the construction part of Theorem 5.0.2. Section 5.4 is devoted to the classification part of Theorem 5.0.2. Actually, we just see that if the subspace V does not satisfy the conditions of (I) or (II), then $S \cdot o$ cannot be a CPC submanifold. In Section 5.5, we analyze if the examples can be realized as singular orbits of cohomogeneity one actions. Finally, in Section 5.6 we provide some further geometric explanations of the examples in the rank 2 cases.

5.1 Motivation and main tools

The concept of CPC submanifold was recently introduced [14], but it is deeply connected to many other objects concerning submanifold theory. Hence, in the following lines we will explain the main motivations for the investigation of CPC submanifolds.

As mentioned above, CPC submanifolds are intimately related to cohomogeneity one actions. Indeed, consider a cohomogeneity one action on a connected complete Riemannian manifold M. If there is a singular orbit of this action, say P, then its principal curvatures do not depend on the normal directions. More precisely, if ξ_1 and ξ_2 are two unit normal vectors of P, at the same point or at two different points, then the principal curvatures of P with respect to ξ_1 and ξ_2 are the same, counted with multiplicities. This is a simple consequence of the homogeneity of the orbit and the fact that the slice representation of the action at a point $p \in P$ acts transitively on the unit sphere in the normal space of P at p.

An obvious consequence is that every singular orbit of a cohomogeneity one action is a CPC, and hence austere and minimal submanifold. Note that the principal curvatures of a homogeneous austere hypersurface do not depend on the normal direction. In other words, the concepts of austere and CPC submanifolds are equivalent in the context of homogeneous hypersurfaces.

Another consequence is that every singular orbit of a cohomogeneity one action is a submanifold with constant principal curvatures. Recall from Section 1.2 that a submanifold P of a Riemannian manifold M has constant principal curvatures if the principal curvatures of P are constant for any parallel normal vector field of P along any piecewise differentiable curve in P. Submanifolds with constant principal curvatures were introduced and studied by Heintze, Olmos and Thorbergsson [58] in the context of isoparametric submanifolds. They proved that a submanifold of a Euclidean space has constant principal curvatures if and only if it is an isoparametric submanifold or a focal manifold of an isoparametric submanifold.

It is interesting to investigate the classification of submanifolds having the above geometric property of singular orbits of cohomogeneity one actions, that is, the classification of CPC submanifolds. Assume that M is a standard real space form, that is, M is the real hyperbolic space $\mathbb{R}H^n$, the Euclidean space $\mathbb{R}^n$, or the sphere S^n, with their standard metrics of constant curvature $-1, 0, +1$ respectively. Let P be a submanifold of M with $\operatorname{codim}(P) \geq 2$. Using Jacobi field theory one can show P is CPC, that is, that its principal curvatures are independent of the normal direction if and only if the tubes (of sufficiently small radii) around P have constant principal curvatures. As it was explained in Section 2.1, according to Cartan [25], a hypersurface of a space of constant curvature has constant principal curvatures if and only if it is isoparametric. Therefore, classifying isoparametric hypersurfaces in constant curvature spaces is equivalent to classifying CPC submanifolds. Recall, also from Section 2.1, that the classification problem for isoparametric hypersurfaces in Euclidean spaces and real hyperbolic spaces was solved by Segre [93] and Cartan [25], respectively. In contrast, the problem for S^n turned out to be very challenging as mentioned in Section 2.1. Interestingly, the CPC property of focal submanifolds of isoparametric hypersurfaces in spheres turns out to play a crucial role in certain approaches to their investigation (e.g. [81, 95]).

One of the implications in the above characterization in spaces of constant curvature was recently generalized by Ge and Tang [54] to arbitrary Riemannian manifolds: let P be a submanifold of a Riemannian manifold M with $\operatorname{codim}(P) \geq 2$ for which the tubes around it (for sufficiently small radii) are isoparametric hypersurfaces with constant principal curvatures. Then the principal curvatures of P are independent of the normal direction. The other implication is not true. In fact, it follows from Theorem 5.0.2 in Chapter 3 that there are tubes around totally geodesic submanifolds (in complex hyperbolic spaces) that are not even isoparametric hypersurfaces.

If $M = \mathbb{R}H^n$ ($n \geq 2$), then the above result by Cartan implies that a CPC submaniold P is congruent to a totally geodesic $\mathbb{R}H^k \subset \mathbb{R}H^n$, $k \in \{0, \ldots, n-1\}$. For the other rank one symmetric spaces, which are the complex hyperbolic spaces $\mathbb{C}H^n$ ($n \geq 2$), the quaternionic hyperbolic spaces $\mathbb{H}H^n$ ($n \geq 2$), and the Cayley hyperbolic plane $\mathbb{O}H^2$, the problem is already much more complicated. Their totally geodesic submanifolds are known from the work by Wolf [108]. In each of the spaces $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$ there exists a homogeneous austere hypersurface [3]. Singular orbits of cohomogeneity one actions on these spaces were described in [5], [17] and [41]. Note that, up to orbit equivalence, the cohomogeneity one actions on $\mathbb{C}H^n$, $\mathbb{H}H^2$ and $\mathbb{O}H^2$ are classified, whereas for $\mathbb{H}H^n$, $n \geq 3$, this is still an open problem. A remarkable discovery in [41] is an 11-dimensional homogeneous CPC submanifold of $\mathbb{O}H^2$ that is not an orbit of a cohomogeneity one action. To our knowledge, this was the only known non-totally geodesic CPC submanifold in an irreducible Riemannian symmetric space of non-compact type that is not an orbit of a cohomogeneity one action.

In the final part of this section, we state some results that will be needed throughout this chapter. We will use the concepts, notations and terminology introduced in Section 1.5.

Recall that $M = G/K$ is an irreducible Riemannian symmetric space of non-compact type, where G is the identity component of the isometry group of M and K is the isotropy group of G at a point $o \in M$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha\right)$ be the induced restricted root space decomposition and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the corresponding Iwasawa decomposition of $\mathfrak{g}$. Recall also that M is isometric to a solvable closed connected subgroup AN of G. Let $\alpha \in \Sigma$ and $\lambda \in \Sigma_0 = \Sigma \cup \{0\}$. Recall that the α-string containing λ is defined as the set of all elements in Σ_0 of the form $\lambda + n\alpha$ with $n \in \mathbb{Z}$. The following result will play an important role from now on. Basically, it relates the dimensions of the root spaces involved in a string.

Lemma 5.1.1. *Let $\alpha, \lambda \in \Sigma^+$ be linearly independent.*

(i) *If the α-string of λ is $\lambda, \lambda + \alpha$, then $A_{\alpha,\lambda} = -1$ and $\dim(\mathfrak{g}_\lambda) = \dim(\mathfrak{g}_{\lambda+\alpha})$.*

(ii) *If the α-string of λ is $\lambda, \lambda+\alpha, \lambda+2\alpha$, then $A_{\alpha,\lambda} = -2$, $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{\lambda+\alpha})$ and $\dim(\mathfrak{g}_\lambda) = \dim(\mathfrak{g}_{\lambda+2\alpha})$.*

Proof. The statements about $A_{\alpha,\lambda}$ follow from Proposition 1.5.1(v). We denote by $s_\alpha(\lambda) = \lambda - A_{\alpha,\lambda}\alpha$ the Weyl reflection of α.

If the α-string containing λ is $\lambda, \lambda+\alpha$, then $A_{\alpha,\lambda} = -1$ and $s_\alpha(\lambda) = \lambda - A_{\alpha,\lambda}\alpha = \lambda+\alpha$. Since the Weyl reflection s_α interchanges λ and $\lambda+\alpha$, we get $\dim(\mathfrak{g}_\lambda) = \dim(\mathfrak{g}_{\lambda+\alpha})$.

Next, assume that the α-string containing λ is $\lambda, \lambda+\alpha, \lambda+2\alpha$. Then $A_{\alpha,\lambda} = -2$ and $s_\alpha(\lambda) = \lambda - A_{\alpha,\lambda}\alpha = \lambda + 2\alpha$, which implies $\dim(\mathfrak{g}_\lambda) = \dim(\mathfrak{g}_{\lambda+2\alpha})$. The only root systems of rank two with α-strings of length 3 and containing only positive roots are B_2 and BC_2. In the B_2-case there is only one such string $\lambda, \lambda+\alpha, \lambda+2\alpha$, namely when α, λ are the simple roots of B_2 and α is the shortest of the two roots. In this case we have $s_\lambda(\alpha) = \lambda + \alpha$, which implies $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{\lambda+\alpha})$. In the BC_2-case there is another such string $\lambda', \lambda'+\alpha', \lambda'+2\alpha'$ with $\lambda' = 2\alpha$ and $\alpha' = \lambda$. In this situation we have $s_{\lambda'}(\alpha') = s_{2\alpha}(\lambda) = 2\alpha + \lambda = \lambda' + \alpha'$, which implies $\dim(\mathfrak{g}_{\alpha'}) = \dim(\mathfrak{g}_{\lambda'+\alpha'})$. □

Recall also from Section 1.5 the Levi-Civita connection of M reads

$$4\langle \nabla_X Y, Z\rangle_{AN} = \langle [X,Y] + (1-\theta)[\theta X, Y], Z\rangle_{B_\theta}. \tag{5.1}$$

In this work, we are interested in a particular class of submanifolds of M. Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ and S the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. We will study the orbit $S \cdot o$, which by definition is a homogeneous submanifold of M. We can identify the tangent space $T_o(S\cdot o)$ with $\mathfrak{s}$ and the normal space $\nu_o(S\cdot o)$ with the orthogonal complement V of $\mathfrak{s}$ in $\mathfrak{a}\oplus\mathfrak{n}$. The shape operator $\mathcal{S}_\xi$ of $S\cdot o$ with respect to a unit normal vector $\xi \in V$ is given by

$$\mathcal{S}_\xi X = -\left(\nabla_X \xi\right)^\top, \tag{5.2}$$

where $X \in \mathfrak{s}$ and $(\cdot)^\top$ denotes the orthogonal projection onto the space $T_o(S\cdot o) \cong \mathfrak{s}$.

In order to simplify some arguments of this chapter, we state a result which will allow us to use the Levi-Civita connection more efficiently.

Lemma 5.1.2. *Let $\lambda \in \Sigma^+$ and $X, Y \in \mathfrak{g}_\lambda$ be orthogonal.*

(i) $[\theta X, X] = 2\langle X, X\rangle_{AN} H_\lambda = \langle X, X\rangle_{B_\theta} H_\lambda$.

(ii) $[\theta X, Y] \in \mathfrak{k}_0 = \mathfrak{g}_0 \ominus \mathfrak{a}$.

(iii) *If $2\lambda \notin \Sigma^+$, then*

$$\langle [\theta X, Y], [\theta X, Z]\rangle_{B_\theta} = 4|\lambda|^2 \langle X, X\rangle_{AN} \langle Y, Z\rangle_{AN}$$

for all $Z \in \mathfrak{g}_\lambda$ orthogonal to X.

(iv) *If $2\lambda \notin \Sigma^+$, then $\nabla_X Y = 0$.*

Proof. Firstly, we have $\theta[\theta X, X] = -[\theta X, X]$. Using the bracket relation in (1.3), the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the facts that $\theta|_\mathfrak{k} = \mathrm{id}_\mathfrak{k}$ and $\theta|_\mathfrak{p} = -\mathrm{id}_\mathfrak{p}$, we deduce that $[\theta X, X] \in \mathfrak{a} = \mathfrak{p} \cap \mathfrak{g}_0$. Now, using (1.1) and the definition of restricted root space, we obtain

$$\langle [\theta X, X], H_\lambda\rangle_{B_\theta} = \langle X, [H_\lambda, X]\rangle_{B_\theta} = |\lambda|^2 \langle X, X\rangle_{B_\theta} = 2|\lambda|^2 \langle X, X\rangle_{AN}.$$

A similar calculation shows that $\langle [\theta X, X], H\rangle_{B_\theta} = 0$ for all H orthogonal to H_λ. Then, we get $[\theta X, X] = 2\langle X, X\rangle_{AN} H_\lambda$, which proves (i).

For (ii), let $H \in \mathfrak{a}$. Clearly, $[\theta X, Y] \in \mathfrak{g}_0$ by (1.3). However, using again (1.1) and the definition of restricted root space, we obtain $\langle [\theta X, Y], H\rangle_{B_\theta} = \lambda(H)\langle Y, X\rangle_{B_\theta} = 0$, which implies $[\theta X, Y] \in \mathfrak{k}_0 = \mathfrak{g}_0 \ominus \mathfrak{a}$.

For (iii), let $Z \in \mathfrak{g}_\lambda$ be orthogonal to X. Then, using (1.1), the Jacobi identity, the assumption that $2\lambda \notin \Sigma^+$, (i), and the definition of restricted root space, we have

$$\begin{aligned}\langle [\theta X, Y], [\theta X, Z]\rangle_{B_\theta} &= -\langle Y, [X, [\theta X, Z]]\rangle_{B_\theta} = \langle Y, [Z, [X, \theta X]]\rangle_{B_\theta} \\ &= \langle Y, [[\theta X, X], Z]\rangle_{B_\theta} = 2|\lambda|^2 \langle X, X\rangle_{AN} \langle Y, Z\rangle_{B_\theta} \\ &= 4|\lambda|^2 \langle X, X\rangle_{AN} \langle Y, Z\rangle_{AN}.\end{aligned}$$

In order to prove (iv), we will use equation (5.1) directly. On the one hand, from (1.3), we obtain that the vectors $[\theta X, Y]$ and $[X, \theta Y]$ both belong to $\mathfrak{g}_0$. Since $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$, we deduce that they have trivial projections onto $\mathfrak{n}$. From (ii) we conclude they have also trivial projections onto $\mathfrak{a}$ and consequently onto $\mathfrak{a} \oplus \mathfrak{n}$. On the other hand, the element $[X, Y]$ vanishes because of (1.3) and the assumption that $2\lambda \notin \Sigma^+$. Then, we deduce

$$4\langle \nabla_X Y, Z\rangle_{AN} = \langle [X, Y] + [\theta X, Y] - [X, \theta Y], Z\rangle_{B_\theta} = 0$$

for all $Z \in \mathfrak{a} \oplus \mathfrak{n}$. This finishes the proof. □

The next result will be used later for calculating principal curvatures.

Lemma 5.1.3. *Let $\gamma \in \Sigma^+$ be the root of minimum level in its non-trivial ν-string, for $\nu \in \Sigma^+$ non-proportional to γ. Let $X \in \mathfrak{g}_\gamma$ and $\xi \in \mathfrak{g}_\nu$ with $\langle \xi, \xi\rangle_{AN} = 1$.*

(i) $\mathrm{ad}(\xi)|_{\mathfrak{g}_\gamma} \colon \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+\nu}$ *is an injective map preserving the inner product up to a positive constant.*

(ii) $[\theta\xi, [\xi, X]] = A_{\nu,\gamma}|\nu|^2 X$.

(iii) $[\theta\xi, [\xi, [\xi, X]]] = (A_{\nu,\gamma+\nu} + A_{\nu,\gamma})|\nu|^2[\xi, X]$.

(iv) *If* $A_{\nu,\gamma} \leq -2$, *then*

$$[\theta\xi, [\xi, [\xi, [\xi, X]]]] = (A_{\nu,\gamma+2\nu} + A_{\nu,\gamma+\nu} + A_{\nu,\gamma})|\nu|^2[\xi, [\xi, X]].$$

Proof. Since γ is the root of minimum level in its ν-string, we have $\gamma - \nu \notin \Sigma$. Since γ and ν are non-proportional, we have $\gamma - \nu \neq 0$. Altogether, we conclude $\gamma - \nu \notin \Sigma_0$.

(ii): Using the Jacobi identity, $\gamma - \nu \notin \Sigma_0$ and Lemma 5.1.2(i), we obtain $[\theta\xi, [\xi, X]] = -[X, [\theta\xi, \xi]] - [\xi, [X, \theta\xi]] = [[\theta\xi, \xi], X] = 2[H_\nu, X] = A_{\nu,\gamma}|\nu|^2 X$.

(i): Let $Y \in \mathfrak{g}_\gamma$. Combining (1.1) with (ii), we obtain

$$\begin{aligned}\langle \mathrm{ad}(\xi)X, \mathrm{ad}(\xi)Y\rangle_{AN} &= -\langle X, \mathrm{ad}(\theta\xi) \circ \mathrm{ad}(\xi)Y\rangle_{AN} = -\langle X, [\theta\xi, [\xi, Y]]\rangle_{AN} \\ &= -|\nu|^2 A_{\nu,\gamma}\langle X, Y\rangle_{AN}.\end{aligned}$$

Since the ν-string of γ is non-trivial, we have $A_{\nu,\gamma} < 0$ and the assertion follows.

(iii): Next, using the Jacobi identity, (ii) and Lemma 5.1.2(i), we deduce

$$\begin{aligned}[\theta\xi, [\xi, [\xi, X]]] &= -[[\xi, X], [\theta\xi, \xi]] - [\xi, [[\xi, X], \theta\xi]] \\ &= 2[H_\nu, [\xi, X]] + [\xi, [\theta\xi, [\xi, X]]] \\ &= (A_{\nu,\gamma+\nu} + A_{\nu,\gamma})|\nu|^2[\xi, X].\end{aligned}$$

(iv): Similar arguments as those used before, together with (iii), give

$$\begin{aligned}[\theta\xi, [\xi, [\xi, [\xi, X]]]] &= -[[\xi, [\xi, X]], [\theta\xi, \xi]] - [\xi, [[\xi, [\xi, X]], \theta\xi]] \\ &= 2[H_\nu, [\xi, [\xi, X]]] + [\xi, [\theta\xi, [\xi, [\xi, X]]]] \\ &= A_{\nu,\gamma+2\nu}|\nu|^2[\xi, [\xi, X]] + (A_{\nu,\gamma+\nu} + A_{\nu,\gamma})|\nu|^2[\xi, [\xi, X]] \\ &= (A_{\nu,\gamma+2\nu} + A_{\nu,\gamma+\nu} + A_{\nu,\gamma})|\nu|^2[\xi, [\xi, X]].\end{aligned}$$

□

5.2 Construction of CPC submanifolds

In this section we construct new examples of CPC submanifolds in the rank 2 non-compact symmetric spaces $SL_3(\mathbb{R})/SO_3$, $SL_3(\mathbb{C})/SU_3$, $SL_3(\mathbb{H})/Sp_3$ ($= SU_6^*/Sp_3$) and E_6^{-26}/F_4. These are precisely the non-compact symmetric spaces whose restricted root system is of type A_2. The new examples will provide the building blocks for further new examples in other non-compact symmetric spaces, via the so-called canonical extension method introduced in [18] and studied further in [47]. We emphasize that the CPC property is not preserved in general under the canonical extension method (an example will be given in the last paragraph of this section). A fundamental ingredient in our investigations will be a decomposition of the tangent space of a CPC submanifold into subspaces that are invariant under the shape operator.

Our construction is based on a suitable choice of a linear subspace V of the vector space $\bigoplus_{\alpha\in\Pi'}\mathfrak{g}_\alpha \subseteq \mathfrak{n}$. The nilpotent subalgebra $\mathfrak{n}$ has a natural gradation that is generated by $\bigoplus_{\alpha\in\Pi}\mathfrak{g}_\alpha$. Thus, if we remove a linear subspace V from $\bigoplus_{\alpha\in\Pi'}\mathfrak{g}_\alpha$, that is, consider the subspace $\mathfrak{n}\ominus V$, we get a subalgebra of $\mathfrak{n}$. We then define the subspace

$$\mathfrak{s} = \mathfrak{a}\oplus(\mathfrak{n}\ominus V)$$

of $\mathfrak{a}\oplus\mathfrak{n}$. Unfortunately, this subspace is in general not a subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$. Assume for the moment that $\mathfrak{s}$ is a subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$ and choose a vector $X\in\mathfrak{s}$ of the form $X=\sum_{\alpha\in\Pi'}X_\alpha$. Let $\beta\in\Pi'$ and $0\neq H\in\mathfrak{a}\ominus\left(\bigoplus_{\alpha\in\Pi\setminus\{\beta\}}\mathbb{R}H_\alpha\right)$. Since $\mathfrak{s}$ is a subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$ and $\mathfrak{a}\subset\mathfrak{s}$, we get $[H,X]=\sum_{\alpha\in\Pi'}[H,X_\alpha]=\sum_{\alpha\in\Pi'}\alpha(H)X_\alpha=\beta(H)X_\beta\in\mathfrak{s}$. Since $H\neq 0$ is orthogonal to the vector spaces $\mathbb{R}H_\alpha$ for all $\alpha\in\Pi\setminus\{\beta\}$, we must have $\beta(H)\neq 0$ and hence $X_\beta\in\mathfrak{s}$. Thus, if $\sum_{\alpha\in\Pi'}X_\alpha\in\mathfrak{s}$, we deduce that $X_\alpha\in\mathfrak{s}$ for all $\alpha\in\Pi'$. Consequently, if $\mathfrak{s}$ is a subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$, then V is of the form

$$V=\bigoplus_{\alpha\in\psi}V_\alpha \tag{5.3}$$

with $V_\alpha\subseteq\mathfrak{g}_\alpha$ and $\psi\subseteq\Pi'$. Without loss of generality, we can assume that $V_\alpha\neq\{0\}$ for each $\alpha\in\psi$.

We assume from now on that $\mathfrak{s}=\mathfrak{a}\oplus(\mathfrak{n}\ominus V)$ is a subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$ and that V is of the form (5.3). Let S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. The orbit $S\cdot o$ of S through o is a connected homogeneous submanifold of the symmetric space $M=G/K\cong AN$. We want to understand when this orbit is a CPC submanifold.

The simplest situation occurs when V is contained in a single root space $\mathfrak{g}_\alpha$, $\alpha\in\Pi'$. Let $m_\alpha=\dim(\mathfrak{g}_\alpha)$ and $k=m_\alpha-\dim(V)$. The orbit through o of the connected closed subgroup of AN with Lie algebra $\mathbb{R}H_\alpha\oplus\mathfrak{g}_\alpha$ is a real hyperbolic space $\mathbb{R}H^{m_\alpha+1}$, embedded in M as a totally geodesic submanifold. The orbit through o of the connected closed subgroup of AN with Lie algebra $\mathbb{R}H_\alpha\oplus(\mathfrak{g}_\alpha\ominus V)$ is a real hyperbolic space $\mathbb{R}H^{k+1}$, embedded in $\mathbb{R}H^{m_\alpha+1}$ as a totally geodesic submanifold. This $\mathbb{R}H^{k+1}$ is the singular orbit of a cohomogeneity one action on $\mathbb{R}H^{m_\alpha+1}$. This cohomogeneity one action admits a canonical extension to a cohomogeneity one action on M (see [18] for details). The singular orbit of this cohomogeneity one action on M, which is the canonical extension of $\mathbb{R}H^{k+1}$, then must be a CPC submanifold since the slice representation at any point of the singular orbit acts transitively on the unit sphere in the normal bundle. We can also give a slightly more complicated argument in this situation, which has the advantage though that we can apply it to more general situations. The generic orbits are homogeneous hypersurfaces, hence have the properties that they are both isoparametric and have constant principal curvatures. By applying the result by Ge and Tang [54] that we mentioned in Section 5.1, we can deduce that the canonical extension of the $\mathbb{R}H^{k+1}$ must be a CPC submanifold. It is this line of argument that we are going to apply for producing our new examples.

Back to the general situation. The orbit $S \cdot o$ is a homogeneous submanifold and therefore it suffices to study its shape operator $\mathcal{S}$ at the point o. We will now investigate the shape operator in Lie algebraic terms by using equation (5.1). In our situation we need to analyze the equation

$$4\langle \nabla_X \xi, Z\rangle_{AN} = \langle [X,\xi] + (1-\theta)[\theta X, \xi], Z\rangle_{B_\theta} \tag{5.4}$$

for unit normal vectors $\xi \in V$, tangent vectors $X \in \mathfrak{s}$ and all $Z \in \mathfrak{a} \oplus \mathfrak{n}$.

We start by choosing $X \in \mathfrak{a} \subset \mathfrak{p}$. Then $\theta X = -X$ and

$$[X,\xi] + [\theta X, \xi] - [X, \theta\xi] = -[X,\theta\xi] \in \bigoplus_{\alpha\in\psi} \mathfrak{g}_{-\alpha}.$$

Hence $[X,\theta\xi]$ has trivial projection onto $\mathfrak{a} \oplus \mathfrak{n}$. Therefore, $\nabla_X \xi = 0$ for all $X \in \mathfrak{a}$ and all normal vectors $\xi \in V$. In other words, for each unit normal vector ξ, 0 is a principal curvature of $S \cdot o$ with respect to ξ and $\mathfrak{a}$ is contained in the 0-eigenspace. This is also clear from a geometric viewpoint. The orbit $A \cdot o$ is a Euclidean space $\mathbb{R}^r$ of dimension $r = \mathrm{rk}(M)$ and embedded in M as a totally geodesic flat submanifold, a so-called maximal flat in M. Since $\mathfrak{a} \subset \mathfrak{s}$, we have $A \cdot o \subset S \cdot o$, and the assertion follows. In particular, the maximal flat $A \cdot o = \mathbb{R}^r$ is a totally geodesic submanifold of $S \cdot o$.

Therefore, we now need to examine the terms involved in (5.4) when $X \in \mathfrak{n} \ominus V$. On the one hand, since $X, \xi \in \mathfrak{n}$, we have $[X,\xi] \in \mathfrak{n}$ and hence $[X,\xi]$ has trivial projection onto $\mathfrak{a}$. On the other hand, we will see that the elements $[\theta X, \xi]$ and $[X, \theta\xi]$ involved in (5.4) have also trivial projections onto $\mathfrak{a}$. Moreover, we will justify that $[\theta X, \xi]$ must have trivial projection onto $\mathfrak{a} \oplus \mathfrak{n}$.

Let $X \in \mathfrak{n} \ominus V$ and decompose X into $X = \sum_{\lambda\in\Sigma^+} X_\lambda$ with $X_\lambda \in \mathfrak{g}_\lambda$. We decompose ξ into $\xi = \sum_{\alpha\in\psi} \xi_\alpha$ with $\xi_\alpha \in V_\alpha$. Let $\alpha \in \psi$ and $\beta \in \Pi$. We will analyze the elements $[\theta X_\beta, \xi_\alpha]$ and $[X_\beta, \theta\xi_\alpha]$. Since $\alpha, \beta \in \Pi$, we have $\pm(\alpha - \beta) \notin \Sigma$. Using (1.3) we deduce $[\theta X_\beta, \xi_\alpha] = 0 = [X_\beta, \theta\xi_\alpha]$ whenever $\alpha \neq \beta$. If $\beta = \alpha$, since $\langle X_\alpha, \xi_\alpha\rangle_{AN} = 0$ for all $\alpha \in \psi$ because of (5.3), we have $[\theta X_\alpha, \xi_\alpha] \in \mathfrak{k}_0$ and $[X_\alpha, \theta\xi_\alpha] \in \mathfrak{k}_0$ by Lemma 5.1.2(ii), and hence they have trivial projections onto $\mathfrak{a} \oplus \mathfrak{n}$. Thus we conclude that $[\theta X_\beta, \xi]$ and $[X_\beta, \theta\xi]$ have trivial projections onto $\mathfrak{a} \oplus \mathfrak{n}$. Let $\lambda \in \Sigma^+ \setminus \Pi$. Then $\alpha - \lambda \notin \Sigma_0^+$ and hence $[\theta X_\lambda, \xi_\alpha] \in \mathfrak{g}_{\alpha-\lambda}$ has trivial projection onto $\mathfrak{a} \oplus \mathfrak{n}$. Altogether this implies that $[\theta X, \xi]$ has trivial projection onto $\mathfrak{a} \oplus \mathfrak{n}$. We also see that $[X_\lambda, \theta\xi_\alpha] \in \mathfrak{g}_{\lambda-\alpha}$ has trivial projection onto $\mathfrak{g}_0$ and, since $\mathfrak{a} \subseteq \mathfrak{g}_0$, also onto $\mathfrak{a}$, which implies that $[X, \theta\xi]$ has trivial projection onto $\mathfrak{a}$. Then the Levi-Civita connection becomes

$$2\langle \nabla_X \xi, H + Y\rangle_{AN} = \langle [X,\xi] - [X,\theta\xi], Y\rangle_{AN}$$

for $H \in \mathfrak{a}$, $Y \in \mathfrak{n}$, $\xi \in V$ and $X \in \mathfrak{n} \ominus V$. We saw above that $[X,\theta\xi] \in \mathfrak{k}_0 \oplus \mathfrak{n}$. Moreover, $0 \neq [X,\theta\xi] \in \mathfrak{k}_0$ is possible only if there exists $\alpha \in \psi$ with $X_\alpha \neq 0 \neq \xi_\alpha$. In this situation, since X_α, ξ_α are orthogonal for each $\alpha \in \psi$, we have $\nabla_{X_\alpha}\xi_\alpha = 0$ by Lemma 5.1.2(iv). Otherwise, the above equation yields

$$2\nabla_X \xi = [X, \xi] - [X, \theta\xi] \tag{5.5}$$

for all $\xi \in V$ and $X \in \mathfrak{n} \ominus V$ with $[X_\alpha, \theta\xi_\alpha] = 0$ for all $\alpha \in \psi$. In particular, if $\xi \in \mathfrak{g}_\alpha$ and $X \in \mathfrak{n} \ominus V$, equations (5.2) and (5.5) imply that the shape operator $\mathcal{S}_\xi$ with respect to ξ of the submanifold $S \cdot o$ can be written as

$$2\mathcal{S}_\xi X = -\left([X, \xi] - [X, \theta\xi]\right)^\top = [(1-\theta)\xi, X]^\top. \tag{5.6}$$

Note that $\theta(\xi - \theta\xi) = -(\xi - \theta\xi)$ and hence $\frac{1}{2}(1-\theta)\xi \in \mathfrak{p}$ is the orthogonal projection of ξ onto $\mathfrak{p}$ with respect to B_θ.

Before considering the examples introduced in Theorem 5.0.2, we will study the behavior of the Levi-Civita connection in terms of the concept of string. Let $\gamma \in \Sigma^+$ be the root of minimum level in its non-trivial ν-string, for $\nu \in \Sigma^+$ non-proportional to γ. For each unit vector $\xi \in \mathfrak{g}_\nu$ we define

$$\phi_\xi = |\nu|^{-1}(-A_{\nu,\gamma})^{-1/2}\operatorname{ad}(\xi)\ ,\ \phi_{\theta\xi} = -|\nu|^{-1}(-A_{\nu,\gamma})^{-1/2}\operatorname{ad}(\theta\xi). \tag{5.7}$$

From Lemma 5.1.3(i),(ii) we easily deduce:

Lemma 5.2.1. *Let $\gamma \in \Sigma^+$ be the root of minimum level in its non-trivial ν-string, for $\nu \in \Sigma^+$ non-proportional to γ. Then:*

(i) *$\phi_\xi|_{\mathfrak{g}_\gamma} \colon \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+\nu}$ is a linear isometry onto $\phi_\xi(\mathfrak{g}_\gamma) = [\xi, \mathfrak{g}_\gamma]$.*

(ii) *$(\phi_{\theta\xi} \circ \phi_\xi)|_{\mathfrak{g}_\gamma} = \operatorname{id}_{\mathfrak{g}_\gamma}$.*

The next result will be useful for calculating principal curvatures explicitly.

Proposition 5.2.2. *Let $\gamma \in \Sigma^+$ be the root of minimum level in its ν-string, for $\nu \in \Sigma^+$ satisfying $A_{\nu,\gamma} = -1$, and $\xi \in \mathfrak{g}_\nu$ be a unit vector with respect to $\langle\cdot,\cdot\rangle_{AN}$. Then ϕ_ξ and $\phi_{\theta\xi}$ are inverse linear isometries when restricted to $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{\gamma+\nu}$, respectively. Moreover, for each $X \in \mathfrak{g}_\gamma$ we have*

$$\nabla_X \xi = -\frac{|\nu|}{2}\phi_\xi(X) \quad \textit{and} \quad \nabla_{\phi_\xi(X)}\xi = -\frac{|\nu|}{2}X.$$

Proof. From Lemma 5.1.1 we deduce $\dim(\mathfrak{g}_\gamma) = \dim(\mathfrak{g}_{\gamma+\nu})$. Lemma 5.2.1 then implies that $\phi_\xi|_{\mathfrak{g}_\gamma} \colon \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+\nu}$ is a linear isometry onto $\mathfrak{g}_{\gamma+\nu}$ and $(\phi_\xi|_{\mathfrak{g}_\gamma})^{-1} = \phi_{\theta\xi}|_{\mathfrak{g}_{\gamma+\nu}}$. Since γ is the root of minimum level in its ν-string and $A_{\nu,\gamma} = -1$, we have $\gamma - \nu \notin \Sigma_0$. Using (5.5), the fact that $\gamma - \nu \notin \Sigma_0$, and then (5.7), we deduce

$$2\nabla_X \xi = [X, \xi] - [X, \theta\xi] = [X, \xi] = -|\nu|\phi_\xi(X),$$

for unit vectors $\xi \in \mathfrak{g}_\nu$ and vectors $X \in \mathfrak{g}_\gamma$. Finally, using (5.5), the fact that $\gamma + 2\nu \notin \Sigma$, (5.7) and then Lemma 5.2.1, we obtain

$$\begin{aligned} 2\nabla_{\phi_\xi(X)}\xi =& [\phi_\xi(X), \xi] - [\phi_\xi(X), \theta\xi] = -[\phi_\xi(X), \theta\xi] \\ =& - |\nu|(\phi_{\theta\xi} \circ \phi_\xi)(X) = -|\nu|X, \end{aligned}$$

for a unit vector $\xi \in \mathfrak{g}_\nu$ and $X \in \mathfrak{g}_\gamma$. □

After these considerations we shall focus now on the examples introduced in Theorem 5.0.2. Consider a symmetric space G/K of non-compact type with at least two simple roots, say α_0 and α_1, that are connected by a single edge in its Dynkin diagram. Consider the subalgebra $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$ with $\psi = \{\alpha_0, \alpha_1\}$ and $V \subseteq \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$. Let $\xi \in V$ be a unit vector and $X \in \mathfrak{g}_\lambda^\top$, where $\mathfrak{g}_\lambda^\top$ denotes the orthogonal projection of $\mathfrak{g}_\lambda$ onto $\mathfrak{n} \ominus V$ for $\lambda \in \Sigma^+$. From (5.6) and (1.3) we obtain

$$\mathcal{S}_\xi X \in (\mathfrak{g}^\top_{\lambda+\alpha_0} \oplus \mathfrak{g}^\top_{\lambda+\alpha_1}) \oplus (\mathfrak{g}^\top_{\lambda-\alpha_0} \oplus \mathfrak{g}^\top_{\lambda-\alpha_1}).$$

This shows that we need to understand how the shape operator $\mathcal{S}$ relates the different root spaces of positive roots.

In order to clarify this situation, we introduce a generalization of the concept of α-string. For $\alpha_0, \alpha_1 \in \Sigma$ and $\lambda \in \Sigma_0$ we define the (α_0, α_1)-string containing λ as the set of all elements in Σ_0 of the form $\lambda + n\alpha_0 + m\alpha_1$ with $n, m \in \mathbb{Z}$. This leads to the following equivalence relation on Σ^+. We say that two roots $\lambda_1, \lambda_2 \in \Sigma^+$ are (α_0, α_1)-related if $\lambda_1 - \lambda_2 = n\alpha_0 + m\alpha_1$ for some $n, m \in \mathbb{Z}$. Therefore, the equivalence class $[\lambda]_{(\alpha_0,\alpha_1)}$ of the root $\lambda \in \Sigma^+$ consists of the elements which may be written as $\lambda + n\alpha_0 + m\alpha_1$ for some $n, m \in \mathbb{Z}$. We will write $[\lambda]$ for this equivalence class, taking into account that this class depends on the roots α_0 and α_1 defining the string. Put $\Sigma^+/\sim$ for the set of equivalence classes. The family $\{[\lambda]\}_{\lambda\in\Sigma^+}$ constitutes a partition of Σ^+.

Using this notation, we can now write

$$\mathcal{S}_\xi \left(\bigoplus_{\gamma\in[\lambda]} \mathfrak{g}^\top_\gamma \right) \subseteq \bigoplus_{\gamma\in[\lambda]} \mathfrak{g}^\top_\gamma \text{ for all } \lambda \in \Sigma^+. \tag{5.8}$$

In other words, for each $\lambda \in \Sigma^+$ the subspace $\bigoplus_{\gamma\in[\lambda]} \mathfrak{g}^\top_\gamma$ is an $\mathcal{S}_\xi$-invariant subspace of the tangent space $\mathfrak{s}$. Clearly, $S \cdot o$ is a CPC submanifold if and only if the eigenvalues of $\mathcal{S}_\xi$ are independent of the unit normal vector ξ when restricted to each of these invariant subspaces $\bigoplus_{\gamma\in[\lambda]} \mathfrak{g}^\top_\gamma$ for every $\lambda \in \Sigma^+$. Thus it suffices to consider the orthogonal decomposition

$$\mathfrak{n} \ominus V = \bigoplus_{\lambda\in\Sigma^+/\sim} \left(\bigoplus_{\gamma\in[\lambda]} \mathfrak{g}^\top_\gamma \right) \tag{5.9}$$

and to study the shape operator when restricted to each of these $\mathcal{S}_\xi$-invariant subspaces. These invariant subspaces will be determined more explicitly in Lemma 5.3.1 by using the concept of (α_0, α_1)-string of λ. However, note that one of them is very easy to determine. Since α_0 and α_1 are simple roots and connected by a single edge in the Dynkin diagram, the (α_0, α_1)-string of α_0 is just the set of roots of a rank 2 symmetric space of non-compact type whose Dynkin diagram is of type A_2. Therefore, studying the shape operator $\mathcal{S}_\xi$ when restricted to the $\mathcal{S}_\xi$-invariant subspace $\bigoplus_{\gamma\in[\alpha_0]} \mathfrak{g}_\gamma^\top$ is equivalent to studying the CPC property of the submanifold $S\cdot o$ in one of the symmetric spaces $SL_3(\mathbb{R})/SO_3$, $SL_3(\mathbb{C})/SU_3$, $SL_3(\mathbb{H})/Sp_3$ or E_6^{-26}/F_4. The remaining part of this section is devoted to the study of the shape operator of $S \cdot o$ when restricted to the vector space $\bigoplus_{\gamma\in[\alpha_0]} \mathfrak{g}_\gamma^\top$, or equivalently, to classifying CPC submanifolds in these rank 2 symmetric spaces under the hypotheses of Theorem 5.0.2.

We restrict now to the rank 2 symmetric spaces of non-compact type whose Dynkin diagram is of type A_2. In this case we have $\Sigma^+ = \{\alpha_0, \alpha_1, \alpha_0 + \alpha_1\}$ and $|\alpha_0| = |\alpha_1| = |\alpha_0 + \alpha_1| = \sqrt{2}$. From Lemma 5.1.1 we see that $\dim(\mathfrak{g}_{\alpha_0}) = \dim(\mathfrak{g}_{\alpha_1}) = \dim(\mathfrak{g}_{\alpha_0+\alpha_1})$. In line with the construction that we explained at the beginning of this section, we consider the subalgebra

$$\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_{\alpha_0} \ominus V_{\alpha_0}) \oplus (\mathfrak{g}_{\alpha_1} \ominus V_{\alpha_1}) \oplus \mathfrak{g}_{\alpha_0+\alpha_1}$$

with $V = V_{\alpha_0} \oplus V_{\alpha_1}$ and $\{0\} \neq V_{\alpha_k} \subseteq \mathfrak{g}_{\alpha_k}$, $k \in \{0, 1\}$. We put $V_k = V_{\alpha_k}$ and $T_k = \mathfrak{g}_{\alpha_k} \ominus V_k$. If U_1, U_2 are linear subspaces of $\mathfrak{g}$, we denote by $[U_1, U_2]$ the linear subspace of $\mathfrak{g}$ spanned by $\{[u_1, u_2] : u_1 \in U_1, u_2 \in U_2\}$. The following result will help us computing the shape operator of $S \cdot o$ explicitly.

Lemma 5.2.3. *Let $0 \neq \xi_k \in V_k$, $k \in \{0, 1\}$. Then*

$$\mathfrak{g}_{\alpha_0+\alpha_1} = \phi_{\xi_0}(V_1) \oplus \phi_{\xi_0}(T_1) = \phi_{\xi_1}(V_0) \oplus \phi_{\xi_1}(T_0) \tag{5.10}$$

are orthogonal decompositions of $\mathfrak{g}_{\alpha_0+\alpha_1}$. Moreover, if $\dim(V_0) = \dim(V_1) = \dim([V_0, V_1])$, *then:*

(i) $\phi_{\xi_0}(V_1) = \phi_{\xi_1}(V_0) = [V_0, V_1]$ *and*
$\phi_{\xi_0}(T_1) = \phi_{\xi_1}(T_0) = [V_0, T_1] = [V_1, T_0]$.

(ii) *If $T_k \neq \{0\}$, then* $\dim(T_k) \geq \dim(V_k)$.

(iii) *The maps $(\phi_{\theta\xi_0} \circ \phi_{\xi_1})|_{T_0} : T_0 \to T_1$ and $(\phi_{\theta\xi_1} \circ \phi_{\xi_0})|_{T_1} : T_1 \to T_0$ are linear isometries and*

$$\mathfrak{s} \ominus (\mathfrak{a} \oplus V) = T_0 \oplus T_1 \oplus [V_0, T_1] \oplus [V_0, V_1]$$

is an orthogonal decomposition of $\mathfrak{s} \ominus (\mathfrak{a} \oplus V)$.

Proof. According to Proposition 5.2.2, we deduce that the maps $\phi_{\xi_0}|_{\mathfrak{g}_{\alpha_1}} : \mathfrak{g}_{\alpha_1} \to \mathfrak{g}_{\alpha_0+\alpha_1}$ and $\phi_{\xi_1}|_{\mathfrak{g}_{\alpha_0}} : \mathfrak{g}_{\alpha_0} \to \mathfrak{g}_{\alpha_0+\alpha_1}$ are linear isometries. Since $\mathfrak{g}_{\alpha_k} = V_k \oplus T_k$ is an orthogonal decomposition by construction, we get (5.10).

Assume from now on that $\dim(V_0) = \dim(V_1) = \dim([V_0, V_1])$. As $\phi_{\xi_0}(V_1) \subseteq [V_0, V_1]$ and $\dim(\phi_{\xi_0}(V_1)) = \dim(V_1) = \dim([V_0, V_1])$, we get $\phi_{\xi_0}(V_1) = [V_0, V_1]$, and analogously, $\phi_{\xi_1}(V_0) = [V_0, V_1]$. From (5.10) we then obtain the other part of (i). From (i) we get

$\dim(T_0) = \dim([T_1, V_0])$. If $\dim(T_1) > 0$ we also get $\dim([T_1, V_0]) \geq \dim(V_0)$ from Proposition 5.2.2. Altogether this implies $\dim(T_0) \geq \dim(V_0)$ if $\dim(T_1) > 0$. Analogously, $\dim(T_1) \geq \dim(V_1)$ if $\dim(T_0) > 0$. Note that $\dim(T_0) = \dim(T_1)$. This proves (ii). Recall that $\phi_{\xi_0}(T_1) = \phi_{\xi_1}(T_0)$ is orthogonal to $\phi_{\xi_0}(V_1) = \phi_{\xi_1}(V_0)$. For $X_0 \in T_0$ and $\eta_1 \in V_1$ we have

$$\langle (\phi_{\theta\xi_0} \circ \phi_{\xi_1})(X_0), \eta_1 \rangle_{AN} = \langle \phi_{\xi_1}(X_0), \phi_{\xi_0}(\eta_1) \rangle_{AN} = 0.$$

Since $\dim(V_0) = \dim(V_1)$ and $\dim(T_0) = \dim(T_1)$ we get that $(\phi_{\theta\xi_0} \circ \phi_{\xi_1})|_{T_0} \colon T_0 \to T_1$ is a linear isometry. This implies (iii). □

The next result provides an algebraic characterization of the CPC property of the orbit $S \cdot o$.

Proposition 5.2.4. *Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ defined by*

$$\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_{\alpha_0} \ominus V_0) \oplus (\mathfrak{g}_{\alpha_1} \ominus V_1) \oplus \mathfrak{g}_{\alpha_0+\alpha_1}$$

and S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. Then the orbit $S \cdot o$ is a CPC submanifold of the symmetric space $G/K = AN$ if and only if $\dim(V_0) = \dim(V_1) = \dim([V_0, V_1])$. *Moreover, if $S \cdot o$ is a CPC submanifold, then its principal curvatures are $\pm\frac{1}{\sqrt{2}}$, both with multiplicity* $\dim(T_0)$, *and* 0 *with multiplicity* $\dim(\mathfrak{g}_{\alpha_0+\alpha_1}) + 2$.

Proof. Assume that the orbit $S \cdot o$ is a CPC submanifold. Let $j, k \in \{0, 1\}$ with $j \neq k$ and $\xi_j \in V_j$ be a unit vector. According to (5.10), the tangent space $\mathfrak{s}$ of $S \cdot o$ at o has the orthogonal decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus T_j \oplus T_k \oplus \phi_{\xi_j}(T_k) \oplus \phi_{\xi_j}(V_k).$$

We saw at the beginning of this section that $\mathcal{S}_{\xi_j}|_{\mathfrak{a}} = 0$. Using Lemma 5.1.2(iv) and Proposition 5.2.2, we get following expression for the shape operator $\mathcal{S}_{\xi_j}$:

$$\sqrt{2}\mathcal{S}_{\xi_j} X = \phi_{\xi_j}(X_{T_k}) + \phi_{\theta\xi_j}(X_{\phi_{\xi_j}(T_k)}),$$

where $X \in \mathfrak{s}$ is a tangent vector and the index to X denotes the orthogonal projection of X onto that space. In particular, $\dim(\ker(\mathcal{S}_{\xi_j})) = 2 + \dim(T_j) + \dim(V_k)$. Since $S \cdot o$ is a CPC submanifold, we have $\dim(\ker(\mathcal{S}_{\xi_j})) = \dim(\ker(\mathcal{S}_{\xi_k}))$ and thus $\dim(T_j) + \dim(V_k) = \dim(T_k) + \dim(V_j)$. On the other hand, we have $\dim(T_j) + \dim(V_j) = \dim(\mathfrak{g}_{\alpha_j}) = \dim(\mathfrak{g}_{\alpha_k}) = \dim(T_k) + \dim(V_k)$. From the previous two equations we easily get $\dim(V_j) = \dim(V_k)$, that is, $\dim(V_0) = \dim(V_1)$ (and then also $\dim(T_0) = \dim(T_1)$).

We now investigate the shape operator $\mathcal{S}_\xi$ with respect to the unit normal vector $\xi = \frac{1}{\sqrt{2}}(\xi_0 + \xi_1)$. Since $\mathcal{S}_\xi = \frac{1}{\sqrt{2}}(\mathcal{S}_{\xi_0} + \mathcal{S}_{\xi_1})$, we get

$$2\mathcal{S}_\xi X = \phi_{\xi_0}(X_{T_1}) + \phi_{\xi_1}(X_{T_0}) + \phi_{\theta\xi_0}(X_{\phi_{\xi_0}(T_1)}) + \phi_{\theta\xi_1}(X_{\phi_{\xi_1}(T_0)}).$$

Since all the ϕ-maps are linear isometries on the corresponding spaces (see Lemma 5.2.1), we obtain

$$\begin{aligned}\ker(\mathcal{S}_\xi) &= \mathfrak{a} \oplus \{X \in T_0 \oplus T_1 : \phi_{\xi_0}(X_{T_1}) + \phi_{\xi_1}(X_{T_0}) = 0\} \\ &\quad \oplus \{X \in \mathfrak{g}_{\alpha_0+\alpha_1} : X_{\phi_{\xi_0}(T_1)} = 0 = X_{\phi_{\xi_1}(T_0)}\} \\ &= \mathfrak{a} \oplus \{\phi_{\theta\xi_0} Y - \phi_{\theta\xi_1} Y \in T_0 \oplus T_1 : Y \in \phi_{\xi_0}(T_1) \cap \phi_{\xi_1}(T_0)\} \\ &\quad \oplus (\phi_{\xi_0}(V_1) \cap \phi_{\xi_1}(V_0)).\end{aligned}$$

Since $S \cdot o$ is a CPC submanifold, then $\dim(\ker(\mathcal{S}_{\xi_j})) = \dim(\ker(\mathcal{S}_\xi))$ and therefore

$$\dim(T_j) + \dim(V_k) = \dim(\phi_{\xi_0}(T_1) \cap \phi_{\xi_1}(T_0)) + \dim(\phi_{\xi_0}(V_1) \cap \phi_{\xi_1}(V_0)).$$

Again, since all the ϕ-maps are linear isometries on the corresponding spaces, this is possible only when $\phi_{\xi_0}(T_1) = \phi_{\xi_1}(T_0)$ and $\phi_{\xi_0}(V_1) = \phi_{\xi_1}(V_0)$. As $\xi_0 \in V_0$ and $\xi_1 \in V_1$ are arbitrary unit vectors, this implies in particular that $\dim([V_0, V_1]) = \dim(V_0) = \dim(V_1)$.

Conversely, assume that $\dim([V_0, V_1]) = \dim(V_0) = \dim(V_1)$. Let ξ be a unit normal vector of $S \cdot o$ at o. There exist unit vectors $\xi_0 \in V_0$, $\xi_1 \in V_1$ and $\varphi \in [0, \frac{\pi}{2}]$ so that $\xi = \cos(\varphi)\xi_0 + \sin(\varphi)\xi_1$. From Lemma 5.2.3 we have the orthogonal decomposition

$$\begin{aligned}\mathfrak{s} &= \mathfrak{a} \oplus T_0 \oplus (\phi_{\theta\xi_0} \circ \phi_{\xi_1})(T_0) \oplus \phi_{\xi_1}(T_0) \oplus [V_0, V_1] \\ &= \mathfrak{a} \oplus T_0 \oplus T_1 \oplus [V_1, T_0] \oplus [V_0, V_1]\end{aligned} \tag{5.11}$$

of the tangent space $\mathfrak{s}$ of $S \cdot o$ at o. As shown above, we have

$$\sqrt{2}\mathcal{S}_{\xi_j} X = \phi_{\xi_j}(X_{T_k}) + \phi_{\theta\xi_j}(X_{\phi_{\xi_j}(T_k)}).$$

This implies

$$\begin{aligned}\sqrt{2}\mathcal{S}_\xi X = &\cos(\varphi)(\phi_{\xi_0}(X_{T_1}) + \phi_{\theta\xi_0}(X_{\phi_{\xi_0}(T_1)})) \\ &+ \sin(\varphi)(\phi_{\xi_1}(X_{T_0}) + \phi_{\theta\xi_1}(X_{\phi_{\xi_1}(T_0)})).\end{aligned}$$

We immediately see that $\mathcal{S}_\xi$ vanishes on $\mathfrak{a} \oplus [V_0, V_1]$. Next, consider the vectors $0 \neq X \in T_0$, $\phi_{\xi_1}(X) \in [V_1, T_0] = [V_0, T_1]$ and $\phi_{\theta\xi_0}(\phi_{\xi_1}(X)) \in T_1$. The 3-dimensional subspace of $\mathfrak{s}$ spanned by $X, \phi_{\xi_1}(X), \phi_{\theta\xi_0}(\phi_{\xi_1}(X))$ is $\mathcal{S}_\xi$-invariant and the matrix representation of $\mathcal{S}_\xi$ with respect to the basis $X, \phi_{\xi_1}(X), \phi_{\theta\xi_0}(\phi_{\xi_1}(X))$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sin(\varphi) & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) \\ 0 & \cos(\varphi) & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are 0 and $\pm\frac{1}{\sqrt{2}}$. It follows that $S \cdot o$ is a CPC submanifold of AN. The statement about the principal curvatures and their multiplicities also follows from this calculation. □

The previous result implies that the codimension of a CPC submanifold is even. However, as we will see in the next result, there are further constraints on the codimension.

Corollary 5.2.5. *Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ defined by*

$$\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_{\alpha_0} \ominus V_0) \oplus (\mathfrak{g}_{\alpha_1} \ominus V_1) \oplus \mathfrak{g}_{\alpha_0+\alpha_1}$$

and S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. Assume that $S \cdot o$ is a CPC submanifold of $G/K = AN$.

(i) If $G/K = SL_3(\mathbb{R})/SO_3$, then $S \cdot o$ has codimension 2.

(ii) If $G/K = SL_3(\mathbb{C})/SU_3$, then $S \cdot o$ has codimension 2 or 4.

(iii) If $G/K = SL_3(\mathbb{H})/Sp_3$, then $S \cdot o$ has codimension 2, 4 or 8.

(iv) If $G/K = E_6^{-26}/F_4$, then $S \cdot o$ has codimension 2, 4, 8 or 16.

Proof. According to Proposition 5.2.4, for (i) and (ii) there is nothing to prove since the dimensions of the root spaces are 1 and 2 respectively. In the cases (iii) and (iv) the dimensions of the root spaces are 4 and 8 respectively, and therefore we need to exclude the possibility for codimension 6 in case (iii) and for codimensions 6, 10, 12 and 14 in case (iv). The codimensions 10, 12 and 14 in case (iv) cannot occur by Proposition 5.2.4 and Lemma 5.2.3(ii). It remains to investigate the possibility for codimension 6 in cases (iii) and (iv). In this situation we have $\dim(V_0) = \dim(V_1) = \dim([V_0, V_1]) = 3$.

Let η_1, η_2, η_3 be an orthonormal basis of V_1 and ξ_1 be a unit vector in V_0. The vector $\xi_2 = (\phi_{\theta\eta_2} \circ \phi_{\eta_3})(\xi_1)$ is non-zero by means of Proposition 5.2.2. On the one hand, using again Proposition 5.2.2, we obtain

$$\begin{aligned}\langle \xi_1, \xi_2 \rangle_{AN} &= \langle \xi_1, (\phi_{\theta\eta_2} \circ \phi_{\eta_3})(\xi_1) \rangle_{AN} = \langle \phi_{\eta_2}(\xi_1), \phi_{\eta_3}(\xi_1) \rangle_{AN} \\ &= \langle \phi_{\xi_1}(\eta_2), \phi_{\xi_1}(\eta_3) \rangle_{AN} = \langle \eta_2, \eta_3 \rangle_{AN} = 0.\end{aligned}$$

On the other hand, we have $\phi_{\eta_2}(\xi_2) = (\phi_{\eta_2} \circ \phi_{\theta\eta_2} \circ \phi_{\eta_3})(\xi_1) = \phi_{\eta_3}(\xi_1)$. From Lemma 5.1.2(ii) we have $[\eta_3, \theta\eta_2] \in \mathfrak{k}_0 \subseteq \mathfrak{k}$. Since $\theta|_{\mathfrak{k}} = \mathrm{id}_{\mathfrak{k}}$ we have $[\eta_3, \theta\eta_2] = [\theta\eta_3, \eta_2]$. Using this and the Jacobi identity we get

$$\phi_{\eta_3}(\xi_2) = (\phi_{\eta_3} \circ \phi_{\theta\eta_2} \circ \phi_{\eta_3})(\xi_1) = -(\phi_{\eta_2} \circ \phi_{\theta\eta_3} \circ \phi_{\eta_3})(\xi_1) = -\phi_{\eta_2}(\xi_1).$$

To sum up, having in mind definition (5.7), we have shown that $\phi_{\xi_2}(\eta_2) = \phi_{\xi_1}(\eta_3)$ and $\phi_{\xi_2}(\eta_3) = -\phi_{\xi_1}(\eta_2)$. Since $\phi_{\xi_1}(V_1)$ and $\phi_{\xi_2}(V_1)$ must be the same vector space by Proposition 5.2.4 and Lemma 5.2.3(i), we conclude that $\phi_{\xi_1}(\eta_1)$ is either $\phi_{\xi_2}(\eta_1)$ or $-\phi_{\xi_2}(\eta_1)$, which implies that $\phi_{\eta_1}(\xi_1)$ is either $\phi_{\eta_1}(\xi_2)$ or $-\phi_{\eta_1}(\xi_2)$. Since $\langle \xi_1, \xi_2 \rangle_{AN} = 0$, this contradicts the injectivity of ϕ_{η_1} (see Proposition 5.2.2). This concludes the proof. □

We want to derive a more geometric characterization of the CPC property. For this, we first prove an auxiliary result.

Lemma 5.2.6. *Let $X, Y \in \mathfrak{g}_{\alpha_0+\alpha_1}$ be orthonormal (and then $G/K \neq SL_3(\mathbb{R})/SO_3$). Then:*

(i) *The linear map $\frac{1}{4}\operatorname{ad}([\theta X, Y])$ defines a complex structure on the vector space $\mathbb{R}X \oplus \mathbb{R}Y$ spanned by X and Y.*

(ii) *The linear map $\frac{1}{2}\operatorname{ad}([\theta X, Y])$ defines complex structures on the vector spaces $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{\alpha_1}$.*

(iii) *Let $X, Y, Z \in \mathfrak{g}_{\alpha_0+\alpha_1}$ be orthonormal (then G/K is either $SL_3(\mathbb{H})/Sp_3$ or E_6^{-26}/F_4). Define $J_1 = \frac{1}{2}\operatorname{ad}([\theta X, Y])$, $J_2 = \frac{1}{2}\operatorname{ad}([\theta X, Z])$ and $J_3 = J_1 \circ J_2$. Then $\{J_1, J_2, J_3\}$ defines quaternionic structures on the vector spaces $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{\alpha_1}$.*

Proof. (i): First, using the Jacobi identity, $2(\alpha_0 + \alpha_1) \notin \Sigma$ and Lemma 5.1.2(i), we obtain

$$[[\theta X, Y], X] = -[[X, \theta X], Y] = [[\theta X, X], Y] = 2|\alpha_0 + \alpha_1|^2 Y = 4Y. \tag{5.12}$$

According to Lemma 5.1.2(ii) we have $[\theta X, Y] \in \mathfrak{k}_0 \subseteq \mathfrak{k}$. Since $\theta|_{\mathfrak{k}} = \operatorname{id}_{\mathfrak{k}}$ we have $[\theta X, Y] = [X, \theta Y]$. Together with (5.12), we deduce $[[\theta X, Y], Y] = [[X, \theta Y], Y] = -[[\theta Y, X], Y] = -4X$. Thus we have $(\frac{1}{4}\operatorname{ad}([\theta X, Y]))^2 = -\operatorname{id}$ on $\mathbb{R}X \oplus \mathbb{R}Y$.

