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DOCTORAL THESIS

Morse Theory on Finite Spaces

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Chapter 1
Introduction

Topology and Dynamics

The relation between the topology of a space and the kind of dynamics which can be defined on it is a classical and popular topic of research. Let us briefly discuss this. On the one hand, for a space X, the dynamics defined on X provide topological information about X. For an example of this phenomena, consider a gradient vector field on a smooth manifold X with only one critical point. The vector field provides a homotopy equivalence between the manifold and a point; a way to deform the manifold to a point. Conversely, consider some dynamics and a manifold X. There exist topological conditions on X so that dynamics can be realized on X. For instance, it is not possible to define a gradient vector field with exactly two critical points on the torus. The one who first noticed the examples we just presented was Morse ([127]), who related the "potential" or "height" functions (the ones corresponding to gradient vector fields) on manifolds with the topology (homology in fact) of the manifold. His observations were the beginning of what is now referred to as Morse theory.

Not only Morse realized about this strong connection between Topology and Dynamics. At more or less the same time as Morse's work, Lusternik and Schnirelmann ([114]) were trying to understand the topology of manifolds through some class of well-behaved functions defined on them. The results they obtained were more general, in the sense that they are more widely applicable, but also provide, in general, less information about the topology of manifolds. In fact, both Morse theory and this Lusternik-Schnirelmann theory may be seen as complementary. Another theory which relates Dynamics and Topology, and can be seen as complementary to Morse theory is the so called Conley theory, introduced by Conley ([48]). This theory allows us to deal with much wilder dynamics on the space under study than Morse theory at the expense of providing less topological information in return.

Morse theory

Morse theory evolved rapidly due to its connections and applicability in different problems (see for example [33, 34, 155–158]). In particular, not long after Morse's first exposition, topologists became aware that Morse theory did not only provide information about the homology of the manifold under study, but also its homotopy type. This led to the so called Structure Theorem of Morse theory:

Theorem (Structure Theorem of Morse theory)**.** *Let X be a compact manifold and let $f\colon X \to \mathbb{R}$ be a Morse function. Then, X is homotopy equivalent to a CW-complex with exactly one p-cell for each critical point of index p.*

Basically, this result guarantees that Morse functions (equivalently, their associated gradient dynamics) contain the homotopical information about the manifold. In fact, the homotopical information is localized or concentrated at the critical points. For a classical exposition of Morse theory we recommend [32, 35–37, 102, 125].

Let us review our previous discussion with a particular example, which, as to follow the steps of every classic exposition of Morse theory, it will be a torus. Consider a torus X in $\mathbb{R}^3$ as pictured in Figure 1.0.1 and let $f\colon X \to \mathbb{R}$ be the height function given by $f(x,y,z) = z$.

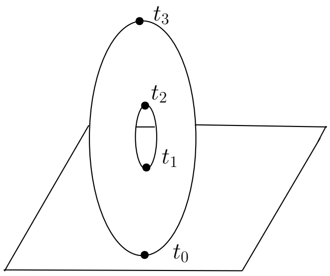


Figure 1.0.1: Torus in $\mathbb{R}^3$ tangent to the plane $z = 0$.

Observe that this function induces a gradient vector field going "upwards". Our objective is to relate the dynamics to the topology of the torus, so first observe that there are four critical points for the map f. A global minimum of f, a global maximum of f, and two saddle points. Now let us scan the torus using our function f, that is, let us consider a filtration of X induced by f. We denote $X_t = f^{-1}((-\infty, t])$.

Let us say that the critical values of f are denoted by $t_0 < t_1 < t_2 < t_3$. Observe that if $t < t'$ are two real numbers in $[t_i, t_{i+1})$ for $i \in \{0,1,2\}$, then $X_{t'}$ deforms to X_t (the deformation may be performed following the gradient flow in the opposite direction). Hence, it only remains to understand what happens when we reach a critical value of f. In order to do that, we need to recall that the *index of a critical point* p of f is just the dimension of the maximal subspace of the tangent space T_pX on which the Hessian of f is negative definite. Observe that in this example, when we reach t_0 we just add a 0-cell, when we reach both t_1 and t_2 we adjunction one 1-cell for each of the critical points. Finally, when we reach t_3, we adjunction a 2-cell.

Morse's original approach to this did not take homotopy into account. He just proved that the homology does not change while critical points are not reached and that when a critical point is reached, the change in homology is under control. That way one can consider a filtration on the manifold. Let us restrict our attention to the torus for simplicity:

$$X_{t_{-1}} = \emptyset \hookrightarrow X_{t_0} \hookrightarrow \cdots \hookrightarrow X_{t_j} \hookrightarrow X_{t_{j+1}} \hookrightarrow \cdots \hookrightarrow X_{t_n} = X. \tag{1.1}$$

Moreover, the filtration can be chosen so that for every j, there is at most one critical point in $X_{t_{j+1}} - X_{t_j}$. That way, the homology of the pair $H_*(X_{t_{j+1}}, X_{t_j})$ is either trivial or it is controlled. In fact, Morse proved that if the homology is not trivial, then $H_*(X_{t_{j+1}}, X_{t_j})$ is trivial except for the dimension equal to the index of the critical point and that in that dimension it is the infinite abelian group with one generator (see the exposition [141]). As a consequence, he managed to relate the dynamics and the topology of X by means of the *(strong) Morse inequalities*:

Theorem (Morse inequalities)**.** *Let X be a manifold and let $f\colon X \to \mathbb{R}$ be a Morse function. We denote by m_i the number of critical points of index i and by b_i the Betti number of dimension i. Then,*

1. *For every $i \geq 0$ and domain of coefficients:*

$$m_i - m_{i-1} + \cdots + (-1)^i m_0 \geq b_i - b_{i-1} + \cdots + (-1)^i b_0.$$

2. *$m_i \geq b_i$ for every i.*

3. *The Euler-Poincaré Characteristic satisfies a Poincaré-Hopf Theorem:*

$$\chi(X) = \sum_{i=0}^{\dim(X)} (-1)^i b_i = \sum_{i=0}^{\dim(X)} (-1)^i m_i.$$

The next thing to do after studying the relation between topology and smooth functions with isolated non-degenerate critical points (Morse functions), is to allow the critical sets to be submanifolds. For an example of this situation consider the torus X in $\mathbb{R}^3$ depicted in Figure 1.0.2 and the function $f\colon X \to \mathbb{R}$ given by $f(x, y, z) = z$. That is the idea which led Bott to develop

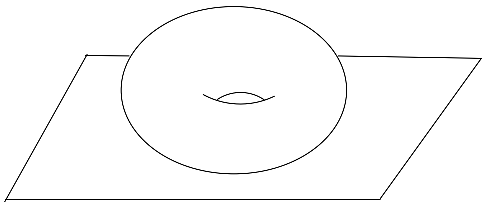


Figure 1.0.2: Torus in $\mathbb{R}^3$ tangent to the plane $z = 0$.

what is now called Morse-Bott theory ([32, 134]).

So far, we have mentioned the most classical approach to Morse theory. Nevertheless, the idea of localizing the homological or homotopical information at the critical points flourished notably. In fact, that led to at least another two approaches to Morse theory. One of them consists in constructing a chain complex, the *Morse complex*, using the flow of the gradient vector field ([134]). The homology of this complex is isomorphic to the homology of the manifold. The other one consists in using the flow of the gradient vector field to construct a topological loop-free category $\mathcal{C}$ (just a small category satisfying an extra condition) whose geometric realization recovers the homotopy of the manifold. This approach was initiated in an unpublished preprint of Cohen, Jones, and Segal, and it is now gaining popularity (see [40]).

Lusternik-Schnirelmann category and related invariants

Lusternik-Schnirelmann theory approaches the study of the relation between the critical sets of a smooth function $f\colon X \to \mathbb{R}$ (not necessarily Morse) defined on a manifold and the topology of the manifold. In order to study the topology of the manifold, homology is not used. Instead, Lusternik-Scnirelmann introduced what now is called the *Lusternik-Schnirelmann category*.

Let X be a topological space. The *Lusternik-Schnirelmann category* (LS category), denoted by $\mathrm{cat}(X)$, is the minimum number of open sets which cover X and such that each of them is contractible in X. If there is not a finite covering satisfying this property, we say that the category is infinite. For example, it is easy to convince ourselves that a sphere has category 2. In general, it is not difficult to realise that compact manifolds have finite category. However, the computation of the Lusternik-Schnirelmann category for manifolds is a very hard problem (see for example [118, 119]).

The LS category is a widely studied homotopic invariant since it appears in many areas of Mathematics (see for instance [49] for an exposition). However, the LS category is just one of the many invariants one may define to study topological properties of a space X. More generally, a categorical invariant of a space X is usually defined as the smallest number of open sets that cover X and that satisfy certain properties, such as being elementary in a certain sense, for instance acyclic or contractible (see [83, 89] for more examples). More generally, and analogously, a categorical invariant of an object X (such as a simplicial complex) can be defined as the smallest number of subobjects needed to cover X and that verify certain properties (see for example [64,65,117,170]). In vague terms, such an invariant provides a certain measure of the complexity of an object. For instance, the Lusternik-Schnirelmann category measures, in a particular manner, how far is a space from being contractible. Notice that categorical does not have here the same meaning as when speaking of Category Theory.

For a topological space X, there are other related homotopic invariants which are related to the LS category and have applications in Engineering or Robotics. One of those is the *topological complexity* of X, denoted by $\mathrm{TC}(X)$, and introduced by Farber in 2003 ([61]). Roughly speaking, $\mathrm{TC}(X)$ measures the instabilities of any motion planning algorithm in X (see the survey [62]).

In recent years, a lot of new invariants related to the category and topological complexity have appeared (see for instance [131,139,140,150,180]). They have a wide range of applications, from motion planning and robotics to social choice (see [42]). The interest in studying all of them at the same time, so we could understand better their relationships, led us to introduce another invariant which unifies them all, that is the *homotopic distance between continuous maps*. This invariant rapidly received attention (see for example [29–31]).

Discrete structures and computation

The emerging fields of Applied and Computational Topology (see [58–60,88,100,146,181,182]) have resulted in a huge popularization of discrete and finite contexts, where algorithms to perform concrete and complete calculations can be developed. In the following diagram we represent different settings in the finite context, along with their connections and their relation to classical objects such as manifolds.

$$
\begin{tikzcd}
\text{Small Cat} \arrow[r, "\mathcal{N}"] \arrow[d, shift left, "\mathrm{sd}"] & \text{Simplicial sets} \\
\text{Loop-free Cat} \arrow[r, "\mathcal{N}"] \arrow[u, hook, shift left] \arrow[d, shift left, "\mathrm{sd}"] & \text{Regular } \Delta\text{-Complexes} \arrow[u, hook] \arrow[dl, "\mathcal{X}"'] \\
\text{Posets} \arrow[u, hook, shift left] \arrow[r, shift left, "\mathcal{K}"] & \text{Simplicial Complexes} \arrow[u, hook] \arrow[l, shift left, "\mathcal{X}"] \\
\text{Chain Complexes} \arrow[u] & \text{Manifolds} \arrow[u]
\end{tikzcd}
\tag{1.2}
$$

We proceed with a brief discussion of the diagram.

First of all, (smooth) manifolds can be triangulated to give simplicial complexes in such a way that some invariants of the manifold can be related to analogous or associated invariants in the simplicial complex ([81, 82]). We will develop this idea further in the next subsection.

Simplicial complexes and partially ordered sets (posets) are related, as was first observed by Alexandroff ([3]) and later McCord ([123, 124]), by what are now sometimes referred as McCord or Alexandroff-McCord functors. Moreover, McCord realized ([123,124]) that this functors induce weak homotopy equivalences. This relation can be extended to more general cell complexes such as regular $\Delta-$complexes, which, roughly speaking, are just simplicial complexes whose simplices are not completely determined by their set of vertices. Regular $\Delta-$complexes play an important role since the nerve of a loop-free category (a small category such that the only arrows with the same source and target are identities) is a regular $\Delta-$complex.

Posets are examples of loop-free categories. Conversely, every loop-free category can be subdivided to obtain a poset (as it happens with regular $\Delta-$complexes and simplicial complexes). This same relation between posets and loop-free categories repeats itself for loop-free categories and small categories. It is worth mentioning as well that it is possible to associate a simplicial set to every small category by the well-known nerve functor. For a detailed treatment of the relations we have mentioned so far, we recommend the works of May ([121, 122]), Tanaka ([171, 172]) and Del Hoyo ([53]).

Finally, every chain complex with a distinguished basis can be seen as a poset. We will explain this in detail in Chapter 3.

Topology and Dynamics in discrete structures

The recent interest in discrete structures motivated the development of combinatorial analogues of some techniques and results in the setting of manifolds. To mention some examples:

- Leinster ([22, 111]) developed the notion of Euler-Poincaré characteristic for categories. This allowed Tanaka to develop a theory of integration against the Euler-Poincaré characteristic in the setting of finite spaces ([167, 168]), extending the previous theory for the context of simplicial complexes and manifolds by Ghrist and Baryshnikov ([17, 18, 50]).

- A notion of curvature and several classical results from Riemannian Geometry were also adapted to finite contexts (see [27, 80, 82, 91, 162, 163, 177, 178]) with applications to life sciences (see for example [16, 99, 148, 149, 159, 160, 179]).

- The combinatorial Laplacian introduced by Eckmann ([57]) was recovered in the last twenty years and further studied ([80, 96, 103, 120, 138, 161]).

- Combinatorial Hodge theory has attracted a lot of interest recently (see [2, 80]).

- Barmak and Minian have revitalised the study of finite topological spaces (posets), first studied by Alexandroff ([3]), McCord ([123, 124]), Stong ([164, 165]) and May ([122]) by means of showing the particularities of some constructions of Algebraic Topology in that context ([6–15]).

- Curry, Ghrist, and Hansen have developed a combinatorial approach to cosheaves and sheaves on complexes ([51, 92, 93]).

We will now be focusing our attention in the combinatorial analogues of Morse Theory, dynamics, and Lusternik-Schnirelmann type invariants on the objects considered in Diagram (1.2).

First, Fernández, Macías, Minuz, and Vilches developed combinatorial analogues of the Lusternik-Schnirelmann category and topological complexity for simplicial complexes ([64, 65, 68]). Tanaka developed a notion of Lusternik-Schnirelmann category for simplicial complexes ([166]) and a notion of topological complexity for posets ([169]).

When it comes to Morse theory, there have been lots of different proposals of Morse theories in the discrete context:

- Bestvina and Brady developed a Morse Theory for cell complexes ([23]) in order to solve a problem in Geometric Group Theory.

- Banchoff developed a Morse theory for polyhedrons taking as the starting point the notion of curvature ([4, 5]).

- D. Fernández, Macías, Scoville, and Vilches proposed a theory oriented to control the strong homotopy type of simplicial complexes ([67]).

- X. Fernández developed a theory to take into account n-deformations of cell complexes in order to study the Andrews-Curtis Conjecture (see [69, 70]).

- Brown developed a Morse theory for simplicial sets and obtained a Structure Theorem in this context ([38, 39]).

- Forman developed a purely combinatorial Morse theory for simplicial complexes and regular CW-complexes ([73–79, 81, 82]). He proved some Structure Theorems for this theory (see Chapter 4). He also extended this theory to allow more general critical objects (as an analogue of Morse-Bott theory). Moreover, he proved that the Witten approach to Morse theory can be carried out in this context. Furthermore, he also showed how to define a Morse complex in this setting.

- Following Forman's theory, many researchers tried to extend and to improve his work. Chari realized that some notions of Forman's theory could be stated in terms of the face poset associated to the simplicial complex under study ([43]) and that Freij developed an equivariant version of the theory ([85]).

- Very recently, Nanda, Tamaki, and Tanaka proved that it is possible to define a category from a generalization of the flow induced by a Morse function (in the sense of Forman) on a simplicial complex and that this category records the weak homotopy of the simplicial complex ([132]). That is, they proved that the classifying space of the "flow category" is homotopy equivalent to the simplicial complex.

- Kozlov ([105] and Minian ([126]) developed Morse theories for posets. Later, some of Minian's results were extended to the infinite case by Kukiela ([107]).

- Kozlov ([104, 105]) and Skoldberg ([153]) developed Morse theories for chain complexes. They are called *algebraic Morse theories*. Later, Donau developed an equivariant version of Kozlov's Morse theory ([54]).

- There have been some attempts by Fernández, Macías, Scoville, and Vilches ([67]), by Knudson and Johnson ([98]), by Aaronson and Scoville ([1]), and by Tanaka ([170]) to relate combinatorial notions of Lusternik-Schnirelmann category with a combinatorial Morse theory to obtain a Lusternik-Schnirelmann Theorem.

As we mentioned earlier, strongly related to Morse theory is Conley theory. A combinatorial analogue of Conley theory in the setting of simplicial complexes and posets was developed by Lipiński, Kubica, Mrozek and Wanner ([112]). This is the continuation of the work of Mrozek on a combinatorial Conley theory for a certain family of chain complexes ([128]) and of the work of Kaczynski, Mrozek and Wanner ([176]) on combinatorial dynamics.

Our work in context: the problems we address in this thesis and its organization

Our main objective is the study of Morse theory and related homotopical invariants, such as the homotopic distance, Lusternik-Schnirelmann category and topological complexity, on finite contexts.

Chapter 2 is the starting point of our study. We both recall some brief preliminaries and state new definitions which facilitate our posterior work. We have tried to distinguish between the results already in the literature and original ones. We may highlight some of the latter ones. In the first part of the chapter, we prove that the Euler-Poincaré characteristic admits a reinterpretation for a certain class of posets in terms of the heights of the elements (Theorem 2.15). This theorem plays an important role not only in its own but also for the latter curvature related results and Morse inequalities. We also prove a form of Poincaré duality for a family of posets (Theorem 2.79). In particular, the statement holds for finite closed homology manifolds (see Definition 2.76).

Another achievement of the chapter is to formulate and explain Diagram (1.3) (see Subsection 2.2.7), which relates different families of posets and cell complexes.

$$\begin{tikzcd}
 & & T_0\text{-FTop} \cong \text{Pos} \arrow[ddddr, "\mathcal{K}"] & \\
\text{T-wPos} \arrow[urr, hook] & \text{CPos} \arrow[r, hook] & \text{D-wPos} \arrow[u, hook] & \text{h-rCW} \arrow[dl, "\mathcal{X}"] \\
 & \text{haPos} \arrow[ul, hook] \arrow[u, hook] & \text{h-rPos} \arrow[u, hook] & \text{rCW} \arrow[u, hook] \arrow[dl, "\mathcal{X}"] \\
 & & \text{HaPos} \arrow[ul, hook] \arrow[u, hook] & \text{SimC} \arrow[u, hook] \\
 & & \text{FChMan} \arrow[u, hook] \arrow[r, shift left, "\mathcal{K}"] & \text{ChMan} \arrow[u, hook] \arrow[l, shift left, "\mathcal{X}"]
\end{tikzcd} \tag{1.3}$$

The second part of the chapter is devoted to the study of curvature. We begin with the definition of a combinatorial Laplacian and the statement of a combinatorial Hodge theorem for posets (Theorem 2.84). This is used to develop the relation between homology of posets and their curvature. In particular, we prove that if the *Ricci curvature* is positive at every point, then the homology in dimension one vanishes (Corollary 2.92). Finally, we state a Gauss-Bonnet theorem (Theorem 2.94).

In Chapter 3 we begin the study of Morse theory on posets. We define the notion of *tame Morse function* (Definition 3.13), which is an improvement of the Morse functions, in the sense that facilitates the derivation of some results.

We also prove a classical theorem of calculus in this context stating that global minimums for Morse functions are critical points (Theorem 3.34). We then relate combinatorial vector fields or matchings (dynamics) with Morse functions. First, we show how to obtain gradient matchings from Morse functions (Theorem 3.42). Then, we prove that it is possible to go the opposite way and integrate grandient matchings to obtain tame Morse functions (Theorem 3.47).

After that, we prove that Morse theory may be seen as a theory of nice filtrations for a space (Theorem 3.51). We also show that for this nice theory of filtration of a space, we may restrict our attention to tame Morse functions instead of just Morse ones (Theorem 3.52).

Finally, we prove a result about the cancellation of critical points (Theorem 3.53), which is identical to the case in Discrete Morse theory for cell complexes.

Chapter 4 is devoted to studying the Structure Theorems of Morse theory in this context. First, we give a negative answer to the following two conjectures:

Conjecture 4.6 Let X be well-behaved poset and let $f\colon X \to \mathbb{R}$ be a well-behaved Morse function on X. Then, X is homotopy equivalent to a poset with exactly one element of height p for each critical point of index p.

Conjecture 4.8 Let X be well-behaved poset and let $f\colon X \to \mathbb{R}$ be a well-behaved Morse function on X. Then, X is homotopy equivalent to a CW-complex with exactly one one p-cell for each critical point of index p.

This leads us to looking for weaker Structure Theorems. We prove two invariance results (Theorems 4.19 and 4.21) and a collapsing theorem (Theorem 4.25). We also prove an adjunction theorem (Theorem 4.23).

As a consequence of our results, we prove several theorems. First, we improve Forman's Structure Theorem by extending it to more general cell complexes (Theorem 4.30). Furthermore, we prove a strengthening of the Morse inequalities in this context (Theorem 4.37).

In Chapter 5 we study more general dynamics by developing a Morse-Bott theory for posets. We prove the Structure Theorems in this context (Theorems 5.12 and 5.13). As a consequence, we obtain several versions of the Morse-Bott inequalities (see for example Theorem 5.23). To finish the chapter, we introduce a new notion of Lusternik-Schnirelmann category and we prove a Lusternik-Schnirelmann category for Morse-Bott functions in this context (Theorem 5.33).

Moreover, as a consequence of Remarks 3.18 and 3.31, Theorems 3.42, 3.47 and 3.52, all Morse theory and Morse-Bott theory we have developed so far, apply to the three contexts: the algebraic setting on Lefschetz-Kubica-Mrozek complexes, the discrete setting on h-regular CW-complexes and the poset context.

Chapter 6 addresses the extension of Morse(-Bott) theory to loop-free categories. The purpose of the chapter is to move upwards in Diagram (1.4).

$$
\begin{tikzcd}
\text{Small Cat} \arrow[r, "\mathcal{N}"] \arrow[d, shift left, "\mathrm{sd}"] & \text{Simplicial sets} \\
\text{Loop-free Cat} \arrow[u, hook, shift left] \arrow[r, "\mathcal{N}"] \arrow[d, shift left, "\mathrm{sd}"] & \text{Regular } \Delta\text{-Complexes} \arrow[u, hook] \arrow[dl, "\mathcal{X}"'] \\
\text{Posets} \arrow[u, hook, shift left] \arrow[r, shift left, "\mathcal{K}"] & \text{Simplicial Complexes} \arrow[u, hook] \arrow[l, shift left, "\mathcal{X}"] \\
\text{Chain Complexes} \arrow[u] & \text{Manifolds} \arrow[u, hook]
\end{tikzcd}
\tag{1.4}
$$

We prove a Structure Theorem in this setting (Theorem 6.31). As a consequence, we prove the Strong and Weak Morse-Bott inequalities (Theorems 6.34 and 6.35).

Finally, Chapter 7 is devoted to the study of two notions of *homotopic distance between functors* and *weak homotopic distance between functors*, which are the natural counterpart to the homotopic distance between maps in the discrete setting. We show that these notions generalize Tanaka's definition of Lusternik-Schnirelmann category for small categories ([170]) (see Proposition 7.15). Later we prove that it is possible to define the categorical complexity of a small category by means of our notion of homotopic distance between functors (Theorem 7.17). Then we prove several properties. For example, we compare different notions of homotopic distance in several contexts (Proposition 7.36). Finally, we restrict our attention to the setting of posets and prove several properties. We mention two of them. One relating different notions of homotopic distance in the setting of posets (Theorem 7.48). And the second one showing that this notions approximate their continuous counterparts (Theorem 7.51).

Part I

Combinatorial Algebraic Topology and Morse Theory on posets

Chapter 2
Posets and cellular structures

In order to develop topological techniques for posets, we need first to endow them with a natural topology. That is the beginning of this chapter. We both recall some preliminaries from the literature and introduce novel notions, trying to make clear which ones are original. Moreover, we prove classical results of Algebraic Topology on manifolds for this context. Furthermore, we establish relations between different classes of posets (some new and others previously defined in the literature). We summarize them in Diagram (2.1):

$$
\begin{tikzcd}
 & & T_0\text{-FTop} \cong \text{Pos} \arrow[dddr] & \\
\text{T-wPos} \arrow[urr, hook] & \text{CPos} \arrow[r, hook] & \text{D-wPos} \arrow[u, hook] & \text{h-rCW} \arrow[dl, "\mathcal{K}"'] \\
 & \text{haPos} \arrow[ul, hook] \arrow[u, hook] & \text{h-rPos} \arrow[u, hook] & \text{rCW} \arrow[u, hook] \arrow[dl, "\mathcal{X}"'] \\
 & & \text{HaPos} \arrow[ul, hook] \arrow[u, hook] & \text{SimC} \arrow[u, hook] \\
 & & \text{FChMan} \arrow[u, hook] \arrow[r, shift left, "\mathcal{K}"] & \text{ChMan} \arrow[u, hook] \arrow[l, shift left, "\mathcal{X}"]
\end{tikzcd}
\tag{2.1}
$$

2.1 Topology of Posets

2.1.1 Topologizing Posets

Our first goal is to endow partially ordered sets with topologies. We reproduce a construction due to Alexandroff ([3]) and we refer the reader to [6, Sections 1.1 and 1.2] for the proofs.

Recall that a topological space (X, τ) is T_0 if for any two distinct points $x, y \in X$, there is either an open subset U containing x but not y, or an open subset V containing y but not x.

Let X be a finite set endowed with a partial order relation, that is, X is a finite *poset*. Then, X is a T_0 finite topological space with basis the sets $\{U_x\}_{x \in X}$ where

$$U_x = \{w \colon w \leq x\}.$$

We may refer to this basis as the *minimal basis of* X.

Conversely, let X be a finite T_0 topological space. For each $x \in X$, we denote by U_x the intersection of all the open sets that contain x. There is a partial order on the set X by declaring: $x \leq y$ if and only if $U_x \subset U_y$.

Moreover, these correspondences are mutually inverses of each other. Furthermore, let X and Y be both finite T_0 topological spaces and posets (with the associated structures), then a set map $f\colon X \to Y$ is continuous if and only if it is order preserving, that is, if $x \leq y$ then $f(x) \leq f(y)$.

To sum up, in modern language we may state that there is an isomorphism of categories between the category of finite posets with order preserving maps and the category of finite T_0 topological spaces and continuous maps.

Additionally, homotopies can be studied combinatorially. Let X and Y be two finite posets. Then we can endow the finite set Y^X with the point-wise order: $f \leq g$ if $f(x) \leq g(x)$ for every $x \in X$.

Proposition 2.1. *Let $f, g\colon X \to Y$ be two order-preserving maps between posets. Then f and g are homotopic, $f \simeq g$ if and only if there is a sequence of order-preserving maps*

$$f = f_0 \leq f_1 \geq f_2 \leq \cdots \leq f_n = g.$$

Corollary 2.2. *Let X be a finite poset with a maximum or a minimum. Then, X is contractible.*

Proof. Suppose that X has a maximum since the other case is analogous. If X has a maximum $x' \in X$, then the identity map of X is less than or equal to the constant map at x'. □

2.1.2 Homology of posets

Homology for posets is a widely studied topic (see for example: [41, 45, 52, 55, 90, 136]). We shall consider a special kind of posets called cellular. These are of interest since a “cellular homology theory” can be defined on them. They were first introduced by Farmer ([63]) and then recovered by Minian ([126]) and Cianci and Ottina ([45]). Farmer’s definition is more general while Minian’s one is more adequate for our purposes. That is why we present the latter one.

We need to introduce some terminology.

Definition 2.3. A *chain* in a poset X is a subset $C \subseteq X$ such that if $x, y \in C$, then either $x \leq y$ or $y \leq x$.

Definition 2.4. The *height* of a poset X is the maximum length of the chains in X, where the chain $x_0 < x_1 < \cdots < x_n$ has length n. The height $h(x)$ of an element $x \in X$ is the height of $U_x = \{w \in X : w \leq x\}$ with the induced order.

Definition 2.5. A poset X is said to be *homogeneous* of degree n, $\deg(X) = n$, if all maximal chains in X have length n. A poset is *graded* if U_x is homogeneous for every $x \in X$. In that case, the *degree* of x, denoted by $\deg(x)$, is its height. We will denote both the height and degree of an element by superscripts, for example $x^{(p)}$.

Let X be a finite poset, $x, y \in X$. If $x < y$ and there is no $z \in X$ such that $x < z < y$, we write $x \prec y$.

For $x \in X$ we define $\widehat{U}_x := \{w \in X \colon w < x\}$ as well as $F_x := \{y \in X : y \geq x\}$ and $\widehat{F}_x := \{y \in X : y > x\}$.

Let X be a poset. We denote by $H_*(X)$ the singular homology of X. Unless stated otherwise, homology will be considered with coefficients in a principal ideal domain.

Definition 2.6 ([126])**.** The poset X is *cellular* if it is graded and for every $x \in X$, $\widehat{U}_x$ has the homology of a $(p-1)$-sphere, $\mathbb{S}^{p-1}$, where p is the degree of x.

Remark 2.7. In Definition 2.6, by the homology of a $(p-1)$-sphere, we mean that $\widehat{U}_x$ has reduced trivial homology in dimensions different from $p-1$ and has $H_{p-1}(U_x)$ isomorphic to the free module with one generator over the coefficient ring.

We recall the construction due to Farmer ([63]) and Minian ([126]) of a "cellular homology theory" for cellular posets.

Definition 2.8. Given a finite graded poset X, we define $X^{(p)}$ as the subposet of elements of degree less than or equal to p, i.e.

$$X^{(p)} = \{x \in X \colon \deg(x) \leq p\}.$$

Given the cellular poset X, there is a natural filtration by the degree

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n)} = X$$

which allows to define a *cellular chain complex* (C_*, d) as follows:

$$C_p(X) = H_p(X^{(p)}, X^{(p-1)}) = \bigoplus_{\deg(x)=p} H_{p-1}(\widehat{U}_x),$$

which is a free abelian group with one generator for each element of X of degree p. The differential $d \colon C_p(X) \to C_{p-1}(X)$ is defined as the composition

$$H_p(X^{(p)}, X^{(p-1)}) \xrightarrow{\partial} H_{p-1}(X^{(p-1)}) \xrightarrow{j} H_{p-1}(X^{(p-1)}, X^{(p-2)})$$

where j is the canonical map induced by the inclusion and ∂ is the conecting homomorphism coming from the long exact sequence associated to the pair $(X^{(p)}, X^{(p-1)})$. It can be shown (see [126]) that the differential can be written as

$$d(x) = \sum_{w \prec x} \epsilon(x, w) w$$

where the incidence number $\epsilon(x, w)$ is the degree of the map

$$\widetilde{\partial} \colon \mathbb{Z} = H_{p-1}(\widehat{U}_x) \to H_{p-2}(\widehat{U}_w) = \mathbb{Z}$$

(see [126]).

Theorem 2.9 ([126, Theorem 3.7])**.** *Let X be a cellular poset. Then*

$$H_*(C_*(X)) \cong H_*(X).$$

Remark 2.10. A new approach for studying homology of posets has been developed more recently by Cianci and Ottina ([45]). They weakened the hypothesis needed in Theorem 2.9.

2.1.3 Homologically admissible posets

We present the notion of homologically admissible posets recently introduced by Minian ([126]). Recall that the *Hasse diagram* of a poset X is a graph whose vertices are the points of X and whose edges are the ordered pairs (x, y) such that $x \prec y$. We denote by $\mathcal{H}(X)$ the Hasse diagram associated to the poset X.

Definition 2.11. Let X be a poset. An edge $(w, x) \in \mathcal{H}(X)$ is homologically admissible if $\widehat{U}_x - \{w\}$ is acyclic. A poset is *homologically admissible* if all its edges are homologically admissible.

The importance of homologically admissible posets, lies, partially, in the following result.

Lemma 2.12 ([126, Remark 3.9])**.** *If (w, x) is a homologically admissible edge of a cellular poset X, then the incidence number $\epsilon(x, w)$ is 1 or -1.*

Lemma 2.13 ([126])**.** *Let X be a poset. If X is homologically admissible, then it is cellular.*

2.1.4 Euler-Poincaré Characteristic

We introduce the Euler-Poincaré characteristic in this setting since we will use it later on. Recall that for a topological space X, its Euler-Poincaré characteristic is defined as:

$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$$

if the sum converges.

We introduce an alternative, and more suited definition for computations. This definition is also introduced by Bloch ([27]) in a different context in order to prove a Gauss-Bonnet Theorem in the setting of posets.

Definition 2.14. Let X be a finite graded poset of degree n. Denote by $X^{(=p)}$ the elements of degree p in X. The *graded Euler-Poincaré characteristic* of X is defined as the number:

$$\chi_g(X) = \sum_{p=0}^{n} (-1)^p \# X^{(=p)}.$$

Observe that as a consequence of Minian's result (Theorem 2.9), the standard homological argument (see for example [95, p. 146-147]) proves:

Theorem 2.15. *Let X be a finite cellular poset. Then,*

$$\chi_g(X) = \chi(X). \tag{2.2}$$

We provide an elementary combinatorial proof. We will show this by induction both on the degree of the poset and the number of elements in each degree. We assume that all spaces are connected for simplicity. If not, we would consider each path-component independently. We start with an immediate lemma.

Lemma 2.16. *Assume that X is a finite cellular poset such that $\deg(X) = 0$, then Equation (2.2) holds for X.*

Lemma 2.17. *Assume that X is a finite cellular poset such that $\deg(X) = p+1$, $X^{(=p+1)} = \{x\}$ and Equation (2.2) holds for $X^{(p)}$, then Equation (2.2) holds for $X^{(p+1)}$.*

Proof. First, consider the open subsets: $U = X^{(p)}$ and $V = U_x$ and apply the Inclusion-Exclusion Principle for the Euler-Poincaré characteristic:

$$\begin{aligned}\chi(X) &= \chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V) \\ &= \chi_g(U) + \chi(U_x) - \chi(\hat{U}_x)\end{aligned}$$

Second, notice that $\chi_g(X) = \chi_g(U) + (-1)^{\deg(x)}$. There are two cases to consider depending on the parity of $\deg(x)$. In both of them

$$(-1)^{\deg(x)} = \chi(U_x) - \chi(\hat{U}_x) = 1 - \chi(\mathbb{S}^{\deg(x)-1}). \quad \square$$

We state a useful observation as a lemma:

Lemma 2.18. *The graded Euler-Poincaré characteristic satisfies the Inclusion-Exclusion Principle for open sets, that is: if U and V are open sets in X, then $\chi_g(U \cup V) = \chi_g(U) + \chi_g(V) - \chi_g(U \cap V)$.*

Lemma 2.19. *Assume that X is a finite cellular poset such that $\deg(X) = k$, $\#X^{(=p)} \geq 2$ and Equation (2.2) holds for $X^{(p)} - \{x\}$ where $\deg(x) = p$. Then Equation (2.2) holds for $X^{(p)}$.*

Proof. Let us denote by $\{x_1, \ldots, x_n = x\}$ the elements of degree k of X. Consider the open subsets:

$$U = X^{(p-1)} \cup (\cup_{i=1}^{n-1} U_{x_i}) = X^{(p)} - \{x\} \quad \text{and} \quad V = U_{x_n}.$$

Applying the Inclusion-Exclusion Principle for the Euler-Poincaré characteristic we obtain:

$$\begin{aligned}\chi(X) &= \chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V) \\ &= \chi(\cup_{i=1}^{n-1} U_{x_i}) + \chi(U_x) - \chi(\cup_{i=1}^{n-1} U_{x_i} \cap U_x) \\ &= \chi(\cup_{i=1}^{n-1} U_{x_i}) + \chi(U_x) - \chi(\cup_{i=1}^{n-1} \hat{U}_{x_i} \cap \hat{U}_x) \\ &= \chi_g(\cup_{i=1}^{n-1} U_{x_i}) + \chi_g(U_x) - \chi_g(\cup_{i=1}^{n-1} \hat{U}_{x_i} \cap \hat{U}_x) \quad \text{(Lemma 2.17)} \\ &= \chi_g(U \cup V) \quad \text{(Lemma 2.18)}\end{aligned} \quad \square$$

Combining the previous lemmas we obtain the desired result.

Remark 2.20. The good behavior of the Euler-Poincaré characteristic in the setting of finite posets suggests the study of the Lefschetz number. This was initiated by Bilski in [24].

2.1.5 (Weakly) Homotopically Admissible Posets

We recall the notion of homotopically admissible posets which were also introduced by Minian ([126]). They are a homotopical analogue to the homologically admissible posets.

