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TESE DE DOUTORAMENTO

The geometry of weakly-Einstein manifolds

Rodrigo Mariño Villar

ESCOLA DE DOUTORAMENTO INTERNACIONAL
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Resumo e conclusións

Unha pregunta clásica en xeometría de Riemann é a seguinte: Dada unha variedade diferenciable n-dimensional M, existe algunha estrutura Riemanniana que se poida considerar a "mellor" nela?

As métricas Einstein son un bo candidato a ser ditas métricas xa que aparecen coma métricas críticas para o funcional de Hilbert-Einstein

$$\mathcal{E} : g \mapsto \int_M \tau dvol_g,$$

restrinxido a métricas de volume constante, sendo τ a curvatura escalar. Pódese ver que o gradiente deste funcional vén dado por $\nabla\mathcal{E} = -\rho + \frac{\tau}{2}g$, onde ρ denota o tensor de Ricci. Debido ao Teorema de Gauss-Bonnet en dimensión dous, obtense a indentidade universal

$$\rho = \frac{\tau}{2}g,$$

que non é máis que a condición Einstein en dita dimensión.

É posible xeneralizar esta idea para dimensións maiores. En dimensión catro, se tomamos o funcional

$$\mathcal{F} : g \mapsto \int_M 4(||R||^2 - 4||\rho||^2 + \tau^2) dvol_g,$$

sendo R é o tensor de curvatura, pódese calcular o seu gradiente e buscar métricas críticas coma no caso anterior. Utilizando o Teorema de Gauss-Bonnet-Chern, Berger obtivo en [4] a seguinte identidade de curvatura en dimensión catro,

$$\left(\check{R} - \frac{\|R\|^2}{4}g\right) + \tau\left(\rho - \frac{\tau}{4}g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4}g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right) = 0, \qquad (1.7)$$

onde $\check{R}$, $\check{\rho}$ e $R[\rho]$ son os campos de tensores simétricos de tipo $(0,2)$ que veñen dados por $\check{R}_{ij} = R_{iabc}R_j{}^{abc}$, $\check{\rho}_{ij} = \rho_{ia}\rho^a{}_j$ e $R[\rho]_{ij} = R_{iabj}\rho^{ab}$. Nesta situación, se asumimos que a métrica é Einstein, entón todos os termos agrupados en paréntese da Ecuación (1.7) se anulan. Agora ben, a pregunta que nos atende é xusto a oposta, se algunha das expresións entre parénteses non correspondente á expresión da condición Einstein se anula, é a métrica Einstein? Polo tanto, aparece así a cuestión sobre que sucede cando algún destes tensores é un múltiplo da métrica e esta non é Einstein. Isto lévanos a definir o que chamaremos as condicións *debilmente Einstein*.

Definición 1.11 Unha variedade de Riemann ,(M, g), dise

1. $\check{R}$-Einstein se $\check{R} = \frac{\|R\|^2}{n} g$.
2. $\check{\rho}$-Einstein se $\check{\rho} = \frac{\|\rho\|^2}{n} g$.
3. $R[\rho]$-Einstein se $R[\rho] = \frac{\|\rho\|^2}{n} g$.

Ademais dise *debilmente Einstein* se verifica algunha das condicións anteriores.

Estas condicións xogan un papel importante no estudo de métricas críticas para funcionais cuadráticos da curvatura. Se tomamos os funcionais $\mathcal{S}(g) = \int_M \tau^2 dvol_g$ e $\mathcal{F}_t(g) = \int_M \{\|\rho\|^2 + t\tau^2\} dvol_g$ restrinxidos a métricas de volume constante, unha métrica é $\mathcal{S}$-crítica se e só se se satisfai a ecuación

$$2\left(Hess_\tau - \frac{\Delta\tau}{4} g\right) - 2\tau\left(\rho - \frac{\tau}{4} g\right) = 0, \tag{2.7}$$

e analogamente, unha métrica é $\mathcal{F}_t$-crítica se e só se se satisfai

$$Hess_\tau - \Delta\rho + 2t\left(Hess_\tau - \frac{\Delta\tau}{4} g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right) - 2t\tau\left(\rho - \frac{\tau}{4} g\right) = 0. \tag{2.8}$$

Pódese ver nestas ecuacións a relación que gardan as condicións debilmente Einstein co estudo de métricas críticas para estes funcionais.

Durante o desenvolvemento da primeira parte deste traballo, analizamos o comportamento das condicións debilmente Einstein en diferentes contextos. A súa elección non é arbitraria e o obxectivo de estudo de cada un destes campos irase motivando antes de presentar os resultados obtidos.

Por outra banda, as condicións debilmente Einstein teñen interese máis aló delas mesmas. A identidade (1.7) pódese escribir equivalentemente coma

$$\check{R} - \frac{\|R\|^2}{4} g = \frac{1}{3}\tau\rho_0 + 2W[\rho_0], \tag{5.1}$$

onde $\rho_0 = \rho - \frac{\tau}{4} g$, $W[\rho_0]_{ij} = W_{iabj}\rho_0^{ab}$ e W é o tensor de Weyl. A condición $W[\rho_0] = 0$, que é equivalente a $W[\rho] = 0$ xa que o tensor de Weyl non ten traza, claramente xeneraliza a condición Einstein, pero ademais tamén o fai coa de ser localmente conformemente chá ($W = 0$). Así mesmo, séguese de (5.1) que $W[\rho] = 0$ e ser $\check{R}$-Einstein son equivalentes se e só se a curvatura escalar é cero. Outro aspecto a ter en conta é que tensor de Bach redúcese á $W[\rho]$ se a métrica considerada ten $\operatorname{div}_1 \operatorname{div}_4 W = 0$ (en particular, se o tensor de Weyl é harmónico, e en xeral, cando o tensor de Cotton teña diverxencia cero). Ao longo da memoria, chamaremos a esta condición a condición Einstein xeneralizada e será estudada na súa segunda parte.

No primeiro capítulo, introducimos algúns resultados básicos e técnicos sobre xeometría de Riemann, topoloxía e álxebra que utilizamos ao longo do traballo, tales coma o Teorema de

Gauss-Bonnet-Chern ou as bases de Gröbner. Introducimos tamén a definición de métrica debilmente Einstein e mostramos a existencia de exemplos que satisfán todas as condicións debilmente Einstein sen ser métricas de Einstein.

Empezaremos a nosa análise no contexto das variedades localmente conformemente chás. É un resultado coñecido que se unha variedade de dimensión maior ou igual que catro satisfai esta propiedade, entón o seu tensor de Weyl é identicamente nulo. Ademais, se engadimos a condición de ser Einstein, entón esta última implica que a variedade é de curvatura seccional constante xa que o tensor de curvatura está totalmente determinado polo tensor de Ricci, que neste caso, será un múltiplo escalar da métrica polo Lema de Schur. Polo tanto, as condicións debilmente Einstein son un paso intermedio de cara a clasificación das variedades localmente coformemente chás. Debido a isto, no segundo capítulo, próbase o seguinte resultado.

Teorema 2.2 Sexa (M, g) unha variedade de Riemann localmente conformemente chá. Entón (M, g) é

1. $\check{R}$-Einstein se e só se cumpre que
   - (i) $dimM = 4$ e (M, g) ten curvatura escalar nula.
   - (ii) $dimM \neq 4$ e
     - (ii.a) (M, g) é localmente homotética a un produto deformado da forma $\mathcal{I} \times_f N(c)$, con métrica $g = dt^2 + f^2 g_N$, onde $\mathcal{I}$ é un intervalo real e $(N(c), g_N)$ é unha variedade de curvatura seccional constante $c \in \{0, \pm 1\}$. Ademais a función de deformación vén dada por
       - (ii.a.1) $f(t)^2 = t^2 - 1$, se $c = 1$, e $\mathcal{I} = (1, +\infty)$,
       - (ii.a.2) $f(t)^2 = t$, se $c = 0$, e $\mathcal{I} = (0, +\infty)$,
       - (ii.a.3) $f(t)^2 = 1 - t^2$, se $c = -1$, e $\mathcal{I} = (-\infty, 1)$.
     - (ii.b) (M, g) é localmente simétrica e localmente homotética a un produto $M = N_1^m(c) \times N_2^m(-c)$, onde $m \geq 2$.
2. $R[\rho]$-Einstein se e só se é un produto da forma de 1.(ii.b).
3. $\check{\rho}$-Einstein se e só se é un produto da forma de 1.(ii.b) ou localmente un produto deformado $\mathcal{I} \times_f \mathbb{R}^{n-1}$, con
$$f(t) = \left(\frac{2(n-1)\,(at+b)}{n} \right)^{\frac{n}{2(n-1)}},$$
   con $t \in (\frac{-b}{a}, +\infty)$ e $a, b \in \mathbb{R}$.

Obtemos, deste xeito, novos exemplos de métricas debilmente Einstein diferentes das coñecidas ata o de agora. A clasificación faise salvo homotecia, dado que as condicións debilmente Einstein son invariantes por estas.

A diferenza das métricas Einstein, as debilmente Einstein non verifican un análogo ao Lema de Schur. Nas métricas dadas no Teorema 2.2, a norma do tensor de curvatura ou a do tensor de Ricci non teñen por que ser necesariamente constantes. De feito, un produto deformado $\check{R}$-Einstein de dimensión cinco coa métrica dada polo Teorema 2.2-1.$(ii.a,1)$ ten norma do tensor de curvatura

$$\|R\|^2 = \frac{40}{(t^2-1)^4},$$

que é claramente non constante.

No que respecta ás métricas críticas para funcionais cuadráticos da curvatura, o caso trivial debilmente Einstein $N^2(c) \times N^2(-c)$ é sempre $\mathcal{S}$-crítica e $\mathcal{F}_t$-crítica para $t \in \mathbb{R}$, polo que é omitido ao longo da análise. Para os casos restantes temos o seguinte resultado.

Teorema 2.15 Sexa (M, g)unha variedade de Riemann localmente conformemente chá debilmente Einstein de dimensión catro. Entón

1. Se (M, g) é $\check{R}$-Einstein, entón é $\mathcal{S}$-crítica e $\mathcal{F}_t$-crítica se e só se
$$\Delta\rho = -2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right).$$
2. Se (M, g) é $\check{\rho}$-Einstein, entón $\mathcal{F}_t$-crítica para $t = -\frac{1}{3}$.

Outro contexto axeitado onde estudar estas condicións é o das hipersuperficies. Polo Teorema de Nash [57], toda variedade de Riemann é unha subvariedade de $\mathbb{R}^N$ para algún $N \in \mathbb{N}$. O problema reside en saber cando esta pode ser unha subvariedade de codimensión mínima, é dicir, unha hipersuperficie. As hipersuperficies de Einstein foron clasificadas por Fialkow [25]. No terceiro capítulo, analizamos as condicións debilmente Einstein en hipersuperficies en espazos forma, obtendo unha clasificación parcial en función das curvaturas principais, que son os autovalores do operador forma. Cando estes autovalores son constantes, a hipersuperficie dise isoparamétrica. Cecil e Ryan recollen en [15] unha clasificación deste tipo de hipersuperficies (dependendo do número de curvaturas principais) cando o espazo ambiente é un espazo forma real (o espazo euclídeo, a esfera ou o espazo hiperbólico). Ademais, cando se teñen dúas curvaturas principais (non necesariamente constantes), sendo unha delas simple e función da outra, do Carmo e Dajczer [13] proban que esta é unha hipersuperficie de revolución. O seguinte resultado recapitula os exemplos obtidos.

Teorema 3.1 Sexa (M, g) unha hipersuperficie nun espazo forma real $\mathbb{Q}_c^{n+1}$, con $c = 0, \pm 1$, con dúas curvaturas principais. Se (M, g) é debilmente Einstein entón é o produto de dúas esferas, o produto dunha esfera e dun espazo hiperbólico ou unha hipersuperficie de revolución sobre unha curva perfil.

No capítulo cuarto, centrámonos no estudo das métricas homoxéneas e a súa relación coas propiedades debilmente Einstein. As métricas homoxéneas debilmente Einstein en dimensión tres teñen un compartamento totalmente diferente ao que terán as de dimensión catro. Caeiro-Oliveira proba en [11] que unha variedade homoxénea de dimensión tres é $\check{R}$-Einstein se e só se o seu operador de Ricci é de rango un, e polo tanto, é isométrica a un grupo de Lie unimodular cuxa álxebra de Lie está determinada por

$$[e_1, e_2] = (\lambda_1 + \lambda_2)e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2.$$

Ademais, tamén proba que unha variedade homoxénea de dimensión tres é $\check{\rho}$-Einstein se e só se ten operador de Ricci dado por $Q_\rho = \mathrm{diag}[\lambda, \lambda, -\lambda]$, o que implica que ten curvatura seccional constante ou é homotética ao grupo de Heisenberg cunha xeometría Nil_3. Por último, proba que unha variedade homoxénea de dimensión tres é $R[\rho]$-Einstein se e só se o seu operador de Ricci é $Q_\rho = \mathrm{diag}[\lambda, \lambda, 2\lambda]$, e consecuentemente, é isométrica a unha esfera de Berger determinada pola álxebra de Lie

$$[e_1, e_2] = \frac{4}{3}\lambda e_3, \quad [e_2, e_3] = \lambda e_1, \quad [e_3, e_1] = \lambda e_2.$$

Como xa indicamos, o resultado en dimensión catro non segue a mesma liña. Arias-Marco e Kowalski [1] clasificaron as variedades homoxéneas de dimensión catro que cumpren a condición $\check{R}$-Einstein. Neste traballo, conseguimos mellorar este resultado, mostrando que as métricas homoxéneas $\check{R}$-Einstein constitúen unha única clase homotética. Para a condición $\check{\rho}$-Einstein, a casuística é moito máis rica. Atopamos tanto exemplos con curvatura escalar nula coma unha familia uniparamétrica de clases homotéticas de variedades homoxéneas $\check{\rho}$-Einstein, entre outros. A condición $R[\rho]$-Einstein é moito máis ríxida e tan só permite que unha métrica cumpla esta condición se é simétrica, e polo tanto, o produto de dúas variedades de curvatura seccional constante e oposta unha da outra ou unha variedade Einstein. Estes resultados xeneralizan o traballo de Jensen [45], que clasificou as métricas homoxéneas Einstein de dimensión catro, mostrando que eran simétricas. De forma máis concisa, os resultados están recollidos no seguinte teorema.

Teorema 4.2 Sexa (M, g) unha variedade homoxénea simplemente conexa e de dimensión catro. Entón

(1) (M, g) é $\check{R}$-Einstein e non simétrica se e só se é homotética a un grupo de Lie $\mathbb{R} \ltimes \mathbb{R}^3$ con métrica invariante pola esquerda determinada pola álxebra de Lie

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_4, e_3] = -e_3,$$

onde $\{e_1, \dots, e_4\}$ é unha base ortonormal.

(2) (M, g) é $\check{\rho}$-Einstein e non simétrica se e só se é homotética a un dos seguintes grupos de Lie:

(2.a) O grupo de Lie $SU(2) \times \mathbb{R}$ con métrica invariante pola esquerda determinada pola álxebra de Lie

$$\begin{aligned} &[e_1, e_2] = (4 \pm 2\sqrt{2})e_3, \quad [e_2, e_3] = (3 \pm 2\sqrt{2})e_1, \quad [e_3, e_1] = e_2, \\ &[e_4, e_1] = -e_2, \qquad\qquad [e_4, e_2] = (3 \pm 2\sqrt{2})e_1, \end{aligned}$$

onde $\{e_1, \dots, e_4\}$ é unha base ortonormal.

(2.b) O grupo de Lie $\mathbb{R}\ltimes H^3$ con métrica invariante pola esquerda determinada pola álxebra de Lie

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \frac{1}{2}e_1, \quad [e_2, e_4] = -\frac{1}{2}e_2,$$

onde $\{e_1, \dots, e_4\}$ é unha base ortonormal.

(2.c) O grupo de Lie $\mathbb{R}\ltimes\mathbb{R}^3$ con métrica invariante pola esquerda determinada pola álxebra de Lie

$$\begin{aligned}
[e_4, e_1] &= e_1 - \frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_2 + \frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_3,\\
[e_4, e_2] &= \frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_1 + \alpha e_2 + \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_3,\\
[e_4, e_3] &= -\frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_1 - \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_2 - \tfrac{\alpha}{\alpha+1}e_3,
\end{aligned}$$

onde $\{e_1, \dots, e_4\}$ é unha base ortonormal e $\alpha \in (-1, 1)$, $\alpha \neq -\frac{1}{2}$, $\alpha \neq 0$.

(3) (M, g) é $R[\rho]$-Einstein se e só se é simétrica.

Respecto ás métricas críticas para funcionais cuadráticos da curvatura, como no caso homoxéneo a curvatura escalar é constante, a condición de ser $\mathcal{S}$-crítica redúcese a ter curvatura escalar nula ou ser Einstein, polo que esta condición redúcese a unha comprobación directa. O resultado obtido é o seguinte.

Teorema 4.12 Sexa (M, g) unha variedade de Riemann homoxénea simplemente conexa debilmente Einstein de dimensión catro. Entón,

1. (M, g) é $\mathcal{S}$-crítica se e só se é homotética ao grupo de Lie $SU(2)\times\mathbb{R}$ con métrica invariante pola esquerda determinada pola álxebra de Lie

$$\begin{aligned}
&[e_1, e_2] = (4 \pm 2\sqrt{2})e_3, && [e_2, e_3] = (3 \pm 2\sqrt{2})e_1, && [e_3, e_1] = e_2,\\
&[e_4, e_1] = -e_2, && [e_4, e_2] = (3 \pm 2\sqrt{2})e_1.
\end{aligned}$$

   onde $\{e_1, \dots, e_4\}$ é unha base ortonormal.

2. (M, g) é $\mathcal{F}_t$-crítica se e só se $t = -\frac{3}{4}$ e (M, g) é homotética ao grupo de Lie $\mathbb{R} \times \mathbb{R}^3$ con métrica invariante pola esquerda determinada pola álxebra de Lie

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_4, e_3] = -e_3,$$

   onde $\{e_1, \dots, e_4\}$ é unha base ortonormal.

A segunda parte de memoria céntrase no estudo das variedades Einstein xeneralizadas e o fluxo renormalizado de segunda orde.

Recordamos que as variedades Einstein xeneralizadas estendían non só a condición de Einstein, senón tamén a condición de ser localmente conformemente chá. Ao igual que as variedades

Einstein homoxéneas foron clasificadas por Jensen, as localmente conformemente chás homoxéneas foron clasificadas por Takagi [63], que mostrou que eran de curvatura seccional constante ou produtos $\mathbb{R} \times N(c)$ ou $N(c) \times N(-c)$.

O estudo da condición Einstein xeneralizada no marco das métricas homoxéneas de dimensión catro faise no capítulo cuarto e está resumido no seguinte resultado.

Teorema 5.2 Sexa (M, g) unha variedade homoxénea, non simétrica e de dimensión catro. Entón o campo de tensores $W[\rho]$ anúlase se e só se (M, g) é homotética ao produto semi-directo $\mathbb{R} \ltimes H^3$ do grupo de Heisenberg con métrica invariante pola esquerda determinada pola álxebra de Lie

$$[e_1, e_2] = e_3,\ [e_4, e_1] = \mu e_1,\ [e_4, e_2] = -\frac{1}{2\mu} e_2,\ [e_4, e_3] = (\mu - \frac{1}{2\mu}) e_3,$$

con $0 < \mu \leq \frac{1}{\sqrt{2}}$, onde $\{e_1, e_2, e_3, e_4\}$ é unha base ortonormal.

Esta familia uniparamétrica de clases homotéticas contén exemplos especialmete relevantes como é o caso dos espazos 3-simétricos, que corresponden ao valor $\mu = \frac{1}{2}$.

As métricas $\mathcal{F}_t$-criticas tamén están relacionadas coa propiedade $W[\rho] = 0$ xa que a Ecuación (2.8) pódese escribir como

$$\Delta\rho - (1 + 2t)\nabla^2\tau + \frac{1 + 4t}{2}\Delta\tau g + 2\left(t + \frac{2}{3}\right)\tau\rho_0 - 2\check{\rho}_0 + 2W[\rho] = 0,$$

polo que tamén se estudan as métricas críticas que sexan Einstein xeneralizadas.

Teorema 5.9 Sexa (M, g) unha variedade de Riemann homoxénea simplemente conexa, Einstein xeneralizada e de dimensión catro. Entón, (M, g) é $\mathcal{F}_t$-crítica se e só se $t = -\frac{1}{2}$ e (M, g) é homotética ao grupo de Lie $\mathbb{R} \ltimes H^3$ con métrica invariante pola esquerda determinada pola álxebra de Lie

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\frac{1}{2} e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = \frac{1}{2} e_3,$$

onde $\{e_1, \dots, e_4\}$ é unha base ortonormal.

Tamén é interesante estudar esta condición no contexto dos fluxos. Defínese o fluxo renormalizado de segunda orde (ou fluxo RG2) coma

$$\frac{\partial}{\partial_t} g_t = -2\rho - \frac{\alpha}{2}\check{R}, \tag{6.1}$$

onde $\alpha \in \mathbb{R}$. Este fluxo obtense coma a aproximación de orde dous do fluxo de Ricci e está relacionado co tensor $\check{R}$ ([21], [33], [34], [35]). No fluxo de Ricci, toda solución auto-similar (é dicir, unha solución do fluxo da forma $g(t) = \sigma(t)\phi_t^* g$, onde σ é unha función real tal que $\sigma(0) = 1$ e ϕ_t é unha familia uniparamétrica de difeomorfismos de M) é un solitón de Ricci (unha métrica satisfacendo $\frac{1}{2}\mathcal{L}_X g + \rho = \lambda g$, onde X é un campo de vectores e $\mathcal{L}$ é a derivada de Lie). O contrario tamén se dá. Sen embargo, mentras que o primeiro é certo para o fluxo RG2, o segundo tan só o é cando o solitón é estable ($\lambda = 0$), xa que, mentras que o fluxo de Ricci mantense por homotecias, o fluxo RG2 non o fai [64].

Neste contexto, e tendo en conta (5.1), compróbase que se $W[\rho] = 0$, entón, se τ é non nula, $\rho + \frac{\alpha}{4}\check{R}$ é un múltiplo da métrica para $\alpha = -\frac{12}{\tau}$ e polo tanto dá lugar a unha solución auto-similar do fluxo RG2. Se $\tau = 0$, entón a métrica e $\check{R}$-Einstein. Se estamos na situación homoxénea, o único exemplo posible é o obtido por Arias-Marco e Kowalski, pero este non ten curvatura escalar nula, polo tanto non se podería dar este caso.

No capítulo seis, estudamos puntos fixos xenuínos de dito fluxo para métricas homoxéneas de dimensión catro. Chámaselle punto fixo xenuíno de RG2 a unha métrica que satisfai $\rho + \frac{\alpha}{4}\check{R} = 0$. É salientable o feito de que, se tomamos trazas nesta última igualdade, obtense que $\alpha = -4\tau\|R\|^{-2}$. O seguinte resultado recolle toda a casuística posible.

Teorema 6.4 Unha variedade homoxénea simplemente conexa de dimensión catro é un punto fixo xenuíno para o fluxo RG2 se e só se é Einstein, un produto $\mathbb{R} \times N^3(c)$, un produto $\mathbb{R}^2 \times N^2(c)$ ou homotética ao grupo de Lie $SU(2) \times \mathbb{R}$ con métrica invariante pola esquerda

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = \tfrac{4}{3}e_2,$$

onde $\{e_1, \dots, e_4\}$ é unha base ortonormal $\mathfrak{su}(2) \times \mathbb{R}$.

Ademais, estudamos tamén puntos fixos xeométricos, é dicir, solucións fixas do fluxo salvo difeomorfismos e homotecias. Wears deu en [64] a clasificación dos solitóns RG2 de dimensión tres no caso unimodular. Neste traballo, estendemos este resultado ao caso non unimodular e tamén demostramos o seguinte teorema, onde estudamos a existencia de solitóns RG2 alxébricos en Grupos de Lie homoxéneos de dimensión catro irreducibles. Se temos un grupo de Lie de dimensión tres H, entón podemos construír un grupo de Lie de dimensión catro $G = \mathbb{R} \times H$, e polo tanto, se H é un solitón, tamén o será G. Reciprocamente, se temos un grupo de Lie de dimensión catro G e a hipótese extra de ter un campo de vectores paralelo invariante pola esquerda, entón G pode escribirse coma $\mathbb{R} \times H$. Entón, se G é un solitón, tamén o será H. Así, se o grupo non é irreducible, basta con estudar o caso de dimensión tres. Cabe destacar que todo solitón alxébrico dá lugar a un solitón, mentras que o recíproco é un resultado aberto.

Teorema 6.6 Un grupo de Lie simplemente conexo non Einstein irreducible de dimensión catro G é un solitón alxébrico estable RG2 se e só se é homotético a un grupo de Lie determinado polas seguintes álxebras de Lie, onde $\{e_1, \dots, e_4\}$ é unha base ortonormal:

1. $\mathbb{R} \ltimes \mathfrak{e}(1,1)$, con constante $\alpha = \frac{2}{\kappa^2+1}$, dada por

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$

onde $\kappa > 0$, $\kappa \neq 1$.

2. $\mathbb{R} \ltimes \mathfrak{h}^3$, con constante $\alpha = 2$, dada por

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_1,$$
$$[e_2, e_4] = \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_2, \quad [e_3, e_4] = \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_3,$$

onde $\kappa \in [-1, 1)$.

3. $\mathbb{R} \ltimes \mathfrak{h}^3$, con constante $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$, dada por

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa} e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right) e_3,$$

onde $\kappa \in \left(0, \frac{1}{2}\right]$, $\kappa \neq \frac{1}{2}\sqrt{2-\sqrt{3}}$.

4. $\mathbb{R} \ltimes \mathfrak{r}^3$, con constante $\alpha = \frac{2(\kappa^2+\delta^2+1)}{\kappa^4+\delta^4+1}$, dada por

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2, \quad [e_3, e_4] = \delta e_3,$$

onde $(\kappa, \delta) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \backslash \{(1, 1)\}$.

5. $\mathbb{R} \ltimes \mathfrak{r}^3$, con constante $\alpha = \frac{2}{\kappa^2+p^2}$, dada por

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2 + h e_3, \quad [e_3, e_4] = -h e_2 + p e_3,$$

onde os parámetros p e h veñen dados por $p = \frac{1}{2}\left(1 + \sqrt{1 - 4\kappa(\kappa - 1)}\right)$ e $h = \left(\frac{\kappa^2(2p^2+1)+p^2-1}{2(\kappa-p)^2}\right)^{\frac{1}{2}}$, para calquera $\kappa \in (0, 1)$.

Todos os exemplos anteriores cumplen que $\rho + \frac{\alpha}{4}\check{R} = 0$. Sen embargo, non son métricas chás (nen sequera Ricci chás). Este resultado está en contraste cos solitóns de Ricci, para os cales os solitóns estables homoxéneos son métricas chás.

En canto a estudo das métricas críticas nestes dous últimos contextos, obtéñense os dous seguintes resultados.

Teorema 6.20 Sexa (M, g) un punto fixo xenuíno do fluxo renormalizado de segunda orde homoxéneo. Entón, (M, g) é $\mathcal{F}_t$-crítico se e só se:

1. (M, g) é homotético a $\mathbb{R}^2 \times N^2(c)$ e $t = -\frac{1}{2}$.
2. (M, g) é homotético a $\mathbb{R} \times N^3(c)$ e $t = -\frac{1}{3}$.

Teorema 6.21 Sexa (M, g) un solitón estable RG2 homoxéneo de dimensión catro. Entón, (M, g) é $\mathcal{F}_t$-crítico se e só se:

1. (M, g) é homotético a $\mathbb{R} \ltimes E(1, 1)$ con álxebra de Lie

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$

onde $\kappa > 0$, $\kappa \neq 1$, e $t = -\frac{1+\kappa^2}{1+3\kappa^2}$.

2. (M, g) é homotético a $\mathbb{R} \ltimes H^3$ con álxebra de Lie

$$\begin{aligned} [e_1, e_2] &= e_3, & [e_1, e_4] &= \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_1, \\ [e_2, e_4] &= \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_2, & [e_3, e_4] &= \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_3, \end{aligned}$$

onde $\kappa \in [-1, 1)$, e $t = -\frac{3(1+\kappa+\kappa^2)}{2(5+\kappa(8+5\kappa))}$.

3. (M, g) é homotético a $\mathbb{R} \ltimes \mathbb{R}^3$ con álxebra de Lie

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2, \quad [e_3, e_4] = \delta e_3,$$

onde $(\kappa, \delta) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \backslash \{(1, 1)\}$, e $t = -\frac{1+\delta^2+\kappa^2}{2(1+\delta+\delta^2+\kappa+\delta\kappa+\kappa^2)}$.

Summary and conclusions

A classical question in Riemannian geometry is the following: Given a smooth n-dimensional manifold M, is there any Riemannian structure that can be considered the "best" on it?

Einstein metrics arise as a good candidate to be those metrics due to they appear as critical metrics to the Hilbert-Einstein functional

$$\mathcal{E} : g \mapsto \int_M \tau dvol_g,$$

restricted to constant volume metrics, being τ is the scalar curvature. One can see that the gradient of this functional is given by $\nabla\mathcal{E} = -\rho + \frac{\tau}{2}g$, where ρ denotes the Ricci tensor. Due to the Gauss-Bonnet Theorem in dimension two, we obtain the universal curvature identity

$$\rho = \frac{\tau}{2}g,$$

which is no more than the Einstein condition in this dimension.

One can generalize the same for higher dimension. In dimension four, if we take the functional

$$\mathcal{F} : g \mapsto \int_M 4(||R||^2 - 4||\rho||^2 + \tau^2) dvol_g,$$

being R the curvature tensor, one can compute its gradient and look for critical metrics as in the previous case. Using the Gauss-Bonnet-Chern Theorem, Berger obtained in [4] the four-dimensional curvature identity

$$\left(\check{R} - \frac{\|R\|^2}{4}g\right) + \tau\left(\rho - \frac{\tau}{4}g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4}g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right) = 0, \tag{1.7}$$

where $\check{R}$, $\check{\rho}$ and $R[\rho]$ are the symmetric tensor fields of type $(0,2)$ given by $\check{R}_{ij} = R_{iabc}R_j{}^{abc}$, $\check{\rho}_{ij} = \rho_{ia}\rho^a{}_j$ and $R[\rho]_{ij} = R_{iabj}\rho^{ab}$. In this situation, if we assume that the metric is Einstein, then every term into brackets of (1.7) vanish. However, the questions that concerns us is the converse, if the any of the expressions into brackets which does not correspond to the Einstein one vanishes, is the metric Einstein? Thus, it arises the question about what happens when these tensors are a multiple of the metric and they are not Einstein. This leads us to define what we call from now the *weakly-Einstein conditions*.

Definition 1.11 A Riemannian manifold (M, g) is said to be

1. $\check{R}$-Einstein if $\check{R} = \frac{\|R\|^2}{n} g$.
2. $\check{\rho}$-Einstein if $\check{\rho} = \frac{\|\rho\|^2}{n} g$.
3. $R[\rho]$-Einstein if $R[\rho] = \frac{\|\rho\|^2}{n} g$.

Moreover, it is said *weakly-Einstein* if any of the conditions above is satisfied.

These conditions play an important role in the study of critical metrics for quadratic curvature functionals. If we take the functionals $\mathcal{S}(g) = \int_M \tau^2 dvol_g$ e $\mathcal{F}_t(g) = \int_M \{\|\rho\|^2 + t\tau^2\} dvol_g$ restricted to constant volume metrics, a metric is said to be $\mathcal{S}$-critical if and only if the equation

$$2\left(Hess_\tau - \frac{\Delta\tau}{4} g\right) - 2\tau\left(\rho - \frac{\tau}{4} g\right) = 0, \tag{2.7}$$

is satisfied. Analogously, a metric is said to be $\mathcal{F}_t$-critical if and only if

$$Hess_\tau - \Delta\rho + 2t\left(Hess_\tau - \frac{\Delta\tau}{4} g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right) - 2t\tau\left(\rho - \frac{\tau}{4} g\right) = 0, \tag{2.8}$$

is satisfied. One can see in these equations the relation between weakly-Einstein conditions and the study of critical metrics for these functionals.

During the development of the first part of this work, we analyse the behaviour of the weakly-Einstein conditions in different fields. This choice is not arbitrary and the purpose to study these fields is motivated before introducing the results obtained.

On the other hand, weakly-Einstein conditions have interest beyond themselves. Identity (1.7) can be written equivalently as

$$\check{R} - \frac{\|R\|^2}{4} g = \frac{1}{3}\tau\rho_0 + 2W[\rho_0], \tag{5.1}$$

where $\rho_0 = \rho - \frac{\tau}{4} g$, $W[\rho_0]_{ij} = W_{iabj}\rho_0^{ab}$ and W is the Weyl tensor. $W[\rho_0] = 0$ condition, which is equivalent to $W[\rho] = 0$ since the Weyl tensor is traceless, clearly generalizes Einstein condition, but it also generalizes locally conformally flat one ($W = 0$). Moreover, it follows from (5.1) that $W[\rho] = 0$ and being $\check{R}$-Einstein are equivalent if and only if the scalar curvature is vanishing. Another point to take into account is that the Bach tensor reduces to $W[\rho]$ if the considered metric has $\operatorname{div}_1 \operatorname{div}_4 W = 0$ (in particular, if the Weyl tensor is harmonic, and in general, when the Cotton tensor has vanishing divergence). Throughout the memoir, we call this condition generalized Einstein and we study it in the second part.

In the first chapter, we introduce some basic and technical results about Riemannian geometry, topology and algebra that we use along the work, such as the Gauss-Bonnet-Chern Theorem and Gröbner basis. We also introduce the definition of weakly-Einstein metrics and show the existence of examples that fulfil every weakly-Einstein condition without satisfying the Einstein one.

We start our analysis in the context of locally conformally flat weakly-Einstein manifolds. It is well known that if a manifold of dimension greater or equal than four satisfies this property, then its Weyl tensor vanishes identically. Moreover, if we add the Einstein condition, in that case this last one implies that the manifold is of constant sectional curvature due to the curvature tensor is totally determined by the Ricci tensor. Therefore, weakly-Einstein conditions are an intermediate step towards the classification of locally conformally flat manifolds. Because of that, in the second chapter, the following result is proved.

Theorem 2.2 Let (M, g) be a locally conformally flat Riemannian manifold. Then (M, g) is

1. $\check{R}$-Einstein if and only if one of the following holds
   - (i) $dimM = 4$ and (M, g) has vanishing scalar curvature.
   - (ii) $dimM \neq 4$ and
     - (ii.a) (M, g) is locally homothetic to a warped product of the form $\mathcal{I} \times_f N(c)$, with metric $g = dt^2 + f^2 g_N$, where $\mathcal{I}$ is a real interval and $(N(c), g_N)$ is a manifold of constant sectional curvature $c \in \{0, \pm 1\}$. Furthermore the warping function is given by
       - (ii.a.1) $f(t)^2 = t^2 - 1$, if $c = 1$, and $\mathcal{I} = (1, +\infty)$,
       - (ii.a.2) $f(t)^2 = t$, if $c = 0$, and $\mathcal{I} = (0, +\infty)$,
       - (ii.a.3) $f(t)^2 = 1 - t^2$, if $c = -1$, and $\mathcal{I} = (-\infty, 1)$.
     - (ii.b) (M, g) is locally symmetric and locally isometric to a product $M = N_1^m(c) \times N_2^m(-c)$, where $m \geq 2$.
2. $R[\rho]$-Einstein if and only if it is a product as in 1.(ii.b).
3. $\check{\rho}$-Einstein if and only if it is a product as in 1.(ii.b) or locally a warped product $\mathcal{I} \times_f \mathbb{R}^{n-1}$, with
$$f(t) = \left(\frac{2(n-1)\,(at+b)}{n} \right)^{\frac{n}{2(n-1)}},$$
with $t \in (\frac{-b}{a}, +\infty)$ and $a, b \in \mathbb{R}$.

We obtain, in this way, new examples of weakly-Einstein metrics different from the ones known so far. The classification is done up to homothety as the weakly-Einstein conditions are invariant by them.

Unlike Einstein metrics, weakly-Einstein ones do not verify a Schur's Lemma analogous. In the metrics from Theorem 2.2, the norms of the curvature tensor and the Ricci tensor need not to be necessarily constant. In fact, a five-dimensional $\check{R}$-Einstein warped product with metric given by Theorem 2.2-1.$(ii.a.1)$ has norm of the curvature tensor

$$\|R\|^2 = \frac{40}{(t^2-1)^4},$$

which is clearly non-constant.

Regarding critical metrics for quadratic curvature functionals in this part, the trivial weakly-Einstein case of a product $N^2(c) \times N^2(-c)$ is always $\mathcal{S}$-critical and $\mathcal{F}_t$-critical for $t \in \mathbb{R}$, and hence, it will be omitted during the analysis. For the remaining cases we have the following result.

Theorem 2.15 Let (M, g) be a four-dimensional locally conformally flat Riemannian manifold. Then

1. If (M, g) is $\check{R}$-Einstein, then it is $\mathcal{S}$-critical and it is $\mathcal{F}_t$-critical if and only if $\Delta\rho = -2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right)$.
2. If (M, g) is $\check{\rho}$-Einstein, then it is $\mathcal{F}_{-\frac{1}{3}}$-critical.

Another suitable field where one can study these conditions is the hypersurface one. Due to Nash Theorem [57], every Riemannian manifold is a submanifold of $\mathbb{R}^N$ for some $N \in \mathbb{N}$. The problem is trying to know when this can be a submanifold with minimum codimension, namely, a hypersurface. Einstein hypersurfaces were classified by Fialkow [25]. In the third chapter, we analyse weakly-Einstein conditions in space forms, obtaining a partial classification depending on the principal curvatures, which are the eigenvalues of the shape operator. When these eigenvalues are constant, the hypersurface is said to be isoparametric. Cecil and Ryan summarize in [15] the classification of these sort of hypersurfaces (depending on the number of principal curvatures) when the ambient space is a real space form (the Euclidean space, the sphere or the hyperbolic space). Furthermore, when one has two principal curvatures (not necessarily constant), being one of them simple and a function of the other, do Carmo and Dajczer [13] prove that this is a rotation hypersurface. The next result gather the examples obtained.

Theorem 3.1 Let (M, g) be a hypersurface in a real space form $\mathbb{Q}_c^{n+1}$, with $c = 0, \pm 1$, with two principal curvatures. If (M, g) is weakly-Einstein, then it is a product of two spheres, a product of a sphere and a hyperbolic space or a rotation hypersurface over some profile curve.

In chapter four, we focus on the study of four-dimensional homogeneous metrics. Three-dimensional weakly-Einstein homogeneous metrics have a totally different behaviour than the four-dimensional ones. Caeiro-Oliveira proved in [11] that a three-dimensional homogeneous manifolds is $\check{R}$-Einstein if and only if its Ricci operator has rank one, and thus, is isometric to an unimodular Lie group whose Lie algebra is determined by

$$[e_1, e_2] = (\lambda_1 + \lambda_2)e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2.$$

Moreover, he also proved that a three-dimensional homogeneous manifold is $\check{\rho}$-Einstein if and only if it has Ricci operator given by $Q_\rho = \operatorname{diag}[\lambda, \lambda, -\lambda]$, which implies that it has constant sectional curvature or it is homothetic to the Heisenberg group with Nil_3 geometry. Lastly, he proves that a three-dimensional homogeneous manifold is $R[\rho]$-Einstein if and only if its Ricci operator is $Q_\rho = \operatorname{diag}[\lambda, \lambda, 2\lambda]$, and consequently, it is isometric to a Berger sphere determined by the Lie algebra

$$[e_1, e_2] = \frac{4}{3}\lambda e_3, \quad [e_2, e_3] = \lambda e_1, \quad [e_3, e_1] = \lambda e_2.$$

As we have said, four-dimensional case does not behave in the same way. Arias-Marco and Kowalski [1] classified four-dimensional homogeneous manifolds satisfying $\check{R}$-Einstein condition. In this work, we are able to improve this result working up to homothetic class, showing that the $\check{R}$-Einstein are a single homothetic class. For the $\check{\rho}$-Einstein condition, the casuistic is greater. We find both, examples with vanishing scalar curvature or a uniparametric family of homothetic classes of $\check{\rho}$-Einstein homogeneous manifolds, among others. The $R[\rho]$-Einstein condition is much more rigid and it only allows that a metric fulfils this condition if it is symmetric, and thus, the product of two manifolds of constant sectional curvature, one opposite of each other, or an Einstein manifold. These results generalize Jensen's work [45], who classified four-dimensional Einstein metrics, which are symmetric. In brief, the results are gathered in the following Theorem.

Theorem 4.2 Let (M, g) be a four-dimensional simply connected homogeneous manifold. Then

(1) (M, g) is $\check{R}$-Einstein and non-symmetric if and only if it is homothetic to the Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ with left-invariant metric determined by the Lie algebra

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_4, e_3] = -e_3,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

(2) (M, g) is $\check{\rho}$-Einstein and non-symmetric if and only if it is homothetic to one of the following:

(2.a) The Lie group $SU(2) \times \mathbb{R}$ with left-invariant metric determined by the Lie algebra

$$\begin{aligned} &[e_1, e_2] = (4 \pm 2\sqrt{2})e_3, \quad &&[e_2, e_3] = (3 \pm 2\sqrt{2})e_1, \quad [e_3, e_1] = e_2, \\ &[e_4, e_1] = -e_2, &&[e_4, e_2] = (3 \pm 2\sqrt{2})e_1, \end{aligned}$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

(2.b) The Lie group $\mathbb{R} \ltimes H^3$ with left-invariant metric determined by the Lie algebra

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \frac{1}{2}e_1, \quad [e_2, e_4] = -\frac{1}{2}e_2,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

(2.c) The Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ with left-invariant metric determined by the Lie algebra

$$\begin{aligned} [e_4, e_1] &= e_1 - \frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_2 + \frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_3, \\ [e_4, e_2] &= \frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_1 + \alpha e_2 + \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_3, \\ [e_4, e_3] &= -\frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_1 - \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_2 - \tfrac{\alpha}{\alpha+1}e_3, \end{aligned}$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis and $\alpha \in (-1, 1)$, $\alpha \neq -\frac{1}{2}$, $\alpha \neq 0$.

(3) (M, g) is $R[\rho]$-Einstein if and only if it is symmetric.

Regarding the analysis of critical metrics for quadratic curvature functionals, due to in the homogeneous setting the scalar curvature is constant, the condition to be $\mathcal{S}$-critical reduces to either having vanishing scalar curvature or being Einstein, therefore, this condition reduces to a direct proof. The obtained result is the following.

Theorem 4.12 Let (M, g) be an simply connected homogeneous four dimensional weakly-Einstein Riemannian manifold. Then,

1. (M, g) is $\mathcal{S}$-critical if and only if it is homothetic the Lie group $SU(2) \times \mathbb{R}$ with left-invariant metric determined by the Lie algebra

$$\begin{aligned} &[e_1, e_2] = (4 \pm 2\sqrt{2})e_3, \quad [e_2, e_3] = (3 \pm 2\sqrt{2})e_1, \quad [e_3, e_1] = e_2, \\ &[e_4, e_1] = -e_2, \qquad\qquad\quad [e_4, e_2] = (3 \pm 2\sqrt{2})e_1. \end{aligned}$$

   where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

2. (M, g) is $\mathcal{F}_t$-critical if and only if $t = -\frac{3}{4}$ and (M, g) is homothetic to the Lie group $\mathbb{R} \times \mathbb{R}^3$ with left-invariant metric determined by the Lie algebra

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_4, e_3] = -e_3,$$

   where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

Second part of the memoir is devoted to the study of generalized Einstein manifolds and the two-loop renormalization flow.

Recall that generalized Einstein manifolds not only generalized the Einstein condition, but the locally conformally flat one too. Like homogeneous Einstein manifolds were classified by Jensen, homogeneous locally conformally flat ones were classified by Takagi [63], who showed that they were of constant sectional curvature or products $\mathbb{R} \times N(c)$ or $N(c) \times N(-c)$.

The study of generalized Einstein condition into four-dimensional homogeneous metrics setting is carried out in chapter four and it is summarized as follows.

Theorem 5.2 Let (M, g) be a non-symmetric four-dimensional homogeneous manifold. Then the tensor field $W[\rho]$ vanishes if and only if (M, g) is homothetic to a semi-direct product $\mathbb{R} \ltimes H^3$ of the Heisenberg group with left-invariant metric determined by the Lie algebra

$$[e_1, e_2] = e_3, \; [e_4, e_1] = \mu e_1, \; [e_4, e_2] = -\frac{1}{2\mu} e_2, \; [e_4, e_3] = (\mu - \frac{1}{2\mu}) e_3,$$

with $0 < \mu \leq \frac{1}{\sqrt{2}}$, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

This uniparametric family of homothetic classes includes specially relevant examples such as the case of 3-symmetric spaces, which corresponds to the value $\mu = \frac{1}{2}$.

$\mathcal{F}_t$-critical metrics are also related to $W[\rho] = 0$ property since (2.8) can be written like

$$\Delta\rho - (1+2t)\nabla^2\tau + \frac{1+4t}{2}\Delta\tau g + 2\left(t+\frac{2}{3}\right)\tau\rho_0 - 2\check{\rho}_0 + 2W[\rho] = 0,$$

so that generalized Einstein critical metrics are also studied.

Theorem 5.9 Let (M, g) be an simply connected homogeneous four dimensional generalized Einstein Riemannian manifold. Then, (M, g) is $\mathcal{F}_t$-critical if and only if $t = -\frac{1}{2}$ and (M, g) is homothetic to the Lie group $\mathbb{R} \ltimes H^3$ with left-invariant metric determined by the Lie algebra

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\frac{1}{2}e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = \frac{1}{2}e_3,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

It is also interesting studying this condition in the context of flows. It is defined the two-loop renormalization flow (or RG2 flow) by

$$\frac{\partial}{\partial_t} g_t = -2\rho - \frac{\alpha}{2}\check{R}, \tag{6.1}$$

where $\alpha \in \mathbb{R}$. This flow is given as a second order approximation of the Ricci flow and it is related with the $\check{R}$ tensor field ([21], [33], [34], [35]). In the Ricci flow, every self-similar solution (namely, a solution for the flow given by $g(t) = \sigma(t)\phi_t^* g$, where σ is a real valued function such that $\sigma(0) = 1$ and ϕ_t a uniparametric family of diffeomorphisms of M) is a Ricci soliton (a metric satisfying $\frac{1}{2}\mathcal{L}_X g + \rho = \lambda g$, where X is a vector field and $\mathcal{L}$ is de Lie derivative). The converse also holds. However, while the first is true for RG2 flow, the second is only true if the soliton is steady $(\lambda = 0)$ since the Ricci flow is preserved under homotheties, whereas the RG2 flow does not [64].

In this context, and taking into account (5.1), one can check that if $W[\rho] = 0$, then, if τ is not vanishing, $\rho + \frac{\alpha}{4}\check{R}$ is a multiple of the metric for $\alpha = -\frac{12}{\tau}$ and thus it leads us to a self-similar solution of RG2 flow. If $\tau = 0$, then the metric is $\check{R}$-Einstein. If we are in the homogeneous setting, the only possible example is the one given by Arias-Marco and Kowalski, but this one has non-vanishing scalar curvature, hence this case cannot occur.

In chapter six, we study genuine fixed points for this flow among four-dimensional homogeneous metrics. A metric is said a genuine fixed point for the RG2 flow if it satisfies $\rho + \frac{\alpha}{4}\check{R} = 0$. It is remarkable the fact that, if we take traces in this last identity, one obtains that $\alpha = -4\tau\|R\|^{-2}$. The following result gathers all the possible casuistic.

Theorem 6.4 A simply connected four-dimensional homogeneous manifold is a genuine fixed point of the RG2 flow if and only if it is Einstein, a product $\mathbb{R} \times N^3(c)$, a product $\mathbb{R}^2 \times N^2(c)$ or homothetic to the Lie group $SU(2) \times \mathbb{R}$ with left-invariant metric

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = \tfrac{4}{3}e_2,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis of $\mathfrak{su}(2) \times \mathbb{R}$.

In addition, we also study geometrical fixed points, i.e., fixed solutions of the flow up to diffeomorphisms and homotheties. Wears gave in [64] a classification of RG2 solitons in dimension three in the unimodular setting. In this work, we extend this result to the non-unimodular case and we also prove the following theorem, where we study the existence of algebraic RG2 solitons in four-dimensional irreducible homogeneous Lie groups. If one has a three-dimensional Lie group H, then you can construct a four-dimensional Lie group $G = \mathbb{R} \times H$, and so, if H is a soliton, G will be as well. Conversely, if one has a four-dimensional Lie group and the extra hypothesis of having a left-invariant parallel vector field, then G can be given by $\mathbb{R} \times H$. Hence, if G is a soliton, then H is as well. Thus, if the group is not irreducible, it is enough studying the three-dimensional case. Notice that every algebraic soliton is a soliton, whereas the converse is an open problem.

Theorem 6.6 A simply connected non-Einstein four-dimensional irreducible Lie group G is an RG2 algebraic steady soliton if and only if it is homothetic to one of the Lie groups determined by the following Lie algebras, where $\{e_1, \dots, e_4\}$ is an orthonormal basis:

1. $\mathbb{R} \ltimes \mathfrak{e}(1,1)$, for a coupling constant $\alpha = \frac{2}{\kappa^2+1}$, given by
$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$
where $\kappa > 0$, $\kappa \neq 1$.

2. $\mathbb{R} \ltimes \mathfrak{h}^3$, for a coupling constant $\alpha = 2$, given by
$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_1,$$
$$[e_2, e_4] = \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_2, \quad [e_3, e_4] = \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_3,$$
where $\kappa \in [-1, 1)$.

3. $\mathbb{R} \ltimes \mathfrak{h}^3$, for a coupling constant $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$, given by
$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa} e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right) e_3,$$
where $\kappa \in \left(0, \frac{1}{2}\right]$, $\kappa \neq \frac{1}{2}\sqrt{2-\sqrt{3}}$.

4. $\mathbb{R} \ltimes \mathfrak{r}^3$, for a coupling constant $\alpha = \frac{2(\kappa^2+\delta^2+1)}{\kappa^4+\delta^4+1}$, given by
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2, \quad [e_3, e_4] = \delta e_3,$$
where $(\kappa, \delta) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \backslash \{(1, 1)\}$.

5. $\mathbb{R} \ltimes \mathfrak{r}^3$, for a coupling constant $\alpha = \frac{2}{\kappa^2+p^2}$, given by
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2 + h e_3, \quad [e_3, e_4] = -h e_2 + p e_3,$$
where the parameters p and h are given by $p = \frac{1}{2}\left(1 + \sqrt{1 - 4\kappa(\kappa-1)}\right)$ and $h = \left(\frac{\kappa^2(2p^2+1)+p^2-1}{2(\kappa-p)^2}\right)^{\frac{1}{2}}$, for any $\kappa \in (0, 1)$.

Every example from Theorem 6.6 satisfies that $\rho+\frac{\alpha}{4}\check{R}=0$. However, they are not flat metrics (not even Ricci flat). This result is in sharp contrast with Ricci solitons, where homogeneous steady Ricci solitons are flat metrics.

Regarding the study of critical metrics in this last two situations, the following results are obtained.

Theorem 6.20 Let (M,g) be a four-dimensional homogeneous fixed point for the two-loop renormalization flow. Then, it is $\mathcal{F}_t$-critical if and only if

1. (M,g) is homothetic to $\mathbb{R}^2\times N^2(c)$ and $t=-\frac{1}{2}$.
2. (M,g) is homothetic to $\mathbb{R}\times N^3(c)$ and $t=-\frac{1}{3}$.

Theorem 6.21 Let (M,g) be a four-dimensional simply connected homogeneous RG2 algebraic steady soliton. Then, it is $\mathcal{F}_t$-critical if and only if

1. (M,g) is homothetic to $\mathbb{R}\ltimes E(1,1)$ with Lie algebra
$$[e_1,e_3]=e_2,\quad [e_2,e_3]=e_1,\quad [e_1,e_4]=\kappa e_1,\quad [e_2,e_4]=\kappa e_2,$$
where $\kappa>0$, $\kappa\neq 1$, and $t=-\frac{1+\kappa^2}{1+3\kappa^2}$.
2. (M,g) is homothetic to $\mathbb{R}\ltimes H^3$ with Lie algebra
$$\begin{aligned}[e_1,e_2]&=e_3, & [e_1,e_4]&=\tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_1,\\ [e_2,e_4]&=\tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_2, & [e_3,e_4]&=\tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_3,\end{aligned}$$
where $\kappa\in[-1,1)$, and $t=-\frac{3(1+\kappa+\kappa^2)}{2(5+\kappa(8+5\kappa)))}$.
3. (M,g) is homothetic to $\mathbb{R}\ltimes\mathbb{R}^3$ with Lie algebra
$$[e_1,e_4]=e_1,\quad [e_2,e_4]=\kappa e_2,\quad [e_3,e_4]=\delta e_3,$$
where $(\kappa,\delta)\in\{(x,y)\in\mathbb{R}^2; x\in(0,1], 0\neq y\leq x\}\backslash\{(1,1)\}$, and $t=-\frac{1+\delta^2+\kappa^2}{2(1+\delta+\delta^2+\kappa+\delta\kappa+\kappa^2)}$.

Chapter 1
Preliminaries

In this chapter we introduce the notation to be used throughout this work. Firstly, we give some basic definitions about Riemannian geometry and then present some basic results needed in subsequent chapters.

1.1 Riemannian geometry

First of all, we fix some notation. We mainly follow the books of Kühnel [47] and Lee [53].

1.1.1 Tensors and metrics

Let M^n be a n-dimensional smooth manifold. Let $\mathcal{T}_s^r(M)$ be the space of tensor fields of type (r,s) on M. If M has local coordinates $(x^1,\dots,x^n)$, then an element $T\in\mathcal{T}_s^r(M)$ can be locally written as

$$T=T_{i_1\dots i_r}^{j_1\dots j_s}\partial_{x^{j_1}}\otimes\cdots\otimes\partial_{x^{j_s}}\otimes dx^{i_1}\otimes\cdots\otimes dx^{i_r},$$

where ∂_{x^i} denotes the coordinate vector fields and dx^i denotes the dual locally defined 1-forms.

A n-dimensional smooth manifold endowed with a symmetric positive-definite $(0,2)$-tensor field g is called a Riemannian manifold.

Whenever we have a Riemannian manifold, the Levi-Civita connection is determined by the Koszul formula

$$\begin{aligned}2g(\nabla_XY,Z)=&Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)\\&-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),\end{aligned}$$

for $X,Y,Z\in\mathfrak{X}(M)$.

The metric allows us to rewrite every (r,s)-tensor field as a $(0,r+s)$-tensor field and vice versa. In particular, let $T=T_{ij}dx^i\otimes dx^j$ be a $(0,2)$-tensor field, one can define a $(1,1)$-tensor field Q_T by the relation $T(X,Y)=g(Q_T(X),Y)$. Thus, the local coordinates of T and $Q_T={Q_T}_j^i\partial_{x^i}\otimes dx^j$ are related by $T_{ij}={Q_T}_j^k g_{ki}$.

1.1.2 Curvature

Let (M,g) be a n-dimensional Riemannian manifold. The $(1,3)$-tensor field, given by

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X.Y]}Z,$$

is called the curvature tensor of M. One can define the $(0,4)$ curvature tensor as

$$R(X,Y,Z,T) = g(R(X,Y)Z,T).$$

This tensor field satisfies the following identities.

$$R(X,Y,Z,T) = -R(Y,X,Z,T) = -R(X,Y,T,Z),$$

$$R(X,Y,Z,T) = R(Z,T,X,Y),$$

$$R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0,$$

$$(\nabla_X R)(Y,Z,T,U) + (\nabla_Y R)(Z,X,T,U) + (\nabla_Z R)(Y,X,T,U) = 0.$$

We say that a $(0,4)$-tensor field on a vector space V is an algebraic curvature tensor if it satisfies the first three identities above. We define the standard algebraic curvature tensor by

$$R_0(X,Y,Z,T) = g(Y,Z)g(X,T) - g(X,Z)g(Y,T).$$

The sectional curvature of a plane $\Pi = \text{span}\{X,Y\}$ is defined by

$$K(\Pi) = \frac{R(X,Y,Y,X)}{R_0(X,Y,Y,X)}.$$

If (M,g) satisfies that $K(\Pi) = c$ for every plane $\Pi \subset TM$ at each point $p \in M$, then we say that M has constant sectional curvature. Moreover, its curvature tensor is determined by the standard curvature tensor as follows.

$$R(X,Y,Z,T) = cR_0(X,Y,Z,T).$$

A complete and simply connected Riemannian manifold with constant sectional curvature is called a space form. They are isometric to the sphere $\mathbb{S}^n$, the hyperbolic space $\mathbb{H}^n$ and the euclidean space $\mathbb{R}^n$, depending on whether the sectional curvature is positive, negative or vanishing, respectively.

(M,g) is said to be flat if every point has a neighborhood that is isometric to an euclidean space. This condition is equivalent to having vanishing curvature tensor (see [53]).

We now define the Ricci tensor, given by

$$\rho(X,Y) = \text{tr}\{Z \mapsto R(Z,X)Y\} = \sum_{i=1}^{n} g(R(E_i,X)Y,E_i) = \sum_{i=1}^{n} R(E_i,X,Y,E_i),$$

where $\{E_1,\ldots,E_n\}$ is a local orthonormal frame.

We define the Ricci operator as the $(1,1)$-tensor field given by

$$\rho(X,Y) = g(Q_\rho X,Y),$$

and define the scalar curvature of M by

$$\tau = \operatorname{tr} Q_\rho.$$

In dimension two, the scalar curvature determines the curvature tensor, whereas in dimension three, the Ricci tensor does.

A metric g is said to be Einstein if and only if the Ricci tensor is a multiple of the metric, i.e.,

$$\rho = \frac{\tau}{n} g.$$

By Schur's Lemma (see [53]), if dimension of M is greater that two, one has that if the metric is Einstein, then τ is constant.

1.1.3 Differential operators

In this section, we introduce some differential operators naturally associated to the Riemannian structure. Let $f : M \to \mathbb{R}$ be a real function. We define the gradient, which we also denote by ∇, as

$$g(\nabla f, X) = X(f).$$

Moreover we define the Hessian operator by

$$hess_f(X) = \nabla_X \nabla f,$$

and its corresponding $(0, 2)$ symmetric tensor field

$$Hess_f(X, Y) = g(hess_f(X), Y).$$

It immediately follows that

$$Hess_f(X, Y) = XY(f) - (\nabla_X Y)(f).$$

From this, we also define the Laplacian of f as

$$\Delta f = \operatorname{tr} hess_f.$$

Let T be a $(0, s)$-tensor field, the divergence of T is the $(0, s-1)$-tensor field given by

$$\operatorname{div} T(X_1, \ldots, X_{s-1}) = \sum_{i=1}^{n} (\nabla_{E_i} T)(X_1, \ldots, X_{s-1}, E_i),$$

where $\{E_i\}$ is a locally define orthonormal frame.

1.1.4 Locally conformally flat manifolds

Let A and B be two $(0,2)$ symmetric tensor fields. We define the Kulkarni-Nomizu product as

$$\begin{aligned} A\cdot B(X,Y,Z,T) =& A(Y,Z)B(X,T)-A(X,Z)B(Y,T)\\ &+A(X,T)B(Y,Z)-A(Y,T)B(X,Z). \end{aligned}$$

Remark 1.1. The standard curvature tensor can be written as $R_0=\frac{1}{2}g\cdot g$.

Define the Schouten tensor

$$S=\frac{1}{n-2}\left(\rho-\frac{\tau}{2(n-1)}g\right),$$

and thus, the Weyl tensor is given by

$$W=R-S\cdot g.$$

Equivalently, one has

$$\begin{aligned} W(X,Y,Z,T) =& R(X,Y,Z,T)\\ &+\frac{\tau}{(n-1)(n-2)}\{g(Y,Z)g(X,T)-g(X,Z)g(Y,T)\}\\ &-\frac{1}{n-2}\{\rho(Y,Z)g(X,T)-\rho(X,Z)g(Y,T)\\ &\qquad+\rho(X,T)g(Y,Z)-\rho(Y,T)g(X,Z)\}. \end{aligned}$$

We denote the Weyl tensor of type $(1,3)$ by $\mathcal{W}$. We also introduce the Cotton tensor,

$$C(X,Y,Z)=(n-2)\{(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)\}.$$

The Cotton tensor measures the symmetry of the derivative of the Schouten tensor. Moreover, the next result gives a relation between the Cotton and the Weyl tensors.

Lemma 1.2 ([53]). *Let (M,g) be a n-dimensional Riemannian manifold such that $n\geq 4$. Then*

$$\operatorname{div}W=\frac{n-3}{n-2}C.$$

Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds and let $\Phi:M_1\to M_2$ be a smooth map between them. We say that Φ is a conformal map if there exists a non-zero function $\sigma:M_1\to\mathbb{R}$ such that for all $p\in M_1$,

$$g_{2\Phi(p)}(\Phi_*(p)X,\Phi_*(p)Y)=\sigma^2(p)g_{1p}(X,Y).$$

If σ is constant (respectively $\sigma = 1$), we say that Φ is an homothety (respectively, an isometry). If we have a conformal map between M_1 and M_2 (respectively, an homothety or an isometry), we say that M_1 and M_2 are conformally related (respectively, homothetic or isometric). We refer to σ as the conformal factor.

A Riemannian manifold (M, g) is said to be locally conformally flat if for each point of the manifold, there exist a neighbourhood where it is conformally related to a flat space. If the conformal factor is defined in all M, then M is said to be conformally flat.

Conformal transformations are strongly related with the Weyl tensor.

Lemma 1.3 ([47]). *Let (M_1, g_1) and (M_2, g_2) be two conformally related Riemannian manifolds and let W_1 and W_2 be their Weyl tensors respectively. Then, $\mathcal{W}_1 = \mathcal{W}_2$ and $W_2 = e^{2\sigma} W_1$, being $\sigma : M_1 \to \mathbb{R}$ the conformal factor. In particular, if a Riemannian manifold is locally conformally flat, then $W = 0$.*

One can check that if $dim M = 3$, then $W = 0$ automatically. Thus, locally conformal flatness condition behaves different in dimension three.

Theorem 1.4 ([47]). *Let (M, g) be a n-dimensional Riemannian manifold. Then,*

- *If $n \geq 4$, M is locally conformally flat if and only if $W = 0$.*
- *If $n = 3$, M is locally conformally flat if and only if $C = 0$.*

Remark 1.5. Notice that if $n \geq 4$ and locally conformally flat, then $W = 0$, and since the Cotton tensor was a multiple of the divergence of W, $C = 0$, and thus, $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$. The converse does not hold in general.

1.2 Homogeneous spaces

We recall the classification of four-dimensional homogeneous spaces given by Bérard-Bergery.

A Riemannian manifold (M, g) is said to be homogeneous if for each pair of points $p, q \in M$, there exists an isometry Φ of (M, g) such that $\Phi(p) = q$. Consequently, every homogeneous space is complete.

Every Lie group with a left-invariant metric is homogeneous as the translation maps are isometries. Moreover, every symmetric space is homogeneous.

In dimension three, a complete simply connected manifold is homogeneous if and only if it is symmetric or a Lie group (see [54]). Since every symmetric space is locally symmetric, then the Ricci operator is parallel and either $Q_\rho = \kappa \operatorname{Id}$, with $\kappa \in \mathbb{R}$ or $Q_\rho = \operatorname{diag}[0, \kappa, \kappa]$ and (M, g) is isometric to a product $\mathbb{R} \times N(\kappa)$.

Three-dimensional Lie groups with a left-invariant metric are divided into two classes: the unimodular and the non-unimodular. Let us give a brief explanation first.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and let $\varphi : \mathfrak{h} \to \operatorname{Der}(\mathfrak{g})$ be an homomorphism from $\mathfrak{h}$ to the derivations of $\mathfrak{g}$. One can prove that there exist a Lie algebra structure in $\mathfrak{g} \oplus \mathfrak{h}$ such that $[v_1, v_2] = [v_1, v_2]_\mathfrak{g}$, $[w_1, w_2] = [w_1, w_2]_\mathfrak{h}$ and $[v, w] = \varphi(w)(v)$ for all $v_i, v \in \mathfrak{g}$ and $w_i, w \in \mathfrak{h}$.

With this bracket, $\mathfrak{g} \oplus \mathfrak{h}$ is said to be the semidirect product of $\mathfrak{g}$ by $\mathfrak{h}$ via φ and it is denoted by $\mathfrak{g} \rtimes_{\varphi} \mathfrak{h}$. This concept can be extended to Lie groups. One can see that the correspondent Lie algebra of the semidirect product $G \rtimes H$, where G, and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, is the semidirect product $\mathfrak{g} \rtimes \mathfrak{h}$.

Milnor showed in [56] that there exist an orthonormal basis $\{e_1, e_2, e_3\}$ where the Lie algebra of the unimodular case can be written as

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_1, e_3] = \lambda_2 e_2, \quad [e_2, e_3] = \lambda_3 e_3,$$

where the sign of λ_i determined the algebra. Trough the memoir, we take the simply connected Lie group associated to each Lie algebra. The casuistic is summarized as follows.

λ_1	λ_2	λ_3	Lie Group
$+$	$+$	$+$	$SU(2)$
$+$	$+$	$-$	$\widetilde{SL}(2,\mathbb{R})$
$+$	$+$	0	$\widetilde{E}(2)$
$+$	$-$	0	$E(1,1)$
$+$	0	0	H_3
0	0	0	$\mathbb{R}^3$

The description of each Lie group from the table is the following.

- $SU(2)$ is the group of unitary 2×2 matrices with determinant one. Its Lie algebra is $\mathfrak{su}(2)$, which is made up of the anti-hermitian traceless 2×2 matrices.

- $\widetilde{SL}(2,\mathbb{R})$ is the universal covering of the group of real 2×2 matrices with determinant one. Its Lie algebra is $\mathfrak{sl}(2,\mathbb{R})$, which are all the real traceless 2×2 matrices. We denote it by $SL(2,\mathbb{R})$.

- $\widetilde{E}(2)$ is the universal covering of the group of rigid motions into the Euclidean plane, which has Lie algebra $\mathfrak{e}(2)$, the euclidean algebra, given by the semidirect product $\mathfrak{r}^2 \rtimes \mathfrak{r}$, determined by an endomorphism of $\mathfrak{r}^2$ with complex eigenvalues. We denote it by $E(2)$.

- $E(1,1)$ is the group of rigid motions into the Minkowski plane. Its Lie algebra is $\mathfrak{e}_1$, the Poincaré algebra, given by the semidirect product $\mathfrak{r}^2 \rtimes \mathfrak{r}$, determined by an endomorphism of $\mathfrak{r}^2$ with real eigenvalues.

- H_3 is the Heisenberg group, which is made up of the real 3×3 matrices given by

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

whose Lie algebra, $\mathfrak{h}_3$, is the algebra of upper triangular matrices with vanishing diagonal.

Milnor also described the non-unimodular case as the three-dimensional Lie groups with Lie algebra $\mathbb{R}_{e_1} \ltimes \mathfrak{g}_1$, given by

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0,$$

such that the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

has trace $\alpha + \delta = 2$ and $\mathfrak{g}_1 = \{e_2, e_3\}$ is the abelian subalgebra.

1.2.1 Four-dimensional homogeneous spaces

Bérard-Bergery, in [3], proved the following.

Theorem 1.6. *Let M be a four-dimensional homogeneous, simply connected Riemannian manifold. Then M is either symmetric or isometric to a Lie group with a left-invariant metric. If it is a Lie group, it is either $SU(2) \times \mathbb{R}$ or $SL(2, \mathbb{R}) \times \mathbb{R}$ or a solvable Lie group, which are $\mathbb{R} \ltimes E(2)$, $\mathbb{R} \ltimes E(1, 1)$, $\mathbb{R} \ltimes H^3$ and $\mathbb{R} \ltimes \mathbb{R}^3$.*

Groups from Theorem 1.6 can be constructed from their Lie algebras in a generic way. The idea is taking a basis on a three-dimensional Lie algebra and then expanding it. In order to do that, we take a fourth vector and make a new bracket depending on the three-dimensional algebra derivations. The calculation lead us to a new orthonormal basis for a four-dimensional Lie algebra. For the solvable ones, the construction is as follows.

Take $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{g}_3$, a extension of the unimodular three-dimensional algebra $\mathfrak{g}_3$. Take an inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{g}$. Following Milnor's work [56], there exist an orthonormal basis of $\mathfrak{g}_3$, $\{v_1, v_2, v_3\}$, such that the Lie algebra $\mathfrak{g}_3$ is given by

$$[v_2, v_3] = \lambda_1 v_1, \quad [v_3, v_2] = \lambda_2 v_2, \quad [v_1, v_2] = \lambda_3 v_3, \tag{1.1}$$

and take $\mathbb{R} = span\{v_4\}$, which does not need to be orthogonal to $\mathfrak{g}_3$. Now we define the Lie brackets as

$$[v_i, v_j] = [v_i, v_j]_{\mathfrak{g}_3}, \quad [v_4, v_i] = Dv_i, \ i, j = 1, 2, 3,$$

where D is a derivation of $\mathfrak{g}_3$. Lastly, we normalize the basis. Take $k_i = \langle v_4, v_i \rangle$ and make a new basis for $\mathfrak{g}$, $\{e_1, e_2, e_3, e_4\}$, where $e_i = v_i, \ i = 1, 2, 3$ and $e_4 = \frac{1}{R}\{v_4 - \sum_{i=1}^{3} k_i v_i\}$, being $R = \sqrt{1 - \sum_{i=1}^{3} k_i^2}$. Now we have an orthonormal basis for a four-dimensional Lie algebra for any of the solvable semi-direct products from the Theorem 1.6. Once we know the brackets of the algebra, we can obtain all its geometrical properties constructing the Christoffel symbols using the Koszul formula.