(ii): Let $W \in \mathfrak{g}_{\alpha_k}$ for $k \in \{0, 1\}$. Using the Jacobi identity, the equations (1.3), (5.12) and $[\theta X, Y] = [X, \theta Y]$, and Lemma 5.1.2(i), we obtain

$$\begin{aligned}
[[\theta X, Y], [[\theta X, Y], W]] &= -[[[[\theta X, Y], W], \theta X], Y] \\
&= [[[W, \theta X], [\theta X, Y]], Y] + [[[\theta X, [\theta X, Y]], W], Y] \\
&= [[[W, \theta X], [X, \theta Y]], Y] - [[\theta[[\theta X, Y], X], W], Y] \\
&= -[[\theta Y, [[W, \theta X], X]], Y] - 4[[\theta Y, W], Y] \\
&= [[\theta Y, [[\theta X, X], W]], Y] - 4[[\theta Y, Y], W] \\
&= 2[[\theta Y, [H_{\alpha_0+\alpha_1}, W]], Y] - 8[H_{\alpha_0+\alpha_1}, W] \\
&= 2[[\theta Y, W], Y] - 8W = 2[[\theta Y, Y], W] - 8W \\
&= 4[H_{\alpha_0+\alpha_1}, W] - 8W = 4W - 8W = -4W.
\end{aligned}$$

(iii): With analogous arguments as above, we obtain

$$\begin{aligned}
[[\theta X, Y], [[\theta X, Z], W]] &= -[[[[\theta X, Z], W], \theta X], Y] \\
&= [[[W, \theta X], [\theta X, Z]], Y] + [[[\theta X, [\theta X, Z]], W], Y] \\
&= [[[W, \theta X], [X, \theta Z]], Y] - [[\theta[[\theta X, Z], X], W], Y] \\
&= -[[\theta Z, [[W, \theta X], X]], Y] - 4[[\theta Z, W], Y] \\
&= [[\theta Z, [[\theta X, X], W]], Y] - 4[[\theta Z, Y], W] \\
&= 2[[\theta Z, [H_{\alpha_0+\alpha_1}, W]], Y] + 4[[\theta Y, Z], W] \\
&= 2[[\theta Z, W], Y] + 4[[\theta Y, Z], W] \\
&= -2[[\theta Y, Z], W] + 4[[\theta Y, Z], W] = 2[[\theta Y, Z], W].
\end{aligned}$$

Using the previous equality and $[\theta Y, Z] = [Y, \theta Z]$, we deduce

$$\begin{aligned}[][\theta X, Y], [[\theta X, Z], W]] &= 2[[\theta Y, Z], W] = -2[[\theta Z, Y], W] \\ &= -[[\theta X, Z], [[\theta X, Y], W]].\end{aligned}$$

Now define $J_1 = \frac{1}{2}\operatorname{ad}([\theta X, Y])$ and $J_2 = \frac{1}{2}\operatorname{ad}([\theta X, Z])$. We just proved $(J_1 \circ J_2)|_{\mathfrak{g}_{\alpha_k}} = -(J_2 \circ J_1)|_{\mathfrak{g}_{\alpha_k}}$. Hence, using (ii) and defining $J_3 = J_1 \circ J_2$, the result follows. □

Remark 5.2.7. We state here a generalization of Lemma 5.2.6 to arbitrary symmetric spaces of non-compact type. Assume that $\lambda \in \Sigma^+$ with $2\lambda \notin \Sigma^+$. Then every 2-dimensional subspace $\mathbb{R}X \oplus \mathbb{R}Y$ of $\mathfrak{g}_\lambda$, with $X, Y \in \mathfrak{g}_\lambda$ orthonormal, can be viewed as a complex vector space with complex structure $\frac{1}{2|\lambda|^2}\operatorname{ad}([\theta X, Y])$. Furthermore, each 4-dimensional subspace of $\mathfrak{g}_\lambda$ can be described as a quaternionic subspace. Choose $X, Y, Z \in \mathfrak{g}_\lambda$ orthonormal. First, using $\theta|_{\mathfrak{k}} = \mathrm{id}_{\mathfrak{k}}$ and the Jacobi identity, we deduce

$$\begin{aligned}[][\theta X, Y], Z] &= [[X, \theta Y], Z] = -[[\theta Y, Z], X] = -[[Y, \theta Z], X] \\ &= [[\theta Z, X], Y] = [[Z, \theta X], Y] = -[[\theta X, Y], Z],\end{aligned} \tag{5.13}$$

which implies $[[\theta X, Y], Z] = 0$. Let W be a 4-dimensional subspace of $\mathfrak{g}_\lambda$ and $X, Y, Z, T \in W$ be orthonormal. Then J_1, J_2, J_3 with

$$\begin{aligned} J_1 &= \frac{1}{2|\lambda|^2}(\operatorname{ad}([\theta X, Y] + \operatorname{ad}([\theta Z, T])), \\ J_2 &= \frac{1}{2|\lambda|^2}(\operatorname{ad}([\theta X, Z] - \operatorname{ad}([\theta Y, T])), \\ J_3 &= J_1 \circ J_2 \end{aligned}$$

is a quaternionic structure on W.

If we think about our symmetric spaces of type A_2 in terms of matrices, we have canonical real, complex, quaternionic or octonionic structures on the root spaces. More precisely, the Iwasawa decomposition $G = KAN$ gives

$$G/K = AN = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} : \begin{matrix} x_{11}, x_{22}, x_{33} \in \mathbb{R}; \\ x_{12}, x_{13}, x_{23} \in \mathbb{F}; \\ x_{11}x_{22}x_{33} = 1 \end{matrix} \right\}$$

with

$$\mathbb{F} = \begin{cases} \mathbb{R} & \text{if } G/K = SL_3(\mathbb{R})/SO_3, \\ \mathbb{C} & \text{if } G/K = SL_3(\mathbb{C})/SU_3, \\ \mathbb{H} & \text{if } G/K = SL_3(\mathbb{H})/Sp_3, \\ \mathbb{O} & \text{if } G/K = E_6^{-26}/F_4. \end{cases}$$

The x_{12}- and x_{23}-entries correspond (on Lie algebra level) to the root spaces $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{\alpha_1}$ respectively, and the x_{13}-entry corresponds to the root space $\mathfrak{g}_{\alpha_0+\alpha_1}$. The standard examples of CPC submanifolds in these symmetric spaces are given by

$$\left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} : \begin{array}{l} x_{11}, x_{22}, x_{33} \in \mathbb{R}; \\ x_{12}, x_{23} \in \mathbb{F} \ominus \mathbb{F}_0; \\ x_{13} \in \mathbb{F}; x_{11}x_{22}x_{33} = 1 \end{array} \right\}$$

with $\mathbb{F}_0 \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $\mathbb{F}_0 \subseteq \mathbb{F}$. If $\mathbb{F}_0 = \mathbb{F}$, we get the totally geodesic submanifolds

$$\begin{aligned} &\mathbb{R}H^2 \times \mathbb{R} \subset SL_3(\mathbb{R})/SO_3, \\ &\mathbb{R}H^3 \times \mathbb{R} \subset SL_3(\mathbb{C})/SU_3, \\ &\mathbb{R}H^5 \times \mathbb{R} \subset SL_3(\mathbb{H})/Sp_3, \\ &\mathbb{R}H^9 \times \mathbb{R} \subset E_6^{-26}/F_4. \end{aligned}$$

In all other cases the submanifold is not totally geodesic. The following result makes this more precise.

Theorem 5.2.8. *Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ defined by*

$$\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_{\alpha_0} \ominus V_0) \oplus (\mathfrak{g}_{\alpha_1} \ominus V_1) \oplus \mathfrak{g}_{\alpha_0+\alpha_1},$$

$V_0, V_1 \neq \{0\}$, and S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. Then $S \cdot o$ is a CPC submanifold if and only if one of the following statements holds:

(i) $V_0 \oplus V_1 = \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$*; or*

(ii) $V_0 \oplus V_1$ *is a proper subset of* $\mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$ *and*

- (a) V_0 *and* V_1 *are isomorphic to* $\mathbb{R}$*; or*
- (b) V_0 *and* V_1 *are isomorphic to* $\mathbb{C}$ *and there exists* $T \in \mathfrak{k}_0$ *such that* $\mathrm{ad}(T)$ *defines complex structures on* V_0 *and* V_1 *and vanishes on* $[V_0, V_1]$*; or*
- (c) V_0 *and* V_1 *are isomorphic to* $\mathbb{H}$ *and there exists a subset* $\mathfrak{l} \subseteq \mathfrak{k}_0$ *such that* $\mathrm{ad}(\mathfrak{l})$ *defines quaternionic structures on* V_0 *and* V_1 *and vanishes on* $[V_0, V_1]$.

Proof. Assume that $S \cdot o$ is a CPC submanifold. From Proposition 5.2.4 we have $\dim(V_0) = \dim(V_1) = \dim([V_0, V_1])$. Recall that $T_k = \mathfrak{g}_{\alpha_k} \ominus V_k$, $k \in \{0, 1\}$, and hence $\dim(T_0) = \dim(T_1)$. If $\dim(T_0) \leq 1$, we have (i) or (iia). Assume that $\dim(T_0) \geq 2$. From Lemma 5.2.3 we get $[V_0, T_1] = [V_1, T_0]$ and $\dim([V_0, T_1]) = \dim(T_0) \geq 2$. Note that $[V_0, T_1] \subseteq \mathfrak{g}_{\alpha_0+\alpha_1}$. Thus, using elements in $[V_0, T_1]$, we can construct complex structures (following Lemma 5.2.6(ii) if $\dim(T_0) = 2$) or quaternionic structures (following Lemma 5.2.6(iii) if $\dim(T_0) > 2$) on $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{\alpha_1}$. From (5.13) we deduce that these structures vanish on $[V_0, V_1]$. Thus it remains to check that these structures can be restricted to V_0 and V_1. In other words, we need to verify that $\langle [[\theta X, Y], \xi_k], Z_k \rangle_{AN} = 0$ for $X, Y \in [V_0, T_1] = [V_1, T_0]$, $\xi_k \in V_k$ and $Z_k \in T_k$. Let $j \in \{0, 1\}$ with $j \neq k$. There exist $L_j \in T_j$ and $\eta_j \in V_j$ so that $X = \phi_{\xi_k}(L_j)$

and $Y = \phi_{Z_k}(\eta_j)$. Then, using the Jacobi identity, the fact that $\langle\cdot,\cdot\rangle_{B_\theta}$ is θ-invariant, (1.1) and Proposition 5.2.2, we obtain

$$\begin{aligned}\langle[[\theta X, Y], \xi_k], Z_k\rangle_{B_\theta} &= -\langle[[\xi_k, \theta X], Y], Z_k\rangle_{B_\theta}\\ &= \langle[\xi_k, \theta X], [Z_k, \theta Y]\rangle_{B_\theta} = \langle[\theta\xi_k, X], [\theta Z_k, Y]\rangle_{B_\theta}\\ &= 2\langle\phi_{\theta\xi_k}\circ\phi_{\xi_k}(L_j), \phi_{\theta Z_k}\circ\phi_{Z_k}(\eta_j)\rangle_{B_\theta} = 2\langle L_j, \eta_j\rangle_{B_\theta} = 0,\end{aligned}$$

which implies that (iib) or (iic) holds.

Conversely, if (i) or (iia) holds, then $S\cdot o$ is a CPC submanifold by Proposition 5.2.4. For case (iib), put $J = \mathrm{ad}(K)$ with $K \in \mathfrak{k}_0$. By assumption, we can write $V_k = \mathbb{R}X_k \oplus \mathbb{R}JX_k$ with $0 \neq X_k \in V_k$. Then $[V_0, V_1]$ is spanned by $[X_0, X_1]$, $[JX_0, JX_1]$, $[X_0, JX_1]$, $[JX_0, X_1]$. Since $J = \mathrm{ad}(K)$ is a derivation and vanishes on $[V_0, V_1]$, we have

$$\begin{aligned}0 &= J[X_0, X_1] = [JX_0, X_1] + [X_0, JX_1],\\ 0 &= J^2[X_0, X_1] = [J^2X_0, X_1] + 2[JX_0, JX_1] + [X_0, J^2X_1]\\ &= 2([JX_0, JX_1] - [X_0, X_1]),\end{aligned}$$

which implies $\dim([V_0, V_1]) = 2$. Thus $S\cdot o$ is a CPC submanifold by Proposition 5.2.4. In case (iic) we can write $J_\nu = \mathrm{ad}(K_\nu)$, $K_\nu \in \mathfrak{k}_0$, $\nu = 1, 2, 3$, for the quaternionic structure. Then V_k is spanned by X_k, J_1X_k, J_2X_k, J_3X_k with $0 \neq X_k \in V_k$. As above, we get $[J_\nu X_0, X_1] = -[X_0, J_\nu X_1]$ and $[J_\nu X_0, J_\nu X_1] = [X_0, X_1]$. For $\nu, \mu \in \{1, 2, 3\}$ with $\nu \neq \mu$ we have $J_\nu J_\mu = \pm J_\rho$ and hence $[J_\nu X_0, J_\mu X_1] = [J_\nu^2 X_0, J_\nu J_\mu X_1] = \pm[X_0, J_\rho X_1]$. Altogether this implies $\dim([V_0, V_1]) = 4$ and from Proposition 5.2.4 we conclude that $S\cdot o$ is a CPC submanifold. □

This finishes the proof of Theorem 5.0.2 for the symmetric spaces of non-compact type $SL_3(\mathbb{R})/SO_3$, $SL_3(\mathbb{C})/SU_3$, $SL_3(\mathbb{H})/Sp_3$ and E_6^{-26}/F_4. Recall that this is equivalent to characterize the CPC property of the shape operator $\mathcal{S}_\xi$ of the examples we constructed in a general symmetric space G/K, when it is restricted to the $\mathcal{S}_\xi$-invariant subspace $\bigoplus_{\gamma\in[\alpha_0]}\mathfrak{g}_\gamma^\top$.

As we mentioned at the beginning of this section, all the examples of Theorem 5.0.2 can be described as canonical extensions of CPC submanifolds in the above four symmetric spaces. As was shown in [47], several geometric properties of submanifolds are preserved via canonical extensions. However, the CPC property is not preserved in general by canonical extension. For example, the maximal flat $A\cdot o \cong \mathbb{R}^2$ is a totally geodesic submanifold of $SL_3(\mathbb{R})/SO_3$. However, its canonical extension to the symmetric space $SL_4(\mathbb{R})/SO_4$ is not even austere. For this reason we need to analyze more thoroughly the shape operator of the examples described in Theorem 5.0.2.

5.3 Canonical extensions of CPC submanifolds

In this section we calculate the shape operator of the canonical extensions of the examples that we investigated in the previous section. We will conclude that these canonical extensions are also CPC submanifolds.

The concept of canonical extension was introduced in [18] and studied in the context of cohomogeneity one actions. We refer the reader to [18] for details, but roughly it works as follows. Every subset Φ of Π determines a parabolic subgroup Q_Φ of G. Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be its Langlands decomposition (see Section 6.1). The orbit $B_\Phi = M_\Phi \cdot o$ is a totally geodesic submanifold of M whose rank is equal to the cardinality of Φ. If S is a subgroup of M_Φ, then $SA_\Phi N_\Phi$ is the canonical extension of S from M_Φ to G and the orbit $SA_\Phi N_\Phi \cdot o \subseteq M$ is the canonical extension of the orbit $S \cdot o \subseteq B_\Phi$. If there exist $\alpha_0, \alpha_1 \in \Pi$ so that α_0 and α_1 are connected in the Dynkin diagram of $M = G/K$ by a single edge, and put $\Phi = \{\alpha_0, \alpha_1\}$, then B_Φ is one of the symmetric spaces $SL_3(\mathbb{R})/SO_3$, $SL_3(\mathbb{C})/SU_3$, $SL_3(\mathbb{H})/Sp_3$ or E_6^{-26}/F_4. In Theorem 5.2.8 we classified the CPC submanifolds of B_Φ of the form $S \cdot o$, where $\mathfrak{s} = \mathfrak{a} \oplus ((\mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}) \ominus V) \oplus \mathfrak{g}_{\alpha_0+\alpha_1}$. In this section we will prove that the canonical extension of $S \cdot o \subset B_\Phi$ to the symmetric space $M = G/K$ is a CPC submanifold if and only if $S \cdot o$ is a CPC submanifold of B_Φ.

Let G/K be a symmetric space of non-compact type, with at least two simple roots α_0 and α_1 connected by a single edge in its Dynkin diagram. Our approach for constructing new examples was to take a subspace $V \subset \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$ and define the subalgebra $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$. Let S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. We are interested in the geometry of the submanifold $S \cdot o$ of $AN \cong G/K$.

Let $\xi \in V$ be a unit normal vector. As we clarified in Section 5.2, the subspaces $\bigoplus_{\gamma\in[\lambda]} \mathfrak{g}_\gamma^\top$ in the orthogonal decomposition

$$\mathfrak{n} \ominus V = \bigoplus_{\lambda\in\Sigma^+/\sim} \left(\bigoplus_{\gamma\in[\lambda]} \mathfrak{g}_\gamma^\top \right)$$

are all $\mathcal{S}_\xi$-invariant. Therefore, $S \cdot o$ is a CPC submanifold of M if and only if for all unit normal vectors ξ the shape operator $\mathcal{S}_\xi$ has the same eigenvalues when restricted to each of these subspaces. We clarified this in Theorem 5.2.8 for the invariant subspace $\bigoplus_{\gamma\in[\alpha_0]} \mathfrak{g}_\gamma^\top$. In this section we will clarify this for the remaining subspaces in the above decomposition. The following result explains the above decomposition in more detail.

Lemma 5.3.1. *Let Σ be the root system of a symmetric space of non-compact type with at least two simple roots α_0 and α_1 connected by a single edge in the Dynkin diagram. Then the equivalence class $[\lambda]$ of a positive root $\lambda \in \Sigma^+ \backslash (\mathbb{R}\alpha_0 \oplus \mathbb{R}\alpha_1)$, which has minimum level in its (α_0, α_1)-string, can be described as follows (with $k \in \{0, 1\}$ and indices modulo 2):*

(i) $[\lambda] = \{\lambda\}$, *if* $\langle \lambda, \alpha_0 \rangle = 0 = \langle \lambda, \alpha_1 \rangle$.

(ii) $[\lambda] = \{\lambda, \lambda + \alpha_k, \lambda + \alpha_k + \alpha_{k+1}\}$, *if* $|\alpha_k| \geq |\lambda|$ *and* $\langle \lambda, \alpha_k \rangle \neq 0$.

(iii) $[\lambda] = \{\lambda, \lambda + \alpha_k, \lambda + \alpha_k + \alpha_{k+1}, \lambda + 2\alpha_k, \lambda + 2\alpha_k + \alpha_{k+1}, \lambda + 2\alpha_k + 2\alpha_{k+1}\}$, *if* $|\alpha_k| < |\lambda|$ *and* $\langle \lambda, \alpha_k \rangle \neq 0$.

Proof. Since $\lambda, \alpha_0, \alpha_1$ are linearly independent, they generate a root system $R \subseteq \Sigma$ of rank 3. We can assume that $\lambda, \alpha_0, \alpha_1$ are positive roots in R.

If R is reducible, we must have $R \cong A_2 \oplus A_1$ with A_2 generated by α_0 and α_1 and A_1 generated by λ. It is clear that this is equivalent to $[\lambda] = \{\lambda\}$ and $\langle \lambda, \alpha_k \rangle = 0$ for $k \in \{0, 1\}$, which corresponds to case (i).

If R is irreducible, then R is isomorphic to A_3, B_3, C_3 or BC_3. The result follows by inspecting these rank 3 root systems case by case and taking into account that λ has minimum level in its (α_0, α_1)-string. If $R \cong A_3$ or $R \cong B_3$, we get (ii). If $R \cong C_3$, we get (iii). Finally, if $R \cong BC_3$, then λ is either reduced or non-reduced. If λ is reduced, we get (ii), and if λ is non-reduced, we get (iii). □

In view of Lemma 5.3.1 we have to investigate three cases.

Case (i): $[\lambda] = \{\lambda\}$. From (5.6) and (1.3) we see that $\mathcal{S}_\xi$ vanishes on $\mathfrak{g}_\lambda = \mathfrak{g}_\lambda^\top$.

Case (ii): $[\lambda] = \{\lambda, \lambda + \alpha_k, \lambda + \alpha_k + \alpha_{k+1}\}$. We consider the subspace

$$\mathfrak{g}_\lambda \oplus \mathfrak{g}_{\lambda+\alpha_k} \oplus \mathfrak{g}_{\lambda+\alpha_k+\alpha_{k+1}} \subseteq \mathfrak{s}.$$

We write $\xi = \cos(\varphi)\xi_k + \sin(\varphi)\xi_{k+1}$ with $\varphi \in [0, \frac{\pi}{2}]$, $\xi_k \in V_k$ and $\xi_{k+1} \in V_{k+1}$. Note that $A_{\alpha_k,\lambda} = -1$ and $A_{\alpha_{k+1},\lambda+\alpha_k} = -1$. For the pairs $(\gamma, \nu) = (\lambda, \alpha_k)$ and $(\gamma, \nu) = (\lambda+\alpha_k, \alpha_{k+1})$ we obtain from Proposition 5.2.2 that $\mathfrak{g}_{\lambda+\alpha_k} = \phi_{\xi_k}(\mathfrak{g}_\lambda)$ and $\mathfrak{g}_{\lambda+\alpha_k+\alpha_{k+1}} = (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(\mathfrak{g}_\lambda)$. Let $0 \neq X_\lambda \in \mathfrak{g}_\lambda$. From (5.6) and (1.3), together with the fact that $\lambda + \alpha_{k+1} \notin \Sigma$, we get

$$\mathcal{S}_{\xi_{k+1}} X_\lambda = \mathcal{S}_{\xi_k}(\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda) = 0.$$

For the pair $(\gamma, \nu) \in \{(\lambda, \alpha_k), (\lambda + \alpha_k, \alpha_{k+1})\}$, we deduce from (5.2) and Proposition 5.2.2 that

$$\begin{aligned}
\mathcal{S}_{\xi_k} X_\lambda &= -\left(\nabla_{X_\lambda}\xi_k\right)^\top = \frac{|\alpha_0|}{2}\phi_{\xi_k}(X_\lambda),\\
\mathcal{S}_{\xi_k}\phi_{\xi_k}(X_\lambda) &= -\left(\nabla_{\phi_{\xi_k}(X_\lambda)}\xi_k\right)^\top = \frac{|\alpha_0|}{2}X_\lambda,\\
\mathcal{S}_{\xi_{k+1}}\phi_{\xi_k}(X_\lambda) &= -\left(\nabla_{\phi_{\xi_k}(X_\lambda)}\xi_{k+1}\right)^\top = \frac{|\alpha_0|}{2}(\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda),\\
\mathcal{S}_{\xi_{k+1}}(\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda) &= -\left(\nabla_{(\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda)}\xi_{k+1}\right)^\top = \frac{|\alpha_0|}{2}\phi_{\xi_k}(X_\lambda).
\end{aligned}$$

Thus, the 3-dimensional vector space spanned by the vectors X, $\phi_{\xi_k}(X)$ and $(\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X)$ is $\mathcal{S}_\xi$-invariant. It follows that the matrix representation of $\mathcal{S}_\xi$ is given by $\dim(\mathfrak{g}_\lambda)$ blocks

$$\frac{|\alpha_0|}{2}\begin{pmatrix} 0 & \cos(\varphi) & 0 \\ \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & \sin(\varphi) & 0 \end{pmatrix}$$

with respect to the decomposition $\mathfrak{g}_\lambda \oplus \phi_{\xi_k}(\mathfrak{g}_\lambda) \oplus (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(\mathfrak{g}_\lambda)$. An elementary calculation shows that $\mathcal{S}_\xi$ restricted to $\mathfrak{g}_\lambda \oplus \mathfrak{g}_{\lambda+\alpha_k} \oplus \mathfrak{g}_{\lambda+\alpha_k+\alpha_{k+1}}$ has the three eigenvalues 0 and $\pm\frac{|\alpha_0|}{2}$, all of them with multiplicity $\dim(\mathfrak{g}_\lambda)$. Thus we established that the eigenvalues of $\mathcal{S}_\xi$ are independent of the choice of ξ for case (ii). Note that cases (i) and (ii) together already settle the problem if G/K is a symmetric space whose Dynkin diagram is of type A_r, B_r, D_r, E_6, E_7 or E_8.

Case (iii): $[\lambda] = \{\lambda, \lambda+\alpha_k, \lambda+\alpha_k+\alpha_{k+1}, \lambda+2\alpha_k, \lambda+2\alpha_k+\alpha_{k+1}, \lambda+2\alpha_k+2\alpha_{k+1}\}$. We consider the subspace

$$\mathfrak{g}_\lambda \oplus \mathfrak{g}_{\lambda+\alpha_k} \oplus \mathfrak{g}_{\lambda+2\alpha_k} \oplus \mathfrak{g}_{\lambda+\alpha_k+\alpha_{k+1}} \oplus \mathfrak{g}_{\lambda+2\alpha_k+\alpha_{k+1}} \oplus \mathfrak{g}_{\lambda+2\alpha_k+2\alpha_{k+1}} \subseteq \mathfrak{s}.$$

We need to understand better the behavior of the Levi-Civita connection when restricted to this subspace. As we did in Proposition 5.2.2, we will calculate the Levi-Civita connection using the map ϕ_ξ defined in (5.7).

Proposition 5.3.2. *Let $\gamma \in \Sigma^+$ be the root of minimum level in its ν-string, for $\nu \in \Sigma^+$ non-proportional to γ satisfying $A_{\nu,\gamma} = -2$. Let $\xi \in \mathfrak{g}_\nu$ be a unit vector with respect to $\langle \cdot, \cdot \rangle_{AN}$ and $X \in \mathfrak{g}_\gamma$. Then:*

(i) $\nabla_X \xi = -\frac{|\nu|}{\sqrt{2}} \phi_\xi(X)$;

(ii) $\nabla_{\phi_\xi(X)} \xi = -\frac{|\nu|}{\sqrt{2}} (X + \phi_\xi^2(X))$;

(iii) $\nabla_{\phi_\xi^2(X)} \xi = -\frac{|\nu|}{\sqrt{2}} \phi_\xi(X)$;

(iv) $\phi_\xi^2|_{\mathfrak{g}_\gamma} : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+2\nu}$ *is a linear isometry;*

(v) $\nabla_W \xi = 0$ *for all* $W \in \mathfrak{g}_{\gamma+\nu} \ominus \phi_\xi(\mathfrak{g}_\gamma)$.

Proof. Using (5.5) and (5.7) we easily obtain $\nabla_X \xi = -\frac{|\nu|}{\sqrt{2}} \phi_\xi(X)$. The same arguments together with Lemma 5.2.1 show that

$$\nabla_{\phi_\xi(X)} \xi = \frac{1}{2}([\phi_\xi(X), \xi] - [\phi_\xi(X), \theta\xi]) = -\frac{|\nu|}{\sqrt{2}} \left(\phi_\xi^2(X) + X\right).$$

Note that $A_{\nu,\gamma+\nu} = 0$. Thus, combining (5.5), (5.7) and the fact that $\gamma + 3\nu$ is not a root with Lemma 5.1.3(iii), we obtain

$$\nabla_{\phi_\xi^2(X)} \xi = -\frac{1}{2}[\phi_\xi^2(X), \theta\xi] = \frac{1}{4|\nu|^2}[\theta\xi, [\xi, [\xi, X]]] = -\frac{|\nu|}{\sqrt{2}} \phi_\xi(X).$$

Moreover, using again Lemma 5.1.3(iii), we deduce

$$\begin{aligned} \langle \phi_\xi^2(Y), \phi_\xi^2(Z) \rangle_{AN} &= \frac{1}{4|\nu|^4} \langle [\xi, [\xi, Y]], [\xi, [\xi, Z]] \rangle_{AN} \\ &= -\frac{1}{4|\nu|^4} \langle [\xi, Y], [\theta\xi, [\xi, [\xi, Z]]] \rangle_{AN} = \frac{1}{2|\nu|^2} \langle [\xi, Y], [\xi, Z] \rangle_{AN} \\ &= \langle \phi_\xi(Y), \phi_\xi(Z) \rangle_{AN} = \langle Y, Z \rangle_{AN} \end{aligned}$$

for $Y, Z \in \mathfrak{g}_\gamma$. It is then clear that ϕ_ξ^2 is an injective linear map preserving the inner product when restricted to $\mathfrak{g}_\gamma$. Furthermore, from Lemma 5.1.1 we know that $\dim(\mathfrak{g}_\gamma) =$

$\dim(\mathfrak{g}_{\gamma+2\nu})$, and thus $\phi_\xi^2|_{\mathfrak{g}_\gamma} : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+2\nu}$ is a linear isometry. Note that Lemma 5.1.3(iii) for $A_{\nu,\gamma} = -2$ is equivalent to $(\phi_{\theta\xi} \circ \phi_\xi^2)|_{\mathfrak{g}_\gamma} = \phi_\xi|_{\mathfrak{g}_\gamma}$. Then, we deduce that $\phi_\xi(\mathfrak{g}_\gamma) = \phi_{\theta\xi}(\mathfrak{g}_{\gamma+2\nu})$. To complete the proof, fix a vector $W \in \mathfrak{g}_{\gamma+\nu} \ominus \phi_\xi(\mathfrak{g}_\gamma) = \mathfrak{g}_{\gamma+\nu} \ominus \phi_{\theta\xi}(\mathfrak{g}_{\gamma+2\nu})$. On the one hand, we have $\langle \phi_\xi(W), Y\rangle_{AN} = \langle W, \phi_{\theta\xi}(Y)\rangle_{AN} = 0$ for all $Y \in \mathfrak{g}_{\gamma+2\nu}$. On the other hand, $\langle \phi_{\theta\xi}(W), Z\rangle_{AN} = \langle W, \phi_\xi(Z)\rangle_{AN} = 0$ for all $Z \in \mathfrak{g}_\gamma$. This implies $\nabla_W \xi = 0$ for all $W \in \mathfrak{g}_{\gamma+\nu} \ominus \phi_\xi(\mathfrak{g}_\gamma)$, which finishes the proof. □

Let $\xi \in V$ be a unit vector and, as above, write $\xi = \cos(\varphi)\xi_k + \sin(\varphi)\xi_{k+1}$. We first study the shape operator $\mathcal{S}_\xi$ on the subspace

$$\begin{aligned} &\mathfrak{g}_\lambda \oplus \phi_{\xi_k}(\mathfrak{g}_\lambda) \oplus \phi_{\xi_k}^2(\mathfrak{g}_\lambda) \oplus (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(\mathfrak{g}_\lambda) \\ &\quad \oplus (\phi_{\xi_{k+1}} \circ \phi_{\xi_k}^2)(\mathfrak{g}_\lambda) \oplus (\phi_{\xi_{k+1}}^2 \circ \phi_{\xi_k}^2)(\mathfrak{g}_\lambda). \end{aligned} \tag{5.14}$$

Let $0 \neq X_\lambda \in \mathfrak{g}_\lambda$. First, using (5.6), (1.3) and the fact that neither $\lambda + \alpha_{k+1}$ nor $\lambda + 2\alpha_{k+1} + \alpha_k$ are roots, we deduce

$$\mathcal{S}_{\xi_{k+1}} X_\lambda = \mathcal{S}_{\xi_k} (\phi_{\xi_{k+1}}^2 \circ \phi_{\xi_k}^2)(X_\lambda) = 0.$$

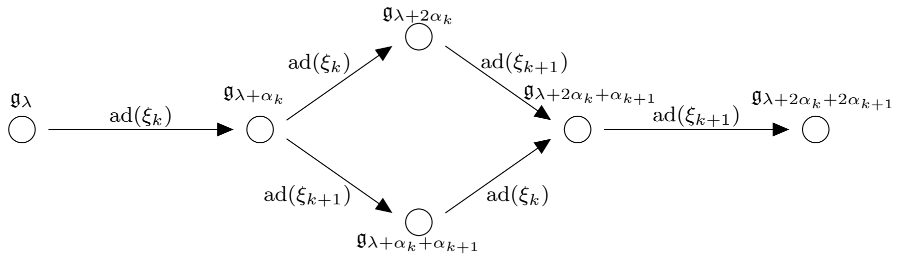
We will analyze the α_k-string of λ and the α_{k+1}-string of $\lambda + 2\alpha_k$ simultaneously. Let $\mu \in \{\lambda, \lambda + 2\alpha_k\}$ and define $r(\mu) = k$ if $\mu = \lambda$ and $r(\mu) = k+1$ otherwise. Put $X_\mu = X_\lambda$ if $\mu = \lambda$ and $X_\mu = \phi_{\xi_k}^2(X_\lambda)$ otherwise. Using (5.2) and Proposition 5.3.2 we obtain

$$\begin{aligned} \mathcal{S}_{\xi_{r(\mu)}} X_\mu &= -\left(\nabla_{X_\mu} \xi_{r(\mu)}\right)^\top = \frac{|\alpha_0|}{\sqrt{2}} \phi_{\xi_{r(\mu)}}(X_\mu), \\ \mathcal{S}_{\xi_{r(\mu)}} \phi_{\xi_{r(\mu)}}(X_\mu) &= -\left(\nabla_{\phi_{\xi_{r(\mu)}}(X_\mu)} \xi_{r(\mu)}\right)^\top = \frac{|\alpha_0|}{\sqrt{2}} (X_\mu + \phi_{\xi_{r(\mu)}}^2(X_\mu)), \\ \mathcal{S}_{\xi_{r(\mu)}} \phi_{\xi_{r(\mu)}}^2(X_\mu) &= -\left(\nabla_{\phi_{\xi_{r(\mu)}}^2(X_\mu)} \xi_{r(\mu)}\right)^\top = \frac{|\alpha_0|}{\sqrt{2}} \phi_{\xi_{r(\mu)}}(X_\mu). \end{aligned}$$

Note that $A_{\alpha_{k+1},\lambda+\alpha_k} = A_{\alpha_k,\lambda+\alpha_k+\alpha_{k+1}} = -1$. Then, using (5.2) and Proposition 5.2.2 for the pair $(\gamma,\nu) \in \{(\lambda+\alpha_k, \alpha_{k+1}), (\lambda+\alpha_k+\alpha_{k+1}, \alpha_k)\}$, we deduce

$$\begin{aligned} \mathcal{S}_{\xi_{k+1}} \phi_{\xi_k}(X_\lambda) &= \frac{|\alpha_0|}{2} (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda), \\ \mathcal{S}_{\xi_{k+1}} (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda) &= \frac{|\alpha_0|}{2} \phi_{\xi_k}(X_\lambda), \\ \mathcal{S}_{\xi_k} (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda) &= \frac{|\alpha_0|}{2} (\phi_{\xi_k} \circ \phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda), \\ \mathcal{S}_{\xi_k} (\phi_{\xi_k} \circ \phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda) &= \frac{|\alpha_0|}{2} (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda). \end{aligned}$$

So far we calculated the shape operator $\mathcal{S}_\xi$ on the subspace in (5.14). However, all this information is not conclusive as $(\phi_{\xi_k} \circ \phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda)$ and $(\phi_{\xi_{k+1}} \circ \phi_{\xi_k}^2)(X_\lambda)$ both belong to $\mathfrak{g}_{\lambda+2\alpha_k+\alpha_{k+1}}$, but we do not know how they are related. Consider the (α_k, α_{k+1})-string containing λ:



(Note that the nodes in this diagram represent root spaces and not roots.) The problem is that it is not clear whether or not the square diagram in the middle is commutative. More precisely, we do not yet understand the behavior of the vector $\phi_{\xi_k}(X_\lambda)$ depending on the part of the diagram it follows. In terms of brackets, the key point is to understand the relation between $[[\phi_{\xi_k}(X_\lambda), \xi_k], \xi_{k+1}]$ and $[[\phi_{\xi_k}(X_\lambda), \xi_{k+1}], \xi_k]$. Using (5.7) and the Jacobi identity twice, we obtain

$$\begin{aligned}
\sqrt{2}|\alpha_0|[\phi_{\xi_k}(X_\lambda), [\xi_{k+1}, \xi_k]] &= -[[X_\lambda, \xi_k], [\xi_{k+1}, \xi_k]] \\
&= [[\xi_k, [\xi_{k+1}, \xi_k]], X_\lambda] + [[[\xi_{k+1}, \xi_k], X_\lambda], \xi_k] \\
&= [[[\xi_{k+1}, \xi_k], X_\lambda], \xi_k] \\
&= -[[[X_\lambda, \xi_{k+1}], \xi_k], \xi_k] - [[[\xi_k, X_\lambda], \xi_{k+1}], \xi_k] \\
&= -[[[\xi_k, X_\lambda], \xi_{k+1}], \xi_k] = -\sqrt{2}|\alpha_0|[[\phi_{\xi_k}(X_\lambda), \xi_{k+1}], \xi_k].
\end{aligned}$$

Using the last equality and writing $Y = \phi_{\xi_k}(X_\lambda)$ for the sake of simplicity, we deduce

$$\begin{aligned}
2|\alpha_0|^2(\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(Y) &= [\xi_{k+1}, [\xi_k, Y]] \\
&= -([\xi_k, [Y, \xi_{k+1}]] + [Y, [\xi_{k+1}, \xi_k]]) \\
&= [[Y, \xi_{k+1}], \xi_k] - [Y, [\xi_{k+1}, \xi_k]] \\
&= [[Y, \xi_{k+1}], \xi_k] + [[Y, \xi_{k+1}], \xi_k] = 2[\xi_k, [\xi_{k+1}, Y]] \\
&= 4|\alpha_0|^2(\phi_{\xi_k} \circ \phi_{\xi_{k+1}})(Y),
\end{aligned}$$

which proves that the diagram is commutative up to a constant. In particular, we established that the vector space spanned by the vectors

$$\begin{gathered}
X_\lambda,\ \phi_{\xi_k}(X_\lambda),\ \phi^2_{\xi_k}(X_\lambda),\ (\phi_{\xi_{k+1}} \circ \phi_{\xi_k})(X_\lambda), \\
(\phi_{\xi_{k+1}} \circ \phi^2_{\xi_k})(X_\lambda),\ (\phi^2_{\xi_{k+1}} \circ \phi^2_{\xi_k})(X_\lambda)
\end{gathered}$$

is $\mathcal{S}_\xi$-invariant. Therefore, the matrix representation of the shape operator $\mathcal{S}_\xi$ on that subspace is given by $\dim(\mathfrak{g}_\lambda)$ blocks of the form

$$\frac{|\alpha_0|}{2}\begin{pmatrix} 0 & \sqrt{2}\cos(\varphi) & 0 & 0 & 0 & 0 \\ \sqrt{2}\cos(\varphi) & 0 & \sqrt{2}\cos(\varphi) & \sin(\varphi) & 0 & 0 \\ 0 & \sqrt{2}\cos(\varphi) & 0 & 0 & \sqrt{2}\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & 0 & 0 & \cos(\varphi) & 0 \\ 0 & 0 & \sqrt{2}\sin(\varphi) & \cos(\varphi) & 0 & \sqrt{2}\sin(\varphi) \\ 0 & 0 & 0 & 0 & \sqrt{2}\sin(\varphi) & 0 \end{pmatrix}$$

with respect to the decomposition in (5.14). A straightforward calculation shows that the eigenvalues of $\mathcal{S}_\xi$ are $\pm|\alpha_0|, \pm\frac{|\alpha_0|}{2}, 0$, each of them with multiplicity $\dim(\mathfrak{g}_\lambda)$, except 0, which has multiplicity $2\dim(\mathfrak{g}_\lambda)$.

Finally, from Lemma 5.1.1 and Lemma 5.1.3 we see that

$$\begin{aligned}\dim(\mathfrak{g}_\lambda) &= \dim(\mathfrak{g}_{\lambda+2\alpha_k}) = \dim(\mathfrak{g}_{\lambda+2\alpha_k+2\alpha_{k+1}}) \\ &\leq \dim(\mathfrak{g}_{\lambda+\alpha_k}) = \dim(\mathfrak{g}_{\lambda+\alpha_k+\alpha_{k+1}}) = \dim(\mathfrak{g}_{\lambda+2\alpha_k+\alpha_{k+1}}),\end{aligned}$$

where indices are modulo 2. Define $U = \mathfrak{g}_{\lambda+\alpha_k} \ominus \phi_{\xi_k}(\mathfrak{g}_\lambda)$. We still need to analyze the behavior of $\mathcal{S}_\xi$ on the vector space

$$U \oplus \phi_{\xi_{k+1}}(U) \oplus (\phi_{\xi_k} \circ \phi_{\xi_{k+1}})(U).$$

Let $0 \neq X \in U$. On the one hand, using (5.2) and Proposition 5.3.2, we obtain

$$\mathcal{S}_{\xi_k} X = \mathcal{S}_{\xi_{k+1}}(\phi_{\xi_k} \circ \phi_{\xi_{k+1}})(X) = 0.$$

Note that $A_{\alpha_{k+1},\lambda+\alpha_k} = -1$ and $A_{\alpha_k,\lambda+\alpha_k+\alpha_{k+1}} = -1$. On the other hand, for the pair $(\lambda,\nu) \in \{(\lambda+\alpha_k, \alpha_{k+1}), ((\lambda+\alpha_k+\alpha_{k+1}, \alpha_k))\}$ we obtain

$$\begin{aligned}\mathcal{S}_\xi(X) &= \frac{|\alpha_0|}{2}\sin(\varphi)\phi_{\xi_{k+1}}(X), \\ \mathcal{S}_\xi\phi_{\xi_{k+1}}(X) &= \frac{|\alpha_0|}{2}(\sin(\varphi)X + \cos(\varphi)(\phi_{\xi_k} \circ \phi_{\xi_{k+1}})(X)), \\ \mathcal{S}_\xi(\phi_{\xi_k} \circ \phi_{\xi_{k+1}})(X) &= \frac{|\alpha_0|}{2}\cos(\varphi)\phi_{\xi_{k+1}}(X),\end{aligned}$$

using (5.2) and Proposition 5.2.2. Taking into account that the vector space generated by the vectors $X, \phi_{\xi_{k+1}}(X), (\phi_{\xi_k} \circ \phi_{\xi_{k+1}})(X)$ is $\mathcal{S}_\xi$-invariant, the matrix representation of $\mathcal{S}_\xi$ on $U \oplus \phi_{\xi_{k+1}}(U) \oplus (\phi_{\xi_k} \circ \phi_{\xi_{k+1}})(U)$ is given by $(\dim(\mathfrak{g}_{\lambda+\alpha_k}) - \dim(\mathfrak{g}_\lambda))$ blocks of the form

$$\frac{|\alpha_0|}{2}\begin{pmatrix} 0 & \sin(\varphi) & 0 \\ \sin(\varphi) & 0 & \cos(\varphi) \\ 0 & \cos(\varphi) & 0 \end{pmatrix}.$$

The eigenvalues are 0, $\frac{|\alpha_0|}{2}$ and $-\frac{|\alpha_0|}{2}$, each of them with multiplicity $\dim(\mathfrak{g}_{\lambda+\alpha_k}) - \dim(\mathfrak{g}_\lambda)$. Altogether we have now established that the canonical extensions are also CPC submanifolds.

5.4 The classification

In this section we finish the classification in Theorem 5.0.2. We will show that if $S \cdot o$ is a CPC submanifold of $M = G/K$, then it must be one of the examples presented in Theorem 5.0.2. More precisely, we will prove that if $S \cdot o$ is a CPC submanifold, then either $V \subseteq \mathfrak{g}_\alpha$ for some $\alpha \in \Pi'$ or there exist $\alpha_0, \alpha_1 \in \Pi'$ with $A_{\alpha_0,\alpha_1} = A_{\alpha_1,\alpha_0} = -1$ and $V \subseteq \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$. Together with Theorem 5.2.8 this finishes the classification part of Theorem 5.0.2. We start with a result about the principal curvatures of the submanifold $S \cdot o$. Recall that, according to (5.3), we can write $V = \bigoplus_{\alpha\in\psi} V_\alpha$, where V_α is a non-trivial subspace of $\mathfrak{g}_\alpha$ for each $\alpha \in \psi$.

Proposition 5.4.1. *Let $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$ be a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ with $V = \oplus_{\alpha\in\psi} V_\alpha$ and $\psi \subseteq \Pi'$. Let $\gamma \in \Sigma^+$ be the root of minimum level in its ν-string, for $\nu \in \psi$ non-proportional to γ. Let I be the set of roots in the ν-string of γ. Consider the restriction of the shape operator $\mathcal{S}_\xi$ of $S \cdot o$ to the vector space $\bigoplus_{\alpha\in I} \mathfrak{g}_\alpha^\top$, where ξ is a unit vector in V_ν.*

(i) *If $A_{\nu,\gamma} = -1$, then $\pm\frac{|\nu|}{2}$ are principal curvatures, both with multiplicity* $\dim(\mathfrak{g}_\gamma^\top)$.

(ii) *If $A_{\nu,\gamma} = -2$, then $\pm|\nu|$ are principal curvatures, both with multiplicity* $\dim(\mathfrak{g}_\gamma^\top)$, *and $\pm\frac{|\nu|}{\sqrt{2}}$ are principal curvatures, both with multiplicity* $\dim(V_\gamma)$.

Proof. Assume first that $A_{\nu,\gamma} = -1$. In this case the ν-string of γ consists of $\gamma, \gamma+\nu$. Since $\gamma + \nu \notin \Pi$, we have $\mathfrak{g}_{\gamma+\nu}^\top = \mathfrak{g}_{\gamma+\nu}$. Let $\xi \in V_\nu$ be a unit vector and consider the restriction of the shape operator $\mathcal{S}_\xi$ to $\mathfrak{g}_\gamma^\top \oplus \mathfrak{g}_{\gamma+\nu}$. From (5.2) and Proposition 5.2.2 we get

$$\mathcal{S}_\xi X = -(\nabla_X \xi)^\top = \frac{|\nu|}{2}\phi_\xi(X),$$
$$\mathcal{S}_\xi \phi_\xi(X) = -(\nabla_{\phi_\xi(X)}\xi)^\top = \frac{|\nu|}{2}X$$

for $X \in \mathfrak{g}_\gamma^\top$. Then the 2-dimensional vector space spanned by $X, \phi_\xi(X)$ is $\mathcal{S}_\xi$-invariant for all $0 \neq X \in \mathfrak{g}_\gamma^\top$ and all unit vectors $\xi \in V_\nu$. Thus the matrix representation of $\mathcal{S}_\xi$ on $\mathfrak{g}_\gamma^\top \oplus \phi_\xi(\mathfrak{g}_\gamma^\top)$ consists of $\dim(\mathfrak{g}_\gamma^\top)$ blocks of the form

$$\frac{|\nu|}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Finally, let $Y \in \phi_\xi(V_\gamma)$ and write $Y = \phi_\xi(\eta)$ with $\eta \in V_\gamma$. From (5.2) and Proposition 5.2.2 we obtain

$$\mathcal{S}_\xi Y = \mathcal{S}_\xi \phi_\xi(\eta) = -(\nabla_{\phi_\xi(\eta)}\xi)^\top = \frac{|\nu|}{2}\eta^\top = 0.$$

Therefore, $\pm\frac{|\nu|}{2}$ are the non-zero principal curvatures of $\mathcal{S}_\xi$ on $\mathfrak{g}_\gamma^\top \oplus \mathfrak{g}_{\gamma+\nu}$, and both have multiplicity $\dim(\mathfrak{g}_\gamma^\top)$. This proves (i).

Now assume that $A_{\nu,\gamma} = -2$. Then the ν-string of γ consists of $\gamma, \gamma+\nu, \gamma+2\nu$. Since $\gamma+\nu$ and $\gamma+2\nu$ are not simple roots, we have $\mathfrak{g}_{\gamma+\nu}^\top = \mathfrak{g}_{\gamma+\nu}$ and $\mathfrak{g}_{\gamma+2\nu}^\top = \mathfrak{g}_{\gamma+2\nu}$. Let ξ be a unit vector in V_ν and consider the restriction of the shape operator $\mathcal{S}_\xi$ to $\mathfrak{g}_\gamma^\top \oplus \mathfrak{g}_{\gamma+\nu} \oplus \mathfrak{g}_{\gamma+2\nu}$. Let $X \in \mathfrak{g}_\gamma^\top$. From (5.2) and Proposition 5.3.2 we obtain

$$\begin{aligned}
\mathcal{S}_\xi X &= -\left(\nabla_X \xi\right)^\top = \frac{|\nu|}{\sqrt{2}}\phi_\xi(X),\\
\mathcal{S}_\xi \phi_\xi(X) &= -\left(\nabla_{\phi_\xi(X)}\xi\right)^\top = \frac{|\nu|}{\sqrt{2}}(\phi_\xi^2(X)+X)^\top = \frac{|\nu|}{\sqrt{2}}(\phi_\xi^2(X)+X),\\
\mathcal{S}_\xi \phi_\xi^2(X) &= -\left(\nabla_{\phi_\xi^2(X)}\xi\right) = \frac{|\nu|}{\sqrt{2}}\phi_\xi(X).
\end{aligned}$$

Thus the 3-dimensional vector space spanned by $X, \phi_\xi(X), \phi_\xi^2(X)$ is $\mathcal{S}_\xi$-invariant for all $0 \neq X \in \mathfrak{g}_\gamma^\top$ and all unit vectors $\xi \in V_\nu$. Thus the matrix representation of $\mathcal{S}_\xi$ on $\mathfrak{g}_\gamma^\top \oplus \phi_\xi(\mathfrak{g}_\gamma^\top) \oplus \phi_\xi^2(\mathfrak{g}_\gamma^\top)$ consists of $\dim(\mathfrak{g}_\gamma^\top)$ blocks of the form

$$\frac{|\nu|}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}.$$

This shows that $\pm|\nu|$ are principal curvatures of the shape operator $\mathcal{S}_\xi$ with multiplicities at least $\dim(\mathfrak{g}_\gamma^\top)$. There are two other cases to analyze. Assume that $X \in \phi_\xi(V_\gamma)$ and write $X = \phi_\xi(\eta)$ with $\eta \in V_\gamma$. From (5.2) and Proposition 5.3.2 we deduce

$$\begin{aligned}
\mathcal{S}_\xi X = \mathcal{S}_\xi\phi_\xi(\eta) &= -\left(\nabla_{\phi_\xi(\eta)}\xi\right)^\top = \frac{|\nu|}{\sqrt{2}}(\phi_\xi^2(\eta)+\eta)^\top = \frac{|\nu|}{\sqrt{2}}\phi_\xi^2(\eta),\\
\mathcal{S}_\xi\phi_\xi^2(\eta) &= -\left(\nabla_{\phi_\xi^2(\eta)}\xi\right) = \frac{|\nu|}{\sqrt{2}}\phi_\xi(\eta).
\end{aligned}$$

So the 2-dimensional vector space spanned by $\phi_\xi(\eta), \phi_\xi^2(\eta)$ is $\mathcal{S}_\xi$-invariant for all $0 \neq \eta \in V_\gamma$ and all unit vectors $\xi \in V_\nu$. Thus the matrix representation of $\mathcal{S}_\xi$ on $\phi_\xi(V_\gamma) \oplus \phi_\xi^2(V_\gamma)$ consists of $\dim(V_\gamma)$ blocks of the form

$$\frac{|\nu|}{\sqrt{2}}\begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}.$$

Consequently, $\pm|\nu|$ and $\pm\frac{|\nu|}{\sqrt{2}}$ are principal curvatures with multiplicities at least $\dim(\mathfrak{g}_\gamma^\top)$ and $\dim(V_\gamma)$, respectively. Finally, assume that $X \in \mathfrak{g}_{\gamma+\nu} \ominus \phi_\xi(\mathfrak{g}_\gamma)$. From (5.2) and Proposition 5.3.2 we deduce

$$\mathcal{S}_\xi X = -(\nabla_X \xi)^\top = 0.$$

This finishes the proof. □

We will now show that if $S \cdot o$ is a CPC submanifold, then all roots in ψ must have the same length. We will start by investigating the symmetric spaces G_2^2/SO_4 and $G_2^{\mathbb{C}}/G_2$.

Proposition 5.4.2. *Let $M = G/K$ be a symmetric space of non-compact type whose Dynkin diagram is of type G_2. Let α_0 and α_1 be its simple roots. Let S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s} = \mathfrak{s} \oplus (\mathfrak{n} \ominus V)$, where $V \subseteq \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$ has non-trivial projection onto $\mathfrak{g}_{\alpha_k}$ for $k \in \{0, 1\}$. Then $S \cdot o$ cannot be a CPC submanifold of M.*

Proof. We can assume $|\alpha_0| > |\alpha_1|$ and hence $|\alpha_0|^2 = 6$ and $|\alpha_1|^2 = 2$. The α_1-string of α_0 consists of $\alpha_0, \alpha_0 + \alpha_1, \alpha_0 + 2\alpha_1, \alpha_0 + 3\alpha_1$ and we have $A_{\alpha_1,\alpha_0} = -3$. Let $\xi_k \in V_{\alpha_k}$ be a unit vector and $k \in \{0,1\}$. We will determine a principal curvature of the shape operator $\mathcal{S}_{\xi_1}$ that cannot be a principal curvature of the shape operator $\mathcal{S}_{\xi_0}$. Note that $[\xi_1, \xi_0] \in \mathfrak{g}_{\alpha_0+\alpha_1}$ is tangent to $S \cdot o$. Using (5.6) and Lemma 5.1.3(ii) we deduce

$$\begin{aligned} 2\mathcal{S}_{\xi_1}[\xi_1, \xi_0] &= -([[\xi_1, \xi_0], \xi_1] - [[\xi_1, \xi_0], \theta\xi_1])^\top \\ &= [\xi_1, [\xi_1, \xi_0]]^\top - [\theta\xi_1, [\xi_1, \xi_0]]^\top \\ &= [\xi_1, [\xi_1, \xi_0]] - A_{\alpha_1,\alpha_0}|\alpha_1|^2 \xi_0^\top = [\xi_1, [\xi_1, \xi_0]]. \end{aligned}$$

Note that $A_{\alpha_1,\alpha_0+\alpha_1} = -1$. From (5.6) and Lemma 5.1.3(iii), we obtain

$$\begin{aligned} 2\mathcal{S}_{\xi_1}[\xi_1, [\xi_1, \xi_0]] &= -([[\xi_1, [\xi_1, \xi_0]], \xi_1] - [[\xi_1, [\xi_1, \xi_0]], \theta\xi_1])^\top \\ &= [\xi_1, [\xi_1, [\xi_1, \xi_0]]]^\top - [\theta\xi_1, [\xi_1, [\xi_1, \xi_0]]]^\top \\ &= [\xi_1, [\xi_1, [\xi_1, \xi_0]]] + 8[\xi_1, \xi_0]. \end{aligned}$$

Finally, since $A_{\alpha_1,\alpha_0+2\alpha_1} = 1$, from (5.6) and Lemma 5.1.3(iv) we conclude

$$\begin{aligned} 2\mathcal{S}_{\xi_1}[\xi_1, [\xi_1, [\xi_1, \xi_0]]] &= -[[\xi_1, [\xi_1, [\xi_1, \xi_0]]], \xi_1] + [[\xi_1, [\xi_1, [\xi_1, [\xi_1, \xi_0]]]], \theta\xi_1]^\top \\ &= [\xi_1, [\xi_1, [\xi_1, [\xi_1, \xi_0]]]]^\top - [\theta\xi_1, [\xi_1, [\xi_1, [\xi_1, [\xi_1, \xi_0]]]]^\top \\ &= 6[\xi_1, [\xi_1, \xi_0]]. \end{aligned}$$

Therefore, the 3-dimensional vector space spanned by the three vectors $\mathrm{ad}(\xi_1)\xi_0$, $\mathrm{ad}^2(\xi_1)\xi_0$ and $\mathrm{ad}^3(\xi_1)\xi_0$ is $\mathcal{S}_{\xi_1}$-invariant. The corresponding matrix representation of $\mathcal{S}_{\xi_1}$ on that subspace is

$$\begin{pmatrix} 0 & 4 & 0 \\ \frac{1}{2} & 0 & 3 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

The principal curvatures of $\mathcal{S}_{\xi_1}$ on this subspace are $\pm\sqrt{7/2}$ and 0. If $S \cdot o$ is a CPC submanifold, then $\sqrt{7/2}$ must also be a principal curvature of the shape operator $\mathcal{S}_{\xi_0}$. However, since $|\alpha_0| > |\alpha_1|$, we deduce from Proposition 1.5.1 that $|A_{\alpha_0,\mu}| \leq 1$ for all $\mu \in \Sigma$. According to Proposition 5.4.1(i) all the non-trivial principal curvatures of $\mathcal{S}_{\xi_0}$ are $\pm\sqrt{3/2}$. Therefore $S \cdot o$ cannot be a CPC submanifold. □

We now prove a similar result for symmetric spaces of non-compact type whose Dynkin diagram is not of type G_2.

Proposition 5.4.3. *Let $M = G/K$ be a symmetric space of non-compact type whose Dynkin diagram is not of type G_2. Let S be the connected closed subgroup of AN whose Lie algebra is $\mathfrak{s} = \mathfrak{s} \oplus (\mathfrak{n} \ominus V)$, where $V = \bigoplus_{\alpha\in\psi} V_\alpha$. If $S \cdot o$ is a CPC submanifold of M, then all roots in ψ must have the same length.*

Proof. Assume that there are two roots $\alpha_0, \alpha_1 \in \psi$ with different length and that $|\alpha_0| > |\alpha_1|$. Then we have $|\alpha_0| = \sqrt{2}|\alpha_1|$ and there exists $\lambda \in \Sigma^+$ with $A_{\alpha_1,\lambda} = -2$. Then λ is the root of minimum level in its non-trivial α_1-string, which consists of $\lambda, \lambda + \alpha_1, \lambda + 2\alpha_1$. Let $\xi_1 \in V_{\alpha_1}$ be a unit vector. Consider the restriction of the shape operator $\mathcal{S}_{\xi_1}$ to the tangent projection of the root spaces of the α_1-string of λ. From Proposition 5.4.1(ii) we see that the non-zero principal curvatures of $\mathcal{S}_{\xi_1}$ are $\pm|\alpha_1|$, both with multiplicity $\dim(\mathfrak{g}_\lambda^\top)$, and $\pm|\alpha_1|/\sqrt{2}$, both with multiplicity $\dim(V_\lambda)$. In particular, the submanifold $S \cdot o$ is not totally geodesic. There exists $\gamma \in \Sigma^+$ such that its α_0-string is non-trivial, because otherwise the shape operator $\mathcal{S}_{\xi_0}$ with respect to a unit vector $\xi_0 \in V_{\alpha_0}$ vanishes, which contradicts that $S \cdot o$ is a CPC submanifold. Without loss of generality we can assume that γ is the root of minimum level in its α_0-string. Since α_0 is a long root, Proposition 1.5.1 implies $A_{\alpha_0,\gamma} = -1$. From Proposition 5.4.1(i) we see that the non-zero principal curvatures of $\mathcal{S}_{\xi_0}$ are $\pm|\alpha_0|/2$, both with multiplicity $\dim(\mathfrak{g}_\gamma^\top)$. But $|\alpha_1| \neq |\alpha_0|/2$. Since $S \cdot o$ is a CPC submanifold, it follows that $|\alpha_1|$ cannot be a principal curvature and hence $\dim(\mathfrak{g}_\lambda^\top) = 0$. In other words, $V_\lambda = \mathfrak{g}_\lambda$ and λ is a simple root connected to α_1 by a single edge.

We put $\alpha_2 = \lambda$ and define the normal vector $\xi = \cos(\varphi)\xi_1 + \sin(\varphi)\xi_2$, where $\xi_k \in V_{\alpha_k}$ for $k \in \{1, 2\}$. Note that α_1, α_2 generate a root system of type B_2 $(= C_2)$. Therefore, according to (5.6) and (1.3), the vector space

$$\mathfrak{g}_{\alpha_1}^\top \oplus \mathfrak{g}_{\alpha_2}^\top \oplus \mathfrak{g}_{\alpha_1+\alpha_2}^\top \oplus \mathfrak{g}_{2\alpha_1+\alpha_2}^\top = \mathfrak{g}_{\alpha_1}^\top \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2}$$

is $\mathcal{S}_\xi$-invariant. We will now investigate the shape operator $\mathcal{S}_\xi$ on this subspace. In fact, studying the principal curvatures of $\mathcal{S}_\xi$ when restricted to this subspace is equivalent to studying the principal curvatures of $S \cdot o$ as a submanifold of a rank 2 symmetric space whose Dynkin diagram is of type B_2. First note that the α_2-string containing α_1 consists of $\alpha_1, \alpha_1 + \alpha_2$ and the α_1-string containing α_2 consists of $\alpha_2, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1$. We will use Proposition 5.4.1 for both cases. On the one hand, the non-zero principal curvatures of $\mathcal{S}_{\xi_2}$ are $\pm|\alpha_2|/2$, both with multiplicity $\dim(\mathfrak{g}_{\alpha_1}^\top)$. On the other hand, since $\mathfrak{g}_{\alpha_2} = V_{\alpha_2}$, the non-zero principal curvatures of $\mathcal{S}_{\xi_1}$ are $\pm|\alpha_1|/\sqrt{2}$, both with multiplicity $\dim(\mathfrak{g}_{\alpha_2}) = \dim(V_{\alpha_2})$. This implies that $\dim(\mathfrak{g}_{\alpha_2}) = \dim(\mathfrak{g}_{\alpha_1}^\top)$ is a necessary condition for $S \cdot o$ to be a CPC submanifold. Since $V_{\alpha_1} \neq \{0\}$ by assumption, we get $\dim(\mathfrak{g}_{\alpha_1}) > \dim(\mathfrak{g}_{\alpha_2})$. This means, according to [7, p. 337], that $S \cdot o$ must be contained in the symmetric space $SO^o_{r,r+n}/SO_rSO_{r+n}$, where $\dim(\mathfrak{g}_{\alpha_2}) = 1$ and $\dim(\mathfrak{g}_{\alpha_1}) = n$. Since $\dim(\mathfrak{g}_{\alpha_2}) = \dim(\mathfrak{g}_{\alpha_1}^\top)$, V_{α_1} must be an $(n-1)$-dimensional subspace of $\mathfrak{g}_{\alpha_1}$. Let $\xi_2 \in V_{\alpha_2}$ and $X \in \mathfrak{g}_{\alpha_1}^\top$. From (5.2) and Proposition 5.2.2 we deduce

$$\mathcal{S}_{\xi_2} X = \frac{|\alpha_2|}{2}\phi_{\xi_2}(X) \quad \text{and} \quad \mathcal{S}_{\xi_2}\phi_{\xi_2}(X) = \frac{|\alpha_2|}{2}X.$$

Now consider $\xi_1 \in V_{\alpha_1}$. Since $\phi^2_{\xi_1}|_{\mathfrak{g}_{\alpha_2}} : \mathfrak{g}_{\alpha_2} \to \mathfrak{g}_{2\alpha_1+\alpha_2}$ is a linear isometry, we have $\mathfrak{g}_{\alpha_2+2\alpha_1} = \mathbb{R}\phi^2_{\xi_1}(\xi_2)$. Recall that $\phi_{\xi_2}|_{\mathfrak{g}_{\alpha_1}} : \mathfrak{g}_{\alpha_1} \to \mathfrak{g}_{\alpha_1+\alpha_2}$ is also linear isometry. Then, using (1.1) and combining definition (5.7) with Lemma 5.1.3(ii),(i), we obtain

$$\begin{aligned}\langle(\phi_{\xi_1} \circ \phi_{\xi_2})(X), \phi^2_{\xi_1}(\xi_2)\rangle_{AN} &= \langle\phi_{\xi_2}(X), \phi_{\xi_1}(\xi_2)\rangle_{AN} = -\langle\phi_{\xi_2}(X), \phi_{\xi_2}(\xi_1)\rangle_{AN} \\ &= \langle X, \xi_1\rangle_{AN} = 0.\end{aligned}$$

Therefore, using (5.2) and Proposition 5.2.2, we obtain

$$\mathcal{S}_{\xi_1}\phi_{\xi_2}(X) = \frac{|\alpha_1|}{\sqrt{2}}(\phi_{\xi_1} \circ \phi_{\xi_2}(X))^\top = 0.$$

Since $\xi = \cos(\varphi)\xi_1 + \sin(\varphi)\xi_2$ with $\xi_k \in V_{\alpha_k}$ and $k \in \{1, 2\}$, we deduce

$$\mathcal{S}_\xi(X + \phi_{\xi_2}(X)) = \sin(\varphi)\frac{|\alpha_2|}{2}(X + \phi_{\xi_2}(X)),$$

which shows that $S \cdot o$ cannot be a CPC submanifold. □

In order to finish this section, we just need to prove that $S \cdot o$ is not a CPC submanifold whenever there are at least two orthogonal roots in ψ. One of the consequences of [12] is that $S \cdot o$ is not a CPC submanifold when ψ has exactly two orthogonal simple roots. The next result settles the general case.