Definition 2.21. Let X be a poset. An edge $(w, x) \in \mathcal{H}(X)$ is weakly homotopically admissible if $\widehat{U}_x - \{w\}$ is weakly contractible (i.e. has trivial homotopy groups). A poset is *weakly homotopically admissible* if all its edges are weakly homotopically admissible.

We introduce an original weaker notion which has the advantage of being completely determined with a computer (see [6, Section 2.4]).

Definition 2.22. Let X be a poset. An edge $(w, x) \in \mathcal{H}(X)$ is 1-weakly homotopically admissible if $\widehat{U}_x - \{w\}$ is simply connected. A poset is *1-weakly homotopically admissible* if all its edges are 1-weakly homotopically admissible.

Remark 2.23. Observe that by Hurewicz theorem (see [95, Theorem 4.32]), a homologically admissible and 1-weakly homotopically admissible space is weakly homotopically admissible.

We also introduce a stronger notion than weakly homotopically admissible:

Definition 2.24. Let X be a poset. An edge $(w, x) \in \mathcal{H}(X)$ is homotopically admissible if $\widehat{U}_x - \{w\}$ is contractible. A poset is *homotopically admissible* if all its edges are homotopically admissible.

2.1.6 Two-wide and down-wide posets

We define two classes of posets which will play a key role in the later development of Morse theory. Moreover, we relate them to homologically admissible posets.

The first notion, the one of *two-wide* posets, was already present, to some extent, despite being less general, in the work of Björner, Las Vergnas, Sturmfels, White and Ziegler (see [25, Corollary 4.7.12]). Later, and independently, it was also used by Bloch ([28]). We developed it independently.

Definition 2.25. A poset X is *two-wide* if for any $x, z, y \in X$ such that $x \prec z \prec y$, there is some $z' \in X$ such that $z' \neq z$ and $x \prec z' \prec y$.

Remark 2.26. Let X be a finite poset. If X is two-wide, then for any pair of elements $x, y \in X$ such that $x < y$ and $x \not\prec y$, we have $\#\{z \colon x \leq z \leq y\} \geq 4$.

We now introduce the notion of *down-poset*, which is another property that the family of posets we are interested in satisfies.

Definition 2.27. Given a poset X and $x \in X$, we define the *down-incidence number* of x as the cardinality of the set $\partial x = \{y \in X : y \prec x\}$. The poset X is *down-wide* if $\#\partial x \neq 1$ for every x in X.

We relate the properties of being homologically admissible with those of being two-wide and down-wide. We will prove that homologically admissible posets are down-wide and two-wide. In order to do so, we will use cellular homology with coefficients in the field with two elements, which we denote by $\mathbb{Z}_2$. Observe that for $\mathbb{Z}_2$ coefficients the differential $d\colon C_p(X) \to C_{p-1}(X)$ simplifies to the formula: $d(x) = \sum_{w \prec x} w$.

Proposition 2.28. *Let X be a poset. If X is homologically admissible, then it is down-wide.*

Proof. For any $x^{(1)}$ and $w \prec x$, since $\widehat{U}_x - \{w\}$ is acyclic, then $\widehat{U}_x - \{w\}$ is not empty, so there exists $w' \neq w$ such that $w' \prec x$. For any $x^{(p)}$ with $p > 1$, suppose that there is a unique $w \in X$ such that $w \prec x$. Using cellular homology $\mathbb{Z}_2$ coefficients:

$$d^2(x) = d(w) = \sum_{q \prec w} q \neq 0$$

which is a contradiction. □

Proposition 2.29. *Let X be a poset. If X is homologically admissible, then it is two-wide.*

Proof. Suppose there are elements $x \prec z \prec y$. We have to show that there is some $z' \neq z$ such that $x \prec z' \prec y$. Using cellular homology with $\mathbb{Z}_2$ coefficients:

$$dy = z + \sum_{\tilde{z} \neq z,\, \tilde{z} \prec y} \tilde{z}$$

and:

$$dz = x + \sum_{\tilde{x} \neq x,\, \tilde{x} \prec z} \tilde{x}.$$

Since $d^2 = 0$:

$$0 = d^2 y = dz + \sum_{\tilde{z} \neq z,\, \tilde{z} \prec y} d\tilde{z} = x + \sum_{\tilde{x} \neq x,\, \tilde{x} \prec z} \tilde{x} + \sum_{\tilde{z} \neq z,\, \tilde{z} \prec y} d\tilde{z}.$$

Since this equation holds, there must be some $z' \neq z$ such that

$$dz' = x + \sum_{\tilde{x} \neq x,\, \tilde{x} \prec z'} \tilde{x},$$

that is, there is some $z' \neq z$ such that $x \prec z' \prec y$. □

2.1.7 Local to global isomorphism theorems

Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous map. It induces maps in homology and between homotopy groups. Now we recall some results which provide sufficient conditions on the local behavior of the induced maps to obtain conclusions about their global properties. This kind of results have been widely studied in the literature (see for example [7, 20, 21, 26, 44, 143, 154, 174]). We begin with a definition taken from [6] to set the notion of "local" we will use.

Definition 2.30. Let X be a set, and suppose $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a *basis-like covering* of X if it satisfies the following two conditions:

1. $\bigcup_{B \in \mathcal{B}} B = X$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Theorem 2.31 (McCord, [124, Theorem 6])**.** *Let $f \colon X \to Y$ be a continuous map between topological spaces. Suppose there is a basis like open cover $\mathcal{U}$ of Y such that for every $U \in \mathcal{U}$, the restriction*

$$f_{|f^{-1}(U)} \colon f^{-1}(U) \to U$$

is a weak homotopy equivalence. Then $f \colon X \to Y$ is a weak homotopy equivalence.

Since for finite topological spaces the open sets of the minimal basis are contractible (Corollary 2.2), we obtain the following consequence in the context of finite posets:

Theorem 2.32 (McCord-Quillen)**.** *Let $f \colon X \to Y$ be a continuous map between posets. If for each $y \in Y$, $f^{-1}(U_y)$ is weakly contractible (i.e., it has trivial homotopy groups), then $f \colon X \to Y$ is a weak homotopy equivalence.*

Remark 2.33. In Proposition 2.32, the consequence holds as well with the hypothesis: for each $y \in Y$, $f^{-1}(F_y)$ is weakly contractible.

There is a homological version of this result, which we now state:

Theorem 2.34 (Homological McCord-Quillen, [15, Theorem 2.1])**.** *Let $f \colon X \to Y$ be a continuous map between posets. If for each $y \in Y$, $f^{-1}(U_y)$ is acyclic, then $f \colon X \to Y$ induces isomorphisms in homology.*

Remark 2.35. In Theorem 2.34, the consequence holds as well with the hypothesis: for each $y \in Y$, $f^{-1}(F_y)$ is acyclic.

2.1.8 Combinatorial homotopy and homology on posets

The strong connection between posets and T_0 spaces suggests that the combinatorics of the poset should control, to some extent, its topology. Now we address the study of the homotopy of T_0 spaces. In order to do so, we introduce some new terminology. We follow the works of Stong ([164]), Barmak and Minian ([10], [6]) and Osaki ([137]). The idea is to detect, for an arbitrary poset X, a special class of points which can be removed without changing the strong, simple, weak homotopy type or homology of X.

Definition 2.36. A point x in a finite poset X is a *down strong homotopic removable point* if the set $\hat{U}_x = U_x - \{x\}$ has a maximum. Dually, $x \in X$ is an *up strong homotopic removable point* if the set $\hat{F} = F_x - \{x\}$ has a minimum. In any of these cases we say that x is a *strong homotopic removable point* of X.

Remark 2.37. Other authors ([6, 122, 164]) refer to strong homotopic removable points as *beat points* or *linear and colinear points*.

Proposition 2.38 ([6, Proposition 1.3.4])**.** *Let X be a finite poset and let $x \in X$ be a strong homotopic removable point. Then, $X - \{x\}$ is a strong deformation retract of X.*

Definition 2.39. A point x in a finite poset X is a *down simple homotopic removable point* if $\hat{U}_x = U_x - \{x\}$ is contractible. Dually, $x \in X$ is an *up simple homotopic removable point* if $\hat{F} = F_x - \{x\}$ is contractible. In any of these cases we say that x is a *simple homotopic removable point* of X.

Definition 2.40. A point x in a finite poset X is a *γ-point* if $\widehat{C}_x = (U_x \cup F_x) - \{x\}$ is homotopically trivial.

We introduce now two original notions of removable points:

Definition 2.41. A point x in a finite poset X is a *down weak homotopic removable point* if $\hat{U}_x = U_x - \{x\}$ is weakly contractible (i.e., it has trivial homotopy groups). Dually, $x \in X$ is an *up weak homotopic removable point* if $\hat{F} = F_x - \{x\}$ is weakly contractible. In any of these cases we say that x is a *weak homotopic removable point* of X.

Proposition 2.42. *Let X be a finite poset and let $x \in X$ be a weak homotopic removable point. Then, $i\colon X - \{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

Proof. Apply Proposition 2.32. □

Definition 2.43. A point x in a finite poset X is a *down homological removable point* if $\hat{U}_x = U_x - \{x\}$ is acyclic. Dually, $x \in X$ is an *up homological removable point* if $\hat{F} = F_x - \{x\}$ is acyclic. In any of these cases we say that x is a *homological removable point* of X.

Proposition 2.44. *Let X be a finite poset and let $x \in X$ be a homological removable point. Then, $i\colon X - \{x\} \hookrightarrow X$ induces an isomorphism in homology.*

Proof. Apply Theorem 2.34. □

We introduce some notation.

Definition 2.45. Let X be a finite poset and $x \in X$ a strong homotopic removable point. We say that there is an *elementary strong collapse* from X to $X - \{x\}$ and we denote it by $X \overset{eS}{\searrow} X - \{x\}$. Conversely, we say that there is an *elementary strong expansion* from $X - \{x\}$ to X and we denote it by $X - \{x\} \overset{eS}{\nearrow} X$. If there is a sequence of posets $X_0, \ldots, X_n$ such that for each i, $X_i \overset{eS}{\searrow} X_{i+1}$, then we say that X_0 *strongly collapses* to X_n and we write $X_0 \overset{S}{\searrow} X_n$. Conversely, we say that X_n *strongly expands* to X_0, denoted $X_0 \overset{S}{\nearrow} X_n$. We say that X_0 and X_n are *strongly homotopy equivalent* if there is a sequence of posets $X_0, \ldots, X_n$ such that for each i, $X_i \overset{eS}{\searrow} X_{i+1}$ or $X_i \overset{eS}{\nearrow} X_{i+1}$.

We define the elementary simple (respectively weak homotopic, γ and homological) collapses in the case that x is a simple homotopic removable point (respectively a weak homotopic removable point, a γ-point and a homological removable point) as expected, and we denote them by $X \overset{es}{\searrow} X - \{x\}$, $X \overset{ew}{\searrow} X - \{x\}$, $X \overset{e\gamma}{\searrow} X - \{x\}$ and $X \overset{eh}{\searrow} X - \{x\}$, respectively. We denote the associated elementary expansions, collapses and expansions as expected.

Proposition 2.46. *Let X be a finite poset and $x \in X$.*

(1) If $x \in X$ a strong homotopic removable point, then it is a simple homotopic removable point.

(2) If $x \in X$ a simple homotopic removable point, then it is a weak homotopic removable point.

(3) If $x \in X$ a weak homotopic removable point, then it is a homological removable point and a γ-point.

Proof. First, (1) follows from Corollary 2.2. Second, (2) is a consequence of the homotopy invariance of homotopy groups. Third, (3) follows from Hurewicz Theorem (see for example [95, Theorem 4.37]) and from [6, Proposition 6.2.12]. $\square$

2.2 Cellular structures and their relation to posets

The reader may see [6, 10, 72, 87, 95, 105, 122] for a detailed exposition and examples of the non original notions we present.

2.2.1 Abstract Simplicial Complexes

Definition 2.47. A *finite abstract simplicial complex* K consists of a pair of sets:

1. a finite set of objects, $V(K)$, called *vertices*,
2. a (finite) set, $S(K)$, of (finite) non-empty subsets of $V(K)$, called *simplices*,

satisfying the following conditions:

1. if $\tau \subset V(K)$ is a simplex and $\sigma \subset \tau$, $\sigma \neq \emptyset$, then σ is also a simplex;
2. for every vertex $v \in V(K)$, the singleton $\{v\}$ is a simplex.

From now on, by abstract simplicial complex we mean finite abstract simplicial complex.

Remark 2.48. A subset of the vertices $\sigma \subset V(K)$ is a simplex if and only if all its subsets are elements of $S(K)$.

Example 2.49. Consider the abstract simplicial complex $K = (V(K), S(K))$ where $V(K) = \{a, b, c\}$ and $S(K) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$.

From now on we may omit the vertex set of the simplicial complexes and just write the set of simplices.

Example 2.50 ([105, Definition 9.1]). Consider a finite graph $G = (V, E)$ where we allow cycles. We can associate an abstract simplicial complex $K = (V(K), S(K))$ to G as follows:

1. The vertices of K are the vertices of G.
2. The n-simplices for $n \geq 1$ are $(n+1)$-tuples $(v_0, \ldots, v_n)$, of vertices such that for each $0 \leq i < j \leq n$, there is an edge in G between v_i and v_j.

Given a simplicial complex K and two of its simplices: τ and σ, if $\sigma \subset \tau$, then σ is said to be a *face* of τ. If $\sigma \in S(K)$ has $p+1$ elements it is said to be a *p-simplex*. The set of p-simplices of K is denoted by $K^{(=p)}$. The *dimension* of K is the largest p such that $K^{(=p)}$ is non-empty.

Definition 2.51. Given two simplicial complexes K and L, we say that K is a *subcomplex* of L if:

1. $V(K) \subset V(L)$
2. $S(K) \subset S(L)$.

We say that K is a *full subcomplex* if K is a subcomplex and whenever all the vertices of a simplex in L are in $V(K)$, then the simplex is in K.

Definition 2.52. Given two simplicial complexes K and L, a *simplicial map* between them is a map $f : V(K) \to V(L)$ such that whenever $\sigma \subseteq V(K)$ belongs to $S(K)$, $f(\sigma)$ belongs to $S(L)$.

2.2.2 Abstract Ordered Simplicial Complexes

Definition 2.53. An ordering of an abstract simplicial complex K is a partial order on the vertices of K, $V(K)$, which restricts to a total order on each simplex, that is, on each element of $S(K)$. An *ordered simplicial complex* is an abstract simplicial complex with an ordering. Ordered simplicial complexes will be denoted the same way as abstract simplicial complexes.

Definition 2.54. Let K and L be ordered simplicial complexes. An ordered simplicial map $f\colon K \to L$ is a simplicial map $f\colon K \to L$ such that $f : V(K) \to V(L)$ is an ordering preserving map between the posets of vertices.

Remark 2.55. The definition of an ordered abstract simplicial complex suggests a relation between posets and simplicial complexes. We will study this later on.

Example 2.56. Coming back to Example 2.50, we can now keep track of the directions in the graph by using ordered simplicial complexes. In this way, we do not loose information. Consider a finite directed graph $G = (V, E)$ where we allow cycles but we do not allow that given any pair of elements x, y, we have both edges (x, y) and (y, x). We can associate an ordered abstract simplicial complex $K = (V(K), S(K))$ to G as follows:

1. The vertices of K are the vertices of G.
2. The n-simplices for $n \geq 1$ are $(n+1)$-tuples $(v_0, \ldots, v_n)$, of vertices such that for each $0 \leq i < j \leq n$, there is an edge in G directed from v_i to v_j.

This procedure has been used in order to study Neuroscience by means of Topology (see for instance [146]).

2.2.3 Geometric Simplicial Complexes

Definition 2.57. Given $n+1$ points $v_0, \ldots, v_n$ in general position in $\mathbb{R}^m$. We define the *geometric n-simplex* σ generated by $v_0, v_1, \ldots, v_n$ to be the set:

$$\sigma = [v_0, \ldots, v_n] = \{t_0 v_0 + \cdots + t_n v_n \in \mathbb{R}^m : \sum_i t_i = 1 \text{ and } t_i \geq 0 \ \forall i\}.$$

If the $n+1$ points $v_0, \ldots, v_n$ are the endpoints of the standard unit basis of $\mathbb{R}^{n+1}$, then the resulting simplex is called the *standard geometric n-simplex*.

Let σ be a geometric n-simplex generated by $v_0, v_1, \ldots, v_n$. A *face* of σ is a geometric k-simplex generated by a subset of cardinality $k+1$ of $\{v_0, v_1, \ldots, v_n\}$.

Definition 2.58. A geometric finite simplicial complex is a finite collection K of geometric simplices in an Euclidean space $\mathbb{R}^n$, satisfying:

1. If $\sigma \in K$, then all the faces of σ are in K.
2. $\sigma, \beta \in K \Rightarrow \sigma \cap \beta \in K$. This means that the intersection of each pair of simplices in K is the empty set or a common face to both simplices.

If we endow K with the subspace topology inherited from $\mathbb{R}^n$, then we obtain a topological space which we denote by $|K|$. Any topological space that is homeomorphic to $|K|$ is called the *geometric realization* of K.

Let K and L be two geometric simplicial complexes. A *simplicial map* $f\colon K \to L$ is a set map between the set of vertices of K and the set of vertices of L which takes simplices to simplices. This map $f\colon K \to L$ induces a continuous map between the geometric realizations $|f|\colon |K| \to |L|$ as follows:

$$f\Big(\sum_{v_i \in V(K)} t_i \cdot v_i\Big) = \sum_{v_i \in V(K)} t_i \cdot f(v_i).$$

Observe that for any abstract simplicial complex K, we can associate a geometric simplicial complex in $\mathbb{R}^{V(K)}$ which is the union of standard simplices in $\mathbb{R}^{V(K)}$ for all $\sigma \in K$. In this way, we can associate a topological space to any finite abstract simplicial complex. Conversely, for any geometric simplicial complex, we can define an abstract simplicial complex by just considering the collection of simplices and vertices. As a consequence of this observation, we may abuse of notation and refer to topological properties of a simplicial complex when what we really mean are the topological properties of its geometric realization.

2.2.4 Remarkable classes of CW-complexes

We begin with some notation. We denote the unitary closed n-disk by $\mathbb{D}^n$ and its boundary, the $n-1$-sphere, by $\mathbb{S}^{n-1}$. Recall that a *finite CW-complex* is a topological space X constructed inductively by gluing cells. That is, we begin with a finite discrete set of points, $X^{(0)}$. Then, for each $n \geq 1$, $X^{(n)}$ is constructed from $X^{(n-1)}$ as follows. Let

$$\{\varphi_\alpha \colon \mathbb{S}^{n-1}_\alpha \to X^{(n-1)}\}_\alpha$$

be a finite family of continuous maps, which we call *attaching* or *gluing maps*. Then $X^{(n)}$ is the quotient space of the disjoint union $X^{(n-1)} \sqcup_\alpha \mathbb{D}^n_\alpha$ under the identifications $x \sim \varphi_\alpha(s)$ for $s \in \partial\mathbb{D}^n_\alpha = \mathbb{S}^{n-1}_\alpha$. We stop this process at a finite N which we call the *dimension* of the CW-complex and $X = X^N$. There are maps $\phi_\alpha \colon \mathbb{D}^n_\alpha \to X$ defined as the composition

$$\mathbb{D}^n_\alpha \hookrightarrow X^{(n)} \cup_\alpha \mathbb{D}^n_\alpha \to X^n \hookrightarrow X$$

which we refer to as *characteristic maps*. The image of the characteristic maps ϕ_α are the *closed cells* $\bar{e}^n_\alpha$ of the CW-complex.

A *subcomplex* of a CW-complex X is a closed subspace $A \subset X$ that is a union of closed cells of X. For each n, $X^{(n)}$ is a subcomplex of X and is called the n-skeleton.

Let X and Y be two finite CW-complexes, a continuous map $f \colon X \to Y$ is *cellular* if $f(X^{(n)}) \subseteq Y^{(n)}$ for every $n \in \mathbb{N}$.

We present two remarkable classes of CW-complexes which will play an important role in this work. Both notions are related to the flexibility we allow for the attaching maps. We begin with the classical notion of regular CW-complex.

Definition 2.59. A CW-complex X is *regular* if the attaching maps are homeomorphisms onto their images.

Example 2.60. The geometric realization of simplicial complexes are examples of regular CW-complexes.

Recently, Barmak and Minian ([10]) introduced the more flexible class of h-regular CW-complex.

Definition 2.61. A CW-complex X is *h-regular* if the attaching maps are homotopy equivalences onto their images.

2.2.5 Alexandroff-McCord functors and weak homotopy equivalences

We begin by quoting a result from Barmak's thesis:

Theorem 2.62 ([6, Corollary 2.3.4])**.** *Let X be a connected and non contractible T_1-space. Then X does not have the same homotopy type as any finite space.*

Due to Theorem 4.10, given a T_1-space, such as a manifold, we no longer can expect to carry all its homotopical information in an associated finite space (poset), at least by means of a homotopy equivalence. Therefore, this motivates us to try to keep as much topological information as possible by a weaker notion than homotopy equivalence.

We begin by recalling a construction due to Alexandroff ([3]).

Definition 2.63. Let X be a finite poset. We define the (ordered) simplicial complex $\mathcal{K}(X)$, called the *order complex* of X, as the simplicial complex whose simplices are the nonempty chains of X. Furthermore, given a continuous map $f\colon X \to Y$ between finite posets, we define a simplicial map $\mathcal{K}(f)\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ given by $\mathcal{K}(f)(x) = f(x)$.

Definition 2.64. Let K be a finite h-regular CW-complex. We define the finite poset $\mathcal{X}(K)$, called the face poset of K, as the poset of of cells of K ordered by inclusion. Given a cellular map $\phi\colon K \to L$ we define a continuous map $\mathcal{X}(\phi)\colon \mathcal{X}(K) \to \mathcal{X}(L)$ given by $\mathcal{X}(\phi)(\sigma) = \phi(\sigma)$ for each simplex σ of K.

Recall that given a finite poset X, a point x in the geometric realization of $\mathcal{K}(X)$, $|\mathcal{K}(X)|$, is a convex combination $x = t_1x_1 + t_2x_2 + \cdots + t_nx_n$ where $\sum t_i = 1$, $t_i > 0$ for all i and $x_1 < x_2 < \ldots < x_n$ is a chain in X. We define the *support* of x as $\operatorname{supp}(x) = \{x_1, \ldots, x_n\}$.

Definition 2.65. Let X be a finite poset. We define the $\mathcal{K}$-McCord map $\mu_X\colon |\mathcal{K}(X)| \to X$ by $\mu_X(x) = \min(\operatorname{supp}(x))$.

Let K be a finite simplicial complex. We denote its *barycentric subdivision* $\operatorname{sd}(K)$ as the simplicial complex $\mathcal{K}\mathcal{X}(K)$. For a simplicial map $\phi\colon K \to L$ we define a simplicial map $\operatorname{sd}(\phi) = \mathcal{K}\mathcal{X}(\phi)$. Denote by $\mathrm{s}_K\colon |\operatorname{sd}(K)| \to |K|$ the linear homeomorphism defined by $s_K(\sigma) = \mathrm{b}(\sigma)$ where $\mathrm{b}(\sigma)$ denotes the barycenter of σ. Analogously, for a finite poset X, we define its first barycentric subdivision $\operatorname{sd}(X)$, as $\mathcal{X}\mathcal{K}(X)$.

Definition 2.66. Let K be a finite simplicial complex, we define the $\mathcal{X}$-McCord map $\mu_K = \mu_{\mathcal{X}}(X)\mathrm{s}_K^{-1}\colon |K| \to \mathcal{X}(K)$.

Combining [6, Theorem 1.4.6, Remark 1.4.7, Corollary 1.4.8, Corollary 1.4.9, Theorem 1.4.12, Proposition 1.4.13, Corollary 1.4.15] we have:

Theorem 2.67. *(1) The $\mathcal{K}$-McCord map μ_X is a weak homotopy equivalence for every finite poset X.*

(2) If $f\colon X \to Y$ is a continuous map between finite posets, the following diagram commutes:

$$\begin{array}{ccc} |\mathcal{K}(X)| & \xrightarrow{|\mathcal{K}(f)|} & |\mathcal{K}(Y)| \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_Y} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

(3) Let $f\colon X \to Y$ be a map between finite T_0-spaces. Then f is a weak homotopy equivalence if and only if $|\mathcal{K}(f)|\colon |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ is a homotopy equivalence.

(4) The $\mathcal{X}$-McCord map μ_K is a weak homotopy equivalence for every finite simplicial complex K.

(5) Let $\phi\colon K \to L$ be a simplicial map between finite simplicial complexes. Then the following diagram commutes up to homotopy:

$$\begin{array}{ccc} |K| & \xrightarrow{|\phi|} & |L| \\ \big\downarrow{\scriptstyle \mu_K} & & \big\downarrow{\scriptstyle \mu_L} \\ \mathcal{X}(K) & \xrightarrow{\mathcal{X}(\phi)} & \mathcal{X}(L). \end{array}$$

(6) Let $\phi\colon K \to L$ be a simplicial map between finite simplicial complexes. Then $|\phi|$ is a homotopy equivalence if and only if $\mathcal{X}(\phi)\colon \mathcal{X}(K) \to \mathcal{X}(L)$ is a weak homotopy equivalence.

Part (4) of Theorem 2.67 was generalized to the context of h-regular CW-complexes by Barmak and Minian (see [10, Theorem 4.7]):

Theorem 2.68. *Let K be a finite h-regular CW-complex. Then, there exists a map $f_K\colon K \to \mathcal{X}(K)$ (see [10, Theorem 4.7] for the definition) which is a weak homotopy equivalence.*

Remark 2.69. Since weak homotopy equivalences induce isomorphisms in homology and cohomology ([95, Proposition 4.21]), Theorem 2.67 provides an alternative approach to define homology for posets. That is, one may define the homology of a poset X as the homology of the associated simplicial complex $\mathcal{K}(X)$. This is called *poset homology* and is dealt with, for example, in [175].

2.2.6 Finite models and manifolds

Definition 2.70. We say that the poset X is a *finite model* for the topological space Y if the geometric realization $|\mathcal{K}(X)|$ of the simplicial complex $\mathcal{K}(X)$ is homotopy equivalent to Y.

Barmak and Minian introduced in [10] a class of posets which model the h-regular CW-complexes.

Definition 2.71. A finite poset X is called an *h-regular poset* if for every $x \in X$, the set $\hat{U}_x$ is a finite model of $\mathbb{S}^{n-1}$, where n is the height of x.

As a consequence of the definition of h-regular posets and cellular posets, it follows that:

Proposition 2.72. *Let X be a cellular or an h-regular finite poset. Then, X does not have down homological removable points or down weak homotopical removable points, respectively. Moreover, cellular and a h-regular finite posets are down-wide.*

Proof. First, by definition of cellular and h-regular finite posets, for $x \in X$, $\hat{U}_x$ is not acyclic or weak homotopically trivial, respectively. As a consequence, if $x \in X$ is of height greater than or equal to one, then $\hat{U}_x$ is not contractible, so it can not have a maximum, that is, x is down-wide. $\square$

For a given poset X, the *dual poset* X^{op} has the same elements as X but the arrows are reversed, that is, $x \leq y$ in X if and only if $y \leq x$ in X^{op}. Due to the correspondence between posets and finite spaces, this also defines the notion of *dual space*.

Observe that $\mathcal{K}(X)$ is the same simplicial complex as $\mathcal{K}(X^{\mathrm{op}})$. Since weak homotopy equivalences induce isomorphisms in homology and cohomology ([95, Proposition 4.21]), there is a strong relation between the (co)homology of the space X, that of the geometric realization of its order complex $|\mathcal{K}(X)|$, and that of its dual space X^{op}:

Proposition 2.73. *Let X be a finite poset. Then X, $\mathcal{K}(X) = \mathcal{K}(X^{\mathrm{op}})$ and X^{op} have isomorphic homology and isomorphic cohomology with $\mathbb{Z}_2$ coefficients.*

Definition 2.74. The poset X is *homologically bi-admissible* if both X and X^{op} are homologically admissible.

Let K be a finite simplicial complex and σ a simplex of K. The *link* of σ is the subcomplex of K whose simplices are

$$\{\beta \in K \colon \beta \cap \sigma = \emptyset \text{ and } \beta \cup \sigma \in K\}.$$

Definition 2.75. A finite simplicial complex K is a *closed homology manifold* of dimension n if the link of every simplex has the homology of the sphere $\mathbb{S}^{n-k-1}$, where k is the dimension of the simplex.

Definition 2.76 ([126])**.** A poset X is a *finite closed homology manifold* of dimension n if its order complex $\mathcal{K}(X)$ is a closed homology manifold of dimension n.

For a detailed treatment of finite (closed homology) manifolds see [126, 133]. The following result provides examples of homologically bi-admissible posets.

Proposition 2.77. *(1) Let X be a finite closed homology manifold of dimension n, then it is homologically bi-admissible.*

(2) Let K be a closed homology manifold of dimension n, then its face poset $\mathcal{X}(K)$ is homologically bi-admissible.

Proof. First of all, if X is a finite closed homology manifold, then it is homologically admissible (see [126]). Moreover, since $\mathcal{K}(X) = \mathcal{K}(X^{\mathrm{op}})$, by the definition of finite closed homology manifold, X^{op} is homologically admissible too.

Second, K is a closed homology manifold of dimension n if and only if

$$H_k(|K|, |K| \setminus \{x\}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k = n \\ 0 & \text{else} \end{cases}$$

for every $x \in |K|$ (see [130]). Therefore, if K is a closed homology manifold of dimension n, then $\mathcal{K}\mathcal{X}(K)$ is a closed homology manifold, so $\mathcal{X}(K)$ is homologically bi-admissible by (1). $\square$

Remark 2.78. Observe that, in particular, the face posets of combinatorial manifolds (see [130] for the definition) are homologically bi-admissible.

As a consequence, the theory developed for homologically bi-admissible posets can be applied to study the topology of triangulable homology or combinatorial manifolds by means of their triangulations. In particular, we can prove a version of Poincaré Duality for $\mathbb{Z}_2$ coefficients.

In order to do so, we make the following observation. For a homologically admissible poset, cellular cohomology with coefficients in a field can be defined in the same way as cellular homology and it also holds that it is isomorphic to singular cohomology. In particular, using coefficients in $\mathbb{Z}_2$, the differential $\delta_{p+1} \colon C^p(X) \to C^{p+1}(X)$ simplifies to the formula: $\delta(x) = \sum_{y \succ x} y$.

Theorem 2.79 ($\mathbb{Z}_2$-Poincaré Duality)**.** *Let X be a finite homologically bi-admissible poset of degree n. Then $H^k(X;\mathbb{Z}_2) \cong H_{n-k}(X;\mathbb{Z}_2)$ for every $k \leq n$. In particular, this holds if X is a finite closed homology manifold of degree n.*

Proof. Observe that for each k, there is an isomorphism between $C_k(X)$ and $C^{n-k}(X^{\mathrm{op}})$ and the differentials d_k and δ_{n-k+1} coincide. □

Remark 2.80. Basak proved in [19], by different methods, a Poincaré Duality result with integer coefficients for a particular class of posets.

2.2.7 Summary of the relations

We summarize the relations between cellular complexes and posets in Diagram (2.3). In order to do so, we introduce the following notation:

- Pos denotes the category of finite posets and order-preserving maps.
- T-wPos denotes the full subcategory of Pos whose objects are two-wide posets.
- CPos denotes the full subcategory of Pos whose objects are cellular posets.
- D-wPos denotes the full subcategory of Pos whose objects are down-wide posets.
- haPos denotes the full subcategory of Pos whose objects are homologically admissible posets.
- h-rPos denotes the full subcategory of Pos whose objects are h-regular posets.
- HaPos denotes the full subcategory of Pos whose objects are homotopically admissible posets.
- FChManPos denotes the full subcategory of Pos whose objects are finite closed homology manifolds.
- h-rCW denotes the category of h-regular CW-complexes and cellular maps between them.
- rCW denotes the full subcategory of h-RCW whose objects are regular CW-complexes.

- SimC denotes the full subcategory of h-RCW whose objects are simplicial complexes.

We refer the reader to [126] for the proofs which we omitted in this presentation.

$$
\begin{tikzcd}
 & & T_0\text{-FTop} \cong \text{Pos} \arrow[dddr] & \\
\text{T-wPos} \arrow[urr, hook] & \text{CPos} \arrow[r, hook] & \text{D-wPos} \arrow[u, hook] & \text{h-rCW} \arrow[dl, "\mathcal{K}"'] \\
 & \text{haPos} \arrow[ul, hook] \arrow[u, hook] & \text{h-rPos} \arrow[u, hook] & \text{rCW} \arrow[u, hook] \arrow[dl, "\mathcal{X}"] \\
 & & \text{HaPos} \arrow[ul, hook] \arrow[u, hook] & \text{SimC} \arrow[u, hook] \\
 & & \text{FChMan} \arrow[u, hook] \arrow[r, shift left, "\mathcal{K}"] & \text{ChMan} \arrow[u, hook] \arrow[l, shift left, "\mathcal{X}"]
\end{tikzcd}
\tag{2.3}
$$

2.2.8 Examples

In this subsection we present some examples which clarify the theory developed earlier.

Example 2.81. The finite model of $\mathbb{R}P^2$ depicted in Figure 5.1.1 (see [6, Example 7.1.1], [94, Proposition 4.1] and [46, p. 138]) is homologically admissible.

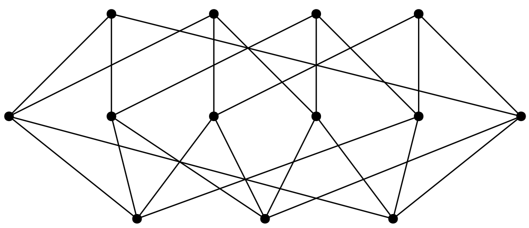


Figure 2.2.1: Example of homologically admissible poset.

Example 2.82. Consider the following example (Figure 2.2.2), an h-regular model of $\mathbb{S}^3$, taken from [126, Fig.2], which is not a graded poset, therefore it is not cellular.

Example 2.83. We will construct a cellular poset which is not h-regular. Let K be a triangulation of the Poincaré homology 3-sphere (see for example [101]). Construct the non-Hausdorff cone of $\mathcal{X}(K)$, i.e. we add a new point x' such that $x' \geq x$ for every $x \in \mathcal{X}(K)$. The non-Hausdorff

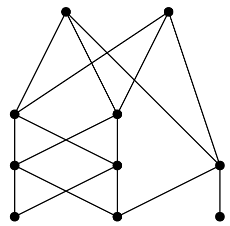


Figure 2.2.2: Example of h-regular space which is not cellular.

cone of $\mathcal{X}(K)$, $\mathbb{C}(\mathcal{X}(K))$ is cellular since it is graded and $\hat{U}_x$ has the homology of a $(\deg(x)-1)$-sphere for every x in $\mathbb{C}(\mathcal{X}(K))$. However, $\mathbb{C}(\mathcal{X}(K))$ is not h-regular since $\hat{U}_{x'}$ has not the same weak homotopy type as a sphere cause it has the weak homotopy type of the Poincaré homology 3-sphere.

2.3 Curvature

This section is devoted to studying some classical results in Differential and Algebraic topology in the setting of posets. In particular, we will state a Hodge decomposition Theorem, some results relating (co)homology and curvature, and a Gauss-Bonnet Theorem.

2.3.1 Hodge decomposition Theorem

This subsection is a generalization of some techniques introduced by Forman ([80]) in the context of complexes to the context of cellular posets, so the presentation will be brief.

First, we state a Combinatorial Hodge Theorem for cellular posets generalizing Forman's result [80, Theorem 2.1]. To do so we will consider coefficients in $\mathbb{R}$. Given a cellular poset X, we endow each $C_p(X,\mathbb{R})$ with an inner product

$$\langle \bullet, \bullet \rangle_p \colon C_p(X,\mathbb{R}) \times C_p(X,\mathbb{R}) \to C_p(X,\mathbb{R})$$

such that the elements of degree p of X are orthogonal. Observe that this amounts to choosing a positive weight ω_x for each p-element x and defining $\langle x, x \rangle = \omega_x$. Denote by

$$\delta_{(p+1)} \colon C_p(X) \to C_{p+1}(X)$$

the adjoint of the cellular boundary operator, i.e.

$$\langle d_{(p+1)} c_{(p+1)}, c'_{(p)} \rangle_p = \langle c_{(p+1)}, \delta_{(p+1)} c'_{(p)} \rangle_{p+1}$$

for $c^{(p+1)} \in C_{p+1}$ and $c'^{(p)} \in C_p$. We define the combinatorial Laplacian

$$\Box_p \colon C_p(X) \to C_p(X)$$

given by

$$\Box_p = d \circ \delta + \delta \circ d. \tag{2.4}$$

The same argument given in [80, Theorem 2.1] proves:

Theorem 2.84 (Combinatorial Hodge Theorem)**.** *Let X be a cellular poset, then:*

- $\ker \Box_p = \ker d_p \cap \ker \delta_{p+1}$.
- $\ker \Box_p \cong H_p(X, \mathbb{R})$.

