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TESE DE DOUTORAMENTO

Homogeneous hypersurfaces and totally geodesic submanifolds

Alberto Rodríguez Vázquez

ESCOLA DE DOUTORAMENTO INTERNACIONAL

PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

SANTIAGO DE COMPOSTELA

ANO 2022

A presente tese foi dirixida por José Carlos Díaz Ramos e Miguel Domínguez Vázquez. Defendida na Universidade de Santiago de Compostela o 30 de setembro de 2022.

Os resultados presentados nesta memoria foron obtidos durante o desfrute dunha axuda FPU17/01030 para a formación do profesorado universitario do Ministerio de Ciencia, Innovación y Universidades e grazas en parte ao financiamento do Ministerio de Ciencia e Innovación del Gobierno de España e a Agencia Estatal de Investigación (incluído cofinanciamento do FEDER) a través dos proxectos MTM2016-75897-P e PID2019-105138GB-C21 e da Consellería de Cultura, Educación e Ordenación Universitaria da Xunta de Galicia, na modalidade de Grupo de Referencia Competitiva con referencia ED431C 2019/10 e Grupo de Excelencia ED431F 2020/04.

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Á memoria da miña avoa

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Agradecementos

Tras case catro anos de traballo duro conseguín rematar esta tese. Porén, se de min dependese, repetiría unha cantidade non numerable de veces este camiño que comecei en setembro de 2018 e que me permitiu coñecer a moitas das persoas que nomearei a continuación.

Por unha banda, gustaríame dar as grazas ós meus directores Carlos e Miguel polo seu tempo, paciencia e o saber facer que me teñen transmitido. Quero extender este agradecemento a todo o grupo de investigación en Matemáticas e, en particular, a Eduardo e a Elena polo seu apoio constante. Síntome especialmente agradecido de ter tido a uns grandes irmáns académicos: Cris, Víctor, Olga, Tomás e Juanma.

Por outra banda, penso que non sería xusto obviar ás persoas das que tanto aprendín nas viaxes polo mundo que realicei durante este período. Refírome a Paco López (UGR), Jürgen Berdnt (KCL), Andreas Kollross (U. Stuttgart) e Carlos Olmos (UNC). Aproveito, tamén, para agradecer a aquelas persoas que desinteresadamente me teñen transmitido coñecementos matemáticos que me permitiron chegar a este punto. Falo, por exemplo, de Alexandrino, Bettiol, De Graaf, González-Álvaro, Gorodski, Klein, Radeschi, Ziller e moitos outros que non collen nas estreitas marxes desta folla.

Xa nun eido máis persoal, quero expresar o meu agradecemento a todas esas persoas que me teñen amosado o seu apoio e coas que teño pasado grandes ratos na Facultade de Matemáticas en Compostela. En particular, e co risco de estarme olvidando de moita xente (que por favor non se enfaden comigo); quero amosar a miña gratitude a: Brais, Dani, David, Jorge Losada, Jorge Albella, Pilar, Óscar, Sebas.

Ademais, quero agradecer especialmente á miña familia e sobre todo ós meus pais; César e Fernanda, por sementar en min a curiosidade, que é o motor principal desta tese. Por último, quero agradecer a Nerea, a miña compañeira de vida; por estar sempre nas boas e nas malas. A súa fe en min nunca vacilou, mesmo nos momentos máis difíciles.

Introdución

A simetría, independemente da amplitude ou limitacións da definición que consideremos, é unha idea a través da cal o home, ao longo dos séculos, tentou comprender e crear orde, beleza e perfección.

Este comentario débese a Hermann Weyl. As súas palabras revelan que a simetría reside no núcleo do coñecemento humano. De feito, se cadra, o campo máis natural para estudar a simetría sexa a xeometría. Felix Klein describiu a xeometría como o estudo das propiedades dun espazo que son invariantes baixo un grupo de transformacións. Dende o punto de vista da xeometría riemanniana, o grupo natural a estudar é o grupo de isometrías. Ademais, a maioría dos obxectos xeométricos que podemos percibir mediante os nosos sentidos pódense describir en termos de curvas e superficies. As subvariedades proporcionan a xeneralización natural destes obxectos a dimensións superiores.

Esta tese de doutoramento trata sobre o estudo de certas clases de subvariedades en presenza de simetría. En concreto, obtívose resultados relativos á teoría de subvariedades en espazos homoxéneos riemannianos con especial énfase nos espazos simétricos. Nesta disertación centrarémonos en dúas das clases máis naturais de subvariedades que poden ser estudadas nas variedades de Riemann. Trátase das hipersuperficies homoxéneas e das subvariedades totalmente xeodésicas. Respecto das primeiras, concluirase a clasificación das hipersuperficies homoxéneas en espazos hiperbólicos cuaterniónicos completando a clasificación das hipersuperficies homoxéneas en espazos simétricos de rango un. En canto ás subvariedades totalmente xeodésicas, derivaremos diferentes clasificacións. En particular, clasificaremos as subvariedades totalmente xeodésicas nos seguintes espazos: en produtos de espazos simétricos de rango un, en espazos simétricos excepcionais e en esferas de Hopf-Berger.

En liñas xerais, un espazo homoxéneo é aquel que presenta o mesmo aspecto en cada punto. Por este motivo, os espazos homoxéneos serven como modelos para moitos tipos diferentes de estruturas xeométricas. En concreto, interésannos aqueles espazos homoxéneos que resultan de considerar accións isométricas, é dicir, accións dun grupo de Lie que preserva a métrica do espazo. Os espazos simétricos constitúen unha clase especial de espazos homoxéneos. Estes aparecen nunha ampla variedade de situacións tanto en Matemáticas como na Física. Un espazo simétrico é unha variedade de Riemann cuxo grupo de isometrías contén unha simetría involutiva en cada punto. Isto implica que estes espazos admiten unha boa descrición en termos de grupos de Lie, e que podemos utilizar ferramentas alxébricas para comprender máis a fondo a súa xeometría. Os espazos simétricos foron clasificados por Élie Cartan na década de 1920 e algúns exemplos son: os espazos euclidianos, a esfera redonda, o espazo hiperbólico, as Grassmannianas, o conxunto de estruturas complexas ortogonais dun espazo vectorial, o conxunto dos produtos escalares dun espazo vectorial, o conxunto de subespazos lagrangianos dun espazo vectorial simpléctico ou os grupos de Lie compactos.

Probablemente, o invariante máis importante nun espazo simétrico é o rango. O rango é a maior dimensión dunha subvariedade propia, chá e totalmente xeodésica. Os espazos simétricos de rango un, xunto cos espazos euclidianos forman, agás cocientes, unha familia privilexiada dentro da xeometría riemanniana, os chamados espazos 2-punto homoxéneos, véxase [168]. Estes defínense como aquelas variedades de Riemann M tales que para cada dous pares de puntos (p_1, p_2) e (q_1, q_2) que satisfán $d(p_1, p_2) = d(q_1, q_2)$, existe unha isometría φ de M tal que $\varphi(p_i) = q_i$ para cada $i \in \{1, 2\}$. Nesta tese, os espazos simétricos de rango un xogarán un papel fundamental.

A continuación, resumimos as contribucións orixinais desta tese, xunto co estado da cuestión dos problemas matemáticos que motivaron as nosas investigacións.

Unha hipersuperficie non isoparamétrica con curvaturas principais constantes. Dúas clases interesantes de hipersuperficies que podemos considerar nunha variedade de Riemann son as hipersuperficies isoparamétricas e as hipersuperficies con curvaturas principais constantes. É ben coñecido [44] que unha hipersuperficie nun espazo forma real é isoparamétrica se e só se ten curvaturas principais constantes. Porén, isto non é certo noutros espazos simétricos. Por exemplo, hai hipersuperficies nos espazo hiperbólicos complexos que son isoparamétricas pero non teñen curvaturas principais constantes [60]. Non obstante, descoñecemos se existe unha hipersuperficie nun espazo simétrico que teña curvaturas principais constantes pero que non sexa isoparamétrica. De feito, un exemplo destas características nin sequera era coñecido no caso xeral das variedades de Riemann. Ademais, a construción dunha hipersuperficie minimal, non isoparamétrica e con curvaturas principais constantes no espazo proxectivo complexo, serviría para construír un contraexemplo (véxase [83]) para a conxectura de Chern, que leva moitos anos aberta, sobre hipersuperficies isoparamétricas en esferas redondas. Esta afirma que unha hipersuperficie pechada minimal e con curvatura escalar constante nunha esfera redonda é isoparamétrica. Nesta tese construímos un exemplo explícito dunha métrica conformemente chá en $\mathbb{R}^n$ que admite unha hipersuperficie totalmente xeodésica (en particular minimal e con curvaturas principais constantes) que non é isoparamétrica. Isto proporciona o primeiro exemplo dunha hipersuperficie non isoparamétrica con curvaturas principais constantes nunha variedade de Riemann. Ademais, amosa que a equivalencia entre isoparametricidade e constancia das curvaturas principais en espazos de curvatura constante non é certa no contexto máis xeral dos espazos conformemente cháns. A idea principal para esta construción foi definir unha métrica conformemente chá en $\mathbb{R}^n$ que admita un hiperplano totalmente xeodésico, pero cun grupo de isometrías suficientemente pequeno para estragar o bo comportamento das hipersuperficies paralelas a tal hiperplano.

Accións de cohomoxeneidade un en espazos simétricos de rango un. Unha acción de cohomoxeneidade un nunha variedade de Riemann M é unha acción isométrica con órbitas principais de codimensión un. As órbitas principais de tal acción chámanse hipersuperficies homoxéneas. O problema de clasificar as accións de cohomoxeneidade un nun espazo dado é un problema clásico na xeometría de subvariedades que se remonta aos tempos de Beniamino Segre [161] e Élie Cartan [44], que clasificaron as accións de cohomoxeneidade un nos espazos euclidianos e nos espazos hiperbólicos reais. Un tempo máis tarde, Kollross clasificou as accións de cohomoxeneidade un en espazos simétricos irreducibles de tipo compacto [113]. Despois deste traballo, Berndt e Tamaru iniciaron un programa para estudar as accións de cohomoxeneidade un en espazos simétricos de tipo non compacto [26, 28, 29]. Usando as ideas desenvolvidas nestes artigos, Berndt e Tamaru [28] foron capaces de clasificar as accións de cohomoxeneidade un nos espazos simétricos de tipo non compacto agás nos espazos hiperbólicos cuaterniónicos.

Vinte anos despois de que Berndt e Brück anunciasen os primeiros exemplos non-triviais de accións de cohomoxeneidade un en espazos hiperbólicos cuaterniónicos en [13], obtivemos a clasificación completa das accións de cohomoxeneidade un nos espazos hiperbólicos cuaterniónicos salvo equivalencia de órbitas. Ademais, como froito do noso estudo, atopamos unha cantidade non numerable de familias isoparamétricas inhomoxéneas de hipersuperficies con curvaturas principais constantes. Estas familias isoparamétricas constitúen as únicas coñecidas en variedades riemannianas, ademais dos famosos exemplos dados por Ferus, Karcher e Münzner en esferas redondas [78] e un exemplo no plano hiperbólico de Cayley [60]. A clasificación de accións de cohomoxeneidade un en espazos hiperbólicos cuaterniónicos redúcese a un problema de álxebra lineal cuaterniónica moi complicado. Este consiste en clasificar subespazos reais dun espazo vectorial euclidiano cuaterniónico $\mathbb{H}^n$ para os que existe un subgrupo de $\mathsf{Sp}_1\mathsf{Sp}_n$ que actúa transitivamente sobre as súas esferas unitarias. Chamamos a estes subespazos protohomoxéneos. En particular, os subespazos protohomoxéneos están intimamente relacionados coa noción de ángulo de Kähler cuaterniónico, que é a xeneralización do concepto de ángulo de Kähler estudado nalgúns traballos recentes (véxase por exemplo [60]).

A idea principal para resolver o problema mencionado anteriormente é clasificar os subespazos con ángulo de Kähler cuaterniónico constante de dimensión menor ou igual a catro e despois construír cada subespazo protohomoxéneo a partir destes. Os ingredientes fundamentais para demostrar isto son certas ferramentas topolóxicas e argumentos usando a teoría de grupos de Lie. Ademais, en cada subespazo protohomoxéneo de dimensión maior que catro, podemos construír unha estrutura de Clifford. Hai dúas clases inequivalentes de Cl_3-módulos irreducibles, e mesturándoas podemos producir subespazos non protohomoxéneos con ángulo de Kähler constante cuxa dimensión é un múltiplo de catro. Estes inducen hipersuperficies inhomoxéneas, isoparamétricas e con curvaturas principales constantes nos espazos hiperbólicos sobre os cuaternios.

Subvariedades totalmente xeodésicas en produtos de espazos simétricos de rango un. O problema de clasificar subvariedades totalmente xeodésicas en espazos simétricos ten sido un tema destacado de investigación en xeometría de subvariedades durante as últimas décadas. Este foi iniciado por Wolf [187] nos anos sesenta, cando clasificou estes obxectos en espazos simétricos de rango un. No caso de rango dous, este problema foi abordado por Chen, Nagano [48, 49] e Klein [107, 108, 109]. De feito, a día de hoxe só temos clasificacións completas en espazos simétricos de rango inferior ou igual a dous. Calquera subvariedade totalmente xeodésica dun espazo simétrico é en si mesma un espazo simétrico. Mesmo nun espazo simétrico irreducible poden existir subvariedades totalmente xeodésicas reducibles. Así, para obter unha clasificación completa das subvariedades totalmente xeodésicas nun espazo simétrico irreducible dado, é necesario dispoñer dunha boa comprensión das subvariedades totalmente xeodésicas dos espazos simétricos reducibles. Nesta tese estendemos o resultado de Wolf aos produtos de espazos simétricos de rango un. Veremos que as subvariedades totalmente xeodésicas de produtos de espazos simétricos de rango un admiten unha boa descrición de natureza combinatoria. En primeiro lugar, introducimos unha pequena modificación dos taboleiros de Young, que chamamos taboleiros de Young adaptados (consúltese a sección §6.2 para a definición), que serán útiles para clasificar as subvariedades totalmente xeodésicas en produtos arbitrarios de espazos simétricos de rango un e para determinar o seu tipo de isometría. En particular, probamos un resultado que dá unha correspondencia entre estes taboleiros de Young adaptados e as subvariedades totalmente xeodésicas semisimples en produtos de espazos simétricos de rango un. Ademais disto, construímos infinitos exemplos de subvariedades totalmente xeodésicas irreducibles en espazos simétricos hermitianos que non son nin totalmente reais nin complexas. Este fenómeno, que se diferencia do que ocorre no caso de rango un, xa fora observado por Klein, quen atopou dúas subvariedades totalmente xeodésicas irreducibles que non eran nin totalmente reais nin complexas en espazos simétricos hermitianos de rango dous. Ambos os dous exemplos atopados por Klein teñen ángulo de Kähler constante igual a $\arccos(1/5)$. Neste aspecto, podemos demostrar que todo número racional en $[0, 1]$ pode ser realizado como o coseno do ángulo de Kähler dalgunha subvariedade totalmente xeodésica en certo espazo simétrico hermitiano.

Subvariedades totalmente xeodésicas en espazos simétricos excepcionais. O seguinte obxectivo desta tese é establecer unha nova estratexia para clasificar as subvariedades totalmente xeodésicas maximais en espazos simétricos irreducibles de rango superior a dous. Para espazos simétricos excepcionais, presentamos unha idea que reduce o problema a algunhas clasificacións coñecidas de subálxebras redutivas de álxebras de Lie simples reais, así como a considerar un pequeno número de casos nos que este método non dá unha resposta completa sobre a maximalidade. Este é precisamente o contido do Teorema da Correspondencia, véxase o Teorema 7.3.3. Grazas a este novo enfoque podemos clasificar as subvariedades totalmente xeodésicas en espazos simétricos excepcionais. Ademais, inspirados polos traballos de Dynkin [74, 73], introducimos un novo invariante para subvariedades totalmente xeodésicas en espazos simétricos, que chamamos índice de Dynkin. Demostramos que o índice de Dynkin determina se dous mergullos totalmente xeodésicos son isométricos. Ademais, demostramos un resultado relacionado coa conxectura do índice establecida e demostrada por Berndt e Olmos (véxase [20, 21, 22, 23, 24]). O índice $i(M)$ dun espazo simétrico irreducible M é a codimensión máis pequena dunha subvariedade totalmente xeodésica propia. A conxectura do índice pódese enunciar do seguinte xeito: todo espazo simétrico irreducible $M \neq \mathsf{G}_2^2/\mathsf{SO}_4, \mathsf{G}_2/\mathsf{SO}_4$ ten unha subvariedade Σ reflectiva totalmente xeodésica tal que $i(M) = \operatorname{codim}(\Sigma)$. Demostramos unha afirmación análoga que nos permite incluír os casos $M = \mathsf{G}_2^2/\mathsf{SO}_4$ e $M = \mathsf{G}_2/\mathsf{SO}_4$. O noso resultado afirma que todo espazo simétrico irreducible M ten unha subvariedade totalmente xeodésica Σ con $i(M) = \operatorname{codim}(\Sigma)$ tal que cada factor irreducible de Σ ten índice de Dynkin un.

Subvariedades totalmente xeodésicas en esferas de Hopf-Berger. Motivados polos nosos resultados en espazos simétricos, iniciamos o estudo das subvariedades totalmente xeodésicas na clase dos espazos homoxéneos con curvatura positiva. É un feito recorrente que, cando se intenta clasificar unha determinada clase de subvariedades, o problema adoita ser máis factible cando se ten asegurada a homoxeneidade extrínseca de tal clase de subvariedades. Ademais, moitos resultados que demostran a homoxeneidade dunha clase de subvariedades teñen sido de gran relevancia, véxase [88, 176]. É ben sabido que as subvariedades totalmente xeodésicas nun espazo homoxéneo son intrinsecamente homoxéneas. Non obstante, non son necesariamente órbitas de subgrupos do grupo de isometrías do espazo homoxéneo ambiente. A familia de espazos homoxéneos de curvatura positiva consta de varios espazos difeomorfos a esferas e espazos proxectivos complexos, outros espazos simétricos de tipo compacto de rango un equipados coas súas métricas simétricas estándar, e algúns outros exemplos esporádicos, véxase [186]. Aínda que as subvariedades totalmente xeodésicas nestes espazos foron amplamente utilizadas para derivar certas propiedades sobre a curvatura destes espazos, véxase [157], carecemos dunha clasificación completa das mesmas. O interese deste problema reside no feito de que unha subvariedade totalmente xeodésica (de dimensión $d \geq 2$) dun espazo homoxéneo de curvatura positiva volve ser un espazo homoxéneo de curvatura positiva.

Nesta tese clasificamos as subvariedades totalmente xeodésicas nunha clase importante de espazos homoxéneos difeomorfos a esferas, en concreto, en boa parte das esferas de Hopf-Berger, que constitúen a familia de esferas homoxéneas obtida reescalando a métrica redonda da espazo total dunha fibración de Hopf na dirección das fibras. Como se mencionou anteriormente, non podemos facer uso da homoxeneidade extrínseca das subvariedades totalmente xeodésicas en espazos homoxéneos, como si ocorre no caso simétrico. Así, o noso estudo precisa dun enfoque moi xeométrico que combina ideas procedentes da teoría xeral de espazos homoxéneos riemannianos, campos de Killing ou xeodésicas pechadas.

Estrutura da tese. Esta tese organízase en dúas partes, ademais dun primeiro capítulo de preliminares. Neste primeiro capítulo introducimos os feitos básicos e a terminoloxía que será utilizada ao longo da tese. Máis precisamente, lembramos algúns feitos coñecidos sobre xeometría de Riemann, sobre a teoría de subvariedades na sección §1.1 e sobre accións isométricas na sección §1.2. A sección §1.3 está dedicada a introducir os espazos ambiente dos problemas que abordaremos nesta tese, a saber, os espazos simétricos e espazos homoxéneos. Finalmente, a sección §1.4 trata dous conceptos básicos que teñen grande relevancia para esta tese: as álxebras de Clifford e as álxebras de Heisenberg xeneralizadas.

A primeira parte desta tese versa sobre o estudo das hipersuperficies homoxéneas, as hipersuperficies isoparamétricas e as hipersuperficies con curvaturas principais constantes.

No capítulo 2, lembramos as nocións de hipersuperficie homoxénea, hipersuperficie isoparamétrica e hipersuperficie con curvaturas principais constantes. Na sección §2.1 explicamos a relación entre estes tipos de hipersuperficies en espazos simétricos de rango un. En particular, nas Figuras 2.2 a 2.6 explicitamos as relacións coñecidas entre estes tres conceptos en espazos simétricos de rango un. Ademais, nas Táboas 2.1 e 2.2 resumimos o estado do actual do problema de clasificación de hipersuperficies homoxéneas, con curvaturas principais constantes ou isoparamétricas en espazos simétricos de rango un. Máis tarde, na sección §2.3 construímos un exemplo de hipersuperficie non isoparamétrica con curvaturas principais constantes nun espazo conformemente chan.

No capítulo 3 ofrecemos un resumo dos principais resultados coñecidos relativos á clasificación de hipersuperficies homoxéneas en espazos simétricos de rango un. Este capítulo comeza motivando o estudo das accións de cohomoxeneidade un en variedades de Riemann. Este tipo de accións está intimamente ligado co estudo das hipersuperficies homoxéneas en variedades de Riemann xa que unha hipersuperficie homoxénea dunha variedade de Riemann M é precisamente unha órbita principal dunha acción de cohomoxeneidade un en M. Seguidamente na sección §3.2 repasamos a clasificación de hipersuperficies homoxéneas en espazos simétricos de tipo compacto, dedicando especial atención ao caso de rango un. Na sección §3.3 introducimos a noción de ángulo de Kähler e outras xeneralizacións desta. A sección §3.4 está consagrada a explicar a teoría das accións de cohomoxeneidade un en espazos simétricos de tipo non compacto de rango un, mentres que na sección §3.5 explicamos brevemente o programa desenvolvido por Berndt e Tamaru para clasificar as accións de cohomoxeneidade un en espazos simétricos de tipo non compacto e rango arbitrario.

No capítulo 4 clasificamos as hipersuperficies homoxéneas en espazos hiperbólicos cuaterniónicos $\mathbb{H}\mathsf{H}^{n+1}$, $n \geq 1$. Neste capítulo hai dúas nocións fundamentais: os subespazos protohomoxéneos e os subespazos con ángulo de Kähler cuaterniónico constante. Estes defínense na sección §4.1. O obxectivo principal deste capítulo é a clasificación dos subespazos protohomoxéneos e o estudo dos subespazos con ángulo de Kähler cuaterniónico constante (seccións §4.2,4.3,4.4,4.5). Como subproduto do noso estudo construímos na sección §4.5 unha cantidade non numerable de familias de hipersuperficies inhomoxéneas que son isoparamétricas e teñen curvaturas principais constantes en $\mathbb{H}\mathsf{H}^{n+1}$, $n \geq 7$.

A segunda parte desta tese versa sobre o estudo de subvariedades totalmente xeodésicas en espazos simétricos e homoxéneos.

No capítulo 5 expoñemos algúns feitos coñecidos sobre subvariedades totalmente xeodésicas. Comezamos este capítulo motivando o estudo desta clase de subvariedades. Posteriormente, ofrecemos unha breve introdución á teoría de subvariedades totalmente xeodésicas en variedades de Riemann (sección §5.1) e espazos simétricos (sección §5.3). Ademais, incluímos unha demostración do feito de que, baixo certas hipóteses pouco restritivas, sempre se pode estender unha subvariedade totalmente xeodésica a unha completa (sección §5.2).

No capítulo 6 clasificamos as subvariedades totalmente xeodésicas en produtos de espazos simétricos de rango un establecendo unha correspondencia entre subvariedades totalmente xeodésicas e taboleiros de Young adaptados. Estes taboleiros de Young adaptados defínense na sección §6.2. Ademais, na sección §6.3, construímos infinitas subvariedades totalmente xeodésicas con ángulo de Kähler constante non trivial nas Grassmanianas complexas. Como consecuencia, demostramos que o conxunto de todos os posibles ángulos de Kähler de subvariedades totalmente xeodésicas dun espazo simétrico hermitiano irreducible é un subconxunto denso do intervalo $[0, \pi/2]$.

No capítulo 7 clasificamos as subvariedades totalmente xeodésicas maximais en espazos simétricos excepcionais. Este resultado baséase fortemente na definición dun novo invariante chamado índice de Dynkin (sección §7.4) e o teorema de Karpelevich sección §7.1). En particular, na sección §7.3 tamén demostramos un teorema de correspondencia que establece unha relación un a un entre subvariedades semisimples totalmente xeodésicas maximais en espazos simétricos e unha determinada clase de subálxebras (Teorema 7.3.3). Na sección §4.6, incluímos as demostracións dos teoremas principais.

No capítulo 8 investigamos as subvariedades totalmente xeodésicas en esferas de Hopf-Berger. A idea é caracterizar unha certa clase de subvariedades totalmente xeodésicas destes espazos que chamamos ben posicionadas (subsección §8.4.1). Entón, demostramos que cada subvariedade totalmente xeodésica de dimensión maior ca un é ben posicionada. Ademais, como subproduto do noso estudo, calculamos o índice de simetría destas esferas homoxéneas (sección §8.5).

Introduction

Symmetry, as wide or narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

This comment is due to Hermann Weyl and reveals that symmetry lies in the very core of human knowledge. Perhaps, the most natural field to study symmetry is geometry. Felix Klein described geometry as the study of those properties of a space that are invariant under a transformation group. From the viewpoint of Riemannian geometry, the natural group to study is the isometry group. Moreover, most geometric objects that we can perceive by means of our senses can be described in terms of curves and surfaces. Submanifolds provide the natural generalization of these objects to higher dimensions.

This Ph.D. thesis deals with the study of certain classes of submanifolds in the presence of symmetry. In particular, we have derived results concerning submanifolds in Riemannian homogeneous spaces with a focus on symmetric spaces.

Roughly speaking, a homogeneous space is one that looks the same at every point. For this reason, homogeneous spaces serve as model spaces for many different types of geometric structures. Specifically, we are interested in homogeneous spaces that result from isometric actions, that is, from Lie group actions that preserve the metric.

Symmetric spaces constitute a special class of homogeneous spaces. They occur in a wide variety of situations in both Mathematics and Physics. A symmetric space is a Riemannian manifold whose group of isometries contains an inversion symmetry at each point. This implies that these spaces admit a nice description in terms of Lie groups, and that we can use algebraic tools to get a deeper understanding of their geometry. Symmetric spaces were classified by Élie Cartan in the 1920s and some examples are: the Euclidean spaces, the round spheres, the hyperbolic spaces, Grassmannians, the set of orthogonal complex structures of a vector space, the set of inner products of a vector space, the set of Lagrangian subspaces of a symplectic vector space, or compact Lie groups.

Probably, the most important invariant in a symmetric space is the rank. The rank is the greatest dimension of a proper, flat, totally geodesic submanifold. Symmetric spaces of rank one together with Euclidean spaces form, up to quotients, a privileged family within Riemannian geometry, the so-called 2-point homogeneous spaces, see [168]. These are defined as those Riemannian manifolds M such that for every two pairs of points (p_1, p_2) and (q_1, q_2) satisfying $d(p_1, p_2) = d(q_1, q_2)$, there is an isometry φ of M such that $\varphi(p_i) = q_i$ for each $i \in \{1, 2\}$. In this thesis, symmetric spaces of rank one will play a fundamental role.

In what follows, we summarize the original contributions of this thesis, along with the state-of-the-art of the mathematical problems that motivated our investigations.

A non-isoparametric hypersurface with constant principal curvatures. Two interesting classes of hypersurfaces of Riemannian manifolds are isoparametric hypersurfaces and hypersurfaces with constant principal curvatures. It is known [44] that a hypersurface in a real space form is isoparametric if and only if it has constant principal curvatures. This is no longer true for other symmetric spaces. For example, there are hypersurfaces in complex hyperbolic spaces that are isoparametric but do not have constant principal curvatures [60]. However, it is not known if there exists a hypersurface of a symmetric space with constant principal curvatures that is not isoparametric. This was not even known for the general setting of Riemannian manifolds. Moreover, the construction of a minimal, non-isoparametric closed hypersurface with constant principal curvatures in the complex projective space would yield a counterexample (see [83]) for the longstanding Chern conjecture on isoparametric hypersurfaces in spheres, which asserts that a minimal closed hypersurface with constant scalar curvature in a round sphere must be isoparametric.

In this thesis we construct an explicit example of a conformally flat metric in $\mathbb{R}^n$ that admits a totally geodesic hypersurface (in particular minimal and with constant principal curvatures) that is not isoparametric. This provides the first example of a non-isoparametric hypersurface with constant principal curvatures in a Riemannian manifold, and it shows that the equivalence between isoparametricity and constancy of the principal curvatures in spaces of constant curvature does not hold in the more general setting of conformally flat spaces. The main idea for this construction was to define a conformally flat metric in $\mathbb{R}^n$ admitting a totally geodesic hyperplane, but with a very small isometry group that spoils the good behavior of the parallel hypersurfaces to such hyperplane.

Cohomogeneity one actions on rank one symmetric spaces. A cohomogeneity one action on a Riemannian manifold M is an isometric action with codimension one principal orbits. The principal orbits of such an action are homogeneous hypersurfaces. The problem of classifying cohomogeneity one actions on a given space is a classical problem in submanifold geometry that traces back to the times of Beniamino Segre [161] and Élie Cartan [44], who classified cohomogeneity one actions on Euclidean and real hyperbolic spaces, respectively. Much later, Kollross classified cohomogeneity one actions on irreducible symmetric spaces of compact type [113]. After this work, Berndt and Tamaru started a program to study cohomogeneity one actions on symmetric spaces of non-compact type [26, 28, 29]. Using the ideas developed in these articles, Berndt and Tamaru [28] were able to classify cohomogeneity one actions on every symmetric space of rank one except on quaternionic hyperbolic spaces.

Twenty years after Berndt and Brück announced the first non-trivial examples of cohomogeneity one actions on quaternionic hyperbolic spaces in [13], we have obtained the full classification of cohomogeneity one actions on quaternionic hyperbolic spaces up to orbit equivalence. Moreover, as a by-product of our proof, we found an uncountable number of examples of inhomogeneous isoparametric families of hypersurfaces with constant principal curvatures. These isoparametric families constitute the only such examples known in Riemannian manifolds, apart from the celebrated Ferus, Karcher, and Münzner hypersurfaces in spheres [78] and one example in the Cayley hyperbolic plane [60].

The classification of cohomogeneity one actions on quaternionic hyperbolic spaces was reduced to a very involved quaternionic linear algebra problem. This one boils down to classifying real subspaces of a quaternionic Euclidean vector space $\mathbb{H}^n$ such that there exists a subgroup of $\mathsf{Sp}_1\mathsf{Sp}_n$ acting transitively on their unit spheres. We call these spaces protohomogeneous subspaces. In particular, protohomogeneous subspaces are intimately related to the notion of quaternionic Kähler angle, which is a generalization of the concept of Kähler angle studied in some recent works (see for instance [60]).

The main idea to solve the problem mentioned above is to classify subspaces with constant quaternionic Kähler angle of dimension less than or equal to four and then build every protohomogeneous subspace out of these. The key ingredients to prove this are certain topological and Lie group theoretic tools. Moreover, for each protohomogeneous subspace of dimension greater than four, one can can construct a Clifford structure on it. There are two inequivalent classes of irreducible Cl_3-modules, and by mixing them we can produce non-protohomogeneous subspaces with constant quaternionic Kähler angle whose dimension is a multiple of four. These induce inhomogeneous isoparametric hypersurfaces with constant principal curvatures in quaternionic hyperbolic spaces.

Totally geodesic submanifolds in products of rank one symmetric spaces. The problem of classifying totally geodesic submanifolds in symmetric spaces has been an outstanding topic of research in submanifold geometry over the last decades. This was started by Wolf [187] in the sixties, when he classified these objects in symmetric spaces of rank one. For rank two, this problem has been addressed by Chen, Nagano [48, 49] and Klein [107, 108, 109]. Indeed, up to date, we only have complete classifications in symmetric spaces of rank less or equal than two.

Any totally geodesic submanifold of a symmetric space is itself a symmetric space. Even on an irreducible symmetric space, there can exist reducible totally geodesic submanifolds. Thus, in order to have a complete classification of totally geodesic submanifolds in a given irreducible symmetric space it is necessary to have a good understanding of totally geodesic submanifolds of reducible symmetric spaces.

We extend Wolf's result to products of rank one symmetric spaces. We will see that the totally geodesic submanifolds of products of rank one symmetric spaces admit a nice combinatorial description. Firstly, we introduce some slight modification of Young tableaux that we call adapted Young tableaux (see Section §6.2 for the definition), which will be useful to classify totally geodesic submanifolds in arbitrary products of symmetric spaces of rank one and to determine their isometry type. We prove a result that gives a correspondence between these adapted Young tableaux and semisimple totally geodesic submanifolds in products of rank one symmetric spaces.

In addition to this, we construct infinitely many examples of irreducible totally geodesic submanifolds in Hermitian symmetric spaces that are neither totally real nor complex. This phenomenon, which differs from what happens in rank one, had already been observed by Klein, who found two irreducible totally geodesic submanifolds that were neither totally real nor complex. Both examples found by Klein have constant Kähler angle $\arccos(\frac{1}{5})$. However, we can prove that every rational number in $[0,1]$ can be realized as the cosine of the constant Kähler angle of some totally geodesic submanifold of some Hermitian symmetric space.

Totally geodesic submanifolds in exceptional symmetric spaces. The next objective of this thesis is to establish a new strategy to classify maximal totally geodesic submanifolds in irreducible symmetric spaces of rank higher than two. For exceptional symmetric spaces, we present an idea that reduces the problem to some known classifications of reductive subalgebras of real simple Lie algebras and to consider a small number of cases where this method does not give a complete answer about maximality. This is precisely the content of the Correspondence Theorem, see Theorem 7.3.3. Thanks to this new approach we are able to classify maximal totally geodesic submanifolds in exceptional symmetric spaces. Furthermore, inspired by the works of Dynkin [74], we introduce a new invariant for totally geodesic submanifolds in symmetric spaces that we call Dynkin index. We will show that the Dynkin index determines if two totally geodesic embeddings are isometric.

Moreover, we prove a result related to the index conjecture established and proved by Berndt and Olmos (see [20, 21, 22, 23, 24]). The index $i(M)$ of an irreducible symmetric space M is the minimal codimension of a proper totally geodesic submanifold. The index conjecture can be stated as follows: every irreducible symmetric space $M \neq \mathsf{G}_2^2/\mathsf{SO}_4, \mathsf{G}_2/\mathsf{SO}_4$ has a reflective totally geodesic submanifold Σ such that $i(M) = \operatorname{codim}(\Sigma)$. We prove an analogous statement that allows us to include the cases $M = \mathsf{G}_2^2/\mathsf{SO}_4$ and $M = \mathsf{G}_2/\mathsf{SO}_4$. Our result states that every irreducible symmetric space M has a totally geodesic submanifold Σ with $i(M) = \operatorname{codim}(\Sigma)$ and such that every irreducible factor of Σ has Dynkin index one.

Totally geodesic submanifolds in Hopf-Berger spheres. Motivated by our results in symmetric spaces, we started the study of totally geodesic submanifolds in the class of homogeneous spaces with positive curvature.

It is a recurring fact that, when one tries to classify a certain class of submanifolds, the problem is usually more feasible when one is ensured the extrinsic homogeneity of such class of submanifolds. Moreover, results that prove the homogeneity of a class of submanifolds have been of great relevance, see [88, 176]. It is well known that totally geodesic submanifolds in a homogeneous space are intrinsically homogeneous. However, they are not necessarily orbits of subgroups of the isometry group of the ambient homogeneous space.

The family of homogeneous spaces of positive curvature consists of several homogeneous spaces diffeomorphic to spheres and complex projective spaces, the rank one symmetric spaces of compact type with their standard symmetric metrics, and some other sporadic examples, see [186]. Although totally geodesic submanifolds in homogeneous spaces with positive curvature have been extensively used to derive curvature properties of these spaces, see [157], we lack a complete classification of them. The interest of this problem relies on the fact that a totally geodesic submanifold (of dimension $d \geq 2$) of a positively curved homogeneous space is again a positively curved homogeneous space.

In this thesis we give a classification of totally geodesic submanifolds in an important class of homogeneous spaces diffeomorphic to spheres, i.e. Hopf-Berger spheres, which constitute the family of homogeneous spheres obtained by rescaling the round metric of the total space of a Hopf fibration in the direction of the fibers. As mentioned before, we cannot make use of the extrinsic homogeneity of the totally geodesic submanifolds in homogeneous spaces as it happens in the symmetric setting. Thus, our study relies on a very geometric approach which combines ideas coming from the general theory of Riemannian homogeneous spaces, Killing vector fields or closed geodesics.

Structure of the thesis. This thesis is organized in two parts with a first chapter of preliminaries.

In this first chapter we introduce the basic facts and terminology that will be used throughout this thesis. More precisely, we recall some well-known facts about Riemannian geometry and submanifold theory in Section §1.1 and about isometric actions in Section §1.2. Furthermore, Section §1.3 is devoted to introducing the ambient spaces of the problems that we will tackle in this thesis, namely, symmetric and homogeneous spaces. Finally, Section §1.4 deals with two basic concepts that have great relevance for this thesis: Clifford and Heisenberg algebras.

The first part of this thesis deals with the study of homogeneous hypersurfaces, isoparametric hypersurfaces, and hypersurfaces with constant principal curvatures.

In Chapter 2, we recall the notions of homogeneous hypersurface, isoparametric hypersurface, and hypersurface with constant principal curvatures. In Section §2.2 we explain the relationship among them in symmetric spaces of rank one. Later, in Section §2.3 we construct an example of a non-isoparametric hypersurface with constant principal curvatures in a conformally flat space.

In Chapter 3 we summarize the known results concerning the classification of homogeneous hypersurfaces in symmetric spaces of rank one. In Section §3.2 we revisit the classification of homogeneous hypersurfaces in symmetric spaces of compact type, specifically focusing on the rank one case. In Section §3.3 we introduce the notion of Kähler angle and other generalizations of it. Section §3.4 is devoted to explaining the theory of cohomogeneity one actions on symmetric spaces of non-compact type and rank one, while in Section §3.5 we briefly explain the program developed by Berndt and Tamaru to classify cohomogeneity one actions on symmetric spaces of non-compact type and higher rank.

In Chapter 4 we classify homogeneous hypersurfaces in quaternionic hyperbolic spaces $\mathbb{H}\mathsf{H}^{n+1}$, $n \geq 1$. There are two fundamental notions along this chapter: protohomogeneous subspaces and subspaces with constant quaternionic Kähler angle. These are defined in Section §4.1. The main goal of this chapter is the classification of protohomogeneous subspaces and the study of subspaces with constant quaternionic Kähler angle (Sections §4.2, 4.3, 4.4 and 4.5). As a by-product of our study we construct in Section §4.6 uncountably many families of inhomogeneous hypersurfaces that are isoparametric and have constant principal curvatures in $\mathbb{H}\mathsf{H}^{n+1}, n \geq 7$.

The second part of this thesis deals with the study of totally geodesic submanifolds in symmetric and homogeneous spaces.

In Chapter 5 we provide some well-known facts about totally geodesic submanifolds. In particular, we briefly present interesting information concerning totally geodesic submanifolds in Riemannian manifolds (Section §5.1) and symmetric spaces (Section §5.3). Moreover, we include a proof of the fact that, under mild assumptions, one can always extend a totally geodesic submanifold to a complete one (Section §5.2).

In Chapter 6 we classify totally geodesic submanifolds in products of rank one symmetric spaces by establishing a correspondence between totally geodesic submanifolds in products of symmetric spaces of rank one and adapted Young tableaux. These adapted Young tableaux are defined in Section §6.2. Moreover, in Section §6.3, we construct infinitely many totally geodesic submanifolds with non-trivial constant Kähler angle in complex Grassmannians. As a consequence, it is proved that the set of all possible Kähler angles of totally geodesic submanifolds of irreducible Hermitian symmetric spaces is a dense subset of the interval $[0, \pi/2]$.

In Chapter 7 we classify maximal totally geodesic submanifolds in exceptional symmetric spaces. This result is strongly based on the definition of a new invariant called Dynkin index (Section §7.4) and the Karpelevich Theorem (Section §7.1). In particular, in Section §7.4 we also prove a correspondence theorem that establishes a one-to-one correspondence between maximal semisimple totally geodesic submanifolds in symmetric spaces and a certain class of subalgebras (Theorem 7.3.3). In Section §7.7 we include the proofs of the main theorems.

In Chapter 8 we classify totally geodesic submanifolds in most Hopf-Berger spheres. The idea is to characterize a nice class of totally geodesic submanifolds of these spaces that we call well-positioned (Subsection §8.4.1). Then, we prove that every totally geodesic submanifold of dimension greater than one is well-positioned. Moreover, as a by-product of our study, we compute the index of symmetry of these homogeneous spheres (Section §8.5).

Chapter 1

Preliminaries

This chapter is entirely devoted to the introduction of the basic concepts and notation that are used in this thesis.

In Section §1.1 we recall some well-known facts about Riemannian geometry and its submanifold theory. In Section §1.2 we define some terminology regarding isometric actions and some notions related to them. Section §1.3 is devoted to introducing the concepts of Riemannian homogeneous spaces and Riemannian symmetric spaces, which will constitute the ambient spaces of the problems that we tackle in this thesis. Finally, Section §1.4 deals with the basic ideas related to Clifford and Heisenberg algebras, which have a strong link with symmetric spaces of rank one, and regularly appear over the course of this work.

1.1 Geometry of Riemannian submanifolds

Some good references to learn about Riemannian geometry are [122], or [156], and to learn about submanifold geometry, one can consult [14], or [55].

Let M be a connected smooth manifold. Throughout this thesis, we will consider smooth manifolds to be second-countable. We denote the tangent space at $p \in M$ by T_pM, the tangent bundle of M by TM, and the set of smooth vector fields of M by $\Gamma(TM)$. If $\mathcal{D}$ is a smooth distribution of M, we denote its set of sections by $\Gamma(\mathcal{D})$.

A Riemannian manifold (M, g) is a connected smooth manifold M equipped with a non-degenerate symmetric bilinear tensor field g of type $(0, 2)$. We will also use $\langle \cdot, \cdot \rangle$ to denote g. The tensor g induces a distance d in M, by considering $d(p, q)$ as the infimum of the lengths of the piecewise smooth curves joining $p \in M$ and $q \in M$, thus turning M into a metric space.

Two Riemannian manifolds (M, g) and (M', g') are (locally) isometric if there exists a (local) diffeomorphism $\varphi\colon M \to M'$ such that $g = \varphi^* g'$, where $\varphi^* g$ denotes the pullback of g by φ. We denote by $\operatorname{Isom}(M)$ the isometry group of M. A vector field $X \in \Gamma(TM)$ is *Killing* if $v \mapsto \nabla_v X$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$, for every $v \in T_pM$ and every $p \in M$, where ∇ denotes the Levi-Civita connection of M. This condition implies that the flow of X is by isometries.

A central notion in Riemannian geometry is curvature. The curvature tensor R measures to which extent M fails to be locally isometric to a Euclidean space. In this thesis we adopt the following convention:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \text{where } X, Y, Z \in \Gamma(TM).$$

We say that a manifold M is flat if its curvature vanishes identically. We will denote by exp the Riemannian exponential map of M, and recall that M is complete if and only if $\exp_p$ is surjective and it is well defined in T_pM for some (and hence for all) $p \in M$.

Given a vector $X_p \in T_pM$ and a piecewise smooth path $\gamma\colon [0,1] \to M$ joining p and $q \in M$, there exists a unique parallel vector field $X\colon [0,1] \to TM$ along γ such that $X(0) = X_p$. The vector $X(1) \in T_qM$ is called the *parallel transport* of X_p to q through γ. If γ is a piecewise smooth loop, we can define a map $P_\gamma\colon T_pM \to T_pM$, which maps every $X_p \in T_pM$ to its parallel transport through γ. The map P_γ is a linear isometry and the set generated by all the maps P_γ, where γ is a piecewise smooth loop based at p, is called the *holonomy group* of M at p and we denote it by $\mathrm{Hol}_p(M)$. The natural action of $\mathrm{Hol}_p(M)$ on T_pM is called the holonomy representation. It is easy to prove, using that M is connected, that $\mathrm{Hol}_p(M)$ is isomorphic to $\mathrm{Hol}_q(M)$ for every $p, q \in M$. Hence, for the sake of brevity we will just write $\mathrm{Hol}(M)$. De-Rham Theorem implies that a complete simply connected Riemannian manifold M whose holonomy representation is reducible is isometric to a product of Riemannian manifolds. In this case, M is said to be *reducible*, and if not it is said to be *irreducible*.

Now we will introduce several different kinds of submanifolds of a Riemannian manifold $\bar{M}$ attending to the relationship of its topology with that of the ambient space $\bar{M}$. We say that M is an *immersed submanifold* of $\bar{M}$ if there exists an isometric immersion $f\colon M \to \bar{M}$. Two immersed submanifolds $f_1\colon M_1 \to M$ and $f_2\colon M_2 \to M$ of $\bar{M}$ are said to be *congruent* in $\bar{M}$ if there exists $\varphi \in \mathrm{Isom}(\bar{M})$ such that $f_2 = \varphi \circ f_1$. It is said that $M \subset \bar{M}$ is an *injectively immersed submanifold* of $\bar{M}$ if M is endowed with a topology (not necessarily the one induced by the ambient space $\bar{M}$) in such a way that the inclusion map $i\colon M \to \bar{M}$ is an isometric smooth immersion. Additionally, an injectively immersed submanifold M of $\bar{M}$ is an *embedded submanifold* of $\bar{M}$ if M has the subspace topology inherited from $\bar{M}$. Finally, an injectively immersed submanifold M is *closed* if the map $i\colon M \to \bar{M}$ is closed. If the ambient space is complete, one has the following chain of strict inclusions

$$\left\{\begin{array}{c}\text{Closed}\\ \text{submanifolds}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{Complete}\\ \text{embedded}\\ \text{submanifolds}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{Embedded}\\ \text{submanifolds}\end{array}\right\} \subsetneq \left\{\begin{array}{c}\text{Injectively}\\ \text{immersed}\\ \text{submanifolds}\end{array}\right\}.$$

The local theories for these kinds of submanifolds are identical. Hence, when we deal with local properties of submanifolds, we will make no distinction between these kinds of submanifolds, and we will just refer to them as submanifolds.

Let M be a connected Riemannian manifold and $f\colon M \to \bar{M}$ be an isometric immersion. Then, for each point $p \in M$, there is an open neighborhood U of $p \in M$ such that $f_{|U}\colon U \to f(U)$ is an isometry. This allows us to identify U with $f(U)$, and by identifying T_pM with a subspace of $T_p\bar{M}$, we have the orthogonal splitting

$$T_p\bar{M} = T_pM \oplus \nu_pM \quad \text{for every } p \in M,$$

where νM denotes the normal bundle of M. Moreover, let us denote by $\bar{\nabla}$ and $\bar{R}$ the Levi-Civita connection and the curvature tensor of $\bar{M}$, respectively. In the next lines, we will state the fundamental equations of submanifolds of first order. The Gauss formula relates the Levi-Civita connections of M and $\bar{M}$ in the following way

$$\bar{\nabla}_X Y = \nabla_X Y + I\!I(X,Y), \quad \text{where } X, Y \in \Gamma(TM),$$

and $I\!I$ denotes the second fundamental form of M. If $I\!I$ vanishes identically, M is said to be *totally geodesic*. For each $\xi \in \Gamma(TM)$, we define the *shape operator* $\mathcal{S}_\xi$ of M by $\langle \mathcal{S}_\xi X, Y\rangle = \langle I\!I(X,Y), \xi\rangle$, for every $X, Y \in \Gamma(TM)$. The *mean curvature vector field* of M is the normal vector field given by $\sum_{i=1}^k I\!I(E_i, E_i)$, for any orthonormal frame $\{E_i\}_{i=1}^k$ of M, where k is the dimension of M. A submanifold M is called *minimal* if its mean curvature vanishes. A hypersurface M of $\bar{M}$ has *constant mean curvature* if its mean curvature vector field has constant length, or equivalently, if the trace of $\mathcal{S}_\xi$ is constant on M, for some unit normal ξ on M.

The Weingarten formula, relates the normal connection $\nabla^\perp$ of M to the Levi-Civita connection of $\bar{M}$ as follows

$$\bar{\nabla}_X \xi = -\mathcal{S}_\xi X + \nabla^\perp_X \xi, \quad \text{where } X \in \Gamma(TM), \quad \text{and } \xi \in \Gamma(\nu M).$$

The fundamental equations of submanifolds of second order are named after Gauss, Codazzi and Ricci, respectively, and are the following ones:

$$\begin{aligned}
\langle \bar{R}(X,Y)Z, W\rangle &= \langle R(X,Y)Z, W\rangle - \langle I\!I(Y,Z), I\!I(X,W)\rangle + \langle I\!I(X,Z), I\!I(Y,W)\rangle,\\
\langle \bar{R}(X,Y)Z, \xi\rangle &= \langle (\nabla^\perp_X I\!I)(Y,Z) - (\nabla^\perp_Y I\!I)(X,Z), \xi\rangle,\\
\langle R^\perp(X,Y)\xi, \eta\rangle &= \langle R(X,Y)\xi, \eta\rangle + \langle [\mathcal{S}_\xi, \mathcal{S}_\eta]X, Y\rangle,
\end{aligned}$$

where $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(\nu M)$. Moreover, $R^\perp$ stands for the curvature tensor of νM, defined by $R^\perp(X,Y)\xi = [\nabla^\perp_X, \nabla^\perp_Y]\xi - \nabla^\perp_{[X,Y]}\xi$, where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$.

In the codimension one case, i.e., M is a hypersurface, the Ricci equation holds trivially and the Gauss and Codazzi equations reduce to

$$\begin{aligned}
\langle \bar{R}(X,Y)Z, W\rangle &= \langle R(X,Y)Z, W\rangle - \langle \mathcal{S}_\xi Y, Z\rangle\langle \mathcal{S}_\xi X, W\rangle + \langle \mathcal{S}_\xi X, Z\rangle\langle \mathcal{S}_\xi Y, W\rangle,\\
\langle \bar{R}(X,Y)Z, \xi\rangle &= \langle (\nabla_X \mathcal{S}_\xi)Y - (\nabla_Y \mathcal{S}_\xi)X, Z\rangle,
\end{aligned}$$

where $X, Y, Z, W \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$.

1.2 Isometric actions

We refer to [5] for a detailed exposition about isometric actions.

Let (M, g) be a Riemannian manifold, $\mathrm{Isom}(M)$ its isometry group, which is known to be a Lie group, and consider an action by isometries of a Lie group G on M. For every $p \in M$ we have a G_p-principal bundle

$$\mathsf{G}_p \to \mathsf{G} \to \mathsf{G}/\mathsf{G}_p \simeq \mathsf{G} \cdot p,$$

where $\mathsf{G} \cdot p = \{g \cdot p : g \in \mathsf{G}\}$ denotes the orbit of G through $p \in M$ and the group $\mathsf{G}_p = \{g \in \mathsf{G} : g \cdot p = p\}$ its isotropy at $p \in M$. We will write $\mathsf{G} \curvearrowright M$ to denote the action of G on M.

Two isometric actions $\mathsf{G}_1 \curvearrowright M_1$ and $\mathsf{G}_2 \curvearrowright M_2$ are *isomorphic* if there exists a Lie group isomorphism $\psi\colon \mathsf{G}_1 \to \mathsf{G}_2$ and an isometry $f\colon M_1 \to M_2$ in such a way that

$$f(gp) = \psi(g)f(p), \quad \text{for all } p \in M \text{ and } g \in \mathsf{G}_1.$$

Let M/G be the set of orbits of the isometric action of the Lie group G on M. In order to ensure that the orbits are closed and thus embedded submanifolds of M, we will require that G acts properly on M. The action of G on M is *proper* if the map $\mathsf{G} \times M \to M \times M$, $(g, p) \mapsto (p, g \cdot p)$, is proper. Since G acts by isometries, this is equivalent to the fact that G is (up to effectivization) a closed subgroup of $\mathrm{Isom}(M)$. See [58] for more information about proper actions. A (closed) embedded submanifold of M is *extrinsically homogeneous* if it is an orbit of a closed subgroup of $\mathrm{Isom}(M)$. From now on, we will always assume that isometric actions are proper.

Two orbits $\mathsf{G} \cdot p$ and $\mathsf{G} \cdot q$, where $p, q \in M$, are *equivalent* if the isotropy subgroups G_p and G_q are conjugate in G. Let us denote by $[\mathsf{G} \cdot p]$ the equivalence class of $\mathsf{G} \cdot p$. A partial ordering $\leq$ on the set of equivalence classes of orbits of G on M can be defined by

$$[\mathsf{G} \cdot p] \leq [\mathsf{G} \cdot q] :\iff \mathsf{G}_q \text{ is conjugate to a subgroup of } \mathsf{G}_p \text{ in } \mathsf{G}.$$

An orbit $\mathsf{G} \cdot p$ is called *principal* if $[\mathsf{G} \cdot p]$ is maximal for $\leq$. All principal orbits are equivalent, and then, they have the same codimension in M, which is called the *cohomogeneity* of the action of G on M. Moreover, the union of the principal orbits is an open and dense subset of M and we can recover all the nearby orbits of the action of G by knowing a principal orbit, see [14, §2.1.8]. Indeed, if $\mathsf{G} \cdot p$ is a principal orbit of a G-action on M and $F \subset M$ is any other orbit of G, there exists a vector $\xi_p \in \nu_p(\mathsf{G} \cdot p)$ such that $\exp_p(\xi_p) = q \in F$, which can be extended to a G-equivariant normal vector field in $\mathsf{G} \cdot p$, since $\mathsf{G} \cdot p$ is principal, and $\mathsf{G} \cdot q = \{\exp_x(\xi_x) : x \in \mathsf{G} \cdot p\}$. An orbit $\mathsf{G} \cdot p$ is called *singular* if its codimension is larger than the cohomogeneity of the action, and is called *exceptional* if its codimension coincides with the cohomogeneity of G but is not principal.

Let $p \in M$ and consider the action of G_p on T_pM. The restriction of this action to $T_p(\mathsf{G} \cdot p)$ is called the *isotropy representation* of the G-action at $p \in M$, and the restriction to $\nu_p(\mathsf{G} \cdot p)$ is called the *slice representation* at $p \in M$. Let $p \in M$ and G be a connected Lie group acting isometrically and properly on M. A *slice* at $p \in M$ is an embedded submanifold S_p of M passing through p that satisfies:

i) $T_pM = T_p(\mathsf{G} \cdot p) \oplus T_p(S_p)$, and $T_qM = T_q(\mathsf{G} \cdot p) + T_q(S_p)$, for every $q \in S_p$.

ii) For every $q \in S_p$ and $g \in \mathsf{G}$, we have $g \cdot q \in S_p$ if and only if $g \in \mathsf{G}_p$.

Moreover, this implies that there exists a G-equivariant diffeomorphism between $\mathsf{G} \cdot S_p$ and the total space of the bundle with fiber S_p

$$S_p \to \mathsf{G} \times_{\mathsf{G}_p} S_p \to \mathsf{G}/\mathsf{G}_p,$$

associated with the G_p-principal bundle described above. The action of G_p on S_p is isomorphic to the slice representation restricted to an open ball of $\nu_p(\mathsf{G}\cdot p)$. Furthermore, the cohomogeneity of the slice representation at every point coincides with the cohomogeneity of the action of G on M, and the orbit $\mathsf{G}\cdot p$ is principal if and only if the slice representation at p is trivial. It was proved in [155] that proper actions always have slices at every point of M.

An important class of isometric actions is constituted by polar actions. A proper isometric action of G on M is *polar* if there exists a complete, closed and embedded submanifold Σ (called section) which intersects all the orbits of the G-action on M orthogonally. It follows that the cohomogeneity of G coincides with the dimension of Σ and it can be proved that Σ is a totally geodesic submanifold of M. See [129] for a detailed proof of these facts. Moreover, if Σ is flat, the action is said to be *hyperpolar*.

1.3 Homogeneous and symmetric spaces

Homogeneity is a central notion in Mathematics. The origin of homogeneous spaces dates back to the emergence of non-Euclidean geometry in the mid-19th century. The geometry of these spaces is quite different from that of the Euclidean spaces that we are accustomed to studying in high school. At this point, the need arises to clarify how to define geometry. Erlangen's program answers this question. This was proposed by Felix Klein in 1872. Basically, geometry was defined as the study of those properties in a space that are invariant under a given transformation group.

Intuitively, a homogeneous space is a space that looks the same at each point. For this reason, homogeneous spaces serve as a model space for various types of geometric structures. In particular, our interest lies in those homogeneous spaces that arise from isometric actions, that is, actions preserving the metric.

A symmetric space is a homogeneous space whose isometry group contains an inversion symmetry at each point. Symmetric spaces arise in a broad diversity of situations in both Mathematics and Physics. Their origin goes back to the following question posed by Cartan in 1926:

Which are the Riemannian manifolds whose curvature tensor R
is preserved by parallel transport along any curve?

This property is equivalent to the equation $\nabla R = 0$, and the spaces satisfying this property are intimately related to symmetric spaces. Indeed, every Riemannian manifold satisfying $\nabla R = 0$ is locally isometric to a symmetric space. Cartan achieved a complete classification of symmetric spaces in [43].

1.3.1 Homogeneous spaces

For a nice introduction to the theory of homogeneous spaces, one can consult [6] or [111, Chapter X]. A Riemannian manifold M is *homogeneous* if there exists some subgroup G of

$\mathrm{Isom}(M)$ such that G acts transitively on M. We fix a point $o \in M$, so M is diffeomorphic to G/K, where $\mathsf{K} = \mathsf{G}_o$ is the isotropy at o, by the map $\Phi\colon \mathsf{G}/\mathsf{K} \to M$ defined by $g\mathsf{K} \to g(o)$. We pull back the metric of M by Φ to G/K, turning Φ into an isometry. Furthermore, the metric $\langle \cdot, \cdot \rangle$ induced in G/K is G-invariant. Homogeneous spaces are analytic Riemannian manifolds, see [35, Lemma 1.1].

For any $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, we can associate a Killing vector field X^* given by $X^*_p = \frac{d}{dt}_{|t=0}(\mathrm{Exp}(tX) \cdot p)$, for every $p \in M$, where Exp denotes the Lie exponential map of G. Homogeneous spaces can be characterized in terms of Killing vector fields as follows. A Riemannian manifold M is homogeneous if for every $p \in M$ and every $v \in T_pM$, there is a Killing field $X \in \Gamma(TM)$ such that $X_p = v$. Riemannian homogeneous spaces G/K always have a reductive decomposition. A *reductive decomposition* is a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is an $\mathrm{Ad}(\mathsf{K})$-invariant subspace of $\mathfrak{g}$. Thus, we have the bracket relations $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$. If we consider the linearization at $o \in M$ of the isotropy action of K on M, we get the *isotropy representation* of the homogeneous space M, which is defined as $k \in \mathsf{K} \mapsto k_{*o} \in \mathsf{GL}(T_oM)$, where k_{*o} denotes the differential of k at $o \in M$. This is equivalent to the adjoint representation of G restricted to K on $\mathfrak{p}$, since $\mathfrak{p}$ and T_oM can be identified by the map which sends $X \in \mathfrak{p}$ to $X^*_o \in T_oM$. A homogeneous space $M = \mathsf{G}/\mathsf{K}$ is *isotropy irreducible* if the isotropy representation is an irreducible representation. If $M = \mathsf{G}/\mathsf{K}$ is isotropy irreducible, Schur's lemma implies that there is a unique G-invariant metric on M up to homothety, and that M is an Einstein manifold. Moreover, M is *strongly isotropy irreducible* if the restriction of the isotropy representation to the connected component of K is irreducible.

Let us denote by $X_\mathfrak{k}$ and $X_\mathfrak{p}$ the projection of $X \in \mathfrak{g}$ onto $\mathfrak{k}$ and $\mathfrak{p}$, respectively. We define a symmetric bilinear map $U\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ by

$$2\langle U(X,Y), Z \rangle = \langle [Z,X]_\mathfrak{p}, Y \rangle + \langle X, [Z,Y]_\mathfrak{p} \rangle,$$

where $X, Y, Z \in \mathfrak{p}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathfrak{p}$ induced by the Riemannian metric on M. The reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is *naturally reductive* if U is identically zero. In particular, if $U \equiv 0$, every geodesic γ of M passing through $o \in M$ is given by to $t \mapsto \mathrm{Exp}(tX)$ with $X \in \mathfrak{p}$, where Exp denotes the Lie exponential map of G. A homogeneous space where every geodesic is an orbit of a 1-parameter subgroup of the isometry group is said to be a *geodesic orbit space*, or, for short, g. o. space. Thus, naturally reductive spaces are g. o. Moreover, we say that the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is *normal homogeneous* if there exists some $\mathrm{Ad}(\mathsf{G})$-invariant inner product q on $\mathfrak{g}$ such that $\langle \cdot, \cdot \rangle = q_{|\mathfrak{p} \times \mathfrak{p}}$ and $\mathfrak{p} = \mathfrak{k}^\perp$, where $\mathfrak{k}^\perp$ denotes the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to q. It turns out that every normal homogeneous reductive decomposition is naturally reductive. Indeed, we have the chain of strict inclusions

$$\left\{ \begin{array}{c} \text{Normal} \\ \text{homogeneous spaces} \end{array} \right\} \subsetneq \left\{ \begin{array}{c} \text{Naturally reductive} \\ \text{homogeneous spaces} \end{array} \right\} \subsetneq \left\{ \begin{array}{c} \text{Geodesic orbit} \\ \text{spaces} \end{array} \right\}.$$

Let us consider the canonical connection ∇^c associated with a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which is the unique G-invariant affine connection on M such that

$$(\nabla^c_{X^*} Y^*)_o = (-[X,Y]_\mathfrak{p})^*_o, \tag{1.1}$$

where $X, Y \in \mathfrak{p}$. We can express the Levi-Civita connection of M at o as

$$(\nabla_{X^*}Y^*)_o = \left(-\frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y)\right)^*_o, \tag{1.2}$$

where $X, Y \in \mathfrak{p}$. The difference tensor D at $o \in M$ is defined as $D = (\nabla - \nabla^c)_o$, and using the identification of T_oM and $\mathfrak{p}$, we have

$$D_X Y = \frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y), \quad \text{for every } X, Y \in \mathfrak{p}. \tag{1.3}$$

Using the formula for the Levi-Civita connection and the identification of $\mathfrak{p}$ with T_oM, we can compute the curvature tensor of M, which is given by

$$\begin{aligned} R_o(X,Y)Z = {} & \frac{1}{2}[Z,[X,Y]_{\mathfrak{p}}]_{\mathfrak{p}} - [[X,Y]_{\mathfrak{k}},Z]]_{\mathfrak{p}} - U(Z,[X,Y]_{\mathfrak{p}}) + \frac{1}{4}[[Z,Y]_{\mathfrak{p}},X]_{\mathfrak{p}} \\ & - \frac{1}{2}U([Z,Y]_{\mathfrak{p}},X) - \frac{1}{2}[U(Z,Y),X]_{\mathfrak{p}} + U(U(Z,Y),X) - \frac{1}{4}[[Z,X]_{\mathfrak{p}},Y]_{\mathfrak{p}} \\ & + \frac{1}{2}U([Z,X]_{\mathfrak{p}},Y) + \frac{1}{2}[U(Z,X),Y]_{\mathfrak{p}} - U(U(Z,X),Y), \end{aligned} \tag{1.4}$$

where $X, Y, Z \in \mathfrak{p}$. We end up this section with a remark that will be useful to compute the covariant derivatives of the curvature tensor at the base point $o \in M = \mathsf{G}/\mathsf{K}$.

Remark 1.3.1. By [42, Proposition 1.4.15] the curvature tensor R of a Riemannian homogeneous space $M = \mathsf{G}/\mathsf{K}$ satisfies $\nabla^c R = 0$, since it is G-invariant. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a reductive decomposition for $M = \mathsf{G}/\mathsf{K}$. Then, using the definition of the difference tensor and the identification of $\mathfrak{p}$ with T_oM we have

$$\begin{aligned} (\nabla_V R)(X,Y,Z) &= ((\nabla_V - \nabla^c_V)R)(X,Y,Z) \\ &= D_V R(X,Y)Z - R(D_V X,Y)Z - R(X,D_V Y)Z - R(X,Y)D_V Z, \end{aligned}$$

where $X, Y, Z, V \in \mathfrak{p}$.

1.3.2 Symmetric spaces

For a detailed exposition of the theory of symmetric spaces one can follow [90], [91], [127], and [128]. For a quicker introduction, we recommend [77] or [193].

Let M be a Riemannian manifold. We say that M is a *symmetric space* if for every point $p \in M$ there exists an isometry $s_p \in \mathrm{Isom}(M)$ such that s_p fixes $p \in M$ and $s_{*p} = -\mathrm{Id}_{T_pM}$. The isometry s_p is called *geodesic reflection* at $p \in M$.

From the definition, we can deduce that symmetric spaces are complete, since geodesics can be extended by using geodesic reflections. This implies that for any $p, q \in M$, there is a geodesic segment γ joining p and q. Thus, s_o, where o is the mid-point of γ, maps p to q, proving that every symmetric space is homogeneous. Let $\mathsf{G} = \mathrm{Isom}^0(M)$ be the connected component of $\mathrm{Isom}(M)$ containing the identity and K be the isotropy of G at $o \in M$.

Symmetric spaces can be characterized in terms of Killing vector fields as follows. A Killing vector field X on a Riemannian manifold M such that $(\nabla X)_p = 0$ is called a *transvection* at $p \in M$. A Riemannian manifold M is symmetric if and only if for every point $p \in M$ and every $v \in T_pM$, there is a transvection $X \in \Gamma(TM)$ with $X_p = v$.

Now consider $\sigma\colon \mathsf{G} \to \mathsf{G}$, given by $g \in \mathsf{G} \mapsto s_o g s_o^{-1} \in \mathsf{G}$. Then, $\mathsf{G}_\sigma^0 \subset \mathsf{K} \subset \mathsf{G}_\sigma$, where G_σ^0 is the connected component of $\mathsf{G}_\sigma := \{g \in \mathsf{G}\colon \sigma(g) = g\}$. The map σ is an involutive automorphism of Lie groups, and its differential $\theta = \sigma_{*e}\colon \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism of Lie algebras. The map θ is called the *Cartan involution* of the symmetric space $M = \mathsf{G}/\mathsf{K}$ and it splits $\mathfrak{g}$ into the sum of the eigenspaces of θ, $\mathfrak{k}$ and $\mathfrak{p}$, associated with the eigenvalues 1 and -1, respectively. This provides a reductive decomposition of $M = \mathsf{G}/\mathsf{K}$ given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is identified with T_oM, and $\mathfrak{k}$ is the Lie algebra of K. The *rank* of a symmetric space M is the dimension of a maximal flat totally geodesic submanifold of M, or equivalently, the dimension of a maximal abelian subspace of $\mathfrak{p}$. Furthermore, in the case of symmetric spaces we have $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. This implies that U vanishes identically, proving that symmetric spaces are naturally reductive homogeneous spaces. Moreover, by Equation (1.4), the curvature tensor of M at o can be expressed as

$$R_o(X, Y)Z = -[[X, Y], Z] \tag{1.5}$$

for $X, Y, Z \in \mathfrak{p} \simeq T_oM$.

Let $\operatorname{ad}_X\colon \mathfrak{g} \to \mathfrak{g}$ the map given by $\operatorname{ad}_X(Y) = [X, Y]$. Now consider $\mathcal{B}_\mathfrak{g}$ be the Killing form of $\mathfrak{g}$, that is,

$$\mathcal{B}_\mathfrak{g}(X, Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y) \quad \text{for } X, Y \in \mathfrak{g}.$$

It follows that $\mathcal{B}_\mathfrak{g}(X, Y) = 0$, for every $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$. If the Lie algebra for which we consider the Killing form is clear from the context, we will simply write $\mathcal{B}$.

A symmetric space $M = \mathsf{G}/\mathsf{K}$ is said to be of *compact type*, of *non-compact type* or of *Euclidean type* if $\mathcal{B}_{|\mathfrak{p}\times\mathfrak{p}}$, the restriction of $\mathcal{B}$ to $\mathfrak{p}$, is negative definite, positive definite or identically zero, respectively. If $M = \mathsf{G}/\mathsf{K}$ is isotropy irreducible, Schur's lemma yields that $\mathcal{B}_{|\mathfrak{p}\times\mathfrak{p}}$ is a multiple of the induced metric on $\mathfrak{p} \simeq T_oM$. Hence, if M is isotropy irreducible, the type is a mutually exclusive property of M. Let $\widetilde{M}$ be the universal covering of M. Then, $\widetilde{M}$ is again a symmetric space and by De-Rham Theorem, $\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k$, where $\widetilde{M}_0$ is isometric to a Euclidean space and $\widetilde{M}_i$ is a simply connected irreducible symmetric space, with $i \in \{1, \dots, k\}$. We say that M is *semisimple* if $\widetilde{M}_0$ is just a point. In this case the Lie algebra $\mathfrak{g}$ is semisimple. Moreover, if M is semisimple, M is irreducible if and only if it is isotropy irreducible.

An important notion, which establishes a relation between symmetric spaces of compact type and non-compact type, is duality. If we restrict our attention to simply connected symmetric spaces, there is a one-to-one correspondence between symmetric spaces of non-compact type and symmetric spaces of compact type. At the Lie algebra level this works as follows. Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of non-compact type and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the reductive decomposition induced by the Cartan involution θ. Consider $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$, the complexification of $\mathfrak{g}$. We can define the subspace $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_\mathbb{C}$, where $i = \sqrt{-1}$. Then $\mathfrak{g}^*$ is a compact Lie algebra and $M^* = \mathsf{G}^*/\mathsf{K}^*$ is a symmetric space of compact type equipped with the Riemannian metric induced by the negative of the Killing form of $\mathfrak{g}^*$, where G^* is the simply connected Lie group with Lie algebra $\mathfrak{g}^*$ and K^* is the connected subgroup of G^* with Lie algebra $\mathfrak{k}$.

1.3.3 Symmetric spaces of non-compact type

The symmetric spaces of non-compact type are of particular relevance for this thesis since many results that we obtain are proved in this setting. See [65] for more details.

Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of non-compact type and consider the Cartan involution θ of $\mathfrak{g}$ induced by the geodesic symmetry at the base point $o \in M$. The reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, induced by θ, is called the *Cartan decomposition* of $\mathfrak{g}$. Let us consider the positive definite inner product on $\mathfrak{g}$ given by

$$\mathcal{B}_\theta(X, Y) = -\mathcal{B}(\theta X, Y) \quad \text{for every } X, Y \in \mathfrak{g}.$$

A useful fact about this inner product is that the adjoint map of $\operatorname{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ with respect to $\mathcal{B}_\theta$ is $-\operatorname{ad}_{\theta X}$ for every $X \in \mathfrak{g}$.

The isotropy representation of M at o is polar and every maximal abelian subspace of $\mathfrak{p}$ is a section for this action. Thus, two maximal abelian subspaces of $\mathfrak{p}$ are conjugate by an element of K. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Moreover, it can be proved that M is simply connected and thus it is diffeomorphic to a Euclidean space, since it has non-positive sectional curvature.

Since $\mathfrak{a} \subset \mathfrak{p}$, every operator $\operatorname{ad}_H \colon \mathfrak{g} \to \mathfrak{g}$ is self-adjoint with respect to $\mathcal{B}_\theta$. Moreover, since $[\operatorname{ad}_{H_1}, \operatorname{ad}_{H_2}] = \operatorname{ad}[H_1, H_2] = 0$, the set $\{\operatorname{ad}_H : H \in \mathfrak{a}\}$ constitutes a commuting family of self-adjoint endomorphisms of $\mathfrak{g}$. Thus, they diagonalize simultaneously. Their common eigenspaces are the *(restricted) root spaces* of $\mathfrak{g}$ and the non-zero eigenvalues (which depend linearly on $H \in \mathfrak{a}$) are the *(restricted) roots* of $\mathfrak{g}$. For each $\lambda \in \mathfrak{a}^*$, we define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \operatorname{ad}_H X = \lambda(H) X \quad \text{for all } H \in \mathfrak{a}\}.$$

Then, any $\lambda \neq 0$ such that $\mathfrak{g}_\lambda \neq 0$ is a root and every $\mathfrak{g}_\lambda \neq 0$ is a root space. It can be checked that

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} \quad \text{for every } \lambda, \mu \in \mathfrak{a}^*.$$

Let Δ denote the set of roots. Then we have the following orthogonal decomposition with respect to $\mathcal{B}_\theta$:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda \right),$$

which is called the *(restricted) root space decomposition* of $\mathfrak{g}$. We have $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, implying that $\lambda \in \Delta$ if and only if $-\lambda \in \Delta$. Additionally, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$.

For each $\lambda \in \Delta$, we define $H_\lambda \in \mathfrak{a}$ as the unique element of $\mathfrak{a}$ satisfying $\mathcal{B}(H_\lambda, H) = \lambda(H)$, for all $H \in \mathfrak{a}$. This induces an inner product on $\mathfrak{a}^*$ given by $\langle \lambda, \mu \rangle = \mathcal{B}(H_\lambda, H_\mu)$, for every $\lambda, \mu \in \mathfrak{a}^*$. Moreover, it can be proved that Δ defines a root system in $\mathfrak{a}^*$, thus satisfying:

i) $\mathfrak{a}^*$ is spanned by Δ,

ii) $n_{\alpha,\beta} = 2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle \in \mathbb{Z}$,

iii) $\beta - n_{\alpha,\beta}\alpha \in \Delta$, for every $\alpha, \beta \in \Delta$.

Now choose a hyperplane in $\mathfrak{a}^*$ such that it does not contain any root. We can define a positivity criterion on Δ by declaring those roots that lie at one of the two half-spaces determined by the hyperplane to be positive. If Δ^+ denotes the set of positive roots, then $\Delta = \Delta^+ \cup (-\Delta^+)$. Furthermore, we can define the set of simple roots Π as the subset of those positive roots which cannot be expressed as the sum of two positive roots. The subspace

$$\mathfrak{n} = \bigoplus_{\lambda\in\Delta^+} \mathfrak{g}_\lambda$$

of $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathfrak{a} \oplus \mathfrak{n}$ is then a solvable subalgebra such that $[\mathfrak{a}\oplus\mathfrak{n}, \mathfrak{a}\oplus\mathfrak{n}] = \mathfrak{n}$. Any two choices of positive criteria on Δ give rise to nilpotent subalgebras $\mathfrak{n}$ which are conjugate by an element of the group $N_{\mathsf{K}}(\mathfrak{a}) = \{k \in \mathsf{K} : \mathrm{Ad}(k)\mathfrak{a} \subset \mathfrak{a}\}$.