As for the non-solvable cases, the construction follows the same idea but we define $[v_4, v_i] = 0$ instead.

1.2.2 Conformal transformations in homogeneous spaces

During this section, we study conformal changes between homogeneous spaces. We have seen that if two Riemannian manifolds are conformally related then their Weyl tensors are proportional. The converse is not true in general, but Hall, in [41], claims the following. Let (M, g) be a four-dimensional Riemannian manifold with Weyl tensor $\mathcal{W}_g$. If g' is other metric on M whose Weyl tensor is such that $\mathcal{W}_g = \mathcal{W}_{g'}$, then g and g' are conformally related.

On the other hand, Kulkarni showed in [48] the following.

Theorem 1.7 ([48])**.** *Let (M, g) and (M', g') be two n-dimensional ($n \geq 3$) Riemannian manifolds and let $\Phi : M \to M'$ be a curvature preserving diffeomorphism. Then Φ is a conformal transformation.*

Moreover, in the homogeneous setting we have the following result.

Lemma 1.8. *Let (M, g) and (M', g') be two n-dimensional homogeneous Riemannian manifolds and let $\Phi : M \to M'$ be a conformal transformation. Then, either Φ is an homothety or (M, g) and (M', g') are locally conformally flat.*

Proof. Let W and W' be the respective Weyl $(0, 4)$-tensor field. Recall Theorem 1.3, which said that if two Riemannian manifolds are conformally related, then their Weyl tensors are related by

$$W = e^{2\sigma} W',$$

and thus

$$\|W\|^2 = e^{4\sigma} \|W'\|^2.$$

Now, notice that these norms have to be constant because we are in a homogeneous space, and since they are related, they are constant if only if both are vanishing, and hence M and M' are locally conformally flat, or σ is constant, and then Φ is an homothety. $\square$

The converse does not hold in general. That is, if two homogeneous manifolds have the same Weyl tensor, they do not need to be homothetic, which shows a contradiction with Hall's statement. One can check the following.

Example 1.9. Let $(G_1, \langle\ ,\ \rangle_1)$ be the Lie group endowed with a inner product and with Lie algebra $\mathfrak{g}_1$ given by

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = 3e_2, \quad [e_3, e_4] = e_3$$

and $(G_2, \langle\ ,\ \rangle_2)$ the Lie group with Lie algebra $\mathfrak{g}_2$ given by

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = 4e_2, \quad [e_3, e_4] = 3e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal frame.

Now take the left-invariant metrics $\langle\ ,\ \rangle_{1^*} = 36\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_{2^*} = 90\langle\ ,\ \rangle_2$. After rescaling, both metrics have scalar curvature $\tau_1 = \tau_2 = -1$ and

$$\begin{aligned}
\mathcal{W}_1(e_1,e_2) = \mathcal{W}_2(e_1,e_2) &= E^1{}_2 - E^2{}_1,\\
\mathcal{W}_1(e_1,e_3) = \mathcal{W}_2(e_1,e_3) &= -2(E^1{}_3 - E^3{}_1),\\
\mathcal{W}_1(e_1,e_4) = \mathcal{W}_2(e_1,e_4) &= E^1{}_4 - E^4{}_1,\\
\mathcal{W}_1(e_2,e_3) = \mathcal{W}_2(e_2,e_3) &= E^2{}_3 - E^2{}_3\\
\mathcal{W}_1(e_2,e_4) = \mathcal{W}_2(e_2,e_4) &= -2(E^2{}_4 - E^4{}_2),\\
\mathcal{W}_1(e_3,e_4) = \mathcal{W}_2(e_3,e_4) &= E^3{}_4 - E^4{}_3,
\end{aligned}$$

where $E^i{}_j$ denotes the matrix with 1 in the position (i,j) and zero otherwise. By Hall's result, then both Lie groups should be homothetic but we have that

$$\|R_1\|^2 = \frac{17}{54} \quad \text{and} \quad \|R_2\|^2 = \frac{169}{675},$$

where R_i is curvature tensor of G_i. This two examples cannot be homothetic as they have different norm of the curvature tensor, which is a necessary condition since the scalar curvature was rescalated previously. This shows that a result like that of Hall cannot be expected for the Weyl tensor.

1.3 The Gauss-Bonnet-Chern Theorem: Weakly-Einstein conditions

We introduce the definition of the main topic of the work, the weakly-Einstein conditions. In order to do that, we give a generalization of the Gauss-Bonnet Theorem for greater dimension.

Let S be a compact surface. The Gauss-Bonnet theorem claims that

$$\int_S K dvol_g = 2\pi\chi(S), \tag{1.2}$$

where K is the Gaussian curvature of S and $\pi\chi(S)$ its Euler characteristic.

Now think of the Gauss-Bonnet integrand as the Riemannian functional $\mathcal{E} : g \mapsto \int_S K dvol_g$, which is known as the Hilbert-Einstein functional. Since the Euler characteristic is one of the simplest topological invariants, then every metric is critical for $\mathcal{E}$. If one takes derivatives in (1.2), we obtain

$$\nabla\mathcal{E} = \rho - \frac{\tau}{2} g = 0.$$

Thus, the two-dimensional identity

$$\rho = \frac{\tau}{2} g,$$

holds for every compact surface.

Chern, in [17], generalized the Gauss-Bonnet Theorem to even higher dimension. In dimension four, the Gauss-Bonnet-Chern integrand is

$$4\int_M (\tau^2 - 4\|\rho\|^2 + \|R\|^2) dvol_g = 32\pi^2\chi(M). \tag{1.3}$$

Again, if we define a functional with the above expression (denote it by $\mathcal{F}$), in order to study its critical metrics, we call

$$\mathcal{S}(g) = \int_M \tau^2 dvol_g, \quad \mathcal{T}(g) = \int_M \|\rho\|^2 dvol_g, \quad \mathcal{R}(g) = \int_M \|R\|^2 dvol_g$$

One has that (see [4], [14])

$$\nabla \mathcal{S} = 2\Delta\tau g + 2Hess\tau - 2\tau\rho + \frac{1}{2}\tau^2 g, \tag{1.4}$$

$$\nabla \mathcal{T} = \bar{\Delta}\rho + Hess\tau + \frac{1}{2}\Delta\tau g - 2R[\rho] + \frac{1}{2}\|\rho\|^2 g, \tag{1.5}$$

$$\nabla \mathcal{R} = 4\bar{\Delta}\rho + 2Hess\tau - 4R[\rho] - 2\check{R} + 4\check{\rho} + \frac{1}{2}\|R\|^2 g, \tag{1.6}$$

where $\bar{\Delta}\rho = \Delta\rho - 2\check{\rho} + 2R[\rho]$ and $\check{R}$, $\check{\rho}$ and $R[\rho]$ are the symmetric tensor fields of type $(0,2)$ given by $\check{R}_{ij} = R_{iabc}R_j{}^{abc}$, $\check{\rho}_{ij} = \rho_{ia}\rho^a{}_j$ and $R[\rho]_{ij} = R_{iabj}\rho^{ab}$. Again, since the Euler characteristic is a topological invariant, then $\nabla\mathcal{F} = 0$ and every metric is critical. Therefore, from (1.4), (1.5) and, (1.6), we obtain

$$\begin{aligned}
\frac{1}{4}\nabla\mathcal{F} =& \nabla\mathcal{S} - 4\nabla\mathcal{T} + \nabla\mathcal{R} \\
=& 2\Delta\tau g + 2Hess\tau - 2\tau\rho + \frac{1}{2}\tau^2 g \\
& - 4\bar{\Delta}\rho - 4Hess\tau - 2\Delta\tau g + 8R[\rho] - 2\|\rho\|^2 g \\
& + 4\bar{\Delta}\rho + 2Hess\tau - 4R[\rho] - 2\check{R} + 4\check{\rho} + \frac{1}{2}\|R\|^2 g \\
=& -2\tau\rho + \frac{1}{2}\tau^2 g + 4R[\rho] - 2\|\rho\|^2 g - 2\check{R} + 4\check{\rho} + \frac{1}{2}\|R\|^2 g = 0,
\end{aligned}$$

Reordering, we obtain the following four-dimensional curvature identity,

$$\left(\check{R} - \frac{\|R\|^2}{4} g\right) + \tau\left(\rho - \frac{\tau}{4} g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4} g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right) = 0. \tag{1.7}$$

This identity was shown by Berger in [4]. Euh, Park and Sekigawa extended it to the non-compact case [23]. Labbi was able to extend this study to higher dimension in [49].

If g is Einstein, then, in (1.7), the second, the third and the forth bracket vanish directly as $\check{\rho} = R[\rho] = \frac{\|\rho\|^2}{4} g = \frac{\tau^2}{16} g$, and therefore, the first one also vanishes as it is the only one remaining. Hence, every four-dimensional Einstein metric satisfies that the tensor fields $\check{R}$, $\check{\rho}$ and $R[\rho]$ are a multiple of the metric.

In this situation, it is a natural question asking what happens the other way round: if one of these three tensor fields is a multiple of the metric, is the metric Einstein? Euh, Park and Sekigawa gave this counterexample.

Example 1.10 ([23]). Let $M_1(c)$ and $M_2(-c)$ be two Riemannian surfaces with Gaussian curvature c and $-c$. Then, the Riemannian product $M_1(c) \times M_2(-c)$ is not Einstein but satisfies that $\check{R}$, $\check{\rho}$ and $R[\rho]$ are a multiple of the metric.

Thus, we define what we are calling the weakly-Einstein conditions.

Definition 1.11. Let (M, g) be a Riemannian manifold. Then we say that

- (M, g) is said to be $\check{R}$*-Einstein* if $\check{R} = \frac{1}{n}\|\rho\|^2 g$.
- (M, g) is said to be $\check{\rho}$*-Einstein* if $\check{\rho} = \frac{1}{n}\|\rho\|^2 g$.
- (M, g) is said to be $R[\rho]$*-Einstein* if $R[\rho] = \frac{1}{n}\|\rho\|^2 g$.

From this starting point, we classify this kind of metrics all along the memoir, assuming that they are not Einstein.

Remark 1.12. If all weakly-Einstein conditions are fulfilled at once, then (1.7) remains

$$\tau\left(\rho - \frac{\tau}{4}g\right) = 0.$$

Therefore, the metric is either Einstein or has vanishing scalar curvature.

1.4 Hypersurfaces in real space forms

Let M^n be a smooth manifold, $(\widetilde{M}^m, \widetilde{g})$ a Riemannian manifold with $n < m$ and $\iota : M \to \widetilde{M}$ the inclusion. If we endow M with the metric $g = \iota^*\widetilde{g}$, (M, g) is said to be a Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$ (which is said to be the ambient space).

The tangent bundle of $\widetilde{M}$ can be split up into a tangent and a normal space respect to M as follows,

$$T\widetilde{M} = TM \oplus NM = TM \oplus (TM)^\perp,$$

where TM is the tangent bundle of M. We define the tangential and normal projections as

$$\pi^T : T\widetilde{M}_{|M} \to TM,$$

$$\pi^\perp : T\widetilde{M}_{|M} \to NM,$$

and for every vector field X, define $X^T = \pi^T X$ and $X^\perp = \pi^\perp X$. Now we can decompose the Levi-Civita connection of $\widetilde{M}$ as its normal and tangent component,

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^T + (\widetilde{\nabla}_X Y)^\perp.$$

We define the normal component as the second fundamental form of M

$$II(X, Y) = (\widetilde{\nabla}_X Y)^\perp.$$

The Levi-Civita connection and the curvature tensor of M and $\widetilde{M}$ are related as follows.

Theorem 1.13 (The Gauss formula, [53]). *Let X, Y be vector fields on M and extend them to $\widetilde{M}$, the following holds in M.*

$$\widetilde{\nabla}_X Y = \nabla_X Y + II(X,Y).$$

Theorem 1.14 (The Gauss equation, [53]). *For any X, Y, Z, T vector fields in M, then the following equation holds*

$$R(X,Y,Z,T) = \widetilde{R}(X,Y,Z,T) + g(II(Y,Z), II(X,T)) - g(II(X,Z), II(Y,T)).$$

Assume now that M is a hypersurface (M has codimension 1) into a real space form ($\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$). Choose a distinguished unit normal vector field and denote it by N. Thus, we define the scalar second fundamental form by

$$h(X,Y) = g(II(X,Y), N),$$

where h is a $(0,2)$-symmetric tensor field. The second fundamental form can be written as $II(X,Y) = h(X,Y)N$. We can get the $(1,1)$-tensor field corresponding to h. Thus, we define the shape operator $\mathcal{S}$ of M by

$$h(X,Y) = g(\mathcal{S}X, Y).$$

The eigenvalues of the shape operator are called principal curvatures and its eigenspaces principal directions. If the principal curvatures are constant functions, then the hypersurface is said to be isoparametric. Moreover, the Gauss formula becomes

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(\mathcal{S}X, Y)N$$

and the Gauss equation

$$R(X,Y,Z,T) = cR_0(X,Y,Z,T) + g(\mathcal{S}Y,Z)g(\mathcal{S}X,T) - g(\mathcal{S}X,Z)g(\mathcal{S}Y,T),$$

where $c = -1, 0, 1$, depending on if we are into the hyperbolic space, the euclidean space or the sphere, respectively.

1.5 Warped products and Codazzi tensors

All definitions and results we give are shown in [53] and [60].

Let (B, g_B) and (F, g_F) be Riemannian manifolds. Let $f : B \to \mathbb{R}$ be a smooth function. A warped product $B \times_f F$ is a Riemannian manifold endowed with a metric

$$g = \pi_B^*(g_B) + (f \circ \pi_B)^2 \pi_F^*(g_F),$$

where $\pi_B : B \times F \to B$ and $\pi_F : B \times F \to F$ are the canonical projections.

B is said to be the base of the product and F, the fibre. The function f is said to be the warping function. Observe that if $f = 1$, then we have a Riemannian product.

The Levi-Civita connection in a warped product is given by the following result.

Proposition 1.15 ([60])**.** *Let $B \times_f F$ be a warped product and X, Y and U, V vector fields along B and F, respectively. One have the following.*

- $\nabla_X Y$ *is the lift of* $\nabla^B_X Y$ *in* B.
- $\nabla_X V = \nabla_V X = \frac{X(f)}{f} V$.
- $(\nabla_U V)^\perp = II(U, V) = -\dfrac{g(U, V)}{f} \nabla f$.
- $(\nabla_U V)^T$ *is the lift of* $\nabla^F_U V$ *on* F.

Now, we define the curvature tensor.

Proposition 1.16 ([60])**.** *Let $B \times_f F$ be a warped product with curvature tensor R and $dimF = d > 1$. If X, Y, Z and U, V, W are vector fields along B and F, respectively, then*

- $R(X, Y)Z$ *is the lift of* $R^B(X, Y)Z$ *on* B.
- $R(X, V)Y = \frac{Hess_f(X,Y)}{f} V$.
- $R(X, Y)V = R(U, V)X = 0$.
- $R(U, X)V = \frac{g(U,V)}{f} hess_f(X)$.
- $R(U, V)W = R^F(U, V)W - \frac{g(\nabla f, \nabla f)}{f^2} R_0(U, V)W$.

Moreover, the Ricci tensor is given by

- $\rho(X, Y) = \rho^B(X, Y) - \frac{d}{f} Hess_f(X, Y)$.
- $\rho(X, V) = 0$.
- $\rho(U, V) = \rho^F(U, V) - g(U, V) \left(\frac{\Delta f}{f} + (d - 1) \frac{g(\nabla f, \nabla f)}{f^2} \right)$.

1.5.1 Codazzi tensors

Let M be a Riemannian manifold such that $dimM \geq 3$. A $(0, 2)$-symmetric tensor field T is said Codazzi if it satisfy the symmetry

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z).$$

Analogously, a $(1, 1)$-tensor field Q_T is Codazzi if

$$(\nabla_X Q_T)Y = (\nabla_Y Q_T)X.$$

Example 1.17.

1. A parallel tensor field, i.e, $\nabla T = 0$, is Codazzi.
2. If (M, g) is locally conformally flat and $dimM = 3$, then the Schouten tensor S is Codazzi by Theorem 1.4. If $dimM \geq 4$, then, by Remark 1.5, S is also Codazzi. Therefore, if (M, g) is locally conformally flat, the Schouten tensor is Codazzi.

1.6 Gröbner basis

During the work, we find systems of polynomials that cannot be solved using the classical methods, so we have to look for another ways to work with them. Gröbner basis give us a really powerful tool for this. The main idea of Gröbner basis is constructing an ideal with the polynomials of a given system, dividing them and eliminating those which are redundant, getting a new system which is equivalent to the first one but with simpler factors. For more details about Gröbner basis, see [22].

Let $\mathbb{F}$ be a field and denote by $\mathbb{F}[x_1, \ldots, x_n]$ the ring of polynomials with variables $x_1, \ldots, x_n$. A monomial ordering in $\mathbb{F}[x_1, \ldots, x_n]$ is a relation on the set of monomials $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, with $\alpha \in \mathbb{Z}^n_{\geq 0}$, denoted by $>$, such that

- The relation is a total ordering in $\mathbb{Z}^n_{\geq 0}$.
- If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.
- The relation is a well-ordering in $\mathbb{Z}^n_{\geq 0}$, i.e., there is a smallest element under the relation.

Example 1.18.

1. Lexicographical order. We say that $\alpha >_{lex} \beta$ if the leftmost nonzero entry in $\alpha - \beta \in \mathbb{Z}^n$ is positive.
2. Graded lexicographical order. We say that $\alpha >_{grlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and $\alpha >_{lex} \beta$, where $|\alpha| = \sum_{i=1}^n \alpha_i$.
3. Graded reverse lexicographical order. We say that $\alpha >_{grevlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost nonzero entry in $\alpha - \beta$ is positive.

Let $p = \sum_\alpha a_\alpha x^\alpha$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$, $a_\alpha \in \mathbb{F}$ and $>$ be a monomial ordering. The multidegree of p is

$$mldeg(p) = \max\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid a_\alpha \neq 0\}.$$

We call leading term of p to

$$LT(p) = a_{mldeg(p)} x^{mldeg(p)}.$$

Theorem 1.19 (The division algorithm, [22]). *Let $>$ be a monomial ordering and let $Q = \{q_1, \ldots, q_s\}$ be an ordered s-tuple of polynomials in $\mathbb{F}[x_1, \ldots, x_n]$. Then every $p \in \mathbb{F}[x_1, \ldots, x_n]$ can be written as*

$$p = f_1 q_1 + \cdots + f_s q_s + r,$$

where $f_i \in \mathbb{F}[x_1, \ldots, x_n]$ and r either can be zero or it is a linear combination, with coefficients in $\mathbb{F}$, of monomials such that none of them are divisible by any of $LT(q_i)$. We call r the remainder of p on the division by Q. In addition, if $q_i f_i \neq 0$, then $mldeg(p) \geq mldeg(q_i f_i)$.

The division algorithm is an important part in Gröbner basis theory since it is essential to prove their existence .

A monomial ideal $I \subset \mathbb{F}[x_1, \ldots, x_n]$ is an ideal such that all its polynomials are finite sums of the form $\sum_\alpha h_\alpha x^\alpha$, where $h_\alpha \in \mathbb{F}[x_1, \ldots, x_n]$. We write $I = \langle x^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0} \rangle$.

Lemma 1.20 ([22]). *Let $I = \langle x^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0} \rangle$ be a monomial ideal. Then, $x^\beta \in I$ if and only if x^α divides x^β.*

Lemma 1.21 ([22]). *Let I be a monomial ideal and let $p \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial. The following are equivalent.*

$i)$ $p \in I$.

$ii)$ *Every term of p are in I.*

$iii)$ *p is a $\mathbb{F}$-linear combination of elements of I.*

One can get a basis of I such that x_i^α does not divide x_j^α whenever $i \neq j$. This basis is unique and we say that is a minimal basis of I.

Now denote by $LT(I)$ the set of leading terms of the elements of I and $\langle LT(I) \rangle$ the monomial ideal generated by $LT(I)$.

Fix a monomial ordering on $\mathbb{F}[x_1, \ldots, x_n]$. A finite subset $\mathcal{G} = \{g_1, \ldots, g_r\}$ of an ideal I (not necessarily monomial) is said to be a Gröbner basis if

$$\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_r) \rangle.$$

Now we shall see that there always exist such a basis.

Proposition 1.22 ([22]). *Let $I \subset \mathbb{F}[x_1, \ldots, x_n]$ be a non-zero ideal (not necessarily monomial). Then there exist polynomials $p_1 \ldots, p_t \in I$ such that*

$$\langle LT(I) \rangle = \langle LT(p_1), \ldots, LT(p_t) \rangle.$$

Theorem 1.23 (Hilbert basis Theorem, [22]). *Every ideal $I \subset \mathbb{F}[x_1, \ldots, x_n]$ has a finite generating set, that is, $I = \langle p_1, \ldots, p_t \rangle$ for some $p_1, \ldots, p_t \in I$.*

Proof. If $I = \{0\}$ the statement is trivial. If it is non-zero, then I has a $\langle LT(I) \rangle$. By proposition 1.22, there are polynomials $p_1 \ldots, p_t \in I$ such that $\langle LT(I) \rangle = \langle LT(p_1), \ldots, LT(p_t) \rangle$. Since all the p_i are in I, then the ideal generated by those is included in I, so let us see the converse.

Let $f \in I$ be any polynomial, we apply the division algorithm. Then we can write

$$f = q_1 p_1 + \cdots + q_s p_s + r,$$

where r is not divisible by any of $LT(p_i)$.

If we proof that $r = 0$, then, by Lemma 1.21, f is a linear combination of elements of $\langle p_1, \ldots, p_s \rangle$ and so f is in that ideal. As f is an arbitrary element of I, then $I \subset \langle p_1, \ldots, p_s \rangle$ and the result follows. Thus, let us see that $r = 0$. We have that

$$r = f - (q_1 p_1 + \cdots + q_s p_s) \in I.$$

Then, if $r \neq 0$, $LT(r) \in \langle LT(I) \rangle = \langle LT(p_1), \ldots, LT(p_t) \rangle$. Then, by Lemma 1.20, $LT(r)$ is divisible by some of the $LT(p_i)$, but r was the remainder, so we get a contradiction and thus, $r = 0$. □

As a consequence of this result we have the following.

Corollary 1.24 ([22]). *Every ideal has a Gröbner basis. Moreover, this basis is also a basis of the ideal.*

Proof. Given a set $G = \{g_1, \ldots, g_r\}$ as in the proof of the previous Theorem is a Gröbner basis by definition. Moreover, the argument given in that proof also assure us that $\langle G \rangle = I$, so G is a basis of I. □

Once we have established what a Gröbner basis is, one may think of its computation. There is different ways to do it, but the original was given by Buchberger, which works as following.

1. Take an ideal $I \subset \mathbb{F}[x_1, \ldots, x_n]$. Take a pair of generators $g_i,\ g_j \in I$ with multidegrees α and β respectively.
2. Calculate
$$S(g_i, g_j) = \frac{x^\gamma}{LT(g_i)} g_i - \frac{x^\gamma}{LT(g_j)} g_j,$$
where $\gamma = max(\alpha_i, \beta_i)$.
3. Compute the reminder of the division of $S(g_i, g_j)$ by all the polynomials of I. If it is not zero, add $S(g_i, g_j)$ to I.
4. Repeat until all the reminders of all $S(g_i, g_j)$ are zero.

Part I

Weakly-Einstein Conditions

Einstein metrics are a large research field in Riemannian geometry. They arise from the study of critical metrics for the Hilbert-Einstein functional, which was given by

$$\mathcal{E} : g \mapsto \int_M \tau_g dvol_g,$$

and try to answer the question about if there exist a "perfect" metric into a Riemannian manifold. Einstein metrics are the best candidate to this as they "distribute" the metric along the manifold homogeneously. Henceforth, this field has been applied to several contexts in differential geometry and physics.

Moreover, being Einstein is a strong condition due to it implies other different properties. For instance, a locally conformally flat Einstein manifold has constant sectional curvature. It also implies the constancy of the scalar curvature due to the Schur Lemma. To see more topics about Einstein manifolds, see [5, 47].

Thus, it is a natural goal trying to weaken the Einstein condition in order to obtain new geometric objects. There are a few known generalizations of this condition.

Since the Ricci tensor of an Einstein metric is parallel, one may think about linear generalizations. Gray, in [37], considered this situation, where they appear metrics with cyclic-parallel Ricci tensor and Codazzi Ricci tensor.

Furthermore, if there is an Einstein metric into the conformal class of a given one, then this metric is called conformally Einstein. Brinkmann [7] introduced necessary and sufficient conditions for this in terms of positive solutions for a differential equation. More generalizations come from the notion of Ricci solitons, which are self-similar solutions of the Ricci flow. A metric is said to be a Ricci soliton if

$$\mathcal{L}_X g + \rho = \lambda g,$$

for some $X \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$, where $\mathcal{L}$ is the Lie derivative. If $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, the soliton is called expanding, steady or shrinking, respectively.

Recall the identity (1.7), given by Berger in [4],

$$\left(\check{R} - \frac{\|R\|^2}{4} g\right) + \tau\left(\rho - \frac{\tau}{4} g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4} g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right) = 0.$$

As we have said in the first chapter, this identity implies that if the metrics is Einstein, then the terms into brackets which are not related to the Einstein condition vanish, and thus, each tensor within is a multiple of the metric. The example given by Euh, Park and Sekigawa showed that the converse does not happen. This shows the existence of $\check{R}$, $\check{\rho}$ and $R[\rho]$-Einstein metrics which are not Einstein. Before that, Gray and Willmore considered the Einstein and the $\check{R}$-Einstein condition together (what they called super-Einstein manifold) into mean-value theorems in Riemannian geometry [38]. Chen and Vanhecke studied this in the context of geodesic spheres and tangent unit bundles [6, 16].

Singer and Thorpe [62] proved that a four-dimensional Riemannian manifold is Einstein if and only if there exists a Singer-Thorpe basis at each point of the manifold, which is an orthonormal basis satisfying

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}, \quad R_{1414} = R_{2323},$$

and $R_{ijjk} = 0$ for all $i, j, k \in \{1, \ldots, 4\}$. Euh, Park and Sekigawa, in [24], gave a first generalization of this, giving explicit conditions for the curvature tensor, which they call a Singer-Thorpe generalized basis. They proved that a four-dimensional Riemannian manifold is $\check{R}$-Einstein if and only if it has an orthonormal basis which satisfies

$$R_{1212}^2 = R_{3434}^2, \quad R_{1313}^2 = R_{2424}^2, \quad R_{1414}^2 = R_{2323}^2,$$

and $R_{ijjk} = 0$ for all $i, j, k \in \{1, \ldots, 4\}$.

Following this path, Arias-Marco and Kowalski, in [1], classified $\check{R}$-Einstein condition for four-dimensional homogeneous Riemannian manifolds, generalizing the classification of four-dimensional homogeneous Einstein metrics given by Jensen [45].

In dimension three, there is an analogous of this identity. Since every three-dimensional manifold can be embedded in a four-dimensional space by taking $M \times \mathbb{R}$, any three-dimensional manifold satisfies that

$$\frac{2}{3}\tau\left(\rho - \frac{\tau}{3}g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{3}g\right) - \left(R[\rho] - \frac{\|\rho\|^2}{3}g\right) = 0.$$

Haji-Badali, Atashpeykar and Zaeim study all weakly-Einstein conditions in dimension three for the Lorentzian setting [39, 40].

During the memoir, we analyse the weakly-Einstein conditions, and then, we study their relations with other generalizations, which will be introduced when necessary for the comfort of the reader.

The aim of this fist part is classifying all three weakly-Einstein conditions in different fields. Our aim is giving geometrical structures satisfying these conditions in order to create new examples of weakly-Einstein manifolds.

Observe that we define weakly-Einstein manifolds in a slightly different way. In [23] and [24], the definition given for the weakly-Einstein condition is what we call $\check{R}$-Einstein, whereas we use the name of weakly-Einstein for all conditions mentioned and then we specify each one by themselves.

We can split this first part into two: In chapter two, we give a classification for weakly-Einstein locally conformally flat manifolds. In order to do that, we take each condition separately and we study the algebraic structure of each tensor, which depends on the Ricci operator. From there, we stablish algebraic conditions, up to dimension, for the Ricci curvatures and then we study the geometric structure via the Schouten tensor, which, in the locally conformally flat case, is Codazzi.

In chapter three, following the same structure, we classify weakly-Einstein hypersurfaces in real space forms ($\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$), which presents a much harder problem than the previous one since the curvature tensor in a space form depends on the shape operator, which gives a quartic equation on the principal curvatures.

The second path follows Arias-Marco and Kowalski's work. We complete the classification of all weakly-Einstein four-dimensional homogeneous Riemannian metrics. For the Lie-group setting, we find non-linear systems of polynomials depending on the structure constants of the group, which is a hard problem using classic methods. Therefore, we use Gröbner basis theory on these systems in order to get "better" polynomials and then we classify each family up to homothety class.

Metrics given in this part are new examples of $\check{R}$ and $\check{\rho}$-Einstein metrics. $R[\rho]$-Einstein condition is much more rigid, so the only non-Einstein example that appears is the product of two manifolds of opposite sectional curvature, which was known already.

Chapter 2
Locally conformally flat weakly-Einstein manifolds

In this chapter, we classify weakly-Einstein locally conformally flat Riemannian manifolds. Some results of this chapter are shown in [28].

2.1 Introduction

In the previous chapter we presented locally conformally flat metrics as those where the Weyl tensor vanishes, unless for dimension three, where the Cotton tensor does. Having this property allows us to simplify the study of the curvature tensor since the Weyl tensor is given by

$$\begin{aligned}W(X,Y,Z,T) =& R(X,Y,Z,T)\\ &+\frac{\tau}{(n-1)(n-2)}\{g(Y,Z)g(X,T)-g(X,Z)g(Y,T)\}\\ &-\frac{1}{n-2}\{\rho(Y,Z)g(X,T)-\rho(X,Z)g(Y,T)\\ &\qquad+\rho(X,T)g(Y,Z)-\rho(Y,T)g(X,Z)\}.\end{aligned}$$

Hence, if it is vanishing, we have that the curvature tensor is given by

$$\begin{aligned}R(X,Y,Z,T) \;=\; & -\frac{\tau}{(n-1)(n-2)}\{g(Y,Z)g(X,T)-g(X,Z)g(Y,T)\}\\ &+\frac{1}{n-2}\{\rho(Y,Z)g(X,T)-\rho(X,Z)g(Y,T)\\ &\qquad+\rho(X,T)g(Y,Z)-\rho(Y,T)g(X,Z)\}.\end{aligned} \tag{2.1}$$

In this situation, the study of algebraic properties of the curvature tensor is simpler as it only depends on the Ricci tensor. In fact, some properties can be easily seen directly. For instance, if the metric is also Einstein, then it is of constant curvature. Other examples shall be shown.

Example 2.1.

1. Any two-dimensional Riemannian manifold is locally conformally flat.

2. Any Riemannian manifold of constant sectional curvature is locally conformally flat.
3. A Riemannian product $M_1(c_1) \times M_2(c_2)$, where $M_i(c_i)$ is a Riemannian manifold of constant sectional curvature c_i, is locally conformally flat if and only if $c_1 = -c_2$ [65].
4. A warped product $\mathbb{R} \times_f N(c)$ is locally conformally flat [8].

During this chapter, we describe weakly-Einstein locally conformally flat metrics in the following way. First of all, we give a description of the algebraic conditions a metric of this kind has to satisfy to be weakly-Einstein. We do this by solving a polynomial equation depending on the Ricci tensor. Then, once we stablish its algebraic structure, we show that the manifold can only realize as items 3 or 4 from Example 2.1 or, for the $\check{R}$-Einstein condition, as a four-dimensional manifold with vanishing scalar curvature.

The main purpose of this chapter is proving the next result.

Theorem 2.2. *Let (M, g) be a locally conformally flat Riemannian manifold. Then (M, g) is*

1. *$\check{R}$-Einstein if and only if one of the following holds*
   - *(i) $dimM = 4$ and (M, g) has vanishing scalar curvature.*
   - *(ii) $dimM \neq 4$ and*
     - *(ii.a) (M, g) is locally homothetic to a warped product of the form $\mathcal{I} \times_f N(c)$, with metric $g = dt^2 + f^2 g_N$, where $\mathcal{I}$ is a real interval and $(N(c), g_N)$ is a manifold of constant sectional curvature $c \in \{0, \pm 1\}$. Furthermore the warping function is given by*
       - *(ii.a.1) $f(t)^2 = t^2 - 1$, if $c = 1$, and $\mathcal{I} = (1, +\infty)$,*
       - *(ii.a.2) $f(t)^2 = t$, if $c = 0$, and $\mathcal{I} = (0, +\infty)$,*
       - *(ii.a.3) $f(t)^2 = 1 - t^2$, if $c = -1$, and $\mathcal{I} = (-\infty, 1)$.*
     - *(ii.b) (M, g) is locally symmetric and locally isometric to a product $M = N_1^m(c) \times N_2^m(-c)$, where $m \geq 2$.*
2. *$R[\rho]$-Einstein if and only if it is a product as in 1.(ii.b).*
3. *$\check{\rho}$-Einstein if and only if it is a product as in 1.(ii.b) or locally a warped product $\mathcal{I} \times_f \mathbb{R}^{n-1}$, with*
$$f(t) = \left(\frac{2(n-1)\,(at+b)}{n} \right)^{\frac{n}{2(n-1)}},$$
with $t \in (\frac{-b}{a}, +\infty)$ and $a, b \in \mathbb{R}$.

2.2 $\check{R}$-Einstein locally conformally flat metrics

Firstly, we may study the three-dimensional case since the locally conformal flatness property is slightly different for this dimension and the curvature tensor of every three-dimensional Riemannian manifold is determined by its Ricci tensor.

2.2.1 Three-dimensional case

The curvature of all three-dimensional Riemannian manifold directly depends on the Ricci operator as the Weyl tensor is always vanishing. It does not matter that it is locally conformally flat or not. Thus, we start taking any three-dimensional Riemannian manifold and try to figure out its algebraic structure assuming that it satisfies the $\check{R}$-Einstein condition. This structure is as follows.

Lemma 2.3. *Let (M, g) be a three-dimensional Riemannian manifold. Then it is $\check{R}$-Einstein if and only if its Ricci operator is of rank one.*

Proof. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame of eigenvectors for the Ricci operator Q_ρ, i.e, an orthonormal frame such that $Q_\rho(e_i) = \lambda_i e_i$, where λ_i is the corresponding eigenvalue of e_i. Since the curvature tensor was given by (2.1), then

$$R_{ijkl} = 2(-\tau + 2(\lambda_i + \lambda_j))\{\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}\},$$

where δ_{ij} is the Kronocker delta. Therefore, $R_{ikkj} = 0$ and if we compute the $\check{R}$ tensor in terms of the frame, we obtain

$$\begin{aligned}\check{R}(e_\alpha, e_\beta) = &\sum_{i,j,k=1}^{3} R(e_\alpha, e_i, e_j, e_k)R(e_\beta, e_i, e_j, e_k)\\ =&\sum_{i=1}^{3} R(e_\alpha, e_i, e_i, e_\alpha)R_(e_\beta, e_i, e_i, e_\alpha)\\ &+\sum_{i=1}^{3} R(e_\alpha, e_i, e_\alpha, e_i)R_(e_\beta, e_i, e_\alpha, e_i).\end{aligned}$$

Since we cannot have three different indices, then $\alpha = \beta$ necessarily. So an orthonormal frame for the Ricci operator diagonalizes the $\check{R}$ operator. Moreover, due to the symmetry $R_{ijkl} = -R_{jikl}$ of the curvature tensor we finally have that

$$\check{R}(e_\alpha, e_\alpha) = 2\sum_{i\neq\alpha} R^2_{\alpha ii\alpha}.$$

Therefore, by (2.1), we have the following.

$$\begin{aligned}\check{R}(e_\alpha, e_\alpha) =&2\sum_{i\neq\alpha}\left(-\frac{\tau}{2} + (\lambda_\alpha + \lambda_i)\right)^2\\ =&2\sum_{i\neq\alpha}\left\{\frac{\tau^2}{4} + (\lambda_\alpha^2 + \lambda_i^2 + 2\lambda_\alpha\lambda_i) - \tau(\lambda_\alpha + \lambda_i)\right\}\\ =&2\left\{\frac{\tau^2}{2} + \|\rho\|^2 + 2\tau\lambda_\alpha - \lambda_\alpha^2 - \tau\{\lambda_\alpha + \tau\}\right\}\\ =&2\|\rho\|^2 - \tau^2 + 2\lambda_\alpha(\tau - \lambda_\alpha).\end{aligned}$$

Now, notice that we want $\check{R}$ to be a multiple of the metric, so we need that $\check{R}(e_\alpha, e_\alpha) = \check{R}(e_\beta, e_\beta)$, for all α, β. Thus, we obtain the system of equations

$$(\lambda_\alpha - \lambda_\beta)\{\tau - (\lambda_\alpha + \lambda_\beta)\} = 0 \tag{2.2}$$

As τ is the sum of the three eigenvalues, system (2.2) holds if and only if the Ricci operator is a multiple of the identity, in which case it is Einstein, or it is of rank one. □

2.2.2 Higher-dimensional case

Now assume that (M, g) is a n-dimensional locally conformally flat Riemannian manifold with $n \geq 4$. We work again with the Ricci operator to obtain an equation which has to satisfy.

Lemma 2.4. *A n-dimensional locally conformally flat Riemannian manifold with $n \geq 4$ is $\check{R}$-Einstein if and only if one of the following holds*

(i) $\dim M = 4$ *and the scalar curvature vanishes, or*

(ii) $\dim M \geq 5$ *and the Ricci operator has exactly two-distinct eigenvalues λ and $\mu = -\frac{(n-4)(n-1)+2m}{(n-4)(n-1)+2(n-m)}\lambda$, where m is the multiplicity of the eigenvalue λ.*

Proof. As M is locally conformally flat, then the curvature tensor is written like (2.1). Thus, the curvature tensor depends on the Ricci tensor. If we compute $\check{R}$, we obtain

$$\begin{aligned}
\check{R}(X,Y) =& \sum_{i,j,k} R(X,e_i,e_j,e_k)R(Y,e_i,e_j,e_k) \\
=& \sum_{i,j,k} \Big(-\frac{\tau}{(n-2)(n-1)}\{g(e_i,e_j)g(X,e_k) - g(X,e_j)(e_i,e_k)\} \\
&+ \frac{1}{(n-2)}\{\rho(e_i,e_j)g(X,e_k) - \rho(X,e_j)g(e_i,e_k) \\
&\qquad + \rho(X,e_k)g(e_i,e_j) - \rho(e_i,e_k)g(X,e_k)\}\Big) \\
&\times\Big(-\frac{\tau}{(n-2)(n-1)}\{g(e_i,e_j)g(Y,e_k) - g(Y,e_j)(e_i,e_k)\} \\
&+ \frac{1}{(n-2)}\{\rho(e_i,e_j)g(Y,e_k) - \rho(Y,e_j)g(e_i,e_k) \\
&\qquad + \rho(Y,e_k)g(e_i,e_j) - \rho(e_i,e_k)g(Y,e_k)\}\Big).
\end{aligned}$$

Expanding each term, we get that

$$\begin{aligned}
\check{R}(X,Y) =& \frac{2}{(n-2)^2}\{(n-4)\rho^2(X,Y) + \frac{2\tau}{(n-1)}\rho(X,Y) \\
&+ \frac{(n-1)\|\rho\|^2 - \tau^2}{(n-1)}g(X,Y)\}.
\end{aligned}$$

Hence, the corresponding operator $Q_{\check{R}}$ is

$$Q_{\check{R}} = \frac{2}{(n-2)^2}\left\{(n-4)Q_\rho^2 + \frac{2\tau}{(n-1)}Q_\rho + \frac{(n-1)\|\rho\|^2-\tau^2}{(n-1)}\,\mathrm{Id}\right\}.$$

Now we compute its trace, which is $||R||^2$,

$$||R||^2 = \operatorname{tr} Q_{\check{R}} = \frac{2}{(n-2)^2}\left\{(n-4)\|\rho\|^2 + \frac{2\tau^2}{(n-1)} + n\frac{(n-1)\|\rho\|^2-\tau^2}{(n-1)}\right\}.$$

Furthermore, (M,g) is $\check{R}$-Einstein if and only if $\check{R} = \frac{||R||^2}{n}g$, then, we obtain the equation

$$(n-4)Q_\rho^2 + \frac{2\tau}{n-1}Q_\rho - \left(\frac{(n-4)}{n}\|\rho\|^2 + \frac{2\tau^2}{n(n-1)}\right)\mathrm{Id} = 0. \tag{2.3}$$

The eigenvalues of the Ricci operator must satisfy this equation, so we can only have two different at most. Notice that not every solution of the equation has to be an eigenvalue in general. Therefore, we can have just one or two, but in the first case, the metric would be Einstein, so we need that they are exactly two.

If $n=4$, then the equation becomes

$$\tau\left(Q_\rho - \frac{\tau}{4}\,\mathrm{Id}\right) = 0,$$

so either the metric is Einstein or it has vanishing scalar curvature.

In higher dimension, due to the Vieta formulae [27], if we have two different eigenvalues, we have the following relation.

$$\lambda + \mu = -\frac{2\tau}{(n-4)(n-1)},$$

with λ and μ eigenvalues of Q_ρ. Since $\tau = m\lambda + (n-m)\mu$, being m the multiplicity of λ, we get to

$$\mu = -\frac{2m+(n-1)(n-4)}{2(n-m)+(n-1)(n-4)}\lambda.$$

This completes the proof. □

2.2.3 Geometric structure

Let us recall the Schouten tensor

$$S = \frac{1}{n-2}\left(\rho - \frac{\tau}{2(n-1)}g\right).$$

In dimension three, a metric is locally conformally flat if and only if S is a Codazzi tensor. Moreover, as we said in Remark 1.5, in higher dimension, if the metric is locally conformally flat, then $W=0$, so the Cotton tensor is also vanishing and thus S is Codazzi.

Now, if we assume that the metric is also $\check{R}$-Einstein, then we have shown that either the Ricci operator is rank one or it has two different eigenvalues, one a scalar multiple of the other. Knowing this, we use the following result on Codazzi tensors given by Merton [55] to finish the classification.

Theorem 2.5 ([55]). *Let (M,g) be a Riemannian manifold such that $dimM \geq 3$ and T a Codazzi tensor on (M,g). Let λ be an eigenfunction of T with eigenspace V_λ. If $dimV_\lambda \geq 2$, then $\nabla\lambda$ is orthogonal to V_λ. In addition, if T has exactly two different eigenfunctions λ and μ with $dimV_\lambda \leq dimV_\mu$, then*

- *(i) M is locally a product if $dimV_\lambda \geq 2$.*
- *(ii) M is locally a warped product with one-dimensional base and non-trivial warping function if and only if*
  - *(ii.a) $dimV_\lambda = 1$,*
  - *(ii.b) the eigenfunction μ is not constant and $\nabla\lambda$ is orthogonal to V_μ.*

The next result includes all possibilities of geometric realizations for the algebraic structures we obtained in the previous Lemma.

Lemma 2.6. *Let (M^n,g) be a $\check{R}$-Einstein locally conformally flat Riemannian manifold with $n \neq 4$. Then*

- *(i) If the Ricci eigenvalues λ and μ have multiplicity greater than one, then (M,g) is locally symmetric and locally isometric to a product $N_1^m(c) \times N_2^m(-c)$.*
- *(ii) If the Ricci curvature λ has multiplicity one, then (M,g) is locally isometric to a warped product of the form $\mathbb{R} \times_f F$, where (F,g_F) is of constant curvature.*

Proof. First, we are focusing on the case of dimension greater than four.

Assume that $dimM \geq 5$. It follows from Lemma 2.4 that the scalar curvature is given by

$$\tau = -\frac{(n-4)(n-1)(n-2m)}{(n-4)(n-1)+2(n-m)}\lambda.$$

Also from the same Lemma, as the Ricci operator had two eigenvalues, we obtain that the Schouten tensor has also two-distinct eigenvalues given by

$$\bar{\lambda} = \frac{2m-3n+4}{4m-2n^2+6n-8}\lambda\,, \quad \bar{\mu} = \frac{2m+n-4}{4m-2n^2+6n-8}\lambda = \frac{2m+n-4}{2m-3n+4}\bar{\lambda}\,,$$

where m is the multiplicity of λ. We call $V_{\bar{\lambda}}$ to the eigenspace associated to $\bar{\lambda}$ (respectively with $\bar{\mu}$). Assume that $dimV_{\bar{\lambda}} \leq dimV_{\bar{\mu}}$.

If $dimV_{\bar{\lambda}} \geq 2$ (and thus, $V_{\bar{\mu}} \geq 2$), since the Schouten is a Codazzi tensor, then, by Theorem 2.5, (M,g) is locally a product and by Theorem 4 from [65], locally conformal flatness implies that $M = N_1(c) \times N_2(-c)$, where $N_i(k)$ is a Riemannian manifold of constant sectional curvature

k. Now let $dimN_i = m_i$. A Riemannian product $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ of manifolds with constant sectional curvature has a $\check{R}$ operator

$$\begin{pmatrix} \check{R}_1 & 0 \\ 0 & \check{R}_2 \end{pmatrix}.$$

Moreover, if $N(k)$ is a Riemannian manifold of constant sectional curvature, then its curvature tensor is given by $R_{N(k)} = kR_0$, and thus, its $\check{R}$ tensor is

$$\check{R}(X,Y) = k^2 \sum_{i,j,k}^{n} R_0(X, e_i, e_j, e_k) R_0(Y, e_i, e_j, e_k) = 2k^2(dimN - 1)g(X,Y),$$

where the $\{e_i\}$ is an orthonormal frame. Therefore, a product $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ is $\check{R}$-Einstein if and only f

$$c_1^2(n_1 - 1) = c_2^2(n_2 - 1).$$

Consequently, if $c_1 = -c_2$, then the product is $\check{R}$-Einstein if and only if $m_1 = m_2$. This completes Assertion 1.(i)

Assume now that $dimV_{\bar{\lambda}} = 1$ and $dimV_{\bar{\mu}} = n - 1$. By Theorem 2.5, $\nabla\bar{\mu}$ is orthogonal to $V_{\bar{\mu}}$ and, as $\bar{\mu}$ is a scalar multiple of $\bar{\lambda}$, then $\nabla\bar{\lambda}$ is orthogonal to $V_{\bar{\mu}}$. Moreover $\bar{\mu}$ cannot be constant because if it is, then $\bar{\lambda}$ is as well, and then, the Ricci operator would be parallel, implying that the manifold would be 0-curvature homogeneous. Takagi [63] showed that $M = \mathbb{R} \times N(c)$ in this situation, so this would imply that $\lambda = 0$ and then the M would be flat. Thus, Theorem 2.5 shows that M is a warped product with one-dimensional base and, due to locally conformal flatness [8], the fiber is of constant sectional curvature.

Assume now that $dimM = 3$. Then, by Lemma 2.3, the Ricci operator is of rank one $Q_\rho = \mathrm{diag}[\kappa, 0, 0]$, for some function κ. Hence, the Schouten tensor has eigenvalues $\lambda = \frac{3}{4}\kappa$ and $\mu = -\frac{1}{4}\kappa$. Then, as S is Codazzi and $dimV_\mu = 2$, $\nabla\mu$ is orthogonal to V_μ and as $\nabla\lambda = -\frac{1}{3}\nabla\mu$, then $\nabla\lambda$ is also orthogonal to V_μ. Moreover κ is not a constant function unless it is vanishing following the same idea as before. Takagi's result would imply that $M = \mathbb{R} \times N^2$ and this would be a contradiction with Q_ρ being of rank one. Then, by Theorem 2.5, (M, g) is locally a warped product of the form $\mathbb{R} \times_f F$ for some surface (F, g_F).

Now, the warped product metric is in the conformal class of a product metric since $dt \otimes dt + f(t)^2 g_F = f(t)^2\{f(t)^{-2}dt \otimes dt + g_F\}$. Hence the product metric $f(t)^{-2}dt \otimes dt + g_F$ is locally conformally flat and thus the corresponding Schouten tensor is Codazzi. Since $\dim F = 2$, its Ricci operator satisfies $Q_\rho^F = \frac{1}{2}\tau^F \,\mathrm{Id}$, and thus the Schouten tensor of the product manifold $(M, f(t)^{-2}dt \otimes dt + g_F)$ is given by $S = \tau^F \,\mathrm{diag}[-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$. Set $X = \partial_t$ and take Y to be a vector field on M tangent to F. Now it follows from the condition Codazzi properties, $(\nabla_X S)Y = (\nabla_Y S)X$, that the scalar curvature τ^F is constant. Hence, since $dimF = 2$, the fibre (F, g_F) is of constant sectional curvature. This completes the proof. $\square$

What is left to determine is if the existence of a warped product which satisfies $\check{R}$-Einstein condition is possible, so we need to give a description of the warping function. If we are able to compute it, then Assertion 1.(ii.a) of Theorem 2.2 follows.

First, in order to do that, we compute $\check{R}$ tensor of a generic warped product.

Lemma 2.7. *Let $B \times_f F$ be a warped product manifold with $\dim F = d$. Then*

(i) $\check{R}(X,Y) = \check{R}^B(X,Y) + \frac{2d}{f^2}\| \mathrm{h}_f \|^2 g(X,Y)$.

(ii) $\check{R}(X,V) = 0$.

(iii) $\check{R}(U,V) = \frac{1}{f^2}\check{R}^F(U,V) - 4\frac{\|\nabla f\|^2}{f^2}\rho^F(U,V) + 2\{\frac{\| \mathrm{h}_f \|^2}{f^2} + (d-1)\frac{\|\nabla f\|^4}{f^4}\}g(U,V)$.

where X, Y, Z (resp., U, V, W) are lifts to M of vector fields on B (resp., vector fields on F).

Proof. Let $\{\bar{e}_1,\dots,\bar{e}_n\} = \{e_1,\dots,e_{n-d},v_1,\dots,v_d\}$ be an orthonormal frame for $B \times_f F$, where $\{e_1,\dots,e_{n-d}\}$ is the lift of an orthonormal frame of (B,g_F) (respectively $\{v_1,\dots,v_d\}$ and (F,g_F)). Recall the curvature formulas for a general warped product from Proposition 1.16. Take $X,Y \in \mathfrak{X}(B)$ and $U,V \in \mathfrak{X}(F)$. Now we have different non-zero combinations of vectors on the basis and the fibre. Thus, we have

$$\begin{aligned}\check{R}(X,Y) = \sum & R(X,e_{i_1},e_{j_1},\bar{e}_k)R(Y,e_{i_1},e_{j_1},\bar{e}_k)\\ &+ R(X,v_{i_2},e_{j_1},\bar{e}_k)R(Y,v_{i_2},e_{j_1},\bar{e}_k)\\ &+ R(X,v_{i_2},v_{j_2},\bar{e}_k)R(Y,v_{i_2},v_{j_2},\bar{e}_k),\end{aligned}$$

and that is

$$\begin{aligned}\check{R}(X,Y) =& \check{R}^B(X,Y) + \sum \frac{Hess_f(X,e_{j_1})}{f}g(v_{i_2},\bar{e}_k)\frac{Hess_f(Y,e_{j_1})}{f}g(v_{i_2},\bar{e}_k)\\ &+ \frac{g(v_{i_2},v_{j_2})}{f}g(hess_f(X),\bar{e}_k)\frac{g(v_{i_2},v_{j_2})}{f}g(hess_f(Y),\bar{e}_k)\\ =& \check{R}^B(X,Y) + \frac{2d}{f^2}\|hf\|^2 g(X,Y),\end{aligned}$$

where the indices $i_1, j_1 = 1,\dots,n-d$, $i_2, j_2 = 1,\dots,d$ and $k = 1,\dots,n$.

Using vectors in the fibre instead of in the basis we obtain

$$\begin{aligned}\check{R}(U,V) = \sum & R(U,e_{i_1},e_{j_1},\bar{e}_k)R(V,e_{i_1},e_{j_1},\bar{e}_k)\\ &+ R(U,e_{i_1},v_{j_2},\bar{e}_k)R(V,e_{i_1},v_{j_2},\bar{e}_k)\\ &+ R(U,v_{i_2},v_{j_2},\bar{e}_k)R(V,v_{i_2},v_{j_2},\bar{e}_k),\end{aligned}$$

and thus,

$$\begin{aligned}\check{R}(U,V) = \sum & \frac{Hess_f(e_{i_1},e_{j_1})}{f} g(U,\bar{e}_k) \frac{Hess_f(e_{i_1},e_{j_1})}{f} g(V,\bar{e}_k) \\ & + \frac{Hess_f(e_{i_1},\bar{e}_k)}{f} g(U,v_{j_2}) \frac{Hess_f(e_{i_1},\bar{e}_k)}{f} g(V,v_{j_2}) \\ & + \left(R^F(U,v_{i_2},v_{j_2},\bar{e}_k) - \frac{\|\nabla f\|^2}{f^2} R_0(U,v_{i_2},v_{j_2},\bar{e}_k) \right) \\ & \times \left(R^F(V,v_{i_2},v_{j_2},\bar{e}_k) - \frac{\|\nabla f\|^2}{f^2} R_0(V,v_{i_2},v_{j_2},\bar{e}_k) \right),\end{aligned}$$

which is

$$\begin{aligned}\check{R}(U,V) = & \frac{1}{f^2}\check{R}^F(U,V) - 4\frac{\|\nabla f\|^2}{f^2}\rho^F(U,V) \\ & + 2\{\frac{\| \mathrm{h}_f \|^2}{f^2} + (d-1)\frac{\|\nabla f\|^4}{f^4}\} g(U,V).\end{aligned}$$

Lastly, $\check{R}(X,U) = 0$ for every X and U due to the warped product curvature. □

Lemma 2.8. *A warped product $\mathcal{I} \times_f N$ with fibre $N(c)$ of constant sectional curvature c is $\check{R}$-Einstein if and only if it is homothetic to one of the following*

(i) $f(t)^2 = t^2 - 1$, *if* $c = 1$, *and* $\mathcal{I} = (1, +\infty)$,

(ii) $f(t)^2 = t$, *if* $c = 0$, *and* $\mathcal{I} = (0, +\infty)$,

(iii) $f(t)^2 = 1 - t^2$, *if* $c = -1$, *and* $\mathcal{I} = (-\infty, 1)$.

Proof. Let (N, g_N) be an $(n-1)$-dimensional Riemannian manifold with constant sectional curvature c. Then $\rho_N = c(n-2)g_N$ and $\check{R}_N = 2c^2(n-2)g_N$. A direct application of Lemma 2.7 shows that a manifold $M = \mathcal{I} \times_f N$ is $\check{R}$-Einstein if and only if

$$2(n-1)\frac{1}{f^2}(f'')^2 = \frac{2}{f^2}\left\{\frac{c^2}{f^2}(n-2) - 2c(n-2)\frac{(f')^2}{f^2} + (f'')^2 + (n-2)\frac{(f')^4}{f^2}\right\},$$

where we used that $g(U,V) = f^2 g_N(U,V)$. From this, one gets

$$(f')^4 - f^2(f'')^2 - 2c(f')^2 + c^2 = 0,$$

which can be written as

$$\left(f'^2 - ff'' - c\right)\left(f'^2 + ff'' - c\right) = 0.$$

On the one hand, a standard calculation shows that the solutions of $f'(t)^2 - f(t)f''(t) - c = 0$ are given by $f(t) = \pm\left(\frac{1}{2}e^{-e^a(b+t)-2a} - \frac{1}{2}ce^{e^a(b+t)}\right)$ if $c \neq 0$ and $f(t) = ae^{bt}$ if $c = 0$ for constants $a, b \in \mathbb{R}$. In both situations, the metric obtained is Einstein, so we discard these cases. On the other hand, the solutions for $f'(t)^2 + f(t)f''(t) - c = 0$, are of the form

$$f(t) = \pm\frac{\sqrt{c^2(b+t)^2 - e^{2a}}}{\sqrt{c}}\,, \quad \text{for } c \neq 0, \quad \text{and } f(t) = a\sqrt{2t-b}\,, \quad \text{for} \quad c = 0.$$

The Ricci operator of $\mathcal{I} \times_f N$ satisfies

$$Q_\rho = h(t)\,\mathrm{diag}[(n-1), -(n-3), \overset{n-1}{\ldots}, -(n-3)],$$

where $h(t) = \left(\frac{ce^a}{e^{2a}-c^2(b+t)^2}\right)^2$ if $c \neq 0$ and $h(t) = \frac{1}{(b-2t)^2}$ if $c = 0$. Now Assertions 1.(ii.a.1) and 1.(ii.a.3) in Theorem 2.2 follows after rescaling. □

2.3 $R[\rho]$-Einstein locally conformally flat metrics

We start, as in the previous section, with the three-dimensional case. We study the algebraic structure of the metric before its geometric one, following previous section's procedure. We obtain an algebraic classification and then we study the geometric realizability.

2.3.1 Three-dimensional case

We proceed as in the previous section. We take an orthonormal basis for the Ricci operator and try to figure out its structure, which is given in the next statement.

Lemma 2.9. *Let (M, g) be a three-dimensional Riemannian manifold. Then it is $R[\rho]$-Einstein if and only if its Ricci operator is given by*

$$Q_\rho = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}.$$

Proof. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame of eigenvectors of the Ricci operator. We are computing $R[\rho]$ tensor. First of all, since $\rho(e_i, e_j) = 0$ if $i \neq j$, $Q_{R[\rho]}$ operator is diagonal. If we compute its components, we obtain

$$\begin{aligned}
R[\rho](e_\alpha, e_\alpha) =& \sum_i R(e_i, e_\alpha, e_\alpha, e_i)\rho(e_i, e_i) \\
=& \sum_i (-\frac{\tau}{2}\{g(e_\alpha, e_\alpha)g(e_i, e_i) - g(e_i, e_\alpha)g(e_\alpha, e_i)\}\lambda_i \\
&+ \{\rho(e_\alpha, e_\alpha)g(e_i, e_i) - \rho(e_i, e_\alpha)g(e_\alpha, e_i) \\
&+ \rho(e_\alpha, e_\alpha)g(e_i, e_i) - \rho(e_i, e_\alpha)g(e_\alpha, e_i)\})\lambda_i \\
=& -2\lambda_i^2 + \frac{3\tau}{2}\lambda_i + \left\{||\rho||^2 - \frac{\tau^2}{2}\right\}\mathrm{Id}\,.
\end{aligned}$$

Now, as we have a diagonal tensor, it is a multiple of the metric if and only if $R[\rho]_{\alpha\alpha}-R[\rho]_{\beta\beta}=0$, for all α,β. Thus, we obtain the equation

$$(\lambda_\alpha-\lambda_\beta)\left(-4(\lambda_\alpha+\lambda_\beta)+3\tau\right)=0.$$

From this system, we obtain that either Q_ρ is a multiple of the metric, then (M,g) is an Einstein manifold, or $Q_\rho=\operatorname{diag}(\lambda,\lambda,2\lambda)$. □

2.3.2 Higher-dimensional case

Let (M,g) be a n-dimensional locally conformally flat Riemannian manifold with $n\geq 4$. The algebraic structure for the Ricci operator is given in the next result.

Lemma 2.10. *A n-dimensional locally conformally flat Riemannian manifold with $n\geq 4$ is $R[\rho]$-Einstein if and only if the Ricci operator has exactly two-distinct eigenvalues λ and $\mu=\frac{2(n-1)-mn}{n(n-m)-2(n-1)}\lambda$, where m is the multiplicity of λ.*

Proof. Let $\{e_1,\dots,e_n\}$ be an orthonormal frame of Ricci eigenvectors. As M is locally conformally flat, then the curvature tensor is as in Equation (2.1). Therefore, if we compute the $R[\rho]$ tensor, we obtain that

$$\begin{aligned}
R[\rho](X,Y)=&\sum_{i,j}R(e_i,X,Y,e_j)\rho(e_i,e_j)\\
=&\sum_{i,j}(-\frac{\tau}{(n-2)(n-1)}\{g(X,Y)g(e_i,e_j)-g(e_i,Y)g(X,e_j)\}\rho(e_i,e_j)\\
&+\frac{1}{(n-2)}\{\rho(X,Y)g(e_i,e_j)-\rho(e_i,Y)g(X,e_j)\\
&\qquad+\rho(X,Y)g(e_i,e_j)-\rho(e_i,Y)g(X,e_j)\})\rho(e_i,e_j),
\end{aligned}$$

which is,

$$\begin{aligned}
R[\rho](X,Y)=&-\frac{2}{(n-2)}\rho^2(X,Y)+\frac{n\tau}{(n-1)(n-2)}\rho(X,Y)\\
&+\left\{\frac{1}{(n-2)}||\rho||^2-\frac{\tau^2}{(n-1)(n-2)}\right\}g(X,Y),
\end{aligned}$$

and thus,

$$Q_{R[\rho]}=-\frac{2}{(n-2)}Q_\rho^2+\frac{n\tau}{(n-1)(n-2)}Q_\rho+\left\{\frac{1}{(n-2)}||\rho||^2-\frac{\tau^2}{(n-1)(n-2)}\right\}\mathrm{Id}\,.$$

Now we want to see when this tensor is a multiply of the identity, so we need that

$$Q_{R[\rho]} - \frac{||\rho||^2}{n} \operatorname{Id} = 0.$$

We obtain the equation

$$-\frac{2}{(n-2)}Q^2 + \frac{n\tau}{(n-1)(n-2)}Q + \left\{ \frac{2}{n(n-2)}||\rho||^2 - \frac{\tau^2}{(n-1)(n-2)} \right\} \operatorname{Id} = 0. \tag{2.4}$$

Thus, since this equation needs to be satisfied by every eigenvalue, then we can have two at most. If we have just one, then the manifold would be Einstein, then, we have exactly two. Moreover, using the Vieta's Formulae, one has that

$$\lambda + \mu = \frac{n\tau}{2(n-1)}, \tag{2.5}$$

and, since $\tau = m\lambda + (n-m)\mu$, both eigenvalues are related by

$$\mu = \frac{2(n-1) - mn}{n(n-m) - 2(n-1)}\lambda. \tag{2.6}$$

□

2.3.3 Geometric structure

As we have two different eigenvalues for the Ricci operator, then the Schouten tensor has also two different. Consequently, we can apply the Merton's result and classify the metric.

Lemma 2.11. *Let (M^n, g) be a $R[\rho]$-Einstein locally conformally flat Riemannian manifold. Then it is locally symmetric and locally isometric to a product $N_1^m(c) \times N_2^m(-c)$.*

Proof. As the Schouten tensor is Codazzi and it has two eigenvalues, we can apply again Theorem 2.5. Now, if $\dim V_{\bar{\lambda}} \geq 2$, then M is locally a product and due to locally conformal flatness it is a product $M^{n_1}(c) \times M^{n_2}(-c)$. One can easily see that a manifold $N(k)$ of constant sectional curvature has $R[\rho]$ tensor as

$$R[\rho](X,Y) = \sum_{i,j} kR_0(e_i, X, Y, e_j)k(n-1)g(e_i, e_j) = k^2(n-1)^2 g(X,Y),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame. If we have a product, then the $R[\rho]$ tensor of the whole manifolds splits into a diagonal depending on the respective tensors of each term, so if $N^{n_1}(c_1) \times N^{n_2}(c_2)$, it is $R[\rho]$-Einstein if and only if

$$c_1^2(n_1 - 1)^2 = c_2^2(n_2 - 1)^2,$$

and, as $c_1 = -c_2$, then $n_1 = n_2$.

If $\dim V_{\bar{\lambda}} = 1$, then $\nabla_{\bar{\lambda}}$ is orthogonal to $V_{\bar{\lambda}}$, but, as $\bar{\mu}$ is a multiple of $\bar{\lambda}$, then $\nabla_{\bar{\mu}}$ is also orthogonal to $V_{\bar{\lambda}}$. Besides, $\bar{\mu}$ is not constant. Otherwise, $\bar{\lambda}$ would be constant as well, which would imply that λ and μ would be constant. Hence we would have a locally conformally flat manifold with constant Ricci curvatures, which is curvature homogeneous, and by Takagi [63], it is locally symmetric. Then M splits as a product of the form $\mathbb{R}\times N(c)$, whose factors corresponds to the Ricci curvatures, so λ would be vanishing and so μ and M would be flat. Thus we have a warped product and due to locally conformal flatness, the fibre has to be of constant sectional curvature. Three-dimensional case follows as in the proof for the $\check{R}$-Einstein condition but changing the algebraic structure of that case for the one in here.

Now, if we are in a warped product $\mathbb{R} \times_f N(c)$, then, by Lemma 1.16 the Ricci operator is written like

$$
\begin{aligned}
Q(\partial_t) &= -(n-1)\frac{f''}{f}\partial_t,\\
Q(X) &= \left((n-2)\frac{c}{f^2} - (n-2)\frac{f'^2}{f^2} - \frac{f''}{f}\right) X.
\end{aligned}
$$

We had that the Ricci eigenvalues were related by $\lambda = (n-1)\mu$, and then, we obtain the relation

$$f'^2 - c = 0,$$

which only have solution if $c \geq 0$, and in that case, it is a linear function, what gives Einstein metrics. Therefore we cannot have $R[\rho]$-Einstein warped products in this way, so this completes the proof. $\square$

Remark 2.12. This last argument to obtain the differential equation for the warping function does also work in the $\check{R}$-Einstein case.

2.4 $\check{\rho}$-Einstein locally conformally flat metrics

In the field of locally conformally flat metrics, the Ricci tensor has no specific representation in terms of other algebraic objects as the curvature tensor does. Moreover, the respective operator for $\check{\rho}$, $Q_{\check{\rho}}$, taking a orthonormal basis for the Ricci tensor, is the square of the Ricci operator. Thus, we have not much information whenever we try to get an equation to study its algebraic structure. Nonetheless, this is enough to get a classification.

2.4.1 Algebraic structure

In a locally conformally flat manifold, while the curvature tensor is determined by the Ricci one, this last one does not have any special property, so we are studying it in general dimension. Its algebraic structure is given by the following.

Lemma 2.13. *Let (M, g) be a n-dimensional Riemannian manifold. Then it is $\check{\rho}$-Einstein if and only if the Ricci operator has two eigenvalues related by $\mu = -\lambda$.*

Proof. We want that $Q_{\check{\rho}} = \frac{\|\rho\|^2}{n} \operatorname{Id}$, and since $Q_{\check{\rho}} = Q_\rho^2$, we have the equation

$$Q_\rho^2 - \frac{\|\rho\|^2}{n} \operatorname{Id} = 0.$$

Again, we can have just two eigenvalues. Now check that there is no coefficient with degree one in the equation. Then, by Vietta's Formula, we have that $\lambda + \mu = 0$, being λ and μ the eigenvalues of the Ricci operator. □

2.4.2 Geometric structure

We use the same arguments to get a full classification for $\check{\rho}$-Einstein condition. In the following result we stablish the geometric structure of manifolds satisfying this condition.

Theorem 2.14. *A locally conformally flat Riemannian manifold is $\check{\rho}$-Einstein if and only if $N^m(c) \times N^m(-c)$ or a warped product $\mathcal{I} \times_f \mathbb{R}^{n-1}$ with*

$$f(t) = \left(\frac{2(n-1)\,(at+b)}{n} \right)^{\frac{n}{2(n-1)}},$$

with $t \in (\frac{-b}{a}, +\infty)$ and $a, b \in \mathbb{R}$.

Proof. Since we assume that the manifold is locally conformally flat, then the Schouten tensor is Codazzi, and since we have two different Ricci eigenvalues, λ and $\mu = -\lambda$, the Schouten tensor has two different eigenvalues as well. Therefore, applying Merton's result, if $\dim V_\lambda \geq 2$ then we have a product $N^{n_1}(c) \times N^{n_2}(-c)$ and the condition to a product of this kind to be $\check{\rho}$-Einstein is that

$$c_1^2(n_1 - 1)^2 = c_2^2(n_2 - 1)^2,$$

so $n_1 = n_2$.

If $\dim V_\lambda = 1$, then we have a warped product $\mathbb{R} \times_f N(c)$ using the same arguments as in previous cases, and as we know that $\mu = -\lambda$, we can use again Lemma 1.16 and obtain the differential equation

$$nff'' + (n-2)f'^2 - (n-2)c = 0.$$

Now, take derivatives on both sides to get

$$nff''' + (3n-4)f'f'' = 0.$$

Since f and f'' cannot be zero as we would have flat metrics, then we have

$$\frac{(4-3n)}{n}\frac{f'}{f} = \frac{f'''}{f''}.$$

Next, we integrate both parts of the equations and get

$$\frac{(4-3n)}{n}\ln f = \ln f'' + K.$$

If we take the exponentials, the equations becomes

$$f^{\frac{(4-3n)}{n}} = e^K f'',$$

and now, multiplying both sides by $2f'$,

$$2e^{-K} f' f^{\frac{(4-3n)}{n}} = 2f'f''.$$

We have standard integrals on both parts, so we get

$$\bar{K}\frac{n}{4-2n} f^{\frac{4-2n}{n}} = f'^2,$$

where $\bar{K} = 2e^{-K}$. Finally, we isolate f'

$$f' = \tilde{K} f^{\frac{2-n}{n}},$$

where $\tilde{K} = \left(\bar{K}\frac{n}{4-2n}\right)^{\frac{1}{2}}$. The solution for this last differential equation is

$$f(t) = \left(\frac{2(n-1)\left(\tilde{K}t + a\right)}{n}\right)^{\frac{n}{2(n-1)}},$$

where $a \in \mathbb{R}$. Thus, we obtain a solution for the derivative of the original equation. Now, if some function is a solution for the original equation, it is a solutions for its derivative, and as we know the solution for this last one, the solution of the original equation needs to be of this form. So if we put this f in the original equation, we get that it is a solution for it if and only if

$$(n-2)c = 0.$$

Therefore, we are in a warped product of the form $\mathbb{R} \times_f \mathbb{R}^{n-1}$ and we have no other possibility here. □

2.5 Critical metrics for the functionals $\mathcal{S}$ and $\mathcal{F}_t$

In this section, we stablish when the metrics obtained along the chapter are critical for the functionals $\mathcal{S} = \int_M \tau^2 dvol_g$ and $\mathcal{F}_t = \int_M \{\|\rho\|^2 + t\tau^2\} dvol_g$.