As it happens in [80], a more explicit formula for $\Box_p$ can be given as follows. In order to simplify the presentation we will assume that all weights ω_x are equal to one, but the same arguments work in the general case. Recall that the incidence number $\epsilon(x, w)$ is zero if $w \not< x$. The same argument of [80, p. 338] shows:

$$\begin{aligned}\Box_p x = \sum_{\deg(x')=p} \Bigg[&\sum_{\deg(y)=p+1} \epsilon(y,x)\epsilon(y,x') \\ &+ \sum_{\deg(w)=p-1} \epsilon(x,w)\epsilon(x',w)\Bigg].\end{aligned} \tag{2.5}$$

Moreover, by construction, $\Box_p$ is a self-adjoint operator with respect to the inner product $\langle \bullet, \bullet \rangle_p$. Therefore, we can construct the symmetric matrix, S, associated to $\Box_p$. Denote by s_{ij} the i, j-element of the matrix, that is $s_{ij} = \langle \Box_p x_i, x_j \rangle$, which by using Equation (2.5) becomes:

$$\begin{aligned} s_{ij} &= s_{ji} \\ &= \sum_{\deg(y)=p+1} \epsilon(y, x_i)\epsilon(y, x_j) + \sum_{\deg(w)=p-1} \epsilon(x_i, w)\epsilon(x_j, w). \end{aligned}$$

2.3.2 Curvature and (co)homology

Now we will define a notion of combinatorial curvature in the context of cellular posets and we will see that, as it happens in the smooth and simplicial settings, (co)homology imposes some constraints on curvature. For a discussion on the intuition behind this combinatorial curvatures in the context of complexes the reader is referred to [80]. First, we define the p-th curvature operator $F_p \colon C_p(X) \to C_p(X)$ given, in matrix form, as the following diagonal matrix:

$$F_p(x_i, x_j) = \begin{cases} 0, & x_i \neq x_j, \\ s_{ij} - \sum_{k \neq i} |s_{ik}|, & x_i = x_j. \end{cases}$$

Definition 2.85. For any p-chain $c \in C_p(X)$, define the *p-th curvature* of c by

$$\mathcal{F}_p(c) = \langle F_p(c), c \rangle.$$

In the case of an element $x \in X$ such that $\deg(x) = p$, we define the *p-th curvature* at x as $\mathcal{F}_p(x) = \langle F_p(x), x \rangle$.

As it happens in [80, Theorem 2.2], we can find an explicit expression for $\mathcal{F}_p(x)$ by straightforward computations:

Theorem 2.86. *Let X be a cellular poset. For any p-element $x \in X$, the p-th curvature function applied to x, $\mathcal{F}_p(x)$, is given by*

$$\begin{aligned}\mathcal{F}_p(x) = &\sum_{\deg(y)=p+1} (\epsilon(y,x))^2 + \sum_{\deg(w)=p-1} (\epsilon(x,w))^2 \\ &- \sum_{x'^{(p)} \neq x} \Big| \sum_{\deg(y)=p+1} \epsilon(y,x)\epsilon(y,x') + \sum_{\deg(w)=p-1} \epsilon(x,w)\epsilon(x',w) \Big|.\end{aligned}$$

It simplifies as follows for homologically admissible posets:

Corollary 2.87. *Let X be a a finite homologically admissible poset X and let $x^{(p)} \in X$, then the p-th curvature satisfies:*

$$\begin{aligned}\mathcal{F}_p(x) =& \#\{y^{(p+1)} \in X \colon y \succ x\} + \#\{w^{(p-1)} \in X \colon w \prec x\} \\ &- \sum_{x'^{(p)} \neq x} \Big| \sum_{\deg(y)=p+1} \epsilon(y,x)\epsilon(y,x') + \sum_{\deg(w)=p-1} \epsilon(x,w)\epsilon(x',w) \Big|.\end{aligned}$$

Proof. Recall from Lemma 2.12 that the incidence numbers of homologically admissible posets are $+1$ or -1, so the equation for the pth curvature follows. $\square$

The operator F_p is said to be ≥ 0 (respectively > 0) if $\mathcal{F}_p(x) \geq 0$ (respectively > 0) for every element $x^{(p)} \in X$.

Definition 2.88. For any $x^{(1)} \in X$, we define the *Ricci curvature* of x by $\mathrm{Ric}(x) = \mathcal{F}_1(x)$.

Definition 2.89. Given a finite cellular poset X and two elements $x^{(p)}, x'^{(p)} \in X$, we say that x and x' are *metric neighbors* if $\langle \square_p x, x' \rangle \neq 0$.

We consider the equivalence relation generated by the metric neighbor relation, i.e: $x^{(p)} \sim x'^{(p)}$ if there are elements of degree p, $x^{(p)} = x_1^{(p)}, \dots, x_k^{(p)} = x'^{(p)}$, such that for each $i = 1, \dots, k-1$, $x_i^{(p)}$ and $x_{i+1}^{(p)}$ are metric neighbors. We denote by $\mathcal{N}(p)$ the number of equivalence classes of elements of degree p. By repeating the arguments of [80] it holds:

Theorem 2.90. *Let X a finite cellular poset. If $F_p \geq 0$, then*

$$\dim H_p(X, \mathbb{R}) \leq \mathcal{N}(p).$$

Consider the equivalence relation generated by the metric neighbor relation. We say that an equivalence class Λ of elements of degree p is *positive* if there is an $x^{(p)} \in \Lambda$ with $\mathcal{F}_p(x) > 0$. Otherwise, we say that Λ is *flat*. We denote by $\mathcal{N}^0(p)$ the number of flat equivalence classes of elements of degree p. The same argument provided in [80] shows:

Theorem 2.91. *Let X be a finite cellular poset. If $F_p \geq 0$, then*

$$\dim H_p(X, \mathbb{R}) \leq \mathcal{N}^0(p).$$

Corollary 2.92. *Let X be a finite cellular poset. If $F_p > 0$, then*

$$H_p(X, \mathbb{R}) = 0.$$

In particular, if $\mathrm{Ric}(x) > 0$ *for every element $x \in X$ of degree 1, then*

$$H_1(X, \mathbb{R}) = 0.$$

2.3.3 Gauss-Bonnet Theorem

In this subsection, we present a Gauss-Bonnet theorem in the setting of posets. We recall the discrete curvature functions on graded posets introduced by Bloch ([27]). The introduction of this new functions was motivated by the failure of Gauss-Bonnet Theorem with the curvature functions developed by Forman. Given a graded poset X and $x^{(p)} \in X$, let

$$A_i(x) = \#\{y \in X\colon x \prec y\}, \quad B_i(x) = \#\{w \in X\colon w \prec x\},$$

$$U_i(x) = \sum_{y \succ x} B_{i+1}(y), \quad \text{and} \quad D_i(x) = \sum_{w \prec x} B_{i-1}(w).$$

Definition 2.93. Let X be a graded poset of degree 2. For each $i \in \{0,1,2\}$, let $R_i\colon X^{(i)} - X^{(i-1)} \to \mathbb{R}$ be defined by:

$$R_0(x) = 1 + \frac{3}{2}A_0(x) - (A_0(x))^2,$$

$$R_1(x) = 1 + 6A_1(x) + \frac{3}{2}B_1(x) - U_1(x) - D_1(x),$$

$$R_2(x) = 1 + 6B_2(x) - (B_2(x))^2.$$

It is shown in [27] that for certain posets, including the face posets of all 2-dimensional simplicial complexes, R_1 equals the combinatorial Ricci curvature introduced in Definition 2.88.

As a consequence of Theorem 2.15 and [27, Theorem 2.4] it follows that cellular posets of degree two satisfy a Gauss-Bonnet Theorem.

Theorem 2.94 (Gauss-Bonnet)**.** *Let X be a finite cellular poset of degree 2. Then*

$$\sum_{\deg(x)=0} R_0(x) - \sum_{\deg(x)=1} R_1(x) + \sum_{\deg(x)=2} R_2(x) = \chi(X).$$

Proof. It was proved in [27, Theorem 2.4] that the result is true for the graded Euler characteristic $\chi_g(X)$ on finite graded posets. From the coincidence of graded Euler characteristic and Euler characteristic for cellular posets (Theorem 2.15) the theorem follows. □

In the smooth category, as a consequence of the Gauss-Bonnet Theorem, it is also true that if the Gaussian curvature of a smooth surface is everywhere positive, then the Euler characteristic of the surface is positive. In our setting we can prove the following. We define the average of the R_1 curvature function following Bloch ([27]):

$$\bar{R}_1 = \frac{1}{\#(X^{(1)} - X^{(0)})} \sum_{\deg(x)=1} R_1(x).$$

Observe that if R_1 is everywhere positive, then $\bar{R}_1$ is positive. We also recall from [27]:

$$\bar{A}_1 = \frac{1}{\#(X^{(1)} - X^{(0)})} \sum_{\deg(x)=1} A_1(x)$$

and

$$\bar{B}_1 = \frac{1}{\#(X^{(1)} - X^{(0)})} \sum_{\deg(x)=1} B_1(x).$$

Due to [27, Remark 2.6], a particular case of [27, Theorem 2.7] can be stated as:

Proposition 2.95. *[27] Let X be a finite graded poset of degree 2. If $\bar{B}_1 = 2$, $\bar{A}_1 \geq 2$ and $\bar{R}_1 > 0$, then $\chi_g(X) > 0$.*

As a consequence we can prove:

Corollary 2.96. *Let X be a finite cellular poset of degree 2. If $\bar{R}_1 > 0$ and the degree one elements are not up strong homotopic removable points, then $\chi(X) > 0$.*

Proof. First, observe that if degree one elements are not up strong homotopic removable points, then $\bar{A}_1 \geq 2$. Second, since X is cellular, $\widehat{U}_x$ has the homology of $\mathbb{S}^0$ for every element x of degree 1, so $\bar{B}_1 = 2$. Therefore, by Proposition 2.95, $\chi_g(X) > 0$. Now the result follows from the equality of graded Euler characteristic and Euler characteristic for cellular posets (Theorem 2.15). $\square$

Chapter 3
Morse functions and matchings on posets

Our approach to Morse theory consists in constructing appropriate filtrations for the objects under study, so we can later extract global information about the object by means of integrating local information in some critical steps of the filtration. Informally, by an appropriate filtration we mean one that allows us to keep track of the changes of some invariant we are interested in as we go up in the sequence of subobjects.

One straightforward way to produce a filtration of a finite poset X is to define an order preserving map $f\colon X \to \mathbb{R}$ and then consider, for each $t \in \mathbb{R}$, the down-set subposet:

$$X_t^f = \bigcup_{f(x)\leq t} U_x.$$

Since the poset is finite, the image of f, $\{t_0, t_1, \dots, t_n\}$, is a finite subset of $\mathbb{R}$ and we have a filtration with a finite sequence of subposets:

$$X_{t_{-1}} = \emptyset \hookrightarrow X_{t_0} \hookrightarrow \cdots \hookrightarrow X_{t_i} \hookrightarrow X_{t_{i+1}} \hookrightarrow \cdots \hookrightarrow X_{t_n} = X. \tag{3.1}$$

This would be an analogous situation to the one in smooth Morse theory for compact manifolds. However, as it happens in the smooth setting, this approach does not guarantee that we obtain control on the resulting filtration. That is why we need to impose some extra condition on the map $f\colon X \to \mathbb{R}$.

In order to come up with a suitable condition, let us think about what we would like to have and then work backwards. We are interested in studying some global invariant in X by means of studying it locally. As it happens in smooth Morse theory, we would like to have two kinds of values for the map f, the regular values and the critical ones. First, if t_{i+1} was to be regular, then $X_{t_i} \hookrightarrow X_{t_{i+1}}$ should not produce any change in the invariant under consideration. For example, for classical Morse theory, $X_{t_i} \hookrightarrow X_{t_{i+1}}$ would be a homotopy equivalence. Second, if t_{i+1} was to be critical, then we should be able to understand and keep track of the change of the invariant when passing from X_{t_i} to $X_{t_{i+1}}$. The idea to achieve this goal in smooth Morse theory is to consider maps which come from some suitable dynamics in the manifold. Those maps are precisely the ones inducing a gradient vector field. So, we will follow the same strategy in this context.

In order to do so, we will adapt the notion of combinatorial vector field introduced by Forman ([74]) from the setting of simplicial complexes to the context of posets. The advantage of this approach is that the points in $X_{t_{i+1}} - X_{t_i}$ satisfy certain conditions which allow us to control passing from X_{t_i} to $X_{t_{i+1}}$. In the next chapters, this condition will be seen to be strongly related to the notion of homological and homotopical admissibility.

In this chapter we introduce the notions of vector field, matching, Morse function and tame Morse function and study the relations between them and some properties. Moreover, we show how these notions in the context of posets unify previous Morse theories on chain complexes (Algebraic Morse theory) and on cell complexes (Forman's Morse theory).

Some of the results in this chapter appeared in our preprint [66].

3.1 The definition of tame Morse function

The first goal of this chapter is to introduce the novel notion of *tame Morse function* (Definition 3.13). In order to do so, we begin by introducing the combinatorial analogues of the gradient vector fields in the smooth setting.

First, we state the definition of combinatorial vector field in a poset. It is a generalization of the notion of combinatorial vector field introduced by Forman ([74]) in the context of simplicial complexes.

Definition 3.1. Let X be a finite poset. A *combinatorial vector field on X is a map*

$$V\colon X \to X \cup \{0\}$$

such that:

1. *If $V(x) \neq 0$, then $x \prec V(x)$.*
2. *If $V(x) = y \neq 0$, then $V(y) = 0$.*
3. *For all $x \in X$, $\#\{V^{-1}(x)\} \leq 1$.*

Remark 3.2. Forman claims in his work that the notion of combinatorial vector field is related to previous work of Duval ([56]). We very recently discovered that it is also related, although it seems Forman was not aware of it, to the work initiated by Brown on collapsing schemes ten years earlier ([38, 39, 47]) as Tochi points out in his dissertation ([173]).

To facilitate the technical work and to provide a combinatorial intuition for the concept of vector field, we introduce the definition of a matching in a poset. The definition of matching for Morse theoretic purposes was first introduced by Chari ([43]) and further developed by Minian ([126]).

Definition 3.3. Let X be a finite poset. A *matching* $\mathcal{M}$ on X is a subset $\mathcal{M} \subseteq X \times X$ such that

- $(x, y) \in \mathcal{M}$ implies $x \prec y$;
- each $x \in X$ belongs to at most one element in $\mathcal{M}$.

The next result guarantees that the notions of combinatorial vector fields and matchings on posets are equivalent.

Lemma 3.4. *Let X be a poset. There is a bijective correspondence between combinatorial vector fields and matchings on posets.*

Proof. Let $V\colon X \to X \cup \{0\}$ be a combinatorial vector field. We define a matching $\mathcal{M}$ on X as follows. A pair $(x, y) \in X \times X$ is in the matching if and only if $V(x) = y$. Conversely, for a matching $\mathcal{M}$ on X, we define a combinatorial vector field $V\colon X \to X \cup \{0\}$ as follows: $V(x) = y$ if and only if $(x, y) \in \mathcal{M}$. □

Therefore, from now on we will use matchings and vector fields interchangeably depending on the context and for convenience. Any definition or result for one notion applies to the other.

We make an observation in the form of a result:

Proposition 3.5. *There is a bijective correspondence between matchings in X and in X^{op} given as follows: to a matching $\mathcal{M}$ in X, we associate a matching $\mathcal{M}^{op}$ in X^{op} as follows, $(x, y) \in \mathcal{M}$ iff $(y, x) \in \mathcal{M}^{op}$.*

We introduce the notion of criticality in matchings. Let $\mathcal{H}(X)$ be the Hasse diagram of a poset X. If $\mathcal{M}$ is a matching in X, write $\mathcal{H}_{\mathcal{M}}(X)$ for the directed graph obtained from $\mathcal{H}(X)$ by reversing the orientations of the edges which are not in $\mathcal{M}$.

Definition 3.6. Let X be a poset and let $\mathcal{M}$ be a matching on X. Any point of $\mathcal{H}(X)$ not incident with an edge of $\mathcal{M}$ is called *critical*. The set of all critical points of $\mathcal{M}$ is denoted by $\operatorname{crit}(\mathcal{M})$. The points which are not critical are called *regular*. The *index* of a critical point is its height.

The following example shows how to visualize matchings on posets.

Example 3.7. In Figure 3.1.1 we exhibit a matching $\mathcal{M}$ on a poset X. The crosses represent critical points, the dashed edges with the circles represent a periodic orbit, and the arrows represent the matched elements.

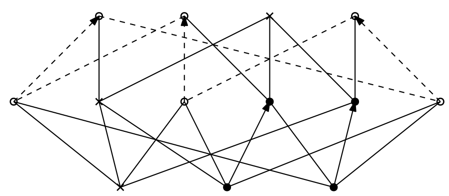


Figure 3.1.1: A homologically admissible poset X with a matching $\mathcal{M}$.

Definition 3.8. Let X be a finite poset and let $\mathcal{M}$ be a matching on X. The matching $\mathcal{M}$ is a *Morse* or *gradient matching* if $\mathcal{H}_{\mathcal{M}}(X)$ is acyclic as a directed graph.

Example 3.9. In Figure 3.1.2 we depict a Morse matching $\mathcal{M}$ on a poset X. The crosses represent critical points and the arrows represent the matched elements.

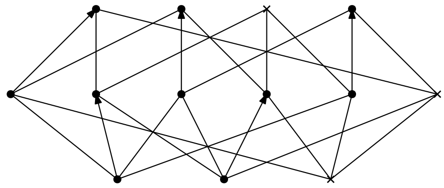


Figure 3.1.2: A homologically admissible poset X with a Morse matching $\mathcal{M}$.

We introduce some terminology which will be used later. Given a matching $\mathcal{M}$ on a poset X, we will decompose X as the disjoint union of three subsets:

$$X = \operatorname{crit}(\mathcal{M}) \sqcup s(\mathcal{M}) \sqcup t(\mathcal{M}).$$

For each edge $(x, y) \in \mathcal{M}$, we say that x is the source of the edge and y is the target. We define the *source of the matching* $s(\mathcal{M})$ as the set whose elements are the sources of the edges in the matching. Analogously, we define the *target of the matching* $t(\mathcal{M})$ as the set whose elements are the targets of the edges in the matching. For convenience, we define the *source* and *target maps* (only defined for elements in the matching $\mathcal{M}$) as follows: given $(x, y) \in \mathcal{M}$, $s(y) = x$ and $t(x) = y$.

Definition 3.10. Let X be a finite poset and let $\mathcal{M}$ be a matching on X. A *generalized $\mathcal{M}$-path* from x to z is a sequence of one of the following two forms:

1. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} = z$
2. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} \succ x_r = z$

such that for each $i \in \{0, \ldots, r-1\}$:

1. $(x_i, y_i) \in \mathcal{M}$,
2. $x_i \neq x_{i+1}$.

In case X is a graded poset, a *$\mathcal{M}$-path of index* p from $x^{(p)}$ to $\tilde{x}^{(p)}$ is a sequence:

$$\gamma\colon x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)} = \tilde{x}$$

such that for each $i \in \{0, \ldots, r-1\}$, it satisfies the same conditions as before.

Using the ideas of Minian ([126, Lemma 3.12]), which are based in the ideas of Forman ([74, Theorem 2.4]), we can prove the following improved integration result for Morse matchings:

Theorem 3.11. *Let X be a finite graded poset and let $\mathcal{M}$ be a a Morse matching on X. Then, there is a set function $f\colon X \to \mathbb{R}$ satisfying:*

(OP) It is order preserving, that is, if $x \leq y$, then $f(x) \leq f(y)$.

(M1) $\#\{y \in X : x \prec y \text{ and } f(x) \geq f(y)\} \leq 1$.

(M2) $\#\{w \in X : w \prec x$ *and* $f(w) \geq f(x)\} \leq 1$.

(E) There do not exist $w, x, y \in X$, $w \prec x \prec y$ *such that* $f(w) \geq f(x) \geq f(y)$.

(SI) It is self-indexing, that is, for every critical element $x^{(p)}$, $f(x) = p$.

(F) If $(x, y) \in \mathcal{M}$, *then* $f(x) = f(y)$.

Proof. First, we define an auxiliary map $l : X \to \mathbb{N}$ given by:

$$\begin{aligned} l(x) = \max\{r : \exists\, &\mathcal{M}\text{-path} \\ &\gamma : x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)}\}. \end{aligned}$$

Second, we define $L = \max_{x \in X} l(x)$. Now, we define the function $f : X \to \mathbb{R}$ inductively on the degree of the poset. Given $x^{(p)} \in X$, we define $f(x)$ as follows:

1. If $x^{(p)}$ is a critical point of $\mathcal{M}$, then $f(x) = p$.
2. If $x \in s(\mathcal{M})$, then
$$f(x) = p + \frac{l(x)}{2L}.$$
Note that this guarantees that
$$p < f(x) \leq p + \frac{1}{2}.$$
3. If $x \in t(\mathcal{M})$, then there exists $w^{(p-1)}$ such that $t(w) = x$ and $f(w)$ was defined in (2). We set $f(x) = f(w)$ and it follows that
$$p - 1 < f(x) \leq p - \frac{1}{2}.$$

By construction, the function $f : X \to \mathbb{R}$ is order preserving (satisfies (OP)), it satisfies (E), it is self-indexing (it satisfies (SI)) and it satisfies (F). It remains to check that f satisfies (M1) and (M2). We split the verification in cases:

First, suppose $x^{(p)}$ is critical. Then, by construction of f, for any $w^{(p-1)}$, it follows $f(w) \leq p - 1 + \frac{1}{2} < p$, and for any $y^{(p+1)}$, it holds $f(y) > p$. Second, assume that $x^{(p)}$ is not critical and $y^{(p+1)} > x$.

1. If $t(x) = y$, then $f(y) = f(x)$, so
$$f(x) \geq f(y).$$
2. If $t(x) \neq y$, we consider several cases again:
   (a) If y is a critical point, then
   $$f(y) = p + 1 > p + 1/2 \geq f(x).$$

(b) If $y \in s(\mathcal{M})$, then

$$f(y) > p + 1 > p + 1/2 \geq f(x).$$

(c) If $y \in t(\mathcal{M})$. Then there exists an unique $\tilde{x}^{(p)} \neq x$ such that $t(\tilde{x}) = y$. Since x is not critical, there are two cases:

i. If $x \in t(\mathcal{M})$, then

$$f(y) = f(\tilde{x}) \geq p > p - 1/2 \geq f(x).$$

ii. If $x \in s(\mathcal{M})$ and $\gamma \colon x \prec \cdots$ is any $\mathcal{M}$-path beginning at x, then

$$\tilde{\gamma} \colon \tilde{x} \prec y \succ x \prec \cdots$$

is a $\mathcal{M}$-path beginning at $\tilde{x}$. Therefore

$$l(\tilde{x}) \geq l(x) + 1,$$

hence

$$f(y) = f(\tilde{x}) > f(x).$$

Third, assume that $x^{(p)}$ is not critical and $w^{(p-1)} < x$. This case is analogous to the second one. □

Theorem 3.11 motivates the notion of *tame Morse function* (Definition 3.13). Moreover, we will add an extra condition (that we will refer as (CI)) to guarantee that when we construct the filtration of the space under study (see Equation (3.1)), it holds that $\#(X_{t_{i+1}} - X_{t_i}) \leq 2$.

First, we recall some notation.

Definition 3.12. Let X be a finite space and let $x \in X$. The set $\{z \in X \colon z \prec x \text{ or } z \succ x\}$ is referred as the set of *adjacent elements* to x.

Let C_m denote the poset whose elements are $\{0, 1, \ldots, m-1, m\}$ and with the order $i \leq i+1$ for $i \in \{0, \ldots, m-1\}$.

Definition 3.13. Let X be a finite poset. A *tame Morse function* is a map $f \colon X \to C_m$ for some $m \in \mathbb{N}$ satisfying the following conditions:

(OP) It is order preserving, that is, if $x \leq y$, then $f(x) \leq f(y)$.

(M1) $\#\{y \in X \colon x \prec y \text{ and } f(x) \geq f(y)\} \leq 1$.

(M2) $\#\{w \in X \colon w \prec x \text{ and } f(w) \geq f(x)\} \leq 1$.

(E) There do not exist $w, x, y \in X$, $w \prec x \prec y$ such that $f(w) \geq f(x) \geq f(y)$.

(CI) For each $x \in X$, if there is an element $z \in X$ such that $f(x) = f(z)$, then z is an adjacent element to x.

We introduce another definition before we pause to explain the notion of tame Morse function.

Definition 3.14. Let X be a finite poset and let $f\colon X \to C_m$ be a tame Morse function. A point $x \in X$ is said to be *critical* if

$$\#\{y \in X\colon x \prec y \text{ and } f(x) \geq f(y)\} = 0$$

and

$$\#\{w \in X\colon w \prec x \text{ and } f(w) \geq f(x)\} = 0.$$

The set of critical points is denoted by $\operatorname{crit} f$. The images of the critical points are called *critical values*. The points (respectively values) which are not critical are said to be *regular points* (respectively *regular values*). The *index* of a critical point is its height.

Remark 3.15. Observe that Condition (F) from Theorem 3.11 was redundant for the definition.

Remark 3.16. Condition (E) is called the *Exclusion condition* and imposes a bond between conditions (M1) and (M2).

Remark 3.17. Condition (CI) encapsulates two ideas and consequences:

(1) The restriction of the function $f\colon X \to C_m$ to its set of critical points

$$f_{|\operatorname{crit} f}\colon \operatorname{crit} f \to C_m$$

is injective.

(2) Combining Conditions (M1), (M2), (E) and (CI) we have the following. For every $t \in C_m$, its preimage $f^{-1}(t)$ consists of at most two points, that is, $\#f^{-1}(t) \leq 2$. If $x \in X$ is critical, then $\#f^{-1}(f(x)) = 1$. If $x \in X$ is not critical, then $\#f^{-1}(f(x)) = 2$.

Remark 3.18. Remark 3.17 guarantees that a poset map with small fibers (see [105, Definition 11.3]) in the sense of Kozlov is a tame Morse function.

The following lemma relates our notion of tameness to Nishinou's one (see [135]) for face posets of cell complexes. In particular, it guarantees that if X is the face poset of an affine complex (see [135]), then our tame Morse functions are also tame according to his notion.

Lemma 3.19. *Let X be a finite poset and let $f\colon X \to C_m$ be a tame Morse function. If $w < x$ and $w \not\prec x$, then $f(w) < f(x)$.*

Proof. It follows from combining Conditions (OP), (M1), (M2) and (E) in our definition of tame Morse function (Definition 3.13). □

We recall the definition of Morse functions for finite posets introduced by Minian ([126]). It is a generalization of Forman's theory ([75, 77]) to the context of posets:

Definition 3.20. Let X be a finite poset. A *Morse function* on X is a set-theoretic function $f\colon X \to \mathbb{R}$ such that, for every $x \in X$, we have

$$\#\{y \in X\colon x \prec y \text{ and } f(x) \geq f(y)\} \leq 1$$

and

$$\#\{w \in X\colon w \prec x \text{ and } f(w) \geq f(x)\} \leq 1.$$

The notion of criticality for Morse functions is the same as the one for tame Morse functions (see Definition 3.14).

Remark 3.21. Tame Morse functions are Morse functions by identifying the chain C_m with a finite subset of $\mathbb{R}$.

Several observations are in order. First, a Morse function does not have to be order-preserving (so they are not necessarily continuous with the poset topologies), so it is not an arrow in the category of posets and order-preserving maps between them. Minian's definition of Morse function may look, at first sight, more general than the notion of tame Morse function. We will compare them later on.

We finish this section by presenting two remarkable contexts where our definitions of Morse function and matching applies.

3.1.1 Forman's Discrete Morse theory

We begin with Forman's definition of Morse function, the basis of his discretization of Morse theory. We follow [1, 73, 75, 77, 82, 102]. We refer the reader interested in more examples or a detailed treatment of the subject to such references.

Definition 3.22. Let K a finite regular CW-complex and denote by $K^{(=p)}$ the set of p-cells of K. A discrete Morse function on X is a map $f\colon \bigcup_p K^{(=p)} \to \mathbb{R}$ satisfying the following two conditions for all $\sigma \in K^{(=p)}$:

1. $\#\{\tau^{(p+1)} > \sigma\colon f(\tau) \leq f(\sigma)\} \leq 1$
2. $\#\{\tau^{(p-1)} < \sigma\colon f(\tau) \geq f(\sigma)\} \leq 1.$

Remark 3.23. Due to the regularity of the attaching maps in the finite CW-complex it holds (see for example [102, Lemma 6.11] or [151, Lemma 2.24]) that a map

$$f\colon \bigcup_p K^{(=p)} \to \mathbb{R}$$

is a discrete Morse function if and only if the following condition holds for all $\sigma \in K^{(=p)}$:

$$\#\{\tau^{(p+1)} > \sigma\colon f(\tau) \leq f(\sigma)\} + \#\{\tau^{(p-1)} < \sigma\colon f(\tau) \geq f(\sigma)\} \leq 1.$$

Remark 3.24. For the sake of simplicity and following the notational convenience in the literature, we will refer to Morse functions on K as $f\colon K \to \mathbb{R}$ without mentioning $\bigcup_p K^{(=p)}$.

In what follows K will denote a finite regular CW-complex.

Definition 3.25. Let $f\colon K \to \mathbb{R}$ be a discrete Morse function on K. A cell σ^p is said to be a *critical cell of index* p if the following condition holds:

$$\#\{\tau^{(p+1)} > \sigma\colon f(\tau) \leq f(\sigma)\} + \#\{\tau^{(p-1)} < \sigma\colon f(\tau) \geq f(\sigma)\} \leq 0.$$

Remark 3.26. Given any finite regular CW-complex K, discrete Morse functions always exist on K. Consider for example $f\colon K \to \mathbb{R}$ given by $f(\sigma) = \dim \sigma$.

The first main result of discrete Morse theory is that discrete Morse functions encode all homotopical information of the space where they are defined (see for example [75, 77]):

Theorem 3.27 (Structure theorem of discrete Morse theory)**.** *Let K be a finite regular CW-complex and let $f\colon K \to \mathbb{R}$ be a discrete Morse function. Then, K is homotopy equivalent to a CW-complex with exactly one p-cell for each critical cell of index p.*

Let K be a regular CW-complex and $f\colon K \to \mathbb{R}$ a discrete Morse function. Then there is an associated Morse function $\mathcal{X}(f)\colon \mathcal{X}(K) \to \mathbb{R}$ given by $\mathcal{X}(f)(\sigma) = f(\sigma)$. We will see later that $\mathcal{X}(f)\colon \mathcal{X}(K) \to \mathbb{R}$ encodes all the homotopical information of $f\colon K \to \mathbb{R}$.

3.1.2 Morse theory on Lefschetz Complexes

We introduce the notion of Lefschetz complex following [106, 110, 128, 129]. Let us denote by R a commutative ring with unity.

Definition 3.28. A Lefschetz complex (X, κ) is a finite set with gradation $X = (X_p)_{p\in\mathbb{N}}$ endowed with a map $\kappa\colon X \times X \to R$ such that $\kappa(x, w) \neq 0$ implies $x \in X_p, w \in X_{p-1}$ and for any $x, w \in X$ it holds that

$$\sum_{y\in X} \kappa(y, x)\kappa(x, w) = 0. \tag{3.2}$$

The elements of X are referred to as *cells* and $\kappa(x, w)$ as the *incidence coefficient* of x, w. If $x \in X_p$, we say that x has *dimension* p.

Let (X, κ) be a Lefschetz complex. Denote by $C^\kappa(X)$ the graded free module over R spanned by X. We define a boundary homomorphism $d^\kappa\colon C^\kappa(X) \to C^\kappa(X)$ on generators $x \in X$ by

$$d^\kappa(x) = \sum_{w\in X} \kappa(x, w)w.$$

By Equation (3.2), it can be proved ([106]) that $d^\kappa \circ d^\kappa = 0$. Hence, $(C^\kappa(X), d^\kappa)$ is a chain complex. Its homology is called *Lefschetz homology* and it is denoted by $H^\kappa(X)$.

Let (X, κ) be a Lefschetz complex. It can be given a partial order structure as follows. For $x, w \in X$, we declare $w \prec x$ if $\kappa(x, w) \neq 0$. We define a partial order $\leq$ on X as the transitive closure of $\prec$.

A Morse theory for Lefschetz complexes was developed, under the name of Algebraic Morse theory, among others, by Kozlov ([104, 105]) and later Donau ([54]). It is build upon the idea of a gradient matching for a chain complex. A *gradient matching* $\mathcal{M}$ for a Lefschetz complex (X,κ) is just a gradient matching on its associated poset.

A Lefschetz complex (X,κ) is *augmentable* if the linear map $\epsilon\colon C^{\kappa}(X)_0 \to R$ defined on the basis elements $x \in X_0$ by $\epsilon(x) = 1$ satisfies $\epsilon \circ d_1^{\kappa} = 0$.

The following result by Kubica and Mrozek ([106, Theorem 3.2]) guarantees that under some hypothesis, Lefschetz homology is isomorphic to singular homology.

Theorem 3.29. *Let (X,κ) be an augmentable Lefschetz complex such that for every $x \in X$, U_x is acyclic. Then the Lefschetz homology of (X,κ) is isomorphic to the singular homology of the T_0 finite space associated to the poset structure on (X,κ).*

Definition 3.30. A Lefschetz complex (X,κ) in the hypothesis of Theorem 3.29 will be called a *Lefschetz-Kubica-Mrozek complex*.

We will say that a Lefschetz-Kubica-Mrozek complex (X,κ) is *homologically admissible* if the associated poset is so. Conversely, observe that for a homologically admissible poset, its cellular chain complex $(C(X),d)$ is a homologically admissible Lefschetz-Kubica-Mrozek complex.

Remark 3.31. As a consequence of the discussion above, we can both do Morse theory and study homological properties of Lefschetz complexes by means of doing Morse theory on the associated posets. Moreover, all the Morse theory developed for homologically admissible posets applies directly to homologically admissible Lefschetz-Kubica-Mrozek complexes.

3.2 Properties of Morse functions and matchings

Our next goal is to study properties of Morse functions, tame Morse functions and Morse matchings, and the relation between them. To do so, we will restrict our attention to certain classes of posets.

First, we begin with an observation to settle a property that the posets we work with should satisfy in order to guarantee that Morse functions on them behave as we expect. Recall that in smooth Morse theory there is a "Minimum Theorem", which asserts that for a smooth Morse function $f\colon M \to \mathbb{R}$ defined on the manifold M, the point where the global minimum of f is attached is a critical point.

In order to provide a sufficient condition on posets to guarantee this result to hold, we recall the notions of local minimum and global minimum for a function.

Definition 3.32. Let X be a finite poset and let $f\colon X \to \mathbb{R}$ be a continuous (i.e order preserving) function. We say that an element $x \in X$ is a *local minimum for* f if for every adjacent element to x, z, $f(z) > f(x)$. We say that $x \in X$ is a *global minimum for* f if for every element $z \in X$, $z \neq x$, $f(z) > f(x)$.

Remark 3.33. Observe that for a finite poset X and a Morse function $f\colon X \to \mathbb{R}$, a global minimum for f is a local minimum for f.

Theorem 3.34 (Minimum Theorem)**.** *Let X be a finite down-wide poset and let $f\colon X \to \mathbb{R}$ be a Morse function on X.*

(1) If an element $x \in X$ is a local minimum for f, then it is a minimal element of X, that is, its height $h(x) = 0$.

(2) Furthermore, if x is a global minimum for f, then it is critical.

Proof. We will prove (1) by contraposition. Let $x \in X$ be a non-minimal element (of height $h(x) \geq 1$). We will see that it is not a local minimum for f. Since the the down-degree of every element of height different from zero is equal or greater than two, then there exist elements $w, w' \in X$, $w \neq w'$, adjacent to x, such that $w, w' \prec x$. By the definition of Morse function, $f(w) < f(x)$ or $f(w') < f(x)$. Hence, x is not a local minimum for f.

It remains to prove (2), that is, if x is a global minimum, then it is critical. We will do it again by contraposition. By (1), $h(x) = 0$. Suppose that x is not critical, so there exists $y \succ x$ (so $h(y) \geq 1$), such that $f(y) \leq f(x)$. Since the down-degree of y is grater than or equal to two, then by the definition of Morse function there exists $x' \prec y$ such that $f(x') < f(y) \leq f(x)$, which contradicts that x is a global minimum. □

We define the notion of local maximum and global maximum for a Morse function analogously.