The *Iwasawa decomposition theorem* (see [98]) states that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is a vector space direct sum. Observe that this sum is neither an orthogonal sum nor a semidirect sum. Let A and N be the connected Lie subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The connected Lie subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ is a semidirect product AN, since $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$. Then, the Iwasawa theorem at the Lie group level states that the multiplication map

$$\mathsf{K} \times \mathsf{A} \times \mathsf{N} \to \mathsf{G}, \quad (k, a, n) \mapsto kan,$$

is an analytic diffeomorphism. Moreover, the Lie groups A and N are simply connected, and thus they and AN are diffeomorphic to Euclidean spaces. The smooth map $\Phi_{|\mathsf{AN}} \to M$ is a diffeomorphism. This allows us to pull back the Riemannian metric on M to AN. Moreover, this metric on AN is left-invariant. Consequently, every symmetric space $M = \mathsf{G}/\mathsf{K}$ of non-compact type is isometric to a solvable Lie group AN equipped with a left-invariant metric. In particular, this shows that M is diffeomorphic to a Euclidean space. By Equation (1.5), M is non-positively curved, and thus M is a Hadamard manifold.

A useful concept related to a symmetric space of non-compact type M is that of ideal boundary. The *ideal boundary* $M(\infty)$ of M is defined as the set of equivalence classes of complete, unit-speed geodesics of M under the relation

$$\gamma_1 \sim \gamma_2 :\Leftrightarrow \{d(\gamma_1(t), \gamma_2(t)) : t \geq 0\} \quad \text{is bounded.}$$

Now, we can introduce with the so-called cone topology on $M \sqcup M(\infty)$, see [75] for more details, in such a way that $M \sqcup M(\infty)$ becomes homeomorphic to a Euclidean closed ball, where M corresponds to its interior and $M(\infty)$ to its boundary. Finally, it is important to notice that the action of G on M can be naturally extended to $M(\infty)$ by taking $g \cdot [\gamma] := [g \cdot \gamma]$.

1.4 Heisenberg algebras and hyperbolic spaces

Generalized Heisenberg algebras are highly significant for this thesis since they are closely related to symmetric spaces of rank one. In particular, symmetric spaces of rank one and of non-compact type constitute a special case of Damek-Ricci spaces, which are solvable Lie groups equipped with a left-invariant metric whose Lie algebras are obtained as certain one-dimensional extensions of generalized Heisenberg algebras. It turns out that this structure is particularly well-suited and relevant to study submanifold geometry in these spaces as was shown in [13, 60] or [70].

1.4.1 Clifford algebras

In this subsection, we fix some notation and recall certain well-known facts related to Clifford algebras. We will mainly follow [120]. Let us start by introducing the notion of Clifford algebra. Let V be a real vector space over and q be a quadratic form on V. Let $T(V) := \bigoplus_{r=0}^{\infty} T^r(V)$ be the tensor algebra of V, where $T^r(V) := V \otimes \overset{r)}{\cdots} \otimes V$ and $T^0(V) = \mathbb{R}$. This is an associative, unitary and graded algebra where $T^k(V)$ is constituted by the homogeneous elements of degree $k \in \mathbb{N}$. Let $T_q(V)$ be the two-sided ideal in $T(V)$ generated by all elements of the form $v \otimes v + q(v)1$, where $v \in V$. We define $\mathsf{Cl}(V, q)$, the *Clifford algebra associated with V and q*, as the quotient algebra

$$\mathsf{Cl}(V, q) := T(V)/T_q(V).$$

Let V and V' be two vector spaces equipped with quadratic forms q and q'. Then, every linear map $f\colon (V, q) \to (V', q')$ such that $f(q(v)) = q'(f(v))$ for every $v \in V$, induces a morphism between $\mathsf{Cl}_q(V)$ and $\mathsf{Cl}_{q'}(V')$ in the natural way.

Two Clifford algebras $\mathsf{Cl}(V, q)$ and $\mathsf{Cl}(V', q')$ with $\dim V = \dim V'$ and such that q and q' have the same signature are isomorphic. Since we will consider only Clifford algebras where q is a positive definite quadratic form, in order to simplify our notation we will write Cl_n or $\mathsf{Cl}(V)$ instead of $\mathsf{Cl}(V, q)$, where V has dimension n.

Let $\mathbb{F}$ be the normed division algebra of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$. Denote by $\mathbb{F}(k)$ the algebra of matrices of order k whose entries are in $\mathbb{F}$. In Table 1.1, we list Clifford algebras Cl_n, where $n \leq 8$. Notice that one has the periodicity isomorphism $\mathsf{Cl}_{n+8} \simeq \mathsf{Cl}_n \otimes \mathsf{Cl}_8$. Hence, it is enough to list Cl_n, with $n \leq 8$, to determine Cl_n for every $n \in \mathbb{N}$.

n	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

Table 1.1: Clifford algebras Cl_n for $n \leq 8$.

It can be proved that every irreducible representation of $\mathbb{F}(k)$ is equivalent to the standard action on $\mathbb{F}^k$, and that $\mathbb{F}(k) \oplus \mathbb{F}(k)$ has exactly two equivalence classes of irreducible representations, given by the standard action of each one of the two factors on $\mathbb{F}^k$, see [120, Theorem 5.6].

Moreover, Cl_n is isomorphic either to $\mathbb{F}(k)$, when $n \not\equiv 3 \mod 4$, or to $\mathbb{F}(k) \oplus \mathbb{F}(k)$, when $n \equiv 3 \mod 4$. As mentioned above, there exists a unique irreducible representation ρ in the first case and exactly two irreducible representations ρ_+ and ρ_- in the second case, up to equivalence. In order to distinguish between ρ_+ and ρ_-, we introduce the volume element of Cl_n. Let us fix an orientation $(e_1, \ldots, e_n)$ in V. Then we define $\omega = e_1 \cdots e_n \in \mathsf{Cl}(V)$ as the volume element of Cl_n. It turns out that $\rho_+(\omega) = \mathrm{Id}$ and $\rho_-(\omega) = -\mathrm{Id}$, when $n \equiv 3 \mod 4$.

1.4.2 Generalized Heisenberg algebras

In what follows, we will introduce the basic concepts needed to define generalized Heisenberg algebras. See [30] for a nice and complete survey on this topic.

Let us consider two non-zero real vector spaces $\mathfrak{v}$ and $\mathfrak{z}$, and $\beta\colon \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ a skew-symmetric bilinear map. We define $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{z}$ and we endow it with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. Moreover, we introduce a linear map $J\colon Z \in \mathfrak{z} \mapsto J_Z \in \mathrm{End}(\mathfrak{v})$ given by

$$\langle J_Z U, V \rangle = \langle \beta(U, V), Z \rangle, \qquad \text{for all } U, V \in \mathfrak{v},\ Z \in \mathfrak{z},$$

and we define a Lie bracket in $\mathfrak{n}$ by

$$[U + X, V + Y] = \beta(U, V), \qquad \text{for all } U, V \in \mathfrak{v},\ X, Y \in \mathfrak{z}.$$

Then, $\mathfrak{n}$ is a two-step nilpotent Lie algebra whose center is $Z(\mathfrak{n}) = \mathfrak{z}$. If, in addition to that, we have $J_Z^2 = -\langle Z, Z \rangle \,\mathrm{id}_{\mathfrak{v}}$ for every $Z \in \mathfrak{z}$, then $\mathfrak{n}$ is said to be a *generalized Heisenberg algebra*, and the associated simply connected nilpotent Lie group N, endowed with the induced left-invariant Riemannian metric, is called a *generalized Heisenberg group*. The more classical notions of Heisenberg algebras and groups are recovered precisely when $\mathfrak{z}$ is one-dimensional.

Let U, $V \in \mathfrak{v}$ and X, $Y \in \mathfrak{z}$. One has the following well-known properties of generalized Heisenberg algebras (see [30, Chapter 3]):

$$\begin{aligned} J_X J_Y + J_Y J_X &= -2\langle X, Y \rangle \,\mathrm{id}_{\mathfrak{v}}, & [J_X U, V] - [U, J_X V] &= -2\langle U, V \rangle X, \\ \langle J_X U, J_X V \rangle &= \langle X, X \rangle \langle U, V \rangle, & \langle J_X U, J_Y U \rangle &= \langle X, Y \rangle \langle U, U \rangle. \end{aligned}$$

In particular, for any unit $Z \in \mathfrak{z}$, J_Z is a complex structure on $\mathfrak{v}$. Moreover, the map $J\colon \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ can be extended to the Clifford algebra $\mathsf{Cl}(\mathfrak{z}, q)$, where q is the quadratic form induced by $\langle \cdot, \cdot \rangle$, in such a way that $\mathfrak{v}$ becomes a Clifford module over $\mathsf{Cl}(\mathfrak{z}, q)$.

1.4.3 Symmetric spaces of rank one and non-compact type

Hurwitz's theorem asserts that any normed real division algebra $\mathbb{F}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. The hyperbolic spaces over these algebras constitute the symmetric spaces of non-compact type and rank one. In other words, if M is a symmetric space of non-compact type and rank one, then M is either a real hyperbolic space $\mathbb{R}\mathsf{H}^{n+1}$, $n \geq 1$, a complex hyperbolic space $\mathbb{C}\mathsf{H}^{n+1}$, $n \geq 1$, a quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^{n+1}$, $n \geq 1$, or the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$. As a symmetric space, any of these manifolds M can be identified with a quotient G/K of Lie groups, where G is the connected component of the identity of the isometry group of M, up to a finite covering, and K is the isotropy subgroup of G corresponding to a certain point $o \in M$ that we fix from now on. Then one can take $\mathsf{G} = \mathsf{SO}^0_{1,n+1}$, $\mathsf{SU}_{1,n+1}$, $\mathsf{Sp}_{1,n+1}$, F_4^{-20} and $\mathsf{K} = \mathsf{SO}_{n+1}$, $\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_{n+1})$, $\mathsf{Sp}_1 \times \mathsf{Sp}_{n+1}$, Spin_9, depending on whether $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, respectively.

We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively, by $\mathcal{B}$ the Killing form of $\mathfrak{g}$, and by θ the Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ induced by θ. We have that $\langle X, Y\rangle = -\mathcal{B}(X, \theta Y)$ is an inner product that restricted to $\mathfrak{p}$ induces a Riemannian metric on G/K that makes G/K isometric to M, up to homothety.

	$\mathbb{R}\mathsf{H}^{n+1}$	$\mathbb{C}\mathsf{H}^{n+1}$	$\mathbb{H}\mathsf{H}^{n+1}$	$\mathbb{O}\mathsf{H}^2$
G	$\mathsf{SO}^0_{1,n+1}$	$\mathsf{SU}_{1,n+1}$	$\mathsf{Sp}_{1,n+1}$	F_4^{-20}
K	SO_{n+1}	$\mathsf{S}(\mathsf{U}_1\mathsf{U}_{n+1})$	$\mathsf{Sp}_1\mathsf{Sp}_{n+1}$	Spin_9
K_0	SO_n	$\mathsf{S}(\mathsf{U}_1\mathsf{U}_n)$	$\mathsf{Sp}_1\mathsf{Sp}_n$	Spin_7
$\mathfrak{g}_\alpha$	$\mathbb{R}^n$	$\mathbb{C}^n$	$\mathbb{H}^n$	$\mathbb{O}$
$\mathfrak{g}_{2\alpha}$	0	$\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}^7$

Table 1.2: Data for each hyperbolic space.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, which is one-dimensional as M has rank one. Then, the corresponding root space decomposition of $\mathfrak{g}$ adopts the form

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

Here, the root space $\mathfrak{g}_0$ splits as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0$ is the Lie algebra of $\mathsf{K}_0 = N_{\mathsf{K}}(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in K, which also normalizes $\mathfrak{g}_\alpha$ and centralizes $\mathfrak{g}_{2\alpha}$. Moreover, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, is an Iwasawa decomposition of $\mathfrak{g}$. When $\mathbb{F} = \mathbb{R}$, we have $\mathfrak{g}_{-2\alpha} = \mathfrak{g}_{2\alpha} = 0$ and $\mathfrak{n}$ is abelian. Otherwise, $\mathfrak{n}$ is only two-step nilpotent. In fact, $\mathfrak{n}$ is isomorphic to the $(2n + 1)$-dimensional Heisenberg algebra when $\mathbb{F} = \mathbb{C}$ and to a certain generalized Heisenberg algebra if $\mathbb{F} \in \{\mathbb{H}, \mathbb{O}\}$. Moreover, $\mathfrak{g}_{2\alpha}$, the center of $\mathfrak{n}$, is equal to the derived algebra of $\mathfrak{n}$, and has dimension 1, 3 or 7 for $\mathbb{F} = \mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, respectively.

In addition to this, we can identify $\mathfrak{g}_\alpha$ with $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$, $\mathbb{O}$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, respectively. Indeed, $\mathfrak{g}_\alpha$ is a Clifford module over Cl_m, where $m = \dim \mathfrak{g}_{2\alpha}$, which is the sum of equivalent Clifford modules if $m = 3$, and is irreducible if $m = 7$. The possibilities for G, K, K_0 and the root spaces corresponding to positive roots are summarized in Table 1.2.

Part I

Homogeneous and isoparametric hypersurfaces

Chapter 2
A non-isoparametric hypersurface with constant principal curvatures

This chapter is devoted, on the one hand, to an exposition of the notions of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures, and their relationship. On the other hand, we exhibit an example of a non-isoparametric hypersurface with constant principal curvatures in a conformally flat ambient space, which is, up to our knowledge, the first such example ever described.

This chapter is organized in the following way. In Section §2.1, we recall some well-known facts about isoparametric hypersurfaces and hypersurfaces with constant principal curvatures. In Section §2.2, we describe the relationship between these two classes of hypersurfaces in symmetric spaces of rank one together with the class of extrinsically homogeneous hypersurfaces. Moreover, we review the classification results concerning isoparametric hypersurfaces and hypersurfaces with constant principal curvatures in these spaces. Finally, in Section §2.3, we construct an example of a non-isoparametric hypersurface with constant principal curvatures. The original result contained in Section §2.3 has given rise to the publication [159].

2.1 Isoparametricity and constancy of the principal curvatures

Let $(\bar{M}, g)$ be a Riemannian manifold. A smooth map $f\colon \bar{M} \to \mathbb{R}$ is an *isoparametric function* if its gradient does not vanish on any open subset of $\bar{M}$ and there are real valued functions $\Phi \in \mathcal{C}^2(\bar{M})$ and $\Psi \in \mathcal{C}^0(\bar{M})$ such that

$$||\nabla f||^2 = \Phi \circ f, \qquad \text{and} \qquad \Delta f = \Psi \circ f, \tag{2.1}$$

where ∇f is the gradient of f and Δf is the Laplacian of f. An isoparametric family is the collection $\{f^{-1}(c) : c \in \mathbb{R}\}$ of the level sets of f. The first identity in Equation (2.1) implies that the level sets of an isoparametric function f are equidistant whereas the second one means that the level sets of f have constant mean curvature.

Let M be an immersed hypersurface of $\bar{M}$ and $p \in M$. Then, there exists some neighborhood U of p in M such that U is an embedded hypersurface with a unit normal vector field ξ, and for every sufficiently small $r > 0$, the equidistant hypersurfaces $U^r = \{\exp_q(r\xi_q) : q \in U\}$ are embedded in $\bar{M}$. We say that a hypersurface M immersed on a

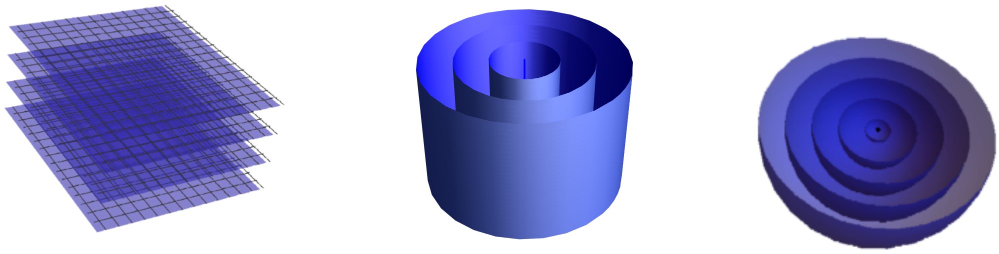

Figure 2.1: Level sets of the isoparametric function $f_k\colon \mathbb{R}^3 \to \mathbb{R}$, defined by $f_k(x_1, x_2, x_3) := \sum_{i=1}^{k} x_i^2$, for each $k \in \{1, 2, 3\}$, respectively.

Riemannian manifold $\bar{M}$ is *isoparametric* if for every $p \in M$ there is an open neighborhood U of p in M such that U and its nearby equidistant hypersurfaces have constant mean curvature. On the one hand, every regular level set of an isoparametric function is an isoparametric hypersurface, and conversely, if M is an isoparametric hypersurface of $\bar{M}$, then for each $p \in M$ there exists an open neighborhood U of p in M such that U is the level set of a certain isoparametric function defined on some open neighborhood of p in $\bar{M}$. On the other hand, the singular level sets of an isoparametric function are called *focal sets* or *focal submanifolds*. Every regular level set of an isoparametric function is a tube around a focal set, and the focal submanifolds are minimal submanifolds of $\bar{M}$, see [82], [182].

The study of isoparametric hypersurfaces has been an influential topic of research for the last century and to the present day. It has revealed multiple connections of the area of submanifold geometry with other areas of Mathematics such as algebraic topology, algebraic geometry, Lie group theory, differential equations or functional analysis. Moreover, there are other related notions that can be understood as generalizations or extensions of the notion of isoparametric hypersurface and whose study has been of special importance in the last decades. For instance: isoparametric submanifolds of arbitrary codimension [176, 88], equifocal submanifolds [53, 39], Dupin hypersurfaces [46, 167], polar actions and polar foliations [131, 114, 18].

Now we will recall the notion of constant principal curvatures. Let ξ be a unit normal vector field defined on some open subset U of an immersed hypersurface M of $\bar{M}$. We say that the function $\lambda\colon U \subset M \to \mathbb{R}$ is a *principal curvature of M associated with ξ* if there exists some vector field $X \in \Gamma(TU)$ such that $\mathcal{S}_\xi X = \lambda X$, where $\mathcal{S}$ is the shape operator of M. If λ is a principal curvature, we denote by $T_\lambda(p)$ the eigenspace associated with $\lambda(p)$ and we call it *principal curvature space associated with* $\lambda(p)$. If $X \in T_\lambda(p)$, with $X \neq 0$, we say that X is a principal direction of λ at $p \in M$. We define the multiplicity of λ in $p \in M$ as $\dim(\mathrm{Ker}(\mathcal{S}_{\xi_p} - \lambda(p)\mathrm{Id}))$. We say that a hypersurface M has constant principal curvatures if for every open subset U of M with a unit normal vector field ξ defined on U, the eigenvalues (and corresponding multiplicities) of the shape operator of U with respect to ξ are constant in U.

Isoparametric hypersurfaces and hypersurfaces with constant principal curvatures are related to the notion of extrinsically homogeneous hypersurfaces. We say that a hypersurface M in $\bar{M}$ is *extrinsically homogeneous* if it is a codimension one orbit of the action of some closed subgroup $\mathsf{H} \subset \mathrm{Isom}(\bar{M})$. From now on, we will simply say homogeneous hypersurface instead of extrinsically homogeneous hypersurface. It can be easily proved that a homogeneous hypersurface is isoparametric and has constant principal curvatures. However, as we will see throughout the thesis, there are multiple examples where the converse does not hold.

The study of the interplay between the three properties defined above is a fundamental question in the field of the geometry of hypersurfaces. For instance, if $\bar{M}$ has constant sectional curvature, then Cartan [44] proved that a hypersurface M of $\bar{M}$ is isoparametric if and only if it has constant principal curvatures.

2.2 Isoparametric hypersurfaces and hypersurfaces with constant principal curvatures in rank one symmetric spaces

The Killing-Hopf theorem establishes that the universal cover of a complete Riemannian manifold of constant sectional curvature is isometric to a round sphere, a hyperbolic space or the Euclidean space, where its sectional curvature is positive, negative or zero, respectively. These spaces constitute the most basic and fundamental examples of Riemannian manifolds.

Let $\bar{M}$ be a simply connected Riemannian manifold of constant sectional curvature. If the sectional curvature of $\bar{M}$ is less or equal than zero, we have complete classifications of isoparametric hypersurfaces or, equivalently, hypersurfaces with constant principal curvatures.

Thus, in $\mathbb{R}^n$ we have the following result proved by Segre [161] in 1938:

Theorem 2.2.1. *A hypersurface of $\mathbb{R}^n$ is isoparametric if and only if it is an open part of one of the following hypersurfaces:*

i) an affine hyperplane $\mathbb{R}^{n-1}$,

ii) a sphere S^{n-1},

iii) a generalized cylinder $\mathsf{S}^k \times \mathbb{R}^{n-k-1}$, where $k \in \{1, \dots, n-2\}$.

In the very same year Cartan [44] classified isoparametric hypersurfaces in real hyperbolic spaces $\mathbb{R}\mathsf{H}^n$.

Theorem 2.2.2. *A hypersurface in a real hyperbolic space $\mathbb{R}\mathsf{H}^n$ is isoparametric if and only if it is an open part of one of the following hypersurfaces:*

i) a totally geodesic $\mathbb{R}\mathsf{H}^{n-1}$, *ii) a tube around a totally geodesic* $\mathbb{R}\mathsf{H}^k$*, where* $k \in \{1, \dots, n-1\}$*,*

iii) a geodesic sphere of $\mathbb{R}\mathsf{H}^n$*,*

iv) a horosphere of $\mathbb{R}\mathsf{H}^n$*.*

An important consequence of these classifications is that every isoparametric hypersurface in $\mathbb{R}^n$ or $\mathbb{R}\mathsf{H}^n$ is an open subset of a homogeneous one. Thus, these theorems provide the classifications of homogeneous hypersurfaces in these spaces. Hence, the relationship between the three properties defined above in these two ambient spaces is locally the one outlined in Figure 2.2.

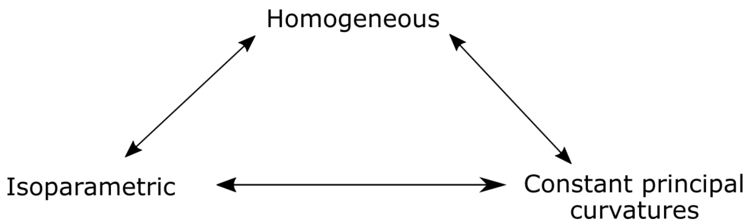

Figure 2.2: Hypersurfaces in $\mathbb{R}^n$ and $\mathbb{R}\mathsf{H}^n$.

The problem in round spheres turned out to be much more complicated. Cartan [44] classified hypersurfaces with $g \in \{1, 2, 3\}$ constant principal curvatures, and with $g = 4$ if all the multiplicities are simple. However, he was not able to solve the general case. Later on, Münzner developed the theory of Cartan further and proved in [139, 140] that the number of distinct principal curvatures of an isoparametric hypersurface must be $g \in \{1, 2, 3, 4, 6\}$. However, there is a main difficulty in the problem of classifiying isoparametric hypersurfaces in S^n: not every isoparametric hypersurface is homogeneous. Surprisingly, in [153], some inhomogeneous examples with $g = 4$ were found.

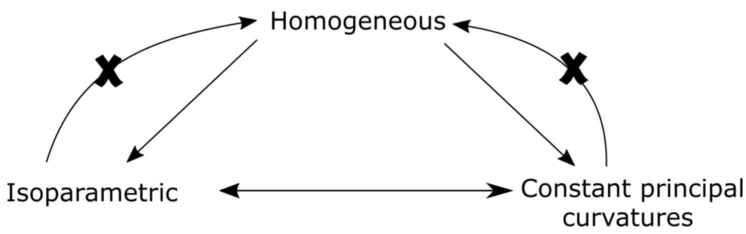

Figure 2.3: Hypersurfaces in S^n.

In 2007-2008, Cecil, Chi, Jensen [45] and Immervoll [97] made substantial progress in the classification of isoparametric hypersurfaces with $g = 4$ distinct principal curvatures. Later, Chi concluded the case $g = 4$ in a series of articles [50, 51, 52]. The last case, $g = 6$, occurs only in S^7 and S^{13}, see [1]. In S^7 such hypersurfaces are homogeneous and they are classified [72, 162]. Miyaoka [134] dealt with the problem in S^{13}, but, as it was explained by Siffert in [162, 163], there seems to be an issue in this article and in the posterior erratum that Miyaoka [135] wrote yielding certain controversy. However, it is believed that every isoparametric hypersurface with $g \in \{1, 2, 3, 6\}$ is homogeneous. We will describe homogeneous hypersurfaces in spheres in Chapter 3.

Let us describe the inhomogeneous isoparametric hypersurfaces with $g = 4$ distinct principal curvatures in round spheres. Let $V = \mathbb{R}^{2n+2}$ be a Euclidean space. We say that an $(m+1)$-tuple $(P_0, \ldots, P_m)$ of real self-adjoint endomorphisms of V is a *Clifford system* in $\mathrm{End}(V)$ if it satisfies:

$$P_i P_j + P_j P_i = 2\delta_{ij}\mathrm{Id},$$

for all $i, j \in \{0, \ldots, m\}$, where δ_{ij} is the Kronecker delta. Let $\mathcal{P}$ be the linear span of a given Clifford system and endow it with the inner product given by $\langle P, P' \rangle = \frac{1}{\dim(V)}\mathrm{tr}(PP')$ for $P, P' \in \mathcal{P}$. Assume that $n - m > 0$. Then, the *FKM foliation* $\mathcal{F}_{\mathcal{P}}$ associated with the Clifford system $(P_0, \ldots, P_m)$ is defined by the level sets of $F_{|\mathsf{S}(V)}$, where $\mathsf{S}(V)$ denotes the unit sphere of V, and $F \colon V \to \mathbb{R}$ is the polynomial:

$$F(x) = \langle x, x \rangle^2 - 2\sum_{i=0}^{m} \langle P_i x, x \rangle^2.$$

By combining multiple results in [45, 50, 51, 52, 97, 153, 154, 167, 173], we have the following:

Theorem 2.2.3. *Let M be an isoparametric hypersurface of $\mathsf{S}^{n-1} \subset \mathbb{R}^n$ with $g = 4$ distinct principal curvatures. Then, M is an open part of a homogeneous hypersurface of S^{n-1} or of a regular leaf of an FKM foliation.*

For spaces with non-constant sectional curvature, the equivalence between isoparametricity and constancy of the principal curvatures is no longer true. In particular, it makes sense to consider some of the Riemannian manifolds with non-constant sectional curvature and simplest curvature tensor such as hyperbolic and projective spaces over a normed division algebra $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. In some of these spaces, we know the existence of isoparametric hypersurfaces that do not have constant principal curvatures. For example, in complex projective spaces, Wang [181] proved the following characterization of isoparametric hypersurfaces with constant principal curvatures:

Theorem 2.2.4. *Let M be an isoparametric hypersurface in $\mathbb{C}\mathsf{P}^n$ with unit normal vector field $\xi \in \Gamma(\nu M)$. Then, the following are equivalent:*

i) M has constant principal curvatures.

ii) $J\xi$ is a principal direction, that is, M is a Hopf real hypersurface.

iii) One focal set of M is a complex submanifold.

In order to provide his example of an isoparametric hypersurface with non-constant principal curvatures, Wang [181] took an inhomogeneous hypersurface in the sphere with $g = 4$ distinct constant principal curvatures in $\mathsf{S}^{8n+7} \subset \mathbb{C}^{4(n+1)}$, with $n \geq 1$, and proved

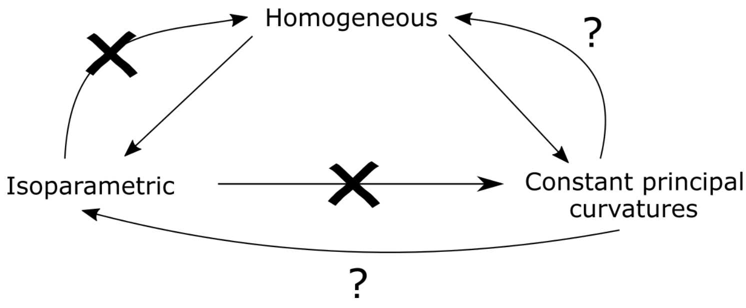

Figure 2.4: Hypersurfaces in $\mathbb{C}\mathsf{P}^n$ and $\mathbb{C}\mathsf{H}^n$.

that its image under the Hopf fibration $\pi\colon \mathsf{S}^{8n+7} \to \mathbb{C}\mathsf{P}^{4n+3}$ does not have complex focal sets.

Furthermore, we know classifications for isoparametric hypersurfaces in $\mathbb{C}\mathsf{P}^n$, with $n \neq 15$ [68], in $\mathbb{H}\mathsf{P}^n$, with $n \neq 7$ [69], and $\mathbb{C}\mathsf{H}^n$, see [64]. It is also known that an isoparametric hypersurface with constant principal curvatures in $\mathbb{C}\mathsf{P}^n$ or $\mathbb{C}\mathsf{H}^n$ is homogeneous. In the first case this follows by combining Theorem 2.2.4 with the classification of Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}\mathsf{P}^n$ [106], and in the second case it follows by the classification in [64].

In $\mathbb{H}\mathsf{H}^n$ or $\mathbb{O}\mathsf{H}^2$ there are examples of inhomogeneous hypersurfaces that are isoparametric and have constant principal curvatures, see Section §4.5 and [60], respectively.

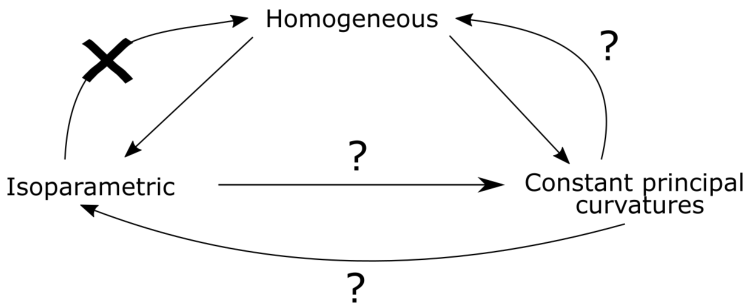

Figure 2.5: Hypersurfaces in $\mathbb{H}\mathsf{P}^n$.

To sum up, the known relations between the three concepts (homogeneity, isoparametricity and constancy of the principal curvatures) for hypersurfaces in symmetric spaces of rank one are explained in Figures 2.2 to 2.6. In $\mathbb{O}\mathsf{P}^2$, no relation is yet known apart from the fact that every homogeneous hypersurface is isoparametric and has constant principal curvatures, which holds for every ambient space.

Furthermore, in Tables 2.1 and 2.2, we summarize the current progress in the classification of isoparametric hypersurfaces in symmetric spaces of rank one. The classification problem for homogeneous hypersurfaces will be discussed in detail in Chapter 3. Finally, it is worth mentioning that the problem of classifying hypersurfaces with constant principal curvatures in symmetric spaces of rank one seems to be really hard, and we only have classifications if we assume that the number of distinct principal curvatures is $g \leq 3$,

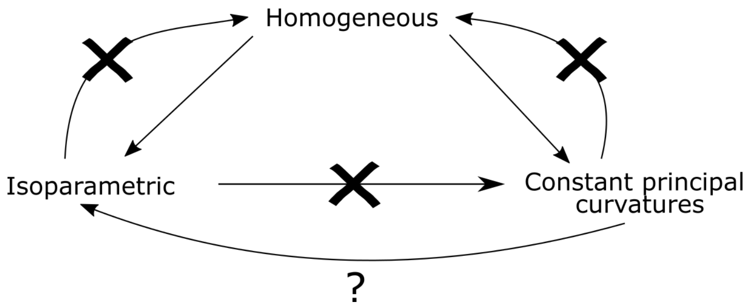

Figure 2.6: Hypersurfaces in $\mathbb{H}\mathsf{H}^n$ and $\mathbb{O}\mathsf{H}^2$.

in $\mathbb{C}\mathsf{P}^n$ (see [170, 171]) or in $\mathbb{C}\mathsf{H}^n$ (see [15, 16]), or if we impose some other hypotheses (see [11, 59, 106, 158]). For the sake of brevity, we will write c.p.c. instead of constant principal curvatures in the tables below, where the tick (respectively, a condition on n or g) means that a complete (respectively, partial) classification has been obtained.

	S^n	$\mathbb{C}\mathsf{P}^n$	$\mathbb{H}\mathsf{P}^n$	$\mathbb{O}\mathsf{P}^2$
Homogeneous	✓	✓	✓	✓
c.p.c.	$n \neq 13$	$g \leq 3$	?	?
Isoparametric	$n \neq 13$	$n \neq 15$	$n \neq 7$	?
Isoparametric + c.p.c.	$n \neq 13$	✓	?	?

Table 2.1: Current progress in the classification of hypersurfaces in symmetric spaces of compact type and rank one.

	$\mathbb{R}\mathsf{H}^n$	$\mathbb{C}\mathsf{H}^n$	$\mathbb{H}\mathsf{H}^n$	$\mathbb{O}\mathsf{H}^2$
Homogeneous	✓	✓	✓	✓
c.p.c.	✓	$g \leq 3$	?	?
Isoparametric	✓	✓	?	?
Isoparametric + c.p.c.	✓	✓	?	?

Table 2.2: Current progress in the classification of hypersurfaces in symmetric spaces of non-compact type and rank one.

2.3 A non-isoparametric hypersurface with constant principal curvatures

In this section we construct a conformally flat metric in $\mathbb{R}^n$ that admits a (non-Riemannian) foliation by totally geodesic, non-isoparametric hyperplanes. Moreover, the metric and the foliation descend to the n-dimensional torus T^n. This provides an example of a non-isoparametric hypersurface with constant principal curvatures in a Riemannian manifold. Also, it shows that the equivalence between isoparametricity and constancy of the principal curvatures in spaces of constant curvature does not hold in the more general setting of conformally flat spaces.

In order to find such a metric, we need the isometry group to be sufficiently small to spoil the good behavior of parallel hypersurfaces. Indeed, if a conformally flat space admits a transitive group of isometries, then it is locally symmetric [172], which would lead us to the apparently outstanding problem of finding such an example in the context of symmetric spaces [65, §6]. On the other hand, we construct the metric so that its isometry group is not too small so as to compute some geodesics explicitly.

2.3.1 The ambient manifold

Let $(x_1, \dots, x_n)$ denote the usual coordinates in $\mathbb{R}^n$ and $(\partial_1, \dots, \partial_n)$ the associated coordinate vector fields. For each $n \geq 2$ we define a metric

$$g_{ij}(x_1, \dots, x_n) := h^2(x_1, \dots, x_n)\delta_{ij},$$

where δ_{ij} is the Kronecker's delta and

$$h(x_1, \dots, x_n) := \prod_{i=1}^{n-1}(2 + \cos(\pi x_i)) \in \mathbb{R}, \text{ for each } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Clearly, g is conformally flat. We will denote $\mathbb{R}^n$ equipped with the metric g by $\bar{M}^n$.

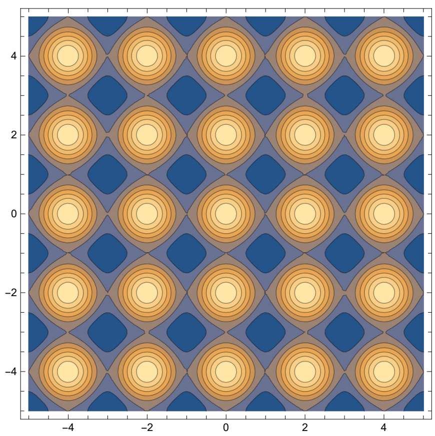

Figure 2.7: Level sets of h on $\mathbb{R}^2$.

Remark 2.3.1. In particular g is invariant under translations of the lattice $2\mathbb{Z}^n$. Hence, our metric g descends to the torus $\mathsf{T}^n = \mathbb{R}^n/(2\mathbb{Z}^n)$.

2.3.2 Christoffel symbols of $\bar{M}^n$

It is known that Christoffel symbols are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \big(g_{jl,i} + g_{li,j} - g_{ij,l} \big),$$

for $i, j, k \in \{1, \ldots, n\}$, where we are using Einstein summation convention and we have denoted the partial derivative with respect to x_i by $_{,i}$. Thus,

$$\Gamma_{ij}^k = \frac{\delta_{jk}}{2h^2} h_{,i}^2 + \frac{\delta_{ki}}{2h^2} h_{,j}^2 - \frac{\delta_{ij}}{2h^2} h_{,k}^2.$$

Now for $n \geq 2$ we have

$$\Gamma_{ii}^i = (\delta_{in} - 1) \frac{\pi \sin(\pi x_i)}{2 + \cos(\pi x_i)}, \qquad \Gamma_{ij}^k = 0, \tag{2.2a}$$

$$\Gamma_{ij}^i = (\delta_{jn} - 1) \frac{\pi \sin(\pi x_j)}{2 + \cos(\pi x_j)}, \qquad \Gamma_{ii}^k = (1 - \delta_{kn}) \frac{\pi \sin(\pi x_k)}{2 + \cos(\pi x_k)}, \tag{2.2b}$$

for mutually distinct $i, j, k \in \{1, \ldots, n\}$.

2.3.3 Some vertical geodesics of $\bar{M}^n$

Let us define

$$\Omega := \{(a_1, \ldots, a_{n-1}, x_n) \in \mathbb{R}^n : a_i \in \mathbb{Z},\ 0 \leq i \leq n-1\}.$$

Let $a = (a_1, \ldots, a_{n-1}, x_n) \in \Omega$ and γ_a be the unit-speed geodesic starting at a with initial direction ∂_n. By the definition of g, the following maps are isometries of $\bar{M}^n$ for each $i = 1, \ldots, n$:

- $\Lambda_i \colon (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n \mapsto (x_1, \ldots, -x_i, \ldots, x_n) \in \mathbb{R}^n$,
- $\Psi_i \colon (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n \mapsto (x_1, \ldots, x_i + 2, \ldots, x_n) \in \mathbb{R}^n$.

Now, for each $i \in \{1, \ldots, n-1\}$, we consider the isometry $\Psi_i^{a_i} \circ \Lambda_i$. Then, we have that $\widetilde{\gamma}_a(t) := \Psi_i^{a_i} \circ \Lambda_i(\gamma_a(t))$ is another geodesic given by

$$\widetilde{\gamma}_a(t) = (\gamma_a^1(t), \ldots, -\gamma_a^i(t) + 2a_i, \ldots \gamma_a^n(t)).$$

But $\widetilde{\gamma}_a(t)$ and $\gamma_a(t)$ have the same initial conditions. Hence, by uniqueness we have that $\gamma_a^i(t) = a_i$ for each $1 \leq i \leq n-1$. Observe that $h(a_1, \ldots, a_{n-1}, x) = 3^\rho$ for any $x \in \mathbb{R}$, where ρ is the number of even entries of $(a_1, \ldots, a_{n-1})$. Thus, since $\gamma_a(t)$ is parametrized by arc length we get that

$$\gamma_a(t) = (a_1, \ldots, a_{n-1}, x_n + 3^{-\rho} t). \tag{2.3}$$

2.3.4 The Jacobi operator

It is clear that $\{\partial_i\}_{i=1}^n$ is an orthogonal global frame for $\bar{M}^n$. We will compute $\bar{R}_{\partial_n}$, the Jacobi operator associated with ∂_n.

All we have to do is to compute the entries $\bar{R}_{innj}$ of the curvature tensor $\bar{R}$ for each $i, j \in \{1, \ldots, n\}$. If $i = n$ or $j = n$, then $\bar{R}_{innj} = 0$. If $i, j \neq n$, then

$$\bar{R}_{innj} = \langle \bar{\nabla}_{\partial_i} \bar{\nabla}_{\partial_n} \partial_n, \partial_j \rangle - \langle \bar{\nabla}_{\partial_n} \bar{\nabla}_{\partial_i} \partial_n, \partial_j \rangle - \langle \bar{\nabla}_{[\partial_i, \partial_n]} \partial_n, \partial_j \rangle.$$

On the one hand

$$\langle \bar{\nabla}_{\partial_i} \bar{\nabla}_{\partial_n} \partial_n, \partial_j \rangle = \langle \bar{\nabla}_{\partial_i} (\Gamma^k_{nn} \partial_k), \partial_j \rangle = \langle \Gamma^k_{nn,i} \partial_k + \Gamma^k_{nn} \Gamma^l_{ik} \partial_l, \partial_j \rangle = h^2 (\Gamma^j_{nn,i} + \Gamma^k_{nn} \Gamma^j_{ik}),$$

and on the other hand

$$\langle \bar{\nabla}_{\partial_n} \bar{\nabla}_{\partial_i} \partial_n, \partial_j \rangle = \langle \bar{\nabla}_{\partial_n} (\Gamma^k_{in} \partial_k), \partial_j \rangle = \langle \Gamma^k_{in,n} \partial_k + \Gamma^k_{in} \Gamma^l_{nk} \partial_l, \partial_j \rangle = h^2 (\Gamma^j_{in,n} + \Gamma^k_{in} \Gamma^j_{nk}).$$

Since $[\partial_i, \partial_j] = 0$, we conclude

$$(\bar{R}_{\partial_n})_{ij} = \begin{cases} h^2 (\Gamma^j_{nn,i} - \Gamma^j_{in,n} + \Gamma^k_{nn} \Gamma^j_{ik} - \Gamma^k_{in} \Gamma^j_{nk}), & \text{if } i, j \neq n \\ 0, & \text{in any other case.} \end{cases} \tag{2.4}$$

2.3.5 The example

Let $\mathcal{F} = \{\mathcal{F}_s\}_{s \in \mathbb{R}}$, where $\mathcal{F}_s = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = s\}$, for each $s \in \mathbb{R}$. It is clear that $\mathcal{F}$ is a foliation of codimension one on $\bar{M}^n$. Let $\mathcal{S}$, $\mathcal{H}$ and $\nu \mathcal{F}_s$ denote the shape operator, the mean curvature and the normal bundle of $\mathcal{F}_s$, respectively. Then, each leaf is totally geodesic since $\partial_n \in \Gamma(\nu \mathcal{F}_s)$, and using (2.2a) and (2.2b), we have that $\langle \mathcal{S}_{\partial n} \partial_i, \partial_j \rangle = -\langle \bar{\nabla}_{\partial_i} \partial_n, \partial_j \rangle = -h^2 \Gamma^j_{in} = 0$, for each $i, j = 1, \ldots, n-1$.

Remark 2.3.2. Again, since $\mathcal{F}$ is invariant by the action of $2\mathbb{Z}^n$, this foliation descends to the torus T^n.

Given any $s \in \mathbb{R}$, let us consider $p \in \mathcal{F}_s$, a unit-speed geodesic $\gamma \colon [0, \varepsilon) \to \bar{M}^n$ with $\gamma(0) = p$ and $\dot{\gamma}(0) \in \nu_p \mathcal{F}_s$ for some $\varepsilon > 0$, and M^r the parallel hypersurface of M at distance $r > 0$ satisfying $\gamma(r) \in M^r$. By the Riccati equation (cf. [86, Equation 3.8]), we have

$$\frac{d}{dr} \mathcal{S}^r_{\dot{\gamma}(r)} = \bar{R}_{\dot{\gamma}(r)} + (\mathcal{S}^r_{\dot{\gamma}(r)})^2, \quad \mathcal{S}^0_{\dot{\gamma}(0)} = \mathcal{S}_{\partial_n},$$

where $\mathcal{S}^r_{\dot{\gamma}(r)}$ is the shape operator of M^r at $\gamma(r)$ with respect to the normal vector $\dot{\gamma}(r)$. Now we take the trace, so

$$\frac{d}{dr} \mathcal{H}^r_{\dot{\gamma}(r)} = \overline{\mathrm{Ric}}(\dot{\gamma}(r), \dot{\gamma}(r)) + ||\mathcal{S}^r_{\dot{\gamma}(r)}||^2, \quad \mathcal{H}^0_{\dot{\gamma}(0)} = \mathcal{H}, \tag{2.5}$$

where $\mathcal{H}^r_{\dot{\gamma}(r)}$ denotes the mean curvature of M^r at $\gamma(r)$, $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\bar{M}^n$ and $|| \cdot ||$ the Hilbert–Schmidt norm of an operator.

Now we prove that no leaf of $\mathcal{F}$ is isoparametric. Let us consider $a \in \mathcal{F}_s \cap \Omega$ for some $s \in \mathbb{R}$. First note that $\dot{\gamma}_a = 3^{-\rho}\partial_n$ by (2.3). By (2.2a) and (2.2b), $\Gamma_{ij}^k(\gamma_a(r)) = 0$ and $\Gamma_{in}^n = 0$. Hence, by (2.4), we have

$$\overline{\mathrm{Ric}}(\dot{\gamma}_a(r), \dot{\gamma}_a(r)) = \sum_{i=1}^{n-1} \Gamma_{nn,i}^i(\gamma_a(r)) = \pi^2(1 - n + \frac{4}{3}\rho),$$

where we recall that ρ is the number of even entries of $(a_1, \dots, a_{n-1})$.

As a consequence, if $a = (0, \dots, 0, s) \in \mathcal{F}_s \cap \Omega$ and $b = (1, \dots, 1, s) \in \mathcal{F}_s \cap \Omega$,

$$\overline{\mathrm{Ric}}(\dot{\gamma}_a(r), \dot{\gamma}_a(r)) = \frac{n-1}{3}\pi^2 > 0 \qquad \text{and} \qquad \overline{\mathrm{Ric}}(\dot{\gamma}_b(r), \dot{\gamma}_b(r)) = (1-n)\pi^2 < 0,$$

for any $r \in \mathbb{R}$.

But in our case, for $r = 0$, we have $||\mathcal{S}_{\dot{\gamma}_a(0)}||^2 = ||\mathcal{S}_{\dot{\gamma}_b(0)}||^2 = 0$. Therefore, by (2.5), we deduce that $\frac{d}{dr}_{|r=0}\mathcal{H}^r_{\dot{\gamma}_a(r)} > 0$ and $\frac{d}{dr}_{|r=0}\mathcal{H}^r_{\dot{\gamma}_b(r)} < 0$. This way we can conclude that, for small $r > 0$, the mean curvature of the parallel hypersurface of $\mathcal{F}_s$ at distance $r > 0$ is not constant. Then, $\mathcal{F}_s$ is not isoparametric.

Chapter 3

Homogeneous hypersurfaces in symmetric spaces

In Chapter 2 we recalled the definition of (extrinsically) homogeneous hypersurface and how these hypersurfaces are related to isoparametric hypersurfaces and hypersurfaces with constant principal curvatures.

The aim of this chapter is to describe the known classification results of homogeneous hypersurfaces in symmetric spaces with a special emphasis on those of rank one.

This chapter is organized in the following way. In Section §3.1, we motivate the study and recall some well-known facts about cohomogeneity one actions. After establishing the relationship between cohomogeneity one actions and homogeneous hypersurfaces, we dedicate Section §3.2 to revise the classification of homogeneous hypersurfaces in symmetric spaces of compact type, specifically focusing on the rank one case. Then, in Section §3.3, we recall the notion of generalized Kähler angle, which will be of enormous relevance for the theory of cohomogeneity one actions on symmetric spaces of non-compact type and rank one. This will be the topic of discussion of Section §3.4. Finally, Section §3.5 is devoted to explaining the program developed by Berndt and Tamaru to classify cohomogeneity one actions on symmetric spaces of non-compact type and higher rank.

3.1 Cohomogeneity one actions

The discipline of geometric analysis uses the tools from the theory of partial differential equations (PDEs) to establish new results in differential geometry. This is due to the fact that many special kinds of geometric structures on a given smooth manifold M are controlled by PDEs.

A possible way to construct these structures is to find a Lie group G acting on M in such a way that the defining PDE is invariant under the action of G. In general the dimension of our problem will be reduced, and it will suffice to construct a solution on a submanifold transversal to the orbits of G on M, since this solution will be transported by the action of G to the rest of M. The simplest scenario happens when G acts transitively on M, and then the PDE turns into an algebraic equation. However, if G acts with cohomogeneity one on M, our initial PDE will be reduced to an ordinary differential equation. These symmetry reduction methods, and particularly cohomogeneity one methods, have been extremely useful and successful on the search for geometric structures on Riemannian manifolds.

An example of application of these cohomogeneity one methods happens when we are looking for nearly Kähler structures. These occur as a distinguished class in the classification of almost Hermitian structures into 16 natural classes, by Gray and Hervella [85]. Recall that a nearly Kähler manifold is an almost Hermitian manifold M with an almost complex structure J such that ∇J is a skew-symmetric $(2,1)$-tensor. For instance, until very recently, the only known complete, simply connected, 6-dimensional nearly Kähler manifolds were:

$$\mathsf{S}^6 = \mathsf{G}_2/\mathsf{SU}_3, \quad \mathsf{S}^3 \times \mathsf{S}^3 = \mathsf{Sp}_1^3/\Delta\mathsf{Sp}_1, \quad \mathbb{C}\mathsf{P}^3 = \mathsf{Sp}_2/(\mathsf{U}_1 \times \mathsf{Sp}_1), \quad F_3 = \mathsf{SU}_3/\mathsf{T}^2,$$

which were constructed in 1968 by Gray and Wolf [190]. All these nearly Kähler structures are homogeneous. A breakthrough in this topic was the construction of the first examples of inhomogeneous nearly Kähler structures on S^6 and $\mathsf{S}^3 \times \mathsf{S}^3$ by Foscolo and Haskins [79] using cohomogeneity one methods.

Another example of the usefulness of these techniques is the construction of the first complete Riemannian manifolds with exceptional holonomies. A complete, simply connected and irreducible Riemannian manifold M that is not locally symmetric satisfies that its holonomy group $\mathrm{Hol}(M)$ is equal to a group in Berger's list:

$$\mathsf{SO}_n, \quad \mathsf{U}_n, \quad \mathsf{SU}_n, \quad \mathsf{Sp}_1\mathsf{Sp}_n, \quad \mathsf{Sp}_n, \quad \mathsf{Spin}_7, \quad \mathsf{G}_2.$$

Initially, Berger [10] included Spin_9 in the list, but it was proved by Alekseevsky [3] that a Riemannian manifold M under the above hypotheses cannot have $\mathrm{Hol}(M) = \mathsf{Spin}_9$. The reason for this particular list was understood thanks to Simons [164] and Olmos [145]. They both proved using different methods that for a Riemannian manifold M under the previous hypotheses, $\mathrm{Hol}(M)$ acts transitively on the unit sphere of T_pM, for every $p \in M$. Hence, by the classification of connected Lie groups acting effectively and transitively on sphere, see [136], $\mathrm{Hol}(M)$ is one of the groups appearing in the above list plus possibly $\mathsf{U}_1\mathsf{Sp}_n$ and Spin_9, but the first one cannot be the holonomy of a Riemannian manifold. At that point it was not still clear if one could still remove Spin_7 and G_2 from the list. However, Bryant [41] gave examples of Riemannian manifolds with holonomy equal to G_2 or Spin_7. The examples were cones over the following homogeneous spaces: the flag manifold $F_3 = \mathsf{SU}_3/\mathsf{T}^2$ and the Berger space $B^7 = \mathsf{SO}_5/\mathsf{SO}_3$, respectively. These metrics turn out to be of cohomogeneity one.

Finally, other contexts where cohomogeneity one actions have appeared are in the study of Einstein metrics and also when looking for Riemannian metrics of positive curvature. On the one hand, in relation to Einstein metrics, Böhm [34] constructed the first examples of inhomogeneous Einstein metrics on spheres. In particular, he endowed S^k, where $k \in \{5,6,7,8,9\}$, with cohomogeneity one Einstein metrics of positive scalar curvature. On the other hand, Eschenburg [76] constructed inhomogeneous spaces of positive sectional curvature. The group $\mathsf{SU}_3 \times \mathsf{SU}_3$ acts on SU_3 by multiplication on the left and right. Let $p \in \mathbb{Z}$. Eschenburg considers the quotient E_p^7 of SU_3 induced by the action of a certain subgroup of $\mathsf{SU}_3 \times \mathsf{SU}_3$ isomorphic to U_1 which is embedded via $f_p \colon \mathsf{U}_1 \to \mathsf{SU}_3 \times \mathsf{SU}_3$, where

$$z \in \mathsf{U}_1 \mapsto (\mathrm{diag}(z,z,z^p), \mathrm{diag}(1,1,z^{p+2})) \in \mathsf{SU}_3 \times \mathsf{SU}_3.$$

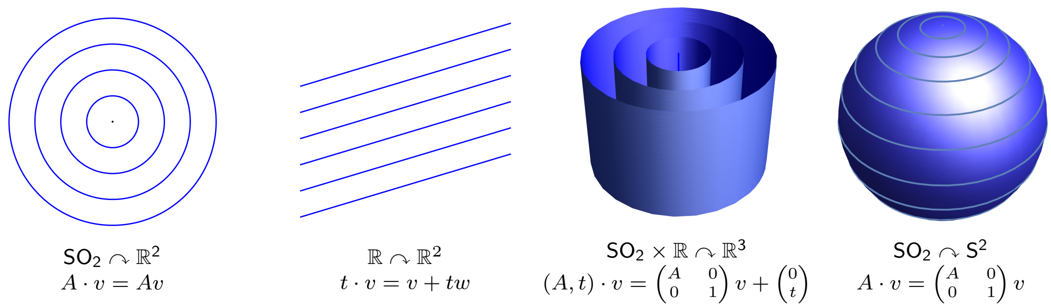

Figure 3.1: Examples of cohomogeneity one actions.

Now, he considers a metric on E_p^7 induced by the positively curved bi-invariant metric on SU_3. This metric is invariant by $\mathsf{SU}_2 \times \mathsf{SU}_2$, which acts on E_p^7 with cohomogeneity one, endowing E_p^7 with a metric of positive sectional curvature when $p \geq 1$.

Keeping the previous discussion in mind, it makes sense to carry out a systematic study of cohomogeneity one actions on Riemannian manifolds. Let (M, g) be a connected Riemannian manifold, and let G be a Lie subgroup of $\mathrm{Isom}(M)$ acting with cohomogeneity one on M. Recall that by a cohomogeneity one action we understand a proper isometric action with codimension one principal orbits. Then, the space of orbits M/G is one of the following ones:

$$\textit{i)}\quad M/\mathsf{G} = \mathbb{R}, \qquad\qquad \textit{ii)}\quad M/\mathsf{G} = [0, +\infty),$$
$$\textit{iii)}\quad M/\mathsf{G} = \mathsf{S}^1, \qquad\qquad \textit{iv)}\quad M/\mathsf{G} = [0, 1].$$

Moreover, it is important to notice that points at the boundary of M/G correspond to non-principal orbits. Thus, they have at most two non-principal orbits. In the cases *i)* and *iii)*, the action of G induces a foliation on M where all the leaves have codimension one. In the case *ii)*, M is G-equivariantly diffeomorphic to a tubular neighborhood of the non-principal orbit $\mathsf{G}\cdot p$, or equivalently, to the total space of the disk bundle $\mathbb{D}_p \to (\mathsf{G}\cdot p) \times_{\mathsf{G}_p} \mathbb{D}_p \to \mathsf{G}/\mathsf{G}_p$, which is the associated bundle to the principal bundle $\mathsf{G}_p \to \mathsf{G} \to \mathsf{G}/\mathsf{G}_p$ with fiber $\mathbb{D}_p$. Finally, in case *iv)*, there are two non-principal orbits $\mathsf{G} \cdot p$ and $\mathsf{G} \cdot q$, and M admits a decomposition as a union of two disk bundles

$$M \cong (\mathsf{G} \times_{\mathsf{G}_p} \mathbb{D}_p) \cup_{\mathsf{G}/\mathsf{K}} (\mathsf{G} \times_{\mathsf{G}_q} \mathbb{D}_q),$$

where $\mathbb{D}_p$ and $\mathbb{D}_q$ are slices at p and q, respectively, K is the isotropy at a point of a principal orbit, and the union of the disks is made along a principal orbit G/K. Some examples of simple cohomogeneity one actions are represented in Figure 3.1.

Our interest in cohomogeneity one actions arises from submanifold geometry. Thus, the study of the geometry of their principal orbits, i.e. homogeneous hypersurfaces, will be our main goal. Hence, it makes sense to consider the following equivalence relation. The actions on M of two subgroups $\mathsf{H}, \mathsf{H}' \subset \mathrm{Isom}(M)$ are *orbit equivalent* if there exists a $\varphi \in \mathrm{Isom}(M)$ such that $\varphi(\mathsf{H} \cdot p) = \mathsf{H}' \cdot \varphi(p)$, for every $p \in M$. Hence, $\varphi \in \mathrm{Isom}(M)$ maps H-orbits to H'-orbits. Then one can pose the following problem:

Classify cohomogeneity one actions on a given
Riemannian manifold M up to orbit equivalence.

This problem turns out to be equivalent to classifying homogeneous hypersurfaces in M up to congruence. As one may expect, the theory of these hypersurfaces will be specially rich on spaces with a large isometry group such as homogeneous spaces, and particularly, symmetric spaces.

3.2 Homogeneous hypersurfaces in symmetric spaces of compact type

In this section, we review the theory of homogeneous hypersurfaces in symmetric spaces of compact type.

Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of $\operatorname{rank}(M) = r \geq 1$, where $\mathsf{G} = \operatorname{Isom}^0(M)$. Since every isometry is uniquely determined by its differential and its image at one point, we have that $\mathsf{K} \subset \mathsf{O}(T_oM)$, where $\mathsf{O}(T_oM)$ denotes the group of linear isometries of T_oM, is a closed subgroup, and it follows that K is compact. Let us consider $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the reductive decomposition of M induced by the geodesic symmetry $s_o \in \operatorname{Isom}(M)$ and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Recall that $\operatorname{rank}(M) = \dim(\mathfrak{a})$. Moreover, it can be proved that $\mathfrak{a}$ meets orthogonally every orbit $\operatorname{Ad}(\mathsf{K})X$, where $X \in \mathfrak{p}$. Hence, K acts polarly on T_oM with a section of dimension r. We say that a real representation $\rho\colon \mathsf{G} \to V$ is an s-representation if it is the isotropy representation of a semisimple symmetric space. Then, s-representations provide examples of polar representations, i.e. polar linear actions on $\mathbb{R}^n$. A converse for this was proved by Dadok [54], and it constitutes a fundamental result in the area.

Theorem 3.2.1. *Every polar representation is orbit equivalent to an s-representation.*

Observe that there are polar representations that are not s-representations. For instance, the standard action of SU_n on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ has as orbits concentric spheres and their common center at $0 \in \mathbb{R}^{2n}$ as a fixed point. This action is orbit equivalent to the isotropy representation of $\mathbb{C}\mathsf{P}^n$. However, it can be checked by using the classification of symmetric spaces that it is not an s-representation.

Since K acts polarly on T_oM with cohomogeneity r, it follows that it acts with cohomogeneity $r-1$ on $\mathsf{S}(T_oM)$, where $\mathsf{S}(T_oM)$ denotes the unit sphere of T_oM. In particular, if M is of rank two, K acts with cohomogeneity one on $\mathsf{S}(T_oM)$. Notice that the isotropy representation is preserved under duality, so we do not get new examples by changing to the compact or non-compact setting. To sum up, the previous discussion provides a method to construct cohomogeneity one actions on round spheres by considering the isotropy representations of symmetric spaces of rank two. Indeed, the next theorem shows that these representations exhaust all the possible cohomogeneity one actions on spheres, see [94].

Theorem 3.2.2. *A hypersurface in S^n is homogeneous if and only if it is a principal orbit of the isotropy representation of a Riemannian symmetric space of rank two.*

Now let us describe the number of distinct principal curvatures g and their multiplicities for homogeneous hypersurfaces in spheres. First of all, as a consequence of Takagi and Takahashi's work [174], $g \in \{1, 2, 3, 4, 6\}$. As we pointed out in Chapter 2, this result was improved by Münzner [139, 140], who proved that the same restriction holds in the more general setting of isoparametric hypersurfaces in spheres. In addition to this, he proved that the principal curvatures of an isoparametric hypersurface in S^n can be written as

$$\lambda_i = \cot \theta_i, \quad \text{where } \theta_i = \theta_1 + \frac{i-1}{g}\pi, \quad \text{and } i \in \{1, \dots, g\}.$$

Furthermore, if the corresponding multiplicities are $m_1, \dots, m_g$, we have $m_i = m_{i+2}$, where the subindices are taken modulo g. Hence, if g is odd, all the multiplicities coincide, and if g is even, it suffices to know m_1 and m_2 to determine all the multiplicities.

The next natural step is to classify homogeneous hypersurfaces in symmetric spaces of compact type and rank one. There have been two different approaches for this problem: one of them is more geometrical and the other one is more topological.

On the one hand, Takagi [172] classified homogeneous hypersurfaces in $\mathbb{C}\mathsf{P}^n$, and D'Atri [56] classified homogeneous hypersurfaces in $\mathbb{H}\mathsf{P}^n$. In both cases they proved that these hypersurfaces are what D'Atri called amenable, and they classified these kind of hypersurfaces. In this setting, this turns out to be equivalent to the notion of curvature-adaptedness. We say that a hypersurface N of M is *curvature adapted* if its shape operator $\mathcal{S}_\xi$, where ξ is a unit normal vector field to N, commutes with the Jacobi operator R_ξ, defined by $R_\xi = R(X, \xi)\xi$, for every tangent vector field X of N, see [11].

On the other hand, Uchida [179] classified cohomogeneity one actions on $\mathbb{C}\mathsf{P}^n$ and Iwata did the same on $\mathbb{H}\mathsf{P}^n$ [99] and $\mathbb{O}\mathsf{P}^2$ [100]. They classified cohomogeneity one actions up to essential isomorphism. Two actions $\mathsf{G} \curvearrowright M$ and $\mathsf{G}' \curvearrowright M'$ are essentially isomorphic if there exist an isomorphism $h\colon \mathsf{G} \to \mathsf{G}'$ and a diffeomorphism $f\colon M \to M'$ such that $f(g \cdot x) = h(g) \cdot f(x)$ for every $g \in \mathsf{G}$ and $x \in M$. Moreover, they do not only classify cohomogeneity one actions on these spaces but in any other space with the same rational cohomology. Their strategy is based on determining the possible rational cohomology of the singular orbits by using the decomposition of M into a union of two disk bundles. Then, they determine the pairs (G, M) using known results about the cohomology of homogeneous spaces and the classification of cohomogeneity one actions on spheres [94].

In what follows, we give an overview of the classification of homogeneous hypersurfaces in $\mathbb{C}\mathsf{P}^n$, $\mathbb{H}\mathsf{P}^n$ and $\mathbb{O}\mathsf{P}^2$.

Let us consider the Hopf projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathsf{P}^n$ and let J be the complex structure of $\mathbb{C}\mathsf{P}^n$ induced by multiplication of i in $\mathbb{C}^{n+1}$. Now, we consider a (possibly reducible) Hermitian symmetric space G/K of rank two, with dimension $\dim \mathsf{G}/\mathsf{K} = 2(n+1)$. Then, one can prove that the isotropy representation of G/K maps $\mathbb{C}$-lines of $T_o(\mathsf{G}/\mathsf{K})$ to $\mathbb{C}$-lines of $T_o(\mathsf{G}/\mathsf{K})$, thus inducing a cohomogeneity one action on $\mathbb{C}\mathsf{P}^n$. Once again, these isotropy representations induce all the possible cohomogeneity one actions on $\mathbb{C}\mathsf{P}^n$. The classification of homogeneous hypersurfaces in $\mathbb{C}\mathsf{P}^n$ was given by Takagi [169].

Theorem 3.2.3. *A hypersurface in* $\mathbb{C}\mathsf{P}^n$ *is homogeneous if and only if it is the image under the Hopf projection* $\pi\colon \mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{C}\mathsf{P}^n$ *of a principal orbit of the isotropy representation of a* $2(n+1)$*-dimensional Hermitian symmetric space of rank two.*

The symmetric spaces whose isotropy representations give rise to these homogeneous hypersurfaces via the Hopf projection are:

$$\mathbb{C}\mathsf{P}^{k+1}\times\mathbb{C}\mathsf{P}^{n-k},\quad \mathsf{G}_2^+(\mathbb{R}^{n+3}),\quad \mathsf{G}_2(\mathbb{C}^{n+3}),\quad \mathsf{SO}_{10}/\mathsf{U}_5,\quad \text{and}\quad \mathsf{E}_6/\mathsf{U}_1\cdot\mathsf{Spin}_{10},$$

where $k\in\{0,\dots,n-1\}$. Here, we denote by $\mathsf{G}_k(\mathbb{F}^n)$ the Grassmannian of k-planes in $\mathbb{F}^n$, where $\mathbb{F}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, and by $\mathsf{G}_k^+(\mathbb{R}^n)$ the Grassmannian of oriented k-planes in $\mathbb{R}^n$.

Let $M=\mathsf{H}\cdot p\subset\mathbb{C}\mathsf{P}^n$ be an orbit of codimension one, where H is some Lie group acting properly and by isometries on $\mathbb{C}\mathsf{P}^n$. Let ξ be a unit normal vector field of M defined around $p\in M$. Notice that, in this case, M is curvature adapted if and only if $J\xi$ is an eigenvector for the shape operator of M, see [11]. We now briefly describe the cohomogeneity one actions on $\mathbb{C}\mathsf{P}^n$ induced (in the way explained above) by the corresponding Hermitian symmetric space of rank two.

The isotropy representation of the Hermitian symmetric space $\mathbb{C}\mathsf{P}^1\times\mathbb{C}\mathsf{P}^n$ induces the isotropy action on $\mathbb{C}\mathsf{P}^n$. The principal orbits of this action are geodesic spheres $\mathsf{S}^{2n-1}=\mathsf{U}_n/\mathsf{U}_{n-1}$, and the singular orbits are a point and a totally geodesic $\mathbb{C}\mathsf{P}^{n-1}$. Moreover, a principal orbit has $g=2$ distinct principal curvatures. The eigenspaces of the shape operator associated with ξ_p are: $\mathbb{R}J\xi_p$ and $T_pM\ominus\mathbb{R}J\xi_p$, which have dimensions 1 and $2n-2$, respectively. By $\mathbb{R}v$ we mean the linear subspace spanned by the vector v, and $\ominus$ denotes orthogonal complement.

The isotropy representation of the Hermitian symmetric space $\mathbb{C}\mathsf{P}^{k+1}\times\mathbb{C}\mathsf{P}^{n-k}$ induces the action of $\mathsf{H}=\mathsf{U}_{k+1}\times\mathsf{U}_{n-k}$ on $\mathbb{C}\mathsf{P}^n$ which has a totally geodesic $\mathbb{C}\mathsf{P}^{k+1}$ and a totally geodesic $\mathbb{C}\mathsf{P}^{n-k}$ as singular orbits, and tubes around any of these totally geodesic submanifolds as principal orbits. These principal orbits are equal to $\mathsf{U}_{k+1}\times\mathsf{U}_{n-k}/(\mathsf{U}_k\times\mathsf{U}_{n-k-1}\times\mathsf{U}_1)$, up to a quotient by a finite subgroup. In this case, the principal orbit $\mathsf{H}\cdot p$ has $g=3$ distinct principal curvatures. Let $q=(q_1,q_2)=\pi^{-1}(p)\in\mathsf{S}^{2n+1}\subset\mathbb{C}^{n+1}$ and assume that $q_i\in V_i\subset\mathbb{C}^{n+1}$ is not zero, where V_1 and V_2 are the complex subspaces invariant under the actions of U_{k+1} and U_{n-k}, respectively. Then, the eigenspaces of the shape operator associated with ξ_p are $\pi_{*q}(V_i)$, for $i\in\{1,2\}$, and the 1-dimensional subspace spanned by $J\xi_p$. The dimensions of $\pi_{*q}(V_1)$ and $\pi_{*q}(V_2)$ are $2k$ and $2(n-k-1)$, respectively.

The isotropy representation of the Hermitian symmetric space $\mathsf{G}_2^+(\mathbb{R}^{n+3})$ induces the action of $\mathsf{H}=\mathsf{SO}_{n+1}$ on $\mathbb{C}\mathsf{P}^n$ which has a totally geodesic submanifold isometric to $\mathbb{R}\mathsf{P}^n$ and a minimal smooth complex quadric $\mathsf{G}_2^+(\mathbb{R}^{n+1})=\mathsf{SO}_{n+1}/(\mathsf{SO}_{n-1}\times\mathsf{SO}_2)$ as singular orbits. The principal orbits are tubes around any of these totally geodesic submanifolds and they are equal to $\mathsf{SO}_{n+1}/\mathsf{SO}_{n-1}$, up to a quotient by a finite subgroup. In this case, any principal orbit $\mathsf{H}\cdot p$ has $g=3$ distinct principal curvatures. Let $q=(q_1,q_2)=\pi^{-1}(p)\in\mathsf{S}^{2n+1}\subset\mathbb{C}^{n+1}$ and assume that $q_i\in W_i\subset\mathbb{C}^{n+1}$ is not zero, where W_1 and W_2 are two orthogonal totally real subspaces of maximal dimension $n+1$ in $\mathbb{C}^{n+1}$ left invariant by the action of H. The eigenspaces of the shape operator associated with ξ_p are $\mathbb{R}J\xi_p$, and $\pi_{*q}W_i$ for $i\in\{1,2\}$. The dimension of $\pi_{*q}W_i$ is $n-1$ for each $i\in\{1,2\}$.

We have seen that the principal orbits of all the actions above can be understood as tubes around reflective submanifolds (see Section §5.3 for the definition), hence totally geodesic submanifolds of $\mathbb{C}\mathsf{P}^n$. The principal orbits of the following actions can be understood as tubes around Kähler submanifolds of $\mathbb{C}\mathsf{P}^n$, hence minimal, but not totally geodesic. They all have $g = 5$ distinct principal curvatures and their multiplicities and associated eigenspaces can be found in [47].

The embedding of the singular orbit $\mathbb{C}\mathsf{P}^1 \times \mathbb{C}\mathsf{P}^k$ of the action induced by $\mathsf{G}_2(\mathbb{C}^{n+3})$ on $\mathbb{C}\mathsf{P}^{2n+1}$ (see Table 3.2) is an example of the so-called Segre embedding of $\mathbb{C}\mathsf{P}^{n_1} \times \mathbb{C}\mathsf{P}^{n_2}$ into $\mathbb{C}\mathsf{P}^{(n_1+1)(n_2+1)-1}$, which is the map given by

$$([z_0, \dots, z_{n_1}], [w_0, \dots, w_{n_2}]) \mapsto [z_0 w_0, \dots, z_0 w_{n_2}, z_1 w_0, \dots, z_1 w_{n_2}, \dots, z_{n_1} w_{n_2}],$$

where $[z_0, \dots, z_{n_1}] \in \mathbb{C}\mathsf{P}^{n_1}$ and $[w_0, \dots, w_{n_2}] \in \mathbb{C}\mathsf{P}^{n_2}$.

The embedding of the singular orbit $\mathsf{G}_2(\mathbb{C}^5)$ (see Table 3.2) of the action induced by $\mathsf{SO}_{10}/\mathsf{U}_5$ on $\mathbb{C}\mathsf{P}^9$ is given by a particular instance of the standard Plücker embedding of $\mathsf{G}_r(\mathbb{C}^n)$ into the projectivization of $\Lambda^r\mathbb{C}^n$. In particular for $r = 2$ and $n = 5$, this is given by $\mathrm{span}\{v_1, v_2\} \mapsto [v_1 \wedge v_2]$, where $\{v_1, v_2\}$ is a basis for a 2-plane in $\mathbb{C}^5$.

Finally, the embedding of the (complex) singular orbit $\mathsf{SO}_{10}/\mathsf{U}_5$ (see Table 3.2) of the action induced by $\mathsf{E}_6/\mathsf{U}_1 \cdot \mathsf{Spin}_{10}$ on $\mathbb{C}\mathsf{P}^{15}$ is rather complicated and is a particular case of the so-called half-spin embedding of $\mathsf{SO}_d/\mathsf{U}_d$ in $\mathbb{C}\mathsf{P}^n$, where $n = 2^{d-1} - 1$, $d \geq 2$, which is a complex embedding. For more details on this embedding, we refer to [47, §7.5].

The classification problem in $\mathbb{H}\mathsf{P}^n$ was solved by Iwata [99] and D'Atri [56], independently. Let us consider the quaternionic Hopf projection $\pi\colon \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{H}\mathsf{P}^n$ and also let $\{J_1, J_2, J_3\}$ be a local basis around some $p \in \mathbb{H}\mathsf{P}^n$ of the quaternionic structure $\mathfrak{J}$ of $\mathbb{H}\mathsf{P}^n$ induced by multiplication of i, j and k in $\mathbb{H}^{n+1}$, respectively.

Theorem 3.2.4. *A hypersurface in $\mathbb{H}\mathsf{P}^n$, $n \geq 2$, is homogeneous if and only if it is the image under the Hopf projection $\pi\colon \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{H}\mathsf{P}^n$ of a principal orbit of the isotropy representation of one of the following symmetric spaces of rank two:*

(1) $\mathbb{H}\mathsf{P}^1 \times \mathbb{H}\mathsf{P}^n$,

(2) $\mathbb{H}\mathsf{P}^{k+1} \times \mathbb{H}\mathsf{P}^{n-k}$, *where* $k \in \{1, \dots, n-2\}$,

(3) $\mathsf{SU}_{n+3}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_{n+1})$.

Remark 3.2.5. One might expect an analogous statement as in Theorem 3.2.3, substituting Hermitian symmetric spaces by quaternionic-Kähler symmetric spaces. However, this is not true since the product of quaternionic projective spaces is clearly not quaternionic-Kähler in general.

Let $M = \mathsf{H} \cdot p \subset \mathbb{H}\mathsf{P}^n$ be an orbit of codimension one of a closed subgroup of isometries of $\mathbb{H}\mathsf{P}^n$ acting with cohomogeneity one. Let ξ be a unit normal vector field of M defined around p. Notice that, in this case, M is curvature adapted if and only if $\mathcal{D}$ is invariant by the shape operator of M, where $\mathcal{D}$ is the maximal subbundle of the tangent bundle of M that is invariant under $\mathfrak{J}$, see [12].

Case (1) in Theorem 3.2.4 induces the action of $\mathsf{H} = \mathsf{Sp}_n\mathsf{Sp}_1$ on $\mathbb{H}\mathsf{P}^n$ given by the isotropy action of $\mathbb{H}\mathsf{P}^n = \mathsf{Sp}_{n+1}/\mathsf{Sp}_n \times \mathsf{Sp}_1$. This action has geodesic spheres $\mathsf{S}^{4n-3} = \mathsf{Sp}_n\mathsf{Sp}_1/\mathsf{Sp}_{n-1}\mathsf{Sp}_1$ as principal orbits, and a point and a totally geodesic $\mathbb{H}\mathsf{P}^{n-1}$ as singular orbits. Moreover, any principal orbit $\mathsf{H} \cdot p$ has $g = 2$ distinct principal curvatures. The eigenspaces of the shape operator associated with ξ_p are $\mathfrak{J}\xi_p$ and $T_pM \ominus \mathfrak{J}\xi_p$, which have dimensions 3 and $4n - 4$, respectively.

The case (2) induces the action of $\mathsf{H} = \mathsf{Sp}_{k+1} \times \mathsf{Sp}_{n-k}$ on $\mathbb{H}\mathsf{P}^n$, which has a totally geodesic $\mathbb{H}\mathsf{P}^{k+1}$ and a totally geodesic $\mathbb{H}\mathsf{P}^{n-k}$ as singular orbits, and tubes around any of these totally geodesic submanifolds as principal orbits. The principal orbits are equal to $\mathsf{Sp}_{k+1} \times \mathsf{Sp}_{n-k}/(\mathsf{Sp}_k \times \mathsf{Sp}_{n-k-1} \times \mathsf{Sp}_1)$, up to a quotient by a finite subgroup. In this case any principal orbit $\mathsf{H} \cdot p$ has $g = 3$ distinct principal curvatures. Let $q = (q_1, q_2) = \pi^{-1}(p) \in \mathsf{S}^{4n+3} \subset \mathbb{H}^{n+1}$ and assume that $q_i \in V_i \subset \mathbb{H}^{n+1}$ is not zero, where V_1 and V_2 are the quaternionic subspaces invariant under the actions of Sp_{k+1} and Sp_{n-k}, respectively. Then, the eigenspaces of the shape operator associated with ξ_p are $\pi_{*q}(V_i)$, for $i \in \{1, 2\}$, and $\mathcal{J}\xi_p$. The dimensions of $\pi_{*q}(V_1)$ and $\pi_{*q}(V_2)$ are $4k$ and $4(n - k - 1)$, respectively.