Proposition 5.4.4. *Let $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$ be a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, for $V = \oplus_{\alpha\in\psi} V_\alpha$ and $\psi \subset \Pi'$. Assume that there are two orthogonal roots $\alpha_0, \alpha_1 \in \psi$. Let S be the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$. Then the submanifold $S \cdot o$ is not a CPC submanifold.*

Proof. In view of Proposition 5.4.2 and Proposition 5.4.3 we can assume that all roots in ψ have the same length. Taking into account the classification of Dynkin diagrams (see e.g. [69]), we deduce that there exist simple roots $\beta_1, \dots, \beta_r \in \Pi$ so that $\alpha_0, \beta_1, \dots, \beta_r, \alpha_1$ corresponds to a Dynkin diagram of type A_{r+2}. We define $\gamma = \sum_i^r \beta_i \in \Sigma^+$. The (α_0, α_1)-string of γ consists of $\gamma, \gamma + \alpha_0, \gamma + \alpha_1, \gamma + \alpha_0 + \alpha_1$. Let $\xi = \cos(\varphi)\xi_0 + \sin(\varphi)\xi_1$ be a unit normal vector with $\xi_k \in V_{\alpha_k}$ and $k \in \{0, 1\}$. Using (5.2) and Proposition 5.2.2, we obtain that the non-trivial part of the matrix representation $\mathcal{S}_\xi$ consists of $\dim(\mathfrak{g}_\gamma^\top)$ blocks of the form

$$\frac{|\alpha_0|}{2}\begin{pmatrix} 0 & \cos(\varphi) & \sin(\varphi) & 0 \\ \cos(\varphi) & 0 & 0 & \sin(\varphi) \\ \sin(\varphi) & 0 & 0 & \cos(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \end{pmatrix}$$

with respect to $X, \phi_{\xi_0}(X), \phi_{\xi_1}(X), (\phi_{\xi_1} \circ \phi_{\xi_0})(X)$ for $X \in \mathfrak{g}_\gamma^\top$. The corresponding eigenvalues are $\pm\sqrt{1 - \sin(2\varphi)}$, both with multiplicity 2. They clearly depend on φ, which cannot happen if $S \cdot o$ is a CPC submanifold. This implies $\mathfrak{g}_\gamma = V_\gamma$ and $\gamma = \beta_1 \in \Pi$.

Let $\xi_\gamma \in V_\gamma$ be a unit vector. Note that $\phi_{\xi_\gamma}(\xi_0) \in \mathfrak{g}_{\alpha_0+\gamma}$ and $(\phi_{\xi_1} \circ \phi_{\xi_\gamma})(\xi_0) \in \mathfrak{g}_{\alpha_0+\gamma+\alpha_1}$ are tangent to $S \cdot o$ at o. Using (5.2) and Proposition 5.2.2, we get $2\mathcal{S}_{\xi_\gamma}\phi_{\xi_\gamma}(\xi_0) = |\gamma|\xi_0^\top = 0$ and

$$\mathcal{S}_{\xi_1}(\phi_{\xi_\gamma}(\xi_0) + (\phi_{\xi_1} \circ \phi_{\xi_\gamma})(\xi_0)) = \frac{|\alpha_1|}{2}(\phi_{\xi_\gamma}(\xi_0) + (\phi_{\xi_1} \circ \phi_{\xi_\gamma})(\xi_0)).$$

Since $\alpha_0 + \alpha_1, \alpha_0 + 2\gamma + \alpha_1 \notin \Sigma$, we deduce from (5.6) and (1.3) that $\mathcal{S}_{\xi_\gamma}(\phi_{\xi_1} \circ \phi_{\xi_\gamma})(\xi_0) = 0$. Thus, if we define $\xi = \cos(\varphi)\xi_1 + \sin(\varphi)\xi_\gamma$, we get

$$\mathcal{S}_\xi(\phi_{\xi_\gamma}(\xi_0) + (\phi_{\xi_1} \circ \phi_{\xi_\gamma})(\xi_0)) = \cos(\varphi)\frac{|\alpha_1|}{2}(\phi_{\xi_\gamma}(\xi_0) + (\phi_{\xi_1} \circ \phi_{\xi_\gamma})(\xi_0)).$$

From this we see that $S \cdot o$ cannot be a CPC submanifold. This finishes the proof. □

5.5 Description of the examples

In this section we show that, with a few basic exceptions, the CPC submanifolds that we introduced in Theorem 5.0.2 are not singular orbits of cohomogeneity one actions.

Recall that α_0 and α_1 are two simple roots and $A_{\alpha_0,\alpha_1} = A_{\alpha_1,\alpha_0} = -1$. Recall also that V is a subspace of $\mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$ with non-trivial projections onto $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{\alpha_1}$ (equivalently $V_0 \neq \{0\} \neq V_1$). We are studying the orbit $S \cdot o$, where S is the connected closed subgroup of AN with Lie algebra $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$. First, assume that $V = \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$. Then $S \cdot o$ is one of the following submanifolds, or a canonical extension to G/K of it:

(i) $\mathbb{R}H^2 \times \mathbb{R} \cong (SL_2(\mathbb{R})/SO_2) \times \mathbb{R} \subset SL_3(\mathbb{R})/SO_3$,

(ii) $\mathbb{R}H^3 \times \mathbb{R} \cong (SL_2(\mathbb{C})/SU_2) \times \mathbb{R} \subset SL_3(\mathbb{C})/SU_3$,

(iii) $\mathbb{R}H^5 \times \mathbb{R} \cong (SL_2(\mathbb{H})/Sp_2) \times \mathbb{R} \subset SL_3(\mathbb{H})/Sp_3$,

(iv) $\mathbb{R}H^9 \times \mathbb{R} \subset E_6^{-26}/F_4$.

These four submanifolds appear in the list [16, Theorem 3.3] of reflective submanifolds and are singular orbits of cohomogeneity one actions. Therefore, their canonical extensions are also singular orbits of cohomogeneity one actions.

We will now see that the remaining submanifolds that we introduced in Theorem 5.0.2 do not admit such a description. One might study them in a rank 2 symmetric space and after that use some tools involving canonical extensions to conclude. However, for the sake of simplicity, we will carry out a direct study to avoid the introduction of these techniques.

Assume that V_k is a proper subspace of $\mathfrak{g}_{\alpha_k}$ for $k \in \{0, 1\}$ and that $\dim(\mathfrak{g}_{\alpha_0+\alpha_1}) \geq 2$. We will assume that $S \cdot o$ is a singular orbit of a cohomogeneity one action and derive a contradiction. Up to now we used the Iwasawa decomposition to identify the tangent space $T_o(S \cdot o)$ of the orbit $S \cdot o$ at o with $\mathfrak{s}$ and the normal space $\nu_o(S \cdot o)$ with V. However, in this section we will use the identification $\mathfrak{p} \cong T_o(G/K)$. This means that we will identify $T_o(S \cdot o)$ and $\nu_o(S \cdot o)$ with the orthogonal projections of $\mathfrak{s}$ and V onto $\mathfrak{p}$, which are $(1-\theta)\mathfrak{s}$ and $(1-\theta)V$ respectively.

If $S \cdot o$ is the singular orbit of a cohomogeneity one action on G/K, then the normalizer $N_K(S \cdot o)$ of $S \cdot o$ in K acts transitively on the unit sphere $\nu_o^1(S \cdot o)$ in $\nu_o(S \cdot o)$. Let $\mathfrak{m}$ be the Lie algebra of $N_K(S \cdot o)$. Then we have $[\mathfrak{m}, \xi] = \nu_o(S \cdot o) \ominus \mathbb{R}\xi$ for each $\xi \in \nu_o^1(S \cdot o)$. Let $\xi_0 \in V_0$ and $\xi_1 \in V_1$ be unit vectors. Taking into account that $\nu_o(S \cdot o) \cong (1-\theta)V$, there exists $Z \in \mathfrak{m}$ so that

$$[Z, (1-\theta)\xi_0] = (1-\theta)\xi_1 \in \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}. \tag{5.15}$$

Consider the orthogonal decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda$ with $\mathfrak{k}_\lambda = \mathfrak{k} \cap (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda})$, and write $Z = Z_0 + \sum_{\lambda \in \Sigma^+} Z_\lambda$ accordingly. On the one hand, we have

$$\begin{aligned} &[Z_\lambda, (1-\theta)\xi_0] \\ &= (1-\theta)[Z_\lambda, \xi_0] \in \mathfrak{g}_{\lambda+\alpha_0} \oplus \mathfrak{g}_{-(\lambda+\alpha_0)} \oplus \mathfrak{g}_{\lambda-\alpha_0} \oplus \mathfrak{g}_{-(\lambda-\alpha_0)} \end{aligned} \tag{5.16}$$

for each $\lambda \in \Sigma^+$. From (5.16) and (5.15), using $[\mathfrak{k}_0, \mathfrak{g}_\lambda] \subseteq \mathfrak{g}_\lambda$ for each $\lambda \in \Sigma^+$, we deduce $[Z_0, (1-\theta)\xi_0] = 0$. Thus, without loss of generality, we can assume that $Z_0 = 0$ and hence $Z = \sum_{\lambda \in \Sigma^+} Z_\lambda$. From (5.16) and (5.15) we also see that $Z_{\alpha_0+\alpha_1} \neq 0$. It is now easy to verify that

$$N_{\mathfrak{k}}(T_o(S \cdot o)) = N_{\mathfrak{k}}(\nu_o(S \cdot o)) \subset \mathfrak{k}_0 \oplus \mathfrak{k}_{\alpha_0+\alpha_1} \oplus \left(\bigoplus_{\lambda \in \{\alpha_0,\alpha_1\}^\perp} \mathfrak{k}_\lambda \right),$$

where $\{\alpha_0, \alpha_1\}^\perp$ denotes the set of positive roots that are orthogonal to both α_0 and α_1. Since $\mathfrak{m} \subset N_{\mathfrak{k}}(T_o(S \cdot o))$, we can thus write $Z = X + \theta X + \sum_{\lambda \in \{\alpha_0,\alpha_1\}^\perp} Z_\lambda$ with $0 \neq X \in \mathfrak{g}_{\alpha_0+\alpha_1}$. Denote by $\mathfrak{l}$ the Lie algebra of $N_G(S \cdot o)$. It is clear that $\mathfrak{s} \subset \mathfrak{l}$ and $Z \in \mathfrak{l}$. Let $Y_1, \ldots, Y_q$ be an orthogonal basis of $\mathfrak{g}_{\alpha_0+\alpha_1} \ominus \mathbb{R}X \subset \mathfrak{s}$, where $q = \dim(\mathfrak{g}_{\alpha_0+\alpha_1}) - 1$. According to Lemma 5.1.2(ii),(iii), the set $\{[Z, Y_i] = [\theta X, Y_i] : i = 1, \ldots q\}$ generates a q-dimensional linear subspace W of $\mathfrak{k}_0$. Since $\mathfrak{l}$ is a subalgebra, we also have $W \subset \mathfrak{l}$ and therefore $W \subset N_{\mathfrak{k}}(T_o(S \cdot o))$. For $0 \neq \eta \in V_0$ we have

$$[[Z, Y_i], (1-\theta)\eta] = (1-\theta)[[Z, Y_i], \eta] = (1-\theta)[[\theta X, Y_i], \eta] \in (1-\theta)V,$$

which is equivalent to $[[\theta X, Y_i], \eta] \in V_0$ for all $i \in \{1, \ldots, q\}$. Note that $[[\theta X, Y_i], \eta] = [Y_i, \theta[\theta\eta, X]] \neq 0$ for all $i \in \{1, \ldots, q\}$ by using twice Proposition 5.2.2, first for $[\theta\eta, X]$ and then for $[Y_i, \theta[\theta\eta, X]]$, taking into account that θ is an isomorphism of Lie algebras. Note also that $\langle [U, L], L \rangle_{B_\theta} = -\langle L, [U, L] \rangle_{B_\theta}$ for all $U \in \mathfrak{k}_0$ and $L \in \mathfrak{n}$, which means that $[U, L]$ is orthogonal to L for all $U \in \mathfrak{k}_0$ and $L \in \mathfrak{n}$. If $\dim(\mathfrak{g}_{\alpha_0+\alpha_1}) = 2$, then $V_0 = \mathbb{R}\eta$ is 1-dimensional and $0 \neq [[\eta, \theta X], Y_1] \in V_0$ is orthogonal to η, which is a contradiction. If $\dim(\mathfrak{g}_{\alpha_0+\alpha_1}) > 2$, we have $0 \neq [[\theta X, Y_i], \eta] \in V_0$ for $i \in \{1, \ldots q\}$. Since $\dim(V_0) \leq \dim(T_0)$ by Proposition 5.2.4 and Lemma 5.2.3(ii), these q vectors must be linearly dependent. Thus

$$0 = \sum_{i=1}^{q} a_i[[\theta X, Y_i], \eta] = \sum_{i=1}^{q}[[\theta X, a_i Y_i], \eta] = [[\theta X, \sum_{i=1}^{q} a_i Y_i], \eta],$$

which contradicts Proposition 5.2.2 by the above argument. These contradictions come from the assumption that the action of $N_K(S \cdot o)$ on $\nu_o^1(S \cdot o)$ is transitive. Therefore, if V_k is a proper subset of $\mathfrak{g}_{\alpha_k}$ for $k \in \{0, 1\}$, then the orbit $S \cdot o$ cannot be the singular orbit of a cohomogeneity one action.

5.6 Further geometric explanations

In this section we present a brief geometric context for some of the algebraic constructions in the previous sections. Consider the inclusions

$$SL_3(\mathbb{R}) \subset SL_3(\mathbb{C}) \subset SL_3(\mathbb{H}) \subset E_6^{-26}.$$

The maximal compact subgroup of E_6^{-26} is F_4 and E_6^{-26}/F_4 is an exceptional Riemannian symmetric space of non-compact type whose root system is of type A_2. We have

$$SL_3(\mathbb{R}) \cap F_4 = SO_3 \ , \ SL_3(\mathbb{C}) \cap F_4 = SU_3 \ , \ SL_3(\mathbb{H}) \cap F_4 = Sp_3.$$

This leads to the totally geodesic embeddings

$$SL_3(\mathbb{R})/SO_3 \subset SL_3(\mathbb{C})/SU_3 \subset SL_3(\mathbb{H})/Sp_3 \subset E_6^{-26}/F_4.$$

The root system of these four Riemannian symmetric spaces G/K is of type A_2 and the multiplicities of their roots are $1, 2, 4, 8$, respectively. These dimensions correspond to the dimensions of the four normed real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. This suggests a close relation between these four symmetric spaces and normed real division algebras.

In fact, we have totally geodesic embeddings of the hyperbolic planes over these four normed real division algebras into these symmetric spaces:

$$\begin{array}{ccccccc} \mathbb{R}H^2 & \subset & \mathbb{C}H^2 & \subset & \mathbb{H}H^2 & \subset & \mathbb{O}H^2 \\ \cap & & \cap & & \cap & & \cap \\ SL_3(\mathbb{R})/SO_3 & \subset & SL_3(\mathbb{C})/SU_3 & \subset & SL_3(\mathbb{H})/Sp_3 & \subset & E_6^{-26}/F_4 \end{array}$$

In each of the four cases, the totally geodesic submanifold $\mathbb{F}H^2$ is reflective and hence there exists a totally geodesic submanifold (which is also reflective) that is perpendicular to the hyperbolic plane. These are

$$\begin{array}{ccccccc} SL_3(\mathbb{R})/SO_3 & \subset & SL_3(\mathbb{C})/SU_3 & \subset & SL_3(\mathbb{H})/Sp_3 & \subset & E_6^{-26}/F_4 \\ \cup & & \cup & & \cup & & \cup \\ \mathbb{R}H^2 \times \mathbb{R} & \subset & \mathbb{R}H^3 \times \mathbb{R} & \subset & \mathbb{R}H^5 \times \mathbb{R} & \subset & \mathbb{R}H^9 \times \mathbb{R} \\ \perp & & \perp & & \perp & & \perp \\ \mathbb{R}H^2 & \subset & \mathbb{C}H^2 & \subset & \mathbb{H}H^2 & \subset & \mathbb{O}H^2 \\ \cap & & \cap & & \cap & & \cap \\ SL_3(\mathbb{R})/SO_3 & \subset & SL_3(\mathbb{C})/SU_3 & \subset & SL_3(\mathbb{H})/Sp_3 & \subset & E_6^{-26}/F_4 \end{array}$$

The products $\mathbb{R}H^k \times \mathbb{R}$ are precisely our orbits $S \cdot o$ for the case when we remove $V = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}$. Thus the normal space $\nu_o(S \cdot o) \cong V$ of $S \cdot o$ at o coincides with the tangent space $T_o\mathbb{F}H^2$ of $\mathbb{F}H^2$ at o for a suitable $\mathbb{F}H^2 \subset G/K$ and where $\mathbb{F}$ is the corresponding division algebra.

Now suppose that V is a proper subspace of $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \cong T_o\mathbb{F}H^2$.

If $\mathbb{F} = \mathbb{C}$, then $V \cong \mathbb{R} \oplus \mathbb{R} \cong T_o\mathbb{R}H^2$ for a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^2 \subset SL_3(\mathbb{C})/SU_3$.

If $\mathbb{F} = \mathbb{H}$, then $V \cong \mathbb{R} \oplus \mathbb{R} \cong T_o\mathbb{R}H^2$ for a totally geodesic $\mathbb{R}H^2 \subset \mathbb{H}H^2 \subset SL_3(\mathbb{H})/Sp_3$, or $V \cong \mathbb{C} \oplus \mathbb{C} \cong T_o\mathbb{C}H^2$ for a totally geodesic $\mathbb{C}H^2 \subset \mathbb{H}H^2 \subset SL_3(\mathbb{H})/Sp_3$.

If $\mathbb{F} = \mathbb{O}$, then $V \cong \mathbb{R} \oplus \mathbb{R} \cong T_o\mathbb{R}H^2$ for a totally geodesic $\mathbb{R}H^2 \subset \mathbb{O}H^2 \subset E_6^{-26}/F_4$, or $V \cong \mathbb{C} \oplus \mathbb{C} \cong T_o\mathbb{C}H^2$ for a totally geodesic $\mathbb{C}H^2 \subset \mathbb{O}H^2 \subset E_6^{-26}/F_4$, or $V \cong \mathbb{H} \oplus \mathbb{H} \cong T_o\mathbb{H}H^2$ for a totally geodesic $\mathbb{H}H^2 \subset \mathbb{O}H^2 \subset E_6^{-26}/F_4$.

In other words, this means that the totally geodesic hyperbolic planes in G/K correspond to the subspaces V that we can remove from $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}$ to obtain our CPC submanifolds.

The submanifolds $S \cdot o$ with V strictly contained in $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}$ are some kind of ruled submanifolds. Here is a description for the simplest case when $G/K = SL_3(\mathbb{C})/SU_3$ and $V \cong \mathbb{R} \oplus \mathbb{R}$. In this case we have the two reflective submanifolds $\mathbb{R}H^3 \times \mathbb{R}$ and $\mathbb{C}H^2$ which are perpendicular to each other at o. Consider the polar action on $\mathbb{C}H^2$ given in (ii)(d) in Theorem 5.0.2 of [11]. The orbit of this polar action through o is a Euclidean plane $\mathbb{R}^2$, embedded in a horosphere of $\mathbb{C}H^2$ (equivalently, the 3-dimensional Heisenberg group N) as a minimal surface. Perpendicular to $\mathbb{R}^2$ at o in $\mathbb{C}H^2$ is a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^2$. Moving this $\mathbb{R}^2$ along $\mathbb{R}H^3 \times \mathbb{R}$ through the action on $\mathbb{R}H^3 \times \mathbb{R}$ by the solvable group S' with $S' \cdot o = \mathbb{R}H^3 \times \mathbb{R}$ arising from the Iwasawa decomposition gives the orbit $S \cdot o$. Thus $S \cdot o$ is foliated by these Euclidean planes. The normal spaces are obtained by moving the totally geodesic $\mathbb{R}H^2$ perpendicular to $\mathbb{R}^2$ in $\mathbb{C}H^2$ along $S \cdot o$. According to Proposition 3.4, the principal curvatures are $\pm 1/\sqrt{2}$ with multiplicity 1 each and 0 with multiplicity 4. The 0-eigenspace at o is the tangent space at o of the totally geodesic $\mathbb{R}H^3 \times \mathbb{R}$, and the other two eigenspaces arise from the non-totally geodesic minimal embedding of $\mathbb{R}^2$.

Chapter 6
Austere submanifolds in classical symmetric spaces

A particularly interesting subclass of minimal submanifolds which, in turn, is broader than the class of CPC submanifolds introduced in the previous chapter, is that of austere submanifolds. Austere submanifolds are defined by an algebraic property that must be satisfied at every point of the submanifold, namely, the principal curvatures (counted with multiplicities) with respect to any unit normal vector are invariant under change of sign.

The aim of this and the following chapter is to establish the classification of austere submanifolds that arise as orbits of the solvable part S_Φ of parabolic subgroups of the isometry group G of an irreducible symmetric space of non-compact type $M \cong G/K$. In order to formalize our result, we need to introduce some terminology and notation. We refer to [18], [47] and [50, Section 2.7] for further information.

Let $M \cong G/K$ be a symmetric space of non-compact type, where G is the connected component of the identity of the isometry group of M, and K is the isotropy group at some base point $o \in M$. We will make use of the concepts and results stated in Section 1.5. Thus, let Σ be the set of restricted roots of the real semisimple Lie algebra $\mathfrak{g}$ of G. Consider a positivity criterion on Σ, and let Σ^+ be the corresponding set of positive roots, and Π the associated system of simple roots.

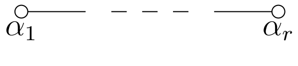
Now take any subset Φ of Π. Let Σ^Φ denote the subset of positive roots in Σ^+ that are not spanned by Φ. Define the abelian and nilpotent Lie algebras

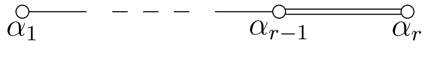
$$\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \ker \alpha \quad \text{and} \quad \mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^\Phi} \mathfrak{g}_\alpha,$$

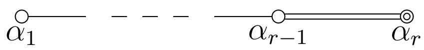
respectively. Then, the direct sum $\mathfrak{s}_\Phi = \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ turns out to be a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Let S_Φ be the connected Lie subgroup of AN whose Lie algebra is $\mathfrak{s}_\Phi$.

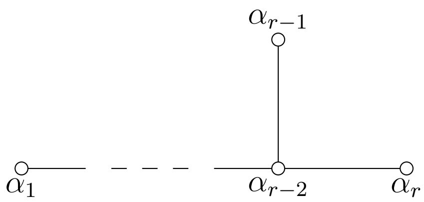
In this chapter we focus on the classification of those homogeneous submanifolds $S_\Phi \cdot o$ that are austere in the irreducible symmetric spaces of non-compact type $M \cong G/K$ with classical Lie group G. The study of the symmetric spaces of exceptional type is postponed to Chapter 7. Thus, the main result of this chapter is the following

Theorem 6.0.1. *Let G/K be a classical symmetric space of non-compact type whose Dynkin diagram adopts one of the following configurations*









where the second diagram can be of type B_r or C_r. Let Φ be a proper subset of the set Π of simple roots of G/K. The submanifold $S_\Phi \cdot o$ is austere if and only if one of the following statements holds:

(i) Φ *is discrete, or*

(ii) $\Phi = \Phi_0$*, satisfying the conditions specified in Table 6.1, or*

(iii) $\Phi = \Phi_0 \cup \Phi_1$*, where* Φ_0 *is orthogonal to* Φ_1 *and they satisfy the conditions specified in Table 6.1 (in the gray row all the roots have the same multiplicity).*

Π	Φ_0	Φ_1
A_r	Symmetric, connected	$\emptyset$
B_r	B_n, $n < r$	Discrete
B_r	$\{\alpha_{r-2}, \alpha_{r-1}\}$	Discrete
C_r	C_n, $n < r$	Discrete
BC_r	BC_n, $n < r$	Discrete
D_r	D_n, $n < r$	Discrete
D_r	$\{\alpha_{r-3}, \alpha_{r-2}, \alpha_{r-1}\}$	Discrete
D_r	$\{\alpha_{r-3}, \alpha_{r-2}, \alpha_r\}$	Discrete

Table 6.1: Classification in classical symmetric spaces.

In the statement, a subset Ψ of the simple system Π is said to be *discrete* if any two roots in Ψ are orthogonal (equivalently, no edge links them in the Dynkin diagram of Π). A subset $\Psi \subset \Pi$ is said to be *connected* if it cannot be expressed as a non-trivial union $\Psi_1 \cup \Psi_2$ where Ψ_1 is orthogonal to Ψ_2 (equivalently, if there is a connected subgraph of the Dynkin diagram of Π whose nodes correspond precisely to the roots in Ψ). Finally, if Π is a simple system of type A_r, then $\Psi \subset \Pi$ is called *symmetric* if $\alpha_i \in \Psi$ implies that $\alpha_{r-i+1} \in \Psi$, for $i \in \{1, \dots, r\}$, with the notation in the statement (equivalently, the set of nodes associated with Ψ in the Dynkin diagram of Π are invariant under the non-trivial involution of such Dynkin diagram).

The action of S_Φ on the symmetric space M is isometric (indeed polar) and free, and all its orbits are mutually congruent. This action is fundamental in the canonical extension method that was introduced in [18] and further investigated in [47], and which was already mentioned in Chapter 5. A key ingredient in such method is the minimality of the S_Φ-orbits on M, which was proved by Tamaru [102] (who also showed that such orbits are Einstein solvmanifolds). Moreover, such orbits are never totally geodesic, unless Φ and $\Pi \setminus \Phi$ are orthogonal sets of roots, which essentially leads to reducible symmetric spaces M.

It is therefore natural to ask the question of which S_Φ-actions have austere orbits, as austerity is a stronger condition than minimality, and much weaker than being totally geodesic. Since all S_Φ-orbits are mutually congruent, in order to analyze their extrinsic geometry we can and will focus on the orbit $S_\Phi \cdot o$ through the base point o.

It is important to notice the following observation regarding the definition of austere submanifold. Because of the nonlinearity of the higher degree symmetric polynomials in the principal curvatures of a submanifold, it is not enough to impose the eigenvalue condition on the shape operator $\mathcal{S}_\xi$ for ξ running over a basis of the normal space at each point. This makes the study of austere submanifolds difficult. In our setting, we will develop several tools to analyze which subsets $\Phi \subset \Pi$ give rise to orbits $S_\Phi \cdot o$ that are austere. The fundamental idea that we introduce and discuss is the concept of Φ-string, which generalizes the classical notion of string in the context of root systems [69, p. 152] and also the concept of (α_0, α_1)-string introduced in Section 5.2. Moreover, we will associate a diagram to each such Φ-string, which will be helpful to understand their structure and, ultimately, determine if $S_\Phi \cdot o$ is austere or not.

In order to prove the above result, this chapter is organized as follows. In Section 6.1 we introduce the general setting for studying the austerity of $S_\Phi \cdot o$. In particular, we prove that it suffices to analyze the shape operator with respect to unit normal vectors in $\mathfrak{a}^\Phi$ (Subsection 6.1.1) and we explain the crucial role that strings will play in this and the following chapter. More precisely, we will generalize the notion of string introduced in Chapter 5 and we will consider a decomposition of the tangent space induced by strings. Moreover, in this case we will associate a diagram to each string (Subsection 6.1.2). Roughly speaking, the austerity of $S_\Phi \cdot o$ will be equivalent to certain symmetry conditions of this diagram (Subsection 6.1.3). In Section 6.2 we consider and inspect particular strings that will appear throughout this chapter and the next and we study their symmetries. As a consequence, we will obtain the first examples of austere submanifolds of the form $S_\Phi \cdot o$. Section 6.3 is completely devoted to finishing the proof of Theorem 6.0.1.

6.1 Φ-strings and their diagrams

In this section we establish the general setup for the study of the extrinsic geometry (and, in particular, the austerity) of the orbits of the form $S_\Phi \cdot o$. We start by recalling some notation and facts regarding parabolic subalgebras of real semisimple Lie algebras. Then, in Subsection 6.1.1 we link the study of the shape operator of $S_\Phi \cdot o$ with the restricted root structure of the symmetric space, and introduce the concept of Φ-string. In Subsection 6.1.2 we associate a diagram to each Φ-string, and explain how to read geometric information of the orbit $S_\Phi \cdot o$ from its associated diagrams. Subsection 6.1.3 is devoted to prove several important necessary and sufficient conditions for the austerity of $S_\Phi \cdot o$.

Let Φ be a proper subset of the set Π of simple roots of a symmetric space G/K of non-compact type. We will denote by Σ_Φ the root subsystem of Σ generated by the simple roots in Φ. Let $\Sigma_\Phi^+ = \Sigma^+ \cap \Sigma_\Phi$ be the set of positive roots spanned by Φ and let Σ^Φ be the set of positive roots of Σ^+ that are not generated by Φ, that is,

$$\Sigma^\Phi = \Sigma^+ \backslash \Sigma_\Phi^+.$$

Define the abelian and the nilpotent subalgebras

$$\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \ker \alpha \quad \text{and} \quad \mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^\Phi} \mathfrak{g}_\alpha,$$

respectively. Note that $\mathfrak{s}_\Phi = \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Let S_Φ be the connected closed subgroup of AN whose Lie algebra is $\mathfrak{s}_\Phi$.

In order to study and understand the geometry of such orbits, we will need to introduce some tools related to parabolic subgroups and parabolic subalgebras. We follow [18]. Consider the reductive and abelian Lie subalgebras

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right) \quad \text{and} \quad \mathfrak{a}^\Phi = \mathfrak{a} \ominus \mathfrak{a}_\Phi = \bigoplus_{\alpha \in \Phi} \mathbb{R} H_\alpha$$

respectively. Then, $\mathfrak{l}_\Phi$ is the centralizer and normalizer of $\mathfrak{a}_\Phi$ in $\mathfrak{g}$. Moreover, we have that

$$\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$$

is a subalgebra of $\mathfrak{g}$, which is called the parabolic subalgebra of $\mathfrak{g}$ associated with the subset Φ of Π. The decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ is usually called the Chevalley decomposition of the parabolic subalgebra $\mathfrak{q}_\Phi$.

We define now the reductive subalgebra $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$. Note that it normalizes $\mathfrak{s}_\Phi$. The decomposition

$$\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$$

is the so-called Langlands decomposition of the parabolic subalgebra $\mathfrak{q}_\Phi$. Consider the subalgebra $\mathfrak{k}_\Phi$ of $\mathfrak{k}$ given by

$$\mathfrak{k}_\Phi = \mathfrak{k} \cap \mathfrak{q}_\Phi = \mathfrak{k}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi^+} \mathfrak{k}_\alpha \right),$$

where $\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, for each $\alpha \in \Sigma_\Phi^+$. Note that $[\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$ and define

$$\mathfrak{b}_\Phi = \mathfrak{m}_\Phi \cap \mathfrak{p} = \mathfrak{a}^\Phi \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi^+} \mathfrak{p}_\alpha \right),$$

which turns out to be a Lie triple system, where $\mathfrak{p}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, for each $\alpha \in \Sigma_\Phi^+$. Note that

$$\mathfrak{g}_\Phi = [\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \oplus \mathfrak{b}_\Phi \tag{6.1}$$

is a semisimple Lie algebra (note also that in [18] $\mathfrak{g}_\Phi$ is used to denote a different subalgebra). Moreover, we have that (6.1) is a Cartan decomposition for the semisimple Lie algebra $\mathfrak{g}_\Phi$. Note also that $\mathfrak{a}^\Phi$ is a maximal abelian subspace of $\mathfrak{b}_\Phi$. Then, we can consider the restricted root decomposition

$$\mathfrak{g}_\Phi = (\mathfrak{g}_\Phi)_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right) = (\mathfrak{g}_\Phi \cap \mathfrak{k}_0) \oplus \mathfrak{a}^\Phi \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right) \tag{6.2}$$

of $\mathfrak{g}_\Phi$ with respect to $\mathfrak{a}^\Phi$, for which Φ is the set of simple roots. Let G_Φ be the connected Lie subgroup of G with Lie algebra $\mathfrak{g}_\Phi$. The orbit $B_\Phi = G_\Phi \cdot o$ of the G_Φ-action on M containing o is a connected totally geodesic submanifold of M with $T_o B_\Phi = \mathfrak{b}_\Phi$. Moreover, if Φ is a non-empty subset of Π then B_Φ is a Riemannian symmetric space of non-compact type with rank $|\Phi|$.

6.1.1 The shape operator of $S_\Phi \cdot o$

All the tools presented above will be necessary in order to simplify our calculations throughout the study of the shape operator and the austerity of the orbit $S_\Phi \cdot o$. Note that we have $\mathfrak{a}^\Phi \subset \nu_o(S_\Phi \cdot o) = \mathfrak{b}_\Phi$. Roughly speaking, the following result states that it suffices to analyze the shape operators of the form $\mathcal{S}_\xi$, with $\xi \in \mathfrak{a}^\Phi$, in order to characterize austere submanifolds of the form $S_\Phi \cdot o$. Following Bryant [22] (cf. [64]), we will say that a linear subspace $\mathcal{S}$ of the space of selfadjoint endomorphisms of a Euclidean space $(V, \langle \cdot, \cdot \rangle)$ is *austere* if each endomorphism in $\mathcal{S}$ has eigenvalues occuring in oppositely signed pairs. Similarly, if $\mathcal{S}$ is as before and W is an $\mathcal{S}$-invariant subspace of V, we will say that $\mathcal{S}$ is *austere when restricted to* W if the eigenvalues of the endomorphisms in $\mathcal{S}$ restricted to W occur in oppositely signed pairs. We will use this terminology to refer to the shape operator $\mathcal{S}$ of a submanifold.

Proposition 6.1.1. *Let Φ be a proper subset of the set of simple roots Π of the symmetric space G/K. Let $\mathcal{S}$ be the shape operator of the submanifold $S_\Phi \cdot o$. Then, the submanifold $S_\Phi \cdot o$ is austere if and only if $\mathcal{S}_\xi$ is austere for all $\xi \in \mathfrak{a}^\Phi$.*

Proof. One of the implications is trivial. Let us see the other one. Since B_Φ is a symmetric space, we can consider the representation

$$K_\Phi \times \mathfrak{b}_\Phi \to \mathfrak{b}_\Phi,$$

which is equivalent to the isotropy representation of B_Φ. Fix an element $\xi \in \mathfrak{b}_\Phi$. Hence, there exists an isometry $g \in K_\Phi$ such that $g_* \xi \in \mathfrak{a}^\Phi$. Note that g preserves S_Φ, $\mathfrak{b}_\Phi = \nu_o(S_\Phi \cdot o)$ and $(1-\theta)\mathfrak{s}_\Phi = T_o(S_\Phi \cdot o)$. Moreover, we have

$$\begin{aligned}\langle \mathcal{S}_{g_*\xi} X, Y \rangle_{B_\theta} &= \langle II(X,Y), g_*\xi \rangle_{B_\theta} = \langle (g_*)^{-1} II(X,Y), \xi \rangle_{B_\theta} \\ &= \langle II((g_*)^{-1}X, (g_*)^{-1}Y), \xi \rangle_{B_\theta} = \langle \mathcal{S}_\xi (g_*)^{-1} X, (g_*)^{-1} Y \rangle_{B_\theta} \\ &= \langle g_* \mathcal{S}_\xi (g_*)^{-1} X, Y \rangle_{B_\theta}\end{aligned}$$

for all $X, Y \in T_o(S_\Phi \cdot o)$ and all $\xi \in \nu_o(S_\Phi \cdot o)$. Hence we have that the principal curvatures of the shape operator $\mathcal{S}_\xi$, with $\xi \in \nu_o(S_\Phi \cdot o)$, coincide with the principal curvatures of the shape operator $\mathcal{S}_\eta$, for some $\eta \in \mathfrak{a}^\Phi$. Hence, the result follows. □

Let us start with the study of the geometry of the submanifold $S_\Phi \cdot o$. Now, we identify the tangent space with $\mathfrak{s}_\Phi = \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$. Take $H_\alpha \in \mathfrak{a}^\Phi$ with $\alpha \in \Phi$ and $B \in \mathfrak{a}_\Phi$. Then, using that $\mathfrak{a}$ is an abelian subalgebra satisfying $\theta|_\mathfrak{a} = -\operatorname{id}$, and recalling (1.6) we deduce

$$4\langle \nabla_B H_\alpha, Z\rangle_{AN} = \langle [B, H_\alpha] + [\theta B, H_\alpha] - [B, \theta H_\alpha], Z\rangle_{B_\theta} = 0, \tag{6.3}$$

for all $Z \in \mathfrak{a} \oplus \mathfrak{n}$. This shows that $\mathcal{S}_\xi \mathfrak{a}_\Phi = 0$ for all $\xi \in \mathfrak{a}^\Phi$. Therefore, we just need to study the shape operator $\mathcal{S}_\xi$ when restricted to $\mathfrak{n}_\Phi$, for all $\xi \in \mathfrak{a}^\Phi$. Take $X_\lambda \in \mathfrak{g}_\lambda$ with $\lambda \in \Sigma^\Phi$. Thus, we have

$$\begin{aligned} 4\langle \nabla_{X_\lambda} H_\alpha, Z\rangle_{AN} &= \langle [X_\lambda, H_\alpha] + [\theta X_\lambda, H_\alpha] - [X_\lambda, \theta H_\alpha], Z\rangle_{B_\theta} \\ &= -2\langle [H_\alpha, X_\lambda], Z\rangle_{B_\theta} = -|\alpha|^2 A_{\alpha,\lambda}\langle X_\lambda, Z\rangle_{B_\theta}. \end{aligned}$$

Take the unit normal vector $\xi = |\alpha|^{-1} H_\alpha$. Then, we have that

$$\mathcal{S}_\xi X_\lambda = \frac{|\alpha|}{2} A_{\alpha,\lambda} X_\lambda, \tag{6.4}$$

where $X_\lambda \in \mathfrak{g}_\lambda \subset \mathfrak{n}_\Phi$. Let Φ be a proper subset of the set of simple roots Π. Let $\xi = \sum_{\alpha\in\Phi} a_\alpha H_\alpha$ be a unit normal vector of the submanifold $S_\Phi \cdot o$. Take $X_\lambda \in \mathfrak{g}_\lambda$ for $\lambda \in \Sigma^\Phi$. Using (6.4) and the linearity of the shape operator we obtain

$$\mathcal{S}_\xi X_\lambda = \sum_{\alpha\in\Phi} a_\alpha \frac{|\alpha|^2}{2} A_{\alpha,\lambda} X_\lambda. \tag{6.5}$$

According to this equation, all the vectors in $\mathfrak{g}_\lambda$ are eigenvectors of $\mathcal{S}_\xi$ with the same principal curvature for each $\lambda \in \Sigma^\Phi$. Thus it makes sense to talk about the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\lambda$ for each root $\lambda \in \Sigma^\Phi$. In most of the cases all the roots in the set Φ will have the same length. Hence, the calculation of the Cartan integers of the form $A_{\alpha,\lambda}$, where $\alpha \in \Phi$ and $\lambda \in \Sigma^\Phi$, is the key point in the investigation of the austerity of the orbit $S_\Phi \cdot o$.

In order to calculate Cartan integers, we reorganize the roots by using a generalization of the concept of α-string (Subsection 1.5.1) and the concept of (α, β)-string (Chapter 5). Let Φ be a proper subset of the set of simple roots Π. Consider a root $\lambda \in \Sigma \cup \{0\}$. We define the Φ-string containing λ as the set of all elements in $\Sigma \cup \{0\}$ of the form $\lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}$ for all $\alpha \in \Phi$. In what follows, we will write $I(\lambda, \Phi)$ to denote the Φ-string of λ, for $\lambda \in \Sigma$.

Let Φ be a proper subset of the set of simple roots Π of the root system Σ. We will say that two roots $\gamma_1, \gamma_2 \in \Sigma^+$ are Φ-related if and only if the element $\gamma_1 - \gamma_2$ is spanned by Φ. This relation is an equivalence relation in Σ^+ and also in Σ^Φ. Let λ be a root in

Σ^Φ. Let $\mathcal{S}$ be the shape operator of the submanifold $S_\Phi \cdot o$. By *the restriction of* $\mathcal{S}$ *to the* Φ*-string of* λ we will refer to the restriction of $\mathcal{S}$ to the vector subspace of $\mathfrak{n}_\Phi \subset \mathfrak{s}_\Phi$

$$\bigoplus_{\alpha \in I(\lambda,\Phi)} \mathfrak{g}_\alpha.$$

Furthermore, we will say that $\mathcal{S}$ is austere when restricted to the Φ-string of λ if for each unit vector ξ normal to $S_\Phi \cdot o$ and each principal curvature μ of $\mathcal{S}_\xi$ when restricted to the Φ-string of λ, then $-\mu$ is also a principal curvature of $\mathcal{S}_\xi$ when restricted to the Φ-string of λ with the same algebraic multiplicity as μ.

Thus, the decomposition into Φ-strings induces a partition of Σ^Φ and we can calculate the shape operator $\mathcal{S}$ of the submanifold $S_\Phi \cdot o$ by calculating its restriction to each one of the Φ-strings. The key point is that these Φ-strings adopt just a few configurations that we can control. This fact motivates the analysis of strings, that will play a crucial role in what follows. In order to determine the Φ-strings, it is essential to understand first how roots are constructed by means of simple roots. The following well-known lemma addresses this question. Roughly speaking, it says that each non-simple positive root can be obtained by adding a simple root to a positive root.

Lemma 6.1.2. [69, p. 204, Exercise 7] *Let* Π *be a set of simple roots of a root system* Σ. *Any* $\lambda \in \Sigma^+$ *can be written in the form*

$$\lambda = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k},$$

where $\lambda_{i_j} \in \Pi$ *and each partial summand from the left is in* Σ^+.

Proof. We prove this by induction on the level $l(\lambda)$ of the positive root $\lambda \in \Sigma^+$. Recall that λ can be written as $\lambda = \sum_{\alpha\in\Pi} n_\alpha \alpha$, for some integers $n_\alpha \geq 0$, for each $\alpha \in \Pi$. The claim is obvious if $l(\lambda) = 1$. Assume that it is true for level $k > 1$ and let $\lambda \in \Sigma^+$ be a positive root with $l(\lambda) = k + 1$. If $\langle \lambda, \alpha \rangle \leq 0$ for all $\alpha \in \Pi$, then we would have

$$0 < \langle \lambda, \lambda \rangle = \langle \lambda, \sum_{\alpha\in\Pi} n_\alpha \alpha \rangle \leq 0.$$

Thus there exists $\alpha \in \Pi$ such that $\langle \lambda, \alpha \rangle > 0$. Then, using Proposition 1.5.1 (iv), we deduce that $\lambda - \alpha \in \Sigma$. Moreover, $\lambda - \alpha \in \Sigma^+$ as $l(\lambda) > 2$ and $l(\alpha) = 1$. So we may write $\lambda = (\lambda - \alpha) + \alpha$ and the result follows by applying the induction hypothesis to $\lambda - \alpha$, which has level k. □

It is also important to understand that the sum of roots spanned by orthogonal subsets of Π cannot be a root. The following result makes this fact precise.

Proposition 6.1.3. *Let* Φ_0, $\Phi_1 \subset \Pi$ *be orthogonal subsets. Let* λ_0, $\lambda_1 \in \Sigma$ *be roots spanned by* Φ_0 *and* Φ_1, *respectively. Then* $\pm\lambda_0 \pm \lambda_1$ *cannot be roots.*

Proof. Put $\lambda_0 = \sum_{\alpha\in\Phi_0} n_\alpha\alpha$ and $\lambda_1 = \sum_{\beta\in\Phi_1} n_\beta\beta$, where n_ν is an integer for all $\nu \in \Phi_0\cup\Phi_1$.

Assume first that λ_0 and λ_1 are positive roots. Then there must exist $\alpha \in \Phi_0$ and $\beta \in \Phi_1$ such that $n_\alpha > 0$ and $n_\beta > 0$. Note that $A_{\gamma_0,\gamma_1} = 0$ for each element γ_k in the span of Φ_k, with $k \in \{0, 1\}$. If $\lambda_0 + \lambda_1$ were a root, using Proposition 1.5.1 (v) together with $A_{\lambda_0,\lambda_1} = 0$ we would obtain that $\lambda_0 - \lambda_1$ and $\lambda_1 - \lambda_0$ are both roots. However, we have that

$$\lambda_0 - \lambda_1 = \sum_{\alpha\in\Phi_0} n_\alpha\alpha + \sum_{\beta\in\Phi_1}(-n_\beta)\beta, \tag{6.6}$$

where $n_\alpha > 0$ for some $\alpha \in \Phi_0$ and $-n_\beta < 0$ for some $\beta \in \Phi_1$. This is a contradiction. Since $\lambda_0 + \lambda_1$ is not a root, from Proposition 1.5.1 (i) neither is $-(\lambda_0 + \lambda_1)$. In particular, equation (6.6) proves that $\lambda_0 - \lambda_1$ is not a root. Again, using Proposition 1.5.1 (i) we deduce that neither is $-\lambda_0 + \lambda_1$.

If λ_0 and λ_1 are both negative, then from Proposition 1.5.1 (i) we have that $-\lambda_0$ and $-\lambda_1$ are positive and we proceed as above.

Finally, let us assume that λ_0 is positive and λ_1 is negative. Then $-\lambda_1$ is positive and proceeding as above we can prove that neither $\lambda_0 - \lambda_1$ nor $\lambda_0 + \lambda_1$ are roots. Using again Proposition 1.5.1 (i) the result follows. □

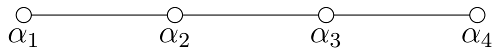
6.1.2 The diagram of a Φ-string

As mentioned above, in order to make a systematic approach to the study and classification of austere submanifolds of the form $S_\Phi \cdot o$, we will consider an orthogonal decomposition of their tangent space. This decomposition comes from a decomposition into Φ-strings of the set Σ^Φ of positive roots not spanned by Φ. Furthermore, we will construct a diagram associated with each Φ-string. These diagrams will allow us to calculate the principal curvatures of the shape operator $\mathcal{S}$ when restricted to each Φ-string very efficiently. Moreover, each symmetric space G/K will admit just a few configurations for its Φ-strings. Thus, the examination of these diagrams will lead us to determine if the submanifold $S_\Phi \cdot o$ is austere or not directly. Roughly speaking, we will need certain symmetry conditions in the diagrams of Φ-strings for $S_\Phi \cdot o$ to be austere. This section is devoted to the explanation of the construction of these diagrams as well as to characterizing the austerity of $S_\Phi \cdot o$ in terms of them.

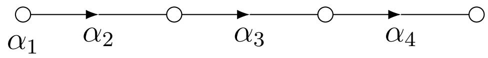
Take $\Phi \subset \Pi$. The construction of the diagram of a Φ-string is as follows. Let $\lambda \in \Phi$ be a root of minimum level in its Φ-string (we will see in Proposition 6.2.1 (i) that it is unique). We will draw a node for each root ν in the Φ-string of λ. We will point out the node λ of minimum level as a starting node. Now, the nodes corresponding to the roots ν_1 and ν_2 in the Φ-string of λ will be connected (by a single line) if and only if $\nu_1 - \nu_2 = \pm\alpha$ for some $\alpha \in \Phi$. In this case, the arrow connecting the node ν_1 to the node ν_2 will have label α. This arrow will be oriented pointing to the highest level root among the two roots that it connects. Fix a node ν in the diagram of the Φ-string of λ. If one considers a path from λ to ν following the arrows, then ν will be the sum of λ and all the labels $\alpha \in \Phi$ associated with the arrows of the chosen path.

In order to clarify the construction of these diagrams, we include two particular examples.

Example 6.1.4. Assume that Π is an A_4 simple system with Dynkin diagram

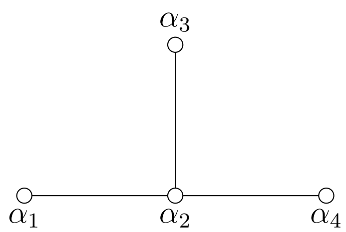


and put $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$. The Φ-string of α_1 consists of the roots α_1, $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3$ and $\alpha_1+\alpha_2+\alpha_3+\alpha_4$. Thus, the diagram of the Φ-string of α_1 is of the form

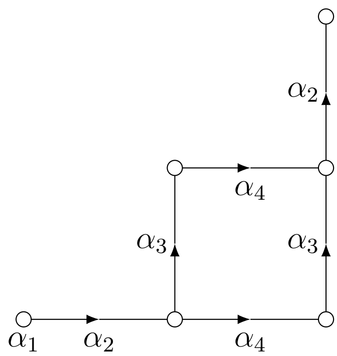


Let us continue with a more interesting example.

Example 6.1.5. Assume that Π is a D_4 simple system with Dynkin diagram



and put $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$. In this case, the Φ string of α_1 consists of the roots α_1, $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4$ and $\alpha_1+2\alpha_2+\alpha_3+\alpha_4$. Thus, we obtain the diagram



for the Φ-string of α_1.

Let us come back to a more general situation. Let $\xi = \sum_{\alpha\in\Phi} a_\alpha H_\alpha$ be a unit normal vector to the submanifold $S_\Phi \cdot o$. Take a root $\nu \in \Sigma^\Phi$ and $X_\nu \in \mathfrak{g}_\nu$. Recall from (6.5) that the principal curvature of the shape operator $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$ is

$$2^{-1} \sum_{\alpha\in\Phi} a_\alpha |\alpha|^2 A_{\alpha,\nu}, \tag{6.7}$$

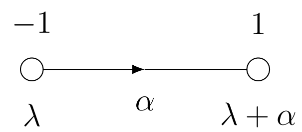
for each $\nu \in \Sigma^{\Phi}$.

Before going on, we will focus on Φ-strings when $\Phi = \{\alpha\}$, for some $\alpha \in \Sigma$. Indeed, we will determine the possibilities for these strings and we will also specify the Cartan integers associated with them.

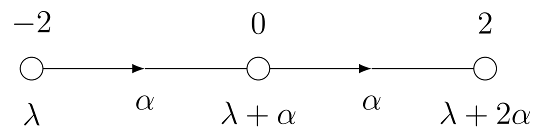
Proposition 6.1.6. *Let α, $\lambda \in \Sigma$ be non-proportional roots. Then, the α-string of λ must adopt one of the following configurations. For each case we draw the diagram of the α-string of λ and above each node ν we write the Cartan integer $A_{\alpha,\nu}$.*

(i) The α-string of λ consists of the root λ. In this case we have that $A_{\alpha,\lambda} = 0$.

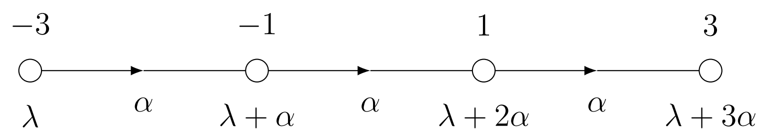
(ii) The α-string of λ consists of the roots λ and $\lambda+\alpha$ and both have the same multiplicity. The diagram of this α-string is of the form:



(iii) The α-string of λ consists of the roots λ, $\lambda+\alpha$ and $\lambda+2\alpha$. The roots λ and $\lambda+2\alpha$ have the same multiplicity. This case just appears when Σ has either a B_r, C_r, BC_r or F_4 Dynkin diagram. The diagram of this α-string is of the form:



(iv) The α-string of λ consists of the roots λ, $\lambda+\alpha$, $\lambda+2\alpha$ and $\lambda+3\alpha$ and all of them have the same multiplicity. This case just appears when Σ has a G_2 Dynkin diagram. The diagram of this α-string is of the form:



Proof. The claims concerning the configuration of the strings come directly from Proposition 1.5.1 (ii), (v). The claims about the multiplicities come from Lemma 5.1.1 and the fact that in G_2 all the roots have the same multiplicity from [7, p. 339]. □

In the following lines, we will explain a method that allows us to calculate the principal curvatures of the shape operator of the submanifold $S_{\Phi} \cdot o$ when restricted to a Φ-string, just by inspecting its diagram. Let $\lambda \in \Sigma^{\Phi}$ be a root of minimum level in its Φ-string and assume that ν belongs to the Φ-string of λ.

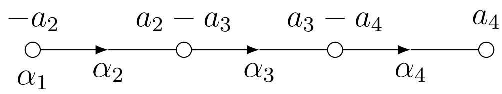
Take a root $\alpha \in \Phi$. First, we will assume that $|\alpha| \geq |\nu|$. From Proposition 1.5.1 (iii), (v) we deduce that $\nu - \alpha$ is a root if and only if $A_{\alpha,\nu} = 1$; we deduce that $\nu + \alpha$ is a root if and only if $A_{\alpha,\nu} = -1$; and we deduce that neither $\nu + \alpha$ nor $\nu - \alpha$ are roots if and only if $A_{\alpha,\nu} = 0$. On the one hand, if $A_{\alpha,\nu} = 1$, then the addend $2^{-1}|\alpha|^2 a_\alpha$ in (6.7) will appear in the expression of the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$. But $A_{\alpha,\nu} = 1$ if and only if there is an arrow with label α in the diagram from the node $\nu - \alpha$ pointing to the node ν. On the other hand, if $A_{\alpha,\nu} = -1$, then the addend $-2^{-1}|\alpha|^2 a_\alpha$ in (6.7) will appear in the expression of the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$. But $A_{\alpha,\nu} = -1$ if and only if there is an arrow with label α in the diagram from the node ν pointing to the node $\nu + \alpha$. Finally, if $A_{\alpha,\nu} = 0$, neither the addend $2^{-1}|\alpha|^2 a_\alpha$ nor the addend $-2^{-1}|\alpha|^2 a_\alpha$ in (6.7) will appear in the expression of the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$. But $A_{\alpha,\nu} = 0$ if and only if there are no arrows from the node ν or reaching the node ν with label α.

In particular, assume that $\nu \in \Sigma^\Phi$ satisfies $|\alpha| \geq |\nu|$ for all $\alpha \in \Phi$. Let Φ_1 be the set of roots $\alpha_1 \in \Phi$ such that there is an arrow with label α_1 in the diagram of the Φ-string of λ from the node $\nu - \alpha_1$ pointing to the node ν. Let Φ_2 be the set of roots $\alpha_2 \in \Phi$ such that there is an arrow with label α_2 in the diagram of the Φ-string of λ from the node ν pointing to the node $\nu + \alpha_2$. Recall that $\xi = \sum_{\alpha \in \Phi} a_\alpha H_\alpha$. Then, from (6.5) we obtain that

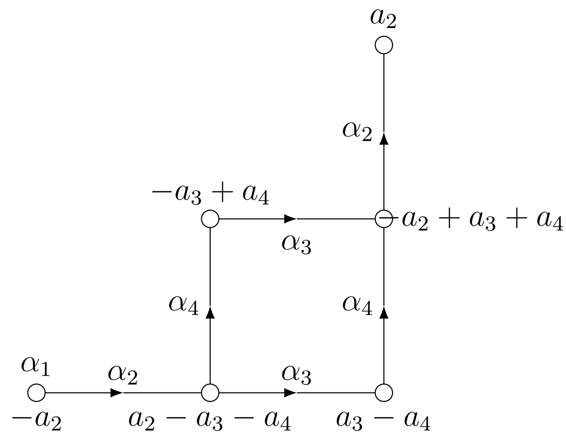
$$\sum_{\alpha_1 \in \Phi_1} a_{\alpha_1} \frac{|\alpha_1|^2}{2} - \sum_{\alpha_2 \in \Phi_2} a_{\alpha_2} \frac{|\alpha_2|^2}{2}$$

is the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$. We will apply this information to the particular examples considered above.

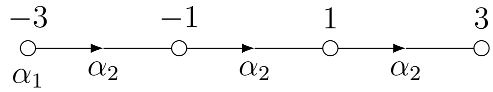
Example 6.1.7 (Continuation of Example 6.1.4). Recall that in Example 6.1.4 we were studying an A_4 simple system with $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$. Put $\xi = \sum_{i=2}^4 a_i H_{\alpha_i}$ for a unit normal vector to the submanifold $S_\Phi \cdot o$. Since all the roots have the same length, put $|\alpha_1|^2 = 2$ for the sake of simplicity. In the following diagram of the Φ-string of α_1, we write the principal curvature associated with each root space at the top of the corresponding node.



Example 6.1.8 (Continuation of Example 6.1.5). Let us also use Example 6.1.5 to clarify these ideas. Recall that in that example we considered a D_4 simple system and $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$. Put $\xi = \sum_{i=2}^4 a_i H_{\alpha_i}$ for a unit normal vector to the submanifold $S_\Phi \cdot o$. Since all the roots have the same length, put $|\alpha_1|^2 = 2$ for the sake of simplicity. Again, we complete the diagram of the Φ-string of α_1 writing the principal curvature associated with each root space.