Corollary 3.35. *Let X be a finite homologically bi-admissible poset and let $f\colon X \to \mathbb{R}$ be a Morse function on X.*

(1) If an element $x \in X$ is a local maximum for f, then it is a maximal element of X.

(2) Furthermore, if x is a global maximum for f, then it is critical.

Proof. Consider the poset X^{op} and the Morse function $f^{\mathrm{op}}\colon X^{\mathrm{op}} \to \mathbb{R}$ given by $f^{\mathrm{op}}(x) = -f(x)$. Now the result follows from Theorem 3.34. □

The next result is a consequence of the fact that the image of a Morse function $f\colon X \to \mathbb{R}$ is a finite subset of $\mathbb{R}$. So we can perturb the function at the critical points to assure that we obtain a new function $f'\colon X \to \mathbb{R}$ such that $f'_{|\mathrm{crit} f'}\colon \mathrm{crit} f' \to \mathbb{R}$ is injective.

Lemma 3.36. *Let X be a finite poset and let $f\colon X \to \mathbb{R}$ be a Morse function on X. Then there exists a Morse function $f'\colon X \to \mathbb{R}$ such that:*

(1) The restriction $f'_{|\mathrm{crit} f'}\colon \mathrm{crit} f' \to \mathbb{R}$ is injective.

(2) The functions share the critical points and agree outside the critical points. That is: For $x \in X$, $x \in \mathrm{crit} f$ if and only if $x \in \mathrm{crit} f'$. And for $x \in X - \mathrm{crit} f$, $f_{|X - \mathrm{crit} f}(x) = f'_{|X - \mathrm{crit} f'}(x)$.

We continue by stating a result that plays the role of two important theorems developed by Forman in the simplicial setting ([75, Theorems 1.2 and 1.3]). In fact, the proof of the following Key Lemma is immediate in this context.

Lemma 3.37 (Key Lemma)**.** *Suppose that X is a finite two-wide poset and there are two elements $w < y$ such that $w \not\prec y$. Then, there are elements $x \neq \tilde{x}$ such that $w \prec x < y$ and $w \prec \tilde{x} < y$.*

Remark 3.38. Observe that Lemma 3.37 does not hold in general for finite posets. As an example, consider the poset of Figure 3.2.1 taking the points labelled as w and y.

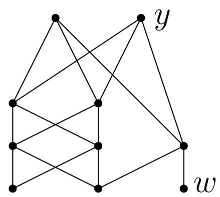


Figure 3.2.1: Lemma 3.37 may not hold for posets which are not two-wide.

Definition 3.39. Given a poset X, a Morse function $f\colon X \to \mathbb{R}$ is said to satisfy the *Exclusion condition* if for every regular point $x \in X$, exactly one of the following conditions holds:

(1) There exists exactly one $y \in X$, $x \prec y$, such that $f(x) \geq f(y)$.

(1) There exists exactly one $w \in X$, $w \prec x$, such that $f(w) \geq f(x)$.

That is, a Morse function $f\colon X \to \mathbb{R}$ satisfies the Exclusion condition if it satisfies condition (E) of the definition of tame Morse function (Definition 3.13).

The following result plays the role, in the context of finite spaces, of the Exclusion Lemma ([75, Lemma 2.5]).

Lemma 3.40 (Exclusion Lemma)**.** *Let X be a finite two-wide poset and $f\colon X \to \mathbb{R}$ a Morse function on X. Then $f\colon X \to \mathbb{R}$ satisfies the Exclusion condition.*

Proof. Let x be a regular point. Since x is not critical, then at least one of the conditions in Definition 3.39 holds. We will see that these conditions are mutually exclusive. By way of contradiction, suppose that both conditions hold and take $x \prec y$ and $w \prec x$ as in Conditions (1) and (2) respectively. Then $w \prec x \prec y$. Since X is two-wide, by the Key Lemma (Lemma 3.37) there exists $x' \neq x$ such that $w \prec x' \prec y$. By the definition of Morse function applied to w with $w \prec x'$, we get $f(w) < f(x')$ since we already have $f(w) \geq f(x)$ and $w \prec x$. Similarly with y and $x' \prec y$, we obtain $f(x') < f(y) \leq f(x)$.

As a consequence we obtain the following chain of inequalities:

$$f(x) \leq f(w) < f(x') < f(y) \leq f(x).$$

So $f(x) < f(x)$, which is a contradiction. $\square$

It is interesting to point out that the Exclusion Lemma does not necessarily hold in general for posets which are not two-wide, as the following example shows.

Example 3.41. Lemma 3.40 may not hold for arbitrary posets. Consider the down-wide model of $\mathbb{S}^3$ depicted in Figure 3.2.1 with the Morse function f represented by the labelling of the points.

As a consequence of our Exclusion Lemma for Morse functions on two-wide posets (Lemma 3.40), we obtain a converse result to Theorem 3.11.

Theorem 3.42. *Let X be a finite poset and let $f\colon X \to \mathbb{R}$ be a Morse function satisfying the Exclusion condition. Then, there exists an associated Morse matching $\mathcal{M}_f$ satisfying that: for $(x, y) \in X \times X$ such that $x \prec y$, $(x, y) \in \mathcal{M}_f$ if and only if $f(x) \geq f(y)$. As a consequence,* $\operatorname{crit}(f) = \operatorname{crit}(\mathcal{M}_f)$.

In particular, given a finite two-wide poset X and a Morse function $f\colon X \to \mathbb{R}$, there exists an associated Morse matching $\mathcal{M}_f$ with $\operatorname{crit}(f) = \operatorname{crit}(\mathcal{M}_f)$ *and $(x, y) \in \mathcal{M}_f$ if and only if $f(x) \geq f(y)$ and $x \prec y$.*

Proof. Define the matching $\mathcal{M}_f$ as follows. Let $(x, y) \in X \times X$ such that $x \prec y$. Then $(x, y) \in \mathcal{M}_f$ if and only if $f(x) \geq f(y)$ (so $\operatorname{crit}(f) = \operatorname{crit}(\mathcal{M}_f)$). It remains to check that the matching $\mathcal{M}_f$ is a Morse matching on X, that is: $\mathcal{H}_{\mathcal{M}_f}(X)$ is acyclic as a directed graph. First, observe that the paths in $\mathcal{H}_{\mathcal{M}_f}(X)$ are just generalized $\mathcal{M}_f$-paths. Now, observe that for any generalized $\mathcal{M}_f$-path of any of the forms:

1. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} = z$
2. $\gamma\colon x = x_0 \prec y_0 \succ x_1 \prec y_1 \succ \cdots \prec y_{r-1} \succ x_r = z$

it holds, respectively:

1. $f(x) = f(x_0) \geq f(y_0) > f(x_1) \geq f(y_1) > \cdots \geq f(y_{r-1}) = f(z)$
2. $f(x) = f(x_0) \geq f(y_0) > f(x_1) \geq f(y_1) > \cdots \geq f(y_{r-1}) > f(x_r) = f(z)$

So, there can not exist loops and $\mathcal{M}_f$ is a Morse matching on X. □

Corollary 3.43. *Let X be a finite graded poset and let $f\colon X \to \mathbb{R}$ be a Morse function satisfying the Exclusion condition. Then, there exists an order preserving and self-indexing Morse function $f'\colon X \to \mathbb{R}$ satisfying the Exclusion condition with* $\operatorname{crit}(f') = \operatorname{crit}(f)$.

Proof. First apply Theorem 3.42 to $f\colon X \to \mathbb{R}$ to obtain a Morse matching $\mathcal{M}_f$ with the same critical set. Then apply Theorem 3.11 to $\mathcal{M}_f$. □

Remark 3.44. As a consequence of the previous discussion, we can establish a correspondence between Morse matchings and order preserving Morse functions satisfying the Exclusion condition on graded posets. However, the correspondence is not bijective since given a Morse function $f\colon X \to \mathbb{R}$, a function $f'\colon X \to \mathbb{R}$ given by $f'(x) = 2f(x)$ is again Morse and both functions share the same associated matching.

3.2.1 Dynamically equivalent Morse functions

Our next goal is to provide a way to reduce the proof of certain results to verifying them for a class of reasonably well-behaved Morse functions, which will be in our case the tame Morse functions.

Definition 3.45. Let X be a finite poset. Let $f, g\colon X \to \mathbb{R}$ be two Morse functions satisfying the Exclusion condition and let $\mathcal{M}_f$ and $\mathcal{M}_g$ be their associated Morse matchings by Theorem 3.42. The functions f and g are *dynamically equivalent* if their associated Morse matchings coincide.

Remark 3.46. Observe that for a Morse function $f\colon X \to \mathbb{R}$ satisfying the Exclusion condition, by applying first Corollary 3.43 and then Lemma 3.36 we obtain a Morse function $f'\colon X \to \mathbb{R}$ satisfying all the conditions of tame Morse function other than Condition (CI). Moreover, the functions f and f' are dynamically equivalent.

The purpose of what follows is to improve the idea of Remark 3.46 in order to be able to find, for a Morse function $f\colon X \to \mathbb{R}$ satisfying the Exclusion condition, a tame Morse function $f'\colon X \to \mathbb{R}$ which is dynamically equivalent to f.

We begin with an alternative approach to integrating matchings. In order to do that, we will incorporate in the proof a heuristic which appears in the works of Kozlov ([105, Theorem 11.2]).

Theorem 3.47. *Let X be a finite poset and let $\mathcal{M}$ be a a Morse matching on X. Then, there is a set function $f\colon X \to C_m$ for some $m \in \mathbb{N}$ satisfying the following conditions:*

(OP) It is order preserving, that is, if $x \leq y$, then $f(x) \leq f(y)$.

(M1) $\#\{y \in X\colon x \prec y$ and $f(x) \geq f(y)\} \leq 1$.

(M2) $\#\{w \in X\colon w \prec x$ and $f(w) \geq f(x)\} \leq 1$.

(E) There do not exist $w, x, y \in X$, $w \prec x \prec y$ such that $f(w) \geq f(x) \geq f(y)$.

(CI) For each $x \in X$, if there is an element $z \in X$ such that $f(x) = f(z)$, then z is an element adjacent to x.

That is, $f\colon X \to C_m$ is a tame Morse function.

Proof. The idea of the proof is the following. First, we will construct a filtration of X by a finite sequence of subposets:

$$X_{-1} = \emptyset \hookrightarrow X_0 \hookrightarrow \cdots \hookrightarrow X_i \hookrightarrow X_{i+1} \hookrightarrow \cdots \hookrightarrow X_m = X. \tag{3.3}$$

satisfying that for every $i \in \{0, \ldots, m\}$, $X_i - X_{i-1}$ is either a critical element or a pair of matched elements. Then, we will define the map $f\colon X \to C_m$ as follows. For every $i \in \{0, \ldots, m\}$ and $x \in X_i - X_{i-1}$, $f(x) = i$. It is clear that the map f satisfies the desired conditions.

Now we show that we can build a filtration as in Equation (3.3). We define X_{-1} as the empty set. Now we apply the following iterative process. For $X_{i-1} \neq X$, a down-set subposet of X, we denote by M_i the set of minimal elements of X in $X - X_{i-1}$. If there is a critical element $c \in M_i$, then we define $X_i = X_{i-1} \cup \{c\}$. If there are no critical elements in M_i, then we proceed as follows. Among all the edges $e \in \mathcal{M}$ such that $s(e) \in M_i$, we pick one satisfying that $(\widehat{U}_{t(e)} - s(e)) \subset X_{i-1}$. Then, we define X_i as the down-set with the elements $X_{i-1} \cup \{s(e), t(e)\}$. Observe that if $X_{i-1} \neq X$ is a down-set subposet of X and there are no critical elements in M_i, then among the $e \in \mathcal{M}$ such that $s(e) \in M_i$, there must be at least one satisfying $(\widehat{U}_{t(e)} - s(e)) \subset X_{i-1}$ since if this was not the case, then $\mathcal{M}$ would not be acyclic. $\square$

Now, combining Theorems 3.42 and 3.47 we obtain the following result:

Corollary 3.48. *Let X be a finite poset and let $f\colon X \to \mathbb{R}$ be a Morse function satisfying the Exclusion condition. Then, there exists a tame Morse function $f'\colon X \to \mathbb{R}$ dynamically equivalent to f.*

Remark 3.49. The same arguments of Remark 3.44 taking now into account Theorem 3.47 prove that there is a correspondence between Morse matchings and tame Morse functions on arbitrary posets.

Furthermore, we can summarize a consequence of the previous discussion as follows:

Corollary 3.50. *Let X be a finite poset and let $f\colon X \to C_m$ be a tame Morse function. Then there is an induced filtration by down-sets:*

$$X_{-1} = \emptyset \hookrightarrow X_0 \hookrightarrow \cdots \hookrightarrow X_i \hookrightarrow X_{i+1} \hookrightarrow \cdots \hookrightarrow X_m = X \tag{3.4}$$

where $X_i = f^{-1}(U_i)$ for $i \in C_m$. Moreover, this filtration satisfies that for every $i \in \{0, \ldots, m\}$, $X_{i+1} - X_i$ is either a critical element or a pair of matched elements.

In fact, Corollary 3.50 has a converse result. As a consequence, Morse theory may be seen as a theory of nice filtrations for a space. We formalize this idea as follows:

Theorem 3.51. *Let X be a finite poset. Then:*

(A) If $f\colon X \to C_m$ is a tame Morse function, then there is an induced filtration by down-sets as in Equation (3.4) satisfying that $X_i = f^{-1}(U_i)$ and that for every $i \in \{0, \ldots, m\}$, $X_i - X_{i-1}$ is either a critical element or a pair of matched elements.

(B) Conversely, if there is a filtration by down-sets as in Equation (3.4) satisfying that for every $i \in \{0, \ldots, m\}$, either $X_i - X_{i-1} = \{x_i\}$ or $X_{i+1} - X_i = \{x_i, y_i\}$ with $x_i \prec y_i$, then there is a function $f\colon X \to C_m$ defined as follows. For every $i \in \{0, \ldots, m\}$ and $x \in X_i - X_{i-1}$, $f(x) = i$. Moreover, this function is tame Morse.

We introduce the following notation: given a finite poset X and a Morse function $f\colon X \to \mathbb{R}$, for each $a \in \mathbb{R}$ we denote

$$X_a^f = \bigcup_{f(x) \leq a} U_x.$$

Observe that for each $a \in \mathbb{R}$, the subposet X_a^f is an open subset of X. When the Morse function f is clear from the context, we simply write X_a for X_a^f. Let us say $f(X) = \{a_j\}_{j=0}^n$ where the $\{a_j\}_{j=0}^n$ are increasingly ordered, then there is a filtration of X induced by f:

$$X_{a_{-1}} = \emptyset \hookrightarrow X_{a_0} \hookrightarrow \cdots \hookrightarrow X_{a_j} \hookrightarrow X_{a_{j+1}} \hookrightarrow \cdots \hookrightarrow X_{a_n} = X \tag{3.5}$$

Since we are interested in constructing right filtrations, we now improve Corollary 3.48 to not only replace Morse functions with dynamically equivalent tame ones, but also such that the induced filtrations by the two maps are compatible in the sense that we will describe below.

Theorem 3.52. *Let X be a finite poset and let $f\colon X \to \mathbb{R}$ be a Morse function satisfying the Exclusion condition. Then, there exists a tame Morse function $f'\colon X \to \mathbb{R}$ dynamically equivalent to f. Moreover, for each $X_{a_j}^f$ in Equation (3.5) there is an $i \in \mathbb{N}$ such that $X_i^{f'}$ is homeomorphic to $X_{a_j}^f$ and the homeomorphism is compatible with the filtration. That is, we have a commutative diagram where the vertical arrows are homeomorphisms:*

$$\begin{tikzcd}
X_{a_0}^f \arrow[r, hook] & \cdots \arrow[r, hook] & X_{a_j}^f \arrow[r, hook] & \cdots \arrow[r, hook] & X_{a_n}^f = X \\
X_0^{f'} \arrow[r, hook] & \cdots \arrow[r, hook] & X_i^{f'} \arrow[u] \arrow[r, hook] & \cdots \arrow[r, hook] & X_m^{f'} = X. \arrow[u]
\end{tikzcd} \tag{3.6}$$

Proof. First, by Theorem 3.42 there exists an associated Morse matching $\mathcal{M}_f$ satisfying that: for $(x,y) \in X \times X$ such that $x \prec y$, $(x,y) \in \mathcal{M}_f$ if and only if $f(x) \geq f(y)$. Now we will work as in Theorem 3.47 but taking into account that we do not only have a matching but also a function f.

Now we show that we can build a filtration as in Equation (3.3). We define X_{-1} as the empty set. Now we apply the following iterative process. For $X_{i-1} \neq X$, a down-set subposet of X, we denote by M_i the set of minimal elements of X in $X - X_{i-1}$. Consider

$$t = \min\{f(c)\colon c \text{ critical and } c \in M_i\}.$$

If there is a unique c where f attains the minimum t, then define $X_i = X_{i-1} \cup \{c\}$. If that is not the case and there are $\{c_1, \ldots, c_k\} \subset M_i$ critical and such that $f(c_l) = t$, then we order them and define for each $l \in \{1 \ldots, k\}$, $X_{i+l-1} = X_{i+l-2} \cup \{c_l\}$.

Assume there are no critical elements in M_i. Consider the set of edges:

$$E = \{e \in \mathcal{M}\colon s(e) \in M_i \text{ and } (\widehat{U}_{t(e)} - s(e)) \subset X_i\}.$$

Consider $t = \min\{f(s(e))\colon e \in E\}$. If there is a unique e where f attains the minimum t, then define X_i as the down-set with the elements $X_{i-1} \cup \{s(e), t(e)\}$. Otherwise, there are $\{e_1, \ldots, e_k\} \subset E$ such that $f(s(e_l)) = t$, then we order them and define for each $l \in \{1 \ldots, k\}$, X_{i+l-1} as the down-set with the elements $X_{i+l-2} \cup \{s(e_l), t(e_l)\}$.

Observe that if $X_i \neq X$ is a down-set subposet of X and there are no critical elements in M_i, then among the $e \in \mathcal{M}$ such that $s(e) \in M_i$, there must be at least one satisfying that $(\widehat{U}_{t(e)} - s(e)) \subset X_{i-1}$ since if this was not the case, then $\mathcal{M}$ would not be acyclic.

Finally, we define the map $f\colon X \to C_m$ as follows. For every $i \in \{0, \ldots, m\}$ and $x \in X_i - X_{i-1}$, $f(x) = i$. It is clear that the map f satisfies the desired conditions. □

3.2.2 Cancelling critical points

Recall that Morse functions and matchings measure in a certain sense the evolution of invariants across a filtration of a chosen space X. Moreover, passing through regular values keep the invariants unchanged. So, it may be desirable to construct Morse functions and matchings with as few critical values as possible. In order to obtain such functions we present an approach consisting in canceling pairs of critical elements. This approach extends to the context of posets known results on smooth manifolds and simplicial complexes.

We present a result, which can be seen as the generalization of [75, Theorem 11.1] to our context.

Theorem 3.53 (Cancelling critical points)**.** *Given a Morse matching $\mathcal{M}$ on a finite graded poset X, assume that $z^{(p+1)}$ and $x^{(p)}$ are critical points such that there is a unique $\mathcal{M}$-path*

$$\gamma\colon z \succ y = x_0 \prec z_0 \succ x_1 \prec z_1 \succ \cdots \prec z_r \succ x_r = x$$

with $y^{(p)} \prec z^{(p+1)}$ (there is no other $\mathcal{M}$-path from any p-face of $z^{(p+1)}$ to $x^{(p)}$). Then there is a matching $\mathcal{M}'$ such that:

- *The set of critical points of $\mathcal{M}'$ is*

$$\mathrm{crit}(\mathcal{M}') = \mathrm{crit}(\mathcal{M}) - \{x, z\}.$$

- *Moreover, $\mathcal{M}' = \mathcal{M}$ except along the unique $\mathcal{M}$-path from ∂z to x.*

Proof. We define $\mathcal{M}'$ as follows:

1. $t_{\mathcal{M}'}(w) = t_{\mathcal{M}}(w)$ if $w \notin \{y, z_0, x_1, z_1, \ldots z_r, x\}$ ($\mathcal{M}' = \mathcal{M}$ except along the unique gradient path from ∂z to x)

2. $t_{\mathcal{M}'}(x_i) = z_{i-1}$, $i = 1, \ldots, r$ (we reverse the gradient path from x to z_0 so x is no longer critical)

3. $t_{\mathcal{M}'}(y) = z$ (we reverse the arrow from y to z so z is no longer critical).

It remains to check that there are no closed $\mathcal{M}'$-paths. We argue by contradiction. Suppose there was a closed $\mathcal{M}'$-path δ.

Claim. *Under the above hypothesis, δ would contain at least one p-element from γ and one p-element not in γ.*

Proof of the Claim. The elements coming from γ can not give a closed $\mathcal{M}'$-path on their own since we have just reversed their arrows. The elements of X which are not in γ can not give a closed $\mathcal{M}$-path since in that case we would also have a closed $\mathcal{M}$-path and $\mathcal{M}$ is a gradient vector field. Therefore in δ we must have at least one p-element in each of their sets. □

Hence, δ would contain a sequence of the form:

$$x_i \prec w_0 \succ s_1 \prec w_1 \succ \cdots \prec w_s \succ x_j$$

where $s \geq 0$, $w_l \neq x_k$, $w_l \neq z_k$, $s_l \neq x_k$, $s_l \neq z_k$, for all l and k. Since $t_{\mathcal{M}'}(w_l) = t_{\mathcal{M}}(w_l)$ and $t_{\mathcal{M}'}(s_l) = t_{\mathcal{M}}(s_l)$ for all l, we have a $\mathcal{M}$-path:

$$w_0 \succ s_1 \prec w_1 \succ \cdots \prec w_s \succ x_j.$$

Let us consider two cases:

1. If $i \neq 0$, then $s_1 \neq x_{i-1}, x_i$ and $s_1 \prec t_{\mathcal{M}'}(x_i) = z_{i-1}$. Therefore, we can define a second gradient $\mathcal{M}$-path $\gamma' \neq \gamma$ from ∂z to x:

$$\begin{aligned}\gamma' \colon y =& x_0 \prec z_0 \succ x_1 \prec \cdots \succ x_{i-1} \prec z_{i-1} \succ s_1 \prec w_1 \succ \cdots \\ &\succ x_j \prec z_j \succ \cdots \succ x_r = x.\end{aligned}$$

   Which is a contradiction.

2. If $i = 0$, then $y = x_0 \neq s_1 \prec t_{\mathcal{M}'}(y) = z$. Therefore, we can define the following $\mathcal{M}$-path:

$$\gamma' \colon z \succ s_1 \prec w_1 \succ \cdots \prec w_s \succ x_j \prec z_j \succ \cdots \succ x_r = x$$

   which is different from γ and also goes from ∂z to x. Then we have a contradiction. □

This result raises the question of what is the minimum number of critical points of a Morse function for a given space X. That is equivalent to studying the relations between dynamics and topology on X. We will see later than Morse inequalities provide a partial answer to this question.

Chapter 4
Structure Theorems for Morse Theory on posets and their consequences

We have studied in the previous chapter the notions of matchings (or vector fields) and Morse functions. Furthermore, we have addressed the relations between them. At this stage, we already know how to construct, for a finite well-behaved poset X, a filtration induced by a (tame) Morse function

$$X_{a_{-1}} = \emptyset \hookrightarrow X_{a_0} \hookrightarrow \cdots \hookrightarrow X_{a_j} \hookrightarrow X_{a_{j+1}} \hookrightarrow \cdots \hookrightarrow X_{a_n} = X \tag{4.1}$$

which satisfies that for every $i \in \{0, \ldots, m\}$, $X_{i+1} - X_i$ is either a critical element or a pair of matched elements. In this chapter we address the study of such filtrations so we can control the evolution of certain topological invariants as we advance on the filtration, that is, the filtrations have a good-topological behavior. We are interested in finding properties about the dynamics that we define on X, that is, properties about the Morse functions or matchings, which result in a good-topological behavior of the filtration.

The organization of the chapter is the following. First, we will study the existence and form of some Structure Theorems for Morse theory for posets. Afterwards, we will obtain some consequences of the preceding results.

Some of the results of this chapter were introduced in our work [66]. However, in that paper we did not use the approach involving tame Morse functions.

4.1 Structure Theorems for Morse Theory on posets

In this section we study the existence and form of some Structure Theorems for Morse theory in this context.

4.1.1 Adding new connected components is critical

We begin with our study of the interplay between dynamics and topology on finite posets. We denote by $b_0(X)$ the number of connected components of X. The following result guarantees that for a tame Morse function $f\colon X \to C_m$, new connected components of X_i (as $i \in C_m$ increases) arise as critical minimal elements.

Proposition 4.1. *Let X be a path-connected finite down-wide poset and let $f\colon X \to C_m$ be a tame Morse function.*

1. *If $b_0(X_i) < b_0(X_{i+1})$, then $i \in C_m$ is critical.*

2. *Furthermore,* $b_0(X_{i+1}) = b_0(X_i) + 1$ *and the critical value* $i+1$ *corresponds to a minimal element of* X.

Proof. We prove the first part by showing the contrapositive. Assume $i+1 \in C_m$ is regular. Then, $X_{i+1} - X_i = \{x, y\}$ with $x \prec y$. Since X is down-wide, there exists $x' \prec y$, $x' \neq x$. By the definition of Morse function, $f(x') < f(y)$ and $x' \in X_i$, so $b_0(X_i) = b_0(X_{i+1})$. For the second part, first observe that if $i+1 \in C_m$ is critical, then $f^{-1}(i+1)$ is just one element x, so $b_0(X_{i+1}) \leq b_0(X_i) + 1$. Finally, it remains to check that x is minimal. If that were not the case, then there would exist a $w \in X_i$ such that $w \prec x$ and $f(w) < f(x)$, so $b_0(X_{i+1}) = b_0(X_i)$. □

As a consequence of Theorem 3.52 we obtain:

Corollary 4.2. *Let* X *be a path-connected finite down-wide poset and let* $f\colon X \to \mathbb{R}$ *be a Morse function satisfying the Exclusion condition. Let* $a, b \in \mathbb{R}$, $a < b$.

1. *If* $b_0(X_a) < b_0(X_b)$*, then there exists a critical value* $c \in (a, b]$.

2. *Furthermore, one of those critical values* $c \in (a, b]$ *corresponds to a minimal element of* X.

Example 4.3. Consider the Morse function represented in Figure 4.1.1. The value 3 must correspond to a critical point since we are adding a new path-component ($b_0(X_3) = b_0(X_2) + 1$). Moreover, the point corresponding to the value 3 is of zero height.

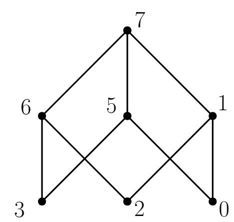


Figure 4.1.1: Regular values and path-components in general posets.

Proposition 4.2 may not hold for arbitrary posets, as the following example shows.

Example 4.4. Consider the Morse function represented in Figure 4.1.2. The value 4 is regular. However, $b_0(X_4) \neq b_0(X_3)$ while there are no critical values in $(3, 4]$.

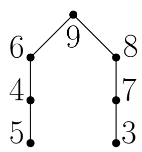


Figure 4.1.2: Regular values and path-components in general posets.

4.1.2 Failure of the expected Structure Theorems

Both in smooth and discrete Morse Theory, manifolds and cell complexes can be recovered up to homotopy equivalence from Morse functions defined on them by means of the so called Structure Theorems of Morse Theory. This results can be stated as follows (see for example [125] and [77]).

Theorem 4.5 (Classical and discrete Structure Theorems).

- *Let X be a compact manifold and let $f\colon X \to \mathbb{R}$ be a Morse function. Then, X is homotopy equivalent to a CW-complex with exactly one p-cell for each critical point of index p.*
- *Let X be a regular finite CW-complex and let $f\colon X \to \mathbb{R}$ be a discrete Morse function. Then, X is homotopy equivalent to a CW-complex with exactly one p-cell for each critical cell of index p.*

One may wonder whether it is possible to prove an analogous result to Theorem 4.5 in the setting of posets, that is:

Conjecture 4.6. *Let X be a well-behaved poset and let $f\colon X \to \mathbb{R}$ be a well-behaved Morse function on X. Then, X is homotopy equivalent to a poset with exactly one element of height p for each critical point of index p.*

The next example shows that Conjecture 4.6 fails.

Example 4.7. Consider the face poset of the simplicial complex depicted in Figure 4.1.3. It does not have the homotopy type of a point since its face poset does not have any strong homotopic removable point and a poset is contractible if and only if it can be contracted by removing strong homotopic removable points (see [6, Example 5.1.12] for a more detailed exposition). However,

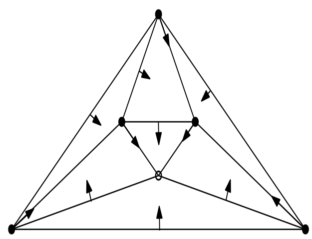


Figure 4.1.3: The Triangle.

there is a Morse function with only one critical point, namely, the Morse function associated to the matching drawn in the figure.

Example 4.7 shows that this Morse theory on posets sees at most the simple homotopy type of the posets but does not recover by any means their (strong) homotopy type.

So, this phenomena leads us to a second conjecture.

Conjecture 4.8. *Let X be well-behaved poset and let $f\colon X \to \mathbb{R}$ be a well-behaved Morse function on X. Then, X is homotopy equivalent to a CW-complex with exactly one one p-cell for each critical point of index p.*

We will see that Conjecture 4.8 fails as well. In order to do so, we recall a result by Barmak.

Definition 4.9. A topological space (X, τ) is: T_1 if for any two distinct points $x, y \in X$, there exist open subsets U and V of X such that: $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$.

Theorem 4.10 ([6, Theorem 2.3.2, Corollary 2.3.4]). *1. Let X be a finite space and let Y be a T_1-space homotopy equivalent to X. Then X is a disjoint union of contractible spaces.*

2. Let X be a connected and non contractible T_1-space. Then X does not have the same homotopy type as any finite space.

As a consequence of Theorem 4.10, and the fact that CW-complexes are T_1 (see [95]), then Conjecture 4.8 fails.

However, a weaker result can be proved. Minian proved that the order complex of X is homotopy equivalent to a CW-complex with exactly one one p-cell for each critical point of index p (see [126]). Nevertheless, we are interested in studying Morse theory intrinsically in the setting of finite posets and to obtain properties about the posets themselves and not just about their associated order complexes.

4.1.3 Structure Theorems for Morse Theory on posets

This subsection is devoted to proving the substitutes of the Structure Theorems for Morse Theory in this context, that is: several invariance and collapsing results and an adjunction theorem. Both the invariance and collapsing theorems guarantee that in the absence of critical values, a certain invariant remains unchanged provided the matching satisfies a reasonable property. The adjunction theorem provides a way to control the change in homotopy type (even homeomorphism) when we reach a critical value.

We begin with a definition extending the ideas of Minian's work ([126]).

Definition 4.11. Let X be a finite poset and $\mathcal{M}$ a matching on X. The matching $\mathcal{M}$ is *homologically admissible* if each element of the matching is homologically admissible. The notions of 1-*weakly homotopically admissible*, *weakly homotopically admissible* and *homotopically admissible* matching are defined analogously.

Definition 4.12. Let X be a finite poset. A Morse function $f\colon X \to \mathbb{R}$ satisfying the Exclusion condition is *homologically admissible*, respectively 1-*weakly homotopically admissible*, *weakly homotopically admissible* and *homotopically admissible* if its associated matching $\mathcal{M}_f$ is so.

Inspired by the notions of h-regular and cellular posets introduced in [126], we present the following definition.

Definition 4.13. Let X be a finite poset and $\mathcal{M}$ a matching on X. The matching $\mathcal{M}$ is *homology-regular* if for every $x^{(p)} \in \mathrm{crit}(\mathcal{M})$, the subspace $\widehat{U}_x$ is graded and has the homology of a sphere $\mathbb{S}^{p-1}$ where p is the height of x. The matching $\mathcal{M}$ is *homotopy-regular* if for every $x^{(p)} \in \mathrm{crit}(\mathcal{M})$, the subspace $\widehat{U}_x$ is a finite model of $\mathbb{S}^{p-1}$ where p is the height of x.

Definition 4.14. Let X be a finite poset. A Morse function $f\colon X \to \mathbb{R}$ satisfying the Exclusion condition is *homology-regular* or *homotopy-regular* if its associated matching $\mathcal{M}_f$ satisfies that property.

Proposition 4.15. *Let X be a finite path-connected down-wide poset and let $f\colon X \to C_m$ be a homologically admissible tame Morse function. If $i+1$ is a regular value, then $i\colon X_i \hookrightarrow X_{i+1}$ induces an isomorphism in all homology modules.*

Proof. Assume that $f^{-1}(i) = \{x, y\}$ with $x \prec y$. Then x is a strong homotopic removable point in X_{i+1}, so $i\colon X_{i+1} - \{w\} \hookrightarrow X_b$ is a homotopy equivalence, so it is enough to prove that $i\colon X_a = X_b - \{w, v\} \hookrightarrow X_b - \{w\}$ induces an isomorhism in homology. By applying the long exact sequence of homology to the pair $(X_b - \{w\}, X_a)$ it follows that $i\colon X_a \hookrightarrow X_b - \{x\}$ induces an isomorphism in all homology groups if and only if $H_*(X_{i+1} - \{x\}, X_i) \cong 0$. As a consequence of the Excision Theorem (see [95, Theorem 2.20]), given two open sets A and B which cover $X_{i+1} - \{x\}$, then there is an isomorphism $H_*(B, A \cap B) \cong H_*(X_{i+1} - \{x\}, A)$. Considering $A = X_i$ and $B = U_y$, it follows that

$$H_*(U_y, \widehat{U}_y - \{x\}) \cong H_*(X_{i+1} - \{x\}, X_i).$$

Since $x \prec y$ is an element in the matching and the matching is homologically admissible, then $\widehat{U}_y - \{x\}$ is acyclic. By applying the long exact sequence of homology to the pair $(U_y, \widehat{U}_y - \{x\})$ and using the fact that U_y is contractible, it follows that $H_*(U_y, \widehat{U}_y - \{x\}) \cong H_*(\widehat{U}_y - \{x\})$, so $H_*(U_y, \widehat{U}_y - \{x\}) \cong 0$ if and only if the element of the matching $w \prec v$ is homologically admissible. $\square$

Remark 4.16. Observe that in the above proof $\widehat{U}_y - \{x\}$ is always non-empty because X is a down-wide poset.

Remark 4.17. Alternatively, Proposition 4.15 can be proved by applying McCord-Quillen Homological Theorem (Theorem 2.34).

Proposition 4.15 can be proved as well without making use of tame Morse functions. However, that approach relies much more on combinatorial arguments. That was what we did in [66]. We introduce here the first part of the argument for comparison and illustration of the advantages of using tame Morse functions. The second part would amount to invoking McCord-Quillen Homological Theorem (Theorem 2.34).

Proposition 4.18. *Let X be a finite path-connected down-wide and two-wide poset. Let $f\colon X \to \mathbb{R}$ be a Morse function. Suppose that $(a, b]$ for $a < b$ contains no critical values of f and contains at most one regular value c. Then, either $X_b = X_a$ or $X_b - X_a = \{v_i, w_i\}_{i=1}^r$, where:*

1. *$f(v_i) = c$ for all i.*
2. *$w_i \prec v_i$ with $f(v_i) \leq f(w_i)$ for all i.*
3. *$\{v_i, w_i\} \cap \{v_j, w_j\} = \emptyset$ for all $i \neq j$.*

4. *For all i, w_i is an up strong homotopic removable point of X_b.*

Proof. Let $V = \{v \in X \colon f(v) = c\}$. Since c is a regular value, for each $v \in V$ there exists:

1. a unique $w \in X$ such that $w \prec v$ and $f(v) \leq f(w)$,
2. or else, a unique $w \in X$ such that $v \prec w$ and $f(w) < f(v)$ (in case $v \prec w$ and $f(w) = f(v)$ we rename v and w to be in the first case).

Observe that by the Exclusion Lemma (Lemma 3.40) exactly one of these two possibilities can hold. During all the proof, we will refer to an arbitrary v_i as v and to its correspondent w_i as w.