Finally, case (3) induces the action of $\mathsf{H} = \mathsf{U}_{n+1}$ on $\mathbb{H}\mathsf{P}^n$, which has as singular orbits a totally geodesic submanifold isometric to $\mathbb{C}\mathsf{P}^n$ and a minimal homogeneous space that is equal to $\mathsf{U}_{n+1}/(\mathsf{U}_{n-1} \times \mathsf{SU}_2)$, up to a quotient by a finite subgroup. Every principal orbit is a tube around any of these singular orbits and it is equal to the homogeneous space $\mathsf{U}_{n+1}/(\mathsf{U}_{n-1} \times \mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_1))$, up to a quotient by a finite subgroup. In this case any principal orbit $\mathsf{H} \cdot p$ has $g = 4$ distinct principal curvatures. Let $q = (q_1, q_2) = \pi^{-1}(p) \in \mathsf{S}^{4n+3} \subset \mathbb{H}^{n+1}$ and assume that $q_i \in W_i \subset \mathbb{H}^{n+1}$ is not zero, where W_1 the subspace invariant under multiplication by the imaginary unit i induced by H and W_2 is its orthogonal complement in $\mathbb{H}^{n+1}$. Observe that H leaves W_i invariant for every $i \in \{1, 2\}$. The eigenspaces of $\mathcal{S}_{\xi_p}$ are $\mathbb{R}J_1\xi_p$, $\operatorname{span}\{J_2\xi_p, J_3\xi_p\}$ and $\pi_{*q}W_i$ for $i \in \{1, 2\}$. The dimension of $\pi_{*q}W_i$ is $2(n - 1)$.

The classification problem in $\mathbb{O}\mathsf{P}^2$ was solved by Iwata [100].

Theorem 3.2.6. *A hypersurface in $\mathbb{O}\mathsf{P}^2$ is homogeneous if and only if it is:*

(1) *a geodesic sphere, or*

(2) *a tube around a totally geodesic $\mathbb{H}\mathsf{P}^2$ in $\mathbb{O}\mathsf{P}^2$.*

Firstly, geodesic spheres of $\mathbb{O}\mathsf{P}^2$ can be regarded as principal orbits of the isotropy action of Spin_9 on $\mathbb{O}\mathsf{P}^2$ and, as homogeneous spaces, they are isomorphic to $\mathsf{S}^{15} = \mathsf{Spin}_9/\mathsf{Spin}_7$ and have $g = 2$ distinct principal curvatures. The singular orbits of this action are a fixed point and its cut locus, namely, a totally geodesic $\mathbb{O}\mathsf{P}^1 = \mathsf{S}^8 = \mathsf{Spin}_9/\mathsf{Spin}_8$.

The second hypersurface can be regarded as a principal orbit of the action of $\mathsf{Sp}_3\mathsf{Sp}_1$ on $\mathbb{O}\mathsf{P}^2$. Any such principal orbit is isomorphic to the homogeneous space $\mathsf{Sp}_3\mathsf{Sp}_1/(\mathsf{Sp}_1 \times \mathsf{Sp}_1 \times \mathsf{Sp}_1)$ and has $g = 4$ distinct principal curvatures (see [141]). The singular orbits of this action are a totally geodesic $\mathbb{H}\mathsf{P}^2$ and a minimal $\mathsf{S}^{11} = \mathsf{Sp}_3\mathsf{Sp}_1/\mathsf{Sp}_2\mathsf{Sp}_1$.

In Table 3.1, we list the homogeneous hypersurfaces of symmetric spaces of compact type and rank one, and their focal sets, up to a quotient by a finite subgroup. These were computed in [118]. We denote by ρ_n, μ_n and ν_n the standard representations of SO_n on $\mathbb{R}^n$, SU_n (or U_n) on $\mathbb{C}^n$ and Sp_n on $\mathbb{C}^{2n}$, respectively. Moreover, we denote by Ad the adjoint representation, by λ_3 the 14-dimensional irreducible representation of Sp_3 of quaternionic type, by λ_4 the 26-dimensional irreducible representation of F_4, by Δ_{10}^+ the half-spin representation of Spin_{10} (see [193, Chapter 5]), and we write $-\theta$ to omit the 1-dimensional trival representation.

In 1998, in a monumental work for the theory of homogeneous hypersurfaces, Kollross [113] classified homogeneous hypersurfaces in irreducible symmetric spaces of compact type. Before stating his result, we will recall some well-known facts about Hermann actions.

Let G be a compact semisimple Lie group equipped with a bi-invariant metric and H and K be closed symmetric subgroups of G. This means that H and K are fixed point sets of involutive automorphisms of G. Therefore, (G, H) and (G, K) are compact symmetric pairs. Under these conditions, we say that a *Hermann action* is the action of $\mathsf{H} \times \mathsf{K}$ on G given by

$$(h, k) \cdot g = hgk^{-1}, \quad h \in \mathsf{H}, k \in \mathsf{K}, g \in \mathsf{G}.$$

Clearly, this action induces a natural action of H on the compact symmetric space G/K. In addition to that, it turns out that the slice representation of $\mathsf{H} \times \mathsf{K}$ on G is the same as the slice representation of the action of H on G/K. Hence, the action $\mathsf{H} \times \mathsf{K}$ on G has cohomogeneity one if and only if the action of H on G/K has cohomogeneity one. Moreover, all Hermann actions are hyperpolar and their totally geodesic orbits are reflective, see [143].

The main idea of the work by Kollross is the following. Let $M = \mathsf{G}/\mathsf{K}$ be an irreducible symmetric space of compact type. We start from the top of the lattice of subalgebras of $\mathfrak{g}$, the Lie algebra of G, downwards until we get a subalgebra corresponding to a connected subgroup of G acting with cohomogeneity one. Once we have achieved such a subgroup we stop going down through that branch and we choose a different branch of the lattice. This works because if $\mathfrak{h}$ is properly contained in a subalgebra $\mathfrak{h}'$ whose corresponding connected subgroup H' of G acts with cohomogeneity k, then H acts with cohomogeneity greater or equal than k. Furthermore, if $\mathsf{H} \subset \mathsf{H}'$ and H' acts with cohomogeneity one, we have that either H acts with the same orbits as H' or H acts with larger cohomogeneity.

Remark 3.2.7. As an example of the methods used by Kollross, we provide a proof of the classification of homogeneous hypersurfaces in $\mathbb{O}\mathsf{P}^2$, which was originally obtained by Iwata via topological arguments.

Let $\mathbb{O}\mathsf{P}^2 = \mathsf{G}/\mathsf{K}$, where $\mathsf{G} = \mathsf{F}_4$ and $\mathsf{K} = \mathsf{Spin}_9$. Let H be a subgroup of G acting with cohomogeneity one. By [150, Proposition 3, p. 45], the complexification of a maximal subalgebra $\mathfrak{h}$ of a simple compact Lie algebra $\mathfrak{g}$ is maximal in $\mathfrak{g} \otimes \mathbb{C}$. Then $\mathfrak{h}$, the Lie algebra of H, is contained in one of the following maximal subalgebras (see [74, Table 12 and Table 39]) of $\mathfrak{f}_4$:

$$\mathfrak{so}_9, \quad \mathfrak{sp}_3 \oplus \mathfrak{sp}_1, \quad \mathfrak{g}_2^1 \oplus \mathfrak{su}_2^8, \quad \mathfrak{su}_2^{156},$$

Symmetric space	Representation	Principal Orbit	Singular Orbits	n	g	(m_1, m_2)
$S^1 \times S^{\ell-1}, \ell \geq 3$	$\rho_{\ell-1}$	$S^{\ell-2}$	$\{*\}$	$\ell - 1$	1	$\ell - 2$
$S^{k+1} \times S^{\ell-k-1}$	$\rho_{k+1} + \rho_{\ell-k-1}$	$S^k \times S^{\ell-k-2}$	$S^k, S^{\ell-k-2}$	$\ell - 1$	2	$(k, \ell - k - 2)$
SU_3/SO_3	$\mathrm{Sym}^2\rho_3 - \theta$	SO_3	$\mathbb{R}P^2$	4	3	1
SU_3	Ad	SU_3/T^2	$\mathbb{C}P^2$	7	3	2
SU_6/Sp_3	$\Lambda^2\nu_3 - \theta$	$Sp_3/(Sp_1)^3$	$\mathbb{H}P^2$	13	3	4
E_6/F_4	λ_4	$F_4/Spin_8$	$\mathbb{O}P^2$	25	3	8
Sp_2	Ad	$SO_5/(SO_2 \times SO_2)$	$SO_5/(SO_2 \times SO_3)$	9	4	$(2, 2)$
SO_{10}/U_5	$(\Lambda^2\mu_5)_{\mathbb{R}}$	$SU_5/(SU_2 \times SU_2)$	$SU_5/(SU_2 \times SU_3), SU_5/SO_5$	19	4	$(4, 5)$
$E_6/Spin_{10}U_1$	$(\mu \otimes_{\mathbb{C}} \Delta^+_{10})_{\mathbb{R}}$	$Spin_{10}/Spin_6$	$Spin_{10}/SU_5, Spin_{10}/Spin_7$	31	4	$(9, 6)$
$SO_{k+2}/(SO_2 \times SO_k), k \geq 3$	$\rho_2 \otimes_{\mathbb{R}} \rho_k$	$S^1 \times SO_k/SO_{k-2}$	$S^1 \times S^{k-1}, SO_k/SO_{k-2}$	$2k - 1$	4	$(1, k - 2)$
$SU_{k+2}/S(U_2 \times U_k), k \geq 2$	$(\mu_2 \otimes_{\mathbb{C}} \mu_k)_{\mathbb{R}}$	$S^2 \times U_k/U_{k-2}$	$S^2 \times S^{2k-1}, U_k/U_{k-2}$	$4k - 1$	4	$(2, 2k - 3)$
$Sp_{k+2}/(Sp_2 \times Sp_k), k \geq 3$	$\nu_2 \otimes_{\mathbb{H}} \nu_k$	$S^4 \times Sp_k/Sp_{k-2}$	$S^4 \times S^{4k-1}, Sp_k/Sp_{k-2}$	$8k - 1$	4	$(4, 4k - 5)$
$Sp_4/(Sp_2 \times Sp_2)$	$\nu_2 \otimes_{\mathbb{H}} \nu_2$	$S^7 \times S^7$	$S^4 \times S^7, Sp_2$	15	4	$(1, 3)$
G_2/SO_4	$(\mathrm{Sym}^3\mu_2 \otimes_{\mathbb{C}} \mu_2)_{\mathbb{R}}$	SO_4	SO_4/SO_2	7	6	$(1, 1)$
G_2	Ad	G_2/T^2	G_2/U_2	13	6	$(2, 2)$

Table 3.1: Homogeneous hypersurfaces in S^n.

Symmetric space	Representation	Principal Orbit	Singular Orbits	n	g	Multiplicities
$\mathbb{C}P^1 \times \mathbb{C}P^\ell, \ell \geq 2$	$\mu_1 + \mu_\ell$	$S^{2\ell-1}$	$\{*\}, \mathbb{C}P^{\ell-1}$	ℓ	2	$(1, 2(\ell - 1))$
$\mathbb{C}P^{k+1} \times \mathbb{C}P^{\ell-k}, 1 \leq k \leq \ell - 2$	$\mu_{k+1} + \mu_{\ell-k}$	$U_{k+1} \times U_{\ell-k-1}/(U_k \times U_{\ell-k-2} \times U_1)$	$\mathbb{C}P^k, \mathbb{C}P^{\ell-k-1}$	ℓ	3	$(1, 2(\ell - 1 - k), 2k)$
$SO_{\ell+3}/(SO_2 \times SO_{\ell+1}), \ell \geq 2$	$\rho_2 \otimes_{\mathbb{R}} \rho_\ell$	$SO_{\ell+1}/SO_{\ell-1}$	$\mathbb{R}P^\ell, SO_{\ell+1}/(SO_{\ell-1}SO_2)$	ℓ	3	$(1, \ell - 1, \ell - 1)$
$SU_{\ell+3}/S(U_2 \times U_{\ell+1}), \ell \geq 2$	$(\mu_2 \otimes_{\mathbb{C}} \mu_{\ell+1})_{\mathbb{R}}$	$\mathbb{C}P^1 \times U_{\ell+1}/(U_{\ell-1}U_1)$	$\mathbb{C}P^1 \times \mathbb{C}P^\ell, U_{\ell+1}/(U_{\ell-1}U_1)$	$2\ell + 1$	5	$(1, 2, 2, 2(\ell - 1), 2(\ell - 1))$
SO_{10}/U_5	$(\Lambda^2\mu_5)_{\mathbb{R}}$	$SU_5/S(U_2 \times U_2)$	$SU_5/S(U_2 \times U_3), SU_5/SO_5U_1$	9	5	$(1, 4, 4, 4, 4)$
$E_6/Spin_{10}U_1$	$(\mu \otimes_{\mathbb{C}} \Delta^+_{10})_{\mathbb{R}}$	$Spin_{10}/Spin_6U_1$	$Spin_{10}/U_5, Spin_{10}/Spin_7U_1$	15	5	$(1, 6, 6, 8, 8)$

Table 3.2: Homogeneous hypersurfaces in $\mathbb{C}P^n$.

Symmetric space	Representation	Principal Orbit	Singular Orbits	g	Multiplicities
$\mathbb{H}P^1 \times \mathbb{H}P^n$	$\nu_1 + \nu_n$	S^{4n-3}	$\{*\}, \mathbb{H}P^{n-1}$	2	$(3, 4n - 6)$
$\mathbb{H}P^{k+1} \times \mathbb{H}P^{n-k}, 1 \leq k \leq n - 2$	$\nu_{k+1} + \nu_{n-k}$	$Sp_{k+1} \times Sp_{n-k}/(Sp_k \times Sp_{n-k-1} \times Sp_1)$	$\mathbb{H}P^k, \mathbb{H}P^{n-k-1}$	3	$(3, 4k, 4(n - k - 1))$
$SU_{n+3}/S(U_2 \times U_{n+1})$	$(\mu_2 \otimes_{\mathbb{C}} \mu_{n+1})_{\mathbb{R}}$	$U_{n+1}/(U_{n-1} \times S(U_1 \times U_1))$	$\mathbb{C}P^n, U_{n+1}/(U_{n-1} \times SU_2)$	4	$(1, 2, 2(n - 1))$

Table 3.3: Homogeneous hypersurfaces in $\mathbb{H}P^n$.

Group acting	Representation	Principal Orbit	Singular Orbits	g	Multiplicities
$Spin_9$	Δ_4	S^{15}	$\{*\}, \mathbb{O}P^1$	2	$(7, 8)$
Sp_3Sp_1	$\lambda_3 \otimes \mu_2$	$Sp_3Sp_1/(Sp_1 \times Sp_1 \times Sp_1)$	$\mathbb{H}P^2, S^{11}$	4	$(4, 4, 3, 4)$

Table 3.4: Homogeneous hypersurfaces in $\mathbb{O}P^2$.

where we indicate with a superscript the Dynkin index of the complexified subalgebra, see Section §7.4. Notice that the subgroups of G corresponding to $\mathfrak{so}_9$ and $\mathfrak{sp}_3 \oplus \mathfrak{sp}_1$ act with cohomogeneity one, since they correspond to the isotropy action of $\mathbb{O}\mathsf{P}^2$, and to the action whose principal orbits are tubes around a totally geodesic submanifold of $\mathbb{O}\mathsf{P}^2$ isometric to $\mathbb{H}\mathsf{P}^2$, respectively. Thus, if we proved that the subgroups corresponding to $\mathfrak{g}_2 \oplus \mathfrak{su}_2^8$ and $\mathfrak{su}_2^{156}$ act with cohomogeneity larger than one, we would be done.

The case of $\mathfrak{su}_2^{156}$ follows easily. Observe that, in general, if H is a subgroup acting on M, we have

$$\dim(\mathsf{H}) \geq \dim \mathsf{H} \cdot p = (\dim \mathsf{H} \cdot p - \dim M) + \dim M = \dim M - \operatorname{cohom}(H \cdot p).$$

Hence, for the case corresponding to a subalgebra $\mathfrak{su}_2^{156}$ we cannot have cohomogeneity one actions by dimensional reasons.

Observe that the previous argument does not apply to the case of $\mathfrak{g}_2 \oplus \mathfrak{su}_2^8$. A possible way to tackle this case is to study the slice representation of the action of the H-action on $\mathbb{O}\mathsf{P}^2$ at the base point $o \in \mathbb{O}\mathsf{P}^2$, which coincides with the cohomogeneity of the action of H on $\mathbb{O}\mathsf{P}^2$. Notice that the isotropy of this action at the base point is equal to $\mathsf{H} \cap \mathsf{K}$. Let us assume that $\mathfrak{h} = \mathfrak{g}_2 \oplus \mathfrak{su}_2^8$. We know that the factor of $\mathfrak{h}$ isomorphic to $\mathfrak{g}_2$ is maximally contained in a subalgebra isomorphic to $\mathfrak{spin}_7$ in $\mathfrak{k} \cong \mathfrak{spin}_9$. However, $\mathfrak{g}_2 \oplus \mathfrak{su}_2^8$ cannot be contained in $\mathfrak{spin}_9$, since otherwise it would not be a maximal semisimple subalgebra of $\mathfrak{f}_4$. Hence, $\mathfrak{h} \cap \mathfrak{k}$ is either equal to $\mathfrak{g}_2$ or to $\mathfrak{g}_2 \oplus \mathfrak{u}_1$. However, these subalgebras have dimension 14 and 15, respectively, and we know that the corresponding connected Lie subgroups cannot act transitively on spheres of dimension $\dim \nu_o(\mathsf{H} \cdot o) - 1 = \dim \mathbb{O}\mathsf{P}^2 - \dim \mathsf{H}/\mathsf{G}_2 - 1 = 12$ and $\dim \nu_o(\mathsf{H}/(\mathsf{G}_2\mathsf{U}_1)) - 1 = 13$, respectively. This proves that the subgroup corresponding to $\mathfrak{g}_2 \oplus \mathfrak{su}_2^8$ acts with cohomogeneity larger than one.

The classification theorem obtained by Kollross can be stated as follows.

Theorem 3.2.8. *Let $M = \mathsf{G}/\mathsf{K}$ be an irreducible symmetric space of compact type. A cohomogeneity one action on M is locally orbit equivalent to one of the following actions:*

(1) *a Hermann action of cohomogeneity one (see Table 3.5), or*

(2) *the action of $\{(g, \bar{g}) : g \in \mathsf{SU}_3\}$ on SU_3, or*

(3) *an action induced by the isotropy representation of a symmetric space of rank two, or*

(4) *one of the seven exceptions corresponding to the action of $\mathsf{H} \times \mathsf{K}$ on G, or the action of H on G/K, where $(\mathsf{H}, \mathsf{K}, \mathsf{G})$ is a triple appearing in Table 3.6.*

3.3 A digression: the generalized Kähler angle

In this section we recall the notion of generalized Kähler angle of a vector with respect to a subspace of a Clifford module introduced in [60]. This notion constituted a generalization

H	G	K
SO_{n+1}	SU_{n+1}	$\mathsf{S}(\mathsf{U}_n \times \mathsf{U}_1)$
$\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_{2n-2})$	SU_{2n}	Sp_n
$\mathsf{S}(\mathsf{U}_3 \times \mathsf{U}_{2n-3})$	SU_{2n}	Sp_n
$\mathsf{S}(\mathsf{U}_{p+q} \times \mathsf{U}_1)$	SU_{p+q+1}	$\mathsf{S}(\mathsf{U}_p \times \mathsf{U}_{q+1})$
SO_{p+q}	SO_{p+q+1}	$\mathsf{SO}_p \times \mathsf{SO}_{q+1}$
$\mathsf{Sp}_n \times \mathsf{Sp}_1$	Sp_{n+1}	U_{n+1}
$\mathsf{Sp}_{p+q} \times \mathsf{Sp}_1$	Sp_{p+q+1}	$\mathsf{Sp}_p \times \mathsf{Sp}_{q+1}$
$\mathsf{SO}_2 \times \mathsf{SO}_{2n-2}$	SO_{2n}	U_n
$\mathsf{SO}_3 \times \mathsf{SO}_{2n-3}$	SO_{2n}	U_n
$\mathsf{SU}_6 \cdot \mathsf{SU}_2$	E_6	F_4
$\mathsf{SO}_{10} \cdot \mathsf{SO}_2$	E_6	F_4
$\mathsf{Sp}_3 \cdot \mathsf{Sp}_1$	F_4	Spin_9

Table 3.5: Actions in item (1) of Theorem 3.2.8.

H	G_2	G_2	U_3	Spin_9	$\mathsf{Sp}_1\mathsf{Sp}_n$	SU_3	SU_3
K	$\mathsf{SO}_3 \times \mathsf{SO}_4$	G_2	G_2	$\mathsf{SO}_2 \times \mathsf{SO}_{14}$	$\mathsf{SO}_2 \times \mathsf{SO}_{4n-2}$	SO_4	SU_3
G	SO_7	SO_7	SO_7	SO_{16}	SO_{4n}	G_2	G_2

Table 3.6: Actions in item (4) of Theorem 3.2.8.

of the notion of Kähler angle of a vector of a real subspace in $\mathbb{C}^n$, see [36] or [13]. This concept will be of great relevance for the study of cohomogeneity one actions on hyperbolic spaces, see Section §3.4. We will start by recalling this older notion before introducing the generalized Kähler angle.

Let us endow $\mathbb{C}^n$ with the inner product given by the real part of its standard Hermitian inner product. Let $V \subset \mathbb{C}^n$ be a real subspace. Furthermore, let us denote by $\pi_V \colon \mathbb{C}^n \to V$ the orthogonal projection onto V, and by $J \colon \mathbb{C}^n \to \mathbb{C}^n$ the linear map given by the multiplication by the imaginary unit $i \in \mathbb{C}$.

The *Kähler angle* of a non-zero vector $v \in V$ with respect to V is given by $\measuredangle(Jv, V)$, the angle between Jv and V. Equivalently, this is the value $\varphi \in [0, \pi/2]$ such that

$$||\pi_V Jv||^2 = \cos^2(\varphi)||v||^2.$$

The real subspace $V \subset \mathbb{C}^n$ has constant Kähler angle $\varphi \in [0, \pi/2]$ if

$$\measuredangle(Jv, V) = \varphi \quad \text{for every } v \in V \setminus \{0\}.$$

Notice that there are two extreme cases. On the one hand, $V \subset \mathbb{C}^n$ has constant Kähler angle 0 if and only if it is complex, i.e. it is invariant under J. On the other hand, $V \subset \mathbb{C}^n$ has constant Kähler angle $\pi/2$ if and only if it is totally real, i.e. $JV \subset V^\perp = \mathbb{C}^n \ominus V$. Thus, the Kähler angle provides a way to measure how a real subspace of $\mathbb{C}^n$ fails to be complex.

Example 3.3.1. Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{C}^2$ and consider the real subspace $V = \operatorname{span}\{e_1, \cos(\varphi)Je_1 + \sin(\varphi)e_2\}$ for some $\varphi \in [0, \pi/2]$. We claim that V has constant Kähler angle φ. Let $v = ae_1 + b(\cos(\varphi)Je_1 + \sin(\varphi)e_2)$, where $a^2 + b^2 = 1$. Now,

$$Jv = aJe_1 + b(-\cos(\varphi)e_1 + \sin(\varphi)Je_2),$$

and then $\pi_V Jv = -b\cos(\varphi)e_1 + a\cos(\varphi)(\cos(\varphi)Je_1 + \sin(\varphi)e_2)$. Consequently,

$$||\pi_V Jv||^2 = \cos^2(\varphi)(b^2 + a^2) = \cos^2(\varphi),$$

proving our claim.

Indeed, every real subspace of constant Kähler angle $\varphi \in [0, \pi/2]$ is equal to the direct sum of copies of the subspace V in Example 3.3.1, up to some transformation $T \in \mathsf{U}_n$. Even more, every real subspace of $\mathbb{C}^n$ can be factorized as a direct sum of real subspaces with constant Kähler angle as it was shown in [61]. Here we include an alternative proof of this fact.

Theorem 3.3.2. *Let $V \subset \mathbb{C}^n$ be a real subspace. Then, V admits an unique orthogonal decomposition given by*

$$V = \bigoplus_{\varphi \in \Phi} V_\varphi,$$

where

i) V_φ has constant Kähler angle $\varphi \in [0, \pi/2]$, and

ii) $\mathbb{C}V_\varphi \perp \mathbb{C}V_\psi$, for every distinct $\varphi, \psi \in \Phi$.

Proof. Let $k = \dim V$ and denote by $\mathsf{Stf}_2(V) = \mathsf{SO}_k/\mathsf{SO}_{k-2}$ the Stiefel manifold of orthonormal 2-frames in V, and consider $f\colon \mathsf{Stf}_2(V) \to \mathbb{R}$, which maps a 2-orthonormal frame $(u, v) \in \mathsf{Stf}_2(V)$ to $\langle Ju, v\rangle$. By compactness, there exists some $(u_1, v_1) \in \mathsf{Stf}_2(V)$ such that $\langle Ju_1, v_1\rangle = \cos(\varphi_1)$ is a maximum for f. Observe that $\cos(\varphi_1) = 0$ if and only if V is totally real. Assume V is not totally real. Then, since u_1 and v_1 are perpendicular, there exists some unique unit vector $w_1 \in V \ominus \operatorname{span}_{\mathbb{R}}\{u_1, Ju_1, Jv_1\}$ that satisfies $v_1 = \cos(\varphi_1)Ju_1 + \sin(\varphi_1)w_1$. An analogous computation as in Example 3.3.1 shows that the subspace $V^1_{\varphi_1} := \operatorname{span}_{\mathbb{R}}\{u_1, v_1\}$ has constant Kähler angle $\varphi_1 \in [0, \pi/2]$.

In what follows, we will prove that $V = V^1_{\varphi_1} \oplus (V \ominus V^1_{\varphi_1})$ is a $\mathbb{C}$-orthogonal direct sum. Let $w \in V \ominus V_{\varphi_1}$, and consider the map $g_w\colon [0, 2\pi] \to \mathbb{R}$, given by $g_w(\theta) := \langle Ju_1, \cos(\theta)v_1 + \sin(\theta)w\rangle$ for each $\theta \in [0, 2\pi]$. Now, since f attains a maximum at (u_1, v_1), g_w attains a maximum at $\theta = 0$. Hence, $\langle Ju_1, w\rangle = 0$ and using an analogous argument one proves that $\langle Jv_1, w\rangle = 0$. Thus, $V = V^1_{\varphi_1} \oplus (V \ominus V^1_{\varphi_1})$ is a $\mathbb{C}$-orthogonal direct sum.

Now we proceed inductively, and since V is finite dimensional, we end up factorizing V as an orthogonal sum $V = \bigoplus_{i=1}^r \bigoplus_{j=1}^s V^j_{\varphi_i}$, where $V^j_{\varphi_i}$ has constant Kähler angle φ_i. Moreover, $\mathbb{C}V^j_{\varphi_i} \perp \mathbb{C}V^{j'}_{\varphi_{i'}}$ if $i \neq i'$ or $j \neq j'$. Consequently, if one sets $V_{\varphi_i} := \bigoplus_{j=1}^s V^j_{\varphi_i}$, for every $i \in \{1, \ldots, r\}$, we have the desired orthogonal decomposition for V. Finally, notice that this decomposition is unique by construction. $\square$

Proposition 3.3.3. *Let $V \subset \mathbb{C}^n$ be a real subspace. Then, the following statements are equivalent:*

i) V has constant Kähler angle.

ii) There is some Lie subgroup $\mathsf{H} \subset \mathsf{U}_n$ acting transitively on the unit sphere of V.

Proof. Let us assume that V has constant Kähler angle equal to $\pi/2$. Then it has dimension $k \leq n$. Now, if we consider $\mathbb{C}V$, the complex span of V, there is some subgroup isomorphic to U_k in U_n which acts transitively on the unit sphere of $\mathbb{C}V$. However, now there is a subgroup O_k in U_k acting transitively on the unit sphere of V.

Now assume that V has constant Kähler angle $\varphi < \pi/2$. By the proof of Theorem 3.3.2, V has even dimension $k \leq 2n$. Hence, the map $P\colon V \to V$ given by $P(v) = \frac{1}{\cos(\varphi)}\pi_V Jv$ defines a complex structure on V, which preserves $\langle\cdot,\cdot\rangle$. Thus, $\mathsf{U}(V) = \{g \in \mathsf{GL}(V) : gP = Pg, g^t g = \mathrm{Id}_V\}$ is a group isomorphic to U_k. Any element $g \in \mathsf{U}(V)$ can be extended to $A \in \mathsf{U}_n$ by defining $A(v_1 + Jv_2) = gv_1 + Jgv_2$ and $Aw = w$, for every v_1, $v_2 \in V$ and $w \in \mathbb{C}^n \ominus \mathbb{C}V$. This shows that $\mathsf{U}(V)$ can be regarded as a subgroup of the normalizer of V in U_n. The group $\mathsf{U}(V)$ acts transitively on the unit sphere of V. This proves that *i)* implies *ii)*.

Now let us prove that *ii)* implies *i)*. Let $v, w \in V \subset \mathbb{C}^n$ be two unit vectors. Then, there is $T \in \mathsf{H} \subset \mathsf{U}_n$ such that $Tv = w$ and $TV \subset V$. Let us assume that v has Kähler angle φ with respect to V. Then,

$$\begin{aligned}\cos^2(\varphi) &= \langle \pi_V Jv, \pi_V Jv\rangle = \langle T\pi_V Jv, T\pi_V Jv\rangle = \langle \pi_V TJv, \pi_V TJv\rangle \\ &= \langle \pi_V JTv, \pi_V JTv\rangle = \langle \pi_V Jw, \pi_V Jw\rangle,\end{aligned}$$

where we have used that $T \in \mathsf{U}_n$ and that it preserves V. Then, the Kähler angle of w with respect to V is also $\varphi \in [0, \pi/2]$, and $V \subset \mathbb{C}^n$ has constant Kähler angle $\varphi \in [0, \pi/2]$. □

Let $\mathcal{M}_{k,n}$ be the moduli space of non-zero real subspaces of real dimension k in $\mathbb{C}^n$ with constant Kähler angle. Then, $\mathcal{M}_{k,n}$ is described in the following table.

$\mathcal{M}_{k,n}$	$1 \leq k \leq n$	$n < k \leq 2n$
k odd	$\{\pi/2\}$	$\emptyset$
k even	$[0, \pi/2]$	$\{0\}$

Table 3.7: Sets of possible Kähler angles for k-dimensional real subspaces of $\mathbb{C}^n$.

In the following lines we will recall the notion of generalized Kähler angle introduced in [60]. Let $\mathfrak{v}$ be a Clifford module over $\mathsf{Cl}(\mathfrak{z}, q)$ and $J\colon \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ the restriction to $\mathfrak{z}$ of the underlying Clifford algebra representation, see Subsection §1.4.1 for some basic facts about Clifford modules or representations. We equip $\mathfrak{z}$ with the inner product induced by polarization of $-q$, and extend it to an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, so that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular, and J_Z is an orthogonal map for each unit $Z \in \mathfrak{z}$. Then, $\mathfrak{n}$ can be naturally endowed with a generalized Heisenberg algebra structure as defined in Subsection §1.4.2.

Let $\mathfrak{w}$ be a subspace of $\mathfrak{v}$. We denote by $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{v}$. For each $Z \in \mathfrak{z}$ and $\xi \in \mathfrak{w}$, we write $J_Z\xi = P_Z\xi + F_Z\xi$, where $P_Z\xi$ and $F_Z\xi$ denote the orthogonal projections of $J_Z\xi$ onto $\mathfrak{w}$ and $\mathfrak{w}^\perp$, respectively.

Then, $\xi \in \mathfrak{w}$, $\xi \neq 0$, is said to have *Kähler angle* $\varphi \in [0, \pi/2]$ with respect to the element $Z \in \mathfrak{z}$ (or with respect to J_Z) and the subspace $\mathfrak{w} \subset \mathfrak{v}$ if $\langle P_Z\xi, P_Z\xi\rangle = \cos^2(\varphi)\langle Z, Z\rangle\langle \xi, \xi\rangle$.

The following theorem, cf. [60, Theorem 3.1], is fundamental to understand the definition of generalized Kähler angle which will be introduced inmediately after.

Theorem 3.3.4. *Let $\mathfrak{w}$ be some vector subspace of $\mathfrak{v}$ and let $\xi \in \mathfrak{w}$ be a non-zero vector. Then there exists an orthonormal basis $\{Z_1, \dots, Z_m\}$ of $\mathfrak{z}$ and a uniquely defined m-tuple $(\varphi_1, \dots, \varphi_m)$ such that:*

(i) φ_i *is the Kähler angle of* ξ *with respect to* J_{Z_i}*, for each* $i = 1, \dots, m$.

(ii) $\langle P_{Z_i}\xi, P_{Z_j}\xi\rangle = \langle F_{Z_i}\xi, F_{Z_j}\xi\rangle = 0$ *whenever* $i \neq j$.

(iii) $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_m \leq \pi/2$.

(iv) φ_1 *is minimal and* φ_m *is maximal among the Kähler angles of* ξ *with respect to all the elements of* $\mathfrak{z}$.

Thus, the *generalized Kähler angle* of ξ with respect to $\mathfrak{w}$ is the m-tuple $(\varphi_1, \dots, \varphi_m)$ satisfying properties (i)-(iv) of Theorem 3.3.4.

Remark 3.3.5. Observe that the Kähler angles $\varphi_1, \dots, \varphi_m$ depend, not only on the subspace $\mathfrak{w}$ of $\mathfrak{v}$, but also on the vector $\xi \in \mathfrak{w}$ and the basis $\{Z_1, \dots, Z_m\}$.

A subspace $\mathfrak{w}$ of $\mathfrak{v}$ has *constant generalized Kähler angle* $(\varphi_1, \dots, \varphi_m)$ if the m-tuple $(\varphi_1, \dots, \varphi_m)$ is independent of the unit vector $\xi \in \mathfrak{w}$.

3.4 Cohomogeneity one actions on hyperbolic spaces

In this section we explain the general theory of cohomogeneity one actions on hyperbolic spaces developed by Berndt and Tamaru. Recall that a quick introduction on hyperbolic spaces (rank one symmetric spaces of non-compact type) was developed in Subsection §1.4.3.

We can distinguish three different classes of cohomogeneity one actions on symmetric spaces of non-compact type and rank one, up to orbit equivalence. It was shown in [13] that any such action has at most one singular orbit. The extrinsic geometry of these singular orbits and their tubes, which are homogeneous hypersurfaces, was studied in [60].

Actions with no singular orbit

Berndt and Tamaru [26] classified actions without singular orbits on hyperbolic spaces $\mathbb{F}\mathsf{H}^n$. They proved that there are exactly two such actions up to orbit equivalence. The Iwasawa decomposition $\mathsf{G} = \mathsf{KAN}$ associated with $\mathbb{F}\mathsf{H}^n$ plays an important role in the description of these actions (we refer to Section §1.4 for notation).

(i) The action of N on $\mathbb{F}\mathsf{H}^n$ has cohomogeneity one. The orbits of this action are mutually congruent horospheres that form a regular Riemannian foliation on $\mathbb{F}\mathsf{H}^n$, called the horosphere foliation.

(ii) Let S be the connected Lie subgroup of AN with Lie algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$, where $\mathfrak{w}$ is a vector subspace of $\mathfrak{g}_\alpha$ of codimension one. The action of S on $\mathbb{F}\mathsf{H}^n$ has cohomogeneity one and its orbits form a regular Riemannian foliation on $\mathbb{F}\mathsf{H}^n$, called the solvable foliation. Different choices of $\mathfrak{w}$ lead to conjugate actions.

Actions with a totally geodesic singular orbit

Berndt and Brück [13] classified cohomogeneity one actions on $\mathbb{F}\mathsf{H}^n$ with a totally geodesic singular orbit F. In order to do so it suffices to classify those totally geodesic submanifolds F whose tubes are homogeneous hypersurfaces. Since totally geodesic submanifolds in symmetric spaces of rank one are classified (see Figure 8.1), it suffices to compute the associated slice representation in order to check when the cohomogeneity of the corresponding action is one.

The tubes around a totally geodesic submanifold F of $\mathbb{F}\mathsf{H}^n$ are homogeneous if and only if F is one of the totally geodesic submanifolds listed below:

(i) $\mathbb{F} = \mathbb{R}$: $F \in \{\text{point}, \mathbb{R}\mathsf{H}^1, \dots, \mathbb{R}\mathsf{H}^{n-1}\}$;

(ii) $\mathbb{F} = \mathbb{C}$: $F \in \{\text{point}, \mathbb{C}\mathsf{H}^1, \dots, \mathbb{C}\mathsf{H}^n, \mathbb{R}\mathsf{H}^{n+1}\}$;

(iii) $\mathbb{F} = \mathbb{H}$: $F \in \{\text{point}, \mathbb{H}\mathsf{H}^1, \dots, \mathbb{H}\mathsf{H}^n, \mathbb{C}\mathsf{H}^{n+1}\}$;

(iv) $\mathbb{F} = \mathbb{O}$: $F \in \{\text{point}, \mathbb{O}\mathsf{H}^1, \mathbb{H}\mathsf{H}^2\}$.

Actions with a non-totally geodesic singular orbit

Berndt and Tamaru [28] gave a construction method of all cohomogeneity one actions with a non-totally geodesic singular orbit in hyperbolic spaces. Such actions only appear if $\mathbb{F} \neq \mathbb{R}$. We recall that K_0 acts on the root space $\mathfrak{g}_\alpha$ by the adjoint representation, and hence, if V is a real subspace of $\mathfrak{g}_\alpha$, $N^0_{\mathsf{K}_0}(V)$ will denote the connected component of the identity of the normalizer of V in K_0.

Theorem 3.4.1. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra of the isometry group of the hyperbolic space $M = \mathbb{F}\mathsf{H}^n$, $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

(i) *Let V be a non-zero vector subspace of $\mathfrak{g}_\alpha$ such that $N^0_{\mathsf{K}_0}(V)$ acts transitively on the unit sphere of V. Denote by $\mathfrak{g}_\alpha \ominus V$ the orthogonal complement of V in $\mathfrak{g}_\alpha$. Then the connected subgroup of G with Lie algebra*

$$N_{\mathfrak{k}_0}(V) \oplus \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus V) \oplus \mathfrak{g}_{2\alpha}$$

acts on M with cohomogeneity one, and the orbit through o is singular, provided that $\dim V \geq 2$. *Furthermore, every cohomogeneity one action on M with a non-totally geodesic singular orbit can be obtained in this way up to orbit equivalence.*

(ii) *Let V and V' be vector subspaces of $\mathfrak{g}_\alpha$ as in item* (i)*, and assume that the corresponding cohomogeneity one actions have non-totally geodesic singular orbits. Then, these actions are orbit equivalent if and only if there exists $k \in \mathsf{K}_0$ such that* $\mathrm{Ad}(k)V = V'$.

In the case of real hyperbolic spaces, we have already seen that homogeneous hypersurfaces have constant principal curvatures and these are classified, see Section §2.2. Thus, we only have to deal with hyperbolic spaces over $\mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

Berndt and Tamaru in [28] classified homogeneous hypersurfaces in $\mathbb{C}\mathsf{H}^n$. Their result can be stated as follows:

Theorem 3.4.2 (Homogeneous hypersurfaces in complex hyperbolic spaces)**.** *A homogeneous hypersurface in $\mathbb{C}\mathsf{H}^n$ is congruent to:*

(i) *a geodesic sphere, or*

(ii) *a tube around a totally geodesic $\mathbb{C}\mathsf{H}^k$ in $\mathbb{C}\mathsf{H}^n$, $k \in \{1, \dots, n-1\}$, or*

(iii) *a tube around a totally geodesic $\mathbb{R}\mathsf{H}^n$ in $\mathbb{C}\mathsf{H}^n$, or*

(iv) *a horosphere, or*

(v) *a ruled homogeneous minimal Lohnherr hypersurface $W^{2n-1}_{\pi/2}$, or one of its equidistant hypersurfaces, or*

(vi) *a tube around a ruled homogeneous minimal Berndt–Brück submanifold W^{2n-k}_{φ}, for $k \in \{2, \dots, n-1\}$, with $\varphi \in (0, \pi/2]$, where k is even if $\varphi = \pi/2$.*

The first three examples arise by considering the dual action of the corresponding cohomogeneity one action in the complex projective space $\mathbb{C}\mathsf{P}^n$, see Section §3.2 for the classification and [115] for a detailed exposition on dual actions. These are orbits of the actions of the following subgroups of $\mathsf{SU}_{1,n}$:

$$\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_n), \quad \mathsf{S}(\mathsf{U}_{1,k} \times \mathsf{U}_{n-k}), \quad \mathsf{SO}^0_{1,n},$$

where $k \in \{1, \dots, n-1\}$, respectively.

The homogeneous hypersurfaces in (iv) correspond to the orbits of the action of the group N, which are isometric to generalized Heisenberg groups, and foliate $\mathbb{C}\mathsf{H}^n$. Moreover, the orbits of this action are principal and congruent to each other.

Now we explain the construction of the submanifolds appearing in items (v) and (vi). Let $\mathfrak{w}$ be a real subspace of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, denote by J the complex structure of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ and recall that $\mathsf{K}_0 \cong \mathsf{U}_{n-1}$. By Proposition 3.3.3, it turns out that $\mathfrak{w}^\perp$ has constant Kähler angle if and only if $N_{\mathsf{K}_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$. It was shown by Berndt and Tamaru [28] that the connected subgroup S of $\mathsf{SU}_{1,n}$ whose Lie algebra is $\mathfrak{s} := N_{\mathfrak{k}_0}(\mathfrak{w}) \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ acts on $\mathbb{C}\mathsf{H}^n$ with cohomogeneity one, where $\mathfrak{w}^\perp \subset \mathfrak{g}_\alpha$ has constant Kähler angle $\varphi \in [0, \pi/2]$. The idea of the proof is the following. Let $N^0_{\mathsf{K}}(\mathsf{S})$ be the connected component of the identity of the normalizer of S in K,

$$N_{\mathsf{K}}(S) = \{k \in \mathsf{K} : k\mathsf{S}k^{-1} \subset \mathsf{S}\},$$

which is formed by elements of K that leave S invariant. Therefore, S is an orbit of the action of $N^0_{\mathsf{K}}(\mathsf{S})\mathsf{S}$ on $\mathbb{C}\mathsf{H}^n$. Moreover, $N^0_{\mathsf{K}}(\mathsf{S})$ acts transitively on the unit sphere of the normal space $\nu_o(S \cdot o)$. Let W^{2n-k}_φ be the orbit of this group through the origin $o \cong e\mathsf{K}$, where k is its codimension.

- If $\varphi = 0$, that is, $\mathfrak{w}^\perp$ is a complex subspace of $\mathfrak{g}_\alpha$, then W^{2n-k}_0 is a totally geodesic complex hyperbolic subspace $\mathbb{C}\mathsf{H}^{n-k'}$, where $k = 2k'$.
- If $\varphi = \pi/2$, then $\mathfrak{w}^\perp$ is a totally real k-dimensional subspace of $\mathfrak{g}_\alpha$. If $k = 1$, the corresponding hypersurface $W^{2n-1}_{\pi/2}$ will be denoted by W^{2n-1} and it is known as the *Lohnherr hypersurface*, see [126]. If $k > 1$, then $W^{2n-k}_{\pi/2}$ is a submanifold of dimension $2n-k$ with totally real normal bundle and rank k. We will write W^{2n-k} in this case.
- If $0 < \varphi < \pi/2$, then k is even (see Table 3.3), and W^{2n-k}_φ is a $(2n-k)$-dimensional submanifold of $\mathbb{C}\mathsf{H}^n$, known as a *Berndt-Brück* submanifold, such that its normal bundle has rank k and constant Kähler angle φ. It can be verified that the construction of these submanifolds does not depend on the choice of the Iwasawa decomposition of G, i.e., all possible choices produce submanifolds that are holomorphically congruent among them.

Finally, it turns out that W^{2n-k}_φ is a ruled and minimal submanifold of $\mathbb{C}\mathsf{H}^n$. A proof of this fact can be found in [17]. It is important to mention that W^{2n-1} can be characterized as the unique hypersurface of $\mathbb{C}\mathsf{H}^n$ that is ruled and has constant principal curvatures, see [71] and [126].

The classification of homogeneous hypersurfaces in $\mathbb{H}\mathsf{H}^n$ constitutes the main original contribution of Chapter 4, and will be developed there.

Now let us deal with the classification of homogeneous hypersurfaces in $\mathbb{O}\mathsf{H}^2$, which was achieved in [28].

Theorem 3.4.3 (Homogeneous hypersurfaces in the Cayley hyperbolic plane)**.** *A homogeneous hypersurface in* $\mathbb{O}\mathsf{H}^2$ *is congruent to:*

(i) *a geodesic sphere, or*

(ii) *a tube around a totally geodesic* $\mathbb{O}\mathsf{H}^1$*, or*

(iii) *a tube around a totally geodesic* $\mathbb{H}\mathsf{H}^2$*, or*

(iv) *a horosphere, or*

(v) *a minimal homogeneous hypersurface* F_1*, or one of its equidistant hypersurfaces, or*

(vi) *a tube around the minimal submanifold* F_k *of codimension* $k \in \{2, 3, 6, 7\}$*, or*

(vii) *a tube around the minimal submanifold* $F_{4,\varphi}$ *of codimension* 4*, for some* $\varphi \in [0, 1]$*.*

The first three items correspond to the actions of the following subgroups of F_4^{-20}:

$$\mathsf{Spin}_9, \quad \mathsf{Spin}_{1,8}^0, \quad \mathsf{Sp}_{1,2}\mathsf{Sp}_1.$$

The action of the group N, the nilpotent part of the Iwasawa decompostion of F_4^{-20}, gives rise to the horosphere foliation in $\mathbb{O}\mathsf{H}^2$.

The group $\mathsf{K}_0 \cong \mathsf{Spin}_7$ acts on $\mathfrak{g}_\alpha = \mathbb{O} \cong \mathbb{R}^8$ by its irreducible 8-dimensional spin representation. Let us denote by F_k the singular orbit of the action on $\mathbb{O}\mathsf{H}^2$ of the connected subgroup of F_4^{-20} with Lie algebra $\mathfrak{s} := N_{\mathfrak{k}_0}(\mathfrak{w}) \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $k = \dim \mathfrak{w}^\perp = 8 - \dim \mathfrak{w}$ and $\mathfrak{w}$ is a subspace that plays the same role as V in Theorem 3.4.1. When $k \in \{1, 2, 3, 4, 6, 7\}$, there is a subspace $\mathfrak{w} \subset \mathfrak{g}_\alpha$ such that $N_{\mathsf{K}_0}(\mathfrak{w})$ acts on $\mathfrak{w}$, and transitively on the unit sphere of $\mathfrak{w}^\perp$. If $k = 5$, there is not such a subspace. Also, the spin representation of dimension 8 induces an action of Spin_7 on the k-plane Grassmannian $\mathsf{G}_k(\mathbb{R}^8)$. For $k \in \{1, 2, 3, 6, 7\}$ the restriction of such action on $\mathsf{G}_k(\mathbb{R}^8)$ to $N_{\mathsf{K}_0}^0(\mathfrak{w})$ is transitive, and for $k = 4$, it is of cohomogeneity one, and therefore there is a one-parameter family of non-orbit equivalent cohomogeneity one actions on $\mathbb{O}\mathsf{H}^2$ with a non-totally geodesic singular orbit of dimension four.

The minimal submanifolds F_k or $F_{4,\varphi}$ appearing in items (v), (vi) and (vi) correspond to the orbit that is obtained via the action of the connected subgroup S of F_4^{-20}, whose Lie algebra is $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}^\perp := \mathfrak{g}_\alpha \ominus \mathfrak{w}$ has dimension $k \in \{1, 2, 3, 6, 7\}$. Moreover, the item (v) corresponds to the solvable foliation.

3.5 Homogeneous hypersurfaces in symmetric spaces of non-compact type and arbitrary rank

For completeness, in this section we summarize some of the known results and techniques for the study of homogeneous hypersurfaces in symmetric spaces of non-compact type of higher rank.

As we have seen in Section §3.4, there is a trichotomy when one studies cohomogeneity one actions on a symmetric space of non-compact type $M = \mathsf{G}/\mathsf{K}$. Namely: the action does not have singular orbits (this case was solved in [26]), the action has a totally geodesic singular orbit (this case was solved in [27]), or the action has a singular orbit that is not totally geodesic. This last case is the hardest. In [29], it was proved that a cohomogeneity one action with such a property should be constructed from one of the following two methods: canonical extension or nilpotent construction. The first one allows us to construct cohomogeneity one actions on symmetric spaces of non-compact type by extending cohomogeneity one actions from some special class of totally geodesic submanifolds called boundary components. The second one is much more involved and is related to a representation theory problem that involves addressing the classification of the so-called protohomogeneous subspaces, see [165] for a more general overview.

Cohomogeneity one actions without singular orbits.

Let H be a subgroup acting on a symmetric space of non-compact type $M = \mathsf{G}/\mathsf{K}$ without singular orbits. Then, the action of H is orbit equivalent to the action of a codimension one subgroup of AN which can be of two different types. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots associated with some root space decomposition of $\mathfrak{g}$, the Lie algebra of G.

On the one hand, consider a one-dimensional subspace ℓ in $\mathfrak{a}$, and let $\mathfrak{h}_\ell := (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$, where $\ominus$ denotes orthogonal complement. Let H_ℓ be the connected subgroup of G with Lie algebra $\mathfrak{h}_\ell$. It turns out that the orbits of this action are congruent to each other and they produce a foliation of M. The resulting foliation of M is known as a *foliation of horospherical type*. Furthermore, the actions of H_ℓ and H'_ℓ are orbit equivalent if and only if there exists an isometry of M that induces a symmetry of the Dynkin diagram of $\mathfrak{g}$ taking ℓ to ℓ'.

On the other hand, consider a one-dimensional subspace ℓ of a root space associated with a simple root $\alpha_i \in \Pi$, and let $\mathfrak{h}_i := \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$. Let H_i be the connected subgroup of G with Lie algebra $\mathfrak{h}_i$. The action of H_i induces a foliation of M with exactly one minimal orbit and is called *foliation of solvable type*. Furthermore, the actions of H_i and H_j are orbit equivalent if and only if there exists an isometry of M that induces a symmetry of the Dynkin diagram of $\mathfrak{g}$ taking α_i to α_j.

Cohomogeneity one actions with totally geodesic singular orbits.

Now we will consider the case when the singular orbit is totally geodesic. This case was treated in [27], where the following theorem was proved.

Theorem 3.5.1. *Let F be a totally geodesic submanifold of an irreducible symmetric space of non-compact type $M = \mathsf{G}/\mathsf{K}$. Then, F arises as the singular orbit of a cohomogeneity one action on M if and only if one of the following possibilities holds:*

(i) *F is a reflective submanifold such that the totally geodesic submanifold given by $F^\perp := \exp_p(\nu_p F)$, where $p \in F$, is a symmetric space of rank one (see [27, Theorem 3.3] for the explicit list), or*

(ii) *F is one of five possible non-reflective totally geodesic submanifolds:*

M	$\mathsf{SO}_{3,7}/\mathsf{SO}_3 \times \mathsf{SO}_7$	$\mathsf{SO}_7(\mathbb{C})/\mathsf{SO}_7$	$\mathsf{G}_2^2/\mathsf{SO}_4$	$\mathsf{G}_2^{\mathbb{C}}/\mathsf{G}_2$
F	$\mathsf{G}_2^2/\mathsf{SO}_4$	$\mathsf{G}_2^{\mathbb{C}}/\mathsf{G}_2$	$\mathbb{C}\mathsf{H}^2$, $\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$

Finally, we focus our attention on those cohomogeneity one actions with a singular orbit that is not totally geodesic. For this purpose it will be necessary to introduce some notions related to parabolic subroups and their Lie algebras. We refer to [14, §13.2], [75, §2.17], or [110, §VII.7] for more information.

Parabolic subgroups

Recall that the transitive action $\mathsf{G} \curvearrowright M$ can be extended to the ideal boundary $M(\infty)$ of $M = \mathsf{G}/\mathsf{K}$, see Subsection §1.3.3. A Lie subgroup Q of G is a *parabolic subgroup* if either $\mathsf{Q} = \mathsf{G}$ or Q is the stabilizer G_x of a point at infinity $x \in M(\infty)$. Moreover, a Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is the Lie algebra of a parabolic subgroup Q of G if it contains a subalgebra of $\mathfrak{g}$ conjugate to $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{k}_0 = N_{\mathfrak{k}}(\mathfrak{a})$.

We will associate a parabolic subalgebra $\mathfrak{q}_\Phi$ to a subset $\Phi \subset \Pi$ of simple roots of $\mathfrak{g}$. Let us consider $\Delta_\Phi = \Delta \cap \operatorname{span}\Phi$, the root subsystem generated by Φ, where Δ is the set of roots of $\mathfrak{g}$, and let $\Delta_\Phi^+ = \Delta^+ \cap \Delta_\Phi$, where we have considered the positivity notion on Δ_Φ induced by the one in Δ, see Subsection §1.3.3. Now, we define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \Bigg(\bigoplus_{\alpha\in\Sigma_\Phi} \mathfrak{g}_\alpha\Bigg), \qquad \mathfrak{a}_\Phi = \bigcap_{\alpha\in\Phi} \ker\alpha, \qquad \mathfrak{n}_\Phi = \bigoplus_{\alpha\in\Sigma^+\setminus\Sigma_\Phi^+} \mathfrak{g}_\alpha.$$

The subalgebra $\mathfrak{l}_\Phi$ is invariant by the Cartan involution θ of $\mathfrak{g}$, and is thus reductive, while $\mathfrak{a}_\Phi$ and $\mathfrak{n}_\Phi$ are abelian and nilpotent, respectively. Moreover, consider $\mathfrak{s}_\Phi$ to be the smallest subalgebra of $\mathfrak{g}$ containing $\bigoplus_{\alpha\in\Delta_\Phi}\mathfrak{g}_\alpha$. Let us denote by L_Φ, A_Φ, N_Φ, and S_Φ the connected subgroups of G with Lie algebras $\mathfrak{l}_\Phi$, $\mathfrak{a}_\Phi$, $\mathfrak{n}_\Phi$, and $\mathfrak{s}_\Phi$, respectively.

In what follows, we use the subalgebras defined above to construct the parabolic subalgebra $\mathfrak{q}_\Phi$ and we define certain decompositions of it. The subalgebra $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. We say that $\mathfrak{q}_\Phi$ is the parabolic subalgebra of $\mathfrak{g}$ associated with the subset $\Phi \subset \Pi$. The decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ is known as the *Chevalley decomposition* of $\mathfrak{q}_\Phi$. The subalgebra $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$ is a reductive subalgebra of $\mathfrak{g}$ that normalizes $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$. Hence, we have a decomposition $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, which is known as the *Langlands decomposition* of $\mathfrak{q}_\Phi$. It turns out that every parabolic subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ is conjugate to one of the subalgebras $\mathfrak{q}_\Phi$, for some $\Phi \subset \Pi$, see [14, Theorem 13.2.1]. Moreover, maximal proper parabolic subalgebras are the ones corresponding to subsets of Φ with $r-1$ elements, where r is the rank of M.

Let $\mathfrak{k}_\alpha = \operatorname{proj}_{\mathfrak{k}}\mathfrak{g}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ and $\mathfrak{p}_\alpha = \operatorname{proj}_{\mathfrak{p}}\mathfrak{g}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ for every $\alpha \in \Delta$, where $\operatorname{proj}_{\mathfrak{k}}$ and $\operatorname{proj}_{\mathfrak{p}}$ denote the orthogonal projection on $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Now we construct a Lie subgroup of G with Lie algebra $\mathfrak{m}_\Phi$. Define the possibly disconnected subgroup $\mathsf{K}_\Phi = Z_{\mathsf{K}}(\mathfrak{a}_\Phi)$ of G that has as Lie algebra $\mathfrak{k}_\Phi = \operatorname{proj}_{\mathfrak{k}}\mathfrak{m}_\Phi$. Then, the subgroup $\mathsf{M}_\Phi = \mathsf{K}_\Phi\mathsf{S}_\Phi$ is a subgroup of G with Lie algebra $\mathfrak{m}_\Phi$.

Now, we remark some geometric properties concerning the subgroups related to the subalgebras that we have defined. It follows from the properties of root spaces that the following subspaces of $\mathfrak{p}$ are Lie triple systems (see Section §5.3)

$$\mathfrak{p}_\Phi = \mathfrak{l}_\Phi \cap \mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta_\Phi} \mathfrak{p}_\alpha, \quad \mathfrak{b}_\Phi = \mathfrak{m}_\Phi \cap \mathfrak{p} = \mathfrak{a}^\Phi \oplus \bigoplus_{\alpha \in \Delta_\Phi} \mathfrak{p}_\alpha,$$

where $\mathfrak{a}^\Phi := \mathfrak{a} \ominus \mathfrak{a}_\Phi$. Since $\mathfrak{b}_\Phi$ is a Lie triple system, $B_\Phi = \mathsf{S}_\Phi \cdot o = \mathsf{M}_\Phi/\mathsf{K}_\Phi$ is a totally geodesic submanifold of M that is known as *boundary component* associated with the subset of simple roots Φ. It turns out that B_Φ is intrinsically a symmetric space of non-compact type and rank $|\Phi|$. Furthermore, $\mathsf{A}_\Phi\mathsf{N}_\Phi$ acts freely and polarly on M, with B_Φ as a section, producing minimal orbits that are all congruent. Moreover, the Langlands decomposition of Q_Φ induces a diffeomorphism at the manifold level, given by

$$\mathsf{A}_\Phi \times \mathsf{N}_\Phi \times B_\Phi \to M, \qquad (a, n, m \cdot o) \mapsto (anm) \cdot o,$$

known as the *horospherical decomposition* of the symmetric space M associated with the subset $\Phi \subset \Pi$ of simple roots.

Canonical extension and nilpotent construction.

It was proved by Berndt and Tamaru in [29] that a cohomogeneity one action with a singular orbit on a symmetric space of non-compact type is obtained by one of the following methods: canonical extension or nilpotent construction.

On the one hand, the idea behind the canonical extension method is to consider a cohomogeneity one action on a boundary component B_Φ associated with $\Phi \subset \Pi$ and then extend it to the whole symmetric space $M = \mathsf{G}/\mathsf{K}$. On the other hand, the nilpotent construction, which is a much more involved method, is equivalent to solving a particular representation theory problem.

Let Φ be a subset of Π and consider its associated boundary component B_Φ. Since S_Φ is equal to the identity component of $\mathrm{Isom}(B_\Phi)$, up to some covering, any isometric action on B_Φ has the same orbits as some connected Lie subgroup H_Φ of S_Φ. Consider the subgroup $\mathsf{H}_\Phi^\Pi := \mathsf{H}_\Phi\mathsf{A}_\Phi\mathsf{N}_\Phi$ of G. Thus, H_Φ^Π acts on M with the same cohomogeneity as the action of H_Φ on B_Φ. We say that H_Φ^Π is the subgroup of G obtained by *canonical extension* of H_Φ. Additionally, if the actions of two connected subgroups of S_Φ are orbit equivalent on B_Φ by an isometry in S_Φ, then their canonical extensions are orbit equivalent on M by an element of G, see [29, Proposition 4.2]. Moreover, the composition of canonical extensions is a canonical extension, see [62, Lemma 4.2].

Let us consider $\Phi = \Pi \setminus \{\alpha_j\}$, for some $\alpha_j \in \Pi$, and consider the dual vector $H^j \in \mathfrak{a}$ of α_j, defined by $\alpha_i(H^j) = \delta_{ij}$. The subalgebra $\mathfrak{n}_\Phi$ admits a natural gradation

$$\mathfrak{n}_\Phi = \bigoplus_{\nu \geq 1} \mathfrak{n}_\Phi^\nu, \qquad \text{where } \mathfrak{n}_\Phi^\nu = \bigoplus_{\alpha(H^j) = \nu} \mathfrak{g}_\alpha.$$

The equality $\alpha(H^j) = \nu$ holds if and only if α has coefficient $\nu \in \mathbb{N}$ in α_j when expressed as a sum of simple roots. Let $\mathfrak{v}$ be a linear subspace of dimension at least 2. Then,

$\mathfrak{n}_{\Phi,\mathfrak{v}} := \mathfrak{n}_\Phi \ominus \mathfrak{v}$ is a Lie subalgebra of $\mathfrak{n}_\Phi$. We denote by $\mathsf{N}_{\Phi,\mathfrak{v}}$ the corresponding connected Lie subgroup of N_Φ. Let us assume that the following conditions hold:

(NC1) $N_{\mathsf{M}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}})$ acts transitively on $B_\Phi = M_\Phi \cdot o$,

(NC2) $N_{\mathsf{K}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N_{\mathsf{K}_\Phi}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}$.

Then, the group

$$\mathsf{H}_{\Phi,\mathfrak{v}} = N^0_{\mathsf{L}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}})\mathsf{N}_{\Phi,\mathfrak{v}} = N^0_{\mathsf{M}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}})\mathsf{A}_\Phi\mathsf{N}_{\Phi,\mathfrak{v}}$$

acts on M with cohomogeneity one and a singular orbit $\mathsf{H}_{\Phi,\mathfrak{v}} \cdot o$. In this case, we say that the action of $\mathsf{H}_{\Phi,\mathfrak{v}}$ on M has been obtained by *nilpotent construction* from the choices Φ and $\mathfrak{v}$. On the one hand, condition (NC1) implies that the orbit $\mathsf{H}_{\Phi,\mathfrak{v}} \cdot o$ contains the boundary component B_Φ, and hence its normal space $\nu_o(\mathsf{H}_{\Phi,\mathfrak{v}} \cdot o)$ can be identified with $\mathfrak{v}$. On the other hand, condition (NC2) means that the slice representation of $\mathsf{H}_{\Phi,\mathfrak{v}}$ on $\nu_o(\mathsf{H}_{\Phi,\mathfrak{v}} \cdot o) \cong \mathfrak{v}$ is of cohomogeneity one on the Euclidean space $\nu_o(\mathsf{H}_{\Phi,\mathfrak{v}} \cdot o)$ and the orbits are concentric spheres. Observe that in the rank one case the condition (NC1) is trivially satisfied, thus one only has to classify subspaces of $\mathfrak{n}^1_\Phi$ satisfying condition (NC2). These subspaces will give rise to the notion of protohomogeneous subspaces that we will study in the particular case of quaternionic hyperbolic spaces in Chapter 4. It must be pointed out that only two examples of cohomogeneity one actions on symmetric spaces of rank higher than one not arising through any other technique have been constructed by the nilpotent construction method. They are both related to G_2.

This approach has been used to obtain the complete classification of the cohomogeneity one actions on several symmetric spaces of rank higher than one, see [29], [19], [165]. Namely, up to date, we have complete classifications on the following symmetric spaces of rank 2:

$$\begin{gathered}\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3, \quad \mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3, \quad \mathsf{SL}_3(\mathbb{H})/\mathsf{Sp}_3, \quad \mathsf{SO}_5(\mathbb{C})/\mathsf{SO}_5, \\ \mathsf{G}^2_2/\mathsf{SO}_4, \quad \mathsf{G}^{\mathbb{C}}_2/\mathsf{G}_2, \quad \mathsf{SO}^0_{2,n}/\mathsf{SO}_2\mathsf{SO}_n, \quad \mathsf{SU}_{2,n}/\mathsf{S}(\mathsf{U}_2\mathsf{U}_n).\end{gathered} \tag{3.1}$$

Thus, the remanining symmetric spaces of non-compact type and rank two are:

$$\mathsf{E}^{-26}_6/\mathsf{F}_4, \quad (\mathsf{Sp}_{2,n}/\mathsf{Sp}_2\mathsf{Sp}_n, n \geq 2), \quad \mathsf{E}^{-14}_6/\mathsf{Spin}_{10}\mathsf{U}_1, \quad \mathsf{SO}^*_{10}/\mathsf{U}_5.$$

Recently, a classification in products of rank one symmetric spaces of non-compact type and in $\mathsf{SL}_n(\mathbb{R})/\mathsf{SO}_n$, for every $n \geq 2$, has been obtained in [62].

Chapter 4
Homogeneous hypersurfaces in $\mathbb{H}\mathsf{H}^n$

The aim of this chapter is to present the classification of homogeneous hypersurfaces in quaternionic hyperbolic spaces, and thus, to conclude the classification of homogeneous hypersurfaces in rank one symmetric spaces. This had been an open problem for more than twenty years. As a by-product of our study, we construct for the first time uncountably many inhomogeneous isoparametric families of hypersurfaces with constant principal curvatures in Riemannian manifolds. These results have been published in [63].

The first main result of this chapter can be stated in terms of quaternionic algebra. We denote by $\mathbb{H}$ the real division algebra of the quaternions, endowed with its standard complex structures i, j and k. Let $\mathbb{H}^n$ be a right quaternionic vector space of dimension n. The compact symplectic group Sp_n is the group of quaternionic matrices (acting on the left on $\mathbb{H}^n$) that preserve the standard quaternionic bilinear form $\sum_{i=1}^n \bar{v}_i w_i$, where $v, w \in \mathbb{H}^n$, and bar denotes conjugation. This bilinear form naturally induces an inner product in $\mathbb{H}^n$ that makes it isometric with the Euclidean space $\mathbb{R}^{4n}$. By $\mathfrak{J}$ we will denote the quaternionic structure of $\mathbb{H}^n$, that is, the subspace of real endomorphisms of $\mathbb{H}^n$ generated by the right multiplications by i, j and k, which can therefore be seen as the Lie algebra of Sp_1.

We also consider the Lie group $\mathsf{Sp}_1\mathsf{Sp}_n = \mathsf{Sp}_1 \times \mathsf{Sp}_n/\mathbb{Z}_2$, which acts on $\mathbb{H}^n$ as $(q, A)\cdot v = Avq^{-1}$. This is an important group in differential geometry, as it arises in Berger's holonomy list, that is, the list of Lie groups which can be realized as the holonomy of irreducible, simply connected and non-locally symmetric Riemannian manifolds. Thus, a Riemannian manifold is called quaternionic Kähler if it has dimension $4n$, is not Ricci-flat, and its holonomy is isomorphic to a subgroup of $\mathsf{Sp}_1\mathsf{Sp}_n$, $n \geq 2$. The simplest examples of symmetric, quaternionic Kähler spaces are the quaternionic projective spaces, and their non-compact duals, the quaternionic hyperbolic spaces. In any case, understanding algebraic properties linked to holonomy groups is a first fundamental step towards the study of more geometric questions, such as those related to curvature (e.g. the celebrated LeBrun-Salamon conjecture [121]) or submanifolds (e.g. the theory of calibrations [40]). Similarly, the problem of submanifold geometry that we address in this chapter relies on a linear algebraic problem that we describe below.

We say that a real subspace V of $\mathbb{H}^n$ is *protohomogeneous* if there exists a connected Lie subgroup of $\mathsf{Sp}_1\mathsf{Sp}_n$ that acts transitively on the unit sphere of V. A protohomogeneous subspace of $\mathbb{H}^n$ has constant quaternionic Kähler angle, which is a particular instance of the notion of subspace with constant generalized Kahler angle of a Clifford module, introduced in Section §3.3. As this concept is central in our study, we shall recall it now in the current quaternionic setting. Let π_V denote the orthogonal projection onto a vector subspace V, and define

$$P_J = \pi_V \circ J, \quad \text{where } J \in \mathfrak{J}.$$

We say that V has *constant quaternionic Kähler angle* $(\varphi_1, \varphi_2, \varphi_3)$, with $\varphi_1 \leq \varphi_2 \leq \varphi_3$, if for any $v \in V$ the symmetric bilinear form

$$L_v : \mathfrak{J} \times \mathfrak{J} \to \mathbb{R}, \quad L_v(J, J') = \langle P_J v, P_{J'} v \rangle,$$

has eigenvalues $\cos^2(\varphi_i)\langle v, v\rangle$, $i \in \{1, 2, 3\}$. We point out here the fact that the bilinear forms L_v, $v \in V$, described above do not necessarily diagonalize simultaneously (although we can prove *a priori* that they do so for protohomogeneous subspaces of dimension greater than or equal to 5, see Corollary 4.3.2, and by classification results for dimension different from 3).

The first main result of this chapter is to classify protohomogeneous subspaces of $\mathbb{H}^n$, up to congruence by elements in $\mathsf{Sp}_1\mathsf{Sp}_n$. We present here the moduli space of such subspaces of dimension k in $\mathbb{H}^n$ by exhibiting their possible quaternionic Kähler angles. In Theorem A, and in what follows, $\sqcup$ denotes disjoint union.

Theorem A. *The moduli space $\mathcal{M}_{k,n}$ of non-zero protohomogeneous subspaces of dimension k in $\mathbb{H}^n$, up to congruence in $\mathsf{Sp}_1\mathsf{Sp}_n$, is described in the following table:*

$\mathcal{M}_{k,n}$	$k \leq n$	$n < k \leq \frac{4n}{3}$	$\frac{4n}{3} < k \leq 2n$	$k > 2n$
$k \equiv 0 \pmod 4$	$(\mathfrak{R}_4^+ \setminus \mathfrak{R}_4^-) \sqcup (\mathfrak{R}_4^- \times \mathbb{Z}_2)$	$\mathfrak{S}$	$\{(0,\varphi,\varphi)\}_{\varphi\in[0,\frac{\pi}{2}]}$	$\{(0,0,0)\}$
$k \equiv 2 \pmod 4$	$\{(\varphi,\frac{\pi}{2},\frac{\pi}{2})\}_{\varphi\in[0,\frac{\pi}{2}]}$	$\{(0,\frac{\pi}{2},\frac{\pi}{2})\}$	$\{(0,\frac{\pi}{2},\frac{\pi}{2})\}$	$\emptyset$
$k \neq 3$ odd	$\{(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})\}$	$\emptyset$	$\emptyset$	$\emptyset$
$k = 3$	$(\mathfrak{R}_3^+ \setminus \mathfrak{R}_3^-) \sqcup (\mathfrak{R}_3^- \times \mathbb{Z}_2)$	$\emptyset$	$\{(\varphi,\varphi,\frac{\pi}{2})\}_{\varphi\in\{0,\frac{\pi}{3}\}}$	$\{(0,0,\frac{\pi}{2})\}$

where $\Lambda = \{(\varphi_1, \varphi_2, \varphi_3) \in [0, \pi/2]^3 : \varphi_1 \leq \varphi_2 \leq \varphi_3\}$, *and*

$$\begin{aligned}
\mathfrak{R}_3^+ &= \{(\varphi, \varphi, \pi/2) \in \Lambda : \varphi \in [0, \pi/2]\},\\
\mathfrak{R}_3^- &= \{(\varphi, \varphi, \pi/2) \in \Lambda : \varphi \in [\pi/3, \pi/2)\},\\
\mathfrak{R}_4^+ &= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) - \cos(\varphi_3) \leq 1\},\\
\mathfrak{R}_4^- &= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) \leq 1,\ \varphi_3 \neq \pi/2\},\\
\mathfrak{S} &= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) + \varepsilon\cos(\varphi_3) = 1,\ \textit{for } \varepsilon = \pm 1\}.
\end{aligned}$$

This classification includes typical examples such as totally real subspaces (precisely those with quaternionic Kähler angle $(\pi/2, \pi/2, \pi/2)$), totally complex subspaces (with quaternionic Kähler angle $(0, \pi/2, \pi/2)$), quaternionic subspaces (with quaternionic Kähler angle $(0, 0, 0)$), subspaces of constant Kähler angle $\varphi \in (0, \pi/2)$ inside a totally complex vector subspace (with quaternionic Kähler angle equal to $(\varphi, \pi/2, \pi/2)$), complexifications of subspaces of constant Kähler angle $\varphi \in (0, \pi/2)$ in a totally complex subspace (with quaternionic Kähler angle $(0, \varphi, \varphi)$), and $\mathfrak{J}v$, $v \in \mathbb{H}^n$, $v \neq 0$ (with quaternionic Kähler angle $(0, 0, \pi/2)$). However, there are some other non-classical examples. Some of them were introduced in [60], but there are some others, which are basically presented and classified in Section §4.4. A basis of these subspaces can be calculated explicitly, but for $\mathfrak{R}_3^\pm$ and $\mathfrak{R}_4^\pm$ its expression is rather long. See Proposition 4.4.3 for $\mathfrak{R}_3^\pm$ and Propositions 4.4.10 and 4.5.1 for $\mathfrak{R}_4^\pm$ to get further details. Furthermore, there are non-congruent subspaces of $\mathbb{H}^n$ with the same Kähler angles. These correspond precisely to the intersections $\mathfrak{R}_3^+ \cap \mathfrak{R}_3^- = \mathfrak{R}_3^-$ and $\mathfrak{R}_4^+ \cap \mathfrak{R}_4^- = \mathfrak{R}_4^-$.

We point out here three main tools that have been essential to obtain this classification. First we use the classical generalization of the hairy ball theorem regarding the possible rank of continuous distributions on spheres [166] in order to reduce the classification problem of real subspaces of $\mathbb{H}^n$ with constant quaternionic Kähler angle to subspaces of dimensions 3 and multiples of 4 (Section §4.2). Secondly, we provide a Lie theoretic argument relying on results by Borel [38] and Montgomery and Samelson [136] on groups acting effectively and transitively on spheres, to prove that, for subspaces of dimension greater or equal than 5, the maps L_v that are used to define quaternionic Kähler angle diagonalize simultaneously (Corollary 4.3.2). In third place, using the previous results, we can show that a protohomogeneous subspace of dimension $4l$ is the sum of protohomogeneous subspaces of dimension 4 with the same quaternionic Kähler angle (Subsection §4.3.2). All this reduces the classification of protohomogeneous subspaces to dimensions 3 and 4. At this stage, we actually obtain the more general classification of real subspaces of dimensions 3 and 4 with constant quaternionic Kähler angle. This is a (hard) problem of linear algebra that is solved in Section §4.4.