First of all, Besse showed in [5] that a metric is $\mathcal{S}$-critical if and only if

$$2\left(Hess_\tau - \frac{\Delta\tau}{4}g\right) - 2\tau\left(\rho - \frac{\tau}{4}g\right) = 0, \tag{2.7}$$

and it is $\mathcal{F}_t$-critical if and only if

$$Hess_\tau - \Delta\rho + 2t\left(Hess_\tau - \frac{\Delta\tau}{4}g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right) - 2t\tau\left(\rho - \frac{\tau}{4}g\right) = 0, \tag{2.8}$$

with $t \in \mathbb{R}$.

We study the four-dimensional case because all curvature functionals are equivalent to this two. The case where (M, g) is $N^2(c) \times N^2(-c)$ is $\mathcal{S}$-critical and $\mathcal{F}_t$-critical for all $t \in \mathbb{R}$ since it has vanishing scalar curvature, then $\mathcal{S}$ is identically zero. Moreover, since this product fulfilled all weakly-Einstein conditions, then Equation (2.8) reduces to $-\Delta\rho = 0$, but $\rho = \operatorname{diag}[c, c, -c, -c]$, with $c \in \mathbb{R}$, so it is vanishing. Later on, when we study each condition, we will omit this example.

Theorem 2.15. *Let (M, g) be a four-dimensional locally conformally flat weakly-Einstein Riemannian manifold. Then*

1. *If (M, g) is $\check{R}$-Einstein, then it is $\mathcal{S}$-critical and $\mathcal{F}_t$-critical if and only if*

$$\Delta\rho = -2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right).$$

2. *If (M, g) is $\check{\rho}$-Einstein, then it is $\mathcal{F}_{-\frac{1}{3}}$-critical.*

Proof. In order to study (2.7) and (2.8), we define the tensors S and F as the (0,2)-tensor field given by the left-hand side of both equations, i.e., the $(0, 2)$-tensor field associated to each gradient, respectively. Thus, a metric is $\mathcal{S}$-critical if and only if S is vanishing (respectively, $\mathcal{F}_t$-critical and F).

First, if (M, g) is four-dimensional and $\check{R}$-Einstein, on the one hand, it has vanishing scalar curvature, so it is trivially $\mathcal{S}$-critical. On the other hand,

$$\mathsf{F} = -\Delta\rho - 2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right),$$

but the only condition the $\check{R}$-Einstein property gave was having vanishing scalar curvature, which admit many different structures for the Ricci tensor, so we just can say that it is $\mathcal{F}_t$-critical if and only if $\Delta\rho = -2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right)$.

If (M, g) is $\check{\rho}$-Einstein, then it is a warped product $\mathcal{I} \times_f \mathbb{R}^3$ with warping function $f(s) = \left(\frac{3(as+b)}{2}\right)^{\frac{2}{3}}$. In this situation, the first component of the tensor S is

$$\mathsf{S}_{11} = -\frac{12a^4}{(b + as)^4},$$

then, it is vanishing if and only if $a = 0$, but in that case f is linear, and thus, M is Einstein.

The non-zero components of F are given by

$$\mathsf{F}_{11} = K\mathsf{F}_{22} = K\mathsf{F}_{33} = K\mathsf{F}_{44} = -\frac{4a^4(3t+1)}{(as+b)^4}$$

where $K = -\frac{2\left(\frac{2}{3}\right)^{\frac{1}{3}}}{(as+b)^{\frac{4}{3}}}$. Thus, M is $\mathcal{F}_t$-critical for $t = -\frac{1}{3}$.

If (M, g) is $R[\rho]$-Einstein, then it is $M^2(c) \times M^2(-c)$, which has been studied already.

$\square$

Remark 2.16. If (M^4, g) is a $\check{R}$-Einstein warped product $\mathcal{I} \times_f N^3(c)$, with warping function $f(s) = \pm\frac{\sqrt{c^2(b+s)^2 - e^{2a}}}{\sqrt{c}}$ if $c \neq 0$ or $f(s) = a\sqrt{2s - b}$ if $c = 0$, which were the warping functions given in Theorem 2.2, then it is $\mathcal{F}_t$-critical.

Chapter 3
Weakly-Einstein hypersurfaces in real space forms

In this chapter, we classify weakly-Einstein metrics in hypersurfaces. In order to do that, we work with the shape operator, which is Codazzi when the ambient space is a real space form. Some results of this chapter can be seen in [28].

3.1 Introduction

Hypersurfaces theory is well known topic in differential geometry. In the first chapter, we introduced some topics about it, showing how the connection and the curvature tensor were constructed and their dependence on the ambient space. Specifically, we show that the curvature tensor of a hypersurface in a space form, i.e, a complete simply connected Riemannian manifold with constant sectional curvature, was given by

$$R(X,Y,Z,T) = cR_0(X,Y,Z,T) + g(\mathcal{S}Y,Z)g(\mathcal{S}X,T) - g(\mathcal{S}X,Z)g(\mathcal{S}Y,T),$$

where $\mathcal{S}$ is the shape operator, which is given as the operator related to the second fundamental form $II(X,Y) = g(\mathcal{S}X,Y)$.

Recall that we have three possibilities for a real space form: it is either $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$ depending on if the constant sectional curvature is vanishing or ± 1 [46]. From now on, whenever we do not specify any of them, we denote these as $\mathbb{Q}_c^{n+1}$, where c is its sectional curvature.

Since the curvature tensor only depends on the shape operator, we can work easily with it as in the locally conformally flat case and try to achieve an algebraic structure for the principal curvatures which allows us to classify the geometry of weakly-Einstein hypersurfaces.

The purpose of this chapter is similar to the previous one. We assume that some of the weakly-Einstein condition is fulfilled, we compute an equation for the principal curvatures and then we see if that structure can realize as any hypersurface. Unfortunately, this case give us much more casuistic and more difficult equations, so we work in general dimension n whenever it is possible, but some results are given for dimension four, which was where the original weakly-Einstein problem came from.

The aim of this chapter is proving the following.

Theorem 3.1. *Let (M,g) be a hypersurface in a space form $\mathbb{Q}_c^{n+1}$, with $c = 0, \pm 1$, with two principal curvatures. If (M,g) is weakly-Einstein, then it is a product of two spheres, a product of a sphere and a hyperbolic space or a rotation hypersurface over some profile curve.*

Notice that this is not a complete classification, but a starting point. Depending on the space form we take, we have different results. In $\mathbb{R}^{n+1}$, we have a complete classification for the $\check{R}$-Einstein condition, whereas in the corresponding $R[\rho]$ and $\check{\rho}$-Einstein ones, we just have partial results depending on the dimension. Analogously, in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$, the same occurs with each condition since the casuistic is much more harder to handle. The result given is just a summary of the examples obtained assuming just two different principal curvatures, but the family of weakly-Einstein hypersurfaces may be larger considering a higher number of these. However, we are able to determine that in some cases there exists no examples. Details on this fact are given along the chapter.

Firstly, we introduce some topics about rotation hypersurfaces. These will help us in further classifications.

3.1.1 Rotation hypersurfaces

Rotation hypersurfaces are a generalization of surfaces of revolution. The main idea of their construction is taking a curve and rotating it using the action of a group. The different copies of the curve are called parallels and the curve made by the action is called meridian. All the results concerning to this section are shown in the work of do Carmo and Dajczer [13].

Let L^n be the set of n-tuples $(x_1, \ldots, x_n)$, P^k the k-dimensional subspace of L^n passing through the origin and $O(P^k)$ the set of orthogonal transformations of L^n that leaves P^k fixed. One can defines a rotation hypersurfaces as follows.

Choose $P^2 \subset P^3$ such that $P^3 \cap \mathbb{Q}_c^{n+1} \neq \emptyset$. Let C be a regular curve in $P^3 \cap \mathbb{Q}_c^{n+1}$ that does not meet P^2. The orbit of C under the action of $O(P^2)$ is called a rotation hypersurface $M \subset \mathbb{Q}_c^{n+1}$ generated by C around P^2.

The analogous example of a surface of revolution would be if we take a curve in the plane ZY (which plays the role of P^3) that does not intersect axis Z (respectively, P^2) and rotate the curve around it.

The parametrizations of these hypersurfaces are given by do Carmo and Dajczer in [13]. They depend on the ambient space and on the type of subspace P^k. Furthermore, in their work, they also give some sufficient conditions for a hypersurface to be a rotation one.

Theorem 3.2 ([13]). *Let $M^n \hookrightarrow \mathbb{Q}_c^{n+1}$ be an arbitrary hypersurface in a real space form with $n \geq 3$. Assume that its principal curvatures satisfy $\kappa_1 = \cdots = \kappa_{n-1} = \lambda$, $\kappa_n = \mu = \mu(\lambda)$ and $\lambda \neq \mu$. Then M is contained in a rotation hypersurface.*

As a consequence of this result we have the following.

Corollary 3.3 ([13]). *Let M^n, with $n \geq 4$, be a conformally flat hypersurface into a real space form. If it has two different principal curvatures λ with multiplicity $n-1$ and $\mu = \mu(\lambda)$. Then M is contained in a rotation hypersurface.*

Remark 3.4. Rotation surfaces can be seen as a warped product. If we take a plane curve $(f(v), 0, g(v))$ and we rotate it over the Z axis, we obtain a parametrization

$$X(v, \theta) = (f(v)\cos\theta, f(v)\sin\theta, g(v)),$$

Compute now the metric tensor, obtaining that $g = dt^2 + f^2 d\theta^2$, which is a warped product metric. We can always take a plane curve like this since we can take rigid motion and move the curve into other plane. This shows that a rotation hypersurface may be seen as a warped product $I \times_{x_1} \mathbb{Q}_c^{n-1}$, where x_1 is the first coordinate of the profile curve as seen in [13].

3.2 Algebraic structure

3.2.1 $\check{R}$-Einstein condition

Let M^n be a hypersurface in $\mathbb{Q}_c^{n+1}$. We consider $\check{R}$-Einstein condition on M. We recall Gauss equation

$$R^M(X,Y,Z,V) = cR^0(X,Y,Z,V) + g(\mathcal{S}Y,Z)g(\mathcal{S}X,V) - g(\mathcal{S}X,Z)g(\mathcal{S}Y,V).$$

Now we shall compute the $\check{R}$ tensor field in terms of the shape operator.

$$\begin{aligned}\check{R}(X,Y) =& \sum_{i,j,k} R(X,e_i,e_j,e_k)R(Y,e_i,e_j,e_k),\\ =& \sum_{i,j,k}\{cg(e_i,e_j)g(X,e_k) - cg(X,e_j)g(e_i,e_k)\}\\ &+ g(\mathcal{S}e_i,e_j)g(\mathcal{S}X,e_k) - g(\mathcal{S}X,e_j)g(e_i,e_k)\}\\ &\times \sum_{i,j,k}\{cg(e_i,e_j)g(Y,e_k) - cg(Y,e_j)g(e_i,e_k)\}\\ &+ g(\mathcal{S}e_i,e_j)g(\mathcal{S}Y,e_k) - g(\mathcal{S}Y,e_j)g(e_i,e_k)\}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\check{R}(X,Y) =& 2c^2(n-1)g(X,Y) + 4c(tr\mathcal{S})g(\mathcal{S}X,Y)\\ &+ (2tr\mathcal{S}^2 - 4c)g(\mathcal{S}^2X,Y) - 2g(\mathcal{S}^4X,Y).\end{aligned}$$

Then the $(1,1)$-tensor field $Q_{\check{R}}$ is given by

$$Q_{\check{R}} = 2(c^2(n-1)\mathrm{Id} + 2c(tr\mathcal{S})\mathcal{S} + (tr\mathcal{S}^2 - 2c)\mathcal{S}^2 - \mathcal{S}^4),$$

and since $\|R\|^2 = trQ_{\check{R}}$, one has that

$$\|R\|^2 = 2(c^2n(n-1) + 2c(tr\mathcal{S})^2 + (tr\mathcal{S}^2 - 2c)tr\mathcal{S}^2 - tr\mathcal{S}^4).$$

Then, as M is $\check{R}$-Einstein if and only if $\check{R} = \frac{\|R\|^2}{n}g$, its principal curvatures must satisfy the equation

$$\mathcal{S}^4 - (\|\mathcal{S}\|^2 - 2c)\mathcal{S}^2 - (2ncH)\mathcal{S} - \frac{1}{n}\{\|\mathcal{S}^2\|^2 - (\|\mathcal{S}\|^2 - 2c)\|\mathcal{S}\|^2 - 2c(nH)^2\}Id = 0.$$

where $H = \frac{1}{n}\operatorname{tr}\mathcal{S}$ is the mean curvature, $\|\mathcal{S}\|^2 = \operatorname{tr}\mathcal{S}^2$ is the length of the shape operator and $\|\mathcal{S}^2\|^2 = \operatorname{tr}\mathcal{S}^4$.

3.2.2 $R[\rho]$-Einstein condition

Assume now that M is $R[\rho]$-Einstein and proceed in the same way. In this case, we have that

$$\rho(X,Y) = \sum_{i,j} R(e_i, X, Y, e_j) = c(n-1)g(X,Y) + g(\mathcal{S}X,Y)nH - g(\mathcal{S}^2X,Y),$$

where $\{e_1,\dots,e_n\}$ is an orthonormal frame. Thus, we have

$$\begin{aligned} R[\rho](X,Y) =& \sum_{i,j} R(e_i,X,Y,e_j)\rho(e_i,e_j) \\ =& \sum_{i,j}\{cg(X,Y)g(e_i,e_j) - cg(e_i,Y)g(X,e_j)\} \\ &+ g(\mathcal{S}X,Y)g(\mathcal{S}e_i,e_j) - g(\mathcal{S}e_i,Y)g(X,e_j)\} \\ &\times \{c(n-1)g(e_i,e_j) + g(\mathcal{S}e_i,e_j)nH - g(\mathcal{S}^2e_i,e_j)\}. \end{aligned}$$

We obtain

$$\begin{aligned} Q_{R[\rho]} =& \mathcal{S}^4 - (\operatorname{tr}\mathcal{S})\mathcal{S}^3 - c(n-2)\mathcal{S}^2 + (c(n-2)\operatorname{tr}\mathcal{S} + \operatorname{tr}\mathcal{S}\operatorname{tr}(\mathcal{S}^2) - \operatorname{tr}(\mathcal{S}^3))\mathcal{S} \\ &+ c(c(n-1)^2 + (\operatorname{tr}\mathcal{S})^2 - \operatorname{tr}(\mathcal{S}^2)), \end{aligned}$$

and as $\operatorname{tr} Q_{R[\rho]} = \|\rho\|^2$, then

$$\begin{aligned} \frac{\|\rho\|^2}{n} =& \frac{1}{n}(\operatorname{tr}(\mathcal{S}^4) - 2\operatorname{tr}\mathcal{S}\operatorname{tr}(\mathcal{S}^3) + (\operatorname{tr}\mathcal{S})^2\operatorname{tr}(\mathcal{S}^2) - 2c(n-1)((\operatorname{tr}\mathcal{S})^2 - \operatorname{tr}(\mathcal{S}^2)) \\ &+ c^2(n-1)^2n). \end{aligned}$$

Finally, M is $R[\rho]$-Einstein if and only if its principal curvatures satisfies the equation

$$\begin{aligned} &\mathcal{S}^4 - (nH)\mathcal{S}^3 - c(n-2)\mathcal{S}^2 + (c(n-2)nH + nH\|\mathcal{S}\|^2 - \operatorname{tr}(\mathcal{S}^3))\mathcal{S} \\ &+ \frac{1}{n}(-\|\mathcal{S}^2\|^2 + 2nH\operatorname{tr}(\mathcal{S}^3) - (nH)^2\|\mathcal{S}\|^2 + c(n-2)\|\mathcal{S}\|^2 - c(n-2)(nH)^2) = 0. \end{aligned}$$

3.2.3 $\check{\rho}$-Einstein condition

As we have computed $\|\rho\|^2$ in the previous section, the only condition we have left is $\check{\rho}$-Einstein one. Thus,

$$
\begin{aligned}
\check{\rho}(X,Y) =& \sum_i \rho(X,e_i)\rho(e_i,Y) \\
=& \sum_i \{c(n-1)g(X,e_i) + g(\mathcal{S}X,e_)nH - g(\mathcal{S}^2X,e_i)\} \\
& \times \{c(n-1)g(Y,e_i) + g(\mathcal{S}Y,e_)nH - g(\mathcal{S}^2Y,e_i)\}.
\end{aligned}
$$

Consequently, we obtain

$$
Q_{\check{\rho}} = \mathcal{S}^4 - 2(\operatorname{tr}\mathcal{S})\mathcal{S}^3 + (\operatorname{tr}(\mathcal{S})^2 - 2c(n-1))\mathcal{S}^2 + 2c\operatorname{tr}\mathcal{S}(n-1)\mathcal{S} + c^2(n-1)^2,
$$

hence, M is $\check{\rho}$-Einstein if and only if the following equation is satisfied

$$
\begin{aligned}
&\mathcal{S}^4 - 2(nH)\mathcal{S}^3 + ((nH)^2 - 2c(n-1))\mathcal{S}^2 + 2cnH(n-1)\mathcal{S} \\
&\frac{1}{n}(\|\mathcal{S}^2\|^2 - 2nH\operatorname{tr}(\mathcal{S}^3) + (nH)^2\|\mathcal{S}^2\|^2 2c(n-1)((nH)^2 - \|\mathcal{S}^2\|^2)) = 0.
\end{aligned}
$$

3.3 Geometrical structure

From the previous section, we have that each weakly-Einstein condition is satisfied if and only if the principal curvatures satisfies an equation of degree four. Recall that every principal curvature must satisfy the equation, but not every solution of the equation may be a principal curvature as the coefficients of it depends on different terms involving the shape operator. Thus, we know that, at most, we can have four different eigenvalues for each one, but we have to think about the possibilities when we have just two or three (if we have one, then the metric is Einstein).

This adds much complexity to the problem that one may think at first. If we assume that we have four different eigenvalues, then we have a lot of different possibilities with the multiplicities of each one, being them simple or greater than one. Adding four different multiplicities to the problem adds four more unknowns to the equation, and thus, it becomes quite hard to manage.

Another possibility is classifying up to dimension. If we are in dimension four, the different casuistic for the multiplicities reduces to just four cases. Therefore, we are classifying cases of three and four different principal curvatures for this dimension, whereas the case of two different can be done in general dimension as the shape operator of hypersurfaces in real space forms is Codazzi (Example 1.17–2), and since we have two eigenvalues, then we can use Merton's result (Theorem 2.5) to classify the submanifold.

We split up this problem in different parts. The first assumption depends on the space form, the second one, on the weakly-Einstein condition and the third one, on the number of principal curvatures.

3.3.1 Hypersurfaces in $\mathbb{R}^{n+1}$

Assume now that $c = 0$. As we have said above, we work in general dimension whenever is possible and we reduce it to four in order to simplify the problem when needed.

$\check{R}$-Einstein

In the next result, we see that this kind of hypersurface are in the same family as the examples given in the previous chapter.

Lemma 3.5. *Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a hypersurface. Then, M is $\check{R}$-Einstein if and only if it is locally isometric to a warped product given in Theorem 2.2. Moreover, M is a rotation hypersurface.*

Proof. We need that $\check{R}(e_i, e_i) - \check{R}(e_j, e_j) = 0$, with $i \neq j$, then we obtain the equation

$$(\lambda_i - \lambda_j)(\lambda_i + \lambda_j)((m_i - 1)\lambda_i^2 + (m_j - 1)\lambda_j^2 + \sum_{i \neq k \neq j} m_k \lambda_k) = 0,$$

where m_α is the multiplicity of the principal curvature λ_α. Thus, this system is only fulfilled if $\mathcal{S}$ has rank one, which would give us an Einstein hypersurfaces, or if we have two principal curvatures, one opposite of each other, i.e,

$$\mathcal{S} = \operatorname{diag}[\lambda, \overset{m}{\ldots}, \lambda, -\lambda, \overset{n-m}{\ldots}, -\lambda].$$

Now, we apply Merton's result as the shape operator of a hypersurface into a space form is Codazzi. If we have both principal curvatures with multiplicity greater that two $(m \geq 2)$, then the gradient of one eigenfunction is orthogonal to its own eigenspace, and as both are multiple of each other, then it is also perpendicular to the other one. Therefore, the only way that this can occur is that the gradient of each principal curvatures is zero and then they are constant eigenfunctions. Therefore we are in an isoparametric hypersurface.

In [15], Cecil and Ryan summarize a classification of isoparametric hypersurfaces in each real space form. Regarding the Euclidean space, a isoparametric hypersurface with two principal curvatures is an open subset of a spherical cylinder $\mathbb{S}^m \times \mathbb{R}^{n-m}$, but this kind of hypersurface has shape operator of the form

$$\mathcal{S} = \operatorname{diag}[\lambda, \overset{m}{\ldots}, k, 0, \overset{n-m}{\ldots}, 0],$$

which is not the structure we have.

On the other hand, if we have one principal curvature with multiplicity one, then, a result by Nishikawa and Maeda in [58] ensure that the hypersurface is locally conformally flat. Then, because of Theorem 2.2 in chapter two, the hypersurface is locally isometric to a warped product with an specific non-trivial warping function. Moreover, by Corollary 3.3, we have a rotation hypersurface. □

$R[\rho]$-Einstein

In contrast with the previous case, this one is much harder to study in general dimension n. Thus, we only work in this case when we assume that we have two principal curvatures, whereas in the other two cases we are working in dimension four.

First, assume that we have two different principal curvatures, call them λ and μ.

Lemma 3.6. *There is no $R[\rho]$-Einstein hypersurface $M^n \hookrightarrow \mathbb{R}^{n+1}$ with two different principal curvatures.*

Proof. The condition $R[\rho](e_i, e_i) - R[\rho](e_j, e_j) = 0$ give us the equation

$$(\lambda(m-1) + \mu(n-m-1))(\lambda^2(m-1) + \mu^2(n-m-1)) = 0,$$

where m is the multiplicity of λ.

If $m \geq 2$, since λ and μ are real eigenfunctions, then this only vanishes if the first bracket does, then we have that

$$\lambda = -\mu \frac{n-m-1}{m-1}.$$

Using Merton's result, we can only have an isoparametric hypersurface in the Euclidean space, so it has to be a open subset of a cylinder $\mathbb{S}^m \times \mathbb{R}^{n-m}$, which has a different shape operator structure. Therefore, this cannot happen. If $m = 1$, then $\mu = 0$ and thus we obtain an Einstein example. □

Remark 3.7. We can also proof the case where $m = 1$ by Nishikawa and Maeda's work [58], since the hypersurface is locally conformally flat and there were no examples of $R[\rho]$-Einstein in that field, as we have seen in the previous chapter.

Now, once we have analysed the case with two principal curvatures, we consider smaller dimension to try to simplify the problem. The advantage of working in dimension four is that, knowing that we have four different principal curvatures at most, then we can only have two possible configurations for the shape operator: either we have three different principal curvatures with $\mathcal{S} = \operatorname{diag}[\lambda, \lambda, \mu, \gamma]$ or four different with $\mathcal{S} = \operatorname{diag}[\lambda, \mu, \gamma, \delta]$. Next Lemma shows that there are no examples in this setting

Lemma 3.8. *There is no four-dimensional $R[\rho]$-Einstein hypersurface $M^4 \hookrightarrow \mathbb{R}^5$ with three or four different principal curvatures.*

Proof. Assume now that we have three principal curvatures in dimension four, call them λ, μ and γ.

Now we work with the condition $R[\rho] - \frac{\|\rho\|^2}{4} g = 0$. This gives us the system of equations

$$\left.\begin{aligned} \gamma^2\left(\lambda^2-\mu^2\right) - 2\gamma\lambda^2\mu + \lambda^2\left(\lambda^2+\mu^2\right) &= 0 \\ \gamma^2\left(-3\lambda^2+2\lambda\mu+\mu^2\right) - 2\gamma\lambda\left(\lambda^2-\lambda\mu+\mu^2\right) + \lambda^2\left(-\lambda^2+2\lambda\mu+\mu^2\right) &= 0 \\ \gamma^2(\lambda-\mu)^2 + 2\gamma\lambda\left(\lambda^2+\lambda\mu+\mu^2\right) + \lambda^2\left(-\left(\lambda^2+2\lambda\mu+3\mu^2\right)\right) &= 0 \end{aligned}\right\}$$

With these three polynomials, we construct a Gröbner basis $\mathcal{G} \subset \mathbb{R}[\lambda, \mu, \gamma]$, with respect to the lexicographic order, in order to solve it. After a thorough analysis of $\mathcal{G}$, we see that the polynomial

$$-\lambda(\gamma-\mu)\left(\gamma\lambda-\gamma\mu+\lambda^2+\lambda\mu\right)$$

is in the basis, and since every solution of the system is also a solution of the basis, then it has to be vanishing. If $\lambda = 0$, then we obtain from the first polynomial of the system that $\gamma^2\mu^2 = 0$, which is not possible since $\lambda \neq \gamma \neq \mu \neq \lambda$. Therefore, we need that $\gamma = \frac{\lambda(\lambda+\mu)}{\lambda-\mu}$. If we add this to construct a new basis $\mathcal{G}_1 = \mathcal{G} \cup \{\gamma(\lambda-\mu)-\lambda(\lambda+\mu)\}$, we have that

$$\gamma^3((\gamma+\mu)^2+2\gamma^2+2\mu^2)$$

is in the basis. The second term only vanishes if $\gamma = \mu = 0$, which is not possible, so we need that $\gamma = 0$, but in that case, we have that the first polynomial of the system becomes $\lambda^2(\lambda^2+\mu^2) = 0$, and then we have no real solution different from zero.

If we assume now four different principal curvatures (λ, μ, γ and δ) and the same condition, we obtain the system

$$\left.\begin{array}{r}
\gamma^2\lambda(\delta+\lambda+\mu)+\delta^2\lambda(\gamma+\lambda+\mu)+\lambda\mu^2(\gamma+\delta+\lambda)-\frac{\|\rho\|^2}{4}=0\\
\gamma^2\mu(\delta+\lambda+\mu)+\delta^2\mu(\gamma+\lambda+\mu)+\lambda^2\mu(\gamma+\delta+\mu)-\frac{\|\rho\|^2}{4}=0\\
\gamma\delta^2(\gamma+\lambda+\mu)+\gamma\lambda^2(\gamma+\delta+\mu)+\gamma\mu^2(\gamma+\delta+\lambda)-\frac{\|\rho\|^2}{4}=0\\
\gamma^2\delta(\delta+\lambda+\mu)+\delta\lambda^2(\gamma+\delta+\mu)+\delta\mu^2(\gamma+\delta+\lambda)-\frac{\|\rho\|^2}{4}=0
\end{array}\right\},$$

where

$$\begin{aligned}
\|\rho\|^2 =& \frac{1}{4}(-\gamma^2(\delta+\lambda+\mu)^2-\delta^2(\gamma+\lambda+\mu)^2-\lambda^2(\gamma+\delta+\mu)^2\\
&+\mu^2\left(-(\gamma+\delta+\lambda)^2\right))
\end{aligned}$$

Now, in order to solve this system, we construct a Gröbner basis $\mathcal{G} \subset \mathbb{R}[\lambda, \mu, \gamma, \delta]$ with respect to the lexicographic order with the system. After a deep analysis of the basis, we find that the polynomial

$$(\gamma-\delta)(\gamma-\mu)(\delta-\mu)(\gamma^2\delta+\gamma\delta^2+\gamma^2\lambda+\gamma\delta\lambda+\delta^2\lambda+\mu(\gamma^2+\gamma\delta+\delta^2))$$

is in $\mathcal{G}$. Since every principal curvature is different from each other, then we need that the last bracket vanishes, and so we have that

$$\mu = -\frac{\gamma^2\delta+\gamma\delta^2+\gamma^2\lambda+\gamma\delta\lambda+\delta^2\lambda}{\gamma^2+\gamma\delta+\delta^2}.$$

Now we take this value and make a new basis to obtain that the polynomial

$$\frac{(\gamma+\delta)\left(\gamma^2\delta^2+\gamma^2\delta\lambda+\gamma^2\lambda^2+\gamma\delta^2\lambda+\gamma\delta\lambda^2+\delta^2\lambda^2\right)}{\gamma^2+\gamma\delta+\delta^2}$$

is in the basis. The second bracket only vanishes if

$$\lambda = \frac{\gamma^2\delta + \gamma\delta^2 \pm \sqrt{-\gamma^2\delta^2\left((\gamma+\delta)^2 + 2\gamma^2 + 2\delta^2\right)}}{2\left(\gamma^2 + \gamma\delta + \delta^2\right)},$$

but this is a complex number unless $\gamma = 0$ or $\delta = 0$, but in both cases $\lambda = 0$. Then we necessarily need that $\gamma = -\delta$. In that case, we obtain that

$$(\delta - \lambda)(\delta + \lambda)(\delta^2 + \lambda^2)$$

is in the basis, but if any of the brackets is zero, then not every principal curvature is different, then we cannot have any solution for this system. $\square$

$\check{\rho}$-Einstein

This part follows as before. First assume two principal curvatures in a n-dimensional hypersurface and then reduce the dimension to four and assume that we have three and four different principal curvatures.

Lemma 3.9. *Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a hypersurface with two principal curvatures. If M is $\check{\rho}$-Einstein, then it is locally isometric to a warped product given in Theorem 2.2. Moreover, M is extrinsically a rotation hypersurface.*

Proof. Assume that we have two principal curvatures. The condition $\check{\rho}(e_i, e_i) - \check{\rho}(e_j, e_j) = 0$, when $i \neq j$, gives the equation

$$\lambda^2((m-1)\lambda + (n-m)\mu)^2 - \mu^2(m\lambda + (n-m-1)\mu)^2 = 0,$$

where m is the multiplicity of λ. If $m \geq 2$, then the solutions of the equation are

$$\lambda = -\frac{n-m-1}{m-1}\mu \quad \text{and} \quad \lambda = \mu\left(\frac{\pm\sqrt{(n-2m)^2 + 4(n-1)} - n}{2(m-1)}\right).$$

Thus, one principal curvature is a multiple of the other and with the same argument as in the previous proofs, the hypersurface is isoparametric and then it has to be a cylinder $\mathbb{S}^m \times \mathbb{R}^{n-m}$, which is not possible.

If $m = 1$, by Nishikawa and Maeda, the hypersurface is locally conformally flat, and thus, the hypersurface is one of the warped products given in Theorem 2.2. Moreover, the equation becomes

$$\mu(\lambda - \mu)(n-2)(n(\lambda + \mu) - 2\mu) = 0,$$

and thus $\lambda = -\frac{n-2}{n}\mu$. By Corollary 3.3, is a rotation hypersurface. $\square$

Lemma 3.10. *There is no four-dimensional $\check{\rho}$-Einstein hypersurface $M^4 \hookrightarrow \mathbb{R}^5$ with three or four different principal curvatures.*

Proof. Assume now that we have three principal curvatures in dimension four, call them λ, μ and γ. Now condition $\check{\rho} - \frac{\|\rho\|^2}{4} g = 0$ gives us the system of equations

$$\left.\begin{array}{r} -\gamma^2(2\lambda+\mu)^2 + 2\lambda^2(\gamma+\lambda+\mu)^2 + \mu^2\left(-(\gamma+2\lambda)^2\right) = 0 \\ -\gamma^2(2\lambda+\mu)^2 - 2\lambda^2(\gamma+\lambda+\mu)^2 + 3\mu^2(\gamma+2\lambda)^2 = 0 \\ 3\gamma^2(2\lambda+\mu)^2 - 2\lambda^2(\gamma+\lambda+\mu)^2 + \mu^2\left(-(\gamma+2\lambda)^2\right) = 0 \end{array}\right\}.$$

Make a Gröbner basis with these polynomials and obtain that

$$\lambda(\gamma-\mu)(\gamma\lambda+\gamma\mu+\lambda\mu)$$

is in the basis. As this has to vanish, then or $\lambda = 0$ or $\gamma = \frac{-\lambda\mu}{\lambda+\mu}$. If the first happens, then the first polynomial of the system becomes $\gamma^2\mu^2 = 0$, but all principal curvatures are different, so $\lambda \neq 0$. If the second happens, then the first polynomial becomes

$$\frac{2\lambda^3(\lambda^2+\lambda\mu-2\mu^2)}{\lambda+\mu} = 0,$$

which only vanishes if $\lambda = \mu$, which is not possible, or if $\lambda = -2\mu$ and in that case $\lambda = \gamma$, so we have no possible solution for this system.

Now, if we assume that we have four different principal curvatures, the we obtain

$$\left.\begin{array}{r} -\gamma^2(\delta+\lambda+\mu)^2 - \delta^2(\gamma+\lambda+\mu)^2 + 3\lambda^2(\gamma+\delta+\mu)^2 + \mu^2\left(-(\gamma+\delta+\lambda)^2\right) = 0 \\ -\gamma^2(\delta+\lambda+\mu)^2 - \delta^2(\gamma+\lambda+\mu)^2 - \lambda^2(\gamma+\delta+\mu)^2 + 3\mu^2(\gamma+\delta+\lambda)^2 = 0 \\ 3\gamma^2(\delta+\lambda+\mu)^2 - \delta^2(\gamma+\lambda+\mu)^2 - \lambda^2(\gamma+\delta+\mu)^2 + \mu^2\left(-(\gamma+\delta+\lambda)^2\right) = 0 \\ -\gamma^2(\delta+\lambda+\mu)^2 + 3\delta^2(\gamma+\lambda+\mu)^2 - \lambda^2(\gamma+\delta+\mu)^2 + \mu^2\left(-(\gamma+\delta+\lambda)^2\right) = 0 \end{array}\right\}$$

Again we are solving the system using Gröbner basis with respect to to the lexicographic order. We have that the polynomial

$$(\gamma-\delta)(\lambda+\mu)(2\gamma\delta+\gamma\lambda+\gamma\mu+\delta\lambda+\delta\mu),$$

is in the basis. So either $\lambda = -\mu$ or $\mu = -\frac{2\gamma\delta+\gamma\lambda+\delta\lambda}{\gamma+\delta}$. In the first case, if we make a new basis with this polynomial, then we obtain that $(\gamma+\delta)\mu^3$ is in the basis, so $\gamma = -\delta$ because if $\mu = 0$ then $\mu = \lambda = 0$. If we have this setting, then the condition reduces to only one polynomial, which is $2(\gamma^4 - \lambda^4) = 0$, which gives no possible solutions.

If $\mu = -\frac{2\gamma\delta+\gamma\lambda+\delta\lambda}{\gamma+\delta}$, we make a new basis and find that

$$-\frac{\gamma\delta\left(\gamma^2+\delta^2\right)\left(\gamma^2+4\gamma\delta+\delta^2\right)\left(\delta\lambda+\gamma(\delta+\lambda)\right)}{(\gamma+\delta)^2}$$

is in the basis. Here we have several possibilities which we analyse separately. We can discard $\gamma = 0$ and $\delta = 0$ since, in that case, $\mu = -\lambda$ and we are in the previous case. If the second bracket vanishes then $\delta = (-2 \pm \sqrt{3})\gamma$, and with this setting the first equation of the system becomes

$$\pm \frac{\left(521 + 296\sqrt{3}\right)}{18}\gamma^4 = 0,$$

which is not possible because if $\gamma = 0$ then $\delta = 0$ and this cannot happen.

If the last bracket is zero, then $\lambda = -\frac{\delta\gamma}{\delta+\gamma}$. Making a new basis with this term we obtain that $\gamma^2\delta^3$ is in the basis, which only vanishes if one of this is zero, but that implies that $\lambda = 0$. Therefore, as we have analysed all possibilities, we have no solution.

□

3.3.2 Hypersurfaces in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$

In this section we assume that $c^2 = 1$. Now equations are harder as the term involving the curvature does not vanish and much more coefficients appear in the equations.

$\check{R}$-Einstein

First, we study the case of two different principal curvatures in a n-dimensional hypersurface and then we take smaller dimension in order to get particular results in the four-dimensional case.

Lemma 3.11. *Let $M^n \hookrightarrow \mathbb{Q}_c^{n+1}$, with $c^2 = 1$, be a $\check{R}$-Einstein hypersurface with two different principal curvatures. Then, it is either a product of two spheres, a product of a hyperbolic space and a sphere or a rotation hypersurface over a curve.*

Proof. Firstly, assume that both principal curvatures, λ and μ, have multiplicity m and $n - m$ respectively, both greater than two. The $\check{R}$-Einstein tensor gives the equation

$$\begin{aligned}&\lambda^3(m-1) + \lambda^2\mu(m-1) + \lambda\left(2c(m-1) + \mu^2(n-m-1)\right)\\ &+ \mu^3(n-m-1) + 2c\mu(n-m-1) = 0.\end{aligned}$$

Thus λ and μ are related, and since the shape operator is Codazzi in space forms, then the hypersurface is isoparametric. The summary given by Cecil and Ryan states that, in $\mathbb{S}^{n+1}$, an isoparametric hypersurface has to be a product of two spheres $\mathbb{S}^m\left(\sin^{-2}\theta\right) \times \mathbb{S}^{n-m}\left(\cos^{-2}\theta\right)$ [15], where $\sin^{-2}\theta$ and $\cos^{-2}\theta$ are the sectional curvature of each sphere, with $\theta \in \left(0, \frac{\pi}{2}\right)$. Recall that a product $N^{n_1}(c_1) \times N^{n_2}(c_2)$ is $\check{R}$-Einstein if and only if $c_1^2(n_1 - 1) = c_2^2(n_1 - 1)$. Then we need that

$$\tan^4\theta = \frac{m-1}{n-m-1.}$$

One can use the same argument in $\mathbb{H}^{n+1}$, where the only possibility is a product $\mathbb{S}^m\left(\sinh^{-2}\theta\right) \times \mathbb{H}^{n-m}\left(-\cosh^{-2}\theta\right)$ [15], where

$$\tanh^4\theta = \frac{m-1}{n-m-1}.$$

Now assume that λ has multiplicity one. We know that the hypersurface is locally conformally flat by [58] and one can see that λ is a function of μ since the condition to be $\check{R}$-Einstein reduces to the polynomial

$$\mu(n-2)(2c+\mu\lambda+\mu^2)=0,$$

so $\mu = 0$, which gives an Einstein metric, or $\lambda = -\frac{2c+\mu^2}{\mu}$. Then, by [13], we have a rotation hypersurface with a plane curve as profile. □

Remark 3.12. In the first case, if $m = n - m$, then we have that $\cos^4\theta = \sin^4\theta$, and since $\theta \in \left(0, \frac{\pi}{2}\right)$, $\cos\theta = \sin\theta$, which gives us an Einstein hypersurface. In the second case we have that $\tanh^4(\theta) = 1$, so the sectional curvatures are vanishing, so this cannot happen. Thus we can discard the cases where both principal curvatures have the same multiplicity.

Assume now that we have four different principal curvatures. We state the following.

Lemma 3.13. *There is no $\check{R}$-Einstein hypersurface $M^4 \hookrightarrow \mathbb{Q}_c^5$, with $c^2 = 1$, with four different principal curvatures.*

Proof. Recall the $\check{R}$-Einstein equation for a hypersurface,

$$\mathcal{S}^4-(\|\mathcal{S}\|^2-2c)\mathcal{S}^2-(2ncH)\mathcal{S}-\frac{1}{n}\{\|\mathcal{S}^2\|^2-(\|\mathcal{S}\|^2-2c)\|\mathcal{S}\|^2-2c(nH)^2\}Id=0.$$

As all the eigenvalues of $\mathcal{S}$ have to satisfy the equation, and due to Vieta's formulae, we have that the sum of all of them are zero, and therefore, $4H = \lambda+\beta+\gamma+\delta = 0$ and M is a minimal hypersurface. The remaining polynomial is biquadratic, so the solutions for it are

$$\lambda_1=\sqrt{\frac{(||\mathcal{S}||^2-2c)+\sqrt{||\mathcal{S}^2||^2-2c(||\mathcal{S}||^2-2c)}}{2}}=-\lambda_2$$

$$\lambda_3=\sqrt{\frac{(||\mathcal{S}||^2-2c)-\sqrt{||\mathcal{S}^2||^2-2c(||\mathcal{S}||^2-2c)}}{2}}=-\lambda_4,$$

and so

$$||\mathcal{S}||^2=2\lambda_1^2+2\lambda_3^2=2(||\mathcal{S}||^2-2c),$$

which gives that $||\mathcal{S}||^2 = 4c$.

On the one hand, c cannot be -1 as the length of the shape operator is a non-negative number. On the other hand, one can see in [18] that the only minimal submanifold into the sphere $\mathbb{S}^5$ with $||\mathcal{S}||^2 = 4$ is the product of two spheres $\mathbb{S}^m\left(\frac{\sqrt{m}}{2}\right) \times \mathbb{S}^{4-m}\left(\frac{\sqrt{4-m}}{2}\right)$, where $m = 1, 2$, which has two distinct principal curvatures. Consequently, we do not have any weakly Einstein hypersurface in this category.

□

Remark 3.14. The case of the hyperbolic space can be seen from another point of view. If we assume that the hypersurface is minimal and $c = -1$, then if we make $\check{R}(e_\alpha, e_\alpha) - \check{R}(e_\beta, e_\beta) = 0$, being e_α, e_β, with $\alpha, \beta = 1, \ldots, n$ an orthonormal basis of principal directions, we obtain that

$$(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta)\{\|\mathcal{S}\|^2 + 2 - (\lambda_\alpha^2 + \lambda_\beta^2)\} = 0.$$

If we develop the third part, we have

$$2 + (m_\alpha - 1)\lambda_\alpha^2 + (m_\beta - 1)\lambda_\beta^2 + \sum_{i \neq \alpha, \beta}^{n} m_i \lambda_i^2 = 0,$$

where m_α denote the multiplicity of λ_α and so on. As $m_\alpha \geq 0$ for all α and the last summation is strictly positive, then the equation can only be solved if $|\lambda_\alpha| = |\lambda_\beta|$. Consequently, a n-dimensional minimal hypersurface in the hyperbolic space is $\check{R}$-Einstein if and only if it has two different principal curvatures, one opposite of each other.

$R[\rho]$-Einstein

In this section we classify the condition for two principal curvatures in the n-dimensional case.

Lemma 3.15. *Let $M^n \hookrightarrow \mathbb{Q}_c^{n+1}$, with $c^2 = 1$, be a hypersurface with two different principal curvatures. If M is $R[\rho]$-Einstein, then it is a product of two spheres if $c = 1$ or a product of a sphere and a hyperbolic space if $c = -1$.*

Proof. The case where one of the principal curvatures has multiplicity one gives us a locally conformally flat example, where we have seen that there is no hypersurface satisfying this condition. Now, assume that both principal curvatures have multiplicities greater than two. The condition $R[\rho](e_\alpha, e_\alpha) - R[\rho](e_\beta, e_\beta) = 0$ gives the following equation.

$$(\lambda - \mu)(\lambda(m-1) + \mu(-m+n-1))\left(c(n-2) + \lambda^2(m-1) + \mu^2(-m+n-1)\right) = 0,$$

where the possible real solutions are

$$\lambda = \pm\frac{\sqrt{-\mu^2(n-m-1) - c(n-2)}}{\sqrt{m-1}} \quad \text{and} \quad \lambda = -\mu\frac{n-m-1}{m-1},$$

and in that case, as the Schouten tensor is Codazzi, we can apply Merton's result. Now, the gradient of each eigenfunction is orthogonal to its own eigenspace due to $m \geq 2$. Taking the first solution, we have that $\nabla\lambda = \frac{\pm(n-m-1)\mu}{\sqrt{m-1}\sqrt{(n-m-1)\mu^2 - c(n-2)}}\nabla\mu$, and if $\mu = 0$, then $\nabla\lambda = 0$ and $\lambda \in \mathbb{R}$, but in that case $\mathcal{S} = \mathrm{diag}[\lambda, \ldots, \lambda, 0, \ldots, 0]$, but there is no isoparametric hypersurface of this form in $\mathbb{H}^{n+1}$. Thus, we may assume that $\mu \neq 0$ and therefore, $\nabla\lambda = 0$ since it is orthogonal to the whole space and then $\lambda \in \mathbb{R}$. Hence the hypersurface is isoparametric. The same happens with the second solution since it is a scalar multiple.

Cecil and Ryan show in [15] that M has to be a product of two spheres $\mathbb{S}^m(\sin^{-2}\theta) \times \mathbb{S}^{n-m}(\cos^{-2}\theta)$, where the brackets are the sectional curvature of each space. As the condition of a product manifold to be $R[\rho]$-Einstein was $c_1^2(n_1-1)^2 = c_2^2(n_2-1)^2$, then we need that

$$\tan^4\theta = \left(\frac{m-1}{n-m-1}\right)^2.$$

The same occurs if we are in the hyperbolic space. In that case, we have a product $\mathbb{S}^m(\sinh^{-2}\theta)\times \mathbb{H}^{n-m}(\cosh^{-2}\theta)$ [15], so the condition to be satisfied is

$$\tanh^4\theta = \left(\frac{m-1}{n-m-1}\right)^2.$$

$\square$

Remark 3.16. Notice that the first solution given in the proof above can be real if and only if $c=-1$. Therefore, we cannot discard it.

$\check{\rho}$-Einstein

Again, we just classify for two principal curvatures in the n-dimensional case.

Lemma 3.17. *Let $M^n \hookrightarrow \mathbb{Q}_c^{n+1}$, with $c^2=1$, be a hypersurface with two different principal curvatures. If M is $\check{\rho}$-Einstein, then it is either a product of two spheres, a product of a hyperbolic space and a sphere or a rotation hypersurface over a curve.*

Proof. Assume first that both principal curvatures, λ and μ, both different, have multiplicities greater than two m and $n-m$ respectively. The condition $\check{\rho}(e_\alpha,e_\alpha)-\check{\rho}(e_\beta,e_\beta)=0$ gives the following equation.

$$(c(n-1)+\lambda(\lambda(m-1)+\mu(n-m)))^2-(c(n-1)+\mu(\lambda m+\mu(n-m-1)))^2=0,$$

which has two solutions given by

$$\begin{aligned}\lambda =& \frac{-n\mu \pm \sqrt{\mu^2\left((n-2m)^2+4(n-1)\right)-8c(m-1)(n-1)}}{2(m-1)} \quad \text{and}\\ \lambda =& -\mu\frac{n-m-1}{m-1}.\end{aligned}$$

In the first case we obtain that

$$\nabla\lambda = \frac{\left(\pm\frac{\left((n-2m)^2+4(n-1)\right)\mu}{\sqrt{((n-2m)^2+4(n-1))\mu^2-8c(m-1)(n-1)}}-n\right)}{2(m-1)}\nabla\mu$$

and the bracket does not vanish for any real value. Thus, since all solutions depends on one principal curvature and using the same arguments as in the previous proof, we obtain the same result using Merton's Theorem.

If λ has multiplicity one, then we have a locally conformally flat hypersurface. Moreover, the equation for the condition remains

$$\mu(n-2)(\lambda-\mu)(2c(n-1)+\mu(n(\lambda+\mu)-2\mu))=0$$

from we can get that

$$\lambda=-\frac{2c(n-1)+\mu^2(n-2)}{n\mu}$$

Thus, by Corollary 3.3 from [13], it is a rotational hypersurface over a curve. □

Chapter 4
Homogeneous four-dimensional weakly-Einstein manifolds

In this chapter, we classify homogeneous weakly-Einstein metrics in dimension four. The main results are shown in [29].

4.1 Introduction

Bérard-Bergery showed in [3] that a four-dimensional homogeneous manifolds is either a Lie group with a left-invariant metric or a symmetric manifold. We have seen in the introduction that we can construct this groups knowing the brackets of a three-dimensional Lie algebra and then extending to a four-dimensional one. Once we know the brackets, we can use the Koszul formula to obtain the Christoffel symbols and then all the geometrical objects involving them as the curvature and the Ricci tensors. Thus, we can construct all the weakly-Einstein tensors, providing a new ambient where we can get new examples. The classification of these metrics is harder since the polynomials involved are in terms of the structure constants of the groups, so we may have many different unknowns. To get through this issue, we use Gröbner Basis.

There are some previous works in this field. Jensen in [45] classified homogeneous four-dimensional manifolds which are a complex or a real space form or a product of two surfaces $M_1(c)\times M_2(c)$ with the same constant sectional curvature. Moreover, Arias-Marco and Kowalski classified in [1] $\check{R}$-Einstein manifolds in the same setting, turning out being a product of two surfaces $M_1(c) \times M_2(-c)$ with opposite curvature or a Lie group with Lie algebra

$$[e_4, e_1] = \alpha e_1, \quad [e_4, e_2] = -\alpha e_2 - \beta e_3, \quad [e_4, e_3] = \beta e_2 - \alpha e_3,$$

with $\{e_1, \ldots, e_4\}$ an orthonormal frame and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$.

The aim of this chapter is to extend the result by Arias-Marco and Kowalski to the other weakly-Einstein conditions. Moreover, our classification is done up to homothetic class. Thus, we also give a class including Arias-Marco and Kowalski's example.

The locally symmetric case is classified as follows.

Lemma 4.1. *Let (M, g) be a non-Einstein locally symmetric four-dimensional manifold. Then the following are equivalent*

(i) (M, g) *is $\check{R}$-Einstein.*

(ii) (M, g) *is $\check{\rho}$-Einstein.*

(iii) (M,g) *is* $R[\rho]$*-Einstein.*

(iv) (M,g) *is homothetic to the product* $M=\mathbb{S}^2\times\mathbb{H}^2$.

Proof. Let (M,g) be a symmetric space. Since the Ricci tensor is parallel, the Ricci curvatures are constant and the corresponding eigenspaces are parallel. Let $\{\kappa_1,\dots,\kappa_4\}$ be the Ricci curvatures. If (M,g) is non-Einstein, then there are at least two-distinct Ricci curvatures since the space splits in each eigenspace. Let κ be a Ricci curvature appearing with multiplicity one. Then the associated eigenspace splits off a one-dimensional manifold so that (M,g) is locally isometric to $\mathbb{R}\times N$ for some three-dimensional symmetric manifold N. Now such a product $\mathbb{R}\times N$ is $\check{R}$-Einstein, $\check{\rho}$-Einstein or $R[\rho]$-Einstein if and only if it is flat. Hence the only possibility is that the Ricci operator has exactly two-distinct Ricci curvatures with multiplicity two. Moreover, in such case the Ricci curvatures κ_1 and κ_2 satisfy $\kappa_1=-\kappa_2$ and the result follows. □

Hence, we focus on the left-invariant metrics in what remains. The purpose is proving the next theorem.

Theorem 4.2. *Let* (M,g) *be a four-dimensional simply connected homogeneous manifold. Then*

(1) (M,g) *is* $\check{R}$*-Einstein and non-symmetric if and only if it is homothetic to the Lie group* $\mathbb{R}\ltimes\mathbb{R}^3$ *with left-invariant metric determined by the Lie algebra*

$$[e_4,e_1]=e_1,\quad [e_4,e_2]=-e_2,\quad [e_4,e_3]=-e_3,$$

where $\{e_1,\dots,e_4\}$ *is an orthonormal basis.*

(2) (M,g) *is* $\check{\rho}$*-Einstein and non-symmetric if and only if it is homothetic to one of the following:*

(2.a) The Lie group $SU(2)\times\mathbb{R}$ *with left-invariant metric determined by the Lie algebra*

$$\begin{aligned}&[e_1,e_2]=(4\pm2\sqrt{2})e_3, && [e_2,e_3]=(3\pm2\sqrt{2})e_1, \quad [e_3,e_1]=e_2,\\ &[e_4,e_1]=-e_2, && [e_4,e_2]=(3\pm2\sqrt{2})e_1,\end{aligned}$$

where $\{e_1,\dots,e_4\}$ *is an orthonormal basis.*

(2.b) The Lie group $\mathbb{R}\ltimes H^3$ *with left-invariant metric determined by the Lie algebra*

$$[e_1,e_2]=e_3,\quad [e_1,e_4]=\frac{1}{2}e_1,\quad [e_2,e_4]=-\frac{1}{2}e_2,$$

where $\{e_1,\dots,e_4\}$ *is an orthonormal basis.*

(2.c) The Lie group $\mathbb{R}\ltimes\mathbb{R}^3$ *with left-invariant metric determined by the Lie algebra*

$$\begin{aligned}[e_4,e_1]&=e_1-\frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_2+\frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_3,\\ [e_4,e_2]&=\frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_1+\alpha e_2+\frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_3,\\ [e_4,e_3]&=-\frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_1-\frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_2-\tfrac{\alpha}{\alpha+1}e_3,\end{aligned}$$

where $\{e_1, \ldots, e_4\}$ *is an orthonormal basis and* $\alpha \in (-1, 1)$, $\alpha \neq -\frac{1}{2}$, $\alpha \neq 0$.

(3) (M, g) *is* $R[\rho]$*-Einstein if and only if it is symmetric.*

Remark 4.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A left-invariant metric is called an algebraic Ricci soliton if

$$\mathfrak{D} = Q_\rho - \lambda \, \mathrm{Id},$$

with $\lambda \in \mathbb{R}$ and $\mathfrak{D}$ is a derivation of the algebra. The metric defined in Theorem 4.2.(1) is the only algebraic Ricci soliton for $\lambda = -3$.

4.1.1 Four-dimensional homogeneous Lie groups

We have shown how these four-dimensional groups can be constructed following the work of Milnor [56]. We have two cases whether the group is unimodular or non-unimodular.

The direct products $SL(2, \mathbb{R}) \times \mathbb{R}$ **and** $SU(2) \times \mathbb{R}$

Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of the Lie algebra $\mathfrak{g}_3$ ($\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{su}(2)$) such that

$$[v_2, v_3] = \lambda_1 v_1, \quad [v_3, v_1] = \lambda_2 v_2, \quad [v_1, v_2] = \lambda_3 v_3, \qquad \lambda_1 \lambda_2 \lambda_3 \neq 0.$$

Here λ_1, λ_2 and λ_3 are all positive for the Lie algebra $\mathfrak{su}(2)$ and $\lambda_1\lambda_2\lambda_3 < 0$ in the $\mathfrak{sl}(2, \mathbb{R})$ case. Take v_4 (not necessarily orthogonal to $\mathfrak{g}_3$) so that $[v_4, v_i] = 0$, for all $i = 1, 2, 3$ and let $\langle \cdot\,, \cdot \rangle$ be an inner product on the four-dimensional algebra. Set $e_k = v_k$, ($k = 1, 2, 3$) and normalize the vector $v_4 - \sum_{i=1}^{3} \langle v_4, v_i \rangle v_i$ so that $\{e_1, \ldots, e_4\}$ is an orthonormal basis with brackets given by

$$\begin{aligned} &[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, && [e_3, e_1] = \lambda_2 e_2, \\ &[e_1, e_4] = \frac{1}{R}\{k_3 \lambda_2 e_2 - k_2 \lambda_3 e_3\}, && [e_2, e_4] = \frac{1}{R}\{k_1 \lambda_3 e_3 - k_3 \lambda_1 e_1\}, \\ &[e_3, e_4] = \frac{1}{R}\{k_2 \lambda_1 e_1 - k_1 \lambda_2 e_2\},, && R > 0, \end{aligned} \tag{4.1}$$

where $k_i = \langle v_4, v_i \rangle$.

The semi-direct products $\mathbb{R}e_4 \ltimes E(1, 1)$ **and** $\mathbb{R}e_4 \ltimes E(2)$

Let $\mathfrak{g}_3$ be either the Poincare algebra $\mathfrak{e}(1, 1)$ or the Euclidean algebra $\mathfrak{e}(2)$. There exist an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathfrak{g}_3$ such that

$$[v_1, v_2] = 0, \quad [v_2, v_3] = \lambda_1 v_1, \quad [v_3, v_1] = \lambda_2 v_2,$$

where $\lambda_1\lambda_2 \neq 0$. Moreover $\lambda_1\lambda_2 > 0$ if $\mathfrak{g}_3 = E(2)$ ($\lambda_1\lambda_2 < 0$ if $\mathfrak{g}_3 = E(1,1)$). Writing a derivation of $\mathfrak{g}_3$ with respect to $\{v_1, v_2, v_3\}$, one has

$$\operatorname{der}(\mathfrak{g}_3) = \left\{ \begin{pmatrix} b & a & c \\ -\dfrac{\lambda_2}{\lambda_1}a & b & d \\ 0 & 0 & 0 \end{pmatrix}; a, b, c, d \in R \right\}.$$

Extend $\{v_1, v_2, v_3\}$ to a basis $\{v_1, v_2, v_3, v_4\}$ so that $\operatorname{ad}(v_4)$ is determined by the derivation and consider an orthonormal basis $\{e_i\}$ so that $e_i = v_i$ for $i = 1, 2, 3$ and e_4 is obtained after normalization of $v_4 - \sum_i k_i v_i$. Then one has the non zero Lie brackets

$$\begin{aligned} &[e_2, e_3] = \lambda_1 e_1, && [e_3, e_1] = \lambda_2 e_2, \\ &[e_4, e_1] = \frac{1}{R}\{be_1 - \lambda_2(\frac{a}{\lambda_1} + k_3)e_2\}, && [e_4, e_2] = \frac{1}{R}\{(a + k_3\lambda_1)e_1 + be_2\}, \\ &[e_4, e_3] = \frac{1}{R}\{(c - k_2\lambda_1)e_1 + (d + k_1\lambda_2)e_2\}, && R > 0, \end{aligned} \tag{4.2}$$

The semi-direct product $\mathbb{R}e_4 \ltimes H^3$

Let $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{h}_3$ be a semi-direct product of $\mathbb{R}$ with the Heisenberg algebra $\mathfrak{h}_3$. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of $\mathfrak{h}_3$ so that

$$[v_1, v_2] = \gamma v_3, \quad [v_2, v_3] = 0, \quad [v_1, v_3] = 0, \qquad \gamma \neq 0.$$

The derivations of $\mathfrak{h}_3$ with respect to the basis $\{v_1, v_2, v_3\}$ are given by

$$\operatorname{der}(\mathfrak{h}_3) = \left\{ \begin{pmatrix} a & c & 0 \\ -c & d & 0 \\ h & f & a+d \end{pmatrix}; a, c, d, h, f \in \mathbb{R} \right\}.$$

Let $\{v_1, v_2, v_3, v_4\}$ be a basis of $\mathfrak{g}$, where $\operatorname{ad}(e_4)$ is determined by a derivation as above. After normalization, as in the previous cases, there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ where the nonzero Lie brackets are given as follows

$$\begin{aligned} &[e_1, e_2] = \gamma e_3, && [e_4, e_1] = \frac{1}{R}\{ae_1 - ce_2 + (h + k_2\gamma)e_3\}, \\ &[e_4, e_3] = \frac{1}{R}(a + d)e_3, && [e_4, e_2] = \frac{1}{R}\{ce_1 + de_2 + (f - k_1\gamma)e_3\}, \quad R > 0. \end{aligned} \tag{4.3}$$

The semi-direct product $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

Let $\mathfrak{r}^3$ be the three-dimensional Abelian Lie algebra. The corresponding algebra of derivations is $\mathfrak{gl}(3, \mathbb{R})$. For any $D \in \mathfrak{gl}(3, \mathbb{R})$, decomposing it into its symmetric and skew-symmetric part, one has that any $D \in \mathfrak{gl}(3, \mathbb{R})$ is conjugate to one of the matrices given by

$$\left\{ \begin{pmatrix} a & -b & -c \\ b & f & -h \\ c & h & p \end{pmatrix}; a, b, c, f, h, p \in \mathbb{R} \right\}.$$

The corresponding semi-direct product $\mathbb{R} \ltimes \mathfrak{r}^3$ expresses in an orthonormal basis $\{e_1, \ldots, e_4\}$ as

$$\begin{aligned} &[e_4, e_1] = \frac{1}{R}(ae_1 + be_2 + ce_3), \qquad [e_4, e_2] = \frac{1}{R}(-be_1 + fe_2 + he_3), \\ &[e_4, e_3] = \frac{1}{R}(-ce_1 - he_2 + pe_3), \quad R > 0. \end{aligned} \tag{4.4}$$

4.2 Left-invariant $\check{\rho}$-Einstein metrics

During the next section we prove Theorem 4.2 (2). All cases follow the same structure. Firstly, we get a system of polynomials related with the condition and then we use Gröbner basis in order to reduce some variables. Finally, we classify up to homothetic class.

4.2.1 The direct products $SL(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

Lemma 4.4. *$SU(2) \times \mathbb{R}$ admits a non-Einstein $\check{\rho}$-Einstein left invariant metric if and only if it is homothetic to the Lie group determined by the Lie algebra given by*

$$\begin{aligned} &[e_1, e_2] = (4 \pm 2\sqrt{2})e_3, \quad [e_1, e_3] = -e_2, \qquad [e_2, e_3] = (3 \pm 2\sqrt{2})e_1 \\ &[e_4, e_1] = -e_2, \qquad [e_4, e_2] = (3 \pm 2\sqrt{2})e_1, \end{aligned} \tag{4.5}$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis. Moreover, $SL(2,\mathbb{R}) \times \mathbb{R}$ does not admit any left-invariant $\check{\rho}$-Einstein metric.