Suppose that we are in the second case, then $f(w) \leq a$ since $f(v)$ is the unique regular value in $(a, b]$ and $f(w) < f(v) \leq b$. Therefore, $X_b = X_a$. So, let us assume now that we are in the first situation. We have to check that $w \notin X_a$, that is, there is no $u \in X$, $w < u$ such that $f(u) \leq a$. Suppose that there exists such an $u \in X$ and we will reach a contradiction. First, observe that $w \not\prec u$ because of the definition of Morse function ($f(u) \leq f(w)$ and $f(v) \leq f(w)$ can not hold simultaneously). So there exists v' such that $w \prec v' < u$. Since X is two-wide, there exists $v'' \neq v'$ such that $w \prec v'' < u$. By the definition of Morse function, since $f(v) \leq f(w)$, it follows that $f(v') > f(w)$ and $f(v'') > f(w)$. Now, by repeating this argument (taking v' instead of w in the first iteration) a finite number of times, we arrive to a contradiction with the definition of Morse function. Therefore, we have proved (1) and (2).

Condition (3) follows as a straightforward consequence of the definition of Morse function. It remains to check assertion (4). That is, we have to see that w is an up strong homotopic removable point in X_b. So, suppose on the contrary that there exists $u \neq v$, such that $w \prec u$ and $u \in X_b$. Then $f(w) < f(u)$ by the definition of Morse function ($w \prec u$ and $f(w) > f(v)$) and therefore $f(u) > a$. By the claim $u \in X_a$, then there exists $z > u$ such that $f(z) \leq a$, but we get $w \prec u < z$ and so $w \in X_a$, which is a contradiction. Then w is an up strong homotopic removable point. □

We can now state the first invariance theorem. It is a homological invariance theorem, which asserts that in the absence of critical values, the homology remains unchanged provided the matching is homologically admissible. This result, combined with the adjunction theorem, is enough to prove the Morse inequalities.

Theorem 4.19 (Invariance theorem for homology)**.** *Let X be a finite path-connected down-wide and two-wide poset and let $f \colon X \to \mathbb{R}$ be a Morse function. Suppose that $(a, b]$ for $a < b$ contains no critical values of f. If f is homologically admissible, then the inclusion $i \colon X_a \hookrightarrow X_b$ induces an isomorphism in homology.*

Proof. It follows by combining Theorem 3.52 and Proposition 4.15. □

Our next goal is to state an invariance theorem for weak homotopy, where again it is necessary to use the topology of the posets rather than their combinatorial properties. We need to add the extra hypothesis that the Morse matching associated to the function f is 1-admissible.

Proposition 4.20. *Let X be a finite path-connected down-wide poset amd let $f\colon X \to C_m$ be a homologically admissible and 1-admissible tame Morse function. If $i+1$ is a regular value, then $i\colon X_i \hookrightarrow X_{i+1}$ is a weak homotopy equivalence.*

Proof. We will apply McCord-Quillen Theorem (Theorem 2.32) to the base $\{U_x\colon x \in X_b - \{w\}\}$. There are two cases to consider:

1. If $x \neq v$, then $i^{-1}(U_x)$ has a maximum and therefore is contractible, so $i_{|i^{-1}(U_x)}\colon i^{-1}(U_x) \to U_x$ is a weak homotopy equivalence.

2. If $x = v$, then $i_{|i^{-1}(U_x)}\colon i^{-1}(U_x) \to U_x$ is the map $i\colon \widehat{U}_v - \{w\} \hookrightarrow U_v$. The subspace U_v is contractible so it is homotopically trivial. Therefore $i\colon \widehat{U}_v - \{w\} \hookrightarrow U_v$ is a weak homotopy equivalence if and only if $\widehat{U}_v - \{w\}$ is homotopically trivial. Now, since $\widehat{U}_v - \{w\}$ is simply connected and acyclic, by Hurewicz Theorem it is homotopically trivial. □

As a consequence of Proposition 4.20 we obtain a second invariance theorem which guarantees that, in the absence of critical values, the weak homotopy type remains unchanged, provided that the matching is 1-admissible and homologically admissible. This result is analogous to [75, Theorem 3.3] in discrete Morse theory and plays the role of [125, Theorem 3.1] in smooth Morse Theory. Note that this result highlights the need of using the topology of posets and not just their combinatorial properties.

Theorem 4.21 (Invariance theorem for weak homotopy)**.** *Let X be a finite path-connected down-wide and two-wide poset and let $f\colon X \to \mathbb{R}$ be a Morse function. Suppose that $(a, b]$ for $a < b$ contains no critical values of f. If f is homologically admissible and 1-admissible, then the inclusion $i\colon X_a \hookrightarrow X_b$ is a weak homotopy equivalence.*

Remark 4.22. Theorem 4.21 does not necessarily hold for arbitrary posets, as Example 4.4 shows.

The following result explains what happens with the homotopy type when we reach critical values. It plays the role of [125, Theorem 3.2] in the case of smooth Morse theory, and [75, Theorem 3.4] in discrete Morse theory. The advantage of introducing tame Morse functions is that this result becomes trivial. In [66] we proved it without using tame Morse functions.

Theorem 4.23 (Adjunction theorem)**.** *Let X be a path-connected finite poset and let $f\colon X \to \mathbb{R}$ be a tame Morse function. Suppose that $f^{-1}(\{i+1\}) = \{x\}$ is a critical element. Then $X_{i+1} = X_i \cup \{x\}$, that is, X_{i+1} is obtained from the poset X_i by adding a new element x.*

As a consequence of Theorem 3.52, we obtain the general result for Morse functions.

Theorem 4.24. *Let X be a path-connected finite two-wide poset and let $f\colon X \to \mathbb{R}$ a Morse function. Suppose that there are no regular values in $(a, b]$ and that there is only one critical value $c \in (a, b]$. That is, $\operatorname{Im} f \cap (a, b] = \{c\}$ is a critical value. Suppose that $f^{-1}(\{c\}) = \{s(e_i), t(e_i)\}_i \cup \{x_j\}_j$ where $\{e_i\}_i$ are the edges of the matched elements by f (the edges of $\mathcal{M}_f$) and the $\{x_j\}_j$ are critical elements. Then $X_b = X_a \cup \{s(e_i), t(e_i)\}_i \cup \{x_j\}_j$.*

We end up the section with a result we discovered recently and which is stronger than the invariance theorems due to Propositions 2.44 and 2.42. It is a collapsing theorem which claims that in the absence of critical values and under suitable hypothesis on the dynamics defined on the poset, there is not only an invariance in some topological invariant, but we can also realize that invariance through elementary collapses.

Theorem 4.25 (Collapsing theorem)**.** *Let X be a finite path-connected down-wide poset, let $f\colon X \to C_m$ be a tame Morse function and let $i+1$ be a regular value.*

(1) If $f\colon X \to C_m$ is homologically admissible (see Definition 4.12), then $X_{i+1} \overset{h}{\searrow} X_i$.

(2) If $f\colon X \to C_m$ is 1-weakly homotopically admissible and homologically admissible, then $X_{i+1} \overset{w}{\searrow} X_i$ and in consequence $X_{i+1} \overset{\gamma}{\searrow} X_i$.

(3) If $f\colon X \to C_m$ is homotopically admissible, then $X_{i+1} \overset{s}{\searrow} X_i$.

Proof. Assume that $f^{-1}(i) = \{x, y\}$ with $x \prec y$. Then x is a strong homotopic removable point in X_{i+1}. Now the result follows from Proposition 2.46. □

As a consequence of Theorem 3.52, we obtain the general result for Morse functions.

Corollary 4.26. *Let X be a finite path-connected down-wide and two-wide poset, let $f\colon X \to C_m$ be a Morse function and let $i+1$ be a regular value.*

(1) If $f\colon X \to C_m$ is homologically admissible, then $X_{i+1} \overset{h}{\searrow} X_i$.

(2) If $f\colon X \to C_m$ is 1-weakly homotopically admissible and homologically admissible, then $X_{i+1} \overset{w}{\searrow} X_i$ and in consequence $X_{i+1} \overset{\gamma}{\searrow} X_i$.

(3) If $f\colon X \to C_m$ is homotopically admissible, then $X_{i+1} \overset{s}{\searrow} X_i$.

4.1.4 An example

In this subsection we work out an example to illustrate the previous results.

Example 4.27. Let us denote by X the finite model of $\mathbb{R}P^2$ depicted in Figure 4.1.4 (see [6, Example 7.1.1], [94, Proposition 4.1] and [46, p. 138]). It can be checked that it is two-wide, down-wide and homologically admissible. Consider the function $f\colon X \to \mathbb{R}$ given by the values depicted at the right side of the elements of X. It is clear that f is a Morse function. We will denote the level subposets by X_t.

We begin the analysis of the level subposets. First, as Proposition 4.2 claims, the minimum value of f, corresponding to the element w_1, is a critical value (we are adding a path-connected component) (see Figure 4.1.5):

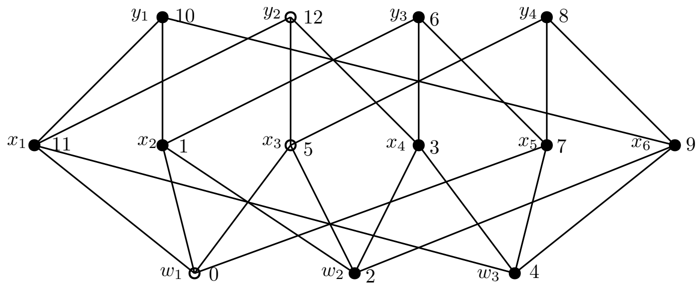


Figure 4.1.4: Morse function on a finite model of $\mathbb{R}P^2$.

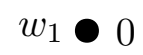
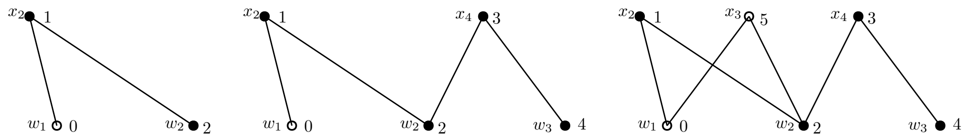
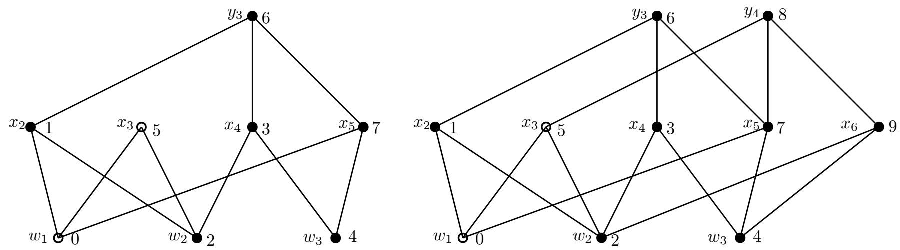


Figure 4.1.5: Level subposet X_0.



(a) (b) (c)

Figure 4.1.6: Level subposets X_1, X_3 and X_5.



(a) (b)

Figure 4.1.7: Level subposets X_6 and X_8.

As we reach the value $t = 1$, the inclusion $i\colon X_0 \to X_1$ induces an isomorphism in homology. Observe that X_1 is contractible by removing strong homotopic removable points (see Figure 4.1.6 (a)).

The situation does not change when we reach the value $t = 3$ since X_3 is still contractible by removing strong homotopic removable points (see Figure 4.1.6 (b)).

The value $t = 5$ is critical and the map $i\colon X_3 \to X_5$ no longer induces an isomorphism in homology. Observe that X_5 has the homotopy type, by removing strong homotopic removable points, of a finite model of the circle $\mathbb{S}^1$ (see Figure 4.1.6 (c)).

The value $t = 6$ is regular and it can be checked that the map $i\colon X_5 \to X_6$ induces an isomorphism in homology (see Figure 4.1.7 (a)).

The situation does not change when we reach the value $t = 8$ (see Figure 4.1.7 (b)) nor the value $t = 10$ (see Figure 4.1.8).

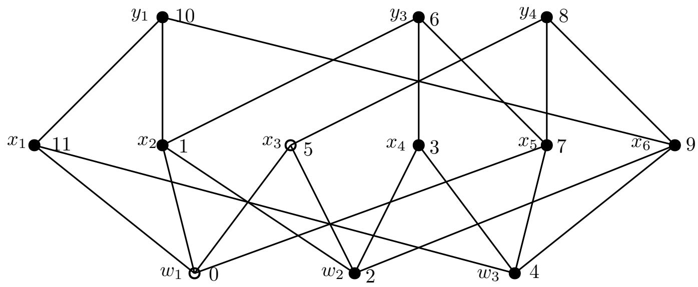


Figure 4.1.8: Level subposet X_{10}.

Finally, we reach the value $t = 12$, which is critical. The map $i\colon X_9 \to X_{12} = X$ induces an isomorphism in homology although $t = 12$ is a critical value.

4.2 Consequences of the Structure Theorems

We now pay attention to the consequences of the theory developed in the previous section.

4.2.1 Improving Forman's Theory

Forman's discrete Morse theory depends on a collapsing theorem for Morse functions ([75, Corollary 3.5]). However, this result has two drawbacks:

1. First, it involves the use of simple homotopy types.
2. Second, it only applies to discrete Morse functions on regular CW-complexes. Later, it has been extended (see for example [102] to h-regular CW-complexes but forcing that non-regular cells have to be critical for the discrete Morse functions.

We provide an alternative proof without that requirements by using our invariance theorems. First, we need to provide a generalization of the notion of discrete Morse function in the sense of Forman (see Definition 3.22) to the setting of h-regular CW-complexes and introduce some notation.

Definition 4.28. Let K a finite h-regular CW-complex and denote by $K^{(=p)}$ the set of p-cells of K. A discrete Morse function on X is a map $f\colon \bigcup_p K^{(=p)} \to \mathbb{R}$ satisfying the following two conditions for all $\sigma \in K^{(=p)}$:

1. $\#\{\tau^{(p+1)} > \sigma \colon f(\tau) \leq f(\sigma)\} \leq 1$
2. $\#\{\tau^{(p-1)} < \sigma \colon f(\tau) \geq f(\sigma)\} \leq 1$.

Let K be a finite h-regular CW-complex and $f\colon K \to \mathbb{R}$ a discrete Morse function. Then, K_a denotes the level subcomplex corresponding to the value a, that is, the subcomplex of K defined as follows:

$$K_a = \bigcup_{f(\tau)<a} \bigcup_{\sigma\leq\tau} \sigma.$$

Observe that for a h-regular CW-complex K endowed with a Morse function $f\colon K \to \mathbb{R}$, there is an induced Morse function on its face poset $\mathcal{X}(f)\colon \mathcal{X}(K) = X \to \mathbb{R}$. Moreover, the face poset functor satisfies $\mathcal{X}(K_a) = \mathcal{X}(K)_a$.

Theorem 4.29 (Improvement of Forman's collapsing result)**.** *Let K be a finite h-regular CW-complex and let $f\colon K \to \mathbb{R}$ be a discrete Morse function such that $\mathcal{X}(K)$ is two-wide and $\mathcal{X}(f)\colon \mathcal{X}(K) \to \mathbb{R}$ is homologically admissible and 1-weakly homotopically admissible. If $(a, b]$ for $a < b$ contains no critical values of f, then K_b is homotopy equivalent to K_a.*

Proof. Observe that $\mathcal{X}(f)\colon \mathcal{X}(K) \to \mathbb{R}$ verifies the hypothesis of the Invariance theorem for weak homotopy (Theorem 4.21). Now the result follows from Theorem 2.67 (6). □

Corollary 4.30. *Let X be a a finite h-regular CW-complex and $f\colon K \to \mathbb{R}$ a discrete Morse function such that $\mathcal{X}(K)$ is two-wide and $\mathcal{X}(f)\colon \mathcal{X}(K) \to \mathbb{R}$ is homologically admissible and 1-weakly homotopically admissible. Then, X is homotopy equivalent to a CW-complex with exactly one p-cell for each critical cell of index p.*

4.2.2 Morse-Pitcher Inequalities

Another consequence of our Structure Theorems of Morse Theory for finite spaces is that we can reproduce the standard argument (presented for example by Milnor or Pitcher [125,141]) to prove the Morse inequalities in this new context of posets. We consider coefficients in a principal ideal domain. We state some auxiliary algebraic results from [125, 141] in order to prove the Morse inequalities.

Let

$$X_{-1} = \emptyset \hookrightarrow X_0 \hookrightarrow \cdots \hookrightarrow X_j \hookrightarrow X_{j+1} \hookrightarrow \cdots \hookrightarrow X_m = X \tag{4.2}$$

be a filtration of a poset X induced by a Morse function $f\colon X \to \mathbb{R}$ (as in Corollary 3.50).

We begin with the definition of sub-additive function:

Definition 4.31. Let S be a map which assigns an integer to each pair of subposets of the Filtration (4.2). The map S is *sub-additive* if for every triad $X_i \subseteq X_j \subseteq X_k$, it holds that:

$$S(X_k, X_i) \leq S(X_k, X_j) + S(X_j, X_i).$$

Proposition 4.32. *Let $X_i \subseteq X_j$ elements of the Filtration (4.2) and k a natural number.*

1. *Let S be a sub-additive map. Then*

$$S(X_m, X_{-1}) \leq \sum_{j=0}^{m} S(X_j, X_{j-1}).$$

2. *Let be $k \leq m$. The map S_k given by*

$$S_k(X_j, X_i) = \operatorname{rank} H_k(X_j, X_i) - \operatorname{rank} H_{k-1}(X_j, X_i) \\ + \operatorname{rank} H_{k-2}(X_j, X_i) - \cdots \pm \operatorname{rank} H_0(X_j, X_i)$$

is sub-additive.

Definition 4.33. Let $f\colon X \to \mathbb{R}$ be a Morse function. We denote by m_i the number of critical points of index i and by b_i the Betti number of dimension i.

Now, Strong Morse inequalities follow from applying Proposition 4.32 to the Filtration (4.2) (see [125] for a detailed proof).

Corollary 4.34 (Strong Morse inequalities)**.** *Let X be a down-wide and two wide poset and let $f\colon X \to \mathbb{R}$ be a Morse function. Suppose that f is homologically admissible and homology-regular. Then, for every $i \geq 0$ and any domain of coefficients:*

$$m_i - m_{i-1} + \cdots + (-1)^i m_0 \geq b_i - b_{i-1} + \cdots + (-1)^i b_0.$$

Corollary 4.35 (Weak Morse inequalities)**.** *Let X be a down-wide and two wide poset and let $f\colon X \to \mathbb{R}$ be a Morse function. Suppose that f is homologically admissible and homology-regular. Then:*

1. *$m_i \geq b_i$ for every i.*
2. *The Euler-Poincaré Characteristic satisfies a Poincaré-Hopf Theorem:*

$$\chi(X) = \sum_{i=0}^{\deg(X)} (-1)^i b_i = \sum_{i=0}^{\deg(X)} (-1)^i m_i.$$

Our next goal is to strengthen the Morse inequalities in the spirit of Pitcher's approach for smooth Morse theory ([141]). In order to proceed, we introduce some notation and an auxiliary result from homological algebra.

Let us denote by $\{c_0, \ldots, c_k, \ldots, c_n\}$ the image of the Morse function $f\colon X \to \mathbb{R}$. Then, there exist real numbers $\{a_k\}_k$ satisfying:

$$c_0 < a_0 < c_1 < a_1 \cdots < c_k < a_k < \cdots c_n < a_n.$$

To simplify our notation, we will denote X_{a_i} by X_i.

We denote the coefficient ring, which is assumed to be a principal ideal domain, by R. As a consequence of the Structure Theorem for finitely generated modules over a principal ideal domain, it follows that:

$$H_i(X) \cong R^{b_i} \oplus \frac{R}{(r_1)} \oplus \cdots \oplus \frac{R}{(r_{\eta_i})}.$$

For the case of relative homology, we use the notation:

$$H_i(X_k, X_{k-1}) \cong R^{b_i^k} \oplus \frac{R}{(r_1)} \oplus \cdots \oplus \frac{R}{(r_{\eta_i^k})}.$$

Set $M_i^k = b_i^k + \eta_i^k + \eta_i^{k-1}$ and

$$M_i = \sum_{k=0} M_i^k. \tag{4.3}$$

Proposition 4.36 ([141, Theorem 14.1]). *Let $(C_*(X), \partial)$ be a free chain complex with singular homology modules $H_i(X)$, $i = 0, 1, \ldots$. Then there exists a free chain complex (L, ∂^L) such that:*

1. *For every $i \geq 0$, the module L_i has rank M_i.*
2. *There exists a monomorphism $i\colon L \hookrightarrow C$ which is a chain map an induces isomorphims in homology.*

Theorem 4.37 (Strengthening of Morse inequalities). *Let X be a down-wide and two wide poset and let $f\colon X \to \mathbb{R}$ be an order preserving Morse function whose associated Morse matching is homologically admissible and homology-regular. Then it holds that:*

1. *For every $i \geq 0$:*
$$m_i \geq b_i + \eta_i + \eta_{i-1}.$$
2. *For every $i \geq 0$:*
$$m_i - m_{i-1} + \cdots + (-1)^i m_0 \geq b_i - b_{i-1} + \cdots + (-1)^i b_0 + \eta_i.$$
Moreover the equality is attained when i is the height of X.

Proof. First, apply Theorem 4.36 to the singular chain complex of X. Now, observe that:

$$\operatorname{rank} L_i = \operatorname{rank} H_i(X) + \operatorname{rank} \operatorname{Im} \partial_{i+1}^L + \operatorname{rank} \operatorname{Im} \partial_i^L.$$

The first set of inequalities follows from observing that $\operatorname{rank} \operatorname{Im} \partial_i^L \geq \eta_{i-1}$. The second set of inequalities follows from taking alternating sums in i. □

Observe that if $\deg(X) = n$, then $\mu_n = 0$ since $H_n(X)$ is a submodule of the free module $C_n(X)$. Moreover, $\mu_0 = 0$ and μ_{-1} is defined as 0.

Example 4.38. Consider the poset of Example 4.27. It is a finite model of the projective plane, so its integer homology is trivial except for $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$ and $H_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. According to Morse inequalities, there is no constrain for a Morse function on the number of critical points in the sense that they would provide the same information than for a one point poset. However, the strengthening of Morse inequalities provide a much better bound: any Morse function must have at least three critical points. This bound is in fact attached by the function provided in Example 4.27.

Chapter 5
Non-gradient dynamics. Morse-Bott Theory

In classical Morse theory, one of the goals after successfully developing Morse theory, was extending Morse theory to more general dynamics, not just gradient vector fields. That was achieved by Bott and it is today commonly known as Morse-Bott theory (see [32, 35–37, 134]). When it comes to discrete Morse theory, Forman managed to extend his theory to Morse-Bott theory and related it to Conley theory (see [74]). In this chapter we pursuit that same goal in the context of posets.

The contents of the chapter are the following. First, we relate arbitrary matchings to a notion of Morse-Bott functions for posets. After that we prove some Structure Theorems which generalize the previous Structure Theorems of last chapter. In order to do so, we only focus on the homological approach to Morse-Bott theory since it is the one needed for proving the Morse-Bott inequalities, which are the next topic of the chapter. Moreover, for practical reasons, we work on homologically admissible posets unless stated otherwise. Then, we introduce a notion of Lusternik-Schnirelmann category and we prove a Lusternik-Schnirelmann theorem. We finish the chapter with a fully computed example which illustrates the main ideas we presented.

Most of the results of this chapter were introduced in our work [71].

5.1 Morse-Bott functions and Morse-Smale matchings

The purpose of this section is to introduce the notion of Morse-Bott function, Morse-Smale matching and to prove an integration result for matchings which relates them to Morse-Bott functions.

We recall the notion of $\mathcal{M}$-path from Definition 3.10. Let X be a finite graded poset. A *$\mathcal{M}$-path of index p* from $x^{(p)}$ to $\tilde{x}^{(p)}$ is a sequence:

$$\gamma\colon x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)} = \tilde{x}$$

such that for each $i \in \{0, \dots, r-1\}$:

1. $(x_i, y_i) \in \mathcal{M}$,
2. $x_i \neq x_{i+1}$.

Definition 5.1. Let X be a finite graded poset. A *$\mathcal{M}$-cycle* γ in $\mathcal{H}_{\mathcal{M}}(X)$ is a closed $\mathcal{M}$-path in $\mathcal{H}_{\mathcal{M}}(X)$.

Recall that the matching $\mathcal{M}$ is a Morse matching if $\mathcal{H}_{\mathcal{M}}(X)$ is acyclic.

5.1.1 Critical subposets

In this subsection we develop the notion of critical subposet (*chain recurrent set*) by means of matchings generalizing the analogous notion introduced by Forman ([74]) in the context of discrete Morse theory.

Definition 5.2. Let $\mathcal{M}$ be a matching on X. We say that $x^{(p)} \in X$ is an element of the *chain recurrent set* $\mathcal{R}$ if one of the following conditions holds:

- x is a critical point of $\mathcal{M}$.
- There is an $\mathcal{M}$-cycle γ in $\mathcal{H}_{\mathcal{M}}(X)$ such that $x \in \gamma$.

The chain recurrent set decomposes into disjoint subsets Λ_i by means of the equivalence relation defined as follows:

1. If x is a critical point, then it is only related to itself.
2. Given $x, y \in \mathcal{R}$, $x \neq y$, $x \sim y$ if there is an $\mathcal{M}$-cycle γ such that $x, y \in \gamma$.

Let $\Lambda_1, \ldots, \Lambda_k$ be the equivalence classes of $\mathcal{R}$. The Λ_i's are called *basic sets*. Each Λ_i consists of either a single critical point of $\mathcal{M}$ or a union of $\mathcal{M}$-cycles.

Example 5.3. Consider the finite model of $\mathbb{R}P^2$ depicted in Figure 5.1.1. There is a critical point which is also a basic set, depicted with a cross. Moreover, the dashed and dotted arrows represent another two basic sets, each consisting of one cycle.

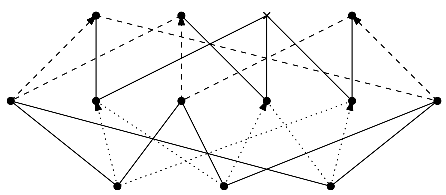


Figure 5.1.1: A finite model of $\mathbb{R}P^2$.

5.1.2 Integration of matchings

When working in the differentiable category, Morse theory generalizes naturally to Morse-Bott Theory. The purpose of this subsection is to generalize the integration result for matchings (Theorem 3.11) to the context of Morse-Bott functions and arbitrary matchings.

Recall that for each edge $(x, y) \in \mathcal{M}$, we say that x is the *source* of the edge and y is the *target*. Moreover, we have also defined the *source* and *target maps* (only defined for elements in the matching $\mathcal{M}$) as follows: given $(x, y) \in \mathcal{M}$, $s(y) = x$ and $t(x) = y$.

Definition 5.4. Let $\mathcal{M}$ be a matching on a finite poset X. A function $f\colon X \to \mathbb{R}$ is a *Morse-Bott* or *Lyapunov function* if it is constant on each basic set and it is a Morse function away from the chain recurrent set. More formally, a function $f\colon X \to \mathbb{R}$ is *Morse-Bott* if it is constant on each basic set and it is a Morse function in $X - \bigcup_{\gamma \in \mathcal{M}-\text{cycles}} \{x \in X : x \in \gamma\}$.

We say that the *critical values* of a Morse-Bott function are the images of the basic sets. The ideas of Forman's proof of [74, Theorem 2.4] generalize to the context of graded posets giving:

Theorem 5.5 (Integration of matchings)**.** *Let X be a finite graded poset and let $\mathcal{M}$ be a matching in X. Then, there exists a Morse-Bott function $f\colon X \to \mathbb{R}$ such that:*

1. *If $x^{(p)} \notin \mathcal{R}$ and $x \prec y^{(p+1)}$, then*

$$\begin{cases} f(x) < f(y), & \text{if } (x,y) \notin \mathcal{M}, \\ f(x) \geq f(y), & \text{if } (x,y) \in \mathcal{M}. \end{cases}$$

2. *If $x^{(p)} \in \mathcal{R}$ and $x \prec y^{(p+1)}$, then*

$$\begin{cases} f(x) = f(y), & \text{if } x \sim y, \\ f(x) < f(y), & \text{if } x \nsim y. \end{cases}$$

Proof. First of all, we extend the equivalence relation $\sim$ to all of X as follows: if $x \notin \mathcal{R}$, then $\{x\}$ is an equivalence class. Second, we define an auxiliary map $d\colon X \to \mathbb{N}$ given by:

$$\begin{aligned} d(x) = \max\{s\colon \exists\, &\mathcal{M}\text{-path} \\ &\gamma\colon x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)} = \tilde{x}^{(p)} \\ &\text{such that the } x_i's \text{ in } \gamma \text{ include elements from exactly} \\ &s \text{ distinct equivalence classes}\}. \end{aligned} \tag{5.1}$$

Third, we define $D = \max_{x\in X} d(x)$. Now, we define the function $f\colon X \to \mathbb{R}$ inductively on the degree of the poset. Given $x^{(p)} \in X$, we define $f(x)$ as follows:

(F1) If x is a critical point of $\mathcal{M}$, then $f(x) = p$.

(F2) If $x \in s(\mathcal{M})$, then

$$f(x) = p + \frac{d(x)}{2D}.$$

Note that this guarantees that

$$p < f(x) \leq p + \frac{1}{2}$$

due to $d(x) \geq 1$ in this case.

(F3) If $x \in t(\mathcal{M})$, then there exists $w^{(p-1)}$ such that $t(w) = x$ and $f(w)$ was defined in (2). We set $f(x) = f(w)$ and it follows that

$$p - 1 \leq f(x) \leq p - \frac{1}{2}.$$

It remains to check that f satisfies the desired properties. We split the verification in cases:

1. Assume that $x^{(p)} \notin \mathcal{R}$ and $x < y^{(p+1)}$.
   (a) If $t(x) = y$, then $f(y) = f(x)$, so
   $$f(x) \geq f(y).$$
   (b) If $t(x) \neq y$, we consider several cases again:
      i. If y is a critical point, then
      $$f(y) = p + 1 > p + 1/2 \geq f(x).$$
      ii. If $y \in s(\mathcal{M})$, then
      $$f(y) > p + 1 > p + 1/2 \geq f(x).$$
      iii. If $y \in t(\mathcal{M})$, then there exists an unique $\tilde{x}^{(p)} \neq x$ such that $t(\tilde{x}) = y$. Since $x \notin \mathcal{R}$, there are two cases:
         A. If $x \in t(\mathcal{M})$, then
         $$f(y) = f(\tilde{x}) \geq p > p - 1/2 \geq f(x).$$
         B. If $x \in s(\mathcal{M})$ and $\gamma\colon x \prec \cdots$ is any $\mathcal{M}$-path beginning at x, then
         $$\tilde{\gamma}\colon \tilde{x} \prec y \succ x \prec \cdots$$
         is a $\mathcal{M}$-path beginning at $\tilde{x}$. Moreover, since $x \notin \mathcal{R}$, x is not an element of any closed $\mathcal{M}$-path. Therefore
         $$d(\tilde{x}) \geq d(x) + 1,$$
         hence
         $$f(y) = f(\tilde{x}) > f(x).$$

2. Assume that $x^{(p)} \in \mathcal{R}$ and
$$\gamma\colon x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots x_r^{(p)} = x_0^{(p)}$$
is a non-stationary closed $\mathcal{M}$-path. Then for each i, j, $0 \leq i, j \leq r-1$, $d(x_i) = d(x_j)$, hence $f(x_i) = f(x_j)$. Moreover, by the definition of f, $f(y_i) = f(x_i)$, so f is constant on each non-stationary closed $\mathcal{M}$-path.

3. Suppose $x^{(p)} \in \mathcal{R}$ and $y^{(p+1)} > x$, $y \nsim x$. We want to prove that $f(y) > f(x)$.
   (a) If y is a critical point, then
   $$f(y) \geq p + 1 > p + 1/2 \geq f(x).$$
   (b) If $y \in s(\mathcal{M})$, then
   $$f(y) \geq p + 1 \geq p + 1/2 \geq f(x).$$

(c) Suppose $y \in t(\mathcal{M})$. Since $y \nsim x$ and $y \neq t(x)$, then there exists a unique $\tilde{x}^{(p)} \neq x^{(p)}$ such that $t(\tilde{x}) = y$.

i. if x is a critical point or $x \in t(\mathcal{M})$,

$$f(y) = f(\tilde{x}) > p \geq f(x).$$

ii. If $x \in s(\mathcal{M})$ and $\gamma \colon x \prec \cdots$ is any $\mathcal{M}$-path starting at x, then

$$\tilde{\gamma} \colon \tilde{x} \prec y \succ x \prec \cdots$$

is a $\mathcal{M}$-path beginning at $\tilde{x}$. Moreover, $\tilde{x}$ is not equivalent to any element of γ, since, otherwise, x and y would be contained in a non-stationary closed path, which contradicts $y \nsim x$. Thus $d(\tilde{x}) \geq d(x) + 1$, which implies

$$f(y) = f(\tilde{x}) > f(x). \quad \square$$

Remark 5.6. Observe that Theorem 5.5 generalizes the analogous integration result for Morse or gradient matchings (Theorem 3.11).

5.1.3 Morse-Smale matchings

In this subsection we generalize the notion of Morse-Smale vector field from the context of simplicial complexes ([74]) to the setting of finite posets.

Let X be a homologically admissible poset and let $\mathcal{M}$ be a matching on X. A $\mathcal{M}$-cycle γ is *prime* if there does not exist a natural number $n > 1$ and a $\mathcal{M}$-cycle $\widetilde{\gamma}$ such that γ is the concatenation of $\widetilde{\gamma}$ n times.

An equivalence relation on the set of $\mathcal{M}$-cycles is defined as follows. Two $\mathcal{M}$-cycles γ and $\widetilde{\gamma}$ are equivalent if $\widetilde{\gamma}$ is the result of varying the starting point of γ (see [74, p. 631] for an example in the setting of simplicial complexes). An equivalence class of $\mathcal{M}$-cycles is called a *closed $\mathcal{M}$-orbit*. The equivalence class of γ is denoted by $[\gamma]$. A closed $\mathcal{M}$-orbit is *prime* if any of the representative cycles is prime. The *index of a closed $\mathcal{M}$-orbit* is defined as the index of any of the representatives.

A special kind of matching which will play an important role is the following one. In a certain sense, it controls the complexity of the chain recurrent set.

Definition 5.7. Let X be a homologically admissible poset. A matching $\mathcal{M}$ on X is a *Morse-Smale matching* if the chain recurrent set $\mathcal{R}$ consists only of critical points and pairwise disjoint prime closed $\mathcal{M}$-orbits.

5.1.4 An example

We finish the section with an example which illustrates its main ideas.

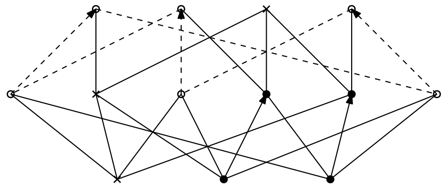


Figure 5.1.2: The homologically admissible poset X with a matching.

Example 5.8. Consider the homologically admissible poset depicted in Figure 5.1.2. We we will denote it by X.

Moreover, in Figure 5.1.2 we also exhibit a matching $\mathcal{M}$ on X. The crosses represent critical points, the dashed edges with the circles represent a periodic orbit, and the arrows represent the matched elements. Observe that the matching $\mathcal{M}$ is Morse-Smale since it only has a prime orbit (therefore disjoint from the others) and three critical points.

We proceed to illustrate how to integrate the matching $\mathcal{M}$ on X following the proof of Theorem 5.5. First, in Figure 5.1.3 we show the values of the map $d\colon X \to \mathbb{N}$ given by Equation (5.1).

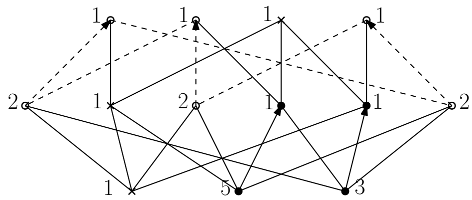


Figure 5.1.3: Values of the map $d\colon X \to \mathbb{N}$.

Observe that $d(x)$ is greater than one only for $x \in s(\mathcal{M})$. We compute $D = \max_{x\in X} d(x) = 5$. Now, we will build a Morse-Bott or Lyapunov function $f\colon X \to \mathbb{R}$. Recall that for $x^{(p)} \in X$, we define $f(x)$ following (F1), (F2) and (F3) in p. 71.

We show the values of the Morse-Bott function $f\colon X \to \mathbb{R}$ in Figure 5.1.4:

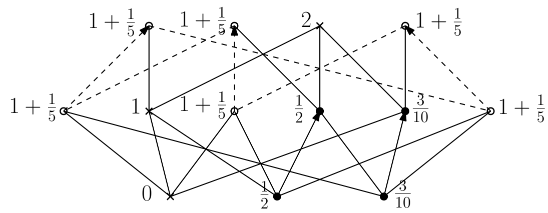


Figure 5.1.4: Values of the map $f\colon X \to \mathbb{R}$.