The first consequence of Theorem A is the classification of cohomogeneity one actions on quaternionic hyperbolic spaces $\mathbb{H}\mathsf{H}^{n+1}$ up to orbit equivalence. In fact, Berndt and Tamaru explained in [28] a method to obtain this classification. This method was explained in Section §3.4 for the general setting of hyperbolic spaces, but we shall particularize it now in the current quaternionic setting. Consider the symmetric pair $(\mathsf{G},\mathsf{K}) = (\mathsf{Sp}_{1,n+1}, \mathsf{Sp}_1 \times \mathsf{Sp}_{n+1})$ representing the symmetric space $\mathbb{H}\mathsf{H}^{n+1}$. We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, which is one-dimensional because $\mathbb{H}\mathsf{H}^{n+1}$ is of rank one. Let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the restricted root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then, $\mathfrak{g}_\alpha$ is isomorphic to a quaternionic vector space $\mathbb{H}^n$ endowed with the standard quaternionic bilinear form, and $\mathsf{K}_0 \cong \mathsf{Sp}_1 \times \mathsf{Sp}_n$, the connected Lie subgroup of G whose Lie algebra is $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k} = N_{\mathfrak{k}}(\mathfrak{a})$, normalizes $\mathfrak{g}_\alpha$ and acts on $\mathfrak{g}_\alpha$ in the canonical way. The classification of cohomogeneity one actions on $\mathbb{H}\mathsf{H}^{n+1}$ can be obtained if we determine the protohomogeneous subspaces V of $\mathfrak{g}_\alpha \cong \mathbb{H}^n$. If V is such a protohomogeneous subspace, we define the Lie subalgebra $\mathfrak{s}_V = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus V) \oplus \mathfrak{g}_{2\alpha}$ of $\mathfrak{g}$, and denote by S_V the connected Lie subgroup of G with Lie algebra $\mathfrak{s}_V$. We recall that $\ominus$ denotes orthogonal complement. Then $N^0_{\mathsf{K}_0}(\mathsf{S}_V)\mathsf{S}_V = N^0_{\mathsf{K}_0}(V)\mathsf{S}_V$ acts on $\mathbb{H}\mathsf{H}^{n+1}$ with cohomogeneity one, where $N^0_{\mathsf{K}_0}(\cdot)$ denotes the connected component of the identity of the normalizer in K_0. Knowing all such subspaces V up to congruence by an element of $\mathsf{Sp}_1\mathsf{Sp}_n$ determines all cohomogeneity one actions on $\mathbb{H}\mathsf{H}^{n+1}$ up to orbit equivalence.

Roughly twenty years after Berndt and Brück [13] announced the first examples of cohomogeneity one actions using this procedure, we obtain the full classification of cohomogeneity one actions on quaternionic hyperbolic spaces up to orbit equivalence as a consequence of Theorem A. Together with the results by Berndt and Tamaru [28], this finishes the classification of cohomogeneity one actions on non-compact symmetric spaces of rank one:

Theorem B. *The moduli space of cohomogeneity one actions on $\mathbb{H}\mathsf{H}^{n+1}$ up to orbit equivalence is given by the disjoint union*

$$\{\mathsf{N}, \mathsf{K}, \mathsf{SU}_{1,n+1}\} \sqcup \bigsqcup_{k=1}^{4n} \mathcal{M}_{k,n}.$$

The actions referenced here are:

(1) N*: the action that produces a horosphere foliation.*

(2) K*: the action that produces a family of geodesic spheres centered at a point.*

(3) $\mathsf{SU}_{1,n+1}$*: the action that produces a family of tubes around a totally geodesic* $\mathbb{C}\mathsf{H}^{n+1}$*.*

(4) $\mathcal{M}_{k,n}$*: the cohomogeneity one actions of the connected Lie subgroups of* $\mathsf{Sp}_{1,n+1}$ *with Lie algebras* $N_{\mathfrak{k}_0}(V) \oplus \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus V) \oplus \mathfrak{g}_{2\alpha}$*, where* V *is a protohomogeneous subspace of dimension* k *of* $\mathfrak{g}_\alpha \cong \mathbb{H}^n$*.*

We note that, in this classification, the action of $\mathsf{Sp}_{1,\ell} \times \mathsf{Sp}_{n+1-\ell} \subset \mathsf{Sp}_{1,n+1}$, which gives tubes around a totally geodesic lower dimensional quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^\ell$, $\ell \in \{1, \dots, n\}$, in $\mathbb{H}\mathsf{H}^{n+1}$, is included in item (4), where in this case V is a quaternionic subspace of $\mathfrak{g}_\alpha \cong \mathbb{H}^n$ (hence, of quaternionic Kähler angle $(0,0,0)$) of real dimension $k = 4(n - \ell + 1)$. Moreover, if we take V a line in $\mathfrak{g}_\alpha$ (i.e. $k = 1$), then $N^0_{\mathsf{K}_0}(V)$ is trivial and we recover the action that gives rise to the so-called solvable foliation [26].

In our study of protohomogeneous subspaces of $\mathbb{H}^n$ we have also encountered non-congruent pairs of subspaces with the same constant quaternionic Kähler angles. Moreover, we prove in Section §4.5 that an $\mathbb{H}$-orthogonal direct sum of subspaces of dimension 4 with the same constant quaternionic Kähler angle is protohomogeneous if and only if any two factors are congruent under an element of Sp_n. However, even if that direct sum is not protohomogeneous, it has constant quaternionic Kähler angle in some cases. Thus, if we take V a non-protohomogeneous subspace with constant quaternionic Kähler angle as above, and denote by S_V the connected subgroup of G whose Lie algebra is $\mathfrak{s}_V = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus V) \oplus \mathfrak{g}_{2\alpha}$, then: (1) since V has constant quaternionic Kähler angle, tubes around $S_V \cdot o$ are isoparametric and have constant principal curvatures by [60, Theorem 4.5], and (2) these tubes are not homogeneous by [28, Theorem 4.1]. Hence, we have the following remarkable consequence:

Theorem C. *There exist uncountably many inhomogeneous isoparametric families of hypersurfaces with constant principal curvatures in* $\mathbb{H}\mathsf{H}^{n+1}$ *with* $n \geq 7$*, up to congruence.*

We recall that the only examples of inhomogeneous isoparametric families of hypersurfaces with constant principal curvatures known so far in any irreducible Riemannian symmetric space are the celebrated examples in spheres by Ferus, Karcher and Münzner [78] and a single example found in the Cayley hyperbolic plane [60]. Thus, this is the first time an uncountable collection of such examples is produced in some symmetric space.

This chapter is organized as follows. The fundamental concept of quaternionic Kähler angle is recalled in Section §4.1 together with some important notation that will be used throughout this chapter. In Section §4.2 we use a generalization of the hairy ball theorem to rule out several possibilities for quaternionic Kähler angles. Then, in Subsection §4.3.1 we prove a simultaneous diagonalization result for subspaces of constant quaternionic Kähler angle. This is used in Subsection §4.3.2 to prove a factorization theorem for protohomogeneous subspaces of dimension multiple of 4. Altogether, this reduces our study to dimensions 3 (Subsection §4.4.1) and 4 (Subsection §4.4.2). The existence of inhomogeneous isoparametric hypersurfaces with constant principal curvatures in quaternionic hyperbolic spaces (Theorem C) is established in Section §4.5. We finally prove Theorems A and B in Section §4.6.

4.1 Quaternionic Kähler angle

We start this section by introducing the main known results concerning the concept of quaternionic Kähler angle, which we recall in this section. Also, we will present some properties and summarize all the examples of subspaces with constant quaternionic Kähler angle known up to the present. The main references for these notions and results are [13], [28], and [60].

The metric and the quaternionic Kähler structure on $\mathbb{H}\mathsf{H}^{n+1}$ induce a positive definite inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_\alpha$ and a quaternionic structure $\mathfrak{J}$ on $\mathfrak{g}_\alpha$, respectively, such that $\mathfrak{g}_\alpha$ is isomorphic to $\mathbb{H}^n$ as a (right) quaternionic Euclidean space. Here, by a quaternionic structure $\mathfrak{J}$ we understand a 3-dimensional vector subspace of $\operatorname{End}_{\mathbb{R}}(\mathbb{H}^n)$, the space of real endomorphisms of $\mathbb{H}^n \cong \mathbb{R}^{4n}$, admitting a basis $\{J_1, J_2, J_3\}$ of orthogonal transformations of $\mathbb{H}^n \cong \mathbb{R}^{4n}$ such that $J_i^2 = -\mathrm{Id}$ and $J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i$, for each $i \in \{1,2,3\}$ (indices modulo 3). Such a basis is called a *canonical basis* of the quaternionic structure $\mathfrak{J}$. Sometimes it is helpful to regard $\mathfrak{J}$ as endowed with a positive definite inner product that makes it isometric to the Euclidean 3-space $\mathbb{R}^3$, and such that the elements of $\mathfrak{J}$ that are orthogonal complex structures of $\mathbb{H}^n$ constitute the unit sphere $\mathsf{S}^2 \subset \mathfrak{J}$ with respect to such inner product. Throughout this chapter, if v is a vector in $\mathbb{H}^n$ and V is a real subspace of $\mathbb{H}^n$ (i.e. a vector subspace of the real vector space $\mathbb{R}^{4n}$ with the underlying real vector space structure of $\mathbb{H}^n$), we denote by $\mathbb{H}v = \mathbb{R}v \oplus \mathfrak{J}v$ and by $\mathbb{H}V = V + \mathfrak{J}V$ the quaternionic spans of $v \in \mathbb{H}^n$ and of $V \subset \mathbb{H}^n$, respectively; sometimes we also write $(\operatorname{Im}\mathbb{H})v$ to refer to $\mathfrak{J}v$.

Theorem 3.4.1, due to Berndt and Tamaru, shows the crucial role played by real subspaces V of $\mathfrak{g}_\alpha \cong \mathbb{H}^n$ and their behavior with respect to K_0 in the classification problem of cohomogeneity one actions on $\mathbb{H}\mathsf{H}^{n+1}$. Note that the effectivization of K_0 on $\mathfrak{g}_\alpha \cong \mathbb{H}^n$ is the Lie group $\mathsf{Sp}_1\mathsf{Sp}_n = (\mathsf{Sp}_1 \times \mathsf{Sp}_n))/\{\pm(1,\mathrm{Id})\}$, which acts in the standard way: $(q, A) \cdot v = Avq^{-1}$, where $q \in \mathsf{Sp}_1$ and $A \in \mathsf{Sp}_n$. Thus, in this subsection we gather some important terminology and useful facts to study real subspaces of a quaternionic Euclidean space, up to congruence by elements of $\mathsf{Sp}_1\mathsf{Sp}_n$.

Firstly, motivated by Theorem 3.4.1, we will say that a real subspace $V \subset \mathbb{H}^n$ is *protohomogeneous* if there is a connected subgroup of $\mathsf{Sp}_1\mathsf{Sp}_n$ that acts transitively on the unit sphere of V. Equivalently, V is protohomogeneous if the connected Lie group $N^0_{\mathsf{Sp}_1\mathsf{Sp}_n}(V)$ acts transitively on the unit sphere of V. Note that protohomogeneous subspaces V of $\mathfrak{g}_\alpha \cong \mathbb{H}^n$ are precisely those inducing cohomogeneity one actions on $\mathbb{H}\mathsf{H}^{n+1}$ via the construction in Theorem 3.4.1(i). We also say that two real subspaces V and W of $\mathbb{H}^n$ are *equivalent* if there exists an element $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV = W$. Observe that, by Theorem 3.4.1(ii), V and W are equivalent protohomogeneous subspaces of $\mathfrak{g}_\alpha \cong \mathbb{H}^n$ if and only if they induce orbit equivalent cohomogeneity one actions on $\mathbb{H}\mathsf{H}^{n+1}$.

Let us now recall a useful description of the action of $\mathsf{Sp}_1\mathsf{Sp}_n$ on $\mathbb{H}^n$. Let us consider $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_n\}$ two $\mathbb{H}$-orthonormal bases of $\mathbb{H}^n$, and let $\{J_1, J_2, J_3\}$ and $\{J_1', J_2', J_3'\}$ be two canonical bases of the quaternionic structure of $\mathbb{H}^n$. Then, there exists a unique $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $T(X_i) = Y_i$ and $TJ_j = J_j'T$ for all $i \in \{1, \dots, n\}$ and all $j \in \{1, 2, 3\}$. Conversely, any $\mathbb{R}$-linear endomorphism of $\mathbb{H}^n$ that maps $\mathbb{H}$-orthonormal bases of $\mathbb{H}^n$ to $\mathbb{H}$-orthonormal bases of $\mathbb{H}^n$ and intertwines canonical bases of the quaternionic structure of $\mathbb{H}^n$ in the above described fashion lies in $\mathsf{Sp}_1\mathsf{Sp}_n$.

Let V be a real vector subspace of the quaternionic Euclidean space $\mathbb{H}^n$. The *Kähler angle* of a non-zero vector $v \in V$ with respect to a non-zero $J \in \mathfrak{J}$ and V is defined to be the angle between Jv and V. Equivalently, it is the value $\varphi \in [0, \pi/2]$ such that $\langle P_J v, P_J v\rangle = \cos^2(\varphi)\langle v, v\rangle$, where $P_J := \pi_V J$ and we denote by π_V the orthogonal projection onto V.

The following lemma was essentially proved by Berndt and Brück [13, Lemma 3]. We state it in a somewhat different form following Theorem 3.3.4, where it was stated in the more general context of subspaces of Clifford modules.

Lemma 4.1.1. *Let V be a real subspace of $\mathbb{H}^n$ and let $v \in V$ be a non-zero vector. Then there exists a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ and a uniquely defined triple $(\varphi_1, \varphi_2, \varphi_3)$, such that:*

(i) *φ_i is the Kähler angle of v with respect to J_i for each $i \in \{1, 2, 3\}$,*

(ii) *$\langle P_i v, P_j v\rangle = 0$ for every $i \neq j$, where $P_i = \pi_V J_i$.*

(iii) *$\varphi_1 \leq \varphi_2 \leq \varphi_3$.*

(iv) *φ_1 is minimal and φ_3 is maximal among the Kähler angles of v with respect to all non-zero elements of $\mathfrak{J}$.*

Indeed, $\{J_1, J_2, J_3\}$ is a basis of $\mathfrak{J}$ with respect to which the symmetric bilinear form

$$L_v\colon \mathfrak{J} \times \mathfrak{J} \to \mathbb{R}, \qquad L_v(J, J') := \langle P_J v, P_{J'} v\rangle,$$

has a diagonal matrix expression with eigenvalues $\cos^2(\varphi_i)\langle v, v\rangle$, $i \in \{1, 2, 3\}$.

The previous lemma allows us to introduce the following definition [13]. If V is a real subspace of $\mathbb{H}^n$, the *quaternionic Kähler angle* of a non-zero vector $v \in V$ with respect to V is the triple $(\varphi_1, \varphi_2, \varphi_3)$ given in Lemma 4.1.1. Sometimes we will also say that $v \in V$

has quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$ with respect to V and to the canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$, in order to specify that the basis $\{J_1, J_2, J_3\}$ is under the conditions of Lemma 4.1.1. A linear subspace V of $\mathbb{H}^n$ is said to have *constant quaternionic Kähler angle* $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$ if the triple $(\varphi_1, \varphi_2, \varphi_3)$ is independent of the non-zero (or by linearity, unit) vector $v \in V$. In this chapter, whenever we use the notation $\Phi(V)$ we will implicitly assume that V has constant quaternionic Kähler angle.

Remark 4.1.2. Note that the $J_i \in \mathfrak{J}$ defined in Lemma 4.1.1 may depend on $v \in V$. This is true, even in the case that V has constant quaternionic Kähler angle. For example $V = \operatorname{Im}\mathbb{H} \subset \mathbb{H}$ has constant quaternionic Kähler angle $\Phi(V) = (0, 0, \pi/2)$, but the basis $\{J_1, J_2, J_3\}$ of Lemma 4.1.1 cannot be chosen independently of $v \in V$. However, we will prove that, under certain hypotheses (see Corollary 4.3.2 or Proposition 4.4.10), the J_i can be chosen independently of $v \in V$. This is one of the crucial results in this chapter.

The following result is known (see [13, p. 229]), but we find it instructive to include a proof.

Lemma 4.1.3. *Let $V \subset \mathbb{H}^n$ be a protohomogeneous subspace. Then, V has constant quaternionic Kähler angle.*

Proof. Let $v \in V$ be a unit vector of quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$ with respect to V and a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$. Thus, $\langle P_i v, P_j v\rangle = \cos^2(\varphi_i)\delta_{ij}$, for $i, j \in \{1, 2, 3\}$, where δ_{ij} stands for Kronecker delta. Let $w \in V$ be a unit vector. Since V is protohomogeneous, there exists $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ that leaves V invariant and satisfies $Tv = w$. By the description of the action of $\mathsf{Sp}_1\mathsf{Sp}_n$ on $\mathbb{H}^n$, there exists a canonical basis $\{J_1', J_2', J_3'\}$ of $\mathfrak{J}$ such that $TJ_i = J_i'T$, for $i \in \{1, 2, 3\}$. Furthermore, since T leaves V invariant, we have that $T\pi_V = \pi_V T$. Hence, $TP_i = P_i'T$ for $i \in \{1, 2, 3\}$, where $P_i' = P_{J_i'} = \pi_V J_i'$. Finally,

$$\langle P_i' w, P_j' w\rangle = \langle P_i' Tv, P_j' Tv\rangle = \langle TP_i v, TP_j v\rangle = \langle P_i v, P_j v\rangle = \cos^2(\varphi_i)\delta_{ij}.$$

Since w is arbitrary, by the last claim of Lemma 4.1.1 we get $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$. $\square$

We now introduce a matrix map that will be very useful in what follows. Let V be a real subspace of $\mathbb{H}^n$ of dimension k, and let $\{J_1, J_2, J_3\}$ be a canonical basis of $\mathfrak{J}$. Then, we define the *Kähler angle map* of V with respect to $\{J_1, J_2, J_3\}$ as the map Ω that sends each unit vector $v \in \mathsf{S}^{k-1} \subset V$ to the symmetric matrix $\Omega(v)$ of order 3 whose (i, j)-entry is given by

$$\Omega(v)_{ij} := \langle P_i v, P_j v\rangle = L_v(J_i, J_j), \tag{4.1}$$

where $P_i = P_{J_i}$, $i \in \{1, 2, 3\}$. A straightforward but important observation is that V has constant quaternionic Kähler angle if and only if the matrices $\Omega(v)$ have the same eigenvalues counted with multiplicities, for any $v \in \mathsf{S}^{k-1}$. In other words, $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$ if and only if the eigenvalues of $\Omega(v)$ are $\cos^2(\varphi_i)$, $i \in \{1, 2, 3\}$, for all unit $v \in V$. This isospectrality property of the Kähler angle map will play a crucial role in this chapter.

Known examples of subspaces with constant quaternionic Kähler angle

We conclude this section by stating some known partial classifications and examples of subspaces V with constant quaternionic Kähler angle in a quaternionic Euclidean space $\mathbb{H}^n$.

In [28], Berndt and Tamaru listed some triples that can arise as constant quaternionic Kähler angles $\Phi(V)$ of non-zero real subspaces V of $\mathbb{H}^n$, and stated the classification of such particular types of subspaces. All the subspaces in this list are protohomogeneous [13, 28]. Such triples are the following:

(1) $\Phi(V) = (\pi/2, \pi/2, \pi/2)$. These are precisely the totally real subspaces of $\mathbb{H}^n$. Recall that a linear subspace $V \subset \mathbb{H}^n$ is totally real if $JV \subset \mathbb{H}^n \ominus V$ for every $J \in \mathfrak{J}$. In this case $\dim_{\mathbb{R}} V \in \{1, 2, \dots, n\}$.

(2) $\Phi(V) = (0, \pi/2, \pi/2)$. These are the totally complex subspaces, that is, the subspaces V of $\mathbb{H}^n$ such that $J_1 V \subset V$ and $JV \subset \mathbb{H}^n \ominus V$ for some complex structure $J_1 \in \mathfrak{J}$ and all $J \in \mathfrak{J}$ perpendicular to J_1. In this case $\dim_{\mathbb{R}} V \in \{2, 4, \dots, 2n\}$.

(3) $\Phi(V) = (0, 0, \pi/2)$. These subspaces are the 3-dimensional subspaces of the form $\mathfrak{J}v = (\operatorname{Im}\mathbb{H})v$ for some non-zero $v \in \mathbb{H}^n$.

(4) $\Phi(V) = (0, 0, 0)$. These are the quaternionic subspaces, that is, the subspaces $V \subset \mathbb{H}^n$ such that $JV \subset V$ for every $J \in \mathfrak{J}$. Hence, $\dim_{\mathbb{R}} V \in \{4, 8, \dots, 4n\}$.

(5) $\Phi(V) = (\varphi, \pi/2, \pi/2)$, $\varphi \in (0, \pi/2)$. Let W be a totally complex subspace of $\mathbb{H}^n$, with $J_1 W \subset W$ for some complex structure $J_1 \in \mathfrak{J}$. Then, a subspace V of $\mathbb{H}^n$ satisfies $\Phi(V) = (\varphi, \pi/2, \pi/2)$ if and only if V is a subspace of some W as before with constant Kähler angle $\varphi \in (0, \pi/2)$ as a subspace of the complex vector space (W, J_1). Thus $\dim_{\mathbb{R}} V \in \{2, 4, \dots, 2[n/2]\}$.

(5) $\Phi(V) = (0, \varphi, \varphi)$. Let W be a totally complex subspace of $\mathbb{H}^n$ such that $J_2 W \subset W$ for some complex structure $J_2 \in \mathfrak{J}$, and let $\tilde{V}$ be a real subspace of (W, J_2) with constant Kähler angle $\varphi \in (0, \pi/2)$. Then, V is a subspace of $\mathbb{H}^n$ with $\Phi(V) = (0, \varphi, \varphi)$ if and only if it is the complexification $V = J_1\tilde{V} \oplus \tilde{V}$ of some $\tilde{V} \subset W$ as before with respect to some complex structure $J_1 \in \mathfrak{J}$ orthogonal to J_2. In this case $\dim_{\mathbb{R}} V \in \{4, 8, \dots, 4[n/2]\}$.

We also recall, as observed in [28, pp. 3434-3435], that:

(i) for each $\ell \in \{1, \dots, n\}$ there exists, up to equivalence, exactly one real subspace V of $\mathbb{H}^n$ with $\dim_{\mathbb{R}} V$ equal to ℓ, 2ℓ or 4ℓ, for each of the types (1), (2) or (4) above, respectively;

(ii) there exists only one subspace V of $\mathbb{H}^n$ of type (3), up to equivalence; and

(iii) for each $\ell \in \{1, \dots, [n/2]\}$ and each $\varphi \in (0, \pi/2)$ there exists exactly one subspace V of $\mathbb{H}^n$ with $\dim_{\mathbb{R}} V = 2\ell$ of type (5), and exactly one subspace V of $\mathbb{H}^n$ with $\dim_{\mathbb{R}} V = 4\ell$ of type (6), up to equivalence.

Berndt and Tamaru conjectured in [28] that these were all the possible subspaces with constant quaternionic Kähler angle, but in [60] new examples of subspaces V of dimension 4 such that $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$ where $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$ were given. These are constructed as follows. Let $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$ with $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$, and consider a 4-dimensional totally real subspace of $\mathbb{H}^n$ and a basis of unit vectors $\{e_0, e_1, e_2, e_3\}$ of it, where $\langle e_0, e_i \rangle = 0$, for $i \in \{1, 2, 3\}$, and

$$\langle e_i, e_{i+1} \rangle = \frac{\cos(\varphi_{i+2}) - \cos(\varphi_i)\cos(\varphi_{i+1})}{\sin(\varphi_i)\sin(\varphi_{i+1})}, \quad i \in \{1, 2, 3\}.$$

For the sake of simplicity let us define $\varphi_0 = 0$ and $J_0 = \mathrm{Id}$. Notice that $\langle J_j e_k, e_l \rangle = 0$ for $j \in \{1, 2, 3\}$ and $k,\ l \in \{0, 1, 2, 3\}$, because $\mathrm{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_3\}$ is a totally real subspace of $\mathbb{H}^n$. Then we can define

$$\xi_k = \cos(\varphi_k) J_k e_0 + \sin(\varphi_k) J_k e_k, \quad k \in \{0, 1, 2, 3\}.$$

(Note that $\xi_0 = e_0$.) We consider the subspace V spanned by these four vectors, for which $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ is an orthonormal basis. Then, $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$. It was also observed in [60] that one can take several copies of these 4-dimensional subspaces to construct subspaces V of $\mathbb{H}^n$ of dimension multiple of 4 with $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$, where $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$. This fact will be proved carefully in Section §4.5 for a broader family of examples that we will provide.

At this point, we find interesting to remark that, unlike the six types of examples known to Berndt and Tamaru in [28], and as we will see in Proposition 4.4.14, we can prove that, for any positive integer k multiple of 4, there are triples $(\varphi_1, \varphi_2, \varphi_3)$ for which there are non-equivalent subspaces V of $\mathbb{H}^n$ with $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$ and $\dim_{\mathbb{R}} V = k$.

4.2 Hairy ball method

In this section we use a topological argument to reduce the classification problem of subspaces V with constant quaternionic Kähler angle in $\mathbb{H}^n$ to the study of subspaces with dimensions 3 and multiples of 4. The idea is to construct a distribution on the unit sphere of the subspace V of $\mathbb{H}^n$, and then use a generalization of the hairy ball theorem to exclude several cases.

Let V be a real subspace of $\mathbb{H}^n$ of real dimension k with constant quaternionic Kähler angle $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$. Let S^{k-1} denote the unit sphere of V. For each $v \in \mathsf{S}^{k-1}$ and $J \in \mathfrak{J}$ we have $\langle P_J v, v \rangle = 0$ and $P_J v \in V$, and thus $P_J v \in T_v \mathsf{S}^{k-1}$. For each $v \in \mathsf{S}^{k-1}$ consider the subspace of $T_v \mathsf{S}^{k-1}$ given by

$$\Delta_v = \{P_J v : J \in \mathfrak{J}\}.$$

Since V has constant quaternionic Kähler angle, the dimension of Δ_v is independent of $v \in \mathsf{S}^{k-1}$. Hence, Δ defines a smooth distribution on the sphere S^{k-1}, and its rank coincides with the number of elements $i \in \{1, 2, 3\}$ such that $\varphi_i \neq \pi/2$.

Steenrod [166] computed the possible ranks of continuous distributions on spheres. We summarize these results in the following statement [166, p. 144, Theorem 27.18].

Theorem 4.2.1. *The sphere S^ℓ does not admit a continuous distribution of rank r if ℓ is even and $1 \leq r \leq \ell - 1$, or if $\ell \equiv 1 \pmod 4$ and $2 \leq r \leq \ell - 2$.*

Now we can state and prove the main result of this section.

Proposition 4.2.2. *Let V be a real subspace of $\mathbb{H}^n$ with constant quaternionic Kähler angle and $\dim_{\mathbb{R}} V = k$. Then:*

(i) *If $k \geq 5$ is odd, then V is a totally real subspace of $\mathbb{H}^n$, that is, it has constant quaternionic Kähler angle $(\pi/2, \pi/2, \pi/2)$.*

(ii) *If $k \equiv 2 \pmod 4$, then V has constant quaternionic Kähler angle $(\varphi, \pi/2, \pi/2)$, for some $\varphi \in [0, \pi/2]$.*

(iii) *If $k = 3$, then V has constant quaternionic Kähler angle $(\varphi, \varphi, \pi/2)$ for some $\varphi \in [0, \pi/2]$.*

Proof. Let us consider the distribution Δ defined above in this section. Recall that, by construction, its rank is at most 3.

Let $k \geq 5$ be odd. Then, Theorem 4.2.1 implies that S^{k-1} does not admit a non-trivial continuous distribution. Thus, the rank of Δ is 0. Hence, by definition of Δ we have $P_J v = 0$ for all $J \in \mathfrak{J}$ and $v \in \mathsf{S}^{k-1}$, which means that $\mathfrak{J}V$ is perpendicular to V. Therefore, V is totally real. This proves (i).

Let now $k \equiv 2 \pmod 4$. Theorem 4.2.1 guarantees that the rank of Δ is 0 or 1. If Δ has rank 1, then for each $v \in \mathsf{S}^{k-1}$ there is, by definition of Δ, a canonical basis $\{J_1^v, J_2^v, J_3^v\}$ of $\mathfrak{J}$ such that $P_1^v v \neq 0$ and $P_2^v v = P_3^v v = 0$, where $P_i^v = P_{J_i^v}$, for $i \in \{1, 2, 3\}$. Hence, v has quaternionic Kähler angle $(\varphi, \pi/2, \pi/2)$ with respect to V and $\{J_1^v, J_2^v, J_3^v\}$, for some $\varphi \in [0, \pi/2)$. Therefore, $\Phi(V) = (\varphi, \pi/2, \pi/2)$, $\varphi \in [0, \pi/2)$. If Δ has rank 0, then V is totally real, as in the proof of (i). Altogether, we have proved (ii).

Let $k = 3$. Then Theorem 4.2.1 implies that the rank of Δ is 0 or 2. If it is 0, then V is totally real. If the rank of Δ is 2, then $\Phi(V) = (\varphi_1, \varphi_2, \pi/2)$, for some $\varphi_1, \varphi_2 \neq \pi/2$. In this case, let us assume that $\varphi_1 \neq \varphi_2$. Then, for each $v \in \mathsf{S}^2 \subset V$ there exist complex structures J_1^v and J_2^v in $\mathfrak{J}$, depending continuously on v, such that v has Kähler angle $\varphi_i \in [0, \pi/2)$ with respect to J_i^v and V, for $i \in \{1, 2\}$. But then $v \mapsto P_1^v v$ would define a non-vanishing continuous vector field on S^2, which contradicts Theorem 4.2.1. Hence, $\varphi_1 = \varphi_2$, which proves (iii). □

In view of Proposition 4.2.2 and the previous partial classification results (Section §4.1), the classification of real subspaces with constant quaternionic Kähler angle is reduced to two main cases: subspaces with dimension $k = 3$, and subspaces with dimension k multiple of 4. The case $k = 3$ (and hence $\Phi(V) = (\varphi, \varphi, \pi/2)$) will be addressed in Subsection §4.4.1 by a direct study. The other case is much more involved and, indeed, we will content ourselves with addressing the subcase $k = 4$ and, for higher dimensions, restricting our attention to protohomogeneous subspaces. Thus, in Section §4.3 we will reduce the study of protohomogeneous subspaces of dimension multiple of 4 to the case of dimension $k = 4$, and in Subsection §4.4.2 we will obtain the classification of subspaces of dimension $k = 4$ with constant quaternionic Kähler angle.

4.3 Factorization of subspaces of dimension multiple of four

In this section we prove that any protohomogeneous subspace of real dimension k multiple of 4 in $\mathbb{H}^n$ can be factorized as an $\mathbb{H}$-orthogonal direct sum of subspaces of dimension 4 with the same constant quaternionic Kähler angle. The first step (Subsection §4.3.1) will be to show, using a Lie group theoretical argument, that the canonical basis of $\mathfrak{J}$ provided by Lemma 4.1.1 is independent of the vector in the subspace V of $\mathbb{H}^n$. Then, using this, one can induce a Clifford module structure on V, which allows us to conclude the factorization result by using the classification of Clifford modules by Atiyah, Bott and Shapiro [7] (Subsection §4.3.2).

4.3.1 Canonical quaternionic structure

Let V be a real subspace of a quaternionic Euclidean space $\mathbb{H}^n$. Assume that V is protohomogeneous. Equivalently, $\mathsf{H}' := N^0_{\mathsf{Sp}_1\mathsf{Sp}_n}(V)$, the connected component of the identity of the normalizer of V in $\mathsf{Sp}_1\mathsf{Sp}_n$, acts transitively on the unit sphere S^{k-1} of V. In particular, V has constant quaternionic Kähler angle by Lemma 4.1.3.

Consider the subgroup H'' of all elements of H' which act trivially on V,

$$\mathsf{H}'' = Z_{\mathsf{Sp}_1\mathsf{Sp}_n}(V) = \{h \in \mathsf{H}' : hv = v, \text{ for all } v \in V\}.$$

This is a closed normal subgroup of H'. Hence, $\mathsf{H} := \mathsf{H}'/\mathsf{H}''$ is a compact connected Lie group. Moreover, the action of H' on V induces an action of H on V, and the latter inherits the basic properties of the former (it is orthogonal and transitive on the unit sphere S^{k-1} of V), but now the H-action is effective.

The compact connected Lie group H acts effectively and transitively on the unit sphere S^{k-1} of V. Montgomery and Samelson [136], and Borel [38], classified compact connected Lie groups acting effectively and transitively on spheres (see also [32, p. 179]). In particular (see [136, Theorem I]), we have that either H is simple or $\mathsf{H} = (\mathsf{H}_1 \times \mathsf{H}_2)/\mathsf{N}$, where H_1, H_2 are connected simple Lie groups and N is a finite normal subgroup of $\mathsf{H}_1 \times \mathsf{H}_2$; moreover, the subgroup of H corresponding to H_1 still acts transitively on S^{k-1}.

Proposition 4.3.1. *Let V be a protohomogeneous real subspace of $\mathbb{H}^n$ of dimension $k \geq 5$. Then, there exists a connected Lie subgroup S of the Sp_n-factor of $\mathsf{Sp}_1\mathsf{Sp}_n$ that acts* *transitively on the unit sphere* S^{k-1} *of* V*. Moreover, the elements of* S *commute with any complex structure* $J \in \mathfrak{J}$*.*

Proof. Let $\mathfrak{h}'$ and $\mathfrak{h}''$ denote the Lie algebras of H' and H'', respectively. Since $\mathfrak{h}'$ is compact, and hence reductive, the ideal $\mathfrak{h}''$ of $\mathfrak{h}'$ admits a complementary ideal $\mathfrak{h}$ of $\mathfrak{h}'$ such that $\mathfrak{h}' = \mathfrak{h} \oplus \mathfrak{h}''$ and $\mathfrak{h} \simeq \mathfrak{h}'/\mathfrak{h}''$. Note that the Lie algebra of $\mathsf{H} = \mathsf{H}'/\mathsf{H}''$ is isomorphic to $\mathfrak{h}$. If $\widehat{\mathsf{H}}$ denotes the connected subgroup of H' with Lie algebra $\mathfrak{h}$, then $\mathsf{H}' = \widehat{\mathsf{H}} \cdot \mathsf{H}''$ and hence $\mathsf{H} = \mathsf{H}'/\mathsf{H}'' \cong \widehat{\mathsf{H}}/(\widehat{\mathsf{H}} \cap \mathsf{H}'')$ is a finite quotient of $\widehat{\mathsf{H}}$.

If H is simple, put $\mathfrak{s} := \mathfrak{h}$. If H is not simple, put $\mathfrak{s} := \mathfrak{h}_1$, where $\mathfrak{h}_1$ is the ideal of $\mathfrak{h}$ whose associated connected Lie subgroup of H still acts transitively on S^{k-1}. Note that, in any case, the connected Lie subgroup S of $\widehat{\mathsf{H}} \subset \mathsf{H}' \subset \mathsf{Sp}_1\mathsf{Sp}_n$ with Lie algebra $\mathfrak{s}$ acts transitively on the unit sphere S^{k-1} of V.

Recall that $\mathfrak{h}'$ is a Lie subalgebra of the direct sum Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$. Consider $\pi_{\mathfrak{sp}(1)} \colon \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \to \mathfrak{sp}(1)$ the projection map onto the first factor, and $\Psi = \pi_{\mathfrak{sp}(1)}|_{\mathfrak{s}} \colon \mathfrak{s} \to \mathfrak{sp}(1)$ its restriction to $\mathfrak{s}$, which is a Lie algebra homomorphism.

Since $\mathrm{Ker}\Psi$ is an ideal of $\mathfrak{s}$ and $\mathfrak{s}$ is simple, we have $\mathrm{Ker}\Psi = 0$ or $\mathrm{Ker}\Psi = \mathfrak{s}$. If $\mathrm{Ker}\Psi = 0$, then $\mathfrak{s}$ is isomorphic to a subalgebra of $\mathfrak{sp}(1)$; but $\dim \mathfrak{sp}(1) = 3$, so S cannot act transitively on S^{k-1}, $k \geq 5$. Hence, $\mathrm{Ker}\Psi = \mathfrak{s}$, and thus, $\mathrm{Im}\Psi = 0$, that is, $\mathfrak{s}$ is contained in the $\mathfrak{sp}(n)$-factor of the Lie algebra of $\mathsf{Sp}_1\mathsf{Sp}_n$. This proves the first part of the claim.

The connected subgroup S of $\mathsf{Sp}_n \subset \mathsf{Sp}_1\mathsf{Sp}_n$ with Lie algebra $\mathfrak{s}$, which acts transitively on the unit sphere of V, commutes with the elements of the Sp_1-factor of $\mathsf{Sp}_1\mathsf{Sp}_n$. Since the quaternionic structure $\mathfrak{J}$ of $\mathbb{H}^n$ is induced precisely by the action of the Sp_1-factor on $\mathbb{H}^n$, we obtain that the elements of S commute with any $J \in \mathfrak{J}$. □

As a consequence, we have

Corollary 4.3.2. *Let* $V \subset \mathbb{H}^n$ *be a protohomogeneous real subspace of dimension* $k \geq 5$ *with constant quaternionic Kähler angle* $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$*. Then, there exists a canonical basis* $\{J_1, J_2, J_3\}$ *of* $\mathfrak{J}$ *such that the Kähler angle of any unit vector* $v \in V$ *with respect to* J_i *and* V *is* φ_i*, for each* $i \in \{1, 2, 3\}$*.*

Proof. It suffices to show that the bilinear form L_v given in Lemma 4.1.1 is independent of $v \in \mathsf{S}^{k-1}$. Indeed, given $v, w \in \mathsf{S}^{k-1}$, there exists $T \in \mathsf{S}$ such that $Tv = w$. Since T commutes with all $J \in \mathfrak{J}$ and preserves V, we have

$$\begin{aligned} L_w(J, J') &= \langle P_J w, P_{J'} w \rangle = \langle \pi_V J T v, \pi_V J' T v \rangle = \langle \pi_V T J v, \pi_V T J' v \rangle \\ &= \langle T \pi_V J v, T \pi_V J' v \rangle = \langle T P_J v, T P_{J'} v \rangle = \langle P_J v, P_{J'} v \rangle = L_v(J, J'), \end{aligned}$$

for all J, $J' \in \mathfrak{J}$. □

4.3.2 Factorization Lemma

Let V be a real subspace of $\mathbb{H}^n$ of constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$ with $\varphi_2 \neq \pi/2$. Assume that there exists a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ such that the Kähler angle of any non-zero vector $v \in V$ with respect to J_i and V is φ_i, for $i \in \{1, 2, 3\}$. Note that, by Corollary 4.3.2, if V is protohomogeneous of dimension at least 5, then the previous assumption holds. In view of Proposition 4.2.2 (and leaving the case $k = 3$ for later), we will assume that $\dim_{\mathbb{R}} V = 4l$ with $l \in \mathbb{N}$.

Let us regard $\mathbb{H}^n$ as a complex vector space $\mathbb{C}^{2n}$ with respect to the complex structure J_i. By [61, p. 1191], we have that $\bar{P}_i := P_i/\cos(\varphi_i) = \pi_V J_i/\cos(\varphi_i)$ leaves V invariant and defines an orthogonal complex structure in V, for each $i \in \{1, 2\}$, and also for $i = 3$ if and only if $\varphi_3 \neq \pi/2$. Furthermore, we can easily check that $\bar{P}_i\bar{P}_j = -\bar{P}_j\bar{P}_i$, for $i \neq j$. Indeed, if $v, w \in V$, then Lemma 4.1.1 yields

$$0 = \langle \bar{P}_i(v + w), \bar{P}_j(v + w)\rangle = \langle \bar{P}_i v, \bar{P}_j w\rangle + \langle \bar{P}_j v, \bar{P}_i w\rangle = -\langle \bar{P}_j\bar{P}_i v, w\rangle - \langle \bar{P}_i\bar{P}_j v, w\rangle.$$

Hence, V has a module structure over the Clifford algebra Cl_3 if $\varphi_3 \neq \pi/2$, or over Cl_2 if $\varphi_3 = \pi/2$. It is well known that there are exactly two inequivalent irreducible Clifford modules over Cl_3, both of dimension 4 (we will denote them by V^0 and V^1), whereas there is exactly one irreducible Cl_2-module up to equivalence, again of dimension 4 (we will denote it by V^0). Moreover, Clifford modules are semisimple. This implies that, if $\varphi_3 \neq \pi/2$, we can decompose V into a direct sum of irreducible Cl_3-modules as follows

$$V = \left(\bigoplus^{l_0} V^0\right) \oplus \left(\bigoplus^{l_1} V^1\right),$$

where $l_0 + l_1 = l$, whereas if $\varphi_3 = \pi/2$ the Cl_2-module V can be decomposed as

$$V = \bigoplus^{l} V^0.$$

The above decompositions can be assumed to be orthogonal because the complex structures $\bar{P}_i$ are orthogonal. This also implies that two different summands are $\mathbb{H}$-orthogonal: if $v, w \in V$ belong to two different summands, $\langle J_k v, w\rangle = \langle P_k v, w\rangle = 0$ by the $\bar{P}_k$-invariance. Finally, since $\bar{P}_i$ leaves each factor V^r ($r \in \{0, 1\}$) invariant, we deduce that each V^r has constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$. This leads us to state the following:

Lemma 4.3.3. *Let V be a real subspace of $\mathbb{H}^n$ of dimension $4l$, with $l \in \mathbb{N}$, and constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$. Assume that there exists a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ such that the Kähler angle of any non-zero $v \in V$ with respect to J_i and V is φ_i, for each $i \in \{1, 2, 3\}$. Then, there is an $\mathbb{H}$-orthogonal decomposition*

$$V = \bigoplus_{r=1}^{l} V_r,$$

where each V_r has dimension 4 and $\Phi(V_r) = (\varphi_1, \varphi_2, \varphi_3)$ as a subspace of $\mathbb{H}^n$.

Conversely, let V be a real subspace of $\mathbb{H}^n$ given by an $\mathbb{H}$-orthogonal direct sum $V := \bigoplus_{r=1}^{l} V_r$, where each V_r has dimension 4, and $\Phi(V_r) = (\varphi_1, \varphi_2, \varphi_3)$. Let $\{J_1, J_2, J_3\}$ be a canonical structure of $\mathfrak{J}$ such that every non-zero vector in V_r has Kähler angle φ_i with respect to J_i and V_r, for each $i \in \{1, 2, 3\}$ and each $r \in \{1, \dots, l\}$. Then, $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$.

Proof. The first assertion has been proved above under the assumption $\varphi_2 \neq \pi/2$. If $\varphi_2 = \pi/2$, the first claim follows from the classification of subspaces V with $\Phi(V) = (\varphi, \pi/2, \pi/2)$, $\varphi \in [0, \pi/2]$ (cf. Section §4.1 and [13, pp. 230-232]).

In order to prove the converse, we first note that $\pi_V(\mathbb{H}V_r) = V_r$, for each $r \in \{1, \dots, l\}$. Indeed, for every $v \in V_r$ and $w \in V_s$, $r \neq s$, $\langle \pi_V J_i v, w \rangle = \langle J_i v, w \rangle = 0$, where in the last equality we have used $\mathbb{H}V_r \perp \mathbb{H}V_s$. Hence, $\pi_V \mathfrak{J}(V_r) \subset V_r$, and since $\pi_V(V_r) = V_r$, we deduce $\pi_V(\mathbb{H}V_r) = V_r$.

Now let $v = \sum_{r=1}^{l} v_r \in V$, with $v_r \in V_r$ for each $r \in \{1, \dots, l\}$. Denoting as usual $P_i = \pi_V J_i$, for each $i \in \{1, 2, 3\}$, we have

$$\begin{aligned} L_v(J_i, J_j) = \langle P_i v, P_j v \rangle &= \sum_{r,s=1}^{l} \langle \pi_V J_i v_r, \pi_V J_j v_s \rangle = \sum_{r=1}^{l} \langle \pi_V J_i v_r, \pi_V J_j v_r \rangle \\ &= \sum_{r=1}^{l} \langle P_i v_r, P_j v_r \rangle = \sum_{r=1}^{l} \cos^2(\varphi_i) \delta_{ij} \|v_r\|^2 = \cos^2(\varphi_i) \delta_{ij} \|v\|^2, \end{aligned}$$

where in the third equality we have used $\pi_V(\mathbb{H}V_r) = V_r$ and $V_r \perp V_s$ for all $r, s \in \{1, \dots, l\}$, and in the fifth one we have used that the quaternionic Kähler angle of v_r with respect to V_r and $\{J_1, J_2, J_3\}$ is $(\varphi_1, \varphi_2, \varphi_3)$. Since $v \in V$ is arbitrary, by Lemma 4.1.1 we conclude that $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$. □

4.4 Low dimensional subspaces with constant quaternionic Kähler angle

As a consequence of Proposition 4.2.2, we only have to study subspaces of dimensions 3 and multiples of 4. The latter can be reduced to studying subspaces of dimension 4 by virtue of Corollary 4.3.2 and Lemma 4.3.3. We devote this section to the classification of (not necessarily protohomogeneous) real subspaces of dimensions $k \in \{3, 4\}$ with constant quaternionic Kähler angle. The main tool that we will use in this section is the isospectrality of the Kähler angle map Ω introduced in Equation (4.1). We start with a lemma that provides an appropriate basis of the subspace.

Lemma 4.4.1. *Let V be a real subspace of $\mathbb{H}^n$ of dimension $k \in \{3, 4\}$ with $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$. Let $e_0 \in V$ be a unit vector. Then, there exists a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ and vectors $e_i \in \mathbb{H}^n \ominus \mathbb{H}e_0$, $i \in \{1, \dots, k-1\}$, such that*

$$\cos(\varphi_i) J_i e_0 + \sin(\varphi_i) J_i e_i, \qquad i \in \{0, \dots, k-1\}, \tag{4.2}$$

constitute an $\mathbb{R}$*-orthonormal basis of* V*, where we put* $J_0 := \mathrm{Id}$ *and* $\varphi_0 = 0$.

Moreover, for each $i \in \{0, \dots, k-1\}$ *with* $\varphi_i \neq \pi/2$*, we have*

$$\bar{P}_i e_0 = \cos(\varphi_i) J_i e_0 + \sin(\varphi_i) J_i e_i,$$

where $\bar{P}_i = P_i / \cos(\varphi_i) = \pi_V J_i / \cos(\varphi_i)$.

Finally, if $\varphi_i = 0$ *we take* $e_i = 0$*, whereas if* $\varphi_i > 0$*, then* e_i *is a unit vector.*

Proof. Let $e_0 \in V$ be a unit vector. By Lemma 4.1.1, there is a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ such that e_0 has Kähler angle φ_i with respect to J_i for $i \in \{1,2,3\}$, and $\langle P_i e_0, P_j e_0 \rangle = \cos^2(\varphi_i)\delta_{ij}$. In particular, $\langle \bar{P}_i e_0, \bar{P}_j e_0 \rangle = 0$ for any i, $j \in \{0, \dots, k-1\}$, $i \neq j$, with φ_i, $\varphi_j \neq \pi/2$.

Fix $i \in \{1,2,3\}$. If $\varphi_i = 0$, then we take $e_i = 0$. Let us assume first that $\varphi_i \in (0, \pi/2)$. By regarding $\mathbb{H}^n$ as a complex vector space $\mathbb{C}^{2n}$ with respect to the complex structure J_i, [13, Lemma 2] yields the existence of a unit vector $e_i \in \mathbb{H}^n \ominus \mathrm{span}_{\mathbb{R}}\{e_0, J_i e_0\}$ satisfying

$$\bar{P}_i e_0 = \cos(\varphi_i) J_i e_0 + \sin(\varphi_i) J_i e_i.$$

We have to see that $e_i \in \mathbb{H}^n \ominus \mathbb{H} e_0$. Observe that $\mathbb{H}^n \ominus \mathrm{span}_{\mathbb{R}}\{e_0, J_i e_0\}$ coincides with the orthogonal sum $(\mathbb{H}^n \ominus \mathbb{H} e_0) \oplus \mathrm{span}_{\mathbb{R}}\{J_{i+1} e_0, J_{i+2} e_0\}$, where indices are taken modulo 3. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
\langle e_i, \, a J_{i+1} e_0 + b J_{i+2} e_0 \rangle &= -\frac{1}{\sin(\varphi_i)} \langle J_i \bar{P}_i e_0 + \cos(\varphi_i) e_0, \, a J_{i+1} e_0 + b J_{i+2} e_0 \rangle \\
&= \frac{1}{\sin(\varphi_i)\cos(\varphi_i)} \left(a \langle P_i e_0, P_{i+2} e_0 \rangle - b \langle P_i e_0, P_{i+1} e_0 \rangle \right) = 0,
\end{aligned}$$

where in the last equality we have used Lemma 4.1.1. Therefore, $e_i \in \mathbb{H}^n \ominus \mathbb{H} e_0$.

Now if $\varphi_2 = \pi/2$, subspaces V with $\Phi(V) = (\varphi, \pi/2, \pi/2)$, $\varphi \in [0, \pi/2]$, are classified (see Section §4.1) and they can be spanned by a basis as in the statement (see [13, p. 232] and note that the $\{e_i\}$ in the statement do not have to be $\mathbb{H}$-orthonormal).

Thus, we finally have to deal with the case $k = 4$, $\varphi_2 \neq \pi/2$, and $\varphi_3 = \pi/2$. Then, by the previous argument, there exists a unit vector $v \in \mathbb{H}^n$ such that $\{e_0, \bar{P}_1 e_0, \bar{P}_2 e_0, v\}$ is an $\mathbb{R}$-orthonormal basis of V, where $\bar{P}_i e_0 = \cos(\varphi_i) J_i e_0 + \sin(\varphi_i) J_i e_i$, $i \in \{1, 2\}$. Recalling the definition of the Kähler angle map (see Equation (4.1)), we have

$$\begin{aligned}
\mathrm{tr}(\Omega(e_0)) &= \sum_{i=1}^{3} \langle P_i e_0, P_i e_0 \rangle \\
&= \sum_{i=1}^{3} \left(\langle P_i e_0, e_0 \rangle^2 + \langle P_i e_0, \bar{P}_1 e_0 \rangle^2 + \langle P_i e_0, \bar{P}_2 e_0 \rangle^2 + \langle P_i e_0, v \rangle^2 \right) \\
&= \cos^2(\varphi_1) + \cos^2(\varphi_2) + \sum_{i=1}^{3} \langle J_i e_0, v \rangle^2,
\end{aligned}$$

where we have used Lemma 4.1.1 and $\bar{P}_i e_0 = P_i e_0 / \cos(\varphi_i)$. Since the quaternionic Kähler angle of V is $\Phi(V) = (\varphi_1, \varphi_2, \pi/2)$, the eigenvalues of $\Omega(e_0)$ are $\cos^2(\varphi_1)$, $\cos^2(\varphi_2)$ and 0, and hence we deduce that $v \in \mathbb{H}^n \ominus \mathbb{H} e_0$. Thus, taking $e_3 = -J_3 v$ yields the result. $\square$

Remark 4.4.2. Whenever $\varphi_1 > 0$, the orthogonality of (4.2) yields $\langle J_3 e_1, e_2\rangle = 0$, and if $k = 4$, also $\langle J_1 e_2, e_3\rangle = \langle J_2 e_3, e_1\rangle = 0$.

4.4.1 Subspaces of dimension three

In this subsection we classify 3-dimensional real subspaces of $\mathbb{H}^n$ with constant quaternionic Kähler angle.

Proposition 4.4.3. *Let $V \subset \mathbb{H}^n$ be a real subspace of dimension 3. Then, V has constant quaternionic Kähler angle if and only if $\Phi(V) = (\varphi, \varphi, \pi/2)$, $\varphi \in [0, \pi/2]$, and for any unit $e_0 \in V$, there is a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ such that*

$$\{e_0, \cos(\varphi) J_1 e_0 + \sin(\varphi) J_1 e_1, \cos(\varphi) J_2 e_0 + \sin(\varphi) J_2 e_2\} \tag{4.3}$$

is an orthonormal basis of V, where, if $\varphi \neq 0$, e_1, e_2 are unit vectors satisfying $e_1, e_2 \in \mathbb{H}^n \ominus \mathbb{H} e_0$, $e_2 \in \mathbb{H}^n \ominus (\operatorname{Im}\mathbb{H}) e_1$, and either $\langle e_1, e_2\rangle = \cos(\varphi)/(\cos(\varphi) - 1)$ with $\varphi \in [\pi/3, \pi/2]$, or $\langle e_1, e_2\rangle = \cos(\varphi)/(\cos(\varphi) + 1)$ with $\varphi \in (0, \pi/2]$.

Proof. By Proposition 4.2.2, we have that $\varphi_1 = \varphi_2 = \varphi \in [0, \pi/2]$ and $\varphi_3 = \pi/2$. Let us assume that V is spanned by the basis described in Lemma 4.4.1 with $k = 3$. If $\varphi = 0$ or $\varphi = \pi/2$ the claim follows from the classification of subspaces with constant quaternionic Kähler angle $(0, 0, \pi/2)$ or $(\pi/2, \pi/2, \pi/2)$; see Section §4.1.

Thus, let us assume $\varphi \in (0, \pi/2)$. Then, for each $l \in \{1, 2\}$ and understanding the subscript $l + 1 \in \{1, 2\}$ modulo 2,

$$\begin{aligned}
\Omega(\bar P_l e_0)_{ij} &= \langle P_i \bar P_l e_0, P_j \bar P_l e_0\rangle \\
&= \langle J_i \bar P_l e_0, e_0\rangle \langle J_j \bar P_l e_0, e_0\rangle + \sum_{r=1}^{2} \langle J_i \bar P_l e_0, \bar P_r e_0\rangle \langle J_j \bar P_l e_0, \bar P_r e_0\rangle, \\
&= \langle \bar P_l e_0, P_i e_0\rangle \langle \bar P_l e_0, P_j e_0\rangle + \langle J_i \bar P_l e_0, \bar P_{l+1} e_0\rangle \langle J_j \bar P_l e_0, \bar P_{l+1} e_0\rangle,
\end{aligned}$$

where in the second equality we have calculated the orthogonal projection of vectors onto V by using the orthonormal basis $\{e_0, \bar P_1 e_0, \bar P_2 e_0\}$ of V. Hence, for $l \in \{1, 2\}$, using Lemma 4.4.1 we have

$$\begin{aligned}
\Omega(\bar P_l e_0)_{ll} &= \cos^2(\varphi) + \langle e_l, J_{l+1} e_{l+1}\rangle^2 \sin^4(\varphi), \\
\Omega(\bar P_l e_0)_{l+1,l} &= \langle e_1, J_1 e_2\rangle \langle e_1, J_2 e_2\rangle \sin^4(\varphi), \\
\Omega(\bar P_l e_0)_{l+1,l+1} &= \langle e_{l+1}, J_l e_l\rangle^2 \sin^4(\varphi), \\
\Omega(\bar P_l e_0)_{13} &= \langle e_2, J_2 e_1\rangle \sin^2(\varphi)(\cos^2(\varphi) + \langle e_1, e_2\rangle \sin^2(\varphi)), \\
\Omega(\bar P_l e_0)_{23} &= -\langle e_1, J_1 e_2\rangle \sin^2(\varphi)(\cos^2(\varphi) + \langle e_1, e_2\rangle \sin^2(\varphi)), \\
\Omega(\bar P_l e_0)_{33} &= (\cos^2(\varphi) + \langle e_1, e_2\rangle \sin^2(\varphi))^2.
\end{aligned} \tag{4.4}$$

Now, since $\Omega(\bar P_l e_0)$ is symmetric with eigenvalues $\cos^2(\varphi)$ (of multiplicity 2) and 0, by the min-max theorem, one obtains

$$0 \le \Omega(\bar P_l e_0)_{ll} \le \cos^2(\varphi), \qquad l \in \{1, 2\}.$$

This implies $\langle e_1, J_2e_2\rangle = \langle e_2, J_1e_1\rangle = 0$, which together with Remark 4.4.2 yields $e_2 \in \mathbb{H}^n \ominus (\operatorname{Im}\mathbb{H})e_1$. Taking again into account the spectrum of $\Omega(\bar{P}_1e_0)$, we have the following relation for its trace,

$$2\cos(\varphi)^2 = \operatorname{tr}(\Omega(\bar{P}_1e_0)) = (\cos^2(\varphi) + \langle e_1, e_2\rangle \sin^2(\varphi))^2 + \cos^2(\varphi).$$

From this and the fact that e_1 and e_2 are unit vectors, we deduce that either $\langle e_1, e_2\rangle = \cos(\varphi)/(\cos(\varphi) - 1)$ where $\varphi \in [\pi/3, \pi/2)$ or $\langle e_1, e_2\rangle = \cos(\varphi)/(1 + \cos(\varphi))$ where $\varphi \in (0, \pi/2)$. This proves the necessity of the statement.

For the converse we take an arbitrary unit vector $v \in V$ which we write as

$$v = x_0e_0 + x_1\big(\cos(\varphi)J_1e_0 + \sin(\varphi)J_1e_1\big) + x_2\big(\cos(\varphi)J_2e_0 + \sin(\varphi)J_2e_2\big).$$

Then, if $\varepsilon \in \{\pm 1\}$ is such that $\langle e_1, e_2\rangle = \cos(\varphi)/(1 + \varepsilon\cos(\varphi))$, we have

$$\Omega(v) = \cos^2(\varphi)\begin{pmatrix} x_0^2 + x_1^2 & x_1x_2 & -\varepsilon x_0x_2 \\ x_1x_2 & x_0^2 + x_2^2 & \varepsilon x_0x_1 \\ -\varepsilon x_0x_2 & \varepsilon x_0x_1 & x_1^2 + x_2^2 \end{pmatrix}.$$

Since v is a unit vector, $x_0^2 + x_1^2 + x_2^2 = 1$, and it is now easy to see that $\Omega(v)$ has a double eigenvalue $\cos^2(\varphi)$, and a simple eigenvalue 0. □

Remark 4.4.4. We will denote by V_+^φ and V_-^φ any real subspace of $\mathbb{H}^n$ constructed as in Proposition 4.4.3, depending on whether $\langle e_1, e_2\rangle = \cos(\varphi)/(\cos(\varphi) + 1)$ for $\varphi \in (0, \pi/2]$, or $\langle e_1, e_2\rangle = \cos(\varphi)/(\cos(\varphi) - 1)$ for $\varphi \in [\pi/3, \pi/2]$, respectively. Note that the subspaces $V_\pm^\varphi$ can be constructed as subspaces of any $\mathbb{H}^n$ with $n \geq 3$. One can easily check that the only one that fits into an $\mathbb{H}^2$ is $V_-^{\pi/3}$ (but it cannot fit into $\mathbb{H}$).

Proposition 4.4.5. *Let V be a subspace of $\mathbb{H}^n$ with constant quaternionic Kähler angle and dimension 3. Then V is protohomogeneous.*

Proof. We know from Proposition 4.4.3 that $\Phi(V) = (\varphi, \varphi, \pi/2)$. We can assume that $\varphi \in (0, \pi/2)$ since, otherwise, V is known to be protohomogeneous (see Section §4.1).

Let $e_0 \in V$ be an arbitrary unit vector. By Lemma 4.1.1 there is a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ such that e_0 has Kähler angle φ with respect to J_1 and J_2, and Kähler angle $\pi/2$ with respect to J_3. In view of Lemma 4.4.1 and Proposition 4.4.3, let us consider the unit vectors $e_i \in \mathbb{H}^n \ominus \mathbb{H}e_0$, $i \in \{1, 2\}$, given by

$$e_i := -(J_i\bar{P}_ie_0 + \cos(\varphi)e_0)/\sin(\varphi), \qquad i \in \{1, 2\}, \tag{4.5}$$

where $\bar{P}_i := \pi_V J_i/\cos(\varphi)$. On the one hand, by (4.5) we have

$$\begin{aligned}\langle e_1, e_2\rangle &= \frac{1}{\sin^2(\varphi)}\langle J_1\bar{P}_1e_0 + \cos(\varphi)e_0, J_2\bar{P}_2e_0 + \cos(\varphi)e_0\rangle \\ &= \frac{1}{\sin^2(\varphi)}\big(\langle J_1\bar{P}_1e_0, J_2\bar{P}_2e_0\rangle - \cos^2(\varphi)\big).\end{aligned} \tag{4.6}$$

On the other hand, again by Proposition 4.4.3, $\langle e_1, e_2\rangle$ can take two possible values. We will first see that, given V, $\langle e_1, e_2\rangle$ is independent of e_0.

Let S^2 denote the unit sphere of V. We define $\Theta\colon \mathsf{S}^2 \to \mathbb{R}$ by $\Theta(e_0) = \langle e_1, e_2\rangle$. We claim that Θ is well defined. Let $\{J_1', J_2', J_3\}$ be another canonical basis of $\mathfrak{J}$ such that e_0 has Kähler angle φ with respect to J_i', $i \in \{1,2\}$, and let $e_i' := -(J_i'\bar{P}_i' e_0 + \cos(\varphi)e_0)/\sin(\varphi)$ where $\bar{P}_i' := \pi_V J_i'/\cos(\varphi)$ for $i \in \{1,2\}$. Then, there is $\theta \in [0, 2\pi)$ such that $J_i' = \cos(\theta)J_i + (-1)^{i+1}\sin(\theta)J_{i+1}$ for $i \in \{1,2\}$ and subscripts modulo 2. Thus,

$$\begin{aligned} J_i'\bar{P}_i' &= (\cos(\theta)J_i + (-1)^{i+1}\sin(\theta)J_{i+1})(\cos(\theta)\bar{P}_i + (-1)^{i+1}\sin(\theta)\bar{P}_{i+1}) \\ &= \cos^2(\theta)J_i\bar{P}_i + \sin^2(\theta)J_{i+1}\bar{P}_{i+1} + (-1)^{i+1}\cos(\theta)\sin(\theta)(J_1\bar{P}_2 + J_2\bar{P}_1). \end{aligned} \tag{4.7}$$

Consequently, using Equation (4.6) twice, and then (4.7), we get, after some calculations,

$$\langle e_1, e_2\rangle - \langle e_1', e_2'\rangle = \frac{1}{\sin^2(\varphi)}\big(\langle J_1\bar{P}_1 e_0, J_2\bar{P}_2 e_0\rangle - \langle J_1'\bar{P}_1' e_0, J_2'\bar{P}_2' e_0\rangle\big) = 0,$$

which implies that Θ is well-defined.

Now note that the assignment $e_0 \in \mathsf{S}^2 \mapsto \operatorname{span}\{J_1, J_2\} \in \mathsf{G}_2(\mathfrak{J})$, where $\mathsf{G}_2(\mathfrak{J})$ is the Grassmannian of 2-planes of $\mathfrak{J} \cong \mathbb{R}^3$, is continuous due to the continuous dependence of the quadratic form $J \in \mathfrak{J} \mapsto L_v(J,J) = \langle P_J v, P_J v\rangle \in \mathbb{R}$ on v. Hence, the map Θ is also continuous. But, as mentioned just after (4.6), $\Theta(\mathsf{S}^2)$ has at most two elements. Therefore, Θ is constant on S^2.

Finally, we prove that V is protohomogeneous. Let e_0, e_0' be arbitrary unit vectors in V. Let $\{J_1, J_2, J_3\}$, $\{J_1', J_2', J_3'\}$ be canonical bases of $\mathfrak{J}$, and e_1, e_2, e_1', e_2' be unit vectors in V such that both (4.3), and (4.3) with e_i' instead of e_i and J_i' instead of J_i, are orthonormal bases of V. Both sets of vectors $\{e_0, e_1, e_2\}$ and $\{e_0', e_1', e_2'\}$ span a totally real subspace of $\mathbb{H}^n$, and since Θ is constant, $\langle e_i, e_j\rangle = \langle e_i', e_j'\rangle$ for all $i,j \in \{0,1,2\}$. It then follows that there exists an element $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $Te_i = e_i'$ for each $i \in \{0,1,2\}$, and $TJ_j = J_j'T$ for each $j \in \{1,2,3\}$. Thus, by (4.5) we get $T\bar{P}_i e_0 = \bar{P}_i' e_0'$ for $i \in \{0,1,2\}$, where $\bar{P}_0 = \bar{P}_0' = \mathrm{Id}$. Therefore, T is an element of $\mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV = V$ and $Te_0 = e_0'$. Since e_0, e_0' are arbitrary, this proves that V is protohomogeneous. □

Finally we show that the two types of subspaces V_+^φ and V_-^φ introduced in Remark 4.4.4 are indeed inequivalent for $\varphi \neq \pi/2$. Recall that V_+^φ is defined for all $\varphi \in (0, \pi/2]$, but V_-^φ only for $\varphi \in [\pi/3, \pi/2]$.

Proposition 4.4.6. *Let $\varphi \in [\pi/3, \pi/2]$. Then there exists $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV_+^\varphi = V_-^\varphi$ if and only if $\varphi = \pi/2$.*

Proof. If $\varphi = \pi/2$, then $V_+^{\pi/2}$ and $V_-^{\pi/2}$ are totally real, therefore equivalent. Let us assume that $\varphi \neq \pi/2$ and that there is $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV_+^\varphi = V_-^\varphi$. By applying an element of $\mathsf{Sp}_1\mathsf{Sp}_n$ if necessary, we can assume that there is a unit vector $e_0 \in V_+^\varphi \cap V_-^\varphi$ and that e_0 has quaternionic Kähler angle $(\varphi, \varphi, \pi/2)$ with respect to both V_+^φ and V_-^φ and a common canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$. Then, by Lemma 4.4.1 and Proposition 4.4.3, $V_\pm^\varphi$ is spanned by the basis $\{e_0, \bar{P}_1^\pm e_0, \bar{P}_2^\pm e_0\}$, where $\bar{P}_i^\pm := \pi_{V_\pm^\varphi} J_i / \cos(\varphi)$, $i \in \{1,2\}$. Moreover,

$$\bar{P}_i^\pm e_0 = \cos(\varphi) J_i e_0 + \sin(\varphi) J_i e_i^\pm, \text{ with } \langle e_1^\pm, e_2^\pm \rangle = \frac{\cos(\varphi)}{\cos(\varphi) \pm 1},$$

and $e_i^\pm \in \mathbb{H}^n \ominus \mathbb{H}e_0$, $i \in \{1,2\}$.

By Proposition 4.4.5, we can assume that $Te_0 = e_0$. Let $J' = TJ_3T^{-1} \in \mathfrak{J}$. Since $J_3 e_0 \in \mathbb{H}^n \ominus V_+^\varphi$, we have $J'e_0 = J'Te_0 = TJ_3e_0 \in \mathbb{H}^n \ominus V_-^\varphi$. This implies $TJ_3 = \varepsilon J_3 T$, where $\varepsilon \in \{-1, 1\}$, because $\pm J_3$ are the only complex structures in $\mathfrak{J}$ that send e_0 to $\mathbb{H}^n \ominus V_-^\varphi$. Therefore, there exists $\theta \in [0, 2\pi)$ such that

$$TJ_i = \varepsilon^i (\cos(\theta) J_i + (-1)^{i+1} \sin(\theta) J_{i+1}) T, \quad i \in \{1,2\}, \quad \text{and} \quad TJ_3 = \varepsilon J_3 T. \tag{4.8}$$

Using (4.8) and $Te_0 = e_0$, we have

$$\begin{aligned} T\bar{P}_1^+ e_0 =& \cos(\varphi) TJ_1 e_0 + \sin(\varphi) TJ_1 e_1^+ \\ =& \varepsilon \cos(\varphi)(\cos(\theta) J_1 e_0 + \sin(\theta) J_2 e_0) \\ &+ \varepsilon \sin(\varphi)(\cos(\theta) J_1 T e_1^+ + \sin(\theta) J_2 T e_1^+). \end{aligned} \tag{4.9}$$

By Proposition 4.4.5, $V_\pm^\varphi$ is protohomogeneous, and note that SO_3 is the only connected subgroup of $\mathsf{Sp}_1\mathsf{Sp}_n \subset \mathsf{SO}_{4n}$ that acts transitively and effectively on the unit sphere of $V_\pm^\varphi$. Thus, we can assume that $T\bar{P}_1^+ e_0 = \varepsilon \bar{P}_1^- e_0$, just by composing T with some element in the isotropy of the action of SO_3 on V_-^φ at e_0. But inserting (4.9) and $\bar{P}_1^- e_0 = \cos(\varphi) J_1 e_0 + \sin(\varphi) J_1 e_1^-$ into the equality $T\bar{P}_1^+ e_0 = \varepsilon \bar{P}_1^- e_0$, and analyzing the $\mathbb{H}e_0$ and $\mathbb{H}^n \ominus \mathbb{H}e_0$ components (note that $e_1^\pm \in \mathbb{H}^n \ominus \mathbb{H}e_0$, $Te_0 = e_0$, and T preserves $\mathbb{H}$-orthonormality) we get $\theta = 0$ and $Te_1^+ = e_1^-$. Moreover, by (4.8) we get $TJ_i = \varepsilon^i J_i T$, $i \in \{1,2,3\}$.

Since $Te_0 = e_0$, $T\bar{P}_1^+ e_0 = \varepsilon \bar{P}_1^- e_0$ and $TV_+^\varphi = V_-^\varphi$, we must have $T\bar{P}_2^+ e_0 = \pm \bar{P}_2^- e_0$. Then, inserting $\bar{P}_2^\pm e_0 = \cos(\varphi) J_2 e_0 + \sin(\varphi) J_2 e_2^\pm$ in the last equality, and using $TJ_2 = J_2 T$, we deduce that $Te_2^+ = e_2^-$. But this jointly with $Te_1^+ = e_1^-$ yields a contradiction with the fact that T is an orthogonal transformation of $\mathbb{H}^n$, because $\langle e_1^+, e_2^+ \rangle \neq \langle e_1^-, e_2^- \rangle$ for all $\varphi \neq \pi/2$. □

4.4.2 Subspaces of dimension four

The aim of this subsection is to classify 4-dimensional real subspaces of $\mathbb{H}^n$ with constant quaternionic Kähler angle.

We start by restricting our attention to subspaces with $\varphi_1 = 0$.

Proposition 4.4.7. *Let $V \subset \mathbb{H}^n$ be a real subspace of dimension 4 with constant quaternionic Kähler angle $(0, \varphi_2, \varphi_3)$. Then, $\varphi_2 = \varphi_3 \in [0, \pi/2]$.*

Proof. First of all, if $\varphi_2 = 0$, then $\varphi_3 = 0$ by a combination of [13, Proposition 9] and the fact that subspaces with $\Phi(V) = (0, 0, \pi/2)$ have dimension 3 (see Section §4.1). Hence, let us assume that $\varphi_2 \neq 0$. Lemma 4.4.1 yields a basis $\{e_0, J_1 e_0, v_2, v_3\}$ of V, where $v_i = \cos(\varphi_i) J_i e_0 + \sin(\varphi_i) J_i e_i$, for certain unit $e_i \in \mathbb{H}^n \ominus \mathbb{H}e_0$, $i \in \{2, 3\}$. Therefore, a computation as in Equations (4.4), for each $i \in \{2, 3\}$, gives

$$\begin{aligned}
\Omega(v_i)_{11} &= \big(\cos(\varphi_2)\cos(\varphi_3) + \langle e_2, e_3\rangle \sin(\varphi_2)\sin(\varphi_3)\big)^2,\\
\Omega(v_i)_{22} &= \cos(\varphi_i)^2 + \langle J_3 e_3, e_2\rangle^2 \sin^2(\varphi_2)\sin^2(\varphi_3),\\
\Omega(v_i)_{33} &= \cos(\varphi_i)^2 + \langle J_2 e_2, e_3\rangle^2 \sin^2(\varphi_2)\sin^2(\varphi_3).
\end{aligned}$$

Hence, by the isospectrality of Ω,

$$0 = \operatorname{tr}(\Omega(v_2)) - \operatorname{tr}(\Omega(v_3)) = 2\cos^2(\varphi_2) - 2\cos^2(\varphi_3),$$

from where we conclude $\varphi_2 = \varphi_3$. □

In view of Proposition 4.4.7, all real subspaces V of $\mathbb{H}^n$ with $\Phi(V) = (0, \varphi_2, \varphi_3)$ actually satisfy $\Phi(V) = (0, \varphi, \varphi)$. Note that such subspaces have been classified (see Section §4.1).

Thus, in the following results we will analyze the case $\varphi_1 > 0$. We consider the basis of V given in Lemma 4.4.1.

Lemma 4.4.8. *Let $V \subset \mathbb{H}^n$ be a real subspace of dimension 4 such that $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$ with $\varphi_1 > 0$. For each $i \in \{1, 3\}$ with $\varphi_i \neq \pi/2$, we have $\langle e_i, J_j e_j\rangle = 0$ for all $j \in \{1, 2, 3\}$.*

Proof. According to Lemma 4.4.1, e_0 has Kähler angle φ_i with respect to V and $J_i \in \mathfrak{J}$ for each $i \in \{1, 2, 3\}$. Let us regard $\mathbb{H}^n$ as a complex Euclidean space $\mathbb{C}^{2n}$ whose complex structure is J_i, for $i \in \{1, 2, 3\}$. By [61, Theorem 2.7] there is a non-empty finite subset $\Psi^i \subset [0, \pi/2]$ such that $V = \bigoplus_{\varphi \in \Psi^i} V^i_\varphi$ is a $\mathbb{C}$-orthonormal decomposition of V and $V^i_\varphi \subset \mathbb{C}^{2n}$ is a real subspace with constant Kähler angle $\varphi \in \Psi^i$. It follows that any non-zero $v \in V^i_\varphi$ has Kähler angle φ with respect to V and J_i, and the minimum (resp. maximum) of Ψ^i coincides with the minimum (resp. maximum) Kähler angle of a non-zero vector $v \in V$ with respect to V and J_i.

We claim that $\varphi_1 \in \Psi^1$. On the one hand, if there existed $\varphi \in \Psi^1$ such that $\varphi < \varphi_1$, then there would be vectors in V whose Kähler angle with respect to V and $J_1 \in \mathfrak{J}$ is $\varphi < \varphi_1$, thus contradicting the minimality of φ_1 by Lemma 4.1.1. On the other hand, if $\varphi > \varphi_1$ for all $\varphi \in \Psi^1$, then we would get a contradiction with the fact that e_0 has Kähler angle φ_1 with respect to J_1. Analogously, we get that $\varphi_3 \in \Psi^3$.

Now assume $\varphi_1 \neq \pi/2$. By [61, p. 1190–1191] and the discussion above, we have a decomposition $V = V_{\varphi_1} \oplus V_{\psi_1}$ into real subspaces of constant Kähler angle with respect to the complex structure J_1, where $\psi_1 \in \Psi^1$ (the possibility $\psi_1 = \varphi_1$ is allowed). We also have that $\bar{P}_1 := \pi_{V_{\varphi_1}} J_1/\cos(\varphi_1) = \pi_V J_1/\cos(\varphi_1)|_{V_{\varphi_1}}$ defines a complex structure on V_{φ_1}. As $e_0 \in V_{\varphi_1}$, we get $V_{\varphi_1} = \operatorname{span}_{\mathbb{R}}\{e_0, \bar{P}_1 e_0\}$. Moreover, $\mathbb{C}V_{\varphi_1} \perp \mathbb{C}V_{\psi_1}$, so $V_{\psi_1} = \operatorname{span}_{\mathbb{R}}\{\bar{P}_2 e_0, \bar{P}_3 e_0\}$, and for $j \in \{2, 3\}$, using Lemma 4.4.1,

$$0 = \langle \bar{P}_1 e_0, J_1 \bar{P}_j e_0\rangle = \sin(\varphi_1)\sin(\varphi_j)\langle e_1, J_j e_j\rangle.$$

Since $\varphi_1 > 0$, we get $\langle e_1, J_j e_j\rangle = 0$. A similar argument works for φ_3, if $\varphi_3 \neq \pi/2$. □

Before addressing the classification, we state a lemma that refines [60, Lemma 5.1].