Proof. Take the algebra given in Equation (4.1) and let $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ be the trace-free $\check{\rho}$-tensor. A straightforward calculation shows that the $\check{\rho}_0$ components are given by

$$\begin{aligned} &16R^4\check{\rho}_{0_{11}} = \mathfrak{C}_{11}, \quad 4R^4\check{\rho}_{0_{12}} = \mathfrak{C}_{12}, \quad 4R^4\check{\rho}_{0_{13}} = \mathfrak{C}_{13}, \quad 4R^3\check{\rho}_{0_{14}} = \mathfrak{C}_{14}, \\ &16R^4\check{\rho}_{0_{22}} = \mathfrak{C}_{22}, \quad 4R^4\check{\rho}_{0_{23}} = \mathfrak{C}_{23}, \quad 4R^3\check{\rho}_{0_{24}} = \mathfrak{C}_{24}, \quad 16R^4\check{\rho}_{0_{33}} = \mathfrak{C}_{33}, \\ &4R^3\check{\rho}_{0_{34}} = \mathfrak{C}_{34}, \quad 16R^4\check{\rho}_{0_{44}} = \mathfrak{C}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{C}_{\mathrm{ij}}$ are determined as follows.

$$\begin{aligned} \mathfrak{C}_{11} = {} & R^4\lambda_1^4 + 4R^4\lambda_1^3\lambda_2 - 10R^4\lambda_1^2\lambda_2^2 + 4k_1^2R^2\lambda_1\lambda_2^3 + 4R^4\lambda_1\lambda_2^3 - 3k_1^4\lambda_2^4 - 2k_1^2R^2\lambda_2^4 + R^4\lambda_2^4 \\ & + k_3^4(\lambda_1 - \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2) + 4R^4\lambda_1^3\lambda_3 + 12R^4\lambda_1^2\lambda_2\lambda_3 - 4k_1^2R^2\lambda_1\lambda_2^2\lambda_3 \\ & - 4R^4\lambda_1\lambda_2^2\lambda_3 + 4k_1^4\lambda_2^3\lambda_3 - 8k_1^2R^2\lambda_2^3\lambda_3 - 12R^4\lambda_2^3\lambda_3 - 10R^4\lambda_1^2\lambda_3^2 - 4k_1^2R^2\lambda_1\lambda_2\lambda_3^2 \\ & - 4R^4\lambda_1\lambda_2\lambda_3^2 - 2k_1^4\lambda_2^2\lambda_3^2 + 20k_1^2R^2\lambda_2^2\lambda_3^2 + 22R^4\lambda_2^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_3^3 + 4R^4\lambda_1\lambda_3^3 \\ & + 4k_1^4\lambda_2\lambda_3^3 - 8k_1^2R^2\lambda_2\lambda_3^3 - 12R^4\lambda_2\lambda_3^3 - 3k_1^4\lambda_3^4 - 2k_1^2R^2\lambda_3^4 + R^4\lambda_3^4 \\ & + k_2^4(\lambda_1 - \lambda_3)^2(\lambda_1^2 + 6\lambda_1\lambda_3 + \lambda_3^2) + 2k_2^2(R^2(\lambda_1 - \lambda_3)(\lambda_1^3 + \lambda_1^2\,(2\lambda_2 + 5\lambda_3) \\ & + \lambda_1(-4\lambda_2^2 + 8\lambda_2\lambda_3 - 5\lambda_3^2) - \lambda_3(4\lambda_2^2 - 6\lambda_2\lambda_3 + \lambda_3^2)) - k_1^2(\lambda_1^2\lambda_2(\lambda_2 - 2\lambda_3) \\ & + \lambda_3^3(-2\lambda_2 + \lambda_3) - 2\lambda_1\lambda_3(\lambda_2^2 - 3\lambda_2\lambda_3 + \lambda_3^2)) + k_3^2(\lambda_1^4 + \lambda_2^2\lambda_3^2 + 2\lambda_1^3(\lambda_2 + \lambda_3) \\ & + 2\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3) - 2\lambda_1^2(2\lambda_2^2 + \lambda_2\lambda_3 + 2\lambda_3^2))) + 2k_3^2(k_1^2(-\lambda_2^3(\lambda_2 - 2\lambda_3) \end{aligned}$$

$$
\begin{aligned}
&+\lambda_1^2(2\lambda_2-\lambda_3)\,\lambda_3+2\lambda_1\lambda_2(\lambda_2^2-3\lambda_2\lambda_3+\lambda_3^2))+R^2(\lambda_1-\lambda_2)(\lambda_1^3+\lambda_1^2(5\,\lambda_2\\
&+2\lambda_3)+\lambda_1(-5\lambda_2^2+8\lambda_2\lambda_3-4\lambda_3^2)-\lambda_2(\lambda_2^2-6\lambda_2\lambda_3+4\lambda_3^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}_{12}=\;&k_1k_2(R^2\lambda_1^2\lambda_2^2-k_1^2\lambda_1\lambda_2^3-4R^2\lambda_1^2\lambda_2\lambda_3-4R^2\lambda_1\lambda_2^2\lambda_3+R^2\lambda_1^2\lambda_3^2+k_1^2\lambda_1\lambda_2\lambda_3^2\\
&+6R^2\lambda_1\lambda_2\lambda_3^2+k_1^2\lambda_2^2\lambda_3^2+R^2\lambda_2^2\lambda_3^2-k_1^2\lambda_3^4-R^2\lambda_3^4-k_2^2(\lambda_1^2-\lambda_3^2)(\lambda_1\lambda_2-\;\lambda_3^2)\\
&+k_3^2(\lambda_1^2\lambda_2^2-\lambda_1^3\lambda_3-\;\lambda_2^3\lambda_3+\lambda_1\lambda_2\lambda_3^2)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}_{13}=\;&k_1k_3(R^2\lambda_1^2\lambda_2^2-k_1^2\lambda_2^4-R^2\lambda_2^4-4R^2\lambda_1^2\lambda_2\lambda_3+k_1^2\lambda_1\lambda_2^2\lambda_3+6R^2\lambda_1\lambda_2^2\lambda_3+R^2\lambda_1^2\lambda_3^2\\
&-4R^2\lambda_1\lambda_2\lambda_3^2+k_1^2\lambda_2^2\lambda_3^2+R^2\lambda_2^2\lambda_3^2-k_1^2\lambda_1\lambda_3^3-k_3^2(\lambda_1^2-\lambda_2^2)(-\lambda_2^2+\lambda_1\,\lambda_3)\\
&+k_2^2(-\lambda_1^3\lambda_2+\lambda_1\lambda_2^2\,\lambda_3+\lambda_1^2\lambda_3^2-\lambda_2\lambda_3^3)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}_{14}=\;&-k_1((R^2(-\lambda_1^2+(\lambda_2-\lambda_3)^2)+k_1^2(\lambda_2-\lambda_3)^2)(\lambda_2-\lambda_3)^2+k2^2(\lambda_1-\lambda_3)(\lambda_1^2\lambda_2\\
&-\lambda_1\lambda_3(\lambda_2+\lambda_3)-\lambda_3(2\,\lambda_2^2-4\lambda_2\lambda_3+\lambda_3^2))+k_3^2(\lambda_1-\lambda_2)(\lambda_1^2\lambda_3-\;\lambda_1\lambda_2(\lambda_2+\lambda_3)\\
&-\lambda_2(\;\lambda_2^2-4\lambda_2\lambda_3+2\lambda_3^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}_{22}=\;&R^4\lambda_1^4+4R^4\lambda_1^3\lambda_2-8k_1^2R^2\lambda_1^2\lambda_2^2-10R^4\lambda_1^2\lambda_2^2+4k_1^2R^2\lambda_1\lambda_2^3+4R^4\lambda_1\lambda_2^3+k_1^4\lambda_2^4\\
&+2k_1^2R^2\lambda_2^4+R^4\lambda_2^4+k_3^4(\lambda_1-\lambda_2)^2(\lambda_1^2+6\lambda_1\lambda_2+\lambda_2^2)-12R^4\lambda_1^3\lambda_3-4R^4\lambda_1^2\lambda_2\lambda_3\\
&+12k_1^2R^2\lambda_1\lambda_2^2\lambda_3+12R^4\lambda_1\lambda_2^2\lambda_3+4k_1^4\lambda_2^3\lambda_3+8k_1^2R^2\lambda_2^3\lambda_3+4R^4\lambda_2^3\lambda_3+8k_1^2R^2\lambda_1^2\lambda_3^2\\
&+22R^4\lambda_1^2\lambda_3^2-4k_1^2R^2\lambda_1\lambda_2\lambda_3^2-4R^4\lambda_1\lambda_2\lambda_3^2-10k_1^4\lambda_2^2\lambda_3^2-20k_1^2R^2\lambda_2^2\lambda_3^2-10R^4\lambda_2^2\lambda_3^2\\
&-12k_1^2R^2\lambda_1\lambda_3^3-12R^4\lambda_1\lambda_3^3+4k_1^4\lambda_2\lambda_3^3+8k_1^2R^2\lambda_2\lambda_3^3+4R^4\lambda_2\lambda_3^3+k_1^4\lambda_3^4+2k_1^2R^2\lambda_3^4\\
&+R^4\lambda_3^4-k_2^4(\lambda_1-\lambda_3)^2(3\lambda_1^2+2\lambda_1\lambda_3+3\lambda_3^2)-2k_2^2(R^2(\lambda_1-\lambda_3)^2(\lambda_1^2-2\lambda_1\lambda_2\\
&+6\lambda_1\lambda_3-2\lambda_2\lambda_3+\lambda_3^2)+k_3^2(\lambda_1^4+6\lambda_1^2\lambda_2\lambda_3+\lambda_2^2\,\lambda_3^2-2\lambda_1^3(\lambda_2+\lambda_3)\\
&-2\lambda_1\lambda_2\lambda_3(\lambda_2+\lambda_3))+k_1^2(\lambda_1^2\lambda_2(\lambda_2-2\lambda_3)+\lambda_3^3(-2\lambda_2+\lambda_3)-2\lambda_1\lambda_3(\lambda_2^2-3\lambda_2\lambda_3\\
&+\lambda_3^2)))+2k_3^2(k_1^2(\lambda_2^2(\lambda_2^2+2\lambda_2\lambda_3-4\lambda_3^2)+2\lambda_1\lambda_2(\lambda_2^2-\lambda_2\lambda_3+\lambda_3^2)+\lambda_1^2(-4\lambda_2^2+2\lambda_2\lambda_3\\
&+\lambda_3^2))+R^2(\lambda_1-\lambda_2)(\lambda_1^3+\lambda_1^2(5\,\lambda_2-6\lambda_3)-\lambda_2(\lambda_2^2+2\lambda_2\lambda_3-4\lambda_3^2)+\lambda_1(-5\lambda_2^2\\
&-8\lambda_2\lambda_3+4\lambda_3^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}_{23}=\;&k_2k_3(-R^2\lambda_1^4+R^2\lambda_1^2\lambda_2^2-k_1^2\lambda_1\lambda_2^3+k_1^2\lambda_1^2\lambda_2\lambda_3+6R^2\lambda_1^2\lambda_2\lambda_3-4R^2\lambda_1\lambda_2^2\lambda_3\\
&+R^2\lambda_1^2\lambda_3^2-4R^2\lambda_1\lambda_2\lambda_3^2+k_1^2\lambda_2^2\lambda_3^2+R^2\lambda_2^2\lambda_3^2-k_1^2\lambda_1\lambda_3^3-k_3^2(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_2\,\lambda_3)\\
&-k_2^2(\lambda_1^2-\lambda_2\lambda_3)(\lambda_1^2-\;\lambda_3^2)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}_{24}=\;&-k_2(R^2\lambda_1^4-R^2\lambda_1^2\lambda_2^2+k_1^2\lambda_1\lambda_2^3+k_2^2(\lambda_1-\lambda_3)^4-4R^2\lambda_1^3\lambda_3-2k_1^2\lambda_1^2\lambda_2\lambda_3-2k_1^2\lambda_1\lambda_2^2\lambda_3\\
&+2R^2\lambda_1\lambda_2^2\lambda_3+2k_1^2\lambda_1^2\lambda_3^2+6R^2\lambda_1^2\lambda_3^2+5k_1^2\lambda_1\lambda_2\lambda_3^2-k_1^2\lambda_2^2\lambda_3^2-R^2\lambda_2^2\lambda_3^2-4k_1^2\lambda_1\lambda_3^3\\
&-4R^2\lambda_1\lambda_3^3+k_1^2\lambda_3^4+R^2\lambda_3^4+k_3^2(\lambda_1-\lambda_2)(\lambda_1^3+\lambda_1^2(\;\lambda_2-4\lambda_3)-\lambda_2^2\lambda_3+\lambda_1\,\lambda_3(\lambda_2\\
&+2\lambda_3))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}_{33}=\;&R^4\lambda_1^4-12R^4\lambda_1^3\lambda_2+8k_1^2R^2\lambda_1^2\lambda_2^2+22R^4\lambda_1^2\lambda_2^2-12k_1^2R^2\lambda_1\lambda_2^3-12R^4\lambda_1\lambda_2^3+k_1^4\lambda_2^4\\
&+2k_1^2R^2\lambda_2^4+R^4\lambda_2^4-k_3^4(\lambda_1-\lambda_2)^2(3\lambda_1^2+2\lambda_1\lambda_2+3\lambda_2^2)+4R^4\lambda_1^3\lambda_3-4R^4\lambda_1^2\lambda_2\lambda_3\\
&-4k_1^2R^2\lambda_1\lambda_2^2\lambda_3-4R^4\lambda_1\lambda_2^2\lambda_3+4k_1^4\lambda_2^3\lambda_3+8k_1^2R^2\lambda_2^3\lambda_3+4R^4\lambda_2^3\lambda_3-8k_1^2R^2\lambda_1^2\lambda_3^2\\
&-10R^4\lambda_1^2\lambda_3^2+12k_1^2R^2\lambda_1\lambda_2\lambda_3^2+12R^4\lambda_1\lambda_2\lambda_3^2-10k_1^4\lambda_2^2\lambda_3^2-20k_1^2R^2\lambda_2^2\lambda_3^2-10R^4\lambda_2^2\lambda_3^2\\
&+4k_1^2R^2\lambda_1\lambda_3^3+4R^4\lambda_1\lambda_3^3+4k_1^4\lambda_2\lambda_3^3+8k_1^2R^2\lambda_2\lambda_3^3+4R^4\lambda_2\lambda_3^3+k_1^4\lambda_3^4+2k_1^2R^2\lambda_3^4\\
&+R^4\lambda_3^4+k_2^4(\lambda_1-\lambda_3)^2(\lambda_1^2+6\lambda_1\lambda_3+\lambda_3^2)-2k_3^2(R^2(\lambda_1-\lambda_2)^2(\lambda_1^2+6\lambda_1\lambda_2+\lambda_2^2\\
&-2\lambda_1\lambda_3-2\lambda_2\lambda_3)+k_1^2(\lambda_2^3(\lambda_2-2\lambda_3)+\lambda_1^2\lambda_3(-2\lambda_2+\;\lambda_3)-2\lambda_1\lambda_2(\lambda_2^2-3\lambda_2\lambda_3\\
&+\lambda_3^2)))+2k_2^2(-k_3^2(\lambda_1^4+6\lambda_1^2\lambda_2\lambda_3+\lambda_2^2\,\lambda_3^2-2\lambda_1^3(\lambda_2+\lambda_3)-2\lambda_1\lambda_2\lambda_3(\lambda_2+\lambda_3))\\
&+R^2(\lambda_1-\lambda_3)(\lambda_1^3+\lambda_1^2(-6\,\lambda_2+5\lambda_3)+\lambda_1(4\lambda_2^2-8\lambda_2\lambda_3-5\lambda_3^2)-\lambda_3(-4\lambda_2^2+2\lambda_2\lambda_3\\
&+\lambda_3^2))+k_1^2(\lambda_1^2(\lambda_2^2+2\lambda_2\lambda_3-4\lambda_3^2)+2\lambda_1\lambda_3(\lambda_2^2-\lambda_2\lambda_3\;+\lambda_3^2)+\lambda_3^2(-4\lambda_2^2+2\lambda_2\lambda_3\\
&+\lambda_3^2))),
\end{aligned}
$$

$$\begin{aligned}\mathfrak{C}_{34} = &-k_3(R^2\lambda_1^4 + k_3^2(\lambda_1-\lambda_2)^4 - 4R^2\lambda_1^3\lambda_2 + 2k_1^2\lambda_1^2\lambda_2^2 + 6R^2\lambda_1^2\lambda_2^2 - 4k_1^2\lambda_1\lambda_2^3 - 4R^2\lambda_1\lambda_2^3\\ &+ k_1^2\lambda_2^4 + R^2\lambda_2^4 - 2k_1^2\lambda_1^2\lambda_2\lambda_3 + 5k_1^2\lambda_1\lambda_2^2\lambda_3 - R^2\lambda_1^2\lambda_3^2 - 2k_1^2\lambda_1\lambda_2\lambda_3^2 + 2R^2\lambda_1\lambda_2\lambda_3^2\\ &- k_1^2\lambda_2^2\lambda_3^2 - R^2\lambda_2^2\lambda_3^2 + k_1^2\lambda_1\lambda_3^3 + k_2^2(\lambda_1-\lambda_3)(\lambda_1^3 - \lambda_2\lambda_3^2 + \lambda_1^2(-4\lambda_2+\lambda_3)+\\ &\lambda_1\lambda_2(2\lambda_2+\lambda_3))),\end{aligned}$$

$$\begin{aligned}\mathfrak{C}_{44} = &-3R^4\lambda_1^4 + 4R^4\lambda_1^3\lambda_2 - 2R^4\lambda_1^2\lambda_2^2 + 4k_1^2R^2\lambda_1\lambda_2^3 + 4R^4\lambda_1\lambda_2^3 + k_1^4\lambda_2^4 - 2k_1^2R^2\lambda_2^4\\ &- 3R^4\lambda_2^4 + k_3^4(\lambda_1-\lambda_2)^2(\lambda_1^2 - 10\lambda_1\lambda_2 + \lambda_2^2) + 4R^4\lambda_1^3\lambda_3 - 4R^4\lambda_1^2\lambda_2\lambda_3 - 4k_1^2R^2\lambda_1\lambda_2^2\lambda_3\\ &- 4R^4\lambda_1\lambda_2^2\lambda_3 - 12k_1^4\lambda_2^3\lambda_3 - 8k_1^2R^2\lambda_2^3\lambda_3 + 4R^4\lambda_2^3\lambda_3 - 2R^4\lambda_1^2\lambda_3^2 - 4k_1^2R^2\lambda_1\lambda_2\lambda_3^2\\ &- 4R^4\lambda_1\lambda_2\lambda_3^2 + 22k_1^4\lambda_2^2\lambda_3^2 + 20k_1^2R^2\lambda_2^2\lambda_3^2 - 2R^4\lambda_2^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_3^3 + 4R^4\lambda_1\lambda_3^3\\ &- 12k_1^4\lambda_2\lambda_3^3 - 8k_1^2R^2\lambda_2\lambda_3^3 + 4R^4\lambda_2\lambda_3^3 + k_1^4\lambda_3^4 - 2k_1^2R^2\lambda_3^4 - 3R^4\lambda_3^4 + k_2^4(\lambda_1-\lambda_3)^2(\lambda_1^2\\ &- 10\lambda_1\lambda_3 + \lambda_3^2) + 2k_2^2(-R^2(\lambda_1-\lambda_3)^2(\lambda_1^2 - 2\lambda_1\lambda_2 + 6\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2) + k_3^2(\lambda_1^4\\ &+ \lambda_2^2\lambda_3^2 - 6\lambda_1^3(\lambda_2+\lambda_3) - 6\lambda_1\lambda_2\lambda_3(\lambda_2+\lambda_3) + 2\lambda_1^2(2\lambda_2^2 + 7\lambda_2\lambda_3 + 2\lambda_3^2)) + k_1^2(\lambda_3^2(4\lambda_2^2\\ &- 6\lambda_2\lambda_3 + \lambda_3^2) - 2\lambda_1\lambda_3(3\lambda_2^2 - 7\lambda_2\lambda_3 + 3\lambda_3^2) + \lambda_1^2(\lambda_2^2 - 6\lambda_2\lambda_3 + 4\lambda_3^2))) - 2k_3^2(R^2(\lambda_1\\ &- \lambda_2)^2(\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3) - k_1^2(\lambda_1^2(4\lambda_2^2 - 6\lambda_2\lambda_3 + \lambda_3^2) - 2\lambda_1\lambda_2(3\lambda_2^2\\ &- 7\lambda_2\lambda_3 + 3\lambda_3^2) + \lambda_2^2(\lambda_2^2 - 6\lambda_2\lambda_3 + 4\lambda_3^2))).\end{aligned}$$

Hence a left-invariant metric is $\check{\rho}$-Einstein if and only if the system $\{\mathfrak{C}_{ij} = 0\}$ has a real solution. First of all, observe that if $\lambda_1 = \lambda_2 = \lambda_3$, then the Ricci operator takes the form $Q_\rho = \frac{1}{2}\operatorname{diag}[\lambda_1^2, \lambda_1^2, \lambda_1^2, 0]$, which shows that it cannot be $\check{\rho}$-Einstein.

Now, we consider the terms k_i from Equation (4.1). Up to a permutation of the basis $\{e_1, e_2, e_3\}$ one may assume that one of the following holds:

$$(i)\ k_1k_2k_3 \neq 0, \qquad (ii)\ k_1 = 0, k_2k_3 \neq 0, \qquad (iii)\ k_1 = k_2 = 0, k_3 \neq 0.$$

We analyze the different possibilities separately.

(i) $k_1k_2k_3 \neq 0$

Let $\mathcal{I} \subset \mathbb{R}[k_1, k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ be the ideal generated by the polynomials $\mathfrak{C}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and a detailed analysis of that basis (using that $k_1k_2k_3 \neq 0$) shows that the polynomial

$$R^4\lambda_1\lambda_2^2\lambda_3^2(\lambda_2^4 - \lambda_2^3\lambda_3 + \lambda_2\lambda_3^3 - \lambda_3^4)$$

belongs to $\mathcal{G}$. Since $R^4\lambda_1\lambda_2^2\lambda_3^2 \neq 0$, one has that $\lambda_3 = \pm\lambda_2$.

Next, we compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{\lambda_3 + \lambda_2\}$ with respect to the lexicographical order so that the polynomial $R^4\lambda_1^2\lambda_3^3$ belongs to $\mathcal{G}_1$, which gives a contradiction since the λ_i are not vanishing. Proceeding in the same way, let $\mathcal{G}_2$ be the ideal generated by $\mathcal{G} \cup \{\lambda_3 - \lambda_2\}$ with respect to the lexicographical order. Then $R^6\lambda_1\lambda_3^4$ belongs to $\mathcal{G}_2$, thus showing that no $\check{\rho}$-Einstein left-invariant metrics may exist.

(ii) $k_1 = 0$**,** $k_2k_3 \neq 0$

We compute a Gröbner basis $\mathcal{G}$ of the ideal generated by the polynomials of the original system $\{\mathfrak{C}_{ij}\} \subset \mathbb{R}[k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ with respect to the lexicographical order so that

$$\mathbf{g}_1 = R^8\lambda_2\lambda_3^3(10\lambda_2^2 + 19\lambda_2\lambda_3 + 10\lambda_3^2) \in \mathcal{G}.$$

Since $R\lambda_2\lambda_3 \neq 0$ and $10\lambda_2^2 + 19\lambda_2\lambda_3 + 10\lambda_3^2 = 0$ has no real solutions, we get that no $\check{\rho}$-Einstein left-invariant metric may exist.

(iii) $k_1 = k_2 = 0$, $k_3 \neq 0$

We compute a Gröbner basis $\mathcal{G}$ of the ideal generated by $\{\mathfrak{C}_{ij}\} \subset \mathbb{R}[\lambda_1, \lambda_2, \lambda_3, k_3, R]$ with respect to the lexicographical order so that $\mathrm{g}_1 = k_3^2 R^8 (k_3^2 - R^2)\lambda_3^7 \in \mathcal{G}$. Since $k_3 R \lambda_3 \neq 0$, the only possible solutions of $\mathrm{g}_1 = 0$ are $k_3 = \pm R$.

Assume $k_3 = R$ and compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{k_3 - R\}$. Since the polynomial $\mathrm{g}_2 = R^4(\lambda_1 + \lambda_2 - \lambda_3)\lambda_3^5 \in \mathcal{G}_1$, one has that $\lambda_3 = \lambda_1 + \lambda_2$. Proceeding exactly in the same way under the assumption $k_3 = -R$, one gets as in the previous case that $\lambda_3 = \lambda_1 + \lambda_2$. Now a straightforward calculation shows that $\mathfrak{C}_{34} = -R^3(\lambda_1 - \lambda_2)^2(\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2)$. If $\lambda_1 = \lambda_2$, then $\mathfrak{C}_{11} = -16R^4\lambda_2^4 \neq 0$. Hence assume $\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2 = 0$, and thus $\lambda_1 = (3 \pm 2\sqrt{2})\lambda_2$. A straightforward calculation now shows that the Ricci operator takes the form

$$Q_\rho = 2\lambda_2^2 \begin{pmatrix} 4 \pm 3\sqrt{2} & 0 & 0 & 0 \\ 0 & -4 \mp 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 3 \pm 2\sqrt{2} & 3 \pm 2\sqrt{2} \\ 0 & 0 & 3 \pm 2\sqrt{2} & -3 \mp 2\sqrt{2} \end{pmatrix},$$

from where it follows that the left-invariant metric determined by $k_3 = R$, $\lambda_1 = \lambda_3 - \lambda_2$, $\lambda_3 = (4 + 2\sqrt{2})\lambda_2$ is $\check{\rho}$-Einstein and non-Einstein. Finally, considering the homothetic basis $e_k = \frac{1}{\lambda_2} e_k$, the Lie algebra structure (4.5) is obtained, thus completing the proof. □

Observe that, in sharp contrast with Lemma 4.4, neither $SL(2, \mathbb{R}) \times \mathbb{R}$ nor $SU(2) \times \mathbb{R}$ admits left-invariant Einstein metrics.

4.2.2 The semi-direct products $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$

Lemma 4.5. $\mathbb{R}e_4 \ltimes E(1,1)$ *and* $\mathbb{R}e_4 \ltimes E(2)$ *do not admit non-Einstein $\check{\rho}$-Einstein left invariant metrics.*

Proof. Take the algebra given in (4.2). To simplify the notation, set $A = \dfrac{a}{\lambda_1} + k_3$, $C = c - k_2\lambda_1$ and $D = d + k_1\lambda_2$. Moreover, since $\lambda_1\lambda_2 \neq 0$, we work with an homothetic basis $\hat{e}_i = \dfrac{1}{\lambda_1} e_i$, so that we may assume $\lambda_1 = 1$.

A straightforward calculation shows that the components of $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ are given by

$$16R^4\check{\rho}_{0_{11}} = \mathfrak{C}_{11}, \quad 4R^4\check{\rho}_{0_{12}} = \mathfrak{C}_{12}, \quad 4R^4\check{\rho}_{0_{13}} = \mathfrak{C}_{13}, \quad 4R^3\check{\rho}_{0_{14}} = \mathfrak{C}_{14},$$

$$16R^4\check{\rho}_{0_{22}} = \mathfrak{C}_{22}, \quad 4R^4\check{\rho}_{0_{23}} = \mathfrak{C}_{23}, \quad 4R^3\check{\rho}_{0_{24}} = \mathfrak{C}_{24}, \quad 16R^4\check{\rho}_{0_{33}} = \mathfrak{C}_{33},$$

$$4R^3\check{\rho}_{0_{34}} = \mathfrak{C}_{34}, \quad 16R^4\check{\rho}_{0_{44}} = \mathfrak{C}_{44},$$

where the polynomials $\mathfrak{C}_{\mathrm{ij}}$ now become

$$\begin{aligned}
\mathfrak{C}_{11} = {} & 16b^4 + C^4 - 2C^2D^2 - 3D^4 + 2C^2R^2 + R^4 + 4C^2R^2\lambda_2 + 4D^2R^2\lambda_2 + 4R^4\lambda_2 \\
& - 8C^2R^2\lambda_2^2 - 2D^2R^2\lambda_2^2 - 10R^4\lambda_2^2 + 4R^4\lambda_2^3 + R^4\lambda_2^4 - 4AbCD(5+\lambda_2) \\
& + A^4(-1+\lambda_2)^2(1+6\lambda_2+\lambda_2^2) + 2A^2(R^2 + 2D^2\lambda_2 + 4R^2\lambda_2 - D^2\lambda_2^2 - 10R^2\lambda_2^2 \\
& + 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(1+2\lambda_2-4\lambda_2^2) + 16b^2(-1+\lambda_2^2)) - 2b^2(7C^2 + 9D^2 \\
& - 16R^2(-1+\lambda_2^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{12} = {} & -A^2CD\lambda_2 + CD(b^2 + C^2 + D^2 - R^2\lambda_2) + Ab(-16b^2(-1+\lambda_2) \\
& - D^2(2+\lambda_2) + C^2(1+2\lambda_2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{13} = {} & 6bC(2b^2 + R^2(-1+\lambda_2)\lambda_2) + A^2bC(-1-2\lambda_2+3\lambda_2^2) + A^3D(\lambda_2 - \lambda_2^3) \\
& - AD(C^2 + 2b^2(-3+5\lambda_2) + \lambda_2(D^2 + R^2(-1+\lambda_2^2)))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{14} = {} & -(A^2D\lambda_2(-1+\lambda_2^2) + AbC(1-4\lambda_2+3\lambda_2^2) + D(C^2 + \lambda_2(8b^2 + D^2 \\
& + R^2(-1+\lambda_2^2)))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{22} = {} & 16b^4 - 3C^4 - 2C^2D^2 + D^4 - 2C^2R^2 - 8D^2R^2 + R^4 + 4C^2R^2\lambda_2 + 4D^2R^2\lambda_2 \\
& + 4R^4\lambda_2 + 2D^2R^2\lambda_2^2 - 10R^4\lambda_2^2 + 4R^4\lambda_2^3 + R^4\lambda_2^4 + 4AbCD(1+5\lambda_2) \\
& + A^4(-1+\lambda_2)^2(1+6\lambda_2+\lambda_2^2) + 2A^2(-4D^2 + R^2 + 2D^2\lambda_2 + 4R^2\lambda_2 + D^2\lambda_2^2 - 10R^2\lambda_2^2 \\
& + 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(-1+2\lambda_2) - 16b^2(-1+\lambda_2^2)) - 2b^2(9C^2 + 7D^2 \\
& + 16R^2(-1+\lambda_2^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{23} = {} & 6bD(2b^2 - R^2(-1+\lambda_2)) - A^2bD(-3+2\lambda_2+\lambda_2^2) + A^3(C - C\lambda_2^2) \\
& + AC(C^2 + R^2 + b^2(10-6\lambda_2) + D^2\lambda_2 - R^2\lambda_2^2),
\end{aligned}$$

$$\mathfrak{C}_{24} = -AbD(3-4\lambda_2+\lambda_2^2) + A^2(C - C\lambda_2^2) + C(8b^2 + C^2 + R^2 + D^2\lambda_2 - R^2\lambda_2^2),$$

$$\begin{aligned}
\mathfrak{C}_{33} = {} & -48b^4 + C^4 + 2C^2D^2 + D^4 + 18b^2(C^2+D^2) + 2C^2R^2 + 8D^2R^2 + R^4 \\
& - 20AbCD(-1+\lambda_2) - 12C^2R^2\lambda_2 - 12D^2R^2\lambda_2 - 12R^4\lambda_2 + 8C^2R^2\lambda_2^2 + 2D^2R^2\lambda_2^2 \\
& + 22R^4\lambda_2^2 - 12R^4\lambda_2^3 + R^4\lambda_2^4 - A^4(-1+\lambda_2)^2(3+2\lambda_2+3\lambda_2^2) - 2A^2(C^2 + R^2 \\
& + 8b^2(-1+\lambda_2)^2 - 2C^2\lambda_2 - 2D^2\lambda_2 + 4R^2\lambda_2 + D^2\lambda_2^2 - 10R^2\lambda_2^2 + 4R^2\lambda_2^3 + R^2\lambda_2^4),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{34} = {} & -(3bCD(-1+\lambda_2) + A^3(-1+\lambda_2)^4 + A(2D^2 + R^2 + 4b^2(-1+\lambda_2)^2 - 4D^2\lambda_2 \\
& - 4R^2\lambda_2 + D^2\lambda_2^2 + 6R^2\lambda_2^2 - 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(1-4\lambda_2+2\lambda_2^2))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{44} = {} & 16b^4 + C^4 + 2C^2D^2 + D^4 + 14b^2(C^2+D^2) - 2C^2R^2 - 3R^4 \\
& + 4AbCD(-1+\lambda_2) + 4C^2R^2\lambda_2 + 4D^2R^2\lambda_2 + 4R^4\lambda_2 - 2D^2R^2\lambda_2^2 - 2R^4\lambda_2^2 \\
& + 4R^4\lambda_2^3 - 3R^4\lambda_2^4 + A^4(-1+\lambda_2)^2(1-10\lambda_2+\lambda_2^2) + 2A^2(4D^2 - R^2 \\
& + 8b^2(-1+\lambda_2)^2 - 6D^2\lambda_2 - 4R^2\lambda_2 + D^2\lambda_2^2 + 10R^2\lambda_2^2 - 4R^2\lambda_2^3 - R^2\lambda_2^4 \\
& + C^2(1-6\lambda_2+4\lambda_2^2)).
\end{aligned}$$

Hence a left-invariant metric determined by (4.2) is $\check{\rho}$-Einstein if and only if the system of equations $\{\mathfrak{C}_{\mathrm{ij}} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[A, b, \lambda_2, C, D, R]$ be the ideal generated by the polynomials $\mathfrak{C}_{\mathrm{ij}}$. We compute a Gröbner basis $\mathcal{G} \subset \mathcal{I}$ with respect to the lexicographic order so that the polynomial $\mathrm{g}_0 = CD^2(C^2+D^2)R^4(5C^2+D^2+R^2)$ belongs to $\mathcal{G}$. Hence $C = 0$ or $D = 0$, since $R > 0$.

Case $C = 0$

Assuming $C = 0$, we compute a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G} \cup \{C\}$ with respect to the lexicographical order. Since $\mathrm{g}_1 = AD^4$ belongs to $\mathcal{G}_1$, one has that either $A = 0$ or $D = 0$.

If we have $C = D = 0$, then $\mathfrak{C}_{12} = -(16Ab^3(-1+\lambda_2))$. Hence $A = 0$, $b = 0$ or $\lambda_2 = 1$. If $A = 0$, then we have that

$$\begin{aligned}
\mathfrak{C}_{11} &= 16b^4 + 32b^2R^2(\lambda_2^2 - 1) + R^4(\lambda_2 - 1)^2(1 + 6\lambda_2 + \lambda_2^2) \\
\mathfrak{C}_{22} &= 16b^4 - 32b^2R^2(\lambda_2^2 - 1) + R^4(\lambda_2^2 - 1)^2(1 + 6\lambda_2 + \lambda_2^2) \\
\mathfrak{C}_{33} &= -48b^4 + R^4(\lambda_2^2 - 1)^2(1 - 10\lambda_2 + \lambda_2^2) \\
\mathfrak{C}_{44} &= 16b^4 - R^4(\lambda_2^2 - 1)^2(3 + 2\lambda_2 + 3\lambda_2^2).
\end{aligned}$$

Now $b = \pm R$ and $\lambda_2 = -1$, leading to an Einstein metric. Assume now that $A \neq 0$ and $b = 0$. Then $\mathfrak{C}_{34} = -(A(A^2 + R^2)(-1+\lambda_2)^4)$ and one has that $\lambda_2 = 1$, leading again to an Einstein metric. Finally, assuming $A \neq 0$, $b \neq 0$ and $\lambda_2 = 1$, we obtain that $\mathfrak{C}_{11} = b^4$, which has no solution since $b \neq 0$.

Next assume $C = A = 0$, $D \neq 0$. Then $\mathfrak{C}_{23} = 6bD(2b^2 - R^2(-1+\lambda_2))$ shows that either $b = 0$ or $\lambda_2 = 1 + \frac{2b^2}{R^2}$. If $b = 0$ then $\mathfrak{C}_{14} = -D\lambda_2(D^2 + R^2(-1+\lambda_2^2))$ and thus $D = \pm\sqrt{R^2(1-\lambda_2^2)}$. Now a straightforward calculation gives

$$\begin{aligned}
\mathfrak{C}_{11} = \mathfrak{C}_{44} &= 2R^4(-1 + 4\lambda_2 - 3\lambda_2^2) \\
\mathfrak{C}_{22} &= 2R^4(-3 + 4\lambda_2 - \lambda_2^2) \\
\mathfrak{C}_{33} &= 2R^4(5 - 12\lambda_2 + 7\lambda_2^2).
\end{aligned}$$

Then $\lambda_2 = 1$ and thus $D = 0$, which is a contradiction.

Assume now $\lambda_2 = 1 + \frac{2b^2}{R^2}$ and $b \neq 0$. Then $\mathfrak{C}_{14} = -\frac{1}{R^4}D(2b^2 + R^2)(4b^4 + 12b^2R^2 + D^2R^2)$. Since $\mathfrak{C}_{14} = 0$ has no real solutions, no $\check{\rho}$-Einstein metrics may occur in this setting.

Case $D = 0$

Assume $C \neq 0$ and compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated $\mathcal{G} \cup \{D\}$ with respect to the lexicographic order. Since the polynomial $\mathrm{g}_2 = AC^4\lambda_2^2$ belongs to $\mathcal{G}_2$, one has that $A = 0$.

A standard calculation now gives $\mathfrak{C}_{13} = 6bC(2b^2 + R^2(-1+\lambda_2)\lambda_2)$, and so either $b = 0$ or $b = \pm\frac{1}{\sqrt{2}}\sqrt{R^2(1-\lambda_2)\lambda_2}$. If $b = 0$, then $\mathfrak{C}_{24} = C(C^2 - R^2(-1+\lambda_2^2))$, and thus $C = \pm R\sqrt{\lambda_2^2 - 1}$. Then the polynomials $\mathfrak{C}_{ij}$ reduce to

$$\begin{aligned}
\mathfrak{C}_{11} &= -2\lambda_2^2(1 - 4\lambda_2 + 3\lambda_2^2) \\
\mathfrak{C}_{22} = \mathfrak{C}_{44} &= -2\lambda_2^2(3 - 4\lambda_2 + \lambda_2^2) \\
\mathfrak{C}_{33} &= 2\lambda_2^2(7 - 12\lambda_2 + 5\lambda_2^2).
\end{aligned}$$

Therefore $\lambda_2 = 1$ and thus $C = 0$, which is a contradiction.

Setting $b = \pm\frac{1}{\sqrt{2}}\sqrt{R^2(1-\lambda_2)\lambda_2}$ and assuming $b \neq 0$, one has that the polynomial $\mathfrak{C}_{24} = C(C^2 + R^2(1 + 4\lambda_2 - 5\lambda_2^2))$. Hence $C = \pm R\sqrt{-1 - 4\lambda_2 + 5\lambda_2^2}$ and thus $\mathfrak{C}_{ii} = \pm R^4\lambda_2(-3 + 13\lambda_2 - 13\lambda_2^2 + 3\lambda_2^3)$, for all $i = 1, 2, 3, 4$. Since $\lambda_2 \neq 0$, the solutions of the equations $\mathfrak{C}_{ii} = 0$ are $\lambda_2 = 1$, $\lambda_2 = 3$ and $\lambda_2 = \dfrac{1}{3}$. The solution $\lambda_2 = 1$ gives $b = 0$ which is a contradiction. Moreover, if $\lambda_2 = 3$ then b cannot be real. Analogously, if $\lambda_2 = \dfrac{1}{3}$ then C is complex. This completes the proof. □

4.2.3 The semi-direct product $\mathbb{R}e_4 \ltimes H^3$

Lemma 4.6. $\mathbb{R}e_4 \ltimes H^3$ *admits a non-Einstein $\check{\rho}$-Einstein left invariant metric if and only if it is homothetic to the Lie group determined by the Lie algebra*

$$[e_1, e_2] = e_3, \qquad [e_4, e_1] = -\frac{1}{2}e_1, \qquad [e_4, e_2] = \frac{1}{2}e_3, \tag{4.6}$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

Proof. Take the algebra of $\mathbb{R} \times H^3$ given in (4.3). In order to simplify the notation, set $F = f - k_1\gamma$ and $H = h + k_2\gamma$. Moreover, since $\gamma \neq 0$, working with an homothetic basis $\hat{e}_i = \dfrac{1}{\gamma}e_i$, so that we may assume $\gamma = 1$. Then the components of the trace free tensor $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ are given by

$$\begin{aligned}
&16R^4\check{\rho}_{0_{11}} = \mathfrak{C}_{11}, \quad 4R^4\check{\rho}_{0_{12}} = \mathfrak{C}_{12}, \quad 4R^4\check{\rho}_{0_{13}} = \mathfrak{C}_{13}, \quad 4R^3\check{\rho}_{0_{14}} = \mathfrak{C}_{14},\\
&16R^4\check{\rho}_{0_{22}} = \mathfrak{C}_{22}, \quad 4R^4\check{\rho}_{0_{23}} = \mathfrak{C}_{23}, \quad 4R^3\check{\rho}_{0_{24}} = \mathfrak{C}_{24}, \quad 16R^4\check{\rho}_{0_{33}} = \mathfrak{C}_{33},\\
&4R^3\check{\rho}_{0_{34}} = \mathfrak{C}_{34}, \quad 16R^4\check{\rho}_{0_{44}} = \mathfrak{C}_{44},
\end{aligned}$$

where the polynomials $\mathfrak{C}_{ij}$ are as follows

$$\begin{aligned}
\mathfrak{C}_{11} = {} & 16a^4 - 48d^4 - 16d^2F^2 - 3F^4 - 28cdFH + 18d^2H^2 - 2F^2H^2 + H^4\\
& + 2c^2(4d^2 + F^2 - H^2) - 2F^2R^2 + 2H^2R^2 + R^4 + 2a^2(4c^2 - 56d^2 - 9F^2 + 16H^2 + 16R^2)\\
& - 4a(4c^2d + 3cFH + 2d(16d^2 + 3F^2 - 7H^2 - 4R^2)),\\
\mathfrak{C}_{12} = {} & 8a^3c - c^2FH + 2a^2(4cd + 5FH) + cd(-8d^2 - 4F^2 + H^2 - 4R^2)\\
& + FH(10d^2 + F^2 + H^2 + R^2) + a(21dFH - c(8d^2 + F^2 - 4(H^2 + R^2))),\\
\mathfrak{C}_{13} = {} & 16a^3H - 2c^2dH + d(12d^2 - F^2)H + a^2(-2cF + 48dH) + a(-14cdF + 2c^2H\\
& + (44d^2 + F^2)H) + cF(-8d^2 + F^2 + H^2),\\
\mathfrak{C}_{14} = {} & 8a^2F + 8adF + 4d^2F + F^3 - 2acH + 2cdH + FH^2 + FR^2,\\
\mathfrak{C}_{22} = {} & -48a^4 - 128a^3d + 16d^4 + 32d^2F^2 + F^4 + 12cdFH - 18d^2H^2 - 2F^2H^2 - 3H^4\\
& + 2a^2(4c^2 - 56d^2 + 9F^2 - 8H^2) + 2c^2(4d^2 - F^2 + H^2) + 32d^2R^2 + 2F^2R^2 - 2H^2R^2\\
& + R^4 - 4a(4c^2d - 7cFH - 2d(7F^2 - 3H^2 + 4R^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_{23} &= 12a^3F + 2c^2dF + a^2(44dF + 8cH) + dF(16d^2 + H^2) - cH(-2d^2 + F^2 + H^2) \\
&\quad + a(-2c^2F + 48d^2F + 14cdH - FH^2),\\
\mathfrak{C}_{24} &= -(2cdF + 4a^2H + a(-2cF + 8dH) + H(8d^2 + F^2 + H^2 + R^2)),\\
\mathfrak{C}_{33} &= 16a^4 + 128a^3d + 16d^4 - 32d^2F^2 + F^4 + 4cdFH - 14d^2H^2 + 2F^2H^2 + H^4 \\
&\quad + 2c^2(-4d^2 + F^2 + H^2) - 32d^2R^2 + 2F^2R^2 + 2H^2R^2 + R^4 - 2a^2(4c^2 - 104d^2 \\
&\quad + 7F^2 + 16H^2 + 16R^2) + 4a(4c^2d - cFH + 2d(16d^2 - 5F^2 - 5H^2 - 8R^2)),\\
\mathfrak{C}_{34} &= (-a + d)FH - c(F^2 + H^2),\\
\mathfrak{C}_{44} &= 16a^4 + 16d^4 + 16d^2F^2 + F^4 + 12cdFH + 14d^2H^2 + 2F^2H^2 + H^4 - 2c^2(4d^2 + F^2 + H^2) \\
&\quad + 2a^2(-4c^2 + 8d^2 + 7F^2 + 8H^2) + 4a(4c^2d - 3cFH + 2d(F^2 + H^2)) - 2F^2R^2 \\
&\quad - 2H^2R^2 - 3R^4.
\end{aligned}$$

Hence a left-invariant metric determined by (4.3) is $\check{\rho}$-Einstein if and only if the system of equations $\{\mathfrak{C}_{\mathrm{ij}} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, H, F, R]$ be the ideal generated by the polynomials $\mathfrak{C}_{\mathrm{ij}}$. We compute a Gröbner basis $\mathcal{G} \subset \mathcal{I}$ with respect to the lexicographic order so that the polynomial $\mathrm{g}_0 = FH(F^2 + H^2 + R^2)^2(15F^2 + 15H^2 + 32R^2)$ belongs to $\mathcal{G}$. Hence $F = 0$ or $H = 0$, since $R > 0$.

Assume $F = 0$ and compute a Gröbner basis $\mathcal{G}_{01}$ of the ideal generated $\mathcal{G} \cup \{F\}$ with respect to the lexicographic order. Since the polynomial $\mathrm{g}_{01} = H^2(H^2 + R^2)(4H^2 + 9R^2)$ belongs to $\mathcal{G}_{01}$, one has that $H = 0$. On the other hand, assuming $H = 0$ and computing a Gröbner basis $\mathcal{G}_{02}$ of the ideal generated $\mathcal{G} \cup \{H\}$ with respect to the lexicographic order, one has that $\mathrm{g}_{02} = F^2(F^2 + R^2)(4F^2 + 9R^2) \in \mathcal{G}_{02}$, thus showing that $F = 0$.

We assume therefore that $F = H = 0$ in what follows. Then one has that

$$\mathfrak{C}_{12} = 4c(a - d)(2a^2 + 4ad + 2d^2 + R^2)$$

and the only possible real zeros are $c = 0$ and $a = d$.

If $c = 0$, then the only remaining equations are

$$\begin{aligned}
\mathfrak{C}_{11} &= 16a^4 - 112a^2d^2 - 128ad^3 - 48d^4 + 32a^2R^2 + 32adR^2 + R^4\\
\mathfrak{C}_{22} &= -48a^4 - 128a^3d - 112a^2d^2 + 16d^4 + 32adR^2 + 32d^2R^2 + R^4\\
\mathfrak{C}_{33} &= 16a^4 + 128a^3d + 16d^4 - 32d^2R^2 + R^4 + 16a^2(13d^2 - 2R^2)\\
&\quad + 64a(2d^3 - dR^2)\\
\mathfrak{C}_{44} &= -3R^4 + 16(a^4 + a^2d^2 + d^4),
\end{aligned}$$

and the only possible solutions for the system are $d = \pm a$ and $2a = \pm R$. The cases corresponding to $d = a$ are Einstein so we assume $d = -a$ and $2a = \pm R$. A straightforward calculation now shows that Ricci operator $Q_\rho = \frac{1}{2}\,\mathrm{diag}[-1, -1, 1, -1]$ and hence the left-invariant metric is $\check{\rho}$-Einstein. This shows 4.6.

Finally, assuming $a = d$, $c \neq 0$, one has that $\mathfrak{C}_{44} = 3(-R^4 + 16a^4)$. Hence $2a = \pm R$, and the corresponding left-invariant metrics are Einstein. □

4.2.4 The semi-direct product $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

Lemma 4.7. $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ *admits a non-Einstein $\check{\rho}$-Einstein left invariant metric if and only if it is homothetic to a Lie group determined by a solvable Lie algebra given by*

$$\begin{aligned} &[e_4, e_1] = e_1 + be_2 + ce_3, && [e_4, e_2] = -be_1 + \alpha e_2 + he_3, \\ &[e_4, e_3] = -ce_1 - he_2 - \tfrac{\alpha}{\alpha+1} e_3, \end{aligned} \tag{4.7}$$

where $\{e_1, \dots, e_4\}$ *is an orthonormal basis and*

$$b = -\frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}, \quad h = \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}, \quad c = \frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}, \quad \alpha \in \mathbb{R}, \alpha \neq 0.$$

Proof. Take the algebra from (4.4). Furthermore, considering the homothetic basis $\bar{e}_k = Re_k$, one may assume $R = 1$ in what follows.

A straightforward calculation shows that $\check{\rho}_0 = \check{\rho} - \frac{1}{4}\|\rho\|^2 g$ is determined by

$$\begin{aligned} &2R^4\check{\rho}_{0_{11}} = \mathfrak{C}_{11}, && R^4\check{\rho}_{0_{12}} = \mathfrak{C}_{12}, && R^4\check{\rho}_{0_{13}} = \mathfrak{C}_{13}, && \check{\rho}_{0_{14}} = 0, \\ &2R^4\check{\rho}_{0_{22}} = \mathfrak{C}_{22}, && R^4\check{\rho}_{0_{23}} = \mathfrak{C}_{23}, && \check{\rho}_{0_{24}} = 0, && R^4\check{\rho}_{0_{33}} = \mathfrak{C}_{33}, \\ &\check{\rho}_{0_{34}} = 0, && 2R^4\check{\rho}_{0_{44}} = \mathfrak{C}_{44}, \end{aligned}$$

where the polynomials $\mathfrak{C}_{ij}$ are given by

$$\begin{aligned} \mathfrak{C}_{11} = &-(-a^4 - b^2f^2 + f^4 + f^2h^2 + f^3p - 2fh^2p - c^2p^2 + 2f^2p^2 + h^2p^2 + fp^3 + p^4 \\ &- 3a^3(f+p) - a^2(b^2 + c^2 + 3fp) + a(2b^2f + f^3 + 2c^2p + f^2p + fp^2 + p^3)), \end{aligned}$$

$$\mathfrak{C}_{12} = -a^3b + a(bf^2 + ch(f-p)) + chp(-f+p) - a^2b(f+p) + bf^2(f+p),$$

$$\mathfrak{C}_{13} = -a^3c + bfh(-f+p) - a^2c(f+p) + cp^2(f+p) + a(bh(f-p) + cp^2),$$

$$\begin{aligned} \mathfrak{C}_{22} = &-(a^4 - b^2f^2 - f^4 - f^2h^2 - 3f^3p + 2fh^2p + c^2p^2 - h^2p^2 + fp^3 + p^4 \\ &+ a^3(f+p) + a(2b^2f - 3f^3 - 2c^2p - 3f^2p + fp^2 + p^3) + a^2(-b^2 + c^2 + p(f+2p))), \end{aligned}$$

$$\mathfrak{C}_{23} = a^2bc - f^3h - f^2hp + hp^3 - a(bc + h(f-p))(f+p) + fp(bc + hp),$$

$$\begin{aligned} \mathfrak{C}_{33} = &-(a^4 + b^2f^2 + f^4 - f^2h^2 + f^3p + 2fh^2p - c^2p^2 - h^2p^2 - 3fp^3 - p^4 \\ &+ a^3(f+p) + a(-2b^2f + f^3 + 2c^2p + f^2p - 3fp^2 - 3p^3) + a^2(b^2 - c^2 + f(2f+p))), \end{aligned}$$

$$\begin{aligned} \mathfrak{C}_{44} = &-(-a^4 + b^2f^2 - f^4 + f^2h^2 + f^3p - 2fh^2p + c^2p^2 - 2f^2p^2 + h^2p^2 + fp^3 - p^4 \\ &+ a^3(f+p) + a^2(b^2 + c^2 - 2f^2 + fp - 2p^2) + a(-2b^2f + f^3 - 2c^2p + f^2p + fp^2 + p^3)). \end{aligned}$$

Further observe from (4.4) that the structure constants are symmetric (up to a change of basis) in the parameters b, c, h as well as in the parameters a, f, p. This clearly influences the polynomials $\mathfrak{C}_{ij}$. We are firstly considering the following four cases showing that none of them supports a $\check{\rho}$-Einstein metric.

$a = f$

Then $\mathfrak{C}_{12} = ch(f-p)^2$. One easily checks that if $f = p$ then the metric is Einstein. Hence $f \neq p$ and either $c = 0$ or $h = 0$.

If $c = 0$, then $\mathfrak{C}_{23} = h(-2f^3-f^2p+2fp^2+p^3)$. Now, if $h = 0$, then $\mathfrak{C}_{44} = (f-p)^2(2f^2+p^2)$, which contradicts $f \neq p$. If $h \neq 0$, then either $p = -2f$ or $p = -f$. Setting $p = -2f$ one gets $\mathfrak{C}_{11} = -9f^2(2f^2 + h^2)$. Hence $f = 0$ and thus $f = p = 0$ which is a contradiction. Finally, if $p = -f$, then one gets $\mathfrak{C}_{11} = -2f^2(f^2 + h^2)$, which also leads to a contradiction.

If $h = 0$ and $c \neq 0$, then $\mathfrak{C}_{13} = c(-2f^3 - f^2p + 2fp^2 + p^3)$ and hence $p = -2f$ or $p = -f$. Proceeding in a completely analogous way as in the previous case with the coefficient $\mathfrak{C}_{22}$ one gets that no $\check{\rho}$-Einstein metrics may exist in this case.

Since the system of polynomial equations is symmetric up to a change of the basis $\{e_k\}$, the previous case also includes the situations $a = p$ and $f = p$. Hence we assume in what follows that $a \neq p$, $a \neq f$ and $p \neq f$.

$a = -f$

Then $\mathfrak{C}_{12} = ch(-f^2 + p^2)$ and, since $a \neq p$, $a \neq f$ and $p \neq f$, one has $c = 0$ or $h = 0$.

If $c = 0$ then $\mathfrak{C}_{23} = hp(-f^2 + p^2)$ and thus $h = 0$ or $p = 0$. If $h = 0$, then we have the system of equations

$$\begin{aligned}
\mathfrak{C}_{11} = \mathfrak{C}_{22} &= 4b^2f^2 - 2f^4 - f^2p^2 - p^4 \\
\mathfrak{C}_{33} &= -4b^2f^2 - 2f^4 - 3f^2p^2 + p^4 \\
\mathfrak{C}_{44} &= -4b^2f^2 + 6f^4 + 5f^2p^2 + p^4,
\end{aligned}$$

which has no real solutions. If $p = 0$, $h \neq 0$ then $\mathfrak{C}_{13} = -2bf^2h$. Hence $b = 0$ and then $\mathfrak{C}_{11} = -f^2(2f^2 + h^2)$, which has no zeroes. This shows that c cannot be zero.

If $h = 0$, $c \neq 0$ then $\mathfrak{C}_{13} = cp(-f^2 + p^2)$ and the only possibility is $p = 0$, but then $\mathfrak{C}_{23} = 2bcf^2$, which implies that $b = 0$ and thus $\mathfrak{C}_{22} = -f^2(c^2 + 2f^2)$, which has no zeroes.

Therefore, we have seen that $a \neq -f$. Hence, by considering appropriate changes on the basis $\{e_k\}$ one has that $a^2 \neq f^2$, $a^2 \neq p^2$, $f^2 \neq p^2$.

$c = 0$

Then $\mathfrak{C}_{12} = -b(a^2 - f^2)(a + f + p)$, and thus either $b = 0$ or $a + f + p = 0$.

If $a+f+p = 0$ then $\mathfrak{C}_{13} = bh(-2f^2 + fp + p^2)$, and hence either $b = 0$, $h = 0$ or $p = -2f$. Note that the latter condition is not possible since $a+f+p = 0$ and $p = -2f$ would give $a = f$, which is a contradiction. If $b = 0$, then we have

$$\begin{aligned}
\mathfrak{C}_{11} &= 2(-f^4 - 2f^3p + fp(h^2 - 2p^2)) - p^2(h^2 + 2p^2) - f1^2(h1^2 + 6p1^2) \\
\mathfrak{C}_{22} = \mathfrak{C}_{33} &= -2f^4 - 4f^3p + f^2(h^2 - 6p^2) + p^2(h^2 - 2p^2) - 2fp(h^2 + 2p^2) \\
\mathfrak{C}_{44} &= -h^2(f-p)^2 + 6(f^2 + fp + p^2)^2,
\end{aligned}$$

which has no real solution. Assuming $h = 0$ and $b \neq 0$, one has

$$\begin{aligned}
\mathfrak{C}_{11} = \mathfrak{C}_{22} &= b^2(2f+p)^2 - 2(f^2+fp+p^2)^2 \\
\mathfrak{C}_{33} &= -b^2(2f+p)^2 - 2(f^2+fp+p^2)^2 \\
\mathfrak{C}_{44} &= -h^2(f-p)^2 + 6(f^2+fp+p^2)^2,
\end{aligned}$$

which again has no real solution.

Set now $b = 0$ and assume $a+f+p \neq 0$. Then $\mathfrak{C}_{23} = -h(a+f+p)(f^2-p^2)$ shows that $h = 0$. Now the system of polynomial equations becomes

$$\begin{aligned}
\mathfrak{C}_{11} &= a^4 - f^4 + 3a^2fp - f^3p - 2f^2p^2 - fp^3 - p^4 \\
&\quad + 3a^3(f+p) - a(f^3+f^2p+fp^2+p^3) \\
\mathfrak{C}_{22} &= -a^4 + f^4 + 3f^3p - fp^3 - p^4 - a^3(f+p) \\
&\quad - a^2p(f+2p) - a(-3f^3 - 3f^2p + fp^2 + p^3) \\
\mathfrak{C}_{33} &= -a^4 - f^4 - f^3p + 3fp^3 + p^4 - a^3(f+p) \\
&\quad - a^2f(2f+p) - a(f^3+f^2p-3fp^2-3p^3) \\
\mathfrak{C}_{44} &= -2a^2(a+f+p)^2 - f^2(a+f+p)^2 \\
&\quad - p^2(a+f+p)^2 + 3(a^2+f^2+p^2)^2,
\end{aligned}$$

and again it has no real solution.

Hence we may assume $c \neq 0$ and, up to a change of basis, one has that $c \neq 0$, $b \neq 0$ and $h \neq 0$.

$a + f + p = 0$

Then $\mathfrak{C}_{12} = -ch(f^2+fp-2p^2)$ and thus $p = f$ or $p = -\frac{1}{2}f$, which contradicts $a^2 \neq f^2$ and $p^2 \neq a^2$.

We now return to the original system $\{\mathfrak{C}_{ij} = 0\}$. Based on the analysis of the previous cases, the polynomial

$$\mathfrak{C}_{12} = -a^3b + a(bf^2 + ch(f-p)) + chp(-f+p) - a^2b(f+p) + bf^2(f+p),$$

gives $b = \dfrac{ch(a-p)(f-p)}{(a^2-f^2)(a+f+p)}$. Substituting b in $\mathfrak{C}_{13}$, it becomes

$$\begin{aligned}
\tfrac{(a+f)(a+f+p)}{c(p-a)}\mathfrak{C}_{13} &= a^4 + f^3p - h^2p^2 + 3a^3(f+p) - f^2(h^2-2p^2) \\
&\quad + a^2(3f^2+7fp+3p^2) + f(2h^2p+p^3) \\
&\quad + a(f^3+5f^2p+5fp^2+p^3).
\end{aligned}$$

Hence $h^2 = \dfrac{(a+f)(a+p)(a+f+p)^2}{(f-p)^2}$. Replacing again in the equations, one gets

$$\begin{aligned}\tfrac{(a+f)(a+f+p)}{h(p-f)}\mathfrak{C}_{23} &= f^4 + 3f^3p - c^2p^2 + 3f^2p^2 + fp^3 + a^3(f+p)\\ &\quad + a^2(-c^2 + 3f^2 + 5fp + 2p^2)\\ &\quad + a(3f^3 + 2c^2p + 7f^2p + 5fp^2 + p^3),\end{aligned}$$

and thus $c^2 = \dfrac{(a+f)(f+p)(a+f+p)^2}{(a-p)^2}$. Now, the only remaining equations are $\mathfrak{C}_{11} = \mathfrak{C}_{22} = \mathfrak{C}_{33} = \mathfrak{C}_{44}$, where

$$\mathfrak{C}_{\mathrm{ii}} = a^3(f+p) + a(f+p)^3 + fp(f^2 + fp + p^2) + a^2(f^2 + 3fp + p^2).$$

The only real solutions of $\mathfrak{C}_{ii} = 0$ are $a = f = 0$ (which gives an Einstein metric) or $p = -\dfrac{af}{a+f}$.

Next observe that setting $\hat{e_1} = e_2$, $\hat{e_2} = e_1$, $\hat{e}_3 = e_3$, $\hat{e}_4 = e_4$ one gets an isometry interchanging a and f. Furthermore since $p \neq a$, $p \neq f$ the constants a, f satisfy $af \neq 0$. Hence considering the homothetic basis $\tilde{e}_k = \frac{1}{a}e_k$ one may assume that $a = 1$. Finally setting $f = \alpha$ one has that $p = -\frac{\alpha}{\alpha+1}$ and the remaining structure constants are given by

$$h = \varepsilon_h \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}, \quad c = \varepsilon_c \frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}, \quad b = -\varepsilon_h\varepsilon_c \frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)},$$

where $\varepsilon_h^2 = \varepsilon_c^2 = 1$. Finally since $e_1^* = -e_1$, $e_2^* = -e_2$ (resp., $e_1^* = -e_1$, $e_3^* = -e_3$ and $e_2^* = -e_2$, $e_3^* = -e_3$) determines an isometry interchanging the signs of c and h (resp., the signs of b and h and the signs of b and c) we may assume $\varepsilon_h = \varepsilon_c = 1$. We recall that the values $\alpha = 0$, $\alpha = -2$, $\alpha = -\frac{1}{2}$ and $\alpha = \pm 1$ are not possible since $a^2 \neq f^2$, $a^2 \neq p^2$ and $f^2 \neq p^2$. Furthermore, the isometry interchanging a and f induces an isometry interchanging α with $\frac{1}{\alpha}$.

Finally observe that the Ricci operator takes the form

$$Q_\rho = \frac{1+\alpha+\alpha^2}{\alpha+1}\begin{pmatrix} -1 & \frac{\alpha}{\alpha+1} & \alpha & 0\\ \frac{\alpha}{\alpha+1} & -\alpha & 1 & 0\\ \alpha & 1 & \frac{\alpha}{\alpha+1} & 0\\ 0 & 0 & 0 & -\frac{1+\alpha+\alpha^2}{\alpha+1}\end{pmatrix}$$

which shows that the left-invariant metrics (4.7) are $\check{\rho}$-Einstein but not Einstein. □

4.3 Left-invariant $R[\rho]$-Einstein metrics

This section follows the exact same procedure as the previous one. We compute a trace free tensor field and check when it satisfies the condition required.

4.3.1 The direct products $SL(2,\mathbb{R})\times\mathbb{R}$ and $SU(2)\times\mathbb{R}$

Lemma 4.8. *$SL(2,\mathbb{R})\times\mathbb{R}$ and $SU(\mathbb{R})\times\mathbb{R}$ do not admit $R[\rho]$-Einstein left invariant metrics.*

Proof. Let $R[\rho]_0$ be the trace-free tensor $R[\rho]_0 = R[\rho] - \frac{1}{4}\|\rho\|^2 g$. A straightforward calculation shows that the components of $R[\rho]_0$ for any left-invariant metric (4.1) are given by

$$
\begin{aligned}
&16R^4R[\rho]_{0_{11}} = \mathfrak{R}_{11}, \quad 8R^4R[\rho]_{0_{12}} = \mathfrak{R}_{12}, \quad 8R^4R[\rho]_{0_{13}} = \mathfrak{R}_{13},\\
&8R^3R[\rho]_{0_{14}} = \mathfrak{R}_{14}, \quad 16R^4R[\rho]_{0_{22}} = \mathfrak{R}_{22}, \quad 8R^4R[\rho]_{0_{23}} = \mathfrak{R}_{23},\\
&8R^3R[\rho]_{0_{24}} = \mathfrak{R}_{24}, \quad 16R^4R[\rho]_{0_{33}} = \mathfrak{R}_{33}, \quad 8R^3R[\rho]_{0_{34}} = \mathfrak{R}_{34},\\
&16R^4R[\rho]_{0_{44}} = \mathfrak{R}_{44},
\end{aligned}
$$

where the polynomials $\mathfrak{R}_{ij}$ are

$$
\begin{aligned}
\mathfrak{R}_{11} = {}& -7R^4\lambda_1^4 + 8R^4\lambda_1^3\lambda_2 + 10R^4\lambda_1^2\lambda_2^2 - 4k_1^2R^2\lambda_1\lambda_2^3 - 16R^4\lambda_1\lambda_2^3 - 3k_1^4\lambda_2^4 + 2k_1^2R^2\lambda_2^4\\
&+ 5R^4\lambda_2^4 - k_3^4(\lambda_1-\lambda_2)^2(7\lambda_1^2 + 6\lambda_1\lambda_2 - 5\lambda_2^2) + 8R^4\lambda_1^3\lambda_3 - 28R^4\lambda_1^2\lambda_2\lambda_3\\
&+ 4k_1^2R^2\lambda_1\lambda_2^2\lambda_3 + 16R^4\lambda_1\lambda_2^2\lambda_3 + 4k_1^4\lambda_2^3\lambda_3 + 8k_1^2R^2\lambda_2^3\lambda_3 + 4R^4\lambda_2^3\lambda_3 + 10R^4\lambda_1^2\lambda_3^2\\
&+ 4k_1^2R^2\lambda_1\lambda_2\lambda_3^2 + 16R^4\lambda_1\lambda_2\lambda_3^2 - 2k_1^4\lambda_2^2\lambda_3^2 - 20k_1^2R^2\lambda_2^2\lambda_3^2 - 18R^4\lambda_2^2\lambda_3^2 - 4k_1^2R^2\lambda_1\lambda_3^3\\
&- 16R^4\lambda_1\lambda_3^3 + 4k_1^4\lambda_2\lambda_3^3 + 8k_1^2R^2\lambda_2\lambda_3^3 + 4R^4\lambda_2\lambda_3^3 - 3k_1^4\lambda_3^4 + 2k_1^2R^2\lambda_3^4 + 5R^4\lambda_3^4\\
&- k_2^4(\lambda_1-\lambda_3)^2(7\lambda_1^2 + 6\lambda_1\lambda_3 - 5\lambda_3^2) + 2k_3^2(-R^2(\lambda_1-\lambda_2)(7\lambda_1^3 - \lambda_1^2(\lambda_2+4\lambda_3)\\
&+ \lambda_2(5\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2) + \lambda_1(-11\lambda_2^2 + 10\lambda_2\lambda_3 - 2\lambda_3^2)) + k_1^2(\lambda_1^2(4\lambda_2 - 5\lambda_3)\lambda_3\\
&+ 2\lambda_1\lambda_2(-\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2) + \lambda_2^2(\lambda_2^2 - 4\lambda_2\lambda_3 + 2\lambda_3^2))) + 2k_2^2(k_1^2(\lambda_1^2\lambda_2(-5\lambda_2\\
&+ 4\lambda_3) + 2\lambda_1\lambda_3(\lambda_2^2 + \lambda_2\lambda_3 - \lambda_3^2) + \lambda_3^2(2\lambda_2^2 - 4\lambda_2\lambda_3 + \lambda_3^2)) + k_3^2(-7\lambda_1^4 + 5\lambda_2^2\lambda_3^2\\
&+ 4\lambda_1^3(\lambda_2+\lambda_3) - 8\lambda_1\lambda_2\lambda_3(\lambda_2+\lambda_3) + 2\lambda_1^2(\lambda_2^2 + 3\lambda_2\lambda_3 + \lambda_3^2)) - R^2(\lambda_1-\lambda_3)(7\lambda_1^3\\
&- \lambda_1^2(4\lambda_2 + \lambda_3) + \lambda_1(-2\lambda_2^2 + 10\lambda_2\lambda_3 - 11\lambda_3^2) + \lambda_3(-4\lambda_2^2 + 2\lambda_2\lambda_3 + 5\lambda_3^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{12} = {}& k_1k_2(R^2\lambda_1^3\lambda_2 - 2R^2\lambda_1^2\lambda_2^2 + 2k_1^2\lambda_1\lambda_2^3 + R^2\lambda_1\lambda_2^3 + 5R^2\lambda_1^2\lambda_2\lambda_3 - 2k_1^2\lambda_1\lambda_2^2\lambda_3\\
&+ 5R^2\lambda_1\lambda_2^2\lambda_3 - 4R^2\lambda_1^2\lambda_3^2 - 2k_1^2\lambda_1\lambda_2\lambda_3^2 - 14R^2\lambda_1\lambda_2\lambda_3^2 - 4k_1^2\lambda_2^2\lambda_3^2 - 4R^2\lambda_2^2\lambda_3^2\\
&+ 2k_1^2\lambda_1\lambda_3^3 + 8R^2\lambda_1\lambda_3^3 + 8k_1^2\lambda_2\lambda_3^3 + 8R^2\lambda_2\lambda_3^3 - 4k_1^2\lambda_3^4 - 4R^2\lambda_3^4\\
&+ 2k_2^2(\lambda_1-\lambda_3)^2(\lambda_1\lambda_2 + (\lambda_2 - 2\lambda_3)\lambda_3) + k_3^2(\lambda_1-\lambda_2)^2((\lambda_2 - 3\lambda_3)\lambda_3 + \lambda_1(\lambda_2 + \lambda_3))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{13} = {}& k_1k_3(-4R^2\lambda_1^2\lambda_2^2 + 2k_1^2\lambda_1\lambda_2^3 + 8R^2\lambda_1\lambda_2^3 - 4k_1^2\lambda_2^4 - 4R^2\lambda_2^4 + R^2\lambda_1^3\lambda_3 + 5R^2\lambda_1^2\lambda_2\lambda_3\\
&- 2k_1^2\lambda_1\lambda_2^2\lambda_3 - 14R^2\lambda_1\lambda_2^2\lambda_3 + 8k_1^2\lambda_2^3\lambda_3 + 8R^2\lambda_2^3\lambda_3 - 2R^2\lambda_1^2\lambda_3^2 - 2k_1^2\lambda_1\lambda_2\lambda_3^2\\
&+ 5R^2\lambda_1\lambda_2\lambda_3^2 - 4k_1^2\lambda_2^2\lambda_3^2 - 4R^2\lambda_2^2\lambda_3^2 + 2k_1^2\lambda_1\lambda_3^3 + R^2\lambda_1\lambda_3^3 + 2k_3^2(\lambda_1-\lambda_2)^2(-2\lambda_2^2\\
&+ \lambda_1\lambda_3 + \lambda_2\lambda_3) + k_2^2(\lambda_1-\lambda_3)^2(\lambda_2(-3\lambda_2 + \lambda_3) + \lambda_1(\lambda_2+\lambda_3))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{14} = {}& k_1(k_3^2(\lambda_1-\lambda_2)(-4\lambda_1\lambda_2^2 + 4\lambda_2^3 + \lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 + 6\lambda_2^2\lambda_3 - 3\lambda_1\lambda_3^2 - 6\lambda_2\lambda_3^2)\\
&+ k_2^2(\lambda_1-\lambda_3)(\lambda_1^2\lambda_2 - 6\lambda_2^2\lambda_3 + 6\lambda_2\lambda_3^2 + 4\lambda_3^3 + \lambda_1(-3\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2))\\
&- 2(\lambda_2-\lambda_3)^2(2R^2(-\lambda_1+\lambda_2+\lambda_3)^2 - k_1^2(\lambda_2+\lambda_3)(\lambda_1 - 2(\lambda_2+\lambda_3)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{22} = {}& 5R^4\lambda_1^4 - 16R^4\lambda_1^3\lambda_2 + 4k_1^2R^2\lambda_1^2\lambda_2^2 + 10R^4\lambda_1^2\lambda_2^2 + 8k_1^2R^2\lambda_1\lambda_2^3 + 8R^4\lambda_1\lambda_2^3 - 7k_1^4\lambda_2^4\\
&- 14k_1^2R^2\lambda_2^4 - 7R^4\lambda_2^4 + k_3^4(\lambda_1-\lambda_2)^2(5\lambda_1^2 - 6\lambda_1\lambda_2 - 7\lambda_2^2) + 4R^4\lambda_1^3\lambda_3 + 4k_1^2R^2\lambda_1^2\lambda_2\lambda_3\\
&+ 16R^4\lambda_1^2\lambda_2\lambda_3 - 28k_1^2R^2\lambda_1\lambda_2^2\lambda_3 - 28R^4\lambda_1\lambda_2^2\lambda_3 + 8k_1^4\lambda_2^3\lambda_3 + 16k_1^2R^2\lambda_2^3\lambda_3\\
&+ 8R^4\lambda_2^3\lambda_3 - 8k_1^2R^2\lambda_1^2\lambda_3^2 - 18R^4\lambda_1^2\lambda_3^2 + 16k_1^2R^2\lambda_1\lambda_2\lambda_3^2 + 16R^4\lambda_1\lambda_2\lambda_3^2 + 10k_1^4\lambda_2^2\lambda_3^2\\
&+ 20k_1^2R^2\lambda_2^2\lambda_3^2 + 10R^4\lambda_2^2\lambda_3^2 + 4k_1^2R^2\lambda_1\lambda_3^3 + 4R^4\lambda_1\lambda_3^3 - 16k_1^4\lambda_2\lambda_3^3 - 32k_1^2R^2\lambda_2\lambda_3^3
\end{aligned}
$$

$$
\begin{aligned}
&- 16R^4\lambda_2\lambda_3^3 + 5k_1^4\lambda_3^4 + 10k_1^2R^2\lambda_3^4 + 5R^4\lambda_3^4 - k_2^4(\lambda_1-\lambda_3)^2(3\lambda_1^2+2\lambda_1\lambda_3+3\lambda_3^2)\\
&+ 2k_2^2(R^2(\lambda_1-\lambda_3)^2(\lambda_1^2-2\lambda_1\lambda_2+6\lambda_1\lambda_3-2\lambda_2\lambda_3+\lambda_3^2) + k_3^2(\lambda_1^4-5\lambda_2^2\lambda_3^2\\
&+ 2\lambda_1^2\lambda_3(\lambda_2+\lambda_3)+2\lambda_1\lambda_2\lambda_3(2\lambda_2+\lambda_3) - 2\lambda_1^3(\lambda_2+2\lambda_3))\\
&+ k_1^2(\lambda_3^3(-2\lambda_2+\lambda_3)+2\lambda_1\lambda_3(2\lambda_2^2+\lambda_2\,\lambda_3-2\lambda_3^2)+\lambda_1^2(-5\lambda_2^2+2\lambda_2\lambda_3+2\lambda_3^2)))\\
&+ 2k_3^2(R^2(\lambda_1-\lambda_2)(5\lambda_1^3+\ \lambda_1^2(-11\lambda_2+2\lambda_3)+\lambda_2(7\lambda_2^2-4\lambda_2\lambda_3-2\lambda_3^2)\\
&- \lambda_1(\lambda_2^2-10\lambda_2\lambda_3+4\lambda_3^2)) + k_1^2(2\lambda_1\lambda_2(2\lambda_2^2+3\lambda_2\lambda_3-4\lambda_3^2)+\lambda_2^2(-7\lambda_2^2+4\lambda_2\lambda_3\\
&+ 2\lambda_3^2)+\lambda_1^2(2\lambda_2^2-8\lambda_2\lambda_3+5\lambda_3^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{23} = {}& k_2k_3(-4R^2\lambda_1^4+8R^2\lambda_1^3\lambda_2-3k_1^2\lambda_1^2\lambda_2^2-4R^2\lambda_1^2\lambda_2^2+k_1^2\lambda_1\lambda_2^3+8R^2\lambda_1^3\lambda_3+6k_1^2\lambda_1^2\lambda_2\lambda_3\\
&- 14R^2\lambda_1^2\lambda_2\lambda_3-k_1^2\lambda_1\lambda_2^2\lambda_3+5R^2\lambda_1\lambda_2^2\lambda_3+k_1^2\lambda_2^3\lambda_3+R^2\lambda_2^3\lambda_3-3k_1^2\lambda_1^2\lambda_3^2-4R^2\lambda_1^2\lambda_3^2\\
&- k_1^2\lambda_1\lambda_2\lambda_3^2+5R^2\lambda_1\lambda_2\lambda_3^2-2k_1^2\lambda_2^2\lambda_3^2-2R^2\lambda_2^2\lambda_3^2+k_1^2\lambda_1\lambda_3^3+k_1^2\lambda_2\lambda_3^3+R^2\lambda_2\lambda_3^3\\
&- 2k_2^2(\lambda_1-\lambda_3)^2(2\lambda_1^2-\lambda_1\,\lambda_2-\lambda_2\lambda_3)-2k_3^2(\lambda_1-\lambda_2)^2(2\lambda_1^2-\lambda_1\,\lambda_3-\lambda_2\lambda_3)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{24} = {}& -k_2(4R^2\lambda_1^4-8R^2\lambda_1^3\lambda_2+3k_1^2\lambda_1^2\lambda_2^2+4R^2\lambda_1^2\lambda_2^2-k_1^2\lambda_1\lambda_2^3+3k_1^2\lambda_1^2\lambda_2\lambda_3+8R^2\lambda_1^2\lambda_2\lambda_3\\
&- k_1^2\lambda_1\lambda_2^2\lambda_3-8R^2\lambda_1\lambda_2^2\lambda_3-6k_1^2\lambda_1^2\lambda_3^2-8R^2\lambda_1^2\lambda_3^2-4k_1^2\lambda_1\lambda_2\lambda_3^2+8R^2\lambda_1\lambda_2\lambda_3^2\\
&+ 4k_1^2\lambda_2^2\lambda_3^2+4R^2\lambda_2^2\lambda_3^2+6k_1^2\lambda_1\lambda_3^3-8k_1^2\lambda_2\lambda_3^3-8R^2\lambda_2\lambda_3^3+4k_1^2\lambda_3^4+4R^2\lambda_3^4\\
&+ 2k_2^2(\lambda_1-\lambda_3)^2(\lambda_1+\lambda_3)(2\,\lambda_1-\lambda_2+2\lambda_3)+k_3^2(\lambda_1-\lambda_2)(4\lambda_1^3+2\lambda_1(\lambda_2-3\lambda_3)\lambda_3\\
&+ \lambda_2(\lambda_2-3\lambda_3)\lambda_3+\lambda_1^2(-4\lambda_2+6\lambda_3))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{33} = {}& 5R^4\lambda_1^4+4R^4\lambda_1^3\lambda_2-8k_1^2R^2\lambda_1^2\lambda_2^2-18R^4\lambda_1^2\lambda_2^2+4k_1^2R^2\lambda_1\lambda_2^3+4R^4\lambda_1\lambda_2^3+5k_1^4\lambda_2^4\\
&+ 10k_1^2R^2\lambda_2^4+5R^4\lambda_2^4-k_3^4(\lambda_1-\lambda_2)^2(3\lambda_1^2+2\lambda_1\lambda_2+3\lambda_2^2)-16R^4\lambda_1^3\lambda_3+4k_1^2R^2\lambda_1^2\lambda_2\lambda_3\\
&+ 16R^4\lambda_1^2\lambda_2\lambda_3+16k_1^2R^2\lambda_1\lambda_2^2\lambda_3+16R^4\lambda_1\lambda_2^2\lambda_3-16k_1^4\lambda_2^3\lambda_3-32k_1^2R^2\lambda_2^3\lambda_3\\
&- 16R^4\lambda_2^3\lambda_3+4k_1^2R^2\lambda_1^2\lambda_3^2+10R^4\lambda_1^2\lambda_3^2-28k_1^2R^2\lambda_1\lambda_2\lambda_3^2-28R^4\lambda_1\lambda_2\lambda_3^2+10k_1^4\lambda_2^2\lambda_3^2\\
&+ 20k_1^2R^2\lambda_2^2\lambda_3^2+10R^4\lambda_2^2\lambda_3^2+8k_1^2R^2\lambda_1\lambda_3^3+8R^4\lambda_1\lambda_3^3+8k_1^4\lambda_2\lambda_3^3+16k_1^2R^2\lambda_2\lambda_3^3\\
&+ 8R^4\lambda_2\lambda_3^3-7k_1^4\lambda_3^4-14k_1^2R^2\lambda_3^4-7R^4\lambda_3^4+k_2^4(\lambda_1-\lambda_3)^2(5\lambda_1^2-6\lambda_1\lambda_3-7\lambda_3^2)\\
&+ 2k_3^2(R^2(\lambda_1-\lambda_2)^2(\lambda_1^2+6\lambda_1\lambda_2+\lambda_2^2-2\lambda_1\lambda_3-2\lambda_2\lambda_3)+k_1^2(\lambda_2^3(\lambda_2-2\lambda_3)\\
&+ \lambda_1^2(2\lambda_2^2+2\lambda_2\lambda_3-5\lambda_3^2)+2\lambda_1\lambda_2(-2\lambda_2^2+\lambda_2\,\lambda_3+2\lambda_3^2)))+2k_2^2(k_3^2(\lambda_1^4-5\lambda_2^2\lambda_3^2\\
&+ 2\lambda_1^2\lambda_2(\lambda_2+\lambda_3)-2\lambda_1^3(2\lambda_2+\lambda_3)+2\lambda_1\lambda_2\lambda_3(\lambda_2+2\lambda_3))+R^2(\lambda_1-\lambda_3)(5\lambda_1^3\\
&+ \lambda_1^2(2\,\lambda_2-11\lambda_3)-\lambda_1(4\lambda_2^2-10\lambda_2\lambda_3+\lambda_3^2)+\lambda_3(-2\,\lambda_2^2-4\lambda_2\lambda_3+7\lambda_3^2))\\
&+ k_1^2(\lambda_3^2(2\lambda_2^2+4\lambda_2\lambda_3-7\lambda_3^2)+\lambda_1^2(5\lambda_2^2-8\lambda_2\lambda_3+2\lambda_3^2)+\lambda_1(-8\lambda_2^2\lambda_3+6\lambda_2\lambda_3^2+4\lambda_3^3))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{34} = {}& -k_3(4R^2\lambda_1^4-6k_1^2\lambda_1^2\lambda_2^2-8R^2\lambda_1^2\lambda_2^2+6k_1^2\lambda_1\lambda_2^3+4k_1^2\lambda_2^4+4R^2\lambda_2^4\\
&+ 2k_3^2(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)(2\,\lambda_1+2\lambda_2-\lambda_3)-8R^2\lambda_1^3\lambda_3+3k_1^2\lambda_1^2\lambda_2\lambda_3\\
&+ 8R^2\lambda_1^2\lambda_2\lambda_3-4k_1^2\lambda_1\lambda_2^2\lambda_3+8R^2\lambda_1\lambda_2^2\lambda_3-8k_1^2\lambda_2^3\lambda_3-8R^2\lambda_2^3\lambda_3+3k_1^2\lambda_1^2\lambda_3^2\\
&+ 4R^2\lambda_1^2\lambda_3^2-k_1^2\lambda_1\lambda_2\lambda_3^2-8R^2\lambda_1\lambda_2\lambda_3^2+4k_1^2\lambda_2^2\lambda_3^2+4R^2\lambda_2^2\lambda_3^2-k_1^2\lambda_1\lambda_3^3\\
&+ k_2^2(\lambda_1-\lambda_3)(4\lambda_1^3+\lambda_1^2(6\,\lambda_2-4\lambda_3)+2\lambda_1\lambda_2(-3\lambda_2+\lambda_3)+\ \lambda_2\lambda_3(-3\lambda_2+\lambda_3))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{44} = {}& -3R^4\lambda_1^4+4R^4\lambda_1^3\lambda_2+4k_1^2R^2\lambda_1^2\lambda_2^2-2R^4\lambda_1^2\lambda_2^2-8k_1^2R^2\lambda_1\lambda_2^3+4R^4\lambda_1\lambda_2^3+5k_1^4\lambda_2^4\\
&+ 2k_1^2R^2\lambda_2^4-3R^4\lambda_2^4+k_3^4(\lambda_1-\lambda_2)^2(5\lambda_1^2+14\lambda_1\lambda_2+5\lambda_2^2)+4R^4\lambda_1^3\lambda_3-8k_1^2R^2\lambda_1^2\lambda_2\lambda_3\\
&- 4R^4\lambda_1^2\lambda_2\lambda_3+8k_1^2R^2\lambda_1\lambda_2^2\lambda_3-4R^4\lambda_1\lambda_2^2\lambda_3+4k_1^4\lambda_2^3\lambda_3+8k_1^2R^2\lambda_2^3\lambda_3+4R^4\lambda_2^3\lambda_3\\
&+ 4k_1^2R^2\lambda_1^2\lambda_3^2-2R^4\lambda_1^2\lambda_3^2+8k_1^2R^2\lambda_1\lambda_2\lambda_3^2-4R^4\lambda_1\lambda_2\lambda_3^2-18k_1^4\lambda_2^2\lambda_3^2-20k_1^2R^2\lambda_2^2\lambda_3^2\\
&- 2R^4\lambda_2^2\lambda_3^2-8k_1^2R^2\lambda_1\lambda_3^3+4R^4\lambda_1\lambda_3^3+4k_1^4\lambda_2\lambda_3^3+8k_1^2R^2\lambda_2\lambda_3^3+4R^4\lambda_2\lambda_3^3+5k_1^4\lambda_3^4\\
&+ 2k_1^2R^2\lambda_3^4-3R^4\lambda_3^4+k_2^4(\lambda_1-\lambda_3)^2(5\lambda_1^2+14\lambda_1\lambda_3+5\lambda_3^2)+2k_3^2(R^2(\lambda_1-\lambda_2)^2(\lambda_1^2\\
&+ 6\lambda_1\lambda_2+\lambda_2^2-4\lambda_1\lambda_3-4\lambda_2\lambda_3+2\lambda_3^2)+k_1^2(\lambda_2^2(5\lambda_2^2+2\lambda_2\lambda_3-4\lambda_3^2)+2\lambda_1\lambda_2(\lambda_2^2\\
&- 5\lambda_2\lambda_3+\lambda_3^2)+\lambda_1^2(-4\,\lambda_2^2+2\lambda_2\lambda_3+5\lambda_3^2)))+2k_2^2(R^2(\lambda_1-\lambda_3)^2(\lambda_1^2-4\lambda_1\lambda_2
\end{aligned}
$$

$$+ 2\lambda_2^2 + 6\lambda_1\lambda_3 - 4\lambda_2\lambda_3 + \lambda_3^2) + k_3^2(5\lambda_1^4 + 5\lambda_2^2\lambda_3^2 + 2\lambda_1^3(\lambda_2 + \lambda_3) + 2\lambda_1\lambda_2\lambda_3(\lambda_2$$
$$+ \lambda_3) - 2\lambda_1^2(2\lambda_2^2 + 5\lambda_2\lambda_3 + 2\lambda_3^2)) + k_1^2(\lambda_1^2(5\lambda_2^2 + 2\lambda_2\lambda_3 - 4\lambda_3^2)$$
$$+ 2\lambda_1\lambda_3(\lambda_2^2 - 5\lambda_2\lambda_3 + \lambda_3^2) + \lambda_3^2(-4\,\lambda_2^2 + 2\lambda_2\lambda_3 + 5\lambda_3^2))).$$

We proceed as in the proof of Lemma 4.4, thus, the metric is $R[\rho]$-Einstein if and only if the $\{\mathfrak{R}_{\mathrm{ij}}\}$ are vanishing.