Remark 6.1.9. Recall that $\Phi \subset \Pi$. We have assumed above that $|\alpha| \geq |\nu|$ for all $(\alpha, \nu) \in \Phi \times \Sigma^\Phi$. This will be the case when studying symmetric spaces of non-compact type with A_r, D_r, E_6, E_7 or E_8 Dynkin diagram. Moreover, it will also apply in some cases when the symmetric space has a Dynkin diagram B_r, C_r, F_4, G_2 or BC_r. However, it is possible to extend the study of the principal curvatures in terms of strings. We will not use such approach, but we include here the main ideas for the sake of completeness. As before, let $\mathcal{S}_\xi$ be the shape operator with respect to ξ and consider the Φ-string of λ, where $\lambda \in \Sigma^\Phi$. First, let us assume that we are considering a B_r, C_r or F_4 Dynkin diagram. Take $\alpha \in \Phi$ such that $|\alpha| < |\nu|$, for some $\nu \in \Sigma^\Phi$ in the Φ-string of λ. Hence, we can assume that $2|\alpha|^2 = |\nu|^2$. From Proposition 1.5.1 (iii) , (v) we deduce that $\nu - \alpha$ and $\nu - 2\alpha$ are roots if and only if $A_{\alpha,\nu} = 2$; we have that $\nu + \alpha$ and $\nu + 2\alpha$ are roots if and only if $A_{\alpha,\nu} = -2$; and we deduce that $A_{\alpha,\nu} = 0$ if and only if either both $\nu + \alpha$ and $\nu - \alpha$ are roots, or none of them is a root. On the one hand, if $A_{\alpha,\nu} = 2$ then the coefficient $|\alpha|^2 a_\alpha$ in (6.7) will appear in the expression of the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$. But $A_{\alpha,\nu} = 2$ if and only if there are arrows with label α in the diagram from the nodes $\nu - 2\alpha$ and $\nu - \alpha$ pointing to the nodes $\nu - \alpha$ and ν respectively. On the other hand, if $A_{\alpha,\nu} = -2$ for some root $\alpha \in \Phi$ then the coefficient $-|\alpha|^2 a_\alpha$ in (6.7) will appear in the expression of the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$. But $A_{\alpha,\nu} = -2$ if and only if there are arrows with label α in the diagram from the nodes ν and $\nu + \alpha$ pointing to the nodes $\nu + \alpha$ and $\nu + 2\alpha$ respectively. Finally, if $A_{\alpha,\nu} = 0$ neither the coefficient $2^{-1}|\alpha|^2 a_\alpha$ in (6.7) nor the coefficient $-2^{-1}|\alpha|^2 a_\alpha$ in (6.7) appears in the expression of the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$. But $A_{\alpha,\nu} = 0$ if and only if one of the following condition holds in the diagram: either there are two arrows with label α, one from the node ν and the other one reaching the node ν, or there are no arrows with label α connected to the node ν. There is just one remaining case to consider. Assume we are in the G_2 case. Put $\Pi = \{\alpha_1, \alpha_2\}$, with $|\alpha_1| > |\alpha_2|$, and $\Phi = \{\alpha_2\}$. Then, the Φ-string of α_1 consists of the roots α_1, $\alpha_1 + \alpha_2$, $\alpha_1 + 2\alpha_2$ and $\alpha_1 + 3\alpha_2$. Put $|\alpha_2| = 1$ for simplicity. We just need to consider the unit normal vector $\xi = H_{\alpha_2}$ to $S_\Phi \cdot o$. Hence, the principal curvatures are



and since $2\alpha_1 + 3\alpha_2$ has trivial Φ-string then $S_\Phi \cdot o$ is austere.

6.1.3 Conditions for the austerity of $S_\Phi \cdot o$

The information provided in the discussion above allows us to compute the principal curvatures of the submanifold $S_\Phi \cdot o$ by inspecting the diagram of the Φ-string of λ, for each $\lambda \in \Sigma^\Phi$. However, this idea can be improved. In fact, it would be better to use these diagrams to deduce if the submanifold $S_\Phi \cdot o$ is austere or not directly. In order to do that, we claim some necessary conditions for $S_\Phi \cdot o$ to be austere in terms of Cartan integers, strings and diagrams.

If Φ is a subset of the set Π, $\gamma \in \Sigma^\Phi$ and ξ is a unit normal vector to $S_\Phi \cdot o$, we denote by $\mu_\xi(\gamma)$ the principal curvature of the shape operator $\mathcal{S}_\xi$ associated with the tangent root space $\mathfrak{g}_\gamma$.

Lemma 6.1.10. *Let Φ be a proper subset of the set of simple roots Π. Let $\mathcal{S}$ be the shape operator of the submanifold $S_\Phi \cdot o$. The following two statements are equivalent:*

(a) *For each root $\gamma \in \Sigma^\Phi$ there exists a root $\nu \in \Sigma^\Phi$ such that $A_{\alpha,\gamma} = -A_{\alpha,\nu}$ for all $\alpha \in \Phi$.*

(b) *For each root $\gamma \in \Sigma^\Phi$ and each unit normal vector ξ to $S_\Phi \cdot o$ there exists a root $\nu_\xi \in \Sigma^\Phi$ such that $\mu_\xi(\gamma) = -\mu_\xi(\nu_\xi)$.*

Therefore:

(i) *If the submanifold $S_\Phi \cdot o$ is austere, then for each $\gamma \in \Sigma^\Phi$ there must exist a root $\nu \in \Sigma^\Phi$ such that $A_{\alpha,\gamma} = -A_{\alpha,\nu}$ for all $\alpha \in \Phi$.*

(ii) *Fix a root $\lambda \in \Sigma^\Phi$. If the shape operator $\mathcal{S}$ of the submanifold $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ, then for each root $\gamma \in I(\lambda, \Phi)$ there must exist a root $\nu \in I(\lambda, \Phi)$ such that $A_{\alpha,\gamma} = -A_{\alpha,\nu}$ for all $\alpha \in \Phi$.*

Proof. From (6.5) we easily deduce that (a) implies (b). Let us see the converse.

If $\Phi = \{\alpha\}$ then there is just one unit normal vector up to sign, namely $\xi = |\alpha|^{-1} H_\alpha$. Hence, $\mu_\xi(\nu) = |\alpha| A_{\alpha,\nu}$ for each $\nu \in \Sigma^\Phi$. Then, if $A_{\alpha,\gamma} \neq -A_{\alpha,\nu}$ for each $\nu \in \Sigma^\Phi$, we deduce that $\mu_\xi(\gamma) \neq -\mu_\xi(\nu)$ for each $\nu \in \Sigma^\Phi$.

Assume now that $\Phi \subset \Pi$ contains at least two roots. Consider a root $\gamma \in \Sigma^\Phi$. If $A_{\alpha,\gamma} = 0$ for all $\alpha \in \Phi$ we are done. Thus, assume that $A_{\alpha,\gamma} \neq 0$ for some $\alpha \in \Phi$. We will assume that for each $\nu \in \Sigma^\Phi$ there exists $\beta \in \Phi$ such that $A_{\beta,\gamma} \neq -A_{\beta,\nu}$ and we will get a contradiction. Let $\xi = \sum_{\alpha \in \Phi} a_\alpha H_\alpha$ be a generic unit normal vector to the submanifold $S_\Phi \cdot o$. Recall from (6.5) that $\mu_\xi(\nu) = 2^{-1} \sum_{\alpha \in \Phi} |\alpha|^2 a_\alpha A_{\alpha,\nu}$ is the principal curvature of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_\nu$, for each $\nu \in \Sigma^\Phi$. Put n for the number of elements in Φ. Recall that we are assuming that for each $\nu \in \Sigma^\Phi$ we do not have $A_{\alpha,\gamma} = -A_{\alpha,\nu}$

for all $\alpha \in \Phi$. Using this and $A_{\alpha,\gamma} \neq 0$ for some $\alpha \in \Phi$, we deduce that the equality $\mu_\xi(\gamma) = -\mu_\xi(\nu)$ is the equation of a hyperplane in $\mathbb{R}^n$ with variables $(a_\alpha)_{\alpha\in\Phi}$, for each $\nu \in \Sigma^\Phi \backslash \{\gamma\}$. Note that the number of roots in $\Sigma^\Phi \backslash \{\gamma\}$ is finite and then we obtain a finite number of hyperplanes. Take a unit normal vector $\eta = \sum_{\alpha\in\Phi} b_\alpha H_\alpha$ with the element $(b_\alpha)_{\alpha\in\Phi} \in \mathbb{R}^n$ outside all these hyperplanes. Hence $\mu_\eta(\gamma) \neq -\mu_\eta(\nu)$ for all $\nu \in \Sigma^\Phi$. This proves that (b) implies (a).

Using the equivalence between (a) and (b) and the definition of austerity, then assertions (i) and (ii) follow. □

In summary, Lemma 6.1.10 claims a necessary condition for $S_\Phi \cdot o$ to be austere, and also a necessary condition for the shape operator to be austere when restricted to a Φ-string. However, in both cases we do not get a characterization of austerity. In fact, we have the following difficulty: we can guarantee that for each principal curvature μ there exists the principal curvature $-\mu$, but we cannot guarantee that the multiplicities of the curvatures μ and $-\mu$ coincide. We will now address this question.

Let A, $B \subset \Sigma$ be subsets of the set of roots. A map $f\colon A \to B$ is said to be a *multiplicity-preserving bijection* (respectively *involution*) if f is a bijection (respectively involution with $A = B$) between A and B satisfying that γ and $f(\gamma)$ have the same multiplicity, for each $\gamma \in A$.

Proposition 6.1.11. *Let Φ be a proper subset of the set of simple roots Π. Let $\mathcal{S}$ be the shape operator of the submanifold $S_\Phi \cdot o$ and let $\lambda \in \Sigma^\Phi$ be a root of minimum level in its Φ-string. Then:*

(i) *If there exists a multiplicity-preserving involution $f\colon \Sigma^\Phi \to \Sigma^\Phi$ such that $A_{\alpha,\gamma} = -A_{\alpha,f(\gamma)}$ for all $(\alpha,\gamma) \in \Phi \times \Sigma^\Phi$, then the submanifold $S_\Phi \cdot o$ is austere. If all the roots in Σ^Φ have the same multiplicity, the converse is true.*

(ii) *If there exists a multiplicity-preserving involution $f\colon I(\lambda,\Phi) \to I(\lambda,\Phi)$ such that $A_{\alpha,\gamma} = -A_{\alpha,f(\gamma)}$ for all $(\alpha,\gamma) \in \Phi \times I(\lambda,\Phi)$, then the shape operator of the submanifold $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ. If all the roots in $I(\lambda,\Phi)$ have the same multiplicity, the converse is true.*

(iii) *Let λ and γ be different roots in Σ^Φ of minimum level in their respective Φ-strings. If there exists a multiplicity-preserving bijection $f\colon I(\lambda,\Phi) \to I(\gamma,\Phi)$ such that $A_{\alpha,\nu} = -A_{\alpha,f(\nu)}$ for all $(\alpha,\nu) \in \Phi \times I(\lambda,\Phi)$, then the shape operator of the submanifold $S_\Phi \cdot o$ is austere when restricted to*

$$\bigoplus_{\nu\in I(\lambda,\Phi)\cup I(\gamma,\Phi)} \mathfrak{g}_\nu .$$

If all the roots in $I(\lambda,\Phi) \cup I(\gamma,\Phi)$ have the same multiplicity, the converse is true.

Moreover, for each f as above, we have that $\mu_\xi(\nu) = -\mu_\xi(f(\nu))$ for all unit vector ξ and all ν in Σ^Φ (in case (i)) or in $I(\lambda,\Phi)$ (in cases (ii) and (iii)).

Proof. Recall that $\mu_\xi(\nu)$ denotes the principal curvature of the shape operator $\mathcal{S}_\xi$ when restricted to $\mathfrak{g}_\nu$, for each $\nu \in \Sigma^\Phi$. Consider roots $\gamma_1, \gamma_2 \in \Sigma^\Phi$. The key point of the proof is to see that $\mu_\xi(\gamma_1) = \mu_\xi(\gamma_2)$ for all unit normal vector ξ if and only if $A_{\alpha,\gamma_1} = A_{\alpha,\gamma_2}$ for all $\alpha \in \Phi$. This is clearly true if Φ consists of just one element as follows from (6.5).

Hence, let us assume that Φ contains at least two different elements. One of the implications follows directly from (6.5). Now, assume that A_{α,γ_1} is distinct from A_{α,γ_2} for some $\alpha \in \Phi$. Using this, we will see that $\mu_\eta(\gamma_1) \neq \mu_\eta(\gamma_2)$ for some unit normal vector η. Indeed, let $\xi = \sum_{\alpha\in\Phi} a_\alpha H_\alpha$ be a generic unit normal vector to the submanifold $S_\Phi \cdot o$. Put n for the number of elements in Φ. Since $A_{\alpha,\gamma_1} \neq A_{\alpha,\gamma_2}$, from (6.5) we deduce that the equality $\mu_\xi(\gamma_1) = \mu_\xi(\gamma_2)$ is the equation of a hyperplane in $\mathbb{R}^n$ with variables $(a_\alpha)_{\alpha\in\Phi}$. Take a unit normal vector $\eta = \sum_{\alpha\in\Phi} b_\alpha H_\alpha$ with the element $(b_\alpha)_{\alpha\in\Phi} \in \mathbb{R}^n$ not in this hyperplane. Then $\mu_\eta(\gamma_1) \neq \mu_\eta(\gamma_2)$. This proves that the principal curvatures of $\mathcal{S}_\xi$ coincide when restricted to the root spaces $\mathfrak{g}_{\gamma_1}$ and $\mathfrak{g}_{\gamma_2}$ for all unit normal vector ξ if and only if $A_{\alpha,\gamma_1} = A_{\alpha,\gamma_2}$ for all $\alpha \in \Phi$.

We define now the following equivalence relation in Σ^Φ: $\gamma_1, \gamma_2 \in \Sigma^\Phi$ are related if and only if $A_{\alpha,\gamma_1} = A_{\alpha,\gamma_2}$ for all $\alpha \in \Phi$. Put $[\gamma]$ for the equivalence class of the root $\gamma \in \Sigma^\Phi$. From the previous paragraph, two roots $\gamma, \gamma' \in \Sigma^\Phi$ are related if and only if $\mu_\xi(\gamma) = \mu_\xi(\gamma')$ for all unit normal vector ξ. Therefore, it makes sense to write $\mu([\gamma])$ for the principal curvature of the shape operator $\mathcal{S}$ when restricted to $\mathfrak{g}_{\gamma'}$, for each $\gamma' \in [\gamma]$. Another key observation is that the multiplicity of the principal curvature $\mu([\gamma])$ is exactly the sum of the multiplicities of all the roots $\gamma' \in [\gamma]$.

Assume first that there exists a multiplicity-preserving involution f satisfying the hypothesis of (i). Then, from (6.5) we have that $\mu([\gamma]) = -\mu([f(\gamma)])$ for each $\gamma \in \Sigma^\Phi$. Since f preserves multiplicities, the result follows.

Let us show the converse with the extra assumption on the multiplicity of the roots.

Consider a root $\gamma \in \Sigma^\Phi$ such that $\mu([\gamma]) \neq 0$. Since the submanifold $S_\Phi \cdot o$ is austere, from Lemma 6.1.10 (i) we deduce that there is a root $\gamma' \in \Sigma^\Phi$ such that $A_{\alpha,\gamma} = -A_{\alpha,\gamma'}$ for all $\alpha \in \Phi$. Moreover, since all the multiplicities of the roots in Σ^Φ are equal and $S_\Phi \cdot o$ is austere, then the classes $[\gamma]$ and $[\gamma']$ must have the same number of elements. Now, consider a set $\Psi \subset \Sigma^\Phi$ satisfying these conditions:

(i) All the roots in Ψ have non-trivial Φ-strings, and

(ii) any two roots $\gamma_1, \gamma_2 \in \Psi$ satisfy that $\mu(\gamma_1) \neq \pm\mu(\gamma_2)$.

Let Ψ' be a set defined in the following way: for each $\gamma \in \Psi$ take exactly one $\gamma' \in \Sigma^\Phi$ such that $\mu([\gamma]) = -\mu([\gamma'])$ to be in Ψ'. Then $|\Psi| = |\Psi'|$ and the pair

$$\left(\bigcup_{\gamma\in\Psi} [\gamma], \bigcup_{\gamma'\in\Psi'} [\gamma'] \right) \tag{6.8}$$

is a partition of the subset of roots in Σ^Φ with non-trivial Φ-string. Moreover, if $\mu \neq 0$ is a principal curvature, then there exists $\nu \in \Psi \cup \Psi'$ such that $\mu = \mu(\nu)$. Now, for each $\gamma \in \Psi$ take the unique element $\gamma' \in \Psi'$ such that $\mu([\gamma]) = -\mu([\gamma'])$ and define a bijection

$f_\gamma\colon [\gamma] \to [\gamma']$, and for each $\gamma' \in \Psi'$ define the bijection $f_{\gamma'}\colon [\gamma'] \to [\gamma]$ given by $f_{\gamma'} = f_\gamma^{-1}$. Recall from the partition in (6.8) that for each $\nu \in \Sigma^\Phi$ with non-trivial Φ-string there must exist a root γ either in Ψ or in Ψ' such that $\nu \in [\gamma]$. Now, consider the map $f\colon \Sigma^\Phi \to \Sigma^\Phi$ defined by

$$f(\nu) = \begin{cases} \nu & \text{if } I(\nu,\Phi) = \{\nu\}, \\ f_\gamma(\nu) & \text{if } I(\nu,\Phi) \neq \{\nu\} \text{ and } \nu \in [\gamma], \end{cases}$$

Note that f is a multiplicity-preserving involution and then (i) follows. The same idea holds in order to prove prove (ii) and (iii). □

The above result will be very useful in order to study the austerity of the shape operator when restricted to each Φ-string and consequently the austerity of the submanifold $S_\Phi \cdot o$. In particular, it makes very easy to check austerity by using the diagram of Φ-strings. Recall that each node in the diagram is connected to other nodes by certain oriented arrows. We will say that two nodes have *opposite arrows* if they are connected with arrows of exactly the same labels but with opposite orientations. More precisely, two nodes ν and ν' have opposite arrows if the labels of the arrows leaving from ν coincide with the labels of the arrows arriving at ν', and the labels of the arrows arriving at ν coincide with the labels of the arrows leaving from ν'. In particular, if one root has trivial Φ-string, we will say that it has opposite arrows with respect to itself.

Corollary 6.1.12. *Let Φ be a proper subset of Π. Assume that $|\alpha| \geq |\nu|$ for all $(\alpha,\nu) \in \Phi \times \Sigma^\Phi$. Let $\mathcal{S}$ be the shape operator of the submanifold $S_\Phi \cdot o$. Then, we have:*

(i) *If there exists a multiplicity-preserving involution $f\colon \Sigma^\Phi \to \Sigma^\Phi$ such that ν and $f(\nu)$ have opposite arrows for each $\nu \in \Sigma^\Phi$, then $S_\Phi \cdot o$ is austere. If all the roots in Σ^Φ have the same multiplicity, the converse is true.*

(ii) *Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. If there exists a multiplicity-preserving involution $f\colon I(\lambda,\Phi) \to I(\lambda,\Phi)$ such that ν and $f(\nu)$ have opposite arrows for each $\nu \in I(\lambda,\Phi)$, then the shape operator $\mathcal{S}$ is austere when restricted to the Φ-string of λ. If all the roots in $I(\lambda,\Phi)$ have the same multiplicity, the converse is true.*

(iii) *Let λ and γ be different roots in Σ^Φ of minimum level in their respective Φ-strings. If there exists a multiplicity-preserving bijection $f\colon I(\lambda,\Phi) \to I(\gamma,\Phi)$ such that ν and $f(\nu)$ have opposite arrows for each $\nu \in I(\lambda,\Phi)$, then the shape operator $\mathcal{S}$ is austere when restricted to*

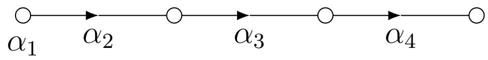
$$\bigoplus_{\nu \in I(\lambda,\Phi)\cup I(\gamma,\Phi)} \mathfrak{g}_\nu.$$

If all the roots in $I(\lambda,\Phi) \cup I(\gamma,\Phi)$ have the same multiplicity, the converse is true.

Proof. Note that $A_{\alpha,\nu} \in \{0,\pm 1\}$ for all $(\alpha,\nu) \in \Phi \times \Sigma^\Phi$ by means of Proposition 1.5.1 (iii). Take a root $\gamma \in \Sigma^\Phi$. Then we have that: $\gamma + \alpha$ is root if and only if $A_{\alpha,\gamma} = -1$; $\gamma - \alpha$ is root if and only if $A_{\alpha,\gamma} = 1$; and neither $\gamma + \alpha$ nor $\gamma - \alpha$ are roots if and only if $A_{\alpha,\gamma} = 0$. Now the claim follows directly from Proposition 6.1.11. □

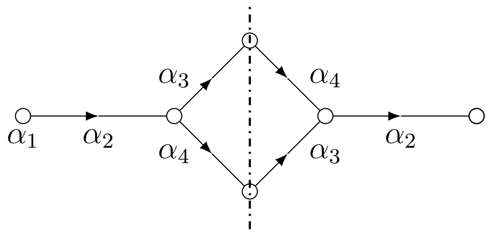
Let us apply the above results to the two examples we have considered above.

Example 6.1.13 (Continuation of Example 6.1.4 and Example 6.1.7)*.* Recall that in Example 6.1.4 we were studying a Φ-string with diagram



Hence, according to the second statement of Corollary 6.1.12 (ii), the shape operator $\mathcal{S}$ of the submanifold $S_\Phi \cdot o$ is not austere when restricted to the Φ-string of α_1. Indeed, the node corresponding to α_1 is connected to just one node by an arrow with label α_2. However, no node has opposite arrows with respect to α_1, that is, there is no node admitting exactly one arrow arriving at it with label α_2.

Example 6.1.14 (Continuation of Example 6.1.5 and Example 6.1.8)*.* In this case, the Φ-string has a diagram of the form



where all the roots have the same multiplicity. Hence, a reflection with respect to the vertical line that interchanges the roots on the line satisfies the conditions of Proposition 6.1.11 (ii). Indeed, it is easy to see that each node is sent to a node with opposite arrows.

In summary, in this section we have seen how to construct the diagram of a Φ-string. Moreover, we have characterized the austerity of the submanifold $S_\Phi \cdot o$ in terms of these diagrams. This fact justifies the crucial role that Φ-strings and their diagrams will play in what follows. In particular, the diagrams will allow us to argue that many examples are not austere by means of Corollary 6.1.12 in a very efficient way. Moreover, these diagrams will also give the hint to construct the map f of Proposition 6.1.11 in order to prove the austerity of $S_\Phi \cdot o$.

6.2 The study of Φ-strings

This section is devoted to the explicit inspection of the configuration of Φ-strings, where Φ is a proper subset of the set Π of simple roots. Indeed, we will start by determining the roots of each Φ-string explicitly, under certain convenient hypotheses on the set Φ. This will allow us to calculate the principal curvatures of the shape operator $\mathcal{S}$ of the submanifold $S_\Phi \cdot o$ when restricted to that Φ-string using the ideas explained in Section 6.1. Moreover, we will use either Proposition 6.1.10 or Proposition 6.1.11 in order to see if the shape operator $\mathcal{S}$ is austere when restricted to the Φ-string under consideration. Altogether, this information will allow us (see Section 6.3) to conclude the classification of the austere submanifolds of the form $S_\Phi \cdot o$ in symmetric spaces G/K with Dynkin diagram A_r, B_r, C_r, D_r or BC_r.

Recall that Φ is a proper subset of the set of simple roots Π. We start with a result focusing on the root of minimum level in a Φ-string. In fact, we are interested in its uniqueness but especially in how to detect when a root $\lambda \in \Sigma^\Phi$ is of minimum level in its Φ-string just by using the Cartan integers of the form $A_{\alpha,\lambda}$, for each $\alpha \in \Phi$.

Proposition 6.2.1. *Let Π be the set of simple roots of the root system Σ. Let Φ be a proper subset of Π and let $\lambda \in \Sigma^\Phi$ be a root of minimum level in its non-trivial Φ-string. Then:*

(i) *The set $\mathrm{span}_{\mathbb{Z}}(\{\lambda\} \cup \Phi) \cap \Sigma$ is a root subsystem of Σ for which $\{\lambda\} \cup \Phi$ is a simple system. Moreover, λ is the unique root of minimum level in its Φ-string.*

(ii) *If Φ is connected and γ is not the root of minimum level in the Φ-string of λ, then there exists a root $\alpha \in \Phi$ such that $\gamma - \alpha$ is a root in the Φ-string of λ.*

(iii) *Assume that Φ is connected and that $|\alpha| \geq |\nu|$, for all $(\alpha, \nu) \in \Phi \times \Sigma^\Phi$. Then, a root $\gamma \in \Sigma^\Phi$ is the root of minimum level in its non-trivial Φ-string if and only if there exists a root $\alpha \in \Phi$ such that $A_{\alpha,\gamma} < 0$ and $A_{\beta,\gamma} = 0$, for all $\beta \in \Phi \backslash \{\alpha\}$.*

Proof. (i): Let $\lambda \in \Sigma^\Phi$ be a root of minimum level in its Φ-string. Since λ is not spanned by Φ, we have that $\Pi_\lambda = \{\lambda\} \cup \Phi$ is a basis for its span. The set $\Sigma \cap \mathrm{span}_{\mathbb{Z}} \Pi_\lambda$ satisfies the three conditions of a root system (see Subsection 1.5.1). We will denote by $\Sigma_\lambda = \Sigma \cap \mathrm{span}_{\mathbb{Z}} \Pi_\lambda$ the new root system and use the positivity criterion in Σ to induce a positivity criterion in Σ_λ. Now, we need to see that $\Pi_\lambda = \{\lambda\} \cup \Phi$ is a simple system for the root system Σ_λ. In other words, we need to see that each root $\alpha \in \Pi_\lambda$ cannot be written as $\alpha = \nu_1 + \nu_2$, for any ν_1, $\nu_2 \in \Sigma_\lambda^+$. In particular, this is true for all $\alpha \in \Phi$, since it is true in the root system Σ and $\Sigma_\lambda \subset \Sigma$. Put $\lambda = \sum_{\beta \in \Pi} m_\beta \beta$. Since λ is not spanned by Φ, we have that $m_\beta > 0$ for some $\beta \in \Pi \backslash \Phi$. In particular, for each ν in the span of Φ we have that

$$-k\lambda + \nu \notin \Sigma^+ \tag{6.9}$$

and thus cannot be an element in Σ_λ^+, for all $k > 0$. Assume now that

$$\lambda = \nu_1 + \nu_2 = \left(n_\lambda^1 \lambda + \sum_{\alpha \in \Phi} n_\alpha^1 \alpha \right) + \left(n_\lambda^2 \lambda + \sum_{\alpha \in \Phi} n_\alpha^2 \alpha \right), \tag{6.10}$$

where ν_1, $\nu_2 \in \Sigma_\lambda^+$ and the coefficient n_ν^k is integer for all $k \in \{1, 2\}$ and $\nu \in \Phi \cup \{\lambda\}$. Since Π_λ is a basis for its span, from (6.10) we deduce that $1 = n_\lambda^1 + n_\lambda^2$ and that $n_\alpha^1 = -n_\alpha^2$, for each $\alpha \in \Phi$. Thus, we can write

$$\lambda = \nu_1 + \nu_2 = \left(n_\lambda \lambda + \sum_{\alpha \in \Phi} n_\alpha \alpha \right) + \left((1 - n_\lambda)\lambda + \sum_{\alpha \in \Phi} (-n_\alpha)\alpha \right). \tag{6.11}$$

Since $n_\lambda \lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha$ and $(1-n_\lambda)\lambda + \sum_{\alpha\in\Phi}(-n_\alpha)\alpha$ must be in Σ^+, we deduce that

$$l(\lambda)n_\lambda + \sum_{\alpha\in\Phi} n_\alpha > 0 \quad \text{and} \quad l(\lambda)(1-n_\lambda) + \sum_{\alpha\in\Phi}(-n_\alpha) > 0.$$

But this is equivalent to the inequality

$$0 < l(\lambda)n_\lambda + \sum_{\alpha\in\Phi} n_\alpha < l(\lambda).$$

Without loss of generality, we can assume that $n_\lambda > 0$ (if not, we would have $1 - n_\lambda > 0$ and rename coefficients). If $n_\lambda > 1$, then $(1-n_\lambda) < 0$ and from (6.9) we deduce that ν_2 cannot be a positive root. Thus, $n_\lambda = 1$ and

$$\lambda = \nu_1 + \nu_2 = \left(\lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha\right) + \left(\sum_{\alpha\in\Phi}(-n_\alpha)\alpha\right). \tag{6.12}$$

Now, if $\sum_{\alpha\in\Phi} n_\alpha < 0$, then λ would not be of minimum level in its Φ-string. Thus $\sum_{\alpha\in\Phi} n_\alpha \geq 0$ and ν_2 cannot be a positive root. This proves that Π_λ is a simple system.

Let $\gamma = \lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha$ be another root of minimum level in the Φ-string of λ. Note that calculating the roots of the Φ-string of λ in the root system Σ is equivalent to studying the roots of the form $n_\lambda \lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha$ with $n_\lambda = 1$ in the root system Σ_λ with simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$. Hence, we think now in the root system Σ_λ. Since $n_\lambda = 1$, we deduce that $n_\alpha \geq 0$ for all $\alpha \in \Phi$. Then we have $\sum_{\alpha\in\Phi} n_\alpha = 0$, which means that $n_\alpha = 0$ for each $\alpha \in \Phi$. Thus $\gamma = \lambda$. This proves the uniqueness of the root of minimum level in a Φ-string and (i) follows.

(ii): We will proceed by induction on the number of roots in Φ. If $|\Phi| = 1$ the result follows from Proposition 1.5.1 directly. In particular, since Φ is a proper subset of Π, if Π is a G_2 simple system then Φ contains exactly one element and the result follows. Thus, we can assume that Π is not a G_2 simple system from now on.

Assume then that our claim is true for Ψ-strings under the hypotheses of (ii), with $|\Psi| = n-1$, and put $|\Phi| = n$. Let $\gamma \in \Sigma^\Phi$ be a root not of minimum level in its Φ-string. We will think of γ in the root system Σ_λ with simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$. According to Lemma 6.1.2, we can write $\gamma = \mu + \nu$, for $\mu \in \Sigma_\lambda^+$ and $\nu \in \Pi_\lambda$. If ν is in Φ, then we are done. Thus, assume that μ is a root spanned by Φ and $\nu = \lambda$. If $\mu \in \Pi$, we are done again.

Thus, assume that μ is a root spanned by Φ with level greater or equal than two (since μ is spanned by Φ the level of μ is the same in Σ and in Σ_λ). Since both μ and $\mu + \lambda = \gamma$ are roots but $\mu - \lambda$ cannot be a root, from Proposition 1.5.1 (v) we deduce that $A_{\lambda,\mu} < 0$. By regarding μ as a root in the root system Σ_Φ with simple set Φ, from Lemma 6.1.2 there must exist a simple root $\alpha \in \Phi$ such that $\mu - \alpha$ is a positive root spanned by Φ. On the one hand, if $A_{\lambda,\mu-\alpha} > 0$, from Proposition 1.5.1 (iv) we would obtain that $\mu - \alpha - \lambda$ is a positive root or zero. This is not possible since λ is not spanned by Φ and $\mu - \alpha$ is a positive root spanned by Φ. On the other hand, if $A_{\lambda,\mu-\alpha} < 0$, from Proposition 1.5.1 (iv) we would obtain that $\mu - \alpha + \lambda = \gamma - \alpha$ is a root and the result follows. Thus, assume that $0 = A_{\lambda,\mu-\alpha} = A_{\lambda,\mu} - A_{\lambda,\alpha}$. Then $A_{\lambda,\mu} = A_{\lambda,\alpha} < 0$. Hence, α is connected to λ in the Dynkin diagram of the simple system Π_λ. Since Φ is connected, if there were another root $\beta \in \Phi$ connected to λ in the Dynkin diagram of Π_λ, we would have a loop. Hence, $\alpha \in \Phi$ is the unique root connected to λ in the Dynkin diagram of the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$. Put $\mu = \sum_{\beta\in\Phi} n_\beta \beta$.

Therefore, from $A_{\lambda,\mu} = A_{\lambda,\alpha} < 0$ and $A_{\beta,\lambda} = 0$ for all $\beta \in \Phi\backslash\{\alpha\}$, we deduce that $n_\alpha = 1$. If $\Phi\backslash\{\alpha\}$ is not connected, then the positive root $\mu - \alpha$ must be spanned by a connected subset of $\Phi\backslash\{\alpha\}$ (see Proposition 6.1.3), since $\mu-\alpha$ is a root and the coefficient corresponding to α in the expression with respect to the simple system Φ is zero. Thus, we can assume that $\mu - \alpha$ is spanned by a connected subset Ψ of $\Phi\backslash\{\alpha\}$. Now, we can write $\mu = \alpha + \sum_{\beta\in\Psi} n_\beta\beta$. Note that μ is a root in the Ψ-string of α. On the one hand, if μ is the root of minimum level in the Ψ-string of α, then $n_\beta = 0$ for all $\beta \in \Psi$ by means of (i). Thus $\mu = \alpha \in \Phi$ and we are done. On the other hand, assume that μ is not the root of minimum level in its Ψ-string. By induction hypothesis, we can take $\beta \in \Psi \subset \Phi\backslash\{\alpha\}$ such that $\mu - \beta$ is a root. Recall that Π is not a G_2 simple system. If μ is proportional to β (and we are consequently in a BC_r root system), then we must have that $\mu = 2\beta$. Hence, we have that

$$A_{\beta,\lambda+\mu} = A_{\beta,\lambda} + 4.$$

Since β is proportional neither to λ nor to $\lambda + \mu$, from Proposition 1.5.1 (ii) we have that $A_{\beta,\lambda}$, $A_{\beta,\lambda+\mu} \in \{0, \pm 1, \pm 2\}$. But then $A_{\beta,\lambda+\mu} = 2$ and from Proposition 1.5.1 (iv) we have that $\lambda+\mu-\beta = \gamma-\beta$ is a root. If β and μ are non-proportional, from Proposition 6.1.6 we deduce that either $A_{\beta,\mu} > 0$ or $\mu + \beta$ is a root and $A_{\beta,\mu+\beta} = 2$. Therefore, since $A_{\beta,\lambda} = 0$, we get either $A_{\beta,\gamma} = A_{\beta,\lambda+\mu} > 0$ or that $\lambda + \mu + \beta$ is a root and $A_{\beta,\lambda+\mu+\beta} = 2$. In both cases we get that $\gamma - \beta$ is a root. This completes the proof of (ii).

(iii): Let $\gamma \in \Sigma^\Phi$ be of minimum level in its Φ-string. If $A_{\alpha,\gamma} = 0$ for all $\alpha \in \Phi$, then $\{\gamma\} \cup \Phi$ would be a reducible system and the Φ-string of γ would be trivial. Hence, there must exist one root $\alpha \in \Phi$ such that $A_{\alpha,\gamma} < 0$. If $A_{\beta,\gamma} < 0$ for some $\beta \in \Phi\backslash\{\alpha\}$, then β and α are connected to γ in the Dynkin diagram of Π_γ. Since Φ is connected, there would be a loop, which is a contradiction. This proves the first implication.

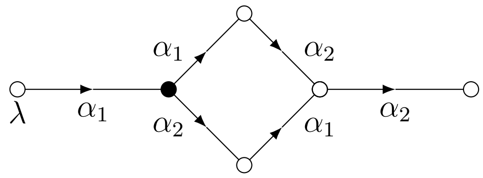
Conversely, assume that γ is not of minimum level in its Φ-string. Hence, from (ii) there must exist a root $\alpha \in \Phi$ such that $\gamma - \alpha$ is also a root. From Proposition 1.5.1 (iii) we get that $A_{\alpha,\gamma} \in \{0, \pm 1\}$. Let us study these possibilities.

If $A_{\alpha,\gamma} = 0$, since $\gamma - \alpha$ is a root, from Proposition 1.5.1 (v) we have that $\gamma + \alpha$ is also a root. But then $A_{\alpha,\gamma+\alpha} \geq 2$ and this contradicts Proposition 1.5.1 (iii), since $|\alpha| \geq |\gamma+\alpha|$ by hypothesis.

If $A_{\alpha,\gamma} = -1$, then from Proposition 1.5.1 (iv) we have that $\gamma + \alpha$ is also a root. Hence, from Proposition 1.5.1 (v) we get that $A_{\alpha,\gamma+\alpha} \geq 2$. Again, this is a contradiction with Proposition 1.5.1 (iii), since $|\alpha| \geq |\gamma + \alpha|$ by hypothesis.

Hence, we deduce that $A_{\alpha,\gamma} = 1$ and the result follows. □

Remark 6.2.2. One of the key tools in order to classify austere submanifolds of the form $S_\Phi \cdot o$ in exceptional symmetric spaces (see Chapter 7) is the characterization provided by Proposition 6.2.1 (iii) for the root of minimum level in a Φ-string. Indeed, in most cases in exceptional symmetric spaces it is very difficult to detect if a root with high level is of minimum level in its Φ-string. Proposition 6.2.1 (iii) addresses this difficulty, and it would be interesting to have a more general characterization. However, it is not true without the assumption on the length of the roots. For example, let Σ be a root system containing an A_2 subsystem, let $\Phi = \{\alpha_1, \alpha_2\} \subset \Pi$ be an A_2 simple system and let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. From Proposition 6.2.1 (i), we have that $\{\lambda\} \cup \Phi$ is a simple system. Assume that it is a C_3 simple system, with $A_{\alpha_1,\lambda} = -2$ and $A_{\alpha_2,\lambda} = 0$. A direct examination of this simple system allows us to deduce that the diagram of the Φ-string of λ is (we include the general calculation of this string in Proposition 6.2.9 (iii))



Note that the root $\lambda + \alpha_1$ is in the Φ-string of λ. We represent it with a black node in the diagram. It is not of minimum level since λ is a root. However, we have that $A_{\alpha_1,\lambda+\alpha_1} = 0$ and $A_{\alpha_2,\lambda+\alpha_1} = -1$. This means that the characterization in Proposition 6.2.1 (iii) is not true without the assumption on the lengths of the roots. However, see Remark 6.2.10 for a partial generalization of such result.

Note that the information provided by Proposition 6.2.1 (iii) in order to detect the root of minimum level of a Φ-string by means of the Cartan integers just addresses the connected case. In the following result, we explain how to calculate a Φ-string of a root when Φ is a non-connected subset of Π. In particular, this idea allows to extend the characterization of the root of minimum level in a Φ-string to the non-connected case.

Corollary 6.2.3. *Let* Π *be the set of simple roots of the root system* Σ. *Let* Φ_0, Φ_1 *be orthogonal connected subsets of* Π *and put* $\Phi = \Phi_0 \cup \Phi_1$. *Let* $\lambda \in \Sigma^\Phi$ *be of minimum level in its Φ-string. Assume that the* Φ_i*-string of* λ *is not trivial for* $i \in \{0, 1\}$. *Then:*

(i) $I(\lambda, \Phi) = \bigcup_{\nu \in I(\lambda,\Phi_i)} I(\nu, \Phi_{i+1})$ *for an arbitrary but fixed* $i \in \{0, 1\}$ *and indices modulo 2. In other words, the roots of the Φ-string of* λ *can be obtained by calculating and taking the union of the* Φ_i*-strings of all the roots in the* Φ_{i+1}*-string of* λ, *for* $i \in \{0, 1\}$ *and indices modulo 2. If* γ *is not the root of minimum level in its Φ-string, then there exists a root* $\alpha \in \Phi$ *such that* $\gamma - \alpha$ *is a root in the Φ-string of* λ.

(ii) *Assume that* $|\alpha| \geq |\nu|$ *for all* $(\alpha, \nu) \in \Phi \times \Sigma^\Phi$. *A root* $\gamma \in \Sigma^\Phi$ *is the root of minimum level in its non-trivial Φ-string if and only if there exists a root* $\alpha_i \in \Phi_i$ *such that* $A_{\alpha_i,\gamma} < 0$ *and* $A_{\beta_i,\gamma} = 0$ *for all* $\beta_i \in \Phi_i \backslash \{\alpha_i\}$, *for each* $i \in \{0, 1\}$. *In other words,* γ *is the root of minimum level in its Φ-string if and only if it is the root of minimum level in its* Φ_i*-string for* $i \in \{0, 1\}$.

Proof. Since Φ_0 and Φ_1 are orthogonal non-empty subsets of Π, we do not need to study the G_2 case.

(i): Consider an arbitrary root γ in the Φ-string of λ. It can be written as $\gamma = \lambda+\gamma_0+\gamma_1$, for γ_i in the span of Φ_i, with $i \in \{0,1\}$. In order to prove (i) it suffices to check that $\lambda+\gamma_0$ and $\lambda+\gamma_1$ are both roots, since then the root γ in the Φ-string of λ can be obtained by calculating the Φ_i-string of the root $\lambda+\gamma_{i+1}$, which is a root in the Φ_{i+1}-string of λ, for some arbitrary but fixed $i \in \{0,1\}$ and indices modulo 2.

We will proceed by induction on the level of γ with respect to the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$. If $l(\gamma) = 1$ then $\gamma = \lambda$ and the result is trivial. Assume that our claim is true for roots in the Φ-string of λ with level $n-1$ and let $\gamma \in I(\lambda, \Phi)$ with $l(\gamma) = n$. From Lemma 6.1.2, we deduce that there must exist a root $\alpha \in \Pi_\lambda$ such that $\gamma - \alpha$ is a root. Since $A_{\alpha_0,\alpha_1} = 0$ for all $\alpha_i \in \Phi_i$ with $i \in \{0,1\}$, if $\alpha = \lambda$ we deduce that $\gamma_j = 0$ for some $j \in \{0,1\}$ and then $\lambda+\gamma_{j+1} = \gamma$ and $\lambda + \gamma_j = \lambda$ (with indices modulo 2) are both roots. Thus, assume that $\alpha \in \Phi_l$ for some fixed $l \in \{0,1\}$. Then $\gamma - \alpha$ is a root in the Φ-string of λ of level $n-1$. By applying the induction hypothesis we deduce that $\lambda+\gamma_l-\alpha$ and $\lambda+\gamma_{l+1}$ are both roots, for some $l \in \{0,1\}$ and indices modulo 2. We just need to see that $\lambda+\gamma_l$ is a root.

Since $\gamma-\alpha$ is a root, Π is not a G_2 simple system and γ and α are non-proportional, from Proposition 6.1.6 we deduce that either $A_{\alpha,\gamma-\alpha} < 0$ or $\gamma - 2\alpha$ is also a root and $A_{\alpha,\gamma-2\alpha} = -2$. If $A_{\alpha,\gamma-\alpha} < 0$, recalling that $A_{\alpha_0,\alpha_1} = 0$ for all $\alpha_i \in \Phi_i$ with $i \in \{0,1\}$, we deduce that $A_{\alpha,\lambda+\gamma_l-\alpha} = A_{\alpha,\gamma-\alpha} < 0$, and then $\lambda+\gamma_l$ is a root using Proposition 1.5.1 (iv). If $\gamma-2\alpha$ is a root we use again the induction hypothesis and deduce that $\lambda+\gamma_l-2\alpha$ is a root. Then we have $A_{\alpha,\lambda+\gamma_l-2\alpha} = A_{\alpha,\gamma-2\alpha} = -2$ and $\lambda+\gamma_l$ is a root by using Proposition 1.5.1 (v).

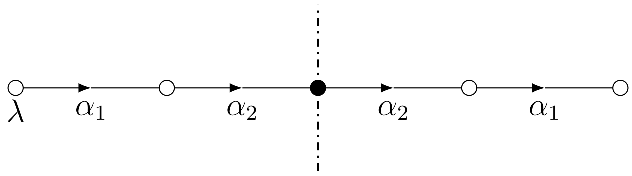
In conclusion, if $\gamma = \lambda+\gamma_0+\gamma_1$ is a root in the Φ-string of λ, then $\lambda+\gamma_0$ and $\lambda+\gamma_1$ are both roots, for γ_i in the span of Φ_i, with $i \in \{0,1\}$. As explained above, γ can be obtained by calculating the Φ_i-string of the root $\lambda+\gamma_{i+1}$, which is a root in the Φ_{i+1}-string of λ, for some arbitrary but fixed $i \in \{0,1\}$ and indices modulo 2.

(ii): Recall that λ is the root of minimum level in its Φ-string. Since $\Phi_i \subset \Phi$, we have that λ is also the root of minimum level in its Φ_i-string, for $i \in \{0,1\}$. Then, from Proposition 6.2.1 (iii) we get one of the implications in (ii). Conversely, assume that $\gamma = \lambda+\gamma_0+\gamma_1$ is a root in the Φ-string of λ satisfying the conditions for the Cartan integers specified in (ii). Recall from the proof of Corollary 6.2.11 that $\lambda+\gamma_i$ is a root in the Φ_i-string of λ with $i \in \{0,1\}$. From Proposition 6.2.1 (iii) we get that $\lambda+\gamma_i$ is the root of minimum level in the Φ_i-string of λ, with $i \in \{0,1\}$ and indices modulo 2. Since the coefficients of the expression of γ_i with respect to Φ_i must be non-negative we deduce that $\gamma_i = 0$ for $i \in \{0,1\}$ and hence $\gamma = \lambda$. □

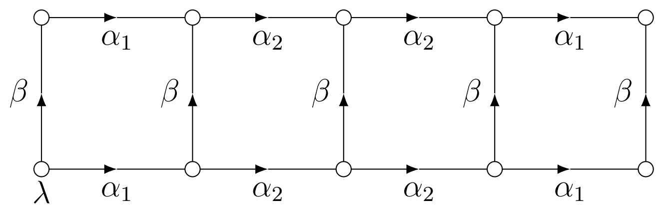
The above result becomes really powerful when combined with Proposition 6.1.11. Indeed, let us consider a particular example using diagrams.

Example 6.2.4. Assume that Σ is a B_r root system and that there are roots α_1, α_2 and β in Π such that α_1 and α_2 span a B_2 simple system orthogonal to β. Put $\Phi_0 = \{\alpha_1, \alpha_2\}$, $\Phi_1 = \{\beta\}$ and $\Phi = \Phi_0 \cup \Phi_1$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Assume that $\Phi \cup \{\lambda\}$ is a B_4 simple system with Dynkin diagram

where $|\beta| = |\lambda| = |\alpha_1| > |\alpha_2|$. A direct examination allows us to argue that the diagram of the Φ_0-string of λ (see Proposition 6.2.9 (iii) for the general calculation of this string) is of the form



From Lemma 5.1.1, we have that all the roots in this string except one corresponding to the central node (the black one) have the same multiplicity. Consider the reflection $f_0 \colon I(\lambda, \Phi_0) \to I(\lambda, \Phi_0)$ with respect to the central node. Note that this central node is then a fixed point with respect to this reflection. The map f_0 satisfies the hypotheses of Proposition 6.1.11 (ii). This means that the shape operator of $S_{\Phi_0} \cdot o$ is austere when restricted to the Φ_0-string of λ. Moreover, the Φ_1-string of λ consists of the roots λ and $\lambda + \beta$. From Lemma 5.1.1, they both have the same multiplicity. Hence, the map $f_1 \colon I(\lambda, \Phi_1) \to I(\lambda, \Phi_1)$ that interchanges both roots (it can be also thought as a reflection) satisfies the hypotheses of Proposition 6.1.11 (ii). This means that the shape operator of $S_{\Phi_1} \cdot o$ is austere when restricted to the Φ_1-string of λ. From Corollary 6.2.3, we deduce that the Φ-string of λ is obtained by calculating the Φ_1-string of each root in the Φ_0-string of λ. Thus, the Φ-string of λ has a diagram of the form



Roughly speaking, the Φ-string of λ consists of several copies of the Φ_k-string of λ, and these copies are parametrized by the Φ_{k+1}-string of λ, for some arbitrary but fixed $k \in \{0, 1\}$ and indices modulo 2. In the beginning, we used row reflection at the bottom of the diagram of the Φ-string of λ to construct a map f_0 satisfying the hypotheses of Proposition 6.1.11 (ii). However, this idea does not work now. In fact, the conditions on the Cartan integers required in Proposition 6.1.11 (ii) are not satisfied by the corresponding nodes of such reflection. However, if we combine the reflections f_0 and f_1 conveniently, we can construct a map satisfying the hypotheses of Proposition 6.1.11 (ii). Note that for each root γ in the Φ_0-string of λ, the Φ_1-string of $f_0(\gamma)$ consist of the roots $f_0(\gamma)$ and $f_0(\gamma) + \beta$. Extend the map f_1 in such a way that it interchanges the roots $f_0(\gamma)$ and $f_0(\gamma) + \beta$ for each γ in the Φ_0-string of λ. Extend also the map f_0 to be a reflection with respect to the central node in the row at the top of the diagram of the Φ-string of λ. Thus, if we consider the composition $f = f_0 \circ f_1$ we obtain an involution satisfying the hypotheses of Proposition 6.1.11 (ii). This means that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

We generalize and make this idea precise, based on the examination of some particular diagrams, in the following

Lemma 6.2.5. *Let Φ_0, Φ_1 be orthogonal subsets of the set of simple roots Π. Put $\Phi = \Phi_0 \cup \Phi_1$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Assume that $f_k \colon I(\lambda, \Phi_k) \to I(\lambda, \Phi_k)$ is a multiplicity-preserving involution satisfying $A_{\alpha,\nu} = -A_{\alpha,f_k(\nu)}$ for all $(\alpha, \nu) \in \Phi_k \times I(\lambda, \Phi_k)$, for each $k \in \{0, 1\}$. Hence, there exists a multiplicity-preserving involution $f \colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ satisfying $A_{\alpha,\nu} = -A_{\alpha,f(\nu)}$ for all $(\alpha, \nu) \in \Phi \times I(\lambda, \Phi)$. In particular, the shape operator of the submanifold $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.*

Proof. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. If the Φ_k-string of λ is trivial for some $k \in \{0, 1\}$, the result is trivial (taking $f = f_{k+1}$ where indices are modulo 2). Let us assume that the Φ_k-string of λ is not trivial for $k \in \{0, 1\}$. Note that λ is also of minimum level in its Φ_k-string for $k \in \{0, 1\}$. Consider an arbitrary root γ in the Φ-string of λ. It can be written as $\gamma = \lambda + \gamma_0 + \gamma_1$, for γ_k in the span of Φ_k, with $k \in \{0, 1\}$. Recall from the proof of Corollary 6.2.3 that $\lambda + \gamma_0$ and $\lambda + \gamma_1$ are both roots.

The map f_k induces an involution $\bar{f}_k \colon A_k \subset \operatorname{span} \Phi_k \to A_k \subset \operatorname{span} \Phi_k$ such that $f_k(\lambda + \gamma_k) = \lambda + \bar{f}_k(\gamma_k)$, where

$$A_k = \{\gamma_k \in \operatorname{span} \Phi_k \ : \ \lambda + \gamma_k \in I(\lambda, \Phi_k)\},$$

for each $k \in \{0, 1\}$. Hence, we have

$$A_{\alpha,\lambda+\gamma_k} = -A_{\alpha,\lambda+\bar{f}_k(\gamma_k)} \tag{6.13}$$

for each $\alpha \in \Phi$ with $k \in \{0, 1\}$. Moreover, we have that the roots $\lambda + \gamma_k$ and $\lambda + \bar{f}_k(\gamma_k)$ have the same multiplicity by assumption.

Recall that λ is of minimum level and Φ_0 is orthogonal to Φ_1. Let $\lambda + \nu_k$ be an arbitrary root in the Φ_k-string of λ for $\nu_k \in \operatorname{span} \Phi_k$. Hence, we have that $\lambda + \nu_k$ is of minimum level in its Φ_{k+1}-string, with $k \in \{0, 1\}$ and indices modulo 2. Indeed, if $\lambda + \nu_k$ is not of minimum level in its Φ_{k+1}-string, from Proposition 6.2.1 (ii) there must exist $\alpha \in \Phi_{k+1}$ such that $\lambda + \nu_k - \alpha$ is a root. Recall from Proposition 6.2.1 (i) that $\Pi_\lambda = \Phi \cup \{\lambda\}$ is a simple system and that each root in the Φ-string of λ must be in the $\mathbb{N} \cup \{0\}$-span of Π_λ. However, the root $\lambda + \nu_k - \alpha$ is not $\mathbb{Z}$-generated with non-negative coefficients by the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$.

Note from the classification of Dynkin diagrams (see [7, p. 337]) that in Φ_0 or in Φ_1 all the roots must have the same length, since they are orthogonal subsets of Π. Taking into account the classification of Dynkin diagrams and the fact that the roots in the $\mathbb{Z}$-span of Π_λ must be contained in Σ, we deduce that in $\Phi_0 \cup \{\lambda\}$ or in $\Phi_1 \cup \{\lambda\}$ or in $\Phi_0 \cup \Phi_1$ all the roots must have the same length. Moreover, if all the roots in $\Phi_0 \cup \Phi_1$ have the same length, using again the classification of the Dynkin diagrams, the fact that Φ_0 and

Φ_1 are orthogonal and that the Φ_k-string of λ is not trivial for $k \in \{0, 1\}$, we deduce that in $\Phi_0 \cup \{\lambda\}$ or in $\Phi_1 \cup \{\lambda\}$ all the roots must have the same length.

Without loss of generality, assume that in the simple system $\Phi_1 \cup \{\lambda\}$ (see Proposition 6.2.1 (i)) all the roots have the same length. Hence, all the roots in the Φ_1-string of λ have the same length (and thus multiplicity), since they lie in the integer span of the simple system $\Phi_1 \cup \{\lambda\}$, where all the roots have the same length.

Consider the roots $\lambda + \nu_1$ and $\lambda + \nu_1'$ in the Φ_1-string of λ, with ν_1, $\nu_1' \in \operatorname{span} \Phi_1$. Recall that they are of minimum level in their Φ_0-strings and that they have the same length. From Proposition 6.2.1 (i) we have that $\Pi^0_{\lambda+\nu_1} = \Phi_0 \cup \{\lambda + \nu_1\}$ and $\Pi^0_{\lambda+\nu_1'} = \Phi_0 \cup \{\lambda + \nu_1'\}$ are both simple systems. Since Φ_0 is orthogonal to Φ_1 and $\lambda+\nu_1$ and $\lambda+\nu_1'$ have the same length, then $A_{\alpha,\lambda+\nu_1} = A_{\alpha,\lambda+\nu_1'}$ and $A_{\lambda+\nu_1,\alpha} = A_{\lambda+\nu_1',\alpha}$ for all $\alpha \in \Phi_0$. Therefore, $\Pi^0_{\lambda+\nu_1}$ and $\Pi^0_{\lambda+\nu_1'}$ have identical Dynkin diagrams. Hence, each root the form $\lambda + \nu_0 + \nu_1$ has the same length (and multiplicity) as the root $\lambda + \nu_0 + \nu_1'$, for $\nu_0 \in \operatorname{span} \Phi_0$ and ν_1, $\nu_1' \in \operatorname{span} \Phi_1$.

This means that the map $f : I(\lambda, \Phi) \to I(\lambda, \Phi)$ defined by

$$f(\lambda + \gamma_0 + \gamma_1) = \lambda + \bar{f}_0(\gamma_0) + \bar{f}_1(\gamma_1),$$

preserves multiplicities. In fact, take $\gamma = \lambda + \gamma_0 + \gamma_1 \in I(\lambda, \Phi)$. We have that $\lambda + \gamma_0 + \gamma_1$ has the same multiplicity (and length) as $\lambda + \gamma_0$; the roots $\lambda + \gamma_0$ and $\lambda + \bar{f}_0(\gamma_0)$ have the same multiplicity by assumption; and $\lambda + \bar{f}_0(\gamma_0)$ and $\lambda + \bar{f}_0(\gamma_0) + \bar{f}_1(\gamma_1)$ have the same multiplicity (and length). This proves that f is a multiplicity-preserving involution.

Finally, take a root $\alpha \in \Phi_k$ for some $k \in \{0, 1\}$. Thus, recalling that Φ_0 and Φ_1 are orthogonal subsets of Φ and using (6.13) we obtain

$$A_{\alpha,f(\gamma)} = A_{\alpha,\lambda+\bar{f}_0(\gamma_0)+\bar{f}_1(\gamma_1)} = A_{\alpha,\lambda+\bar{f}_k(\gamma_k)} = -A_{\alpha,\lambda+\gamma_k} = -A_{\alpha,\lambda+\gamma_0+\gamma_1} = -A_{\alpha,\gamma},$$

for each $\gamma = \lambda+\gamma_0+\gamma_1 \in I(\lambda, \Phi)$. This means that the involution f preserves multiplicities and satisfies the conditions of Proposition 6.1.11 (ii). In particular, this implies that the shape operator of the submanifold $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ. This finishes the proof. □

This result allows us to provide the first class of examples of austere submanifolds of the form $S_\Phi \cdot o$. Recall that a subset Φ of the set Π of simple roots is said to be discrete if $A_{\alpha,\beta} = 0$ for any two distinct roots α, $\beta \in \Phi$.

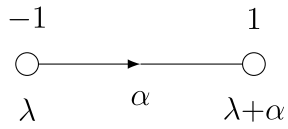
Proposition 6.2.6. *Let Φ be a proper subset of the set of simple roots Π of the root system Σ. If Φ is discrete, the submanifold $S_\Phi \cdot o$ is austere.*

Moreover, for each root $\lambda \in \Sigma^\Phi$ of minimum level in its Φ-string, the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

Proof. Take an arbitrary root $\lambda \in \Sigma^\Phi$ of minimum level in its Φ-string. On the one hand, if this Φ-string is trivial, then $A_{\alpha,\lambda} = 0$ for all $\alpha \in \Phi$. From (6.5) we deduce that $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of the shape operator of the submanifold $S_\Phi \cdot o$.

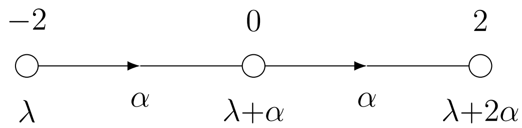
On the other hand, assume that the Φ-string of λ is not trivial. Recall from Proposition 6.2.1 (i) that $\Pi_\lambda = \{\lambda\} \cup \Phi$ is a simple system. Then, the simple root λ is connected to at most three simple roots in the Dynkin diagram of the simple system Π_λ.

Assume first that λ is connected to just one root $\alpha \in \Phi$. Note that the Φ-string of λ consists of the roots $\lambda + \varepsilon\alpha$ for an integer $\varepsilon \in \{0, \dots, -A_{\alpha,\lambda}\}$ (see Proposition 6.1.6). If the Φ-string of λ is



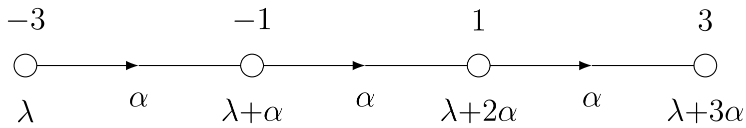
then λ and $\lambda + \alpha$ have the same multiplicity. Note that in the above diagram we write the Cartan integer $A_{\alpha,\nu}$ above the node ν. In this case, consider a function $f_1 \colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ that interchanges λ and $\lambda + \alpha$.

If the Φ-string of λ is



then λ and $\lambda{+}2\alpha$ have the same multiplicity. In this case, consider a function $f_2 \colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ that fixes $\lambda + \alpha$ and that interchanges λ and $\lambda + 2\alpha$.

Finally, if the Φ-string of λ is



all the roots have the same multiplicity. In this case, consider a function $f_3 \colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ that interchanges λ and $\lambda + 3\alpha$, and that interchanges $\lambda + \alpha$ and $\lambda + 2\alpha$.

In conclusion, if the non-trivial Φ-string of λ coincides with the α-string of λ, for some $\alpha \in \Phi$, we can define a multiplicity-preserving involution f satisfying the conditions of Proposition 6.1.11 (ii). Therefore, the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

Assume now that λ is connected to the roots α_0 and α_1 in the Dynkin diagram of the simple system Π_λ. Note that $A_{\alpha_0,\alpha_1} = 0$, since otherwise we would have a loop in the Dynkin diagram of Π_λ. Put $\Phi_0 = \{\alpha_0\}$ and $\Phi_0 = \{\alpha_1\}$. From the above considerations there exist multiplicity-preserving involutions of $I(\lambda, \Phi_0)$ and $I(\lambda, \Phi_1)$ satisfying the hypotheses of Proposition 6.1.11 (ii). Hence, from Lemma 6.2.5 we deduce that there exists a multiplicity-preserving involution of $I(\lambda, \Phi)$ satisfying the hypotheses of Proposition 6.1.11 (ii). In particular, we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

Finally, assume that λ is connected in the Dynkin diagram of Π_λ to three roots, namely α_0, α_1 and α_2. They are mutually orthogonal, since otherwise we would have a loop in the Dynkin diagram of Π_λ. Put $\Phi_0 = \{\alpha_0\}$ and $\Phi_1 = \{\alpha_1, \alpha_2\}$. From the above considerations there exist multiplicity-preserving involutions of $I(\lambda, \Phi_0)$ and $I(\lambda, \Phi_1)$ satisfying the hypotheses of Proposition 6.1.11 (ii). Hence, from Lemma 6.2.5 we deduce that there exists a multiplicity-preserving involution of $I(\lambda, \Phi)$ satisfying the hypotheses of Proposition 6.1.11 (ii). In particular, we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ. This concludes the proof. □

Let G/K be a symmetric space with G_2 Dynkin diagram. Thus, if we take a non-empty proper subset $\Phi \subset \Pi$, it will contain just one root. Hence, Φ is discrete. This allows to deduce a classification result for symmetric spaces of non-compact type with G_2 Dynkin diagram.