Lemma 4.4.9. *Assume* $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$, *and let* $\varepsilon \in \{-1, 1\}$. *Then, there exists a subset* $\{e_1, e_2, e_3\}$ *of unit vectors of* $\mathbb{R}^3$ *with inner products*

$$\langle e_i, e_{i+1}\rangle = \frac{\varepsilon\cos(\varphi_{i+2}) - \cos(\varphi_i)\cos(\varphi_{i+1})}{\sin(\varphi_i)\sin(\varphi_{i+1})} \qquad \text{for each } i \in \{1, 2, 3\}$$

if and only if $\cos(\varphi_1) + \cos(\varphi_2) - \varepsilon\cos(\varphi_3) \leq 1$.

Furthermore, the subspace $\operatorname{span}_{\mathbb{R}}\{e_1, e_2, e_3\}$ *has dimension* 2 *if and only if* $\cos(\varphi_1) + \cos(\varphi_2) + \varepsilon\cos(\varphi_3) = 1$, *and dimension* 3 *otherwise.*

Proof. A subset $\{e_1, e_2, e_3\}$ of the Euclidean space $\mathbb{R}^3$ satisfies the inner product relations in the statement if and only if the associated Gram matrix $G = (\langle e_i, e_j\rangle)_{1\leq i,j\leq 3}$ is positive semi-definite. This happens precisely when all principal minors of G are non-negative; in this proof, by G_{ij} we denote the matrix of order 2 resulting from deleting the i-th row and the j-th column of G. Let $x_i := \cos(\varphi_i)$ for each $i \in \{1, 2, 3\}$. Hence, G is positive semi-definite if and only if $\det G_{ii} \geq 0$ for all $i \in \{1, 2, 3\}$ and $\det G \geq 0$. We compute

$$\det(G) = \frac{(\varepsilon + x_1 - x_2 - x_3)(-\varepsilon + x_1 + x_2 - x_3)(-\varepsilon + x_1 - x_2 + x_3)(\varepsilon + x_1 + x_2 + x_3)}{(1 - x_1^2)(1 - x_2^2)(1 - x_3^2)}.$$

Taking into account that $1 > x_1 \geq x_2 \geq x_3 \geq 0$, one can check that $\det G \geq 0$ if and only if $-1 + x_1 + x_2 - \varepsilon x_3 \leq 0$. Similarly,

$$\det(G_{ii}) = \frac{(1 - x_1^2 - x_2^2 - x_3^2 + 2\varepsilon x_1x_2x_3)}{\prod_{j\neq i}(1 - x_j^2)}, \quad i \in \{1, 2, 3\}.$$

Hence, $\det(G_{ii}) \geq 0$ if and only if

$$1 - x_1^2 - x_2^2 - x_3^2 + 2\varepsilon x_1x_2x_3 \geq 0. \tag{4.10}$$

Now, if $1 > x_1 \geq x_2 \geq x_3 \geq 0$, one can show that (4.10) holds provided that $-1 + x_1 + x_2 - \varepsilon x_3 \leq 0$. This completes the proof of the first claim of the lemma.

Assume that we are in the situation of the first assertion of the statement. Then $\{e_1, e_2, e_3\}$ spans a 3-dimensional subspace if and only if G is positive definite, which in this situation amounts to $\det G > 0$. This happens precisely when $x_1 + x_2 - \varepsilon x_3 < 1$. Hence, the proof of the lemma will be complete if we show that $\operatorname{span}_{\mathbb{R}}\{e_1, e_2, e_3\}$ cannot have dimension 1. Assume this is the case. Then the rank of G is 1. Hence $x_1 = 1 - x_2 + \varepsilon x_3$, and the minor $\det(G_{33})$ vanishes, i.e.

$$0 = \det(G_{33}) = -\frac{2(1 + \epsilon x_3)}{(1 + x_2)(-2 + x_2 - \epsilon x_3)}.$$

Therefore, $x_3 = -\varepsilon$, which yields a contradiction. This finishes the proof. ∎

We are now in position to complete the description of 4-dimensional real subspaces of $\mathbb{H}^n$ with constant quaternionic Kähler angle.

Proposition 4.4.10. *Let $V \subset \mathbb{H}^n$ be a real subspace of dimension 4 and $e_0 \in V$ a unit vector. Then V has constant quaternionic Kähler angle $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$, with $\varphi_1 > 0$, if and only if there is a canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$, $\varepsilon \in \{-1, 1\}$, and unit vectors $e_1, e_2, e_3 \in \mathbb{H}^n \ominus \mathbb{H}e_0$ with $e_i \in \mathbb{H}^n \ominus (\operatorname{Im}\mathbb{H})e_j$, $i, j \in \{1,2,3\}$, such that*

(i) $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$,

(ii) $\cos(\varphi_1) + \cos(\varphi_2) - \varepsilon\cos(\varphi_3) \leq 1$,

(iii) *for all $i \in \{1,2,3\}$ and indices modulo 3,*
$$\langle e_i, e_{i+1}\rangle = \frac{\varepsilon\cos(\varphi_{i+2}) - \cos(\varphi_i)\cos(\varphi_{i+1})}{\sin(\varphi_i)\sin(\varphi_{i+1})},$$

(iv) $\{\cos(\varphi_i)J_ie_0 + \sin(\varphi_i)J_ie_i : i = 0,1,2,3\}$ *is an orthonormal basis of V, where for simplicity we put $\varphi_0 := 0$ and $J_0 := \operatorname{Id}$.*

Moreover, if V is as above, the Kähler angle of any non-zero $v \in V$ with respect to J_i and V is φ_i, for each $i \in \{1,2,3\}$.

Proof. In order to prove the necessity, let us assume that V is spanned by the basis described in Lemma 4.4.1 with $k = 4$. Notice that $\varphi_2 = \pi/2$ implies $\Phi(V) = (\varphi, \pi/2, \pi/2)$; such subspaces are classified (see Section §4.1), and [13, p. 232], together with some straightforward calculations, show that they can be spanned by a basis as above. Thus, we can suppose $\varphi_1, \varphi_2 \in (0, \pi/2)$.

Let us first assume $\varphi_3 \neq \pi/2$. A long but elementary calculation, similar to the one used to obtain Equations (4.4), using the isospectrality of Ω, Remark 4.4.2 and Lemma 4.4.8, yields
$$\begin{aligned}\prod_{j=1}^{3}\cos^2(\varphi_j) &= \det(\Omega(\bar{P}_ie_0))\\ &= \cos^2(\varphi_i)\prod_{\substack{j=1\\ j\neq i}}^{3}(\cos(\varphi_i)\cos(\varphi_j) + \langle e_i, e_j\rangle\sin(\varphi_i)\sin(\varphi_j))^2,\end{aligned}$$
for each $i \in \{1,2,3\}$. This implies for $i \in \{1,2,3\}$,
$$\cos^2(\varphi_{i+2}) = (\cos(\varphi_i)\cos(\varphi_{i+1}) + \langle e_i, e_{i+1}\rangle\sin(\varphi_i)\sin(\varphi_{i+1}))^2. \tag{4.11}$$
Using (4.11), we can also calculate for $i \in \{1,3\}$
$$\sum_{j=1}^{3}\cos^2(\varphi_j) = \operatorname{tr}(\Omega(\bar{P}_ie_0)) = \sum_{j=1}^{3}\cos^2(\varphi_j) + \langle e_2, J_ie_i\rangle^2\sin^2(\varphi_i)\sin^2(\varphi_2),$$

which implies $\langle e_2, J_i e_i\rangle = 0$, $i \in \{1,3\}$. This, along with Remark 4.4.2 and also with Lemma 4.4.8, shows that $e_i \in \mathbb{H}^n \ominus (\operatorname{Im}\mathbb{H})e_j$, $i,j \in \{1,2,3\}$. Furthermore, (4.11) gives rise to the two possible expressions for $\langle e_i, e_{i+1}\rangle$ in the statement (corresponding to $\varepsilon = 1$ or $\varepsilon = -1$). Note that such expressions are incompatible for a fixed V, that is, if for some $i \in \{1,2,3\}$ we have

$$\langle e_i, e_{i+1}\rangle = \frac{\cos(\varphi_{i+2}) - \cos(\varphi_i)\cos(\varphi_{i+1})}{\sin(\varphi_i)\sin(\varphi_{i+1})},$$
$$\langle e_{i+1}, e_{i+2}\rangle = -\frac{\cos(\varphi_i) + \cos(\varphi_{i+1})\cos(\varphi_{i+2})}{\sin(\varphi_{i+1})\sin(\varphi_{i+2})},$$

then one can check that $\det\big(\Omega((e_0 + \bar{P}_{i+1}e_0)/\sqrt{2})\big) = 0$, which gives a contradiction with the assumption $\varphi_3 \neq \pi/2$. Finally, the inequality in item (4.4.10) of the statement follows from Lemma 4.4.9.

Now assume that $\varphi_3 = \pi/2$. Let $\{e_0, \bar{P}_1 e_0, \bar{P}_2 e_0, J_3 e_3\}$ be the orthonormal basis provided by Lemma 4.4.1. A similar computation as in (4.4), using the isospectrality of Ω and Lemma 4.4.8, yields

$$\begin{aligned}\sum_{i=1}^{2}\cos^2(\varphi_i) =\ &\operatorname{tr}(\Omega(\bar{P}_1 e_0)) + \operatorname{tr}(\Omega(\bar{P}_2 e_0)) - \operatorname{tr}(\Omega(J_3 e_3))\\ &= \cos^2(\varphi_1) + \cos^2(\varphi_2) + 2\sin^2(\varphi_1)\sin^2(\varphi_2)\langle J_1 e_1, e_2\rangle^2\\ &\quad + 2\left(\cos(\varphi_1)\cos(\varphi_2) + \sin(\varphi_1)\sin(\varphi_2)\langle e_1, e_2\rangle\right)^2.\end{aligned}$$

Then,

$$\langle e_2, J_1 e_1\rangle = 0 \quad \text{and} \quad \langle e_1, e_2\rangle = -\cot(\varphi_1)\cot(\varphi_2). \tag{4.12}$$

Also, using (4.12), if $i \in \{1,2\}$ we get

$$0 = \det\left(\Omega\left(\frac{1}{\sqrt{2}}e_0 + \frac{1}{\sqrt{2}}\bar{P}_i e_0\right)\right) = \frac{1}{4}\langle e_3, J_i e_i\rangle^2\cos^2(\varphi_1)\cos^2(\varphi_2)\sin^2(\varphi_i).$$

Thus,

$$\langle e_3, J_i e_i\rangle = 0, \quad i \in \{1,2\}. \tag{4.13}$$

Taking into account (4.12) and (4.13), we can calculate

$$\sum_{i=1}^{2}\cos^2(\varphi_i) = \operatorname{tr}(\Omega(\bar{P}_1 e_0)) = \cos^2(\varphi_1) + \langle e_1, e_3\rangle^2\sin^2(\varphi_1),$$
$$\sum_{i=1}^{2}\cos^2(\varphi_i) = \operatorname{tr}(\Omega(\bar{P}_2 e_0)) = \cos^2(\varphi_2) + \sin^2(\varphi_2)(\langle e_2, e_3\rangle^2 + \langle e_2, J_3 e_3\rangle^2),$$

whence

$$\begin{aligned}\langle e_1, e_3\rangle &= \varepsilon\cos(\varphi_2)/\sin(\varphi_1),\\ \cos^2(\varphi_1) &= \sin^2(\varphi_2)(\langle e_2, e_3\rangle^2 + \langle e_2, J_3 e_3\rangle^2),\end{aligned} \tag{4.14}$$

for some $\varepsilon \in \{-1, 1\}$. Using these relations we compute

$$\sum_{i=1}^{2} \cos^2(\varphi_i) = \operatorname{tr}\left(\Omega\left(\frac{1}{\sqrt{2}}\bar{P}_1 e_0 + \frac{1}{\sqrt{2}}\bar{P}_2 e_0\right)\right) = \sum_{i=1}^{2} \cos^2(\varphi_i) + \varepsilon\frac{1}{2}\langle e_2, J_3 e_3\rangle \sin(2\varphi_2),$$

from where (note that we are assuming $\varphi_2 \neq \pi/2$)

$$\langle e_2, J_3 e_3\rangle = 0 \quad \text{and} \quad \langle e_2, e_3\rangle = \varepsilon' \cos(\varphi_1)/\sin(\varphi_2), \tag{4.15}$$

for some $\varepsilon' \in \{-1, 1\}$. Remark 4.4.2, Lemma 4.4.8 and Equations (4.12), (4.13), (4.14) and (4.15) imply $e_i \in \mathbb{H}^n \ominus (\operatorname{Im}\mathbb{H})e_j$, $i, j \in \{1, 2, 3\}$. Furthermore, if $\varepsilon' = -\varepsilon$, we have that 0 is an eigenvalue of $\Omega\left((e_0 + J_3 e_3)/\sqrt{2}\right)$ with double multiplicity, yielding a contradiction with the fact $\varphi_2 \neq \pi/2$. Hence, $\varepsilon' = \varepsilon$ which, along with Lemma 4.4.9, concludes the proof of the necessity in the statement.

The converse implication follows from verifying by direct calculation that the matrix $\Omega(v)$ is diagonal with diagonal entries $\cos^2(\varphi_1)$, $\cos^2(\varphi_2)$, $\cos^2(\varphi_3)$, for any unit v spanned by the basis of V given in the statement. This also proves the final claim of the proposition. □

Remark 4.4.11. In view of Proposition 4.4.10, there can be zero, one or two types of 4-dimensional real subspaces V of $\mathbb{H}^n$ with $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$, $\varphi_1 > 0$, depending on whether the triple $(\varphi_1, \varphi_2, \varphi_3)$ satisfies $\cos(\varphi_1) + \cos(\varphi_2) - \cos(\varphi_3) > 1$, $\cos(\varphi_1) + \cos(\varphi_2) - \cos(\varphi_3) \leq 1 < \cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3)$, or $\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) \leq 1$, respectively. Thus, it will be convenient to denote by V_+ and V_- the subspaces described in Proposition 4.4.10 with $\varepsilon = 1$ or $\varepsilon = -1$, respectively. Note that such subspaces depend on the triple $(\varphi_1, \varphi_2, \varphi_3)$, but we do not specify this in the notation for the sake of simplicity.

Observe that, if $\varphi_3 = \pi/2$, V_+ and V_- are actually equivalent, i.e. there exists $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV_+ = V_-$. Indeed, one can take a $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ that commutes with J_i for all $i \in \{1, 2, 3\}$, fixes each e_i with $i \in \{0, 1, 2\}$, and sends e_3 to $-e_3$. For convenience, from now on we will say that any 4-dimensional real subspace of $\mathbb{H}^n$ with constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \pi/2)$ is of type V_+, and not of type V_-.

In order to encompass all examples of 4-dimensional subspaces with constant quaternionic Kähler angle into the $V_\pm$-notation, we have to consider the case $\varphi_1 = 0$ analyzed in Proposition 4.4.7. Thus, we adopt the convention that any 4-dimensional real subspace V with $\Phi(V) = (0, \varphi, \varphi)$, $\varphi \in [0, \pi/2]$, is of type V_+, and not of type V_-.

Remark 4.4.12. The choice of the $\pm$-notation in Remark 4.4.11 is motivated by certain important property of these subspaces that we now explain. Assume $\varphi_3 \neq \pi/2$. The last claim of Proposition 4.4.10 enables us to reproduce the discussion in Section §4.3 applied to $V = V_\pm$, and hence, $\bar{P}_i = \pi_{V_\pm} J_i/\cos(\varphi_i)$, $i \in \{1, 2, 3\}$, determine a Cl_3-module structure on $V_\pm$, which must be irreducible since $\dim_{\mathbb{R}} V_\pm = 4$. By the classification of Clifford modules, either $\bar{P}_1\bar{P}_2 = \bar{P}_3$ (and hence $\bar{P}_i\bar{P}_{i+1} = \bar{P}_{i+2}$ for all $i \in \{1, 2, 3\}$) or $\bar{P}_1\bar{P}_2 = -\bar{P}_3$ (and hence $\bar{P}_i\bar{P}_{i+1} = -\bar{P}_{i+2}$ for all $i \in \{1, 2, 3\}$). One can easily check using the basis of $V_\pm$ in Proposition 4.4.10 that V_+ satisfies precisely the former relation, whereas V_- satisfies the latter.

Remark 4.4.13. Let V be a real subspace of dimension 4 in $\mathbb{H}^n$, $n \geq 4$, with $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$. If $\varphi_1 = 0$, by Proposition 4.4.7 we have $\Phi(V) = (0, \varphi, \varphi)$ for some $\varphi \in [0, \pi/2]$. In this case, when $\varphi = 0$, V is quaternionic, i.e. $V = \mathbb{H}v$, for some non-zero vector $v \in V$, whereas if $\varphi > 0$, V cannot fit inside a quaternionic line $\mathbb{H}$, but can be placed in some $\mathbb{H}^2$ (see Section §4.1), and thus $\mathbb{H}V = \mathbb{H}^2$.

Now assume $\varphi_1 > 0$. By Proposition 4.4.10 and Lemma 4.4.9, V can be placed in some $\mathbb{H}^3$ if and only if $V = V_+$ and $\cos(\varphi_1) + \cos(\varphi_2) - \cos(\varphi_3) = 1$, or if $V = V_-$ and $\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) = 1$; in this case $\mathbb{H}V = \mathbb{H}^3$. Otherwise we have $\mathbb{H}V = \mathbb{H}^4$.

We end this section by showing that V_+ and V_- are not equivalent.

Proposition 4.4.14. *There does not exist $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV_+ = V_-$.*

Proof. We can assume that $\Phi(V_+) = \Phi(V_-)$ since the quaternionic Kähler angle is preserved by transformations in $\mathsf{Sp}_1\mathsf{Sp}_n$. We also assume $\varphi_3 \neq \pi/2$ in view of Remark 4.4.11. We consider the bases for $V_\pm$ given in Proposition 4.4.10, where we use the notation $e_i^\pm$ accordingly, and assume without loss of generality that the canonical basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ used is the same in both cases.

Let us suppose that there is $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV_+ = V_-$. Denote by π_+ and π_- the orthogonal projections onto V_+ and V_-, respectively. By assumption $T\pi_+ = \pi_- T$. Let $\{J_1', J_2', J_3'\}$ be the canonical basis of $\mathfrak{J}$ given by $J_i' = TJ_iT^{-1}$, $i \in \{1,2,3\}$, and denote $P_i^+ = \pi_+ J_i$ and $P_i' = \pi_- J_i'$, $i \in \{1,2,3\}$. Then, for any unit vector $w \in V_-$ and $i, j \in \{1,2,3\}$, we have

$$\begin{aligned}\langle P_i'w, P_j'w\rangle &= \langle \pi_- TJ_iT^{-1}w, \pi_- TJ_jT^{-1}w\rangle = \langle T\pi_+ J_iT^{-1}w, T\pi_+ J_jT^{-1}w\rangle \\ &= \langle TP_i^+T^{-1}w, TP_j^+T^{-1}w\rangle = \langle P_i^+T^{-1}w, P_j^+T^{-1}w\rangle = \cos^2(\varphi_i)\delta_{ij},\end{aligned}$$

where in the last equality we have used the last claim of Proposition 4.4.10 applied to V_+.

Thus, the canonical basis $\{J_1', J_2', J_3'\}$ of $\mathfrak{J}$ diagonalizes the bilinear form L_w^- (given in Proposition 4.1.1) associated with the subspace V_-, for any unit vector $w \in V_-$. By the last claim of Proposition 4.4.10 applied to V_-, the basis $\{J_1, J_2, J_3\}$ also has this property. Hence, there exists an orthogonal matrix $A \in \mathsf{SO}_3$ such that

$$(J_1', J_2', J_3') = (J_1, J_2, J_3)A$$

and A commutes with the diagonal matrix with diagonal entries $(\varphi_1, \varphi_2, \varphi_3)$. Then V_- coincides with the span of

$$\{e_0^-, \cos(\varphi_1)J_1'e_0^- + \sin(\varphi_1)J_1'e_1^-, \cos(\varphi_2)J_2'e_0^- + \sin(\varphi_2)J_2'e_2^-, \cos(\varphi_3)J_3'e_0^- + \sin(\varphi_3)J_3'e_3^-\},$$

where in this basis we have just changed J_i by J_i' in the original basis of V_-. Since for V_- we had $\bar{P}_1^- \bar{P}_2^- = -\bar{P}_3^-$ by Remark 4.4.12, where $\bar{P}_i^- = \pi_- J_i/\cos(\varphi_i)$, $i \in \{1,2,3\}$, we also have $\bar{P}_1'\bar{P}_2' = -\bar{P}_3'$, where $\bar{P}_i' = P_i'/\cos(\varphi_i) = \pi_- J_i'/\cos(\varphi_i)$, $i \in \{1,2,3\}$.

However, denoting $\bar{P}_i^+ = \pi_+ J_i/\cos(\varphi_i)$, $i \in \{1,2,3\}$, which by Remark 4.4.12 satisfy $\bar{P}_1^+\bar{P}_2^+ = \bar{P}_3^+$, we obtain:

$$\begin{aligned}
\bar{P}_1'\bar{P}_2' &= \frac{1}{\cos(\varphi_1)\cos(\varphi_2)}\pi_- J_1'\pi_- J_2' = \frac{1}{\cos(\varphi_1)\cos(\varphi_2)}\pi_- T J_1 T^{-1}\pi_- T J_2 T^{-1}\\
&= \frac{1}{\cos(\varphi_1)\cos(\varphi_2)} T\pi_+ J_1 \pi_+ J_2 T^{-1} = T\bar{P}_1^+\bar{P}_2^+ T^{-1} = T\bar{P}_3^+ T^{-1}\\
&= \frac{1}{\cos(\varphi_3)} T\pi_+ J_3 T^{-1} = \frac{1}{\cos(\varphi_3)}\pi_- T J_3 T^{-1} = \frac{1}{\cos(\varphi_3)}\pi_- J_3' = \bar{P}_3',
\end{aligned}$$

which leads to a contradiction with $\bar{P}_1'\bar{P}_2' = -\bar{P}_3'$. □

4.5 Inhomogeneous isoparametric hypersurfaces with constant principal curvatures

In this section we investigate when an $\mathbb{H}$-orthogonal sum of copies of the subspaces $V_\pm$ introduced in the previous section gives rise to a protohomogeneous real subspace of $\mathbb{H}^n$. In particular, we obtain subspaces with constant quaternionic Kähler angle that are not protohomogeneous. As a consequence of [60, Theorem 4.5] these subspaces give rise to examples of inhomogeneous isoparametric hypersurfaces with constant principal curvatures in quaternionic hyperbolic spaces.

Let us consider a real subspace V of $\mathbb{H}^n$ such that

(C1) $V = \bigoplus_{r=1}^{l} V_r$, where

(C2) $\dim_{\mathbb{R}} V_r = 4$, for each $r \in \{1,\dots,l\}$,

(C3) V_r and V_s are $\mathbb{H}$-orthogonal for every $r,s \in \{1,\dots,l\}$, $r \neq s$,

(C4) $\Phi(V_r) = (\varphi_1,\varphi_2,\varphi_3)$, for all $r \in \{1,\dots,l\}$,

(C5) $\{J_1,J_2,J_3\}$ is a canonical basis of $\mathfrak{J}$ such that every non-zero vector in V_r, $r \in \{1,\dots,l\}$, has Kähler angle φ_i with respect to J_i for each $i \in \{1,2,3\}$.

Then, Lemma 4.3.3 guarantees that $\Phi(V) = (\varphi_1,\varphi_2,\varphi_3)$. By Proposition 4.4.10, Remark 4.4.11 and Proposition 4.4.14, each factor V_r is either equivalent to V_+ or to V_-. Then, if we define l_+ and l_- as the number of subspaces in the decomposition of V equivalent to V_+ and to V_-, respectively, we have $l = l_+ + l_-$. In this situation we say that the real subspace V has *type* (l_+, l_-).

We claim that the type of V is well defined for real subspaces of $\mathbb{H}^n$ in the conditions (C1-5) above. If $\varphi_3 = \pi/2$, then by Remark 4.4.11 the type of V is $(l,0)$. Let us assume that $\varphi_3 \neq \pi/2$. As usual, we let $\bar{P}_i = \pi_V J_i/\cos(\varphi_i)$, for each $i \in \{1,2,3\}$; since $\mathbb{H}V_r \perp \mathbb{H}V_s$ for $r \neq s$, we have $\bar{P}_i|_{V_r} = \pi_{V_r} J_i/\cos(\varphi_i)$. Thus, it follows from Remark 4.4.12 that $\bar{P}_1\bar{P}_2|_{V_r} = \bar{P}_3|_{V_r}$ or $\bar{P}_1\bar{P}_2|_{V_r} = -\bar{P}_3|_{V_r}$, depending on whether V_r is equivalent to V_+ or V_-, respectively. Hence, $\dim_{\mathbb{R}} \mathrm{Ker}(\bar{P}_1\bar{P}_2 - \bar{P}_3) = 4l_+$ and $\dim_{\mathbb{R}} \mathrm{Ker}(\bar{P}_1\bar{P}_2 + \bar{P}_3) = 4l_-$. Moreover, the type is independent of the canonical basis of $\mathfrak{J}$ chosen. Indeed, if $\{J_1', J_2', J_3'\}$ is another canonical basis satisfying (C5), then there exists an orthogonal matrix $A \in \mathsf{SO}_3$ such that $(J_1', J_2', J_3') = (J_1, J_2, J_3)A$ and commuting with the diagonal matrix with diagonal entries $(\varphi_1, \varphi_2, \varphi_3)$, and one can easily argue (similarly as in the proof of Proposition 4.4.14) that $\mathrm{Ker}(\bar{P}_1'\bar{P}_2' \pm \bar{P}_3') = \mathrm{Ker}(\bar{P}_1\bar{P}_2 \pm \bar{P}_3)$. All in all, the type is well defined. More than that, a slight modification of the previous argument shows that two real subspaces V and W of $\mathbb{H}^n$ in the conditions (C1-5) are equivalent if and only if $\Phi(V) = \Phi(W)$ and their types coincide.

Proposition 4.5.1. *Let V be a real subspace of $\mathbb{H}^n$ satisfying conditions* (C1-5)*. Then, V is protohomogeneous if and only if the type of V is* $(l, 0)$ *or* $(0, l)$*.*

Proof. Assume that V is protohomogenous. By conditions (C1-5), $V = \bigoplus_{r=1}^{l} V_r$, where each factor V_r is equivalent either to V_+ or to V_-. If $\varphi_3 = \pi/2$, by Remark 4.4.11 each factor V_r is equivalent to V_+, whence V has type $(l, 0)$.

Let us suppose that V has type (l^+, l^-) where $l^+, l^- \geq 1$. In this case, $k = 4l \geq 8$. Let $r, s \in \{1, \ldots, l\}$, $r \neq s$, be such that V_r is equivalent to V_+, and V_s is equivalent to V_-. Let v_+ and v_- be unit vectors in V_r and V_s, respectively. Since V is protohomogeneous, there is $T \in \mathsf{Sp}_1\mathsf{Sp}_n$ such that $TV = V$ and $Tv_+ = v_-$. Now, since $k \geq 8$, by Proposition 4.3.1 we can assume that T is such that $TJ_i = J_iT$ for each $i \in \{1, 2, 3\}$. Then,

$$TP_iv_+ = T\pi_V J_i v_+ = \pi_V T J_i v_+ = \pi_V J_i T v_+ = \pi_V J_i v_- = P_i v_-,$$

for each $i \in \{1, 2, 3\}$. Then, T sends the subspace $V_r = \mathrm{span}\{v_+, P_1v_+, P_2v_+, P_3v_+\}$ onto $V_s = \mathrm{span}\{v_-, P_1v_-, P_2v_-, P_3v_-\}$. This yields a contradiction with Proposition 4.4.14.

Now we will prove the converse. Let V be of type $(l, 0)$ or $(0, l)$. We can assume $\varphi_1 > 0$. Otherwise, by Proposition 4.4.7 we have $\Phi(V) = (0, \varphi, \varphi)$ with $\varphi \in [0, \pi/2]$, and then V is protohomogeneous (see Section §4.1). We can also assume $\varphi_2 < \pi/2$. Otherwise, $\Phi(V) = (\varphi, \pi/2, \pi/2)$ for $\varphi \in [0, \pi/2]$, and then V would again be protohomogeneous (see Section §4.1).

As usual, consider the transformations $\bar{P}_i = \pi_V J_i / \cos(\varphi_i)$ for each $i \in \{1, 2, 3\}$ with $\varphi_i \neq \pi/2$, and define $\bar{P}_3 := \bar{P}_1\bar{P}_2$ if $\varphi_3 = \pi/2$. By the characterization of type, we have $\bar{P}_1\bar{P}_2 = \varepsilon\bar{P}_3$, where $\varepsilon = 1$ if the type of V is $(l, 0)$, and $\varepsilon = -1$ if the type of V is $(0, l)$. Thus, taking into account condition (C5) and the discussion in Subsection §4.3.2 (or alternatively by the very definition of V), we deduce that $\{\bar{P}_1, \bar{P}_2, \varepsilon\bar{P}_3\}$ is a canonical basis of a quaternionic structure on V.

Let $v_1, w_1 \in V$ be arbitrary unit vectors. Then, $\{\bar{P}_i v_1\}_{i=0}^3$ and $\{\bar{P}_i w_1\}_{i=0}^3$, where $\bar{P}_0 = \mathrm{Id}$, are $\mathbb{R}$-orthonormal bases for some 4-dimensional subspaces V_{v_1} and V_{w_1} of V, respectively. By construction, V_{v_1} and V_{w_1} are $\bar{P}_i$-invariant for each $i \in \{1, 2, 3\}$, and then $\{\bar{P}_1, \bar{P}_2, \varepsilon\bar{P}_3\}$ is a canonical basis of a quaternionic structure when restricted to V_{v_1} and to V_{w_1}. Moreover, every non-zero vector in V_{v_1} or V_{w_1} has Kähler angle φ_i with respect to J_i, for each $i \in \{1, 2, 3\}$, by Proposition 4.4.10. In conclusion, V_{v_1} and V_{w_1} are both equivalent either to V_+ or to V_-, depending on whether the type of V is $(l, 0)$ or $(0, l)$, respectively.

Proceeding inductively we can choose unit vectors $v_2, w_2, \dots, v_l, w_l$ and define decompositions $V = \bigoplus_{r=1}^{l} V_{v_r}$ and $V = \bigoplus_{r=1}^{l} V_{w_r}$ satisfying (C1-5), and such that $V_{v_r} = \operatorname{span}\{\bar{P}_i v_r\}_{i=0}^{3}$ and $V_{w_r} = \operatorname{span}\{\bar{P}_i w_r\}_{i=0}^{3}$ for each $r \in \{1, \dots, l\}$. Furthermore, all these subspaces V_{v_r} and V_{w_r} are equivalent to either V_+ or to V_-, depending on the type of V. Thus, for each $r \in \{1, \dots, l\}$ there exist $T_r \in \mathsf{Sp}_n \subset \mathsf{Sp}_1\mathsf{Sp}_n$ such that $T_r V_{v_r} = V_{w_r}$, $T_r(\mathbb{H}V_{v_r}) = \mathbb{H}V_{w_r}$, $T_r|_{\mathbb{H}^n \ominus \mathbb{H}V_{v_r}} = \mathrm{Id}$, and $T_r J_i = J_i T_r$ for each $i \in \{1, 2, 3\}$. Now let $e_0 = v_1$, e_1, e_2, e_3 be the unit vectors given in Proposition 4.4.10 for the subspace V_{v_1}, and similarly $f_0 = w_1$, f_1, f_2, f_3 the unit vectors associated with the subspace V_{w_1}. Since $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle$ for all $i, j \in \{0, 1, 2, 3\}$, and both sets of vectors span a totally real subspace of $\mathbb{H}^n$, there exists $T_1' \in \mathsf{Sp}_n \subset \mathsf{Sp}_1\mathsf{Sp}_n$ satisfying, in addition to the properties of the previously constructed T_1, the relations $T_1' e_j = f_j$ for each $j \in \{0, 1, 2, 3\}$ (in particular $T_1' v_1 = w_1$), and $T_1' J_i = J_i T_1'$ for each $i \in \{1, 2, 3\}$. Therefore, the composition $T = T_1' T_2 \dots T_r \in \mathsf{Sp}_n$ satisfies $TV = V$ and $Tv_1 = w_1$, which shows that V is protohomogeneous. □

Remark 4.5.2. We observe that an $\mathbb{H}$-orthogonal direct sum of real subspaces of dimension 4 with the same constant quaternionic Kähler angle (i.e. any subspace V satisfying (C1-4)) is protohomogeneous if and only if any two factors are congruent under an element of Sp_n. The direct implication follows from a combination of the simultaneous diagonalization result in Corollary 4.3.2 (which implies condition (C5)), Lemma 4.3.3 (which guarantees that V has constant quaternionic Kähler angle) and Proposition 4.5.1 (whose proof implies that any two factors are congruent under an element of Sp_n). The converse follows from a direct calculation using the fact that condition (C5) is satisfied if any two factors are congruent under an element of Sp_n, as the elements of Sp_n commute with any $J \in \mathfrak{J}$.

Remark 4.5.3. If we combine Propositions 4.4.10 and 4.5.1, they imply that any 4-dimensional real subspace of $\mathbb{H}^n$ with constant quaternionic Kähler angle is protohomogeneous. Recall that, by Propositions 4.4.3 and 4.4.5, the same happens with any 3-dimensional real subspace of $\mathbb{H}^n$ with constant quaternionic Kähler angle. This, along with Proposition 4.2.2 and the well-known protohomogeneity of subspaces V with $\Phi(V) = (\varphi, \pi/2, \pi/2)$, $\varphi \in [0, \pi/2]$, implies that any real subspace of $\mathbb{H}^n$ with constant quaternionic Kähler angle and dimension $k \neq 4l$, for all $l \in \mathbb{N}$, $l \geq 2$, is protohomogeneous.

An immediate consequence of Proposition 4.5.1, along with Lemma 4.3.3, is the existence of non-protohomogeneous subspaces with constant quaternionic Kähler angle.

Corollary 4.5.4. *A real subspace V of $\mathbb{H}^n$, satisfying conditions* (C1-5) *above in this section and of type (l_+, l_-) with $l_+, l_- \geq 1$, has constant quaternionic Kähler angle but is not protohomogeneous.*

Apart from the purely linear algebraic relevance of the examples described in Corollary 4.5.4, our interest in them stems from the theory of isoparametric hypersurfaces in symmetric spaces of non-compact type, which we briefly describe now in the particular case of the quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^{n+1}$; we refer to [60, 65] for more details.

Following the notation in Subsection §1.3.3, let $M = \mathbb{H}\mathsf{H}^{n+1} = \mathsf{G}/\mathsf{K}$, where $\mathsf{G} = \mathsf{Sp}_{1,n+1}$, and $\mathsf{K} = \mathsf{Sp}_1 \times \mathsf{Sp}_{n+1}$ is the isotropy group at some base point $o \in \mathbb{H}\mathsf{H}^{n+1}$. Let AN be the solvable part of the Iwasawa decomposition of $\mathsf{G} = \mathsf{Sp}_{1,n}$, and $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ its Lie algebra, where $\mathfrak{g}_\alpha \cong \mathbb{H}^n$.

Given any non-zero real subspace V of $\mathfrak{g}_\alpha \cong \mathbb{H}^n$, we define S_V as the connected subgroup of AN with Lie algebra

$$\mathfrak{s}_V = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus V) \oplus \mathfrak{g}_{2\alpha}.$$

Then, by [60, Theorem 4.5], the orbit of S_V through the base point o, together with the distance tubes around it, constitute an isoparametric family of hypersurfaces on $\mathbb{H}\mathsf{H}^{n+1}$:

Theorem 4.5.5. *The tubes of any radius around the submanifold $\mathsf{S}_V \cdot o$ are isoparametric hypersurfaces of $\mathbb{H}\mathsf{H}^{n+1}$. Moreover, they have constant principal curvatures if and only if V has constant quaternionic Kähler angle in $\mathbb{H}^n$.*

As a consequence we get Theorem C.

Proof of Theorem C. The combination of Theorem 3.4.1, Corollary 4.5.4 and Theorem 4.5.5 guarantees the existence of inhomogeneous isoparametric hypersurfaces with constant principal curvatures in quaternionic hyperbolic spaces.

We note that this construction does not provide any such example in $\mathbb{H}\mathsf{H}^{n+1}$ with $n \leq 6$, but it does so for $n \geq 7$. This follows from Corollary 4.5.4 along with the fact that, by Remark 4.4.13, the lowest integer n such that $\mathbb{H}^n$ admits a real subspace V of type (l_+, l_-) with $l_+, l_- \geq 1$ is $n = 7$. Indeed, we can take $V = V_+ \oplus V_- \subset \mathbb{H}^n$ satisfying conditions (C1-5), for any triple $(\varphi_1, \varphi_2, \varphi_3)$, $\varphi_3 \neq \pi/2$, such that $\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) = 1$ if $n = 7$, or such that $\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) \leq 1$ if $n \geq 8$. □

4.6 Proofs of Theorems A and B

In this section we conclude the proof of the classification of protohomogeneous subspaces of any dimension $k > 0$ in $\mathbb{H}^n$ by providing their moduli space.

Proof of Theorem A. We recall from the statement of Theorem A the definition of the sets $\Lambda = \{(\varphi_1, \varphi_2, \varphi_3) \in [0, \pi/2]^3 : \varphi_1 \leq \varphi_2 \leq \varphi_3\}$, and

$$\begin{aligned}
\mathfrak{R}_3^+ &= \{(\varphi, \varphi, \pi/2) \in \Lambda : \varphi \in [0, \pi/2]\},\\
\mathfrak{R}_3^- &= \{(\varphi, \varphi, \pi/2) \in \Lambda : \varphi \in [\pi/3, \pi/2)\},\\
\mathfrak{R}_4^+ &= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) - \cos(\varphi_3) \leq 1\},\\
\mathfrak{R}_4^- &= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) \leq 1,\ \varphi_3 \neq \pi/2\},\\
\mathfrak{S} &= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) + \varepsilon\cos(\varphi_3) = 1,\ \text{for } \varepsilon = \pm 1\}.
\end{aligned}$$

Note that $\mathfrak{R}_3^\pm$ (resp. $\mathfrak{R}_4^\pm$) is the set of possible triples that arise as quaternionic Kähler angles of the 3-dimensional (resp. 4-dimensional) subspaces $V_\pm^\varphi$ (resp. $V_\pm$) introduced in Remark 4.4.4 (resp. Remark 4.4.11). Notice that $\mathfrak{R}_3^- \subset \mathfrak{R}_3^+$, $\mathfrak{R}_4^- \subset \mathfrak{R}_4^+$, $\mathfrak{S} \subset \mathfrak{R}_4^+ \cup \mathfrak{R}_4^-$, and $\mathfrak{R}_4^-$ is precisely the set of triples for which there exist non-protohomogeneous subspaces with constant quaternionic Kähler angle.

Let V be a non-zero protohomogeneous subspace of real dimension k in $\mathbb{H}^n$. The proof of Theorem A follows from the discussion of the following four cases:

(1) *Case* $k \equiv 0 \,(\mathrm{mod}\, 4)$. By Corollary 4.3.2 and Lemma 4.3.3, V satisfies conditions (C1-5) in Section §4.5, and by Proposition 4.5.1, V is of type $(k/4, 0)$ or $(0, k/4)$. Put $V = \bigoplus_{r=1}^{k/4} V_r$ as in (C1). Now, by Remark 4.4.13 we have $\dim_{\mathbb{H}}(\mathbb{H}V_r) \in \{1, 2, 3, 4\}$, depending on the value of the triple $\Phi(V)$. Thus, combining this with the fact that $\mathbb{H}V_r \perp \mathbb{H}V_s = 0$ for $r \neq s$, we can distiguish four subcases of relative sizes of n and k, and determine the possible triples $\Phi(V)$ for each subcase:

   (a) If $k > 2n$, then $\Phi(V) = (0, 0, 0)$.

   (b) If $4n/3 < k \leq 2n$, then $\Phi(V) \in \{(0, \varphi, \varphi) \in \Lambda : \varphi \in [0, \pi/2]\}$.

   (c) If $n < k \leq 4n/3$, then $\Phi(V) \in \mathfrak{S}$.

   (d) Let us assume that $k \leq n$. If V is of type $(k/4, 0)$, then $\Phi(V) \in \mathfrak{R}_4^+$, whereas if V is of type $(0, k/4)$, then $\Phi(V) \in \mathfrak{R}_4^-$. Observe that for each triple $(\varphi_1, \varphi_2, \varphi_3)$ in $\mathfrak{R}_4^-$ (resp. in $\mathfrak{R}_4^+ \setminus \mathfrak{R}_4^-$) we have exactly two (resp. one) inequivalent protohomogeneous subspaces V of dimension k with $\Phi(V) = (\varphi_1, \varphi_2, \varphi_3)$.

(2) *Case k odd,* $k \neq 3$. By Proposition 4.2.2 we have $\Phi(V) = (\pi/2, \pi/2, \pi/2)$. Hence, by the classification of totally real subspaces, we must have $k \leq n$.

(3) *Case* $k \equiv 2 \,(\mathrm{mod}\, 4)$. By Proposition 4.2.2 we have $\Phi(V) = (\varphi, \pi/2, \pi/2)$, for some $\varphi \in [0, \pi/2]$. These examples are classified (see Section §4.1). Thus, we must have $k \leq 2n$. Furthermore, when $n < k \leq 2n$, we have $\Phi(V) = (0, \pi/2, \pi/2)$, whereas when $k \leq n$ we have $\Phi(V) = (\varphi, \pi/2, \pi/2)$, for some $\varphi \in [0, \pi/2]$.

(4) *Case* $k = 3$. By Proposition 4.2.2, $\Phi(V) = (\varphi, \varphi, \pi/2)$ for some $\varphi \in [0, \pi/2]$. If $n \geq 3$, Propositions 4.4.3 and 4.4.6 guarantee that, for each triple $(\varphi, \varphi, \pi/2)$ in $\mathfrak{R}_3^-$ (resp. in $\mathfrak{R}_3^+ \setminus \mathfrak{R}_3^-$) we have exactly two (resp. one) inequivalent subspaces with $\Phi(V) = (\varphi, \varphi, \pi/2)$. By Remark 4.4.4, if $n = 2$, we have $\Phi(V) \in \{(0, 0, \pi/2), (\pi/3, \pi/3, \pi/2)\}$, whereas if $n = 1$, $\Phi(V) = (0, 0, \pi/2)$. □

Proof of Theorem B. This follows from combining Theorem A with the theory of cohomogeneity one actions on symmetric spaces of non-compact type and rank one (cf. Section §3.4). We just have to note that the action producing the solvable foliation (resp. the action with a totally geodesic singular orbit $\mathbb{H}\mathsf{H}^\ell$, $\ell \in \{1, \dots, n\}$) can be recovered by the method that yields the actions with a non-totally geodesic singular orbit by taking V as a 1-dimensional subspace of $\mathfrak{g}_\alpha \cong \mathbb{H}^n$ (resp. by taking V as a quaternionic subspace $\mathbb{H}^{n-\ell+1}$ in $\mathfrak{g}_\alpha \cong \mathbb{H}^n$). □

Part II

Totally geodesic submanifolds

Chapter 5
Totally geodesic submanifolds

Intuitively, a submanifold of a Riemannian manifold is totally geodesic if it curves as the ambient space. If we accept the second fundamental form as a way of measuring how complicated the extrinsic geometry of a submanifold is, it turns out that totally geodesic submanifolds are those with the simplest one, i.e. vanishing second fundamental form.

Totally geodesic submanifolds play a fundamental role in Riemannian geometry. To begin with, apart from their intrinsic interest, their use has had a great impact, not only on the geometry of submanifolds, but also in areas of geometry closer to topology, such as the study of spaces with positive curvature, or even in number theory (the study of arithmetic groups), as we briefly discuss below.

Alexandrov's Theorem [2] states that an embedded hypersurface in the Euclidean space has constant mean curvature if and only if it is a round sphere. This theorem was generalized in several directions, see e.g. [92, 93] for some examples in the context of symmetric spaces. The method that Alexandrov used to prove this theorem was to reflect a given hypersurface with constant mean curvature with respect to totally geodesic hyperplanes and use a maximum principle for elliptic operators. The use of totally geodesic hypersurfaces in this situation is crucial since it guarantees that the reflection is an isometry.

A map $f\colon N \to M$ between (not necessarily connected) manifolds is said to be *ℓ-connected* if $\pi_i(f)\colon \pi_i(N) \to \pi_i(M)$, the induced map between the i-th homotopy groups, is an isomorphism for every $i < \ell$ and a surjection for $i = \ell$. Let M be a compact Riemannian manifold with positive curvature and dimension n. A classical result in the area of spaces with positive curvature is Frankel's Theorem, see [80], which states that two compact totally geodesic submanifolds of M with dimensions n_1 and n_2 must intersect provided that $n_1 + n_2 \geq n$. Moreover, Frankel proved that a smooth compact totally geodesic embedding $f\colon N \to M$, where $2 \dim N \geq n$, must be 1-connected, see [81]. In 2003, Wilking [184] generalized this fact by proving the so-called connectedness principle for manifolds with positive curvature. This states that if N is a compact embedded totally geodesic submanifold of M of codimension k, then the inclusion map $N \hookrightarrow M$ is $(n-2k+1)$-connected. This principle has been shown to constitute a fundamental tool to prove rigidity results in spaces with positive curvature, see e.g. [104, 185].

In the context of hyperbolic geometry, or more generally, locally symmetric spaces of non-compact type, totally geodesic submanifolds also appear to be highly significant. Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of non-compact type. Recall that a discrete subgroup Γ of G is said to be a *lattice* if the quotient $\Gamma\backslash\mathsf{G}$ has finite volume. This implies that if Γ acts freely on M, the space $\Gamma\backslash M$ is a locally symmetric space of non-compact type with finite volume. An important notion in this context is that of arithmetic subgroup, which leads to interesting examples of locally symmetric spaces. Intuitively, a subgroup is arithmetic if all its points have integer coordinates such as $\mathsf{SL}_2(\mathbb{Z})$ in $\mathsf{SL}_2(\mathbb{R})$; see [137, Chapter 5] for a precise definition of arithmetic lattice. Let Γ be a lattice of $\mathsf{SO}^0_{1,n}$, $M = \mathbb{R}\mathsf{H}^n = \mathsf{SO}^0_{1,n}/\mathsf{SO}_n$, and let $N = \Gamma\backslash M$ be its associated locally symmetric space, where $n \geq 2$. Bader, Fisher, Miller, and Stover [8] proved that if N contains infinitely many properly immersed closed maximal totally geodesic submanifolds of dimension at least two, then Γ is arithmetic. They also proved a similar result in the case that $M = \mathbb{C}\mathsf{H}^n$, see [9].

Taking the previous discussion into account, it makes sense to carry out a systematic study of totally geodesic submanifolds in Riemannian manifolds. As we will see, totally geodesic submanifolds are naturally linked to the presence of isometries. Hence, the theory of totally geodesic submanifolds is specially rich on homogeneous spaces, and particularly on symmetric ones. The study and classification of totally geodesic submanifolds in given Riemannian manifolds will be the main goal of the rest of this thesis.

This chapter aims to give an overview of the basic concepts related to totally geodesic submanifolds in Riemannian manifolds. It is structured as follows. In Section §5.1 we recall some well-known facts about totally geodesic submanifolds in Riemannian manifolds. Later, in Section §5.2, we discuss the existence and the uniqueness of totally geodesic submanifolds, and we prove that under certain circumstances a totally geodesic submanifold can be extended to a complete one. Finally, in Section §5.3 we review some background related to totally geodesic submanifolds in symmetric spaces and we discuss the most important results concerning totally geodesic submanifolds in symmetric spaces.

5.1 Totally geodesic submanifolds in Riemannian manifolds

Let $\bar{M}$ and M be connected Riemannian manifolds and $f\colon M \to \bar{M}$ an isometric immersion. We denote by $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\bar{M}$ and M, respectively. Recall that $f\colon M \to \bar{M}$ is a totally geodesic immersion in $\bar{M}$ if its second fundamental form II vanishes identically. The following lemma expresses a series of relevant equivalences (cf. [147, Proposition 13, p. 104]).

Lemma 5.1.1. *Let M be a connected immersed submanifold of $\bar{M}$. Then, the following statements are equivalent:*

i) M is totally geodesic.

ii) If α is a curve in M and $v \in T_{\alpha(0)}M$, the parallel transport of v along α is the same for M and $\bar{M}$.

iii) Every geodesic of M is a geodesic of $\bar{M}$.

iv) The geodesic γ_v of $\bar{M}$ with initial conditions $\dot{\gamma}(0) = p$ and $\dot{\gamma}_v(0) = v \in T_pM$ satisfies that $\gamma_v(t) \in M$ for every $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

From now on, unless otherwise stated, totally geodesic submanifolds will be understood as immersed submanifolds of the ambient space.

Let us denote by $\overline{\exp}$ the exponential map of $\bar{M}$. Let us consider two totally geodesic submanifolds M_1 and M_2 of $\bar{M}$. Moreover, assume that $T_pM_1 = T_pM_2$ for some $p \in M_1 \cap M_2$. Then, by Lemma 5.1.1, there exists some open neighborhood U of $0 \in T_pM_1 = T_pM_2$ such that $\overline{\exp}_p(U) \subset M_1 \cap M_2$. Moreover, if M_i is complete for each $i \in \{1, 2\}$, we have that $M_1 = \overline{\exp}_p T_pM_1 = \overline{\exp}_p T_pM_2 = M_2$ (since every geodesic of M_i is a geodesic in $\bar{M}$). This proves the following useful lemma.

Lemma 5.1.2. *Let M_i be a totally geodesic submanifold of $\bar{M}$, where $i \in \{1, 2\}$. If $T_pM_1 = T_pM_2$ for some $p \in M_1 \cap M_2$, then M_1 and M_2 coincide around a neighborhood of $p \in \bar{M}$. Furthermore, if M_1 and M_2 are complete, then*

$$M_1 = \overline{\exp}_p T_pM_1 = \overline{\exp}_p T_pM_2 = M_2.$$

Let us consider two totally geodesic submanifolds M_1 and M_2 of $\bar{M}$ intersecting at $p \in M_1 \cap M_2$. By Lemma 5.1.1, we can find a small neighborhood U_i of 0 in $T_pM_i \subset T_p\bar{M}$ such that $\overline{\exp}_p(U_i) \subset M_i$, for each $i \in \{1, 2\}$. Thus,

$$\overline{\exp}_p(U_1 \cap U_2) \subset \overline{\exp}_p(U_1) \cap \overline{\exp}_p(U_2) \subset M_1 \cap M_2$$

is a chain of inclusions of open subsets of $M_1 \cap M_2$. This shows the following.

Lemma 5.1.3. *Let M_i be a totally geodesic submanifold of $\bar{M}$, where $i \in \{1, 2\}$. Then, for any $p \in M_1 \cap M_2$, there is an open neighborhood of p in $M_1 \cap M_2$ that is an embedded totally geodesic submanifold of $\bar{M}$. In particular, every connected component of $M_1 \cap M_2$ is a totally geodesic submanifold of $\bar{M}$.*

The next result tells us that a way to construct totally geodesic submanifolds is by using the isometry group of the ambient space $\bar{M}$ (cf. [112, Chapter II, Theorem 5.1]).

Theorem 5.1.4. *Let $\bar{M}$ be a Riemannian manifold and let $S \subset \mathrm{Isom}(\bar{M})$ be a subset. Then, every connected component of*

$$\mathrm{Fix}(S) := \{p \in \bar{M} : \varphi(p) = p \text{ for every } \varphi \in S\}$$

is a totally geodesic closed submanifold of $\bar{M}$.

Proof. Let $p \in \mathrm{Fix}(S)$ and take $V = \{v \in T_p\bar{M} : \varphi_{*p}v = v$ for every $\varphi \in S\}$. Now choose a normal neighborhood U of $\bar{M}$ around p. We claim that $U \cap \mathrm{Fix}(S) = \overline{\exp}_p(\overline{\exp}_p^{-1}(U) \cap V)$. Notice that this implies that every connected component of $\mathrm{Fix}(S)$ is an embedded submanifold of $\bar{M}$, since V is a linear subspace of $T_p\bar{M}$.

On the one hand, let us consider a geodesic γ_v starting at $p = \gamma(0)$ with $\dot{\gamma}(0) = v \in \overline{\exp}_p^{-1}(U) \cap V$. Thus, since $v \in V$, the uniqueness of geodesics, and the fact that isometries map geodesics to geodesics imply that $\varphi \circ \gamma = \gamma$ for every $\varphi \in S$. This proves that $\overline{\exp}_p(\overline{\exp}_p^{-1}(U) \cap V) \subset U \cap \mathrm{Fix}(S)$, since U is a normal neighborhood.

On the other hand, assume that $q \in U \cap \mathrm{Fix}(S)$ and that there is not any geodesic γ_v starting at p with initial velocity in $\overline{\exp}_p^{-1}(U) \cap V$ reaching q. However, since U is a normal neighborhood there exists a unique minimizing-length geodesic γ joining p and q. Since γ cannot have initial velocity in $\overline{\exp}_p^{-1}(U) \cap V$, there is some isometry $\varphi \in S$ such that $\varphi \circ \gamma$ is a geodesic different from γ but connecting p and q. Then, we get a contradiction with the uniqueness of γ, and $U \cap \mathrm{Fix}(S) \subset \overline{\exp}_p(\overline{\exp}_p^{-1}(U) \cap V)$.

Thus, for every $p \in \mathrm{Fix}(S)$ there is a neighborhood U of $p \in \bar{M}$ such that $U \cap \mathrm{Fix}(S)$ is a submanifold of $\bar{M}$. Hence, every geodesic of $\bar{M}$ with intial conditions in $\mathrm{Fix}(S)$ stays for a while in $\mathrm{Fix}(S)$, proving that $\mathrm{Fix}(S)$ consists of an union of totally geodesic submanifolds. By definition, $\mathrm{Fix}(S)$ is closed. Therefore, every connected component of $\mathrm{Fix}(S)$ is a totally geodesic closed embedded submanifold of $\bar{M}$. □

Although the above result shows that the existence of totally geodesic submanifolds is linked to the existence of isometries, there are totally geodesic submanifolds that are not fixed points of a set of isometries. Furthermore, it is known that "generic" Riemannian manifolds do not admit totally geodesic submanifolds of dimension greater than 1, see [142].

Theorem 5.1.4 suggests that in a Riemannian manifold with few isometries there are few totally geodesic submanifolds. This implies that an interesting setting to study totally geodesic submanifolds is that of homogeneous spaces, which have a large group of isometries. A useful characterization of these spaces in terms of Killing fields is the following.

Lemma 5.1.5. *Let M be a connected Riemannian manifold. Then, the following statements are equivalent:*

i) M is homogeneous.

ii) There exists some point $p \in M$ such that T_pM is spanned by Killing vector fields of M evaluated at p.

iii) For every $p \in M$, T_pM is spanned by Killing vector fields of M evaluated at p.

Proof. Let $M = \mathsf{G}/\mathsf{K}$ be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresponding to some point $o \in M$. Notice that $\mathfrak{p}$ is identified with T_oM. Thus, the Killing vector fields induced by elements of $\mathfrak{p}$, when evaluated at o, span T_oM.

If M is a Riemannian manifold and there is a point $p \in M$ such that T_pM is spanned by the Killing vector fields evaluated at p, then there exists some open neighborhood U of p such that every point in U lies on an integral curve of a Killing vector field. This implies that the orbit of p by the action of the isometry group is open. However, since it is also closed, we have that the isometry group acts transitively on M. □

The next proposition shows that totally geodesic submanifolds of homogeneous spaces are again homogeneous spaces.

Proposition 5.1.6. *Let $\bar{M}$ be a homogeneous Riemannian manifold and let M be a complete totally geodesic submanifold of $\bar{M}$. Then, M is homogeneous.*

Proof. Let X be a Killing vector field of $\bar{M}$. Thus, for each $p \in M$, we have the orthogonal decomposition

$$X(p) = X(p)_{T_pM} + X(p)_{\nu_pM} \quad \text{for every } p \in M,$$

where $X(p)_{T_pM}$ and $X(p)_{\nu_pM}$ denote the orthogonal projections of $X(p)$ to T_pM and ν_pM, respectively. Since X is a Killing vector field of $\bar{M}$, for each $Y \in \Gamma(TM)$, we have

$$0 = \langle \bar{\nabla}_Y X, Y\rangle = \langle \bar{\nabla}_Y X_{TM}, Y\rangle + \langle \bar{\nabla}_Y X_{\nu M}, Y\rangle = \langle \nabla_Y X_{TM}, Y\rangle,$$

since M is totally geodesic. Thus, the tangential projection of a Killing vector field of $\bar{M}$ to M, when restricted to M, is a Killing vector field of M, since M is complete. The tangent space of $\bar{M}$ at every point of $\bar{M}$ is generated by Killing fields of $\bar{M}$, implying that the tangent space of M at every point is generated by projecting these Killing vector fields. Hence, by Lemma 5.1.5, M is homogeneous. □

Notice that complete totally geodesic submanifolds of $\bar{M}$ are intrinsically homogeneous, but they are not necessarily extrinsically homogeneous. However, the connected components of the fix point set of any collection of isometries are extrinsically homogeneous submanifolds, see [14, Lemma 9.1.1]. This shows once again that the fixed points sets of isometries provide the most natural and well-behaved examples of totally geodesic submanifolds.

5.2 On the existence and uniqueness of totally geodesic submanifolds

In this section we discuss the existence and the uniqueness of totally geodesic submanifolds, and we prove that under certain hypotheses a totally geodesic submanifold can be extended to a complete one.

The next result was proved by Cartan. A proof of it can be consulted in [14, p. 274]. In this section $B_\varepsilon(0)$ denotes the ball of radius $\varepsilon > 0$ around the origin in a tangent space $T_p\bar{M}$ of an ambient manifold $\bar{M}$.

Theorem 5.2.1. *Let $\bar{M}$ be a Riemannian manifold, $p \in \bar{M}$ and V be a linear subspace of $T_p\bar{M}$. Then, there exists a totally geodesic submanifold M of $\bar{M}$ with $p \in M$ and $T_pM = V$ if and only if there exists some $\varepsilon > 0$ such that for every geodesic $\gamma\colon [0,1] \to \bar{M}$ with $\gamma(0) = p$ and $\dot{\gamma}(0) \in V \cap B_\varepsilon(0)$, the Riemannian curvature tensor of $\bar{M}$ at $\gamma(1)$ preserves the parallel transport of V along γ from p to $\gamma(1)$.*

Our intention for this subsection is to enhance Theorem 5.2.1 by proving a global version of it, see Lemma 5.2.5. From now on we assume that $\bar{M}$ is an analytic Riemannian manifold. We denote the Grassmann bundle of k-planes of $T\bar{M}$ by $\mathsf{G}_k(T\bar{M})$ and the injectivity radius of $\bar{M}$ at p by $\operatorname{inj}(p)$. We mainly follow [14, §10.3].

Lemma 5.2.2. *Let $\bar{M}$ be an analytic Riemannian manifold and let M be a totally geodesic submanifold of $\bar{M}$ passing through $p \in \bar{M}$. Then, $\overline{\exp}_p(B_\delta(0) \cap T_pM)$ is an embedded totally geodesic submanifold of $\bar{M}$ for every $\delta \in (0, \mathrm{inj}(p))$.*

Proof. Let $V = T_pM$. By Theorem 5.2.1, there exists some $\varepsilon > 0$ such that for every geodesic $\gamma_v \colon [0,1] \to \bar{M}$ with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v \in B_\varepsilon(0) \cap T_pM$, the Riemannian curvature tensor of $\bar{M}$ at $\gamma(1)$ preserves the parallel transport of V along γ from p to $\gamma(1)$.

Now consider the geodesic $\widehat{\gamma}_v \colon [0,1] \to \bar{M}$ with $\widehat{\gamma}_v(0) = p$ and with initial velocity $v \in (B_\delta(0) \setminus B_\varepsilon(0)) \cap T_pM$. We extend $X, Y, Z \in V$ and $\xi \in V^\perp := T_p\bar{M} \ominus V$ to parallel vector fields along $\widehat{\gamma}_v$. Thus, the map $\Phi \colon [0,1] \to \mathbb{R}$, given by $t \in [0,1] \mapsto \langle \bar{R}(X(t), Y(t)), Z(t), \xi(t) \rangle$ is analytic. However, by the uniqueness of parallel transport, Φ restricted to $[0, \frac{\varepsilon}{2||v||})$ is identically zero, which implies that Φ is identically zero by the analiticity of Φ. Consequently, the result follows by Theorem 5.2.1. □

Let $f_i \colon M_i \to \bar{M}$, with $i \in \{1, 2\}$, be an isometric immersion. We say that f_1 and f_2 are *equivalent* if there exists an isometry $\varphi \colon M_1 \to M_2$ such that $f_1 = f_2 \circ \varphi$. Notice that every isometric immersion $f \colon M \to \bar{M}$ induces a smooth map $\widetilde{f} \colon M \to \mathsf{G}_k(T\bar{M})$, given by $\widetilde{f}(p) = f_{*p}(T_pM)$, for every $p \in M$, where $k = \dim M$. The isometric immersion (M, f) is said to be *compatible* if M is connected and $\widetilde{f}$ is injective. Since totally geodesic submanifolds are locally determined by its tangent space at some point, see Lemma 5.1.2, every compatible totally geodesic isometric immersion f is completely determined, up to equivalence, by the image of $\widetilde{f}$ in $\mathsf{G}_k(T\bar{M})$.

Let $\mathfrak{T}$ be the collection of all the equivalence classes of compatible totally geodesic immersions into $\bar{M}$. We define a partial order $\preceq$ in $\mathfrak{T}$ in the following way. We write $(M_1, f_1) \preceq (M_2, f_2)$ if there exists an injective local isometry $i \colon M_1 \to M_2$ such that $f_1 = f_2 \circ i$. If this happens, we say that (M_2, f_2) *extends* (M_1, f_1). By Lemma 5.1.2, we have $(M_1, f_1) \preceq (M_2, f_2)$ if and only if $\widetilde{f}_1(M_1) \subset \widetilde{f}_1(M_2)$.

For each $V \in \mathsf{G}_k(T\bar{M})$, we denote by $\mathcal{F}_V$ the set of totally geodesic immersions $f \colon M \to \bar{M}$ from a connected Riemannian manifold M into $\bar{M}$ with $V \in \widetilde{f}(M)$. Moreover, we define $\mathcal{G}_V := \bigcup_{f \in \mathcal{F}_V} \widetilde{f}(M) \subset \mathsf{G}_k(T\bar{M})$.

The following lemma is a technical one, see [14, §10.3] for a proof.

Lemma 5.2.3. *Following the notation above, let us assume that $\mathcal{G}_V \neq \emptyset$ and consider a set of compatible totally geodesic isometric immersions $\{f_i \colon M_i \to \bar{M}\}_{i \in I}$ such that $\mathcal{G}_V = \bigcup_{i \in I} \widetilde{f}_i(M_i)$. Furthermore, let $Y = \bigsqcup_{i \in I} M_i$ and consider the equivalence relation $\sim$ in Y such that*

$$p_i \sim p_j \quad \text{if } p_i \in M_i,\ p_j \in M_j \text{ and } \widetilde{f}_i(p_i) = \widetilde{f}_j(p_j), \text{ where } i, j \in I.$$

Then, the following statements hold:

i) $\widehat{M} = Y/\sim$ *is a connected smooth manifold.*

ii) The map $g \colon \widehat{M} \to \bar{M}$, $[p_i] \mapsto f_i(p_i)$ *is a compatible totally geodesic immersion.*

iii) $(\widehat{M}, g)$ *extends* (M_i, f_i) *for every* $i \in I$.

iv) $(\widehat{M}, g)$ *is maximal for* $\preceq$.

The isometric immersion $g\colon \widehat{M} \to \bar{M}$ constructed in Lemma 5.2.3 is, up to equivalence, the unique maximal compatible totally geodesic isometric immersion into $\bar{M}$ with $V \in \widetilde{g}(\widehat{M})$.

Lemma 5.2.4. *Let* M *be a compatible totally geodesic submanifold of an analytic and complete Riemannian manifold* $\bar{M}$. *Then,* $\widehat{M}$, *the extension of* M *given by Lemma 5.2.3, is complete.*

Proof. By Lemma 5.2.3, there exists an extension of (M, f) given by $(\widehat{M}, g)$. We will prove that $(\widehat{M}, g)$ is complete.

Let us proceed by contradiction. Thus, we will assume that $\gamma\colon [0, b) \to \widehat{M}$ is a unit speed geodesic in $\widehat{M}$ that cannot be further extended beyond $b > 0$. Now consider the geodesic $g \circ \gamma$ in $\bar{M}$. Since $\bar{M}$ is complete, there exists $q = \lim_{t \to b^-} g(\gamma(t)) \in \bar{M}$. Notice that $q \notin g(\widehat{M})$. Furthermore, there exists a uniformly normal neighborhood U around q in $\bar{M}$. Then, there is some $\varepsilon > 0$ such that the injectivity radius satisfies $\mathrm{inj}(p) \geq \varepsilon$ for every $p \in U$. Now take t_0 such that $b - t_0 < \varepsilon$ and $q' = g \circ \gamma(t_0) \in U$. Since $q' \in U$, by Lemma 5.2.2, there is a totally geodesic submanifold N of $\bar{M}$ passing through q' which also contains q. However, by Lemma 5.2.3, N is extended by $\widehat{M}$ contradicting the assumption that $q \notin g(\widehat{M})$. □

Let k be a non-negative integer. The k-th covariant derivative of the curvature tensor $\bar{R}$ denoted by $\bar{\nabla}^k \bar{R}$ is a $(1, k+3)$-tensor. A subspace $V \subset T_p\bar{M}$ is invariant under $(\bar{\nabla}^k \bar{R})_p$ if

$$(\bar{\nabla}^k \bar{R})(V_1, \ldots, V_k, X, Y, Z) \in V$$

for every $X, Y, Z, V_1, \ldots, V_k \in V$.

Theorem 5.2.5. *Let* $\bar{M}$ *be an analytic complete Riemannian manifold,* $p \in \bar{M}$ *and* V *a linear subspace of* $T_p\bar{M}$. *There exists a complete totally geodesic submanifold* M *of* $\bar{M}$ *such that* $p \in M$ *and* $\overline{\exp}_p(V) = M$ *if and only if* $(\bar{\nabla}^k \bar{R})_p$ *leaves* V *invariant for every* $k \geq 0$.

Proof. Let us extend $X, Y, Z \in V$ and $\xi \in V^\perp := T_p\bar{M} \ominus V$ to parallel vector fields along an arbitrary geodesic $\gamma\colon [0,1] \to \bar{M}$ starting at $p \in \bar{M}$. Then, if V is invariant under $(\bar{\nabla}^k \bar{R})_p$ for every $k \geq 0$ we have

$$\frac{d}{dt}^{k}_{|t=0} \bar{R}(X(t), Y(t), Z(t), \xi(t)) = 0, \quad \text{for every } k \geq 0.$$

By the analiticity of $\bar{M}$, this shows that $\bar{R}(X(t), Y(t), Z(t), \xi(t)) = 0$ for every $t \in [0,1]$. Then, by Theorem 5.2.1, there exists a totally geodesic submanifold N of $\bar{M}$ with $T_pN = V$ defined locally around $p \in N$. Now, this totally geodesic submanifold N can be extended to a complete totally geodesic submanifold M by Lemma 5.2.3 and Lemma 5.2.4.

Conversely, if M is a totally geodesic submanifold of $\bar{M}$, Gauss formula together with Gauss and Codazzi equations imply that T_pM is invariant under $(\bar{\nabla}^k \bar{R})_p$ for every $k \geq 0$ and $p \in M$. □

5.3 Totally geodesic submanifolds in symmetric spaces

The purpose of this section is to recall some well-known facts and results related to totally geodesic submanifolds in a very particular case of ambient spaces: symmetric spaces.

Let $M = \mathsf{G}/\mathsf{K}$ be a connected Riemannian symmetric space, where $\mathsf{G} = \mathrm{Isom}^0(M)$ is the connected component of the identity of the isometry group of M and the Lie group $\mathsf{K} = \{g \in \mathsf{G} : g \cdot o = o\}$ is the isotropy at some point $o \in M$. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathcal{B}_\mathfrak{g}$ be the Killing form of $\mathfrak{g}$, which is defined as $\mathcal{B}_\mathfrak{g}(X,Y) = \mathrm{tr}(\mathrm{ad}_X \, \mathrm{ad}_Y)$ for $X, Y \in \mathfrak{g}$, where ad stands for the adjoint representation of $\mathfrak{g}$.

We consider the geodesic symmetry s_o at the base point $o \in M$; it gives rise to an involutive automorphism σ of G defined by $\sigma(g) = s_o g s_o$ whose differential at the identity σ_{*e} is denoted by θ. The map θ is a Lie algebra automorphism of $\mathfrak{g}$, and $\mathfrak{g}$ decomposes as the direct sum of vector spaces $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the fixed point set of θ and $\mathfrak{p}$ is the eigenspace of θ corresponding to the eigenvalue -1. In case $-\mathcal{B}_\mathfrak{g}(\theta X, Y)$ is positive definite, this splitting is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to θ, and the involution θ is called a *Cartan involution* of $\mathfrak{g}$.

We recall that for any real semisimple Lie algebra there exists a Cartan involution, and any two Cartan involutions in a real semisimple Lie algebra differ by an inner automorphism.

Recall from Subsection §1.3.2 that a symmetric space is irreducible if the universal cover $\widetilde{M}$ of M, which is again a symmetric space, is not isometric to a non-trivial product of symmetric spaces. Moreover, a symmetric space is said to be of compact type, non-compact type or Euclidean type if $\mathcal{B}_\mathfrak{g}|_{\mathfrak{p}\times\mathfrak{p}}$, the restriction of the Killing form $\mathcal{B}_\mathfrak{g}$ to $\mathfrak{p}$, is negative definite, positive definite or identically zero, respectively. When M is of compact type, then M is compact with non-negative sectional curvature and G is a compact semisimple Lie group. If M is of non-compact type, then M is diffeomorphic to $\mathbb{R}^n$, for some $n \geq 2$, and G is a non-compact semisimple Lie group. The universal cover of a symmetric space splits as a Riemannian product

$$\widetilde{M} = M_0 \times M_+ \times M_-,$$

where M_0, which is called the *flat factor*, is isometric to a Euclidean space, and M_+ and M_- are simply connected symmetric spaces of compact and non-compact type, respectively. It is said that M is semisimple if M_0 is a point.

Let Σ be a connected totally geodesic submanifold of a symmetric space $M = \mathsf{G}/\mathsf{K}$. By the homogeneity of M, we can assume without loss of generality that $o \in \Sigma$. By Theorem 5.2.5 and the fact that symmetric spaces have parallel curvature tensor, a totally geodesic submanifold Σ of M with $o \in \Sigma$ and $V = T_o\Sigma \subset T_oM$ exists if and only if $V \subset T_oM$ is *curvature invariant*. This means that $R_o(V,V)V \subset V$, where R is the Riemannian curvature tensor of M. Recall that using the identification of $\mathfrak{p}$ and T_oM, we can write the curvature tensor of M at o as

$$R_o(X,Y)Z = -[[X,Y],Z], \quad \text{for } X, Y, Z \in T_oM.$$

Thus, a subspace $V \subset \mathfrak{p}$ is curvature invariant if and only if $[[X,Y],Z] \in V$ for every $X,Y,Z \in V$. A subspace V of $\mathfrak{p}$ with this property is called a *Lie triple system* in $\mathfrak{p}$. Hence, there is a one-to-one correspondence between Lie triple systems V in $\mathfrak{p}$ and complete totally geodesic submanifolds Σ in M containing $o \in M$. In this chapter we consider only complete totally geodesic submanifolds since every totally geodesic submanifold of a symmetric space can be extended to a complete one. Furthermore, if V is a Lie triple system in $\mathfrak{p}$ such that its orthogonal complement in $\mathfrak{p}$ is also a Lie triple system, we say that V is a *reflective Lie triple system* and the corresponding totally geodesic submanifold is called *reflective*. A submanifold of M is reflective if and only if it is a connected component of the fixed point set of an involutive isometry, see [125].

As we saw in Proposition 5.1.6, complete totally geodesic submanifolds of homogeneous spaces are (intrinsically) homogeneous. However, in the setting of symmetric spaces, we have that complete totally geodesic submanifolds are extrinsically homogeneous. To obtain a homogeneous presentation of a totally geodesic submanifold from a Lie triple system we proceed as follows. Let $V \subset \mathfrak{p}$ be a Lie triple system in $\mathfrak{p}$. Define $\mathfrak{g}' := [V,V] \oplus V \subset \mathfrak{k} \oplus \mathfrak{p}$, which is clearly a subalgebra of $\mathfrak{g}$ since V is a Lie triple system. It turns out that if we consider G', the connected Lie subgroup of G with Lie algebra $\mathfrak{g}'$, then the G'-orbit through $o \in M$ is a totally geodesic submanifold $\Sigma \subset M$ with $T_o\Sigma = V$, see [14, Proposition 11.1.2]. This shows that if $\Sigma \subset M$ is a totally geodesic submanifold passing through o, then $\mathfrak{p}_\Sigma := T_o\Sigma$ is a Lie triple system in $\mathfrak{p}$ and we can define

$$\mathfrak{k}_\Sigma := [\mathfrak{p}_\Sigma, \mathfrak{p}_\Sigma], \qquad \mathfrak{g}_\Sigma := \mathfrak{k}_\Sigma \oplus \mathfrak{p}_\Sigma.$$

Then $\mathfrak{k}_\Sigma \subset \mathfrak{k}$ and $\mathfrak{g}_\Sigma \subset \mathfrak{g}$ are subalgebras and if we consider the connected Lie subgroups $\mathsf{G}_\Sigma \subset \mathsf{G}$ with Lie algebra $\mathfrak{g}_\Sigma$ and $\mathsf{K}_\Sigma \subset \mathsf{K}$ with Lie algebra $\mathfrak{k}_\Sigma$, then $\Sigma = \mathsf{G}_\Sigma/\mathsf{K}_\Sigma$ as homogeneous spaces. Moreover, every totally geodesic submanifold Σ of M is invariant under s_p for every $p \in \Sigma$. Thus every totally geodesic submanifold $\Sigma \subset M$ is a symmetric space with respect to its induced Riemannian metric. A totally geodesic submanifold of a symmetric space is said to be *semisimple* if it is a semisimple symmetric space. Finally, observe that if V is a Lie triple system in $\mathfrak{p}$, then iV is a Lie triple system in $i\mathfrak{p}$, where i is the imaginary unit (see Subsection §1.3.2). This means that a totally geodesic submanifold Σ of G/K containing o corresponds to a totally geodesic submanifold Σ^* of the dual symmetric space $\mathsf{G}^*/\mathsf{K}^*$, and Σ^* is dual to Σ. Hence, when studying totally geodesic submanifolds it will not be restrictive to assume that our ambient symmetric space is either of compact type or of non-compact type.

In the compact setting, the problem of determining the topology of the symmetric space that corresponds to a particular Lie triple system is, in general, not straightforward: the Lie triple system determines the symmetric space only up to local isometry. In the non-compact setting, however, we do not have this difficulty: any complete totally geodesic submanifold of a symmetric space of non-compact type is simply connected.

The problem of classifying totally geodesic submanifolds in Riemannian symmetric spaces has been a relevant and outstanding topic of research in submanifold geometry in the last decades. It was started in 1963 when, in his seminal paper, Wolf [187] classified totally geodesic submanifolds in symmetric spaces of rank one. For rank two this problem has been addressed by Chen and Nagano [48, 49] and Klein [107, 108, 109], who found some omissions in the works of Chen and Nagano and completed the classification. Up to now, there are only complete classifications in symmetric spaces of rank less than three.