Assuming $\lambda_1 = \lambda_2 = \lambda_3$, one has that 01 $R[\rho]_0 = \frac{1}{16}\,\mathrm{diag}(\lambda_1^4, \lambda_1^4, \lambda_1^4, -3\lambda_1^4)$, thus showing that the metric is not $R[\rho]$-Einstein since $\lambda_1\lambda_2\lambda_3 \neq 0$. Now, we consider the terms k_i at (4.1). Up to a permutation of the basis $\{e_1, e_2, e_3\}$ one may assume that one of the following holds:

$$(i)\ k_1k_2k_3 \neq 0, \qquad (ii)\ k_1 = 0, k_2k_3 \neq 0, \qquad (iii)\ k_1 = k_2 = 0, k_3 \neq 0.$$

We analyse the different possibilities separately.

(i) $k_1k_2k_3 \neq 0$

Let $\mathcal{I} \subset \mathbb{R}[k_1, k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ be the ideal generated by the polynomials $\mathfrak{R}_{ij}$ (after simplification due to $k_1k_2k_3 \neq 0$) and compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}$ with respect to the lexicographical order. Since the polynomial $\mathrm{g}_1 = R^4\lambda_1\lambda_2^2\lambda_3^2(\lambda_2^2 - \lambda_3^2)$ belongs to $\mathcal{G}_1$ one has that $\lambda_2^2 - \lambda_3^2 = 0$. Computing a Gröbner basis $\mathcal{G}_{11}$ of the ideal generated by $\mathcal{G}_1 \cup \{\lambda_2^2 - \lambda_3^2\}$, one has that $R^4\lambda_1^2\lambda_3^3$ belongs to $\mathcal{G}_{11}$, which is a contradiction.

(ii) $k_1 = 0$, $k_2k_3 \neq 0$

We firstly simplify the polynomials $\mathfrak{R}_{23}$, $\mathfrak{R}_{24}$ and $\mathfrak{R}_{34}$ by using that $k_2k_3 \neq 0$ and then compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{I} \subset \mathbb{R}[k_2, k_3, \lambda_1, \lambda_2, \lambda_3, R]$ generated by the polynomials $\mathfrak{R}_{ij}$ (after simplification). Since $\mathrm{g}_2 = R^6\lambda_2^2\lambda_3^3(\lambda_2 - \lambda_3) \in \mathcal{G}_2$ one has that $\lambda_2 = \lambda_3$. Computing a Gröbner basis $\mathcal{G}_{21}$ of the ideal generated by $\mathcal{G}_2 \cup \{\lambda_2 - \lambda_3\}$, one has that $R^4\lambda_1\lambda_3^4$ belongs to $\mathcal{G}_{21}$, which is a contradiction.

(iii) $k_1 = k_2 = 0$, $k_3 \neq 0$

Let $\mathcal{I} \subset \mathbb{R}[k_3, \lambda_1, \lambda_2, \lambda_3, R]$ be the ideal generated by the polynomials $\mathfrak{R}_{ij}$ (after simplifying $\mathfrak{R}_{34}$ due to $k_3 \neq 0$) and compute a Gröbner basis $\mathcal{G}_3$ of $\mathcal{I}$ with respect to the lexicographical order. Since the polynomial $\mathrm{g}_3 = R^6\lambda_2^2\lambda_3^3(\lambda_2^2 - \lambda_3^2)^2$ belongs to $\mathcal{G}_3$ one has that $\lambda_2^2 - \lambda_3^2 = 0$. Computing a Gröbner basis $\mathcal{G}_{31}$ of the ideal generated by $\mathcal{G}_3 \cup \{\lambda_2^2 - \lambda_3^2\}$, one has that $R^6\lambda_1^2\lambda_3^3$ belongs to $\mathcal{G}_{31}$, which is a contradiction.

Hence the Lie algebra structure (4.5) does not support any $R[\rho]$-Einstein metric. □

4.3.2 The semi-direct products $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$

Lemma 4.9. *Any left-invariant $R[\rho]$-Einstein metric in $\mathbb{R}e_4 \ltimes E(1,1)$ or $\mathbb{R}e_4 \ltimes E(2)$ is Einstein.*

Proof. Let $R[\rho]_0$ be the trace-free tensor $R[\rho]_0 = R[\rho] - \frac{1}{4}\|\rho\|^2 g$. A straightforward calculation shows that the components of $R[\rho]_0$ for any left-invariant metric (4.2) are

$$\begin{array}{lll} 16R^4 R[\rho]_{0_{11}} = \mathfrak{R}_{11}, & 8R^4 R[\rho]_{0_{12}} = \mathfrak{R}_{12}, & 8R^4 R[\rho]_{0_{13}} = \mathfrak{R}_{13}, \\ 8R^3 R[\rho]_{0_{14}} = \mathfrak{R}_{14}, & 16R^4 R[\rho]_{0_{22}} = \mathfrak{R}_{22}, & 8R^4 R[\rho]_{0_{23}} = \mathfrak{R}_{23}, \\ 8R^3 R[\rho]_{0_{24}} = \mathfrak{R}_{24}, & 16R^4 R[\rho]_{0_{33}} = \mathfrak{R}_{33}, & 4R^3 R[\rho]_{0_{34}} = \mathfrak{R}_{34}, \\ 16R^4 R[\rho]_{0_{44}} = \mathfrak{R}_{44}, & & \end{array}$$

where the polynomials $\mathfrak{R}_{\mathrm{ij}}$ are given by

$$\begin{aligned} \mathfrak{R}_{11} = {} & 16b^4 - 7C^4 - 10C^2D^2 - 3D^4 - 14C^2R^2 - 7R^4 + 8C^2R^2\lambda_2 - 4D^2R^2\lambda_2 + 8R^4\lambda_2 \\ & + 4C^2R^2\lambda_2^2 + 2D^2R^2\lambda_2^2 + 10R^4\lambda_2^2 - 16R^4\lambda_2^3 + 5R^4\lambda_2^4 + 8AbCD(-1+2\lambda_2) \\ & - 2b^2(9C^2 - 3D^2 + 8R^2(-1+\lambda_2)\lambda_2) + A^4(-1+\lambda_2)^2(-7 - 6\lambda_2 + 5\lambda_2^2) \\ & + 2A^2(-7R^2 + 8b^2(-1+\lambda 2) - 2D^2\lambda_2 + 8R^2\lambda_2 + D^2\lambda_2^2 + 10R^2\lambda_2^2 - 16R^2\lambda_2^3 \\ & + 5R^2\lambda_2^4 + C^2(-7 + 4\lambda_2 + 2\lambda_2^2)), \end{aligned}$$

$$\begin{aligned} \mathfrak{R}_{12} = {} & -(A^2CD(-1+\lambda_2)^2 + 6A^3b(-1+\lambda_2)^3 + CD(12b^2 + 2C^2 + 2D^2 + R^2 - 2R^2\lambda_2 \\ & + R^2\lambda_2^2) + Ab(-7D^2 - 6R^2 + 8b^2(-1+\lambda_2) + 5D^2\lambda_2 + 18R^2\lambda_2 - 18R^2\lambda_2^2 \\ & + 6R^2\lambda_2^3 + C^2(-5 + 7\lambda_2))), \end{aligned}$$

$$\begin{aligned} \mathfrak{R}_{13} = {} & -4A^3D(-1+\lambda_2)^2\lambda_2 + 4A^2bC(2 - 3\lambda_2 + \lambda_2^2) + bC(24b^2 + 7C^2 + 7D^2 + 7R^2 + 4R^2\lambda_2 \\ & - 11R^2\lambda_2^2) - AD(-2D^2 + 4D^2\lambda_2 + 4R^2\lambda_2 - 8R^2\lambda_2^2 + 4R^2\lambda_2^3 + 4b^2(1+\lambda_2) \\ & + C^2(-1 + 3\lambda_2)), \end{aligned}$$

$$\begin{aligned} \mathfrak{R}_{14} = {} & -4A^2D(-1+\lambda_2)^2\lambda_2 + AbC(1 - 16\lambda_2 + 15\lambda_2^2) + D(C^2 + 2D^2 - 3C^2\lambda_2 - 4D^2\lambda_2 \\ & - 4R^2\lambda_2 + 8R^2\lambda_2^2 - 4R^2\lambda_2^3 + 2b^2(1+\lambda_2)), \end{aligned}$$

$$\begin{aligned} \mathfrak{R}_{22} = {} & 16b^4 - 3C^4 - 10C^2D^2 - 7D^4 + 2C^2R^2 + 4D^2R^2 + 5R^4 + 2b^2(3C^2 - 9D^2 \\ & + 8R^2(-1+\lambda_2)) + 8AbCD(-2+\lambda_2) - 4C^2R^2\lambda_2 + 8D^2R^2\lambda_2 - 16R^4\lambda_2 \\ & - 14D^2R^2\lambda_2^2 + 10R^4\lambda_2^2 + 8R^4\lambda_2^3 - 7R^4\lambda_2^4 - A^4(-1+\lambda_2)^2(-5 + 6\lambda_2 + 7\lambda_2^2) \\ & - 2A^2(C^2(-1+2\lambda_2) + D^2(-2 - 4\lambda_2 + 7\lambda_2^2) + (-1+\lambda_2)(8b^2\lambda_2 \\ & + R^2(5 - 11\lambda_2 - \lambda_2^2 + 7\lambda_2^3)))), \end{aligned}$$

$$\begin{aligned} \mathfrak{R}_{23} = {} & 4A^3C(-1+\lambda_2)^2 + 4A^2bD(1 - 3\lambda_2 + 2\lambda_2^2) + bD(24b^2 + 7C^2 + 7D^2 - 11R^2 + 4R^2\lambda_2 \\ & + 7R^2\lambda_2^2) + AC(3D^2 + 4R^2 - 2C^2(-2+\lambda_2) - D^2\lambda_2 - 8R^2\lambda_2 + 4R^2\lambda_2^2 + 4b^2(1+\lambda_2)), \end{aligned}$$

$$\begin{aligned} \mathfrak{R}_{24} = {} & 4A^2C(-1+\lambda_2)^2 + AbD(15 - 16\lambda_2 + \lambda_2^2) - C(-3D^2 - 4R^2 + 2C^2(-2+\lambda_2) + D^2\lambda_2 \\ & + 8R^2\lambda_2 - 4R^2\lambda_2^2 + 2b^2(1+\lambda_2)), \end{aligned}$$

$$\begin{aligned} \mathfrak{R}_{33} = {} & -48b^4 + 5C^4 + 10C^2D^2 + 5D^4 + 10C^2R^2 - 8D^2R^2 + 5R^4 - 2b^2(C^2 + D^2 \\ & - 8R^2(-1+\lambda_2)^2) + 12AbCD(-1+\lambda_2) + 4C^2R^2\lambda_2 + 4D^2R^2\lambda_2 + 4R^4\lambda_2 \\ & - 8C^2R^2\lambda_2^2 + 10D^2R^2\lambda_2^2 - 18R^4\lambda_2^2 + 4R^4\lambda_2^3 + 5R^4\lambda_2^4 - A^4(-1+\lambda_2)^2(3 + 2\lambda_2 \\ & + 3\lambda_2^2) + 2A^2(2D^2 + R^2 - 8b^2(-1+\lambda_2)^2 - 4D^2\lambda_2 + 4R^2\lambda_2 + D^2\lambda_2^2 - 10R^2\lambda_2^2 \\ & + 4R^2\lambda_2^3 + R^2\lambda_2^4 + C^2(1 - 4\lambda_2 + 2\lambda_2^2)), \end{aligned}$$

$$\mathfrak{R}_{34} = -(-6bCD(-1+\lambda_2) + 2A^3(-1+\lambda_2^2)^2 + A(-3D^2 + 2R^2 + 8b^2(-1+\lambda_2)^2 + 3D^2\lambda_2$$

$$
\begin{aligned}
&+ 2D^2\lambda_2^2 - 4R^2\lambda_2^2 + 2R^2\lambda_2^4 + C^2(2 + 3\lambda_2 - 3\lambda_2^2))),\\
\mathfrak{R}_{44} = {}& 16b^4 + 5C^4 + 10C^2D^2 + 5D^4 + 14b^2(C^2 + D^2) + 2C^2R^2 + 4D^2R^2 - 3R^4\\
&- 36AbCD(-1 + \lambda_2) - 8C^2R^2\lambda_2 - 8D^2R^2\lambda_2 + 4R^4\lambda_2 + 4C^2R^2\lambda_2^2 + 2D^2R^2\lambda_2^2\\
&- 2R^4\lambda_2^2 + 4R^4\lambda_2^3 - 3R^4\lambda_2^4 + A^4(-1 + \lambda_2)^2(5 + 14\lambda_2 + 5\lambda_2^2) + 2A^2(-4D^2 + R^2\\
&+ 16b^2(-1 + \lambda_2)^2 + 2D^2\lambda_2 + 4R^2\lambda_2 + 5D^2\lambda_2^2 - 10R^2\lambda_2^2 + 4R^2\lambda_2^3 + R^2\lambda_2^4\\
&+ C^2(5 + 2\lambda_2 - 4\lambda_2^2)).
\end{aligned}
$$

Let $\mathcal{I} \subset \mathbb{R}[A, b, \lambda_2, C, D, R]$ be the ideal generated by the polynomials $\{\mathfrak{R}_{ij}\}$ and compute a Gröbner basis $\mathcal{G}_0 \subset \mathcal{I}$ with respect to the lexicographic order. Since the polynomial $\mathrm{g}_0 = D^3(C^2 + D^2)^2 \in \mathcal{G}_0$ one has that $D = 0$. Computing a Gröbner basis $\mathcal{G}_{01}$ of the ideal $\mathcal{G}_0 \cup \{D\}$, one has that $C^5\lambda_2^2 \in \mathcal{G}_{01}$, and thus $C = 0$. Once again, compute a Gröbner basis $\mathcal{G}_{011}$ of the ideal $\mathcal{G}_{01} \cup \{C\}$ to see that $Ab^4 \in \mathcal{G}_{011}$. Therefore either $A = 0$ or $b = 0$. We analyze both possibilities separately.

$A = 0$

Compute a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{G}_{011} \cup \{A\}$ to see that $R^4(\lambda_2 - 1)^4(\lambda_2 + 1) \in \mathcal{G}_1$. Hence $\lambda_2 = \pm 1$. Setting $\lambda_2 = 1$. If $\lambda_2 = 1$, then the $R[\rho]$-Einstein equations reduce to $b = 0$, in which case the metric is Einstein. Moreover, if $\lambda_2 = -1$, then the $R[\rho]$-Einstein equations reduce to $b^2 = R^2$ and the metric is Einstein.

$b = 0$

Compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{G}_{011} \cup \{b\}$ to see that $R^2(A^2 + R^2)(\lambda_2^2 - 1)^2 \in \mathcal{G}_2$. Hence $\lambda_2 = \pm 1$. If $\lambda_2 = 1$, then the resulting metric is Einstein while in the case $\lambda_2 = -1$ the $R[\rho]$-Einstein equations become $\frac{1}{R^4}(A^2 + R^2)^2 = 0$ which has no solutions with $R > 0$. □

4.3.3 The semi-direct product $\mathbb{R}e_4 \ltimes H^3$

Lemma 4.10. *Any left-invariant $R[\rho]$-Einstein metric in $\mathbb{R}e_4 \ltimes H^3$ is Einstein.*

Proof. Let $R[\rho]_0$ be the trace-free tensor $R[\rho]_0 = R[\rho] - \frac{1}{4}\|\rho\|^2 g$. A straightforward calculation shows that the components of $R[\rho]_0$ for any left-invariant metric (4.3) are

$$
\begin{aligned}
&16R^4R[\rho]_{0_{11}} = \mathfrak{R}_{11}, \quad 4R^4R[\rho]_{0_{12}} = \mathfrak{R}_{12}, \quad 8R^4R[\rho]_{0_{13}} = \mathfrak{R}_{13},\\
&4R^3R[\rho]_{0_{14}} = \mathfrak{R}_{14}, \quad 16R^4R[\rho]_{0_{22}} = \mathfrak{R}_{22}, \quad 8R^4R[\rho]_{0_{23}} = \mathfrak{R}_{23},\\
&4R^3R[\rho]_{0_{24}} = \mathfrak{R}_{24}, \quad 16R^4R[\rho]_{0_{33}} = \mathfrak{R}_{33}, \quad 8R^3R[\rho]_{0_{34}} = \mathfrak{R}_{34},\\
&16R^4R[\rho]_{0_{44}} = \mathfrak{R}_{44},
\end{aligned}
$$

where the polynomials $\mathfrak{R}_{\mathrm{ij}}$ are given by

$$
\begin{aligned}
\mathfrak{R}_{11} = {}& 16a^4 - 48d^4 - 16d^2F^2 - 3F^4 + 12cdFH - 2d^2H^2 + 2F^2H^2 + 5H^4\\
&- 2c^2(4d^2 + F^2 + H^2) + 16d^2R^2 + 2F^2R^2 + 10H^2R^2 + 5R^4 - 2a^2(4c^2 + 8d^2
\end{aligned}
$$

$$
\begin{aligned}
&- 3F^2 + 8R^2) + 4a(4c^2 d - cFH - 2d(8d^2 + F^2 + 3H^2 - R^2)),\\
\mathfrak{R}_{12} &= 8a^3 c + a^2(-4cd + 2FH) - cd(8d^2 + 2F^2 + 3H^2 - 3R^2) + 2FH(d^2 + F^2 + H^2 + R^2)\\
&\quad + a(-6dFH + c(4d^2 + 3F^2 + 2H^2 - 3R^2)),\\
\mathfrak{R}_{13} &= 8a^3 H + a^2(-8cF + 12dH) - cF(8d^2 + 2F^2 + 2H^2 + R^2) + aH(28d^2 + 5F^2\\
&\quad + 6(H^2 + R^2)) + dH(24d^2 + 8F^2 + 7(H^2 + R^2)),\\
\mathfrak{R}_{14} &= -a^2 F + 8d^2 F - 3cdH + 2a(5dF + 3cH) + 2F(F^2 + H^2 + R^2),\\
\mathfrak{R}_{22} &= -48a^4 - 64a^3 d + 16d^4 + 5F^4 + 4cdFH + 6d^2H^2 + 2F^2H^2 - 3H^4 - 2c^2(4d^2 + F^2\\
&\quad + H^2) - 16d^2R^2 + 10F^2R^2 + 2H^2R^2 + 5R^4 - 2a^2(4c^2 + 8d^2 + F^2 + 8H^2 - 8R^2)\\
&\quad + 4a(4c^2 d - 3cFH - 2d(3F^2 + H^2 - R^2)),\\
\mathfrak{R}_{23} &= 24a^3 F + 8d^3 F + 8cd^2 H + 4a^2(7dF + 2cH) + cH(2F^2 + 2H^2 + R^2) + dF(6F^2 + 5H^2\\
&\quad + 6R^2) + aF(12d^2 + 7F^2 + 8H^2 + 7R^2),\\
\mathfrak{R}_{24} &= 6cdF - 8a^2 H - a(3cF + 10dH) + H(d^2 - 2(F^2 + H^2 + R^2)),\\
\mathfrak{R}_{33} &= -(-16a^4 - 32a^3 d - 16d^4 + 16d^2F^2 + 7F^4 - 20cdFH + 18d^2H^2 + 14F^2H^2 + 7H^4\\
&\quad + 2c^2(4d^2 + F^2 + H^2) + 2a^2(4c^2 - 8d^2 + 9F^2 + 8H^2) - 4a(4c^2 d + 8d^3 - 5cFH\\
&\quad - 4d(F^2 + H^2)) + 14F^2R^2 + 14H^2R^2 + 7R^4),\\
\mathfrak{R}_{34} &= 4a^2 c - dFH + a(-8cd + FH) + c(4d^2 + F^2 + H^2),\\
\mathfrak{R}_{44} &= 16a^4 + 32a^3 d + 16d^4 + 32d^2F^2 + 5F^4 - 36cdFH + 14d^2H^2 + 10F^2H^2 + 5H^4\\
&\quad + 6c^2(4d^2 + F^2 + H^2) + 2a^2(12c^2 + 8d^2 + 7F^2 + 16H^2) + 2F^2R^2 + 2H^2R^2\\
&\quad - 3R^4 + 4a(-12c^2 d + 9cFH + 4d(2d^2 + 3F^2 + 3H^2 - R^2)).
\end{aligned}
$$

Therefore $\mathbb{R}e_4 \ltimes H$ is $R[\rho]$-Einstein if and only if the system of equations $\{\mathfrak{R}_{\mathrm{ij}} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, H, F, R]$ be the ideal generated by the polynomials $\{\mathfrak{R}_{ij}\}$ and compute a Gröbner basis $\mathcal{G}_0 \subset \mathcal{I}$ with respect to the lexicographic order. Since the polynomial $\mathrm{g}_0 = FHR^2(F^2 + H^2 + R^2)(2H^2 + R^2)^3 \in \mathcal{G}_0$ one has that $FH = 0$. Computing a Gröbner basis $\mathcal{G}_{01}$ of the ideal $\mathcal{G}_0 \cup \{FH\}$ with respect to the lexicographic order, one has that the polynomials $HR^2(H^2+R^2)^2$ and $FR^2(F^2+R^2)^2$ belong to $\mathcal{G}_{01}$, and hence $F = H = 0$. Computing again a Gröbner basis $\mathcal{G}_{011}$ of the ideal $\mathcal{G}_{01} \cup \{F, H\}$ with respect to the lexicographic order, one has that the polynomial $(4d^2 - R^2)R^6 \in \mathcal{G}_{011}$. Once again we compute a Gröbner basis $\mathcal{G}_{0111}$ of the ideal $\mathcal{G}_{011} \cup \{(4d^2 - R^2\}$ and get that the polynomial $(a - d)R^4 \in \mathcal{G}_{0111}$. Thus $a = d$ and the resulting metric is Einstein. □

4.3.4 The semi-direct product $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

Lemma 4.11. *Any left-invariant $R[\rho]$-Einstein metric on $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ is Einstein.*

Proof. We proceed as in Lemma 4.7 and consider a left-invariant metric on $\mathbb{R}e_4 \ltimes \mathbb{R}^3$ given as in (4.4). Then the components of the trace-free tensor $R[\rho]_0$ become

$$
\begin{aligned}
&2R^4 R[\rho]_{0_{11}} = \mathfrak{R}_{11}, \quad R^4 R[\rho]_{0_{12}} = \mathfrak{R}_{12}, \quad R^4 R[\rho]_{0_{13}} = \mathfrak{R}_{13}, \quad R[\rho]_{0_{14}} = 0,\\
&2R^4 R[\rho]_{0_{22}} = \mathfrak{R}_{22}, \quad R^4 R[\rho]_{0_{23}} = \mathfrak{R}_{23}, \quad R[\rho]_{0_{24}} = 0, \qquad 2R^4 R[\rho]_{0_{33}} = \mathfrak{R}_{33},\\
&R[\rho]_{0_{34}} = 0, \qquad 2R^4 R[\rho]_{0_{44}} = \mathfrak{R}_{44},
\end{aligned}
$$

where the polynomials $\mathfrak{R}_{ij}$ are given by

$$\begin{aligned}
\mathfrak{R}_{11} &= -(-a^4 + b^2f^2 + f^4 + f^2h^2 + f^3p - 2fh^2p + c^2p^2 + 2f^2p^2 + h^2p^2 + fp^3 + p^4 \\
&\quad + a^3(f+p) + a^2(b^2 + c^2 - 2f^2 + fp - 2p^2) - a(2b^2f + f^3 + 2c^2p + f^2p + fp^2 + p^3)), \\
\mathfrak{R}_{12} &= -(b(a-f)(a^2 - af + f^2 + p^2)), \\
\mathfrak{R}_{13} &= -(c(a-p)(a^2 + f^2 - ap + p^2)), \\
\mathfrak{R}_{22} &= -(a^4 + b^2f^2 - f^4 + f^2h^2 + f^3p - 2fh^2p + c^2p^2 - 2f^2p^2 + h^2p^2 - fp^3 + p^4 \\
&\quad + a^3(-f+p) + a^2(b^2 + c^2 - 2f^2 - fp + 2p^2) + a(-2b^2f + f^3 - 2c^2p + f^2p - fp^2 + p^3)), \\
\mathfrak{R}_{23} &= -(h(f-p)(a^2 + f^2 - fp + p^2)), \\
\mathfrak{R}_{33} &= -(a^4 + b^2f^2 + f^4 + f^2h^2 + a^3(f-p) - f^3p - 2fh^2p + c^2p^2 - 2f^2p^2 + h^2p^2 + fp^3 \\
&\quad - p^4 + a^2(b^2 + c^2 + 2f^2 - fp - 2p^2) + a(-2b^2f + f^3 - 2c^2p - f^2p + fp^2 + p^3)), \\
\mathfrak{R}_{44} &= a^4 + 3b^2f^2 + f^4 + 3f^2h^2 + f^3p - 6fh^2p + 3c^2p^2 - 2f^2p^2 + 3h^2p^2 + fp^3 + p^4 \\
&\quad + a^3(f+p) + a^2(3b^2 + 3c^2 - 2f^2 - fp - 2p^2) + a(-6b^2f + f^3 - 6c^2p - f^2p - fp^2 + p^3).
\end{aligned}$$

Hence a left-invariant metric (4.4) is $R[\rho]$-Einstein if and only if the system of polynomial equations $\{\mathfrak{R}_{ij} = 0\}$ is satisfied. Let $\mathcal{I} \subset \mathbb{R}[a, b, c, f, h, p, R]$ be the ideal generated by the polynomials $\{\mathfrak{R}_{ij}\}$ and compute a Gröbner basis $\mathcal{G}_0$ with respect to the lexicographic order. Since the polynomial $\mathrm{g}_0 = p^7(f^3 - p^3)$ belongs to $\mathcal{G}_0$, one has that either $p = 0$ or $p = f$. We analyze the two cases separately.

$p = 0$

Compute a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{G}_0 \cup \{p\}$ to see that $f^7 \in \mathcal{G}_1$, and thus $f = 0$. Compute again a Gröbner basis $\mathcal{G}_{11}$ of the ideal $\mathcal{G}_1 \cup f$. It now becomes that the new Gröbner basis is given by

$$\mathcal{G}_{11} = \{p, f, a^2(b^2 + c^2), a^3c, a^2b, a^4\}.$$

Hence $a = 0$ and the resulting metric is flat.

$p = f, p \neq 0$

Compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{G}_0 \cup \{p - f\}$ to see that $p^6(a - p) \in \mathcal{G}_2$. Since $p \neq 0$, one has $p = a$. The resulting metric is now flat, which finishes the proof. □

4.4 Homothety classes

Assertion (3) in Theorem 4.2 follows at once from last section. In order to prove Assertion (1), we recall that the $(1, 3)$-Weyl curvature operator is invariant by conformal transformations. Indeed, if two Riemannian metrics are conformally equivalent (i.e., $\widetilde{g} = e^{2\sigma}g$), then their Weyl curvature operators $\widetilde{W}$ and W are equal. The converse does not hold in general.

Furthermore, since Takagi proved in [63] that homogeneous locally conformally flat metrics are symmetric, non-symmetric conformally related homogeneous metrics are necessarily homothetic.

Thus, an homothety between two conformally related metrics can be obtained from the work of Kulkarni [48] just considering the curvature endomorphisms $R(e_i, e_j)$ of the Lie algebra.

Arias-Marco and Kowalski showed in [1] that any non-symmetric homogeneous $\check{R}$-Einstein metric is determined by the Lie algebra structure

$$[e_4, e_1] = \alpha e_1, \quad [e_4, e_2] = -\alpha e_2 - \beta e_3, \quad [e_4, e_3] = \beta e_2 - \alpha e_3, \tag{4.8}$$

where $\alpha \neq 0$ and β are constants and $\{e_1, \dots, e_4\}$ is an orthonormal basis. Now, an explicit calculation shows that the endomorphisms $R(e_i, e_j)$ of any metric (4.8) satisfy

$$\begin{aligned} R(e_1, e_2) &= \alpha^2 (E^1{}_2 - E^2{}_1), & R(e_1, e_3) &= \alpha^2 (E^1{}_3 - E^3{}_1), \\ R(e_1, e_4) &= \alpha^2 (E^4{}_1 - E^1{}_4), & R(e_2, e_3) &= \alpha^2 (E^3{}_2 - E^2{}_3) \\ R(e_2, e_4) &= \alpha^2 (E^2{}_4 - E^4{}_2), & R(e_3, e_4) &= \alpha^2 (E^3{}_4 - E^4{}_3) \end{aligned}$$

where $E^i{}_j$ denotes the matrix with 1 in the position (i, j) and zero otherwise. This shows that these endomorphisms are independent of the parameter β in Equation (4.8).

Hence the Lie group determined by (4.8) is homothetic (but not isomorphically homothetic) to the Lie group determined by the Lie algebra

$$[e_4, e_1] = \alpha e_1, \quad [e_4, e_2] = -\alpha e_2, \quad [e_4, e_3] = -\alpha e_3. \tag{4.9}$$

Finally, the Lie groups determined by (4.9) are all isomorphically homothetic to the Lie group structure in Theorem 4.2-(1) by taking the change $\bar{e}_i = \frac{1}{\alpha} e_i$, thus showing Assertion (1).

The Riemannian structure corresponding to Theorem 4.2-(2.a) has zero scalar curvature, while the scalar curvature in the cases (2.b) and (2.c) is strictly negative, so the Lie group corresponding to Assertion (2.a) cannot be homothetic to any of (2.b) or (2.c) in Theorem 4.2. Furthermore, the structure defined by Assertion (2.b) satisfies

$$\tau = -1, \quad \|\rho\|^2 = 1, \quad \|R\|^2 = 3.$$

On the other hand, replacing the metric $\langle \cdot\,, \cdot \rangle_\alpha$ in Assertion (2.c) by the homothetic one $\langle \cdot\,, \cdot \rangle^*_\alpha = \frac{2(1+\alpha+\alpha^2)^2}{(1+\alpha)^2} \langle \cdot\,, \cdot \rangle_\alpha$, a straightforward calculation shows that

$$\tau_\alpha = -1, \quad \|\rho_\alpha\|^2 = 1, \quad \|R_\alpha\|^2 = \frac{3 + \alpha(1+\alpha)(9 + \alpha(1+\alpha)(7 + 3\alpha(1+\alpha)))}{(1+\alpha+\alpha^2)^3}.$$

Now, any homothety between the metric in Assertion (2.b) and any of the metrics in Assertion (2.c) must induce an isometry between the metric (2.b) and some metric $\langle \cdot\,, \cdot \rangle^*_\alpha$. Hence it must preserve the norm of the curvature tensor, and thus $\|R_\alpha\|^2 = 3$ for some α. A straightforward calculation now shows that $\|R_\alpha\|^2 = 3$ if and only if either $\alpha = 0$ or $\alpha = -1$ and none of these cases may occur. Therefore the structure in Assertion (2.b) cannot be homothetic to any of the structures given by Assertion (2.c).

4.5 Critical metrics for $\mathcal{S}$ and $\mathcal{F}_t$

Now we consider each family of metrics obtained, determining whether or not are $\mathcal{S}$ or $\mathcal{F}_t$-critical. We just consider the non-symmetric case since the symmetric one was studied in chapter two. Since we work with homogeneous spaces, then the Ricci tensor is parallel and the scalar curvature is constant. Recall that a metric is $\mathcal{S}$-critical if and only if

$$2\left(Hess_\tau - \frac{\Delta\tau}{4}g\right) - 2\tau\left(\rho - \frac{\tau}{4}g\right) = 0. \tag{4.10}$$

Therefore, an homogeneous metric is $\mathcal{S}$-critical if and only if it is Einstein or it has vanishing scalar curvature.

All critical metrics are considered in the next result.

Theorem 4.12. *Let (M, g) be a homogeneous four dimensional weakly-Einstein Riemannian manifold. Then,*

1. *(M, g) is $\mathcal{S}$-critical if and only if it is homothetic to the Lie group $SU(2)\times\mathbb{R}$ with left-invariant metric determined by the Lie algebra*

$$\begin{aligned} &[e_1, e_2] = (4 \pm 2\sqrt{2})e_3, \quad [e_2, e_3] = (3 \pm 2\sqrt{2})e_1, \quad [e_3, e_1] = e_2,\\ &[e_4, e_1] = -e_2, \qquad\qquad [e_4, e_2] = (3 \pm 2\sqrt{2})e_1. \end{aligned}$$

   where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

2. *(M, g) is $\mathcal{F}_t$-critical if and only if $t = -\frac{3}{4}$ and (M, g) is homothetic to the Lie group $\mathbb{R}\times\mathbb{R}^3$ with left-invariant metric determined by the Lie algebra*

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_4, e_3] = -e_3,$$

   where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

Proof. Recall that a Riemannian metric is $\mathcal{F}_t$-critical if and only if

$$Hess_\tau - \Delta\rho + 2t\left(Hess_\tau - \frac{\Delta\tau}{4}g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4}g\right) - 2t\tau\left(\rho - \frac{\tau}{4}g\right) = 0. \tag{4.11}$$

Define the $(0, 2)$-tensor field F as the tensor field given by the left-hand side of the equation. Thus, a metric is $\mathcal{F}_t$-critical if and only if $\mathsf{F} = 0$.

Recall that the non-symmetric cases from Theorem 4.2 were homothetic to the following Lie algebras

(1) The Lie group $\mathbb{R}\ltimes\mathbb{R}^3$ with left-invariant metric determined by the Lie algebra

$$[e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_4, e_3] = -e_3.$$

(2.a) The Lie group $SU(2) \times \mathbb{R}$ with left-invariant metric determined by the Lie algebra

$$[e_1, e_2] = (4 \pm 2\sqrt{2})e_3, \quad [e_2, e_3] = (3 \pm 2\sqrt{2})e_1, \quad [e_3, e_1] = e_2,$$
$$[e_4, e_1] = -e_2, \qquad [e_4, e_2] = (3 \pm 2\sqrt{2})e_1.$$

(2.b) The Lie group $\mathbb{R} \ltimes H^3$ with left-invariant metric determined by the Lie algebra

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \frac{1}{2}e_1, \quad [e_2, e_4] = -\frac{1}{2}e_2.$$

(2.c) The Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ with left-invariant metric determined by the Lie algebra

$$[e_4, e_1] = e_1 - \frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_2 + \frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_3,$$
$$[e_4, e_2] = \frac{\alpha(1+\alpha+\alpha^2)}{(\alpha+1)^2(\alpha-1)}e_1 + \alpha e_2 + \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_3,$$
$$[e_4, e_3] = -\frac{\alpha(1+\alpha+\alpha^2)}{2\alpha+1}e_1 - \frac{(1+\alpha+\alpha^2)}{\alpha(\alpha+2)}e_2 - \tfrac{\alpha}{\alpha+1}e_3.$$

and $\alpha \in (-1, 1)$, $\alpha \neq -\frac{1}{2}$, $\alpha \neq 0$.

Now, we consider each algebra separately.

The non-zero components of F for (1) are $\mathsf{F}_{33} = \mathsf{F}_{44} = -4(3+4t)$, so it is critical for $t = -\frac{3}{4}$.

The algebra from $(2.a)$ cannot be $\mathcal{F}_t$-critical since the component $\mathsf{F}_{11} = 16(89 \pm 63\sqrt{2})$, which never vanishes.

The metric from $(2.b)$ is never critical since $\mathsf{F}_{44} = -\frac{t}{2}$ and $\mathsf{F}_{11} = -1 - \frac{t}{2}$, so this cannot be zero simultaneously.

Finally, the family of metrics from $(2.c)$ is never critical. One has that

$$\mathsf{F}_{44} = -\frac{2(2+t)(1+\alpha+\alpha^2)^4}{(1+\alpha)^4},$$

and then, $t = -2$. However, with this setting, we have that

$$\mathsf{F}_{11} = -\frac{4\left(\alpha^2+\alpha+1\right)^3\left(\alpha^3-3\alpha-1\right)\left(\alpha^2(\alpha+3)-1\right)}{(\alpha-1)(\alpha+1)^5(2\alpha+1)}.$$

The first bracket only has complex solutions. If the second vanishes, then F_{14} is zero if and only if $205+\alpha(724+385\alpha) = 0$ and F_{24} is zero if and only if $887+2\alpha(1567+835\alpha) = 0$, which cannot vanish simultaneously. The same happens if the third bracket of F_{11} is vanishing. In that case F_{14} is zero if and only if $-134+\alpha(46+385\alpha) = 0$ and F_{24} if and only if $-6001+\alpha(2084+17279\alpha) = 0$. Thus, F cannot be vanishing and the metric cannot be $\mathcal{F}_t$-critical.

Regarding $\mathcal{S}$-critical metrics, the only Lie algebra given in Theorem 4.2 with vanishing scalar curvature is $(2.a)$ and the results follows. $\square$

Part II

Two-loop renormalization group flow

The study of flows has been a research topic that has grown a lot of attention in differential geometry over the years as it has many different applications in other fields. The most famous one is the Ricci flow. Let (M, g) be a n-dimensional Riemannian manifold. The Ricci flow is the family of metrics g_t, $t \geq 0$, $g_0 = g$, evolving respect to the equation

$$\frac{\partial}{\partial t} g_t = -2\rho_{g_t},$$

where ρ_{g_t} is the Ricci tensor according to the metric $g(t)$. Ricci flow was introduced by Hamilton in [42]. Hamilton proved that any closed three-dimensional manifold with positive Ricci curvatures is diffeomorphic to a quotient of a three-sphere under a finite group of isometries. For this, Hamilton considered the evolution of the metric under the Ricci flow. Over the years, many results were obtained using the same technique, such that the Thurston's geometrization conjecture, which provide a classification of closed three-dimensional manifolds. To see more topics and examples about Ricci flow, see [19, 20].

On the other hand, another well studied topic in Riemannian geometry is the concept of Ricci soliton. A Riemannian manifold (M, g) is called a Ricci soliton if there exists an smooth vector field X such that the equation

$$\mathcal{L}_X g + \rho = \lambda g,$$

is satisfied for some real constant λ. To see more about Ricci solitons, we address to [12].

Clearly, Ricci solitons represents a generalization of Einstein manifolds. Furthermore, a Ricci soliton is a self- similar solution for the Ricci flow, i.e., a solution of the form $g_t = \sigma(t)\psi_t^* g$, where ψ_t, with $\psi_0 = \mathrm{Id}$, is a family of diffeomorphisms. Because of this, both topics are strongly related.

From this point of view, one may think of other ways to generalize the Einstein condition. Identity (1.7) can be seen as (see [5])

$$\check{R} - \frac{\|R\|^2}{4} g = \frac{1}{3}\tau\rho_0 + 2W[\rho_0],$$

where $\rho_0 = \rho - \frac{\tau}{n} g$ is the traceless Ricci tensor and $W[\rho_0]$ is defined as $R[\rho]$. This last tensor is interesting by itself. If the metric is Einstein, then $W[\rho_0] = 0$ automatically, but not the converse. For instance, a $\check{R}$-Einstein with zero scalar curvature has vanishing $W[\rho_0]$ tensor. Thus, this condition generalizes the Einstein one. Moreover, if the metric is locally conformally flat, then $W[\rho_0]$ is vanishing as well, so it also generalize this property.

The main aim of this second part of the memoir is studying generalizations of Einstein metrics in four-dimensional homogeneous manifolds using the same techniques as in chapter four. For that purpose, in chapter five we study the condition $W[\rho] = 0$, which is the same as $W[\rho_0] = 0$. We point out some interesting geometric properties that this examples has.

In chapter six we introduce the two-loop renormalization flow, which is a second-order variation of the Ricci flow [33–35], given by

$$\frac{\partial}{\partial t} g_t = -2\rho_{g_t} - \frac{\alpha}{2}\check{R}_{g_t},$$

where $\alpha \in \mathbb{R}$. This topic has awaken much interest in Physics, applying this flow in black holes metrics [50, 51]. In this chapter, we study solitons and fixed points to this flow, given a complete classification up to homothety.

Notice that these two conditions are related. If $W[\rho] = 0$, then by the given identity, the metric is directly a self-similar solution for the two-loop flow for a specific constant α, so $W[\rho] = 0$ has physical meaning by itself.

Chapter 5
Generalized Einstein condition

We introduce Generalized Einstein manifolds and then we classify homogeneous generalized Einstein manifolds in dimension four. The results shown in this chapter can be seen in [31].

5.1 Introduction

All along the memoir, we have been studying conditions for the weakly-Einstein tensors to be a multiple of the metric one by one, so it is a natural question to see what happens when they interact with each other. Recall identity (1.7)

$$\left(\check{R} - \frac{\|R\|^2}{4} g\right) + \tau\left(\rho - \frac{\tau}{4} g\right) - 2\left(\check{\rho} - \frac{\|\rho\|^2}{4} g\right) - 2\left(R[\rho] - \frac{\|\rho\|^2}{4} g\right) = 0.$$

Moreover, Besse showed that (see [5])

$$\check{R} - \frac{\|R\|^2}{4} g = \frac{1}{3}\tau\rho_0 + 2W[\rho_0], \tag{5.1}$$

where $W[\rho_0]_{ij} = W_{aijb}\rho_0^{ab}$ and $\rho_0 = \rho - \frac{\tau}{n} g$ is the traceless Ricci tensor. Using both identities one can see that

$$W[\rho_0] = \frac{1}{3}\left(2\check{R}_0 - \check{\rho}_0 - R[\rho]_0\right),$$

where $\check{R}_0 = \check{R} - \frac{\|R\|^2}{n} g$, $\check{\rho}_0 = \rho - \frac{\|\rho\|^2}{n} g$ and $R[\rho]_0 = R[\rho] - \frac{\|\rho\|^2}{n} g$. Thus, we are interested in studying when $W[\rho_0]$ is a multiple of the metric. Since this tensor is traceless, the condition we are looking for is whenever it is vanishing. Moreover, this is the same as checking when $W[\rho]$, constructed analogously, is vanishing. We call this condition generalized Einstein. Regarding identity (5.1), it is clear that if the metric is Einstein, then $W[\rho_0]$ is vanishing, so these conditions clearly generalize the Einstein one. It is also clear that if the metric is locally conformally flat, then this tensor is also zero, so this also generalized locally conformally flat metrics. Consequently, we are interested in metrics satisfying that $W[\rho] = 0$ which are not Einstein nor locally conformally flat.

Remark 5.1. In dimension four, the condition $\check{R}$-Einstein is equivalent to $W[\rho] = 0$ if and only if the scalar curvature is vanishing.

The main purpose of this chapter is proving the following theorem.

Theorem 5.2. *Let (M, g) be a non-symmetric four-dimensional simply connected homogeneous manifold. Then the tensor field $W[\rho]$ vanishes if and only if (M, g) is homothetic to a semi-direct product $\mathbb{R} \ltimes H^3$ of the Heisenberg group with left-invariant metric determined by the Lie algebra*

$$[e_1, e_2] = e_3,\ [e_4, e_1] = \mu e_1,\ [e_4, e_2] = -\frac{1}{2\mu}e_2,\ [e_4, e_3] = (\mu - \frac{1}{2\mu})e_3, \quad 0 < \mu \leq \frac{1}{\sqrt{2}},$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

5.2 Generalized Einstein four-dimensional homogeneous manifolds

Using again the result by Bérard-Bergery [3], we have that a four-dimensional homogeneous manifold is either symmetric or a Lie group. We study the symmetric ones first.

Lemma 5.3. *Let (M, g) be a four-dimensional symmetric space. Then $W[\rho] = 0$ if and only if (M, g) is Einstein or locally conformally flat.*

Proof. Let (M, g) be a four-dimensional symmetric space, then its Ricci operator can have either one or two eigenvalues. If it has one, (M, g) is Einstein and $W[\rho] = 0$ trivially. Assume now that it has two Ricci eigenvalues. Then (M, g) splits isometrically due to the parallelizability of its eigenspaces. If one of the eigenvalues has multiplicity one, then (M, g) is isometric to $\mathbb{R} \times N(c)$, which is locally conformally flat. If both has multiplicity two, then the manifold splits as a product of two surfaces $N_1(c_1) \times N_2(c_2)$. Now, the condition that needs to be fulfilled in order to get $W[\rho] = 0$ is $c_1^2 = c_2^2$, and consequently, the metric is Einstein if we take the positive root and locally conformally flat if we take the negative one. □

The nonsymmetric case give us the example given en Theorem 5.2. The proof of this is given in a case by case analysis in each homogeneous Lie group with a left-invariant metric. The condition $W[\rho] = 0$ reduces to solving a polynomial system on the structure constant of each group, where we use again Gröbner basis to solve it.

5.2.1 The direct products $SL(2, \mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

Lemma 5.4. *Let G be a product $SL(2, \mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$. Then G does not admit any non-symmetric left-invariant metric with $W[\rho] = 0$.*

Proof. Take the algebra given in (4.1). A long but straightforward calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{aligned}
&12R^4 W[\rho]_{11} = \mathfrak{W}_{11}, \quad 24R^4 W[\rho]_{12} = \mathfrak{W}_{12}, \quad 24R^4 W[\rho]_{13} = \mathfrak{W}_{13},\\
&24R^3 W[\rho]_{14} = \mathfrak{W}_{14}, \quad 12R^4 W[\rho]_{22} = \mathfrak{W}_{22}, \quad 24R^4 W[\rho]_{23} = \mathfrak{W}_{23},\\
&24R^3 W[\rho]_{24} = \mathfrak{W}_{24}, \quad 12R^4 W[\rho]_{33} = \mathfrak{W}_{33}, \quad 24R^3 W[\rho]_{34} = \mathfrak{W}_{34},\\
&12R^4 W[\rho]_{44} = \mathfrak{W}_{44},
\end{aligned}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
\mathfrak{W}_{11} = {} & -4(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3)k_1^4-(\lambda_1-\lambda_3)^3(2\lambda_1+3\lambda_3)k_2^4\\
& -(\lambda_1-\lambda_2)^3(2\lambda_1+3\lambda_2)k_3^4-(\lambda_1-\lambda_3)(3\lambda_1+\lambda_3)(2\lambda_2^2-\lambda_3^2-\lambda_2\lambda_3)k_1^2k_2^2\\
& +(\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)(\lambda_2-\lambda_3)(\lambda_2+2\lambda_3)k_1^2k_3^2-(4\lambda_1^4-3\lambda_1^3(\lambda_2+\lambda_3)\\
& +2\lambda_1^2(\lambda_2^2+\lambda_3^2-5\lambda_2\lambda_3)-6\lambda_2^2\lambda_3^2+7\lambda_1\lambda_2(\lambda_2+\lambda_3)\lambda_3)k_2^2k_3^2\\
& +(\lambda_2-\lambda_3)^2(3\lambda_1^2-(\lambda_2-\lambda_3)^2-2\lambda_1(\lambda_2+\lambda_3))R^2k_1^2\\
& -(\lambda_1-\lambda_3)(4\lambda_1^3+6\lambda_3^3-\lambda_1^2(3\lambda_2+2\lambda_3)+\lambda_1(2\lambda_2^2-8\lambda_3^2+7\lambda_2\lambda_3)\\
& -3\lambda_2^2\lambda_3)R^2k_2^2-(\lambda_1-\lambda_2)((2\lambda_1-3\lambda_2)\lambda_3^2+\lambda_1(7\lambda_2-3\lambda_1)\lambda_3\\
& +2(\lambda_1-\lambda_2)^2(2\lambda_1+3\lambda_2))R^2k_3^2-\{2\lambda_1^4-3(\lambda_2^2-\lambda_3^2)^2-3\lambda_1^3(\lambda_2+\lambda_3)\\
& -\lambda_1^2(\lambda_2-3\lambda_3)(3\lambda_2-\lambda_3)+7\lambda_1(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)\}R^4,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{12} = {} & 2(\lambda_2-\lambda_3)(7\lambda_3^3-2\lambda_1\lambda_2^2-(3\lambda_1+\lambda_2)\lambda_3^2-\lambda_1\lambda_2\lambda_3)k_1^3k_2\\
& -2(\lambda_1-\lambda_3)(2\lambda_1^2\lambda_2+(3\lambda_2-7\lambda_3)\lambda_3^2+\lambda_1(\lambda_2+\lambda_3)\lambda_3)k_1k_2^3\\
& -\{(5\lambda_1^2+5\lambda_2^2-16\lambda_1\lambda_2)\lambda_3^2\\
& +3(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_2^2)\lambda_3+\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2-8\lambda_1\lambda_2)\}k_1k_2k_3^2\\
& -\{14\lambda_3^4-16(\lambda_1+\lambda_2)\lambda_3^3+2(\lambda_1^2+\lambda_2^2+9\lambda_1\lambda_2)\lambda_3^2\\
& +\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2-8\lambda_1\lambda_2)+\lambda_1\lambda_2(\lambda_1+\lambda_2)\lambda_3\}R^2k_1k_2,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{13} = {} & -2(\lambda_2-\lambda_3)(\lambda_2^2(7\lambda_2-\lambda_3)-\lambda_1(3\lambda_2^2+2\lambda_3^2+\lambda_2\lambda_3))k_1^3k_3\\
& -\{\lambda_1^3(3\lambda_2+\lambda_3)+\lambda_1^2(\lambda_2-\lambda_3)(5\lambda_2+8\lambda_3)\\
& -\lambda_1(16\lambda_2^2-\lambda_3^2-3\lambda_2\lambda_3)\lambda_3+\lambda_2(5\lambda_2+3\lambda_3)\lambda_3^2\}k_1k_2^2k_3\\
& -2(\lambda_1-\lambda_2)(2\lambda_1^2\lambda_3-\lambda_2^2(7\lambda_2-3\lambda_3)+\lambda_1\lambda_2(\lambda_2+\lambda_3))k_1k_3^3\\
& -\{\lambda_1^3\lambda_3-\lambda_1(16\lambda_2^3-\lambda_3^3-18\lambda_2^2\lambda_3-\lambda_2\lambda_3^2)\\
& +\lambda_1^2(2\lambda_2^2-8\lambda_3^2+\lambda_2\lambda_3)+2\lambda_2^2(7\lambda_2^2+\lambda_3^2-8\lambda_2\lambda_3)\}R^2k_1k_3,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{14} = {} & -2(\lambda_2-\lambda_3)^2(7\lambda_2^2+7\lambda_3^2-3\lambda_1(\lambda_2+\lambda_3)+10\lambda_2\lambda_3)k_1^3\\
& +(\lambda_1-\lambda_3)(14\lambda_3^3-2(\lambda_1-\lambda_2)\lambda_3^2-\lambda_1(3\lambda_1+5\lambda_2)\lambda_2+2(2\lambda_1-5\lambda_2)\lambda_2\lambda_3)k_1k_2^2\\
& -(\lambda_1-\lambda_2)((3\lambda_1^2-2\lambda_2^2-4\lambda_1\lambda_2)\lambda_3+2(\lambda_1-7\lambda_2)\lambda_2^2+5(\lambda_1+2\lambda_2)\lambda_3^2)k_1k_3^2\\
& -2((\lambda_2-\lambda_3)^2)(\lambda_1^2+7\lambda_2^2+7\lambda_3^2-8\lambda_1(\lambda_2+\lambda_3)+10\lambda_2\lambda_3)R^2k_1,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{22} = {} & -(\lambda_2-\lambda_3)^3(2\lambda_2+3\lambda_3)k_1^4-4(\lambda_1-\lambda_3)^2(\lambda_1^2+\lambda_3^2+\lambda_1\lambda_3)k_2^4\\
& +(\lambda_1-\lambda_2)^3(3\lambda_1+2\lambda_2)k_3^4-(\lambda_1-\lambda_3)(2\lambda_1+\lambda_3)(\lambda_2-\lambda_3)(3\lambda_2+\lambda_3)k_1^2k_2^2\\
& -(\lambda_1^2(2\lambda_2^2-6\lambda_3^2+7\lambda_2\lambda_3)+\lambda_2^2(4\lambda_2^2+2\lambda_3^2-3\lambda_2\lambda_3)\\
& -\lambda_1\lambda_2(3\lambda_2^2-7\lambda_3^2+10\lambda_2\lambda_3))k_1^2k_3^2\\
& -(\lambda_1-\lambda_2)(\lambda_1+3\lambda_2)(\lambda_1-\lambda_3)(\lambda_1+2\lambda_3)k_2^2k_3^2\\
& -(\lambda_2-\lambda_3)(\lambda_1^2(2\lambda_2-3\lambda_3)+2(\lambda_2-\lambda_3)^2(2\lambda_2+3\lambda_3)\\
& -\lambda_1\lambda_2(3\lambda_2-7\lambda_3))R^2k_1^2-(\lambda_1-\lambda_3)^2(\lambda_1^2-3\lambda_2^2+\lambda_3^2+2\lambda_1\lambda_2-2\lambda_1\lambda_3\\
& +2\lambda_2\lambda_3)R^2k_2^2+(\lambda_1-\lambda_2)\{2(\lambda_1-\lambda_2)^2(3\lambda_1+2\lambda_2)-(3\lambda_1-2\lambda_2)\lambda_3^2\\
& +(7\lambda_1-3\lambda_2)\lambda_2\lambda_3\}R^2k_3^2+(\lambda_1-\lambda_2+\lambda_3)\{3\lambda_1^3-\lambda_1^2(4\lambda_2+3\lambda_3)\\
& +(\lambda_2-\lambda_3)^2(2\lambda_2+3\lambda_3)-\lambda_1(\lambda_2^2+3\lambda_3^2-8\lambda_2\lambda_3)\}R^4,
\end{aligned}$$

$$\mathfrak{W}_{23} = -2(\lambda_1-\lambda_3)(\lambda_1^2(7\lambda_1-3\lambda_2)-2\lambda_2\lambda_3^2-\lambda_1(\lambda_1+\lambda_2)\lambda_3)k_2^3k_3$$

$$
\begin{aligned}
&- 2(\lambda_1 - \lambda_2)(7\lambda_1^3 - \lambda_1^2(\lambda_2 + 3\lambda_3) - 2\lambda_2^2\lambda_3 - \lambda_1\lambda_2\lambda_3)k_2k_3^3 \\
&- (\lambda_1^2(5\lambda_2^2 + 5\lambda_3^2 - 16\lambda_2\lambda_3) + 3\lambda_1(\lambda_2 + \lambda_3)(\lambda_2^2 + \lambda_3^2) \\
&+ \lambda_2(\lambda_2^2 + \lambda_3^2 - 8\lambda_2\lambda_3)\lambda_3)k_1^2k_2k_3 - (14\lambda_1^4 - 16\lambda_1^3(\lambda_2 + \lambda_3) \\
&+ 2\lambda_1^2(\lambda_2^2 + \lambda_3^2 + 9\lambda_2\lambda_3) + \lambda_2(\lambda_2^2 + \lambda_3^2 - 8\lambda_2\lambda_3)\lambda_3 + \lambda_1\lambda_2(\lambda_2 + \lambda_3)\lambda_3)R^2k_2k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{24} = &-2(\lambda_1 - \lambda_3)^2(7\lambda_1^2 + 7\lambda_3^2 - 3\lambda_1\lambda_2 + 10\lambda_1\lambda_3 - 3\lambda_2\lambda_3)k_2^3 \\
&- (\lambda_2 - \lambda_3)(5\lambda_1^2(\lambda_2 + 2\lambda_3) + \lambda_1(3\lambda_2^2 - 2\lambda_3^2 - 4\lambda_2\lambda_3) + 2(\lambda_2 - 7\lambda_3)\lambda_3^2)k_1^2k_2 \\
&- (\lambda_1 - \lambda_2)(14\lambda_1^3 - 2\lambda_1^2(\lambda_2 - \lambda_3) + 2\lambda_1(2\lambda_2 - 5\lambda_3)\lambda_3 - \lambda_2(3\lambda_2 + 5\lambda_3)\lambda_3)k_2k_3^2 \\
&- 2(\lambda_1 - \lambda_3)^2(7\lambda_1^2 + \lambda_2^2 + 7\lambda_3^2 - 8\lambda_1\lambda_2 + 10\lambda_1\lambda_3 - 8\lambda_2\lambda_3)R^2k_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{33} = &(\lambda_2 - \lambda_3)^3(3\lambda_2 + 2\lambda_3)k_1^4 + (\lambda_1 - \lambda_3)^3(3\lambda_1 + 2\lambda_3)k_2^4 \\
&- 4(\lambda_1 - \lambda_2)^2(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2)k_3^4 + (-4\lambda_3^4 + 3(\lambda_1 + \lambda_2)\lambda_3^3 + 6\lambda_1^2\lambda_2^2 \\
&- 2(\lambda_1^2 + \lambda_2^2 - 5\lambda_1\lambda_2)\lambda_3^2 - 7\lambda_1(\lambda_1 + \lambda_2)\lambda_2\lambda_3)k_1^2k_2^2 \\
&+ (\lambda_1 - \lambda_2)(2\lambda_1 + \lambda_2)(\lambda_2 - \lambda_3)(\lambda_2 + 3\lambda_3)k_1^2k_3^2 \\
&- (\lambda_1 - \lambda_2)(\lambda_1 + 2\lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 + 3\lambda_3)k_2^2k_3^2 \\
&+ (\lambda_2 - \lambda_3)(-\lambda_1^2(3\lambda_2 - 2\lambda_3) + 2(3\lambda_2 + 2\lambda_3)(\lambda_2 - \lambda_3)^2 \\
&+ \lambda_1(7\lambda_2 - 3\lambda_3)\lambda_3)R^2k_1^2 + (\lambda_1 - \lambda_3)(6\lambda_1^3 - 8\lambda_1^2\lambda_3 + (2\lambda_2^2 + 4\lambda_3^2 - 3\lambda_2\lambda_3)\lambda_3 \\
&- \lambda_1(\lambda_2 - 2\lambda_3)(3\lambda_2 - \lambda_3))R^2k_2^2 - (\lambda_1 - \lambda_2)^2((\lambda_1 - \lambda_2)^2 - 3\lambda_3^2 \\
&+ 2(\lambda_1 + \lambda_2)\lambda_3)R^2k_3^2 - (2\lambda_3^4 - 3(\lambda_1^2 - \lambda_2^2)^2 - 3(\lambda_1 + \lambda_2)\lambda_3^3 \\
&+ 7(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)\lambda_3 - (\lambda_1 - 3\lambda_2)(3\lambda_1 - \lambda_2)\lambda_3^2)R^4,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{34} = &-2(\lambda_1 - \lambda_2)^2(7\lambda_1^2 + 7\lambda_2^2 + 10\lambda_1\lambda_2 - 3(\lambda_1 + \lambda_2)\lambda_3)k_3^3 \\
&+ (\lambda_2 - \lambda_3)(5\lambda_1^2(2\lambda_2 + \lambda_3) - 2\lambda_2^2(7\lambda_2 - \lambda_3) - \lambda_1(2\lambda_2^2 - 3\lambda_3^2 + 4\lambda_2\lambda_3))k_1^2k_3 \\
&- (\lambda_1 - \lambda_3)(14\lambda_1^3 + 2\lambda_1^2(\lambda_2 - \lambda_3) - 2\lambda_1\lambda_2(5\lambda_2 - 2\lambda_3) - \lambda_2(5\lambda_2 + 3\lambda_3)\lambda_3)k_2^2k_3 \\
&- 2(\lambda_1 - \lambda_2)^2(7\lambda_1^2 + 7\lambda_2^2 + \lambda_3^2 + 10\lambda_1\lambda_2 - 8(\lambda_1 + \lambda_2)\lambda_3)R^2k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{44} = &3(\lambda_2^2 - \lambda_3^2)^2k_1^4 + 3(\lambda_1^2 - \lambda_3^2)^2k_2^4 + 3(\lambda_1^2 - \lambda_2^2)^2k_3^4 \\
&+ (6\lambda_3^4 + 6\lambda_1^2\lambda_2^2 - 3(\lambda_1 + \lambda_2)^2\lambda_3^2)k_1^2k_2^2 \\
&+ 3(2\lambda_2^4 - \lambda_1^2(\lambda_2^2 - 2\lambda_3^2) - \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2^2\lambda_3)k_1^2k_3^2 \\
&+ (6\lambda_1^4 - 3\lambda_1^2(\lambda_2 + \lambda_3)^2 + 6\lambda_2^2\lambda_3^2)k_2^2k_3^2 \\
&+ (\lambda_2 - \lambda_3)^2(2\lambda_1^2 - (\lambda_2 - \lambda_3)^2 - \lambda_1(\lambda_2 + \lambda_3))R^2k_1^2 \\
&- (\lambda_1 - \lambda_3)^2(\lambda_1^2 - 2\lambda_2^2 + \lambda_3^2 + \lambda_1(\lambda_2 - 2\lambda_3) + \lambda_2\lambda_3)R^2k_2^2 \\
&- (\lambda_1 - \lambda_2)^2((\lambda_1 - \lambda_2)^2 - 2\lambda_3^2 + (\lambda_1 + \lambda_2)\lambda_3)R^2k_3^2 \\
&- 4(\lambda_1^4 - \lambda_1^3(\lambda_2 + \lambda_3) + \lambda_1^2\lambda_2\lambda_3 - \lambda_1(\lambda_2 - \lambda_3)^2(\lambda_2 + \lambda_3) \\
&+ (\lambda_2 - \lambda_3)^2(\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3))R^4\,.
\end{aligned}
$$

Since $\lambda_1\lambda_2\lambda_3 \neq 0$, assume $\lambda_1 = 1$ just working with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_1}e_i$. Now, $W[\rho]$ vanishes if and only if the structure constants in the Lie algebra (4.1) satisfy the system of polynomial equations $\{\mathfrak{W}_{ij} = 0\}$. Let $\mathcal{I}_1 \subset \mathbb{R}[k_1, k_2, k_3, R, \lambda_2, \lambda_3]$ be the ideal generated by the polynomials $\mathfrak{W}_{ij}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order and a detailed analysis of that basis shows that the polynomials

$$
\begin{aligned}
\mathbf{g}_{11} &= R^6\lambda_2^3\lambda_3^4(\lambda_3 - 1)^3(\lambda_3 + 1) \qquad \text{and} \\
\mathbf{g}_{12} &= -R^6\lambda_2^3\lambda_3^3(\lambda_3 - 1)^2(3\lambda_3^2 + \lambda_3 - 2\lambda_2 - 2)
\end{aligned}
$$

belong to $\mathcal{G}_1$. Since the zero sets of $\{\mathfrak{W}_{ij} = 0\}$ and $\mathcal{I}_1 = \langle \mathfrak{W}_{ij} \rangle = \langle \mathcal{G}_1 \rangle$ coincide, then necessarily $\lambda_3 = 1$.