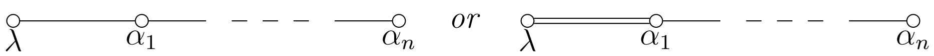
Corollary 6.2.7. *Let G/K be a symmetric space of non-compact type with G_2 Dynkin diagram. Let Φ be a proper subset of the set Π of simple roots. Then, the submanifold $S_\Phi \cdot o$ is austere.*

6.2.1 Study of Φ-strings of classical type

After the results of the previous subsections, we are ready to start the study of most of the Φ-strings that will appear throughout the classification of the austere submanifolds of the form $S_\Phi \cdot o$. This is the approach that we will follow. We will fix a subset Φ of the set Π of simple roots. For each Φ-string in Σ^Φ we will consider the root λ of minimum level. From Proposition 6.2.1 (i) we have that $\Pi_\lambda = \{\lambda\} \cup \Phi$ is a simple system. Then, we will start a case-by-case examination of the Φ-string of λ depending on the Dynkin diagram of the simple system Π_λ. In order to do that, we will calculate the number of roots in the Φ-string of λ by using the knowledge about the number of positive roots spanned by Φ and by Π_λ [69, p. 684]. Then, we will construct all these roots explicitly and draw the diagram of the Φ-string. Finally, using Proposition 6.1.11 or Corollary 6.1.12 we will check in which cases the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

Let us start with the simplest case.

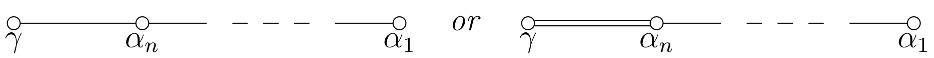
Proposition 6.2.8. *Let $\Phi = \{\alpha_1, \dots, \alpha_n\} \subset \Pi$ be a connected subset of the set of simple roots of the root system Σ. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Assume that Φ is an A_n simple system and that the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$ has either an A_{n+1} or a B_{n+1} Dynkin diagram of the form*



(6.14)

The shape operator of $S_\Phi \cdot o$ is austere when restricted to $\bigoplus_{\alpha \in I(\lambda,\Phi)} \mathfrak{g}_\alpha$ if and only if $n = 1$.

Moreover, let $\gamma \in \Sigma^\Phi \backslash I(\lambda, \Phi)$ be of minimum level in its Φ-string. Assume that the simple system Π_γ has either an A_{n+1} or a B_{n+1} Dynkin diagram of the form

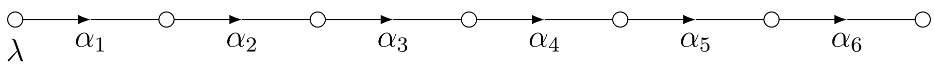


(6.15)

Then, the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to the vector space $\bigoplus_{\alpha\in I(\lambda,\Phi)}\mathfrak{g}_\alpha$ are exactly the opposite to the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to $\bigoplus_{\alpha\in I(\gamma,\Phi)}\mathfrak{g}_\alpha$, for each unit normal vector ξ to the submanifold $S_\Phi\cdot o$. In particular, if λ and γ have the same multiplicity, then $\mathcal{S}$ is austere when restricted to $\bigoplus_{\alpha\in I(\lambda,\Phi)\cup I(\gamma,\Phi)}\mathfrak{g}_\alpha$.

Proof. If $n=1$, our assertion follows from Proposition 6.2.6. Thus, put $n>1$. First, we will study the roots of the Φ-string of λ. This is equivalent to studying the positive roots of the form $n_\lambda\lambda+\sum_{i=1}^n n_i\alpha_i$ $\mathbb{Z}$-spanned by the simple system $\Pi_\lambda=\{\lambda\}\cup\Phi$, with $n_\lambda=1$. Then, the number of roots in the Φ-string of λ is the number of positive roots spanned by an A_{n+1} or B_{n+1} simple system, minus the number of roots with $n_\lambda\geq 2$ in $n_\lambda\lambda+\sum_{i=1}^n n_i\alpha_i$ and minus the number of positive roots spanned by Φ (since $n_\lambda=0$ in this case). In both cases we obtain $|I(\lambda,\Phi)|=n+1$ (see for example [69, p. 684]).

Since Φ is an A_n system, we have that $\alpha(l)=\sum_{i=1}^l\alpha_i$ is a root for each $l\in\{1,\dots,n\}$. Note that, for both possible Dynkin diagrams of Π_λ, we have that $A_{\alpha_1,\lambda}=-1$ and $A_{\alpha_i,\lambda}=0$ for all $\alpha_i\in\Phi$, with $i\geq 2$. Thus, $A_{\alpha(l),\lambda}=-1$ for each $l\in\{1,\dots,n\}$. From Proposition 1.5.1 (iv), we deduce that $\lambda+\alpha(l)$ is a root, for each $l\in\{1,\dots,n\}$. Hence, the Φ-string of λ consists of the root λ and the n roots of the form $\lambda+\alpha(l)$, for $l\in\{1,\dots,n\}$. Therefore, the diagram of the Φ-string of λ (we put $n=6$ for simplicity) is of the form



Using n times Lemma 5.1.1 (first for α and λ, and for α_{l+1} and $\lambda+\alpha(l)$, with $l\in\{1,\dots,n-1\}$) we deduce that all the roots in $I(\lambda,\Phi)$ have the same multiplicity. Since $n>1$, it is easy to see that the shape operator $\mathcal{S}$ is not austere when restricted to the Φ-string of λ, by virtue of Corollary 6.1.12 (ii). Indeed, there is just one arrow connected to the node λ. This arrow has label α_1. Since none of the nodes is connected to just one arrow with label α_1, we deduce that the shape operator of $S_\Phi\cdot o$ cannot be austere when restricted to the Φ-string of λ.

For the sake of completeness, we also include the calculation of the principal curvatures. Put $\lambda_l=\lambda+\alpha(l)$ for $l\in\{1,\dots,n\}$ and $\lambda_0=\lambda$. Note first that

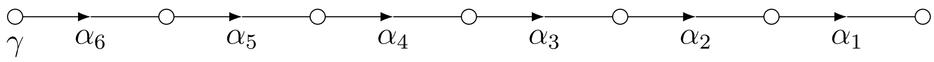
$$A_{\alpha_i,\lambda_l}=\begin{cases}-1 & \text{if } i=l+1,\\ 1 & \text{if } i=l,\\ 0 & \text{otherwise.}\end{cases}\tag{6.16}$$

Let $\xi=\sum_{i=1}^n a_iH_{\alpha_i}$ be a unit normal vector to $S_\Phi\cdot o$. Using (6.16) together with (6.5) we get that the principal curvature $\mu(\lambda_l)$ associated with the root space $\mathfrak{g}_{\lambda_l}$ is

$$\mu(\lambda_l) = \begin{cases} -\dfrac{|\alpha_1|^2}{2} a_1 & \text{if } \ l = 0, \\ \dfrac{|\alpha_1|^2}{2}(a_l - a_{l+1}) & \text{if } \ 0 < l < n, \\ \dfrac{|\alpha_1|^2}{2} a_n & \text{if } \ l = n. \end{cases} \tag{6.17}$$

Note that the principal curvatures are the same for the first and the second type of Φ-string in (6.14). Let us be more precise. Note that $|\alpha| \geq |\lambda|$ for all $\alpha \in \Phi$ in both cases in (6.14). Hence, by virtue of Proposition 1.5.1 (iii) we have $A_{\alpha,\lambda} \in \{0 \pm 1\}$ for all $\alpha \in \Phi$ in both cases in (6.14). Hence, from (6.5) we deduce that the principal curvatures do not depend on the case in (6.14). However, the multiplicity of λ might be different depending on the case in (6.14).

The same arguments as those used above hold for calculating the roots in the Φ-string of γ. This string consists of the roots γ and $\gamma_l = \gamma + \sum_{i=0}^{l} \alpha_{n-i}$ for each $l \in \{0, \dots, n-1\}$. The picture of the Φ-string of γ (we put $n = 6$ for simplicity) is of the form



Let us calculate the principal curvatures of $\mathcal{S}_\xi$ when restricted to the Φ-string of γ. Using (6.16) and (6.5) we get that the principal curvature $\mu(\gamma_l)$ of $\mathcal{S}_\xi$ associated with the root space $\mathfrak{g}_{\gamma_l}$ is

$$\mu(\gamma_l) = \begin{cases} \dfrac{|\alpha_1|^2}{2} a_1 & \text{if } \ l = n-1, \\ \dfrac{|\alpha_1|^2}{2}(a_{n-l} - a_{n-l-1}) & \text{if } \ -1 < l < n-1, \\ -\dfrac{|\alpha_1|^2}{2} a_n & \text{if } \ l = -1. \end{cases} \tag{6.18}$$

Note that these are exactly the opposite to the principal curvatures of $\mathcal{S}_\xi$ when restricted to the Φ-string of λ.

Finally, assume that γ and λ have the same multiplicity. Since all the roots in $I(\alpha, \Phi)$ have the same multiplicity as α, with $\alpha \in \{\lambda, \gamma\}$, then all the roots in $I(\lambda, \Phi) \cup I(\gamma, \Phi)$ have the same multiplicity. Hence, the map $f\colon I(\lambda, \Phi) \to I(\gamma, \Phi)$ defined by $f(\lambda_l) = \gamma_{n-l-1}$ for each $l \in \{0, \dots, n\}$ satisfies the conditions of Proposition 6.1.11 (iii). This means that if γ and λ have the same multiplicity, then $\mathcal{S}$ is austere when restricted to

$$\bigoplus_{\nu \in I(\lambda,\Phi) \cup I(\gamma,\Phi)} \mathfrak{g}_\nu.$$

Note that this claim is true even if λ is of the first type in (6.14) and γ of the second type in (6.15), provided that they have the same multiplicity. □

Consider a symmetric space of non-compact type G/K with A_r Dynkin diagram, for some $r \in \mathbb{N}$. Let $\Phi = \{\alpha_n, \dots, \alpha_m\}$ be a connected subset of the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$, with $n < m$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. From Proposition 6.2.1 (i) we have that $\Pi_\lambda = \Phi \cup \{\lambda\}$ is a simple system. This Φ-string can be trivial. Since $A_{\alpha,\lambda} = 0$ for all $\alpha \in \Phi$ in this case we have that

$$\bigoplus_{\nu \in I(\lambda,\Phi)} \mathfrak{g}_\nu$$

is contained in the 0-eigenspace of the shape operator of $S_\Phi \cdot o$. If the Φ-string of λ is not trivial, the root system with simple system Π_λ is contained in Σ, which has an A_r Dynkin diagram. Note that if Π_λ is not of type A, then there will be at least (see [69, p. 684]) one positive root $\gamma = \sum_{\alpha \in \Pi_\lambda} n_\alpha \alpha$ with $n_\beta \geq 2$ for some $\beta \in \Pi_\lambda$. But this root would be in the A_r system $\mathbb{Z}$-generated by Π and then $0 \leq n_\alpha \leq 1$ for all $\alpha \in \Pi_\lambda$. Thus we get a contradiction. Hence, if the Φ-string of λ is not trivial, then Π_λ must be an A_{m-n+2} simple system.

Recall that we are considering the vector subspace

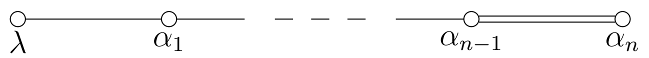
$$\bigoplus_{\alpha \in \Sigma^\Phi} \mathfrak{g}_\alpha.$$

of the tangent space to $S_\Phi \cdot o$. Recall also that we are considering a decomposition of this subspace induced by strings. These strings can be of three types: trivial, of the first type in (6.14) or of the first type in (6.15). Note that all the roots in Σ have the same multiplicity. Thus, combining Proposition 6.2.8 with Lemma 6.1.10, we deduce that the submanifold $S_\Phi \cdot o$ is austere if and only if the number of roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_n,\nu_1} = -1$ coincides with the number of roots $\nu_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_m,\nu_2} = -1$.

This idea of counting roots of minimum level and their multiplicities will hold for the rest of the cases. However, the Φ-strings involved will be more complicated than those we have studied in Proposition 6.2.8. Hence, before stating the classification result for a symmetric space of non-compact type with A_r simple system, we continue investigating Φ-strings that will appear in the rest of the cases.

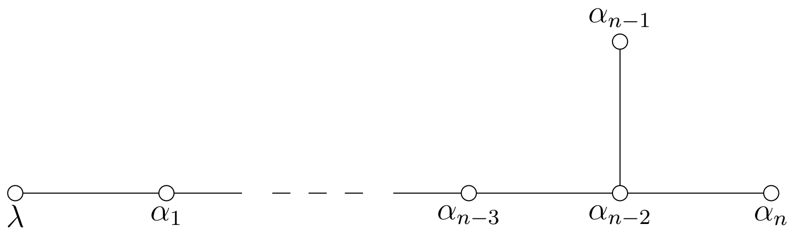
Proposition 6.2.9. *Let Φ be a connected subset of the set of simple roots Π of the root system Σ. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. If one of the following conditions holds, then the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.*

(i) *Φ is a simple system with B_n Dynkin diagram and the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$ has a B_{n+1} Dynkin diagram of the form*



(ii) *Φ is a simple system with C_n Dynkin diagram and the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$ has a C_{n+1} Dynkin diagram of the form*

(iii) Φ *is a simple system with* D_n *Dynkin diagram and the simple system* $\Pi_\lambda = \{\lambda\} \cup \Phi$ *has a* D_{n+1} *Dynkin diagram of the form*

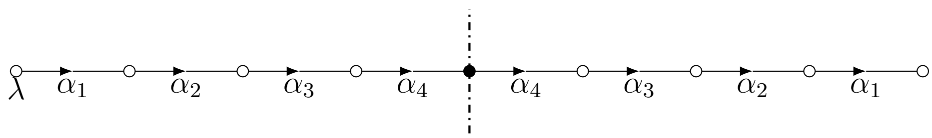


Proof. (i): First we will determine the roots of the Φ-string of λ. In order to do that, we will study the positive roots of the form $n_\lambda \lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha$ $\mathbb{Z}$-spanned by the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$, with $n_\lambda = 1$. Then, the number of roots in the Φ-string of λ is the number of positive roots spanned by a B_{n+1} simple system, minus the number of positive roots spanned by Φ (since $n_\lambda = 0$ in this case and $n_\lambda \geq 2$ does not occur [69, p. 684]). We obtain

$$|I(\lambda,\Phi)| = (n+1)^2 - n^2 = 2n+1.$$

The root $\alpha^0(l) = \sum_{i=1}^{l} \alpha_i$ is spanned by Φ for each $l \in \{1,\ldots,n\}$. Since $A_{\lambda,\alpha^0(l)} = -1$ for each $l \in \{1,\ldots,n\}$, then from Proposition 1.5.1 (iv) we deduce that $\lambda_0(l) = \lambda + \alpha^0(l)$ is a root in the Φ-string of λ for each $l \in \{1,\ldots,n\}$. Since $|\lambda| = |\alpha^0(l)|$ for each $l \in \{1,\ldots,n-1\}$, then from Proposition 1.5.1 (iii) and $A_{\lambda,\alpha^0(l)} = -1$ we deduce that $A_{\alpha^0(l),\lambda} = -1$ for each $l \in \{1,\ldots,n-1\}$. Hence, using Lemma 5.1.1 for $\alpha^0(l)$ and λ, for each $l \in \{1,\ldots,n-1\}$, we deduce that all the roots of the form $\lambda_0(l)$ have the same multiplicity as λ, for each $l \in \{1,\ldots,n-1\}$.

Moreover, the root $\alpha^1(k) = \alpha^0(n) + \sum_{j=0}^{n-(k+1)} \alpha_{n-j}$ is spanned by Φ for each $k \in \{1,\ldots,n-1\}$. Since $A_{\lambda,\alpha^1(k)} = -1$ for each $k \in \{1,\ldots,n-1\}$, then from Proposition 1.5.1 (iv) we get that $\lambda_1(k) = \lambda + \alpha^1(k)$ is a root in the Φ-string of λ for each $k \in \{1,\ldots,n-1\}$. Note that $|\alpha^1(k)| = |\lambda|$ for each $k \in \{1,\ldots,n-1\}$. Hence, Proposition 1.5.1 (iii) and $A_{\lambda,\alpha^1(k)} = -1$ yield $A_{\alpha^1(k),\lambda} = -1$ for each $k \in \{1,\ldots,n-1\}$. Using Lemma 5.1.1 for $\alpha^1(k)$ and λ, for each $k \in \{1,\ldots,n-1\}$, we deduce that all the roots of the form $\lambda_1(k)$ have the same multiplicity as λ, for each $k \in \{1,\ldots,n-1\}$. Put $\lambda_0(0) = \lambda$. Moreover, we have that $A_{\alpha_1,\lambda+\alpha^1(1)} = -1$ and from Proposition 1.5.1 (iv) we get that $\lambda + \alpha^1(k) + \alpha_1$ is also a root (again with the same multiplicity as λ, by virtue of Lemma 5.1.1). Put $\lambda_1(0)$ for this root. The root $\lambda_0(n)$ and the roots of the form $\lambda_\varepsilon(l)$, with $\varepsilon \in \{0,1\}$ and $l \in \{0,\ldots,n-1\}$, give rise to the $2n+1$ roots of the Φ-string of λ. The diagram of this Φ-string (with $n=4$ for simplicity) is of the form



Recall that all the roots in the Φ-string of λ except $\lambda_0(n)$ (the black node in the above diagram) have the same multiplicity as λ. Hence, the map $f \colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ defined by

$$f(\lambda_\varepsilon(l)) = \begin{cases} \lambda_\varepsilon(l) & \text{if } (\varepsilon, l) = (0, n), \\ \lambda_{1-\varepsilon}(l) & \text{otherwise,} \end{cases}$$

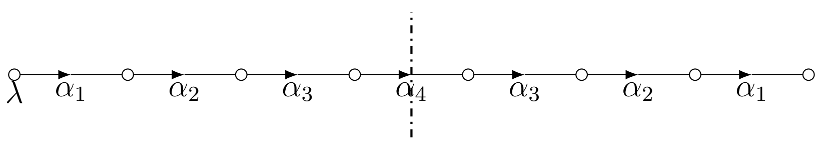
satisfies the conditions of Proposition 6.1.11 (ii). The map f is induced by a reflection with respect to the vertical line in the above diagram.

(ii): We will study the positive roots of the form $n_\lambda \lambda + \sum_{\alpha \in \Phi} n_\alpha \alpha$ $\mathbb{Z}$-spanned by the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$, with $n_\lambda = 1$. Then, the number of roots in the Φ-string of λ is the number of positive roots spanned by a C_{n+1} simple system, minus the number of positive roots spanned by Φ (since $n_\lambda = 0$ for such roots) and minus the number of roots with $n_\lambda \geq 2$ (there is just one root satisfying this condition [69, p. 684]). We obtain

$$|I(\lambda, \Phi)| = (n+1)^2 - n^2 - 1 = 2n.$$

The root $\alpha^0(l) = \sum_{i=1}^{l} \alpha_i$ is spanned by Φ for each $l \in \{1, \dots, n-1\}$. Since $A_{\lambda, \alpha^0(l)} = -1$ for each $l \in \{1, \dots, n-1\}$, then from Proposition 1.5.1 (iv) we deduce that $\lambda_0(l) = \lambda + \alpha^0(l)$ is a root in the Φ-string of λ for each $l \in \{1, \dots, n-1\}$. Note that $|\alpha^0(l)| = |\lambda|$ for each $l \in \{1, \dots, n-1\}$. Hence, from Proposition 1.5.1 (iii) and $A_{\lambda, \alpha^0(l)} = -1$ for each $l \in \{1, \dots, n-1\}$, we deduce that $A_{\alpha^0(l), \lambda} = -1$ for each $l \in \{1, \dots, n-1\}$. Using Lemma 5.1.1 for $\alpha^0(l)$ and λ, for each $l \in \{1, \dots, n-1\}$, we deduce that all the roots of the form $\lambda_0(l)$ have the same multiplicity as λ, for each $l \in \{1, \dots, n-1\}$.

Moreover, the root $\alpha^1(k) = \alpha^0(n-1) + \sum_{j=0}^{n-(k+1)} \alpha_{n-j}$ is spanned by Φ for each $k \in \{0, \dots, n-1\}$. Note that $|\alpha^1(k)| = |\lambda|$ if $k \neq 0$ and $|\alpha^1(0)| > |\lambda|$. Using Proposition 1.5.1 (iii) and $A_{\lambda, \alpha^1(k)} < 0$ for each $k \in \{0, \dots, n-1\}$, we deduce that $A_{\alpha^1(k), \lambda} = -1$ for each $k \in \{0, \dots, n-1\}$. Then, from Proposition 1.5.1 (iv) we get that $\lambda_1(k) = \lambda + \alpha^1(k)$ is a root in the Φ-string of λ for each $k \in \{0, \dots, n-1\}$. Put $\lambda_0(0) = \lambda$. Furthermore, using Lemma 5.1.1 for $\alpha^1(k)$ and λ, for each $k \in \{0, \dots, n-1\}$, we deduce that $\lambda_1(k)$ has the same multiplicity as λ, for each $k \in \{0, \dots, n-1\}$. Thus, the roots of the form $\lambda_\varepsilon(l)$, for $\varepsilon \in \{0, 1\}$ and $l \in \{0, \dots, n-1\}$, give rise to the $2n$ roots of the Φ-string of λ. Note that all of them have the same multiplicity as λ. Its diagram (with $n = 4$) is of the form



Hence, the map $f \colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ defined by $f(\lambda_\varepsilon(l)) = \lambda_{1-\varepsilon}(l)$ satisfies the conditions of Proposition 6.1.11 (ii). The map f is induced by the reflection with respect to the vertical line in the above diagram.

(iii): We will study the positive roots of the form $n_\lambda \lambda + \sum_{\alpha \in \Phi} n_\alpha \alpha$ $\mathbb{Z}$-spanned by the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$, with $n_\lambda = 1$. Then the number of roots in the Φ-string of λ is the number of positive roots spanned by a D_{n+1} simple system, minus the number of positive roots spanned by Φ (since $n_\lambda = 0$ in this case and there are not positive roots satisfying $n_\lambda \geq 2$ [69, p. 684]). We obtain

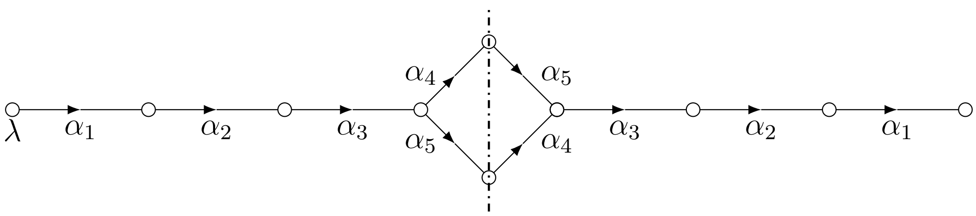
$$|I(\lambda, \Phi)| = (n+1)n - n(n-1) = 2n.$$

Since we are studying a D_{n+1} simple system, then all the roots of the Φ-string of λ have the same multiplicity. In particular, from Proposition 1.5.1 (iv) we have that $A_{\nu_1,\nu_2} = A_{\nu_2,\nu_1}$ for all ν_1, $\nu_2 \in \Sigma$.

Put $\alpha^0(l) = \sum_{i=1}^{l} \alpha_i$ for each $l \in \{1, \dots, n\}$ and $\alpha^0(n+1) = \alpha^0(n) - \alpha_{n-1}$. All these roots are spanned by Φ. Since $A_{\lambda,\alpha^0(l),} = -1 = A_{\alpha^0(l),\lambda}$ for each $l \in \{1, \dots, n+1\}$, from Proposition 1.5.1 (iv) we deduce that $\lambda_0(l) = \lambda + \alpha^0(l)$ is a root in the Φ-string of λ for each $l \in \{1, \dots, n+1\}$.

Moreover, the root $\alpha^1(k) = \alpha^0(n) + \sum_{j=2}^{n-(k+1)} \alpha_{n-j}$ is spanned by Φ for each $k \in \{1, \dots, n-3\}$. Since $A_{\alpha^1(k),\lambda} = -1$ for each $k \in \{1, \dots, n-3\}$, then from Proposition 1.5.1 (iv) we get that $\lambda_1(k) = \lambda + \alpha^1(k)$ is a root in the Φ-string of λ for each $k \in \{1, \dots, n-3\}$. Furthermore, we have that $A_{\alpha_1,\lambda_1(1)} = -1$ and then from Proposition 1.5.1 (iv) we get that $\lambda_1(0) = \lambda_1(1) + \alpha_1$ is a root in the Φ-string of λ.

Put $\lambda_0(0) = \lambda$. Then we have that $\lambda_0(n-2)$, $\lambda_0(n-1)$, $\lambda_0(n)$, $\lambda_0(n+1)$ and the roots of the form $\lambda_\varepsilon(l)$, with $\varepsilon \in \{0, 1\}$ and $l \in \{0, \dots, n-3\}$, give rise to the $2n$ roots of the Φ-string of λ. Its diagram (with $n = 5$ for simplicity) is of the form



Since we are studying a D_{n+1} simple system, then all the roots of the Φ-string of λ have the same multiplicity. The map $f \colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ defined by

$$f(\lambda_\varepsilon(l)) = \begin{cases} \lambda_\varepsilon(n) & \text{if } (\varepsilon, l) = (0, n-2), \\ \lambda_\varepsilon(n-2) & \text{if } (\varepsilon, l) = (0, n), \\ \lambda_\varepsilon(n+1) & \text{if } (\varepsilon, l) = (0, n-1), \\ \lambda_\varepsilon(n-1) & \text{if } (\varepsilon, l) = (0, n+1), \\ \lambda_{1-\varepsilon}(l) & \text{otherwise,} \end{cases}$$

satisfies the conditions of Proposition 6.1.11 (ii). The map f is induced by the composition of reflections with respect to the horizontal and vertical axes in the above diagram. This finishes the proof. □

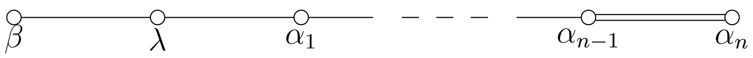
Remark 6.2.10. It is very interesting to remark that although the characterization given in Proposition 6.2.1 (iii) is not true without the assumption on the lengths, it can be extended for some particular cases. Indeed, assume the hypotheses and notations of the proof of

Proposition 6.2.9 (i). We have calculated explicitly the roots of the Φ-string of λ. Note that λ is the unique root in $I(\lambda, \Phi)$ such that there exists a root $\alpha \in \Phi$ satisfying $A_{\alpha,\lambda} = -1$ and $A_{\beta,\lambda} = 0$ for all $\beta \in \Phi \backslash \{\alpha\}$. Hence, the characterization given in Proposition 6.2.1 (iii) is also true when Φ is an B_n simple system and $\Phi \cup \{\lambda\}$ is a B_{n+1} simple system, with $n \geq 1$.

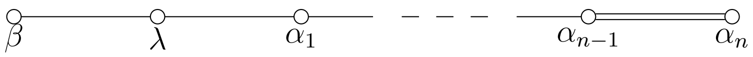
Combining Proposition 6.2.6, Proposition 6.2.9 and Lemma 6.2.5 we directly obtain the following

Corollary 6.2.11. *Let Φ be a proper subset of the set of simple roots Π of the root system Σ. Let $\lambda \in \Sigma^{\Phi}$ be of minimum level in its Φ-string. If one of the following conditions holds, then the shape operator of $S_{\Phi} \cdot o$ is austere when restricted to the Φ-string of λ.*

(i) *Φ is a reducible simple system with $A_1 \oplus B_n$ Dynkin diagram and the simple system $\Pi_{\lambda} = \{\lambda\} \cup \Phi$ has a B_{n+2} Dynkin diagram of the form*



(ii) *Φ is a reducible simple system with $A_1 \oplus C_n$ Dynkin diagram and the simple system $\Pi_{\lambda} = \{\lambda\} \cup \Phi$ has a C_{n+2} Dynkin diagram of the form*



(iii) *Φ is a reducible simple system with $A_1 \oplus D_n$ Dynkin diagram and the simple system $\Pi_{\lambda} = \{\lambda\} \cup \Phi$ has a D_{n+2} Dynkin diagram of the form*

This result allows us to construct a large family of examples of austere submanifolds in symmetric spaces of non-compact type G/K with Dynkin diagram B_r, C_r or D_r.

Proposition 6.2.12. *Let Φ_0, Φ_1 be orthogonal subsets of the set of simple roots Π and put $\Phi = \Phi_0 \cup \Phi_1$. Assume that Φ_1 is discrete and the pair (Π, Φ_0) of simple systems is of one of the following types:*

$$\{(B_r, B_n), (C_r, C_n), (D_r, D_n)\},$$

with $n < r$. Then, the submanifolds $S_{\Phi_0} \cdot o$, $S_{\Phi_1} \cdot o$ and $S_{\Phi_0 \cup \Phi_1} \cdot o$ are austere.

Proof. Let us denote by $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}$ the shape operators of $S_{\Phi_0} \cdot o$, $S_{\Phi_1} \cdot o$ and $S_\Phi \cdot o$, respectively. Recall from Proposition 6.2.1 (i) that $\Pi_\lambda = \Phi \cup \{\lambda\}$, $\Phi_0 \cup \{\lambda\}$ and $\Phi_1 \cup \{\lambda\}$ are simple systems. Let us study all the possibilities:

(a) If the Φ_i-string of λ is trivial for $i \in \{0, 1\}$, then from Corollary 6.2.3 (i) we have that the Φ-string of λ is also trivial. Hence, the subspace $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}$.

(b) Assume that the Φ_0-string of λ is not trivial and that the Φ_1-string of λ is trivial. This implies firstly that $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of $\mathcal{S}_1$. Moreover, due to Corollary 6.2.3 (i) we have that the Φ_0-string of λ coincides with the Φ-string of λ. According to our hypothesis, we have that Π_λ has either a B_{n+1}, C_{n+1} or D_{n+1} Dynkin diagram. These cases have been studied in Proposition 6.2.9 and we have that $\mathcal{S}_0$ and $\mathcal{S}$ are austere when restricted to the Φ_0-string of λ and to the Φ-string of λ, respectively. At this point, we deduce that $S_{\Phi_0} \cdot o$ is austere.

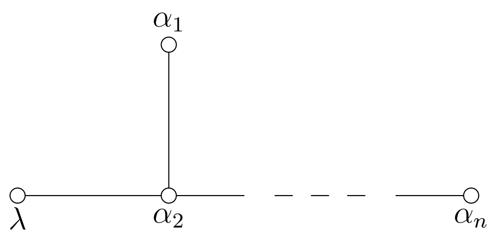
(c) Assume that the Φ_0-string of λ is trivial and that the Φ_1-string of λ is not trivial. This implies firstly that $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of $\mathcal{S}_0$. Moreover, due to Corollary 6.2.3 (i) we have that the Φ-string of λ coincides with the Φ_1-string of λ. Since Φ_1 is discrete, from Proposition 6.2.6 we deduce that $\mathcal{S}_1$ and $\mathcal{S}$ are austere when restricted to the Φ_1-string of λ and to the Φ-string of λ, respectively. At this point, we deduce that $S_{\Phi_1} \cdot o$ is austere.

(d) Assume that the Φ_i-string of λ is not trivial, for $i \in \{0, 1\}$. According to the classification of Dynkin diagrams [7, p. 337], we have that Π_λ must be of type B_{n+2}, C_{n+2} or D_{n+2}. All these cases have been investigated in Corollary 6.2.11 and we have that $\mathcal{S}$ is austere when restricted to the Φ-string of λ.

Since all the cases have been considered, the result follows. □

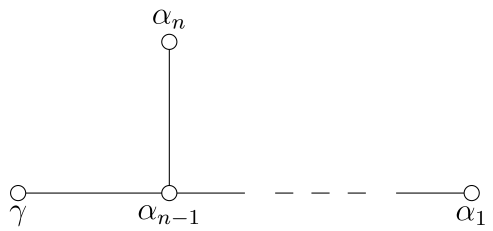
Let us continue with the study of the different kinds of Φ-strings that will appear when we address classification results.

Proposition 6.2.13. *Let $\Phi = \{\alpha_1, \dots, \alpha_n\} \subset \Pi$ be a connected subset of the set of simple roots of the root system Σ. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Assume that Φ is an A_n simple system and the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$ has a D_{n+1} Dynkin diagram of the form*



The shape operator of $S_\Phi \cdot o$ is austere when restricted to $\bigoplus_{\alpha\in I(\lambda,\Phi)} \mathfrak{g}_\alpha$ if and only if $n = 3$.

Moreover, let $\gamma \in \Sigma^\Phi \backslash I(\lambda, \Phi)$ be of minimum level in its Φ-string. Assume that the simple system $\Pi_\gamma = \{\gamma\} \cup \Phi$ has a D_{n+1} Dynkin diagram of the form



Then, the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to the vector space $\bigoplus_{\alpha\in I(\lambda,\Phi)} \mathfrak{g}_\alpha$ are exactly the opposite to the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to $\bigoplus_{\alpha\in I(\gamma,\Phi)} \mathfrak{g}_\alpha$, for each unit normal vector ξ to the submanifold $S_\Phi \cdot o$. In particular, $\mathcal{S}$ is austere when restricted to $\bigoplus_{\alpha\in I(\lambda,\Phi)\cup I(\gamma,\Phi)} \mathfrak{g}_\alpha$.

Proof. If $n = 3$, the shape operator is austere when restricted to the Φ-string of λ by means of Proposition 6.2.9 (iii). Thus, put $n > 3$. First, we will study the roots of the Φ-string of λ. In order to do that, we will study the positive roots of the form $n_\lambda\lambda + \sum_{i=1}^n n_i\alpha_i$ $\mathbb{Z}$-spanned by the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$, with $n_\lambda = 1$. Then, the number of roots in the Φ-string of λ is the number of positive roots spanned by a D_{n+1} simple system, minus the number of positive roots spanned by Φ (since $n_\lambda = 0$ for such roots) and minus the number of positive roots spanned by the D_{n+1} system with $n_\lambda \geq 2$ (there are no roots satisfying this condition [69, p. 684]). Therefore

$$|I(\lambda, \Phi)| = (n + 1)n - \frac{n(n + 1)}{2} = \frac{n(n + 1)}{2}.$$

Note that all the roots in the $\Phi\backslash\{\alpha_1\}$-string of λ are contained in the Φ-string of λ. Since λ is the root of minimum level in its Φ-string, then it is also the root of minimum level in its $\Phi\backslash\{\alpha_1\}$-string. From Proposition 6.2.8 we get that the $\Phi\backslash\{\alpha_1\}$-string of λ consists of the root λ and the roots of the form

$$\lambda_l = \lambda + \sum_{i=2}^{l} \alpha_i,$$

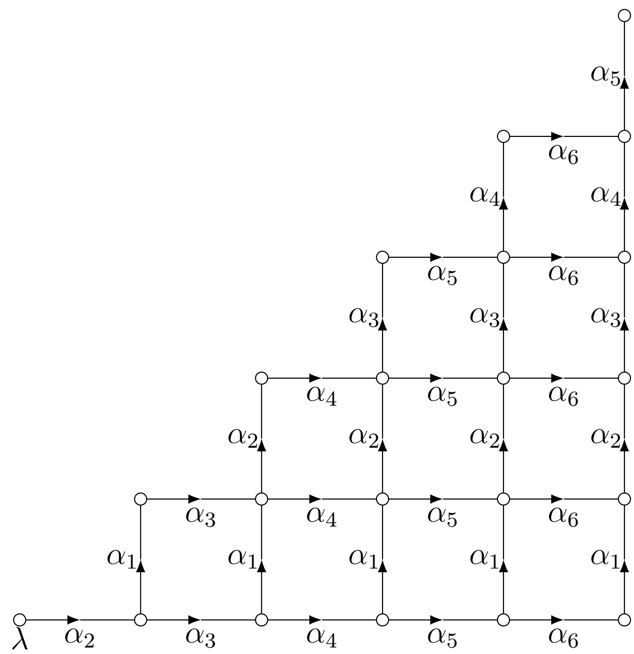
for each $l \in \{2, \dots, n\}$. The root λ_l is the root of minimum level in its $(\alpha_1, \dots, \alpha_{l-1})$-string for each $l \in \{2, \dots, n\}$, as follows from Proposition 6.2.1 (iii) and the facts that $A_{\alpha_1,\lambda_l} < 0$ and $A_{\alpha_i,\lambda_l} = 0$ for each $i \in \{2, \dots, l-1\}$. Since $(\alpha_1, \dots, \alpha_{l-1})$ is an A_{l-1} system then $\{\alpha_1, \dots, \alpha_{l-1}\} \cup \{\lambda_l\}$ is an A_l simple system for each $l \in \{2, \dots, n\}$. Using again Proposition 6.2.8 we obtain that the $(\alpha_1, \dots, \alpha_{l-1})$-string of λ_l consists of the root λ_l and the roots of the form

$$\lambda_l(k) = \lambda_l + \sum_{j=1}^{k} \alpha_j,$$

for each $k \in \{1, \dots, l-1\}$ and $l \in \{2, \dots, n\}$. Put $\lambda_1 = \lambda$ and $\lambda_l(0) = \lambda_l$, for $l \in \{1, \dots, n\}$. Then, the number of roots of the form $\lambda_l(k)$ with $l \in \{1, \dots n\}$ and $k \in \{0, \dots, l-1\}$ is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

and thus we have calculated all the roots in the Φ-string of λ. Hence, the diagram of this Φ-string (we put $n = 6$ for simplicity) is of the form



In the row in the bottom we draw the roots of the form λ_l, that is, the roots of the $\Phi\backslash\{\alpha_1\}$-string of λ. Above the root λ_l we draw the roots of the form $\lambda_l(k)$, that is, the roots of the $(\alpha_1, \dots, \alpha_{l-1})$-string of λ_l. It is easy to see that the shape operator of $\mathcal{S}$ is not austere when restricted to the Φ-string of λ by using Corollary 6.1.12. Indeed, there is just one arrow connected to the node λ. This arrow has label α_2. None of the nodes is connected to just one arrow with label α_2: there is just one node connected to exactly one arrow, but this arrow has label α_{n-1} in general (label α_5 in the above diagram since $n = 6$). Thus, we deduce that the shape operator of $S_\Phi \cdot o$ cannot be austere when restricted to the Φ-string of λ, if $n > 3$.

For the sake of completeness, we also include the calculation of the principal curvatures. Recall from (6.5) that in order to calculate the principal curvatures we need to determine the Cartan integers. In this case we have

$$A_{\alpha_m,\lambda_l(k)} = \begin{cases} 1 & \text{if } m = k, \quad \text{or} \quad m = l \text{ and } k \neq l-1, \\ -1 & \text{if } m = k+1 \text{ and } k \neq l-1, \quad \text{or} \quad m = l+1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.19}$$

Consider the normal vector $\xi = \sum_{i=1}^{n} a_i H_{\alpha_i}$. Since all the roots in Π_λ have the same length, put $|\alpha_1| = 2$ for simplicity. From (6.19) we deduce that the principal curvature $\mu_l(k)$ associated with the root space $\mathfrak{g}_{\lambda_l(k)}$, with $l \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, l-1\}$, is

$$\mu_l(k) = \begin{cases} a_l + a_k - a_{l+1} - a_{k+1} & \text{if } 2 < l < n,\ 0 < k < l-1, \\ a_k - a_{l+1} & \text{if } 2 \le l < n,\ k = l-1, \\ a_l + a_k - a_{k+1} & \text{if } l = n,\ 1 \le k \le n-2, \\ a_l - a_{l+1} - a_{k+1} & \text{if } 2 \le l \le n-1,\ k = 0, \\ a_n - a_1 & \text{if } l = n,\ k = 0, \\ -a_2 & \text{if } l = 1,\ k = 0, \\ a_{n-1} & \text{if } l = n,\ k = n-1. \end{cases} \tag{6.20}$$

Now we will prove the second claim in the statement of this proposition. Let $\gamma \in \Sigma^\Phi \backslash I(\lambda, \Phi)$ be of minimum level in its Φ-string satisfying $A_{\alpha_{n-1},\gamma} = -1$. In order to define later a bijection f between $I(\lambda, \Phi)$ and $I(\gamma, \Phi)$, we include the roots of the Φ-string of γ explicitly. The $\Phi \backslash \{\alpha_n\}$-string of γ consists of the root γ and the roots of the form

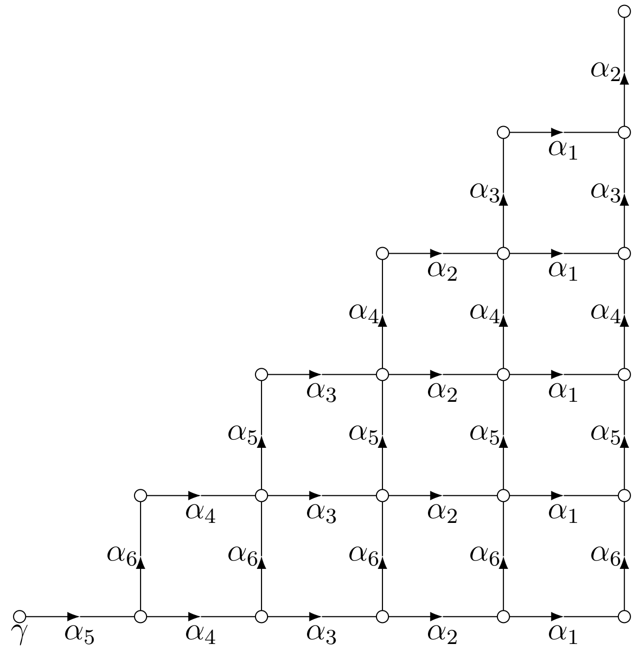
$$\gamma_l = \gamma + \sum_{i=1}^{l} \alpha_{n-i},$$

with $l \in \{1, \ldots, n-1\}$. Note that γ_l is of minimum level in its $(\alpha_{n-l+1}, \ldots, \alpha_n)$-string by means of Proposition 6.2.1 (i). Hence, using Proposition 6.2.8 we will calculate the $(\alpha_{n-l+1}, \ldots, \alpha_n)$-string of γ_l for each $l \in \{1, \ldots, n-1\}$. Thus we have that the roots of the form

$$\gamma_l(k) = \gamma_l + \sum_{j=0}^{k} \alpha_{n-j}$$

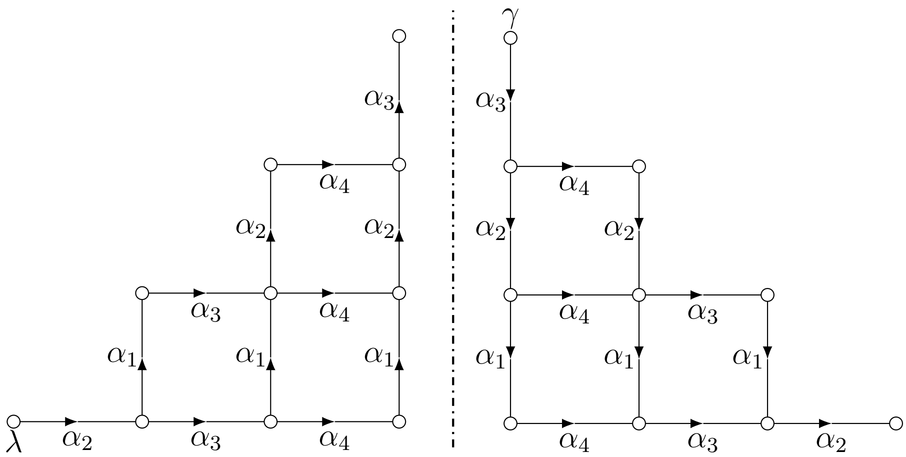
for $k \in \{0, \ldots, l-1\}$ belong to the Φ-string of γ. Put $\gamma_0 = \gamma$ and $\gamma_l(-1) = \gamma_l$, with $l \in \{0, \ldots, n-1\}$. Thus the Φ-string of γ consists of the roots $\gamma_l(k)$ for $l \in \{0, \ldots, n-1\}$ and $k \in \{-1, \ldots, l-1\}$.

Then, the Φ-string of γ has a diagram (which we draw for simplicity for the case $n = 6$) of the form



Recall that, by assumption, λ and γ are connected to a simple root by the same number of edges in the Dynkin diagrams of Π_λ and Π_γ. Hence γ and λ have the same multiplicity. Moreover, note that Π_λ and Π_γ are both D_{n+1} simple systems. This means that all the roots $\mathbb{Z}$-spanned by Π_λ and Π_γ have the same multiplicity. In particular, all the roots in $I(\lambda, \Phi)$ and $I(\gamma, \Phi)$ have the same multiplicity as λ and γ.

Hence, the map $f\colon I(\lambda, \Phi) \to I(\gamma, \Phi)$ defined by $f(\lambda_l(k)) = \gamma_{n-k-1}(n-l-1)$ satisfies the conditions specified in Proposition 6.1.11 (iii). This map is induced by the reflection with respect to a central vertical axis separating the diagrams drawn below (with $n = 4$ for simplicity):

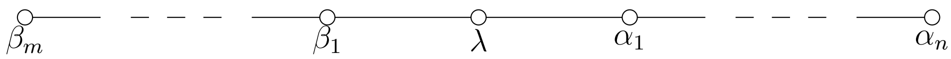


In terms of diagrams it is very easy to check that the condition explained in Corollary 6.1.12 (iii) is satisfied. If we reflect one diagram to the other we can see that each node is sent to a node with opposite arrows. This concludes the proof. □

Remark 6.2.14. Let Φ be a proper subset of the set Π of simple roots. Let λ, $\gamma \in \Sigma^\Phi$ be roots of minimum level in their corresponding different Φ-strings. Assume that the Φ-string of λ is described in Proposition 6.2.8 and that the Φ-string of γ is described in Proposition 6.2.13. Then, for some unit normal vector ξ of $S_\Phi \cdot o$ there is a principal curvature of $\mathcal{S}_\xi$ when restricted to the Φ-string of γ such that its opposite cannot a principal curvature of $\mathcal{S}_\xi$ when restricted to the Φ-string of λ. This is a simple consequence of the assertions on the Cartan integers in the proofs of Proposition 6.2.8 and Proposition 6.2.13 combined with Lemma 6.1.10 (i).

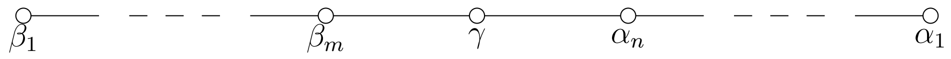
In most of the results above we have dealt with a connected subset Φ of the set Π of simple roots. The following two propositions can be thought as generalizations of Proposition 6.2.8 to the non-connected case.

Proposition 6.2.15. *Let $\Phi \subset \Pi$ be a proper subset of the set of simple roots of the root system Σ. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Assume that the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$ has an A_{m+n+1} Dynkin diagram of the form*



The shape operator of $S_\Phi \cdot o$ is austere when restricted to $\bigoplus_{\alpha \in I(\lambda,\Phi)} \mathfrak{g}_\alpha$ if and only if $n = m = 1$.

Moreover, let $\gamma \in \Sigma^\Phi \backslash I(\lambda,\Phi)$ be of minimum level in its Φ-string. Assume that the simple system $\Pi_\gamma = \{\gamma\} \cup \Phi$ has an A_{m+n+1} Dynkin diagram of the form



Then, the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to the vector space $\bigoplus_{\alpha \in I(\lambda,\Phi)} \mathfrak{g}_\alpha$ are the opposite to the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to $\bigoplus_{\alpha \in I(\gamma,\Phi)} \mathfrak{g}_\alpha$, for each unit normal vector ξ to the submanifold $S_\Phi \cdot o$. In particular, $\mathcal{S}$ is austere when restricted to $\bigoplus_{\alpha \in I(\lambda,\Phi) \cup I(\gamma,\Phi)} \mathfrak{g}_\alpha$.

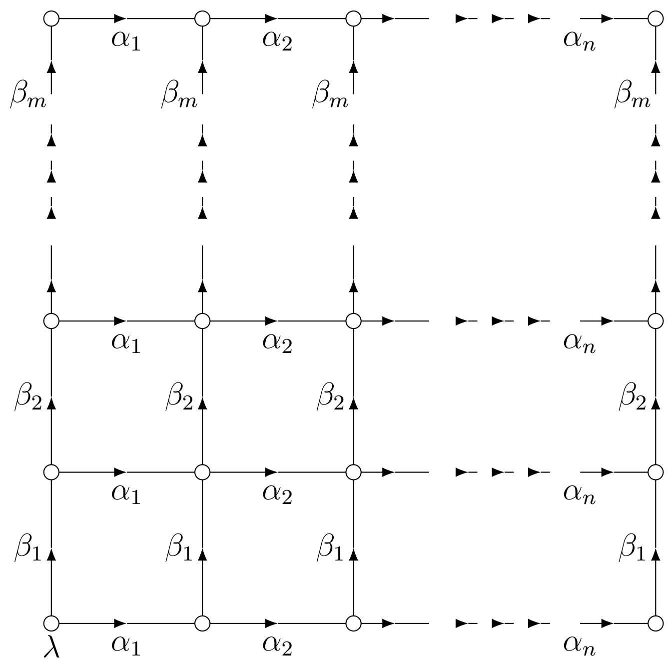
Proof. If $n = m = 1$ the result follows from Proposition 6.2.6. Then, assume that $n > 1$ or $m > 1$. Put $\Phi_0 = \{\alpha_1, \dots, \alpha_n\}$ and $\Phi_1 = \{\beta_1, \dots, \beta_m\}$. Define $\alpha^\lambda(l) = \sum_{i=1}^{l} \alpha_i$ for each $l \in \{1, \dots, n\}$ and $\alpha^\lambda(0) = 0$. From the proof of Proposition 6.2.8 we know that the Φ_0-string of λ consists of the roots

$$\lambda_l = \lambda + \alpha^\lambda(l),$$

with $l \in \{0, \dots, n\}$. From Corollary 6.2.3 (i) we know that we can obtain the Φ-string of λ by calculating the Φ_1-string of λ_l for each $l \in \{0, \dots, n\}$. Put $\beta^\lambda(0) = 0$ and $\beta^\lambda(k) = \sum_{i=1}^{k} \beta_i$ for each $k \in \{1, \dots, m\}$. Thus, using again Proposition 6.2.8, we obtain that the Φ-string of λ consists of the roots

$$\lambda(l,k) = \lambda + \alpha^\lambda(l) + \beta^\lambda(k)$$

for $l \in \{0, \ldots, n\}$ and $k \in \{0, \ldots, m\}$. The diagram of this Φ-string is of the form

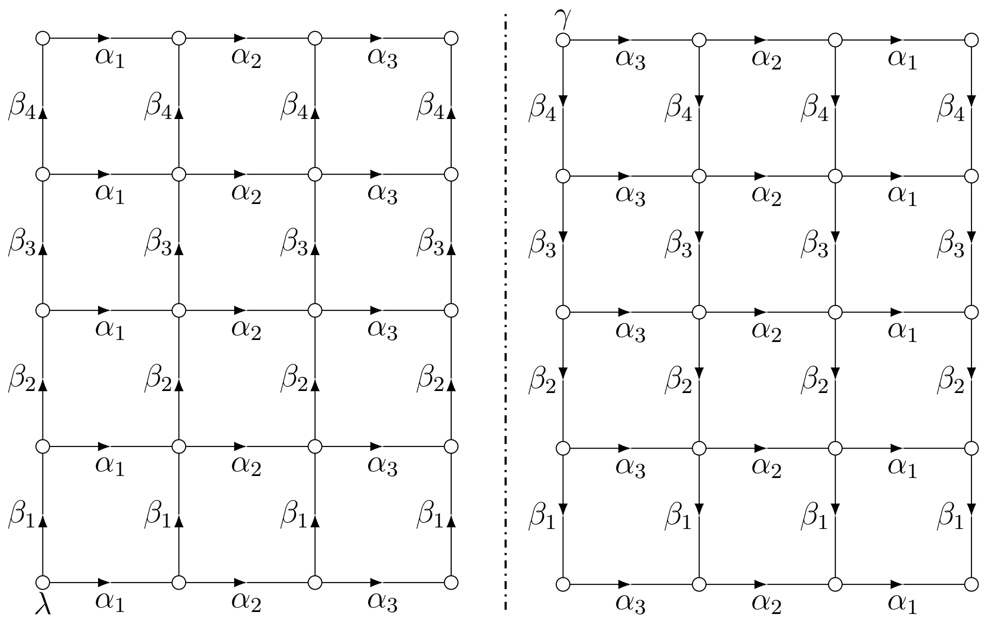


There are two arrows connected to the node λ. One has label α_1 and the other one has label β_1. There is just another node connected to exactly two arrows (located at the top right-hand side of the diagram). However, its arrows have labels α_n and β_m. Thus, from Corollary 6.1.12 we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ if and only if $m = n = 1$.

Let us focus on the Φ-string of γ. Put $\alpha^\gamma(l) = \sum_{i=0}^{l} \alpha_{n-i}$ for each $l \in \{0, \ldots, n-1\}$ and $\beta^\gamma(k) = \sum_{i=0}^{k} \beta_{m-i}$ for each $k \in \{0, \ldots, m-1\}$. For simplicity, write $\alpha^\gamma(-1) = \beta^\gamma(-1) = 0$. Thus, the Φ-string of γ consists of the roots

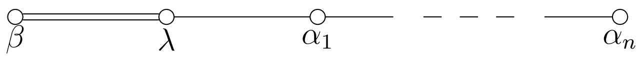
$$\gamma(l, k) = \gamma + \alpha^\gamma(l) + \beta^\gamma(k),$$

for $l \in \{-1, \ldots, n-1\}$ and $k \in \{-1, \ldots, m-1\}$. Note that all the roots have the same multiplicity. Hence, the map $f\colon I(\lambda, \Phi) \to I(\gamma, \Phi)$ defined by $f(\lambda(l, k)) = \gamma(n-l-1, m-k-1)$ satisfies the conditions of Proposition 6.1.11 (iii). This map is induced by the reflection with respect to the central vertical axis of the diagram below (which we draw for the case $(n, m) = (3, 4)$ for simplicity):



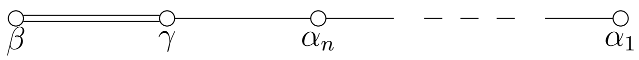
If we reflect one diagram to the other we can see that each node is sent to a node with opposite arrows. This concludes the proof. □

Proposition 6.2.16. *Let $\Phi \subset \Pi$ be a proper subset of the set of simple roots of the root system Σ. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Assume that the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$ has a B_{n+2} Dynkin diagram of the form*



The shape operator of $S_\Phi \cdot o$ is austere when restricted to $\bigoplus_{\alpha \in I(\lambda,\Phi)} \mathfrak{g}_\alpha$ if and only if $n = 1$.

Moreover, let $\gamma \in \Sigma^\Phi \backslash I(\lambda, \Phi)$ be of minimum level in its Φ-string. Assume that the simple system $\Pi_\gamma = \{\gamma\} \cup \Phi$ has a B_{n+2} Dynkin diagram of the form



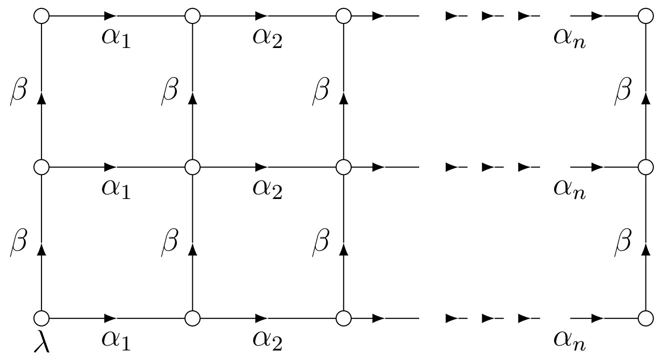
Then, the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to the vector space $\bigoplus_{\alpha \in I(\lambda,\Phi)} \mathfrak{g}_\alpha$ are exactly the opposite to the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to $\bigoplus_{\alpha \in I(\gamma,\Phi)} \mathfrak{g}_\alpha$, for each unit normal vector ξ to the submanifold $S_\Phi \cdot o$. In particular, $\mathcal{S}$ is austere when restricted to $\bigoplus_{\alpha \in I(\lambda,\Phi) \cup I(\gamma,\Phi)} \mathfrak{g}_\alpha$.

Proof. If $n = 1$ this result follows directly from Proposition 6.2.6. Thus, put $n > 1$. Using Proposition 6.2.8, we have that the $(\alpha_1, \dots, \alpha_n)$-string of λ consists of the root λ and the roots of the form $\lambda_l = \lambda + \sum_{i=1}^{l} \alpha_i$, for each $l \in \{1, \dots, n\}$. For simplicity, put $\lambda_0 = \lambda$.

Thus, according to Corollary 6.2.3, Proposition 1.5.1 (v) and Proposition 6.2.8, we deduce that the Φ-string of λ consists of the roots of the form

$$\lambda(l, \varepsilon) = \lambda_l + \varepsilon\beta,$$

for $l \in \{0, \ldots, n\}$ and $\varepsilon \in \{0, 1, 2\}$. The diagram of this Φ-string is of the form



Note that

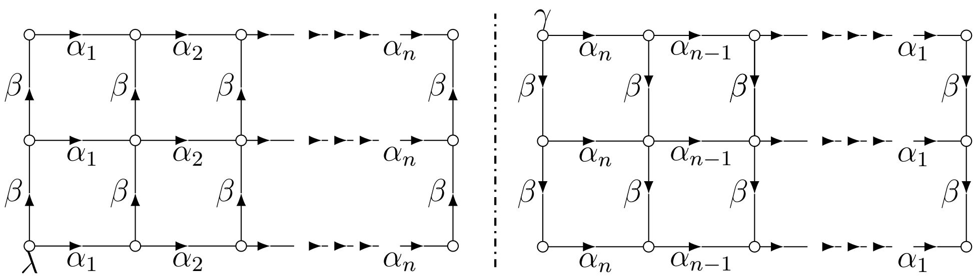
$$A_{\nu,\lambda(l,\varepsilon)} = \begin{cases} -2 & \text{if } (\nu, \varepsilon) = (\beta, 0), \\ -1 & \text{if } \nu = \alpha_{l+1}, \\ 0 & \text{if } (\nu, \varepsilon) = (\beta, 1) \text{ or } \nu \notin \{\alpha_{l+1}, \alpha_l\}, \\ 1 & \text{if } \nu = \alpha_l, \\ 2 & \text{if } (\nu, \varepsilon) = (\beta, 2). \end{cases} \tag{6.21}$$

In particular, using Lemma 5.1.1 and (6.21) we deduce, on the one hand, that all the roots $\lambda(l, \varepsilon)$ with $\varepsilon \in \{0, 2\}$ have the same multiplicity (first and last rows in the diagram). On the other hand, they also imply that all the roots of the form $\lambda(l, 1)$ (second row in the diagram) have the same multiplicity. Put $n > 1$. Thus $(A_{\beta,\lambda}, A_{\alpha_1,\lambda}) = (-2, -1)$ and $A_{\nu,\lambda} = 0$ for all $\nu \in \Phi \backslash \{\beta, \alpha_1\}$. If $\mathcal{S}$ were austere when restricted to the Φ-string of λ, from Lemma 6.1.10 (ii), there should exist a root λ' in the Φ-string of λ satisfying $(A_{\beta,\lambda'}, A_{\alpha_1,\lambda'}) = (2, 1)$ and $A_{\nu,\lambda'} = 0$ for all $\nu \in \Phi \backslash \{\beta, \alpha_1\}$. From (6.21), a root satisfying these conditions does not exist.

Let us focus on the diagram of the Φ-string of γ. Put $\gamma_l = \gamma + \sum_{i=0}^{l} \alpha_{n-i}$ for each $l \in \{0, \ldots, n-1\}$ and put $\gamma_{-1} = \gamma$. Thus, according to Corollary 6.2.3 and Proposition 6.2.8 the Φ-string of γ consists of the roots

$$\gamma(l, k) = \gamma_l + \varepsilon\beta,$$

for $l \in \{-1, \ldots, n-1\}$ and $\varepsilon \in \{0, 1, 2\}$. Recall the information about multiplicities given by Lemma 5.1.1 and (6.21). Hence, the map $f\colon I(\lambda, \Phi) \to I(\gamma, \Phi)$ defined by $f(\lambda(l, \varepsilon)) = \gamma(n - l - 1, 2 - \varepsilon)$ satisfies the conditions of Proposition 6.1.11 (iii). This map is induced by the reflection with respect to a central vertical axis between the diagrams we draw below.