However, classification results for some special kinds of totally geodesic submanifolds are known. A complete, totally geodesic, proper submanifold is said to be *maximal* if it is maximal with respect to the inclusion among complete, totally geodesic, proper submanifolds. For example, Ikawa and Tasaki [96] proved that a totally geodesic submanifold in a simple compact Lie group equipped with a bi-invariant metric is maximal if and only if it is a maximal subgroup or a Cartan embedding. Moreover, in [96] they also studied totally geodesic submanifolds of maximal rank. Another special class of totally geodesic submanifolds is that of reflective submanifolds. They are totally geodesic submanifolds that arise as connected components of the fixed point set of an involutive isometry. Reflective submanifolds have been classified by Leung in [123, 124]. Moreover, Lagrangian totally geodesic submanifolds of Hermitian symmetric spaces were classified by Jaffee [101, 102] and independently by Leung [125]. More recently, Mashimo [132] classified totally geodesic surfaces in classical symmetric spaces.

An important invariant of a symmetric space is the index. The *index* of a symmetric space M, denoted by $i(M)$, is the minimal codimension of a proper totally geodesic submanifold of M. We will say that a totally geodesic submanifold Σ of M *realizes the index* of M if Σ has codimension equal to $i(M)$. In a series of papers, Berndt, Olmos and Rodríguez [20, 21, 22, 23, 24] computed the index of irreducible symmetric spaces. In particular, they proved the so-called index conjecture, which can be stated as follows. Let M be an irreducible symmetric space of non-compact type different from $\mathsf{G}_2^2/\mathsf{SO}_4$. Then, there is some reflective submanifold Σ of M whose codimension equals the index of M.

Chapter 6
Totally geodesic submanifolds in products of rank one symmetric spaces

The classification of totally geodesic submanifolds in symmetric spaces was started in 1963 by Wolf [187], who classified these objects in symmetric spaces of rank one. Throughout this chapter, we extend Wolf's result to products of rank one symmetric spaces. The contents of this chapter have given rise to the paper [160].

We start by studying totally geodesic submanifolds in a product of two symmetric spaces of rank one. In particular, we classify totally geodesic submanifolds in simply connected reducible symmetric spaces of rank two. In the non-compact setting these are products of hyperbolic spaces $\mathbb{F}\mathsf{H}^n$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

Moreover, we will introduce some slight modification of Young tableaux that we call adapted Young tableaux (see Section §6.2 for the definition), which will be useful to classify totally geodesic submanifolds in arbitrary products of symmetric spaces of rank one and to determine their isometry type via Corollary 6.2.11. Young tableaux have been used to classify irreducible representations of the symmetric group and they provide an effective way to gain understanding of a given irreducible representation. We prove a result (Proposition 6.2.12) that gives a correspondence between these adapted Young tableaux and semisimple totally geodesic submanifolds in products of rank one symmetric spaces. This leads to the following theorem, which, via duality, gives a classification of totally geodesic submanifolds in arbitrary products of symmetric spaces of rank one.

Theorem A. *Let $M = M_1 \times \cdots \times M_r$, where M_i is a symmetric space of non-compact type and rank one for each $i \in \{1, \dots, r\}$.*

Then, a submanifold Σ of M is totally geodesic if and only if $\Sigma = \Sigma_0 \times \Sigma_T$, where Σ_T is a semisimple totally geodesic submanifold corresponding to a Young tableau T adapted to $M_{\sigma(1)} \times \cdots \times M_{\sigma(k)}$, Σ_0 is a flat totally geodesic submanifold of $M_{\sigma(k+1)} \times \cdots \times M_{\sigma(r)}$, σ is any permutation of $\{1, \dots, r\}$, and $k \in \{1, \dots, r\}$.

In 1977, Chen and Nagano [48, 49] gave a classification of maximal totally geodesic submanifolds in irreducible symmetric spaces of rank two. In this classification there were some examples missing that were found by Klein in a series of papers [107, 108, 109]. In view of the known results, Theorem A provides the first classification of (not necessarily maximal) totally geodesic submanifolds in some symmetric space of rank higher than two.

A special class of symmetric spaces where totally geodesic submanifolds can be studied is that of Hermitian symmetric spaces. On the one hand, we can use the notion of Kähler angle (see Section §4.1) to measure how a submanifold fails to be complex in a Hermitian symmetric space. For example, a totally geodesic submanifold is complex or totally real if and only if it has constant Kähler angle equal to 0 or to $\pi/2$, respectively. On the other hand, complex totally geodesic submanifolds in Hermitian symmetric spaces were classified by Ihara in [95], and real forms, which constitute a particular type of totally real and totally geodesic submanifolds, were studied and classified by Jaffe [101, 102] and Leung [124, 125].

While in complex projective spaces (the Hermitian symmetric spaces of compact type and rank one) a totally geodesic submanifold is either complex or totally real, in the rank two case the situation is more involved. Klein [107, 109] found two examples of irreducible totally geodesic submanifolds that are neither complex nor totally real, one example in the complex quadric and another one in the complex 2-plane Grassmannian. These examples have non-trivial constant Kähler angle, i.e. they have Kähler angle different from 0 and $\pi/2$, and they have been the only known totally geodesic submanifolds with non-trivial constant Kähler angle up to the present. In particular, these two examples have constant Kähler angle equal to $\arccos(1/5)$.

In this chapter, we will also give a method to construct infinitely many examples of irreducible totally geodesic submanifolds with non-trivial constant Kähler angle in irreducible Hermitian symmetric spaces of higher rank. This method will rely on the construction of certain totally geodesic submanifolds contained in a product of Hermitian symmetric spaces. Clearly, $\mathbb{C}^n$, $n \geq 2$, is a Hermitian symmetric space where every $\varphi \in [0, \pi/2]$ can be realized as the constant Kähler angle of some totally geodesic submanifold. Therefore, we exclude from our study flat Hermitian symmetric spaces. Let $\mathcal{J}_r$ be the set of Kähler angles of totally geodesic submanifolds in irreducible non-flat Hermitian symmetric spaces of rank r, and $\mathcal{J} = \bigcup_{r\geq 1} \mathcal{J}_r$ the set of Kähler angles of totally geodesic submanifolds in non-flat irreducible Hermitian symmetric spaces. As a consequence of the classification results in rank one and two, we have

$$\mathcal{J}_1 = \{0, \pi/2\}, \quad \mathcal{J}_2 = \{0, \arccos(1/5), \pi/2\}.$$

Consequently, an interesting problem is to compute $\mathcal{J}$ or $\mathcal{J}_r$. In this chapter, we will prove the following result.

Theorem B. *Let $\mathcal{J}$ be the set of Kähler angles of totally geodesic submanifolds in non-flat irreducible Hermitian symmetric spaces. Then $\mathcal{J}$ contains a dense subset of* $[0, \pi/2]$.

This chapter is structured as follows. In Section §6.1, we introduce the notion of diagonal totally geodesic submanifold. Then, we give a structure result for totally geodesic submanifolds of maximal rank in reducible symmetric spaces. In Section §6.2, we classify totally geodesic submanifolds in reducible symmetric spaces of rank two. Furthermore, we introduce the notion of adapted Young tableau, which is used to classify totally geodesic submanifolds in products of an arbitrary number of rank one symmetric spaces (Theorem A). Finally, in Section §6.3, we construct totally geodesic submanifolds with non-trivial constant Kähler angle in complex Grassmannians, and prove Theorem B.

6.1 Diagonal totally geodesic submanifolds

In what follows, we introduce the notions of k-diagonal linear subspace and of k-diagonal totally geodesic submanifold.

Let V be a vector space equipped with a positive definite scalar product and let us choose subspaces $V_i \subset V$ such that $V = \bigoplus_{i=1}^r V_i$ is an orthogonal decomposition of V. We say that a subspace $W \subset V$ is *k-diagonal* with respect to the above decomposition if there is a collection of indexes $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$ such that every non-zero element of W has non-trivial projection onto V_l if and only if $l = i_j$ for some $j \in \{1, \ldots, k\}$. For instance, let $V := \mathbb{R}^3$ and consider $V_i := \operatorname{span}\{e_i\}$, where $\{e_i\}_{i=1}^3$ is the canonical basis of $\mathbb{R}^3$. Then, we have that $W = \operatorname{span}\{e_1 + e_2\}$ is a 2-diagonal subspace and $W' = \operatorname{span}\{e_1 + e_2, e_1 + e_3\}$ is not k-diagonal for any $k \in \{1, 2, 3\}$.

Let M be a complete connected Riemannian manifold. Then, by the De-Rham Theorem, the universal cover $\widetilde{M}$ of M splits as a Riemannian product $\widetilde{M} = M_0 \times M_1 \times \cdots \times M_r$, where M_0 a Euclidean space and M_i is a connected, complete, irreducible, simply connected and non-flat Riemannian manifold for each $i \in \{1, \ldots, r\}$. Moreover, this decomposition is unique up to order. Let $p = (p_0, \ldots, p_r) \in \widetilde{M}$, $\Sigma \subset M$ a submanifold, and $\pi\colon \widetilde{M} \to M$ the universal covering map such that $q = \pi(p) \in \Sigma$. Then, $T_p\widetilde{M} = \bigoplus_{i=0}^r T_{p_i} M_i$, and since π_{*p} is a linear isometry, we have an orthogonal decomposition of T_qM given by

$$T_qM = \bigoplus_{i=0}^{r} \pi_{*p} T_{p_i} M_i. \tag{6.1}$$

Then, we say that Σ is *k-diagonal* at $q \in \Sigma$ if $T_q\Sigma \subset T_qM$ is k-diagonal with respect to the decomposition in Equation (6.1), and we say that Σ is *diagonal* at q if Σ is k-diagonal at q for some $k > 1$. It is easy to check that this definition does not depend on $p \in \pi^{-1}(q)$.

If our ambient space is homogeneous and Σ is extrinsically homogeneous, we say that $\Sigma \subset M$ is *k-diagonal* if there is some point $q \in \Sigma$ satisfying that $T_q\Sigma \subset T_qM$ is k-diagonal with respect to the decomposition in Equation (6.1). Observe that if the previous property holds for some point in Σ, then it holds for every point in Σ.

In the present chapter, our interest is on Riemannian symmetric spaces, so let $M = \mathsf{G}/\mathsf{K}$ be a simply connected Riemannian symmetric space. Hence, by De-Rham Theorem, $M = M_0 \times \cdots \times M_r$, where M_0 is isometric to some Euclidean space and M_i is a simply connected, irreducible, semisimple symmetric space for each $i \in \{1, \ldots, r\}$. Moreover, we can identify T_pM with

$$\mathfrak{p} := \bigoplus_{i=0}^{r} \mathfrak{p}_i, \tag{6.2}$$

where $\mathfrak{p}_i$ is a Lie triple system in $\mathfrak{p}$, which is identified with the tangent space $T_{p_i}M_i$ of the i-th factor in the decomposition of M. Observe that $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$, for each $i, j \in \{0, \ldots, r\}$, $i \neq j$, and that $\mathfrak{p}_0$ is identified with the flat factor in the decomposition of M given by De-Rham Theorem, so $[\mathfrak{p}_0, \mathfrak{p}_0] = 0$. Furthermore, we can define

$$\mathfrak{k}_i := [\mathfrak{p}_i, \mathfrak{p}_i], \qquad \mathfrak{g}_i := \mathfrak{k}_i \oplus \mathfrak{p}_i,$$

where $i = 1, \dots, r$. Then, $\mathfrak{k}_i$ and $\mathfrak{g}_i$ are the Lie algebras corresponding to the isotropy and isometry groups of M_i, respectively. On $\mathfrak{p}$ (and hence, on each $\mathfrak{p}_i$) we will consider the inner product $\langle \cdot, \cdot \rangle$ induced from the metric of M (resp. from the metric of M_i) via the identification $\mathfrak{p} \cong T_o M$ (resp. $\mathfrak{p}_i \cong T_{o_i} M_i$).

Let $\Sigma \subset M$ be a totally geodesic submanifold through the point $o \in \Sigma \subset M$. Then, there is a Lie triple system $\mathfrak{p}_\Sigma \subset \mathfrak{p}$ corresponding to Σ. Notice that if $\mathfrak{a}_\Sigma$ is a maximal abelian subspace of $\mathfrak{p}_\Sigma$, then it is contained in a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. By the discussion above, since totally geodesic submanifolds in symmetric spaces are extrinsically homogeneous, a totally geodesic submanifold $\Sigma \subset M$ is k-diagonal if and only if $\mathfrak{p}_\Sigma \subset \mathfrak{p}$ is k-diagonal with respect to the decomposition in (6.2).

Notice that $\operatorname{proj}_i \mathfrak{p}_\Sigma$ is a Lie triple system in $\mathfrak{p}_i$, where $\operatorname{proj}_i \colon \mathfrak{p} \to \mathfrak{p}_i$ is the orthogonal projection onto $\mathfrak{p}_i$. Indeed, given $X, Y, Z \in \mathfrak{p}_\Sigma$, we have

$$[[\operatorname{proj}_i X, \operatorname{proj}_i Y], \operatorname{proj}_i Z] = \operatorname{proj}_i [[X, Y], Z] \in \operatorname{proj}_i \mathfrak{p}_\Sigma,$$

since the projection $\operatorname{proj}_i \colon \mathfrak{g} \to \mathfrak{g}_i$ is a Lie algebra homomorphism and $\mathfrak{p}_\Sigma$ is a Lie triple system. Hence, $\mathfrak{g}^i_\Sigma := \operatorname{proj}_i \mathfrak{p}_\Sigma \oplus [\operatorname{proj}_i \mathfrak{p}_\Sigma, \operatorname{proj}_i \mathfrak{p}_\Sigma]$ is a Lie subalgebra of $\mathfrak{g}_i$. Let G_Σ and G^i_Σ be the connected subgroups of G with Lie algebras $\mathfrak{g}_\Sigma := \mathfrak{p}_\Sigma \oplus [\mathfrak{p}_\Sigma, \mathfrak{p}_\Sigma]$ and $\mathfrak{g}^i_\Sigma$, respectively. Therefore, by [14, Proposition 11.1.2], we have that $\Sigma = \mathsf{G}_\Sigma \cdot o$ and $\operatorname{proj}_i \Sigma = \operatorname{proj}_i(\mathsf{G}_\Sigma \cdot o) = \mathsf{G}^i_\Sigma \cdot o$ are totally geodesic submanifolds in M and M_i, respectively, where by proj_i we also denote the projection $M \to M_i$. Thus, we obtain the following useful lemma.

Lemma 6.1.1. *Let $M = M_0 \times \cdots \times M_r$ be a product of simply connected symmetric spaces and Σ be a totally geodesic submanifold of M. Then, $\operatorname{proj}_i \Sigma$ is a totally geodesic submanifold of M_i for each $i \in \{0, \dots, r\}$.*

The following result gives a sufficient condition for a totally geodesic submanifold $\Sigma \subset M$ to be not diagonal, namely $\operatorname{rank} \Sigma = \operatorname{rank} M$.

Proposition 6.1.2. *Let $M = M_1 \times \cdots \times M_r$ be a product of simply connected irreducible symmetric spaces and $\Sigma \subset M$ a totally geodesic submanifold with the same rank as M. Then, $\Sigma = \Sigma_1 \times \cdots \times \Sigma_r$, where $\Sigma_i \subset M_i$ is a totally geodesic submanifold.*

Proof. Let $\mathfrak{p}_\Sigma$ be the Lie triple system in $\mathfrak{p}$ corresponding to Σ. We clearly have $\mathfrak{p}_\Sigma \subset \bigoplus_{i=1}^r \operatorname{proj}_i \mathfrak{p}_\Sigma$. Let $X_j \in \operatorname{proj}_j \mathfrak{p}_\Sigma$ for some $j \in \{1, \dots, r\}$, and $X \in \mathfrak{p}_\Sigma$ such that $\operatorname{proj}_j X = X_j$. Let $\mathfrak{a}_\Sigma \subset \mathfrak{p}_\Sigma$ be a maximal abelian subspace containing X. Since Σ has the same rank as M, $\mathfrak{a}_\Sigma$ is a maximal abelian subspace of $\mathfrak{p}$. But every maximal abelian subspace of $\mathfrak{p}$ is the sum of maximal abelian subspaces $\mathfrak{a}_i$ of $\mathfrak{p}_i$. Thus, $X = \sum_{i=1}^r X_i$, where $X_i \in \mathfrak{a}_i \subset \mathfrak{a}_\Sigma \subset \mathfrak{p}_\Sigma$ and $\mathfrak{a}_i$ is a maximal abelian subspace of $\mathfrak{p}_i$. In particular, X_j belongs to $\mathfrak{p}_\Sigma$. Since $j \in \{1, \dots, r\}$ was arbitrary, we have $\mathfrak{p}_\Sigma = \bigoplus_{i=1}^r \operatorname{proj}_i \mathfrak{p}_\Sigma$. By Lemma 6.1.1, we have $\Sigma = \Sigma_1 \times \cdots \times \Sigma_r$, where each $\Sigma_i := \operatorname{proj}_i \Sigma$ is a totally geodesic submanifold of M_i, for each $i \in \{1, \dots, r\}$. □

6.2 Totally geodesic submanifolds in products of symmetric spaces of rank one

In this section, we give a classification of totally geodesic submanifolds in products of symmetric spaces of rank one.

Let us recall the classification of totally geodesic submanifolds in symmetric spaces of rank one. Let M be a symmetric space of non-compact type and rank one. Then, M is either a real hyperbolic space $\mathbb{R}\mathsf{H}^n$, $n \geq 2$, a complex hyperbolic space $\mathbb{C}\mathsf{H}^n$, $n \geq 2$, a quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^n$, $n \geq 2$, or the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$. We use the notation $\mathbb{F}\mathsf{H}^n$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $n = 2$ if $\mathbb{F} = \mathbb{O}$. Furthermore, the metric of $\mathbb{F}\mathsf{H}^n$ is such that its sectional curvature is equal to $-c$ in the real case, or pinched between $-c$ and $-c/4$ in the other cases, for some $c > 0$. In this case we will write $\mathbb{F}\mathsf{H}^n(c)$ and simply $\mathbb{F}\mathsf{H}^n$ when the minimal absolute value of the sectional curvatures is equal to 1. Wolf [187] classified totally geodesic submanifolds in symmetric spaces of rank one and compact type. Hence, by duality we obtain the list of proper, non-flat, totally geodesic submanifolds of symmetric spaces of non-compact type and rank one up to congruence (see Table 6.1).

$\mathbb{R}\mathsf{H}^n$		
	$\mathbb{R}\mathsf{H}^k$	$1 \leq k \leq n-1$
$\mathbb{C}\mathsf{H}^n$		
	$\mathbb{C}\mathsf{H}^k$	$2 \leq k \leq n-1$
	$\mathbb{R}\mathsf{H}^k$	$1 \leq k \leq n$
	$\mathbb{R}\mathsf{H}^2(4)$	
$\mathbb{H}\mathsf{H}^n$		
	$\mathbb{H}\mathsf{H}^k$	$2 \leq k \leq n-1$
	$\mathbb{C}\mathsf{H}^k$	$2 \leq k \leq n$
	$\mathbb{R}\mathsf{H}^k$	$2 \leq k \leq n$
	$\mathbb{R}\mathsf{H}^k(4)$	$1 \leq k \leq 4$
$\mathbb{O}\mathsf{H}^2$		
	$\mathbb{H}\mathsf{H}^2$	
	$\mathbb{C}\mathsf{H}^2$	
	$\mathbb{R}\mathsf{H}^2$	
	$\mathbb{R}\mathsf{H}^k(4)$	$1 \leq k \leq 8$

Table 6.1: Totally geodesic submanifolds of dimension $d \geq 2$ in symmetric spaces of non-compact type and rank one, up to congruence.

Notice that in $\mathbb{C}\mathsf{H}^2$ there are two non-congruent totally geodesic submanifolds homothetic to $\mathbb{R}\mathsf{H}^2$: $\mathbb{R}\mathsf{H}^2$ and $\mathbb{R}\mathsf{H}^2(4)$ (which are totally real and complex in $\mathbb{C}\mathsf{H}^2$, respectively). Also, in $\mathbb{H}\mathsf{H}^3$ there are two non-congruent totally geodesic submanifolds homothetic to $\mathbb{R}\mathsf{H}^3$: $\mathbb{R}\mathsf{H}^3$ and $\mathbb{R}\mathsf{H}^3(4) \subset \mathbb{R}\mathsf{H}^4(4)$. Finally, in $\mathbb{H}\mathsf{H}^4$ there are two non-congruent totally geodesic submanifolds homothetic to $\mathbb{R}\mathsf{H}^4$: $\mathbb{R}\mathsf{H}^4$ and $\mathbb{R}\mathsf{H}^4(4)$.

Now, we will set the following notation for the rest of this section. Let $M = M_1 \times \cdots \times M_r$, where $M_i = \mathsf{G}_i/\mathsf{K}_i = \mathbb{F}_i\mathsf{H}^{n_i}(c_i)$ is a symmetric space of non-compact type and rank one for each $i \in \{1, \ldots, r\}$. Let $o = (o_1, \ldots, o_r) \in M$. Hence, we can identify T_oM with a Lie triple system $\mathfrak{p}$ such that $\mathfrak{p} = \bigoplus_{i=1}^r \mathfrak{p}_i$, where $\mathfrak{p}_i$ is identified with $T_{o_i}\mathbb{F}_i\mathsf{H}^{n_i}(c_i)$. This implies that $\mathfrak{g}_i = \mathfrak{p}_i \oplus [\mathfrak{p}_i, \mathfrak{p}_i]$ is the Lie algebra of G_i.

Lemma 6.2.1. *Let $M = M_1 \times \cdots \times M_r$, where each M_i is a simply connected, irreducible symmetric space. Let Σ be an irreducible, non-flat, totally geodesic submanifold of M. Then:*

i) Σ is k-diagonal for some $k \in \{1, \ldots, r\}$ in M.

Moreover, if M is of non-compact type, the following statements hold:

ii) There is some permutation σ of $\{1, \ldots, r\}$ such that Σ is a totally geodesic submanifold of $N_{\sigma(1)} \times \cdots \times N_{\sigma(k)}$, where $N_{\sigma(j)} := \operatorname{proj}_{\sigma(j)} \Sigma$, $j \in \{1, \ldots, r\}$, is a totally geodesic submanifold of $M_{\sigma(j)}$ for every $j \in \{1, \ldots, k\}$.

iii) The embedding of Σ is given by $\Psi\colon \Sigma \to M$, $\Psi(p) = (\Psi_1(p), \ldots, \Psi_r(p))$, where each map $\Psi_j := \operatorname{proj}_j\colon \Sigma \to N_{\sigma(j)}$ is a homothety for every $j \in \{1, \ldots, k\}$, and $\Psi_l\colon \Sigma \to N_{\sigma(l)}$ is a constant map for every $l \in \{k+1, \ldots, r\}$.

Proof. Let Σ be an irreducible, non-flat, totally geodesic submanifold of M. Let $\operatorname{proj}_i\colon \mathfrak{g}_\Sigma \to \mathfrak{g}_i$ be the i-th orthogonal projection onto $\mathfrak{g}_i$ for $i \in \{1, \ldots, r\}$. We will prove that proj_i is either the zero map or injective. Since Σ is an irreducible symmetric space, its isotropy representation is irreducible. Furthermore, since Σ is semisimple by our assumptions, we have that $\mathfrak{k}_\Sigma := [\mathfrak{p}_\Sigma, \mathfrak{p}_\Sigma]$ is the Lie algebra of the isotropy group of Σ. However, $\operatorname{Ker}\operatorname{proj}_{i|\mathfrak{p}_\Sigma} \subset \mathfrak{p}_\Sigma$ is an invariant subspace under the isotropy representation of Σ since

$$\operatorname{proj}_i[Z, X] = [\operatorname{proj}_i Z, \operatorname{proj}_{i|\mathfrak{p}_\Sigma} X] = 0,$$

where $X \in \operatorname{Ker}\operatorname{proj}_{i|\mathfrak{p}_\Sigma}$ and $Z \in \mathfrak{k}_\Sigma$. Thus, $\operatorname{Ker}\operatorname{proj}_i = 0$ or $\operatorname{Ker}\operatorname{proj}_{i|\mathfrak{p}_\Sigma} = \mathfrak{p}_\Sigma$. This implies $\mathfrak{p}_\Sigma \subset \mathfrak{p}_{\sigma(1)} \oplus \cdots \oplus \mathfrak{p}_{\sigma(k)}$ for some $k \in \{1, \ldots, r\}$ and some permutation σ of $\{1, \ldots, r\}$ such that $\operatorname{Ker}\operatorname{proj}_{\sigma(j)|\mathfrak{p}_\Sigma} = 0$ for $j \in \{1, \ldots, k\}$ and $\operatorname{Ker}\operatorname{proj}_{\sigma(l)|\mathfrak{p}_\Sigma} = \mathfrak{p}_\Sigma$ for $l \in \{k+1, \ldots, r\}$. For a non-zero $X \in \mathfrak{p}_\Sigma$, we have $\operatorname{proj}_{\sigma(j)} X \neq 0$ if and only if $j \in \{1, \ldots, k\}$. Hence, every non-zero element X in $\mathfrak{p}_\Sigma$ can be written as $X = \sum_{j=1}^k X_j$, where each $X_j \in \mathfrak{p}_{\sigma(j)}$ is non-zero. Thus, Σ is k-diagonal and we have proved *i)*.

Furthermore, $\mathfrak{p}_\Sigma \subset \operatorname{proj}_{\sigma(1)} \mathfrak{p}_\Sigma \oplus \cdots \oplus \operatorname{proj}_{\sigma(k)} \mathfrak{p}_\Sigma$, where each $\operatorname{proj}_{\sigma(j)} \mathfrak{p}_\Sigma$ is a Lie triple system of $\mathfrak{p}_\Sigma$ by Lemma 6.1.1. Moreover, $\operatorname{Ker}\operatorname{proj}_{\sigma(j)}$ is an ideal of $\mathfrak{g}_\Sigma$. For every $j \in \{1, \ldots, k\}$ we have $\operatorname{Ker}\operatorname{proj}_{\sigma(j)|\mathfrak{p}_\Sigma} = 0$, which implies that $\operatorname{Ker}\operatorname{proj}_{\sigma(j)} = 0$, since $\mathfrak{g}_\Sigma$ is simple as Σ is an irreducible symmetric space of non-compact type. Thus, $\operatorname{proj}_{\sigma(j)}\colon \mathfrak{g}_\Sigma \to \operatorname{proj}_{\sigma(j)} \mathfrak{g}_\Sigma$ is a Lie algebra isomorphism for each $j \in \{1, \dots, k\}$. Hence, $\operatorname{proj}_{\sigma(j)} \mathfrak{g}_\Sigma = \operatorname{proj}_{\sigma(j)} \mathfrak{p}_\Sigma \oplus [\operatorname{proj}_{\sigma(j)} \mathfrak{p}_\Sigma, \operatorname{proj}_{\sigma(j)} \mathfrak{p}_\Sigma]$ is the Lie algebra of the isometry group of $N_{\sigma(j)}$, the totally geodesic submanifold of $M_{\sigma(j)}$ associated with the Lie triple system $\operatorname{proj}_{\sigma(j)} \mathfrak{p}_\Sigma$ in $\mathfrak{p}_{\sigma(j)}$. Additionally, taking into account that $\operatorname{Ker} \operatorname{proj}_{\sigma(j)|_{\mathfrak{p}_\Sigma}} = \mathfrak{p}_\Sigma$ if and only if $l \in \{k+1, \dots, r\}$, we have that Σ projects onto a point in $M_{\sigma(l)}$ if and only if $l \in \{k+1, \dots, r\}$. Therefore, we have that $\Sigma \subset N_{\sigma(1)} \times \cdots \times N_{\sigma(k)}$, where $N_{\sigma(j)}$ is a totally geodesic submanifold of $M_{\sigma(j)}$, which proves *ii)*.

Finally, we will prove that $\operatorname{proj}_{\sigma(i)}$ is a homothety between Σ and $N_{\sigma(i)}$. Let us fix some $i \in \{1, \dots, k\}$ and let us consider the inner product in $\mathfrak{p}_\Sigma$ given by $(X, Y)_i := \langle \operatorname{proj}_{\sigma(i)} X, \operatorname{proj}_{\sigma(i)} Y\rangle$, for $X, Y \in \mathfrak{p}_\Sigma$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathfrak{p}$ induced by its identification with T_oM. Let $g \in \mathsf{K}_\Sigma$, where K_Σ is the connected Lie subgroup of G_Σ with Lie algebra $\mathfrak{k}_\Sigma$. Then, $g = \prod_{i=1}^{k} g_{\sigma(i)}$ for some $g_{\sigma(i)} \in \pi_i \mathsf{K}_\Sigma$, where $\pi_i\colon \mathsf{G}_1 \times \cdots \times \mathsf{G}_r \to \mathsf{G}_i$ is the projection onto the i-th factor. Thus, for any $X, Y \in \mathfrak{p}_\Sigma$,

$$
\begin{aligned}
(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y)_i &= \langle \operatorname{proj}_{\sigma(i)} \operatorname{Ad}(g)X, \operatorname{proj}_{\sigma(i)} \operatorname{Ad}(g)Y\rangle \\
&= \langle \operatorname{proj}_{\sigma(i)} \operatorname{Ad}(g_{\sigma(i)})X, \operatorname{proj}_{\sigma(i)} \operatorname{Ad}(g_{\sigma(i)})Y\rangle \\
&= \langle \operatorname{Ad}(g_{\sigma(i)}) \operatorname{proj}_{\sigma(i)} X, \operatorname{Ad}(g_{\sigma(i)}) \operatorname{proj}_{\sigma(i)} Y\rangle \\
&= \langle \operatorname{proj}_{\sigma(i)} X, \operatorname{proj}_{\sigma(i)} Y\rangle = (X, Y)_i,
\end{aligned}
$$

where we have used that $\operatorname{Ad}(g_{\sigma(i)})$ is a linear isometry for $\langle \cdot, \cdot \rangle$ which leaves $\operatorname{proj}_{\sigma(i)} \mathfrak{p}_\Sigma$ invariant, since $g_{\sigma(i)}$ belongs to the isotropy of $N_{\sigma(i)}$ for each $i \in \{1, \dots, k\}$. Hence, $(\cdot, \cdot)_i$ is a K_Σ-invariant inner product in $\mathfrak{p}_\Sigma$. Moreover, since the isotropy representation of Σ is irreducible by assumption, Schur Lemma implies that $\operatorname{proj}_{\sigma(i)}$ is a homothety between the Lie triple systems $\mathfrak{p}_\Sigma$ and $\operatorname{proj}_{\sigma(i)} \mathfrak{p}_\Sigma$ for each $i \in \{1, \dots, k\}$. In addition to that, $\operatorname{proj}_{\sigma(i)}\colon \mathfrak{p}_\Sigma \to \operatorname{proj}_{\sigma(i)} \mathfrak{p}_\Sigma$ preserves the sectional curvature since it preserves the Lie bracket. Thus, by [189, Theorem 1.9.2] we have that $\operatorname{proj}_{\sigma(i)}\colon \Sigma \to \operatorname{proj}_{\sigma(i)} \Sigma$ is an affine diffeomorphism since Σ and $\operatorname{proj}_{\sigma(i)} \Sigma$ are simply connected. Now let $p \in \Sigma$, γ be a path in Σ joining o and p and $\widetilde{\gamma} := \operatorname{proj}_{\sigma(i)} \gamma$. Let P_γ and $P_{\widetilde{\gamma}}$ be the parallel transports along to γ and $\widetilde{\gamma}$, respectively. Since $\operatorname{proj}_{\sigma(i)}$ is affine, we have ${\operatorname{proj}_{\sigma(i)}}_{*p} = P_{\widetilde{\gamma}} \circ {\operatorname{proj}_{\sigma(i)}}_{*o} \circ P_\gamma^{-1}$ for every $p \in M$. However, P_γ and $P_{\widetilde{\gamma}}$ are isometries and ${\operatorname{proj}_{\sigma(i)}}_{*o}$ is a homothety since $\operatorname{proj}_{\sigma(i)}\colon \mathfrak{p}_\Sigma \to \operatorname{proj}_{\sigma(i)} \mathfrak{p}_\Sigma$ is a homothety. Thus, ${\operatorname{proj}_{\sigma(i)}}_{*p}$ is also a homothety for every $p \in M$ and it turns out that $\operatorname{proj}_i\colon \Sigma \to \operatorname{proj}_{\sigma(i)} \Sigma$ is a homothety. Consequently, we have proved *iii)*. □

Remark 6.2.2. Observe that this lemma admits a converse. Let $M = M_1 \times \cdots \times M_r$, where each M_i is an irreducible symmetric space of non-compact type. Let Σ be a Riemannian manifold and consider the embedding $\Psi\colon \Sigma \to M$, $\Psi(p) = (\Psi_1(p), \dots, \Psi_r(p))$, where each map $\Psi_j\colon \Sigma \to N_j$ is either a homothety or a constant map, and N_j is any totally geodesic submanifold of M_j, $j \in \{1, \dots, r\}$. Then, $\Psi(\Sigma)$ is a totally geodesic submanifold of $N_1 \times \cdots \times N_r$, since homotheties carry geodesics into geodesics. Therefore, $\Psi(\Sigma)$ is a totally geodesic submanifold of M.

Remark 6.2.3. One important consequence of the previous result that deserves to be highlighted is the following. Let Σ be an irreducible, non-flat, r-diagonal totally geodesic submanifold of $M = M_1 \times \cdots \times M_r$, where M_i is an irreducible symmetric space of non-compact type for each $i \in \{1, \ldots, r\}$. Then, with the usual notation, $\mathfrak{p}_\Sigma \subset \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$ is an r-diagonal Lie triple system, and we can define a Lie algebra isomorphism $\Phi_i := \operatorname{proj}_i \colon \mathfrak{g}_\Sigma \to \operatorname{proj}_i \mathfrak{g}_\Sigma$, which sends $\mathfrak{p}_\Sigma$ onto the Lie triple system $\operatorname{proj}_i \mathfrak{p}_\Sigma$ in $\mathfrak{p}_i$. Therefore, $\mathfrak{p}_\Sigma = \{\sum_{i=1}^r \varphi_i X : X \in \operatorname{proj}_1 \mathfrak{p}_\Sigma\}$, where $\varphi_i := \Phi_i \Phi_1^{-1} \colon \operatorname{proj}_1 \mathfrak{g}_\Sigma \to \operatorname{proj}_i \mathfrak{g}_\Sigma$ is a Lie algebra isomorphism sending $\operatorname{proj}_1 \mathfrak{p}_\Sigma$ onto $\operatorname{proj}_i \mathfrak{p}_\Sigma$ for each $i \in \{1, \ldots, r\}$. Let $s \in \{1, \ldots, r\}$. Notice that $\mathfrak{p}_{\Sigma^s} = \{\sum_{i=1}^s \varphi_i X : X \in \operatorname{proj}_1 \mathfrak{p}_\Sigma\}$ is an s-diagonal Lie triple system in $\mathfrak{p}$ such that $\mathfrak{g}_{\Sigma^s}$ is isomorphic to $\mathfrak{g}_\Sigma$ and then Σ^s, the totally geodesic submanifold of M corresponding to $\mathfrak{p}_{\Sigma^s}$, is homothetic to Σ.

Proposition 6.2.4. *Let Σ_1, Σ_2 be r-diagonal, non-flat, irreducible, totally geodesic submanifolds in $M = M_1 \times \cdots \times M_r$, where each M_i is an irreducible symmetric space of non-compact type homothetic to both Σ_1 and Σ_2. Then, there is $g \in \operatorname{Isom}(M_1) \times \cdots \times \operatorname{Isom}(M_r) \subset \operatorname{Isom}(M)$ such that $g\Sigma_1 = \Sigma_2$.*

Proof. Let $\mathfrak{p}_{\Sigma_1} = \{\sum_{i=1}^k \varphi_i X : X \in \mathfrak{p}_1\}$, $\mathfrak{p}_{\Sigma_2} = \{\sum_{i=1}^k \psi_i X : X \in \mathfrak{p}_1\}$ and $\mathfrak{g}_{\Sigma_j} := \mathfrak{p}_{\Sigma_j} \oplus [\mathfrak{p}_{\Sigma_j}, \mathfrak{p}_{\Sigma_j}]$, for $j \in \{1, 2\}$, where $\varphi_i, \psi_i \colon \mathfrak{g}_1 \to \mathfrak{g}_i$ are Lie algebra isomorphisms sending $\mathfrak{p}_1$ onto $\mathfrak{p}_i$ for $i \in \{2, \ldots, r\}$, and $\varphi_1 = \psi_1 = \operatorname{Id}_{\mathfrak{g}_1}$, where $\operatorname{Id}_{\mathfrak{g}_1}$ is the identity map of $\mathfrak{g}_1$. Here we are using Remark 6.2.3 along with the assumption that each M_i is an irreducible symmetric space of non-compact type homothetic to Σ_1 and Σ_2.

Let $\sigma_i := \psi_i \varphi_i^{-1} \in \operatorname{Aut}(\mathfrak{g}_i)$. First of all, observe that $\sigma_{i|_{\mathfrak{p}_i}}$ is a linear isometry of $\mathfrak{p}_i$ since $\sigma_i \in \operatorname{Aut}(\mathfrak{g}_i)$ and the inner product on $\mathfrak{p}_i$ is the restriction of the Killing form of $\mathfrak{g}_i$, up to scaling. Furthermore, $\sigma_{i|_{\mathfrak{p}_i}}$ preserves the curvature tensor of M_i at o_i since this is given by Lie brackets. Hence, $\sigma_{i|_{\mathfrak{p}_i}}$ is a linear isometry of $\mathfrak{p}_i$ that preserves sectional curvature at o_i. Thus, by [189, Corollary 2.3.14], $\sigma_{i|_{\mathfrak{p}_i}}$ extends to an isometry $g_i \in \operatorname{Isom}(M_i)$ that fixes $o_i \in M_i$, since it leaves $\mathfrak{p}_i$ invariant. Then, $\sigma_i = \operatorname{Ad}(g_i)$ and if $g := \prod_{i=1}^r g_i$, where g_1 is the identity element of $\operatorname{Isom}(M_1)$, we obtain

$$\begin{aligned} \operatorname{Ad}(g)^{-1}\mathfrak{g}_{\Sigma_2} &= \prod_{i=1}^r \operatorname{Ad}(g_i)^{-1}\mathfrak{g}_{\Sigma_2} = \left\{ \sum_{i=1}^r \operatorname{Ad}(g_i)^{-1}\psi_i X : X \in \mathfrak{g}_1 \right\} \\ &= \left\{ \sum_{i=1}^r \varphi_i X : X \in \mathfrak{g}_1 \right\} = \mathfrak{g}_{\Sigma_1}, \end{aligned}$$

where we have used $\sigma_i = \psi_i \varphi_i^{-1}$ and $\operatorname{Ad}(g_i)_{|\mathfrak{g}_j} = \operatorname{Id}_{\mathfrak{g}_j}$ for $i \neq j$. Therefore, there is a $g \in \operatorname{Isom}(M_1) \times \cdots \times \operatorname{Isom}(M_r) \subset \operatorname{Isom}(M)$ such that $g\mathsf{G}_{\Sigma_1} = \mathsf{G}_{\Sigma_2} g$. Consequently, $g\Sigma_1 = g\mathsf{G}_{\Sigma_1} \cdot o = \mathsf{G}_{\Sigma_2} g \cdot o = \mathsf{G}_{\Sigma_2} \cdot o = \Sigma_2$, since g fixes o. □

Lemma 6.2.5. *Let Σ be a totally geodesic submanifold of $M = M_1 \times \cdots \times M_r$, where M_i is a symmetric space of non-compact type and rank one, for each $i \in \{1, \ldots, r\}$. Moreover, let $\Sigma = \Sigma_1 \times \Sigma_2$, where Σ_1 is irreducible and not flat. Then, if $\operatorname{proj}_i \Sigma_1$ and $\operatorname{proj}_j \Sigma_2$ have positive dimension, we have that $i \neq j$.*

Proof. Let $\Sigma = \Sigma_1 \times \Sigma_2$ be a totally geodesic submanifold of M, where Σ_1 is irreducible and not flat. Let $\mathfrak{p}_\Sigma = \mathfrak{p}_{\Sigma_1} \oplus \mathfrak{p}_{\Sigma_2} \subset \mathfrak{p} = \bigoplus_{i=1}^r \mathfrak{p}_i$ be the corresponding Lie triple system. Fix some $i \in \{1, \dots, r\}$. Let us suppose that $\widehat{\mathfrak{p}}_j := \mathrm{proj}_i \, \mathfrak{p}_{\Sigma_j}$ has positive dimension for both $j = 1$ and $j = 2$. By Lemma 6.2.1, we have that $\dim \widehat{\mathfrak{p}}_1 > 1$, since Σ_1 is irreducible and not flat. Thus, we can choose $X \in \widehat{\mathfrak{p}}_1$ and $Y \in \widehat{\mathfrak{p}}_2$ spanning a 2-plane in $\mathfrak{p}_i$. Moreover, the sectional curvature sec of M_i is given by

$$\sec(X, Y) = -\frac{\langle [[X, Y], Y], X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}.$$

In particular, $[\widehat{\mathfrak{p}}_1, \widehat{\mathfrak{p}}_2] \neq 0$, since M_i has negative sectional curvature. However, we have $[\widehat{\mathfrak{p}}_1, \widehat{\mathfrak{p}}_2] = [\mathrm{proj}_i \, \mathfrak{p}_{\Sigma_1}, \mathrm{proj}_i \, \mathfrak{p}_{\Sigma_2}] = \mathrm{proj}_i [\mathfrak{p}_{\Sigma_1}, \mathfrak{p}_{\Sigma_2}] = 0$, since Σ is a Riemannian product of the symmetric spaces Σ_1 and Σ_2. Therefore, we obtain a contradiction with the assumption that both $\widehat{\mathfrak{p}}_1$ and $\widehat{\mathfrak{p}}_2$ have positive dimension. □

Remark 6.2.6. It is important to notice that the previous lemma is not true when the ambient space is a product of irreducible symmetric spaces of rank greater than one. For instance, one can find a totally geodesic submanifold Σ homothetic to $\mathbb{R}\mathsf{H}^2 \times \mathbb{R}\mathsf{H}^2$ in $M = M_1 \times M_2$, with $M_i = \mathsf{SO}_{2,4}/(\mathsf{SO}_2 \times \mathsf{SO}_4)$ for each $i \in \{1, 2\}$, such that both factors of Σ have non-trivial projection onto each factor of M. Thus, applying [107, Theorem 4.1 and §5] and duality, there is a totally geodesic submanifold $\Sigma_i^1 \times \Sigma_i^2 \subset M_i$, where Σ_i^1 and Σ_i^2 are mutually isometric real hyperbolic planes, for each $i \in \{1, 2\}$. Now we can consider $\widehat{\Sigma}^j$ a 2-diagonal totally geodesic submanifold in $\Sigma_1^j \times \Sigma_2^j$ for each $j \in \{1, 2\}$. Thus, $\Sigma := \widehat{\Sigma}^1 \times \widehat{\Sigma}^2$ is a totally geodesic submanifold homothetic to $\mathbb{R}\mathsf{H}^2 \times \mathbb{R}\mathsf{H}^2$ in M such that both factors of Σ have non-trivial projection onto each factor of M.

Proposition 6.2.7. *Let M_1 and M_2 be irreducible symmetric spaces of compact and non-compact type, respectively. If $\Sigma \subset M_1 \times M_2$ is a diagonal totally geodesic submanifold, then Σ is flat.*

Proof. Let Σ be a totally geodesic submanifold of $M_1 \times M_2$. By De-Rham Theorem, we have that the universal covering of Σ is $\widetilde{\Sigma} = \Sigma_0 \times \Sigma_1 \times \cdots \times \Sigma_s$, where Σ_0 is flat and each Σ_i is an irreducible semisimple symmetric space. Moreover, $\mathfrak{p}_\Sigma = \bigoplus_{i=0}^s \mathfrak{p}_{\Sigma_i}$, where $\mathfrak{p}_{\Sigma_i} \subset \mathfrak{p}_\Sigma \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2$ is a Lie triple system corresponding to the irreducible symmetric space Σ_i, for each $i \in \{0, \dots, s\}$, and $\mathfrak{p}_1, \mathfrak{p}_2$ are the Lie triple systems corresponding to M_1 and M_2, respectively.

Let us fix some $i \in \{1, \dots, s\}$. Then Σ_i is semisimple, and we have that $\mathfrak{k}_{\Sigma_i} := [\mathfrak{p}_{\Sigma_i}, \mathfrak{p}_{\Sigma_i}]$ is the Lie algebra of the isotropy of Σ_i, and $\mathfrak{g}_{\Sigma_i} := \mathfrak{k}_{\Sigma_i} \oplus \mathfrak{p}_{\Sigma_i}$ is the Lie algebra of the isometry group of Σ_i. Now, we define $\varphi_{ij} \colon \mathfrak{g}_{\Sigma_i} \to \mathfrak{g}_j$, where $\varphi_{ij} X = \mathrm{proj}_j X$ for each $i \in \{1, \dots, s\}$ and $j \in \{1, 2\}$, $\mathfrak{g}_j$ is the Lie algebra of the isometry group of M_j, and $\mathrm{proj}_j \colon \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_j$ is the projection map. Notice that $\operatorname{Ker} \varphi_{ij|\mathfrak{p}_{\Sigma_i}} \subset \mathfrak{p}_{\Sigma_i}$ is an invariant subspace for the isotropy representation of Σ_i. Since Σ_i is irreducible, we have $\operatorname{Ker} \varphi_{ij|\mathfrak{p}_{\Sigma_i}} = 0$ or $\operatorname{Ker} \varphi_{ij|\mathfrak{p}_{\Sigma_i}} = \mathfrak{p}_{\Sigma_i}$. Moreover, as $\mathfrak{p}_{\Sigma_i}$ is diagonal by assumption, $\operatorname{Ker} \varphi_{ij|\mathfrak{p}_{\Sigma_i}} = 0$ for every $j \in \{1, 2\}$. On the one hand, if Σ_i is a compact simple Lie group, $\mathfrak{k}_{\Sigma_i}$ is simple, and hence, $\operatorname{Ker} \varphi_{ij|\mathfrak{k}_{\Sigma_i}} = 0$

or $\operatorname{Ker}\varphi_{ij|\mathfrak{k}_{\Sigma_i}} = \mathfrak{k}_{\Sigma_i}$, since $\operatorname{Ker}\varphi_{ij|\mathfrak{k}_{\Sigma_i}}$ is an ideal of $\mathfrak{k}_{\Sigma_i}$. In any other case, $\mathfrak{g}_{\Sigma_i}$ is simple. Then, we have $\operatorname{Ker}\varphi_{ij} = 0$ or $\operatorname{Ker}\varphi_{ij} = \mathfrak{g}_{\Sigma_i}$. However, $\operatorname{Ker}\varphi_{ij|\mathfrak{p}_{\Sigma_i}} = 0$ and this implies that $\operatorname{Ker}\varphi_{ij} = 0$.

To sum up, for each $j \in \{1, 2\}$, we have $\operatorname{Ker}\varphi_{ij} = 0$ or $\operatorname{Ker}\varphi_{ij} = \mathfrak{k}_{\Sigma_i}$. Let us assume that $\operatorname{Ker}\varphi_{ij} = \mathfrak{k}_{\Sigma_i}$ for some $j \in \{1, 2\}$. In this case $\varphi_{ij}\mathfrak{g}_{\Sigma_i} = \varphi_{ij}\mathfrak{p}_{\Sigma_i}$ is an abelian Lie algebra. Since $\mathfrak{g}_{\Sigma_i}$ is semisimple and $\mathfrak{g}_{\Sigma_i} \subset \varphi_{ij}\mathfrak{g}_{\Sigma_i} \oplus \varphi_{ik}\mathfrak{g}_{\Sigma_i}$, where $k \in \{1, 2\} \setminus \{j\}$, we have

$$\mathfrak{g}_{\Sigma_i} = [\mathfrak{g}_{\Sigma_i}, \mathfrak{g}_{\Sigma_i}] \subset [\varphi_{ij}\mathfrak{g}_{\Sigma_i} \oplus \varphi_{ik}\mathfrak{g}_{\Sigma_i}, \varphi_{ij}\mathfrak{g}_{\Sigma_i} \oplus \varphi_{ik}\mathfrak{g}_{\Sigma_i}] \subset [\varphi_{ik}\mathfrak{g}_{\Sigma_i}, \varphi_{ik}\mathfrak{g}_{\Sigma_i}] \subset \varphi_{ik}\mathfrak{g}_{\Sigma_i} \subset \mathfrak{g}_k.$$

Therefore, we obtain a contradiction with the assumption that $\mathfrak{p}_\Sigma$ is diagonal. Now let us assume that $\operatorname{Ker}\varphi_{ij} = 0$ for every $j \in \{1, 2\}$. This implies that $\mathfrak{g}_{\Sigma_i}$ and $\varphi_{ij}\mathfrak{g}_{\Sigma_i}$ are isomorphic for every $j \in \{1, 2\}$. In particular, $\varphi_{ij}\mathfrak{g}_{\Sigma_i}$ is simple for every $i \in \{1, \dots, s\}$ and $j \in \{1, 2\}$, and $\varphi_{i1}\mathfrak{g}_{\Sigma_i}$ is isomorphic to $\varphi_{i2}\mathfrak{g}_{\Sigma_i}$. Now, as $\varphi_{i1}\mathfrak{g}_{\Sigma_i}$ is a subalgebra of $\mathfrak{g}_1$, we have that $\varphi_{i1}\mathfrak{g}_{\Sigma_i}$ is a compact Lie algebra. Moreover, $\varphi_{i2}\mathfrak{g}_{\Sigma_i} = \varphi_{i2}\mathfrak{p}_{\Sigma_i} \oplus [\varphi_{i2}\mathfrak{p}_{\Sigma_i}, \varphi_{i2}\mathfrak{p}_{\Sigma_i}]$, where $\varphi_{i2}\mathfrak{p}_{\Sigma_i}$ is a Lie triple system, is not a compact Lie algebra since it is simple and then it is the Lie algebra of the isometry group of an irreducible symmetric space of non-compact type (see discussion above Lemma 6.2.1).

Consequently, we obtain a contradiction with the existence of (non-trivial) irreducible semisimple factors of Σ, which yields our result. □

We now introduce a notation that will be useful in what follows. Let M and Σ be two symmetric spaces. We will write $(\Sigma) \leq M$ if M contains a totally geodesic submanifold isometric to Σ.

A simply connected, reducible symmetric space M of rank 2 is a product of two simply connected irreducible symmetric spaces of rank one, M_1 and M_2. Let Σ be a totally geodesic submanifold in $M = M_1 \times M_2$. Notice that if Σ is reducible, then it has maximal rank and Proposition 6.1.2 implies that $\Sigma = \Sigma_1 \times \Sigma_2$, where Σ_i is a totally geodesic submanifold of M_i for each $i \in \{1, 2\}$. Now, if Σ is irreducible, it must be either a geodesic or a semisimple totally geodesic submanifold. Let us assume that Σ is an irreducible semisimple totally geodesic submanifold of M. Then, by Lemma 6.2.1 *i*), Σ is either 1-diagonal or 2-diagonal. If Σ is 1-diagonal, clearly $\Sigma = \Sigma_i \times \{p_j\}$, where Σ_i is a totally geodesic submanifold of M_i and $p_j \in M_j$ for distinct $i, j \in \{1, 2\}$.

Let us assume that M_1 is of compact type and M_2 is of non-compact type. By Proposition 6.2.7, $M = M_1 \times M_2$ has no diagonal totally geodesic submanifolds of dimension greater than one. Hence, a 2-diagonal totally geodesic submanifold is a geodesic.

Let us assume that M_1 is flat and M_2 is of non-compact type. We can suppose that Σ is an irreducible semisimple 2-diagonal totally geodesic submanifold. Thus, every non-zero vector in $\mathfrak{p}_\Sigma$ is of the form $X = X_1 + X_2$, where $X_i \in \mathfrak{p}_i$ is a non-zero vector in $\mathfrak{p}_i$ for each $i \in \{1, 2\}$. However, since Σ is semisimple, then $\mathfrak{p}_\Sigma$ has dimension greater than one. Moreover, $\dim \operatorname{proj}_1 \mathfrak{p}_\Sigma = 1$. Hence, there exists a non-zero vector X' in $\mathfrak{p}_\Sigma \cap \operatorname{proj}_2 \mathfrak{p}_\Sigma$, contradicting the assumption that Σ is 2-diagonal.

To sum up the above discussion: if M_1 and M_2 have opposite types or one of them is flat, then every totally geodesic submanifold Σ in $M = M_1 \times M_2$ is either a geodesic or equal to $\Sigma_1 \times \Sigma_2$, where $\Sigma_i \subset M_i$ is a totally geodesic submanifold for each $i \in \{1, 2\}$. In view of the argumentation above, by duality we will assume that both factors in M are of non-compact type. In this case, we have the following result.

Theorem 6.2.8. *Let $M_i := \mathbb{F}_i\mathsf{H}^{n_i}(c_i)$ be a symmetric space of non-compact type and rank one for $i = 1, 2$. Given positive numbers c_1', c_2', we define the quantity $c = \frac{c_1' c_2'}{c_1' + c_2'}$. Then, $\Sigma \subset M_1 \times M_2$ is a totally geodesic submanifold if and only if it is equal to one in the list below:*

i) A geodesic in $M_1 \times M_2$.

ii) A product $\Sigma_1 \times \Sigma_2$, where $\Sigma_i \subset M_i$ is a totally geodesic submanifold for $i \in \{1, 2\}$.

iii) A totally geodesic diagonal $\mathbb{F}\mathsf{H}^n(c)$, with $\mathbb{F} \neq \mathbb{R}$ and $c_i' = c_i$, whenever $(\mathbb{F}\mathsf{H}^n(c_i)) \leq M_i$ for every $i \in \{1, 2\}$.

iv) A totally geodesic diagonal $\mathbb{R}\mathsf{H}^n(c)$, with $c_i' \in \{c_i, \frac{c_i}{4}\}$, whenever $(\mathbb{R}\mathsf{H}^n(c_i')) \leq M_i$ for every $i \in \{1, 2\}$.

Remark 6.2.9. The diagonal embeddings in items *iii)* and *iv)* are the ones described in Lemma 6.2.1 *iii)*. These are of the form $\Psi\colon \Sigma \to M_1 \times M_2$, $p \in \Sigma \mapsto (\Psi_1(p), \Psi_2(p))$, where Ψ_i is a homothety between Σ and some totally geodesic submanifold N_i of M_i for each $i \in \{1, 2\}$ (see Remark 6.2.2).

Proof. Let us assume that Σ has rank two. Then, by Proposition 6.1.2, $\Sigma = \Sigma_1 \times \Sigma_2$, where $\Sigma_i \subset M_i$ is totally geodesic for each $i = 1, 2$, which corresponds to item *ii)* in the statement.

Now assume that Σ has rank one. Then it is either a geodesic, which corresponds to item *i)*, or it is semisimple. In this latter case, Σ must be isometric to $\mathbb{F}\mathsf{H}^n(c)$ for some $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $c > 0$. If Σ is not diagonal, it is 1-diagonal by Lemma 6.2.1 and it must be congruent to $\Sigma_1 \times \{p_2\}$ or to $\{p_1\} \times \Sigma_2$, where $\Sigma_i \subset \mathbb{F}_i\mathsf{H}^{n_i}(c_i)$ is a totally geodesic submanifold and $p_i \in M_i$ for $i = 1, 2$. This corresponds to item *ii)* in the statement. Moreover, if it is diagonal, it is 2-diagonal. Then, by Lemma 6.2.1 and the classification of totally geodesic submanifolds in the rank one symmetric spaces (see Table 6.1), we have that $\Sigma \subset \mathbb{F}\mathsf{H}^n(c_1') \times \mathbb{F}\mathsf{H}^n(c_2')$, for some positive numbers c_1' and c_2'. Let us further assume that $\mathbb{F} \neq \mathbb{R}$. Hence, by the classification in rank one, $c_i' = c_i$ for $i = 1, 2$. We will prove that the sectional curvature of Σ satisfies

$$\sec(X, Y) \in \left[-\frac{c_1 c_2}{c_1 + c_2}, -\frac{c_1 c_2}{4(c_1 + c_2)} \right],$$

for any $X, Y \in \mathfrak{p}_\Sigma$ spanning a 2-plane. Let $\mathfrak{p}_i' := \operatorname{proj}_i \mathfrak{p}_\Sigma$ be a Lie triple system associated with $T_{o_i}\mathbb{F}\mathsf{H}^n(c_i)$, where $o_i \in \mathbb{F}\mathsf{H}^n(c_i)$. Moreover, consider the Lie algebra of the isometry group of $\mathbb{F}\mathsf{H}^n(c_i)$, which is $\mathfrak{g}_i' := \mathfrak{p}_i' \oplus [\mathfrak{p}_i', \mathfrak{p}_i']$. Then, the Lie triple system corresponding to Σ is $\mathfrak{p}_\Sigma = \{X_1 + \varphi X_1 \colon X_1 \in \mathfrak{p}_1'\}$ for some Lie algebra isomorphism $\varphi\colon \mathfrak{g}_1' \to \mathfrak{g}_2'$ that sends $\mathfrak{p}_1'$ onto $\mathfrak{p}_2'$ (see Remark 6.2.3). Now let $\langle \cdot, \cdot \rangle_1$ be the metric of $\mathbb{F}\mathsf{H}^n(1)$, and $\sec_1(\cdot, \cdot)$

its sectional curvature. We can regard the induced metric of $\mathbb{F}\mathsf{H}^n(c_i)$ on $\mathfrak{p}_i'$ as a positive multiple of $\langle\cdot,\cdot\rangle_1$. Hence, we can write the metric of Σ at (o_1, o_2) as

$$\langle\cdot,\cdot\rangle := \lambda_1\langle \mathrm{proj}_1 \cdot, \mathrm{proj}_1 \cdot\rangle_1 + \lambda_2\langle \mathrm{proj}_2 \cdot, \mathrm{proj}_2 \cdot\rangle_1,$$

for some $\lambda_1, \lambda_2 > 0$. Let $X = X_1 + \varphi X_1, Y = Y_1 + \varphi Y_1 \in \mathfrak{p}_\Sigma$, where $X_1, Y_1 \in \mathfrak{p}_1'$ satisfy $\langle X_1, X_1\rangle_1 = \langle Y_1, Y_1\rangle_1 = 1$ and $\langle X_1, Y_1\rangle_1 = 0$. Moreover, we have

$$\begin{aligned}
\langle [[X, Y], Y], X\rangle &= \lambda_1\langle [[X_1, Y_1], Y_1], X_1\rangle_1 + \lambda_2\langle [[\varphi X_1, \varphi Y_1], \varphi Y_1], \varphi X_1\rangle_1 \\
&= \lambda_1\langle [[X_1, Y_1], Y_1], X_1\rangle_1 + \lambda_2\langle \varphi[[X_1, Y_1], Y_1], \varphi X_1\rangle_1 \\
&= -(\lambda_1 + \lambda_2)\sec_1(X_1, Y_1), \\
\langle X, X\rangle &= \lambda_1\langle X_1, X_1\rangle_1 + \lambda_2\langle \varphi X_1, \varphi X_1\rangle_1 = \lambda_1 + \lambda_2, \\
\langle Y, Y\rangle &= \lambda_1\langle Y_1, Y_1\rangle_1 + \lambda_2\langle \varphi Y_1, \varphi Y_1\rangle_1 = \lambda_1 + \lambda_2, \\
\langle X, Y\rangle &= \lambda_1\langle X_1, Y_1\rangle_1 + \lambda_2\langle \varphi X_1, \varphi Y_1\rangle_1 = 0,
\end{aligned}$$

where we have used that φ preserves $\langle\cdot,\cdot\rangle_1$ since φ preserves the Killing form. Thus, the sectional curvature at $(o_1, o_2) \in \Sigma$ of the 2-plane spanned by $\{X, Y\}$ is given by

$$\sec(X, Y) = \frac{\sec_1(X_1, Y_1)}{\lambda_1 + \lambda_2} = \frac{c_1c_2}{c_1 + c_2}\sec_1(X_1, Y_1) \in \left[-\frac{c_1c_2}{c_1 + c_2}, -\frac{c_1c_2}{4(c_1 + c_2)}\right],$$

since $\lambda_i = 1/c_i$ for each $i \in \{1, 2\}$, because λg has sectional curvature $\frac{1}{\lambda}\sec$, when g is a Riemannian metric, sec its sectional curvature and λ a positive number. Hence, Σ must be isometric to a diagonal $\mathbb{F}\mathsf{H}^n(\frac{c_1c_2}{c_1+c_2})$ whenever $(\mathbb{F}\mathsf{H}^n(c_i)) \leq M_i$ for every $i \in \{1, 2\}$ and $\mathbb{F} \neq \mathbb{R}$. This corresponds to item *iii)* in the statement.

Now let us assume that $\mathbb{F} = \mathbb{R}$. Again, by Lemma 6.2.1 and the classification of totally geodesic submanifolds in symmetric spaces of non-compact type and rank one (see Table 6.1), we have $\Sigma \subset \mathbb{R}\mathsf{H}^n(c_1') \times \mathbb{R}\mathsf{H}^n(c_2')$, where $c_i' \in \{c_i, c_i/4\}$ is such that $(\mathbb{R}\mathsf{H}^n(c_i')) \leq M_i$ for every $i \in \{1, 2\}$. A similar computation as above yields that the sectional curvature of Σ is equal to $-\frac{c_1'c_2'}{c_1'+c_2'}$. Then, Σ is isometric to $\mathbb{R}\mathsf{H}^n(\frac{c_1'c_2'}{c_1'+c_2'})$, which corresponds to item *iv)* in the statement. □

Remark 6.2.10. Notice that unlike in the irreducible rank one case, if M is a reducible space of rank two then we can find mutually isometric diagonal totally geodesic submanifolds Σ_1 and Σ_2 which are not congruent in M. This implies that the hypothesis of M_i being homothetic to Σ_1 and Σ_2 for each $i \in \{1, 2\}$ in Proposition 6.2.4 is crucial. Let us consider $M = M_1 \times M_2$, where $M_1 = \mathbb{C}\mathsf{H}^2$ and $M_2 = \mathbb{C}\mathsf{H}^3$. Let us assume that there is some $\varphi \in \mathrm{Isom}(M)$ such that $\varphi\Sigma_1 = \Sigma_2$ where

$$\begin{aligned}
\Sigma_1 &:= \mathbb{R}\mathsf{H}^2(4/5) \subset L_1 := \mathbb{R}\mathsf{H}^2(4) \times \mathbb{R}\mathsf{H}^2 \subset \mathbb{C}\mathsf{H}^2 \times \mathbb{C}\mathsf{H}^2 \subset M_1 \times M_2, \\
\Sigma_2 &:= \mathbb{R}\mathsf{H}^2(4/5) \subset L_2 := \mathbb{R}\mathsf{H}^2 \times \mathbb{R}\mathsf{H}^2(4) \subset \mathbb{C}\mathsf{H}^2 \times \mathbb{C}\mathsf{H}^2 \subset M_1 \times M_2.
\end{aligned}$$

Clearly, each isometry of $M_1 \times M_2$ must preserve both factors since M_1 and M_2 are not isometric. Then, we have $\Sigma_2 \subset \varphi L_1 \cap L_2$. However, since $\mathbb{R}\mathsf{H}^2(4)$ and $\mathbb{R}\mathsf{H}^2$ are complex and totally real submanifolds in $\mathbb{C}\mathsf{H}^2$, respectively, we have $\varphi L_1 \cap L_2 \subset \mathbb{R}\times\mathbb{R}$, where $\mathbb{R}\times\mathbb{R}$ is a maximal totally geodesic flat submanifold of $M_1 \times M_2$. Moreover, since the intersection of totally geodesic submanifolds is totally geodesic, this implies that $\Sigma_2 \subset \varphi L_1 \cap L_2 \subset \mathbb{R}\times\mathbb{R} \subset M_1 \times M_2$, which contradicts the fact that Σ_2 is not flat and proves that such φ cannot exist.

Let us recall that the elementary symmetric polynomial e_k of order $k \in \{0, \dots, n\}$ in n variables is defined as $e_k(X_1, \dots, X_n) = \sum_{1\le i_1<\ldots<i_k\le n} X_{i_1}\cdots X_{i_k}$. Then, we have the following generalization of Theorem 6.2.8 to the case of diagonal totally geodesic submanifolds in arbitrary products of symmetric spaces of rank one.

Corollary 6.2.11. *Let $M = M_1 \times \cdots \times M_r$, where each $M_i = \mathbb{F}_i\mathsf{H}^{n_i}(c_i)$ is a symmetric space of non-compact type and rank one for $i \in \{1, \dots, r\}$. Given positive numbers $\{c'_i\}_{i=1}^r$, we define the quantity*

$$c := \frac{\prod_{i=1}^r c'_i}{e_{r-1}(c'_1, \dots, c'_r)},$$

where e_{r-1} is the elementary symmetric polynomial of degree $r-1$ in r variables.

If Σ is a non-flat, irreducible, r-diagonal, totally geodesic submanifold in M, then it is isometric to one in the list below:

i) $\mathbb{R}\mathsf{H}^n(c)$, with $c'_i \in \{c_i, \frac{c_i}{4}\}$, whenever $(\mathbb{R}\mathsf{H}^n(c'_i)) \le M_i$ for every $i \in \{1, \dots, r\}$.

ii) $\mathbb{F}\mathsf{H}^n(c)$, with $\mathbb{F} \neq \mathbb{R}$ and $c'_i = c_i$, whenever $(\mathbb{F}\mathsf{H}^n(c_i)) \le M_i$, for every $i \in \{1, \dots, r\}$.

Proof. Let $\Sigma \subset M$ be a non-flat, r-diagonal, irreducible, totally geodesic submanifold. We will proceed by induction on r. The statement is true for $r = 1$ by the classification of totally geodesic submanifolds in symmetric spaces of non-compact type and rank one (see Table 6.1). Let us assume that it is true for $r-1$ factors and we will prove it for r. Observe that Σ is contained in $\widehat{\Sigma}\times\widehat{\Sigma}'$ where $\widehat{\Sigma}$ is the projection of Σ onto $M_1 \times \cdots \times M_{r-1}$ and $\widehat{\Sigma}'$ is the projection of Σ onto M_r. By Lemma 6.1.1, we have that $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ are totally geodesic submanifolds of M. Furthermore, by Lemma 6.2.1 *iii)* and Remark 6.2.3, we have that $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ are homothetic to Σ. Moreover, $\widehat{\Sigma}$ is $(r-1)$-diagonal in M, since Σ is r-diagonal in M, and by induction hypothesis, $\widehat{\Sigma}$ is isometric to $\mathbb{F}\mathsf{H}^n(\widetilde{c})$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, where

$$\widetilde{c} = \frac{\prod_{i=1}^{r-1} c'_i}{e_{r-2}(c'_1, \dots, c'_{r-1})},$$

with $c'_i \in \{c_i, c_i/4\}$ for each $i \in \{1, \dots, r-1\}$. Let us assume that $\mathbb{F} = \mathbb{R}$, since the result follows similarly in the other cases. Then, by Theorem 6.2.8, since Σ is 2-diagonal in $\widehat{\Sigma}\times\widehat{\Sigma}'$, and $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ are of rank one, the sectional curvature of Σ is equal to the opposite of

$$c = \frac{\widetilde{c}\, c'_r}{\widetilde{c} + c'_r} = \frac{\prod_{i=1}^r c'_i}{e_{r-1}(c'_1, \dots, c'_r)},$$

where we have used $e_{r-1}(X_1, \dots, X_r) = e_{r-1}(X_1, \dots, X_{r-1}) + e_{r-2}(X_1, \dots, X_{r-1})X_r$, for arbitrary variables $X_1, \dots, X_r$. □

Now we will provide the classification of totally geodesic submanifolds in products of symmetric spaces of rank one by introducing a combinatorial object that we call adapted Young tableau.

We first recall the well-known notions of partition of a positive integer and of Young diagram. Let $r \geq 1$ be a positive integer. Then, a *partition* of r is a vector $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$ such that $r = \sum_{j=1}^{k} \lambda_j$ and $\lambda_1 \geq \ldots \geq \lambda_k \geq 1$. For each partition λ we associate a *Young diagram*. This is a collection of boxes with λ_j boxes in the j-th row, for each $j \in \{1, \ldots, k\}$.

Now we will introduce the notion of Young tableau adapted to a product M of symmetric spaces of non-compact type and rank one. Let us consider $M = \mathbb{F}_1\mathsf{H}^{n_1}(c_1) \times \cdots \times \mathbb{F}_r\mathsf{H}^{n_r}(c_r)$, where $n_i \geq 2$, $c_i > 0$ and $\mathbb{F}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ for each $i \in \{1, \ldots, r\}$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of r and let us consider its Young diagram. We will add to the m-th box in the j-th row a totally geodesic inclusion $\mathbb{F}'_{i_{j,m}}\mathsf{H}^{n'_{i_{j,m}}}(c'_{i_{j,m}}) \subset \mathbb{F}_{i_{j,m}}\mathsf{H}^{n_{i_{j,m}}}(c_{i_{j,m}})$, for every $j \in \{1, \ldots, k\}$ and $m \in \{1, \ldots, \lambda_j\}$, where $\bigsqcup_{j=1}^{k}\{i_{j,1}, \ldots, i_{j,\lambda_j}\} = \{1, \ldots, r\}$. Furthermore, we require all totally geodesic submanifolds appearing in the j-th row to be mutually homothetic. A Young diagram with this information will be called *Young tableau adapted to M*. See Figure 6.1 for various examples of this.

$\mathbb{R}\mathsf{H}^3(c_1) \subset \mathbb{R}\mathsf{H}^3(c_1)$	$\mathbb{R}\mathsf{H}^3(c_2/4) \subset \mathbb{C}\mathsf{H}^3(c_2)$	$\mathbb{R}\mathsf{H}^3(c_3) \subset \mathbb{H}\mathsf{H}^3(c_3)$	$\mathbb{R}\mathsf{H}^3\left(\frac{c_1c_2c_3}{c_1c_2+4c_1c_3+c_2c_3}\right)$

$\mathbb{C}\mathsf{H}^2(c_2) \subset \mathbb{C}\mathsf{H}^3(c_2)$	$\mathbb{C}\mathsf{H}^2(c_3) \subset \mathbb{H}\mathsf{H}^3(c_3)$	$\mathbb{C}\mathsf{H}^2\left(\frac{c_2c_3}{c_2+c_3}\right)$
$\mathbb{R}\mathsf{H}^2(c_1) \subset \mathbb{R}\mathsf{H}^3(c_1)$	$\mathbb{R}\mathsf{H}^2(c_1)$	

$\mathbb{R}\mathsf{H}^3(c_1) \subset \mathbb{R}\mathsf{H}^3(c_1)$	$\mathbb{R}\mathsf{H}^3(c_1)$
$\mathbb{R}\mathsf{H}^3(c_2/4) \subset \mathbb{C}\mathsf{H}^3(c_2)$	$\mathbb{R}\mathsf{H}^3(c_2/4)$
$\mathbb{R}\mathsf{H}^4(c_3) \subset \mathbb{H}\mathsf{H}^3(c_3)$	$\mathbb{R}\mathsf{H}^4(c_3)$

Figure 6.1: Three examples of Young tableaux adapted to the product $M = \mathbb{R}\mathsf{H}^3(c_1) \times \mathbb{C}\mathsf{H}^3(c_2) \times \mathbb{H}\mathsf{H}^3(c_3)$, along with the irreducible factors of the corresponding totally geodesic submanifolds presented at the end of each row (see Proposition 6.2.12). Notice that the isometry type of these totally geodesic submanifolds can be computed using Corollary 6.2.11.

Proposition 6.2.12. *Let $M = M_1 \times \cdots \times M_r$, where M_i is a symmetric space of non-compact type and rank one for each $i \in \{1, \ldots, r\}$. Then, the following statements hold:*

i) For each Young tableau T adapted to M we can attach a set $\mathcal{S}(T)$ of semisimple totally geodesic submanifolds Σ_T of M that have non-trivial projection onto each factor of M.

ii) If Σ_T and $\widetilde{\Sigma}_T$ belong to $\mathcal{S}(T)$, then Σ_T is isometric to $\widetilde{\Sigma}_T$.

iii) If Σ is a semisimple totally geodesic submanifold of M that has non-trivial projection onto each factor of M, then it is equal to some $\Sigma_T \in \mathcal{S}(T)$ for some Young tableau T adapted to M.

Proof. First of all, we will see how to construct a totally geodesic submanifold of M from a Young tableau adapted to M. Let T be a Young tableau adapted to M and let us assume that it has k rows. Let us further assume that it has λ_j boxes in the j-th row. Namely,

$$\mathbb{F}'_{i_{j,1}}\mathsf{H}^{n'_{i_{j,1}}}(c'_{i_{j,1}}) \subset \mathbb{F}_{i_{j,1}}\mathsf{H}^{n_{i_{j,1}}}(c_{i_{j,1}}), \dots, \mathbb{F}'_{i_{j,\lambda_j}}\mathsf{H}^{n'_{i_{j,\lambda_j}}}(c'_{i_{j,\lambda_j}}) \subset \mathbb{F}_{i_{j,\lambda_j}}\mathsf{H}^{n_{i_{j,\lambda_j}}}(c_{i_{j,\lambda_j}})$$

are the labels in the boxes in the j-th row, where we have $\bigsqcup_{j=1}^{k}\{i_{j,1},\dots,i_{j,\lambda_j}\} = \{1,\dots,r\}$. Let $\mathfrak{p}_{i_{j,m}}$ be a Lie triple system corresponding to $M_{i_{j,m}} = \mathbb{F}_{i_{j,m}}\mathsf{H}^{n_{i_{j,m}}}(c_{i_{j,m}})$, for each $m \in \{1,\dots,\lambda_j\}$. Then, for each $m \in \{1,\dots,\lambda_j\}$, there is some Lie triple system $\mathfrak{p}'_{i_{j,m}} \subset \mathfrak{p}_{i_{j,m}}$ that corresponds to the totally geodesic embedding in the m-th box of the j-th row of T. Notice that, by construction of T, these totally geodesic submanifolds are mutually homothetic. Let us define $\mathfrak{g}'_{i_{j,m}} := \mathfrak{p}'_{i_{j,m}} \oplus [\mathfrak{p}'_{i_{j,m}},\mathfrak{p}'_{i_{j,m}}]$. Clearly, for any fixed $j \in \{1,\dots,k\}$, all these Lie algebras $\mathfrak{g}'_{i_{j,m}}$, $m \in \{1,\dots,\lambda_j\}$, are mutually isomorphic because they are the Lie algebras of the isometry groups of mutually homothetic semisimple symmetric spaces. Let $\varphi^j_{1,m}\colon \mathfrak{g}'_{i_{j,1}} \to \mathfrak{g}'_{i_{j,m}}$ be a Lie algebra isomorphism sending $\mathfrak{p}'_{i_{j,1}}$ into $\mathfrak{p}'_{i_{j,m}}$ for $m \in \{2,\dots,\lambda_j\}$, and $\varphi^j_{1,1}$ be the identity map of $\mathfrak{g}'_{i_{j,1}}$. Now, we define $\widehat{\mathfrak{p}}_j := \left\{\sum_{m=1}^{\lambda_j}\varphi^j_{1,m}X : X \in \mathfrak{p}'_{i_{j,1}}\right\}$ for each $j \in \{1,\dots,k\}$. Then, $\widehat{\mathfrak{p}}_j$ is a Lie triple system in $\mathfrak{p}$ that is λ_j-diagonal.