Next, we compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\mathcal{G}_1 \cup \{\lambda_3 - 1\} \subset \mathbb{R}[k_1, k_2, k_3, R, \lambda_2, \lambda_3]$ with respect to the lexicographical order, obtaining that the polynomial $\mathrm{g}_{21} = R^6 \lambda_2^2(\lambda_2 - 1)$ belongs to $\mathcal{G}_2$. Hence, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and a straightforward calculation shows that the manifold is locally symmetric, which finishes the proof. □

5.2.2 The semi-direct products $\mathbb{R} \ltimes E(1,1)$ and $\mathbb{R} \ltimes E(2)$

Lemma 5.5. *Let G be a semi-direct product $\mathbb{R} \ltimes E(1,1)$ or $\mathbb{R} \ltimes E(2)$. Then G does not admit any non-symmetric left-invariant metric with $W[\rho] = 0$.*

Proof. Take the algebra given in (4.2) and the same simplifications as before. A long but standard calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{array}{lll} 12R^4 W[\rho]_{11} = \mathfrak{W}_{11}, & 24R^4 W[\rho]_{12} = \mathfrak{W}_{12}, & 24R^4 W[\rho]_{13} = \mathfrak{W}_{13}, \\ 24R^3 W[\rho]_{14} = \mathfrak{W}_{14}, & 12R^4 W[\rho]_{22} = \mathfrak{W}_{22}, & 24R^4 W[\rho]_{23} = \mathfrak{W}_{23}, \\ 24R^3 W[\rho]_{24} = \mathfrak{W}_{24}, & 12R^4 W[\rho]_{33} = \mathfrak{W}_{33}, & 12R^3 W[\rho]_{34} = \mathfrak{W}_{34}, \\ 12R^4 W[\rho]_{44} = \mathfrak{W}_{44}, & & \end{array}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned} \mathfrak{W}_{11} = & -(A^2+R^2)^2(2\lambda_1^4 - 3\lambda_2^4 - 3\lambda_1^3\lambda_2 + 7\lambda_1\lambda_2^3 - 3\lambda_1^2\lambda_2^2) \\ & - (A^2(8b^2+4C^2-3D^2) - (4b^2-4C^2+3D^2)R^2)\lambda_1^2 \\ & + (A^2(4b^2-2C^2-D^2) - (8b^2+2C^2+D^2)R^2)\lambda_2^2 \\ & + (A^2+R^2)(4b^2+3C^2-2D^2)\lambda_1\lambda_2 - AbCD(21\lambda_1 - 9\lambda_2) \\ & + b^2(4C^2-5D^2) - 2(C^2+D^2)(C^2+2D^2), \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{12} = & \, 10Ab(A^2+R^2)(\lambda_1^3 - \lambda_2^3 - 3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2) \\ & + CD(A^2+R^2)(\lambda_1^2 + \lambda_2^2 - 8\lambda_1\lambda_2) \\ & + Ab(24b^2+13C^2+D^2)\lambda_1 - Ab(24b^2+C^2+13D^2)\lambda_2 \\ & + 2CD(9b^2+2(C^2+D^2)), \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{13} = & -2AD(A^2+R^2)(7\lambda_2^3 + \lambda_1^2\lambda_2 - 8\lambda_1\lambda_2^2) \\ & + 3bC(2A^2+3R^2)\lambda_1^2 + 9bC(2A^2-R^2)\lambda_2^2 - 24A^2bC\lambda_1\lambda_2 \\ & + 3AD(8b^2-C^2+2D^2)\lambda_1 - AD(24b^2+5C^2+14D^2)\lambda_2 \\ & + 9bC(C^2+D^2), \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{14} = & -2D(A^2+R^2)(7\lambda_2^3 + \lambda_1^2\lambda_2 - 8\lambda_1\lambda_2^2) - 3AbC(\lambda_1^2 - 9\lambda_2^2 + 8\lambda_1\lambda_2) \\ & + 3(2D^3+2b^2D-C^2D)\lambda_1 + D(6b^2-5C^2-14D^2)\lambda_2, \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{22} = & \, (A^2+R^2)^2(3\lambda_1^4 - 2\lambda_2^4 - 7\lambda_1^3\lambda_2 + 3\lambda_1\lambda_2^3 + 3\lambda_1^2\lambda_2^2) \\ & + (A^2(4b^2-C^2-2D^2) - (8b^2+C^2+2D^2)R^2)\lambda_1^2 \\ & - (A^2(8b^2-3C^2+4D^2) - (4b^2+3C^2-4D^2)R^2)\lambda_2^2 \\ & + (A^2+R^2)(4b^2-2C^2+3D^2)\lambda_1\lambda_2 - 3AbCD(3\lambda_1 - 7\lambda_2) \end{aligned}$$

$$
\begin{aligned}
&\quad - b^2(5C^2 - 4D^2) - 2(C^2 + D^2)(2C^2 + D^2),\\
\mathfrak{W}_{23} &= 2AC(A^2 + R^2)(7\lambda_1^3 - 8\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2)\\
&\quad + 9bD(2A^2 - R^2)\lambda_1^2 + 3bD(2A^2 + 3R^2)\lambda_2^2 - 24A^2bD\lambda_1\lambda_2\\
&\quad + AC(24b^2 + 14C^2 + 5D^2)\lambda_1 - 3AC(8b^2 + 2C^2 - D^2)\lambda_2\\
&\quad + 9bD(C^2 + D^2),\\
\mathfrak{W}_{24} &= 2C(A^2 + R^2)(7\lambda_1^3 - 8\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2) + 3AbD(9\lambda_1^2 - \lambda_2^2 - 8\lambda_1\lambda_2)\\
&\quad + (14C^3 - 6b^2C + 5CD^2)\lambda_1 - 3C(2(b^2 + C^2) - D^2)\lambda_2,\\
\mathfrak{W}_{33} &= -(4A^4 + A^2R^2 - 3R^4)(\lambda_1^4 + \lambda_2^4)\\
&\quad + 2(A^2 + R^2)(2A^2\lambda_1^3\lambda_2 + 2A^2\lambda_1\lambda_2^3 - 3R^2\lambda_1^2\lambda_2^2)\\
&\quad - (A^2(12b^2 + C^2 - 2D^2) - 3(2C^2 - D^2)R^2)\lambda_1^2\\
&\quad - (A^2(12b^2 - 2C^2 + D^2) + 3(C^2 - 2D^2)R^2)\lambda_2^2\\
&\quad + A^2(24b^2 - C^2 - D^2)\lambda_1\lambda_2 + 6AbCD(\lambda_1 - \lambda_2)\\
&\quad + 3(C^2 + D^2)^2,\\
\mathfrak{W}_{34} &= -A(A^2 + R^2)(7\lambda_1^4 + 7\lambda_2^4 - 4\lambda_1^3\lambda_2 - 4\lambda_1\lambda_2^3 - 6\lambda_1^2\lambda_2^2)\\
&\quad - A(12b^2 + 7C^2 - 5D^2)\lambda_1^2 - A(12b^2 - 5C^2 + 7D^2)\lambda_2^2\\
&\quad + A(24b^2 - C^2 - D^2)\lambda_1\lambda_2 - 9bCD(\lambda_1 - \lambda_2),\\
\mathfrak{W}_{44} &= (3A^2 - 4R^2)(A^2 + R^2)(\lambda_1^4 + \lambda_2^4)\\
&\quad + 2(A^2 + R^2)(2R^2\lambda_1^3\lambda_2 + 2R^2\lambda_1\lambda_2^3 - 3A^2\lambda_1^2\lambda_2^2)\\
&\quad + (A^2(16b^2 + 6C^2 - 3D^2) + (4b^2 - C^2 + 2D^2)R^2)\lambda_1^2\\
&\quad + (A^2(16b^2 - 3C^2 + 6D^2) + (4b^2 + 2C^2 - D^2)R^2)\lambda_2^2\\
&\quad - (32A^2b^2 + (8b^2 + C^2 + D^2)R^2)\lambda_1\lambda_2 + 24AbCD(\lambda_1 - \lambda_2)\\
&\quad + (b^2 + 3(C^2 + D^2))(C^2 + D^2)\,.
\end{aligned}
$$

Since $\lambda_1\lambda_2 \neq 0$, we work with a homothetic basis $\hat{e}_i = \frac{1}{\lambda_1}e_i$ so that we may assume $\lambda_1 = 1$. The $W[\rho_0]$-tensor field vanishes if and only if the structure constants in Equation ((4.2)) satisfy the system of polynomial equations $\{\mathfrak{W}_{ij} = 0\}$, where $\mathfrak{W}_{ij} \in \mathbb{R}[A, b, C, D, R, \lambda_2]$. We compute a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{I}_1 = \langle \mathfrak{W}_{ij} \rangle$ with respect to the graded lexicographical order and a detailed analysis of that basis shows that the polynomial

$$\mathbf{g}_{11} = D(32b^2 + 5C^2 + 5D^2)(9D^4 + 16b^2D^2 + 128b^2R^2 + 9C^2D^2)$$

belongs to $\mathcal{G}_1$. Thus, necessarily $D = 0$. Now, we compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\mathcal{G}_1 \cup \{D\} \subset \mathbb{R}[A, b, C, D, R, \lambda_2]$ with respect to the graded lexicographical order obtaining that the polynomials

$$\mathbf{g}_{21} = C^2(A^2 + C^2 + R^2)^2 \quad \text{and} \quad \mathbf{g}_{22} = Ab^4(\lambda_2 - 1)$$

belong to $\mathcal{G}_2$. Thus, $C = 0$ and we are led to the cases $\lambda_2 = 1$, $b = 0$, or $A = 0$. If $\lambda_2 = 1$ then a straightforward calculation shows the manifold is locally symmetric. If $b = 0$ then

$$
\begin{aligned}
\mathfrak{W}_{11} &= (A^2 + R^2)^2(\lambda_2 - 1)^3(3\lambda_2 + 2) \qquad \text{and}\\
\mathfrak{W}_{22} &= -(A^2 + R^2)^2(\lambda_2 - 1)^3(2\lambda_2 + 3)\,.
\end{aligned}
$$

Since $\lambda_2 = 1$ was discussed previously, we conclude that $W[\rho]$ does not vanish in this case. Finally, if $A = 0$ then we have $\mathfrak{W}_{33} = 3R^4(\lambda_2^2 - 1)^2$. Since $\lambda_2 = 1$ was considered previously, it follows that $\lambda_2 = -1$. This leads to $\mathfrak{W}_{11} = -8(b^2 - R^2)R^2$, which implies $b = \pm R$ and a standard calculation shows the manifold is Einstein and locally symmetric. This finishes the proof. □

5.2.3 The semi-direct product $\mathbb{R} \ltimes H^3$

Lemma 5.6. *Let G be a semi-direct product $\mathbb{R} \ltimes H^3$. Then G admits a non-Einstein left-invariant metric with $W[\rho] = 0$ if and only if it is homothetic to the left-invariant metric determined by*

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\mu e_1, \quad [e_2, e_4] = \frac{1}{2\mu} e_2, \quad [e_3, e_4] = \left(\frac{1}{2\mu} - \mu\right) e_3,$$

with $\mu \in \left(0, \frac{1}{\sqrt{2}}\right]$ and where $\{e_1, \ldots, e_4\}$ is an orthonormal basis.

Remark 5.7. Let $(G_1, \langle\,\cdot\,,\,\cdot\,\rangle_1)$ and $(G_2, \langle\,\cdot\,,\,\cdot\,\rangle_2)$ be two Lie groups with negative scalar curvature τ_1 and τ_2, respectively. For $i = 1, 2$, let $\langle\,\cdot\,,\,\cdot\,\rangle_i^* = -\tau_i\langle\,\cdot\,,\,\cdot\,\rangle_i$ so that the scalar curvature of the normalized metric $\langle\,\cdot\,,\,\cdot\,\rangle_i^*$ is $\tau_i^* = -1$. Now, one has that $(G_1, \langle\,\cdot\,,\,\cdot\,\rangle_1)$ and $(G_2, \langle\,\cdot\,,\,\cdot\,\rangle_2)$ are homothetic if and only if the normalized metrics $\langle\,\cdot\,,\,\cdot\,\rangle_i^*$ are isometric. In this case one has that $\|\rho_1^*\| = \|\rho_2^*\|$ and $\|R_1^*\| = \|R_2^*\|$, or equivalently, $\tau_1^{-2}\|\rho_1\|^2 = \tau_2^{-2}\|\rho_2\|^2$ and $\tau_1^{-2}\|R_1\|^2 = \tau_2^{-2}\|R_2\|^2$. The failure of any of these relations therefore implies that the left-invariant metrics $\langle\,\cdot\,,\,\cdot\,\rangle_i$ correspond to different homothetical classes.

Now, a standard calculation shows that left-invariant metrics in this Lemma corresponding to different values of the parameter α are never homothetical since $\tau = -\frac{3(4\alpha^4 - 3\alpha^2 + 1)}{2\alpha^2}$ and $\|R\|^2 = \frac{48\alpha^8 - 40\alpha^6 + 35\alpha^4 - 10\alpha^2 + 3}{4\alpha^4}$.

Proof. Take the algebra (4.3) and the same simplifications as in the previous chapters. A straightforward calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{aligned}
&12R^4W[\rho]_{11} = \mathfrak{W}_{11}, \quad 12R^4W[\rho]_{12} = \mathfrak{W}_{12}, \quad 24R^4W[\rho]_{13} = \mathfrak{W}_{13},\\
&12R^3W[\rho]_{14} = \mathfrak{W}_{14}, \quad 12R^4W[\rho]_{22} = \mathfrak{W}_{22}, \quad 24R^4W[\rho]_{23} = \mathfrak{W}_{23},\\
&12R^3W[\rho]_{24} = \mathfrak{W}_{24}, \quad 6R^4W[\rho]_{33} = \mathfrak{W}_{33}, \quad 8R^3W[\rho]_{34} = \mathfrak{W}_{34},\\
&12R^4W[\rho]_{44} = \mathfrak{W}_{44},
\end{aligned}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\begin{aligned}
\mathfrak{W}_{11} = &-16a^3d - 8a^2d^2 - (5F^2 - 4H^2 + 8\gamma^2R^2)a^2 - 3H^2c^2 - 12F^2d^2\\
&- 12FH(ac + cd) - (4(3F^2 + H^2) - 2\gamma^2R^2)ad\\
&- (F^2 + H^2 + \gamma^2R^2)(4F^2 - 3(H^2 + \gamma^2R^2)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{12} = &-20a^2cd + 20acd^2 + 3FH(4a^2 - c^2 + 4d^2) + (2F^2 + 14H^2 - \gamma^2R^2)ac\\
&+ 5FHad - (2(7F^2 + H^2) - \gamma^2R^2)cd + 7FH(F^2 + H^2 + \gamma^2R^2),
\end{aligned}$$

$$\mathfrak{W}_{13} = 24Ha^3 + 12F(a^2c - 2cd^2) + 4H(5a^2d + 3ac^2 - 3c^2d + 3ad^2) - 4Facd$$

$$
\begin{aligned}
&+ H(13F^2 + 10(H^2 + \gamma^2 R^2))a + F(4(F^2 + H^2) + \gamma^2 R^2)c \\
&+ 3H(2F^2 + 3(H^2 + \gamma^2 R^2))d,
\end{aligned}
$$

$$
\mathfrak{W}_{14} = -3\gamma F(a^2 - 4d^2) + \gamma(12Hac + 14Fad - 3Hcd) + 7\gamma F(F^2 + H^2 + \gamma^2 R^2),
$$

$$
\begin{aligned}
\mathfrak{W}_{22} = &-16ad^3 - 8a^2d^2 - 12H^2a^2 - 3F^2c^2 + (4F^2 - 5H^2 - 8\gamma^2 R^2)d^2 \\
&+ 12FH(ac + cd) - (4(F^2 + 3H^2) - 2\gamma^2 R^2)ad \\
&+ (F^2 + H^2 + \gamma^2 R^2)(3F^2 - 4H^2 + 3\gamma^2 R^2),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{W}_{23} = &\ 24Fd^3 + 12H(2a^2c - cd^2) + 4F(3a^2d - 3ac^2 + 3c^2d + 5ad^2) + 4Hacd \\
&+ 3F(3F^2 + 2H^2 + 3\gamma^2 R^2)a - H(4(F^2 + H^2) + \gamma^2 R^2)c \\
&+ F(10F^2 + 13H^2 + 10\gamma^2 R^2)d,
\end{aligned}
$$

$$
\mathfrak{W}_{24} = -3\gamma H(4a^2 - d^2) - \gamma(3Fac + 14Had - 12Fcd) - 7\gamma H(F^2 + H^2 + \gamma^2 R^2),
$$

$$
\begin{aligned}
\mathfrak{W}_{33} = &\ 4a^3d + 4ad^3 - 6a^2c^2 - 6c^2d^2 + 12ac^2d + 2(F^2 - 2H^2 + \gamma^2 R^2)a^2 \\
&- 2(2F^2 - H^2 - \gamma^2 R^2)d^2 - 9FH(ac - cd) + (F^2 + H^2 - 2\gamma^2 R^2)ad \\
&- (F^2 + H^2 + \gamma^2 R^2)^2,
\end{aligned}
$$

$$
\mathfrak{W}_{34} = 4\gamma(a^2c + cd^2) - 8\gamma acd - \gamma FH(a - d) - \gamma(F^2 + H^2)c,
$$

$$
\begin{aligned}
\mathfrak{W}_{44} = &\ 8a^3d + 8ad^3 + 12a^2c^2 + 16a^2d^2 + 12c^2d^2 - 24ac^2d + (F^2 + 16H^2 + 4\gamma^2 R^2)a^2 \\
&+ 3(F^2 + H^2)c^2 + (16F^2 + H^2 + 4\gamma^2 R^2)d^2 + 18FH(ac - cd) + 14(F^2 + H^2)ad \\
&+ (F^2 + H^2 + \gamma^2 R^2)(3(F^2 + H^2) - 4\gamma^2 R^2).
\end{aligned}
$$

Note that since $\gamma \neq 0$, one may work with a homothetic basis $\hat{e}_i = \frac{1}{\gamma} e_i$ so that we may assume $\gamma = 1$. Now, $W[\rho]$ vanishes if and only if the structure constants in the Lie algebra (4.3) satisfy the system of polynomial equations $\{\mathfrak{W}_{ij} = 0\}$. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, F, H, R]$ be the ideal generated by the polynomials $\mathfrak{W}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded lexicographical order and we get that the polynomials

$$
\begin{aligned}
\mathrm{g}_1 &= H^3(F^2 + H^2 + R^2)(d^2 + F^2 + H^2 + R^2), \\
\mathrm{g}_2 &= F^2R^2(F^2 + R^2)(4d^2 + F^2 + R^2) \\
&\quad - H^2R^2(2F^4 + F^2(7H^2 + 6R^2) + 4(H^2 + R^2)(d^2 + H^2 + R^2)) \qquad \text{and} \\
\mathrm{g}_3 &= 4c(a - d)^2 - FH(a - d) - (F^2 + H^2)c,
\end{aligned}
$$

belong to $\mathcal{G}$. From g_1 we get $H = 0$ and hence g_2 leads to $F = 0$. Now, g_3 implies that either $d = a$ or $c = 0$. If $d = a$ then $\mathfrak{W}_{11} = -3(8a^4 + 2a^2R^2 - R^4)$, from where we obtain $a = \pm\frac{R}{2}$ and a standard calculation shows the manifold is Einstein and locally symmetric. Now, if $c = 0$ then $\{\mathfrak{W}_{ij} = 0\}$ reduces to

$$
\begin{aligned}
\mathfrak{W}_{11} &= 3R^4 - 2a(4ad(2a + d) + (4a - d)R^2), \\
\mathfrak{W}_{22} &= -(4d(a + 2d) - 3R^2)(2ad + R^2), \\
\mathfrak{W}_{33} &= (2(a^2 + d^2) - R^2)(2ad + R^2), \\
\mathfrak{W}_{44} &= 4((a + d)^2 - R^2)(2ad + R^2).
\end{aligned}
$$

$\mathfrak{W}_{11}$ implies that a must be non-null. Moreover, since $d = a$ was discussed previously, the expressions of $\mathfrak{W}_{33}$ and $\mathfrak{W}_{44}$ easily leads to $d = -\frac{R^2}{2a}$. Thus, we get a non-Einstein manifold with $W[\rho_0] = 0$ and setting $\mu = \frac{a}{R} \neq 0$ the left-invariant metric is given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\mu e_1, \quad [e_2, e_4] = \frac{1}{2\mu} e_2, \quad [e_3, e_4] = \left(\frac{1}{2\mu} - \mu\right) e_3.$$

Note that the replacement $e_4 \mapsto -e_4$ defines an isometry which interchanges μ and $-\mu$. Hence, one may assume $\mu > 0$ without loss of generality. Moreover, $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, -e_4)$ defines an isometry interchanging μ and $\frac{1}{2\mu}$ which shows that one may restrict the parameter to $\mu \in \left(0, \frac{1}{\sqrt{2}}\right]$, finishing the proof. □

5.2.4 The semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$

Lemma 5.8. *Let G be a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$. Then G does not admit any non-symmetric left-invariant metric with $W[\rho] = 0$.*

Proof. Take the algebra (4.4). A long but straightforward calculation shows that the components $W[\rho]_{ij}$ of the $W[\rho]$-tensor field are determined by

$$\begin{aligned} &3R^4 W[\rho]_{11} = \mathfrak{W}_{11}, \quad 3R^4 W[\rho]_{12} = \mathfrak{W}_{12}, \quad 3R^4 W[\rho]_{13} = \mathfrak{W}_{13},\\ &3R^4 W[\rho]_{22} = \mathfrak{W}_{22}, \quad 3R^4 W[\rho]_{23} = \mathfrak{W}_{23}, \quad 3R^4 W[\rho]_{33} = \mathfrak{W}_{33},\\ &3R^4 W[\rho]_{44} = \mathfrak{W}_{44}, \end{aligned}$$

where the coefficients $\mathfrak{W}_{ij}$ are polynomials on the structure constants given by

$$\mathfrak{W}_{11} = a^4 - (f+p)a^3 + (f^2 + p^2 - fp)a^2 - (f-p)^2(f^2 + 3h^2 + p^2 + fp),$$

$$\begin{aligned} \mathfrak{W}_{12} = &-2a^3 b + (3f + p)a^2 b + (p^2 - 3f^2)ab + 3h(f-p)ac\\ &+ f(f-p)(2f+p)b - 3hp(f-p)c, \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{13} = &-2a^3 c + (f + 3p)a^2 c + 3h(f-p)ab + (f^2 - 3p^2)ac\\ &+ 3fh(p-f)b - p(f-p)(f+2p)c, \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{22} = &-a^4 - 3a^2c^2 + pa^3 + 6pac^2 + f^2a^2 - 3p^2c^2\\ &- (f^3 - p^3 + f^2 p)a + (f-p)(f^3 + p^3 + fp^2), \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{23} = &\,3a^2bc - 3(f+p)abc + h(f-p)a^2 + 3fpbc\\ &+ h(f^2 - p^2)a - h(f-p)(2f^2 + 2p^2 - fp), \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{33} = &-a^4 - 3a^2b^2 + fa^3 + 6fab^2 + p^2a^2 - 3f^2b^2\\ &+ (f^3 - p^3 - fp^2)a - (f-p)(f^3 + p^3 + f^2 p), \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_{44} = &\,a^4 + 3a^2b^2 + 3a^2c^2 - 6fab^2 - 6pac^2 - (2f^2 + 2p^2 - fp)a^2 + 3f^2b^2 + 3p^2c^2\\ &+ fp(f+p)a + (f-p)^2((f+p)^2 + 3h^2)\,. \end{aligned}$$

$W[\rho]$ vanishes if and only if the structure constants satisfy the system of polynomial equations $\{\mathfrak{W}_{ij} = 0\}$, where $\mathfrak{W}_{ij} \in \mathbb{R}[a, b, c, f, h, p]$. We compute a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I} =$

$\langle \mathfrak{W}_{ij} \rangle$ with respect to the lexicographical order and a detailed analysis of that basis shows that the polynomials

$$\mathrm{g}_1 = h^4(f-p)^3(c^2+h^2+p^2) \qquad \text{and}$$
$$\mathrm{g}_2 = -fp(f-p)^3(3h^4 - f^3p - fp^3 - f^2p^2)$$

belong to $\mathcal{G}$. Hence, we are led to the cases $p = f$, $h = p = 0$, or $h = f = 0$.

If $p = f$ then $\mathfrak{W}_{11} = a^2(a-f)^2$ and $\mathfrak{W}_{44} = (a-f)^2(a^2+3(b^2+c^2)+2af)$. Thus, either $f = a$ or $a = b = c = 0$. In the first case the manifold is Einstein and in both cases the manifold is locally symmetric. If $p \neq f$ and $h = p = 0$ then $\mathfrak{W}_{33} = -(a-f)^2(a^2+3b^2+f^2+af)$ implies $f = a$ and, in that case, $\mathfrak{W}_{22} = -3a^2c^2$. Hence, $f = a$ and $c = 0$, which implies that the manifold is locally symmetric. Finally, if $0 \neq p \neq f$ and $h = f = 0$ then $\mathfrak{W}_{22} = -(a-p)^2(a^2+3c^2+p^2+ap)$ implies $p = a$ and, in that case, $\mathfrak{W}_{33} = -3a^2b^2$. Thus, $p = a$ and $b = 0$, from where it follows that the manifold is locally symmetric, finishing the proof. □

5.2.5 Geometric properties

The family of metrics given in Theorem 5.2, which is homothetic to $\mathbb{R} \ltimes H^3$ with Lie algebra

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\mu e_1, \quad [e_2, e_4] = \frac{1}{2\mu} e_2, \quad [e_3, e_4] = \left(\frac{1}{2\mu} - \mu\right) e_3,$$

with $\mu \in \left(0, \frac{1}{\sqrt{2}}\right]$, satisfies some different interesting geometric properties depending on the value of μ. For example, it was shown by Lauret [52] that an algebraic Ricci soliton is a Ricci soliton, and the converse was proved by Jablonski [44] for the homogeneous setting. Recall that a left-invariant metric on a Lie group is a algebraic Ricci soliton if $\mathfrak{D} = Q_\rho - \lambda \operatorname{Id}$ is a derivation. Then, a standard computation shows that the left invariant metric of Theorem 5.2 is a Ricci soliton if and only if $\lambda = -\frac{3}{2}$ and $\mu = \frac{1}{2}$. In this case, the geometric structure is Kähler and it corresponds to the nonsymmetric homogeneous Kähler Ricci soliton [10].

Critical metrics

The generalized Einstein condition appears naturally in the study of critical metrics. Taking the functional $\mathcal{F}_t$, one can check that a metric g is critical for this functional if and only if

$$\Delta\rho - (1+2t)\nabla^2\tau + \frac{1+4t}{2}\Delta\tau g + 2\left(t + \frac{2}{3}\right)\tau\rho_0 - 2\check{\rho}_0 + 2W[\rho] = 0.$$

One the one hand, one has the following.

Theorem 5.9. *Let (M, g) be an homogeneous four dimensional generalized Einstein Riemannian manifold. Then, (M, g) is $\mathcal{F}_t$-critical if and only if $t = -\frac{1}{2}$ and (M, g) is homothetic to the Lie group $\mathbb{R} \ltimes H^3$ with left-invariant metric determined by the Lie algebra*

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\frac{1}{2} e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = \frac{1}{2} e_3,$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis.

Proof. Recall that we defined as F $(0, 2)$-tensor field given by the left-hand side of the equation above. Then, the left-invariant metric given in Theorem 5.2 is critical if and only if $\mathsf{F} = 0$. One can see that the component

$$\mathsf{F}_{11} = \frac{3 + 9t - 2(4 + 21t)\mu^2 + 3(1 + 23t)\mu^4 - 24t\mu^6 - 16(1 + 3t)\mu^8}{8\mu^4},$$

which is vanishing if

$$t = \frac{-16\mu^8 + 3\mu^4 - 8\mu^2 + 3}{48\mu^8 + 24\mu^6 - 69\mu^4 + 42\mu^2 - 9}.$$

Using this value of t, one gets that

$$\mathsf{F}_{44} = \frac{-8\mu^6 + 6\mu^4 + 3\mu^2 - 1}{4\mu^4 + 5\mu^2 - 3},$$

and this only has real solutions if $\mu = \pm 1$ or $\mu = \pm\frac{1}{2}$, and since $\mu \in (0, \frac{1}{\sqrt{2}}]$, $\mu = \dfrac{1}{2}$ necessarily. With this setting, $t = -\frac{1}{2}$ and the metric is critical for $\mathcal{F}_{-\frac{1}{2}}$, so the result follows. □

Remark 5.10. This case corresponds again to the Kähler structure.

Remark 5.11. If we study the functional $\mathcal{S}$, this metric cannot be critical for it as the family given in Theorem 5.2 has always strictly negative scalar curvature $\tau = -\frac{3(4\mu^4 - 3\mu^2 + 1)}{2\mu^2}$ and four-dimensional homogeneous manifolds are $\mathcal{S}$-critical if and only if they are Einstein or have vanishing scalar curvature.

Symplectic structures

Let $\{e^1, \dots, e^4\}$ be the dual basis of 1-forms of the basis in Theorem 5.2 and let

$$E_1^{\pm} = e^1 \wedge e^2 \pm e^3 \wedge e^4, \quad E_2^{\pm} = e^1 \wedge e^3 \pm e^4 \wedge e^2, \quad E_3^{\pm} = e^1 \wedge e^4 \pm e^2 \wedge e^3,$$

be the associated self-dual and anti-self-dual 2-forms. These are defined by the self and anti-self-dual Weyl operators, which have three different eigenvalues in our metric (in fact, they are the opposite of each other, i.e., $W^+ = -W^-$)

Now we want to compute the exterior derivatives $dE_i^{\pm}$. For this, we recall that $de^i = \sum_{i=1<j}^4 a_{ij} e^i \wedge e^j$, with $a_{ij} \in \mathbb{R}$. Moreover, $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega[X, Y]$, with $\omega \in \Lambda^1(M)$ a one-form. Thus, as we know the Lie brackets of the metric, we obtain that

$$de^1 = \mu e^1 \wedge e^4, \quad de^2 = \frac{1}{2\mu} e^4 \wedge e^2, \quad de^3 = \left(\mu - \frac{1}{2\mu}\right) e^3 \wedge e^4 - e^1 \wedge e^2, \quad de^4 = 0.$$

Using that the exterior derivative satisfies that $d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta$, for $\omega, \eta \in \Lambda^1$, then we finally obtain that

$$\begin{aligned} dE_1^{\pm} &= \left(\tfrac{1}{2\mu} - \mu \mp 1\right) e^4 \wedge e^1 \wedge e^2, \\ dE_2^{\pm} &= \left(\tfrac{1}{2\mu} - 2\mu\right) e^4 \wedge e^1 \wedge e^3, \\ dE_3^{\pm} &= \left(\pm\tfrac{1}{\mu} \mp \mu\right) e^4 \wedge e^2 \wedge e^3. \end{aligned}$$

Now take

$$\theta_1^{\pm} = \left(\frac{1}{2\mu} - \mu \mp 1\right) e^4, \quad \theta_2 = \left(\frac{1}{2\mu} - 2\mu\right) e^4, \quad \theta_3^{\pm} = \left(\pm\frac{1}{\mu} \mp \mu\right) e^4,$$

and thus,

$$dE_1^{\pm} = \theta_1^{\pm} \wedge E_1^{\pm}, \quad dE_2^{\pm} = \theta_2 \wedge E_2^{\pm}, \quad dE_3^{\pm} = \theta_3^{\pm} \wedge E_3^{\pm},$$

Therefore, all $\{E_i^{\pm}\}$ define locally conformally symplectic structures on M. In addition, one can easily check that a two-form Ω is closed, i.e., $d\Omega = 0$, if and only if $\Omega = E_2^{\pm}$ with $\mu = \frac{1}{2}$ and $\Omega = E_1^{+}$ with $\mu = \frac{1}{2}(\sqrt{3} - 1)$. This follows since the $E_i^{\pm}$ are closed if only if $\theta_i^{\pm} = 0$ and $dE_i^{\pm}$ are linearly independent. The rest of the values for μ where $\theta_i \pm = 0$ are out of the range of μ, so whenever we take an homothety to be into the correspondent interval, we finish in the same two values given. Notice that the first value give us back the Kähler case. The case where $dE_1^{+} = 0$ has Ricci operator $Q_\rho = \mathrm{diag}[\sqrt{3} - \frac{3}{2}, -\sqrt{3} - \frac{3}{2}, -\frac{3}{2}, 3]$ and $W^{\pm} = \mathrm{diag}[\mp\frac{1}{2}, \pm\frac{1}{4}(1 + \sqrt{3}), \pm\frac{1}{4}(1 - \sqrt{3})]$.

Chapter 6
Fixed points and steady solitons for the two-loop renormalization group flow

In this chapter, we classify fixed points and steady solitons for the renormalization group flow. The results of this chapter are shown in [30].

6.1 Introduction

The problem of constructing a metric with a distinguish property is a main topic in differential geometry. In this direction, Hamilton [42] and Friedan [26] introduced Ricci flow. Let g_t be a one-parameter family of metrics into a manifold Riemannian manifold M, then we say that g_t is a solution to the Ricci flow if it satisfies the equation

$$\frac{\partial}{\partial_t} g_t = -2\rho_{g_t},$$

On the other hand, we also have the so called Ricci solitons, also introduced by Hamilton [43]. A Riemannian manifold (M, g) is a Ricci soliton if there exists a smooth vector field X such that the equation

$$\frac{1}{2}\mathcal{L}_X g + \rho = \lambda g,$$

is satisfied, where $(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$ represents the Lie derivative of the metric in the direction of X and $\lambda \in \mathbb{R}$. Moreover, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, then we say that (M, g) is a expanding, steady or shrinking Ricci soliton, respectively.

Remark 6.1. If there exist a real smooth function such that $X = \nabla f$, then we say that (M, g) is a gradient Ricci soliton.

Remark 6.2. Notice that if $X = 0$, the we obtain the Einstein condition. Thus, a Ricci soliton is a generalization of Einstein manifolds.

Given a one-parameter family of diffeomorphisms on M, call it ψ_t, with $\psi_0 = \mathrm{Id}$, a solution of the form $g_t = \sigma(t)\psi_t^* g$, where $\sigma(t)$ is a real-valued function, is said to be a self-similar solution. Now we have that any self-similar solution to the Ricci flow is a Ricci soliton just considering the one-parameter family ψ_t as the generator of the vector field [19].

Example 6.3.

1. **The cigar soliton.** In dimension two, Hamilton discovered in [43] the first complete non-compact steady soliton with metric

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

2. **Algebraic Ricci solitons.** Lauret showed in [52] that any left-invariant metric such that $Q_\rho - \beta \,\mathrm{Id}$, with $\beta \in \mathbb{R}$, is a derivation of the corresponding Lie Algebra gives an Ricci soliton metric.

Over the years, this topic has taken a lot of attention from many points of view. Recently, in the field of physics, a second-order approximation of the Ricci flow seemed to take attention. This was called the two-loop renormalization flow.

6.1.1 Two-loop renormalization group flow

The two-loop renormalization flow (or $RG2$ flow) appears as a perturbation of the Ricci flow and it is given by

$$\frac{\partial}{\partial_t} g_t = RG[g], \tag{6.1}$$

where$RG[g] = -2\rho - \frac{\alpha}{2}\check{R}$ and α is a positive coupling constant.

On the one hand, one aim of this chapter is studying genuine fixed points of (6.1), i.e, metrics satisfying $\rho + \frac{\alpha}{4}\check{R} = 0$.

In dimension two the condition reduces to constant negative curvature. In dimension three, they were studied by Gimre, Guenther and Isenberg in [33], where they showed solutions with Ricci curvatures $Q_\rho = -2\,\mathrm{diag}[\frac{1}{\alpha}, \alpha, 0]$ or $Q_\rho = -2\,\mathrm{diag}[\frac{2}{\alpha}, \alpha, \frac{1}{\alpha}]$. Einstein metrics are genuine fixed points of this flow in dimension four since if the Ricci tensor is a multiple of the metric, the $\check{R}$ tensor is as well. Therefore, we focus on the non-Einstein cases. Moreover, tracing $RG[g]$ for fixed points, one may see that $\tau + \frac{\alpha}{4}\|R\|^2 = 0$ and hence $\alpha = -4\tau\|R\|^{-2}$.

This flow has been applied in the study of black holes metrics, analysing how they evolved along it and also for the study of entropy, which has been stated as monotonic along this same flow. The reason to use this flow in such cases is that the singularities appearing in the study of other flows disappear in RG2, being this a better approximation to higher curvature effects [50, 51].

In the homogeneous setting, we obtain the following.

Theorem 6.4. *A simply connected four-dimensional homogeneous manifold is a genuine fixed point of the RG2 flow if and only if it is Einstein, a product* $\mathbb{R} \times N^3(c)$*, a product* $\mathbb{R}^2 \times N^2(c)$ *or homothetic to the Lie group* $SU(2) \times \mathbb{R}$ *with left-invariant metric*

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = \tfrac{4}{3} e_2,$$

where $\{e_1, \dots, e_4\}$ *is an orthonormal basis of* $\mathfrak{su}(2) \times \mathbb{R}$.

The above result is in sharp contrast with the geometry of the Ricci flow, since genuine fixed points of the Ricci flow are Ricci-flat manifolds.

On the other hand, we also focus on the study of what we call RG2 solitons. All the terminology for Ricci solitons follows for this changing the Ricci tensor for the tensor field $\rho + \frac{\alpha}{4}\check{R}$. Thus, we say that (M, g) is a a expanding, steady or shrinking RG2 soliton if there exist a smooth vector field X such that

$$\frac{1}{2}\mathcal{L}_X g + RG[g] = \lambda g.$$

Moreover, any self-similar solution to the RG2 flow is a RG2 soliton. Since the two terms comprising $RG[g]$ behave differently under homotheties ($\rho[\kappa g] = \rho[g]$ and $\check{R}[\kappa g] = \frac{1}{\kappa}\check{R}[g]$), one has that the converse holds only for steady solitons, in which case ψ_t is the one-parameter group of diffeomorphisms associated to the vector field X determined by the soliton equation $\mathcal{L}_X g + RG[g] = 0$ and $g(t) = \psi_t^* g$ is a self-similar solution [64].

Remark 6.5. The condition $W[\rho]$ has its own importance in this setting. It follows from (5.1) that a four-dimensional generalized Einstein metric with $\tau \neq 0$ satisfies

$$\rho - \frac{3}{\tau}\check{R} = \frac{1}{4}\left(\tau - \frac{3}{\tau}\|R\|^2\right) g,$$

so this provides a self-similar solution of the RG2 flow with $\alpha = -\frac{12}{\tau}$. Thus, this condition has application in physics. Thus, the family of metrics given in Theorem 5.2 gives a self-similar solution for the RG2 flow for the coupling constant $\alpha = -\frac{12}{\tau} = \frac{8\mu^2}{4\mu^4 - 3\mu^2 + 1}$.

Let G be a Lie group with left-invariant metric $\langle \cdot, \cdot \rangle$ and let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ denote the corresponding Lie algebra. An RG2 algebraic soliton is a derivation of the Lie algebra $\mathfrak{g}$ given by $\mathfrak{D} = Q_{RG[g]} - \beta\,\mathrm{Id}$, where $Q_{RG[g]}$ is the $(1,1)$-tensor field metrically equivalent to $RG[g]$ and $\beta \in \mathbb{R}$. RG2 algebraic solitons give rise to RG2 solitons.

Let $\langle \cdot, \cdot \rangle^* = \kappa\langle \cdot, \cdot \rangle$ be a homothetic deformation of a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then

$$Q_\rho^* + \frac{\kappa\alpha}{4}Q_{\check{R}}^* = \frac{1}{\kappa}Q_\rho + \frac{\kappa\alpha}{4\kappa^2}Q_{\check{R}} = \frac{1}{\kappa}\left(Q_\rho + \frac{\alpha}{4}Q_{\check{R}}\right)$$

and thus $\mathfrak{D} = Q_\rho + \frac{\alpha}{4}Q_{\check{R}}$ is a derivation of the Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ with coupling constant α if and only if $\mathfrak{D}^* = Q_\rho^* + \frac{\kappa\alpha}{4}Q_{\check{R}}^*$ is a derivation of the Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle^*)$ with constant $\kappa\alpha$. To study four-dimensional RG2 algebraic steady solitons, we work up to homothety in what follows in order to simplify the examples obtained.

Let H be a Lie group with a left-invariant metric determined by an inner product $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ and let $G = \mathbb{R} \times H$ be the product Lie group with product left-invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = dt \otimes dt \oplus \langle \cdot, \cdot \rangle_{\mathfrak{h}}$. Since $Q_{RG[g]_{\mathfrak{g}}} = 0 \oplus Q_{RG[g]_{\mathfrak{h}}}$, one has that if $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ is an RG2 algebraic steady soliton then $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ is a soliton as well. Conversely, assume that a complete and simply connected Lie group G with left-invariant metric is an RG2 algebraic steady soliton. Furthermore, assume that there exists a parallel left-invariant vector field on G. Then G breaks a one-dimensional factor so that it splits isometrically as $G = \mathbb{R} \times N$, where N is a complete and simply connected three-dimensional homogeneous manifold. Hence N is either symmetric (in which case G is also a symmetric space) or N is isometric to a Lie group H. Respectively, the tensor field RG also splits as $RG_{\mathfrak{g}} = 0 \oplus RG_{\mathfrak{h}}$ and so does the corresponding $(1,1)$-tensor field. Hence, if G is an RG2 algebraic steady soliton then so is H just considering the derivation determined by $Q_{RG[g]_{\mathfrak{h}}}$.

During this chapter, we analyse RG2 algebraic steady solitons on four-dimensional irreducible Lie groups, since otherwise it reduces to the three-dimensional case, which is studied in the next section. The main result is stated as follows.

Theorem 6.6. *A simply connected non-Einstein four-dimensional irreducible Lie group G is an RG2 algebraic steady soliton if and only if it is homothetic to one of the Lie groups determined by the following Lie algebras, where $\{e_1, \dots, e_4\}$ is an orthonormal basis:*

1. $\mathbb{R} \ltimes \mathfrak{e}(1,1)$, *for a coupling constant $\alpha = \frac{2}{\kappa^2+1}$, given by*
$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$
where $\kappa > 0$, $\kappa \neq 1$.
2. $\mathbb{R} \ltimes \mathfrak{h}^3$, *for a coupling constant $\alpha = 2$, given by*
$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = \frac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_1,$$
$$[e_2, e_4] = \frac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_2, \quad [e_3, e_4] = \frac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_3,$$
where $\kappa \in [-1, 1)$.
3. $\mathbb{R} \ltimes \mathfrak{h}^3$, *for a coupling constant $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$, given by*
$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa} e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right) e_3,$$
where $\kappa \in \left(0, \frac{1}{2}\right]$, $\kappa \neq \frac{1}{2}\sqrt{2-\sqrt{3}}$.
4. $\mathbb{R} \ltimes \mathfrak{r}^3$, *for a coupling constant $\alpha = \frac{2(\kappa^2+\delta^2+1)}{\kappa^4+\delta^4+1}$, given by*
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2, \quad [e_3, e_4] = \delta e_3,$$
where $(\kappa, \delta) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \setminus \{(1, 1)\}$.
5. $\mathbb{R} \ltimes \mathfrak{r}^3$, *for a coupling constant $\alpha = \frac{2}{\kappa^2+p^2}$, given by*
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2 + h e_3, \quad [e_3, e_4] = -h e_2 + p e_3,$$
where the parameters p and h are given by $p = \frac{1}{2}\left(1 + \sqrt{1 - 4\kappa(\kappa-1)}\right)$ and $h = \left(\frac{\kappa^2(2p^2+1)+p^2-1}{2(\kappa-p)^2}\right)^{\frac{1}{2}}$, for any $\kappa \in (0, 1)$.

The above result is in sharp in contrast with the Ricci flow case where steady homogeneous Ricci solitons are Ricci-flat. Moreover, the Lie groups corresponding to cases (2) and (4) are expanding algebraic Ricci solitons, whereas Lie groups of cases (1), (3) and (5) are not Ricci solitons. It follows from sections 6.2.1–6.2.4 that all metrics in Theorem 6.6 represent different homothetical classes. Thus, these results shows some differences between both flows.

In brief, the main target of this chapter is proving Theorems 6.4 and 6.6.

6.1.2 Three-dimensional RG2 algebraic steady solitons

Three-dimensional RG2 algebraic steady solitons have been classified by Wears in the unimodular case [64] (see also [36]). This can be easily summarized as follows:

Lemma 6.7 ([64]). *Let G be a three-dimensional unimodular Lie group. Then G is a non-symmetric RG2 algebraic steady soliton if and only if it is homothetic to one of the following Lie groups:*

1. *The Lie group $E(1,1)$ with a left-invariant metric given by:*
   (a) *the Lie algebra structure $(\lambda_1,\lambda_2,\lambda_3)=(1,-1,0)$, where $\alpha=2$, or*
   (b) *the Lie algebra structure $(\lambda_1,\lambda_2,\lambda_3)=(3,-1,0)$, where $\alpha=\frac{1}{4}$.*
2. *The Heisenberg group H^3 with a left-invariant metric given by the eigenvalues $(\lambda_1,\lambda_2,\lambda_3)=(0,0,1)$, where $\alpha=\frac{8}{3}$.*
3. *The special unitary group $SU(2)$ with a left-invariant metric determined by $(\lambda_1,\lambda_2,\lambda_3)=(1,\frac{4}{3},1)$, where $\alpha=-\frac{9}{2}$.*

Remark 6.8. Metrics corresponding to case $(1.a)$ are algebraic Ricci solitons for $\lambda=-2$ (i.e., $Q+2\mathrm{Id}$ is a derivation), while metrics corresponding to case (1.b) are not. Moreover, the Heisenberg Lie group is an algebraic Ricci soliton for $\lambda=-\frac{3}{2}$ while the special unitary group does not admit any non-Einstein Ricci soliton.

In addition to the previous RG2 algebraic steady solitons, there are some non-unimodular ones, which can be described as follows:

Lemma 6.9. *Let G be a three-dimensional non-unimodular Lie group. Then G is a non-symmetric RG2 algebraic steady soliton if and only if it is homothetic to a left-invariant metric determined by the Lie algebra $\mathfrak{g}=\operatorname{span}\{e_1,e_2,e_3\}$ given by*

$$[e_1,e_2]=(\xi+1)e_2+(\xi+1)\eta e_3\,,\quad [e_1,e_3]=(\xi-1)\eta e_2-(\xi-1)e_3,$$

where $\{e_1,e_2,e_3\}$ is an orthonormal basis and one of the following holds:

1. *$\eta=0$, $\xi>0$ and $\xi\neq 1$, for a coupling constant $\alpha=\frac{2(\xi^2+1)}{(\xi^2+6)\xi^2+1}$.*
2. *$\eta>0$ and $\xi=1\pm\frac{\eta}{\sqrt{\eta^2+1}}$, for a coupling constant $\alpha=\frac{1}{2}\left(1\mp\frac{\eta}{\sqrt{\eta^2+1}}\right)$.*

Proof. Following Milnor [56], any non-symmetric left-invariant metric on a non-unimodular Lie group is determined by Lie brackets

$$[e_1,e_2]=(\xi+1)e_2+(\xi+1)\eta e_3,\quad [e_1,e_3]=(\xi-1)\eta e_2-(\xi-1)e_3,$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis and $\eta \geq 0$, $\xi > 0$, excluding the case $\eta = 0$, $\xi = 1$. A straightforward calculation shows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the following polynomials vanish identically:

$$\begin{aligned}
\mathfrak{D}_{212} &= (\xi+1)(\alpha(\eta^2+1)^2\xi^4 + 2(\eta^2+1)(\alpha(2\eta^2+3)-1)\xi^2 + \alpha - 2),\\
\mathfrak{D}_{313} &= (1-\xi)(\alpha(\eta^2+1)^2\xi^4 + 2(\eta^2+1)(\alpha(2\eta^2+3)-1)\xi^2 + \alpha - 2),\\
\mathfrak{D}_{213} &= \eta(\xi+1)(\alpha(\eta^2+1)\xi^2 + 2\alpha(\eta^2+1)\xi + \alpha - 2)((\eta^2+1)(\xi+2)\xi + 1),\\
\mathfrak{D}_{312} &= \eta(\xi-1)(\alpha(\eta^2+1)\xi^2 - 2\alpha(\eta^2+1)\xi + \alpha - 2)((\eta^2+1)(\xi-2)\xi + 1)\,.
\end{aligned}$$

Computing a Gröbner basis $\mathcal{G}$ of the ideal generated by the polynomials $\mathfrak{D}_{ijk} \in \mathbb{R}[\xi, \eta, \alpha]$ above with respect to the lexicographical order, one gets that the polynomials $\mathrm{g}_1 = \eta(\alpha-2)(\xi^2-4(\alpha-1)^2)$ and $\mathrm{g}_2 = \eta(\eta^2+1)(4\alpha(\alpha-1)(\eta^2+1)+1)\xi$ belong to the basis. Hence g_1 leads to the following cases: $\alpha = 2$, $\eta = 0$ and $\xi^2 = 4(\alpha-1)^2$.

Setting $\alpha = 2$, since $\xi > 0$ one easily gets that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is never a derivation of the Lie algebra. Assuming $\eta = 0$ one has that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation if and only if $((\xi^2+6)\xi^2+1)\alpha - 2(\xi^2+1) = 0$, which corresponds to Assertion (1).

Assume now that $\xi^2 = 4(\alpha-1)^2$ and $\eta > 0$. In this case, the polynomial g_2 leads to $4\alpha(\alpha-1)(\eta^2+1)+1 = 0$ and a straightforward calculation shows that these two conditions suffice for $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ being a derivation. The first equation implies that $\alpha = 1 + \frac{\varepsilon}{2}\xi$, where $\varepsilon^2 = 1$. Then the second equation above becomes $\varepsilon(\eta^2+1)(\varepsilon\xi+2)\xi + 1 = 0$. If $\varepsilon = 1$ then $\xi = -1 \pm \frac{\eta}{\sqrt{\eta^2+1}}$ and thus $\xi < 0$. If $\varepsilon = -1$, then $\xi = 1 \pm \frac{\eta}{\sqrt{\eta^2+1}}$ and Assertion (2) follows. □

Remark 6.10. Left-invariant metrics given in Lemma 6.9 define different homothetical classes. First, note that RG2 algebraic steady solitons corresponding to Assertion (1) are also algebraic Ricci solitons for a derivation $Q + 2(\xi^2+1)\,\mathrm{Id}$, while RG2 algebraic steady solitons corresponding to Assertion (2) are not Ricci solitons (see, for example, [2]).

Let $(G_1, \langle\cdot,\cdot\rangle_1)$ and $(G_2, \langle\cdot,\cdot\rangle_2)$ be two Lie groups with negative scalar curvature τ_1 and τ_2, respectively. For $i = 1, 2$, let $\langle\cdot,\cdot\rangle_i^* = -\tau_i\langle\cdot,\cdot\rangle_i$ so that the scalar curvature of the normalized metric $\langle\cdot,\cdot\rangle_i^*$ is $\tau_i^* = -1$. Now, one has that $(G_1, \langle\cdot,\cdot\rangle_1)$ and $(G_2, \langle\cdot,\cdot\rangle_2)$ are homothetic if and only if the normalized metrics $\langle\cdot,\cdot\rangle_i^*$ are isometric. In this case one has that $\|\rho_1^*\| = \|\rho_2^*\|$ and $\|R_1^*\| = \|R_2^*\|$, or equivalently, $\tau_1^{-2}\|\rho_1\|^2 = \tau_2^{-2}\|\rho_2\|^2$ and $\tau_1^{-2}\|R_1\|^2 = \tau_2^{-2}\|R_2\|^2$. The failure of any of these relations therefore implies that the left-invariant metrics $\langle\cdot,\cdot\rangle_i$ correspond to different homothetical classes.

Now, a standard calculation shows that left-invariant metrics in Assertion (1) corresponding to different values of the parameter ξ are never homothetical since $\tau = -2(\xi^2+3)$ and $\|R\|^2 = 4(3\xi^4 + 10\xi^2 + 3)$. The same result holds for metrics in Assertion (2), where $\tau = -4\left(\eta^2 + 2 - \eta\sqrt{\eta^2+1}\right)$ and $\|R\|^2 = 16(5\eta^2+4)\left(2\eta^2 + 1 \pm 2\eta\sqrt{\eta^2+1}\right)$.

6.2 Four-dimensional RG2 algebraic steady solitons

We work with the same procedure as in the previous chapter. We compute a system of polynomials that have to be satisfied to fulfil the condition wanted and then we use Gröbner basis to solve it.

6.2.1 The direct products $SL(2,\mathbb{R})\times\mathbb{R}$ and $SU(2)\times\mathbb{R}$

Lemma 6.11. *Let G be a product $SL(2,\mathbb{R})\times\mathbb{R}$ or $SU(2)\times\mathbb{R}$. Then G admits a non-symmetric $RG2$ algebraic steady soliton if and only if it is homothetic to the Lie group $SU(2)\times\mathbb{R}$ determined by*

$$[e_1,e_2]=e_3,\qquad [e_2,e_3]=e_1,\qquad [e_3,e_1]=\tfrac{4}{3}e_2,$$

for a coupling constant $\alpha=-\frac{9}{2}$, where $\{e_1,\dots,e_4\}$ is an orthonormal basis. Moreover, it is a fixed point for the $RG2$ renormalization group.

Proof. Take the algebra given in (4.1). Since $\lambda_1\lambda_2\lambda_3\neq 0$, assume $\lambda_1=1$ just working with the homothetic metric determined by $\hat e_i=\frac{1}{\lambda_1}e_i$. Let $\mathfrak{D}=Q+\frac{\alpha}{4}Q_{\check R}$. Then $\mathfrak{D}$ is a derivation of the Lie algebra if and only if all terms $\mathfrak{D}_{ijk}=\langle\mathfrak{D}[e_i,e_j]-[\mathfrak{D}e_i,e_j]-[e_i,\mathfrak{D}e_j],e_k\rangle$ vanish. The components $\mathfrak{D}_{ijk}$ can be obtained directly from the expressions of the Ricci tensor and the $\check R$-tensor and can be written as

$$\begin{array}{llll}
16R^4\mathfrak{D}_{211}=\mathfrak{P}_{211}, & 16R^4\mathfrak{D}_{212}=\mathfrak{P}_{212}, & 16R^4\mathfrak{D}_{213}=\mathfrak{P}_{213}, & 16R^3\mathfrak{D}_{214}=\mathfrak{P}_{214},\\
16R^4\mathfrak{D}_{311}=\mathfrak{P}_{311}, & 16R^4\mathfrak{D}_{312}=\mathfrak{P}_{312}, & 16R^4\mathfrak{D}_{313}=\mathfrak{P}_{313}, & 16R^3\mathfrak{D}_{314}=\mathfrak{P}_{314},\\
16R^5\mathfrak{D}_{411}=\mathfrak{P}_{411}, & 16R^5\mathfrak{D}_{412}=\mathfrak{P}_{412}, & 16R^5\mathfrak{D}_{413}=\mathfrak{P}_{413}, & 16R^4\mathfrak{D}_{414}=\mathfrak{P}_{414},\\
16R^4\mathfrak{D}_{321}=\mathfrak{P}_{321}, & 16R^4\mathfrak{D}_{322}=\mathfrak{P}_{322}, & 16R^4\mathfrak{D}_{323}=\mathfrak{P}_{323}, & 16R^3\mathfrak{D}_{324}=\mathfrak{P}_{324},\\
16R^5\mathfrak{D}_{421}=\mathfrak{P}_{421}, & 16R^5\mathfrak{D}_{422}=\mathfrak{P}_{422}, & 16R^5\mathfrak{D}_{423}=\mathfrak{P}_{423}, & 16R^4\mathfrak{D}_{424}=\mathfrak{P}_{424},\\
16R^5\mathfrak{D}_{431}=\mathfrak{P}_{431}, & 16R^5\mathfrak{D}_{432}=\mathfrak{P}_{432}, & 16R^5\mathfrak{D}_{433}=\mathfrak{P}_{433}, & 16R^4\mathfrak{D}_{434}=\mathfrak{P}_{434},
\end{array}$$

where $\mathfrak{P}_{ijk}$ are polynomials associated to the coefficients $\mathfrak{D}_{ijk}$ which are given by

$$\begin{aligned}
\mathfrak{P}_{211}={}&-k_1k_3\lambda_3(-R^2(8+5\alpha\lambda_2^4+(16-3\alpha)\lambda_3+4\alpha\lambda_3^2-5\alpha\lambda_3^3-2\alpha\lambda_2^3(4+3\lambda_3)\\
&+\lambda_2^2(-8+\alpha\lambda_3(8+\lambda_3))+\lambda_2(-16+\alpha(3-4\lambda_3+5\lambda_3^2)))\\
&+\alpha(k_3^2(-5\lambda_2^4+\lambda_2(-3+\lambda_3)+\lambda_3-4\lambda_2^2\lambda_3+2\lambda_2^3(4+\lambda_3))\\
&+k_1^2(-5\lambda_2^4+2\lambda_2(1-2\lambda_3)\lambda_3+\lambda_3^2(-3+4\lambda_3)+\lambda_2^3(4+6\lambda_3)\\
&-\lambda_2^2(-1+4\lambda_3+\lambda_3^2))+k_2^2(\lambda_2^2(5+\lambda_3-2\lambda_3^2)+\lambda_3(3-4\lambda_3+5\lambda_3^2)\\
&-\lambda_2(5+2\lambda_3^2+\lambda_3^3)))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{212}={}&-k_2k_3\lambda_3(R^2(-8(-1+\lambda_2^2+2\lambda_2(-1+\lambda_3))+\alpha(-1+\lambda_3)\,(5+3\lambda_2^3\\
&-\lambda_3-4\lambda_2^2\lambda_3+\lambda_2(-8+5\lambda_3^2)))+\alpha(k_3^2(-1+\lambda_2)(5+\lambda_2^2(-3+\lambda_3)\\
&-2\lambda_3+\lambda_2(-3+2\lambda_3))-k_2^2(-1+\lambda_3)(-5+\lambda_3+\lambda_2^2(1+3\lambda_3)\\
&-4\lambda_2(-1+\lambda_3^2))+k_1^2(-\lambda_3^2(2+\lambda_3)+\lambda_2^3(-5+3\lambda_3)+\lambda_2^2(5-4\lambda_3^2)\\
&+\lambda_2\lambda_3(1-2\lambda_3+5\lambda_3^2)))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{213}={}&\lambda_3(k_2^4\alpha(-1+\lambda_3)^2(-2\lambda_2(1+\lambda_3)+(1+3\lambda_3)^2)+R^4(1+\lambda_2^2\\
&+2\lambda_2(-1+\lambda_3)+2\lambda_3-3\lambda_3^2)(8+\alpha(1+\lambda_2^2+2\lambda_2(-1+\lambda_3)+2\lambda_3\\
&-3\lambda_3^2))+\alpha(2k_3^4(-1+\lambda_2)^2(3+2\lambda_2+3\lambda_2^2)\\
&+k_1^4(\lambda_2-\lambda_3)^2(\lambda_2^2+\lambda_2(-2+6\lambda_3)+\lambda_3(-2+9\lambda_3))\\
&+k_1^2k_3^2(-1+\lambda_2)(\lambda_2^2+7\lambda_2^3-5\lambda_2(-2+\lambda_3)\lambda_3-\lambda_3(1+12\lambda_3)))
\end{aligned}$$

$$
\begin{aligned}
&+ R^2(k_3^2\alpha(-1+\lambda_2)^2(7+2\lambda_2+7\lambda_2^2-5\lambda_3^2)\\
&+ 2k_1^2(\lambda_2-\lambda_3)(\alpha\lambda_2^3+\ \alpha\lambda_2^2(-3+5\lambda_3)+\lambda_2(4+\alpha-4\alpha\lambda_3+3\alpha\lambda_3^2)\\
&- \lambda_3(-12+\alpha-7\alpha\lambda_3+9\alpha\lambda_3^2)))+k_2^2(2R^2(-1+\lambda_3)(-4(1+3\lambda_3)\\
&+ \alpha(-1+\lambda_3)(\lambda_2^2\ +(1+3\lambda_3)^2-\lambda_2(3+7\lambda_3)))\\
&+ \alpha(k_3^2(-1+\lambda_2)(-7\ +\lambda_2^2\lambda_3+5\lambda_3^2+\lambda_2(-1-10\lambda_3+12\lambda_3^2))\\
&+ k_1^2(\lambda_2^3+2\lambda_2^2(-2+3\lambda_3+\lambda_3^2)+2\lambda_3^2(1-7\lambda_3+9\lambda_3^2)\\
&+ \lambda_2(1+6\lambda_3-4\lambda_3^2-14\lambda_3^3))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{214} = {}& k_3\lambda_3(R^2(-1+\lambda_2)^2(-8+\alpha(5+5\lambda_2^2-6\lambda_2(-1+\lambda_3)-6\lambda_3+\lambda_3^2))\\
&+ \alpha(k_3^2(-1+\ \lambda_2)^2(5+5\lambda_2^2-2\lambda_2(-3+\lambda_3)-2\lambda_3)\\
&+ k_2^2(-1+\lambda_3)(-5+\lambda_3+\lambda_2(-2+\ \lambda_3)\lambda_3+\lambda_2^2(3+2\lambda_3))\\
&+ k_1^2(5\lambda_2^4-6\lambda_2^3\lambda_3+\lambda_3^2(2+\lambda_3)+\ \lambda_2(\lambda_3-3\lambda_3^2)+\lambda_2^2(-3+2\lambda_3+\lambda_3^2)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{311} = {}& k_1k_2\lambda_2(\alpha(k_3^2(-\lambda_2^3(-5+\lambda_3)+5(-1+\lambda_3)\lambda_3+\lambda_2(3+\ \lambda_3^2)\\
&- 2\lambda_2^2(2+\lambda_3+\lambda_3^2))+k_1^2(4\lambda_2^3+\lambda_3^2+4\lambda_3^3-5\lambda_3^4-\lambda_2^2(3+4\lambda_3+\lambda_3^2)\\
&+ 2\lambda_2\lambda_3(1-2\lambda_3+3\lambda_3^2))+k_2^2(-3\lambda_3+8\lambda_3^3-5\lambda_3^4+\lambda_2(1+\lambda_3\\
&- 4\lambda_3^2+2\lambda_3^3)))+R^2(-8+5\alpha\lambda_2^3+(16-3\alpha)\lambda_3+8\lambda_3^2+8\alpha\lambda_3^3-5\alpha\lambda_3^4\\
&- \alpha\lambda_2^2(4+5\lambda_3+\lambda_3^2)+\lambda_2(-16+\alpha(3\ +4\lambda_3-8\lambda_3^2+6\lambda_3^3)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{312} = {}& -\lambda_2(k_3^4\alpha(-1+\lambda_2)^2(1+9\lambda_2^2-2\lambda_2(-3+\lambda_3)-2\lambda_3)+R^4(3\lambda_2^2-(-1+\lambda_3)^2\\
&- 2\lambda_2(1+\lambda_3))(-8+\alpha(3\lambda_2^2-\ (-1+\lambda_3)^2-2\lambda_2(1+\lambda_3)))\\
&+ \alpha(2k_2^4(-1+\ \lambda_3)^2(3+2\lambda_3+3\lambda_3^2)+k_1^4(\lambda_2-\lambda_3)^2(9\lambda_2^2+(-2+\ \lambda_3)\lambda_3\\
&+ \lambda_2(-2+6\lambda_3))+k_1^2k_2^2(-1+\lambda_3)(-\lambda_2^2(12+5\lambda_3)+\lambda_3^2(1+7\lambda_3)\\
&+ \lambda_2(-1+10\lambda_3)))+k_3^2(2R^2(-1+\lambda_2)(-4(1+3\lambda_2)\\
&+ \alpha(-1+\lambda_2)(1+9\lambda_2^2+\lambda_2(6-7\lambda_3)-3\lambda_3+\lambda_3^2))\\
&+ \alpha(k_2^2(-1+\ \lambda_3)(-7-\lambda_3+\lambda_2(-10+\lambda_3)\ \lambda_3+\lambda_2^2(5+12\lambda_3))\\
&+ k_1^2(18\lambda_2^4+2\lambda_2^2(-1+\lambda_3)^2-14\lambda_2^3(1+\lambda_3)+6\lambda_2\lambda_3(1+\lambda_3)\\
&+ \lambda_3(1\ -4\lambda_3+\lambda_3^2))))+R^2(k_2^2\alpha(-1+\lambda_3)^2(7-5\lambda_2^2+2\lambda_3+7\lambda_3^2)\\
&+ 2k_1^2(\lambda_2-\lambda_3)(9\alpha\lambda_2^3-\ \alpha\lambda_2^2(7+3\lambda_3)+\lambda_2(-12+\alpha+4\alpha\lambda_3-5\alpha\lambda_3^2)\\
&- \lambda_3(4+\alpha(1-3\lambda_3+\lambda_3^2))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{313} = {}& k_2k_3\lambda_2(R^2(-8(-1+2(-1+\lambda_2)\lambda_3+\lambda_3^2)+\alpha\ (-1+\lambda_2)(5-8\lambda_3+5\lambda_2^2\lambda_3\\
&+ 3\lambda_3^3-\lambda_2(1+4\lambda_3^2)))+\alpha(k_2^2(-1+\lambda_3)(5-3\lambda_3-3\lambda_3^2\\
&+ \lambda_2(-2+2\lambda_3+\lambda_3^2))+k_3^2(-1+\lambda_2)(5-4\lambda_3+4\lambda_2^2\lambda_3-\lambda_3^2\\
&- \lambda_2(1+3\lambda_3^2))+k_1^2(-5(-1+\lambda_3)\lambda_3^2+\lambda_2^3(-1+5\lambda_3)-2\lambda_2^2(1+\lambda_3\\
&+ 2\lambda_3^2)+\lambda_2(\lambda_3+3\lambda_3^3)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{314} = {}& -k_2\lambda_2(R^2(-1+\lambda_3)^2(-8+\alpha(5+\ \lambda_2^2+6\lambda_3+5\lambda_3^2-6\lambda_2(1+\lambda_3)))\\
&+ \alpha(k_3^2(-1+\ \lambda_2)(-5+\lambda_2-2\lambda_2\lambda_3+\lambda_2^2\lambda_3+3\lambda_3^2+2\lambda_2\lambda_3^2)\\
&+ k_2^2(-1+\lambda_3)^2(5+6\lambda_3+5\lambda_3^2-2\lambda_2(1+\lambda_3))\\
&+ k_1^2(\lambda_2^3+\lambda_2^2(2-3\lambda_3+\lambda_3^2)+\lambda_3^2(-3+5\lambda_3^2)+\lambda_2(\lambda_3+2\lambda_3^2-6\lambda_3^3)))),
\end{aligned}
$$

$$
\mathfrak{P}_{411} = -k_1k_2k_3(\lambda_2-\lambda_3)(\alpha(k_3^2(\lambda_2(3-4\lambda_3)+\lambda_3+\lambda_2^3(5+4\lambda_3)-\lambda_2^2(4+5\lambda_3))
$$