5.2 Structure Theorems for Morse-Bott functions

The purpose of this Section is to prove the Structure Theorems of Morse theory for Morse-Bott functions on posets and obtain some consequences.

5.2.1 Structure Theorems

In what follows, we extend the equivalence relation defined in the chain recurrent set $\mathcal{R}$ to all of X by saying that a point which is not critical is an equivalence class on its own. We will make an abuse of notation and we will refer to the set of elements of the equivalence classes by the equivalence classes.

Definition 5.9. Given a finite poset X, a point $x \in X$ and a matching $\mathcal{M}$ on X, we define:

$$\partial[x] = \{w \in X \colon w \prec \tilde{x} \text{ for some } \tilde{x} \sim x \text{ but } w \nsim \tilde{x}\}.$$

Example 5.10. Consider the poset depicted in Figure 5.1.1. In Figure 5.2.1 we show $\partial[x]$ for any x in the dashed cycle of Figure 5.1.1.

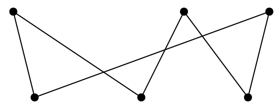


Figure 5.2.1: Example of $\partial[x]$.

The lemma below follows from the definition of matching:

Lemma 5.11. *Let γ be a cycle of index p and let $u^{(p-1)} \in X$, $\tilde{v}^{(p)} \in X$, $w^{(p+1)} \in X$ and $r^{(p+2)} \in X$ such that $u, \tilde{v}, w, r \notin \gamma$. Then the following holds:*

$$t(u) \notin \gamma,\ t(\tilde{v}) \notin \gamma,\ s(w) \notin \gamma \ \textit{and}\ \ s(r) \notin \gamma.$$

We introduce the following definition: given a finite poset X and a Morse-Bott function $f \colon X \to \mathbb{R}$, for each $a \in \mathbb{R}$ we write

$$X_a = \bigcup_{f(x) \leq a} U_x.$$

Our next result is a homological collapsing theorem for Morse-Bott functions. As a consequence of Lemma 5.11, the elements of a cycle can not be connected by arrows with elements which are not in the cycle. Therefore, the result below follows from Theorem 4.19.

Theorem 5.12. *Let X be a finite homologically admissible poset and let $f \colon X \to \mathbb{R}$ be a Morse-Bott function. If $(a, b]$ contains no critical values, then $i \colon X_a \hookrightarrow X_b$ induces an isomorphism in homology.*

In this generalized context, we also have a result which explains what happens when a critical value is reached.

Theorem 5.13. *Let X be a finite homologically admissible poset and let $f \colon X \to \mathbb{R}$ be a Morse-Bott function. If $f(x) \in (a, b]$ is a critical value which corresponds to a unique basic set and there are no other values of f in $(a, b]$, then $X_b = X_a \cup [x]$.*

Proof. There are two cases to consider. First, assume that $[x]$ is a critical point, then the results reduces to Theorem 4.23. So, assume $[x]$ is a cycle of index p. Let $\tilde{f}\colon X/\sim\to\mathbb{R}$ denote the function induced by f on the set of equivalence classes. We may assume that $\tilde{f}$ is injective, that $\tilde{f}([x])>a$ and that the only critical subposet in $f^{-1}((a,b])$ is $[x]$.

Since $[x]$ is a cycle and $f(x)$ is a critical value, then given $y^{(p+1)}\succ\tilde{x}$ and $y\notin[x]$ with $\tilde{x}\in[x]$, we have $f(y)>f(\tilde{x})$. Hence, $f(y)>b$ and Lemma 5.11 guarantees that $f(z)>b$ for every $z>\tilde{x}$, $z\notin[x]$. Therefore, $[x]\cap X_a=\emptyset$. Given any:

- $w^{(p-1)}\prec\tilde{x}^{(p)}$, $\tilde{x}\in[x]$ and $w\notin[x]$ or
- $w^{(p)}\prec\tilde{x}^{(p+1)}$, $\tilde{x}\in[x]$ and $w\notin[x]$,

due to the criticality of $[x]$, it holds that $f(w)<f(\tilde{x})$. Therefore $f(w)\leq a$ and $w\in X_a$. Hence $\partial[x]\subset X_a$. That is, $X_b=X_a\cup[x]$. □

Remark 5.14. The hypothesis referring to $f(x)\in(a,b]$ corresponding to a unique basic set can always be reached by perturbing the Morse-Bott function, so it is not restrictive.

5.2.2 Morse-Bott inequalities

In this subsection we generalize Morse-Bott inequalities from the context of cell complexes ([74, Theorem 3.1]) to the setting of posets. This result can be seen as a combinatorial analogue of a theorem due to Conley ([84, Theorem 1.2] [48]). Again, we assume that our coefficients are any principal ideal domain R. From now on the poset X is assumed to be homologically admissible.

Given a subposet $Y\subset X$ we denote by $\bar{Y}$ the subposet $\cup_{x\in Y}U_x$ and by $\dot{Y}=\bar{Y}-Y$.

Definition 5.15. For each $k\geq 0$, we define

$$m_k=\sum_{\text{basic sets }\Lambda_i}\operatorname{rank}H_k(\bar{\Lambda}_i,\dot{\Lambda}_i).$$

Observe that in the particular case we have a Morse matching, the basic sets are just critical points and m_k is the number of critical points of index k.

Lemma 5.16. *If the index of the basic set Λ_i is p, then $H_k(\bar{\Lambda}_i,\dot{\Lambda}_i)=0$ unless $k=p,p+1$. Moreover, if Λ_i is just a critical point $x^{(p)}$, then $H_k(\bar{\Lambda}_i,\dot{\Lambda}_i)=0$ for $k\neq p$ and the ring of coefficients, R, for $k=p$.*

Proof. For convenience, during the proof we will denote $\Lambda_i=\Lambda$. Since all the posets involved are cellular we can use cellular homology. Consider the Homology Long Exact Sequence for the pair $(\bar{\Lambda}, \dot{\Lambda})$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_p(\bar{\Lambda}) \arrow[r] & H_p(\bar{\Lambda}, \dot{\Lambda}) \arrow[r] & H_{p-1}(\dot{\Lambda}) \arrow[dll, out=-10, in=170] & \\
& H_{p-1}(\bar{\Lambda}) \arrow[r, "j"] & H_{p-1}(\bar{\Lambda}, \dot{\Lambda}) \arrow[r, "\partial"] & H_{p-2}(\dot{\Lambda}) \arrow[dll, out=-10, in=170, "\cong"] & \\
& H_{p-2}(\bar{\Lambda}) \arrow[r] & H_{p-2}(\bar{\Lambda}, \dot{\Lambda}) \arrow[r] & H_{p-3}(\dot{\Lambda}) \arrow[r] & \cdots
\end{tikzcd}$$

First of all, the homomorphism $H_k(\dot{\Lambda}) \to H_k(\bar{\Lambda})$ is an isomorphism for $k \leq p-2$, so $H_k(\bar{\Lambda}, \dot{\Lambda}) = 0$ for $k \leq p-2$. Second, we have that:

$$H_{p-1}(\bar{\Lambda}, \dot{\Lambda}) = \mathrm{Ker}\partial = \mathrm{Im}j \cong \frac{H_{p-1}(\bar{\Lambda})}{\mathrm{Ker}j} \cong \frac{H_{p-1}(\bar{\Lambda})}{\mathrm{Im}i}.$$

Third, the homomorphism $H_{p-1}(\dot{\Lambda}) \to H_{p-1}(\bar{\Lambda})$ induced by the inclusion is surjective by the construction of cellular homology. Therefore $H_{p-1}(\bar{\Lambda}, \dot{\Lambda}) = 0$. Fourth, if Λ is just a critical point $x^{(p)}$, then

$$H_k(\bar{\Lambda}, \dot{\Lambda}) = H_k(U_x, \widehat{U}_x)$$

and by cellularity of X and the Homology Long Exact Sequence for the pair $(U_x, \widehat{U}_x)$ the result follows. $\square$

We denote by b_k the Betti number in degree k with coefficients in R. Following the the ideas involved in the proof of [74, Theorem 3.1] and using our previous Theorems 5.12 and 5.13 we obtain the Strong Morse-Bott inequalities:

Theorem 5.17 (Strong Morse-Bott inequalities)**.** *Let X be a homologically admissible poset and let $\mathcal{M}$ be a matching on X. Then, for every $k \geq 0$:*

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0.$$

Proof. First of all, by Theorem 5.5 we can integrate the matching $\mathcal{M}$ to obtain a Morse-Bott function $f\colon X \to \mathbb{R}$. Second, we can perturb the function $f\colon X \to \mathbb{R}$ as it is done in [74, p. 643] so it remains Morse-Bott and the following condition is satisfied: for any $c \in \mathbb{R}$, if $f^{-1}(c) \neq \emptyset$, then either:

1. $f^{-1}(c) = \{x, y\}$ such that $\{x, y\}$ are not in any basic set and $t(x) = y$
   or

2. $f^{-1}(c) = \Lambda$ for some basic set Λ.

For each $c \in \mathbb{R}$ we define:

$$m_k(c) = \sum_{\text{basic sets } \Lambda_i \text{ with } f(\Lambda_i) \leq c} \operatorname{rank} H_k(\bar{\Lambda}_i, \dot{\Lambda}_i)$$

and

$$b_k(c) = \operatorname{rank} H_k(X_c).$$

It is enough to check that for any $c \in \mathbb{R}$ the sets $\{m_k(c), b_k(c)\}$ satisfy the inequalities since considering c big enough we have $X_c = X$ and the result follows. We prove that the sets $\{m_k(c), b_k(c)\}$ satisfy the inequalities by induction. First, consider c small enough so $X_c = \emptyset$, then the result follows trivially since $m_k(c) = b_k(c) = 0$. Now let c increase. We have to study what happens when c reaches each of the finitely many values of f. Suppose that $c \in (a, b)$ is the only element of the image of f in the interval $[a, b]$, so that $m_k(c) = m_k(b)$ and $b_k(c) = b_k(b)$, and that the inequalities follow for the sets $\{m_k(a), b_k(a)\}$. If $f^{-1}(c) \neq \emptyset$, then there are two cases to consider:

1. Suppose $f^{-1}(c) = \{x, y\}$ such that $\{x, y\}$ are not in any basic set and $t(x) = y$. Then by definition of the $m_k(c)$ it follows that $m_k(a) = m_k(b)$ since there are not new basic sets whose images by f are in the interval (a, b). By Theorem 5.12, the inclusion $i\colon X_a \hookrightarrow X_b$ induces an isomorphism in homology so $b_k(a) = b_k(b)$. Therefore the inequalities hold for the sets $\{m_k(b), b_k(b)\}$.
2. Suppose $f^{-1}(c) = \Lambda_i$ for some basic set Λ_i. Observe that X_c is covered by the open subsets X_b and Λ_i, so by Excision Theorem it follows that the inclusion

$$(\bar{\Lambda}_i, \dot{\Lambda}_i) = (\bar{\Lambda}_i, X_b \cap \bar{\Lambda}_i) \hookrightarrow (X_c, X_b)$$

induces isomorphisms in homology $H_k(\bar{\Lambda}_i, \dot{\Lambda}_i) \cong H_k(X_c, X_b)$. Now, observe that

$$m_k(c) - m_k(b) = \operatorname{rank} H_k(\bar{\Lambda}_i, \dot{\Lambda}_i),$$

so,

$$m_k(c) - m_k(b) = \operatorname{rank} H_k(X_c, X_b).$$

Finally, the result follows as in the Morse case by applying Proposition 4.32. □

Corollary 5.18 (Weak Morse-Bott inequalities)**.** *Let X be a homologically admissible poset and let $\mathcal{M}$ be a matching on X. Then:*

1. *For every $k \geq 0$, $m_k \geq b_k$;*
2. *The Euler-Poincaré Characteristic satisfies a Poincaré-Hopf Theorem:*

$$\chi(X) = \sum_{k=0}^{\deg(X)} (-1)^k b_k = \sum_{k=0}^{\deg(X)} (-1)^k m_k.$$

5.2.3 Morse-Smale matchings

Our next aim is both to extend the results of Forman about Morse-Smale matchings from the setting of complexes ([75, Section 7]) to the setting of posets and to improve some of the results even in the case of simplicial or regular CW-complexes.

Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse-Smale matching on X. We denote by c_k the number of critical points of index k and by A_k the number of prime closed $\mathcal{M}$-orbits of index k .

Recall that the coefficient ring R is a principal ideal domain. Hence, the Structure Theorem for finitely generated modules over a principal ideal domain guarantees that:

$$H_k(X) \cong R^{b_k} \oplus \frac{R}{(r_1)} \oplus \cdots \oplus \frac{R}{(r_{\eta_k})}. \tag{5.2}$$

Making use of our Pitcher strengthening of Morse inequalities for gradient matchings we obtain the following improvement of [75, Theorem 7.1], taking torsion into account, measured by η_k.

Theorem 5.19. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse-Smale matching on X. Let the coefficient ring R be a principal ideal domain. Then, for every $k \geq 0$:*

$$A_k + \sum_{i=0}^{k} (-1)^i c_{k-i} \geq \eta_k + \sum_{i=0}^{k} (-1)^i b_{k-i}.$$

Proof. We begin with a matching $\mathcal{M}_0 = \mathcal{M}$ and we iterate the following procedure. Given a closed $\mathcal{M}_i$-orbit $\{[\gamma]_i\}_i$:

$$\gamma_i \colon x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)},$$

we define a new matching $\mathcal{M}_{i+1} = \mathcal{M}_i - (x_0, y_0)$. We iterate this process until there are no closed orbits left. We call the obtained matching $\mathcal{M}^*$. Observe that $\mathcal{M}^*$ is acyclic or Morse and

$$m_p^* = c_p + A_p + A_{p-1},$$

where m_p^* denotes the number of critical points of index p of the matching $\mathcal{M}^*$. By our strengthening of Morse inequalities (Theorem 4.37), it holds that:

$$\sum_{i=0}^{k} (-1)^i m_{k-i} \geq \eta_k + \sum_{i=0}^{k} (-1)^i b_{k-i},$$

which implies the result we want to prove. □

Definition 5.20. Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse-Smale matching on X. Endow each element of X with an orientation. Let γ be an $\mathcal{M}$-path

$$\gamma \colon x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)}.$$

We define the multiplicity of γ by

$$\prod_{i=0}^{r-1} -\langle d_{(p+1)}y_i, x_i\rangle_p \langle d_{(p+1)}y_i, x_{i+1}\rangle_p$$

where d is the cellular boundary operator and $\langle \bullet\,, \bullet\,\rangle_p$ is the inner product on $C_p(X)$ such that the elements of degree p of X are mutually orthogonal.

Remark 5.21. Observe that the multiplicity of a path is always 1 or -1 due to Lemma 2.12.

The generalization of [74, Lemma 4.6] to our context is straightforward. As a consequence, [74, Theorem 7.3] is generalised to our setting with the same proof, which we omit due to its length and that it would be a mere reproduction of a presentation which we would not be able to improve:

Theorem 5.22. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse-Smale matching on X. Let the coefficient ring R be a principal ideal domain. Let λ_i denote a basic set consisting of a single closed orbit $[\gamma]$ of index p. Then:*

$$H_k(\bar{\Lambda}_i, \dot{\Lambda}_i) \cong 0 \textit{ for } k \neq p, p+1.$$

$$H_p(\bar{\Lambda}_i, \dot{\Lambda}_i) \cong \begin{cases} R & \textit{if } m(\gamma) = 1 \\ \frac{R}{2R} & \textit{if } m(\gamma) = -1 \end{cases}$$

$$H_{p+1}(\bar{\Lambda}_i, \dot{\Lambda}_i) \cong \begin{cases} R & \textit{if } m(\gamma) = 1 \\ 0 & \textit{if } m(\gamma) = -1 \end{cases}$$

Therefore, combining the Strong Morse-Bott inequalities (Theorem 6.34) with Theorem 5.22 and taking into account that fields have no torsion, we obtain a generalization of [74, Corollary 7.4] to our setting:

Theorem 5.23. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse-Smale matching on X. Let the coefficients ring be the field $\mathbb{R}$. Denote by A'_p the number of closed $\mathcal{M}$-orbits of index p and multiplicity 1. Then, for every $k \geq 0$:*

$$A'_k + \sum_{i=0}^{k} (-1)^i c_{k-i} \geq \sum_{i=0}^{k} (-1)^i b_{k-i}(\mathbb{R}).$$

Remark 5.24. While [74, Corollary 7.4] refined [74, Theorem 7.2], Theorem 5.23 does not refine our improved Theorem 5.19. They are complementary results.

5.3 Homological Lusternik-Schnirelmann Theorem

The purpose of this section is to prove a Lusternik-Schnirelmann Theorem for general matchings and a suitable definition of homological category.

5.3.1 Homological chain category

Let (C_*, ∂) denote a free chain complex of abelian groups. It is *bounded* if only finitely many of the C_p are non zero. Moreover, if each term C_p is finitely generated, then we define the *rank* of C_* as $\operatorname{rank}(C_*) = \sum_p \operatorname{rank}(C_p)$.

Definition 5.25. Let (C_*, ∂) be a free chain complex of abelian groups. We define its *homological chain category*

$$\operatorname{hccat}(C_*) = \inf \left\{ \begin{array}{c} \operatorname{rank}(B_*)\colon B_* \text{ is a bounded complex and there is a mo-} \\ \text{nomorphism } i\colon B_* \hookrightarrow C_* \text{ which is a quasi-isomorphism} \end{array} \right\}.$$

Let X be a topological space. We denote by $S_*(X)$ its singular chain complex. For all the definitions that follow we consider coefficients in $\mathbb{Z}$.

Definition 5.26. Let X be a topological space. We define its *homological chain category* as $\operatorname{hccat}(X) = \operatorname{hccat}(S_*(X))$.

We introduce a homological lower bound for $\operatorname{hccat}(X)$ analogous to the Pitcher strengthening of Morse inequalities (Theorem 4.37). Recall from Equation (5.2) the definition of η_k.

Proposition 5.27. *Let X be a topological space with finitely generated homology. Then*

$$\sum_k b_k + 2\sum_k \eta_k \leq \operatorname{hccat}(X).$$

Proof. Let us denote by (B_*, ∂) a bounded chain complex whose homology is isomorphic to $H_*(X)$. By some arguments from linear algebra (see, for example [142, Theorem 4.11]), we have $b_k + \eta_k + \eta_{k-1} \leq \operatorname{rank}(B_k)$. Now the result follows by a sum indexed by the dimension. □

Corollary 5.28. *Let X be a homologically admissible poset or a CW-complex with finitely generated homology. Then*

$$\chi(X) \leq \operatorname{hccat}(X).$$

In fact, the bound given by Proposition 5.27 is the best possible as a consequence of the following result due to Pitcher ([141, Lemma 13.2]).

Proposition 5.29. *Let (C_*, ∂) be a free chain complex with singular homology modules $H_k(X) \cong R^{b_k} \oplus \frac{R}{(r_1)} \oplus \cdots \oplus \frac{R}{(r_{\eta_k})}$, $k = 0, 1, \ldots$. Then there exists a free chain complex (L, ∂^L) such that:*

1. *For every $k \geq 0$, the group L_k has rank $b_k + \eta_k + \eta_{k-1}$.*
2. *There exists a monomorphism $i\colon L \hookrightarrow C$ which is a chain map.*
3. *The monomorphism $i\colon L \hookrightarrow C$ is a quasi-isomorphism.*

Corollary 5.30. *Let X be a topological space with finitely generated homology. Then*

$$\operatorname{hccat}(X) = \sum_k b_k + 2\sum_k \eta_k.$$

Moreover, observe that a topological X is acyclic if and only if $\operatorname{hccat}(X) = 1$.

As a consequence of [49, Example 1.33] we have the following result relating the homological chain category to the Lusternik-Schnirelmann category:

Proposition 5.31. *Let K be a simply connected CW-complex with finitely generated homology groups such that there exists $n \in \mathbb{N}$ satisfying $H_n(K) \neq 0$ and $H_p(K) = 0$ for $p > n$. Then*

$$\operatorname{cat}(K) \leq \operatorname{hccat}(K).$$

The result does not necessarily hold if we remove the simply connectedness hypothesis, as the following example shows:

Example 5.32. Consider the Poincaré homology 3-sphere, which we denote by M. Observe that $\operatorname{hccat}(M) = \operatorname{hccat}(\mathbb{S}^3) = 2$. However, $\operatorname{cat}(M) \geq 3$ ([83]).

5.3.2 Homological Lusternik-Schnirelmann Theorem

In this subsection we state and prove a Lusternik-Schnirelmann Theorem for the homological chain category and general matchings on posets.

Theorem 5.33. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse-Smale matching on X. Then*

$$\operatorname{hccat}(X) \leq \sum_{\text{basic sets } \Lambda_i} \operatorname{hccat}(\Lambda_i).$$

In particular, if $\mathcal{M}$ is a Morse matching on X, then $\operatorname{hccat}(X)$ is a lower bound for the number of critical elements of $\mathcal{M}$.

Proof. We will define another Morse matching $\mathcal{M}^*$ by perturbing $\mathcal{M}$. The idea is to replace each prime closed orbit by two critical points. This will be achieved by removing exactly one of the edges of the matching in each closed orbit. By repeating the technique used in the proof of Theorem 5.19, we obtain a Morse matching $\mathcal{M}^*$ satisfying

$$m_p^* = c_p + A_p + A_{p-1},$$

where m_p^* denotes the number of critical points of index p of the matching $\mathcal{M}^*$ (see p. 79 for the definition of A_p).

Recall that $C_*(X)$ denotes the cellular chain complex of X. We define a map $V\colon C_p(X) \to C_{p+1}(X)$ as follows:

$$V(x) = \begin{cases} -\epsilon(y,x)y, & \text{if there exists } y \in X \text{ with } (x,y) \in \mathcal{M}^*, \\ 0, & \text{otherwise.} \end{cases}$$

Following the ideas of Minian ([126]), define the discrete flow operator $\phi\colon C_p(X) \to C_p(X)$ as

$$\phi = \mathrm{id} + dV + Vd.$$

The ϕ-invariant chains

$$C_p^{\phi}(X) = \{c \in C_p(X)\colon \phi(c) = c\}$$

form a well-defined subcomplex of $(C_*(X), d)$ ([126]). Moreover, the inclusion of $(C_*^{\phi}(X), d)$ into $(C_*(X), d)$ induces isomorphisms in homology and $C_p^{\phi}(X)$ is isomorphic to the free abelian group spanned by the critical p-elements of X ([126]). As a consequence:

$$\mathrm{hccat}(C_*(X)) \leq \sum_p m_p^* = \sum_p c_p + A_p + A_{p-1}. \tag{5.3}$$

There are two kinds of basic sets for $\mathcal{M}$: critical points and disjoint closed $\mathcal{M}$-orbits. Observe that if Λ_i is a critical point, then $\mathrm{hccat}(\Lambda_i) = 1$ while if Λ_i is a closed orbit, then $\mathrm{hccat}(\Lambda_i) = 2$. So, from Equation (5.3), it follows that:

$$\mathrm{hccat}(C_*(X)) \leq \sum_{\text{basic sets } \Lambda_i} \mathrm{hccat}(\Lambda_i).$$

Finally, observe that $\mathrm{hccat}(X) = \mathrm{hccat}(C_*(X))$ due to the isomorphism between cellular homology and singular homology for cellular posets (Theorem 2.9). □

Remark 5.34. In the proof of Theorem 5.33, Equation (5.3) could also be derived as a consequence of combining our Pitcher strengthening of Morse-inequalities (Theorem 4.37) applied to the matching $\mathcal{M}^*$ with Corollary 5.30.

As a consequence, we obtain the following corollary:

Corollary 5.35. *Let X be a homologically admissible poset and let $f\colon X \to \mathbb{R}$ be a Morse function. Then $\mathrm{hccat}(X)$ is a lower bound for the number of critical points of f.*

Remark 5.36. Let K be a simplicial complex or, more generally, a regular CW-complex K. Recall that its face poset $\Delta(K)$ is a homologically admissible poset. Moreover, the chain complex $C_\bullet(\Delta(K), d)$ where d is the cellular boundary operator coincides with the chain complex $C_\bullet(K, \partial)$ where ∂ is the cellular -or simplicial in case K is a simplicial complex- boundary operator. Therefore $\mathrm{hccat}(\Delta(K)) = \mathrm{hccat}(K)$. Hence, we have in particular a simplicial homological Lusternik-Schnirel- mann Theorem.

5.4 A worked out example

In this section, we work out a full example to illustrate the main ideas in Sections 5.2 and 5.3. We recover the poset, the Morse-Smale matching and the Morse-Bott function from Example 5.8.

We insert again Figure 5.1.2 with the matching and Figure 5.1.4 depicting the values of the Morse-Bott function.

Recall that the crosses represent critical points, the dashed edges with the circles represent a periodic orbit, and the arrows represent the matched elements.

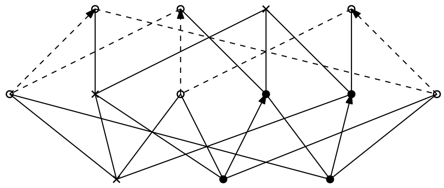


Figure 5.4.1: The homologically admissible poset X with a matching.

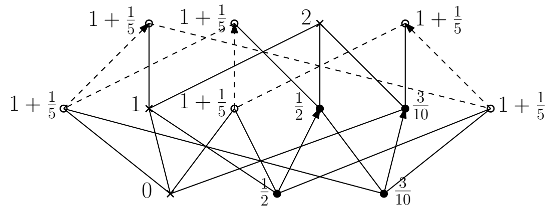


Figure 5.4.2: Values of the map $f\colon X \to \mathbb{R}$.

5.4.1 Structure Theorems

Recall that given the poset X and the Morse-Bott function $f\colon X \to \mathbb{R}$, for each $t \in \mathbb{R}$ we have the level subposet

$$X_t = \bigcup_{f(x)\le t} U_x.$$

We will illustrate the Structure Theorems in this context, (Theorems 5.12 and 5.13), which describe the changes in the level subposets X_t as $t \in \mathbb{R}$ increases. We begin the analysis of the level subposets.

First, the minimum value of f is attached at a critical element (see Figure 5.4.3 (a)):

As we reach the value $t = 3/10$, the inclusion $i\colon X_0 \to X_{3/10}$ induces an isomorphism in homology. Observe that $X_{3/10}$ is acyclic (see Figure 5.4.3 (b)).

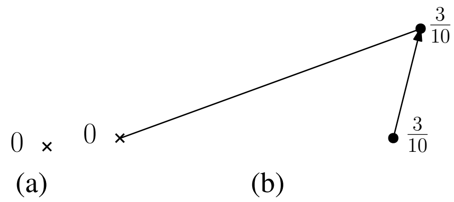
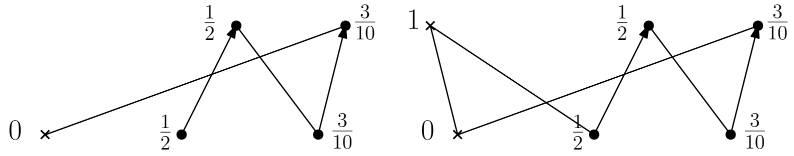


Figure 5.4.3: Level subposets X_0 and $X_{3/10}$.

Next, we reach the value $t = 1/2$ (see Figure 5.4.4 (a)). The inclusion $i\colon X_{3/10} \to X_{1/2}$ induces an isomorphism in homology and $X_{1/2}$ is still acyclic.

As we reach the value $t = 1$ (see Figure 5.4.4 (b)), which is critical, the inclusion $i\colon X_{1/2} \to X_1$ no longer induces and isomorphism in homology. The level subposet X_1 has the same homology as $\mathbb{S}^1$.

The next value is $t = 1 + 1/5$ (see Figure 5.4.5). Despite corresponding to a basic set, the homology does not change.



(a) (b)

Figure 5.4.4: Level subposets $X_{1/2}$ and X_1.

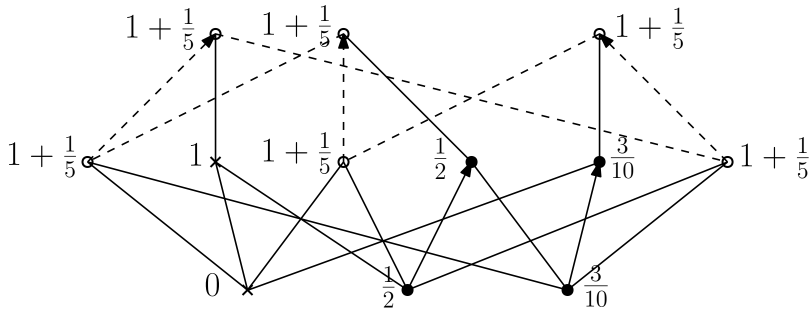


Figure 5.4.5: Level subposet $X_{1+1/5}$.

Finally, we reach the value $t = 2$ (see Figure 5.1.4), which is critical and produces a change in the homology since $X_2 = X$ has homology: $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, $H_1(X;\mathbb{Z}) \cong \mathbb{Z}_2$ and $H_k(X;\mathbb{Z}) \cong 0$ for $k \geq 2$.

5.4.2 Homological inequalities

In this last subsection, we provided the explicit computations which are necessary to check the homological inequalities (Theorems 6.34, 5.19 and 5.33). First of all, $\mathrm{hccat}(X) = 3$. We provide the remaining information in tables (Tables 5.1, 5.2 and 5.3) assuming we work with integer coefficients. We introduce some notation regarding the basic sets: Λ_0 denotes the critical point of degree zero, Λ_1 denotes the critical point of degree one, Λ_2 denotes the orbit and Λ_3 denotes the critical point of degree two.

$H_k(\bar{\Lambda}_i, \dot{\Lambda}_i)$	0	1	2	$k \geq 3$
Λ_0	$\mathbb{Z}$	0	0	0
Λ_1	0	$\mathbb{Z}$	0	0
Λ_2	0	$\mathbb{Z}$	$\mathbb{Z}$	0
Λ_3	0	0	$\mathbb{Z}$	0

Table 5.1: Computation of $H_k(\bar{\Lambda}_i, \dot{\Lambda}_i)$.

$H_k(\Lambda_i)$	0	1	2	$k \geq 3$
Λ_0	$\mathbb{Z}$	0	0	0
Λ_1	$\mathbb{Z}$	0	0	0
Λ_2	$\mathbb{Z}$	$\mathbb{Z}$	0	0
Λ_3	$\mathbb{Z}$	0	0	0

Table 5.2: Computation of $H_k(\Lambda_i)$.

	0	1	2	3	$i \geq 4$
β_i	1	0	0	0	0
η_i	0	1	0	0	0
c_i	1	1	1	0	0
A_i	0	1	0	0	0
m_i	1	3	1	0	0
$\mathrm{hccat}(\Lambda_i)$	1	1	1	2	1

Table 5.3: Computation of several homological and combinatorial invariants.

Part II

Combinatorial Algebraic Topology and Morse Theory on loop-free and small categories

Chapter 6

Morse theory on loop-free categories

The purpose of this chapter is to move upwards in the diagram that we first presented in the Introduction:

$$\begin{tikzcd}
\text{Small Cat} \arrow[rr, "\mathcal{N}"] \arrow[d, shift left, "\mathrm{sd}"] & & \text{Simplicial sets} \\
\text{Loop-free Cat} \arrow[rr, "\mathcal{N}"] \arrow[u, hook, shift left] \arrow[d, shift left, "\mathrm{sd}"] & & \text{Regular } \Delta\text{-Complexes} \arrow[u, hook] \arrow[dll, "\mathcal{X}"'] \\
\text{Posets} \arrow[rr, shift left, "\mathcal{K}"] \arrow[u, hook, shift left] & & \text{Simplicial Complexes} \arrow[u, hook] \arrow[ll, shift left, "\mathcal{X}"] \\
\text{Chain. Complexes} \arrow[u] & & \text{Manifolds} \arrow[u]
\end{tikzcd}$$

and to generalize Morse theory to the context of loop-free categories. This is motivated by the fact that recently some constructions in Combinatorial Algebraic Topology have been carried out in the setting of loop-free categories (see [105, 171, 172]).

The chapter is organized as follows. First, we briefly recall some notions in the context of small categories which we use both in this chapter and the next one. We also introduce some original ideas such as cellular categories. Then we develop from scratch a Morse theory on this setting in order to prove the Morse inequalities. We have tried to keep the presentation as simple and brief as possible.

Some of the results of this chapter, in particular the Morse theory for loop-free categories was introduced in our work ([113]) as an attempt to solve an open question by John ([97]).

6.1 Topology of small categories

In this section we recall some notions related to homotopy, weak homotopy and homology on the setting of small categories. For more details on some classical definitions and constructions we refer the reader to [108, 109, 121, 126, 143, 152].

6.1.1 Homotopies between functors

Recall that a category is said to be small if its arrows form a set. Given a small category $\mathcal{C}$, we denote by $\mathrm{Ob}(\mathcal{C})$ its set of objects, by $\mathrm{Arr}(\mathcal{C})$ its set of arrows and by $\mathcal{C}(c, c')$ the set of arrows between the objects c and c'. Moreover, we define two maps $t, s\colon \mathrm{Arr}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ which send an arrow to its target (codomain) and source (domain), respectively.

All categories will be assumed to be small and all functors will be assumed to be covariant unless stated otherwise. We begin by introducing the notion of homotopy between functors [108, 109].

Definition 6.1. The *interval category* $\mathcal{I}_m$ of length $m \geq 0$ consists of $m+1$ objects with zigzag arrows of the form

$$0 \to 1 \leftarrow 2 \to \cdots \to (\leftarrow) m.$$

Alternatively, the interval category $\mathcal{I}_m$ can be defined in the following way: the objects of $\mathcal{I}_m$ are the non-negative integers $0, 1, \ldots, m$ and the arrows, other than the identities, are defined as follows. Given two distinct objects r and s in $\mathcal{I}_m$, there is exactly one arrow from r to s if r is even and $s = r - 1$ or $s = r + 1$, and no arrows otherwise.

Given two small categories $\mathcal{C}$ and $\mathcal{D}$ we denote its product by $\mathcal{C} \times \mathcal{D}$. Recall that the objects of $\mathcal{C} \times \mathcal{D}$ are pairs of objects in $\mathcal{C}$ and objects in $\mathcal{D}$, and its arrows are pairs of arrows in $\mathcal{C}$ and arrows in $\mathcal{D}$.

Definition 6.2. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between small categories. We say that F and G are *homotopic* if there exists a functor $H\colon \mathcal{C} \times \mathcal{I}_m \to \mathrm{D}$, called a homotopy (of length m), such that $H_0 = F$ and $H_m = G$, for some $m \geq 0$.

Alternatively, the notion of homotopy between functors can be defined as follows. The definitions are equivalent.

Definition 6.3. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between small categories. We say that F and G are *homotopic*, $F \simeq G$, if there is a finite sequence of functors $F_0, \ldots, F_m\colon \mathcal{C} \to \mathcal{D}$, with $F_0 = F$ and $F_m = G$, such that for each $i \in \{0, \ldots, m-1\}$ there is a natural transformation between F_i and F_{i+1} or between F_{i+1} and F_i.

Homotopies can be concatenated ([170]) and therefore, the homotopy relation between functors defined above is an equivalence relation. It also holds that the relation behaves well with respect to compositions, i.e., if $F \simeq F'$ and $G \simeq G'$, then $F \circ G \simeq F' \circ G'$ whenever $F \circ F'$ and $G \circ G'$ make sense.

In order to state the next definition we briefly recall the definition of the classifying space functor B from small categories to topological spaces. Given the small category $\mathcal{C}$, its *nerve* $\mathrm{N}\mathcal{C}$ is a simplicial set (see [86] for a definition) whose m-simplices are composable m-tuples of arrows in $\mathcal{C}$:

$$c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m.$$

The face maps are obtained by composing or deleting arrows and the degenerate maps are obtained by inserting identities. An m-simplex of $\mathrm{N}\mathcal{C}$ is called *non-degenerate* if it includes no identity. Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ between small categories, we define $\mathrm{N}F\colon \mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$ as follows: if $c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m$ is a m-simplex in $\mathrm{N}\mathcal{C}$, then

$$\mathrm{N}F(c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m) = F(c_0) \xrightarrow{F(\alpha_1)} \cdots \xrightarrow{F(\alpha_m)} F(c_m).$$

The *classifying space* $\mathrm{B}\mathcal{C}$ is then the geometric realization $|\mathrm{N}\mathcal{C}|$ of the simplicial set $\mathrm{N}\mathcal{C}$. Moreover, $\mathrm{B}\mathcal{C}$ is a CW-complex with one m-cell for each non-degenerate m-simplex of $\mathrm{N}\mathcal{C}$. This construction is functorial (see [121, 143]), because given a map $\phi\colon K \to L$ between simplicial sets, its geometric realization is a continuous map between topological spaces $|\phi|\colon |K| \to |L|$. The *classifying space functor* is defined as the composition of the nerve functor with the geometric realization functor.