We perform this process for each row $j \in \{1,\dots,k\}$ to define $\mathfrak{p}_T := \bigoplus_{j=1}^{k}\widehat{\mathfrak{p}}_j$. By construction, we have that $[\widehat{\mathfrak{p}}_j,\widehat{\mathfrak{p}}_{j'}] = 0$ for $j \neq j'$ in $\{1,\dots,k\}$. Hence, $\mathfrak{p}_T$ is a Lie triple system in $\mathfrak{p}$ and we will denote by $\Sigma_T = \Sigma_1 \times \cdots \times \Sigma_k$ its corresponding totally geodesic submanifold, where Σ_j is the totally geodesic submanifold corresponding to $\widehat{\mathfrak{p}}_j$. Consequently, for a Young tableau T adapted to M, we have constructed a semisimple totally geodesic submanifold Σ_T of M, that has non-trivial projection onto each factor of M. However, notice that this construction depends on the Lie triple system $\mathfrak{p}'_{i_{j,m}}$ in the subspace $\mathfrak{p}_{i_{j,m}}$ and on the Lie algebra isomorphism $\varphi^j_{1,m}$ that we chose, and if we choose different Lie triple systems and isomorphisms, we get different semisimple totally geodesic submanifolds with non-trivial projection onto each factor of M. Hence, for each Young tableau T adapted to M we can attach a set $\mathcal{S}(T)$ which is equal to the set of all the totally geodesic submanifolds that can be constructed from T through the process described above. This proves *i)*.

Now we will check that all totally geodesic submanifolds in $\mathcal{S}(T)$ are mutually isometric. First of all, observe that two totally geodesic submanifolds of a symmetric space of rank one are congruent if and only if they are isometric. Hence, the totally geodesic embedding $\mathbb{F}'_{i_{j,m}}\mathsf{H}^{n'_{i_{j,m}}}(c'_{i_{j,m}}) \subset \mathbb{F}_{i_{j,m}}\mathsf{H}^{n_{i_{j,m}}}(c_{i_{j,m}})$ represents a congruence class of totally geodesic embeddings in $M_{i_{j,m}} = \mathbb{F}_{i_{j,m}}\mathsf{H}^{n_{i_{j,m}}}(c_{i_{j,m}})$. Let $N_{i_{j,m}}$ and $\widetilde{N}_{i_{j,m}}$ be congruent totally geodesic submanifolds in $M_{i_{j,m}}$ corresponding to the inclusion in the m-th box of the j-th row for each $m \in \{1,\dots,\lambda_j\}$ and $j \in \{1,\dots,r\}$. Then, there is a isometry $\varphi_{i_{j,m}}$ of $M_{i_{j,m}}$ such that $\varphi_{i_{j,m}}N_{i_{j,m}} = \widetilde{N}_{i_{j,m}}$. Following the above procedure, we obtain two different λ_j-diagonal totally geodesic submanifolds, namely,

$$\Sigma_j \subset N_{i_{j,1}} \times \cdots \times N_{i_{j,\lambda_j}} \quad \text{and} \quad \widetilde{\Sigma}_j \subset \widetilde{N}_{i_{j,1}} \times \cdots \times \widetilde{N}_{i_{j,\lambda_j}},$$

such that Σ_j and $\widetilde{\Sigma}_j$ are homothetic to $N_{i_{j,m}}$ and $\widetilde{N}_{i_{j,m}}$ for every $m \in \{1, \dots, \lambda_j\}$ and for each $j \in \{1, \dots, k\}$. Now let

$$\varphi_j := (\varphi_{i_{j,1}}, \dots, \varphi_{i_{j,\lambda_j}}) \colon M_{i_{j,1}} \times \cdots \times M_{i_{j,\lambda_j}} \to M_{i_{j,1}} \times \cdots \times M_{i_{j,\lambda_j}}.$$

Then, φ_j is an isometry of $M_{i_{j,1}} \times \cdots \times M_{i_{j,\lambda_j}}$ for every $j \in \{1, \dots, k\}$. Moreover, $\varphi_j \Sigma_j$ is a non-flat, irreducible, λ_j-diagonal totally geodesic submanifold in $\widetilde{N}_{i_{j,1}} \times \cdots \times \widetilde{N}_{i_{j,\lambda_j}}$. By Proposition 7.3.1, there is $\psi_j \in \mathrm{Isom}(\widetilde{N}_{i_{j,1}}) \times \cdots \times \mathrm{Isom}(\widetilde{N}_{i_{j,\lambda_j}})$ such that $\psi_j \varphi_j \Sigma_j = \widetilde{\Sigma}_j$. Consequently, $\Sigma = \Sigma_1 \times \cdots \times \Sigma_k$ is isometric to $\widetilde{\Sigma} = \widetilde{\Sigma}_1 \times \cdots \times \widetilde{\Sigma}_k$. This proves *ii)*.

Let $\Sigma \subset M$ be a semisimple totally geodesic submanifold that has non-trivial projection onto each factor of M. Let $k \leq r$ be the rank of Σ. By combining De-Rham Theorem, Lemma 6.2.1 and Lemma 6.2.5, we can ensure the existence of a partition $\{\{i_{j,1}, \dots, i_{j,\lambda_j}\} : j \in \{1, \dots, k\}\}$ of $\{1, \dots, r\}$ satisfying:

- $\Sigma = \Sigma_1 \times \cdots \times \Sigma_k$, where Σ_j has rank one for every $j \in \{1, \dots, k\}$.
- $\Sigma_j \subset M$ is λ_j-diagonal for every $j \in \{1, \dots, k\}$.
- $r = \sum_{j=1}^{k} \lambda_j$.
- $\Sigma_j \subset N_j := N_{i_{j,1}} \times \cdots \times N_{i_{j,\lambda_j}}$, where $N_{i_{j,m}} \subset M_{i_{j,m}}$ is a totally geodesic submanifold homothetic to Σ_j for each $j \in \{1, \dots, k\}$ and $m \in \{1, \dots, \lambda_j\}$.

Then, up to some reordering, we can assume that $(\lambda_1, \dots, \lambda_k)$ is a partition of r. Let T be the Young diagram associated with $(\lambda_1, \dots, \lambda_k)$. Then, Σ_j projects non-trivially exactly onto λ_j factors of M. We fill the λ_j boxes of the j-th row of T with the data corresponding to the totally geodesic inclusion resulting from projecting Σ_j in each of these factors. Therefore, we can find a Young tableau T adapted to M such that $\Sigma = \Sigma_1 \times \cdots \times \Sigma_k$ is equal to some Σ_T in $\mathcal{S}(T)$. This proves *iii)*. □

We are now ready to prove the first main theorem of this chapter.

Proof of Theorem A. By Proposition 6.2.12, it follows that $\Sigma_0 \times \Sigma_T$ is a totally geodesic submanifold of M, where Σ_T is a semisimple totally geodesic submanifold corresponding to a Young tableau T adapted to $M_{\sigma(1)} \times \cdots \times M_{\sigma(k)}$, Σ_0 is a flat totally geodesic submanifold of $M_{\sigma(k+1)} \times \cdots \times M_{\sigma(r)}$, σ is any permutation of $\{1, \dots, r\}$, and $k \in \{1, \dots, r\}$.

Let Σ be a totally geodesic submanifold of M. By De-Rham Theorem we have $\Sigma = \Sigma_0 \times \Sigma_1$, where Σ_0 is flat and Σ_1 is semisimple. Then, Σ_1 projects non-trivially onto $M_{\sigma(1)} \times \cdots \times M_{\sigma(k)}$ for some $k \in \{1, \dots, r\}$ and some permutation σ of $\{1, \dots, r\}$. Thus, by Lemma 6.2.5, we have $\Sigma_0 \subset M_{\sigma(k+1)} \times \cdots \times M_{\sigma(r)}$. Now, by Proposition 6.2.12, there is a Young tableau T adapted to $M_{\sigma(1)} \times \cdots \times M_{\sigma(k)}$ such that Σ_1 is equal to some Σ_T in $\mathcal{S}(T)$. Therefore, Σ is equal to $\Sigma_0 \times \Sigma_T$ as submanifolds in M. □

6.3 Totally geodesic submanifolds in Hermitian symmetric spaces

In this section we construct infinitely many examples of totally geodesic submanifolds in Hermitian symmetric spaces that have constant Kähler angle different from 0 or $\pi/2$.

We start by recalling the notion of Kähler angle (see Section §4.1). Let us equip the complex vector space $\mathbb{C}^n$ with $\langle \cdot, \cdot \rangle$, the scalar product given by considering the real part of its standard Hermitian scalar product, and denote the complex structure of $\mathbb{C}^n$ by J (multiplication by the imaginary unit). Furthermore, let us consider a real vector subspace $V \subset \mathbb{C}^n$ and the orthogonal projection $\pi_V \colon \mathbb{C}^n \to V$ onto V. The Kähler angle of a non-zero $v \in V$ with respect to V is defined as the value $\varphi \in [0, \pi/2]$ such that $\langle \pi_V Jv, \pi_V Jv \rangle = \cos^2(\varphi)\langle v, v \rangle$. We say that a real subspace $V \subset \mathbb{C}^n$ has *constant Kähler angle* $\varphi \in [0, \pi/2]$ if the Kähler angle of every non-zero vector $v \in V$ is φ. In particular, $V \subset \mathbb{C}^n$ has constant Kähler angle equal to 0 if and only if it is a complex subspace, and it has constant Kähler angle equal to $\pi/2$ if and only if it is totally real. Also, a submanifold Σ in a Kähler manifold M is said to have constant Kähler angle $\varphi \in [0, \pi/2]$ if the tangent space of Σ at each point is a subspace with constant Kähler angle φ in the corresponding tangent space of M. In the setting of Hermitian symmetric spaces, since totally geodesic submanifolds are homogeneous and the isometries that belong to the connected component of the identity of the isometry group are holomorphic (see [90, Chapter VIII, §4]), the previous property needs to be checked only at one point.

Now we will recall some known facts about complex Grassmannians that will be useful in this section. Let $M = \mathsf{G}_k(\mathbb{C}^{n+k})$ be the Grassmannian of complex k-planes in $\mathbb{C}^{n+k}$. Then, we have $M = \mathsf{G}/\mathsf{K}$, with $\mathsf{G} = \mathsf{SU}_{n+k}$ and $\mathsf{K} = \mathsf{S}(\mathsf{U}_k \times \mathsf{U}_n)$. We can decompose $\mathfrak{su}_{n+k}$ as $\mathfrak{su}_{n+k} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$\begin{aligned} \mathfrak{k} &= \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) : A \in \mathfrak{u}_k, B \in \mathfrak{u}_n, \operatorname{tr} A + \operatorname{tr} B = 0 \right\}, \\ \mathfrak{p} &= \left\{ \left(\begin{array}{c|c} 0 & X \\ \hline -X^* & 0 \end{array} \right) : X \in \mathcal{M}_{k,n}(\mathbb{C}) \right\}. \end{aligned} \tag{6.3}$$

Furthermore, $M = \mathsf{G}/\mathsf{K}$ is a Hermitian symmetric space. Then, $\mathfrak{p}$ inherits a complex structure J which is given by $\operatorname{ad}_{Z|\mathfrak{p}}$ for some $Z \in Z(\mathfrak{k})$ (see [110, Theorem 7.117]). Hence, we can identify $\mathfrak{p}$ with the complex vector space $\mathbb{C}^k \otimes \mathbb{C}^n$ by the usual isomorphism with matrices with complex entries.

For $k = 1$, we have $\mathsf{G}_1(\mathbb{C}^{n+1})$, which is the complex projective space $\mathbb{C}\mathsf{P}^n$. In this case the Lie algebra of the isometry group is $\mathfrak{su}_{n+1}$. Let $\Theta \colon \mathfrak{su}_{n+1} \to \mathfrak{su}_{n+1}$ be such that $X \in \mathfrak{su}_{n+1}$ is mapped to $\overline{X} \in \mathfrak{su}_{n+1}$, the complex conjugate of X. Clearly, Θ is a Lie algebra automorphism of $\mathfrak{su}_{n+1}$ and preserves $\mathfrak{p}$. Furthermore, let $\{e_1, \dots, e_n\}$ be the canonical $\mathbb{C}$-orthonormal basis for $\mathfrak{p} \equiv \mathbb{C}^n$. Observe that we have

$$\Theta e_i = e_i, \qquad \Theta J e_i = -J e_i, \tag{6.4}$$

for every $i = 1, \dots, n$.

Theorem 6.3.1. *Let* $M = \mathbb{C}\mathsf{P}^n \times \overset{k)}{\cdots} \times \mathbb{C}\mathsf{P}^n$, $n \geq 1$. *Then, for each* $s \in \{0, \dots, k\}$, *there is a* k*-diagonal totally geodesic submanifold homothetic to* $\mathbb{C}\mathsf{P}^n$ *in* M *with constant Kähler angle* $\varphi \in [0, \pi/2]$ *satisfying*

$$\cos(\varphi) = \left| \frac{2s - k}{k} \right|.$$

Proof. First of all, let $\mathfrak{p}_i$ be the Lie triple system corresponding to the i-th factor of M. Then, $\mathfrak{g}_i = \mathfrak{p}_i \oplus \mathfrak{k}_i \simeq \mathfrak{su}_{n+1}$, where $\mathfrak{k}_i := [\mathfrak{p}_i, \mathfrak{p}_i]$. Moreover, consider the complex structure $J = (J_1, \dots, J_k)$ of $\bigoplus_{i=1}^{k} \mathfrak{p}_i$, where J_i is the complex structure induced by some element in $Z(\mathfrak{k}_i)$. Let $\sigma_i \colon \mathfrak{g}_1 \to \mathfrak{g}_i$ be a Lie algebra isomorphism for $i \in \{2, \dots, k\}$, and $\sigma_1 := \mathrm{Id}_{\mathfrak{g}_1}$. Furthermore, let us assume that σ_i restricted to $\mathfrak{k}_1$ induces an isomorphism between $\mathfrak{k}_1$ and $\mathfrak{k}_i$. Thus, we have $\sigma_i \operatorname{ad}_Z X = \operatorname{ad}_{\sigma_i Z} \sigma_i X$, for each $X \in \mathfrak{p}_1$ and $Z \in Z(\mathfrak{k}_1)$. In particular, by the discussion above and taking into account that $\dim Z(\mathfrak{k}_i) = 1$, we have $\sigma_i J_1 X = \pm J_i \sigma_i X$, for every $X \in \mathfrak{p}_1$, where we denote by J_1 and J_i the complex structures of both $\mathfrak{p}_1$ and $\mathfrak{p}_i$, respectively. We will declare $\sigma_i = \sigma_i^+$, when $\sigma_i J_1 (\sigma_i)^{-1} = J_i$, and $\sigma_i = \sigma_i^-$, when $\sigma_i J_1 (\sigma_i)^{-1} = -J_i$. Additionally, notice that $\sigma_i^- = \sigma_i^+ \circ \Theta$. Let us consider

$$\mathfrak{p}_{\Sigma^s} := \left\{ \sum_{i=1}^{s} \sigma_i^+ X + \sum_{i=s+1}^{k} \sigma_i^- X : X \in \mathfrak{p}_1 \right\},$$

for each $s \in \{0, \dots, k\}$. Clearly, this is a Lie triple system and it must correspond to some totally geodesic submanifold Σ^s of M that is homothetic to $\mathbb{C}\mathsf{P}^n$ by Corollary 6.2.11 via duality (see also Remark 6.2.3). Let us choose a $\mathbb{C}$-orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathfrak{p}_1$ satisfying Equation (6.4). Then, taking into account that σ_i^+ restricted to $\mathfrak{k}_1$ gives an isomorphism onto $\mathfrak{k}_i$, and Equation (6.4), we can express $\mathfrak{p}_{\Sigma^s}$ as $\operatorname{span}_{\mathbb{R}}\{v_i, w_i\}_{i=1}^n$, where

$$v_i := \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \sigma_j^+ e_i, \qquad w_i := \frac{1}{\sqrt{k}} \sum_{j=1}^{s} \sigma_j^+ J_1 e_i - \frac{1}{\sqrt{k}} \sum_{j=s+1}^{k} \sigma_j^+ J_1 e_i$$

constitute an orthonormal basis of $\mathfrak{p}_{\Sigma^s}$. Let $v \in \mathfrak{p}_{\Sigma^s}$ be a unit vector. Then, we can write $v = \sum_{i=1}^{n} (a_i v_i + b_i w_i)$, where $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are real numbers satisfying that $\sum_{i=1}^{n} (a_i^2 + b_i^2) = 1$. Now we have

$$\pi_{\mathfrak{p}_{\Sigma^s}} Jv = \pi_{\mathfrak{p}_{\Sigma^s}} \left(\sum_{j=1}^{n} (a_i J v_i + b_i J w_i) \right) = \sum_{i=1}^{n} (a_i \pi_{\mathfrak{p}_{\Sigma^s}} J v_i + b_i \pi_{\mathfrak{p}_{\Sigma^s}} J w_i),$$

where $\pi_{\mathfrak{p}_{\Sigma^s}}$ denotes the orthogonal projection onto $\mathfrak{p}_{\Sigma^s}$. Moreover,

$$\pi_{\mathfrak{p}_{\Sigma^s}} J v_i = \sum_{j=1}^{n} (\langle J v_i, v_j \rangle v_j + \langle J v_i, w_j \rangle w_j) = \langle J v_i, w_i \rangle w_i$$

$$\pi_{\mathfrak{p}_{\Sigma^s}} J w_i = \sum_{j=1}^{n} (\langle J w_i, v_j \rangle v_j + \langle J w_i, w_j \rangle w_j) = \langle J w_i, v_i \rangle v_i,$$

for each $i \in \{1, \dots, n\}$, where we have used that σ_j^+ is a linear isometry, since all the factors in M are mutually isometric. Furthermore, using again that σ_j^+ is a linear isometry that commutes with J_j, we get

$$\langle Jv_i, w_i \rangle = \frac{1}{k} \left(\sum_{j=1}^{s} \langle J_j \sigma_j^+ e_i, \sigma_j^+ J_1 e_i \rangle - \sum_{j=s+1}^{k} \langle J_j \sigma_j^+ e_i, \sigma_j^+ J_1 e_i \rangle \right) = \frac{2s-k}{k}.$$

Consequently, combining these equations, we conclude

$$\begin{aligned} \langle \pi_{\mathfrak{p}_{\Sigma^s}} Jv, \pi_{\mathfrak{p}_{\Sigma^s}} Jv \rangle &= \sum_{i=1}^{n} (a_i^2 \langle Jv_i, w_i \rangle^2 + b_i^2 \langle Jw_i, v_i \rangle^2) = \sum_{i=1}^{n} (a_i^2 + b_i^2) \langle Jv_i, w_i \rangle^2 \\ &= \left(\frac{2s-k}{k} \right)^2. \end{aligned}$$

Since v is an arbitrary unit vector in $\mathfrak{p}_{\Sigma^s}$, we have that Σ^s has constant Kähler angle equal to $\varphi \in [0, \pi/2]$, where $\cos(\varphi) = \left| \frac{2s-k}{k} \right|$. □

Thus, Theorem 6.3.1 gives a method to construct totally geodesic submanifolds with constant Kähler angle in Hermitian symmetric spaces. We only need to find a product of complex projective spaces embedded in a totally geodesic way and use Theorem 6.3.1. In particular, in the complex Grassmannians these products are very abundant as the following lemma shows.

Lemma 6.3.2. *Let $(n_1, \dots, n_k)$ be a partition of n. Then, there is a complex totally geodesic submanifold homothetic to $\mathbb{C}\mathsf{P}^{n_1} \times \cdots \times \mathbb{C}\mathsf{P}^{n_k}$ in $\mathsf{G}_k(\mathbb{C}^{n+k})$.*

Proof. Let $\Sigma = \mathbb{C}\mathsf{P}^{n_1} \times \cdots \times \mathbb{C}\mathsf{P}^{n_k}$ and $(n_1, \dots, n_k)$ a partition of n. This means that $n_1 \geq \ldots \geq n_k$ and $\sum_{i=1}^{k} n_i = n$. Now, for each $i \in \{1, \dots, k\}$, we define the subspace

$$\mathfrak{p}_i := \operatorname{span}_{\mathbb{C}} \{ e_i \otimes e_{1+\sum_{j=1}^{i-1} n_j}, \dots, e_i \otimes e_{\sum_{j=1}^{i} n_j} \},$$

where $\{e_l \otimes e_m\}_{l,m=1}^{k,n}$ is the canonical basis of $\mathbb{C}^k \otimes \mathbb{C}^n$.

Let us define $\mathfrak{p}_\Sigma := \bigoplus_{i=1}^{k} \mathfrak{p}_i$. We will see that $\mathfrak{p}_\Sigma$ is a Lie triple system corresponding to Σ. Using the Lie bracket of $\mathfrak{su}_{n+k}$ and the description of $\mathfrak{p}$ in Equation (6.3), it can be checked that $\mathfrak{k}_i := [\mathfrak{p}_i, \mathfrak{p}_i] \simeq \mathfrak{s}(\mathfrak{u}_1 \times \mathfrak{u}_{n_i})$ and that $\mathfrak{p}_i$ is a $\mathfrak{k}_i$-module. Furthermore, it can be seen that $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ is isomorphic to $\mathfrak{su}_{n_i+1}$. This implies that $\mathfrak{p}_i$ is a Lie triple system corresponding to $\mathbb{C}\mathsf{P}^{n_i}$. Moreover, it is clear that $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$ for $i \neq j$. Observe that $\mathfrak{p}_i$ is invariant under ad_Z, where $Z \in Z(\mathfrak{k})$, and hence every $\mathfrak{p}_i$ is invariant under the complex structure of $\mathsf{G}_k(\mathbb{C}^{n+k})$. Furthermore, it is easy to check that there is some $k \in \mathsf{K}$ such that $\operatorname{Ad}(k)$ interchanges the rows of $\mathfrak{p}$, see Equation (6.3). Consequently, $\mathfrak{p}_\Sigma$ is also a Lie triple system whose associated totally geodesic submanifold of $\mathsf{G}_k(\mathbb{C}^{n+k})$ is homothetic to $\mathbb{C}\mathsf{P}^{n_1} \times \cdots \times \mathbb{C}\mathsf{P}^{n_k}$ and invariant under the complex structure of $\mathsf{G}_k(\mathbb{C}^{n+k})$. □

We are now ready to prove the second main theorem of this chapter.

Proof of Theorem B. Using Lemma 6.3.2 and Theorem 6.3.1, given any $q \in [0,1] \cap \mathbb{Q}$ and $m \in \mathbb{N}$, there exist $k, n \in \mathbb{N}$ such that there is a totally geodesic submanifold homothetic to $\mathbb{C}\mathsf{P}^m$ with constant Kähler angle $\arccos(q) \in [0, \pi/2]$ in $\mathsf{G}_k(\mathbb{C}^{n+k})$. This implies our result. □

Chapter 7
Totally geodesic submanifolds in exceptional symmetric spaces

In the present chapter, we classify maximal totally geodesic submanifolds in exceptional symmetric spaces. The contents of this chapter have given rise to the paper [116].

A complete, totally geodesic, proper submanifold is said to be maximal if it is maximal with respect to the inclusion among complete, totally geodesic, proper submanifolds. Recall that every totally geodesic submanifold in a symmetric space can be extended to a complete totally geodesic submanifold. Since duality preserves totally geodesic submanifolds, it suffices to obtain the classification in the non-compact setting. According to the classification by Cartan [90], the irreducible symmetric spaces of non-compact type consist of several infinite families and the 17 exceptional spaces given in Table 7.1.

G_2-type	$\mathsf{G}_2^2/\mathsf{SO}_4$	$\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$			
F_4-type	$\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1$	$\mathsf{F}_4^{-20}/\mathsf{Spin}_9$	$\mathsf{F}_4(\mathbb{C})/\mathsf{F}_4$		
E_6-type	$\mathsf{E}_6^6/\mathsf{Sp}_4$	$\mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1$	$\mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1$	$\mathsf{E}_6^{-26}/\mathsf{F}_4$	$\mathsf{E}_6(\mathbb{C})/\mathsf{E}_6$
E_7-type	$\mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1$	$\mathsf{E}_7^{-25}/\mathsf{E}_6\mathsf{U}_1$	$\mathsf{E}_7^7/\mathsf{SU}_8$	$\mathsf{E}_7(\mathbb{C})/\mathsf{E}_7$	
E_8-type	$\mathsf{E}_8^{-24}/\mathsf{E}_7\mathsf{Sp}_1$	$\mathsf{E}_8^8/\mathsf{SO}_{16}$	$\mathsf{E}_8(\mathbb{C})/\mathsf{E}_8$		

Table 7.1: Exceptional symmetric spaces of non-compact type.

Theorem A. *Let $M = \mathsf{G}/\mathsf{K}$ be an irreducible exceptional symmetric space of non-compact type. Let Σ be a maximal totally geodesic submanifold of M. Then Σ is isometric to one of the spaces listed in Tables 7.5, 7.6, 7.7, 7.8, 7.9 at the end of this chapter. Conversely, every space listed in these tables can be isometrically embedded as a maximal totally geodesic submanifold of M.*

Since non-semisimple maximal totally geodesic submanifolds of irreducible symmetric spaces have been classified by Berndt, Olmos [20], it suffices to study semisimple maximal totally geodesic submanifolds. The proof of Theorem A uses a correspondence between maximal semisimple totally geodesic submanifolds and certain subalgebras of the Lie algebra of the isometry group. Note that a maximal semisimple totally geodesic submanifold needs not be maximal, since it might be contained in a non-semisimple maximal totally geodesic submanifold. While there is no classification of the relevant subalgebras, we construct a set which contains all of them. Then it remains to decide which of these subalgebras give rise to maximal totally geodesic submanifolds. In order to do so, we develop some criteria for maximality.

We would like to remark that our methods can be used to classify maximal totally geodesic submanifolds in symmetric spaces whose isometry group has rank less or equal than eight. Therefore, one could list all maximal totally geodesic submanifolds up to isometry in symmetric spaces with isometry group of rank less or equal than eight. However, in this thesis, we content ourselves with listing the classification in the exceptional symmetric spaces.

Onishchik [148] introduced an invariant of symmetric spaces concerning totally geodesic submanifolds called index, which is defined as the minimal codimension of a totally geodesic proper submanifold. Berndt, Olmos and Rodríguez [20, 21, 22, 23, 24] have computed the index $i(M)$ of every irreducible symmetric space M. In particular, they proved what they called the *Index Conjecture* [20]. This conjecture states that in an irreducible symmetric space of non-compact type $M \neq \mathsf{G}_2^2/\mathsf{SO}_4$, there is some reflective submanifold Σ of M whose codimension equals the index of M.

Generalizing a notion introduced by Dynkin [74], in this chapter we define the Dynkin index of certain semisimple subalgebras of simple real Lie algebras, see Definition 7.4.2. We use this to characterize the isometry types of totally geodesic embeddings of semisimple symmetric spaces into irreducible symmetric spaces. This characterization allows us to derive a result analogous to the Index Conjecture:

Theorem B. *Let M be an irreducible symmetric space of non-compact type. Then, there is some totally geodesic submanifold Σ in M with $i(M) = \operatorname{codim}(\Sigma)$ such that the Dynkin index of the semisimple part of the Lie algebra of the isometry group of Σ equals* $(1, 1, \dots, 1)$.

This chapter is organized as follows. We revisit different formulations of the Karpelevich-Mostow theorem in Section §7.1. Some useful facts about the complexification of Lie subalgebras of a real Lie algebra are recalled in Section §7.2. In Section §7.3, we prove a correspondence between maximal semisimple totally geodesic submanifolds in symmetric spaces of non-compact type and certain subalgebras of the isometry algebra of the ambient space. Then we specialize our study of totally geodesic submanifolds of symmetric spaces to the setting of exceptional symmetric spaces. In Section §7.4, we introduce an invariant for certain semisimple totally geodesic submanifolds in symmetric spaces of non-compact type that we call Dynkin index, which characterizes these submanifolds up to isometries. In Section §7.5, we classify maximal totally geodesic submanifolds in exceptional symmetric spaces whose isometry group is absolutely simple, and in Section §7.6 we deal with the case when the isometry group is not absolutely simple. Section §7.7 contains the proofs of the two main results (Theorem A and B above).

7.1 Karpelevich-Mostow Theorem

A fundamental result in the study of totally geodesic submanifolds in symmetric spaces of non-compact type is known as the Karpelevich Theorem [103], see also [138] and [66].

Theorem 7.1.1. *Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of non-compact type. Then any connected semisimple subgroup $\mathsf{H} \subset \mathsf{G}$ acts on M with a totally geodesic orbit.*

An equivalent, more algebraic formulation, see [150, Corollary 1, p. 46], is the following.

Theorem 7.1.2. *Let $f: \mathfrak{h} \to \mathfrak{g}$ be a homomorphism of real semisimple Lie algebras and let a Cartan decomposition $\mathfrak{h} = \mathfrak{k}' \oplus \mathfrak{p}'$ be given. Then there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $f(\mathfrak{k}') \subset \mathfrak{k}$ and $f(\mathfrak{p}') \subset \mathfrak{p}$.*

A subalgebra $\mathfrak{h}$ of a real semisimple Lie algebra $\mathfrak{g}$ is called *canonically embedded* in $\mathfrak{g}$ with respect to some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ if $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. This is equivalent to $\mathfrak{h}$ being θ-invariant, where θ is the Cartan involution associated with the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

An *algebraic group* over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is an affine algebraic variety G over $\mathbb{K}$ endowed with a group structure for which the map $\mathsf{G} \times \mathsf{G} \to \mathsf{G}$, $(x, y) \mapsto xy^{-1}$ is polynomial. It turns out that an algebraic group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. An *algebraic subgroup* of an algebraic group G is a closed subgroup of G (in the Zariski topology). An algebraic subgroup is itself an algebraic group. A Lie algebra $\mathfrak{g}$ is *algebraic* if it is the Lie algebra of some irreducible algebraic subgroup G of $\mathsf{GL}_n(\mathbb{K})$, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In particular, semisimple Lie algebras are algebraic, see [151, p. 138]. Let G be an algebraic group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with Lie algebra $\mathfrak{g}$. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *algebraic* if there exists an algebraic subgroup H of G with Lie algebra $\mathfrak{h}$. For any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there is a smallest algebraic subalgebra $\mathfrak{h}^a$ of $\mathfrak{g}$ containing $\mathfrak{h}$, called the *algebraic closure* of $\mathfrak{h}$ in $\mathfrak{g}$. An algebraic subalgebra $\mathfrak{h}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is said to be *reductive* if it is a reductive Lie algebra and its center consists of semisimple elements, i.e. of elements $X \in \mathfrak{h}$ for which the linear map ad_X is a diagonalizable endomorphism of the vector space $\mathfrak{h}$.

Remark 7.1.3. Let $\mathfrak{g}$ be a semisimple Lie algebra. By [152, Chapter 1, §6.2, Theorem 6.2], if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $[\mathfrak{h}, \mathfrak{h}] = [\mathfrak{h}^a, \mathfrak{h}^a]$. Let $\mathfrak{l}$ be a maximal proper Lie subalgebra of $\mathfrak{g}$. Then $[\mathfrak{l}^a, \mathfrak{l}^a] = [\mathfrak{l}, \mathfrak{l}] \neq \mathfrak{g}$. Hence $\mathfrak{l}^a \neq \mathfrak{g}$, and thus by maximality $\mathfrak{l}^a = \mathfrak{l}$. Hence a maximal proper Lie subalgebra of a semisimple Lie algebra is algebraic.

A more general version of Theorem 7.1.2 can be formulated as follows, see [152, Theorem 3.6, Chapter 6].

Theorem 7.1.4 (Karpelevich-Mostow). *An algebraic subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ is reductive if and only if it is canonically embedded in $\mathfrak{g}$ with respect to some Cartan decomposition of $\mathfrak{g}$.*

7.2 Complexification of subalgebras

Let $\mathfrak{g}$ be a real Lie algebra. Recall that its *complexification* is defined by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Conversely, the *realification* $\mathfrak{h}_{\mathbb{R}}$ of a complex Lie algebra $\mathfrak{h}$ is defined as the real Lie algebra obtained from $\mathfrak{h}$ by restricting the scalars to the reals. Furthermore, if $\mathfrak{g}$ is a complex Lie algebra, then a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_\mathbb{R}$ is called a *real form* of $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$ and $\mathfrak{g}_0 \cap i\mathfrak{g}_0 = 0$, where i is the imaginary unit. We say that a Lie algebra is of *non-compact type* if it is semisimple and all of its simple ideals are non-compact Lie algebras. We say that a subalgebra of a Lie algebra $\mathfrak{g}$ is *maximal of non-compact type* if it is maximal among all proper subalgebras of non-compact type of $\mathfrak{g}$.

A real Lie algebra $\mathfrak{g}$ is called *absolutely simple* if it is simple and its complexification $\mathfrak{g}_\mathbb{C}$ is a simple complex Lie algebra. In case $\mathfrak{g}$ is an absolutely simple real Lie algebra, $\mathfrak{g}$ is a real form of the simple complex Lie algebra $\mathfrak{g}_\mathbb{C}$. In case $\mathfrak{g}$ is a simple, but not absolutely simple, real Lie algebra, $\mathfrak{g}$ is the realification of a simple complex Lie algebra.

Remark 7.2.1. In later sections of this chapter, we do not sometimes distinguish in our notation between a complex Lie algebra and its realification when it is clear from the context which Lie algebra structure we consider.

Remark 7.2.2. Recall from the classification of Riemannian symmetric spaces that there are two classes of irreducible Riemannian symmetric spaces of non-compact type. If G is the connected component of the isometry group of such a $M = \mathsf{G}/\mathsf{K}$, we distinguish between the following two cases:

- Symmetric spaces of *type III*: $\mathfrak{g}$ is *absolutely simple.*
- Symmetric spaces of *type IV*: $\mathfrak{g}$ is *simple, but not absolutely simple.*

Lemma 7.2.3. *Let $\mathfrak{g}$ be a simple real Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then the following statements hold:*

i) If $\mathfrak{h}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ is a maximal reductive algebraic subalgebra, then $\mathfrak{h} \subset \mathfrak{g}$ is a maximal reductive algebraic subalgebra.

ii) If $\mathfrak{h}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ is a maximal semisimple subalgebra, then $\mathfrak{h} \subset \mathfrak{g}$ is a maximal semisimple subalgebra.

Proof. By [150, §2, Proposition 2(i)], $\mathfrak{h}_\mathbb{C}$ is a semisimple Lie algebra if and only if $\mathfrak{h}$ is a semisimple Lie algebra. Thus, $[\mathfrak{h}, \mathfrak{h}]$ is semisimple if and only if $[\mathfrak{h}, \mathfrak{h}]_\mathbb{C}$ is semisimple. Since $[\mathfrak{h}, \mathfrak{h}]_\mathbb{C} \oplus Z(\mathfrak{h})_\mathbb{C} = [\mathfrak{h}_\mathbb{C}, \mathfrak{h}_\mathbb{C}] \oplus Z(\mathfrak{h}_\mathbb{C})$, $\mathfrak{h}$ is a reductive Lie algebra if and only if $\mathfrak{h}_\mathbb{C}$ is a reductive Lie algebra. Moreover, $\mathrm{ad}_{Z(\mathfrak{h})}$ consists of semisimple elements if and only if $\mathrm{ad}_{Z(\mathfrak{h}_\mathbb{C})}$ does. Consequently, we have proved that $\mathfrak{h} \subset \mathfrak{g}$ is a reductive algebraic subalgebra if and only if $\mathfrak{h}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ is a reductive algebraic subalgebra.

Let us assume that $\mathfrak{h}_\mathbb{C}$ is a maximal reductive algebraic (resp. semisimple) subalgebra of $\mathfrak{g}_\mathbb{C}$, and that there is some reductive algebraic (resp. semisimple) subalgebra $\mathfrak{l} \subsetneq \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{g}$. Then $\mathfrak{h}_\mathbb{C} \subset \mathfrak{l}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ and $\mathfrak{l}_\mathbb{C}$ is reductive (resp. semisimple). Since $\mathfrak{h}_\mathbb{C}$ is a maximal reductive algebraic (resp. maximal semisimple) subalgebra, we have that $\mathfrak{h}_\mathbb{C} = \mathfrak{l}_\mathbb{C}$. Therefore, $\mathfrak{h}$ and $\mathfrak{l}$ have the same dimension and we conclude that $\mathfrak{h} = \mathfrak{l}$. □

We say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is a *regular* subalgebra if $\mathfrak{h}$ is normalized by some Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Following Dynkin [74], we say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an *R-subalgebra* if it is contained in a regular proper subalgebra, and we say that it is an *S-subalgebra* if it is not contained in a regular proper subalgebra. If $\mathfrak{g}$ is not absolutely simple, its maximal subalgebras of non-compact type are described by Lemma 7.2.4, which is similar to [73, Appendix to Chapter 1, Theorem 1.6].

Lemma 7.2.4. *Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{R}}$ be a subalgebra that is maximal among the subalgebras of non-compact type. Then $\mathfrak{h}$ coincides with one of the following:*

i) a maximal semisimple regular subalgebra of $\mathfrak{g}$,

ii) a maximal S-subalgebra of $\mathfrak{g}$,

iii) a non-compact real form of $\mathfrak{g}$.

Conversely, all of the above are maximal subalgebras of non-compact type of $\mathfrak{g}_{\mathbb{R}}$.

Proof. Let $\mathfrak{h}$ be a subalgebra that is maximal among the subalgebras of non-compact type of $\mathfrak{g}_{\mathbb{R}}$. For all subalgebras of $\mathfrak{g}_{\mathbb{R}}$ we have that $\mathfrak{h} + i\mathfrak{h}$ is a complex subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_0 := \{X \in \mathfrak{h} \colon \lambda X \in \mathfrak{h} \text{ for all } \lambda \in \mathbb{C}\} = \mathfrak{h} \cap i\mathfrak{h}$ is an ideal of $\mathfrak{h} + i\mathfrak{h}$, see [73, Appendix to Chapter 1]. If $\mathfrak{h} + i\mathfrak{h} = \mathfrak{g}$, then it follows that $\mathfrak{h}_0$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, it follows that $\mathfrak{h}_0 = 0$ and hence that $\mathfrak{h}$ is a real form of $\mathfrak{g}$. Obviously, $\mathfrak{h}$ is not a compact real form.

Now we may assume that $\mathfrak{h}+i\mathfrak{h} \neq \mathfrak{g}$. Then $\mathfrak{h}+i\mathfrak{h}$ is contained in some maximal subalgebra $\widetilde{\mathfrak{h}}$ of $\mathfrak{g}$. Consider first the case where there is a maximal regular reductive subalgebra $\widetilde{\mathfrak{h}}$ containing $\mathfrak{h} + i\mathfrak{h}$. If the subalgebra $\widetilde{\mathfrak{h}}$ of the complex Lie algebra $\mathfrak{g}$ is semisimple, then it is automatically a subalgebra of non-compact type of $\mathfrak{g}_{\mathbb{R}}$. Hence, in this case we have $\mathfrak{h} = \widetilde{\mathfrak{h}}$ and $\mathfrak{h}$ is one of the subalgebras given in [74, Table 12]. On the other hand, if the subalgebra $\widetilde{\mathfrak{h}}$ of $\mathfrak{g}$ is non-semisimple, then $\mathfrak{h} + i\mathfrak{h}$ is contained in one of the subalgebras in [74, Table 12a] and since these are maximal semisimple regular subalgebras and also subalgebras of non-compact type in $\mathfrak{g}_{\mathbb{R}}$, by maximality, $\mathfrak{h}$ is conjugate to one of them.

It remains the case when $\mathfrak{h}+i\mathfrak{h}$ is not contained in a regular subalgebra of $\mathfrak{g}$. Then it is contained in a maximal S-subalgebra of $\mathfrak{g}$. It follows from [74, Theorem 7.3], that every non-semisimple subalgebra of a complex semisimple Lie algebra is an R-subalgebra. Therefore we know that S-subalgebras of $\mathfrak{g}$ are semisimple and hence they are also subalgebras of non-compact type in $\mathfrak{g}_{\mathbb{R}}$. By maximality, it follows now that $\mathfrak{h}$ is a maximal S-subalgebra of $\mathfrak{g}$ in this case.

Let us now show that the converse holds. Since the real forms of $\mathfrak{g}$ are simple, they are either compact or of non-compact type. The subalgebras in *i*) and, by [74, No. 24, Theorem 7.3], in *ii*) are maximal semisimple subalgebras of $\mathfrak{g}$ and they are of non-compact type, hence maximal subalgebras of non-compact type. □

7.3 Maximal semisimple totally geodesic submanifolds

We prove below a theorem that establishes a one-to-one correspondence between maximal semisimple totally geodesic submanifolds in symmetric spaces of non-compact type and certain subalgebras of the Lie algebra of the isometry group of the ambient space.

A result of Alekseevsky and Di Scala [4, Proposition 5.5] states that in a symmetric space of non-compact type $M = \mathsf{G}/\mathsf{K}$, if the action of a subgroup of G has two totally geodesic orbits, they are isomorphic as homogeneous spaces. Improving on this result, we prove the following statement which provides uniqueness up to congruence for the existence statement in Theorem 7.1.1. However, we note that, in this result, the group acting upon is not assumed to be semisimple.

Proposition 7.3.1. *Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of non-compact type and $\mathsf{H} \subset \mathsf{G}$ be a connected Lie subgroup. Then all totally geodesic H-orbits are congruent in M.*

Proof. Let $\mathsf{H} \cdot p$ and $\mathsf{H} \cdot q$ be two totally geodesic orbits of the action of H on M. By [4, Proposition 5.5], we have that $\mathsf{H} \cdot p$ and $\mathsf{H} \cdot q$ have the same dimension.

Let us suppose that $\mathsf{H} \cdot p$ is a point. Then, by the homogeneity of M, we have that $\mathsf{H} \cdot p$ and $\mathsf{H} \cdot q$ are congruent.

Now, let us assume that $\dim(\mathsf{H} \cdot p) > 0$. By [4, Proposition 5.5] there exists an H-invariant totally geodesic submanifold N of M isometric to a Riemannian product $N = \mathsf{H} \cdot p \times \mathbb{R}$ such that $\mathsf{H} \cdot p, \mathsf{H} \cdot q \subset N$. Moreover, let γ be the geodesic of N starting at $p \in \mathsf{H} \cdot p$ whose initial velocity $\dot{\gamma}(0)$ lies in the orthogonal complement of $T_p(\mathsf{H} \cdot p)$ in $T_p N$. This is also a geodesic in M since N is totally geodesic. Furthermore, γ hits $\mathsf{H} \cdot q$ at some point $q' \in M$, since $\exp\colon \nu(\mathsf{H} \cdot p) \to M$ is a diffeomorphism, see [115, Proposition 3.5 *i)*]. We may assume $q' = \gamma(1)$. Let $x = \gamma(\frac{1}{2}) \in N$. Then s_x, the geodesic reflection of M at x, maps $p \in N$ to $q' \in \mathsf{H} \cdot q$ and preserves N since N is a totally geodesic submanifold in M containing x. Also, its differential sends $v \in T_p(\mathsf{H} \cdot p)$ to $s_{x*} v \in T_{q'} N$ and

$$0 = \langle v, \dot{\gamma}(0) \rangle = \langle s_{x*} v, s_{x*} \dot{\gamma}(0) \rangle = -\langle s_{x*} v, \dot{\gamma}(1) \rangle.$$

This implies that $s_x(\mathsf{H} \cdot p)$ is a totally geodesic submanifold of N passing through $q' \in N$ and orthogonal to γ. The tangent space to any orbit of H is generated by Killing vector fields induced by H. Let X be a Killing vector field induced by the action of H. Then ∇X is skew-symmetric, where ∇ stands for the Levi-Civita connection of N (recall that N is H-invariant). Thus $\langle \nabla_{\dot{\gamma}} X, \dot{\gamma} \rangle = 0$, which implies that $\langle X, \dot{\gamma} \rangle$ is constant. Since γ is orthogonal to $\mathsf{H} \cdot p$ at p, it follows that γ is also orthogonal to $\mathsf{H} \cdot q$ at q'. Now we have shown that $s_x(\mathsf{H} \cdot p)$ and $\mathsf{H} \cdot q$, which both have codimension one in N, are totally geodesic submanifolds of N passing through q' with the same tangent space. This shows that $\mathsf{H} \cdot p$ and $\mathsf{H} \cdot q$ are congruent in M via s_x. ☐

Remark 7.3.2. Notice that the above result works only for symmetric spaces of non-compact type. For instance, the standard action of SO_2 on S^2 has two different isometry classes of totally geodesic orbits, namely, the equator and the poles.

Let $M = \mathsf{G}/\mathsf{K}$ be an irreducible symmetric space of non-compact type, where G is the connected component of the identity of the isometry group of M and K is the isotropy subgroup G_o at $o \in M$. Let Σ be a (complete) totally geodesic submanifold of M passing through $o \in M$.

As it can be deduced from the discussion in [22, §2], if Σ is semisimple, then the Lie algebra $\mathfrak{g}_\Sigma := [\mathfrak{p}_\Sigma, \mathfrak{p}_\Sigma] \oplus \mathfrak{p}_\Sigma$, where $\mathfrak{p}_\Sigma = T_o\Sigma$, is isomorphic to the Lie algebra of the isometry group of Σ, and $\mathfrak{g}_\Sigma$ is a semisimple Lie algebra in this case. Recall that we say that a semisimple Lie algebra is of non-compact type if each of its simple ideals is non-compact. If $\mathfrak{h}$ is a Lie algebra of non-compact type, then there is some symmetric space $\Sigma_\mathfrak{h}$ of non-compact type such that the Lie algebra of its isometry group is $\mathfrak{h}$. In particular, if we consider a Cartan decomposition $\mathfrak{h} = \mathfrak{k}_\mathfrak{h} \oplus \mathfrak{p}_\mathfrak{h}$, we have $\mathfrak{k}_\mathfrak{h} = [\mathfrak{p}_\mathfrak{h}, \mathfrak{p}_\mathfrak{h}]$.

Our ultimate aim would be to to classify maximal totally geodesic submanifolds in M. Note that Berndt and Olmos classified in [20] the maximal totally geodesic submanifolds of M that are non-semisimple. So it remains to find those maximal totally geodesic submanifolds of M that are semisimple.

Our approach will consist in classifying first maximal semisimple totally geodesic submanifolds (i.e. the totally geodesic submanifolds that are maximal among the semisimple ones) and then discarding those that are contained in a non-semisimple totally geodesic submanifold. As announced, we will be able to carry out these tasks for the exceptional symmetric spaces, although many of the results in this chapter hold in more generality.

We now prove a theorem that establishes a correspondence between maximal semisimple totally geodesic submanifolds of M and subalgebras that are maximal among subalgebras of non-compact type of $\mathfrak{g}$.

Theorem 7.3.3 (Correspondence Theorem). *Let $M = \mathsf{G}/\mathsf{K}$ be an irreducible symmetric space of non-compact type and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra that is maximal among the subalgebras of non-compact type of $\mathfrak{g}$. Then there is some $p \in M$ such that $\Sigma = \mathsf{H} \cdot p$ is a maximal semisimple totally geodesic submanifold of M, where $\mathsf{H} \subset \mathsf{G}$ is the connected subgroup of G with Lie algebra $\mathfrak{h}$.*

Conversely, if Σ is a maximal semisimple totally geodesic submanifold of M, then there is a subalgebra $\mathfrak{g}_\Sigma$ of $\mathfrak{g}$ that is maximal among subalgebras of non-compact type of $\mathfrak{g}$ such that $\mathsf{G}_\Sigma \cdot p = \Sigma$ for some $p \in M$, where G_Σ is the connected subgroup of G with Lie algebra $\mathfrak{g}_\Sigma$.

Proof. Let $\mathfrak{h}$ be a maximal subalgebra of non-compact type of $\mathfrak{g}$ and H be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. By Theorem 7.1.2, we may assume that $\mathfrak{h}$ is canonically embedded with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Clearly, $\Sigma := \mathsf{H} \cdot o$ is then a semisimple totally geodesic submanifold of M and we claim that it is maximal among semisimple totally geodesic submanifolds. Let us assume that there is a semisimple totally geodesic submanifold $\widetilde{\Sigma}$ such that $\Sigma \subset \widetilde{\Sigma} \subset M$. We may assume $o \in \widetilde{\Sigma}$. Then $\widetilde{\Sigma}$ is of non-compact type. Let $\mathfrak{h} = \mathfrak{k}_\mathfrak{h} \oplus \mathfrak{p}_\mathfrak{h}$ be the Cartan decomposition of $\mathfrak{h}$, where $\mathfrak{k}_\mathfrak{h} = \mathfrak{k} \cap \mathfrak{h}$ and $\mathfrak{p}_\mathfrak{h} = \mathfrak{p} \cap \mathfrak{h}$. It follows from our assumption that there is a Lie triple system $\mathfrak{p}_{\widetilde{\Sigma}}$ such that $\mathfrak{p}_\mathfrak{h} \subset \mathfrak{p}_{\widetilde{\Sigma}} \subset \mathfrak{p}$. Thus, $\mathfrak{k}_\mathfrak{h} = [\mathfrak{p}_\mathfrak{h}, \mathfrak{p}_\mathfrak{h}] \subset [\mathfrak{p}_{\widetilde{\Sigma}}, \mathfrak{p}_{\widetilde{\Sigma}}] =: \mathfrak{k}_{\widetilde{\Sigma}}$ and $\mathfrak{h} \subset \mathfrak{g}_{\widetilde{\Sigma}} := \mathfrak{k}_{\widetilde{\Sigma}} \oplus \mathfrak{p}_{\widetilde{\Sigma}}$. However, $\mathfrak{g}_{\widetilde{\Sigma}}$ is of non-compact type, since $\widetilde{\Sigma}$ is of non-compact type, and $\mathfrak{h}$ is maximal among subalgebras of $\mathfrak{g}$ of non-compact type. It follows that $\mathfrak{h} = \mathfrak{g}_{\widetilde{\Sigma}}$ and $\widetilde{\Sigma} = \Sigma$.

Let Σ be a maximal semisimple totally geodesic submanifold passing through $o \in M$ and let $\mathfrak{p}_\Sigma$ be the tangent space to Σ at o. Consider $\mathfrak{g}_\Sigma := [\mathfrak{p}_\Sigma, \mathfrak{p}_\Sigma] \oplus \mathfrak{p}_\Sigma$. Clearly, $\mathfrak{g}_\Sigma$ is of non-compact type since it is isomorphic to the Lie algebra of the isometry group of Σ. We will prove that $\mathfrak{g}_\Sigma$ is maximal among subalgebras of non-compact type of $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ of non-compact type such that $\mathfrak{g}_\Sigma \subset \mathfrak{h} \subset \mathfrak{g}$. We prove that $\mathfrak{g}_\Sigma = \mathfrak{h}$. We apply Theorem 7.1.2 to $\mathfrak{g}_\Sigma \subset \mathfrak{h}$ to find a Cartan decomposition for $\mathfrak{h}$ such that

$$\mathfrak{h} = \mathfrak{k}_\mathfrak{h} \oplus \mathfrak{p}_\mathfrak{h} \qquad \text{satisfying} \qquad \mathfrak{k}_\Sigma := [\mathfrak{p}_\Sigma, \mathfrak{p}_\Sigma] \subset \mathfrak{k}_\mathfrak{h}, \qquad \mathfrak{p}_\Sigma \subset \mathfrak{p}_\mathfrak{h}.$$

Applying Theorem 7.1.2 once more to $\mathfrak{h} \subset \mathfrak{g}$, we find a Cartan decomposition for $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{p}' \qquad \text{satisfying} \qquad \mathfrak{k}_\Sigma \subset \mathfrak{k}_\mathfrak{h} \subset \mathfrak{k}', \qquad \mathfrak{p}_\Sigma \subset \mathfrak{p}_\mathfrak{h} \subset \mathfrak{p}'.$$

Now since any two Cartan involutions in a real semisimple Lie algebra differ by an inner automorphism, there is some $g \in \mathsf{G}$ such that $\mathfrak{p}' = \mathrm{Ad}(g)\mathfrak{p}$. Thus $\mathrm{Ad}(g^{-1})\mathfrak{p}_\Sigma$ and $\mathrm{Ad}(g^{-1})\mathfrak{p}_\mathfrak{h}$ are Lie triple systems in $\mathfrak{p}$ since $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Ad}(g^{-1})\mathfrak{p}_\Sigma \subset \mathrm{Ad}(g^{-1})\mathfrak{p}_\mathfrak{h} \subset \mathfrak{p}$. Let H and G_Σ be the Lie subgroups of G with Lie algebras $\mathfrak{h}$ and $\mathfrak{g}_\Sigma$, respectively. If we consider $p := g \cdot o \in M$, we have that $\mathsf{G}_\Sigma \cdot p$ and $\mathsf{H} \cdot p$ are totally geodesic submanifolds in M, since $g^{-1}\mathsf{G}_\Sigma g \cdot o$ and $g^{-1}\mathsf{H}g \cdot o$ are totally geodesic (as $\mathrm{Ad}(g^{-1})\mathfrak{p}_\Sigma$ and $\mathrm{Ad}(g^{-1})\mathfrak{p}_\mathfrak{h}$ are Lie triple systems in $\mathfrak{p}$). Furthermore, $\mathsf{G}_\Sigma \cdot p$ is contained in $\mathsf{H} \cdot p$. However, $\Sigma = \mathsf{G}_\Sigma \cdot o$ is maximal among the semisimple totally geodesic submanifolds passing through $o \in M$. Hence, by Proposition 7.3.1, $\mathsf{G}_\Sigma \cdot p$ is maximal among the semisimple totally geodesic submanifolds passing through $p \in M$. Then $\mathsf{G}_\Sigma \cdot p = \mathsf{H} \cdot p$. Therefore, $\mathfrak{p}_\mathfrak{h} = \mathfrak{p}_\Sigma$ and consequently $\mathfrak{g}_\Sigma = \mathfrak{h}$, since $\mathfrak{g}_\Sigma$ and $\mathfrak{h}$ are of non-compact type. □

7.4 Dynkin index and totally geodesic submanifolds

In this section, we extend the definition of the Dynkin index of a simple subalgebra of a simple complex Lie algebra to certain classes of semisimple subalgebras of simple real Lie algebras, and we use it to characterize isometry classes of totally geodesic submanifolds.

Recall that we denote by $\mathcal{B}_\mathfrak{g}$ the Killing form of a Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of roots with respect to $\mathfrak{a}$. It is shown in [90, Chapter III, Theorem 4.2] that the restriction of $\mathcal{B}_\mathfrak{g}$ to $\mathfrak{a}$ is non-degenerate, and that we hence may identify each root $\alpha \in \Delta$ with a vector $H_\alpha \in \mathfrak{g}$ such that $\alpha(H) = \mathcal{B}_\mathfrak{g}(H, H_\alpha)$ for all $H \in \mathfrak{a}$. It follows from [90, Chapter III, Theorem 4.4(i)] that the numbers $q_\alpha := \mathcal{B}_\mathfrak{g}(H_\alpha, H_\alpha)$ are positive for all $\alpha \in \Delta$. Let $q := \max\{q_\alpha : \alpha \in \Delta\}$ and define the bilinear form $Q_\mathfrak{g}$ on $\mathfrak{g}$ by

$$Q_\mathfrak{g} := \frac{2}{q}\mathcal{B}_\mathfrak{g},$$

i.e. $Q_\mathfrak{g}$ is the multiple of the Killing form normalized so that the square of the length of the longest root equals 2. Let us recall from [74, §2] Dynkin's definition of the index of a simple subalgebra of a simple complex Lie algebra. Let $\mathfrak{h}$ and $\mathfrak{g}$ be simple complex Lie algebras and let $f\colon \mathfrak{h} \to \mathfrak{g}$ be a Lie algebra monomorphism. By Schur's Lemma, the number $\mathrm{ind}_D(f)$, given by

$$\mathrm{ind}_D(f) \cdot Q_{\mathfrak{h}}(X,Y) = Q_{\mathfrak{g}}(f(X), f(Y)) \quad \text{for all } X, Y \in \mathfrak{h}, \tag{7.1}$$

is well defined. It is called the *Dynkin index* of the subalgebra $f(\mathfrak{h})$ in $\mathfrak{g}$. Dynkin proved in [74, Theorem 2.2] that it is a positive integer. If $\mathfrak{h} \subset \mathfrak{g}$ is a complex subalgebra of the complex Lie algebra $\mathfrak{g}$, we will write $\mathrm{ind}_D(\mathfrak{h}, \mathfrak{g})$, or $\mathrm{ind}_D(\mathfrak{h})$ when the embedding is clear from the context, for the Dynkin index of the inclusion map.

Now let us mention the multiplicative property of the Dynkin index, see [74, §2, No. 7]. Observe that given the inclusions $\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \mathfrak{g}$ of complex simple Lie algebras, then

$$\mathrm{ind}_D(\mathfrak{h}_1, \mathfrak{g}) = \mathrm{ind}_D(\mathfrak{h}_1, \mathfrak{h}_2) \cdot \mathrm{ind}_D(\mathfrak{h}_2, \mathfrak{g}).$$

We would like to extend the definition of the Dynkin index to (semi)simple subalgebras of simple real Lie algebras.

Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$ be a semisimple complex subalgebra of the simple complex Lie algebra $\mathfrak{g}$, where $\mathfrak{h}_s$ is a simple complex ideal for every $s \in \{1, \ldots, n\}$. We define

$$\mathrm{ind}_D(\mathfrak{h}) := (\mathrm{ind}_D(\mathfrak{h}_1), \ldots, \mathrm{ind}_D(\mathfrak{h}_n)).$$

For an absolutely simple real Lie algebra $\mathfrak{g}$, we define the Dynkin index of a semisimple subalgebra $\mathfrak{h}$ as the Dynkin index of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. This is well defined since $\mathfrak{h}_{\mathbb{C}}$ is semisimple, see the proof of Lemma 7.2.3. In order to define also the Dynkin index for a semisimple subalgebra of a simple real Lie algebra that is not absolutely simple, i.e. whose complexification is not simple, we prove the following.

Lemma 7.4.1. *Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{h}$ be a semisimple subalgebra of the realification $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$. Then $\mathfrak{h} + i\mathfrak{h}$ is a semisimple subalgebra of $\mathfrak{g}$.*

Proof. Notice that $\mathfrak{h} + i\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \cap i\mathfrak{h}$ is an ideal of $\mathfrak{h} + i\mathfrak{h}$, see [73, Appendix to Chapter 1, Lemma 1.1]. Hence $\mathfrak{h} \cap i\mathfrak{h}$ is also an ideal of the subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$, where the $\mathfrak{h}_j$ are the simple ideals of $\mathfrak{h}$. Since $\mathfrak{h}_j$ is simple, we have either $\mathfrak{h}_j \cap i\mathfrak{h}_j = \mathfrak{h}_j$, and then $\mathfrak{h}_j + i\mathfrak{h}_j = \mathfrak{h}_j$ is simple, or $\mathfrak{h} \cap i\mathfrak{h} = 0$, in which case $\mathfrak{h}_j$ is a simple real form of the complex Lie algebra $\mathfrak{h}_j + i\mathfrak{h}_j$. Note also that $i\mathfrak{h}_j \cap \mathfrak{h}_k = 0$ for $j \neq k$. Indeed, we have $[i\mathfrak{h}_j \cap \mathfrak{h}_k, \mathfrak{h}_k] \subset i[\mathfrak{h}_j, \mathfrak{h}_k] = 0$ for $j \neq k$. Thus we may assume, by renumbering the $\mathfrak{h}_j$, if necessary, that there is a $k \in \{0, \ldots, n\}$ such that multiplication by i maps $\mathfrak{h}_j$ to itself if and only if $j < k$. It follows that

$$\mathfrak{h} + i\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_{k-1} \oplus (\mathfrak{h}_k + i\mathfrak{h}_k) \oplus \cdots \oplus (\mathfrak{h}_n + i\mathfrak{h}_n).$$

Hence $\mathfrak{h} + i\mathfrak{h}$ is semisimple. □

Definition 7.4.2. *Let $\mathfrak{g}$ be a simple real Lie algebra and let $\mathfrak{h}$ be a semisimple subalgebra of $\mathfrak{g}$.*

i) If $\mathfrak{g}$ is absolutely simple, then we define

$$\operatorname{ind}_D(\mathfrak{h},\mathfrak{g}) := \operatorname{ind}_D(\mathfrak{h}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}}).$$

ii) If $\mathfrak{g} = \mathfrak{l}_{\mathbb{R}}$ is the realification of a simple complex Lie algebra $\mathfrak{l}$ and $\mathfrak{h}$ is a complex subalgebra or a real form of $\mathfrak{l}$, then we define

$$\operatorname{ind}_D(\mathfrak{h},\mathfrak{g}) := \operatorname{ind}_D(\mathfrak{h}+i\mathfrak{h},\mathfrak{l}).$$

Furthermore, let $\mathfrak{h}$ and $\mathfrak{h}'$ be two semisimple subalgebras of $\mathfrak{g}$ for which the Dynkin index is defined. We say that $\mathfrak{h}$ and $\mathfrak{h}'$ are isometric *if there is a Lie algebra isomorphism $\varphi\colon \mathfrak{h}\to\mathfrak{h}'$ such that each simple ideal of $\mathfrak{h}$ is mapped onto a simple ideal of $\mathfrak{h}'$ of the same Dynkin index.*

Remark 7.4.3. Note that, with this definition, for a simple subalgebra $\mathfrak{h}$ of an absolutely simple real Lie algebra $\mathfrak{g}$ the Dynkin index of the subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ is either a natural number (in case $\mathfrak{h}$ is absolutely simple) or, by the following lemma, a pair of natural numbers (in case $\mathfrak{h}$ is not absolutely simple).

Lemma 7.4.4. *Let $\mathfrak{g}$ be an absolutely simple real Lie algebra and let $\mathfrak{h}$ be a simple, but not absolutely simple subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_1\oplus\mathfrak{h}_2$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two isomorphic simple subalgebras of $\mathfrak{g}_{\mathbb{C}}$ of equal Dynkin index.*

Proof. We may assume that $\mathfrak{h}$ is canonically embedded with respect to a Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ by Theorem 7.1.2. Let θ be the corresponding Cartan involution. Since $\mathfrak{h}$ is simple, but not absolutely simple, it is isomorphic to the realification of a simple complex Lie algebra and since $\mathfrak{h}$ is canonically embedded with respect to the above Cartan decomposition, it follows that $\mathfrak{h}\cap\mathfrak{k}$ is a compact real form of this simple complex Lie algebra. In particular, θ, restricted to $\mathfrak{h}$, is a non-trivial involutive automorphism of $\mathfrak{h}$.

It is well known that the complexification of $\mathfrak{f}_{\mathbb{R}}$, where $\mathfrak{f}$ is a complex Lie algebra, is isomorphic to $\mathfrak{f}\oplus\bar{\mathfrak{f}}$, where $\bar{\mathfrak{f}}$ denotes the complex conjugate Lie algebra of $\mathfrak{f}$, see e.g. [150, §2, Proposition 3]. Note that the complex Lie algebras that have a real form are isomorphic to their complex conjugates via an antilinear map. This shows that $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_1\oplus\mathfrak{h}_2$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are isomorphic.

Let $\tau\colon \mathfrak{g}_{\mathbb{C}}\to\mathfrak{g}_{\mathbb{C}}$ be the map defined by $\tau(X+iY) = X-iY$ for $X,Y\in\mathfrak{g}$. It is straightforward to check that τ is an automorphism of $\mathfrak{g}_{\mathbb{R}}$. The map τ obviously leaves $\mathfrak{h}_{\mathbb{C}}$ invariant and it acts on $\mathfrak{h}_{\mathbb{C}}$ as an automorphism of the real Lie algebra $(\mathfrak{h}_{\mathbb{C}})_{\mathbb{R}}$. Since $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are isomorphic simple ideals of $(\mathfrak{h}_{\mathbb{C}})_{\mathbb{R}}$, we have either $\tau(\mathfrak{h}_1) = \mathfrak{h}_1$ or $\tau(\mathfrak{h}_1) = \mathfrak{h}_2$. Assume we are in the former case. Then we also have $\tau(\mathfrak{h}_2) = \mathfrak{h}_2$. The fixed point set of the involution τ on $\mathfrak{h}_{\mathbb{C}}$ is the direct sum $\mathfrak{k}_1\oplus\mathfrak{k}_2$, where $\mathfrak{k}_j$ is a proper subalgebra of $\mathfrak{h}_j$ for $j=1,2$. However, the fixed point set of the action of τ on $\mathfrak{h}_{\mathbb{C}}$ coincides with the simple Lie algebra $\mathfrak{h}$ and we have arrived at a contradiction.

We have shown that $\tau(\mathfrak{h}_1) = \mathfrak{h}_2$. Since $\mathfrak{h}_1$ is simple, there are vectors $X,Y\in\mathfrak{h}_1$ such that $\mathcal{B}_{\mathfrak{h}_1}(X,Y)\neq 0$ and we have by [150, § 2, Proposition 2(ii)]

$$\operatorname{ind}_D(\mathfrak{h}_1,\mathfrak{g}) = \frac{Q_{\mathfrak{g}}(X,Y)}{Q_{\mathfrak{h}_1}(X,Y)} = \frac{\overline{Q_{\mathfrak{g}}(\tau(X),\tau(Y))}}{\overline{Q_{\mathfrak{h}_2}(\tau(X),\tau(Y))}} = \overline{\operatorname{ind}_D(\mathfrak{h}_2,\mathfrak{g})}.$$

Since $\mathrm{ind}_D(\mathfrak{h}_2, \mathfrak{g})$ is a natural number, it follows that the subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ have the same Dynkin index. □

Hence, we have defined the Dynkin index for every semisimple subalgebra $\mathfrak{h}$ of an absolutely simple real Lie algebra $\mathfrak{g}$. Also, we have defined the Dynkin index for every semisimple complex subalgebra $\mathfrak{h}$ of the realification of a complex simple Lie algebra $\mathfrak{g}$ and for its real forms. In both cases we denote it by $\mathrm{ind}_D(\mathfrak{h}, \mathfrak{g})$, or $\mathrm{ind}_D(\mathfrak{h})$, when the embedding is clear from the context. It immediately follows from this definition that the Dynkin index is one if $\mathfrak{h}$ is a real form of the simple complex Lie algebra $\mathfrak{g}$. Indeed, the following result shows that isometric real forms of $\mathfrak{g}$ induce congruent, and hence isometric, totally geodesic submanifolds of M.

Lemma 7.4.5. *Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space, where G is a simple complex Lie group. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be isomorphic real forms of a complex simple Lie algebra $\mathfrak{g}$. Then, the totally geodesic orbits of H and H' in M are all congruent in M, where H and H' are the connected subgroups of G with Lie algebras $\mathfrak{h}$ and $\mathfrak{h}'$, respectively.*

Proof. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be isomorphic real forms in $\mathfrak{g}$. Then, there is a Lie algebra isomorphism $f\colon \mathfrak{h} \to \mathfrak{h}'$. By complexifying, this map extends to a Lie algebra automorphism $\widetilde{f}$ of $\mathfrak{g}$. Let us fix some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We can assume without loss of generality that $\mathsf{H} \cdot o$ is a totally geodesic submanifold in M. This implies that there is a Cartan decomposition $\mathfrak{h} = \mathfrak{p}_{\mathfrak{h}} \oplus \mathfrak{k}_{\mathfrak{h}}$ such that $\mathfrak{p}_{\mathfrak{h}} \subset \mathfrak{p}$ and $\mathfrak{k}_{\mathfrak{h}} \subset \mathfrak{k}$. By Theorem 7.1.2, there is a Cartan decomposition $\mathfrak{p}_{\mathfrak{h}'} \oplus \mathfrak{k}_{\mathfrak{h}'} = \mathfrak{h}'$ such that $\widetilde{f}(\mathfrak{p}_{\mathfrak{h}}) = \mathfrak{p}_{\mathfrak{h}'}$ and $\widetilde{f}(\mathfrak{k}_{\mathfrak{h}}) = \mathfrak{k}_{\mathfrak{h}'}$.

Furthermore, two Cartan decompositions of $\mathfrak{g}$ are conjugate in $\mathrm{Int}(\mathfrak{g})$. Hence there is some $g \in \mathsf{G}$ such that $\varphi := \mathrm{Ad}(g) \circ \widetilde{f} \in \mathrm{Aut}(\mathfrak{g})$, $\varphi(\mathfrak{p}) = \mathfrak{p}$ and $\varphi(\mathfrak{h}) \subset \mathfrak{g}$ is a real form of $\mathfrak{g}$ conjugate to $\mathfrak{h}'$ in $\mathfrak{g}$. Furthermore, φ preserves the curvature tensor of M at o since this is given by Lie brackets. Hence, φ is a linear isometry of $\mathfrak{p}$ that preserves sectional curvature at o. Thus, by [189, Corollary 2.3.14], φ extends to an isometry $k \in \mathrm{Isom}(M)$ that fixes $o \in M$, since it leaves $\mathfrak{p}$ invariant. Thus, we have that $k(\mathsf{H} \cdot o) = g\mathsf{H}'g^{-1} \cdot q$, for certain $q \in M$, which implies that there is a totally geodesic orbit of H which is congruent to a totally geodesic orbit of H'. Consequently, by Proposition 7.3.1, the totally geodesic orbits of H and H' are all congruent in M. □

We have defined the Dynkin indices of certain semisimple subalgebras of simple real Lie algebras in such a way that isometry classes of subalgebras of the isometry algebra of an irreducible symmetric space of non-compact type correspond to isometry classes of totally geodesic submanifolds. This is the content of the following theorem.

Theorem 7.4.6. *Let $M = \mathsf{G}/\mathsf{K}$ be an irreducible symmetric space of non-compact type. Let Σ_1, Σ_2 be two semisimple totally geodesic submanifolds containing $o \in M$. For $j = 1, 2$ let $\mathfrak{p}_j = T_o\Sigma_j$ and let $\mathfrak{g}_{\Sigma_j} = [\mathfrak{p}_j, \mathfrak{p}_j] \oplus \mathfrak{p}_j \subset \mathfrak{g}$. Assume that one of the following holds:*

i) *$\mathfrak{g}$ is absolutely simple.*

ii) *$\mathfrak{g}$ is complex and $\mathfrak{g}_{\Sigma_j}$ is a complex subalgebra or a real form of $\mathfrak{g}$ for each $j \in \{1, 2\}$.*

Then Σ_1 *and* Σ_2 *are isometric if and only if* $\mathfrak{g}_{\Sigma_1}$ *and* $\mathfrak{g}_{\Sigma_2}$ *are isometric subalgebras of* $\mathfrak{g}$.

Proof. It suffices to prove the statement in the case when the $\mathfrak{g}_{\Sigma_j}$, $j = 1, 2$, are simple. Let us consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since any two Cartan involutions of $\mathfrak{g}$ differ by an inner automorphism of $\mathfrak{g}$, we may assume by Theorem 7.1.2 that $\mathfrak{g}_{\Sigma_1}$ and $\mathfrak{g}_{\Sigma_2}$ are both canonically embedded into $\mathfrak{g}$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then we have $\Sigma_j = \exp_o(\mathfrak{g}_{\Sigma_j} \cap \mathfrak{p})$, where $\exp_o$ denotes the Riemannian exponential map of M at the point o.

If we assume that Σ_1 and Σ_2 are isometric, it follows that there is an isomorphism $\varphi\colon \mathfrak{g}_{\Sigma_1} \to \mathfrak{g}_{\Sigma_2}$. Conversely, if $\mathfrak{g}_{\Sigma_1}$ and $\mathfrak{g}_{\Sigma_2}$ are isometric subalgebras of $\mathfrak{g}$, they are isomorphic by definition.

Since M, Σ_1 and Σ_2 are irreducible symmetric spaces, their invariant Riemannian metrics are unique up to scaling by a constant factor. Therefore, we may assume that M is endowed with the G-invariant metric induced by the Killing form of $\mathfrak{g}$ and we know that the invariant Riemannian metrics induced on Σ_j are obtained from the metrics induced by the Killing form of $\mathfrak{g}_{\Sigma_j}$ by applying a constant scaling factor c_j to this metric. Thus it remains to show that $c_1 = c_2$ if and only if the Dynkin indices of the subalgebras $\mathfrak{g}_{\Sigma_1}$ and $\mathfrak{g}_{\Sigma_2}$ of $\mathfrak{g}$ are equal. To prove this, we consider separately the two cases where the Lie algebra $\mathfrak{g}$ is absolutely simple and where it is not absolutely simple, since the Dynkin indices of subalgebras are defined differently in these two cases. Note that the Killing form of a real form of a complex Lie algebra is given by restricting the Killing form of its complexification, see [150, §2, Proposition 2]. On the other hand, the Killing form of the realification $\mathfrak{h}_{\mathbb{R}}$ of a complex Lie algebra $\mathfrak{h}$ is given by

$$\mathcal{B}_{\mathfrak{h}_{\mathbb{R}}}(X,Y) = 2\ \mathrm{Re}(\mathcal{B}_{\mathfrak{h}}(X,Y)).$$

First assume that $\mathfrak{g}$ is absolutely simple; then the Dynkin index of $\mathfrak{g}_{\Sigma_j}$ is defined as the Dynkin index of the subalgebra $(\mathfrak{g}_{\Sigma_j})_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. In this case, the Lie algebras $\mathfrak{g}$, $\mathfrak{g}_{\Sigma_1}$, $\mathfrak{g}_{\Sigma_2}$ are real forms of the Lie algebras $\mathfrak{g}_{\mathbb{C}}$, $(\mathfrak{g}_{\Sigma_1})_{\mathbb{C}}$, $(\mathfrak{g}_{\Sigma_2})_{\mathbb{C}}$ and hence their Killing forms are given by restricting the Killing forms of their complexifications. When $\mathfrak{g}_{\Sigma_1}$ and $\mathfrak{g}_{\Sigma_2}$ have simple complexifications, it is immediately clear that $\mathrm{ind}_D(\mathfrak{g}_{\Sigma_1}) = \mathrm{ind}_D(\mathfrak{g}_{\Sigma_2})$ if and only if $c_1 = c_2$. In case the complexifications are not simple, we know by Lemma 7.4.4 that they both are a direct sum of two isometric subalgebras of $\mathfrak{g}_{\mathbb{C}}$, and it follows that $c_1 = c_2$ if and only if their Dynkin indices agree.

Now assume that $\mathfrak{g}$ is not absolutely simple, i.e. it is the realification of a simple complex Lie algebra. In this case, the Dynkin index of $\mathfrak{g}_{\Sigma_j}$ is defined as the Dynkin index of the subalgebra $\mathfrak{g}_{\Sigma_j} + i\mathfrak{g}_{\Sigma_j}$ of $\mathfrak{g}$, viewed as a complex Lie algebra. Now, by hypothesis, the Lie algebra $\mathfrak{g}_{\Sigma_1} \simeq \mathfrak{g}_{\Sigma_2}$ is either a complex subalgebra or a real form. In the first case, $\mathfrak{g}_{\Sigma_j}$ is a simple subalgebra of $\mathfrak{g}$, viewed as a complex Lie algebra and we conclude again that $c_1 = c_2$ if and only the $\mathrm{ind}_D(\mathfrak{g}_{\Sigma_1}) = \mathrm{ind}_D(\mathfrak{g}_{\Sigma_2})$. In the second case, the result follows from Lemma 7.4.5. □

Definition 7.4.7. *Let* $\mathfrak{g}$ *be the Lie algebra of the isometry group of an irreducible symmetric space of non-compact type. We define* $\mathcal{L}(\mathfrak{g})$ *as the set of isometry classes of maximal subalgebras of non-compact type of* $\mathfrak{g}$.