$$
\begin{aligned}
&+ k_1^2(\lambda_3^2(-3+4\lambda_3) + \lambda_2^3(4+5\lambda_3) - \lambda_2^2(3+4\lambda_3+2\lambda_3^2) \\
&+ \lambda_2\lambda_3(-2-4\lambda_3+5\lambda_3^2)) + k_2^2(\lambda_3(3-4\lambda_3+5\lambda_3^2) + \lambda_2(1-4\lambda_3 \\
&- 5\lambda_3^2+4\lambda_3^3))) + R^2(-8+(-16+3\alpha)\lambda_3 - 4\alpha\lambda_3^2+5\alpha\lambda_3^3 \\
&+ 5\alpha\lambda_2^3(1+\lambda_3) - 2\alpha\lambda_2^2(2+4\lambda_3+\lambda_3^2) + \lambda_2(-8(2+\ \lambda_3)+\alpha(3-8\lambda_3^2+5\lambda_3^3)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{412} = {}& k_3(k_3^4\alpha\lambda_2(1+2\lambda_2-3\lambda_2^2)^2 + R^4(-1+\lambda_2)\lambda_2(-8(1+3\lambda_2-2\lambda_3) \\
&+ \alpha(-1+9\lambda_2^3+2\lambda_3-5\lambda_3^2+4\lambda_3^3-\lambda_2^2(3+10\lambda_3)+\lambda_2(-5+8\lambda_3-3\lambda_3^2))) \\
&+ \alpha(k_1^4\lambda_2(9\ \lambda_2^4+\lambda_3^3+4\lambda_2\lambda_3^3-2\lambda_2^3(1+5\lambda_3)+\lambda_2^2(\lambda_3-3\lambda_3^2)) \\
&- k_1^2k_2^2(\lambda_3^3(2+\lambda_3)+\lambda_2^3(-12+11\lambda_3)+\lambda_2\lambda_3^2(-2+5\lambda_3-10\lambda_3^2) \\
&+ \lambda_2^2(-1+5\lambda_3+\lambda_3^3)) + k_2^4(-1+\lambda_3)(-(-5+\lambda_3)\lambda_3 \\
&+ \lambda_2(-6-4\lambda_3-3\lambda_3^2+9\lambda_3^3))) + k_3^2(R^2(-1+\lambda_2)\lambda_2(-8(1+3\lambda_2) \\
&+ \alpha(-1+\lambda_2)(2+18\lambda_2^2-2\lambda_3+\lambda_3^2-2\lambda_2(-6+5\lambda_3))) \\
&+ \alpha(k_1^2\lambda_2(18\ \lambda_2^4+\lambda_3(1+\lambda_3)+4\lambda_2\lambda_3(1+\lambda_3)-2\lambda_2^3(7+5\lambda_3) \\
&+ \lambda_2^2(2-7\lambda_3+\lambda_3^2)) + k_2^2(\lambda_2^2(7-12\lambda_3)\lambda_3+\lambda_3(-5+2\lambda_3) \\
&+ \lambda_2(7+6\lambda_3-6\lambda_3^2)+\lambda_2^3(-5-12\lambda_3+18\lambda_3^2)))) \\
&+ R^2(k_1^2\lambda_2(18\alpha\lambda_2^4-2\alpha\lambda_2^3(7+10\lambda_3)+\lambda_3(8+\alpha(4-3\lambda_3)\lambda_3) \\
&+ \lambda_2^2(-24+\ \alpha(2+19\lambda_3-6\lambda_3^2))+2\lambda_2\lambda_3(8+\alpha(-3-\lambda_3+4\lambda_3^2))) \\
&+ k_2^2(8(1+\lambda_2(2-3\lambda_3))\lambda_3+\alpha(-1+\lambda_3)(5\lambda_2^3-(-5+\lambda_3)\lambda_3 \\
&- \lambda_2^2(-3+\ \lambda_3)\lambda_3+\lambda_2(-7-7\lambda_3-7\lambda_3^2+10\lambda_3^3)))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{413} = {}& -k_2(k_3^4\alpha(-1+\lambda_2)(\lambda_2(5-4\lambda_3)-6\lambda_3+9\lambda_2^3\lambda_3-\lambda_2^2(1+3\lambda_3)) \\
&+ k_3^2(R^2(10\alpha\lambda_2^4\lambda_3+\alpha\lambda_3(7-5\lambda_3^2)-\alpha\lambda_2^3(1+17\lambda_3+\lambda_3^2) \\
&+ \lambda_2^2(-24\lambda_3+\alpha(6+4\lambda_3^2))+\lambda_2(8+16\lambda_3+\alpha(-5-3\lambda_3^2+5\lambda_3^3))) \\
&+ \alpha(k_1^2(2\lambda_2^2\ \lambda_3+\lambda_2^4(-1+10\lambda_3)-\lambda_2\lambda_3^2(5+11\lambda_3)+\lambda_3^2(1+12\lambda_3) \\
&- \lambda_2^3(2+5\lambda_3+\lambda_3^2))+k_2^2(7\lambda_3-5\lambda_3^3+\lambda_2(-5+6\lambda_3+7\lambda_3^2-12\lambda_3^3) \\
&+ 2\lambda_2^2(1-3\lambda_3-6\lambda_3^2+9\lambda_3^3)))) + \lambda_3(k_2^4\alpha(1+2\lambda_3-3\lambda_3^2)^2 \\
&+ k_1^4\alpha(-3\lambda_2^2\lambda_3^2+\lambda_2^3(1+4\lambda_3)+\lambda_3^3(-2+9\lambda_3)+\lambda_2(\lambda_3^2-10\lambda_3^3)) \\
&+ k_1^2R^2(\alpha\lambda_2^3(-3+8\lambda_3)-2\alpha\lambda_2^2(-2+\lambda_3+3\lambda_3^2)+2\lambda_3^2(-12+\alpha-7\alpha\lambda_3 \\
&+ 9\alpha\lambda_3^2)+\lambda_2(8+(16-6\alpha)\lambda_3+19\alpha\lambda_3^2-20\alpha\lambda_3^3)) \\
&+ R^4(-1+\lambda_3)(8(-1+2\lambda_2-3\lambda_3)+\alpha(4\lambda_2^3+(-1+\ \lambda_3)(1+3\lambda_3)^2 \\
&- \lambda_2^2(5+3\lambda_3)+\lambda_2(2+8\lambda_3-10\lambda_3^2))) + k_2^2(k_1^2\alpha(\lambda_2^2(1+4\lambda_3+\lambda_3^2) \\
&+ 2\lambda_3^2(1-7\lambda_3+9\lambda_3^2)+\lambda_2(1+4\lambda_3-7\lambda_3^2-10\lambda_3^3)) \\
&+ R^2(-1+\lambda_3)(-8(1+3\lambda_3)+\alpha(-1+\lambda_3)\ (\lambda_2^2+2(1+3\lambda_3)^2 \\
&- 2\lambda_2(1+5\lambda_3)))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{414} = {}& -k_2k_3(\lambda_2-\lambda_3)(\alpha(k_3^2(-1+\lambda_2)(5\ -2\lambda_3+4\lambda_2^2\lambda_3+\lambda_2(-1+3\lambda_3)) \\
&+ k_2^2(-1+\lambda_3)(5-\lambda_3+\lambda_2(-2+3\lambda_3+4\lambda_3^2)) + k_1^2(-\lambda_3^2(2+\lambda_3) \\
&+ \lambda_2^3(-1+5\lambda_3)-2\lambda_2^2(1-\lambda_3+\lambda_3^2)+\lambda_2\ \lambda_3(-6+2\lambda_3+5\lambda_3^2))) \\
&+ R^2(8-8\lambda_2\lambda_3+\alpha(-5+6\lambda_3+5\lambda_2^3\lambda_3-\lambda_3^2-\lambda_2^2(1+2\lambda_3+2\lambda_3^2) \\
&+ \lambda_2(6-9\lambda_3-2\lambda_3^2+5\lambda_3^3)))),
\end{aligned}
$$

$$
\mathfrak{P}_{321} = -24R^4+9R^4\alpha+16R^4\lambda_2-12R^4\alpha\lambda_2+8R^4\lambda_2^2-5k_1^2R^2\alpha\lambda_2^2-2R^4\alpha\lambda_2^2
$$

$$
\begin{aligned}
&+ 4R^4\alpha\lambda_2^3 + 6k_1^4\alpha\lambda_2^4 + 7k_1^2R^2\alpha\lambda_2^4 + R^4\alpha\lambda_2^4\\
&+ k_3^4\alpha(-1+\lambda_2)^2(9+\lambda_2^2 - 2\lambda_2(-3+\lambda_3) - 2\lambda_3) + 16R^4\lambda_3 - 12R^4\alpha\lambda_3\\
&- 16R^4\lambda_2\lambda_3 + 10k_1^2R^2\alpha\lambda_2\lambda_3 + 20R^4\alpha\lambda_2\lambda_3 - 4R^4\alpha\lambda_2^2\lambda_3 - 8k_1^4\alpha\lambda_2^3\lambda_3\\
&- 12k_1^2R^2\alpha\lambda_2^3\lambda_3 - 4R^4\alpha\lambda_2^3\lambda_3 + 8R^4\lambda_3^2 - 5k_1^2R^2\alpha\lambda_3^2 - 2R^4\alpha\lambda_3^2\\
&- 4R^4\alpha\lambda_2\lambda_3^2 + 4k_1^4\alpha\lambda_2^2\lambda_3^2 + 10k_1^2R^2\alpha\lambda_2^2\lambda_3^2 + 6R^4\alpha\lambda_2^2\lambda_3^2 + 4R^4\alpha\lambda_3^3\\
&- 8k_1^4\alpha\lambda_2\lambda_3^3 - 12k_1^2R^2\alpha\lambda_2\lambda_3^3 - 4R^4\alpha\lambda_2\lambda_3^3 + 6k_1^4\alpha\lambda_3^4 + 7k_1^2R^2\alpha\lambda_3^4 + R^4\alpha\lambda_3^4\\
&+ k_2^4\alpha(-1+\lambda_3)^2(-2\lambda_2(1+\lambda_3) + (3+\lambda_3)^2) + k_3^2(k_1^2\alpha(7\lambda_2^4 - 6\lambda_2^3\lambda_3\\
&+ \lambda_3^2(12+\lambda_3) - \lambda_2\lambda_3(7+11\lambda_3) - \lambda_2^2(5-10\lambda_3+\lambda_3^2))\\
&+ 2R^2(-1+\lambda_2)(4(3+\lambda_2) + \alpha(-1+\lambda_2)(9+6\lambda_2+\lambda_2^2 - 7\lambda_3\\
&- 3\lambda_2\lambda_3 + \lambda_3^2))) + k_2^2(2R^2(-1+\lambda_3)(4(3+\lambda_3) + \alpha(-1+\lambda_3)(\lambda_2^2 + (3+\lambda_3)^2\\
&- \lambda_2(7+3\lambda_3))) + \alpha(k_1^2(\lambda_2^3 + \lambda_2\lambda_3(-7+10\lambda_3 - 6\lambda_3^2)\\
&- \lambda_2^2(-12+11\lambda_3+\lambda_3^2) + \lambda_3^2(-5+7\lambda_3^2)) + k_3^2(\lambda_2^3\lambda_3 + \lambda_2^2(2+6\lambda_3 - 4\lambda_3^2)\\
&+ 2(9-7\lambda_3+\lambda_3^2) + \lambda_2(-14-4\lambda_3+6\lambda_3^2+\lambda_3^3))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{322} = k_1k_2(&-\alpha(k_1^2(\lambda_2^3(1-3\lambda_3) + \lambda_2^2\lambda_3 + (2-5\lambda_3)\lambda_3^3 + 4\lambda_2\lambda_3^2(-1+2\lambda_3))\\
&+ k_2^2(-1+\lambda_3)(\lambda_3^2 - 5\lambda_3^3 + \lambda_2^2(3+\lambda_3) + 4\lambda_2(-1+\lambda_3^2)) - k_3^2(\lambda_3(1+2\lambda_3)\\
&+ \lambda_2^3(-3+5\lambda_3) + \lambda_2^2(4-5\lambda_3^2) - \lambda_2(5-2\lambda_3+\lambda_3^2)))\\
&+ R^2(\lambda_2^2(8-4\alpha(-1+\lambda_3)) + 3\alpha\lambda_2^3(-1+\lambda_3)\\
&+ \lambda_3^2(-8+\alpha-6\alpha\lambda_3+5\alpha\lambda_3^2) - \lambda_2(-1+\lambda_3)(16+\alpha(-5+8\lambda_3^2))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{323} = k_1k_3(&k_2^2\alpha(\lambda_3(-5+4\lambda_3-3\lambda_3^2) - \lambda_2^2(-2+\lambda_3+5\lambda_3^2) + \lambda_2(1+2\lambda_3+5\lambda_3^3))\\
&+ \alpha(k_3^2(-1+\lambda_2)(5\lambda_2^3 + (4-3\lambda_3)\lambda_3 - \lambda_2\lambda_3^2 - \lambda_2^2(1+4\lambda_3))\\
&+ k_1^2(5\lambda_2^4 + 4\lambda_2^2\lambda_3 - \lambda_3^3 + \lambda_2\lambda_3^2(-1+3\lambda_3) - 2\lambda_2^3(1+4\lambda_3)))\\
&+ R^2(5\alpha\lambda_2^4 - 2\alpha\lambda_2^3(3+4\lambda_3) + \lambda_2^2(-8+\alpha+8\alpha\lambda_3) + \lambda_3(8(2+\lambda_3)\\
&+ \alpha(-5+4\lambda_3-3\lambda_3^2)) + \lambda_2\lambda_3(-16+\alpha(5-4\lambda_3+3\lambda_3^2))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{324} = k_1(&-8R^2(\lambda_2-\lambda_3)^2 + \alpha(k_3^2(-1+\lambda_2)(-\lambda_2^2 + 5\lambda_2^3 + \lambda_2(2-3\lambda_3)\lambda_3\\
&- \lambda_3(1+2\lambda_3)) - k_2^2(-1+\lambda_3)(\lambda_2 - 2\lambda_2\lambda_3 + \lambda_3^2 - 5\lambda_3^3 + \lambda_2^2(2+3\lambda_3))\\
&+ (\lambda_2-\lambda_3)^2(R^2(1+5\lambda_2^2 + 6\lambda_2(-1+\lambda_3) - 6\lambda_3 + 5\lambda_3^2)\\
&+ k_1^2(5\lambda_2^2 + \lambda_3(-2+5\lambda_3) + \lambda_2(-2+6\lambda_3)))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{421} = -k_3(&-24R^4 + 9R^4\alpha + 16R^4\lambda_2 - 12R^4\alpha\lambda_2 + 8R^4\lambda_2^2 - 5k_1^2R^2\alpha\lambda_2^2\\
&- 2R^4\alpha\lambda_2^2 + 4R^4\alpha\lambda_2^3 + 6k_1^4\alpha\lambda_2^4 + 7k_1^2R^2\alpha\lambda_2^4 + R^4\alpha\lambda_2^4\\
&+ k_3^4\alpha(-3+2\lambda_2+\lambda_2^2)^2 + 16R^4\lambda_3 - 10R^4\alpha\lambda_3 + 16k_1^2R^2\lambda_2\lambda_3 - 16R^4\lambda_2\lambda_3\\
&+ 5k_1^2R^2\alpha\lambda_2\lambda_3 + 18R^4\alpha\lambda_2\lambda_3 + 8k_1^2R^2\lambda_2^2\lambda_3 - 3k_1^2R^2\alpha\lambda_2^2\lambda_3 - 6R^4\alpha\lambda_2^2\lambda_3\\
&- 2k_1^4\alpha\lambda_2^3\lambda_3 - 2R^4\alpha\lambda_2^3\lambda_3 - 5k_1^4\alpha\lambda_2^4\lambda_3 - 5k_1^2R^2\alpha\lambda_2^4\lambda_3 - 24k_1^2R^2\lambda_3^2\\
&- 3R^4\alpha\lambda_3^2 + 4k_1^2R^2\alpha\lambda_2\lambda_3^2 - 2R^4\alpha\lambda_2\lambda_3^2 - k_1^4\alpha\lambda_2^2\lambda_3^2 + 5R^4\alpha\lambda_2^2\lambda_3^2 + 6k_1^4\alpha\lambda_2^3\lambda_3^2\\
&+ 6k_1^2R^2\alpha\lambda_2^3\lambda_3^2 - k_1^2R^2\alpha\lambda_3^3 + 4R^4\alpha\lambda_3^3 - 12k_1^4\alpha\lambda_2\lambda_3^3 - 17k_1^2R^2\alpha\lambda_2\lambda_3^3\\
&- 4R^4\alpha\lambda_2\lambda_3^3 - k_1^4\alpha\lambda_2^2\lambda_3^3 - k_1^2R^2\alpha\lambda_2^2\lambda_3^3 + 9k_1^4\alpha\lambda_3^4 + 10k_1^2R^2\alpha\lambda_3^4\\
&+ k_2^4\alpha(-1+\lambda_3)(-9+\lambda_3+4\lambda_3^2 + \lambda_2(2+\lambda_3+\lambda_3^2)) + k_3^2(k_1^2\alpha(\lambda_2^4(7-5\lambda_3)\\
&+ 18\lambda_3^2 - 12\lambda_2\lambda_3(1+\lambda_3) + 2\lambda_2^3\lambda_3(3+\lambda_3) + \lambda_2^2(-5+7\lambda_3-6\lambda_3^2))\\
&+ R^2(-1+\lambda_2)(8(3+\lambda_2) + \alpha(-1+\lambda_2)(18+2\lambda_2^2 - 2\lambda_2(-6+\lambda_3)
\end{aligned}
$$

$$
\begin{aligned}
&- 10\lambda_3 + \lambda_3^2))) + k_2^2(R^2(8(-3 + (2 + \lambda_2)\lambda_3)\\
&+ \alpha(-1 + \lambda_3)(\lambda_2^2(-2 + 4\lambda_3) + \lambda_2(14 - 5\lambda_3 - 3\lambda_3^2) + 2(-9 + \lambda_3 + 4\lambda_3^2)))\\
&+ \alpha(k_3^2(18 - 10\lambda_3 + \lambda_2^3\lambda_3 + \lambda_3^2 + \lambda_2^2(2 + 4\lambda_3 + \lambda_3^2) + \lambda_2(-14 - 7\lambda_3 + 4\lambda_3^2))\\
&+ k_1^2(\lambda_2^3 + \lambda_3^3(-1 + 10\lambda_3) + \lambda_2^2(12 - 5\lambda_3 + 2\lambda_3^2 - 2\lambda_3^3)\\
&- \lambda_2\lambda_3(11 + 5\lambda_3^2 + \lambda_3^3))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{422} = {}& -k_1k_2k_3(-1 + \lambda_3)(\alpha(k_3^2(-5\lambda_2(-1 + \lambda_3) + 4\lambda_3 - 4\lambda_2^2(1 + \lambda_3)\\
&+ \lambda_2^3(3 + \lambda_3)) + k_1^2(5\lambda_2(-1 + \lambda_3)\lambda_3^2 + 4\lambda_3^3 - 4\lambda_2^2\lambda_3(1 + \lambda_3)\\
&+ \lambda_2^3(1 + 3\lambda_3)) - k_2^2(-4\lambda_2(-1 + \lambda_3)^2(1 + \lambda_3) + \lambda_3(-5 + 2\lambda_3 - 5\lambda_3^2)\\
&+ \lambda_2^2(3 + 2\lambda_3 + 3\lambda_3^2))) + R^2(3\alpha\lambda_2^3(1 + \lambda_3) - 4\lambda_2^2(2 + \alpha + \alpha\lambda_3^2)\\
&+ \lambda_2(1 + \lambda_3)(-16 + \alpha(5 - 13\lambda_3 + 5\lambda_3^2)) + \lambda_3(-8 + \alpha(5 - 2\lambda_3 + 5\lambda_3^2)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{423} = {}& k_1(k_3^4\alpha(-1 + \lambda_2)(-9\lambda_3 + 3\lambda_2\lambda_3 + \lambda_2^2(1 + 4\lambda_3) + \lambda_2^3(-5 + 6\lambda_3))\\
&- k_2^4\alpha(-1 + \lambda_3)\lambda_3(\lambda_3(4 + \lambda_3 - 9\lambda_3^2) + \lambda_2(1 + \lambda_3 + 2\lambda_3^2))\\
&+ (\lambda_2 - \lambda_3)\lambda_3(k_1^4\alpha(\lambda_2 - \lambda_3)(\lambda_2 + 3\lambda_3)^2 + k_1^2R^2(2\alpha\lambda_2^3 + 2\alpha\lambda_2^2(-1 + 5\lambda_3)\\
&+ \lambda_2(8 + \alpha - 8\alpha\lambda_3 + 6\alpha\lambda_3^2) - \lambda_3(-24 + \alpha - 10\alpha\lambda_3 + 18\alpha\lambda_3^2))\\
&+ R^4(8(-2 + \lambda_2 + 3\lambda_3) + \alpha(-4 + \lambda_2^3 + 3\lambda_3 + 10\lambda_3^2 - 9\lambda_3^3 + \lambda_2^2(-2 + 5\lambda_3)\\
&+ \lambda_2(5 - 8\lambda_3 + 3\lambda_3^2)))) + k_3^2(k_1^2\alpha(18\lambda_3^3 - 12\lambda_2\lambda_3^2(1 + \lambda_3) + \lambda_2^3(2 + 6\lambda_3)\\
&+ \lambda_2^4(-5 + 7\lambda_3) + \lambda_2^2\lambda_3(-6 + 7\lambda_3 - 5\lambda_3^2))\\
&+ R^2(6\alpha\lambda_2^3 - (24 + \alpha(-10 + \lambda_3))\lambda_3 + \alpha\lambda_2^4(-5 + 7\lambda_3) - \lambda_2^2(-8 + \alpha\\
&+ 3\alpha\lambda_3^2 + 5\alpha\lambda_3^3) + \lambda_2\lambda_3(16 + \alpha(-17 + 4\lambda_3 + 5\lambda_3^2))))\\
&+ k_2^2(k_3^2\alpha(-(-10 + \lambda_3)\lambda_3 + \lambda_2^3\lambda_3^2 - \lambda_2(1 + 5\lambda_3 + 11\lambda_3^3) + \lambda_2^2(-2 + 2\lambda_3\\
&- 5\lambda_3^2 + 12\lambda_3^3)) + \lambda_3(k_1^2\alpha(\lambda_2^3 + \lambda_2\lambda_3(4 - 7\lambda_3 - 14\lambda_3^2)\\
&+ \lambda_2^2(1 + 4\lambda_3 + 2\lambda_3^2) + \lambda_3^2(1 - 10\lambda_3 + 18\lambda_3^2)) + R^2(2\alpha\lambda_2^2(2 - 3\lambda_3 + \lambda_3^2)\\
&+ 2\lambda_3(8 - 12\lambda_3 + \alpha(4 - 3\lambda_3 - 10\lambda_3^2 + 9\lambda_3^3)) + \lambda_2(8 - \alpha(3 + 2\lambda_3\\
&- 19\lambda_3^2 + 14\lambda_3^3)))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{424} = {}& k_1k_3(-1 + \lambda_3)(k_2^2\alpha(\lambda_3(-5 + 2\lambda_3 - 5\lambda_3^2) + 2\lambda_2^2(1 + 3\lambda_3 + \lambda_3^2)\\
&+ \lambda_2(1 - 2\lambda_3 - 2\lambda_3^2 + \lambda_3^3)) + \alpha(k_3^2(-1 + \lambda_2)(5\lambda_2^3 + 4\lambda_3 + 3\lambda_2\lambda_3\\
&- \lambda_2^2(1 + 2\lambda_3)) + k_1^2(5\lambda_2^4 + \lambda_2\lambda_3^2 - 4\lambda_3^3 + \lambda_2^2\lambda_3(5 + \lambda_3) - 2\lambda_2^3(1 + 3\lambda_3)))\\
&+ R^2(5\alpha\lambda_2^4 - 6\alpha\lambda_2^3(1 + \lambda_3) + 2\alpha\lambda_2\lambda_3(1 + \lambda_3)\\
&+ \lambda_3(8 + \alpha(-5 + 2\lambda_3 - 5\lambda_3^2)) + \lambda_2^2(-8 + \alpha(1 + 9\lambda_3 + \lambda_3^2)))),\\
\mathfrak{P}_{431} = {}& k_2(-24R^4 + 9R^4\alpha + 16R^4\lambda_2 - 10R^4\alpha\lambda_2 - 24k_1^2R^2\lambda_2^2 - 3R^4\alpha\lambda_2^2\\
&- k_1^2R^2\alpha\lambda_2^3 + 4R^4\alpha\lambda_2^3 + 9k_1^4\alpha\lambda_2^4 + 10k_1^2R^2\alpha\lambda_2^4 + 16R^4\lambda_3 - 12R^4\alpha\lambda_3\\
&+ 16k_1^2R^2\lambda_2\lambda_3 - 16R^4\lambda_2\lambda_3 + 5k_1^2R^2\alpha\lambda_2\lambda_3 + 18R^4\alpha\lambda_2\lambda_3 + 4k_1^2R^2\alpha\lambda_2^2\lambda_3\\
&- 2R^4\alpha\lambda_2^2\lambda_3 - 12k_1^4\alpha\lambda_2^3\lambda_3 - 17k_1^2R^2\alpha\lambda_2^3\lambda_3 - 4R^4\alpha\lambda_2^3\lambda_3 + 8R^4\lambda_3^2\\
&- 5k_1^2R^2\alpha\lambda_3^2 - 2R^4\alpha\lambda_3^2 + 8k_1^2R^2\lambda_2\lambda_3^2 - 3k_1^2R^2\alpha\lambda_2\lambda_3^2 - 6R^4\alpha\lambda_2\lambda_3^2\\
&- k_1^4\alpha\lambda_2^2\lambda_3^2 + 5R^4\alpha\lambda_2^2\lambda_3^2 - k_1^4\alpha\lambda_2^3\lambda_3^2 - k_1^2R^2\alpha\lambda_2^3\lambda_3^2 + 4R^4\alpha\lambda_3^3 - 2k_1^4\alpha\lambda_2\lambda_3^3\\
&- 2R^4\alpha\lambda_2\lambda_3^3 + 6k_1^4\alpha\lambda_2^2\lambda_3^3 + 6k_1^2R^2\alpha\lambda_2^2\lambda_3^3 + 6k_1^4\alpha\lambda_3^4 + 7k_1^2R^2\alpha\lambda_3^4 + R^4\alpha\lambda_3^4\\
&- 5k_1^4\alpha\lambda_2\lambda_3^4 - 5k_1^2R^2\alpha\lambda_2\lambda_3^4 + k_2^4\alpha(-3 + 2\lambda_3 + \lambda_3^2)^2\\
&+ k_3^4\alpha(-1 + \lambda_2)(-9 + 2\lambda_3 + \lambda_2(1 + \lambda_3) + \lambda_2^2(4 + \lambda_3))
\end{aligned}
$$

$$
\begin{aligned}
&- k_3^2(k_1^2\alpha(\lambda_2^4(-10+\lambda_3) - 2\lambda_2^2\lambda_3^2 - \lambda_3^2(12+\lambda_3) + \lambda_2\lambda_3(11+5\lambda_3)\\
&+ \lambda_2^3(1+5\lambda_3+2\lambda_3^2)) + R^2(-8(-3+\lambda_2(2+\lambda_3))\\
&+ \alpha(-1+\lambda_2)(\lambda_2^2(-8+3\lambda_3) + \lambda_2(-2+5\lambda_3-4\lambda_3^2) + 2(9-7\lambda_3+\lambda_3^2))))\\
&+ k_2^2(R^2(-1+\lambda_3)(8(3+\lambda_3) + \alpha(-1+\lambda_3)(\lambda_2^2 + 2(3+\lambda_3)^2 - 2\lambda_2(5+\lambda_3)))\\
&+ \alpha(k_1^2(\lambda_3^2(-5+7\lambda_3^2) + \lambda_2\lambda_3(-12+7\lambda_3+6\lambda_3^2-5\lambda_3^3)\\
&+ 2\lambda_2^2(9-6\lambda_3-3\lambda_3^2+\lambda_3^3)) + k_3^2(2(9-7\lambda_3+\lambda_3^2) + \lambda_2^2(1+4\lambda_3+\lambda_3^2)\\
&+ \lambda_2(-10-7\lambda_3+4\lambda_3^2+\lambda_3^3))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{432} = &-k_1(k_2^4\alpha(-1+\lambda_3)(\lambda_3^2 - 5\lambda_3^3 + \lambda_2(-9+3\lambda_3+4\lambda_3^2+6\lambda_3^3))\\
&- k_2^2(\alpha(k_1^2(\lambda_2^2(12-7\lambda_3)\lambda_3 + \lambda_3^3(-2+5\lambda_3) + \lambda_2\lambda_3^2(6-6\lambda_3-7\lambda_3^2)\\
&+ \lambda_2^3(-18+12\lambda_3+5\lambda_3^2)) + k_3^2(\lambda_2^3(11-12\lambda_3)\lambda_3 + \lambda_3(1+2\lambda_3)\\
&+ \lambda_2(-10+5\lambda_3-2\lambda_3^2) + \lambda_2^2(1+5\lambda_3^2-\lambda_3^3))) + R^2(5\alpha\lambda_2^3(-1+\lambda_3)\lambda_3\\
&+ \alpha\lambda_2^2(1-4\lambda_3+3\lambda_3^2) + \lambda_3^2(-8+\alpha-6\alpha\lambda_3+5\alpha\lambda_3^2) + \lambda_2(24-16\lambda_3\\
&+ \alpha(-10+17\lambda_3-7\lambda_3^4)))) + \lambda_2(k_3^4\alpha(4\lambda_2+9\lambda_2^4+\lambda_2^2(-3+\lambda_3)+\lambda_3\\
&- 2\lambda_2^3(5+\lambda_3)) + (\lambda_2-\lambda_3)(k_1^4\alpha(\lambda_2-\lambda_3)(3\lambda_2+\lambda_3)^2\\
&+ k_1^2R^2(18\alpha\lambda_2^3 - 2\alpha\lambda_2^2(5+3\lambda_3) + \lambda_2(-24+\alpha+8\alpha\lambda_3-10\alpha\lambda_3^2)\\
&- \lambda_3(8+\alpha-2\alpha\lambda_3+2\alpha\lambda_3^2)) + R^4(-8(-2+3\lambda_2+\lambda_3) + \alpha(4+9\lambda_2^3-5\lambda_3\\
&+ 2\lambda_3^2 - \lambda_3^3 - \lambda_2^2(10+3\lambda_3) + \lambda_2(-3+8\lambda_3-5\lambda_3^2))))\\
&+ k_3^2(k_1^2\alpha(18\lambda_2^4 + 4\lambda_2\lambda_3(1+\lambda_3) + \lambda_3^2(1+\lambda_3) - 2\lambda_2^3(5+7\lambda_3)\\
&+ \lambda_2^2(1-7\lambda_3+2\lambda_3^2)) + R^2(18\alpha\lambda_2^4 - 2\alpha\lambda_2^3(10+7\lambda_3)\\
&+ \lambda_3(8+\alpha(-3+4\lambda_3)) + \lambda_2^2(-24+\alpha(-6+19\lambda_3+2\lambda_3^2))\\
&- 2\lambda_2(-8+\alpha(-4+\lambda_3+3\lambda_3^2)))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{433} = &\, k_1k_2k_3(-1+\lambda_2)(\alpha(k_1^2(\lambda_3^3 + \lambda_2\lambda_3^2(-4+3\lambda_3) - \lambda_2^2\lambda_3(5+4\lambda_3)\\
&+ \lambda_2^3(4+5\lambda_3)) + k_3^2((4-3\lambda_3)\lambda_3 + \lambda_2^3(5+4\lambda_3) + \lambda_2(5-4\lambda_3-2\lambda_3^2)\\
&- \lambda_2^2(2+4\lambda_3+3\lambda_3^2)) + k_2^2(\lambda_3(5-4\lambda_3+3\lambda_3^2)\\
&+ \lambda_2(4-5\lambda_3-4\lambda_3^2+\lambda_3^3))) + R^2(5\alpha\lambda_2^3(1+\lambda_3) - 2\alpha\lambda_2^2(1+4\lambda_3+2\lambda_3^2)\\
&+ \lambda_3(-8(2+\lambda_3) + \alpha(5-4\lambda_3+3\lambda_3^2)) + \lambda_2(-8(1+2\lambda_3) + \alpha(5-8\lambda_3+3\lambda_3^3)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{434} = &\, k_1k_2(-1+\lambda_2)(\alpha(k_1^2(4\lambda_2^3 + (2-5\lambda_3)\lambda_3^3 - \lambda_2^2\lambda_3(1+\lambda_3)\\
&+ \lambda_2\lambda_3^2(-5+6\lambda_3)) + k_2^2(-1+\lambda_3)(\lambda_3^2 - 5\lambda_3^3 + \lambda_2(-4-3\lambda_3+2\lambda_3^2))\\
&- k_3^2(\lambda_2^3(-5+\lambda_3) + \lambda_3(1+2\lambda_3) + 2\lambda_2^2(1-\lambda_3+\lambda_3^2)\\
&+ \lambda_2(-5-2\lambda_3+6\lambda_3^2))) + R^2(5\alpha\lambda_2^3 - \alpha\lambda_2^2(2+2\lambda_3+\lambda_3^2)\\
&- \lambda_3^2(-8+\alpha-6\alpha\lambda_3+5\alpha\lambda_3^2) + \lambda_2(-8+\alpha(5-2\lambda_3-9\lambda_3^2+6\lambda_3^3)))),
\end{aligned}
$$

The expressions of the Ricci tensor and the $\check{R}$-tensor imply that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ holds true. We consider separately the cases corresponding to different possibilities (up to rotation) in the constants k_1, k_2 and k_3 as follows.

$k_1k_2k_3 \neq 0$

Since all the k_i's and λ_i's are different from zero, we simplify (when possible) the polynomials $\{\mathfrak{P}_{ijk}\} \subset \mathbb{R}[k_1, k_2, k_3, R, \alpha, \lambda_2, \lambda_3]$. Constructing a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\{\mathfrak{P}_{ijk}\}$ with respect to the graded reverse lexicographical order we get that the polynomials $\mathbf{g}_{11} = (\lambda_3 - 1)R^4$ and $\mathbf{g}_{12} = (\lambda_2 - 1)R^4$ belong to $\mathcal{G}_1$. Thus $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and hence the manifold is symmetric.

$k_1 = 0$ **and** $k_2k_3 \neq 0$

Proceeding as in the previous case, compute a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\} \subset \mathbb{R}[k_2, k_3, R, \alpha, \lambda_2, \lambda_3]$ with respect to the lexicographical order. Since the polynomials $\mathbf{g}_{21} = (\lambda_3 - 1)^2R^4$ and $\mathbf{g}_{22} = (\lambda_2 + \lambda_3 - 2)R^4$ belong to $\mathcal{G}_2$, one has that $\lambda_3 = \lambda_2 = 1$, which corresponds to the previous situation.

$k_1 = k_2 = 0$

Simplifying the polynomials $\{\mathfrak{P}_{ijk}\}$ when possible as in the previous cases and computing a Gröbner basis $\mathcal{G}_3$ of the ideal generated by $\{\mathfrak{P}_{ijk}\} \subset \mathbb{R}[k_3, R, \alpha, \lambda_2, \lambda_3]$ with respect to the graded reverse lexicographical order, one gets that the polynomial $\mathbf{g}_{31} = k_3^3(\lambda_2 - 1)^2R^2$ belongs to $\mathcal{G}_3$. Hence, either $k_3 = 0$ or $\lambda_2 = 1$ and, in both cases, e_4 determines a parallel left-invariant vector field. Now, a direct calculation shows that, in this case, any non-symmetric RG2 algebraic steady soliton is determined by Lemma 6.7-(3), obtaining the case given in Lemma 6.11. Finally, the tensor field $RG[g]$ vanishes, which finishes the proof. □

6.2.2 The semi-direct products $\mathbb{R}e_4 \ltimes E(1,1)$ and $\mathbb{R}e_4 \ltimes E(2)$

Recall that any Einstein metric is a genuine fixed point of the RG2 flow. Moreover, the product manifold $\mathbb{R} \times E(1,1)$ is an RG2 algebraic steady soliton just considering the RG2 algebraic steady solitons in Lemma 6.7-(1). Henceforth we focus on the irreducible non-Einstein case.

Lemma 6.12. *Let G be a semi-direct product $\mathbb{R} \ltimes E(1,1)$ or $\mathbb{R} \ltimes E(2)$. Then G admits a non-Einstein irreducible $RG2$ algebraic steady soliton if and only if it is homothetic to the Lie group $\mathbb{R} \ltimes E(1,1)$ determined by*

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$

where $\kappa > 0$, $\kappa \neq 1$ and for a coupling constant $\alpha = \frac{2}{\kappa^2+1}$. Here $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. Moreover, these metrics are never algebraic Ricci solitons.

Proof. Take the algebra given in (4.2). Since $\lambda_1\lambda_2 \neq 0$, we work with a homothetic basis $\hat{e}_i = \frac{1}{\lambda_1}e_i$ so that we may assume $\lambda_1 = 1$. The expressions of the Ricci tensor and the $\check{R}$-tensor imply that $\mathfrak{D} = Q + \frac{\alpha}{4}Q_{\check{R}}$ is a derivation of the Lie algebra if and only if the system of polynomial equations

$$\begin{aligned}
&16R^4\mathfrak{D}_{211}=\mathfrak{P}_{211}, && 16R^4\mathfrak{D}_{212}=\mathfrak{P}_{212}, && \mathfrak{D}_{213}=0, && \mathfrak{D}_{214}=0,\\
&16R^4\mathfrak{D}_{311}=\mathfrak{P}_{311}, && 16R^4\mathfrak{D}_{312}=\mathfrak{P}_{312}, && 16R^4\mathfrak{D}_{313}=\mathfrak{P}_{313}, && 16R^3\mathfrak{D}_{314}=\mathfrak{P}_{314},\\
&16R^5\mathfrak{D}_{411}=\mathfrak{P}_{411}, && 16R^5\mathfrak{D}_{412}=\mathfrak{P}_{412}, && 16R^5\mathfrak{D}_{413}=\mathfrak{P}_{413}, && 16R^4\mathfrak{D}_{414}=\mathfrak{P}_{414},\\
&16R^4\mathfrak{D}_{321}=\mathfrak{P}_{321}, && 16R^4\mathfrak{D}_{322}=\mathfrak{P}_{322}, && 16R^4\mathfrak{D}_{323}=\mathfrak{P}_{323}, && 16R^3\mathfrak{D}_{324}=\mathfrak{P}_{324},\\
&16R^5\mathfrak{D}_{421}=\mathfrak{P}_{421}, && 16R^5\mathfrak{D}_{422}=\mathfrak{P}_{422}, && 16R^5\mathfrak{D}_{423}=\mathfrak{P}_{423}, && 16R^4\mathfrak{D}_{424}=\mathfrak{P}_{424},\\
&16R^5\mathfrak{D}_{431}=\mathfrak{P}_{431}, && 16R^5\mathfrak{D}_{432}=\mathfrak{P}_{432}, && 16R^5\mathfrak{D}_{433}=\mathfrak{P}_{433}, && 16R^4\mathfrak{D}_{434}=\mathfrak{P}_{434},
\end{aligned}$$

holds true, where $\mathfrak{P}_{ijk}\in\mathbb{R}[A,b,\lambda_2,C,D,R,\alpha]$ are the polynomials associated to the coefficients $\mathfrak{D}_{ijk}$ which are given by

$$\begin{aligned}
\mathfrak{P}_{211}={}&b(C^3\alpha(-9+2\lambda_2)+AbD\alpha(-15+18\lambda_2+\lambda_2^2)+C(\alpha(2b^2(-7+\lambda_2)\\
&-D^2(6+\lambda_2)+A^2(-9+8\lambda_2+\lambda_2^2))+R^2(32+\alpha(-9+8\lambda_2+\lambda_2^2)))),\\
\mathfrak{P}_{212}={}&b(A^2D\alpha\lambda_2(1+8\lambda_2-9\lambda_2^2)+AbC\alpha(-1-18\lambda_2+15\lambda_2^2)\\
&+D(2D^2\alpha+2b^2\alpha(1-7\lambda_2)+32R^2\lambda_2-9D^2\alpha\lambda_2+R^2\alpha\lambda_2+8R^2\alpha\lambda_2^2\\
&-9R^2\alpha\lambda_2^3-C^2(\alpha+6\alpha\lambda_2))),\\
\mathfrak{P}_{311}={}&A^2CD\alpha\lambda_2(-3+4\lambda_2-5\lambda_2^2)+A^3b\alpha(-3+2\lambda_2+\lambda_2^2)^2\\
&+CD(R^2(8+(16-3\alpha)\lambda_2+4\alpha\lambda_2^2-5\alpha\lambda_2^3)-\alpha(C^2\lambda_2+2b^2(-5+3\lambda_2)\\
&+D^2(-3+4\lambda_2)))+Ab(R^2(-3+2\lambda_2+\lambda_2^2)(8+\alpha(-3+2\lambda_2+\lambda_2^2))\\
&-\alpha(2D^2(1+2\lambda_2)+C^2(-9+4\lambda_2-5\lambda_2^2)+4b^2(-7+6\lambda_2+\lambda_2^2))),\\
\mathfrak{P}_{312}={}&-C^2D^2\alpha+2D^4\alpha+24D^2R^2\lambda_2-8R^4\lambda_2-6C^4\alpha\lambda_2-12C^2D^2\alpha\lambda_2\\
&-9D^4\alpha\lambda_2-7C^2R^2\alpha\lambda_2-2D^2R^2\alpha\lambda_2-R^4\alpha\lambda_2-16R^4\lambda_2^2+14D^2R^2\alpha\lambda_2^2\\
&-4R^4\alpha\lambda_2^2+24R^4\lambda_2^3+5C^2R^2\alpha\lambda_2^3-18D^2R^2\alpha\lambda_2^3+2R^4\alpha\lambda_2^3+12R^4\alpha\lambda_2^4\\
&-9R^4\alpha\lambda_2^5-A^4\alpha\lambda_2(1+2\lambda_2-3\lambda_2^2)^2+AbCD\alpha(-1-18\lambda_2+35\lambda_2^2)\\
&-2b^2\alpha(2(3C^2+R^2(-1+\lambda_2)^2)\lambda_2+D^2(-1+5\lambda_2))\\
&-A^2\lambda_2(2R^2(-1-2\lambda_2+3\lambda_2^2)(-4+\alpha(-1-2\lambda_2+3\lambda_2^2))\\
&+\alpha(C^2(7-5\lambda_2^2)+4b^2(-1-6\lambda_2+7\lambda_2^2)+2D^2(1-7\lambda_2+9\lambda_2^2))),\\
\mathfrak{P}_{313}={}&\lambda_2(A^3C\alpha(5-6\lambda_2+\lambda_2^2)+A^2bD\alpha(7-10\lambda_2+3\lambda_2^2)\\
&+AC(\alpha(b^2(12-8\lambda_2)+C^2(5-2\lambda_2)+D^2(2+\lambda_2))\\
&+R^2(-8+\alpha(5-6\lambda_2+\lambda_2^2)))+2bD(2(3b^2+C^2+D^2)\alpha\\
&+R^2(-12+\alpha(-1-\lambda_2+2\lambda_2^2)))),\\
\mathfrak{P}_{314}={}&\lambda_2(C^3\alpha(5-2\lambda_2)-AbD\alpha(-9+8\lambda_2+\lambda_2^2)+C(\alpha(-2b^2(-1+\lambda_2)\\
&+D^2(2+\lambda_2)+A^2(5-6\lambda_2+\lambda_2^2))+R^2(-8+\alpha(5-6\lambda_2+\lambda_2^2)))),\\
\mathfrak{P}_{411}={}&-A^4b\alpha(-3+2\lambda_2+\lambda_2^2)^2+A^3CD\alpha\lambda_2(3-4\lambda_2+5\lambda_2^2)\\
&+ACD(R^2(-8+(-16+3\alpha)\lambda_2-4\alpha\lambda_2^2+5\alpha\lambda_2^3)+\alpha(34b^2(-1+\lambda_2)\\
&+C^2\lambda_2+D^2(-3+4\lambda_2)))\\
&-A^2b(R^2(-3+2\lambda_2+\lambda_2^2)(8+\alpha(-3+2\lambda_2+\lambda_2^2))\\
&+\alpha(8b^2(5-6\lambda_2+\lambda_2^2)+2C^2(9-5\lambda_2+2\lambda_2^2)+D^2(-1-8\lambda_2+5\lambda_2^2)))
\end{aligned}$$

$$
\begin{aligned}
&- b(16b^4\alpha + 9C^4\alpha + 4b^2(3(2C^2 + D^2)\alpha + R^2(-8 + \alpha(-1 + \lambda_2)^2)) \\
&+ D^2(5D^2\alpha + R^2(-8 + \alpha - 4\alpha\lambda_2 + 5\alpha\lambda_2^2)) \\
&- C^2(-14D^2\alpha + R^2(32 + \alpha(-9 + 6\lambda_2 + \lambda_2^2)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{412} = &-12b^3CD\alpha + 16Ab^4\alpha\lambda_2 - bCD(4R^2(-6 + \alpha(-1 + \lambda_2)^2) \\
&+ \alpha(4(C^2 + D^2) + A^2(3 - 26\lambda_2 + 39\lambda_2^2))) + A(6C^4\alpha\lambda_2 + D^4\alpha(-2 + 9\lambda_2) \\
&+ C^2\alpha(D^2(1 + 12\lambda_2) - (A^2 + R^2)\lambda_2(-7 + 5\lambda_2^2)) \\
&+ 2D^2\lambda_2(A^2\alpha(1 - 7\lambda_2 + 9\lambda_2^2) + R^2(-12 + \alpha - 7\alpha\lambda_2 + 9\alpha\lambda_2^2)) \\
&+ (A^2 + R^2)\lambda_2(-1 - 2\lambda_2 + 3\lambda_2^2)(A^2\alpha(-1 - 2\lambda_2 + 3\lambda_2^2) \\
&+ R^2(-8 + \alpha(-1 - 2\lambda_2 + 3\lambda_2^2)))) + 2Ab^2(D^2\alpha(-4 + 11\lambda_2) \\
&+ \lambda_2(8R^2(-2 + \alpha(-1 + \lambda_2)^2) + \alpha(7C^2 + 4A^2(1 - 6\lambda_2 + 5\lambda_2^2)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{413} = &\ 12b^4C\alpha + 8Ab^3D\alpha(1 - 3\lambda_2) - A^2C\lambda_2(\alpha(C^2(5 - 2\lambda_2) + D^2(2 + \lambda_2) \\
&+ A^2(5 - 6\lambda_2 + \lambda_2^2)) + R^2(-8 + \alpha(5 - 6\lambda_2 + \lambda_2^2))) \\
&+ b^2C(-2R^2(12 + \alpha(-2 + \lambda_2 + \lambda_2^2)) + \alpha(4(C^2 + D^2) \\
&+ A^2(3 - 22\lambda_2 + 15\lambda_2^2))) + AbD(D^2\alpha(2 - 9\lambda_2) - C^2(\alpha + 6\alpha\lambda_2) \\
&+ \lambda_2(-8A^2\alpha(-1 + \lambda_2)^2 + R^2(32 + \alpha + 8\alpha\lambda_2 - 9\alpha\lambda_2^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{414} = &-A^3C\alpha\lambda_2(5 - 6\lambda_2 + \lambda_2^2) - 2A^2bD\alpha\lambda_2(5 - 7\lambda_2 + 2\lambda_2^2) \\
&- bD(-2D^2\alpha + 2b^2\alpha(-1 + \lambda_2) - 8R^2\lambda_2 + 5D^2\alpha\lambda_2 + R^2\alpha\lambda_2 - 6R^2\alpha\lambda_2^2 \\
&+ 5R^2\alpha\lambda_2^3 + C^2(\alpha + 2\alpha\lambda_2)) - AC(b^2\alpha(1 + 10\lambda_2 - 11\lambda_2^2) \\
&+ \lambda_2(\alpha(C^2(5 - 2\lambda_2) + D^2(2 + \lambda_2)) + R^2(-8 + \alpha(5 - 6\lambda_2 + \lambda_2^2)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{321} = &-24R^4 + 12b^2D^2\alpha + 6D^4\alpha + 4b^2R^2\alpha - 5D^2R^2\alpha + 9R^4\alpha \\
&+ C^4\alpha(9 - 2\lambda_2) + 16R^4\lambda_2 - 8b^2R^2\alpha\lambda_2 - 12R^4\alpha\lambda_2 + 8R^4\lambda_2^2 + 4b^2R^2\alpha\lambda_2^2 \\
&+ 7D^2R^2\alpha\lambda_2^2 - 2R^4\alpha\lambda_2^2 + 4R^4\alpha\lambda_2^3 + R^4\alpha\lambda_2^4 + A^4\alpha(-3 + 2\lambda_2 + \lambda_2^2)^2 \\
&- AbCD\alpha(-35 + 18\lambda_2 + \lambda_2^2) + C^2(\alpha(-2b^2(-5 + \lambda_2) + D^2(12 + \lambda_2)) \\
&+ 2R^2(-12 + \alpha(9 - 7\lambda_2 + \lambda_2^2))) \\
&+ A^2(2R^2(-3 + 2\lambda_2 + \lambda_2^2)(4 + \alpha(-3 + 2\lambda_2 + \lambda_2^2)) \\
&+ \alpha(2C^2(9 - 7\lambda_2 + \lambda_2^2) - 4b^2(-7 + 6\lambda_2 + \lambda_2^2) + D^2(-5 + 7\lambda_2^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{322} = &\ A^3b\alpha(1 + 2\lambda_2 - 3\lambda_2^2)^2 + A^2CD\alpha(5 - 4\lambda_2 + 3\lambda_2^2) \\
&+ CD(\alpha(D^2 + b^2(6 - 10\lambda_2) + C^2(4 - 3\lambda_2)) + R^2(-8(2 + \lambda_2) \\
&+ \alpha(5 - 4\lambda_2 + 3\lambda_2^2))) \\
&+ Ab(R^2(-1 - 2\lambda_2 + 3\lambda_2^2)(-8 + \alpha(-1 - 2\lambda_2 + 3\lambda_2^2)) \\
&+ \alpha(-2C^2\lambda_2(2 + \lambda_2) + 4b^2(-1 - 6\lambda_2 + 7\lambda_2^2) + D^2(5 - 4\lambda_2 + 9\lambda_2^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{323} = &-12b^3C\alpha + 4Ab^2D\alpha(-2 + 3\lambda_2) + bC(2R^2(12 + \alpha(-2 + \lambda_2 + \lambda_2^2)) \\
&- \alpha(4(C^2 + D^2) + A^2(3 - 10\lambda_2 + 7\lambda_2^2))) + AD(D^2\alpha(-2 + 5\lambda_2) \\
&+ C^2(\alpha + 2\alpha\lambda_2) + \lambda_2(A^2\alpha(1 - 6\lambda_2 + 5\lambda_2^2) + R^2(-8 + \alpha - 6\alpha\lambda_2 + 5\alpha\lambda_2^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}_{324} = &-8DR^2\lambda_2 + \alpha(AbC(1 + 8\lambda_2 - 9\lambda_2^2) + A^2D\lambda_2(1 - 6\lambda_2 + 5\lambda_2^2) \\
&+ D(C^2 - 2D^2 + 2b^2(-1 + \lambda_2) + 2C^2\lambda_2 + 5D^2\lambda_2 + R^2\lambda_2 - 6R^2\lambda_2^2 + 5R^2\lambda_2^3)),
\end{aligned}
$$

$$
\mathfrak{P}_{421} = -4bCD((3b^2 + C^2 + D^2)\alpha + R^2(-6 + \alpha(-1 + \lambda_2)^2))
$$

$$
\begin{aligned}
&- A^5\alpha(-3+2\lambda_2+\lambda_2^2)^2 - A^2bCD\alpha(39-26\lambda_2+3\lambda_2^2)\\
&- A^3(2R^2(-3+2\lambda_2+\lambda_2^2)(4+\alpha(-3+2\lambda_2+\lambda_2^2))\\
&+ \alpha(2C^2(9-7\lambda_2+\lambda_2^2)+8b^2(5-6\lambda_2+\lambda_2^2)+D^2(-5+7\lambda_2^2)))\\
&- A(-24R^4+16b^4\alpha+6D^4\alpha-5D^2R^2\alpha+9R^4\alpha\\
&+ 2b^2(\alpha(7D^2+C^2(11-4\lambda_2))+8R^2(-2+\alpha(-1+\lambda_2)^2))\\
&+ C^4\alpha(9-2\lambda_2)+16R^4\lambda_2-12R^4\alpha\lambda_2+8R^4\lambda_2^2+7D^2R^2\alpha\lambda_2^2-2R^4\alpha\lambda_2^2\\
&+ 4R^4\alpha\lambda_2^3+R^4\alpha\lambda_2^4+C^2(D^2\alpha(12+\lambda_2)+2R^2(-12+\alpha(9-7\lambda_2+\lambda_2^2))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{422} = {}&-A^4b\alpha(1+2\lambda_2-3\lambda_2^2)^2 - A^3CD\alpha(5-4\lambda_2+3\lambda_2^2)\\
&+ ACD(\alpha(-D^2+34b^2(-1+\lambda_2)+C^2(-4+3\lambda_2))+R^2(8(2+\lambda_2)\\
&+ \alpha(-5+4\lambda_2-3\lambda_2^2)))\\
&- A^2b(R^2(-1-2\lambda_2+3\lambda_2^2)(-8+\alpha(-1-2\lambda_2+3\lambda_2^2))\\
&+ \alpha(-C^2(-5+8\lambda_2+\lambda_2^2)+8b^2(1-6\lambda_2+5\lambda_2^2)+2D^2(2-5\lambda_2+9\lambda_2^2)))\\
&- b(16b^4\alpha+5C^4\alpha+4b^2(3(C^2+2D^2)\alpha+R^2(-8+\alpha(-1+\lambda_2)^2))\\
&+ C^2(14D^2\alpha+R^2(-8+\alpha(5-4\lambda_2+\lambda_2^2)))\\
&+ D^2(9D^2\alpha+R^2(-32+\alpha(-1-6\lambda_2+9\lambda_2^2))))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{423} = {}&8A^3bC\alpha(-1+\lambda_2)^2 - A^4D\alpha\lambda_2(1-6\lambda_2+5\lambda_2^2) - A^2D(-2D^2\alpha-8R^2\lambda_2\\
&+ 5D^2\alpha\lambda_2+R^2\alpha\lambda_2-6R^2\alpha\lambda_2^2+5R^2\alpha\lambda_2^3+C^2(\alpha+2\alpha\lambda_2)\\
&+ b^2\alpha(-15+22\lambda_2-3\lambda_2^2)) + AbC(\alpha(C^2(9-2\lambda_2)-8b^2(-3+\lambda_2)\\
&+ D^2(6+\lambda_2))-R^2(32+\alpha(-9+8\lambda_2+\lambda_2^2)))\\
&+ 2b^2D(2(3b^2+C^2+D^2)\alpha+R^2(-12+\alpha(-1-\lambda_2+2\lambda_2^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{424} = {}&-2b^3C\alpha(-1+\lambda_2) - Ab^2D\alpha(-11+10\lambda_2+\lambda_2^2)\\
&+ bC(R^2(-8+\alpha(5-6\lambda_2+\lambda_2^2))+\alpha(C^2(5-2\lambda_2)+D^2(2+\lambda_2)\\
&+ 2A^2(2-7\lambda_2+5\lambda_2^2))) - AD(D^2\alpha(-2+5\lambda_2)+C^2(\alpha+2\alpha\lambda_2)\\
&+ \lambda_2(A^2\alpha(1-6\lambda_2+5\lambda_2^2)+R^2(-8+\alpha-6\alpha\lambda_2+5\alpha\lambda_2^2))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{431} = {}&A^4C\alpha(-9+10\lambda_2+3\lambda_2^2-4\lambda_2^3) + A^3bD\alpha(-3-\lambda_2+3\lambda_2^2+\lambda_2^3)\\
&+ AbD(-\alpha(5(D^2+C^2(8-7\lambda_2))+4b^2(1+\lambda_2))\\
&+ R^2(1+\lambda_2)(8+\alpha(-3+2\lambda_2+\lambda_2^2))) - A^2C(\alpha(D^2\lambda_2(-1+10\lambda_2)\\
&+ C^2(18-10\lambda_2+\lambda_2^2)+b^2(39-50\lambda_2+15\lambda_2^2))+2R^2(-12+8\lambda_2\\
&+ \alpha(9-10\lambda_2-3\lambda_2^2+4\lambda_2^3))) - C(-24D^2R^2-24R^4+12b^4\alpha+9C^4\alpha\\
&+ 9D^4\alpha+9R^4\alpha+16R^4\lambda_2-D^2R^2\alpha\lambda_2-10R^4\alpha\lambda_2+10D^2R^2\alpha\lambda_2^2\\
&- 3R^4\alpha\lambda_2^2+4R^4\alpha\lambda_2^3+C^2(18D^2\alpha+R^2(-24+\alpha(18-10\lambda_2+\lambda_2^2)))\\
&+ 2b^2(10(C^2+D^2)\alpha+R^2(-12+\alpha(7-10\lambda_2+3\lambda_2^2)))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{432} = {}&A^4D\alpha\lambda_2(-4+3\lambda_2+10\lambda_2^2-9\lambda_2^3) + A^3bC\alpha(-1-3\lambda_2+\lambda_2^2+3\lambda_2^3)\\
&- A^2D(D^2\alpha-C^2\alpha(-10+\lambda_2)+16R^2\lambda_2-10D^2\alpha\lambda_2+8R^2\alpha\lambda_2-24R^2\lambda_2^2\\
&+ 18D^2\alpha\lambda_2^2-6R^2\alpha\lambda_2^2-20R^2\alpha\lambda_2^3+18R^2\alpha\lambda_2^4+b^2\alpha(15-50\lambda_2+39\lambda_2^2))\\
&+ AbC(R^2(1+\lambda_2)(-8+\alpha(-1-2\lambda_2+3\lambda_2^2))+\alpha(4b^2(1+\lambda_2)\\
&+ 5(C^2\lambda_2+D^2(-7+8\lambda_2)))) - D(-24D^2R^2+12b^4\alpha+9C^4\alpha+9D^4\alpha\\
&+ D^2R^2\alpha+C^2(18D^2\alpha-R^2(24+\alpha(-10+\lambda_2)))+16R^4\lambda_2-10D^2R^2\alpha\lambda_2
\end{aligned}
$$

$$\begin{aligned}
&+ 4R^4\alpha\lambda_2 - 24R^4\lambda_2^2 + 18D^2R^2\alpha\lambda_2^2 - 3R^4\alpha\lambda_2^2 - 10R^4\alpha\lambda_2^3 + 9R^4\alpha\lambda_2^4 \\
&+ 2b^2(10(C^2+D^2)\alpha + R^2(-12 + \alpha(3 - 10\lambda_2 + 7\lambda_2^2)))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{433} = {}& 12b^3(C^2+D^2)\alpha - 20Ab^2CD\alpha(-1+\lambda_2) \\
&- ACD(-1+\lambda_2)(\alpha(4(C^2+D^2) + A^2(5 - 2\lambda_2 + 5\lambda_2^2)) \\
&+ R^2(-8 + \alpha(5 - 2\lambda_2 + 5\lambda_2^2))) \\
&+ b(4C^4\alpha + D^2(\alpha(4D^2 + A^2(7 - 10\lambda_2 + 3\lambda_2^2)) \\
&+ R^2(-24 + \alpha(-2 - 2\lambda_2 + 4\lambda_2^2))) \\
&+ C^2(-2R^2(12 + \alpha(-2 + \lambda_2 + \lambda_2^2)) + \alpha(8D^2 + A^2(3 - 10\lambda_2 + 7\lambda_2^2)))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{434} = {}& -(-1+\lambda_2)(4C^3D\alpha + AbD^2\alpha(9+\lambda_2) - AbC^2\alpha(1+9\lambda_2) \\
&+ CD(\alpha(4(b^2+D^2) + A^2(5 - 2\lambda_2 + 5\lambda_2^2)) + R^2(-8 + \alpha(5 - 2\lambda_2 + 5\lambda_2^2)))),
\end{aligned}$$

Thus, we look for solutions of the system $\{\mathfrak{P}_{ijk} = 0\}$. For that, we compute a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I} = \langle \mathfrak{P}_{ijk} \rangle$ with respect to the graded reverse lexicographical order and a detailed analysis of that basis shows that the polynomials

$$\mathbf{g}_1 = D^3(4b^2 + 25R^2)R^4, \qquad \mathbf{g}_2 = C(D^2 + C^2\lambda_2)R^4,$$
$$\mathbf{g}_3 = b\left\{3CDR^2 + Ab((\alpha+2)D^2 + 4(\lambda_2 - 1)R^2)\right\}R^2$$

belong to $\mathcal{G}$. Thus $C = D = 0$ and $4Ab^2(\lambda_2 - 1)R^4 = 0$, so we have three different possibilities corresponding to $b = 0$, $A = 0$ or $\lambda_2 = 1$. We consider the three situations separately.

$b = 0$

Constructing a Gröbner basis $\mathcal{G}_1$ of the ideal $\mathcal{G} \cup \{b\} \subset \mathbb{R}[A, b, \lambda_2, R, \alpha]$ with respect to the lexicographical order, one gets that the polynomial

$$\mathbf{g}_{11} = (\lambda_2 - 1)(\lambda_2 + 1)(\lambda_2 + 3)(3\lambda_2 + 1)\lambda_2(A^2 + R^2)R^2$$

belongs to $\mathcal{G}_1$. This shows that λ_2 must take one of the different values $\lambda_2 = 1$, $\lambda_2 = -1$, $\lambda_2 = -3$ or $\lambda_2 = -\frac{1}{3}$. If $\lambda_2 = 1$ then the metric is Einstein. We analyse the other three cases separately.

$\underline{\lambda_2 = -1}$. Considering the coefficient $R^4\mathfrak{D}_{312} = (A^2\alpha + (\alpha - 2)R^2)(A^2 + R^2)$, one has that $\alpha = \frac{2R^2}{A^2+R^2}$ and a straightforward calculation shows that, in this case, $\mathfrak{D}$ is a derivation of the Lie algebra. Moreover, setting $\gamma = -\frac{A}{R}$ one has the Lie algebra structure

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \gamma e_2, \quad [e_2, e_4] = \gamma e_1,$$

with $\alpha = \frac{2}{\gamma^2+1}$. A standard calculation shows that $v = e_4 - \gamma e_3$ determines a parallel left-invariant vector field on G. Therefore, G is a reducible RG2 algebraic steady soliton and one easily checks that it is obtained as a product extension of Lemma 6.7–(1.a).

$\underline{\lambda_2 = -3}$. Since $R^4\mathfrak{D}_{312} = 48(4A^2\alpha + (4\alpha - 1)R^2)(A^2 + R^2)$, we have $\alpha = \frac{R^2}{4(A^2+R^2)}$ and a straightforward calculation shows that, in this case, $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra. In this situation, setting $\kappa = -\frac{A}{R}$ one has

$$[e_1, e_3] = 3e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = 3\kappa e_2, \quad [e_2, e_4] = \kappa e_1,$$

with $\alpha = \frac{1}{4(\kappa^2+1)}$. Now, a direct calculation shows that $\ker \mathrm{Q} = \mathrm{span}\{e_4 - \kappa e_3\}$ and $v = e_4 - \kappa e_3$ is a parallel left-invariant vector field on G. Therefore, G is a reducible RG2 algebraic steady soliton which is obtained as a product extension of Lemma 6.7–(1.b).

$\lambda_2 = -\frac{1}{3}$. The coefficient $81R^4\mathfrak{D}_{321} = 16(4A^2\alpha + (4\alpha - 9)R^2)(A^2 + R^2)$ implies that $\alpha = \frac{9R^2}{4(A^2+R^2)}$ and a straightforward calculation shows that, in this case, $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra. Setting $\kappa = -\frac{A}{R}$ we are in the previous case just considering the homothety determined by $(e_1, e_2, e_3, e_4) \mapsto 3(e_2, e_1, e_3, e_4)$.

$A = 0$ **and** $b \neq 0$

Compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{G} \cup \{A\}$ with respect to the lexicographical order in $\mathbb{R}[R, A, b, \alpha, \lambda_2]$. We get that the polynomial $\mathbf{g}_{21} = (\lambda_2 - 1)(\lambda_2 + 1)^2\lambda_2\alpha^2 b^7$ belongs to $\mathcal{G}_2$ and thus $(\lambda_2 - 1)(\lambda_2 + 1) = 0$. If $\lambda_2 = 1$ then the manifold is symmetric and isometric to a product $\mathbb{R} \times N(c)$, where $N(c)$ is a space of constant negative curvature. On the other hand, if $\lambda_2 = -1$, then the coefficient $R^2\mathfrak{D}_{312} = (\alpha - 2)R^2 + b^2\alpha$ and thus $\alpha = \frac{2R^2}{b^2+R^2}$. A straightforward calculation shows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ defines an RG2 algebraic steady soliton where, setting $\kappa = -\frac{b}{R} \neq 0$, the left-invariant metric is determined by

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$

with $\alpha = \frac{2}{\kappa^2+1}$. Note that the replacement $e_4 \mapsto -e_4$ defines an isometry which interchanges κ and $-\kappa$. Hence, one may assume $\kappa > 0$ without loss of generality. Also, observe that the Ricci operator has eigenvalues $Q = -2\,\mathrm{diag}[\kappa^2, \kappa^2, 1, \kappa^2]$ and thus the metric is Einstein if and only if $\kappa^2 = 1$. Moreover, a direct calculation shows that these metrics are irreducible. Furthermore, the metric is a Ricci soliton if and only $Q + 2\,\mathrm{Id}$ is a derivation, which may occur if and only if $\kappa(\kappa^2 - 1) = 0$. Hence, it is a Ricci soliton if and only if it is Einstein. We conclude that these metrics correspond to the ones given in Lemma 6.12.

$\lambda_2 = 1$ **and** $bA \neq 0$

In this case the manifold is symmetric and isometric to a product $\mathbb{R} \times N(c)$, where $N(c)$ is a space of constant negative curvature, which finishes the proof. □

Remark 6.13. Left-invariant metrics determined by Lemma 6.12 define different homothetical classes for any $\kappa > 0$, $\kappa \neq 1$. This is obtained proceeding as in Remark 6.10 since $\tau = -(6\kappa^2+2)$ and $\|R\|^2 = 4(3\kappa^4 + 2\kappa^2 + 3)$.

6.2.3 The semi-direct product $\mathbb{R}e_4 \ltimes H^3$

In addition to Einstein metrics and symmetric products, $\mathbb{R} \times H^3$ is an RG2 algebraic steady soliton just considering the RG2 algebraic steady solitons in Lemma 6.7-(2). Henceforth we focus on the irreducible non-Einstein case.

Lemma 6.14. *Let G be a semi-direct product $\mathbb{R} \ltimes H^3$. Then G admits an irreducible non-Einstein $RG2$ algebraic steady soliton if and only if it is homothetic to one of the following, where $\{e_1, \dots, e_4\}$ is an orthonormal basis:*

1. *The left-invariant metric determined by*

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_1,$$
$$[e_2, e_4] = \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_2, \quad [e_3, e_4] = \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_3,$$

where $\kappa \in [-1, 1)$ and for a coupling constant $\alpha = 2$.

2. *The left-invariant metric determined by*

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa} e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right) e_3,$$

where $\kappa \in \left(0, \frac{1}{2}\right]$, $\kappa \neq \frac{1}{2}\sqrt{2-\sqrt{3}}$, and for a coupling constant $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$.

Moreover, metrics in case (1) are algebraic Ricci solitons whereas left-invariant metrics (2) are not Ricci solitons.

Remark 6.15. Left-invariant metrics in Lemma 6.14 corresponding to different values of the parameter κ determine different homothetical classes. The scalar curvature and the norm of the Ricci tensor of left-invariant metrics in Assertion (1) are given by $\tau = -\frac{5\kappa^2+8\kappa+5}{\kappa^2+\kappa+1}$ and $\|\rho\|^2 = -\frac{3}{2}\tau$, while for metrics in Assertion (2) one has $\tau = -\frac{48\kappa^4-16\kappa^2+3}{8\kappa^2}$ and $\|\rho\|^2 = \frac{768\kappa^8-512\kappa^6+224\kappa^4-32\kappa^2+3}{64\kappa^4}$. Now, proceeding as in Remark 6.10, a standard calculation shows that left-invariant metrics in Assertion (1) corresponding to different values of κ are never homothetic and the same holds true for left-invariant metrics in Assertion (2).