Definition 6.4. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between small categories. We say that F and G are *weak homotopic*, denoted $F \simeq_w G$ if the maps $\mathrm{B}F, \mathrm{B}G\colon \mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{D}$ are homotopic.

Definition 6.5. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be a homotopy equivalence (respectively a weak homotopy equivalence) if there exists another functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq 1_{\mathcal{C}}$ (respectively $G \circ F \simeq_w 1_{\mathcal{C}}$) and $F \circ G \simeq 1_{\mathcal{D}}$ (respectively $F \circ G \simeq_w 1_{\mathcal{D}}$). Under these circumstances we say that the categories $\mathcal{C}$ and $\mathcal{D}$ are homotopy equivalent (respectively weak homotopy equivalent).

Recall that the classifying space functor preserves homotopies. That is, if two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ are homotopic, then the induced maps $\mathrm{B}F, \mathrm{B}G\colon \mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{D}$ on the classifying spaces are also homotopic. Therefore, homotopy equivalence between categories implies weak homotopy equivalence. However, the converse does not hold as the following example given by Minian ([126]) shows.

Example 6.6. Consider a category $\mathcal{N}$ whose objects are the non-negative integers and the arrows, other than the identities, are defined as follows. If r and s are two distinct objects in $\mathcal{N}$, there is exactly one arrow from r to s if r is even and $s = r - 1$ or $s = r + 1$ and no arrows otherwise. Assume there is a functor $F\colon \mathcal{N} \to \mathcal{N}$ such that $F \simeq 1_{\mathcal{N}}$. We claim that there exists a non-negative integer n_0 such that $F(n) = n$, for all $n \geq n_0$, in particular F is not constant and the category $\mathcal{N}$ is not contractible.

Let us prove the claim. First, note that if there exists a natural transformation $G \Rightarrow 1_{\mathcal{N}}$ or $1_{\mathcal{N}} \Rightarrow G$, then G fixes the odd numbers, that is, $G(n) = n$ for n odd. As a consequence, it follows that $G(m) = m$ for every $m > 0$. By repeating a similar argument it can be deduced that if there exists a natural transformation $G' \Rightarrow G$ or $G \Rightarrow G'$, then G' fixes all natural numbers larger than 1. Repeating this argument it follows that if $F \simeq 1_{\mathcal{N}}$, then there exists a non negative integer n_0 such that $F(n) = n$ for all $n \geq n_0$ as we claimed.

However, the category $\mathcal{N}$ is weak contractible since $\mathrm{B}\mathcal{N}$ is homotopy equivalent to $[0, +\infty)$ and the map $\mathrm{B}\mathcal{N} \to \mathrm{B}(0)$ is a homotopy equivalence of topological spaces.

Example 6.7. When the small categories $\mathcal{C}$ and $\mathcal{D}$ are partially ordered sets seen as finite topological spaces [6] (see Section 7.4) and $F, G\colon \mathcal{C} \to \mathcal{D}$ are order preserving maps, then the notion of homotopy between functors is equivalent to the usual notion of homotopy in the context of topological spaces (see [145] or Proposition 2.1).

We recall a notion of "connectedness" for categories (see [144]):

Definition 6.8. A small category $\mathcal{C}$ is said to be *connected* if for any pair of objects c, c' there is a finite sequence of zigzag arrows joining them:

$$c = c_0 \to c_1 \leftarrow c_2 \to \cdots \to (\leftarrow) c_m = c'.$$

Equivalently, a small category $\mathcal{C}$ is said to be connected if for any pair of objects c, c' there is a functor F from some interval category $\mathcal{I}_m$ to $\mathcal{C}$ such that $F(0) = c$ and $F(m) = c'$.

Definition 6.9. Given two small categories $\mathcal{C}$ and $\mathcal{D}$, a functor $d_0 \colon \mathcal{C} \to \mathcal{D}$ is a *constant functor* onto the object d_0 of $\mathcal{D}$ if $d_0 \colon \mathcal{C} \to \mathcal{D}$ takes every object of $\mathcal{C}$ to d_0 and every arrow to the identity arrow of d_0.

Remark 6.10. Observe that a small category $\mathcal{C}$ is connected if and only if any pair of constant functors $c_0, c_1 \colon \mathcal{C} \to \mathcal{C}$ onto objects c_0 and c_1 are homotopic.

From now on all categories are assumed to be connected.

Remark 6.11. When particularized to the context of posets, Definition 6.8 corresponds to the notion of order-connectedness, which is equivalent to topological connectedness for the associated finite spaces (see [6]).

Definition 6.12. A small category $\mathcal{C}$ is said to be *contractible* if the identity functor is homotopic to a constant functor onto an object.

We state a useful result.

Proposition 6.13. *Suppose the functor $F \colon \mathcal{C} \to \mathcal{D}$ between small categories has a left or right adjoint $G \colon \mathcal{D} \to \mathcal{C}$. Then $F \colon \mathcal{C} \to \mathcal{D}$ is a homotopy equivalence. In particular, when $\mathcal{C}$ has an initial or terminal object, $\mathcal{C}$ is contractible.*

Proof. If G is a right adjoint to F, then there are two natural transformations $1_{\mathcal{C}} \Rightarrow G \circ F$ and $F \circ G \Rightarrow 1_{\mathcal{D}}$ (see [147, Section 4.2]). Therefore F is a homotopy equivalence. $\square$

6.1.2 Homological Quillen's Theorem A

In this subsection we state a homological version of Quillen Theorem A for small categories. For a detailed presentation the reader is referred to [105, 121, 143, 152, 172, 173].

We recall the notions of left and right homotopy fibers due to Quillen.

Definition 6.14. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor and let d be an object of $\mathcal{D}$. The *left homotopy fiber* F/d of F is the small category whose objects are:

$$\mathrm{Ob}(F/d) = \{(c, g) \in \mathrm{Ob}(\mathcal{C}) \times \mathcal{D}(F(c), d)\}$$

and whose arrows are:

$$F/d((c, g), (c', g')) = \{f \in \mathcal{C}(c, c') \colon g' \circ F(f) = g\}.$$

Dually, the *right homotopy fiber* d/F of F is the small category whose objects are:

$$\mathrm{Ob}(d/F) = \{(c, g) \in \mathrm{Ob}(\mathcal{C}) \times \mathcal{D}(d, F(c))\}$$

and whose arrows are:

$$d/F((c, g), (c', g')) = \{f \in \mathcal{C}(c, c')\colon F(f) \circ g = g'\}.$$

We define the homology (with coefficients in a principal ideal domain) of small categories as the homology of the associated objects by the nerve functor. We now state a homological version of Quillen Theorem A. For the proof we refer the reader to [173]:

Theorem 6.15 (Homological Theorem A)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If all the left homotopy fibers or all the right homotopy fibers are homologically trivial, then F induces an isomorphism $H_*(F)\colon H_*(\mathcal{C}) \to H_*(\mathcal{D})$ in homology.*

6.1.3 Homology of loop-free categories

We begin by recalling the concept of loop-free category (see [105]).

Definition 6.16. A small category $\mathcal{C}$ is *loop-free* or *acyclic* if it satisfies the following two conditions:

1. Only the identity arrows have inverses.

2. Any arrow from an object to itself is an identity.

From now on, we will assume that all loop-free categories are finite, that is, their set of arrows are finite.

We recall a construction for loop-free categories which simplifies the computation of homology. We refer the reader to [171], [105] and [95, Appendix] for the notion of regular Δ-complexes or regular trisps. They are just a class of regular CW-complexes where the cells are simplices. The difference between regular Δ-complexes and simplicial complexes is that in a regular Δ-complexes two simplices can be generated by the exact same vertices while this is not possible for a simplicial complex.

Definition 6.17 ([105, 172])**.** Let $\mathcal{C}$ be an loop-free category. Its *order complex* $\mathcal{K}(\mathcal{C})$ is a Δ-complex (or regular trisp) whose m-simplices are the composable m-tuples of arrows in $\mathcal{C}$ not including identities. For an object c, the face map d_c is given by composing arrows at c or by deleting the arrows starting or ending at c.

Observe that for a loop-free category $\mathcal{C}$, it is equivalent to compute the homology as $H \circ \mathcal{K}(\mathcal{C})$ or as $H \circ \mathcal{N}(\mathcal{C})$.

6.1.4 Cellular categories

We begin by introducing a grading for our loop-free categories. In order to do so, we will add an extra assumption (to make our work easier) to the notion of graded loop-free category which appears in [105]. Then, we will introduce the notion of cellular categories. It can be seen as an extension of the concepts of cellular posets defined in [63] and [126].

Definition 6.18. Let $\mathcal{C}$ be a loop-free category and $c \in \mathrm{Ob}(\mathcal{C})$. We say that c is a *minimal object* of $\mathcal{C}$ (or that c is minimal, for short) if c is not the target of any non identity arrow of $\mathcal{C}$, that is, $t^{-1}(c) = \{\mathrm{id}_c\}$. Dually, we say that c is a *maximal object* of $\mathcal{C}$ (or that c is maximal, for short) if c is not the source of any non identity arrow of $\mathcal{C}$, that is, $s^{-1}(c) = \{\mathrm{id}_c\}$.

Definition 6.19. We call an arrow *indecomposable* if it can not be represented as a composition of two non identity arrows. A loop-free category $\mathcal{C}$ is called *graded* if there is a map $r\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}$ such that:

1. Whenever $m\colon c \to c'$ is a non identity indecomposable arrow, we have $r(c') = r(c) + 1$.
2. Moreover, if c is a minimal object of $\mathcal{C}$, then $r(c) = 0$.

For an object c, $r(c)$ will be referred as its degree. Moreover, we will write $c^{(r(c))}$.

Moreover, for a graded category $\mathcal{C}$, $\mathcal{C}^k$ denotes the full subcategory of $\mathcal{C}$ whose objects are the objects of $\mathcal{C}$ of degree less than or equal to k.

We introduce some definitions that will be necessary later.

Definition 6.20. Let $\mathcal{C}$ be a loop-free category and let c be an object of $\mathcal{C}$. Let $\mathcal{C} - \{c\}$ denote the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C} - \{c\}) = \mathrm{Ob}(\mathcal{C}) - \{c\}$.

Definition 6.21. Let $\mathcal{C}$ be a loop-free category and let c be an object of $\mathcal{C}$. Then U_c is the full subcategory of C whose objects are $s(t^{-1}(c))$. Moreover, we define $\widehat{U}_c = U_c - \{c\}$.

Definition 6.22. Let $\mathcal{C}$ be a graded loop-free category. It is said to be *cellular* if for each $c \in \mathrm{Ob}(\mathcal{C})$, $\widehat{U}_c$ has the homology of a wedge of n_c $(r(c) - 1)$-spheres for some $n_c \geq 1$.

Example 6.23. We provide an example of a cellular category in Figure 6.1.1. We do not include the identity arrows in the picture.

6.2 Morse theory on loop-free categories

This section is devoted to extending Morse theory to loop-free categories. By doing that, we answer a question by T. John ([97]).

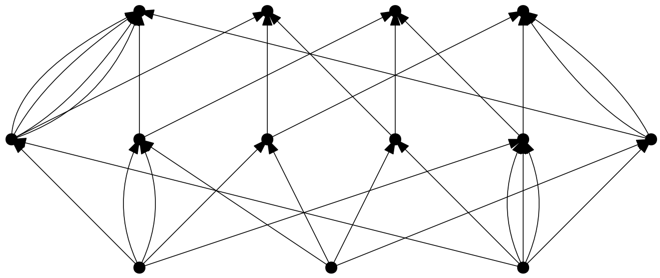


Figure 6.1.1: Example of a cellular category.

6.2.1 Vector fields

We begin with the definition of vector field for loop-free categories. Given an arrow $f \in \mathrm{Arr}(\mathcal{C})$, we denote $\mathrm{Arr}(f) = \mathcal{C}(s(f), t(f))$.

Definition 6.24. Let $\mathcal{C}$ be a graded loop-free category. A *vector field* $\mathcal{V}$ on $\mathcal{C}$ is a subset of the non identity indecomposable arrows of $\mathcal{C}$ satisfying the following conditions:

1. If $f, g \in \mathcal{V}$, then $s(f) \neq t(g)$.
2. If $f \in \mathcal{V}$ and $\#\mathrm{Arr}(f) = 1$, then $s^{-1}(s(f)) \cap \mathcal{V} = \{f\}$ and $t^{-1}(t(f)) \cap \mathcal{V} = \{f\}$.
3. If $f \in \mathcal{V}$ and $\#\mathrm{Arr}(f) > 1$, then $\#(\mathcal{V} \cap \mathrm{Arr}(f)) \leq \#\mathrm{Arr}(f) - 1$.

Remark 6.25. Condition (2) encodes the idea that closed orbits can not interact with a gradient like flow. Condition (3) forces closed orbits in a dynamical sense to be closed in the graph representation. This intuition will become clear later.

The set $\{f \in \mathcal{V} \colon \#\mathrm{Arr}(f) = 1\}$ will be referred as the *gradient like part* of the vector field.

Definition 6.26. Let $\mathcal{C}$ be a graded loop-free category and $\mathcal{V}$ a vector field on $\mathcal{C}$. The vector field $\mathcal{V}$ is *homologically admissible* if for every arrow f in the gradient like part of $\mathcal{V}$, the subcategory $\widehat{U}_{t(f)} - \{s(f)\}$ is homologically trivial.

Given a loop-free category $\mathcal{C}$, let us denote by $\mathcal{H}(\mathcal{C})$ the directed graph defined as follows. The elements of $\mathcal{H}(\mathcal{C})$ are the objects of $\mathcal{C}$ while the edges of $\mathcal{H}(\mathcal{C})$ are the indecomposable non identity arrows of $\mathcal{C}$. If $\mathcal{V}$ is a vector field on $\mathcal{C}$, write $\mathcal{H}_{\mathcal{V}}(\mathcal{C})$ for the directed graph obtained from $\mathcal{H}(\mathcal{C})$ by reversing the orientations of the edges which are not in $\mathcal{V}$. Any node of $\mathcal{H}(\mathcal{C})$ not incident with any edge of $\mathcal{V}$ is called *critical*.

Definition 6.27. Let $\mathcal{V}$ be a vector field on a loop-free graded category $\mathcal{C}$ and let $c^{(k)}, \tilde{c}^{(k)} \in \mathrm{Ob}(\mathcal{C})$ be two objects of $\mathcal{C}$. A $\mathcal{V}$*-path*, γ, of index k from $c^{(k)}$ to $\tilde{c}^{(k)}$ is a sequence:

$$(c^{(k)} = x_0^{(k)}, y_0^{(k+1)}, x_1^{(k)}, y_1^{(k+1)}, \dots, y_{r-1}^{(k+1)}, x_r^{(k)} = \tilde{c}^{(k)})$$

with $r \geq 1$ such that for each $i = 0, 1, \dots, r-1$:

1. There is a $f_i \in \mathcal{V}$ such that $f_i \colon x_i \to y_i$,
2. There is a $g_i \in \mathrm{Arr}(\mathcal{C}) - \mathcal{V}$ such that $g_i \colon y_i \to x_{i+1}$.

A $\mathcal{V}$*-cycle* γ in $\mathcal{H}_{\mathcal{V}}(X)$ is a closed $\mathcal{V}$-path in $\mathcal{H}_{\mathcal{V}}(X)$ seen as a directed graph.

6.2.2 Critical subcategories

The notion of chain recurrent set defined for the setting of posets in Chapter 5 generalizes naturally to this new context as we show below.

Definition 6.28. Let $\mathcal{V}$ be a vector field on a graded loop-free category $\mathcal{C}$. We say that $c^{(k)} \in \mathcal{C}$ is an object of the *chain recurrent set* $\mathcal{R}$ if one of the following conditions holds:

- c is a critical element of $\mathcal{V}$.
- There is a $\mathcal{V}$-cycle γ in $\mathcal{H}_{\mathcal{V}}(X)$ such that $c \in \gamma$.

The chain recurrent set decomposes into disjoint subsets Λ_i by means of the equivalence relation defined as follows:

1. If c is a critical element, then it is only related to itself.
2. Given $c, c' \in \mathcal{R}$ not critical, $c \neq c'$, $c \sim c'$ if there is a cycle γ such that $c, c' \in \gamma$.

Let $\Lambda_1, \ldots, \Lambda_k$ be the equivalence classes of $\mathcal{R}$. The $\Lambda_i' s$ are called *basic sets*. Each Λ_i consists either of a single critical element of $\mathcal{V}$ or a union of cycles, each of which has the same index. We write $\Lambda_i^{(k)}$ and say that Λ_i has *index* k if Λ_i consists of a critical point of index k or a union of closed paths of index k.

Example 6.29. In Figure 6.2.1 we provide a representation of $\mathcal{H}_{\mathcal{V}}(\mathcal{C})$ for a vector field $\mathcal{V}$ on the cellular category of Example 6.23. The orange arrow is the gradient like part. The pink object is a critical element. The other colors (green, blue and red) represent different basic sets.

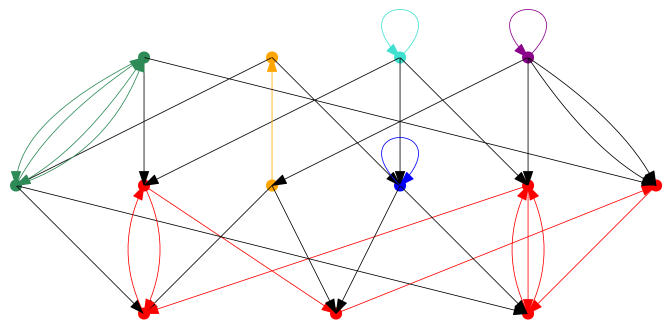
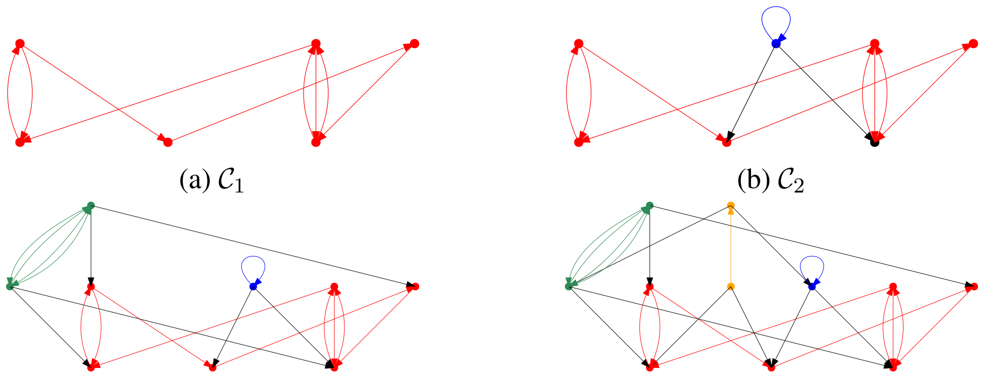


Figure 6.2.1: Example of a vector field.

6.2.3 Filtration induced by a vector field

Let $\mathcal{C}$ be a finite graded loop-free category and $\mathcal{V}$ a vector field on $\mathcal{C}$. We show how this vector field induces a filtration on $\mathcal{C}$:

$$\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_i \hookrightarrow \mathcal{C}_{i+1} \hookrightarrow \cdots \hookrightarrow \mathcal{C}_n = \mathcal{C}$$



(a) $\mathcal{C}_1$ (b) $\mathcal{C}_2$ (c) $\mathcal{C}_3$ (d) $\mathcal{C}_4$

Figure 6.2.2: Filtration induced by a vector field.

where each $\mathcal{C}_i$ is a full subcategory of $\mathcal{C}$. Basically, we follow the same idea as in Theorem 3.47. We define $\mathcal{C}_0$ as the empty category. Now we apply the following iterative process. For $\mathcal{C}_i$ a full subcategory of $\mathcal{C}$, we denote by M_i the set of minimal elements of $\mathcal{C}$ in $\mathcal{C}-\mathcal{C}_i$. If there is a critical element $c \in M_i$, then we define $\mathcal{C}_i$ as the full subcategory with the object c. If there are no critical elements in M_0 and there is a $\mathcal{V}$-cycle γ such that $\mathrm{Ob}(\bigcup_{c\in\gamma} \widehat{U}_c) \subset \mathrm{Ob}(\mathcal{C}_i)$, then we define $\mathcal{C}_{i+1}$ as the full subcategory with the objects in γ. If there are no critical elements nor $\mathcal{V}$-cycles satisfying the stated conditions in M_0, then we proceed as follows. Among all the arrows f in the gradient part of $\mathcal{V}$ such that $s(f) \in M_i$, we pick one satisfying that $\mathrm{Ob}(\widehat{U}_{t(f)}) \subset \mathrm{Ob}(\mathcal{C}_i)$. Then we define $\mathcal{C}_{i+1}$ as the full subcategory with the objects $\mathrm{Ob}(\mathcal{C}_i) \cup \{s(f), t(f)\}$.

Example 6.30. We illustrate in Figures 6.2.2a, 6.2.2b, 6.2.2c, 6.2.2d and 6.2.1 the procedure presented above for the vector field and category of Example 6.29.

6.2.4 A homological invariance theorem

In order to prove the Morse inequalities, we need the following homological invariance theorem:

Theorem 6.31 (Homological invariance theorem)**.** *Let $\mathcal{C}$ be a finite cellular category and $\mathcal{V}$ a homologically admissible vector field on $\mathcal{C}$. Consider a filtration*

$$\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_i \hookrightarrow \mathcal{C}_{i+1} \hookrightarrow \cdots \hookrightarrow \mathcal{C}_n = \mathcal{C}$$

as constructed in Subsection 6.2.3. If there is an f in the gradient part of $\mathcal{V}$ such that $\mathcal{C}_{i+1}$ is the full subcategory with the objects $\mathrm{Ob}(\mathcal{C}_i) \cup \{s(f), t(f)\}$, then $\mathcal{C}_i \hookrightarrow \mathcal{C}_{i+1}$ induces an isomorphism in homology.

Proof. First of all, each right homotopy fiber of the inclusion functor $i\colon \mathcal{C}_{i+1} - \{s(f)\} \hookrightarrow \mathcal{C}_{i+1}$ has an initial object, so it is homologically trivial since it is contractible by Proposition 6.13. Therefore, by the Homological Theorem A (Theorem 6.15), $i\colon \mathcal{C}_{i+1} - \{s(f)\} \hookrightarrow \mathcal{C}_{i+1}$ induces an isomorphism in homology. Now, consider the inclusion:

$$i\colon \mathcal{C}_{i+1} - \{s(f), t(f)\} \hookrightarrow \mathcal{C}_{i+1} - \{s(f)\}.$$

The result would follow if we proved that

$$H_*(\mathcal{C}_{i+1} - \{s(f)\}, \mathcal{C}_{i+1} - \{s(f), t(f)\}) \cong 0.$$

Apply the Excision theorem to the subcomplexes $\mathcal{K}(U_{t(f)})$ and $\mathcal{K}(\mathcal{C}_{i+1} - \{s(f), t(f)\})$ to obtain the isomorphism:

$$H_*(U_{t(f)}, \widehat{U}_{t(f)} - \{s(f)\}) \cong H_*(\mathcal{C}_{i+1} - \{s(f)\}, \mathcal{C}_{i+1} - \{s(f), t(f)\}).$$

Observe that $U_{t(f)}$ has a terminal object $t(f)$, so $\mathcal{K}(U_{t(f)})$ is homologically trivial. Since the vector field $\mathcal{V}$ is homologically admissible, then $\widehat{U}_{t(f)} - \{s(f)\}$ is homologically trivial. By the homology long exact sequence of the pair $(\mathcal{K}(U_{t(f)}), \mathcal{K}(\widehat{U}_{t(f)} - \{s(f)\}))$, it follows that

$$H_*(\mathcal{K}(U_{t(f)}), \mathcal{K}(\widehat{U}_{t(f)} - \{s(f)\})) \cong 0$$

and we obtain the desired result. □

6.2.5 The Morse inequalities

We now prove the Morse inequalities in the context of loop-free categories.

Given a subcategory $\mathrm{D} \hookrightarrow \mathcal{C}$ we denote by $\bar{\mathrm{D}}$ the full subcategory with objects $\mathrm{Ob}(\bar{\mathrm{D}}) = \cup_{d \in \mathrm{D}} U_d$ and by $\dot{\mathrm{D}}$ the full subcategory with objects $\mathrm{Ob}(\dot{\mathrm{D}}) = \mathrm{Ob}(\bar{\mathrm{D}}) - \mathrm{Ob}(\mathrm{D})$.

Definition 6.32. For each $k \geq 0$, we define

$$m_k = \sum_{\text{basic sets } \Lambda_i} \operatorname{rank} H_k(\bar{\Lambda}_i, \dot{\Lambda}_i).$$

Proposition 6.33. *If the index of a basic set Λ is k, then $H_i(\bar{\Lambda}, \dot{\Lambda}) = 0$ unless $i = k, k+1$. Moreover, if Λ is just a critical point $x^{(k)}$, then $H_i(\bar{\Lambda}, \dot{\Lambda}) = 0$ for $i \neq k$.*

Proof. Observe that both $\bar{\Lambda}$ and $\dot{\Lambda}$ are cellular categories. In fact: $\bar{\Lambda} = \bar{\Lambda}^{k+1}$ and $\dot{\Lambda} = \bar{\Lambda}^{k-1}$. By Excision and a standard argument it follows that

$$H_i(\bar{\Lambda}^l, \bar{\Lambda}^{l-1}) = \bigoplus_{\deg(c)=l,\, c \in \bar{\Lambda}} H_{i-1}(\widehat{U}_c).$$

In particular, $H_i(\bar{\Lambda}^l, \bar{\Lambda}^{l-1}) = 0$ for $i \neq l$. Now the result follows from the Long Exact Sequence for the triple $(\bar{\Lambda}^{k+1}, \bar{\Lambda}^k, \bar{\Lambda}^{k-1})$. □

We denote by b_k the Betti number of dimension k with coefficients in a principal domain.

Theorem 6.34 (Strong Morse-Bott inequalities)**.** *Let $\mathcal{C}$ be a finite cellular category and let $\mathcal{V}$ be a homologically admissible vector field on $\mathcal{C}$. Then, for every $k \geq 0$:*

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0.$$

Proof. Consider a filtration of $\mathcal{C}$ as described in Subsection 6.2.3:

$$\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_i \hookrightarrow \mathcal{C}_{i+1} \hookrightarrow \cdots \hookrightarrow \mathcal{C}_n = \mathcal{C}.$$

We will check that the inequalities hold for every i, that is (writing m_k instead of $m_k(\mathcal{C}_i)$ and b_k instead of $b_k(\mathcal{C}_i)$):

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0$$

for every $\mathcal{C}_i$. We argue by induction on i. For C_0 it holds trivially. Assume it holds for $\mathcal{C}_i$ and let us show that then it also holds for $\mathcal{C}_{i+1}$. There are two cases to consider:

1. There is an f in the gradient part of $\mathcal{V}$ such that $\mathcal{C}_{i+1}$ is the full subcategory with the objects $\mathrm{Ob}(\mathcal{C}_i) \cup \{s(f), t(f)\}$. Then, by the Homological invariance Theorem (Theorem 6.31), $\mathcal{C}_i \hookrightarrow \mathcal{C}_{i+1}$ induces an isomorphism in homology. Therefore $b_k(\mathcal{C}_i) = b_k(\mathcal{C}_{i+1})$ for all k. Moreover, $m_k(\mathcal{C}_i) = m_k(\mathcal{C}_{i+1})$ by definition.

2. The subcategory $\mathcal{C}_{i+1}$ is the full subcategory whose objects are the union of the objects of $\mathcal{C}_i$ and the elements of a basic set Λ. Then $m_k(\mathcal{C}_{i+1}) - m_k(\mathcal{C}_i) = \operatorname{rank} H_k(\bar{\Lambda}, \dot{\Lambda})$. By excision, it follows that: $H_k(\mathcal{C}_{i+1}, \mathcal{C}_i) \cong H_k(\bar{\Lambda}, \dot{\Lambda})$. Now the result is obtained by the same arguments as in the case of posets. □

Corollary 6.35 (Weak Morse-Bott inequalities)**.** *Let $\mathcal{C}$ be a finite cellular category and let $\mathcal{V}$ be a homologically admissible vector field on $\mathcal{C}$. Then:*

1. *For every $k \geq 0$, $m_k \geq b_k$;*
2. $\chi(\mathcal{C}) = \sum_{i=0}(-1)^k b_k = \sum_{i=0}(-1)^k m_k$.

Chapter 7
Homotopic distance on small categories

This thesis is devoted to the study of Morse theory in finite contexts. Therefore, it is also worth studying topological invariants which are strongly related to Morse theory as we did with a notion of Lusternik-Schnirelmann category in Chapter 5. In this chapter we present the notion of *homotopic distance between functors* which we introduced in [117]. We expect that this invariant will prove to be related to Morse theory due to the relation we found between the two notions in the non-finite setting and which we recall in Section 7.1.

This chapter is organized as follows. First, we briefly present the *homotopic distance between maps* and its relation to Morse theory in order to motivate the material from the rest of the chapter. Second, we introduce the definition of *homotopic distance between functors*, its first properties, and some examples. In particular, we show that it generalizes a notion of *categorical Lusternik-Schnirelmann category* introduced by Tanaka ([170]). Third, we further study the properties of the distance. Finally, we focus our work in the settings of posets.

The results of this chapter were published in [117].

7.1 Homotopic distance and Morse theory

In this first section we motivate the results of the rest of the chapter by showing that their continuous counterparts in the non-finite setting are strongly connected to Morse theory. We recall the notion of *homotopic distance* between maps which we introduced in [115] and some results from our works [115] and [116].

Let $f, g\colon X \to Y$ be two continuous maps.

Definition 7.1. The *homotopic distance* $\mathrm{D}(f, g)$ between f and g is the least integer $n \geq 0$ such that there exists an open covering $\{U_0, \ldots, U_n\}$ of X with the property that the restrictions of the maps to each open subset are homotopic, $f_{|U_j} \simeq g_{|U_j}$ for all $j = 0, \ldots, n$. If there is no such covering, we define $\mathrm{D}(f, g) = \infty$.

Notice that:

1. $\mathrm{D}(f, g) = \mathrm{D}(g, f)$.

2. $\mathrm{D}(f, g) = 0$ if and only if the maps f, g are homotopic.

3. The homotopic distance only depends on the homotopy class. That is, if $f \simeq f'$ and $g \simeq g'$, then $\mathrm{D}(f, g) = \mathrm{D}(f', g')$.

The interest of homotopic distance relies in the fact that it generalizes well-known and widely studied invariants, such as the Lusternik-Schnirelmann category and topological complexity among many others (see [115] and [116] for a detailed exposition). There is a relative notion introduced in [116] which we recall now.

Definition 7.2. Let $f, g \colon X \to Y$ be two continuous maps, and let $A \subset X$ be a subspace. The *subspace distance* between the two maps f, g on A, denoted by $\mathrm{D}_X(A; f, g)$, is defined as the distance between the restrictions of f, g to A, that is,

$$\mathrm{D}_X(A; f, g) := \mathrm{D}(f_{|A}, g_{|A}).$$

Obviously, when $A = X$ we recover the usual homotopic distance. Moreover, observe that $\mathrm{D}_X(A; f, g) = \mathrm{D}(f \circ i_A, g \circ i_A)$, where $i_A \colon A \subset X$ is the inclusion.

The relation between homotopic distance and Morse theory is explicit in the following result ([116, Theorem 4.5]):

Theorem 7.3. *Let $\Phi \colon M \to \mathbb{R}$ be a Morse-Bott function on a compact smooth manifold M. Let $c_1 < \ldots < c_p$ be its critical values, and let $\Sigma_i = \Phi^{-1}(c_i) \cap \mathrm{crit}\Phi$ be the set of critical points in the level $\Phi = c_i$. If $f, g \colon M \to Y$ are two continuous maps, then*

$$\mathrm{D}(f, g) + 1 \leq \sum_{i=1}^{p} (\mathrm{D}_M(\Sigma_i; f, g) + 1).$$

This lead us to believe that it is worth studying an adaptation of the homotopic distance to finite settings and that there should be a relation with some Morse theory in the latter context.

7.2 Homotopic distance between functors

In this section, we introduce the definition of the homotopic distance between functors, some of its first properties, and some remarkable examples.

7.2.1 Definition of categorical distance

We begin by introducing a suitable notion of covers for categories in order to define a notion of categorical distance between functors. The idea is to think of covers of the classifying space, in order to cover the arrows of the category. This notion was introduced by Tanaka ([170]).

Definition 7.4. A collection of subcategories $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ of a category $\mathcal{C}$ is a *geometric cover of* $\mathcal{C}$ if for every sequence of composable arrows $f_1, \ldots, f_n$ in $\mathcal{C}$, there exists an index $\lambda \in \Lambda$ such that every f_i belongs to $\mathcal{U}_\lambda$.

Proposition 7.5 ([170]). *Let $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be a collection of subcategories of a category $\mathcal{C}$. This is a geometric cover if and only if the collection of subcomplexes $\{\mathrm{B}\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ covers $\mathrm{B}\mathcal{C}$.*

Now we introduce our definition of "distance":

Definition 7.6. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between small categories. The *categorical homotopic distance* $\mathrm{cD}(F, G)$ between F and G is the least integer $n \geq 0$ such that there exists a geometric cover $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ of $\mathcal{C}$ with the property that $F_{|\mathcal{U}_j} \simeq G_{|\mathcal{U}_j}$, for all $j = 0, \ldots, n$. If there is no such covering, we define $\mathrm{cD}(F, G) = \infty$.

Example 7.7. Any finite group $\mathcal{G}$ can be seen as a category with only one object, where the arrows are the elements of G. Then it can be checked easily that if $\mathcal{G}$ is a non trivial abelian group and $F, G\colon \mathcal{G} \to \mathcal{G}$ are two functors, that is, two group homomorphisms, then $cD(F, G) = \infty$ unless $F = G$.

It is easy to prove that some properties of the homotopic distance for continuous maps also hold for the categorical homotopical distance:

Proposition 7.8. *The following properties for the categorical homotopic distance hold:*

1. $\mathrm{cD}(F, G) = \mathrm{cD}(G, F)$.
2. $\mathrm{cD}(F, G) = 0$ *if and only if the functors* F, G *are homotopic.*
3. *The categorical homotopic distance only depends on the homotopy class, that is, if* $F \simeq F'$ *and* $G \simeq G'$ *then* $\mathrm{cD}(F, G) = \mathrm{cD}(F', G')$.
4. *Given two functors* $F, G\colon \mathcal{C} \to \mathcal{D}$ *and a finite geometric covering* $\mathcal{U}_0, \ldots, \mathcal{U}_n$ *of* $\mathcal{C}$,
$$\mathrm{cD}(F, G) \leq \sum_{k=0}^{n} \mathrm{cD}(F_{|\mathcal{U}_k}, G_{|\mathcal{U}_k}) + n.$$
5. *A small category* $\mathcal{C}$ *is connected if and only if the categorical homotopic distance between any pair of constant functors is zero.*

Definition 7.9. The *weak categorical homotopic distance* $\mathrm{wcD}(F, G)$ between F and G is the least integer $n \geq 0$ such that there exists a geometric covering $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ of $\mathcal{C}$ with the property that $\mathrm{B}F_{|\mathrm{B}\mathcal{U}_j} \simeq \mathrm{B}G_{|\mathrm{B}\mathcal{U}_j}$, for all $j = 0, \ldots, n$. If there is no such covering, we define $\mathrm{wcD}(F, G) = \infty$.

The weak homotopic distance satisfies the analogous statements to Properties 1–4 from Proposition 7.8. Moreover, both the weak categorical and the categorical distance behave well with respect to duality:

Proposition 7.10. *Given a functor* $F\colon \mathcal{C} \to \mathcal{D}$ *we can define* $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. *Moreover, if there is a natural transformation between* F *and* G, *then there is a natural transformation between* G^{op} *and* F^{op}. *Therefore,* $\mathrm{cD}(F, G) = \mathrm{cD}(F^{\mathrm{op}}, G^{\mathrm{op}})$. *Notice that* $\mathrm{B}F = \mathrm{B}F^{\mathrm{op}}$. *Hence,* $\mathrm{wcD}(F, G) = \mathrm{wcD}(F^{\mathrm{op}}, G^{\mathrm{op}})$.

7.2.2 Examples

Recall that all categories are assumed to be small and connected.