This concludes the proof. □

6.3 The classification in classical spaces

In this section we will classify the austere submanifolds of the form $S_\Phi \cdot o$ in symmetric spaces G/K with A_r, B_r, C_r, BC_r and D_r Dynkin diagram, where Φ is a subset of the set Π of simple roots. The information about Φ-strings given in Section 6.2 suffices in order to obtain these classifications. We start with an easy but very useful lemma.

Lemma 6.3.1. *Let Φ_0 be a subset of the set of simple roots Π. Assume that $S_{\Phi_0} \cdot o$ is not an austere submanifold. Take a subset $\Phi_1 \subset \Pi$ orthogonal to Φ_0. Put $\Phi = \Phi_0 \cup \Phi_1$. Then $S_\Phi \cdot o$ is not austere.*

Proof. In this proof we will write $\mathcal{S}$ for the shape operator of $S_\Phi \cdot o$ and $\mathcal{S}^{\Phi_0}$ for the shape operator of $S_{\Phi_0} \cdot o$. Since $S_{\Phi_0} \cdot o$ is not austere, there exists a unit normal vector ξ to the submanifold $S_{\Phi_0} \cdot o$ such that the shape operator $\mathcal{S}^{\Phi_0}_\xi$ of $S_{\Phi_0} \cdot o$ is not austere. Note that ξ is also a unit normal vector to the submanifold $S_\Phi \cdot o$ and that

$$T_o(S_\Phi \cdot o) \oplus \left(\bigoplus_{\alpha \in \Sigma_{\Phi_1}} \mathfrak{g}_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Phi_1} \mathbb{R} H_\alpha \right) = T_o(S_{\Phi_0} \cdot o).$$

Moreover, we have that $\mathcal{S}_\xi$ is the restriction of the shape operator $\mathcal{S}^{\Phi_0}_\xi$ to the vector space

$$\left(\bigoplus_{\alpha \in \Sigma^\Phi} \mathfrak{g}_\alpha \right) \oplus \mathfrak{a}_\Phi.$$

In other words, we have that

$$\mathcal{S}_\xi = \mathcal{S}^{\Phi_0}_\xi |_{\left(\bigoplus_{\alpha \in \Sigma^\Phi} \mathfrak{g}_\alpha\right) \oplus \mathfrak{a}_\Phi}.$$

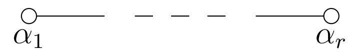
Recall from (6.3) that $\mathcal{S}_{\eta_1} \mathfrak{a}_\Phi = 0$ for all $\eta_1 \in \mathfrak{a}^\Phi$ and $\mathcal{S}^{\Phi_0}_{\eta_2} \mathfrak{a}_{\Phi_0} = 0$ for all $\eta_2 \in \mathfrak{a}^{\Phi_0}$. From (6.5) and $A_{\alpha_0, \alpha_1} = 0$ for all $\alpha_i \in \Phi_i$, with $i \in \{0, 1\}$, we deduce that $\mathcal{S}^{\Phi_0}_\xi$ is zero when restricted to $\mathfrak{g}_\alpha$, for each $\alpha \in \Sigma_{\Phi_1}$. Thus, the non-zero principal curvatures of $\mathcal{S}_\xi$ and $\mathcal{S}^{\Phi_0}_\xi$ coincide. Since $S_{\Phi_0} \cdot o$ is not austere, then $S_\Phi \cdot o$ cannot be austere. □

Remark 6.3.2. Consider the Dynkin diagrams of two simple systems Π and Π' of the same type. Let $f\colon \Sigma \to \Sigma'$ be a bijection satisfying $A_{\alpha,\beta} = A_{f(\alpha),f(\beta)}$ for all α, $\beta \in \Pi$. Let $\Phi \subset \Pi$ and assume that the multiplicity of ν is proportional by a constant factor c (note that c does not depend on ν) to the multiplicity of $f(\nu)$, for all $\nu \in \Sigma$. Then, the submanifold $S_\Phi \cdot o$ is austere if and only the submanifold $S_{f(\Phi)} \cdot o$ is austere. This is because the decomposition of Σ^Φ into Φ-strings is equivalent to the decomposition of $\Sigma^{f(\Phi)}$ into $f(\Phi)$-strings and multiplicities are preserved up to multiplication by a constant factor c. Roughly speaking, this means that it suffices to study Dynkin diagrams instead of symmetric spaces. In particular, in spaces where all the roots have same length we just need to care about the Dynkin diagram.

6.3.1 Symmetric spaces of type A_r

Let us start with the classifications of the simplest cases, that is, symmetric spaces G/K with A_r Dynkin diagram.

Proposition 6.3.3. *Let Φ be a proper subset of the set of simple roots Π of a symmetric space of non-compact type G/K with A_r Dynkin diagram*



Then, the orbit $S_\Phi \cdot o$ is austere if and only if one of the following statements holds:

(i) *Φ is discrete, or*

(ii) *$\Phi = \{\alpha_n, \dots \alpha_m\}$ is a connected and symmetric subset of $\Pi = \{\alpha_1, \dots \alpha_r\}$, that is, $r - m = n - 1$.*

Proof. Consider first a particular case. Let $\Phi = \{\alpha_n, \dots, \alpha_m\}$ be a connected subset of the set of simple roots $\Pi = \{\alpha_1, \dots \alpha_r\}$ of the symmetric space G/K, with $n < m$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

From Proposition 6.2.1 (i), we have that $\Pi_\lambda = \Phi \cup \{\lambda\}$ is a simple system. This Φ-string can be trivial. Since $A_{\alpha,\lambda} = 0$ for all $\alpha \in \Phi$ in this case we have that $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of the shape operator of $S_\Phi \cdot o$.

Assume that the Φ-string of λ is not trivial. Then, the roots in the root system with the simple system Π_λ must be contained in Σ, which has an A_r Dynkin diagram. Note that if Π_λ is not of type A, then there will be at least [69, p. 684] one positive root $\gamma = \sum_{\alpha \in \Pi_\lambda} n_\alpha \alpha$ with $n_\beta \geq 2$ for some $\beta \in \Pi_\lambda$. But this root would be in the A_r system $\mathbb{Z}$-generated by Π and then $0 \leq n_\alpha \leq 1$ for all $\alpha \in \Pi_\lambda$. Thus we get a contradiction. Hence, if the Φ-string of λ is not trivial, then Π_λ must be an A_{m-n+2} simple system.

Then, the principal curvatures of the shape operator of $S_\Phi \cdot o$ when restricted to this Φ-string have been determined in Proposition 6.2.8. Since all the roots in Σ have the same multiplicity, from Proposition 6.2.8 we deduce that $S_\Phi \cdot o$ is austere if and only if the number of roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_n,\nu_1} = -1$

coincides with the number of roots $\nu_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_m,\nu_2} = -1$.

On the one hand, note that the

$$\frac{(n-2)(n-1) + (r-m-1)(r-m)}{2}$$

positive roots generated by the reducible simple system

$$\{\alpha_1, \dots, \alpha_{n-2}\} \cup \{\alpha_{m+2}, \dots, \alpha_r\} \equiv A_{n-2} \oplus A_{r-m-1}$$

have trivial Φ-string. Moreover, also the $(n-1)(r-m)$ roots of the form $\sum_{i=k}^{l} \alpha_i$ with $k < n < m < l$ have trivial Φ-string. On the other hand, we have that $\sum_{i=l}^{n-1} \alpha_i$ is the root of minimum level in its Φ-string by means of Proposition 6.2.1 (i) for each $l \in \{1, \dots, n-1\}$. This Φ-string is of the first type in (6.14), for each $l \in \{1, \dots, n-1\}$. Finally, we have that $\sum_{i=m+1}^{l} \alpha_i$ is of minimum level in its Φ-string by means of Proposition 6.2.1 (i) for each $l \in \{m+1, \dots, r\}$. This Φ-string is of the first type in (6.15), for each $l \in \{m+1, \dots, r\}$. If P is a simple system, we denote by $|P|$ the number of positive roots spanned by P. Recall that Φ is an A_{m-n+1} simple system. Then we have

$$\begin{aligned}&(n-1)(r-m) + |A_{n-2}| + |A_{r-m-1}| + |A_{m-n+1}| \\ &+ (n-1)(m-n+2) + (r-m)(m-n+2) = |A_r|,\end{aligned}$$

which means that we have studied all the roots in Σ^Φ. In summary, we have $n-1$ roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_n,\nu_1} = -1$ and $r-m$ roots $\nu_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_m,\nu_2} = -1$. Therefore, if $\Phi = \{\alpha_n, \dots, \alpha_m\}$ is a connected subset of Π with $n < m$, then $S_\Phi \cdot o$ is austere if and only if $r - m = n - 1$, that is, if and only if Φ is symmetric in Π.

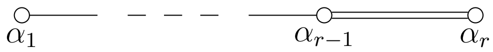
Assume first that Φ_0, $\Phi_1 \subset \Phi$, where Φ_0, Φ_1 are orthogonal connected subsets of Π. If $|\Phi_0|, |\Phi_1| \geq 2$, then Φ_0 or Φ_1 is not symmetric in Π. Thus $S_{\Phi_0} \cdot o$ or $S_{\Phi_1} \cdot o$ is not austere. From Lemma 6.3.1, the submanifold $S_\Phi \cdot o$ is not austere either.

Hence, assume that $\Phi_0 = \{\alpha_n, \dots \alpha_m\}$ and Φ_1 are orthogonal subsets of Π, where Φ_0 is symmetric in Π and Φ_1 is discrete, and put $\Phi = \Phi_0 \cup \Phi_1$. Without loss of generality, assume also $\alpha_l \in \Phi_1$ with $l < n-1$. Let us consider the submanifold $S_{\Phi_0 \cup \{\alpha_l\}}$. Thus, $A_{\alpha,\alpha_l} = 0$ for all $\alpha \in \Phi_0$. Consider the positive root $\lambda = \sum_{i=l+1}^{n} \alpha_i$. Note that $A_{\alpha_l,\lambda} = -A_{\alpha_n,\lambda} = A_{\alpha_{n+1},\lambda} = -1$ and that $A_{\alpha,\lambda} = 0$ for all $\alpha \in \Phi_0 \backslash \{\alpha_n, \alpha_{n+1}\}$. From Lemma 6.1.10 (i) we have that if $S_{\Phi_0 \cup \{\alpha_l\}}$ is austere, then there must exist a root $\gamma \in \Sigma^{\Phi_0 \cup \{\alpha_l\}}$ such that $A_{\alpha_l,\gamma} = -A_{\alpha_n,\gamma} = A_{\alpha_{n+1},\gamma} = 1$ and that $A_{\alpha,\gamma} = 0$ for all $\alpha \in \Phi_0 \backslash \{\alpha_n, \alpha_{n+1}\}$. Put $\gamma = \sum_{i=1}^{r} b_i \alpha_i$. Note that $b_i \in \{0, 1\}$ for all $i \in \{1, \dots, r\}$. Thus, from $A_{\alpha_l,\gamma} = 1$ we deduce $b_l = 1$, from $A_{\alpha_n,\gamma} = -1$ we deduce $b_n = 0$, and from $A_{\alpha_{n+1},\gamma} = 1$ we deduce $b_{n+1} = 1$. But then γ would not be a root. Then $S_{\Phi_0 \cup \{\alpha_l\}}$ is not an austere submanifold. Recall that $\Phi = \Phi_0 \cup \Phi_1$. From Lemma 6.3.1, we conclude that $S_\Phi \cdot o$ is not austere either. $\square$

6.3.2 Symmetric spaces of type B_r

In this subsection we classify the austere submanifolds of the form $S_\Phi \cdot o$ in symmetric spaces of non-compact type with B_r Dynkin diagram. We have:

Proposition 6.3.4. *Let Φ be a proper subset of the set of simple roots Π that has a B_r Dynkin diagram of the form*



Then, the submanifold $S_\Phi \cdot o$ is austere if and only if one of the following conditions holds:

(i) *Φ is a B_n simple subsystem, for $n < r$,*

(ii) *$\Phi = \{\alpha_{r-2}, \alpha_{r-1}\}$ and all the roots in Σ^Φ have the same multiplicity, or*

(iii) *$\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 satisfies the hypotheses of either (i) or (ii), and Φ_1 is a discrete subset of Π orthogonal to Φ_0.*

Proof. Assume first that Φ is connected subset of Π. Thus, according to Proposition 6.2.12, we just need to analyze the case when $\Phi = \{\alpha_n, \dots, \alpha_m\}$ is a subset of the set $\Pi = \{\alpha_1, \dots, \alpha_r\}$ of simple roots, with $n < m < r$.

In Table 6.2 table we gather the following information. In each row we write a family of roots in Σ^Φ of minimum level in their Φ-strings (using Proposition 6.2.1 (iii)). In particular, we also write roots with trivial Φ-string. Moreover, we also specify how many roots of each class we have and which kind of Φ-string they have.

Roots	Conditions	Number of roots	Φ-string
$\sum_{i=l}^{n-1} \alpha_i$	$1 \leq l \leq n-1$	$n-1$	(6.14)
$\sum_{i=l}^{m} \alpha_i + 2\sum_{j=m+1}^{r} \alpha_j$	$1 \leq l \leq n-1$	$n-1$	(6.15)
$\sum_{i=l}^{k} \alpha_i$	$l < n < m < k < r$	$(r-m)(n-1)$	Trivial
$\sum_{i=m+1}^{l} \alpha_i$	$m+1 \leq l \leq r$	$r-m$	(6.15)
$\sum_{i=m+1}^{l} \alpha_i + 2\sum_{j=l+1}^{r} \alpha_j$	$m+2 \leq l \leq r$	$r-m-1$	(6.15)
$\sum_{i=l}^{k} \alpha_i + 2\sum_{j=k+1}^{r} \alpha_j$	$1 \leq l < k \leq n-2$	$(n-2)(n-1)/2$	Trivial
$\sum_{i=l}^{k} \alpha_i + 2\sum_{j=k+1}^{r} \alpha_r$	$1 \leq l \leq n-1$, $m+2 \leq k \leq r$	$(r-m-1)(n-1)$	Trivial

Table 6.2: Some roots of minimum level in their Φ-strings, for Π of B_r type.

It is important to emphasize that all the roots (except one) in Table 6.2 whose Φ-string is of type (6.14) or of type (6.15) have a Φ-string of the first type in (6.14) or of the first type in (6.15), respectively. Indeed, the root (fourth row in Table 6.2 with $l = r$)

$$\gamma = \sum_{i=m+1}^{r} \alpha_i \tag{6.22}$$

has a Φ-string of the second type in (6.14). Hence, γ can have different multiplicity from that of the rest of the roots of minimum level with non-trivial Φ-string we have considered (see [7, p. 337]). According to Remark 6.3.2, we will write d for the multiplicity of γ and 1 for the other multiplicity. Note also that the root

$$\lambda = \alpha_m + 2\sum_{i=m+1}^{r} \alpha_i \in \Sigma^\Phi \tag{6.23}$$

is of minimum level in its Φ-string by means of Proposition 6.2.1 (iii). The Φ-string of λ is of the first type in (6.14) if $|\Phi| = 2$ and described in Proposition 6.2.13 if $|\Phi| \geq 3$.

The reducible simple system

$$A_{n-2} \oplus B_{r-m-1} \cong \{\alpha_1, \ldots, \alpha_{n-2}\} \cup \{\alpha_{m+2}, \ldots, \alpha_r\}$$

spans $(n-2)(n-1)/2+(r-m-1)^2$ roots in Σ^Φ with trivial Φ-string. Since Φ is an A_{m-n+1} subsytem, it spans $(m-n+1)(m-n+2)/2$ positive roots. Moreover, note that each Φ-string described in Proposition 6.2.8 consists of $m-n+2$ roots and the Φ-string of the root λ described in (6.23) contains $(m-n+2)(m-n+1)/2$ roots (see Proposition 6.2.13 if $|\Phi| \geq 3$). Hence we can see that

$$\begin{aligned} &2(n-1)(m-n+2) + (r-m)(n-1) + (r-m)(m-n+2) \\ &+ (r-m-1)(m-n+2) + (n-2)(n-1) + (r-m-1)^2 \\ &+ (r-m-1)(n-1) + (m-n+1)(m-n+2) = r^2, \end{aligned}$$

and this means that we have considered all the roots in Σ and then in Σ^Φ.

According to the data in Table 6.2, the root defined in (6.23) and the root defined in (6.22), we have that if the Φ-string of a root in Σ^Φ is not trivial, then it has been studied in Proposition 6.2.8 or in Proposition 6.2.13. Hence, from Proposition 6.2.8, Proposition 6.2.13 and Remark 6.2.14, we deduce that if the submanifold $S_\Phi \cdot o$ is austere, it must happen that $|\Phi| \leq 3$ and the number of roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_n,\nu_1} = -1$ coincides with the number of roots $\nu_2 \in \Sigma^\Phi$ (counted with multiplicities) of minimum level in their Φ-strings satisfying $A_{\alpha_m,\nu_2} = -1$.

Hence, using again the data in Table 6.2, the root defined in (6.23) and the root defined in (6.22), we deduce that the submanifold $S_\Phi \cdot o$ is austere if and only if

$$n-1 = n-1+r-m-1+r-m-1+d+\varepsilon,$$

where $\varepsilon = 1$ if $|\Phi| = 2$, since then the root in (6.23) has a Φ-string of the first type in (6.14), and $\varepsilon = 0$ otherwise. Since $m < r$ by assumption, and $d > 0$, we deduce that $S_\Phi \cdot o$ is austere if and only if $d = 1$, $\varepsilon = 1$ and $m = r-1$. In other words, $S_\Phi \cdot o$ is austere if and only if $\Phi = \{\alpha_{r-2}, \alpha_{r-1}\}$ and all the roots in Σ have the same multiplicity.

Let us assume that Φ contains at least two connected orthogonal components Φ_0 and Φ_1. If $|\Phi_i| > 2$ for $i \in \{0,1\}$, then either Φ_0 or Φ_1 does not satisfy neither (i) nor (ii) and using the above considerations together with Lemma 6.3.1 we deduce that $S_\Phi \cdot o$ is not austere.

Hence, let us assume that $\Phi = \Phi_0 \cup \Phi_1$, with Φ_1 discrete and orthogonal to Φ_0, where Φ_0 satisfies the conditions specified in (i) or in (ii). If Φ_0 is a B_n simple subsytem, then $S_\Phi \cdot o$ is austere, as follows from Proposition 6.2.12. Hence let us assume that $\Phi_0 = \{\alpha_{r-2}, \alpha_{r-1}\}$.

Take α_l, $\alpha_k \in \Phi$, with $l < k$. Let $\nu = \sum_{i=1}^r a_i\alpha_i \in \Sigma^\Phi$ be of minimum level in its Φ-string satisfying $A_{\alpha_l,\nu} = A_{\alpha_k,\nu} = -1$. Then we have that

$$(a_{i-1}, a_i, a_{i+1}) \in \{(1,0,0), (0,0,1), (1,1,2)\}. \tag{6.24}$$

Hence, either $\nu = \sum_{i=l+1}^{k-1} \alpha_i$ or $\nu = \sum_{i=l+1}^{k} \alpha_i + \sum_{j=k+1}^{r} 2\alpha_j$. In both cases we have that $A_{\alpha,\nu} = 0$ for any $\alpha \in \Phi \setminus \{\alpha_l, \alpha_k\}$, since ν is of minimum level in its Φ-string.

Let $\lambda \in \Sigma^{\Phi}$ be of minimum level in its Φ-string. If the Φ-string of λ is not trivial we basically have three cases to study possibilities.

First, assume that the Φ_0-string of λ is trivial. Then, the Φ-string of λ coincides with the Φ_1-string of λ. From Proposition 6.2.6 we deduce that $\mathcal{S}$ is austere when restricted to the Φ-string of λ.

Now, assume that the Φ_i-string of λ is not trivial for each $i \in \{0,1\}$. Hence, from the above calculations we deduce that the Φ-string of λ coincides with the $\Phi_0 \cup \{\alpha_l\}$-string of λ, for some $\alpha_l \in \Phi_1$. There are exactly two roots in Σ^{Φ} with non-trivial $\Phi_0 \cup \{\alpha_l\}$-string: the root $\gamma_1 = \sum_{i=l+1}^{r-3} \alpha_i$ and the root $\gamma_2 = \sum_{i=l+1}^{r-1} \alpha_i + 2\alpha_r$ are of minimum level in their respective Φ-strings, as follows Proposition 6.2.1 (iii). These Φ-strings have been studied in Proposition 6.2.15. Since γ_1 and γ_2 have the same multiplicity, from Proposition 6.2.15 we deduce that $\mathcal{S}$ is austere when restricted to

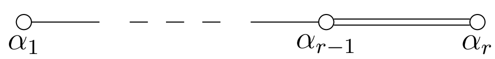
$$\bigoplus_{\nu \in I(\gamma_1,\Phi) \cup I(\gamma_2,\Phi)} \mathfrak{g}_{\nu}.$$

This means that the roots with non-trivial Φ_i-string, for $i \in \{0,1\}$, are organized by pairs and the shape operator is austere when restricted to the union of both strings. Note that in each pair, one of the roots has a Φ_0-string of the first type in (6.14) and the other one has Φ_0-string of the first type in (6.15). Hence this means that the number of roots $\nu_1 \in \Sigma^{\Phi}$ of minimum level in their Φ_0-strings (with trivial Φ_1-string) satisfying $A_{\alpha_{r-1},\nu_1} = -1$ equals the number the number of roots $\nu_2 \in \Sigma^{\Phi}$ of minimum level in their Φ_0-strings (with trivial Φ_1-string) satisfying $A_{\alpha_{r-2},\nu_2} = -1$. This is because for each $\alpha \in \Phi_1$ there are two roots in Table 6.2 whose string change. This was the last case we needed to study. Hence, $S_{\Phi} \cdot o$ is austere. □

6.3.3 Symmetric spaces of types C_r and BC_r

The next step consists in studying the austere submanifolds of the form $S_{\Phi} \cdot o$ in symmetric spaces G/K of non-compact type with Dynkin diagram of the form C_r and BC_r.

Proposition 6.3.5. *Let Φ be a proper subset of the set of simple roots Π of a symmetric space of non-compact type G/K with C_r Dynkin diagram*



Then $S_{\Phi} \cdot o$ is austere if and only if one of the following statements holds:

(i) *Φ is a C_n simple system system, with $n < r$,*

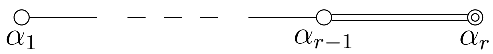
(ii) *Φ is discrete,*

(iii) *$\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 is a C_n simple system orthogonal to the discrete subset Φ_1. In other words, Φ_0 satisfies (i) and Φ_1 satisfies (ii) and is orthogonal to Φ_0.*

Proof. Let Φ be an A_n subsystem of the C_r simple system Π, with $2 \leq n < r$. If we prove that $S_\Phi \cdot o$ is not austere then the result follows using Proposition 6.2.12 and Lemma 6.3.1. Take $\alpha_m, \alpha_{m+1} \in \Phi$ two connected roots. Note that $m+1 < r$, since by assumption Φ is an A_n simple system. Consider the root $\lambda = \alpha_r + \sum_{i=m+2}^{r-1} 2\alpha_i$ if $m+1 < r-1$ and the root $\lambda = \alpha_r$ otherwise. Then $(A_{\alpha_m,\lambda}, A_{\alpha_{m+1},\lambda}) = (0,-2)$. From Lemma 6.1.10 (i) we deduce that if $S_\Phi \cdot o$ is austere, then there must exist a positive root $\gamma \in \Sigma^\Phi$ such that $(A_{\alpha_m,\gamma}, A_{\alpha_{m+1},\gamma}) = (0,2)$. Put $\gamma = \sum_{i=1}^r b_i \alpha_i$. From $A_{\alpha_{m+1},\gamma} = 2$ we deduce that $b_{m+2} = b_m = 0$ and $b_{m+1} = 1$, which implies that $\gamma = \alpha_{m+1}$. But this contradicts $A_{\alpha_m,\gamma} = 0$. Thus, the submanifold $S_\Phi \cdot o$ cannot be austere. This finishes the proof. □

Let us study the BC_r case, which is very similar to C_r. In principle, we can have different kinds of Φ-strings here. However, with the Φ-strings we have already studied and some other general considerations it suffices to obtain a classification in the BC_r case.

Proposition 6.3.6. *Let Φ be a proper subset of the set of simple roots Π of a symmetric space of non-compact type G/K with BC_r Dynkin diagram*



Then $S_\Phi \cdot o$ is austere if and only if one of the following statements holds:

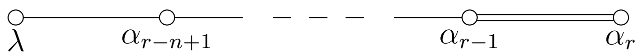
(i) Φ *is a BC_n simple system system, with $n < r$,*

(ii) Φ *is discrete,*

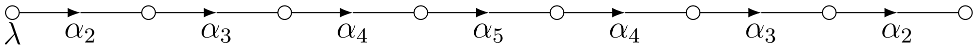
(iii) $\Phi = \Phi_0 \cup \Phi_1$, *where Φ_0 is a BC_n simple system orthogonal to the discrete subset Φ_1. In other words, Φ_0 satisfies (i) and Φ_1 satisfies (ii) and is orthogonal to Φ_0.*

Proof. First, let Φ be a BC_n subsystem of the BC_r simple system Π, with $n < r$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. If this Φ-string is trivial, then $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of the shape operator of $S_\Phi \cdot o$. If the Φ-string of λ is not trivial, then from Proposition 6.2.1 (i) we deduce that $\Pi_\lambda = \{\lambda\} \cup \Phi$ is a simple system. Note that α_r and $2\alpha_r$ are both in the integer span of Π_λ. Hence, Π_λ must be a BC_{n+1} simple system, since it is the unique root system containing double roots (see [7, p. 339]).

Now, we need to determine the roots of the Φ-string of λ and calculate the eigenvalues of the shape operator when restricted to $\mathfrak{g}_\nu$ for each $\nu \in I(\lambda, \Phi)$. However, note that we can think C_{n+1} as a subsystem of BC_{n+1}. Put Π_C for this C_{n+1} simple system, which has a Dynkin diagram of the form



We have studied this simple system and the Φ-string of λ in it in Proposition 6.2.9 (ii). The Φ-string of λ in Π_C has a diagram of the form (with $r = 5$ and $n = 4$ for simplicity)



Moreover, note that the roots $\mathbb{Z}$-spanned by the BC_{n+1} simple system that are not $\mathbb{Z}$-spanned by C_{n+1} simple system are $2\lambda + 2\sum_{k=r-n+1}^{r} \alpha_k$ and those of the form $2\sum_{k=l}^{r} \alpha_k$, for each $l \in \{r-n+1, \ldots, r\}$ (see [7, p. 339]). It is clear that none of them belongs to the Φ-string of λ. Hence, from Proposition 6.2.9 (ii) we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ. In particular, from the proof of that result, we have that there exists a multiplicity-preserving involution of $I(\lambda, \Phi)$ satisfying the conditions of Proposition 6.1.11 (ii).

Now, assume that $\Phi_0 \subset \Pi$ is a BC_n simple system orthogonal to the discrete subset $\Phi_1 \subset \Phi$, and put $\Phi = \Phi_0 \cup \Phi_1$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. If it is trivial, then $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of the shape operator of $S_\Phi \cdot o$. If it is not trivial, then from Proposition 6.2.1 (i) we deduce that Π_λ is a simple system and there are three possibilities:

(a) The Φ_0-string of λ is not trivial and the Φ_1-string of λ is trivial. Hence, from the above considerations we deduce that Π_λ is a BC_{n+1} simple system and there exists a multiplicity-preserving involution of $I(\lambda, \Phi_0) = I(\lambda, \Phi)$ satisfying the conditions of Proposition 6.1.11 (ii). Hence, the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ by means of Proposition 6.1.11 (ii).

(b) The Φ_0-string of λ is trivial and the Φ_1-string of λ is not trivial. From the proof of Proposition 6.2.6 we deduce the existence of a multiplicity-preserving involution of $I(\lambda, \Phi_1) = I(\lambda, \Phi)$ satisfying the conditions of Proposition 6.1.11 (ii). Hence, the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ by means of Proposition 6.1.11 (ii).

(c) The Φ_0-string of λ and the Φ_1-string of λ are not trivial. From the above considerations we deduce that $\Phi_0 \cup \{\lambda\}$ is a BC_{n+1} simple system. Hence, Π_λ must be a BC_{n+2} simple system (see the classification of Dynkin diagrams [7, p. 337]). Also from the above considerations we deduce the existence of multiplicity-preserving involutions of $I(\lambda, \Phi_0)$ and $I(\lambda, \Phi_1)$ which satisfy the conditions of Proposition 6.1.11 (ii), respectively. Hence, from Lemma 6.2.5 we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

Finally, let Φ be an A_n subsystem of the BC_r simple system Π, with $2 \leq n < r$. If we prove that $S_\Phi \cdot o$ is not austere then the result follows using Lemma 6.3.1. Take $\alpha_m, \alpha_{m+1} \in \Phi$ two connected roots. Note that $m+1 < r$, since by assumption Φ is an A_n simple system. Consider the root $\lambda = 2\sum_{i=m+2}^{r} \alpha_i$. Then $(A_{\alpha_m,\lambda}, A_{\alpha_{m+1},\lambda}) = (0, -2)$. From Lemma 6.1.10 (i) we deduce that if $S_\Phi \cdot o$ is austere, then there must exist a positive root $\gamma \in \Sigma^\Phi$ such that $(A_{\alpha_m,\gamma}, A_{\alpha_{m+1},\gamma}) = (0, 2)$. Put $\gamma = \sum_{i=1}^{r} a_i \alpha_i$. From $A_{\alpha_{m+1},\gamma} = 2$ we deduce that

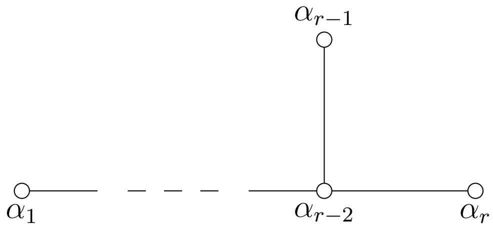
$$(a_m, a_{m+1}, a_{m+2}) \in \{(0, 1, 0), (0, 2, 2)\}.$$

If γ were α_{m+1}, it would be spanned by Φ, which is a contradiction. Thus, let us assume that $(a_m, a_{m+1}, a_{m+2}) = (0, 2, 2)$. Hence, $\gamma = 2\sum_{k=m+1}^{r} \alpha_k$ or $\gamma = \alpha_r + 2\sum_{k=m+1}^{r-1} \alpha_k$. But then $A_{\alpha_m,\gamma} = -2$. Thus, the submanifold $S_\Phi \cdot o$ cannot be austere. This finishes the proof. □

6.3.4 Symmetric spaces of type D_r

Finally, let us consider symmetric spaces G/K of non-compact type with D_r Dynkin diagram.

Proposition 6.3.7. *Let Φ be a proper subset of the set of simple roots Π of a symmetric space of non-compact type G/K with D_r Dynkin diagram*



Then, the submanifold $S_\Phi \cdot o$ is austere if and only if one of the following conditions holds:

(i) Φ *is a D_n simple subsystem, for $n < r$,*

(ii) $\Phi = \{\alpha_{r-3}, \alpha_{r-2}, \alpha_{r-1}\}$ *or* $\Phi = \{\alpha_{r-3}, \alpha_{r-2}, \alpha_r\}$*, or*

(iii) $\Phi = \Phi_0 \cup \Phi_1$*, where Φ_0 satisfies the hypotheses of either (i) or (ii), and Φ_1 is a discrete subset of* Π *orthogonal to* Φ_0*.*

Proof. Assume first that Φ is connected subset of Π. Thus, according to Proposition 6.2.12, we just need to analyze the case when $\Phi = \{\alpha_n, \dots, \alpha_m\}$ is a connected subset of the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ with $n < m < r$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its non-trivial Φ-string. Then $\Pi_\lambda = \Phi \cup \{\lambda\}$ is a simple system as follows from Proposition 6.2.1 (i). Note that the roots $\mathbb{Z}$-spanned by Π_λ must be in Σ and that all the roots in Σ have the same length. Hence, Π_λ is either an A_{m-n+2} simple system or a D_{m-n+2} simple system. Then, the Φ-string of λ has been studied in Proposition 6.2.8 or in Proposition 6.2.13. Moreover, if $S_\Phi \cdot o$ is austere, from Remark 6.2.14 we have that the number of roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_n,\nu_1} = -1$ must coincide with the number of roots $\nu_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_m,\nu_2} = -1$. Define the set

$$\Psi_1 = \left\{ \sum_{i=l}^{n-1} \alpha_i \ : \ 1 \leq l \leq n-1 \right\}. \tag{6.25}$$

Take an arbitrary root $\nu \in \Psi_1$. Note that ν is of minimum level in its non-trivial Φ-string and $A_{\alpha_n,\nu} = -1$.

First let us assume that $m \le r-2$. Let $\gamma \in \Sigma^\Phi$ be of minimum level in its Φ-string satisfying $A_{\alpha_n,\gamma} = -1$. Note that then $A_{\alpha,\gamma} = 0$ for all $\alpha \in \Phi \setminus \{\alpha_n\}$, as follows from Proposition 6.2.1 (iii). Put $\gamma = \sum_{i=1}^r a_i\alpha_i$ with $0 \le a_i \le 2$. Using $A_{\alpha_n,\gamma} = -1$ we deduce that

$$(a_{n-1}, a_n, a_{n+1}) \in \{(1,0,0),(0,0,1),(1,1,2)\}. \tag{6.26}$$

Recall $A_{\alpha,\gamma} = 0$ for all $\alpha \in \Phi \setminus \{\alpha_n\}$. Using this, we get that either $\gamma \in \Psi_1$ or $|\Phi| = 2$ (equivalently $m = n+1$) and

$$\gamma = \alpha_m + 2\sum_{j=m+1}^{r-2} \alpha_j + \alpha_{r-1} + \alpha_r$$

if $m < r-2$, or $\gamma = \alpha_{r-2} + \alpha_{r-1} + \alpha_r$ otherwise. Hence, the number of roots $\nu \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_n,\nu} = -1$ is at most n. Define

$$\left\{ \sum_{i=l}^{m} \alpha_i + 2\sum_{j=m+1}^{r-2} \alpha_j + \alpha_{r-1} + \alpha_r \;:\; 1 \le l \le n-1 \right\} \cup \{\alpha_{m+1}, \alpha_{m+1} + \alpha_{m+2}\}$$

if $m < r-2$ and the set of roots of the form

$$\left\{ \sum_{i=l}^{r} \alpha_i \;:\; 1 \le l \le n-1 \right\} \cup \{\alpha_{m+1}, \alpha_{m+2}\}$$

otherwise ($m = r-2$ since we are assuming $m \le r-2$). In each one of the above sets there are $n+1$ roots that are of minimum level in their corresponding Φ-strings. Note that all these Φ-strings are different to each other. Hence, the number of roots $\nu \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_m,\nu} = -1$ is at least $n+1$. Therefore, if $S_\Phi \cdot o$ is an austere submanifold, then $m = r-1$.

Put $\Phi = \{\alpha_{r-2}, \alpha_{r-1}\}$. Hence, each $\nu \in \Psi_1$ and α_r are of minimum level in their Φ-strings and $A_{\alpha_{r-2},\nu} = A_{\alpha_{r-2},\alpha_r} = -1$. Let $\gamma = \sum_{i=1}^r a_i\alpha_i \in \Sigma^\Phi$ be of minimum level in its Φ-string satisfying $(A_{\alpha_{r-2},\gamma}, A_{\alpha_{r-1},\gamma}) = (0,-1)$. Then we deduce that

$$(a_{r-3}, a_{r-2}, a_{r-1}, a_r) \in \{(1,1,0,1)\}.$$

This means that γ is generated by the A_{r-1} simple subsystem $\Pi \setminus \{\alpha_{r-1}\}$ and also that $(a_{r-3}, a_{r-2}, a_r) = (1,1,1)$ in this subsystem. Hence, the number of roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_{r-2},\nu_1} = -1$ is at least $r-2$, but the number of roots $\nu_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_{r-1},\nu_2} = -1$ is at most $r-3$. Hence, we deduce that $S_\Phi \cdot o$ is not austere if $\Phi = \{\alpha_{r-2}, \alpha_{r-1}\}$.

Finally, assume that $\Phi = \{\alpha_{r-3}, \alpha_{r-2}, \alpha_{r-1}\}$. Recall that the $n-1 = r-4$ roots in Ψ_1 defined in (6.25) are of minimum level in their Φ-strings and $A_{\alpha_{r-3},\lambda} = -1$ for each $\lambda \in \Psi_1$. Define the set

$$\Psi_2 = \{\lambda + \alpha_{r-3} + \alpha_{r-2} + \alpha_r \;:\; \lambda \in \Psi_1\}. \tag{6.27}$$

Note that the $n-1=r-4$ roots in Ψ_2 defined in (6.27) are of minimum level in their Φ-strings and $A_{\alpha_{r-1},\gamma}=-1$ for each $\gamma\in\Psi_2$. According to Proposition 6.2.8, we have that the shape operator $\mathcal{S}$ is austere when restricted to

$$\bigoplus_{\nu\in I(\Psi_1,\Phi)\cup I(\Psi_2,\Phi)}\mathfrak{g}_\nu.$$

Note that α_r is of minimum level in its Φ-string, which has been studied in Proposition 6.2.9 (iii). Hence, the shape operator $\mathcal{S}$ is austere when restricted to the Φ-string of α_r. The roots spanned by the $A_{r-5}\equiv\{\alpha_1,\dots,\alpha_{r-5}\}$ simple subsystem and the ones of the form

$$\sum_{i=l}^{k}\alpha_i+2\sum_{j=k+1}^{r-2}\alpha_j+\alpha_{r-1}+\alpha_r$$

with $1\le l<k\le n-2=r-5$ have trivial Φ-string. Note that Φ spans 6 positive roots, the Φ-string of α_r consists of 6 elements, and that the Φ-string of ν consists of 4 elements for each $\nu\in\Psi_1\cup\Psi_2$. Hence, we get

$$4(r-4)+4(r-4)+6+6+(r-5)(r-4)/2+(r-5)(r-4)/2=r(r-1).$$

Thus, we have considered all the roots in Σ and then in Σ^Φ. Therefore, if $\Phi=\{\alpha_n,\dots,\alpha_m\}$ is a connected subset of the set of simple roots Π with $n<m<r$, then $S_\Phi\cdot o$ is austere if and only if $\Phi=\{\alpha_{r-3},\alpha_{r-2},\alpha_{r-1}\}$.

Let us assume that Φ contains at least two connected orthogonal components Φ_0 and Φ_1. If $|\Phi_i|>2$ for $i\in\{0,1\}$, then either Φ_0 or Φ_1 does not satisfy neither (i) nor (ii) and using the above considerations together with Lemma 6.3.1 we deduce that $S_\Phi\cdot o$ is not austere.

Hence, let us assume that $\Phi=\Phi_0\cup\Phi_1$, with Φ_1 discrete and orthogonal to Φ_0, where Φ_0 satisfies the conditions specified in (i) or in (ii). If Φ_0 is a D_n simple subsytem, then $S_\Phi\cdot o$ is austere, as follows from Proposition 6.2.12. Hence let us assume that $\Phi_0=\{\alpha_{r-3},\alpha_{r-2},\alpha_{r-1}\}$.

Let $\lambda\in\Sigma^\Phi$ be of minimum level in its Φ-string. If the Φ_0-string of λ is trivial, then $\mathcal{S}$ is austere when restricted to the Φ-string of λ by means of Proposition 6.2.6.

Note that the Φ-string of α_r is described in Proposition 6.2.9 (iii), since α_r is orthogonal to all the roots in $\Pi\setminus\{\alpha_r,\alpha_{r-1}\}$. Hence, from Proposition 6.2.9 (iii) we deduce that $\mathcal{S}$ is austere when restricted to the Φ-string of α_r.

Note that if a root $\lambda\in\Sigma^\Phi$ has non-trivial Φ_0-string, then $\lambda\in\Psi_1\cup\Psi_2$. In addition, assume that λ has non-trivial Φ_1-string. Hence, taking into account that Φ_1 is discrete and the form of the roots in Ψ_1 and Ψ_2, we deduce that the Φ-string of λ coincides with the $\Phi_0\cup\{\alpha\}$-string of λ, for some $\alpha\in\Phi_1$.

Now, take a root $\alpha_l\in\Phi_1$. Then, consider the roots

$$\lambda=\sum_{i=l+1}^{r-4}\alpha_i\qquad\text{and}\qquad\gamma=\sum_{i=l+1}^{r-2}\alpha_i+\alpha_r.$$

Note that

$$(A_{\alpha_l,\lambda}, A_{\alpha_{r-3},\lambda}, A_{\alpha_{r-2},\lambda}, A_{\alpha_{r-1},\lambda}) = (-1,-1,0,0)$$

and that

$$(A_{\alpha_l,\gamma}, A_{\alpha_{r-3},\gamma}, A_{\alpha_{r-2},\gamma}, A_{\alpha_{r-1},\gamma}) = (-1,0,0,-1).$$

Hence, from Corollary 6.2.3 (ii) we get that λ and γ are of minimum level in their Φ-strings. Their Φ-strings have been studied in Proposition 6.2.15 and according to it we have that $\mathcal{S}$ is austere when restricted to

$$\bigoplus_{\nu \in I(\lambda,\Phi)\cup I(\gamma,\Phi)} \mathfrak{g}_\nu.$$

Therefore for each $\alpha_l \in \Phi_1$ there are exactly one root in Ψ_1 and one root in Ψ_2 whose strings change with respect to the case when Φ was connected. But the shape operator $\mathcal{S}$ is austere when restricted to the union of both strings (not to each one separately). Moreover, the number of roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $(A_{\alpha,\nu_1}, A_{\alpha_{r-3},\nu_1}) = (0,-1)$ coincides with the number of roots $\nu_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $(A_{\alpha,\nu_2}, A_{\alpha_{r-1},\nu_2}) = (0,-1)$. Hence, $S_\Phi \cdot o$ is austere. □

Finally, Theorem 6.0.1 follows from combining Proposition 6.3.3, Proposition 6.3.4, Proposition 6.3.5, Proposition 6.3.6 and Proposition 6.3.7.

Chapter 7

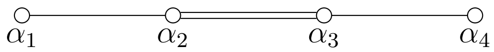
Austere submanifolds in exceptional symmetric spaces

This chapter is devoted to the classification of austere submanifolds of the form $S_\Phi \cdot o$ in exceptional symmetric spaces of non-compact type, that is, non-compact symmetric spaces whose Dynkin diagram is of type E_6, E_7, E_8, F_4 or G_2. Indeed, the main purpose of this chapter is to prove the following

Theorem 7.0.8. *Let Φ be a proper subset of the set of simple roots Π of a symmetric space of non-compact type G/K. Then*

(a) If Π has a G_2 Dynkin diagram, then $S_\Phi \cdot o$ is austere.

(b) If Π has an F_4 Dynkin diagram of the form



with $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4|$, then the submanifold $S_\Phi \cdot o$ is austere if and only if one of the following statements holds:

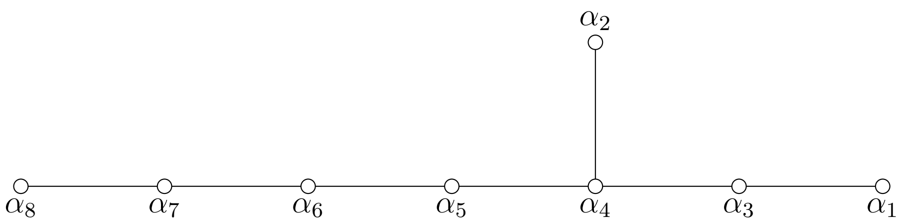
(i) *Φ is a discrete subset of Π, or*

(ii) *Φ is a B_n simple subsystem for $n \in \{2, 3\}$, that is, $\Phi = \{\alpha_2, \alpha_3\}$ or $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$, or*

(iii) *Φ is a C_3 simple subsystem, that is, $\Phi = \{\alpha_1, \alpha_2, \alpha_3\}$, or*

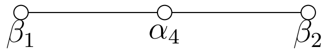
(iv) *$\Phi = \{\alpha_3, \alpha_4\}$ and all the roots in Σ have the same multiplicity.*

(c) If Π has an E_6, E_7 or E_8 Dynkin diagram contained in the diagram



then the submanifold $S_\Phi \cdot o$ is austere if and only if one of the following conditions holds:

(i) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$*, where* Φ_0 *is an* A_3 *simple subsystem containing the root* α_4 *as a central root in its Dynkin diagram, that is, the simple subsystem* Φ_0 *has a Dynkin diagram of the form*



for $(\beta_1, \beta_2) \in \{(\alpha_3, \alpha_5), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5)\}$*, and* $\Phi_1 = \{\beta\}$ *is orthogonal to* Φ_0*, where* $\beta \neq \alpha_1$*, and* $\beta \neq \alpha_6$ *if* $\Pi \equiv E_6$*, or*

(ii) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$*, where* Φ_0 *is a* D_4 *simple subsystem and* Φ_1 *is a discrete subset of* Π *orthogonal to* Φ_0*, or*

(iii) Π *is an* E_6 *simple system and* $\Phi = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ *is an* A_5 *simple subsystem, or*

(iv) Π *is an* E_7 *or* E_8 *simple system and* $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ *is a* D_5 *simple subsystem, or*

(v) Π *is an* E_7 *or* E_8 *simple system and* $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ *is a* D_6 *simple subsystem, or*

(vi) Π *is an* E_8 *simple system and* $\Phi = \Pi \backslash \{\alpha_8\}$ *is an* E_7 *simple subsystem, or*

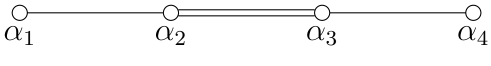
(vii) Φ *is discrete.*

Note that the claim concerning the G_2 case is the content of Proposition 6.2.7. Hence, this chapter is devoted to the study of the rest of the cases and it is organized as follows. In Section 7.1 we will inspect the austere submanifolds of the form $S_\Phi \cdot o$ in a non-compact symmetric space with F_4 Dynkin diagram. Although it is an exceptional symmetric space, the approach we will follow is very similar to the one utilized for classical symmetric spaces in Chapter 6. Since E_6 and E_7 can be thought of as contained in E_8, in the rest of the chapter we will address the E_8 case directly. However, in Section 7.2 we derive a classification for the E_6 case and in Section 7.3 we derive a classification for the E_7 case. Finally, in Section 7.4 we will analyze the remaining particular cases and we will conclude the classification for the E_8 case.

7.1 F_4 case

This section is devoted to classifying the austere submanifolds of the form $S_\Phi \cdot o$ in non-compact symmetric spaces with F_4 Dynkin diagram. Although new examples of Φ-strings that we did not analyze in Chapter 6 will arise in the classification, we will address these new cases directly.

Proposition 7.1.1. *Let* Φ *be a proper subset of the set of simple roots* Π *of a symmetric space* G/K *of non-compact type with* F_4 *Dynkin diagram of the form*



with $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4|$. *Then, the submanifold* $S_\Phi \cdot o$ *is austere if and only if one of the following statements holds:*

(i) Φ *is a discrete subset of* Π, *or*

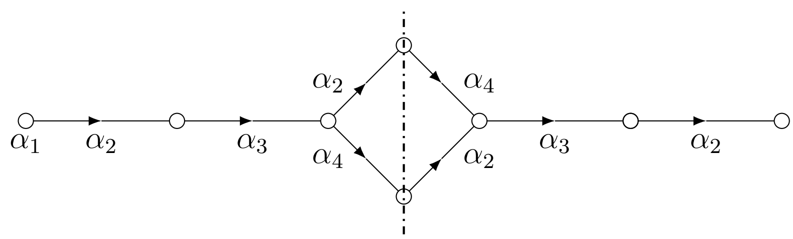
(ii) Φ *is a* B_n *simple subsystem for* $n \in \{2, 3\}$, *that is,* $\Phi = \{\alpha_2, \alpha_3\}$ *or* $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$, *or*

(iii) Φ *is a* C_3 *simple subsystem, that is,* $\Phi = \{\alpha_1, \alpha_2, \alpha_3\}$, *or*

(iv) $\Phi = \{\alpha_3, \alpha_4\}$ *and all the roots in* Σ *have the same multiplicity.*

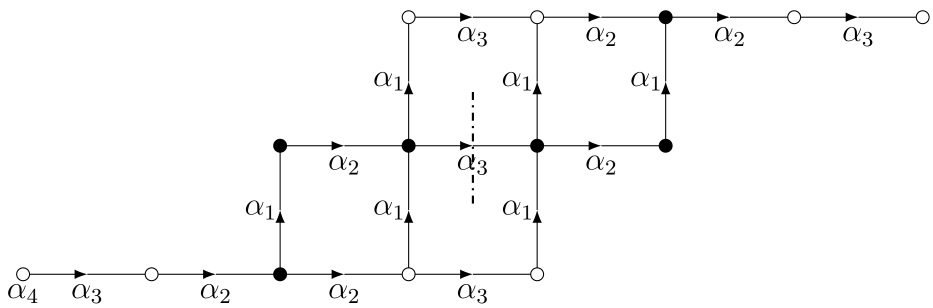
Proof. Consider first the $B_2 \cong C_2$ system $\Phi = \{\alpha_2, \alpha_3\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its non-trivial Φ-string. Then, $\{\lambda\} \cup \Phi$ is a simple system by means of Proposition 6.2.1 (i). Since $|\alpha_2| < |\alpha_3|$, this simple system must have B_3 or C_3 Dynkin diagram. According to Proposition 6.2.9 (i)-(ii), the shape operator $\mathcal{S}$ of the submanifold $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ. Thus, $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_3\}$.

Put now $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$. Let us study the Φ-string of α_1. Since the simple system $\{\alpha_1\} \cup \Phi$ has an F_4 Dynkin diagram, the Φ-string will have the number of positive roots spanned by a F_4 simple system (24), minus the number of positive roots spanned by Φ (9 since Φ has a B_3 Dynkin diagram) and minus the number of positive roots with coefficient corresponding to α_1 greater or equal than 2. Using [69, p. 691], we deduce that the Φ-string of α_1 consists of 8 roots. In fact, it has a diagram of the form



From Lemma 5.1.1 we have that all the roots in this Φ-string have the same multiplicity. Hence, a map induced by a reflection with respect to the vertical line (interchanging the roots on the line) satisfies the hypotheses of Corollary 6.1.11 (ii). Then the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of α_1. In order to conclude this case, note that $2\alpha_1 + 2\alpha_2 + \alpha_3$ is another root of minimum level in its Φ-string, by virtue of Remark 6.2.10. Note that $\Pi_{2\alpha_1+2\alpha_2+\alpha_3} = \{2\alpha_1 + 2\alpha_2 + \alpha_3\} \cup \Phi$ is a simple system by means of Proposition 6.2.1 (iii). According to the Cartan integers $A_{2\alpha_1+2\alpha_2+\alpha_3,\alpha}$ and $A_{\alpha,2\alpha_1+2\alpha_2+\alpha_3}$ we deduce that $\Pi_{2\alpha_1+2\alpha_2+\alpha_3}$ is a B_4 simple system. Hence, the Φ-string of $2\alpha_1 + 2\alpha_2 + \alpha_3$ has been described in Proposition 6.2.9 (i) and the shape operator $\mathcal{S}$ of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of $2\alpha_1 + 2\alpha_2 + \alpha_3$. Note that the Φ-string of α_1 contains 8 elements, the Φ-string of $2\alpha_1 + 2\alpha_2 + \alpha_3$ contains 7 elements and Φ spans 9 positive roots. These are the 24 positive roots of an F_4 root system. This proves that if (ii) holds, then $S_\Phi \cdot o$ is an austere submanifold.

Put now $\Phi = \{\alpha_1, \alpha_2, \alpha_3\}$. The Φ-string of α_4 will have the number of positive roots spanned by an F_4 simple system, minus the number of positive roots spanned by Φ (9 since Φ has a C_3 Dynkin diagram) and minus the number of positive roots with coefficient corresponding to α_4 greater or equal than 2. Using [69, p. 691], we deduce that the Φ-string of α_1 consists of 14 roots. In fact, it has a diagram of the form



From Lemma 5.1.1, roots with nodes of the same colour have the same multiplicity. Consider the involution of $I(\alpha_4, \Phi)$ induced by the composition of the reflections of the above diagram with respect to the central horizontal axis and with respect to the central vertical axis. This map satisfies the hypotheses of Corollary 6.1.11 (ii). Since the Φ-string of α_4 consists of 14 roots, Φ spans 9 positive roots and $2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4$ has trivial Φ-string, we have considered the 24 positive roots generated by the F_4 simple system. This proves that if (iii) holds, then $S_\Phi \cdot o$ is an austere submanifold.

Let us consider the case $\Phi = \{\alpha_3, \alpha_4\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its non-trivial Φ-string. Then, $\{\lambda\} \cup \Phi$ is a simple system by means of Proposition 6.2.1 (i). Since Φ contains the largest roots, then $\{\lambda\} \cup \Phi$ must be either an A_3 simple system or a B_3 simple system. In both cases, these Φ-strings have been described in Proposition 6.2.8. We just need to check that the number of roots $\nu \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_3,\nu} = -1$ coincides (counted with multiplicities) with the number of roots $\gamma \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_4,\gamma} = -1$. From Proposition 6.2.1 (iii), the roots α_2, $\alpha_1+\alpha_2$ and $2\alpha_1+4\alpha_2+2\alpha_3+\alpha_4$ are of the first type. The roots $2\alpha_2+\alpha_3$, $\alpha_1+2\alpha_2+\alpha_3$ and $2\alpha_1+2\alpha_2+\alpha_3$ are of the second type. Note that

$$\begin{aligned} 2|\alpha_2|^2 &= 2|\alpha_1+\alpha_2|^2 = 2|\alpha_1+2\alpha_2+\alpha_3|^2 \\ &= |2\alpha_1+4\alpha_2+2\alpha_3+\alpha_4|^2 = |2\alpha_2+\alpha_3|^2 = |2\alpha_1+2\alpha_2+\alpha_3|^2. \end{aligned}$$

Each of these 6 Φ-string contains 3 roots, Φ spans three positive roots and the roots α_1, $\alpha_1+3\alpha_2+2\alpha_3+\alpha_4$, $2\alpha_1+3\alpha_2+2\alpha_3+\alpha_4$ have trivial Φ-string. All the positive roots of a F_4 system have been considered. Hence, if $\Phi = \{\alpha_3, \alpha_4\}$, then $S_\Phi \cdot o$ is austere if and only if all the roots have the same multiplicity.

Thus, we need to consider the case $\Phi = \{\alpha_1, \alpha_3, \alpha_4\}$. Put $\lambda = 2\alpha_2 + \alpha_3$. We have that $A_{\alpha_1,\lambda} = -2$, $A_{\alpha_4,\lambda} = -1$ and $A_{\alpha_3,\lambda} = 0$. If $S_\Phi \cdot o$ is austere, then from Lemma 6.1.10 (i) there must exist a root $\gamma \in \Sigma^\Phi$ such that $A_{\alpha_1,\gamma} = 2$, $A_{\alpha_4,\gamma} = 1$ and $A_{\alpha_3,\gamma} = 0$. Put

$\gamma = \sum_{i=1}^{4} a_i \alpha_i$. Then we will have:

$$\begin{cases} 2 & = A_{\alpha_1,\gamma} = 2a_1 - a_2, \\ 0 & = A_{\alpha_3,\gamma} = -a_2 + 2a_3 - a_4, \\ 1 & = A_{\alpha_4,\gamma} = -a_3 + 2a_4. \end{cases}$$

In particular we get $3a_4 = 2a_1$ and $a_2 = 3a_4 - 2$. Then $a_1 = 3k$ for some $k \in \mathbb{N}$. From [69, p. 691] we deduce that $a_1 = 0$. But this means that $a_4 = 0$ and then $a_2 = -2$. Since γ must be a positive root, this is a contradiction. Hence, the submanifold $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_1, \alpha_3, \alpha_4\}$.

Finally, put $\Phi = \{\alpha_1, \alpha_2\}$. Note that α_3 and $\alpha_3 + \alpha_4$ are roots in Σ^Φ, with the same multiplicity, and of minimum level in their corresponding Φ-strings. We have that $(A_{\alpha_1,\lambda}, A_{\alpha_2,\lambda}) = (0, -2)$ for $\lambda \in \{\alpha_3, \alpha_3 + \alpha_4\}$. If $S_\Phi \cdot o$ is austere, from Proposition 6.1.10 (i), there must exist a root $\gamma \in \Sigma^\Phi$ such that $(A_{\alpha_1,\gamma}, A_{\alpha_2,\gamma}) = (0, 2)$. Put $\gamma = \sum_{i=1}^{4} a_i \alpha_i$. Then we will have:

$$\begin{cases} 0 & = A_{\alpha_1,\gamma} = 2a_1 - a_2, \\ 2 & = A_{\alpha_2,\gamma} = -a_1 + 2a_2 - 2a_3. \end{cases}$$

Thus $2a_1 = a_2$ and $3a_1 - 2a_3 = 2$. Hence, we deduce that $(a_1, a_2, a_3) = (2, 4, 2)$. The unique root under these conditions is

$$2\alpha_1 + 4\alpha_2 + 2\alpha_3 + \alpha_4.$$

Note that it has the same multiplicity as α_3 and $\alpha_3 + \alpha_4$. Thus, $S_\Phi \cdot o$ cannot be austere. Now, Lemma 6.3.1 finishes the proof. □

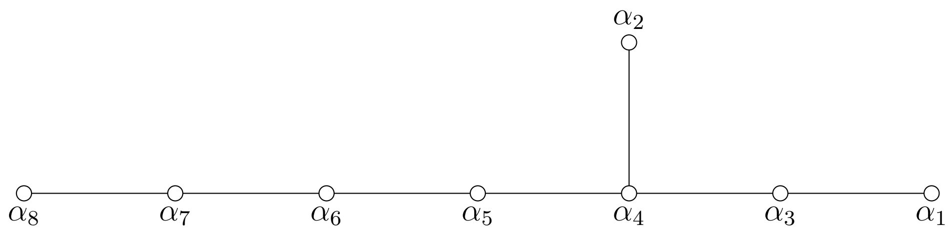
7.2 E_6 case

Let us focus now on the symmetric spaces G/K with E_6, E_7 or E_8 Dynkin diagram. As explained above, since E_6 and E_7 can be thought of as contained in E_8, in the rest of the chapter we will address the E_8 case directly. Again, we will need to study new classes of Φ-strings. However, this is the general procedure that we will follow. We will fix a connected subset Φ of the set of simple roots Π. Then, we will determine all the roots $\lambda \in \Sigma^\Phi$ that are of minimum level in their Φ-strings by means of Proposition 6.2.1 (iii). We include the determination of the roots $\lambda \in \Sigma^\Phi$ with trivial Φ-string. Note that when all the roots in Σ have the same multiplicity, as it is the case now [7, p. 338], then $\lambda \in \Sigma^\Phi$ has trivial Φ-string if and only if $A_{\alpha,\lambda} = 0$ for all $\alpha \in \Phi$.

In some cases, it turns out that these Φ-strings have been studied either in Proposition 6.2.8, Proposition 6.2.9 (iii), Proposition 6.2.13 or Proposition 6.2.15 (we will also combine these results with Lemma 6.2.5). If not, we will address the study of such string directly. Therefore, using these results we will be able to deduce whether the submanifold $S_\Phi \cdot o$ is austere. Note that we will determine all the roots that are of minimum level in their corresponding Φ-strings and that we know the number of roots of each Φ-string. We also know the number of positive roots spanned by Φ. Hence, it will be very easy to check that we are considering the whole tangent space to the submanifold $S_\Phi \cdot o$.

In general, we will start by assuming that Φ is connected. Under this assumption, if $S_\Phi \cdot o$ is not an austere submanifold, then $S_{\Phi\cup\Psi} \cdot o$ cannot be austere if $\Psi \subset \Pi$ is orthogonal to Φ, by virtue of Lemma 6.3.1. Thus, just in the cases when $S_\Phi \cdot o$ is austere (there are not so many examples), we will continue examining the submanifolds of the form $S_{\Phi\cup\Psi} \cdot o$, for $\Psi \subset \Pi$ orthogonal to Φ.

Recall that since E_6 and E_7 can be thought of as contained in E_8, we will address the E_8 case directly. Put



for the Dynkin diagram of the simple system Π.