Proof. Take the algebra given in (4.3). Note that since $\gamma \neq 0$, one may work with a homothetic basis $\hat{e}_i = \frac{1}{\gamma} e_i$ so that we may assume $\gamma = 1$. It follows from the expressions obtained for the Ricci tensor and for the $\check{R}$-tensor that $\mathfrak{D} = Q + \frac{\alpha}{4} Q_{\check{R}}$ is a derivation of the Lie algebra if and only if the terms

$$\begin{array}{llll}
16R^4\mathfrak{D}_{211} = \mathfrak{P}_{211}, & 16R^4\mathfrak{D}_{212} = \mathfrak{P}_{212}, & 16R^4\mathfrak{D}_{213} = \mathfrak{P}_{213}, & 16R^3\mathfrak{D}_{214} = \mathfrak{P}_{214}, \\
16R^4\mathfrak{D}_{311} = \mathfrak{P}_{311}, & 16R^4\mathfrak{D}_{312} = \mathfrak{P}_{312}, & 16R^4\mathfrak{D}_{313} = \mathfrak{P}_{313}, & \mathfrak{D}_{314} = 0, \\
16R^5\mathfrak{D}_{411} = \mathfrak{P}_{411}, & 16R^5\mathfrak{D}_{412} = \mathfrak{P}_{412}, & 16R^5\mathfrak{D}_{413} = \mathfrak{P}_{413}, & 16R^4\mathfrak{D}_{414} = \mathfrak{P}_{414}, \\
16R^4\mathfrak{D}_{321} = \mathfrak{P}_{321}, & 16R^4\mathfrak{D}_{322} = \mathfrak{P}_{322}, & 16R^4\mathfrak{D}_{323} = \mathfrak{P}_{323}, & \mathfrak{D}_{324} = 0, \\
16R^5\mathfrak{D}_{421} = \mathfrak{P}_{421}, & 16R^5\mathfrak{D}_{422} = \mathfrak{P}_{422}, & 16R^5\mathfrak{D}_{423} = \mathfrak{P}_{423}, & 16R^4\mathfrak{D}_{424} = \mathfrak{P}_{424}, \\
16R^5\mathfrak{D}_{431} = \mathfrak{P}_{431}, & 16R^5\mathfrak{D}_{432} = \mathfrak{P}_{432}, & 16R^5\mathfrak{D}_{433} = \mathfrak{P}_{433}, & 16R^4\mathfrak{D}_{434} = \mathfrak{P}_{434},
\end{array}$$

are vanishing, where $\mathfrak{P}_{ijk} \in \mathbb{R}[a, c, d, H, F, R, \alpha]$ are the polynomials associated to the coefficients $\mathfrak{D}_{ijk}$ given by

$$\begin{aligned}
\mathfrak{P}_{211} &= 2c^2(2a+d)H\alpha + cF(4(a^2+4ad+6d^2+F^2+H^2)\alpha + R^2(-16+5\alpha)) \\
&\quad + H(-4a^3\alpha - 16a^2d\alpha - d(4R^2(-6+\alpha) + (12d^2+3F^2+4H^2)\alpha) \\
&\quad + a((-30d^2+H^2)\alpha + R^2(8+\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{212} &= -12a^3F\alpha - 4d^3F\alpha - 4cd^2H\alpha - 6a^2(5dF+4cH)\alpha + dF((4c^2+F^2)\alpha \\
&\quad + R^2(8+\alpha)) - cH(4(F^2+H^2)\alpha + R^2(-16+5\alpha)) - a(4F^3\alpha + 16cdH\alpha \\
&\quad + F(4R^2(-6+\alpha) + (-2c^2+16d^2+3H^2)\alpha)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{213} &= -24R^4 + 16a^2c^2\alpha - 32a^3d\alpha - 32ac^2d\alpha - 32a^2d^2\alpha + 16c^2d^2\alpha - 32ad^3\alpha \\
&\quad + 9F^4\alpha + 34c(a-d)FH\alpha + 9H^4\alpha + 4a^2R^2\alpha + 28adR^2\alpha + 4d^2R^2\alpha \\
&\quad + 9R^4\alpha + 2H^2((14a^2+12ad+5d^2)\alpha + 3R^2(-4+3\alpha)) \\
&\quad + 2F^2((5a^2+12ad+14d^2+9H^2)\alpha + 3R^2(-4+3\alpha)),
\end{aligned}$$

$$\mathfrak{P}_{214} = -(4a^2c + dFH - a(8cd+FH) + c(4d^2-F^2-H^2))\alpha,$$

$$\mathfrak{P}_{311} = -a(4a^2c + dFH - a(8cd+FH) + c(4d^2-F^2-H^2))\alpha,$$

$$\mathfrak{P}_{312} = c(4a^2c + dFH - a(8cd+FH) + c(4d^2-F^2-H^2))\alpha,$$

$$\begin{aligned}
\mathfrak{P}_{313} &= -8cHR^2 + 14a^3F\alpha + 28d^3F\alpha - 6cd^2H\alpha + 2a^2(25dF+8cH)\alpha \\
&\quad + dF((4c^2+9(F^2+H^2))\alpha + 3R^2(-8+3\alpha)) + a(9F^3\alpha + 22cdH\alpha \\
&\quad + F((-4c^2+60d^2+9H^2)\alpha + R^2(-32+9\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{411} &= -16a^5\alpha - 32a^4d\alpha + 4c^3FH\alpha + 4c^2d(-8R^2+4d^2\alpha+5F^2\alpha) \\
&\quad - 4a^3(R^2(-8+\alpha) + (8c^2+12d^2+3F^2+2H^2)\alpha) \\
&\quad + dH^2(4R^2(-6+\alpha) + (12d^2+3F^2+4H^2)\alpha) - cFH(9(4d^2+F^2+H^2)\alpha \\
&\quad + 2R^2(-12+5\alpha)) - 4a^2(8d^3\alpha + 12cFH\alpha + d(R^2(-8+\alpha) \\
&\quad - 2(6c^2-3F^2+H^2)\alpha)) + a(8F^2R^2 - 8H^2R^2 - 16d^4\alpha \\
&\quad - 5F^4\alpha - 4cdFH\alpha - 5F^2H^2\alpha - H^4\alpha - 5F^2R^2\alpha - H^2R^2\alpha - \\
&\quad 4d^2(R^2(-8+\alpha) + (6F^2-5H^2)\alpha) + 4c^2(8R^2 - (8d^2+2F^2+5H^2)\alpha)),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{412} &= 40a^4c\alpha - 34c^2dFH\alpha + 8a^3(4cd+3FH)\alpha + 2c^3(8d^2+3F^2+H^2)\alpha \\
&\quad + dFH((4d^2-F^2)\alpha - R^2(8+\alpha)) + 2a^2(8c^3\alpha + 15dFH\alpha \\
&\quad + c(-40R^2 + 32d^2\alpha + 4F^2\alpha + 31H^2\alpha)) + c(8d^4\alpha + 2d^2(4R^2(-2+\alpha) \\
&\quad + (9F^2+10H^2)\alpha) + H^2(9(F^2+H^2)\alpha + 2R^2(-12+5\alpha))) \\
&\quad + a(9F^3H\alpha + 4cd(-8c^2+8H^2+R^2)\alpha + FH(2(c^2+17d^2+4H^2)\alpha \\
&\quad + R^2(-32+9\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{413} &= -4c^3dF\alpha - 16a^4H\alpha - 4a^3(5cF+4dH)\alpha \\
&\quad - c^2H(-8R^2 + (20d^2+F^2+H^2)\alpha) + cdF(4R^2(-2+\alpha) \\
&\quad + (4d^2+5F^2+40H^2)\alpha) - 2a^2(12cdF\alpha + 20H^3\alpha + H(8R^2(-2+\alpha) \\
&\quad + (18c^2+16d^2+11F^2)\alpha)) - a(-4c^3F\alpha - 36c^2dH\alpha \\
&\quad + dH((16d^2+35F^2+36H^2)\alpha + 8R^2(-2+5\alpha)) \\
&\quad + cF((32d^2+6F^2+41H^2)\alpha + R^2(-40+6\alpha))) \\
&\quad - H(12d^4\alpha + 3(F^2+H^2+R^2)(3(F^2+H^2)\alpha + R^2(-8+3\alpha)) \\
&\quad + d^2((39F^2+20H^2)\alpha + 2R^2(-12+7\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{414} &= 2a^3F\alpha - 8c^2dF\alpha + dFH^2\alpha + 8a^2(2dF + 3cH)\alpha \\
&\quad + cH(2(3d^2 + 2(F^2 + H^2))\alpha + R^2(-8 + 5\alpha)) + a(5F^3\alpha + 6cdH\alpha \\
&\quad + F(2(c^2 + 6d^2 + 2H^2)\alpha + R^2(-8 + 5\alpha))),
\end{aligned}$$

$$\mathfrak{P}_{321} = -c(4a^2c + dFH - a(8cd + FH) + c(4d^2 - F^2 - H^2))\alpha,$$

$$\mathfrak{P}_{322} = -d(4a^2c + dFH - a(8cd + FH) + c(4d^2 - F^2 - H^2))\alpha,$$

$$\begin{aligned}
\mathfrak{P}_{323} &= 4c^2(-a + d)H\alpha + 2cF(-4R^2 + (-3a^2 + 11ad + 8d^2)\alpha) \\
&\quad - H(28a^3\alpha + 60a^2d\alpha + a((50d^2 + 9(F^2 + H^2))\alpha + 3R^2(-8 + 3\alpha)) \\
&\quad + d((14d^2 + 9(F^2 + H^2))\alpha + R^2(-32 + 9\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{421} &= -8a^4c\alpha + 4a^3FH\alpha + 2c^2dFH\alpha - 2c^3(8d^2 + F^2 + 3H^2)\alpha \\
&\quad + dFH((24d^2 + 8F^2 + 9H^2)\alpha + R^2(-32 + 9\alpha)) \\
&\quad - 2a^2(8c^3\alpha - 17dFH\alpha + c(4R^2(-2 + \alpha) + (32d^2 + 10F^2 + 9H^2)\alpha)) \\
&\quad - a(32cdF^2\alpha + 4cd(-8c^2 + 8d^2 + R^2)\alpha + FH((34c^2 - 30d^2 + H^2)\alpha \\
&\quad + R^2(8 + \alpha))) - c(40d^4\alpha + d^2(-80R^2 + 62F^2\alpha + 8H^2\alpha) \\
&\quad + F^2(9(F^2 + H^2)\alpha + 2R^2(-12 + 5\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{422} &= -16a^4d\alpha + 4a^3(4c^2 - 8d^2 + 3F^2)\alpha - 4c^3FH\alpha \\
&\quad - 4c^2d(-8R^2 + (8d^2 + 5F^2 + 2H^2)\alpha) + cFH(3(16d^2 + 3(F^2 + H^2))\alpha \\
&\quad + 2R^2(-12 + 5\alpha)) + a(-32d^4\alpha + 4cdFH\alpha + 4c^2(-8R^2 + 12d^2\alpha + 5H^2\alpha) \\
&\quad - 4d^2(R^2(-8 + \alpha) - 2(F^2 - 3H^2)\alpha) + F^2(4R^2(-6 + \alpha) + (4F^2 + 3H^2)\alpha)) \\
&\quad - 4a^2(12d^3\alpha - 9cFH\alpha + d(R^2(-8 + \alpha) + (8c^2 - 5F^2 + 6H^2)\alpha)) \\
&\quad - d(16d^4\alpha + F^4\alpha + 4d^2(R^2(-8 + \alpha) + (2F^2 + 3H^2)\alpha) \\
&\quad + F^2(5H^2\alpha + R^2(8 + \alpha)) + H^2(5H^2\alpha + R^2(-8 + 5\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{423} &= -12a^4F\alpha - 4c^3dH\alpha - 4a^3(4dF + cH)\alpha \\
&\quad - c^2F(-8R^2 + (36d^2 + F^2 + H^2)\alpha) + cdH((20d^2 + 41F^2 + 6H^2)\alpha \\
&\quad + R^2(-40 + 6\alpha)) - F(16d^4\alpha + 2d^2(8R^2(-2 + \alpha) + (20F^2 + 11H^2)\alpha) \\
&\quad + 3(F^2 + H^2 + R^2)(3(F^2 + H^2)\alpha + R^2(-8 + 3\alpha))) \\
&\quad - a(16d^3F\alpha - 24cd^2H\alpha + cH(4R^2(-2 + \alpha) + (-4c^2 + 40F^2 + 5H^2)\alpha) \\
&\quad + dF((-36c^2 + 36F^2 + 35H^2)\alpha + 8R^2(-2 + 5\alpha))) - a^2(20F^3\alpha - 32cdH\alpha \\
&\quad + F((20c^2 + 32d^2 + 39H^2)\alpha + 2R^2(-12 + 7\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{424} &= 2c^2(4a - d)H\alpha + cF(2(3a^2 + 3ad + 2(6d^2 + F^2 + H^2))\alpha \\
&\quad + R^2(-8 + 5\alpha)) - H(16ad^2\alpha + 2d^3\alpha + aF^2\alpha + d((12a^2 + 4F^2 + 5H^2)\alpha \\
&\quad + R^2(-8 + 5\alpha))),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{431} &= -4c^3dF\alpha - 4a^3(3cF - 4dH)\alpha \\
&\quad - 4a^2(8cdF + 3c^2H - 8d^2H)\alpha + c^2H(8R^2 + (-4d^2 + F^2 + H^2)\alpha) \\
&\quad + d^2H(4R^2(-6 + \alpha) + (12d^2 + 3F^2 + 4H^2)\alpha) - cdF(4R^2(-6 + \alpha) \\
&\quad + (28d^2 + 3F^2 + 4H^2)\alpha) + a(4c^3F\alpha - 4c^2dH\alpha - cF(4R^2(-6 + \alpha) \\
&\quad + (40d^2 + 4F^2 + 3H^2)\alpha) + dH(4R^2(-4 + \alpha) + (32d^2 + 5F^2 + 4H^2)\alpha)),
\end{aligned}$$

$$\mathfrak{P}_{432} = 12a^4F\alpha + 4a^3(8dF + 7cH)\alpha + a^2(4F^3\alpha + 40cdH\alpha + F(4R^2(-6 + \alpha)$$

$$
\begin{aligned}
&+(-4c^2+32d^2+3H^2)\alpha))+c(-4c^2dH\alpha\\
&+cF(8R^2+(-12d^2+F^2+H^2)\alpha)+dH(4R^2(-6+\alpha)\\
&+(12d^2+3F^2+4H^2)\alpha))+a(16d^3F\alpha+32cd^2H\alpha+cH(4R^2(-6+\alpha)\\
&+(4c^2+4F^2+3H^2)\alpha)+dF(4R^2(-4+\alpha)+(-4c^2+4F^2+5H^2)\alpha)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}_{433}={}&-16a^5\alpha-48a^4d\alpha-4a^3(R^2(-8+\alpha)+2(2c^2+10d^2+3F^2+5H^2)\alpha)\\
&-4a^2(20d^3\alpha+9cFH\alpha+d(2R^2(-8+\alpha)+(-4c^2+17F^2+20H^2)\alpha))\\
&-d(16d^4\alpha+9F^4\alpha-36cdFH\alpha+4d^2(R^2(-8+\alpha)\\
&+2(2c^2+5F^2+3H^2)\alpha)\\
&+F^2(2(4c^2+9H^2)\alpha+3R^2(-8+3\alpha))+H^2(9H^2\alpha+R^2(-32+9\alpha)))\\
&-a(48d^4\alpha+9F^4\alpha+4d^2(2R^2(-8+\alpha)+(-4c^2+20F^2+17H^2)\alpha)\\
&+H^2((8c^2+9H^2)\alpha+3R^2(-8+3\alpha))+F^2(18H^2\alpha+R^2(-32+9\alpha))),
\end{aligned}
$$

$$
\mathfrak{P}_{434}=(a+d)(4a^2c+dFH-a(8cd+FH)+c(4d^2-F^2-H^2))\alpha,
$$

We construct a Gröbner basis $\mathcal{G}$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\}$ with respect to the lexicographical order and we get that the polynomial $\mathbf{g}_1 = d^4FHR^2$ is in the basis. Therefore, we have three possibilities which we analyse separately.

$d=0$

Constructing a Gröbner basis $\mathcal{G}_1$ of the ideal generated by $\mathcal{G}\cup\{d\}\subset\mathbb{R}[a,c,d,H,F,R,\alpha]$ with respect to the lexicographical order, one has that the polynomials $\mathbf{g}_{11}=aHR^4$ and $\mathbf{g}_{12}=aFR^4$ are in $\mathcal{G}_1$. Thus, $a=0$ or $F=H=0$, $a\neq 0$.

$\underline{a=0.}$ We construct another Gröbner basis $\mathcal{G}_1'$ of the ideal generated by the previous basis and the new polynomial, namely, $\mathcal{G}_1\cup\{a\}\subset\mathbb{R}[a,c,d,H,F,R,\alpha]$ with respect to the lexicographical order and the polynomials $\mathbf{g}_{11}'=cFR^2$ and $\mathbf{g}_{12}'=cHR^2$ belong to $\mathcal{G}_1'$. Hence, either $c=0$ or $F=H=0$ and a standard calculation shows that $v=-\frac{F}{R}e_1+\frac{H}{R}e_2+e_4$ is a left-invariant parallel vector field on G in any case. Therefore, in this case any RG2 algebraic steady soliton is reducible and one easily checks that it is obtained as a product extension of Lemma 6.7-(2).

$\underline{F=H=0 \text{ and } a\neq 0.}$ Since $4R^4\mathfrak{D}_{131}=a^3c\alpha$, we get $c=0$ and thus $4R^5\mathfrak{D}_{343}=a^3(4a^2\alpha+(\alpha-8)R^2)$, which shows that $\alpha=\frac{8R^2}{4a^2+R^2}$. Now, a straightforward calculation shows that $\mathfrak{D}=Q+\frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if $a=\varepsilon\frac{\sqrt{3}}{2}R$, with $\varepsilon^2=1$. In this case, $\alpha=2$ and the left-invariant metric is determined by

$$
[e_1,e_2]=e_3,\quad [e_1,e_4]=-\varepsilon\tfrac{\sqrt{3}}{2}e_1,\quad [e_3,e_4]=-\varepsilon\tfrac{\sqrt{3}}{2}e_3\,.
$$

Note that the replacement $e_4\mapsto -e_4$ defines an isometry which interchanges $\varepsilon=-1$ with $\varepsilon=1$. Moreover, a direct calculation shows that this metric is never Einstein and that it is irreducible. Furthermore, a straightforward calculation shows that $Q+\frac{3}{2}\operatorname{Id}$ is a derivation of the Lie algebra and thus an algebraic Ricci soliton. Thus, taking $\varepsilon=-1$, the above left-invariant metric determines an RG2 algebraic steady soliton which corresponds to Assertion (1) with $\kappa=0$.

$H = 0$, $d \neq 0$

Computing a Gröbner basis $\mathcal{G}_2$ of the ideal generated by $\mathcal{G} \cup \{H\} \subset \mathbb{R}[a, c, d, H, F, R, \alpha]$ with respect to the lexicographical order, one has that the polynomials $\mathbf{g}_{21} = dFR^4(12F^2 + 7R^2)$ and $\mathbf{g}_{22} = (a - d)c^3R^4$ are in $\mathcal{G}_2$. Hence, $F = 0$ and either $a = d$ or $c = 0$, $a \neq d$.

<u>$F = 0$, $a = d$.</u> Construct a new Gröbner basis $\mathcal{G}_2'$ of the ideal generated by $\mathcal{G}_2 \cup \{F, a - d\} \subset \mathbb{R}[a, c, d, H, F, R, \alpha]$ with respect to the lexicographical order. We get that the polynomial $\mathbf{g}_{21}' = (\alpha + 4)(\alpha - 2)(3\alpha - 8)^2R^8$ is in $\mathcal{G}_2'$ and hence either $\alpha = -4$, $\alpha = 2$ or $\alpha = \frac{8}{3}$. In the first case, $\alpha = -4$, we get $R^5\mathfrak{D}_{141} = -9a^3(4a^2 + R^2)$ which cannot vanish. If $\alpha = 2$ then we get $2R^5\mathfrak{D}_{141} = 9a^3(4a^2 - R^2)$, from where $a = \pm\frac{1}{2}R$ and the metric is Einstein. If $\alpha = \frac{8}{3}$ then $R^5\mathfrak{D}_{411} = 4a^3(R^2 - 6a^2)$ from where $a = \pm\frac{1}{\sqrt{6}}R$. Then $\mathfrak{D}_{123} = -\frac{5}{9}$, which shows that no RG2 algebraic steady soliton may exist in this setting.

<u>$F = 0$, $c = 0$ and $a \neq d$.</u> First we determine α using the component $\mathfrak{D}_{242}$. In particular,

$$4R^5\mathfrak{D}_{242} = d(a^2 + d^2 + ad)(4(a^2 + d^2 + ad)\alpha + R^2(\alpha - 8)),$$

which implies that $\alpha = \frac{8R^2}{4(a^2+d^2+ad)+R^2}$. A straightforward calculation shows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if $(4ad + R^2)(4(a^2 + d^2 + ad) - 3R^2) = 0$ and thus $R = \frac{2}{\sqrt{3}}\sqrt{a^2 + d^2 + ad}$ or $a = -\frac{R^2}{4d}$.

In the first case, $R = \frac{2}{\sqrt{3}}\sqrt{a^2 + d^2 + ad}$, the left-invariant metric determined by the Lie algebra structure

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = -\tfrac{a\sqrt{3}}{2\sqrt{a^2+d^2+ad}}e_1,$$
$$[e_2, e_4] = -\tfrac{d\sqrt{3}}{2\sqrt{a^2+d^2+ad}}e_2, \quad [e_3, e_4] = -\tfrac{(a+d)\sqrt{3}}{2\sqrt{a^2+d^2+ad}}e_3$$

is an RG2 algebraic steady soliton with $\alpha = 2$. Recall that $d \neq 0$ and note that the replacement $e_4 \mapsto -e_4$ defines an isometry between (a, d) and $(-a, -d)$. Hence, assuming $d > 0$, setting $\kappa = \frac{a}{d}$ and applying the homothety determined by $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, -e_4)$ we obtain

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_1,$$
$$[e_2, e_4] = \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_2, \quad [e_3, e_4] = \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}}e_3\,.$$

Since $a \neq d$, we have $\kappa \neq 1$. Moreover, the metrics corresponding to the parameters κ and $\frac{1}{\kappa}$ are isometric. Indeed, $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, e_4)$ if $\kappa > 0$ and $(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, -e_3, -e_4)$ if $\kappa < 0$ determine the corresponding isometries. Hence, we may assume $\kappa \in [-1, 1)$. Furthermore, a direct calculation shows that these metrics are never Einstein and that they are irreducible. Finally, a straightforward calculation shows that $Q + \frac{3}{2}\,\mathrm{Id}$ is a derivation of the Lie algebra and thus these metrics are algebraic Ricci solitons. We conclude that these metrics correspond to Assertion (1).

In the second case above, assuming $a = -\frac{R^2}{4d}$, we set $\kappa = \frac{R}{4d}$. Thus, $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$ and the left-invariant metric determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa}e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right)e_3$$

is an RG2 algebraic steady soliton. Note that $\kappa \neq 0$. Moreover, replacing $e_4 \mapsto -e_4$ we may assume $\kappa > 0$, and $(e_1, e_2, e_3, e_4) \mapsto (e_2, -e_1, e_3, -e_4)$ defines an isometry interchanging κ and $\frac{1}{4\kappa}$ which shows that, without loss of generality, one may restrict the parameter to $\kappa \in \left(0, \frac{1}{2}\right]$. A direct calculation shows that these metrics are never Einstein and that they are irreducible. Finally, the metrics above are algebraic Ricci solitons if and only if $16\kappa^4 - 16\kappa^2 + 1 = 0$ (i.e., $\kappa = \frac{1}{2}\sqrt{2-\sqrt{3}}$), in which case $Q + \frac{3}{2}\,\mathrm{Id}$ is a derivation. A straightforward calculation shows that, taking the homothetical case $\kappa = -\frac{1}{2}\sqrt{2-\sqrt{3}}$, it corresponds to the special case of Assertion (1) for the value $\kappa = -(2+\sqrt{3})^{-1}$. Therefore, these metrics correspond to Assertion (2).

$F = 0$**,** $dH \neq 0$

Construct a Gröbner basis $\mathcal{G}_3$ of the ideal generated by $\mathcal{G} \cup \{F\} \subset \mathbb{R}[a, c, d, H, F, R, \alpha]$ with respect to the lexicographical order. Since the polynomial $\mathrm{g}_{31} = dH(12H^2 + 7R^2)R^4$ belongs to $\mathcal{G}_3$ it follows that no RG2 algebraic steady solitons may exist in this setting, finishing the proof. $\square$

6.2.4 The semi-direct product $\mathbb{R}e_4 \ltimes \mathbb{R}^3$

In addition to Einstein metrics and symmetric products, $\mathbb{R} \ltimes \mathbb{R}^3$ is an RG2 algebraic steady soliton just considering the RG2 algebraic steady solitons in Lemma 6.9. Henceforth we focus on the irreducible non-Einstein case.

Lemma 6.16. *Let G be a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$. Then G admits an irreducible non-Einstein RG2 algebraic steady soliton if and only if it is homothetic to one of the following, where $\{e_1, \dots, e_4\}$ is an orthonormal basis:*

1. *The left-invariant metric determined by*

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = f e_2, \quad [e_3, e_4] = p e_3, \quad \text{with } \alpha = \tfrac{2(f^2+p^2+1)}{f^4+p^4+1},$$

where $(f, p) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \backslash \{(1, 1)\}$.

2. *The left-invariant metric determined by*

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = f e_2 + h e_3, \quad [e_3, e_4] = -h e_2 + p e_3,$$

where the parameters p and h are given by $p = \frac{1}{2}\left(1 + \sqrt{1 - 4f(f-1)}\right)$ and $h = \left(\frac{f^2(2p^2+1)+p^2-1}{2(f-p)^2}\right)^{\frac{1}{2}}$, with coupling constant $\alpha = \frac{2}{f^2+p^2}$ and $f \in (0, 1)$.

Furthermore, Lie groups in case (1) *are algebraic Ricci solitons whereas left-invariant metrics* (2) *are never Ricci solitons.*

Remark 6.17. Left-invariant metrics in Lemma 6.16 define distinct homothetic classes for different values of the parameters in each assertion. For left-invariant metrics in Assertion (1) we have

$$\begin{array}{rcl} \tau &=& -2(f^2+p^2+fp+f+p+1), \\ \|\rho\|^2 &=& -(f^2+p^2+1)\tau, \\ \|R\|^2 &=& 4(f^4+p^4+f^2p^2+f^2+p^2+1). \end{array}$$

Proceeding as in Remark 6.10, a straightforward calculation shows that any left-invariant metric in Assertion (1) with $p \neq -f-1$ is never homothetic to any other metric in Assertion (1). For $p=-f-1$ we cannot use the same argument since $\tau^{-2}\|\rho\|^2 = 1$ and $\tau^{-2}\|R\|^2 = 3$. In this case, for $p=-f-1$, we use the self-dual and the anti-self-dual Weyl curvature operators, given by

$$W^+ = W^- = \tfrac{1}{3}\operatorname{diag}[-(f+4)f-1,\, -(f-2)f+2,\, 2(f+1)f-1]\,.$$

Now, a straightforward calculation shows that two different left-invariant metrics in Assertion (1) with $p=-f-1$ are never homothetic since, after rescaling to have scalar curvature -1, the set of eigenvalues of the above operators never coincides.

For Assertion (2), we get that two different left-invariant metrics are never homothetic proceeding as in Remark 6.10 and using that

$$\begin{array}{rcl} \tau &=& -5f-4-(f+2)\sqrt{1-4f(f-1)}, \\ \|\rho\|^2 &=& -2f^4+4f^2+17f+\frac{13}{2}+(2f^2+6f+\frac{11}{2})\sqrt{1-4f(f-1)}\,. \end{array}$$

Proof. Take the Lie algebra given in (4.4). We consider the diagonal matrix $\operatorname{diag}[a,f,p]$ in the decomposition of elements of $\operatorname{der}(\mathfrak{r}^3)$ and we analyse by separate the cases of null and non-null determinant.

$afp=0$

In this case at least one of a, f and p must be zero. Thus, without loss of generality, we may assume $a=0$. Moreover, one may work with a homothetic basis $\hat{e}_i = Re_i$ so that we may assume $R=1$. A key observation in this case is that if $b=c=0$ then e_1 determines a parallel left-invariant vector field. Hence, if $b=c=0$ and G admits an RG2 algebraic steady soliton then G splits as a product $\mathbb{R}\times H$, where H corresponds to the non-unimodular Lie group determined by the Lie algebra $\mathfrak{h} = \operatorname{span}\{e_2,e_3,e_4\}$ with

$$[e_2,e_4] = -fe_2 - he_3, \quad [e_3,e_4] = he_2 - pe_3,$$

and the RG2 algebraic steady solitons are determined by Lemma 6.9.

Otherwise, the expressions obtained for the Ricci tensor and for the $\check{R}$-tensor imply that $\mathfrak{D} = Q + \frac{\alpha}{4}Q_{\check{R}}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{D}_{ijk} = 0\}$ holds true, where $\mathfrak{D}_{ijk} \in \mathbb{R}[f,b,c,h,p,\alpha]$ are the polynomials given by the components $\mathfrak{D}_{ijk}$, where the only non-zero ones are given by

$$\mathfrak{D}_{411} = a^3+2ab^2+2ac^2-2b^2f+af^2-2c^2p+ap^2-\tfrac{1}{2}(a^5+4a^3(b^2+c^2)$$

$$
\begin{aligned}
&- 6a^2(b^2f + c^2p) + a(f(4b^2f + f^3 - 4bch + 2fh^2) + 4h(bc - fh)p \\
&+ 2(2c^2 + h^2)p^2 + p^4) - 2(b^2f^3 + c^2p^3 + bch(-f^2 + p^2)))\alpha,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{412} = \tfrac{1}{2}&(2(3a^2b + ch(f - 2p) + b(f^2 - fp + p^2) + a(ch + b(-2f + p))) \\
&- (3a^4b + 2b^3f^2 + a^3(-2bf + ch) + ch(f^3 - 2f^2p + 2fp^2 - 2p^3) \\
&+ b(f^4 + 2f^2h^2 - f(c^2 + 3h^2)p + (3c^2 - f^2 + h^2)p^2 + p^4) \\
&+ a(-4b^3f + cf^2h + b(-2f^3 - fh^2 + c^2(f - 5p) + h^2p)) \\
&+ a^2(2b^3 - cfh + b(2(c^2 + f^2) + p^2)))\alpha),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{413} = \tfrac{1}{2}&(2(3a^2c + acf + cf^2 - abh + 2bfh - (2ac + cf + bh)p + cp^2) \\
&- (3a^4c + b^2cf(3f - p) - a^3(bh + 2cp) + bh(2f^3 - 2f^2p + 2fp^2 - p^3) \\
&+ c(f^4 - 3fh^2p + 2(c^2 + h^2)p^2 + p^4 + f^2(h - p)(h + p)) + a^2(2b^2c + bhp \\
&+ c(2c^2 + f^2 + 2p^2)) - a(b^2c(5f - p) + bhp^2 + c(-fh^2 + 4c^2p + h^2p + 2p^3)))\alpha),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{421} = \tfrac{1}{2}&(-2b(3f^2 + fp + p^2) + 2a(ch + b(2f + p)) + a^4b\alpha + 2b^3f^2\alpha \\
&- a^3(2bf + ch)\alpha - a(f(4b^3 - cfh + b(c^2 + 2f^2 - h^2)) \\
&+ b(3c^2 + h^2)p + 2chp^2)\alpha + b(3f^4 + 2f^2h^2 + f(c^2 - 5h^2)p \\
&+ (c^2 + f^2 + 3h^2)p^2 + p^4)\alpha + ch(2f - 4p - f^3\alpha + 2p^3\alpha) \\
&+ a^2(-2b + (2b^3 + 2b(c^2 + f^2) - ch(f - 2p) - bp^2)\alpha)),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{422} = &-2ab^2 + 2b^2f + f^3 + 2fh^2 - 2h^2p + fp^2 + a^3b^2\alpha - \tfrac{1}{2}a^4f\alpha \\
&+ af(3b^2f - 2bch + 2c^2p)\alpha - \tfrac{1}{2}(f^3(4b^2 + f^2 + 4h^2) - 2fh(2bc + 3fh)p \\
&+ 2(c^2f + bch + 2fh^2)p^2 - 2h^2p^3 + fp^4)\alpha + a^2(f - (2b^2 + c^2)f\alpha + bch\alpha),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{423} = \tfrac{1}{2}&(2(a^2h + a(-2bc + h(f - p)) + bc(f + p) + h(3f^2 - 2fp + p^2)) \\
&- (-2a^3bc + a^4h + b^2fh(2f + p) + bc(f^3 - f^2p + fp^2 + p^3) \\
&+ h(3f^4 + 2f^2h^2 - f(c^2 + 2f^2 + 4h^2)p + 2(c^2 + f^2 + h^2)p^2 - 2fp^3 + p^4) \\
&- a(-c^2h(f - 3p) + 2bcp^2 + b^2h(5f + p)) \\
&+ a^2(3b^2h + 2bcp + h(c^2 + f^2 - p^2)))\alpha),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{431} = \tfrac{1}{2}&(-2(a^2c + bh(-2f + p) + c(f^2 + fp + 3p^2) + a(bh - c(f + 2p))) \\
&+ (a^4c + b^2cf(f + p) + a^3(bh - 2cp) + bh(-2f^3 + p^3) + c(f^4 - 5fh^2p \\
&+ 2(c^2 + h^2)p^2 + 3p^4 + f^2(3h^2 + p^2)) + a^2(2b^2c + bh(-2f + p) \\
&+ c(2c^2 - f^2 + 2p^2)) - a(b^2c(3f + p) + bh(-2f^2 + p^2) \\
&+ c(fh^2 + 4c^2p - h^2p + 2p^3)))\alpha),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{432} = \tfrac{1}{2}&(-2h(f^2 - 2fp + 3p^2) + 2a^3bc\alpha + a^4h\alpha + b^2fh(2f - p)\alpha \\
&+ h(f^4 + 2f^2h^2 + f(c^2 - 2f^2 - 4h^2)p + 2(c^2 + f^2 + h^2)p^2 \\
&- 2fp^3 + 3p^4)\alpha + bc(2(f + p) - (f^3 + f^2p - fp^2 + p^3)\alpha) \\
&- a(2h(-f + p) + b^2h(3f - p)\alpha + c^2h(f + 5p)\alpha \\
&- 2bc(-2 + f^2\alpha)) + a^2(-2bcf\alpha + h(-2 + (b^2 + 3c^2 - f^2 + p^2)\alpha))),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_{433} = &-2ac^2 - 2fh^2 + a^2p + 2c^2p + f^2p + 2h^2p + p^3 - \tfrac{1}{2}(-2(a^3c^2 - a^2bch \\
&+ f^2h(bc + fh)) + (a^4 + 2a^2(b^2 + 2c^2) - 4ab(bf + ch) \\
&+ f(2b^2f + f^3 + 4bch + 4fh^2))p - 6(ac^2 + fh^2)p^2 + 4(c^2 + h^2)p^3 + p^5)\alpha,
\end{aligned}
$$

We start with a Gröbner basis $\mathcal{G}_1$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\}$ with respect to the lexicographical order and we get that the polynomial $\mathrm{g}_{11} = p^2(p^2\alpha - 2)c^3$ belongs to $\mathcal{G}_1$. Therefore, we have three possibilities which we analyse separately.

$\underline{p = 0}$. Constructing a Gröbner basis $\mathcal{G}_1'$ of the ideal generated by $\mathcal{G}_1 \cup \{p\}$,which is an ideal in $\mathbb{R}[f, b, c, h, p, \alpha]$, with respect to the lexicographical order, one has that the polynomials $\mathrm{g}_{11}' = fh(b^2 + h^2)$ and $\mathrm{g}_{12}' = bf(b^2 + h^2)$ belong to $\mathcal{G}_1'$. If $f = 0$ then the metric is Einstein. If $f \neq 0$, then $b = h = 0$ and we get $\mathfrak{D}_{422} = -\frac{1}{2}f^3(f^2\alpha - 2)$. Thus, $\alpha = \frac{2}{f^2}$ and this case is symmetric, and thus reducible since it is not Einstein.

$\underline{\alpha = \frac{2}{p^2}, p \neq 0}$. We construct a Gröbner basis $\mathcal{G}_1''$ of the ideal generated by $\mathcal{G}_1 \cup \{p^2\alpha - 2\} \subset \mathbb{R}[c, b, h, p, \alpha, f]$ with respect to the graded reverse lexicographical order and the polynomials $\mathrm{g}_{11}'' = cf(b^2 + c^2)$ and $\mathrm{g}_{12}'' = c(b^2f + c^2p - fh^2 + h^2p)$ belong to $\mathcal{G}_1''$. Hence, necessarily $c = 0$. Moreover, the polynomials $\mathrm{g}_{13}'' = bf^2(b^2 + c^2)$ and $\mathrm{g}_{14}'' = b(f - p)(f^2 - 2h^2 + fp)$ also belong to $\mathcal{G}_1''$. Thus, $b = 0$ and G is reducible or otherwise $f = h = 0$ and the manifold is symmetric.

$\underline{c = 0, p \neq 0, \alpha \neq \frac{2}{p^2}}$. Constructing a Gröbner basis $\mathcal{G}_1'''$ of the ideal generated by $\mathcal{G}_1 \cup \{c\} \subset \mathbb{R}[f, b, c, h, p, \alpha]$ with respect to the lexicographical order, one has that the polynomial $\mathrm{g}_{11}''' = b^2p^2(p^2\alpha - 2)$ belongs to $\mathcal{G}_1'''$. Thus, necessarily $b = 0$ and G is reducible.

$afp \neq 0$

Without loss of generality, one may work with a homothetic basis $\hat{e}_i = \frac{R}{a}e_i$ so that we may assume $R = a = 1$. A key observation in this case is that the cases $b = c = 0$, $c = h = 0$ and $b = h = 0$ are homothetic. Indeed, considering $(e_1, e_2, e_3, e_4) = \frac{1}{p}(e_3, e_2, e_1, e_4)$ the case $c = h = 0$ reduces to $b = c = 0$. Analogously, considering $(e_1, e_2, e_3, e_4) = \frac{1}{f}(e_2, e_1, e_3, e_4)$ the case $b = h = 0$ reduces to $b = c = 0$.

Using the expressions obtained for the Ricci tensor and for the $\check{R}$-tensor it follows that $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ is a derivation of the Lie algebra if and only if the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ holds true, where $\mathfrak{P}_{ijk} \in \mathbb{R}[b, c, f, h, p, \alpha]$ are the polynomials given by the components $\mathfrak{D}_{ijk}$ (which we omit for the sake of brevity). Now we construct a Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\{\mathfrak{P}_{ijk}\}$ with respect to the lexicographical order and we get that the polynomial $\mathrm{g}_{21} = ch(\alpha - 2)^4(3\alpha - 2)(\alpha^2 - 2\alpha + 4)$ belongs to $\mathcal{G}_2$. Therefore, we have four possibilities which we analyse separately.

$\underline{c = 0}$. Constructing a Gröbner basis $\mathcal{G}_2'$ of the ideal generated by $\{\mathfrak{P}_{ijk}\} \cup \{c\} \subset \mathbb{R}[h, b, c, p, \alpha, f]$ with respect to the lexicographical order, one has that the polynomials $\mathrm{g}_{21}' = bfh(f - 1)$ and $\mathrm{g}_{22}' = bh(\alpha - 2)$ belong to $\mathcal{G}_2'$. Hence, we are led to the cases $b = 0$, $h = 0$ and $f = 1$, $\alpha = 2$.

$\underline{b = 0}$. In this case, we construct a Gröbner basis $\widetilde{\mathcal{G}}_2'$ of the ideal generated by $\mathcal{G}_2' \cup \{b\} \subset \mathbb{R}[h, b, c, p, \alpha, f]$ with respect to the graded reverse lexicographical order. We get that the polynomial $\widetilde{\mathrm{g}}_{21}' = h(f - p)^2(f^2 + p^2 - f - p)$ belongs to $\widetilde{\mathcal{G}}_2'$.

If $h = 0$ then we get $2\,\mathfrak{D}_{411} = -(f^4 + p^4 + 1)\alpha + 2(f^2 + p^2 + 1)$. Therefore, $\alpha = \frac{2(f^2+p^2+1)}{f^4+p^4+1}$ and the left-invariant metric given by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = -fe_2, \quad [e_3, e_4] = -pe_3$$

is an RG2 algebraic steady soliton. The metric is Einstein if and only if $f = p = 1$. Since the isometry $(e_1, e_2, e_3, e_4) \mapsto (e_1, e_3, e_2, e_4)$ transforms (f, p) into (p, f), we may assume that $p \leq f$. Moreover, $(e_1, e_2, e_3, e_4) \mapsto \frac{1}{f}(e_2, e_1, e_3, e_4)$ defines an homothety between (f, p) and $(\frac{1}{f}, \frac{p}{f})$. Therefore, we may assume without loss of generality that (f, p) belongs to the set $\{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \setminus \{(1, 1)\}$. Furthermore, a direct calculation shows that these metrics are irreducible and a straightforward calculation shows that $Q + (f^2 + p^2 + 1)\mathrm{Id}$ is a derivation of the Lie algebra and thus an algebraic Ricci soliton. Finally, the isometry $e_4 \mapsto -e_4$ shows that these metrics correspond to Assertion (1).

If $p = f$ and $h \neq 0$ then we get $2\,\mathfrak{D}_{411} = -(2f^4 + 1)\alpha + 2(2f^2 + 1)$. Therefore, $\alpha = \frac{2(2f^2+1)}{2f^4+1}$ and the left-invariant metric given by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = -f e_2 - h e_3, \quad [e_3, e_4] = h e_2 - f e_3$$

is an RG2 algebraic steady soliton. The metric is Einstein if and only if $f = 1$. Moreover, a straightforward calculation shows that $Q + (2f^2 + 1)\mathrm{Id}$ is a derivation of the Lie algebra and thus an algebraic Ricci soliton. A direct calculation shows that the curvature tensor of type $(1, 3)$ does not depend on h and hence it follows from the work of Kulkarni [48] that this case is homothetic (although not a homothetically isomorphic) to the case in Assertion (1) when $p = f$.

If $f^2 + p^2 - f - p = 0$ and $p \neq f$, $h \neq 0$, then we get

$$2\,\mathfrak{D}_{411} = -(f^4 + p^4 + 2(f - p)^2 h^2 + 1)\alpha + 2(f^2 + p^2 + 1)$$

which implies $\alpha = \frac{2(f^2+p^2+1)}{f^4+p^4+2(f-p)^2h^2+1}$. Now, a straightforward calculation shows that

$$\tfrac{f^2+p^2+1}{\alpha}\,\mathfrak{D}_{422} = h^2(f - p)(2(f - p)^2 h^2 - f^2(2p^2 + 1) - p^2 + 1)\,.$$

Since $h \neq 0$ and $p \neq f$, it follows that $h = \tilde{\varepsilon}\left(\frac{f^2(2p^2+1)+p^2-1}{2(f-p)^2}\right)^{\frac{1}{2}}$, with $\tilde{\varepsilon}^2 = 1$. On the other hand, since $f^2 + p^2 - f - p = 0$, we get $p = \frac{1}{2}\left(1 + \varepsilon\sqrt{1 - 4f(f - 1)}\right)$, with $\varepsilon^2 = 1$. For this choice of h and p we have $\alpha = \frac{2}{f^2+p^2}$ and the left-invariant metric given by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = -f e_2 - h e_3, \quad [e_3, e_4] = h e_2 - p e_3 \tag{6.2}$$

is an RG2 algebraic steady soliton. A direct calculation shows that these metrics are never Einstein. Note that a substitution of $e_3 \mapsto -e_3$ is an isometry which interchanges $\tilde{\varepsilon} = -1$ by $\tilde{\varepsilon} = 1$. Hence, we take $\tilde{\varepsilon} = 1$ and to ensure that the structure constants are real we take $f \in \left(0, \frac{1+\sqrt{2}}{2}\right] \setminus \{1\}$ if $\varepsilon = 1$, and $f \in \left(1, \frac{1+\sqrt{2}}{2}\right]$ if $\varepsilon = -1$. Consider now a pair (ε, f) so that $\varepsilon = -1$ and define a corresponding pair $\left(\varepsilon = 1, \frac{1}{2}(1 - \sqrt{1 - 4f(f - 1)}\right)$. It now follows that $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_3, e_2, e_4)$ determines an isometry between the two cases above which shows that one may assume $\varepsilon = 1$ without loss of generality. Moreover, one can specialize $f \in (0, 1)$. To do this, if $f \in \left(1, \frac{1+\sqrt{2}}{2}\right]$, one has that $\frac{1}{2}(1 + \sqrt{1 - 4f(f - 1)}) \in (0, 1)$ and repeating the same change of basis as above we get that both cases are isometric. Finally, a straightforward calculation shows that these metrics are irreducible and that they are never an algebraic Ricci soliton. Thus we conclude that this case corresponds to Assertion (2) after the replacement $e_4 \mapsto -e_4$.

$\underline{h = 0, b \neq 0}$. Since $c = h = 0$, this case reduces to the case $c = b = 0$.

$\underline{f = 1, \alpha = 2, bh \neq 0}$. In this case, we have $\mathfrak{D}_{413} = bhp^2(p-1)$. Since $bhp \neq 0$, it follows that $p = 1$ and the metric is Einstein.

$\underline{h = 0, c \neq 0}$. We construct a Gröbner basis $\mathcal{G}_2''$ of the ideal generated from the previous basis $\mathcal{G}_2 \cup \{h\} \subset \mathbb{R}[b, c, f, h, p, \alpha]$ with respect to the lexicographical order and we get that the polynomials $\mathrm{g}_{21}'' = bcp(p-1)$ and $\mathrm{g}_{22}'' = bcf(f-1)$ belong to $\mathcal{G}_2''$. If $b \neq 0$ then $f = p = 1$ and the corresponding metric is Einstein. Otherwise, $b = h = 0$ which reduces to the case $c = b = 0$.

$\underline{\alpha = 2, ch \neq 0}$. Constructing a Gröbner basis $\mathcal{G}_2'''$ of the ideal generated by $\mathcal{G}_2 \cup \{\alpha - 2\} \subset \mathbb{R}[b, c, f, h, p, \alpha]$ with respect to the lexicographical order, one has that the polynomials $\mathrm{g}_{21}''' = cp^2(p-1)^2$ and $\mathrm{g}_{22}''' = hp(f-p)(f+p-1)$ belong to $\mathcal{G}_2'''$. Hence, it follows that $p = f = 1$ and the corresponding metric is Einstein.

$\underline{\alpha = \frac{2}{3}, ch \neq 0}$. Constructing a Gröbner basis $\mathcal{G}_2''''$ of the ideal generated by $\mathcal{G}_2 \cup \{3\alpha - 2\} \subset \mathbb{R}[b, c, f, h, p, \alpha]$ with respect to the lexicographical order, one has that the polynomial $\mathrm{g}_{21}'''' = ch^2$ belongs to $\mathcal{G}_2''''$. Since $ch \neq 0$ there is no solution in this case, which finishes the proof.

□

6.3 Fixed points and homothety classes

Along this section we finish the proofs of both main theorems in this chapter.

6.3.1 The proof of Theorem 6.4

First of all, recall that if the symmetric $(0,2)$-tensor field $RG = -2\rho - \frac{\alpha}{2}\check{R}$ comes from a fixed point, then α necessarily satisfies $\tau + \frac{\alpha}{4}\|R\|^2 = 0$. Hence the manifold is flat or otherwise $\alpha = -4\tau\|R\|^{-2}$.

Let (M, g) be a complete and simply connected homogeneous four-dimensional manifold. Then it is isometric to a symmetric space or to a Lie group with a left-invariant metric. The analysis of left-invariant metrics on Lie groups was carried out through sections 6.2.1 to 6.2.4. In each case, all possible derivations of the form $\mathfrak{D} = Q + \frac{\alpha}{4}\check{Q}$ are given, showing that $\mathfrak{D} = 0$ if and only if the metric is Einstein or a product $\mathbb{R}^k \times N^{4-k}(c)$ for $k = 1, 2$, unless it corresponds to the left-invariant metric on $SU(2) \times \mathbb{R}$ given in Lemma 6.11 and determined by the Lie algebra

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = \tfrac{4}{3}e_2,$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis of $\mathfrak{su}(2) \times \mathbb{R}$.

On the other hand, if (M, g) is a non-Einstein symmetric space, then it splits as a product $N_1^k(c_1) \times N_2^{4-k}(c_2)$, where $k = 1, 2$, and $N_i^\ell(c_i)$ is a space of constant curvature c_i. If $k = 1$, the resulting manifold is isometric to $\mathbb{R} \times N^3(c)$ and it satisfies $RG[g] = 0$. If $k = 2$, we compute the tensor field $RG[g]$ for a product $N_1^2(c_1) \times N_2^2(c_2)$, with coupling constant $\alpha = -\frac{4\tau}{\|R\|^2}$. An explicit calculation shows that the $(1,1)$-tensor field $Q - \frac{\tau}{\|R\|^2}\check{Q}$ takes the form

$$\frac{c_1 c_2}{c_1^2 + c_2^2} \operatorname{diag}[c_2 - c_1, c_2 - c_1, c_1 - c_2, c_1 - c_2] \,.$$

Hence, assuming $c_1 \neq c_2$, one has that $\rho - \frac{\tau}{\|R\|^2}\check{R} = 0$ if and only if $c_1 c_2 = 0$, which finishes the proof.

Remark 6.18. Products $\mathbb{R}^k \times N(c)$ are rigid gradient Ricci solitons [61]. On the contrary, the product Lie group $SU(2) \times \mathbb{R}$, it is not a Ricci soliton (see, for instance, [2]) whereas it is an RG2 steady soliton,.

6.3.2 The proof of Theorem 6.6

The result follows at once from lemmas 6.11, 6.12, 6.14, and 6.16. Moreover, all metrics corresponding to each assertion in Theorem 6.6 represent different homothetical classes as shown in remarks 6.13, 6.15, and 6.17. Next we show that no metrics corresponding to different assertions in Theorem 6.6 may be homothetic. First, recall all the algebras obtained

1. $\mathbb{R} \ltimes \mathfrak{e}(1,1)$, for a coupling constant $\alpha = \frac{2}{\kappa^2+1}$, given by

$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$

where $\kappa > 0$, $\kappa \neq 1$.

2. $\mathbb{R} \ltimes \mathfrak{h}^3$, for a coupling constant $\alpha = 2$, given by

$$\begin{aligned} &[e_1, e_2] = e_3, && [e_1, e_4] = \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_1, \\ &[e_2, e_4] = \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_2, && [e_3, e_4] = \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_3, \end{aligned}$$

where $\kappa \in [-1, 1)$.

3. $\mathbb{R} \ltimes \mathfrak{h}^3$, for a coupling constant $\alpha = \frac{32\kappa^2}{16\kappa^4+1}$, given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = -\tfrac{1}{4\kappa} e_2, \quad [e_3, e_4] = \left(\kappa - \tfrac{1}{4\kappa}\right) e_3,$$

where $\kappa \in \left(0, \frac{1}{2}\right]$, $\kappa \neq \frac{1}{2}\sqrt{2 - \sqrt{3}}$.

4. $\mathbb{R} \ltimes \mathfrak{r}^3$, for a coupling constant $\alpha = \frac{2(\kappa^2+\delta^2+1)}{\kappa^4+\delta^4+1}$, given by

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2, \quad [e_3, e_4] = \delta e_3,$$

where $(\kappa, \delta) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \backslash \{(1, 1)\}$.

5. $\mathbb{R} \ltimes \mathfrak{r}^3$, for a coupling constant $\alpha = \frac{2}{\kappa^2+p^2}$, given by
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2 + h e_3, \quad [e_3, e_4] = -h e_2 + p e_3,$$
where the parameters p and h are given by $p = \frac{1}{2}\left(1 + \sqrt{1 - 4\kappa(\kappa - 1)}\right)$ and $h = \left(\frac{\kappa^2(2p^2+1)+p^2-1}{2(\kappa-p)^2}\right)^{\frac{1}{2}}$, for any $\kappa \in (0, 1)$.

Cases (1) and (3). In Case (1), in addition to τ and $\|R\|^2$ determined in Remark 6.13, one has $\|\rho\|^2 = 12\kappa^4 + 4$. In Case (3), in addition to τ and $\|\rho\|^2$ already computed in Remark 6.15, we have $\|R\|^2 = \frac{16\kappa^2(48\kappa^6-16\kappa^4+14\kappa^2-1)+3}{64\kappa^4}$.

Now, a straightforward calculation following Remark 6.10 and using the invariants τ, $\|\rho\|^2$ and $\|R\|^2$ shows that left-invariant metrics corresponding to cases (1) and (3) in Theorem 6.6 are never homothetic.

Cases (1) and (5). In Case (1), we consider the invariants τ and $\|R\|^2$ determined in Remark 6.13 and in Case (5) we consider τ determined in Remark 6.17 and
$$\|R\|^2 = -2\left(2(\kappa - 1)\kappa^3 - \kappa^2 - 8\kappa - 3\right) + (2(\kappa + 2)\kappa + 6)\sqrt{1 - 4(\kappa - 1)\kappa}\,.$$
A straightforward calculation following Remark 6.10 now shows that left-invariant metrics corresponding to cases (1) and (5) in Theorem 6.6 are never homothetic.

Cases (3) and (5). We proceed as in Remark 6.10 using the invariants τ and $\|\rho\|^2$ previously determined and a straightforward calculation shows that left-invariant metrics corresponding to cases (3) and (5) in Theorem 6.6 are never homothetic.

Secondly, we analyse the cases in Theorem 6.6 which are Ricci solitons, i.e., cases (2) and (4). We start using the self-dual and the anti-self-dual Weyl curvature operators. In particular, for a left-invariant metric corresponding to Case (4), we have
$$W^+ = W^- = \tfrac{1}{6}\,\mathrm{diag}[-\mu_{42} - \mu_{43}, \mu_{42}, \mu_{43}],$$
where
$$\begin{aligned}\mu_{42} &= -2\kappa^2 + \delta^2 + (\kappa - 2)\delta + \kappa + 1,\\ \mu_{43} &= \kappa^2 + \delta^2 - (2\kappa - 1)\delta + \kappa - 2\,.\end{aligned}$$
For a left-invariant metric corresponding to Case (2), we have
$$W^\pm = \tfrac{1}{6(\kappa^2+\kappa+1)}\,\mathrm{diag}[-\mu_{22}^\pm - \mu_{23}^\pm, \mu_{22}^\pm, \mu_{23}^\pm],$$
where
$$\begin{aligned}\mu_{22}^\pm &= \kappa^2 + 1 + \tfrac{\kappa}{2}(5 \pm 3\sqrt{3(\kappa^2 + \kappa + 1)}),\\ \mu_{23}^\pm &= \kappa^2 + \tfrac{5\kappa}{2} + 1 \pm \tfrac{3}{2}\sqrt{3(\kappa^2 + \kappa + 1)}\,.\end{aligned}$$
Hence, if a left-invariant metric corresponding to Case (2) is homothetic to a left-invariant metric in Case (4), then $W^+ = W^-$, and comparing term by term, a straightforward calculation shows that this occurs if and only $\kappa = -1$ (in that case, the Weyl curvature operator must have an eigenvalue $\frac{1}{6(\kappa^2+\kappa+1)}\mu_{23}^+$ with multiplicity two). Therefore, a left-invariant metric in Case (2) with $\kappa \neq -1$ is never homothetic to a left-invariant metric in Case (4). Finally set $\kappa = -1$ in Case (2) to have $\tau = -2$, $\|\rho\|^2 = 3$ and $\|R\|^2 = 8$. Proceeding as in Remark 6.10, one has that no metric corresponding to Case (4) may be homothetic just using the expressions in Remark 6.17.

6.4 Locally conformally flat fixed points.

In this section we classify fixed points in the context of locally conformally flat manifolds.

Theorem 6.19. *Let (M, g) be a n-dimensional locally conformally flat fixed point for the two-loop renormalization group flow with coupling constant α. Then*

1. *If $n \neq 4$, then (M, g) is homothetic to a product $M_1^m(c) \times M_2^m(-c)$ with $n = 2m$ or to a warped product $\mathbb{R} \times_f N(c)$ with non trivial warping function satisfying*

$$\alpha(n-2)((n-6)n+6)\left(f'^2 - c\right) + \alpha((n-4)(n-2)n-4)ff'' + 2(n-2)^2 f^2 = 0.$$

2. *If $n = 4$, then $\|R\|^2 = \|\rho\|^2 = \frac{\tau^2}{3}$.*

Proof. Let us recall, on the one hand, that a fixed points for the $RG2$ flow is given by a metric fulfilling $\rho + \frac{\alpha}{4}\check{R} = 0$. On the other hand, we have seen in chapter two that the operator related to $\check{R}$ is given

$$Q_{\check{R}} = \frac{2}{(n-2)^2}\left\{(n-4)Q_\rho^2 + \frac{2\tau}{(n-1)}Q_\rho + \frac{(n-1)\|\rho\|^2 - \tau^2}{(n-1)}\,\mathrm{Id}\right\}.$$

Combining these two identities, one can get that a metric of this kind is a fixed point if $Q_\rho + \frac{\alpha}{4}Q_{\check{R}} = 0$, which is

$$Q_\rho + \frac{\alpha}{2(n-2)^2}\left\{(n-4)Q_\rho^2 + \frac{2\tau}{(n-1)}Q_\rho + \frac{(n-1)\|\rho\|^2 - \tau^2}{(n-1)}\,\mathrm{Id}\right\} = 0,$$

and then,

$$\frac{\alpha(n-4)}{2(n-2)}Q_\rho^2 + \frac{\alpha\tau + (n-1)(n-2)^2}{(n-1)(n-2)^2}Q_\rho + \frac{\alpha((n-1)\|\rho\|^2 - \tau^2)}{2(n-2)^2(n-1)}\,\mathrm{Id} = 0. \tag{6.3}$$

Now we have two different possibilities depending on the dimension. If $n \neq 4$, then, as in chapter two, we have a quadratic equation on the Ricci operator, so we have two Ricci eigenvalues, called them λ and μ, related by

$$\lambda + \mu = -\frac{2(\alpha\tau + (n-1)(n-2)^2)}{\alpha(n-1)(n-2)(n-4)},$$

where this relation is given by Vietta's formulae. Thus, we have two eigenvalues, one a multiple of the other, and as the Schouten tensor is Codazzi and it has also two eigenvalues, one a multiple of the other, then we have either a warped product $\mathbb{R} \times_f N(c)$, with f a non trivial real warping function and $N(c)$ an $(n-1)$-dimensional Riemannian manifold of constant curvature or a Riemannian product $M^m(c) \times M^m(-c)$, such that $n = 2m$. In order to determine the function f, assume that λ has multiplicity one. Since $\tau = \lambda + (n-1)\mu$, then both are related by

$$\lambda + \mu = \frac{2\left(\alpha(\lambda + \mu(n-1)) + (n-1)(n-2)^2\right)}{\alpha(n-4)(n-2)(n-1)},$$

and using the formulas from 1.16 for the Ricci operator, we get that f must satisfy the differential equation

$$\alpha(n-2)((n-6)n+6)\left(f'^2-c\right)+\alpha((n-4)(n-2)n-4)ff''+2(n-2)^2f^2=0$$

If $n=4$, then equation (6.3) becomes

$$12(\alpha\tau+12)Q_\rho+\alpha(3\|\rho\|^2-\tau^2)\,\mathrm{Id}=0.$$

Since this is a lineal equation, this only can have one solution, and then, the Ricci operator has only one eigenvalue, so the metric is Einstein as long as the equation is not identically zero. In order to have that, we need that $\alpha=-\frac{12}{\tau}$ and $\|\rho\|=\frac{\tau^2}{3}$. Moreover, recall that $\alpha=-4\tau\|R\|^{-2}$, then $\|R\|^2=\frac{\tau^2}{3}$, and hence $\|R\|^2=\|\rho\|^2$. □

6.5 Critical metrics

As in previous chapters, we are studying case by case every example obtained and see when they fulfil the conditions to be critical for the functionals $\mathcal{S}$ and $\mathcal{F}_t$.

Notice that all metrics obtained along the chapter are summarized in section 6.3. Since we are working with homogeneous manifolds, then (M,g) is $\mathcal{S}$-critical if and only if it is Einstein or has vanishing scalar curvature. The symmetric examples from Theorem 6.4 fulfil this if and only if they are flat: The Lie algebra given in the same Theorem has positive scalar curvature $\tau=\frac{16}{9}$, so it cannot be $\mathcal{S}$-critical. Lie algebras from Theorem 6.6 have strictly non-zero scalar curvature, so there is none example of $\mathcal{S}$-critical metrics. However, we obtain the following for $\mathcal{F}_t$.

Theorem 6.20. *Let (M,g) be a four-dimensional fixed point for the two-loop renormalization flow. Then, it is $\mathcal{F}_t$-critical if and only if*

1. *(M,g) is homothetic to $\mathbb{R}^2\times N^2(c)$ and $t=-\frac{1}{2}$.*
2. *(M,g) is homothetic to $\mathbb{R}\times N^3(c)$ and $t=-\frac{1}{3}$.*

Proof. Recall that we define the tensor F as the $(0,2)$-tensor field corresponding to the functional $\mathcal{F}_t$ and that if $\mathsf{F}=0$, then the metric is $\mathcal{F}_t$-critical.

Take $\mathbb{R}^2\times N^2(c)$. The non-zero components of F are given by

$$\mathsf{F}_{11}=-16\mathsf{F}_{22}=-16\mathsf{F}_{33}=-16\mathsf{F}_{44}=c^2(1+2t).$$

Therefore, this is critical if and only if $c=0$, which means that the metric is flat, or if $t=-\frac{1}{2}$. Analogously, if we take $\mathbb{R}\times N^3(c)$, where the non-zero components of F are given by

$$\mathsf{F}_{11}=\frac{16}{3}\mathsf{F}_{22}=\frac{16}{3}\mathsf{F}_{33}=\frac{16}{3}\mathsf{F}_{44}=6c^2(1+3t),$$

and then it is critical if and only if $c=0$ or $t=-\frac{1}{3}$.

Regarding the left-invariant metric on $SU(2)\times\mathbb{R}$, one has that $\mathsf{F}_{11}=\mathsf{F}_{22}=-\frac{16}{27}$, so this is never critical. □

Theorem 6.21. *Let (M, g) be a four-dimensional RG2 algebraic steady soliton. Then, it is $\mathcal{F}_t$-critical if and only if*

1. *(M, g) is homothetic to $\mathbb{R} \ltimes E(1,1)$ with Lie algebra*
$$[e_1, e_3] = e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = \kappa e_1, \quad [e_2, e_4] = \kappa e_2,$$
where $\kappa > 0$, $\kappa \neq 1$, and $t = -\frac{1+\kappa^2}{1+3\kappa^2}$.
2. *(M, g) is homothetic to $\mathbb{R} \ltimes H^3$ with Lie algebra*
$$\begin{aligned} [e_1, e_2] &= e_3, & [e_1, e_4] &= \tfrac{\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_1, \\ [e_2, e_4] &= \tfrac{\kappa\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_2, & [e_3, e_4] &= \tfrac{(\kappa+1)\sqrt{3}}{2\sqrt{\kappa^2+\kappa+1}} e_3, \end{aligned}$$
where $\kappa \in [-1, 1)$, and $t = -\frac{3(1+\kappa+\kappa^2)}{2(5+\kappa(8+5\kappa)))}$.
3. *(M, g) is homothetic to $\mathbb{R} \ltimes \mathbb{R}^3$ with Lie algebra*
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = \kappa e_2, \quad [e_3, e_4] = \delta e_3,$$
where $(\kappa, \delta) \in \{(x, y) \in \mathbb{R}^2; x \in (0, 1], 0 \neq y \leq x\} \backslash \{(1, 1)\}$, and $t = -\frac{1+\delta^2+\kappa^2}{2(1+\delta+\delta^2+\kappa+\delta\kappa+\kappa^2)}$.

Proof. We analyse case by case every item from Theorem 6.6, studying when the respective F tensor vanishes.

Lie algebra from Theorem 6.6 .(1) gives the following F-tensor.

$$\mathsf{F}_{11} = \mathsf{F}_{22} = -\frac{1}{3}\mathsf{F}_{33} = \mathsf{F}_{44} = -2(\kappa^2 - 1)(1 + t + \kappa^2 + 3t\kappa^2).$$

Since $\kappa > 0$, $\kappa \neq 1$ and the second bracket is linear in t, these are vanishing if $t = -\frac{1+\kappa^2}{1+3\kappa^2}$ and the first statement of from Theorem 6.21 follows.

Take now Theorem 6.6 .(2). The non-zero components of F are given by

$$\begin{aligned} \mathsf{F}_{11} = -\mathsf{F}_{22} = & \frac{3\left(\kappa^2 - 1\right)\left(3\left(\kappa^2 + \kappa + 1\right) + 2(\kappa(5\kappa + 8) + 5)t\right)}{4\left(\kappa^2 + \kappa + 1\right)^2}, \\ \mathsf{F}_{33} = -\mathsf{F}_{44} = & \frac{(\kappa - 1)^2\left(3\left(\kappa^2 + \kappa + 1\right) + 2(\kappa(5\kappa + 8) + 5)t\right)}{4\left(\kappa^2 + \kappa + 1\right)^2}. \end{aligned}$$

Since $\kappa \in [-1, 1)$, then, this is only vanishing if $t = -\frac{3\left(\kappa^2+\kappa+1\right)}{2(\kappa(5\kappa+8)+5)}$, and statement (2) follows.

Take now Theorem 6.6 .(3). In this case, we have that

$$\mathsf{F}_{44} = \frac{\left(-768\kappa^8 + 256\kappa^6 - 96\kappa^4 + 16\kappa^2 - 3\right)t - \left(1 - 16\kappa^4\right)^2}{128\kappa^4},$$

and then, since this is linear in t, it is vanishing if $t = -\frac{\left(1-16\kappa^4\right)^2}{768\kappa^8-256\kappa^6+96\kappa^4-16\kappa^2+3}$. For this value, we obtain that the non-zero components left are

$$\begin{aligned}
\mathsf{F}_{11} =& \frac{(4\kappa^2+1)\,(16\kappa^4-16\kappa^2+1)}{2(16\kappa^4+1)},\\
\mathsf{F}_{22} =& \frac{(4\kappa^2+1)\,(16\kappa^4-16\kappa^2+1)}{8\kappa^2(16\kappa^4+)},\\
\mathsf{F}_{33} =& -\frac{(4\kappa^2+1)^2\,(16\kappa^4-16\kappa^2+1)}{8\kappa^2\,(16\kappa^4+1)}.
\end{aligned}$$

Then F can only be vanishing if $16\kappa^4-16\kappa^2+1=0$, whose solutions are $\kappa = \frac{\sqrt{2\pm\sqrt{3}}}{2}$ and $\kappa = -\frac{\sqrt{2\pm\sqrt{3}}}{2}$. Recall that $\kappa \in \left(0, \frac{1}{2}\right]$, $\kappa \neq \frac{1}{2}\sqrt{2-\sqrt{3}}$, therefore, non of these values are in the correspondent interval and then F is never vanishing.

We study now Theorem 6.6 .(4). The non-vanishing components of F are given by

$$\begin{aligned}
\mathsf{F}_{11} &= \left(\delta^2+\delta(\kappa-1)+(\kappa-1)\kappa-1\right)\left(\delta^2+\kappa^2+2t\left(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1\right)+1\right),\\
\mathsf{F}_{22} &= \left(\delta^2-\delta\kappa+\delta-\kappa(\kappa+1)+1\right)\left(\delta^2+\kappa^2+2t\left(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1\right)+1\right),\\
\mathsf{F}_{33} &= -\left(\delta^2+\delta(\kappa+1)-\kappa(\kappa+1)-1\right)\left(\delta^2+\kappa^2+2t\left(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1\right)+1\right),\\
\mathsf{F}_{44} &= -\left(\delta^2-\delta(\kappa+1)+(\kappa-1)\kappa+1\right)\left(\delta^2+\kappa^2+2t\left(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1\right)+1\right).
\end{aligned}$$

First brackets in every F_{ii} can only vanish at the same time if $\kappa=\delta=1$, which is not a possible value for (κ,δ). Hence, the system is fulfilled if and only if the second bracket vanishes, and since it is linear on t, this happens when $t = -\frac{\delta^2+\kappa^2+1}{2(\delta^2+\delta\kappa+\delta+\kappa^2+\kappa+1)}$ and $\mathsf{F}=0$.

Finally, take Theorem 6.6 .(5). A straightforward calculation shows that

$$\begin{aligned}
\mathsf{F}_{11} =& \kappa((-\kappa^3+((\sqrt{1-4(\kappa-1)\kappa}+2))\kappa+\sqrt{1-4(\kappa-1)\kappa}))\\
&+\frac{1}{4}((2\kappa-\sqrt{1-4(\kappa-1)\kappa}-7))+((\kappa((2((\sqrt{1-4(\kappa-1)\kappa}-1))\\
&+\kappa((-2\kappa(\kappa+1)+3\sqrt{1-4(\kappa-1)\kappa}+7))))-2((\sqrt{1-4(\kappa-1)\kappa}+2))))t,\\
\mathsf{F}_{44} =& \frac{1}{4}(2\kappa((-2\kappa(((4-3\kappa)\kappa+\sqrt{1-4(\kappa-1)\kappa}))+2\sqrt{1-4(\kappa-1)\kappa}-3))\\
&-5\sqrt{1-4(\kappa-1)\kappa}+4((\kappa((2((\sqrt{1-4(\kappa-1)\kappa}-1))\\
&+\kappa((-2\kappa+\kappa((-2\kappa(\kappa+1)+3\sqrt{1-4(\kappa-1)\kappa}+7))))\\
&-2((\sqrt{1-4(\kappa-1)\kappa}+2))))t-3))).
\end{aligned}$$

From F_{11} we obtain that

$$t = \frac{2\kappa\left(2\kappa^3-2\left(\sqrt{1-4(\kappa-1)\kappa}+2\right)\kappa-2\sqrt{1-4(\kappa-1)\kappa}-1\right)+\sqrt{1-4(\kappa-1)\kappa}+7}{2\left(\sqrt{1-4(\kappa-1)\kappa}\kappa+\kappa-2\right)\left(2\left(\sqrt{1-4(\kappa-1)\kappa}+2\right)+\kappa\left(\sqrt{1-4(\kappa-1)\kappa}+5\right)\right)}.$$

Analogously, from F_{44} we get that

$$\begin{aligned}t =&\frac{1}{f(\kappa)}(\kappa((\kappa((2\kappa((\kappa((6\kappa^2-8\kappa+7\sqrt{1-4(\kappa-1)\kappa}-15))-13\sqrt{1-4(\kappa-1)\kappa}+38))\\&-3\sqrt{1-4(\kappa-1)\kappa}-47))+13\sqrt{1-4(\kappa-1)\kappa}-3))-7\sqrt{1-4(\kappa-1)\kappa}-1).\end{aligned}$$

where $f(\kappa) = 8(\kappa-1)^2\left(\kappa^2+\kappa+1\right)(\kappa(\kappa+2)+3)$. Thus, if these two values are different, then the system cannot vanish. Both values for t are the same if and only if

$$(\kappa-1)\left(2\kappa\left(\kappa\left(-2(\kappa-1)\kappa+\sqrt{1-4(\kappa-1)\kappa}+1\right)+1\right)+\sqrt{1-4(\kappa-1)\kappa}-1\right)=0,$$

which only has real solutions if $\kappa = 0$ or $\kappa = 1$, which are not possible values for κ since $\kappa \in (0,1)$ and the result follows. $\square$

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