We begin by restating the concept of categorical Lusternik-Schnirelmann category introduced by Tanaka ([170]) as a particular case of the more general notion of categorical homotopic distance:

Definition 7.11. Let $\mathcal{C}$ be a small category. A subcategory $\mathcal{U}$ is *categorical in* $\mathcal{C}$ if the inclusion functor is homotopic to a constant functor onto an object. The *(normalized) categorical Lusternik-Schnirelmann category* $\mathrm{ccat}(\mathcal{C})$ is the least integer $n \geq 0$ such that there exists a geometric cover of $\mathcal{C}$ formed by $n+1$ categorical subcategories. If there is no such an integer we set $\mathrm{ccat}(\mathcal{C}) = \infty$.

Given a small category $\mathcal{C}$ and a subcategory $\mathcal{U}$, observe that the inclusion functor of the subcategory $\mathcal{U}$ in $\mathcal{C}$ is just the identity functor $1_\mathcal{C}$ of $\mathcal{C}$ restricted to $\mathcal{U}$. As a consequence, the definition of the categorical Lusternik-Schnirelmann category can be reformulated by means of the following result, which shows that our categorical homotopic distance between functors generalizes Tanaka's definition of the categorical Lusternik-Scnirelmannn category [170]:

Proposition 7.12. *The LS-category of* $\mathcal{C}$ *is the categorical homotopic distance between the identity* $1_\mathcal{C}$ *of* $\mathcal{C}$ *and any constant functor, that is* $\mathrm{cat}(\mathcal{C}) = \mathrm{cD}(1_\mathcal{C}, *)$.

More generally, we define the categorical Lusternik-Schnirelmann category of a functor:

Definition 7.13. The *(weak) categorical Lusternik-Schnirelmann category of the functor* $F\colon \mathcal{C} \to \mathcal{D}$ is the (weak) categorical distance bewteen F and a constant functor, $\mathrm{ccat}(F) = \mathrm{cD}(F, *)$.

Example 7.14. Observe that a subcategory $\mathcal{U}$ is categorical if and only if the restriction of the diagonal functor $\Delta_\mathcal{C}\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is homotopic to a constant functor. Hence, the category of the diagonal functor $\Delta_\mathcal{C}\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ equals $\mathrm{ccat}(X)$.

Given a base object $c_0 \in \mathcal{C}$ we define the inclusion functors $i_1, i_2\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ as $i_1(c) = (c, c_0)$ and $i_2(c) = (c_0, c)$.

Proposition 7.15. *The categorical LS-category of* $\mathcal{C}$ *equals the categorical homotopic distance between* i_1 *and* i_2*, that is,* $\mathrm{ccat}(\mathcal{C}) = \mathrm{cD}(i_1, i_2)$.

Proof. First, we show that $\mathrm{cD}(i_1, i_2) \leq \mathrm{ccat}(X)$. Assume that a subcategory $\mathcal{U}$ of $\mathcal{C}$ is categorical and let $H\colon \mathcal{U} \times \mathcal{I}_m \to \mathcal{C}$ be the homotopy between the inclusion functor and the constant functor to $c_0 \in \mathcal{C}$, i.e. $H(c, 0) = c$ and $H(c, 1) = c_0$. We define a homotopy $H'\colon \mathcal{U} \times \mathcal{I}_{2m} \to \mathcal{C}$ between $(i_1)_{|\mathcal{U}}$ and $(i_2)_{|\mathcal{U}}$ (by concatenation) as

$$H'(c, i) = \begin{cases} \big(H(c, i), c_0\big) & \text{if } 0 \leq i \leq m, \\ \big(c_0, H(c, 2m-i)\big) & \text{if } m \leq i \leq 2m. \end{cases}$$

Note that:

$$H'(c, 0) = \big(H(c, 0), c_0\big) = (c, c_0) = i_1(c)$$

while

$$H'(c,2m) = \big(c_0, H(c,0)\big) = (c_0, c) = i_2(c).$$

Second, we show that $\mathrm{ccat}(X) \leq \mathrm{cD}(i_1, i_2)$. Assume that there is a homotopy $H\colon \mathcal{U} \times \mathcal{I}_m \to \mathcal{C} \times \mathcal{C}$ between $(i_1)_{|\mathcal{U}}$ and $(i_2)_{|\mathcal{U}}$, i.e., $H(c,0) = (c, c_0)$ and $H(x,1) = (c_0, c)$. Let $p_1 \circ H$ be the first component of H. Then $p_1 \circ H$ is a homotopy between the inclusion functor of $\mathcal{U}$ and the constant functor onto c_0. □

Therefore, our notion of categorical homotopic distance generalizes the categorical Lusternik-Schnirelmann category introduced by Tanaka [170].

Motivated by the approach adopted in [64] to the Discrete Topological Complexity in the setting of simplicial complexes, we define the complexity of a category as follows:

Definition 7.16. A subcategory $\mathcal{U}$ of $\mathcal{C} \times \mathcal{C}$ is a *Farber subcategory* if there exists a functor $F\colon \mathcal{U} \to \mathcal{C}$ such that $\Delta \circ F \simeq i_{\mathcal{U}}$ where $i_{\mathcal{U}}$ is the inclusion functor. The *(normalized) categorical complexity of* $\mathcal{C}$, $\mathrm{cTC}(\mathcal{C})$, is the least integer $n \geq 0$ such that there exists a geometric cover of $\mathcal{C}$ formed by $n+1$ Farber subcategories. If there is no such an integer we set $\mathrm{cTC}(\mathcal{C}) = \infty$.

Theorem 7.17. *The categorical complexity of a small category $\mathcal{C}$ is the categorical homotopic distance between the two projections $p_1, p_2\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, that is,* $\mathrm{cTC}(\mathcal{C}) = \mathrm{cD}(p_1, p_2)$.

Proof. We will prove that a subcategory $\mathcal{U}$ of $\mathcal{C} \times \mathcal{C}$ is a Farber subcategory if and only if the projection functors are homotopic in $\mathcal{U}$. First, assume that there exists a functor $F\colon \mathcal{U} \to \mathcal{C}$ such that $\Delta \circ F \simeq i_{\mathcal{U}}$. Let us denote the homotopy between $\Delta \circ F$ and $i_{\mathcal{U}}$ by $H\colon \mathcal{U} \times \mathcal{I}_m \to \mathcal{C} \times \mathcal{C}$, where

$$H_0(c_1, c_2) = (\Delta \circ F)(c_1, c_2) = (F(c_1, c_2), F(c_1, c_2))$$

and $H_1(c_1, c_2) = (c_1, c_2)$. We define a homotopy $H'\colon \mathcal{U} \times I_{2m} \to \mathcal{C}$ between the projection functors as follows:

$$H'(c_1, c_2, i) = \begin{cases} p_1 \circ H(c_1, c_2, m-i) & \text{if } 0 \leq i \leq m, \\ p_2 \circ H(c_1, c_2, i-m) & \text{if } m \leq i \leq 2m. \end{cases}$$

Conversely, assume that the projection functors are homotopic in $\mathcal{U}$ through a homotopy $H'\colon \mathcal{U} \times \mathcal{I}_m \to \mathcal{C}$ where $H'_0 = p_1$ and $H'_m = p_2$ and we will prove that there exists a functor $F\colon \mathcal{U} \to \mathcal{C}$ such that $\Delta \circ F \simeq i_{\mathcal{U}}$. Define $F = p_1$. Now, the homotopy between $\Delta \circ F$ and $i_{\mathcal{U}}$ is given by $G\colon \mathcal{U} \times \mathcal{I}_m \to \mathcal{C} \times \mathcal{C}$, where $G(c_1, c_2, m) = (c_1, H'(c_1, c_2, m))$. □

7.3 Properties

We prove several elementary properties, beginning with the behaviour of the homotopic distance under compositions. Several properties of ccat and cTC can be deduced from our general results.

7.3.1 Compositions

Proposition 7.18. *Let be functors* $F, G\colon \mathcal{C} \to \mathcal{D}$ *and* $H : \mathcal{D} \to \mathcal{E}$*. Then*

$$\mathrm{cD}(H \circ F, H \circ G) \leq \mathrm{cD}(F, G).$$

Proof. Let $\mathrm{cD}(F, G) \leq n$ and let $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ be a geometric covering of $\mathcal{C}$ with $F_j = F_{|U_j}$ homotopic to $G_j = G_{|\mathcal{U}_j}$. Then

$$(H \circ F)_j = H \circ F_j \simeq H \circ G_j = (H \circ G)_j,$$

so $\mathrm{cD}(H \circ F, H \circ G) \leq n$. □

Corollary 7.19. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor. Then* $\mathrm{ccat}(F) \leq \mathrm{ccat}(\mathcal{C})$.

Proof. Take $1_\mathcal{C}$ and a constant functor c_0 from $\mathcal{C}$ to $\mathcal{C}$. Then $\mathrm{cD}(F \circ 1_\mathcal{C}, F(c_0)) \leq \mathrm{cD}(1_\mathcal{C}, c_0)$. □

Proposition 7.20. *Let be functors* $F, G\colon \mathcal{C} \to \mathcal{D}$ *and* $H : \mathcal{E} \to \mathcal{C}$*. Then*

$$\mathrm{cD}(F \circ H, G \circ H) \leq \mathrm{cD}(F, G).$$

Proof. Let $\mathrm{cD}(F, G) \leq n$ and let $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ be a geometric covering of $\mathcal{C}$ with $F_j \simeq G_j : \mathcal{U}_j \to \mathcal{D}$. Since $\mathcal{C}$ is a small category, for each $\mathcal{U} = \mathcal{U}_j$ we can define the subcategory $H^{-1}(\mathcal{U})$ where

$$\mathrm{Ob}(H^{-1}(\mathcal{U})) = \{e \in \mathrm{Ob}(\mathcal{E})\colon H(e) \in \mathrm{Ob}(\mathcal{U})\}$$

and if $e, e' \in \mathrm{Ob}(H^{-1}(\mathcal{U}))$ then

$$\mathrm{Arr}(e, e') = \{\alpha \in \mathrm{Arr}(\mathcal{E})\colon H(\alpha) \in \mathrm{Arr}_\mathcal{U}(h(e), h(e'))\}.$$

Consider the geometric covering of $\mathcal{E}$ whose elements are $V_j = H^{-1}(\mathcal{U}_j)$. The restriction $H_j\colon V_j \to \mathcal{C}$ can be written as the composition of $\bar{H}_j\colon V_j \to U_j$, where $\bar{H}_j(c) = H(c)$, and the inclusion I_j of U_j in $\mathcal{C}$. Then we have that

$$(F \circ H)_j = F_j \circ \bar{H}_j \simeq G_j \circ \bar{H}_j = G \circ I_j \circ \bar{H}_j = G \circ H_j = (G \circ H)_j,$$

hence $\mathrm{cD}(F \circ H, G \circ H) \leq n$. □

Corollary 7.21. *Given a functor* $F : \mathcal{C} \to \mathcal{D}$*, then* $\mathrm{ccat}(F) \leq \mathrm{ccat}(\mathcal{D})$.

Proof. Take $1_\mathcal{D}$ and a constant functor d_0 from $\mathcal{D}$ to $\mathcal{D}$. Then $\mathrm{cD}(1_\mathcal{D} \circ F, d_0 \circ F) \leq \mathrm{cD}(1_\mathcal{D}, y_0)$. □

Corollaries 7.19 and 7.21 can be extended to the categorical distance as follows:

Corollary 7.22. *Let* $F, G\colon \mathcal{C} \to \mathcal{D}$ *be functors. Then*

$$\mathrm{cD}(F, G) + 1 \leq (\mathrm{ccat}(F) + 1)(\mathrm{ccat}(G) + 1).$$

Proof. Denote by d_0 a constant functor from $\mathcal{C}$ to $\mathcal{D}$. Assume that $\mathrm{ccat}(F) = \mathrm{cD}(F, d_0) \leq m$, $\mathrm{ccat}(G) = \mathrm{cD}(G, d_0) \leq n$ and let $\{\mathcal{U}_0, \ldots, \mathcal{U}_m\}$, $\{\mathcal{V}_0, \ldots, \mathcal{V}_n\}$ be the corresponding geometric coverings of $\mathcal{C}$. The subcategories $W_{i,j} = \mathcal{U}_i \cap \mathcal{V}_j$ (where the intersection means the intersections of the sets of objects and intersections of the sets of arrows) form a geometric cover of $\mathcal{C}$. Moreover, $F \simeq d_0 \simeq G$ on $W_{i,j}$, so $cD(F, G) \leq m \cdot n$ - 1. The result follows. □

Corollary 7.23. $\mathrm{ccat}(\mathcal{C}) \leq \mathrm{cTC}(\mathcal{C})$.

Proof. In Proposition 7.20 consider the inclusion functors $i_1, i_2 \colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, so

$$\mathrm{cD}(*, 1_{\mathcal{C}}) = \mathrm{cD}(p_1 \circ i_2, p_2 \circ i_2) \leq \mathrm{cD}(p_1, p_2).$$

□

7.3.2 Domain and codomain

Proposition 7.24. *Assume that $F, G \colon \mathcal{C} \to \mathcal{D}$ are two functors between small categories. If at least one of the categories $\mathcal{C}$ or $\mathcal{D}$ have an initial or terminal object, then* $\mathrm{cD}(F, G) = 0$.

Proof. Recall from Proposition 6.13 that a category $\mathcal{E}$ that has an initial or terminal object, is contractible. Since any pair of functors which contractible domain or codomain are homotopic, it is $\mathrm{cD}(F, G) = 0$. □

Remark 7.25. The converse of Proposition 7.24 does not hold. Recall that a poset P, when seen as a category (see Section 7.4), has a terminal object x if and only if x is the unique maximal element of P and a dual statement applies to initial elements. Consider the poset $P = \{x, y, z, w, t\}$ with the order:

- $x \leq z, w, t$,
- $y \leq z, w, t$,
- and $z \leq w, t$.

It is contractible and therefore the distance between any pair of functors is zero. However, it has no terminal nor initial objects.

Theorem 7.26. *Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be two functors. Then*

$$\mathrm{cD}(F, G) \leq \mathrm{ccat}(C).$$

Proof. It is enough to prove that

$$\mathrm{cD}(F, G) = \mathrm{cD}(F \circ 1_{\mathcal{C}}, G \circ 1_{\mathcal{C}}) \leq \mathrm{cD}(1_{\mathcal{C}}, c_0) = \mathrm{ccat}(X).$$

Assume $\mathrm{cD}(1_{\mathcal{C}}, c_0) = n$, and let $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ be a geometric covering for $\mathcal{C}$ such that, for all j, $1_{|\mathcal{U}_j} \simeq (c_0)_{|\mathcal{U}_j}$ by a homotopy $H \colon \mathcal{U}_j \times \mathcal{I}_m \to \mathcal{C}$. Let us define the homotopy $H' \colon \mathcal{U}_j \times I_{2m} \to \mathcal{D}$ as follows:

$$H'(c, t) = \begin{cases} F \circ \mathcal{H}(c, i), & \text{if } 0 \leq i \leq m, \\ G \circ \mathcal{H}(c, 2m - i), & \text{if } m \leq i \leq 2m. \end{cases}$$

Hence, $\mathrm{cD}(F \circ 1_{\mathcal{C}}, G \circ 1_{\mathcal{C}}) \leq n$. □

What follows is the categorical version of a well known result from Farber [61].

Corollary 7.27. $\mathrm{cTC}(\mathcal{C}) \leq \mathrm{ccat}(\mathcal{C} \times \mathcal{C})$.

Proof. In Theorem 7.26 take the functors $p_1, p_2 \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Then $\mathrm{cTC}(\mathcal{C}) = \mathrm{cD}(p_1, p_2) \leq \mathrm{ccat}(\mathcal{C} \times \mathcal{C})$. □

7.3.3 Triangle Inequality

The Triangle Inequality does not hold in general in this context as Example 7.29 shows. However, it holds by adding certain restrictions:

Proposition 7.28. *Let $F, G, H \colon \mathcal{C} \to \mathcal{D}$ be functors between $\mathcal{C}$ and $\mathcal{D}$ such that* $\mathrm{ccat}(\mathcal{C}) \leq 2$. *Then*

$$\mathrm{cD}(F, H) \leq \mathrm{cD}(F, G) + \mathrm{cD}(G, H).$$

Proof. First, notice that if two of the three functors are homotopic, then the result holds automatically, so assume that there is no pair of homotopic functors among F, G and H. Since $\mathrm{cD}(F, H) \leq \mathrm{ccat}(\mathcal{C})$ (Corollary 7.26), the result follows. □

However, the Triangle inequality does not hold in general as the following example, communicated to us by Barmak, shows:

Example 7.29. Let S be the finite space corresponding to the poset depicted in Figure 7.3.1. Observe that it is a finite model of the circle.

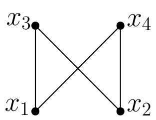


Figure 7.3.1: The finite topological space S.

Consider the finite space $X = S \times S$ and the continuous maps $f, g, h \colon X \to X$ given by $f = \mathrm{id}_X$, $g = \mathrm{id}_S \times c$ and $h = c \times c$ where $c \colon S \to S$ is a constant map. Recall that for any finite space Y and $y \in Y$, the subspace U_y is contractible. Therefore $\{S \times U_{x_1}, S \times U_{x_2}\}$ is an open cover of X such that the restrictions of f and g to each of the members of the cover are homotopic. This proves that $\mathrm{D}(f, g) \leq 1$. A symmetrical argument shows that $\mathrm{D}(g, h) \leq 1$. However $\mathrm{D}(f, h) = \mathrm{cat}(X) \geq 3$ [169, Example 3.5]. Therefore, the maps f, g and h do not satisfy the triangle inequality.

7.3.4 Invariance

We now prove the homotopy invariance of the homotopic distance.

Corollary 7.30. *1. Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be functors and let $\alpha \colon \mathcal{D} \to \mathcal{D}'$ be a functor with a left homotopy inverse. Then*

$$\mathrm{cD}(\alpha \circ F, \alpha \circ G) = \mathrm{cD}(F, G).$$

2. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors and let $\beta\colon \mathcal{C}' \to \mathcal{C}$ be a functor with a right homotopy inverse. Then*
$$\mathrm{cD}(F \circ \beta, G \circ \beta) = \mathrm{cD}(F, G).$$

Proof. We prove 1 since 2 is analogous. By Proposition 7.18,

$$\mathrm{cD}(F, G) \geq \mathrm{cD}(\alpha \circ F, \alpha \circ G) \geq \mathrm{cD}(\beta \circ \alpha \circ F, \beta \circ \alpha \circ G).$$

But $\beta\circ\alpha \simeq 1_{\mathcal{D}}$ implies $\beta\circ\alpha\circ F \simeq F$ and $\beta\circ\alpha\circ G \simeq G$, hence $\mathrm{cD}(\beta\circ\alpha\circ F, \beta\circ\alpha\circ G) = \mathrm{cD}(F, G)$ because the distance only depends on the homotopy class. □

Proposition 7.31. *Assume $\alpha\colon \mathcal{C} \to \mathcal{C}'$ and $\beta\colon \mathcal{D} \to \mathcal{D}'$ are homotopy equivalences between small categories, connecting the functors $F\colon \mathcal{C} \to \mathcal{D}$ (resp. G) and $F'\colon \mathcal{C}' \to \mathcal{D}'$ (resp. G'), that is, the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] \arrow[d, "\alpha"] & \mathcal{D} \arrow[d, "\beta"] \\
\mathcal{C}' \arrow[r, shift left, "F'"] \arrow[r, shift right, "G'"'] & \mathcal{D}'
\end{tikzcd}$$

Then $\mathrm{cD}(F, G) = \mathrm{cD}(F', G')$.

Proof. We denote the homotopic inverse of β by β'. Then from Corollary 7.30 it follows:

$$\mathrm{cD}(F, G) = \mathrm{cD}(F \circ \alpha, G \circ \alpha) = \mathrm{cD}(\beta' \circ F \circ \alpha, \beta' \circ G \circ \alpha). \quad \square$$

Corollary 7.32. $\mathrm{ccat}(\mathcal{C})$ *and* $\mathrm{cTC}(\mathcal{C})$ *are homotopy invariant.*

Note that Corollary 7.30 generalizes the homotopy invariance of ccat stated by Tanaka in [170].

7.3.5 Products

We study now the behaviour of the categorical homotopic distance under products.

Theorem 7.33. *Given $F, G\colon \mathcal{C} \to \mathcal{D}$ and $F', G'\colon \mathcal{C}' \to \mathcal{D}'$, then*

$$\mathrm{cD}(F \times F', G \times G') + 1 \leq \big(\mathrm{cD}(F, G) + 1\big) \cdot \big(\mathrm{cD}(F', G') + 1\big).$$

Proof. Given geometric coverings $\{\mathcal{U}_0, \ldots, \mathcal{U}_m\}$ and $\{\mathcal{V}_0, \ldots, \mathcal{V}_n\}$ of $\mathcal{C}$ and $\mathcal{C}'$, respectively, such that $F_{|\mathcal{U}_i} \simeq G_{|\mathcal{U}_i}$ and $F'_{|\mathcal{V}_j} \simeq G'_{|\mathcal{V}_j}$, then it can be checked that $\{\mathcal{U}_i \times \mathcal{V}_j\}$ is a geometric cover of $\mathcal{C} \times \mathcal{C}'$ such that $F \times F'_{|\mathcal{U}_i \times \mathcal{V}_j} \simeq G \times G'_{|\mathcal{U}_i \times \mathcal{V}_j}$. □

Set $F\colon \mathcal{C} \to \mathcal{C}$ and $F'\colon \mathcal{C}' \to \mathcal{C}'$ to be the identity functors and $G\colon \mathcal{C} \to \mathcal{C}$ and $G'\colon \mathcal{C}' \to \mathcal{C}'$ to be constant functors. Then, we obtain the following corollary:

Corollary 7.34. *It holds that:*

$$\mathrm{ccat}(\mathcal{C}\times\mathcal{C}')+1 \leq (\mathrm{ccat}(\mathcal{C})+1)\cdot(\mathrm{ccat}(\mathcal{C}')+1).$$

Hence, Theorem 7.33 generalizes the product inequality proved by Tanaka [170] for the categorical LS-category.

Corollary 7.35. *It holds that:*

$$\mathrm{cTC}(\mathcal{C}\times\mathcal{C}')+1 \leq (\mathrm{cTC}(\mathcal{C})+1)\cdot(\mathrm{cTC}(\mathcal{C}')+1).$$

Proof. Set $F\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and $F'\colon \mathcal{C}'\times\mathcal{C}'\to\mathcal{C}'$ to be the projection functors onto the first factor and $G\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and $G'\colon \mathcal{C}'\times\mathcal{C}'\to\mathcal{C}'$ to be the projection functors onto the second factor. □

7.3.6 Relationship between homotopic distances

Ordinary homotopic distance between continuous maps and the two notions of categorical homotopic distance that we have defined so far are related by the following result:

Proposition 7.36. *Given two functors $F,G\colon \mathcal{C}\to\mathcal{D}$, then*

$$\mathrm{D}(\mathrm{B}F,\mathrm{B}G)\leq \mathrm{wcD}(F,G)\leq \mathrm{cD}(F,G).$$

Proof. It is well known that given any subcomplex Y of a CW-complex X, there exists an open neighborhood U of Y in X such that Y is a deformation retract of U (see for example [95]). Since deformation retracts are homotopy equivalences and the homotopic distance is invariant under homotopies, by Proposition 7.5 we have

$$\mathrm{D}(\mathrm{B}F,\mathrm{B}G)\leq \mathrm{wcD}(F,G).$$

Finally, the fact that the classifying space functor preserves homotopies guarantees the inequality $\mathrm{wcD}(F,G)\leq \mathrm{cD}(F,G)$. □

Remark 7.37. The difference between the categorical homotopic distance and the weak categorical homotopic distance can be arbitrarily large, as Example 6.6 illustrates.

7.4 The context of posets

Recall that a finite poset can be seen both as a small category and as a finite topological space. In this way, order preserving maps between them can be seen both as functors and as continuous maps. Therefore, given two functors between posets $F,G\colon P\to Q$, it makes sense to study both their homotopic distance as continuous maps and their categorical homotopic distance as functors. We devote this section to the study of homotopic distance between order preserving maps.

We recall a classic result from Barmak's book ([6]). In order to state it, we recall the notion of contiguity. Two simplicial maps $\varphi,\psi\colon K\to L$ are said to be *contiguous* if for every simplex $\sigma\in$

K, $\varphi(\sigma) \cup \psi(\sigma)$ is a simplex of L. Two simplicial maps $\varphi, \psi \colon K \to L$ *lie in the same contiguity class* if there exists a sequence $\varphi = \varphi_1, \ldots, \varphi_n = \psi$ such that φ_i and φ_{i+1} are contiguous for every $0 \leq i < n$.

Lemma 7.38. *Given two homotopic continuous maps between posets $F, G \colon P \to P'$, then the simplicial maps $\mathcal{K}(F)$ and $\mathcal{K}(G)$ are in the same contiguity class. Conversely, let $\varphi, \phi \colon K \to L$ be simplicial maps which lie in the same contiguity class. Then $\chi(\varphi) \simeq \chi(\phi) \colon \chi(K) \to \chi(L)$ (are homotopic).*

7.4.1 Homotopy equivalences in posets

Stong ([164]) showed that for any given finite poset P there exists a unique subposet (up to isomorphism) $P' \subset P$, called the *core of* P, satisfying the following two conditions:

1. P' is a deformation retract of P;
2. no proper subposet of P' is a deformation retract of P.

Under these circumstances P' is called a *minimal* poset.

As a consequence of the homotopy invariance of the distance, it follows that in order to compute the (categorical) homotopic distance between functors $F, G \colon P \to Q$, where P and Q are posets, it is enough to study the (categorical) homotopic distance between the associated functors $F', G' \colon P' \to Q'$ between the cores.

Corollary 7.39. *Given two functors $F, G \colon P \to Q$ between two finite posets P, Q, let P' (respectively, Q') be the core of P (resp., of Q). Denote by $F', G' \colon P' \to Q'$ the compositions of F and G with the equivalences $P \simeq P'$ and $Q \simeq Q'$. Then:*

$$\mathrm{cD}(F, G) = \mathrm{cD}(F', G')$$

and

$$\mathrm{D}(F, G) = \mathrm{D}(F', G').$$

Therefore, from now on, we can restrict our attention to minimal spaces.

7.4.2 Coverings and bounds

We begin with a lemma which relates the notions of geometric cover and open cover of posets.

Lemma 7.40. *If P is a finite poset and $\{U_0, \ldots, U_m\}$ is an open cover of P, then $\{U_0, \ldots, U_m\}$ is also a geometric cover.*

Proof. Let $x_0 \leq \cdots \leq x_n$ be a sequence of composable arrows in P. Since there is a U_k such that $x_n \in U_k$, by definition of the open sets in P it is $U_{x_n} \subset U_k$, so $x_0, \ldots, x_n \in U_k$. □

The following results help us to implement the computation of the (categorical) homotopic distance when the domain is a finite poset by reducing the number of open covers we have to test.

Proposition 7.41. *Let P and Q be two finite posets and let $F, G\colon P \to Q$ be two functors. Then, $\mathrm{D}(F,G) = m$ if and only if there exists an open cover $\{U_0, \ldots, U_m\}$ whose elements are of the form*

$$U_i = U_{x_0} \cup \cdots \cup U_{x_{i_n}},$$

where the x_k are maximal elements (with respect to the order relation in P) and such that the restrictions of F and G to each U_i are homotopic.

Proof. Given $x_k \in P$, the basic open subset U_{x_k} is contractible (Proposition 7.24). It is clear that an open cover $\{U_i\}$ whose elements are of the form $U_i = U_{x_0} \cup \cdots \cup U_{x_{i_m}}$ where the x_k are maximal elements is a geometric cover and since the poset P is finite, so is the cover and $\mathrm{D}(F,G)$. Now, we will prove that it is enough to consider such coverings. Given an open cover $\{V_i\}_{i=0}^m$ such that $F_{|V_i} \simeq G_{|V_i}$ we will obtain a cover formed by unions of maximal basic open sets with at most $m+1$ elements. Suppose that $V_i = \{x_1, \ldots, x_k\}$. Among the elements of V_i pick the ones which are maximal elements of P. Suppose these maximal elements are $\{x_{i_0}, \ldots, x_{i_l}\}$. Then define $U_i = U_{x_{i_0}} \cup \cdots \cup U_{x_{i_l}}$. Note that $U_i \subset V_i$. It can be checked that the covering $\{U_i\}$ constructed by this procedure satisfies that $F_{|U_i} \simeq G_{|U_i}$ and it is a cover of the required form. □

The proof of the following result is similar.

Proposition 7.42. *Let P and Q be two finite posets and let $F, G\colon P \to Q$ be two functors. Then, $\mathrm{cD}(F,G) = m$ if and only if there exists an open cover $\{U_0, \ldots, U_m\}$ whose elements are of the form*

$$U_i = C_{i_0} \cup \cdots \cup C_{i_n},$$

where the C_{i_k} are maximal chains and such that the restrictions of F and G to each U_i are homotopic.

As a consequence of the previous two results, we can given an upper bound for the categorical homotopic distance.

Corollary 7.43. *Let P and Q be two finite posets and let $F, G\colon P \to Q$ be two functors. Then $\mathrm{D}(F,G)$ and $\mathrm{cD}(F,G)$ are less than or equal to the number of maximal elements of P minus one. Furthermore, $\mathrm{cD}(F,G)$ is less than or equal to the number of minimal elements minus one.*

Example 7.44. This example shows that the upper bound of Corollary 7.43 is sharp. Consider the poset P depicted in Figure 7.4.1 and the functors (order preserving maps) $1_P, F\colon P \to P$ where F is given by $F(x_1) = x_2$, $F(x_2) = x_1$, $F(x_3) = x_4$ and $F(x_4) = x_3$. The functors 1_P and F are not homotopic (see [6, Lemma 2.1.1]), so by Corollary 7.43, $\mathrm{D}(F,G) = 1$.

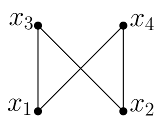


Figure 7.4.1: A finite model of $\mathbb{S}^1$.

7.4.3 Relations with other homotopic distances

In [64, 65, 68], simplicial versions of LS category and topological complexity were given by replacing the notion of homotopic continuous maps with that of contiguous simplicial maps.

In the same vein, a notion of distance between simplicial maps can be defined.

Definition 7.45. The *contiguity distance* $\mathrm{sD}(\varphi, \psi)$ between two simplicial maps $\varphi, \psi \colon K \to L$ is the least integer $n \geq 0$ such that there exists a covering of K by subcomplexes $K_0, \ldots, K_n$ such that the restrictions $\varphi_{|K_j}, \psi_{|K_j} \colon K_j \to L$ are in the same contiguity class, for all $j = 0, \ldots, n$. If there is no such covering, we define $\mathrm{sD}(f, g) = \infty$.

This notion of contiguity distance generalizes those of simplicial LS category $\mathrm{scat}(K)$ and discrete topological complexity $\mathrm{sTC}(K)$ defined in [1, 64, 65, 68, 115]:

Example 7.46. Given two simplicial complexes K and L, denote by $K \prod L$ their categorical product [105]. The contiguity distance between the projections $p_1, p_2 \colon K \prod K \to K$ equals $\mathrm{sTC}(K)$, as follows from [64, Theorem 3.4].

Example 7.47. The simplicial LS category of a simplicial map between simplicial complexes $\varphi \colon K \to L$, denoted $\mathrm{scat}(\varphi)$ [1], is the contiguity distance $\mathrm{sD}(\varphi, v_0)$ where $v_0 \colon K \to L$ is a constant simplicial map.

The following result relates the notions we have been working with so far.

Theorem 7.48. *Given order preserving maps between finite posets $F, G \colon P \to Q$, then*

$$\mathrm{D}(\mathrm{B}F, \mathrm{B}G) \leq \mathrm{wcD}(F, G) \leq \mathrm{sD}(\mathcal{K}(F), \mathcal{K}(G)) \leq \mathrm{cD}(F, G) \leq \mathrm{D}(F, G).$$

Proof. From Proposition 7.36 we already have the inequalities:

$$\mathrm{D}(\mathrm{B}F, \mathrm{B}G) \leq \mathrm{wcD}(F, G) \leq \mathrm{cD}(F, G).$$

First, we prove that:

$$\mathrm{cD}(F, G) \leq D(F, G).$$

Suppose that $D(F, G) = n$ with an open covering $\{U_0, \ldots, U_n\}$. Because of Lemma 7.40 the collection $\{U_0, \ldots, U_n\}$ is also a geometric covering.

Now we show that:

$$\mathrm{sD}(\mathcal{K}(F), \mathcal{K}(G)) \leq \mathrm{cD}(F, G).$$

Recall that a collection $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ of subcategories of a category $\mathcal{C}$, is a geometric cover if and only if the collection of subcomplexes $\{\mathrm{B}\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ covers $\mathrm{B}\mathcal{C}$ (Proposition 7.5). Now, the inequality

$$\mathrm{sD}(\mathcal{K}(F), \mathcal{K}(G)) \leq \mathrm{cD}(F, G)$$

follows from Lemma 7.38. Finally, the inequality

$$\mathrm{wcD}(F, G) \leq \mathrm{sD}(\mathcal{K}(F), \mathcal{K}(G))$$

follows from the fact that if two simplicial maps are in the same contiguity class, then their geometric realizations are homotopic ([6]). □

As a consequence of Theorem 7.48 we obtain several results relating existing notions for LS categories:

Corollary 7.49. *Given a poset P, we have*

$$\mathrm{cat}(|\mathcal{K}(P)|) \leq \mathrm{wccat}(P) \leq \mathrm{scat}(\mathcal{K}(P)) \leq \mathrm{ccat}(P) \leq \mathrm{cat}(P),$$

where $|\mathcal{K}(P)|$ denotes the geometric realization of the simplicial complex $\mathcal{K}(P)$.

Therefore, Theorem 7.48 generalizes the results of Tanaka ([170]) regarding the categorical LS-category.

7.4.4 Subdivisions

We recall that given a poset P, its barycentric subdivision can be defined as $\mathrm{sd}(P) = (\chi \circ \mathcal{K})(P)$ (see [6]). Moreover, this construction is functorial. We have seen that $\mathrm{cD}(F, G) \leq \mathrm{D}(F, G)$ (Theorem 7.48). By subdividing the domain we can reverse this inequality.

Lemma 7.50. *Given two order preserving maps $F, G\colon P \to Q$ between finite posets, we have*

$$\mathrm{D}(\mathrm{sd}(F), \mathrm{sd}(G)) \leq \mathrm{cD}(F, G).$$

Proof. Assume that $\mathrm{cD}(F, G) = n$ with the geometric cover $\{\mathcal{U}_i\}$. Note that $\mathcal{K}(U_i)$ is a subcomplex of $\mathcal{K}(P)$ and therefore $\mathrm{sd}(U_i) = \chi \circ \mathcal{K}(U_i)$ is an open subset of $\mathrm{sd}(P)$. Moreover, $\{\mathrm{sd}(\mathcal{U}_i)\}$ is a cover of $\mathrm{sd}(P)$. Finally, since $F_{|\mathcal{U}_i} \simeq G_{|\mathcal{U}_i}$, from Lemma 7.38 follows that $F_{|\mathrm{sd}(\mathcal{U}_i)} \simeq G_{|\mathrm{sd}(\mathcal{U}_i)}$. As a consequence, $\mathrm{D}(\mathrm{sd}(F), \mathrm{sd}(G)) \leq \mathrm{cD}(F, G)$. □

Moreover, the inequality becomes an equality after enough subdivisions:

Proposition 7.51. *Given two order preserving maps $F, G\colon P \to Q$ between finite posets, there exists a natural number k such that the k-iterated barycentric subdivision stabilizes the distances, that is,*

$$\mathrm{D}(\mathrm{sd}^k(F), \mathrm{sd}^k(G)) = \mathrm{cD}(\mathrm{sd}^k(F), \mathrm{sd}^k(G)).$$

Proof. From Theorem 7.48 and Lemma 7.50 it follows that:

$$\mathrm{cD}(\mathrm{sd}(F), \mathrm{sd}(G)) \leq \mathrm{D}(\mathrm{sd}(F), \mathrm{sd}(G)) \leq \mathrm{cD}(F, G).$$

Therefore,

$$\lim_{k\to\infty} \mathrm{D}(\mathrm{sd}^k(F), \mathrm{sd}^k(G)) = \lim_{k\to\infty} \mathrm{cD}(\mathrm{sd}^k(F), \mathrm{sd}^k(G)). \quad \square$$

Observe that both Lemma 7.50 and Proposition 7.51 generalize the corresponding results in the context of posets by Tanaka for the categorical LS-category ([170]).

Remark 7.52. Section 7.4 could be generalized both to the context of preordered sets and to acyclic categories and most results would hold.

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