Let us start with a very particular case. Assume that $\Phi = \Phi_0 \cup \Phi_1$, where $\Phi_0 \subset \Pi$ determines a D_4 simple subsystem orthogonal to the discrete simple system $\Phi_1 \subset \Pi$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its non-trivial Φ-string. If such Φ-string is trivial then $\mathfrak{g}_\lambda$ is contained in the 0-eigenspace of the shape operator. Hence, assume that the Φ-string of λ is not trivial. From Proposition 6.2.1 (i) we have that $\{\lambda\} \cup \Phi$ is a simple system. There are three possibilities here:

(a) The Φ_0-string of λ is not trivial and the Φ_1-string of λ is trivial. Hence the Φ-string of λ coincides with the Φ_0-string of λ and the simple system $\Phi_0 \cup \{\lambda\}$ (see Proposition 6.2.1 (i)) has a Dynkin diagram of type D_5, according to the classification of Dynkin diagrams [7, p. 337]. From Proposition 6.2.9 (iii), we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

(b) The Φ_0-string of λ is trivial and the Φ_1-string of λ is not trivial. Hence the Φ-string of λ coincides with the Φ_1-string of λ. Since Φ_1 is discrete, from Proposition 6.2.6 we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

(c) The Φ_0-string of λ is not trivial and the Φ_1-string of λ is not trivial. Hence the Φ-string of λ is of type D_6, according to the classification of Dynkin diagrams [7, p. 337]. From Corollary 6.2.11 (iii), we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ.

Hence, the submanifold $S_\Phi \cdot o$ is austere. Then, we have the following

Proposition 7.2.1. *Let Φ_0, $\Phi_1 \subset$ be orthogonal subsets of the set of simple roots Π. Assume that Φ_0 is a D_4 simple subsystem and Φ_1 is discrete. Then, the submanifolds $S_{\Phi_0} \cdot o$, $S_{\Phi_1} \cdot o$ and $S_{\Phi_0\cup\Phi_1} \cdot o$ are austere.*

In the following pages we will study the cases where $\Phi \subset \Pi$ is an A_n simple subsystem, with $n \leq 4$.

7.2.1 Φ containing a component of type A_2

In this subsection we consider the case that Φ is an A_2 simple subsystem, and conclude that no subset Φ of Π containing a connected component of type A_2 gives rise to an austere orbit $S_\Phi \cdot o$.

Let $\Phi = \{\beta_1, \beta_2\}$ be an A_2 simple system. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. From Proposition 6.2.1 (i) we have that $\Pi_\lambda = \{\lambda\} \cup \Phi$ is a simple system. Since the Φ-string of λ is not trivial and all the roots in Σ have the same length, then Π_λ has an A_3 Dynkin diagram. Since all the roots have the same multiplicity, from Proposition 6.2.8 we deduce that $S_\Phi \cdot o$ is austere if and only if the number of roots $\lambda_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_1,\lambda_1} = -1$ coincides with the number of roots $\lambda_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_2,\lambda_2} = -1$.

Since most of the following examples could be thought in symmetric spaces G/K with either E_6, E_7 or E_8 Dynkin diagram, we will use the following notation. When we have to point out the number of roots of minimum level under some properties, we write either (x, y, z), (x, y) or x. On the one hand, the first coordinate will denote the number of roots under certain condition in an E_8 simple system. On the other hand, the second coordinate, if it exists, will denote the number of roots under certain condition in an E_7 simple system. Finally, the third coordinate, if it exists, will denote the number of roots under certain condition in an E_6 simple system.

As explained above, in the following list of roots we will determine and write roots in Σ^Φ of minimum level in their Φ-strings (including roots with trivial Φ-string). In order to simplify notations, except for the roots with trivial Φ-string, we will just specify the non-zero Cartan integers.

Let us start with the case-by-case analysis. In what follows, and in order to write each root with respect to the simple system Π explicitly, we will use the notation of [69, Appendix C]. Moreover, for each possible Φ, $\lambda \in \Sigma^\Phi$ will denote a root of minimum level in its Φ-string.

Put $\Phi = \{\alpha_7, \alpha_8\}$. This example just makes sense in E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following 17 roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011110 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0011221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 10 roots:

$$\begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = A_{\alpha_8,\lambda} = 0$, then λ is one of the following 36 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001100 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000110 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0001211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1233431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}.$$

Since the number of roots $\lambda_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_7,\lambda_1} = -1$ does not coincide with the number of roots $\lambda_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\alpha_8,\lambda_2} = -1$, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_7, \alpha_8\}$.

Put $\Phi = \{\alpha_6, \alpha_7\}$. This example appears in E_7 and E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(16, 10)$ roots:

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0001210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112211 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1112221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(11, 5)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(36, 15)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000100 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111100 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1111211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_6, \alpha_7\}$.

The rest of the examples with $\Phi \equiv A_2$ can be thought in the symmetric spaces with Dynkin diagram E_6, E_7 or E_8.

Put $\Phi = \{\alpha_5, \alpha_6\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(15, 9, 6)$ roots:

$$\begin{pmatrix} 0 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(12, 6, 3)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = 0$, then λ is one of the following $(36, 15, 6)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111100 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111110 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123421 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_5, \alpha_6\}$.

Put $\Phi = \{\alpha_4, \alpha_5\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(14, 8, 5)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(13, 7, 4)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011221 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233431 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1233432 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = 0$, then λ is one of the following $(36, 15, 6)$ roots:

$$\binom{0}{0000001}, \binom{0}{0100000}, \binom{0}{1000000}, \binom{0}{1100000}, \binom{0}{0011110}, \binom{1}{0011100},$$
$$\binom{0}{0111110}, \binom{1}{0111100}, \binom{0}{0011111}, \binom{0}{1111110}, \binom{1}{1111100}, \binom{0}{0111111},$$
$$\binom{0}{1111111}, \binom{1}{0001221}, \binom{1}{0122210}, \binom{1}{1122210}, \binom{1}{0122211}, \binom{1}{1222210},$$
$$\binom{1}{1122211}, \binom{2}{0012321}, \binom{1}{1222211}, \binom{2}{0112321}, \binom{2}{1112321}, \binom{1}{1233321},$$
$$\binom{2}{0123431}, \binom{2}{1123431}, \binom{2}{0123432}, \binom{2}{1223431}, \binom{2}{1123432}, \binom{2}{1223432},$$
$$\binom{3}{1234531}, \binom{2}{1234542}, \binom{3}{1234532}, \binom{3}{1245642}, \binom{3}{1345642}, \binom{3}{2345642}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_4, \alpha_5\}$.

Put $\Phi = \{\alpha_3, \alpha_4\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(13, 7, 4)$ roots:

$$\binom{0}{0000001}, \binom{1}{0001100}, \binom{1}{0011100}, \binom{1}{0111100}, \binom{1}{1111100}, \binom{1}{0012211},$$
$$\binom{1}{0112211}, \binom{1}{1112211}, \binom{1}{0122211}, \binom{1}{1122211}, \binom{1}{1222211}, \binom{2}{1234421},$$
$$\binom{3}{1234532}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(14, 8, 5)$ roots:

$$\binom{1}{0000000}, \binom{0}{0001000}, \binom{0}{0011000}, \binom{0}{0111000}, \binom{0}{1111000}, \binom{1}{0001111},$$
$$\binom{1}{0011111}, \binom{1}{0111111}, \binom{1}{1111111}, \binom{2}{0123321}, \binom{2}{1123321}, \binom{2}{1223321},$$
$$\binom{2}{1233321}, \binom{2}{1234432}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = 0$, then λ is one of the following $(36, 15, 6)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1122321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1233432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_3, \alpha_4\}$.

Put $\Phi = \{\alpha_1, \alpha_3\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(12, 6, 3)$ roots:

$$\begin{pmatrix} 1 \\ 0001210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(15, 9, 6)$ roots:

$$\begin{pmatrix} 0 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111100 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123421 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1223421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}.$$

- If $A_{\alpha_1,\lambda} = A_{\alpha_3,\lambda} = 0$, then λ is one of the following $(36, 15, 6)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1122321 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223321 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_1, \alpha_3\}$.

Put $\Phi = \{\alpha_2, \alpha_4\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(13, 7, 4)$ roots:

$$\begin{pmatrix} 0 \\ 0001110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(14, 8, 5)$ roots:

$$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0012221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1234431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_4,\lambda} = 0$, then λ is one of the following $(36, 15, 6)$ roots:

$$\begin{array}{l}
\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \\
\begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011221 \end{pmatrix}, \\
\begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112211 \end{pmatrix}, \\
\begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \\
\begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233431 \end{pmatrix}, \\
\begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.
\end{array}$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_2, \alpha_4\}$.

The above information, together with Lemma 6.3.1, allows us to deduce the following

Proposition 7.2.2. *Let Φ be a proper subset of the set of simple roots Π. Assume that Π is either an E_6, E_7 or E_8 simple system. If Φ has a connected component that consists of two roots, then $S_\Phi \cdot o$ is not austere.*

7.2.2 Extreme roots of Φ-strings

The Φ-strings that will appear in this chapter are quite difficult to write explicitly (as we used to do in Section 6.2). In order to avoid long and complicated calculations, we introduce a nice property that Φ-strings have. As usual, let Φ be a subset of the set Π of simple roots and let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Let γ be a root in the Φ-string of λ. We will say that γ is *extreme* in the Φ-string of λ if $\gamma + \alpha$ is not a root, for each $\alpha \in \Phi$. This notion can be thought as a generalization of the concept of extreme root [91, p. 65] to Φ-strings.

Remark 7.2.3. Let Φ be a connected subset of Π. Note that if $\gamma = \lambda + m_1\alpha_1 + \dots m_k\alpha_k$ is extreme in its Φ-string, then $\gamma + \nu$ is not a root for any element ν spanned by Φ with positive level. In fact, assume first that Φ is connected and take ν with positive minimum level among the elements spanned by Φ such that $\gamma + \nu$ is a root. Since γ is extreme, then the level of ν must be greater or equal than two. Since $\gamma + \nu$ is not the root of minimum level in the Φ-string of λ, from Proposition 6.2.1 (ii) we deduce that there must exist a simple root $\alpha \in \Phi$ such that $\gamma + \nu - \alpha$ is also a root in the Φ-string of λ. Then $\nu - \alpha$ is an element spanned by Φ with less level than ν such that $\gamma + (\nu - \alpha)$ is a root, which is a contradiction.

Proposition 7.2.4. *Let Π be the set of simple roots of the root system Σ. Assume that all the roots in Σ have the same length. Let Φ be a proper connected subset of Π. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its non-trivial Φ-string. Then:*

(i) *The extreme root in the Φ-string of λ is unique.*

(ii) *A root γ is extreme in the Φ-string of λ if and only if there exists a root $\alpha \in \Phi$ such that $A_{\alpha,\gamma} > 0$ and $A_{\beta,\gamma} = 0$ for all $\beta \in \Phi \backslash \{\alpha\}$.*

Proof. Since all the roots in Σ have the same length, from Proposition 1.5.1 (iii) we have that $A_{\alpha,\beta} \in \{0, \pm 1\}$ for any distinct roots α, $\beta \in \Sigma$. Recall from Proposition 6.2.1 (i) that $\Pi_\lambda = \{\lambda\} \cup \Phi$ is a simple system. Let $\gamma \in \Sigma^\Phi$ be an extreme root in the Φ-string of λ. If $A_{\alpha,\gamma} < 0$ for some $\alpha \in \Phi$, then from Proposition 1.5.1 (iv) we get that $\gamma + \alpha$ is a root, which is a contradiction with the extreme character of γ. Thus $A_{\alpha,\gamma} \geq 0$ for all $\alpha \in \Phi$. Put $\gamma = \lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha$. If $n_\beta = 0$ for some $\beta \in \Phi$, then $A_{\beta,\gamma} \leq 0$. Together with $A_{\alpha,\gamma} \geq 0$ for all $\alpha \in \Phi$, we deduce that $A_{\beta,\gamma} = 0$. Define $\Phi_1 = \{\alpha \in \Phi \ : \ n_\alpha = 0\}$ and $\Phi_2 = \{\alpha \in \Phi \ : \ n_\alpha \neq 0\}$. Since $A_{\alpha,\gamma} = 0$ for all $\alpha \in \Phi_1$, we obtain that $A_{\alpha,\nu} = 0$ for all $\alpha \in \Phi_1$ and all $\nu \in \Phi_2$. This contradicts the connectedness of Φ. Thus, if $\gamma = \lambda + \sum_{\alpha\in\Phi} n_\alpha \alpha$ is an extreme root in the Φ-string of λ, then $n_\alpha > 0$ for all $\alpha \in \Phi$.

(i): Let γ_0, $\gamma_1 \in \Sigma^\Phi$ be extreme roots in the Φ-string of λ. Note that if $A_{\gamma_0,\gamma_1} > 0$ we are done. Indeed, from Proposition 1.5.1 (iv), we get that $\gamma_k - \gamma_{k+1}$ is either a positive root spanned by Φ or zero, for some $k \in \{0, 1\}$ and indices modulo 2. If it is not zero, then we can write $\gamma_k = \gamma_{k+1} + (\gamma_k - \gamma_{k+1})$. But then γ_k would not be an extreme root in the Φ-string of λ. Then $\gamma_0 - \gamma_1 = 0$ and we are done.

Thus, we will prove that $A_{\gamma_0,\gamma_1} > 0$ proceeding by induction on the number of elements in Φ. If $|\Phi| = 1$ the result follows from Proposition 1.5.1 directly. We will assume that our claim is true for all subsets Ψ of Π with $|\Psi| \leq n - 1$. Let $\Phi \subset \Pi$ with $|\Phi| = n$ and let us see that the claim is also true for Φ. Let γ_0, $\gamma_1 \in \Sigma^\Phi$ be extreme roots in the Φ-string of λ. Let us write

$$\gamma_k = \lambda + \sum_{\alpha\in\Phi} n^k_\alpha \alpha, \tag{7.1}$$

where $n^k_\alpha \in \mathbb{N}$ for all $\alpha \in \Phi$ and $k \in \{0, 1\}$. Therefore, we obtain

$$A_{\gamma_k,\gamma_{k+1}} = A_{\gamma_k,\lambda} + \sum_{\alpha\in\Phi} n^{k+1}_\alpha A_{\gamma_k,\alpha}, \tag{7.2}$$

for each $k \in \{0, 1\}$ and indices modulo 2. Since the root γ_k is not of minimum level in the Φ-string of λ, from Proposition 6.2.1 (ii) there must exist a root $\beta_k \in \Phi$ such that $\gamma_k - \beta_k$ is a root for each $k \in \{0, 1\}$. The element $\gamma_k + \beta_k$ cannot be a root since γ_k is extreme for each $k \in \{0, 1\}$. Hence, from Proposition 1.5.1 (v), we deduce that $A_{\beta_k,\gamma_k} > 0$, for each $k \in \{0, 1\}$.

Recall that $n^k_\alpha > 0$ and $A_{\gamma_k,\alpha} \geq 0$ for all $\alpha \in \Phi$ and all $k \in \{0, 1\}$. Now, assume that $A_{\gamma_k,\lambda} \geq 0$ for some $k \in \{0, 1\}$. Using $A_{\beta_k,\gamma_k} > 0$ together with (7.2) we deduce that $A_{\gamma_0,\gamma_1} > 0$ and we are done.

Hence, put $A_{\gamma_0,\lambda} = A_{\gamma_1,\lambda} = -1$. Recall from Proposition 6.2.1 (i) that λ is connected to just one root in the Dynkin diagram of the simple system $\{\lambda\} \cup \Phi$. Put α_1 for this root. Then $A_{\lambda,\alpha_1} = -1$ and we obtain that

$$-1 = A_{\lambda,\gamma_k} = 2 - n^k_{\alpha_1}$$

and we deduce that $n^k_{\alpha_1} = 3$ for $k \in \{0,1\}$. Hence, from [69, p. 684-685] we obtain that the simple system $\Pi_\lambda = \{\lambda\} \cup \Phi$ is of type E_6, E_7 or E_8. Since the roots $\mathbb{Z}$-spanned by Π_λ must be in Σ, we deduce that Π is of type E_7 or E_8

Assume first that $n^k_{\beta_k} > 1$ for some $k \in \{0,1\}$. Then, using $A_{\gamma_{k+1},\lambda} = -1$, $A_{\gamma_{k+1},\alpha} \geq 0$ for all $\alpha \in \Phi$ and (7.2) we get $A_{\gamma_0,\gamma_1} > 0$ and the result follows.

Thus, assume that $n^k_{\beta_k} = 1$ for each $k \in \{0,1\}$. Recall that $\gamma_k - \beta_k$ is a positive root for each $k \in \{0,1\}$ in the simple system Π_λ. Then, we can write it with respect to Π_λ, that is,

$$\gamma_k - \beta_k = m_\lambda \lambda + \sum_{\alpha \in \Phi} m_\alpha \alpha,$$

with integers m_λ, $m_\alpha \geq 0$ for each $\alpha \in \Phi$. Since $n^k_{\beta_k} = 1$ in (7.1) and λ is simple in Π_λ, we deduce that $m_{\beta_k} = 0$. From Proposition 6.1.3 and the fact that $m_\alpha > 0$ for all $\alpha \in \Phi \backslash \{\beta_k\}$, we deduce that $\Phi\backslash\{\beta_k\}$ is a connected subset of Φ with $k \in \{0,1\}$.

Since $\Phi\backslash\{\beta_k\} \subset \Phi$, we have that λ is the root of minimum level in its $\Phi\backslash\{\beta_k\}$-string. Thus, from Proposition 6.2.1 (i) we have that $\{\lambda\} \cup \Phi\backslash\{\beta_k\}$ is a simple system that spans the root $\gamma_k - \beta_k$ for each $k \in \{0,1\}$. Since the coefficient of α_1 in the expression of $\gamma_k - \beta_k$ with respect to the simple system $\{\lambda\} \cup \Phi\backslash\{\beta_k\}$ is 3, we deduce that $\{\lambda\} \cup \Phi\backslash\{\beta_k\}$ must be either an E_6 or an E_7 simple system. This determines the root β_k with $k \in \{0,1\}$. In other words, we can write $\beta = \beta_0 = \beta_1$. This is because if Π is an E_8 (respectively E_7) simple system there is just one root $\nu \in \Pi$ such that $\Pi\backslash\{\nu\}$ is an E_7 (respectively E_6) simple system.

Consider now the root $\gamma_k - \beta$ in the $\Phi\backslash\{\beta\}$-string of λ, for $k \in \{0,1\}$. If it is not extreme in such string for some $k \in \{0,1\}$, then there must exist a root $\alpha \in \Phi\backslash\{\beta\}$ such that $\gamma_k - \beta + \alpha$ is a root. Since all the roots in Σ have the same length and $\gamma_k - \beta + \alpha$ is a root, we have that $-1 = A_{\alpha,\gamma_k-\beta}$ (see Proposition 1.5.1). Hence, from

$$-1 = A_{\alpha,\gamma_k-\beta} = A_{\alpha,\gamma_k} - A_{\alpha,\beta}$$

we deduce that $A_{\alpha,\gamma_k} = -1$. Thus $\gamma_k + \alpha$ would be a root due to Proposition 1.5.1 (iv) and γ_k would not be an extreme root in the Φ-string of λ. Thus, we deduce that $\gamma_0 - \beta$ and $\gamma_1 - \beta$ are extreme in the $\Phi\backslash\{\beta\}$-string of λ. Applying the induction hypothesis, we obtain that $\gamma_0 - \beta = \gamma_1 - \beta$. The result now follows.

(ii): Recall that $A_{\nu_1,\nu_2} \in \{0, \pm 1\}$ for any distinct ν_1, $\nu_2 \in \Sigma$. Assume first that γ is an extreme root in the Φ-string of λ. If $A_{\alpha,\gamma} < 0$, then $\gamma + \alpha$ is a root due to Proposition 1.5.1 (iv) and γ would not be extreme in the Φ-string of λ. Thus $A_{\alpha,\gamma} \geq 0$ for all $\alpha \in \Phi$. Recall that the Φ-string of λ is not trivial. Thus, γ is not of minimum level the Φ-string of λ. Now, from Proposition 6.2.1 (ii) we get that there must exist $\alpha_1 \in \Phi$ such that $\gamma - \alpha_1$ is a root. Since $\gamma + \alpha_1$ cannot be a root, from Proposition 1.5.1 (v) we deduce that $A_{\alpha_1,\gamma} > 0$. Assume that $A_{\alpha_2,\gamma} > 0$ for some $\alpha_2 \in \Phi \backslash \{\alpha_1\}$. Put $\bar{\alpha} = \sum_{\alpha\in\Phi} \alpha$. Note that since Φ is a simple subsystem of Π, then the sum of its simple roots is a root (see [69, p. 684]). Recalling that $A_{\alpha,\gamma} \geq 0$ for all $\alpha \in \Phi$ we obtain that

$$A_{\bar{\alpha},\gamma} = \sum_{\alpha\in\Phi} A_{\alpha,\gamma} \geq 2,$$

which is a contradiction with Proposition 1.5.1 (iii).

Conversely, if ν is not extreme in the Φ-string of λ, then there must exist $\alpha \in \Phi$ such that $\nu + \alpha$ is a root. If $\nu - \alpha$ were a root, then $A_{\alpha,\nu-\alpha} \leq -2$, which is a contradiction with Proposition 1.5.1 (iii). Thus, from Proposition 1.5.1 (v) we deduce that $A_{\alpha,\nu} = -1$. This concludes the proof. □

Combining Corollary 6.2.3 and Proposition 7.2.4 we obtain the following

Corollary 7.2.5. *Let $\Phi = \Phi_0 \cup \Phi_1$ be a proper subset of the set Π, where Φ_0 is orthogonal to Φ_1 and both are connected. Assume that all the roots in Σ have the same length. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its non-trivial Φ_i-string, for each $i \in \{0,1\}$. Let $\gamma \in \Sigma^\Phi$ be a root in the Φ-string of λ. The root γ is extreme in the Φ-string of λ if and only if there exists a root $\alpha_i \in \Phi_i$ such that $A_{\alpha_i,\gamma} = 1$ and $A_{\beta,\gamma} = 0$ for all $\beta \in \Phi_i \backslash \{\alpha_i\}$ for each $i \in \{0,1\}$.*

Proposition 7.2.4 and Corollary 7.2.5 have a very nice consequence that will avoid many calculations in order to justify that the shape operator of $S_\Phi \cdot o$ is not austere when restricted to a Φ-string. Roughly speaking, it allows us to argue that the map f given in Corollary 6.1.11 interchanges extreme roots with roots of minimum level.

Corollary 7.2.6. *Let Σ be a root system whose roots have the same multiplicity. Let Φ be a connected subset of the set of simple roots Π. Denote by J_0 the set of roots $\lambda \in \Sigma^\Phi$ of minimum level in their non-trivial Φ-strings and by J_1 the set of extreme roots $\gamma \in \Sigma^\Phi$ in their non-trivial Φ-strings. Then:*

(i) *If $S_\Phi \cdot o$ is austere, then there exists a bijection $f \colon J_0 \to J_1$ such that $A_{\alpha,\nu} = -A_{\alpha,f(\nu)}$ for all $(\alpha,\nu) \in \Phi \times J_0$.*

(ii) *Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string, and let $\bar{\lambda} \in \Sigma^\Phi$ be the extreme root of the Φ-string of λ. If $\mathcal{S}$ is austere when restricted to the Φ-string of λ, then $A_{\alpha,\lambda} = -A_{\alpha,\bar{\lambda}}$ for all $\alpha \in \Phi$.*

(iii) *If $\mathcal{S}$ is austere when restricted to the non-trivial Φ-string of λ for some $\lambda \in \Sigma^\Phi$, then the number of elements in the Φ-string of λ is even.*

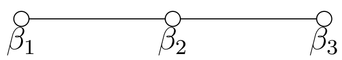
Proof. (i): Since $S_\Phi \cdot o$ is austere and all the roots have the same multiplicity, from Corollary 6.1.11 (i) we deduce that there exists a multiplicity-preserving involution $f \colon \Sigma^\Phi \to \Sigma^\Phi$ such that $A_{\alpha,\nu} = -A_{\alpha,f(\nu)}$ for all $(\alpha,\nu) \in \Phi \times \Sigma^\Phi$. We just need to see that f interchanges roots of minimum level and extreme roots. In other words, it suffices to see that $f(J_0) \subset J_1$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its non-trivial Φ-string. Since Φ is connected and the Φ-string of λ is not trivial, from Proposition 6.2.1 (iii) we deduce that there exists a root $\alpha \in \Phi$ such that $A_{\alpha,\lambda} = -1$ and that $A_{\nu,\lambda} = 0$ for all $\nu \in \Phi \backslash \{\alpha\}$. Let γ be a root that is not extreme in its non-trivial Φ-string. Thus, there exists a root $\beta \in \Phi$ such that $\gamma + \beta$ is a root in the Φ-string of γ. Hence, from Proposition 1.5.1 (iii)-(v) we have that $A_{\beta,\gamma} = -1$. Since $A_{\beta,\gamma} = -1$ and $A_{\nu,\lambda} \leq 0$ for each $\nu \in \Phi$, we have that $f(\lambda) \neq \gamma$, for each root $\gamma \in \Sigma^\Phi$ that is not extreme in its non-trivial Φ-string. Finally, let $\gamma' \in \Sigma^\Phi$ be a root with trivial Φ-string. This means that $A_{\nu,\gamma'} = 0$ for all $\nu \in \Phi$. Hence, we have that $-1 = A_{\alpha,\lambda} \neq -A_{\alpha,\gamma'} = 0$ and $f(\lambda) \neq \gamma'$.

(ii): Since the shape operator $\mathcal{S}$ is austere when restricted to the Φ-string of λ, from Corollary 6.1.11 (ii) we deduce that there must exist a multiplicity-preserving involution $f\colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ such that $A_{\alpha,\nu} = -A_{\alpha,f(\nu)}$ for all $(\alpha, \nu) \in \Phi \times I(\lambda, \Phi)$. Now, substituting Σ^Φ by $I(\lambda, \Phi)$ and proceeding as in (i) the result follows.

(iii): Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Recall that since $\mathcal{S}$ is austere when restricted to the Φ-string of λ, from Corollary 6.1.11 (ii) we deduce that there must exist a multiplicity-preserving involution $f\colon I(\lambda, \Phi) \to I(\lambda, \Phi)$ such that $A_{\alpha,\nu} = -A_{\alpha,f(\nu)}$ for all $(\alpha, \nu) \in \Phi \times I(\lambda, \Phi)$. Take a root $\nu \in \Sigma^\Phi$ in the Φ-string of λ. Since the Φ-string of λ is not trivial, ν cannot be extreme and of minimum level simultaneously. Hence, from Proposition 6.2.1 (i) and the definition of extreme root we deduce that there exists a root $\alpha \in \Phi$ such that $\nu - \alpha$ or $\nu + \alpha$ is a root. Hence, we deduce that $A_{\alpha,\nu} \neq 0$ for some $\alpha \in \Phi$. Thus, we have that $A_{\alpha,\nu} \neq -A_{\alpha,\nu}$ for some $\alpha \in \Phi$. This means that the involution f cannot have fixed points. Then, the number of roots in $I(\lambda, \Phi)$ must be even. □

7.2.3 Φ containing a component of type A_3

We will start now the study of the submanifolds $S_\Phi \cdot o$ when $\Phi = \{\beta_1, \beta_2, \beta_3\}$ is an A_3 simple subsystem with Dynkin diagram



Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Note that all the roots have the same multiplicity. If $\{\lambda\} \cup \Phi$ is a D_4 system, that is, if $A_{\beta_2,\lambda} = -1$, from Proposition 6.2.9 (iii) we deduce that the shape operator of $S_\Phi \cdot o$ is austere when restricted to the Φ-string of λ. Otherwise, $\{\lambda\} \cup \Phi$ will be an A_4 simple system and the Φ-string of λ has been described in Proposition 6.2.8. Hence, the submanifold $S_\Phi \cdot o$ will be austere if and only if the number of roots $\lambda_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_1,\lambda_1} = -1$ coincides with the number of roots $\lambda_3 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_3,\lambda_3} = -1$.

Let us start with the case-by-case analysis.

Put $\Phi = \{\alpha_6, \alpha_7, \alpha_8\}$. This examples just makes sense in E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following 11 roots:

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0001210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following 5 roots:

$$\begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 5 roots:

$$\begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = A_{\alpha_8,\lambda} = 0$, then λ is one of the following 20 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000100 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234431 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_6, \alpha_7, \alpha_8\}$.

Put $\Phi = \{\alpha_5, \alpha_6, \alpha_7\}$. This example makes sense only in E_7 and E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(10, 6)$ roots:

$$\begin{pmatrix} 0 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(5, 3)$ roots:

$$\begin{pmatrix} 1 \\ 0001210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(6, 2)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(20, 7)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_5, \alpha_6, \alpha_7\}$.

The rest of the possibilities with $\Phi \equiv A_3$ arise in all the symmetric spaces with Dynkin diagram E_6, E_7 or E_8.

Put $\Phi = \{\alpha_4, \alpha_5, \alpha_6\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(9, 5, 3)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(5, 3, 2)$ roots:

$$\begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(7, 3, 1)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223431 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1223432 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = 0$, then λ is one of the following $(20, 7, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_4, \alpha_5, \alpha_6\}$.

Put $\Phi = \{\alpha_3, \alpha_4, \alpha_5\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(8, 4, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122211 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(5, 3, 2)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(8, 4, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1122321 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = 0$, then λ is one of the following $(20, 7, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_3, \alpha_4, \alpha_5\}$

Put $\Phi = \{\alpha_1, \alpha_3, \alpha_4\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(7, 3, 1)$ roots:

$$\begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(5, 3, 2)$ roots:

$$\begin{pmatrix} 1 \\ 0001100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(9, 5, 3)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_1,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = 0$, then λ is one of the following $(20, 7, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1122321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_1, \alpha_3, \alpha_4\}$.

Put $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(8, 4, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(8, 4, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122211 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(5, 3, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = 0$, then λ is one of the following $(20, 7, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$.

Put $\Phi = \{\alpha_2, \alpha_4, \alpha_5\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(8, 4, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0011110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(5, 3, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(8, 4, 2)$ roots:

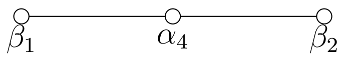
$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1233431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = 0$, then λ is one of the following $(20, 7, 2)$ roots:

$$\begin{pmatrix}0\\0000001\end{pmatrix}, \begin{pmatrix}0\\0100000\end{pmatrix}, \begin{pmatrix}0\\1000000\end{pmatrix}, \begin{pmatrix}0\\1100000\end{pmatrix}, \begin{pmatrix}1\\0001221\end{pmatrix}, \begin{pmatrix}1\\0122210\end{pmatrix},$$
$$\begin{pmatrix}1\\1122210\end{pmatrix}, \begin{pmatrix}1\\0122211\end{pmatrix}, \begin{pmatrix}1\\1222210\end{pmatrix}, \begin{pmatrix}1\\1122211\end{pmatrix}, \begin{pmatrix}1\\1222211\end{pmatrix}, \begin{pmatrix}2\\0123431\end{pmatrix},$$
$$\begin{pmatrix}2\\1123431\end{pmatrix}, \begin{pmatrix}2\\0123432\end{pmatrix}, \begin{pmatrix}2\\1223431\end{pmatrix}, \begin{pmatrix}2\\1123432\end{pmatrix}, \begin{pmatrix}2\\1223432\end{pmatrix}, \begin{pmatrix}3\\1245642\end{pmatrix},$$
$$\begin{pmatrix}3\\1345642\end{pmatrix}, \begin{pmatrix}3\\2345642\end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_4, \alpha_5\}$.

From the above case-by-case analysis it follows that if Φ is an A_3 simple subsystem then $S_\Phi \cdot o$ is austere (in E_6, E_7 or E_8) if and only if Φ contains the root α_4 as a central root, that is, the simple subsystem Φ_0 has a Dynkin diagram of the form

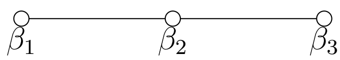


for $(\beta_1, \beta_2) \in \{(\alpha_3, \alpha_5), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5)\}$.

Hence, with this assumption on Φ, we still need to inspect when the submanifold $S_{\Phi\cup\Psi} \cdot o$ is austere, for $\Psi \subset \Pi$ orthogonal to Φ. The cases with $|\Psi| > 3$ will be analyzed later. From the above considerations on the case $\Phi \equiv A_3$, Proposition 7.2.2 and Lemma 6.3.1, we deduce that Ψ must be discrete.

As usual in this section, the approach we will follow is to study all the cases independently.

Basically, there are four types of Φ-strings that will play a crucial role in what follows. More precisely, let $\Phi = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a subset of the simple roots Π. Assume that $\{\beta_1, \beta_2, \beta_3\}$ constitutes an A_3 simple subsystem with Dynkin diagram



and that β_4 is a simple root orthogonal to β_i, with $i \in \{1, 2, 3\}$. As usual, let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

There are several possibilities for the simple system $\{\lambda\} \cup \Phi$. On the one hand, if $(A_{\beta_2,\lambda}, A_{\beta_4,\lambda}) = (-1, 0)$, then the Φ-string of λ has been described in Proposition 6.2.9 (iii) and the shape operator $\mathcal{S}$ is austere when restricted to the Φ-string of λ. On the other hand, if $(A_{\beta_2,\lambda}, A_{\beta_4,\lambda}) = (-1, -1)$, then the Φ-string of λ has been described in Corollary 6.2.11 (iii) and the shape operator $\mathcal{S}$ is austere when restricted to the Φ-string of λ. Assume that $A_{\beta_4,\lambda} = -1$ is the unique non-zero Cartan integer of the form $A_{\alpha,\lambda}$, with

$\alpha \in \Phi$. Then, the Φ-string of λ coincides with the β_4-string of λ. Then, from Proposition 6.2.6 we deduce that the shape operator $\mathcal{S}$ is austere when restricted to the Φ-string of λ.

Now, assume on the one hand that $(A_{\nu,\lambda}, A_{\beta_4,\lambda}) = (-1,-1)$ for some $\nu \in \{\beta_1, \beta_3\}$. This Φ-string has been described in Proposition 6.2.15. On the other hand, assume that $(A_{\nu,\lambda}, A_{\beta_4,\lambda}) = (-1,0)$ for some $\nu \in \{\beta_1, \beta_3\}$. This Φ-string has been described in Proposition 6.2.8. Hence, using now Lemma 6.1.10, Proposition 6.2.15 and Proposition 6.2.8 we deduce the following. The submanifold $S_\Phi \cdot o$ is austere if and only if: the number of roots $\nu_1 \in \Phi$ of minimum level in their Φ-strings satisfying $(A_{\beta_1,\nu_1}, A_{\beta_4,\nu_1}) = (-1,-1)$ coincides with the number of roots $\nu_3 \in \Phi$ of minimum level in their Φ-strings satisfying $(A_{\beta_3,\nu_3}, A_{\beta_4,\nu_3}) = (-1,-1)$, and the number of roots $\gamma_1 \in \Phi$ of minimum level in their Φ-strings satisfying $(A_{\beta_1,\gamma_1}, A_{\beta_4,\gamma_1}) = (-1,0)$ coincides with the number of roots $\gamma_3 \in \Phi$ of minimum level in their Φ-strings satisfying $(A_{\beta_3,\gamma_3}, A_{\beta_4,\gamma_3}) = (-1,0)$.

Put $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_7\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Recall that except for the roots with trivial Φ-string, we just point out the non-zero Cartan integers.

- If $A_{\alpha_3,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 1 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122211 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(4,2)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0011111 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(3,1)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1122321 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(4,2)$ roots:

$$\begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(6,2)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(7,2)$ roots:

$$\begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, the submanifold $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_7\}$.

Put $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_8\}$. This examples makes sense only in E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_3,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0111111 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following 3 roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 6 roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_8,\lambda} = 0$, then λ is one of the following 7 roots:

$$\begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}.$$

Thus, the submanifold $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_8\}$.

Put $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Except for the roots with trivial Φ-string, we just point out the non-zero Cartan integers.

- If $A_{\alpha_2,\lambda} = A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(2,1,1)$ roots:

$$\begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(4,2,0)$ roots:

$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(2,1,0)$ roots:

$$\begin{pmatrix} 1 \\ 0112211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112211 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(4,2,2)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_6,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(3,1,0)$ roots:

$$\begin{pmatrix} 0 \\ 0111000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(6,2,0)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_6,\lambda} = 0$, then λ is one of the following $(7,2,1)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Hence, $S_\Phi \cdot o$ is austere in E_7 and E_8 when $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$, but not in E_6.

Put $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_7\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(4,2)$ roots:

$$\begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122211 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(4,2)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 0 \\ 0011000 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(3,1)$ roots:

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(6, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(7, 2)$ roots:

$$\begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, the submanifold $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_7\}$.
Put $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_8\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 1 \\ 0112211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ the root

$$\begin{pmatrix} 0 \\ 0111000 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following 3 roots:

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 6 roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_8,\lambda} = 0$, then λ is one of the following 7 roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}.$$

Thus, the submanifold $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_8\}$.

Thus, we also need to consider the case $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_8\}$. Let $\lambda \in \Phi$ be the of minimum level in its Φ-string.

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0012211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 2 \\ 1234432 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1235642 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0123321 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_6,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0001111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 0 \\ 0111000 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_6,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_6,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0112211 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix}.$$

- If λ has trivial Φ-string, then it is one of the following 2 roots:

$$\begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}.$$

Consider now the root

$$\gamma = \begin{pmatrix} 2 \\ 1122321 \end{pmatrix}.$$

We have that $(A_{\alpha_2,\gamma}, A_{\alpha_3,\gamma}, A_{\alpha_4,\gamma}, A_{\alpha_6,\gamma}, A_{\alpha_8,\gamma}) = (1, 0, 0, 1, 1)$. Hence, from Corollary 7.2.5, we deduce that γ is the extreme root of its Φ-string. If $S_\Phi \cdot o$ were austere, combining Lemma 6.1.10 and Corollary 7.2.6, there would exist a root $\lambda \in \Sigma^\Phi$ of minimum level in its Φ-string satisfying $(A_{\alpha_2,\lambda}, A_{\alpha_3,\lambda}, A_{\alpha_4,\lambda}, A_{\alpha_6,\lambda}, A_{\alpha_8,\lambda}) = (-1, 0, 0, -1, -1)$. However, we have calculated above all the roots of minimum level in their Φ-string and none of them satisfies such condition. Hence, the submanifold $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_8\}$.

Put $\Phi = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$. By symmetry, we have already considered this case in E_6 above. However, we include its study for the sake of completeness. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(3, 2, 1)$ roots:

$$\begin{pmatrix} 0 \\ 0011110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(2,0,0)$ roots:

$$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_1,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(3,1,0)$ roots:

$$\begin{pmatrix} 1 \\ 0122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_1,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 2 \\ 1233431 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(6,4,2)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}.$$

- If $A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(6,2,0)$ roots:

$$\begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223431 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_1,\lambda} = 0$, then λ is one of the following $(7,2,1)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, the submanifold $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_1\}$.
Put $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_7\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(2,2)$ roots:

$$\begin{pmatrix} 0 \\ 0011110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(4, 0)$ roots:

$$\begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 1122221 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(3, 3)$ roots:

$$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122221 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(2, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011221 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(4, 0)$ roots:

$$\begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(6, 0)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(7, 6)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, the submanifold $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_7\}$.
Put $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_8\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0011110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0122221 \end{pmatrix}$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following 3 roots:

$$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111221 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0011221 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 6 roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122211 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}.$$

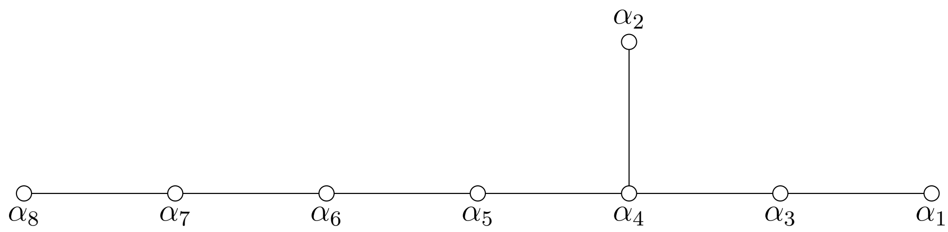
- If $A_{\alpha_2,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_8,\lambda} = 0$, then λ is one of the following 7 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}.$$

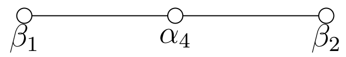
Thus, the submanifold $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_8\}$.

We summarize all this information in the following

Proposition 7.2.7. *Let G/K be a symmetric space of non-compact type with E_6, E_7 or E_8 Dynkin diagram contained in the diagram*



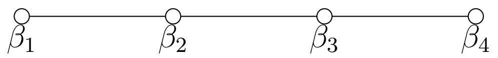
Let Φ be a proper subset of the set Π with a connected component that consists of three roots. Then, the submanifold $S_\Phi \cdot o$ is austere if and only if $\Phi = \Phi_0$ or $\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 is an A_3 simple subsystem containing the root α_4 as a central root in its Dynkin diagram, that is, the simple subsystem Φ_0 has a Dynkin diagram of the form



for $(\beta_1, \beta_2) \in \{(\alpha_3, \alpha_5), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5)\}$ and $\Phi_1 = \{\beta\}$ is orthogonal to Φ_0, where $\beta \neq \alpha_1$, and $\beta \neq \alpha_6$ if $\Pi \equiv E_6$.

7.2.4 Φ containing a component of type A_4

Now, we will assume that $\Phi = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is an A_4 simple subsystem with Dynkin diagram



Recall that all the roots have the same multiplicity. From Proposition 6.2.8, Proposition 6.2.13 and Remark 6.2.14 we deduce that the submanifold $S_\Phi \cdot o$ will be austere if and only if: the number of roots $\lambda_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_1,\lambda_1} = -1$ coincides with the number of roots $\lambda_4 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_4,\lambda_4} = -1$, and the number of roots $\lambda_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_2,\lambda_2} = -1$ coincides with the number of roots $\lambda_3 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_3,\lambda_3} = -1$.

Let us start with the case-by-case analysis.

Put $\Phi = \{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. This example just makes sense in E_8. As usual, let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string, and we continue using the notation used so far in this section.

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following 7 roots:

$$\begin{pmatrix} 0 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following 3 roots:

$$\begin{pmatrix} 1 \\ 0001210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001211 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 1 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 3 roots:

$$\begin{pmatrix} 2 \\ 0123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = A_{\alpha_8,\lambda} = 0$, then λ is one of the following 10 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

Put $\Phi = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. This example just makes sense in E_7 and E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(6, 3)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(3, 2)$ roots:

$$\begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(2, 1)$ roots:

$$\begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(4, 1)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(10, 3)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

The rest of the possibilities with $\Phi \equiv A_4$ appear in all the symmetric spaces with Dynkin diagram E_6, E_7 or E_8.

Put $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(5, 2, 1)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(3, 2, 1)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(2, 1, 1)$ roots:

$$\begin{pmatrix} 1 \\ 0000111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(5, 2, 0)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = 0$, then λ is one of the following $(10, 3, 1)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$.

Put $\Phi = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(4, 1, 0)$ roots:

$$\begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(3, 2, 1)$ roots:

$$\begin{pmatrix} 1 \\ 0011100 \end{pmatrix}, \begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(2, 1, 1)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(6, 3, 1)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix}.$$

- If $A_{\alpha_1,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = 0$, then λ is one of the following $(10, 3, 1)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$.

Put $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(6, 3, 1)$ roots:

$$\begin{pmatrix} 1 \\ 0012210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(4, 1, 0)$ roots:

$$\begin{pmatrix} 1 \\ 0123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 2 \\ 1234421 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(4, 3, 2)$ roots:

$$\begin{pmatrix} 0 \\ 0001000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0011000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111000 \end{pmatrix}.$$

- If $A_{\alpha_1,\lambda} = A_{\alpha_2,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = 0$, then λ is one of the following $(10,3,1)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix},$$
$$\begin{pmatrix} 3 \\ 1235642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Put $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(5,2,0)$ roots:

$$\begin{pmatrix} 0 \\ 0111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(3,2,2)$ roots:

$$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(2,1,0)$ roots:

$$\begin{pmatrix} 1 \\ 0111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(5,2,1)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 1 \\ 0001221 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}.$$

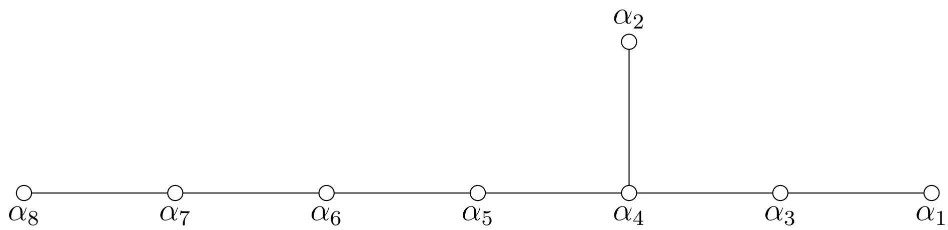
- If $A_{\alpha_2,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = 0$, then λ is one of the following $(10,3,1)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6\}$.

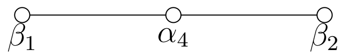
This concludes the study when Φ is an A_4 simple subsystem. Before going on, using the above calculations when Φ is an A_4 simple subsystem, Proposition 7.2.1, Proposition 7.2.2 and Proposition 7.2.7, we summarize all the information in the following

Proposition 7.2.8. *Let Φ be a proper subset of the set of simple roots Π. Assume that Π is an E_6, E_7 or E_8 simple system whose Dynkin diagram is contained in the Dynkin diagram*



Assume that each connected component of Φ *contains at most four elements. Then, the submanifold* $S_\Phi \cdot o$ *is austere if and only if one of the following statements holds:*

(i) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$*, where* Φ_0 *is an* A_3 *simple subsystem containing the root* α_4 *as a central root in its Dynkin diagram, that is, the simple subsystem* Φ_0 *has a Dynkin diagram of the form*



for $(\beta_1, \beta_2) \in \{(\alpha_3, \alpha_5), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5)\}$ *and* $\Phi_1 = \{\beta\}$ *is orthogonal to* Φ_0*, where* $\beta \neq \alpha_1$*, and* $\beta \neq \alpha_6$ *if* $\Pi \equiv E_6$*.*

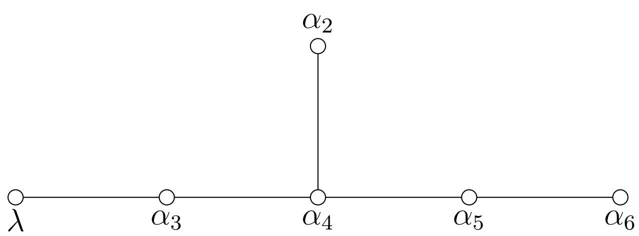
(ii) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$*, where* Φ_0 *is a* D_4 *simple subsystem orthogonal to the discrete subset* Φ_1*.*

(iii) Φ *is discrete.*

7.2.5 The classification in spaces of type E_6

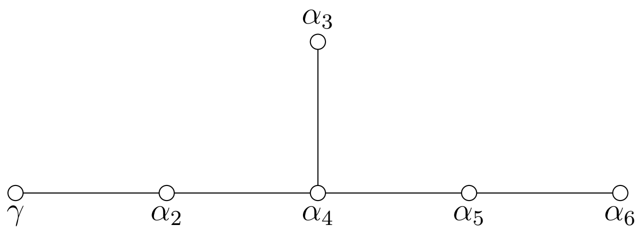
In this subsection we conclude the classification of austere submanifolds of the form $S_\Phi \cdot o$ in symmetric spaces of non-compact type with E_6 Dynkin diagram. However, before that, we still need to analyze a particular class of Φ-strings.

Proposition 7.2.9. *Let* Φ *be a proper subset of the set of simple roots* Π*. Let* $\lambda \in \Sigma^\Phi$ *be of minimum level in its* Φ*-string. Assume that* Φ *is a* D_5 *simple system and* $\{\lambda\} \cup \Phi$ *is an* E_6 *simple system with Dynkin diagram*



Then, the shape operator $\mathcal{S}$ *is not austere when restricted to the* Φ*-string of* λ*.*

Let $\gamma \in \Sigma^\Phi$ *be of minimum in its* Φ*-string. Assume that* $\{\gamma\} \cup \Phi$ *is an* E_6 *simple system with Dynkin diagram*



The principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to $\bigoplus_{\alpha\in I(\lambda,\Phi)}\mathfrak{g}_\alpha$ are exactly the opposite to the principal curvatures of the shape operator $\mathcal{S}_\xi$ when restricted to $\bigoplus_{\alpha\in I(\gamma,\Phi)}\mathfrak{g}_\alpha$, for each unit normal vector ξ to the submanifold $S_\Phi\cdot o$.

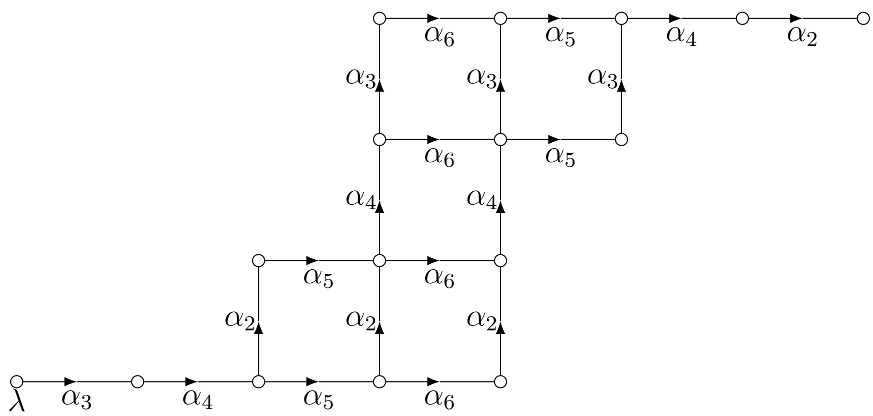
Therefore, the shape operator $\mathcal{S}$ is austere when restricted to $\bigoplus_{\alpha\in I(\lambda,\Phi)\cup I(\gamma,\Phi)}\mathfrak{g}_\alpha$.

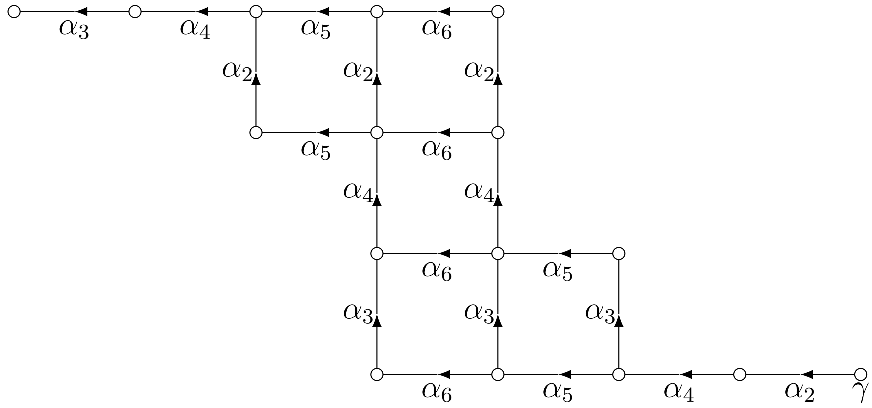
Proof. Note that $A_{\alpha_3,\lambda}=-1$ and $A_{\nu,\lambda}=0$ for all $\nu\in\Phi\backslash\{\alpha_3\}$. The number of roots of the Φ-string of λ equals the number of positive roots spanned by an E_6 simple system, minus the number of positive roots spanned by a D_5 simple system, minus the number of roots with coefficient corresponding to λ greater or equal than 2 (there are no roots satisfying this condition [69, p. 687]). Thus, we have $|I(\lambda,\Phi)|=16$. Consider the root

$$\bar{\lambda}=\begin{pmatrix} 2 \\ 12321 \end{pmatrix},$$

where the coefficients refer to its expression with respect to the simple system $\{\lambda\}\cup\Phi$. We have that $A_{\alpha_2,\bar{\lambda}}=1$ and that $A_{\nu,\bar{\lambda}}=0$ for all $\nu\in\Phi\backslash\{\alpha_2\}$. From Proposition 7.2.4 (ii) we deduce that $\bar{\lambda}$ is the extreme root in the Φ-string of λ. Recall that $A_{\alpha_3,\lambda}=-1$ and $A_{\nu,\lambda}=0$ for all $\nu\in\Phi\backslash\{\alpha_3\}$. Therefore, from Corollary 7.2.6 (ii) we deduce that $\mathcal{S}$ is not austere when restricted the Φ-string of λ.

Now, we draw the diagrams of the Φ-string of λ and of the Φ-string of γ in order to finish the proof.

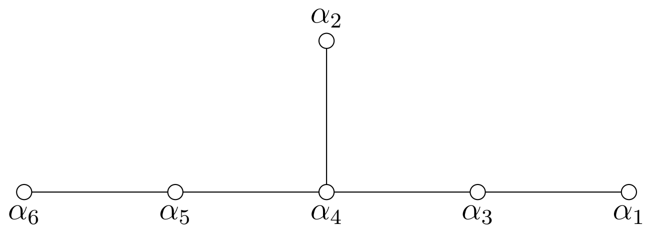




Let f be the bijection induced by the reflection of the above diagrams with respect to an horizontal axis separating them. Recall that all the roots have same multiplicity. Then, f satisfies the conditions of Corollary 6.1.11 (iii). This finishes the proof. □

Finally, we can state and prove the classification result for symmetric spaces of type E_6.

Proposition 7.2.10. *Let G/K be a symmetric space of non-compact type with E_6 Dynkin diagram of the form*



Let Φ be a proper subset of the set Π of simple roots. Then, the submanifold $S_\Phi \cdot o$ is austere if and only if one of the following conditions holds:

(i) *Φ is the A_3 subsystem $\Phi = \{\alpha_3, \alpha_4, \alpha_5\}$, $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$ or $\Phi = \{\alpha_2, \alpha_4, \alpha_5\}$, or*

(ii) *Φ is a D_4 subsystem, that is, $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, or*

(iii) *Φ is an A_5 subsystem, that is, $\Phi = \Pi \backslash \{\alpha_2\}$, or*

(iv) *Φ is discrete.*

Proof. From Proposition 7.2.8, we just need to study the case when Φ has a connected component that consists of five elements.

Assume first that $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Hence, $\alpha_6 \in \Sigma^\Phi$ is of minimum level in its non-trivial Φ-string by means of Proposition 6.2.1 (iii). According to Proposition 7.2.9, the shape operator of $S_\Phi \cdot o$ is not austere when restricted to the Φ-string of α_6. Since the Φ-string of α_6 has 16 roots (see the proof of Proposition 7.2.9) and Φ spans 20 positive roots, we have considered all the roots in Σ^Φ. Hence, if Φ is a D_5 simple subsystem of the E_6 simple system Π, then $S_\Phi \cdot o$ is not austere.

Finally, we only need to examine the case when Φ is an A_5 simple system, that is, $\Phi = \Pi \backslash \{\alpha_2\}$. Then, note that $\{\alpha_2\} \cup \Phi = \Pi$ is an E_6 simple system. Then, the number of roots of the Φ-string of α_2 equals the number of positive roots spanned by an E_6 simple system, minus the number of positive roots spanned by an A_5 simple system, minus the number of positive roots with coefficient corresponding to α_2 greater or equal than 2 (there is just one root satisfying this condition [69, p. 687]). Thus, the Φ-string of α_2 consists of 20 roots. Below, we write explicitly an involution $f\colon I(\alpha_2, \Phi) \to I(\alpha_2, \Phi)$ under the conditions of Corollary 6.1.11 (ii):

$$\begin{array}{cc}
\begin{pmatrix} 1 \\ 00000 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 12321 \end{pmatrix} & \begin{pmatrix} 1 \\ 00100 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 12221 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 01100 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 11221 \end{pmatrix} & \begin{pmatrix} 1 \\ 00110 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 12211 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 11100 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 01221 \end{pmatrix} & \begin{pmatrix} 1 \\ 01110 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 11211 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 00111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 12210 \end{pmatrix} & \begin{pmatrix} 1 \\ 11110 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 01211 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 01210 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 11111 \end{pmatrix} & \begin{pmatrix} 1 \\ 01111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 11210 \end{pmatrix}.
\end{array}$$

Note that the root

$$\begin{pmatrix} 2 \\ 12321 \end{pmatrix}$$

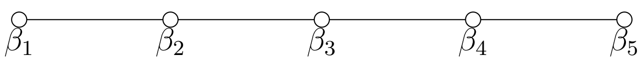
has trivial Φ-string. Thus, all the roots have been considered and $S_\Phi \cdot o$ is austere when $\Phi = \Pi \backslash \{\alpha_2\}$. □

7.3 E_7 case

In the following lines we continue with the study of the austerity of $S_\Phi \cdot o$ in symmetric spaces of non-compact type of exceptional type. As usual, we will analyze the cases E_7 and E_8 simultaneously. In particular, at the end of this section we derive the classification for the E_7 case.

7.3.1 Φ containing a component of type A_5

In this subsection we will assume that Φ is a connected subset of Π with Dynkin diagram



Recall that all the roots in Σ have the same multiplicity. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Then $\{\lambda\} \cup \Phi$ will be an A_6, D_6 or E_6 simple system. Therefore, using Proposition 6.2.8, Proposition 6.2.13, Remark 6.2.14 and the proof of Proposition 7.2.10 we deduce the following. The submanifold $S_\Phi \cdot o$ is austere if and only if: the number of roots $\nu_1 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_1,\nu_1} = -1$ coincides with the number of roots $\nu_5 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_5,\nu_5} = -1$, and the number of roots $\nu_2 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_2,\nu_2} = -1$ coincides with the number of roots $\nu_4 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_4,\nu_4} = -1$.

Put $\Phi = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. This example just makes sense in E_8. As usual, let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 1 \\ 0000110 \end{pmatrix}, \begin{pmatrix} 1 \\ 0000111 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0001221 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 2 \\ 0012321 \end{pmatrix}.$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = A_{\alpha_8,\lambda} = 0$, then λ is one of the following 4 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

Put $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. This example exists in E_7 and E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(3,1)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111111 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0000111 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 2 \\ 1112321 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(3,1)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(4,1)$ roots:

$$\begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

Put $\Phi = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. This example exists in E_6, E_7 and E_8. However, since we already got a classification for E_6 in Proposition 7.2.10, we will only specify the number of roots (satisfying the appropriate conditions) for the cases E_7 and E_8. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(2,0)$ roots:

$$\begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 3 \\ 1234531 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 1 \\ 0111100 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111100 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 2 \\ 1222321 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(4,2)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0112321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}.$$

- If $A_{\alpha_1,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = 0$, then λ is one of the following $(4,1)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 2 \\ 0012321 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ is neither in E_7 nor E_8. Recall from Proposition 7.2.10 that it is austere in E_6.

Put $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(3,0)$ roots:

$$\begin{pmatrix} 0 \\ 1111110 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}, \begin{pmatrix} 2 \\ 1234542 \end{pmatrix}.$$

- If $A_{\alpha_4,\lambda} = -1$, then λ is one of the following $(2,2)$ roots:

$$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}, \begin{pmatrix} 0 \\ 0000011 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 1111221 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 0001221 \end{pmatrix}.$$

- If $A_{\alpha_7,\lambda} = -1$, then λ is one of the following $(3,0)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}.$$

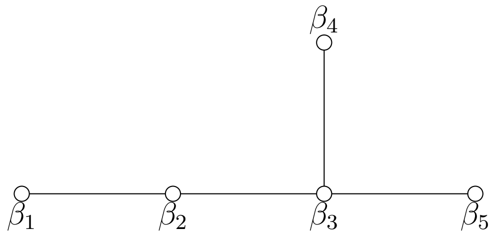
- If $A_{\alpha_2,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = A_{\alpha_7,\lambda} = 0$, then λ is one of the following $(4,3)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

7.3.2 Φ containing a component of type D_5

In this subsection, we will assume Φ is a subset of Π with Dynkin diagram



Recall that all the roots in Σ have the same multiplicity. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string. Then $\{\lambda\} \cup \Phi$ will be a D_6 or an E_6 simple system. Therefore, using Proposition 6.2.9 (iii), Proposition 7.2.9 and Proposition 7.2.10 we deduce the following. The submanifold $S_\Phi \cdot o$ is austere if and only if: the number of roots $\nu_4 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_4,\nu_4} = -1$ coincides with the number of roots $\nu_5 \in \Sigma^\Phi$ of minimum level in their Φ-strings satisfying $A_{\beta_5,\nu_5} = -1$.

Put $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_2,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}, \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}.$$

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following $(2,1)$ roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is one of the following $(3,1)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = A_{\alpha_6,\lambda} = 0$, then λ is one of the following $(6,1)$ roots:

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

Thus, we deduce that $S_\Phi \cdot o$ is austere when $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ both in E_7 and E_8.

Hence, we also need to study the case $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$, which appears only in E_8 type. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_3,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 2 \\ 1223432 \end{pmatrix}$$

- If $A_{\alpha_8,\lambda} = -1$, then λ is one of the following 2 roots:

$$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = A_{\alpha_8,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 0 \\ 0111111 \end{pmatrix}.$$

- If $A_{\alpha_6,\lambda} = A_{\alpha_8,\lambda} = -1$, then is the root

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}.$$

- If λ has trivial Φ-string then it is

$$\begin{pmatrix} 1 \\ 1222210 \end{pmatrix}.$$

Consider now the root

$$\gamma = \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}.$$

We have that $(A_{\alpha_2,\gamma}, A_{\alpha_3,\gamma}, A_{\alpha_4,\gamma}, A_{\alpha_5,\gamma}, A_{\alpha_6,\gamma}, A_{\alpha_8,\gamma}) = (0,1,0,0,0,1)$. Hence, from Corollary 7.2.5 we deduce that γ is the extreme root of its Φ-string. If $S_\Phi \cdot o$ were austere, combining Lemma 6.1.10 and Corollary 7.2.6, there would exist a root $\lambda \in \Sigma^\Phi$ of minimum in its Φ-string satisfying

$$(A_{\alpha_2,\gamma}, A_{\alpha_3,\gamma}, A_{\alpha_4,\gamma}, A_{\alpha_5,\gamma}, A_{\alpha_6,\gamma}, A_{\alpha_8,\gamma}) = (0,-1,0,0,0,-1).$$

However, we have calculated above all the roots of minimum level in their Φ-strings and none of them satisfies such condition. Hence, the submanifold $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$.

Put $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Let $\lambda \in \Sigma^\Phi$ be of minimum level in its Φ-string.

- If $A_{\alpha_1,\lambda} = -1$, then λ is one of the following $(3,1)$ roots:

$$\begin{pmatrix} 1 \\ 0122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}.$$

- If $A_{\alpha_2,\lambda} = -1$, then λ is the root

$$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}.$$

- If $A_{\alpha_5,\lambda} = -1$, then λ is one of the following $(3,2)$ roots:

$$\begin{pmatrix} 0 \\ 0010000 \end{pmatrix}, \begin{pmatrix} 0 \\ 0110000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1110000 \end{pmatrix}.$$

- If $A_{\alpha_1,\lambda} = A_{\alpha_2,\lambda} = A_{\alpha_3,\lambda} = A_{\alpha_4,\lambda} = A_{\alpha_5,\lambda} = 0$, then λ is one of the following $(6,1)$ roots:

$$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 0 \\ 1100000 \end{pmatrix}, \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix}, \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}.$$

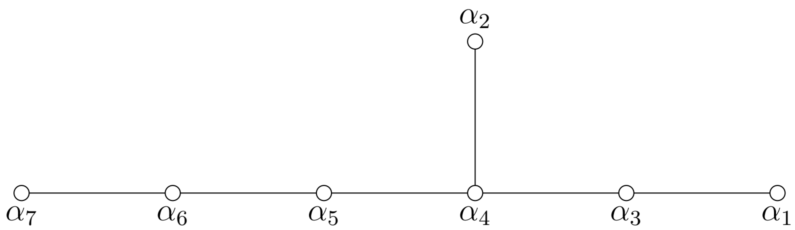
Thus, we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

At this point, we can state the classification of the E_7 case.

7.3.3 The classification in spaces of type E_7

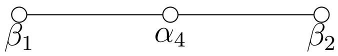
In this subsection we conclude the classification of austere submanifolds of the form $S_\Phi \cdot o$ in symmetric spaces of non-compact type with E_7 Dynkin diagram.

Proposition 7.3.1. *Let G/K be a symmetric space of non-compact type with E_7 Dynkin diagram of the form*



Let Φ be a proper subset of the set Π of simple roots. Then, the submanifold $S_\Phi \cdot o$ is austere if and only if one of the following conditions holds:

(i) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$*, where Φ_0 is an A_3 simple subsystem containing the root α_4 as a central root in its Dynkin diagram, that is, the simple subsystem Φ_0 has a Dynkin diagram of the form*



for $(\beta_1, \beta_2) \in \{(\alpha_3, \alpha_5), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5)\}$ and $\Phi_1 = \{\beta\}$ is orthogonal to Φ_0, where $\beta \neq \alpha_1$, or

(ii) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$*, where Φ_0 is a D_4 simple subsystem and Φ_1 is a discrete subset of Π orthogonal to Φ_0, or*

(iii) $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$*, or*

(iv) Φ *is a D_6 simple subsystem, that is, $\Phi = \Pi \backslash \{\alpha_1\}$, or*

(v) Φ *is discrete.*

Proof. According to the above calculations for the cases where Φ is connected and contains 5 elements (Subsection 7.3.1 and Subsection 7.3.2) and Proposition 7.2.8, we just need to analyze the austerity of $S_\Phi \cdot o$ when Φ is a connected subset of Π that consists of six roots.

Put first $\Phi = \Pi \backslash \{\alpha_7\}$, that is, Φ is an E_6 simple subsystem. The root α_7 is clearly the root of minimum level in its Φ-string. Then, $\Pi_{\alpha_7} = \{\alpha_7\} \cup \Phi$ is an E_7 simple system by means of Proposition 6.2.1 (i). The number of roots of the Φ-string of α_7 is the number of positive roots spanned by an E_7 simple system, minus the number of positive roots spanned by an E_6 simple system Φ, minus the number of roots spanned by an E_7 simple system whose coefficient corresponding to α_7 is greater or equal than two (this last number is zero [69, p. 687]). Thus, we obtain $|I(\lambda, \Phi)| = 27$. From Corollary 7.2.6 (iii) we deduce that the shape operator $\mathcal{S}$ is not austere when restricted to the Φ-string of α_7. Since Π_{α_7}

spans 63 positive roots and Φ spans 36 positive roots, then we have considered all the roots in Σ^Φ. Then, $S_\Phi \cdot o$ is not austere when $\Phi = \Pi \backslash \{\alpha_7\}$.

Put now $\Phi = \Pi\backslash\{\alpha_2\}$. We will gather the following information in a table: each root $\lambda \in \Sigma^\Phi$ of minimum level in its non-trivial Φ-string; the extreme root γ in the Φ-string of λ; the root $\alpha \in \Phi$ such that $A_{\alpha,\lambda} = -1$; the root $\beta \in \Phi$ such that $A_{\beta,\gamma} = 1$; and the number of roots in the Φ-string of λ. We will call the roots α and β in Φ (under the previous conditions) the starting root and the finishing root, respectively. As usual, we will use the notation of [69, Appendix C], but in this case for roots in E_7.

Minimum level	Extreme	Starting	Finishing	Number of roots
$\begin{pmatrix} 1 \\ 000000 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 123321 \end{pmatrix}$	α_4	α_5	35
$\begin{pmatrix} 2 \\ 012321 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 123432 \end{pmatrix}$	α_7	α_1	7

Table 7.1: $\Phi = \Pi\backslash\{\alpha_2\}$.

Since Φ spans 21 positive roots, we have considered all the roots in Σ^Φ. Since α_1 is a finishing root but never a starting root, from Corollary 7.2.6 (ii) we deduce that $S_\Phi \cdot o$ is not austere when $\Phi = \Pi\backslash\{\alpha_2\}$.

Finally, put $\Phi = \Pi\backslash\{\alpha_1\}$. The number of roots in the Φ-string of α_1 equals the number of positive roots spanned by an E_7 simple system, minus the number of positive roots spanned by a D_6 simple system, minus the number of positive roots whose coefficient corresponding to α_1 in an E_7 simple system is greater or equal than 2 (which is just one [69, p. 688]). Thus, $|I(\alpha_1, \Phi)| = 32$. Below, we explicitly write an involution $f\colon I(\alpha_1,\Phi) \to I(\alpha_1,\Phi)$ satisfying the conditions of Corollary 6.1.11 (ii):

$$
\begin{array}{cc}
\begin{pmatrix} 0 \\ 000001 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 123431 \end{pmatrix} & \begin{pmatrix} 0 \\ 000011 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 123421 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 000111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 123321 \end{pmatrix} & \begin{pmatrix} 0 \\ 001111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 122321 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 000111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 123321 \end{pmatrix} & \begin{pmatrix} 0 \\ 011111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 112321 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 001111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 122321 \end{pmatrix} & \begin{pmatrix} 0 \\ 111111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 012321 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 011111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 112321 \end{pmatrix} & \begin{pmatrix} 1 \\ 001211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 122221 \end{pmatrix}
\end{array}
$$

$$\begin{pmatrix} 1 \\ 111111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 012321 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 011211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 112221 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 001221 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 122211 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 111211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 012221 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 012211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 111221 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 011221 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 112211 \end{pmatrix}.$$

Since Φ spans 30 positive roots and the root

$$\begin{pmatrix} 2 \\ 123432 \end{pmatrix}$$

has trivial Φ-string, we have considered all the roots in Σ^Φ. Thus, $S_\Phi \cdot o$ is austere when $\Phi = \Pi \backslash \{\alpha_1\}$. □

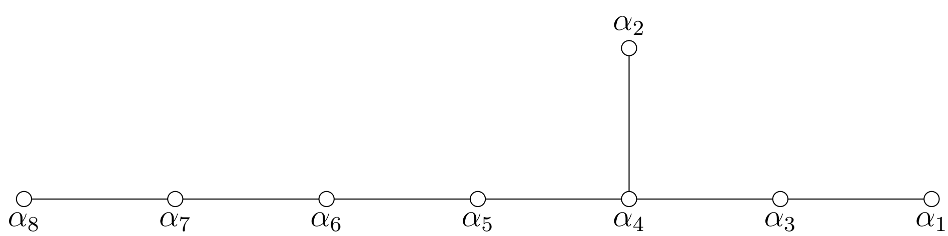
7.4 E_8 case

In this section, we will finish the study of the E_8 case. In order to do this, Corollary 7.2.6 will be the main tool.

In fact, we will address the remaining cases with the following procedure. We will fix a connected subset Φ of Π that consists of 6 or 7 roots. After that, we will gather the following information in a table (as we did in the proof of Proposition 7.3.1): the root $\lambda \in \Sigma^\Phi$ of minimum level in its non-trivial Φ-string; the extreme root γ in the Φ-string of λ; the root $\alpha \in \Phi$ such that $A_{\alpha,\lambda} = -1$ (starting root); the root $\beta \in \Phi$ such that $A_{\beta,\gamma} = 1$ (finishing root); and the number of roots in the Φ-string of λ.

7.4.1 Φ containing a component with 6 roots

Recall that we are studying a symmetric space G/K with E_8 Dynkin diagram of the form



Put $\Phi = \Pi \backslash \{\alpha_2, \alpha_8\}$. This is the corresponding table:

Minimum level	Extreme	Starting	Finishing	Number of roots
$\begin{pmatrix} 3 \\ 1234531 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1345642 \end{pmatrix}$	α_1	α_7	7
$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1111111 \end{pmatrix}$	α_7	α_1	7
$\begin{pmatrix} 2 \\ 0012321 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}$	α_7	α_1	7
$\begin{pmatrix} 1 \\ 1111100 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}$	α_3	α_6	21
$\begin{pmatrix} 2 \\ 1112321 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1234542 \end{pmatrix}$	α_6	α_3	21
$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0123321 \end{pmatrix}$	α_4	α_5	35

Table 7.2: $\Phi = \Pi \backslash \{\alpha_2, \alpha_8\}$.

Since Φ spans 21 positive roots and the root

$$\begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$$

has trivial Φ-string, we have considered all the roots in Σ^{Φ}. Since α_5 is a finishing root but never a starting root, from Corollary 7.2.6 (i) we deduce that $S_{\Phi} \cdot o$ is not austere.

Put $\Phi = \Pi \backslash \{\alpha_1, \alpha_2\}$. This is the corresponding table:

Minimum level	Extreme	Starting	Finishing	Number of roots
$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1111111 \end{pmatrix}$	α_3	α_8	7
$\begin{pmatrix} 3 \\ 1234532 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$	α_3	α_8	7
$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1234542 \end{pmatrix}$	α_8	α_3	7
$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1222210 \end{pmatrix}$	α_4	α_7	21
$\begin{pmatrix} 2 \\ 0012321 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1234531 \end{pmatrix}$	α_7	α_4	21
$\begin{pmatrix} 1 \\ 0000111 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}$	α_5	α_6	35

Table 7.3: $\Phi = \Pi \backslash \{\alpha_1, \alpha_2\}$.

Since Φ spans 21 positive roots and the root

$$\begin{pmatrix} 3 \\ 1234531 \end{pmatrix}$$

has trivial Φ-string, we have considered all the roots in Σ^{Φ}. Since α_6 is a finishing root but never a starting root, from Corollary 7.2.6 (i) we deduce that $S_\Phi \cdot o$ is not austere.

Put $\Phi = \Pi \backslash \{\alpha_1, \alpha_3\}$. This is the corresponding table:

Minimum level	Extreme	Starting	Finishing	Number of roots
$\begin{pmatrix} 2 \\ 0123431 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1234531 \end{pmatrix}$	α_8	α_2	7
$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1234532 \end{pmatrix}$	α_8	α_2	7
$\begin{pmatrix} 2 \\ 1234542 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$	α_2	α_8	7
$\begin{pmatrix} 0 \\ 0000010 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1222210 \end{pmatrix}$	α_4	α_7	21
$\begin{pmatrix} 0 \\ 0000011 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1222211 \end{pmatrix}$	α_4	α_7	21
$\begin{pmatrix} 1 \\ 0001221 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1234421 \end{pmatrix}$	α_6	α_5	35

Table 7.4: $\Phi = \Pi \backslash \{\alpha_1, \alpha_3\}$.

Since Φ spans 21 positive roots and the root

$$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}$$

has trivial Φ-string, we have considered all the positive roots. Since α_5 is a finishing root but never a starting root, from Corollary 7.2.6 (i) we deduce that $S_\Phi \cdot o$ is not austere.

Put $\Phi = \Pi \backslash \{\alpha_7, \alpha_8\}$. This is the corresponding table:

Since Φ spans 36 positive roots and the roots

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix}, \begin{pmatrix} 3 \\ 1345642 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$$

have trivial Φ-string, we have considered all the positive roots. Since α_6 is twice a finishing root and only once a starting root, from Corollary 7.2.6 (i) we deduce that $S_\Phi \cdot o$ is not austere.

Put $\Phi = \Pi \backslash \{\alpha_1, \alpha_8\}$. Consider the roots α_8 and

$$\lambda = \begin{pmatrix} 2 \\ 1123432 \end{pmatrix}.$$

Minimum level	Extreme	Starting	Finishing	Number of roots
$\begin{pmatrix} 0 \\ 0100000 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}$	α_6	α_1	27
$\begin{pmatrix} 0 \\ 1100000 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1123432 \end{pmatrix}$	α_6	α_1	27
$\begin{pmatrix} 1 \\ 1222210 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1245642 \end{pmatrix}$	α_1	α_6	27

Table 7.5: $\Phi = \Pi \backslash \{\alpha_7, \alpha_8\}$.

These two roots are of minimum level in their Φ-strings, as follows easily from Proposition 6.2.1 (iii), since $A_{\alpha_7,\alpha_8} = A_{\alpha_7,\lambda} = -1$ and $A_{\nu,\alpha_8} = A_{\nu,\lambda} = 0$ for all $\nu \in \Phi\backslash\{\alpha_7\}$. Note that $(\Phi, \Phi \cup \{\alpha_8\}, \Phi \cup \{\lambda\}) \equiv (D_6, D_7, D_7)$. Hence, from Proposition 6.2.9 (iii) we have that $\mathcal{S}$ is austere when restricted to the Φ-string of α_8 and the Φ-string of λ. Each one of them consists of 12 roots. Consider the root α_1 and the root

$$\gamma = \begin{pmatrix} 0 \\ 1111111 \end{pmatrix}.$$

They are of minimum level in their Φ-strings by means of Proposition 6.2.1 (iii). Note that $A_{\alpha_1,\alpha_3} = A_{\alpha_2,\gamma} = -1$. The shape operator $\mathcal{S}$ is austere when restricted to

$$\bigoplus_{\alpha \in I(\alpha_1,\Phi)\cup I(\gamma,\Phi)} \mathfrak{g}_\alpha$$

by virtue of Proposition 7.2.9. Each one of the strings consist of 32 roots. Since Φ spans 30 positive roots and the roots

$$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$$

have trivial Φ-string, we have considered all the roots in Σ^Φ. Thus $S_\Phi \cdot o$ is austere.

7.4.2 The classification in spaces of type E_8

Finally, we will analyze the case where Φ is a connected subset of Π that consists of 7 roots, which will allow us to conclude the classification in the E_8 case.

Put $\Phi = \Pi\backslash\{\alpha_1\}$. This is the corresponding table:

Minimum level	Extreme	Starting	Finishing	Number of roots
$\begin{pmatrix} 0 \\ 0000001 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1234531 \end{pmatrix}$	α_3	α_2	64
$\begin{pmatrix} 2 \\ 0123432 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$	α_8	α_8	14

Table 7.6: $\Phi = \Pi \backslash \{\alpha_1\}$.

Since Φ spans 42 positive roots, we have considered all the roots in Σ^Φ. Since α_2 is a finishing root but never a starting root, from Corollary 7.2.6 (i) we deduce that $S_\Phi \cdot o$ is not austere.

Put $\Phi = \Pi \backslash \{\alpha_2\}$. This is the corresponding table:

Minimum level	Extreme	Starting	Finishing	Number of roots
$\begin{pmatrix} 1 \\ 0000000 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1233321 \end{pmatrix}$	α_4	α_6	56
$\begin{pmatrix} 3 \\ 1234531 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$	α_1	α_8	8
$\begin{pmatrix} 2 \\ 0012321 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1234542 \end{pmatrix}$	α_7	α_3	28

Table 7.7: $\Phi = \Pi \backslash \{\alpha_2\}$.

Since Φ spans 28 positive roots, we have considered all the roots in Σ^Φ. Since α_3 is a finishing root but never a starting root, from Corollary 7.2.6 (i) we deduce that $S_\Phi \cdot o$ is not austere.

Finally, let us consider the case $\Phi = \Pi \backslash \{\alpha_8\}$. It is clear that α_8 is the root of minimum level in its Φ-string. Note that $\Phi \cup \{\alpha_8\} = \Pi$ is an E_8 simple system. Thus, the Φ-string of α_8 consists of the number of positive roots spanned by an E_8 simple system, minus those with coefficient corresponding to α_8 greater or equal than two and minus the number of positive roots spanned by the E_7 simple system Φ. Thus, using [69, p. 688], we deduce that the Φ-string of α_8 consists of 56 roots. Below, we write explicitly an involution $f\colon I(\alpha_8, \Phi) \to I(\alpha_8, \Phi)$ under the conditions of Corollary 6.1.11 (ii):

$$\begin{pmatrix} 0 \\ 1000000 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 \\ 1345642 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1100000 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 \\ 1245642 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1110000 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 \\ 1235642 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1111000 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 \\ 1234642 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1111100 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 \\ 1234542 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1111110 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 \\ 1234532 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1111100 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1234542 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1111110 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1234532 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1111111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 \\ 1234531 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1111210 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1234432 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1111111 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1234531 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1112210 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1233432 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1111211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1234431 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1122210 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1223432 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1112211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1233431 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1111221 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1234421 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1222210 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1123432 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1122211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1223431 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1112221 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1233421 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1222211 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1123431 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1122221 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1223421 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1112321 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1233321 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1222221 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1123421 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1122321 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1223321 \end{pmatrix}$$
$$\begin{pmatrix} 2 \\ 1112321 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 1233321 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1222321 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1123321 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1123321 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 \\ 1222321 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1122321 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 1223321 \end{pmatrix}.$$

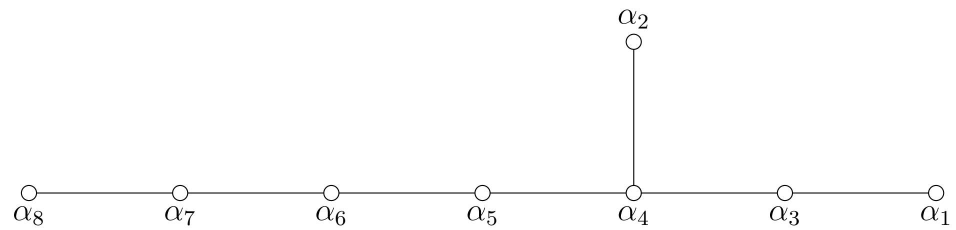
Thus, $\mathcal{S}$ is austere when restricted to the Φ-string of α_8. Since this Φ-string consists of 56 roots, Φ spans 63 positive roots and the root

$$\begin{pmatrix} 3 \\ 2345642 \end{pmatrix}$$

has trivial Φ-string, then we have considered all the roots in Σ^Φ. In conclusion, the submanifold $S_\Phi \cdot o$ is austere when $\Phi = \Pi \backslash \{\alpha_8\}$.

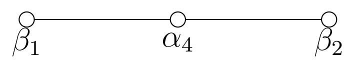
Thus, we can conclude the following

Proposition 7.4.1. *Let G/K be a symmetric space of non-compact type with E_8 Dynkin diagram of the form*



Let Φ be a proper subset of the set Π of simple roots. Then, the submanifold $S_\Phi \cdot o$ is austere if and only if one of the following conditions holds:

(i) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$, *where* Φ_0 *is an* A_3 *simple subsystem containing the root* α_4 *as a central root in its Dynkin diagram, that is, the simple subsystem* Φ_0 *has a Dynkin diagram of the form*



for $(\beta_1, \beta_2) \in \{(\alpha_3, \alpha_5), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5)\}$, *and* $\Phi_1 = \{\beta\}$ *is orthogonal to* Φ_0, *where* $\beta \neq \alpha_1$, *or*

(ii) $\Phi = \Phi_0$ *or* $\Phi = \Phi_0 \cup \Phi_1$, *where* Φ_0 *is a* D_4 *simple subsystem and* Φ_1 *is a discrete subset of* Π *orthogonal to* Φ_0, *or*

(iii) $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ *is a* D_5 *simple subsystem, or*

(iv) $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ *is a* D_6 *simple subsystem, or*

(v) $\Phi = \Pi \backslash \{\alpha_8\}$ *is an* E_7 *simple subsystem, or*

(vi) Φ *is discrete.*

Conclusions and open problems

The first contribution of this thesis is the classification result of isoparametric hypersurfaces in complex hyperbolic spaces proved in Chapter 3. From this classification, we have deduced the following consequences:

- An isoparametric hypersurface in $\mathbb{C}H^2$ is as open part of a homogeneous hypersurface.
- For $n \geq 3$ there are inhomogeneous examples: one family up to congruence for $\mathbb{C}H^3$, and infinitely many for $\mathbb{C}H^n$, $n \geq 4$.
- An isoparametric hypersurface of $\mathbb{C}H^n$ has constant principal curvatures if and only if it is an open part of a homogeneous hypersurface of $\mathbb{C}H^n$.
- The principal curvatures of an isoparametric hypersurface M in $\mathbb{C}H^n$ are pointwise the same as the principal curvatures of a homogeneous hypersurface of $\mathbb{C}H^n$.
- The focal submanifold of an isoparametric hypersurface in $\mathbb{C}H^n$ is locally homogeneous.

In this thesis, we have investigated isoparametric hypersufaces in the semi-Riemannian setting too. Indeed, we focused our attention on anti-De Sitter spaces and we have obtained the following results (see Chapter 4):

- The number of principal curvatures of a spacelike isoparametric hypersurface in the anti-De Sitter space is bounded from above by two.
- Non-totally umbilical spacelike isoparametric hypersurfaces in the anti-De Sitter space H_1^n, $n \geq 3$, are tubes around totally geodesic submanifolds of H_1^n.

Another class of submanifolds we have studied is that of CPC submanifolds (see Chapter 5). Our investigation on CPC submanifolds led us to the following achievements:

- The construction of a large new family of non-totally geodesic CPC submanifolds that do not admit a description as singular orbits of cohomogeneity one actions in symmetric spaces of non-compact type and rank greater than one.
- The development of an original technique based on the examination of the information codified in the root system of each symmetric space that allows us to calculate the geometry of each solvable submanifold in a very efficient way.

Finally, in Chapters 6 and 7 we have investigated the austerity of certain orbits related to the theory of parabolic subgroups. The main achievements of both chapters are:

- The classification of austere orbits of the form $A_\Phi N_\Phi \cdot o$ of symmetric spaces of non-compact type, where $A_\Phi N_\Phi$ is the solvable part of a parabolic subgroup of the isometry group of a symmetric space of non-compact type.
- The generalization of the concept of α-string containing λ [69, p. 152] to subsets Φ of the set of simple roots Π and the explicit determination of most of these Φ-strings.
- The development of a theory that allows to associate a diagram to each Φ-string and to calculate the shape operator of certain solvable submanifolds by looking at these diagrams.

There are still many open problems and questions in view of the above conclusions. Some of these questions stem directly from the above commented results. Others have not been studied in this thesis but can be addressed by using the methods and techniques we have developed. More precisely:

- Cohomogeneity one actions in symmetric spaces of non-compact type have been thoroughly investigated and classifications have been achieved under the following extra assumptions: cohomogeneity one actions that produce regular foliations [15]; cohomogeneity one actions with a totally geodesic singular orbit [16]; and cohomogeneity one actions in rank one symmetric spaces of non-compact type [17]. However, a complete classification is still open. In order to achieve it, the remaining cohomogeneity one actions are those with a non-totally geodesic singular orbit, that is, a non-totally geodesic CPC singular orbit. Hence, the achievements in this thesis concerning CPC submanifols may play a crucial role in order to classify cohomogeneity actions on symmetric spaces G/K of non-compact type. However, it is important to remark that the classification of cohomogeneity one actions would not follow right away from an eventual classification of solvable CPC submanifolds. In fact, we would need to understand and to investigate thoroughly the isometries in K, as well as understanding better a procedure employed successfully to produce cohomogeneity one actions: the nilpotent construction method [18].
- As explained above, we have developed a technique based on the examination of root systems that allows us to use the Levi-Civita connection in a very efficient way. We expect this tool to be used or adapted to address different problems. For instance, we have made remarkable advances in classifying homogeneous hypersurfaces of symmetric spaces of non-compact type that have a natural structure of algebraic Ricci soliton with respect to the induced metric.
- Weakly reflective submanifolds are always examples of austere submanifolds. The converse is not true. We have already achieved some results concerning weakly reflective submanifolds that will allow to check which austere examples in Chapters 6 and 7 are also weakly reflective.

- The canonical extension method was first introduced in [18] and investigated further in [47]. It constitutes a nice tool for constructing new submanifolds from known examples. This procedure preserves some properties that we are interested in, for instance minimality and isoparametricity. However, it does not preserve austerity. Our classification of austere submanifolds of the form $S_\Phi \cdot o$ in symmetric spaces of non-compact should be basically the key in order to make precise when a canonical extension of an austere submanifold is austere.

- Make progress in the classification problem of totally geodesic submanifolds. This problem seems nowadays infeasible in full generality. However, with the algebraic methods utilized in Chapter 5, 6 and 7 we are able to calculate very efficiently the shape operator of many homogeneous submanifolds. These ideas may help to obtain some classification result in certain higher rank symmetric spaces.

Resumen en castellano

La noción de simetría está presente en todos los ámbitos de la ciencia. Esta afirmación debe ser entendida de un modo generalizado. No solo hay simetría en objetos geométricos y formas físicas, sino que aparecen también simetrías en ecuaciones y construcciones teóricas. Cabe destacar aquí las palabras con respecto a la simetría pronunciadas por el premio Nobel P. W. Anderson, que declaró que "es solo un poco exagerado afirmar que la física es el estudio de la simetría".

En la ciencia aplicada existen muy pocos problemas que pueden ser resueltos de manera exacta. No obstante, el rango de problemas que pueden ser resueltos de manera efectiva es a menudo mayor, y la ciencia más teórica lleva años respondiendo a importantes y profundas cuestiones conceptuales y prácticas. Esto sucede así debido a que, en ocasiones, es posible modelar matemáticamente el problema en cuestión. A continuación, se buscan hipótesis simplificadoras, lo cual hace preciso demostrar, normalmente de forma matemática, que tales simplificaciones no influyen, o no en exceso, en la solución del problema. En este sentido, un método de gran efectividad para resolver problemas es aprovechar las simetrías del espacio para reducir el número de grados de libertad de los mismos y convertirlos en algo más manejable.

El objetivo principal de esta tesis es precisamente el estudio, análisis y descripción de ciertos objetos geométricos a través de la observación de sus simetrías.

En la misma línea que Anderson, el matemático Felix Klein describió la geometría como el estudio de aquellas propiedades de un espacio que son invariantes por un grupo de transformaciones (grupo de simetrías) dado. En el seno de la geometría riemanniana, este grupo es el *grupo de isometrías*, esto es, el grupo de transformaciones de una variedad riemanniana determinada que preservan las distancias. La acción de un subgrupo del grupo de isometrías de una variedad dada se denomina *acción isométrica*. La cohomogeneidad de una acción isométrica es la codimensión más baja de sus órbitas. Una órbita cuya codimensión es mayor que la cohomogeneidad de la acción se denomina *órbita singular*. Una órbita de dimensión máxima se denomina *regular*. Una subvariedad se dice *(extrínsecamente) homogénea* si es una órbita de la acción de un subgrupo del grupo de isometrías sobre la variedad ambiente.

El problema de clasificación de hipersuperficies homogéneas en el espacio euclídeo (equivalentemente de acciones de cohomogeneidad uno, salvo equivalencia de órbitas) surge en el seno de la óptica geométrica y se remonta al trabajo de Somigliana [96] a principios del siglo XX. Su trabajo da origen al estudio de uno de los objetos geométricos en los que se centra esta tesis: las *hipersuperficies isoparamétricas*. Una hipersuperficie de una variedad riemanniana se dice isoparamétrica si ella y sus hipersuperficies equidistantes suficientemente próximas tienen curvatura media constante. Las hipersuperficies homogéneas son siempre ejemplos de hipersuperficies isoparamétricas. En la década de 1930, Levi-Civita [75], Segre [93] y Cartan [25, 27, 26] retomaron el estudio de estos objetos desde un punto de vista (más) geométrico. En particular, Cartan [25] demostró que en espacios de curvatura constante una hipersuperficie es isoparamétrica si y solo si sus curvaturas principales son constantes. Además, Segre [93] y Cartan [25] clasificaron estos objetos en el espacios euclídeo e hiperbólico real, respectivamente. Todos los ejemplos conocidos por Cartan tenían una propiedad común: eran homogéneos. Sin embargo, las esferas admiten ejemplos de hipersuperficies isoparamétricas no homogéneas [53]. De hecho, el problema de clasificación de hipersuperficies isoparamétricas en las esferas resultó ser mucho más complejo y sorprendente, llegando como consecuencia de ello a ser incluido por el Medalla Fields Yau en su influyente lista de problemas en geometría [111].

Normalmente, el estudio de las acciones de cohomogeneidad uno se ha enfocado desde el punto de vista de sus órbitas regulares (hipersuperficies homogéneas) o conceptos relacionados con las mismas (hipersuperficies isoparamétricas). Sin embargo, también resulta muy interesante abordar el estudio de las acciones de cohomogeneidad uno centrando nuestra atención en sus órbitas singulares. De hecho, si uno considera una acción de cohomogeneidad uno con una órbita singular en una variedad de Riemann completa y conexa, entonces las curvaturas principales de dicha órbita singular, contadas con multiplicidades, no dependen de las direcciones normales. Resulta realmente interesante investigar la clasificación de las subvaridades que comparten esta propiedad geométrica de las órbitas singulares de las acciones de cohomogeneidad uno. En esta tesis, estas subvariedades se denominan subvariedades CPC. Nótese que las subvariedades CPC tienen curvaturas principales constantes en el sentido introducido por Heintze, Olmos y Thorbergsson en [58], en el contexto de subvariedades isoparamétricas.

Esta relación existente entre acciones de cohomogeneidad uno y subvariedades CPC ha sido generalizada en el resultado que enunciamos a continuación [54]: si M es una subvariedad de una variedad de Riemann de codimensión mayor que uno y los tubos a su alrededor (con radios suficientemente pequeños) son hipersuperficies isoparamétricas con curvaturas principales constantes, entonces la subvariedad M es una subvariedad CPC. Esto indica que las subvariedades CPC juegan un papel crucial en el estudio de las acciones de cohomogeneidad uno y de las hipersuperficies isoparamétricas. En concreto, usando teoría de campos de vectores de Jacobi, es sencillo comprobar que una subvariedad de un espacio forma real es CPC si y solo si los tubos a su alrededor (con radios suficientemente pequeños) son hipersuperficies con curvaturas principales constantes. En otros términos, en los espacios forma reales clasificar hipersuperficies isoparamétricas es equivalente a clasificar subvariedades CPC. Conviene también destacar en este punto que las subvariedades totalmente geodésicas son siempre CPC, y que a su vez las subvariedades CPC son minimales.

Otro de los conceptos que ocupa un lugar central en esta tesis y que está relacionado con los objetos geométricos mencionados hasta ahora es el concepto de subvariedad austera. Una subvariedad M se dice *austera* si, en cada punto, las curvaturas principales (contadas con multiplicidades) con respecto a cualquier vector normal son invariantes tras un cambio de signo. Uno de las principales focos de interés de esta noción viene precisamente de su relación con otros conceptos tales como hipersuperficie isoparamétrica, hipersuperficie homogénea, subvariedad minimal o subvariedad CPC. De hecho, las subvariedades austeras constituyen una clase intermedia entre las subvariedades CPC y las subvariedades minimales. Además, tal y como hemos mencionado anteriormente, los conjuntos focales de familias isoparamétricas de hipersuperficies con curvaturas principales constantes son CPC y, por lo tanto, también austeras. Nótese por último que las hipersuperficies homogéneas austeras son hipersuperficies CPC.

Las subvariedades austeras fueron introducidas por Harvey y Lawson [57] en el contexto de geometrías calibradas. Desde entonces, las subvariedades austeras se han investigado por su propio interés geométrico (véase por ejemplo [22, 38, 64, 36, 62]). De hecho, la condición de ser austera impone una ecuación en derivadas parciales sobredeterminada de segundo orden que implica que la curvatura media se anula, es decir, implica la condición de ser minimal. Nótese que ser minimal y austera equivalen en dimensión dos. Sin embargo, en una dimensión superior a dos, la condición de ser austera resulta mucho más fuerte que la condición de ser minimal.

En esta tesis nos hemos centrado en el estudio de hipersuperficies isoparamétricas, subvariedades CPC y subvariedades austeras en el contexto de los espacios simétricos de tipo no compacto.

De acuerdo con la definición original dada por Cartan [24], un espacio simétrico riemanniano es una variedad de Riemann caracterizada por la propiedad de que la curvatura es invariante mediante el transporte paralelo. Esta definición, a priori geométrica, tuvo el sorprendente efecto de traer a colación, y de manera natural, la teoría de grupos de Lie. En efecto, resulta que los espacios simétricos riemannianos están íntimamente relacionados con los grupos de Lie semisimples. En este sentido, muchos problemas geométricos complicados y planteados sobre espacios simétricos pueden ser traducidos a un lenguaje de álgebra lineal, donde hay herramientas de cálculo más concretas que permiten resolver tal cuestión.

Por esta razón, la familia de espacios simétricos ha sido un agradable entorno de trabajo donde uno puede abordar y comprobar la validez de muchas propiedades de índole geométrica. A menudo, son una interesante fuente de ejemplos y contraejemplos. En concreto, el conjunto de espacios simétricos es una gran familia de espacios que abarca muchos de los más interesantes ejemplos de variedad de Riemann, tales como espacios de curvatura constante, espacios proyectivos e hiperbólicos, grassmannianas o grupos de Lie compactos. Además de desde el punto de vista de la geometría diferencial, los espacios simétricos también se han estudiado desde el punto de vista del análisis global y el análisis armónico, adoptando los espacios simétricos de tipo no compacto una particular relevancia (véase, por ejemplo, [60]). Los espacios simétricos también constituyen una familia de espacios con gran importancia dentro de la teoría de holonomía, constituyendo una clase propia en la clasificación de los grupos de holonomía de Berger.

En cierto sentido, podemos afirmar que hay tres clases de espacios simétricos: espacios euclídeos, espacios simétricos de tipo compacto (cuando el grupo de isometrías es compacto y semisimple) y espacios simétricos de tipo no compacto (cuando el grupo de isometrías es no compacto y semisimple). Existe una dualidad entre los espacios simétricos de tipo compacto y los espacios simétricos de tipo no compacto. Pese a ello, suelen presentar propiedades muy diferentes. Los espacios simétricos de tipo no compacto son difeomorfos a espacios euclídeos y tienen por tanto topología trivial. Por su parte, en los espacios simétricos de tipo compacto la topología suele jugar un papel fundamental.

Todo espacio simétrico de tipo no compacto es isométrico a un grupo de Lie resoluble con una métrica invariante a la izquierda. De hecho, este grupo de Lie, que denotaremos por AN, es la parte resoluble de la descomposicion de Iwasawa del grupo de isometrías del espacio simétrico. Un profundo conocimiento de esta parte resoluble de la descomposición de Iwasawa permite construir, describir e incluso clasificar subvariedades del espacio simétrico con ciertas propiedades de simetría. En efecto, dentro de la teoría de subvariedades, uno puede considerar diferentes e interesantes tipos de subvariedades fijándose en las órbitas de los subgrupos del grupo de Lie resoluble AN. De modo equivalente, uno puede considerar diferentes e interesantes tipos de subvariedades mirando las subálgebras del álgebra de Lie de AN. Esto hace que un buen manejo de la descomposición en espacios de raíces del álgebra de Lie del grupo de isometrías constituya una herramienta fundamental a la hora de estudiar geometría de subvariedades en el contexto de los espacios simétricos de tipo no compacto. Por supuesto, conviene mencionar que no todas las subvariedades de un espacio simétrico de tipo no compacto $M \cong G/K$ (ni siquiera las homogéneas) proceden de un subgrupo de la parte resoluble AN del grupo de isometrías de G/K.

A continuación presentamos los resultados originales de esta tesis.

Hipersuperficies isoparamétricas en el espacio hiperbólico complejo

Una de las principales contribuciones de esta tesis es la *clasificación de hipersuperficies isoparamétricas en el espacio hiperbólico complejo*. En primer lugar, el Capítulo 2 se dedica a la exposición del concepto de hipersuperficie isoparamétrica así como a un breve recorrido por algunos de los resultados más importantes relacionados con dicho concepto. Además, en el Capítulo 2 construimos y describimos geométricamente los ejemplos de hipersuperficies isoparamétricas en el espacio hiperbólico complejo. Después de ello, en el Capítulo 3 clasificamos las hipersuperficies isoparamétricas en $\mathbb{C}H^n$. De hecho, dicha clasificación se reduce a comprobar que cualquier hipersuperficie isoparamétrica del espacio hiperbólico complejo se corresponde con alguno de los ejemplos previamente construidos en el Capítulo 2. Es interesante destacar que, al revés de lo que ocurre en espacios euclídeos o en espacios hipérbolicos reales, en cualquier espacio hiperbólico complejo de dimensión mayor que dos aparecen ejemplos de hipersuperficies isoparamétricas que no son homogéneas [41]. Hasta donde nosotros sabemos y desde la clasificación de Cartan [25] en el año 1938 para espacios hiperbólicos reales, la clasificación recogida en el Capítulo 3 es la primera clasificación de hipersuperficies isoparamétricas en una familia completa de espacios simétricos. Dicha clasificación ha dado lugar a la publicación de los artículos [43] y [44].

El primer paso para la demostración de este resultado consiste en entender el comportamiento de las hipersuperficies isoparamétricas con respecto a la fibración de Hopf. Dicho de un modo más preciso, lo primero que hemos hecho ha sido comprobar que una hipersuperficie del espacio hiperbólico complejo es isoparamétrica si y solo si lo es también su pullback o levantamiento con respecto a la aplicación de Hopf. De este modo, es posible investigar las hipersuperficies isoparamétricas del espacio hiperbólico complejo mediante el estudio de hipersuperficies isoparamétricas lorenztianas en el espacio de anti-De Sitter. Hay dos razones principales que sustentan el comenzar el estudio de las hipersuperficies isoparamétricas del espacio hiperbólico complejo mediante el análisis de sus correspondientes levantamientos lorentzianos: la ecuación de Jacobi es más fácil de resolver en el espacio de anti-De Sitter (dado que tiene curvatura seccional constante) que en el espacio hiperbólico complejo y por tanto es más sencillo tratar con el desplazamiento normal de hipersurficies; además, en el espacio de anti-De Sitter contamos con una generalización de la fórmula de Cartan que permite fundamentalmente obtener cotas para el número de curvaturas principales del levantamiento lorentziano de la hipersuperficie de partida.

De este modo, gran parte del trabajo para la classificación de hipersuperficies isoparamétricas en el espacio hiperbólico complejo se realiza en el espacio de anti-De Sitter, donde resulta más sencillo obtener la información geométrica de la hipersuperficie a través de su operador de configuración y deducir sus implicaciones sobre la hipersuperficie inicial. A continuación, utilizando toda esta información geométrica, probamos un resultado de rigidez en $\mathbb{C}H^n$ que revela aspectos profundos e interesantes de la geometría de los ejemplos. Todos estos argumentos nos permiten probar el siguiente resultado de clasificación:

Teorema 1. *Sea M una hipersuperficie conexa real en el espacio hiperbólico complejo $\mathbb{C}H^n$, $n \geq 2$. Entonces, M es una hipersuperficie isoparamétrica si y solo si M es congruente a una parte abierta de:*

(i) *un tubo alrededor de un espacio hipérbolico complejo totalmente geodésico $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$,*

(ii) *un tubo alrededor de un espacio hiperbólico real totalmente geodésico $\mathbb{R}H^n$,*

(iii) *una horosfera,*

(iv) *una hipersuperficie de Lohnherr reglada minimal homogénea W^{2n-1}, o alguna de sus hipersuperficies equidistantes,*

(v) *un tubo alrededor de una subvariedad reglada minimal homogénea W^{2n-k}_{φ}, construida por Berndt-Brück, para $k \in \{2, \dots, n-1\}$, $\varphi \in (0, \pi/2]$,*

(vi) *un tubo alrededor de una subvariedad reglada minimal homogénea $W_{\mathfrak{w}}$, para algún subespacio propio $\mathfrak{w}$ de $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ tal que $\mathfrak{w}^\perp$, el complemento ortogonal de $\mathfrak{w}$ en $\mathfrak{g}_\alpha$, tiene ángulo de Kähler no constante.*

Los ejemplos (i), (ii), (iii) del resultado que acabamos de enunciar se corresponden con los ejemplos de la lista de Montiel, que son ejemplos de hipersuperficies Hopf homogéneas. Nótese que, con la notación del Teorema 1, de manera natural, podemos pensar en $\mathbb{C}^{k+1}$ incluido en $\mathbb{C}^{n+1}$ (respectivamente $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$); luego también es natural pensar en $\mathbb{C}H^k$

como subvariedad totalmente geodésica de $\mathbb{C}H^n$ (respectivamente $\mathbb{R}H^n \subset \mathbb{C}H^n$). De esta manera, construyendo tubos alrededor de estas dos subvariedades obtenemos ejemplos de hipersuperficies isoparamétricas. Así quedan descritos los ejemplos (i) y (ii) del Teorema 1. Las horosferas, (iii), se construyen mediante la acción de N, la parte nilpotente de la descomposición de Iwasawa del grupo de isometrías de $\mathbb{C}H^n$.

Este resultado de clasificación tiene numerosas e interesantes consecuencias. Destacamos a continuación algunas de las más relevantes. En primer lugar, se deduce de la clasificación que todas las hipersuperficies isoparamétricas con curvaturas principales constantes del espacio hiperbólico complejo son homogéneas. Además, dadas las curvaturas principales (en un punto) de una hipersuperficies isoparamétrica del espacio hiperbólico complejo, existe una hipersuperficie homogénea del espacio hiperbólico complejo que tiene exactamente esas curvaturas principales (constantes). Otra consecuencia interesante es que la subvariedad focal de cualquier hipersuperficie isoparamétrica del espacio hiperbólico complejo es localmente homogénea.

Hipersuperficies isoparamétricas espaciales en el espacio de anti-De Sitter

El concepto de hipersuperficie isoparamétrica también tiene sentido desde el punto de vista de la geometría semi-riemanniana. De hecho, simplemente hay que añadir en la definición dada para el contexto riemanniano que la métrica inducida sea no degenerada. Además, como se sigue del trabajo de Hahn [56], una hipersuperficie en un espacio forma semi-riemanniano es isoparamétrica si y solo si tiene curvaturas principales constantes con multiplicidades algebraicas constantes.

Las hipersuperficies isoparamétricas han sido investigadas también en el contexto de la geometría semi-riemanniana. Además, el abanico de ejemplos parece ser mucho más amplio que en el caso riemanniano. En particular, estos objetos geométricos se suponen clasificados en el espacio de Minkowski por Magid [80], aunque Burth [23] afirma haber encontrado algunos problemas en los argumentos de Magid. Más allá, también se han obtenido resultados interesantes en espacios de De Sitter. De hecho, Nomizu [83] probó que las hipersuperficies isoparamétricas espaciales del espacio de De Sitter son tubos alrededor de subvariedades totalmente geodésicas. La demostración de este resultado se basa en el hecho de que dichas hipersuperficies tienen a los sumo dos curvaturas principales diferentes. Nomizu conjeturó en ese mismo trabajo [83] que en el espacio de anti-De Sitter aparecerían ejemplos de hipersuperficies isoparamétricas espaciales con más de dos curvatures principales.

El principal objetivo del Capítulo 4 es precisamente obtener una respuesta negativa a la conjetura propuesta por Nomizu. En efecto, en el Capítulo 4 probamos que una hipersuperficie isoparamétrica espacial en el espacio de anti-De Sitter tiene a lo sumo dos curvaturas principales diferentes. Para probar esta cota hemos generalizado al contexto semi-riemanniano el trabajo de Ferus [52]. Además, la obtención de tal cota para el número de curvaturas principales nos ha permitido deducir una clasificación para las hipersuperficies isoparamétricas espaciales en los espacios de anti-De Sitter: toda hipersuperficie isoparamétrica espacial no totalmente umbílica en el espacio de anti-De Sitter es un tubo alrededor de una subvariedad totalmente geodésica.

Subvariedades CPC

El Capítulo 5 lo dedicamos al estudio de las subvariedades CPC, es decir, subvariedades cuyas curvaturas principales, contadas con multiplicidades, no dependen de la dirección normal. Arriba hemos enfatizado la importancia de las subvariedades CPC y su relación con muchos otros objetos geométricos de interés tales como las hipersuperficies isoparamétricas, las acciones de cohomogeneidad uno, las subvariedades austeras, las subvariedades totalmente geodésicas o las subvariedades minimales. Sin embargo, no conocemos de la existencia de un estudio profundo y sistemático o con técnicas propias de estas subvariedades en contextos más generales. Esto puede resultar sorprendente dado lo simple y natural que resulta el concepto de subvariedad CPC.

De este modo, el Capítulo 5 se centra en el *desarrollo de una serie de técnicas para construir, describir y clasificar subvariedades CPC en espacios simétricos de tipo no compacto* y rango mayor que uno. Es importante recordar en este punto que tanto las subvariedades totalmente geodésicas como las órbitas singulares de acciones de cohomogeneidad uno son ejemplos de subvariedades CPC. Así, el principal objetivo del Capítulo 5 es el de construir *una nueva y amplia familia de subvariedades CPC que no son totalmente geodésicas y que no admiten una descripción como órbitas singulares de acciones de cohomogeneidad uno.* Hasta donde nosotros sabemos, solo se conocía una subvariedad con estas características en espacios simétricos de tipo no compacto: se trata de un ejemplo 11-dimensional en el plano de Cayley hiperbólico [41]. Los resultados del Capítulo 5 han sido publicados en [14] y, junto con otros resultados, han dado lugar al artículo expositivo [45].

Sea Π el conjunto de raíces simples del sistema de raíces Σ de un espacio simétrico G/K de tipo no compacto. Sea Π' el conjunto de raíces $\alpha \in \Pi$ tales que $2\alpha \notin \Sigma$ (véase la Sección 1.5 para consultar los detalles). A continuación, enunciamos el resultado principal del Capítulo 5.

Teorema 2. *Sea $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus V)$ una subálgebra de $\mathfrak{a} \oplus \mathfrak{n}$ con $V \subseteq \bigoplus_{\alpha\in\Pi'} \mathfrak{g}_\alpha$. Sea S el subgrupo conexo y cerrado de AN cuya álgebra de Lie es $\mathfrak{s}$. Entonces, la órbita $S \cdot o$ es una subvariedad CPC de $M = G/K$ si y solo si se cumple alguna de las siguientes condiciones:*

(I) *Existe una raíz simple $\lambda \in \Pi'$ tal que $V \subset \mathfrak{g}_\lambda$.*

(II) *Existen dos raíces simples y no ortogonales $\alpha_0, \alpha_1 \in \Pi'$ con $|\alpha_0| = |\alpha_1|$ y subespacios $V_0 \subseteq \mathfrak{g}_{\alpha_0}$ y $V_1 \subseteq \mathfrak{g}_{\alpha_1}$ tales que $V = V_0 \oplus V_1$ y se cumple una de las siguientes condiciones:*

   (i) $V_0 \oplus V_1 = \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$*;*

   (ii) $V_0 \oplus V_1$ *es un subconjunto propio de* $\mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{\alpha_1}$ *y*

      (a) V_0 *y* V_1 *son isomorfos a* $\mathbb{R}$*; o*

(b) V_0 *y* V_1 *son isomorfos a* $\mathbb{C}$ *y existe un elemento* $T \in \mathfrak{k}_0$ *tal que* $\operatorname{ad}(T)$ *define estructuras complejas para* V_0 *y* V_1 *y se anula cuando se restringe a* $[V_0, V_1]$*; o*

(c) V_0 *y* V_1 *son isomorfos a* $\mathbb{H}$ *y existe un subespacio* $\mathfrak{l} \subseteq \mathfrak{k}_0$ *tal que* $\operatorname{ad}(\mathfrak{l})$ *define estructuras cuaterniónicas para* V_0 *y* V_1 *y se anula cuando se restringe a* $[V_0, V_1]$.

Además, solo las subvariedades descritas en (I) *y* (II)(i) *son órbitas singulares de acciones de cohomogeneidad uno.*

Para construir y describir esta nueva familia de subvariedades CPC hemos desarrollado un técnica original y muy prometedora basada en traducir geométricamente la información algebraica codificada en el sistema de raíces de cada espacio simétrico de tipo no compacto. Para argumentar esta afirmación o describirla de un modo más preciso, recordemos que la conexión de Levi-Civita constituye una de las herramientas fundamentales de la teoría de subvariedades. En el caso de los espacios simétricos, contamos con potentes herramientas algebraicas que permiten reescribir o expresar de manera algebraica y manejable dicha conexión. Sin embargo, en dicha expresión se relacionan entre sí y de un modo a priori complicado los distintos espacios de raíces. Para desenmarañar esta complicación, hemos rescatado y generalizado el concepto de *α-string de* λ [69, p. 152], donde α y λ denotan dos raíces cualesquiera. De manera informal, podríamos decir que esta generalización del concepto de string nos permite entender mucho mejor cómo la conexión de Levi-Civita relaciona los diferentes espacios de raíces entre sí. Así, resulta mucho más sencillo organizar la información para calcular la geometría (el operador de configuración) de la subvariedad que estemos estudiando.

Subvariedades austeras en espacios simétricos de tipo no compacto

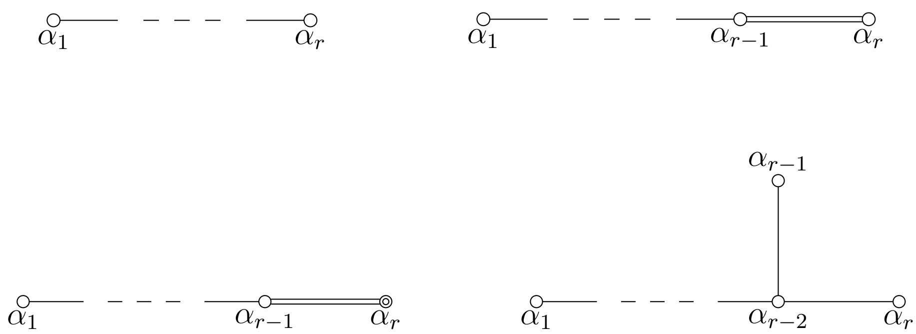
Una de las principales herramientas para el estudio de los espacios simétricos de tipo no compacto y rango mayor que uno se sigue de su descomposición horosférica, que está a su vez relacionada con la teoría de subálgebras parabólicas de álgebras reales semisimples. Estas subálgebras están parametrizadas (salvo conjugación) por los subconjuntos Φ de un conjunto de raíces simples Π del sistema de raíces de un álgebra de Lie real semisimple. Así, dado un espacio simétrico de tipo no compacto $M \cong G/K$, la descomposición horosférica asociada con cada elección $\Phi \subset \Pi$ nos conduce a que M es difeomorfo al producto cartesiano cierta subvariedad totalmente geodésica B_Φ de M, un subgrupo abeliano A_Φ de G y un subgrupo nilpotente N_Φ de G. Además, el subgrupo resoluble conexo $S_\Phi = A_\Phi N_\Phi$ de G actúa libre e isométricamente en M, y todas las órbitas de dicha acción son congruentes entre sí. Tamaru [102] probó que estas órbitas son subvariedades Einstein resolubles desde un punto de vista intrínseco, y subvariedades minimales de M desde el punto de vista de su geometría extrínseca.

En los Capítulos 6 y 7 investigamos bajo qué condiciones las órbitas de S_Φ resultan austeras. En tales capítulos mostramos que esta condición de austeridad está codificada de algún modo en ciertas propiedades algebraicas y combinatorias del par (Π, Φ). Analizar estas propiedades requiere un perfecto entendimiento del sistema de raíces de cada espacio simétrico. Por ello, recurrimos de nuevo y como hicimos en el Capítulo 5, a (una nueva generalización de) la noción de *string*. La clave de este trabajo reside en asociar un diagrama a cada uno de los strings. Este diagrama facilita la comprensión del operador de configuración de la órbita de S_Φ considerada. De hecho, de un modo informal, la austeridad de cada órbita depende de las simetrías de los diagramas de los strings. De este modo, después de probar varios resultados de carácter general sobre los strings y sus diagramas, llevamos a cabo un exhaustivo análisis caso por caso de los sistemas de raíces existentes.

Dada la extensión de este trabajo, hemos dividido su exposición en dos partes. En primer lugar, en el Capítulo 6 concretamos las herramientas y enfoque del problema, introducimos las propiedades fundamentales de los strings y sus diagramas, y clasificamos la órbitas de S_Φ austeras en los espacios simétricos de tipo no compacto clásicos. Finalmente, la clasificación de tales órbitas en los espacios simétricos excepcionales, junto con la presentación de algunas herramientas específicas para su estudio, aparece recogida en el Capítulo 7. Enunciamos a continuación el resultado principal de los Capítulos 6 y 7.

Teorema 3. *Sea G/K un espacio simétrico de tipo no compacto y sea Φ un subconjunto propio del conjunto de raíces simples Π.*

(a) Si el diagrama de Dynkin de Π adopta una de las siguientes configuraciones



donde el segundo diagrama puede ser de tipo B_r o C_r, entonces la subvariedad $S_\Phi \cdot o$ es austera si y solo si se cumple alguna de las siguientes condiciones:

(i) Φ *es discreto, o*

(ii) $\Phi = \Phi_0$ *y satisface las condiciones especificadas en el Cuadro 1, o*

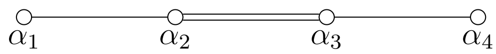
(iii) $\Phi = \Phi_0 \cup \Phi_1$*, donde Φ_0 es ortogonal a Φ_1 y ambos satisfacen las condiciones especificadas en el Cuadro 1 (en la fila de color gris se asume además que todas las raíces de Σ tienen la misma multiplicidad).*

Π	Φ_0	Φ_1
A_r	Simétrico, conexo	$\emptyset$
B_r	B_n, $n < r$	Discreto
B_r	$\{\alpha_{r-2}, \alpha_{r-1}\}$	Discreto
C_r	C_n, $n < r$	Discreto
BC_r	BC_n, $n < r$	Discreto
D_r	D_n, $n < r$	Discreto
D_r	$\{\alpha_{r-3}, \alpha_{r-2}, \alpha_{r-1}\}$	Discreto
D_r	$\{\alpha_{r-3}, \alpha_{r-2}, \alpha_r\}$	Discreto

Cuadro 1: Clasificación en espacios simétricos clásicos.

(b) Si Π *tiene un diagrama de Dynkin de tipo* G_2*, entonces* $S_\Phi \cdot o$ *es austera.*

(c) Si Π *tiene un diagrama de Dynkin de tipo* F_4 *de la forma*



con $|\alpha_1| = |\alpha_2| < |\alpha_3| = |\alpha_4|$*, entonces la subvariedad* $S_\Phi \cdot o$ *es austera si y solo si se cumple una de las siguientes condiciones:*

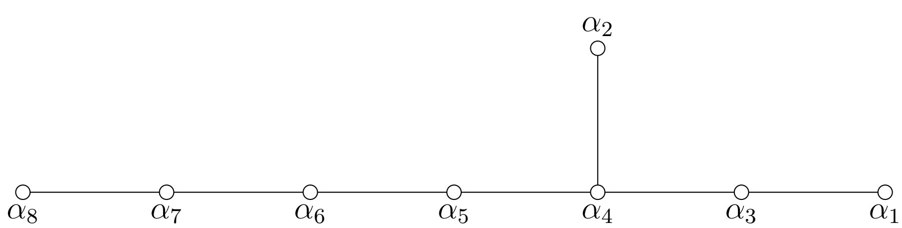
(i) Φ *es un subconjunto discreto de* Π*, o*

(ii) Φ *es un subsistema simple de tipo* B_n *con* $n \in \{2, 3\}$*, equivalentemente,* $\Phi = \{\alpha_2, \alpha_3\}$ *o* $\Phi = \{\alpha_2, \alpha_3, \alpha_4\}$*, o*

(iii) Φ *es un subsistema simple de tipo* C_3*, equivalentemente,* $\Phi = \{\alpha_1, \alpha_2, \alpha_3\}$*, o*

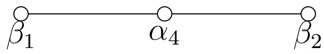
(iv) $\Phi = \{\alpha_3, \alpha_4\}$ *y todas las raíces de* Σ *tienen la misma multiplicidad.*

(d) Si Π *tiene un diagrama de Dynkin de tipo* E_6*,* E_7 *o* E_8 *contenido en el diagrama*



entonces la subvariedad $S_\Phi \cdot o$ *es austera si y solo si se cumple una de las siguientes condiciones:*

(i) $\Phi = \Phi_0$ *o* $\Phi = \Phi_0 \cup \Phi_1$, *donde* Φ_0 *es un subsistema simple de tipo* A_3 *conteniendo la raíz* α_4 *como una raíz central en su diagrama de Dynkin, es decir, el subsistema simple* Φ_0 *tiene un diagrama de Dynkin de la forma*



con $(\beta_1, \beta_2) \in \{(\alpha_3, \alpha_5), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5)\}$ *y* $\Phi_1 = \{\beta\}$ *es ortogonal a* Φ_0, *donde* $\beta \neq \alpha_1$, *y* $\beta \neq \alpha_6$ *si* $\Pi \equiv E_6$, *o*

(ii) $\Phi = \Phi_0$ *o* $\Phi = \Phi_0 \cup \Phi_1$, *donde* Φ_0 *es un subsistema simple de tipo* D_4 *y* Φ_1 *es un subconjunto discreto de* Π *ortogonal a* Φ_0, *o*

(iii) Π *es un sistema simple de tipo* E_6 *y* $\Phi = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ *es un subsistema simple de tipo* A_5, *o*

(iv) Π *es un sistema simple de tipo* E_7 *o* E_8 *y* $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ *es un subsistema simple de tipo* D_5, *o*

(v) Π *es un sistema simple de tipo* E_7 *or* E_8 *y* $\Phi = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ *es un subsistema simple de tipo* D_6, *o*

(vi) Π *es un sistema simple de tipo* E_8 *y* $\Phi = \Pi \backslash \{\alpha_8\}$ *es un subsistema simple de tipo* E_7, *o*

(vii) Φ *es discreto.*

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