Notice that this set is well defined since maximal subalgebras of non-compact type of $\mathfrak{g}$, when $\mathfrak{g}$ is complex, are either complex subalgebras or real forms of $\mathfrak{g}$ by Lemma 7.2.4. We denote the isometry classes of subalgebras by the pairs $(\mathfrak{h}, \mathrm{ind}_D(\mathfrak{h}))$, where the first entry denotes the isomorphism class of the subalgebra and the second entry is its Dynkin index.

Corollary 7.4.8. *Let $\mathfrak{g}$ be the Lie algebra of the isometry group of an irreducible symmetric space of non-compact type M. The set $\mathcal{L}(\mathfrak{g})$ is in one-to-one correspondence with the set of isometry classes of maximal semisimple totally geodesic submanifolds in M.*

Proof. This follows by combining the Correspondence Theorem 7.3.3 and Proposition 7.4.6. □

7.5 Totally geodesic submanifolds in exceptional symmetric spaces of type III

In this section we achieve the classification of maximal totally geodesic submanifolds in $M = \mathsf{G}/\mathsf{K}$ when $\mathfrak{g}$ is the Lie algebra of the isometry group of an exceptional symmetric space M of type III.

To the best of our knowledge, no complete classification of maximal subalgebras of non-compact type is available in the literature so far. The articles of Komrakov [117] and de Graaf and Marrani [57] give lists that, when combined, provide all maximal subalgebras of non-compact type of a real semisimple exceptional Lie algebra $\mathfrak{g}$. On the one hand, the more recent paper of de Graaf and Marrani [57] gives complete lists of maximal reductive subalgebras $\mathfrak{h}$ whose complexification $\mathfrak{h}_{\mathbb{C}}$ is a maximal reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$ for absolutely simple real Lie algebras $\mathfrak{g}$ of rank less or equal than 8. On the other hand, in the article of Komrakov [117], all maximal subalgebras $\mathfrak{h}$ of non-compact simple real Lie algebras whose complexification $\mathfrak{h}_{\mathbb{C}}$ is not a maximal subalgebra of $\mathfrak{g}_{\mathbb{C}}$ are given.

We consider $\widetilde{\mathcal{L}}(\mathfrak{g})$, the result of joining these two lists, deleting the compact ideals of each subalgebra in the union of these lists, and adding the Dynkin index of the resulting subalgebras, which we take from [74, Tables 16, 17, 18, 19, 20, 25]. For the absolutely simple exceptional real Lie algebras, the set $\widetilde{\mathcal{L}}(\mathfrak{g})$ is given by Table 7.2.

Proposition 7.5.1. *Let $\mathfrak{g}$ be the Lie algebra of the isometry group of an irreducible exceptional symmetric space of non-compact type. Then, if $\mathfrak{g}$ is absolutely simple, $\mathcal{L}(\mathfrak{g}) \subset \widetilde{\mathcal{L}}(\mathfrak{g})$.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal subalgebra of non-compact type. Then, since $\mathfrak{h}$ is semisimple, $\mathfrak{h}$ is a reductive algebraic subalgebra by Theorem 7.1.2 and Theorem 7.1.4. Hence, there exists a maximal reductive algebraic subalgebra $\widetilde{\mathfrak{h}} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \widetilde{\mathfrak{h}}$. Observe that $\widetilde{\mathfrak{h}} = \widetilde{\mathfrak{h}}_1 \oplus \widetilde{\mathfrak{h}}_2$, where $\widetilde{\mathfrak{h}}_1$ is a subalgebra of non-compact type and $\widetilde{\mathfrak{h}}_2$ is a compact subalgebra.

Let us show that $\mathfrak{h} \subset \widetilde{\mathfrak{h}}_1$. Let $\widehat{\mathfrak{h}}$ be a simple ideal of $\mathfrak{h}$ and consider $f\colon \widetilde{\mathfrak{h}} \to \widetilde{\mathfrak{h}}_2$, the natural projection of $\widetilde{\mathfrak{h}}$ on $\widetilde{\mathfrak{h}}_2$. Then, f restricted to $\widehat{\mathfrak{h}}$ is injective or zero. If it is injective, $\widetilde{\mathfrak{h}}_2$ has a subalgebra of non-compact type, which leads to a contradiction. Hence, $\mathfrak{h} \subset \widetilde{\mathfrak{h}}_1$.

By definition, $\widetilde{(\mathfrak{h})}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is a reductive algebraic subalgebra. On the one hand, if $\widetilde{(\mathfrak{h})}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is not a maximal reductive subalgebra, in particular it is not maximal and it is given in [117, Theorem 1]. On the other hand, if $\widetilde{(\mathfrak{h})}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is a maximal reductive subalgebra, it is given in [57]. This shows $\mathcal{L}(\mathfrak{g}) \subset \widetilde{\mathcal{L}}(\mathfrak{g})$. □

We prove a criterion that ensures the non-maximality of certain totally geodesic orbits.

Lemma 7.5.2. *Let $M = \mathsf{G}/\mathsf{K}$ be an irreducible symmetric space of non-compact type. Let $\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{g}_1$ be a semisimple subalgebra of $\mathfrak{g}$, which is the direct sum of the ideals $\mathfrak{k}_1$ and $\mathfrak{g}_1$, where $\mathfrak{k}_1$ is compact and $\mathfrak{g}_1$ is of non-compact type. Assume $\mathfrak{h}$ is canonically embedded into $\mathfrak{g}$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the following holds:*

i) The linear subspace $\ell := \mathfrak{h} \cap \mathfrak{p}$ of $\mathfrak{p}$ is a Lie triple system and $L := \exp_o(\ell)$ is a totally geodesic orbit of the H-action on M, where $\exp_o$ *denotes the Riemannian exponential map of M at the point o.*

ii) The connected Lie subgroup K_1 of K corresponding to $\mathfrak{k}_1$ acts effectively on the normal space $N_o(\mathsf{H} \cdot o)$ to the H-orbit at o and trivially on ℓ.

iii) If K_1 contains a subgroup Q and there are vectors $v, w \in N_o(\mathsf{H} \cdot o)$ such that $\mathsf{K}_1 \cdot v \neq v$, $\mathsf{Q} \cdot v = v$, $\mathsf{Q} \cdot w \neq w$, then L is not a maximal totally geodesic submanifold of M.

iv) If K_1 is of rank greater than one or if the action of K_1 on $N_o(\mathsf{H} \cdot o)$ is effectively an SO_3-representation, then L is not a maximal totally geodesic submanifold of M.

Proof. Since $\mathfrak{h}$ is canonically embedded into $\mathfrak{g}$, we have that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \ell$ is a Cartan decomposition of $\mathfrak{h}$. This shows that ℓ is a Lie triple system in $\mathfrak{p}$ and hence its exponential image is a totally geodesic submanifold of M. This proves *i)*.

Observe that K acts effectively on M since it is a subgroup of the isometry group of M. Every isometry f of M is uniquely defined by the image $f(p)$ and the differential f_{*p} for one arbitrary point $p \in M$. In particular, the action of an element in the isotropy group $\mathsf{K} = \mathsf{G}_o$ on M is uniquely determined by the action of its differential on $\mathfrak{p} = T_oM$. Therefore, if $k \in \mathsf{K}$ acts trivially on $\mathfrak{p}$, then $k = e$. Since $\mathfrak{h}$ is canonically embedded, it is invariant under θ. Thus the restriction of θ to $\mathfrak{h}$ is a Lie algebra automorphism and it follows that θ leaves invariant $\mathfrak{k}_1$ and $\mathfrak{g}_1$, since $\mathfrak{k}_1$ is compact and $\mathfrak{g}_1$ is of non-compact type. Hence $\mathfrak{k}_1$ and $\mathfrak{g}_1$ are canonically embedded in $\mathfrak{g}$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and it follows that $\mathfrak{h} \cap \mathfrak{k} = \mathfrak{k}_1 \oplus (\mathfrak{g}_1 \cap \mathfrak{k})$ and $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{g}_1 \cap \mathfrak{p} = \ell$. In particular, the subspaces $\mathfrak{k}_1$ and ℓ of $\mathfrak{g}$ commute. Hence, the connected subgroup K_1 acts trivially on ℓ and effectively on $N_o(\mathsf{H} \cdot o)$, since K_1 is a subgroup of the isometry group of M. This proves *ii)*.

Assume there is a subgroup $\mathsf{Q} \subset \mathsf{K}_1$ as described in part *iii)* of the assertion. Let N be the connected component of the fixed point set of the Q-action on M containing o. Then N is a totally geodesic submanifold of M containing o and $N = \exp_o(F)$, where F is the set of fixed points under the action of Q on $\mathfrak{p}$. It follows that F is a Lie triple system in $\mathfrak{p}$, which contains ℓ, since K_1 acts trivially on ℓ by *ii)*. Since there are vectors $v, w \in N_o(\mathsf{H} \cdot o)$ such that $\mathsf{K}_1 \cdot v \neq v$, $\mathsf{Q} \cdot v = v$, $\mathsf{Q} \cdot w \neq w$, it follows that the dimension of F is strictly greater than the dimension of ℓ and that $F \neq \mathfrak{p}$. Hence N is a proper totally geodesic submanifold containing L and $N \neq M$. This proves *iii)*.

If the action of K_1 on $N_o(\mathsf{H} \cdot o)$ is effectively an SO_3-representation then it is a direct sum of odd-dimensional irreducible real representations of K_1 and a trivial module. Since non-trivial real representations of the abelian group SO_2 are even-dimensional, it follows that if we choose $\mathsf{Q} = \mathsf{SO}_2$, then there are vectors $v, w \in N_o(\mathsf{H} \cdot o)$ as in item *iii)* of the statement.

Let us now assume that the rank of K_1 is greater than one. First note that for the existence of a subgroup Q as described in *iii)*, it suffices that the action of K_1 on $\mathfrak{p}$ has a singular orbit that is not a fixed point of the K_1-action on $\mathfrak{p}$. Indeed, let $v \in \mathfrak{p}$ be contained in such a singular orbit and let Q be the isotropy subgroup $(\mathsf{K}_1)_v$. Then $\mathsf{K}_1 \cdot v \neq v$ and $\mathsf{Q} \cdot v = v$. If there is no vector $w \in N_o(\mathsf{H} \cdot o)$ such that $\mathsf{Q} \cdot w \neq w$, then all isotropy groups contain Q, which implies that Q is a principal isotropy group, a contradiction.

We may now assume that all singular orbits of the action of K_1 on $\mathfrak{p}$ are fixed points. This implies that, as a K_1-module, $\mathfrak{p} = V_0 \oplus V_1$ is a direct sum of a trivial module V_0 and a module V_1 on which K_1 acts in such a fashion that each non-zero vector lies in a principal orbit. Then K_1 acts on the unit sphere in V_1 in such a way that all orbits are of the same dimension. In particular, the orbits comprise a regular homogeneous foliation of the unit sphere in V_1. It follows from [130, Theorem 1.1] that if this foliation is non-trivial (i.e. the action is not transitive on the unit sphere), the orbits are either one-dimensional or three-dimensional and that K_1, which acts effectively on $\mathfrak{p}$, is isomorphic to U_1 or SU_2. Since we have excluded these cases by our hypotheses, we may from now on assume that the action is transitive on the unit sphere.

Hence K_1 is a connected Lie group acting effectively and transitively on a unit sphere. The corresponding presentations of spheres are, by [136],

$$\begin{gathered}\mathsf{S}^n = \mathsf{SO}_{n+1}/\mathsf{SO}_n, \quad \mathsf{S}^{2n+1} = \mathsf{U}_{n+1}/\mathsf{U}_n = \mathsf{SU}_{n+1}/\mathsf{SU}_n,\\ \mathsf{S}^{4n+3} = \mathsf{Sp}_{n+1}\mathsf{Sp}_1/\mathsf{Sp}_n\mathsf{Sp}_1 = \mathsf{Sp}_{n+1}\mathsf{U}_1/\mathsf{Sp}_n\mathsf{U}_1 = \mathsf{Sp}_{n+1}/\mathsf{Sp}_n,\\ \mathsf{S}^6 = \mathsf{G}_2/\mathsf{SU}_3, \quad \mathsf{S}^7 = \mathsf{Spin}_7/\mathsf{G}_2, \quad \mathsf{S}^{15} = \mathsf{Spin}_9/\mathsf{Spin}_7.\end{gathered}$$

By inspection of this list, we see that if the rank of K_1 is two or greater, we may choose Q as the isotropy subgroup of any non-zero vector $v \in \mathfrak{p}$; then $\mathsf{K}_1 \cdot v \neq v$ and $\mathsf{Q} \cdot v = v$ and Q acts non-trivially on the orthogonal complement of v in $\mathfrak{p}$, showing there is a vector $w \in \mathfrak{p}$ such that $\mathsf{Q} \cdot w \neq w$. We have completed the proof of *iv)*. □

Observe that while many subalgebras of $\widetilde{\mathcal{L}}(\mathfrak{g})$ are obtained by removing a compact ideal $\mathfrak{k}_1$ of rank greater than one, hence satisfying the hypotheses of Lemma 7.5.2 *iv)*, there are still some subalgebras of $\widetilde{\mathcal{L}}(\mathfrak{g})$ that cannot be directly treated using this lemma. In order to deal with these cases, we prove the next results.

Lemma 7.5.3. *Let $\mathfrak{g}$ be an exceptional absolutely simple non-compact real Lie algebra. We consider the following maximal subalgebras $\mathfrak{h}$ of real Lie algebras $\mathfrak{g}$ which are the non-compact parts of some subalgebras of $\mathfrak{g}$ with a rank one simple compact factor appearing*

$\mathfrak{g}_2^2$	$(\mathfrak{sl}_3(\mathbb{R}),1),(\mathfrak{su}_{1,2},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_2(\mathbb{R}),(3,1)),(\mathfrak{sl}_2(\mathbb{R}),28)$
$\mathfrak{f}_4^4$	$(\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{sl}_3(\mathbb{R}),(1,2)),(\mathfrak{su}_{1,2}\oplus\mathfrak{su}_{1,2},(1,2)),(\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{su}_3},2),$ $(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sp}_3(\mathbb{R}),(1,1)),(\mathfrak{sp}_{1,2}\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{so}_{4,5},1),(\mathfrak{sl}_2(\mathbb{R}),156),$ $(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{g}_2^2,(8,1)),(\mathfrak{g}_2^2\oplus\cancel{\mathfrak{su}_2},1)$
$\mathfrak{f}_4^{-20}$	$(\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{su}_3},2),(\mathfrak{sp}_{1,2}\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{so}_{1,8},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\cancel{\mathfrak{g}_2},8)$
$\mathfrak{e}_6^6$	$(\mathfrak{so}_{5,5}\oplus\cancel{\mathbb{R}},1),(\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{sl}_3(\mathbb{R}),(1,1,1)),(\mathfrak{su}_{1,2}\oplus\mathfrak{sl}_3(\mathbb{C}),(1,1,1)),$ $(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_6(\mathbb{R}),(1,1)),(\mathfrak{su}_6^*\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{g}_2^2,(2,1)),(\mathfrak{sp}_{2,2},1),$ $(\mathfrak{sp}_4(\mathbb{R}),1),(\mathfrak{f}_4^4,1)$
$\mathfrak{e}_6^2$	$(\mathfrak{so}_{4,6}\oplus\cancel{\mathbb{R}},1),(\mathfrak{so}_{10}^*\oplus\cancel{\mathbb{R}},1),(\mathfrak{sl}_3(\mathbb{C})\oplus\mathfrak{sl}_3(\mathbb{R}),(1,1,1)),(\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{su}_3}\oplus\cancel{\mathfrak{su}_3},1),$ $(\mathfrak{su}_{1,2}\oplus\mathfrak{su}_{1,2}\oplus\mathfrak{su}_{1,2},(1,1,1)),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{su}_{3,3},(1,1)),(\mathfrak{su}_{2,4}\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{su}_{1,2},9),$ $(\mathfrak{sl}_3(\mathbb{R}),9),(\mathfrak{g}_2^2\oplus\cancel{\mathfrak{su}_3},1),(\mathfrak{g}_2^2,3),(\mathfrak{su}_{1,2}\oplus\mathfrak{g}_2^2,(2,1)),(\mathfrak{sp}_{1,3},1),(\mathfrak{sp}_4(\mathbb{R}),1),(\mathfrak{f}_4^4,1)$
$\mathfrak{e}_6^{-14}$	$(\mathfrak{so}_{10}^*\oplus\cancel{\mathbb{R}},1),(\mathfrak{so}_{2,8}\oplus\cancel{\mathbb{R}},1),(\mathfrak{su}_{1,2}\oplus\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{su}_3},(1,1)),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{su}_{1,5},(1,1)),$ $(\mathfrak{su}_{2,4}\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{g}_2},2),(\mathfrak{sp}_{2,2},1),(\mathfrak{f}_4^{-20},1)$
$\mathfrak{e}_6^{-26}$	$(\mathfrak{so}_{1,9}\oplus\cancel{\mathbb{R}},1),(\mathfrak{sl}_3(\mathbb{C})\oplus\cancel{\mathfrak{su}_3},(1,1)),(\mathfrak{su}_6^*\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{sl}_3(\mathbb{R})\oplus\cancel{\mathfrak{g}_2},2),$ $(\mathfrak{sp}_{1,3},1),(\mathfrak{f}_4^{-20},1)$
$\mathfrak{e}_7^7$	$(\mathfrak{e}_6^6\oplus\cancel{\mathbb{R}},1),(\mathfrak{e}_6^2\oplus\cancel{\mathbb{R}},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{so}_{6,6},(1,1)),(\mathfrak{so}_{12}^*\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{sl}_8(\mathbb{R}),1),$ $(\mathfrak{su}_{4,4},1),(\mathfrak{su}_8^*,1),(\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{sl}_6(\mathbb{R}),(1,1)),(\mathfrak{su}_{1,2}\oplus\mathfrak{su}_{3,3},(1,1)),(\mathfrak{su}_{1,5}\oplus\cancel{\mathfrak{su}_3},1),$ $(\mathfrak{sl}_2(\mathbb{R}),231),(\mathfrak{sl}_2(\mathbb{R}),399),(\mathfrak{sl}_3(\mathbb{R}),21),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_2(\mathbb{R}),(15,24))$ $(\mathfrak{sl}_2(\mathbb{R})\oplus\cancel{\mathfrak{g}_2},7),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{g}_2^2,(7,2)),(\mathfrak{sp}_3(\mathbb{R})\oplus\mathfrak{g}_2^2,(1,1)),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{f}_4^4,(3,1))$
$\mathfrak{e}_7^{-25}$	$(\mathfrak{e}_6^{-26}\oplus\cancel{\mathbb{R}},1),(\mathfrak{e}_6^{-14}\oplus\cancel{\mathbb{R}},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{so}_{2,10},(1,1)),(\mathfrak{so}_{12}^*\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{su}_8^*,1),$ $(\mathfrak{su}_{2,6},1),(\mathfrak{su}_{3,3}\oplus\cancel{\mathfrak{su}_3},1),(\mathfrak{su}_{1,2}\oplus\mathfrak{su}_{1,5},(1,1)),(\mathfrak{sp}_3(\mathbb{R})\oplus\cancel{\mathfrak{g}_2},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\cancel{\mathfrak{f}_4},3),$ $(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{f}_4^{-20},(3,1))$
$\mathfrak{e}_7^{-5}$	$(\mathfrak{e}_6^2\oplus\cancel{\mathbb{R}},1),(\mathfrak{e}_6^{-14}\oplus\cancel{\mathbb{R}},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{so}_{12}^*,(1,1)),(\mathfrak{so}_{4,8}\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{su}_{4,4},1),$ $(\mathfrak{su}_{2,6},1),(\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{su}_6^*,(1,1)),(\mathfrak{su}_{1,2}\oplus\mathfrak{su}_{2,4},(1,1)),(\mathfrak{su}_{2,4}\oplus\cancel{\mathfrak{su}_3},1),$ $(\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{su}_6},1),(\mathfrak{su}_{1,2},21),(\mathfrak{sl}_2(\mathbb{R})\oplus\cancel{\mathfrak{su}_2},24),(\mathfrak{g}_2^2\oplus\cancel{\mathfrak{su}_2},2),(\mathfrak{g}_2^2\oplus\cancel{\mathfrak{sp}_3},1),$ $(\mathfrak{sp}_{1,2}\oplus\cancel{\mathfrak{g}_2},1),(\mathfrak{sp}_{1,2}\oplus\mathfrak{g}_2^2,(1,1)),(\mathfrak{f}_4^4\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{f}_4^{-20}\oplus\cancel{\mathfrak{su}_2},1)$
$\mathfrak{e}_8^8$	$(\mathfrak{su}_{1,4}\oplus\cancel{\mathfrak{su}_5},1),(\mathfrak{su}_{1,4}\oplus\mathfrak{su}_{1,4},(1,1)),(\mathfrak{su}_{2,3}\oplus\mathfrak{su}_{2,3},(1,1)),(\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{e}_6^6,(1,1)),$ $(\mathfrak{sl}_5(\mathbb{R})\oplus\mathfrak{sl}_5(\mathbb{R}),(1,1)),(\mathfrak{so}_{8,8},1),(\mathfrak{so}_{16}^*,1),(\mathfrak{su}_{1,8},1),(\mathfrak{su}_{4,5},1),(\mathfrak{sl}_9(\mathbb{R}),1),$ $(\mathfrak{e}_6^{-14}\oplus\cancel{\mathfrak{su}_3},1),(\mathfrak{su}_{1,2}\oplus\mathfrak{e}_6^2,(1,1)),(\mathfrak{e}_7^{-5}\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{e}_7^7,(1,1)),(\mathfrak{so}_{2,3},12),$ $(\mathfrak{so}_{1,4},12),(\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{su}_2},6),(\mathfrak{sl}_2(\mathbb{R})\oplus\cancel{\mathfrak{su}_3},16),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{su}_{1,2},(16,6)),$ $(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_3(\mathbb{R}),(16,6)),(\mathfrak{f}_4^{-20}\oplus\cancel{\mathfrak{g}_2},1),(\mathfrak{f}_4^4\oplus\mathfrak{g}_2^2,(1,1)),$ $(\mathfrak{sl}_2(\mathbb{R}),520),(\mathfrak{sl}_2(\mathbb{R}),760),(\mathfrak{sl}_2(\mathbb{R}),1240),(\mathfrak{g}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{R}),(1,1,8))$
$\mathfrak{e}_8^{-24}$	$(\mathfrak{su}_{1,4}\oplus\mathfrak{su}_{2,3},(1,1)),(\mathfrak{su}_{2,3}\oplus\cancel{\mathfrak{su}_5},1),(\mathfrak{so}_{4,12},1),(\mathfrak{so}_{16}^*,1),(\mathfrak{su}_{3,6},1),(\mathfrak{su}_{2,7},1),$ $(\mathfrak{e}_6^2\oplus\cancel{\mathfrak{su}_3},1),(\mathfrak{su}_{1,2}\oplus\mathfrak{e}_6^{-14},(1,1)),(\mathfrak{su}_{1,2}\oplus\cancel{\mathfrak{e}_6},1),(\mathfrak{sl}_3(\mathbb{R})\oplus\mathfrak{e}_6^{-26},(1,1)),$ $(\mathfrak{e}_7^{-5}\oplus\cancel{\mathfrak{su}_2},1),(\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{e}_7^{-25},(1,1)),(\mathfrak{sl}_3(\mathbb{R})\oplus\cancel{\mathfrak{su}_2},6),(\mathfrak{f}_4^4\oplus\cancel{\mathfrak{g}_2},1),$ $(\mathfrak{g}_2^2\oplus\cancel{\mathfrak{f}_4},1),(\mathfrak{f}_4^{-20}\oplus\mathfrak{g}_2^2,(1,1)),(\mathfrak{g}_2(\mathbb{C})\oplus\cancel{\mathfrak{su}_2},(1,1))$

Table 7.2: $\widetilde{\mathcal{L}}(\mathfrak{g})$ for each exceptional absolutely simple real Lie algebra $\mathfrak{g}$. We include the deleted compact ideals for the convenience of the reader.

in $\widetilde{\mathcal{L}}(\mathfrak{g})$*:*

$$\mathfrak{g}_2^2 \subset \mathfrak{f}_4^4; \quad \mathfrak{sl}_2(\mathbb{R}),\ \mathfrak{g}_2^2,\ \mathfrak{f}_4^4,\ \mathfrak{f}_4^{-20} \subset \mathfrak{e}_7^{-5}; \quad \mathfrak{g}_2(\mathbb{C}),\ \mathfrak{sl}_3(\mathbb{R}) \subset \mathfrak{e}_8^{-24}; \quad \mathfrak{su}_{1,2} \subset \mathfrak{e}_8^8.$$

Let G *be the simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and let* K *be a maximal compact subgroup of* G*. Assume the Lie algebra* $\mathfrak{h}$ *is canonically embedded into* $\mathfrak{g}$ *with respect to the Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$*. Let* H *be the connected closed subgroup of* G *with Lie algebra* $\mathfrak{h}$*. Then a totally geodesic orbit of the* H*-action on the symmetric space* $M = \mathsf{G}/\mathsf{K}$ *is not maximal.*

Proof. The complexifications of all the subalgebras $\mathfrak{h}$ in the statement of the lemma can be found in [74, Table 39, p. 233], where we can also read off the Dynkin indices (given as superscripts in the table) of (the complexifications of) their compact ideals (which are all isomorphic to $\mathfrak{su}_2$). The so-called characteristic representations of the $\mathfrak{su}_2$-summands, i.e. the representations $\mathrm{ad}_{\mathfrak{g}} \ominus \mathrm{ad}_{\mathfrak{su}_2}$ are given in [74, Table 21, p. 186-187]. It follows from this table that they all are direct sums of odd-dimensional irreducible $\mathfrak{su}_2$-representations and a trivial module. Hence, on the level of Lie groups, the corresponding representations are effectively SO_3-representations. Thus the subgroup K_1 as described in Lemma 7.5.2 effectively acts as SO_3 on $\mathfrak{p}$ and it follows from Lemma 7.5.2 *iv)* that a totally geodesic orbit of the H-action on M is not maximal. □

Lemma 7.5.4. *Consider the following subalgebras* $\mathfrak{h}$ *of real Lie algebras* $\mathfrak{g}$ *which are the semisimple parts of some subalgebras of* $\mathfrak{g}$ *with one dimensional center appearing in* $\widetilde{\mathcal{L}}(\mathfrak{g})$*:*

$$\begin{aligned} \mathfrak{so}_{5,5} &\subset \mathfrak{e}_6^6, \qquad & \mathfrak{so}_{9,1} &\subset \mathfrak{e}_6^{-26}, \\ \mathfrak{e}_6^6 &\subset \mathfrak{e}_7^7, \qquad & \mathfrak{e}_6^{-26} &\subset \mathfrak{e}_7^{-25}. \end{aligned}$$

Let G *be the simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and let* K *be a maximal compact subgroup of* G*. Assume the Lie algebra* $\mathfrak{h}$ *is canonically embedded into* $\mathfrak{g}$ *with respect to the Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$*. Let* H *be the connected closed subgroup of* G *with Lie algebra* $\mathfrak{h}$*. Then a totally geodesic orbit of the* H*-action on the symmetric space* $M = \mathsf{G}/\mathsf{K}$ *is not maximal.*

Proof. First note that any subalgebra of the form $\mathfrak{h} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is an abelian 1-dimensional subalgebra of $\mathfrak{g}$ is a reductive algebraic subalgebra. This is because their complexifications are maximal subalgebras of maximal rank of $\mathfrak{g}_{\mathbb{C}}$. Hence, they are algebraic by Remark 7.1.3. By observing that the root spaces of a simple complex Lie algebra are equal to the complexification of the root spaces of certain compact real form, and by using the Borel-de Siebenthal theorem, see [37], we deduce that they are regular subalgebras of $\mathfrak{g}_{\mathbb{C}}$. However, regular subalgebras are clearly canonically embedded with respect to some Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$ and then these are reductive algebraic by Theorem 7.1.4.

Now, notice that $\mathfrak{s}$ is either contained in $\mathfrak{p}$ or in $\mathfrak{k}$. Indeed, θ, the Cartan involution of $\mathfrak{g}$, when restricted to $\mathfrak{h} \oplus \mathfrak{s}$, is a Lie algebra automorphism. Hence, it maps $\mathfrak{s}$ onto $\mathfrak{s}$, since $\mathfrak{s}$ is the center of $\mathfrak{h} \oplus \mathfrak{s}$. However, a one-dimensional subspace invariant under θ must be contained either in $\mathfrak{p}$ or in $\mathfrak{k}$. Let us prove that $\mathfrak{s}$ is contained in $\mathfrak{p}$ in all four cases. We will argue by contradiction, so we assume that $\mathfrak{s} \subset \mathfrak{k}$.

Let $\mathfrak{g} = \mathfrak{e}_6^6$, then $\mathfrak{k} \simeq \mathfrak{sp}_4$. If $\mathfrak{h} \simeq \mathfrak{so}_{5,5}$, then $\mathfrak{h} \cap \mathfrak{k} \simeq \mathfrak{sp}_2 \oplus \mathfrak{sp}_2$. However, $\mathfrak{sp}_4$ does not contain a subalgebra isomorphic to $\mathfrak{sp}_2 \oplus \mathfrak{sp}_2 \oplus \mathbb{R}$, contradicting the assumption $\mathfrak{s} \subset \mathfrak{k}$.

Let $\mathfrak{g} = \mathfrak{e}_6^{-26}$, then $\mathfrak{k} \simeq \mathfrak{f}_4$. If $\mathfrak{h} \simeq \mathfrak{so}_{9,1}$, then $\mathfrak{h} \cap \mathfrak{k} \simeq \mathfrak{so}_9$. However, $\mathfrak{f}_4$ does not contain a subalgebra isomorphic to $\mathfrak{so}_9 \oplus \mathbb{R}$, which contradicts the assumption $\mathfrak{s} \subset \mathfrak{k}$.

Let $\mathfrak{g} = \mathfrak{e}_7^7$, then $\mathfrak{k} \simeq \mathfrak{su}_8$. If $\mathfrak{h} \simeq \mathfrak{e}_6^6$, then $\mathfrak{h} \cap \mathfrak{k} \simeq \mathfrak{sp}_4$. However, $\mathfrak{su}_8$ does not contain a subalgebra isomorphic to $\mathfrak{sp}_4 \oplus \mathbb{R}$, again contradicting $\mathfrak{s} \subset \mathfrak{k}$.

Finally, let $\mathfrak{g} = \mathfrak{e}_7^{-25}$, then $\mathfrak{k} \simeq \mathfrak{e}_6 \oplus \mathbb{R}$. Now, if $\mathfrak{h} \simeq \mathfrak{e}_6^{-26}$, then $\mathfrak{h} \cap \mathfrak{k} \simeq \mathfrak{f}_4$. In addition to that, observe that $\mathfrak{f}_4$ is maximal in $\mathfrak{e}_6$, therefore we have that $\mathfrak{s}$ is equal to the abelian factor of $\mathfrak{k}$. Moreover, the abelian ideal $\mathfrak{s}$ of $\mathfrak{h} \oplus \mathfrak{s}$ acts trivially on $\mathfrak{h} \cap \mathfrak{p}$. However, $\mathfrak{s}$ corresponds to the abelian factor of the isotropy of a Hermitian symmetric space. Therefore it cannot act trivially on any non-trivial subspace of $\mathfrak{p}$.

In all these cases we obtain a contradiction with our assumption $\mathfrak{s} \subset \mathfrak{k}$. Thus, $\mathfrak{s} \subset \mathfrak{p}$. Therefore, by Proposition 7.3.1, we have that every totally geodesic orbit induced by $\mathfrak{h}$ is properly contained in a totally geodesic orbit induced by $\mathfrak{h} \oplus \mathfrak{s}$. □

Theorem 7.5.5. *Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of non-compact type where G is an exceptional Lie group whose Lie algebra is absolutely simple. Let $\Sigma \subset M$ be a maximal totally geodesic submanifold. Then Σ is isometric to one of the spaces listed in Tables 7.5, 7.6, 7.7, 7.8, 7.9. Conversely, every space Σ listed in these tables can be isometrically embedded as a maximal totally geodesic submanifold of M.*

Proof. Let $\Sigma \subset M$ be a maximal totally geodesic submanifold. If Σ is a non-semisimple totally geodesic submanifold, it is one of the examples found by Berndt and Olmos [20], which we include in our tables.

Now assume that Σ is semisimple. Then, by Corollary 7.4.8, there is a corresponding Lie subalgebra $\mathfrak{g}_\Sigma$ given by $\mathcal{L}(\mathfrak{g})$ and, by Proposition 7.5.1, $\mathfrak{g}_\Sigma$ is isometric to some subalgebra appearing in $\widetilde{\mathcal{L}}(\mathfrak{g})$.

We have to decide which elements in $\widetilde{\mathcal{L}}(\mathfrak{g})$ actually give rise to maximal totally geodesic submanifolds. Let us define

$$\widehat{\mathcal{L}}(\mathfrak{g}) := \{(\mathfrak{h}, \mathrm{ind}_D(\mathfrak{h})) \in \widetilde{\mathcal{L}}(\mathfrak{g}) : \mathfrak{h}_\mathbb{C} \text{ is a maximal reductive subalgebra of } \mathfrak{g}_\mathbb{C}\}.$$

By Lemma 7.2.3 and the definition of $\widetilde{\mathcal{L}}(\mathfrak{g})$, the subalgebras given by $\widehat{\mathcal{L}}(\mathfrak{g})$ are exactly those of non-compact type in [57] and [117], i.e. not containing compact ideals. Note that since subalgebras in $\widehat{\mathcal{L}}(\mathfrak{g})$ are semisimple, it follows that they are maximal semisimple subalgebras by Lemma 7.2.3 *ii)*. By Corollary 7.4.8, every subalgebra given by $\widehat{\mathcal{L}}(\mathfrak{g})$ induces a maximal semisimple totally geodesic submanifold of M. Let us assume that $(\mathfrak{g}_\Sigma, \mathrm{ind}_D(\mathfrak{g}_\Sigma)) \in \widehat{\mathcal{L}}(\mathfrak{g})$ does not induce a maximal totally geodesic submanifold. Then $\mathfrak{g}_\Sigma$ must be contained in a reductive non-semisimple subalgebra. However, $\mathfrak{g}_\Sigma$ is a maximal reductive subalgebra by Lemma 7.2.3 *i)*, contradicting our assumption. We have shown that every subalgebra in $\widehat{\mathcal{L}}(\mathfrak{g})$ induces a maximal totally geodesic submanifold and we include those submanifolds in our tables.

As a consequence, it now remains to deal with the subalgebras in the complement

$$\mathcal{C}(\mathfrak{g}) := \widetilde{\mathcal{L}}(\mathfrak{g}) \setminus \widehat{\mathcal{L}}(\mathfrak{g}).$$

We consider each case separately. Note that every subalgebra in $\mathcal{C}(\mathfrak{g})$ is obtained from a larger subalgebra of $\mathfrak{g}$ that contains a compact ideal, see Table 7.2.

Let $\mathfrak{g} = \mathfrak{g}_2^2$ and $M = \mathsf{G}_2^2/\mathsf{SO}_4$. In this case we are done since $\mathcal{C}(\mathfrak{g}) = \emptyset$.

Let $\mathfrak{g} = \mathfrak{f}_4^4$ and $M = \mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1$. In this case $\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{g}_2^2, 1), (\mathfrak{su}_{1,2}, 2), (\mathfrak{sp}_{1,2}, 1)\}$. Moreover, observe that every subalgebra isomorphic to $\mathfrak{sp}_{1,2}$ is a maximal non-compact type subalgebra of $\mathfrak{f}_4^4$. Indeed, if $\mathfrak{h} \simeq \mathfrak{sp}_{1,2}$, then $\mathfrak{h}_{\mathbb{C}} \simeq \mathfrak{sp}_3(\mathbb{C})$. We can deduce by [74, Table 25, p. 199] that $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sp}_3(\mathbb{C})$ is the only proper semisimple subalgebra of $\mathfrak{f}_4(\mathbb{C})$ where $\mathfrak{h}_{\mathbb{C}}$ is properly contained. However, according to Table 7.2, the only real forms in $\mathfrak{g}$ for the embedding $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sp}_3(\mathbb{C}) \subset \mathfrak{f}_4(\mathbb{C})$ are $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sp}_3(\mathbb{R})$ and $\mathfrak{su}_2 \oplus \mathfrak{sp}_{1,2}$. Thus, $\mathfrak{sp}_{1,2}$ is a maximal non-compact type subalgebra of $\mathfrak{f}_4^4$. Then, the corresponding totally geodesic orbit is a maximal semisimple totally geodesic submanifold, which is maximal, since M does not have non-semisimple maximal totally geodesic submanifolds by [20]. Furthermore, by Table 7.2, a subalgebra isometric to $(\mathfrak{su}_{1,2}, 2)$ is contained in $\mathfrak{su}_3 \oplus \mathfrak{su}_{1,2}$ or $\mathfrak{su}_{1,2} \oplus \mathfrak{su}_{1,2}$. Then in the former case, by Lemma 7.5.2 *iv*), the corresponding totally geodesic orbit is not maximal, since $\operatorname{rank}(\mathfrak{su}_3) = 2$, and in the latter case it is obviously not maximal. The totally geodesic orbit corresponding to $(\mathfrak{g}_2^2, 1)$ is not maximal by Lemma 7.5.3.

Let $\mathfrak{g} = \mathfrak{f}_4^{-20}$ and $M = \mathsf{F}_4^{-20}/\mathsf{Spin}_9$. In this case

$$\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{sp}_{1,2}, 1), (\mathfrak{su}_{1,2}, 2), (\mathfrak{sl}_2(\mathbb{R}), 8)\}.$$

Consider a subalgebra isometric to $(\mathfrak{sp}_{1,2}, 1)$. Now, since its complexification cannot be embedded as a subalgebra of the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$, we have that $(\mathfrak{sp}_{1,2}, 1) \in \mathcal{L}(\mathfrak{g})$. Furthermore, by [20], there are no non-semisimple maximal totally geodesic submanifolds in M, so $(\mathfrak{sp}_{1,2}, 1)$ induces a maximal totally geodesic submanifold. On the other hand, by Table 7.2, every subalgebra isometric to $(\mathfrak{su}_{1,2}, 2)$ or $(\mathfrak{sl}_2(\mathbb{R}), 8)$ is such that it can be embedded in the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{su}_{1,2} \subset \mathfrak{su}_3 \oplus \mathfrak{su}_{1,2}, \quad \mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}_2.$$

Thus, by Lemma 7.5.2 *iv*), the corresponding totally geodesic orbits to these subalgebras are not maximal totally geodesic submanifolds.

Let $\mathfrak{g} = \mathfrak{e}_6^6$ and $M = \mathsf{E}_6^6/\mathsf{Sp}_4$. In this case $\mathcal{C}(\mathfrak{g}) := \{(\mathfrak{so}_{5,5}, 1), (\mathfrak{su}_6^*, 1)\}$. Every subalgebra isometric to $(\mathfrak{su}_6^*, 1)$ is a maximal non-compact type subalgebra of $\mathfrak{g}$ since its complexification is not contained in the complexification of any other subalgebra of $\widetilde{\mathcal{L}}(\mathfrak{g})$ except for $\mathfrak{sl}_6(\mathbb{R})$, which clearly does not contain $\mathfrak{su}_6^*$. Thus, $(\mathfrak{su}_6^*, 1)$ induces a maximal totally geodesic submanifold since the corresponding totally geodesic orbit is not contained in a non-semisimple totally geodesic submanifold in the list given by [20]. By Lemma 7.5.4, the totally geodesic orbits corresponding to $\mathfrak{so}_{5,5}$ are not maximal totally geodesic submanifolds in M.

Let $\mathfrak{g} = \mathfrak{e}_6^2$ and $M = \mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1$. In this case

$$\mathcal{C}(\mathfrak{g}) := \{(\mathfrak{so}_{4,6}, 1), (\mathfrak{so}^*_{10}, 1), (\mathfrak{su}_{1,2}, 1), (\mathfrak{su}_{2,4}, 1), (\mathfrak{g}_2^2, 1)\}.$$

By Table 7.2, we have the following embeddings into subalgebras of $\mathfrak{g}$:

$$\mathfrak{su}_{1,2} \subset \mathfrak{su}_3 \oplus \mathfrak{su}_3 \oplus \mathfrak{su}_{1,2}, \quad \mathfrak{g}_2^2 \subset \mathfrak{su}_3 \oplus \mathfrak{g}_2^2.$$

Thus, by Lemma 7.5.2 *iv*), we have that $(\mathfrak{su}_{1,2}, 1), (\mathfrak{g}_2^2, 1)$ do not induce maximal totally geodesic submanifolds. Moreover, the complexification of $\mathfrak{su}_{2,4}$ cannot be contained in the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$ except for $(\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{su}_{3,3}, (1,1))$, which clearly does not contain $\mathfrak{su}_{2,4}$. Therefore, it induces a maximal totally geodesic submanifold since there are no non-semisimple maximal totally geodesic submanifolds in M according to [20]. It follows from Table 7.2 that $(\mathfrak{so}_{4,6}, 1)$ and $(\mathfrak{so}^*_{10}, 1)$ can be properly contained in just one proper reductive subalgebra of $\mathfrak{g}$. Namely,

$$\mathfrak{so}_{4,6} \subset \mathfrak{so}_{4,6} \oplus \mathbb{R} \qquad \mathfrak{so}^*_{10} \subset \mathfrak{so}^*_{10} \oplus \mathbb{R}.$$

Thus, both subalgebras induce maximal totally geodesic submanifolds since there are no non-semisimple maximal totally geodesic submanifolds in M.

Let $\mathfrak{g} = \mathfrak{e}_6^{-14}$ and $M = \mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1$. In this case

$$\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{so}^*_{10}, 1), (\mathfrak{so}_{2,8}, 1), (\mathfrak{su}_{1,2} \oplus \mathfrak{su}_{1,2}, (1,1)), (\mathfrak{su}_{2,4}, 1), (\mathfrak{su}_{1,2}, 2)\}.$$

Moreover, $(\mathfrak{su}_{2,4}, 1)$ is such that its complexification cannot be embedded as a subalgebra of the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$. Thus, $(\mathfrak{su}_{2,4}, 1)$ induces a maximal totally geodesic submanifold, since there are no non-semisimple maximal totally geodesic submanifolds in M according to [20]. Furthermore, by Lemma 7.5.2 *iv*), we have that $(\mathfrak{su}_{1,2} \oplus \mathfrak{su}_{1,2}, (1,1))$ and $(\mathfrak{su}_{1,2}, 2)$ do not induce maximal totally geodesic submanifolds since by Table 7.2, we have the following inclusions into subalgebras of $\mathfrak{g}$:

$$\mathfrak{su}_{1,2} \oplus \mathfrak{su}_{1,2} \subset \mathfrak{su}_3 \oplus \mathfrak{su}_{1,2} \oplus \mathfrak{su}_{1,2} \qquad \mathfrak{su}_{1,2} \subset \mathfrak{su}_{1,2} \oplus \mathfrak{g}_2.$$

Furthermore, by Table 7.2, $(\mathfrak{so}_{2,8}, 1)$ and $(\mathfrak{so}^*_{10}, 1)$ can be properly contained in just one proper reductive subalgebra of $\mathfrak{g}$. Namely,

$$\mathfrak{so}_{2,8} \subset \mathfrak{so}_{2,8} \oplus \mathbb{R} \qquad \mathfrak{so}^*_{10} \subset \mathfrak{so}^*_{10} \oplus \mathbb{R}.$$

However, according to [20], there are no non-semisimple maximal totally geodesic submanifolds in M. Thus, both subalgebras induce maximal totally geodesic submanifolds.

Let $\mathfrak{g} = \mathfrak{e}_6^{-26}$ and $M = \mathsf{E}_6^{-26}/\mathsf{F}_4$. In this case

$$\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{so}_{1,9}, 1), (\mathfrak{sl}_3(\mathbb{C}), (1,1)), (\mathfrak{su}^*_6, 1), (\mathfrak{sl}_3(\mathbb{R}), 2)\}.$$

Note that $(\mathfrak{su}^*_6, 1)$ is such that its complexification cannot be embedded as a subalgebra of the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$. Thus, $(\mathfrak{su}^*_6, 1)$ induces a maximal totally geodesic submanifold, since there is just one non-semisimple maximal totally geodesic submanifold in the list given by [20], which is $\mathbb{R} \times \mathsf{SO}^0_{1,9}/\mathsf{SO}_9$, and it clearly does not contain a totally geodesic $\mathsf{SU}^*_6/\mathsf{Sp}_3$. Furthermore, by Lemma 7.5.2 *iv)*, we have that $(\mathfrak{sl}_3(\mathbb{C}), (1,1))$ and $(\mathfrak{sl}_3(\mathbb{R}), 2)$ do not induce maximal totally geodesic submanifolds, since by Table 7.2, we have the following inclusions into subalgebras of $\mathfrak{g}$:

$$\mathfrak{sl}_3(\mathbb{C}) \subset \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{su}_3, \quad \mathfrak{sl}_3(\mathbb{R}) \subset \mathfrak{sl}_3(\mathbb{R}) \oplus \mathfrak{g}_2.$$

Now by Lemma 7.5.4, $(\mathfrak{so}_{1,9}, 1)$ does not induce maximal totally geodesic submanifolds.

Let $\mathfrak{g} = \mathfrak{e}^7_7$ and $M = \mathsf{E}^7_7/\mathsf{SU}_8$. In this case

$$\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{e}^6_6, 1), (\mathfrak{e}^2_6, 1), (\mathfrak{so}^*_{12}, 1), (\mathfrak{su}_{1,5}, 1), (\mathfrak{sl}_2(\mathbb{R}), 7)\}.$$

By Table 7.2, we have the following inclusions into subalgebras of $\mathfrak{g}$:

$$\mathfrak{su}_{1,5} \subset \mathfrak{su}_3 \oplus \mathfrak{su}_{1,5}, \qquad \mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}_2.$$

Thus, by Lemma 7.5.2 *iv)*, we have that $(\mathfrak{su}_{1,5}, 1)$ and $(\mathfrak{sl}_2(\mathbb{R}), 7)$ do not induce maximal totally geodesic submanifolds. Furthermore, $(\mathfrak{e}^2_6, 1)$ is such that its complexification is not contained in the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$ except for $\mathfrak{e}^6_6$, which clearly does not contain $\mathfrak{e}^2_6$. Thus, $(\mathfrak{e}^2_6, 1)$ gives a maximal totally geodesic submanifold, since by [20], the only non-semisimple maximal totally geodesic submanifold is $\mathbb{R} \times \mathsf{E}^6_6/\mathsf{Sp}_4$, which cannot contain a totally geodesic $\mathsf{E}^2_6/\mathsf{SU}_6\mathsf{Sp}_1$. In addition to that, $(\mathfrak{so}^*_{12}, 1)$ is such that its complexification is not contained in the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$ except for $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_{6,6}$, which clearly does not contain $\mathfrak{so}^*_{12}$. Consequently, $(\mathfrak{so}^*_{12}, 1)$ induces a maximal totally geodesic submanifold in M, since its corresponding totally geodesic submanifold cannot be totally geodesically embedded in $\mathbb{R} \times \mathsf{E}^6_6/\mathsf{Sp}_4$. Finally, by Lemma 7.5.4, we have that $\mathfrak{e}^6_6$ does not induce a maximal totally geodesic submanifold.

Let $\mathfrak{g} = \mathfrak{e}^{-25}_7$ and $M = \mathsf{E}^{-25}_7/\mathsf{E}_6\mathsf{U}_1$. In this case, we have that

$$\mathcal{C}(\mathfrak{g}) := \{(\mathfrak{e}^{-26}_6, 1), (\mathfrak{e}^{-14}_6, 1), (\mathfrak{so}^*_{12}, 1), (\mathfrak{su}_{3,3}, 1), (\mathfrak{sp}_3(\mathbb{R}), 1), (\mathfrak{sl}_2(\mathbb{R}), 3)\}.$$

Notice that $\mathfrak{e}^{-14}_6$ is such that its complexification is not contained in the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$ but $\mathfrak{e}^{-26}_6$ and it is clearly not contained in this one. Moreover, by [20], the only non-semisimple maximal totally geodesic submanifold in M is $\mathbb{R} \times \mathsf{E}^{-26}_6/\mathsf{F}_4$, which cannot contain a totally geodesic $\mathsf{E}^{-14}_6/\mathsf{Spin}_{10}\mathsf{U}_1$. This implies that $(\mathfrak{e}^{-14}_6, 1)$ induces a maximal totally geodesic submanifold of M. Furthermore, by Lemma 7.5.4, $(\mathfrak{e}^{-26}_6, 1)$ cannot induce a maximal totally geodesic submanifold of M. In addition to that, $(\mathfrak{so}^*_{12}, 1)$ is such that its complexification is not contained in the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$ except for $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_{2,10}$, which clearly does not contain $(\mathfrak{so}^*_{12}, 1)$. Thus, $(\mathfrak{so}^*_{12}, 1)$ induces a maximal totally geodesic submanifold, since it is not totally geodesic embedded in $\mathbb{R} \times \mathsf{E}^{-26}_6/\mathsf{F}_4$. Now, we have by Lemma 7.5.2 *iv)* that $(\mathfrak{su}_{3,3}, 1)$, $(\mathfrak{sp}_3(\mathbb{R}), 1)$ and $(\mathfrak{sl}_2(\mathbb{R}), 3)$ cannot give totally geodesic submanifolds since by Table 7.2 they are contained in the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{f}_4, \quad \mathfrak{su}_{3,3} \subset \mathfrak{su}_3 \oplus \mathfrak{su}_{3,3}, \quad \mathfrak{sp}_3(\mathbb{R}) \subset \mathfrak{sp}_3(\mathbb{R}) \oplus \mathfrak{g}_2.$$

Let $\mathfrak{g} = \mathfrak{e}_7^{-5}$ and $M = \mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1$. In this case we have

$$\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{e}_6^2, 1), (\mathfrak{e}_6^{-14}, 1), (\mathfrak{so}_{4,8}, 1), (\mathfrak{su}_{2,4}, 1), (\mathfrak{su}_{1,2}, 1), (\mathfrak{sl}_2(\mathbb{R}), 24), (\mathfrak{g}_2^2, 2), (\mathfrak{g}_2^2, 1), (\mathfrak{sp}_{1,2}, 1), (\mathfrak{f}_4^4, 1), (\mathfrak{f}_4^{-20}, 1)\}.$$

Note that $(\mathfrak{e}_6^2, 1)$ and $(\mathfrak{e}_6^{-14}, 1)$ are such that their complexifications are not contained in the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$ except for each other. However, $\mathfrak{e}_6^2$ cannot be contained in $\mathfrak{e}_6^{-14}$ and viceversa. By [20], there are no non-semisimple maximal totally geodesic submanifolds in M. Then $(\mathfrak{e}_6^2, 1)$ and $(\mathfrak{e}_6^{-14}, 1)$ induce maximal totally geodesic submanifolds in M. In addition to that, $(\mathfrak{so}_{4,8}, 1)$ is such that its complexification is not contained in the complexification of any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$ except for $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_{12}^*$, which clearly cannot contain $\mathfrak{so}_{4,8}$. Thus, $(\mathfrak{so}_{4,8}, 1)$ induces a maximal totally geodesic submanifold in M. Furthermore, by Table 7.2, we have the following inclusions into subalgebras of $\mathfrak{g}$:

$$\mathfrak{su}_{2,4} \subset \mathfrak{su}_3 \oplus \mathfrak{su}_{2,4}, \quad \mathfrak{su}_{1,2} \subset \mathfrak{su}_{1,2} \oplus \mathfrak{su}_6, \quad \mathfrak{g}_2^2 \subset \mathfrak{sp}_3 \oplus \mathfrak{g}_2^2, \quad \mathfrak{sp}_{1,2} \subset \mathfrak{sp}_{1,2} \oplus \mathfrak{g}_2.$$

Therefore, by Lemma 7.5.2 *iv)*, we have that $(\mathfrak{su}_{2,4}, 1)$, $(\mathfrak{su}_{1,2}, 1)$, $(\mathfrak{g}_2^2, 1)$ and $(\mathfrak{sp}_{1,2}, 1)$ do not induce maximal totally geodesic submanifolds. Now by Lemma 7.5.3, we have that $(\mathfrak{sl}_2(\mathbb{R}), 24)$, $(\mathfrak{g}_2^2, 2)$, $(\mathfrak{f}_4^4, 1)$ and $(\mathfrak{f}_4^{-20}, 1)$ do not induce maximal totally geodesic submanifolds.

Let $\mathfrak{g} = \mathfrak{e}_8^8$ and $M = \mathsf{E}_8^8/\mathsf{SO}_{16}$. In this case

$$\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{su}_{1,4}, 1), (\mathfrak{e}_6^{-14}, 1), (\mathfrak{e}_7^{-5}, 1), (\mathfrak{su}_{1,2}, 6), (\mathfrak{f}_4^{-20}, 1), (\mathfrak{sl}_2(\mathbb{R}), 16)\}.$$

Notice that $\mathfrak{e}_7^{-5}$ cannot be contained in any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$. However, by [20], there is no non-semisimple maximal totally geodesic submanifold in M, which implies that $(\mathfrak{e}_7^{-5}, 1)$ induces a maximal totally geodesic submanifold. Now, by Lemma 7.5.3, we have that $(\mathfrak{su}_{1,2}, 6)$ does not induce a maximal totally geodesic submanifold in M. Furthermore, by Table 7.2, we have the following inclusions into subalgebras of $\mathfrak{g}$:

$$\mathfrak{su}_{1,4} \subset \mathfrak{su}_{1,4} \oplus \mathfrak{su}_5, \quad \mathfrak{e}_6^{-14} \subset \mathfrak{e}_6^{-14} \oplus \mathfrak{su}_3, \quad \mathfrak{f}_4^{-20} \subset \mathfrak{f}_4^{-20} \oplus \mathfrak{g}_2, \quad \mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{su}_3.$$

Thus, by Lemma 7.5.2 *iv)*, we have that $(\mathfrak{su}_{1,4}, 1)$, $(\mathfrak{e}_6^{-14}, 1)$, $(\mathfrak{f}_4^{-20}, 1)$ and $(\mathfrak{sl}_2(\mathbb{R}), 16)$ do not induce maximal totally geodesic submanifolds in M.

Let $\mathfrak{g} = \mathfrak{e}_8^{-24}$ and $M = \mathsf{E}_8^{-24}/\mathsf{E}_7\mathsf{Sp}_1$. In this case

$$\mathcal{C}(\mathfrak{g}) = \{(\mathfrak{su}_{2,3}, 1), (\mathfrak{e}_6^2, 1), (\mathfrak{su}_{1,2}, 1), (\mathfrak{e}_7^{-5}, 1), (\mathfrak{sl}_3(\mathbb{R}), 6), (\mathfrak{f}_4^4, 1), (\mathfrak{g}_2^2, 1), (\mathfrak{g}_2(\mathbb{C}), (1, 1))\}.$$

Notice that $\mathfrak{e}_7^{-5}$ cannot be contained in any other subalgebra in $\widetilde{\mathcal{L}}(\mathfrak{g})$. However, by [20], there is no non-semisimple maximal totally geodesic submanifold in M, which implies that $(\mathfrak{e}_7^{-5}, 1)$ induces a maximal totally geodesic submanifold. Now by Lemma 7.5.3, we have that $(\mathfrak{sl}_3(\mathbb{R}), 6)$ and $(\mathfrak{g}_2(\mathbb{C}), (1, 1))$ do not induce maximal totally geodesic submanifolds. Furthermore, by Table 7.2, we have the following embeddings into subalgebras of $\mathfrak{g}$

$$\mathfrak{su}_{2,3} \subset \mathfrak{su}_5 \oplus \mathfrak{su}_{2,3}, \quad \mathfrak{e}_6^2 \subset \mathfrak{su}_3 \oplus \mathfrak{e}_6^2, \quad \mathfrak{su}_{1,2} \subset \mathfrak{su}_{1,2} \oplus \mathfrak{e}_6, \quad \mathfrak{f}_4^4 \subset \mathfrak{f}_4^4 \oplus \mathfrak{g}_2, \quad \mathfrak{g}_2^2 \subset \mathfrak{f}_4 \oplus \mathfrak{g}_2^2.$$

Thus, by Lemma 7.5.2 *iv)*, we have that $(\mathfrak{su}_{2,3}, 1)$, $(\mathfrak{e}_6^2, 1)$, $(\mathfrak{su}_{1,2}, 1)$, $(\mathfrak{f}_4^4, 1)$ and $(\mathfrak{g}_2^2, 1)$ do not induce maximal totally geodesic submanifolds. This concludes the proof. □

7.6 Totally geodesic submanifolds in exceptional symmetric spaces of type IV

In this section we classify maximal totally geodesic submanifolds in exceptional symmetric spaces with complex isometry group. By duality this is equivalent to classifying maximal totally geodesic submanifolds in exceptional compact Lie groups.

Let G/K be a symmetric space of compact type and $\sigma \in \mathrm{Aut}(\mathsf{G})$ be an involutive automorphism of G such that $\mathrm{Fix}^0(\sigma) \subset \mathsf{K} \subset \mathrm{Fix}(\sigma)$, where $\mathrm{Fix}(\sigma)$ is the subset of G that is fixed by σ and $\mathrm{Fix}^0(\sigma)$ is its identity component. The Cartan embedding of G/K into G is the smooth map f given by

$$f\colon \mathsf{G}/\mathsf{K} \to \mathsf{G}, \qquad g\mathsf{K} \mapsto \sigma(g)g.$$

It was shown in [96] that a totally geodesic submanifold in a compact Lie group is maximal if and only if it is a maximal subgroup or a Cartan embedding. However, in this section we give an explicit list of all maximal totally geodesic submanifolds in exceptional symmetric spaces with complex isometry group.

Maximal semisimple regular subalgebras of $\mathfrak{g}$ and maximal S-subalgebras can be found in [74] as mentioned above. Real forms are well known, see e.g. [90]. Therefore, by Lemma 7.2.4, one can obtain the set $\mathcal{L}(\mathfrak{g})$ for $\mathfrak{g}$ a realification of a exceptional simple complex Lie algebra (see Table 7.3 for the explicit list and Definition 7.4.7).

Lemma 7.6.1. *Let $\mathfrak{g}$ be equal to $\mathfrak{e}_6(\mathbb{C})$ or $\mathfrak{e}_7(\mathbb{C})$ and let $\mathfrak{h}$ be one of the following two subalgebras of $\mathfrak{g}$:*

$$\mathfrak{so}_{10}(\mathbb{C}) \subset \mathfrak{e}_6(\mathbb{C}), \qquad \mathfrak{e}_6(\mathbb{C}) \subset \mathfrak{e}_7(\mathbb{C}).$$

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let K be a maximal compact subgroup of G. Assume the Lie algebra $\mathfrak{h}$ is canonically embedded into $\mathfrak{g}$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let H be the connected closed subgroup of G with Lie algebra $\mathfrak{h}$. Then a totally geodesic orbit of the H-action on the symmetric space $M = \mathsf{G}/\mathsf{K}$ is not maximal.

Proof. By [74], there is a reductive subalgebra isomorphic to $\mathfrak{h} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a 1-dimensional complex subalgebra of $\mathfrak{g}$, that is canonically embedded with respect to the Cartan decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ by the same kind of argument as in the first paragraph of the proof of Lemma 7.5.4. Let Σ be the totally geodesic orbit $\mathsf{H} \cdot o$, where H is the connected subgroup of G with Lie algebra $\mathfrak{h}$. Moreover, θ leaves $\mathfrak{h} \oplus \mathfrak{s}$ invariant and maps $\mathfrak{s}$ onto itself, since it is the center of $\mathfrak{h} \oplus \mathfrak{s}$ and θ is a Lie algebra automorphism. Thus $\mathfrak{s}$ is canonically embedded in $\mathfrak{g}$ with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, implying that $\mathfrak{s} = (\mathfrak{k} \cap \mathfrak{s}) \oplus (\mathfrak{p} \cap \mathfrak{s})$.

If $\dim_{\mathbb{R}}(\mathfrak{p} \cap \mathfrak{s}) \neq 0$, then Σ is properly contained in a proper totally geodesic submanifold of M. If $\dim_{\mathbb{R}}(\mathfrak{p} \cap \mathfrak{s}) = 0$, then $\mathfrak{s}$ has dimension two and the rank of $(\mathfrak{s} \oplus \mathfrak{h}) \cap \mathfrak{k}$ is bigger than the rank of $\mathfrak{k}$, which leads to a contradiction in both cases. Indeed, if $\mathfrak{h} \simeq \mathfrak{so}_{10}(\mathbb{C})$, then $\mathfrak{h} \cap \mathfrak{k} \simeq \mathfrak{so}_{10}$ and $\mathfrak{k} \simeq \mathfrak{e}_6$. If $\mathfrak{h} \simeq \mathfrak{e}_6(\mathbb{C})$, then $\mathfrak{h} \cap \mathfrak{k} \simeq \mathfrak{e}_6$ and $\mathfrak{k} \simeq \mathfrak{e}_7$. □

$\mathfrak{g}_2(\mathbb{C})$	$(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), (3,1)), (\mathfrak{sl}_2(\mathbb{C}), 28), (\mathfrak{sl}_3(\mathbb{C}), 1), (\mathfrak{g}_2^2, 1)$
$\mathfrak{f}_4(\mathbb{C})$	$(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sp}_3(\mathbb{C}), (1,1)), (\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}), (1,2)), (\mathfrak{sl}_2(\mathbb{C}), 156), (\mathfrak{so}_9(\mathbb{C}), 1),$ $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{g}_2(\mathbb{C}), (8,1)), (\mathfrak{f}_4^{-20}, 1), (\mathfrak{f}_4^4, 1)$
$\mathfrak{e}_6(\mathbb{C})$	$(\mathfrak{sl}_3(\mathbb{C}), 9), (\mathfrak{g}_2(\mathbb{C}), 3), (\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{g}_2(\mathbb{C}), (2,1)), (\mathfrak{sp}_4(\mathbb{C}), 1), (\mathfrak{f}_4(\mathbb{C}), 1),$ $(\mathfrak{so}_{10}(\mathbb{C}), 1), (\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}), (1,1,1)), (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_6(\mathbb{C}), (1,1)),$ $(\mathfrak{e}_6^6, 1), (\mathfrak{e}_6^2, 1), (\mathfrak{e}_6^{-26}, 1), (\mathfrak{e}_6^{-14}, 1)$
$\mathfrak{e}_7(\mathbb{C})$	$(\mathfrak{sl}_2(\mathbb{C}), 231, 399), (\mathfrak{sl}_3(\mathbb{C}), 21), (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), (15, 24)),$ $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{g}_2(\mathbb{C}), (7,2)), (\mathfrak{sp}_3(\mathbb{C}) \oplus \mathfrak{g}_2(\mathbb{C}), (1,1)), (\mathfrak{e}_6(\mathbb{C}), 1),$ $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_{12}(\mathbb{C}), (1,1)), (\mathfrak{sl}_8(\mathbb{C}), 1), (\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_6(\mathbb{C}), (1,1)),$ $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{f}_4(\mathbb{C}), (3,1)), (\mathfrak{e}_7^{-5}, 1), (\mathfrak{e}_7^7, 1), (\mathfrak{e}_7^{-25}, 1)$
$\mathfrak{e}_8(\mathbb{C})$	$(\mathfrak{sl}_2(\mathbb{C}), 520, 760, 1240), (\mathfrak{so}_5(\mathbb{C}), 12), (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}), (16,6)),$ $(\mathfrak{f}_4(\mathbb{C}) \oplus \mathfrak{g}_2(\mathbb{C}), (1,1)), (\mathfrak{so}_{16}(\mathbb{C}), 1), (\mathfrak{sl}_9(\mathbb{C}), 1), (\mathfrak{sl}_5(\mathbb{C}) \oplus \mathfrak{sl}_5(\mathbb{C}), (1,1)),$ $(\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{e}_6(\mathbb{C}), (1,1)), (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{e}_7(\mathbb{C}), (1,1)), (\mathfrak{e}_8^8, 1), (\mathfrak{e}_8^{-24}, 1)$

Table 7.3: $\mathcal{L}(\mathfrak{g})$ for each exceptional simple complex Lie algebra $\mathfrak{g}$. Notice that we indicate the different isometry classes for a given isomorphism class of a subalgebra by writing all their possible Dynkin indices separated by commas.

Theorem 7.6.2. *Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space with G an exceptional simple complex Lie group. Let Σ be a maximal totally geodesic submanifold of M. Then Σ is isometric to one of the spaces listed in Tables 7.5, 7.6, 7.7, 7.8, 7.9. Conversely, every space listed in these tables can be isometrically embedded as a maximal totally geodesic submanifold of M.*

Proof. Let Σ be a maximal totally geodesic submanifold in M. If Σ is a non-semisimple totally geodesic submanifold, it is one of the examples found by Berndt and Olmos [20], which we include in our tables. Let us now assume that Σ is semisimple. Then, by Corollary 7.4.8, there is some Lie subalgebra $\mathfrak{g}_\Sigma$ of $\mathfrak{g}$ that is isometric to some subalgebra in $\mathcal{L}(\mathfrak{g})$.

Let M be equal to $\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$, $\mathsf{F}_2(\mathbb{C})/\mathsf{F}_4$ or $\mathsf{E}_8(\mathbb{C})/\mathsf{E}_8$. By [20, Corollary 4.3], we have that every maximal totally geodesic submanifold of M is semisimple and therefore $\mathcal{L}(\mathfrak{g})$ is in one-to-one correspondence with the isometry classes of maximal totally geodesic submanifolds of M by Corollary 7.4.8.

Now let $M = \mathsf{E}_6(\mathbb{C})/\mathsf{E}_6$ and $\mathfrak{g} = \mathfrak{e}_6(\mathbb{C})$. Let $(\mathfrak{g}_\Sigma, \mathrm{ind}_D(\mathfrak{g}_\Sigma)) \in \mathcal{L}(\mathfrak{g}) \setminus \{(\mathfrak{so}_{10}(\mathbb{C}), 1)\}$. Let Σ be the corresponding maximal semisimple totally geodesic submanifold. If Σ is not maximal, then it must be contained in $\mathbb{R} \times \mathsf{SO}_{10}(\mathbb{C})/\mathsf{SO}_{10}$, which is the only non-semisimple maximal totally geodesic submanifold in M according to Berndt and Olmos [20]. However, if this is the case, since $\mathfrak{g}_\Sigma$ is semisimple, $\mathfrak{g}_\Sigma$ is contained in $\mathfrak{so}_{10}(\mathbb{C})$, contradicting the fact that $\mathfrak{g}_\Sigma$ is a maximal non-compact type subalgebra in $\mathfrak{g}$. Hence Σ is a maximal totally geodesic submanifold. By Lemma 7.6.1, we have that $(\mathfrak{so}_{10}(\mathbb{C}), 1)$ does not induce a maximal totally geodesic orbit.

Finally, let $M = \mathsf{E}_7(\mathbb{C})/\mathsf{E}_7$ and $\mathfrak{g} = \mathfrak{e}_7(\mathbb{C})$. By a similar argument as above every subalgebra in $\mathcal{L}(\mathfrak{g}) \setminus \{(\mathfrak{e}_6(\mathbb{C}), 1)\}$ induces a maximal totally geodesic submanifold. By Lemma 7.6.1, we have that $(\mathfrak{e}_6(\mathbb{C}), 1)$ does not induce a maximal totally geodesic orbit. □

7.7 Proofs of the main theorems

The aim of this section is to provide the proofs for the main theorems.

Proof of Theorem A. It follows by combining Theorems 7.5.5 and 7.6.2. □

Remark 7.7.1. Notice that in Tables 7.5-7.9 we indicate the different isometry classes for a given homothety class of a totally geodesic embedding by writing all the possible Dynkin indices separated by commas.

Lemma 7.7.2. *Consider the following subalgebras* $\mathfrak{h} \subset \mathfrak{g}$, $k \geq 1$, *given by the embeddings* $\mathfrak{h} \hookrightarrow \mathfrak{g}$, $A \mapsto \begin{pmatrix} A & \\ & 0 \end{pmatrix}$: *i)* $\mathfrak{so}_n(\mathbb{C}) \subset \mathfrak{so}_{n+k}(\mathbb{C}), n \geq 5$; *ii)* $\mathfrak{sl}_n(\mathbb{C}) \subset \mathfrak{sl}_{n+k}(\mathbb{C}), n \geq 2$; *iii)* $\mathfrak{sp}_n(\mathbb{C}) \subset \mathfrak{sp}_{n+k}(\mathbb{C}), n \geq 1$. *Then, the Dynkin index of these subalgebras is* 1. *Moreover, the Dynkin index of the subalgebra* $\mathfrak{so}_4(\mathbb{C}) \subset \mathfrak{so}_5(\mathbb{C})$ *is* $(1,1)$.

Proof. Assume $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{g}$. Then a root system of $\mathfrak{h}$ is a subset of a root system of $\mathfrak{g}$ and we can apply the following simple observation. If $\mathfrak{h}$ contains a root space of $\mathfrak{g}$ corresponding to a longest root with respect to some Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, then the length of the longest root of $\mathfrak{h}$ and the length of the longest root of $\mathfrak{g}$ agree, and the Dynkin index of $\mathfrak{h}$ in $\mathfrak{g}$ is one. This holds in particular if the root systems of both $\mathfrak{h}$ and $\mathfrak{g}$ contain roots of different lengths, which shows that the subalgebras *iii)* and $\mathfrak{so}_n(\mathbb{C}) \subset \mathfrak{so}_{n+2}(\mathbb{C})$, for $n \geq 5$ odd, have Dynkin index one. Using the multiplicative property of the Dynkin index and the chain of inclusions $\mathfrak{so}_n(\mathbb{C}) \subset \mathfrak{so}_{n+1}(\mathbb{C}) \subset \mathfrak{so}_{n+2}(\mathbb{C})$, it now follows inductively that all the subalgebras *i)* have Dynkin index one. The above observation also applies if $\mathfrak{g}$ is of type A_n or D_n, $n \geq 4$, and $\mathfrak{h}$ is a maximal semisimple regular subalgebra, since then all roots of the extended Dynkin diagram are of the same length, see [149, Ch. 1, §3, Table 4]. Using induction and the multiplicative property of the Dynkin index this shows that the subalgebras *ii)* all have Dynkin index one. □

Proof of Theorem B. Examples of totally geodesic submanifolds Σ in irreducible symmetric spaces M with $i(M) = \operatorname{codim}(\Sigma)$ are given in [24, Table 1]. For the exceptional symmetric spaces M, these pairs (M, Σ) are the following:

- $(\mathsf{G}_2^2/\mathsf{SO}_4, \mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3)$, $(\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2, \mathsf{G}_2^2/\mathsf{SO}_4)$, $(\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2, \mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3)$,
- $(\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1, \mathsf{SO}^0_{4,5}/\mathsf{SO}_4 \times \mathsf{SO}_5)$, $(\mathsf{F}_4^{-20}/\mathsf{Spin}_9, \mathsf{SO}^0_{1,8}/\mathsf{SO}_8)$, $(\mathsf{F}_4^{-20}/\mathsf{Spin}_9, \mathsf{Sp}_{1,2}/\mathsf{Sp}_1 \times \mathsf{Sp}_2)$, $(\mathsf{F}_4(\mathbb{C})/\mathsf{F}_4, \mathsf{SO}_9(\mathbb{C})/\mathsf{SO}_9)$,
- $(\mathsf{E}_6^6/\mathsf{Sp}_4, \mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1)$, $(\mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1, \mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1)$, $(\mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1, \mathsf{SO}^*_{10}/\mathsf{U}_5)$, $(\mathsf{E}_6^{-26}/\mathsf{F}_4, \mathsf{F}_4^{-20}/\mathsf{Spin}_9)$, $(\mathsf{E}_6(\mathbb{C})/\mathsf{E}_6, \mathsf{F}_4(\mathbb{C})/\mathsf{F}_4)$,
- $(\mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1, \mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1)$, $(\mathsf{E}_7^{-25}/\mathsf{E}_6\mathsf{U}_1, \mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1)$, $(\mathsf{E}_7^7/\mathsf{SU}_8, \mathbb{R} \times \mathsf{E}_6^6/\mathsf{Sp}_4)$, $(\mathsf{E}_7(\mathbb{C})/\mathsf{E}_7, \mathbb{R} \times \mathsf{E}_6(\mathbb{C})/\mathsf{E}_6)$,
- $(\mathsf{E}_8^{-24}/\mathsf{E}_7\mathsf{Sp}_1, \mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1)$, $(\mathsf{E}_8^8/\mathsf{SO}_{16}, \mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 \times \mathsf{E}_7^7/\mathsf{SU}_8)$, $(\mathsf{E}_8(\mathbb{C})/\mathsf{E}_8, \mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2 \times \mathsf{E}_7(\mathbb{C})/\mathsf{E}_7)$.

Hence, the theorem can be proved in these cases by looking up the Dynkin indices in our Tables 7.5, 7.6, 7.7, 7.8, 7.9.

It remains to show the assertion of the theorem for the classical spaces. We have to consider the pairs of spaces (M, Σ), where $i(M) = \mathrm{codim}(\Sigma)$, given in Table 7.4, where we have partially reproduced the contents of [24, Table 1].

M	Σ	$i(M)$	Conditions
$\mathsf{SU}_{r,r+k}/\mathsf{S}(\mathsf{U}_r \times \mathsf{U}_{r+k})$	$\mathsf{SU}_{r,r+k-1}/\mathsf{S}(\mathsf{U}_r \times \mathsf{U}_{r+k-1})$	$2r$	$r \geq 1, k \geq 1$
$\mathsf{SU}_{r,r}/\mathsf{S}(\mathsf{U}_r \times \mathsf{U}_r)$	$\mathsf{SU}_{r-1,r}/\mathsf{S}(\mathsf{U}_{r-1} \times \mathsf{U}_r)$	$2r$	$r \geq 3$
$\mathsf{SO}^0_{r,r+k}/\mathsf{SO}_r \times \mathsf{SO}_{r+k}$	$\mathsf{SO}^0_{r,r+k-1}/\mathsf{SO}_r \times \mathsf{SO}_{r+k-1}$	r	$r \geq 1, k \geq 1$
$\mathsf{SO}^0_{r,r}/\mathsf{SO}_r \times \mathsf{SO}_r$	$\mathsf{SO}^0_{r-1,r}/\mathsf{SO}_{r-1} \times \mathsf{SO}_r$	r	$r \geq 4$
$\mathsf{Sp}_{2,2}/\mathsf{Sp}_2 \times \mathsf{Sp}_2$	$\mathsf{Sp}_2(\mathbb{C})/\mathsf{Sp}_2$	6	
$\mathsf{Sp}_{r,r+k}/\mathsf{Sp}_r \times \mathsf{Sp}_{r+k}$	$\mathsf{Sp}_{r,r+k-1}/\mathsf{Sp}_r \times \mathsf{Sp}_{r+k-1}$	$4r$	$r \geq 1, k \geq 1$
$\mathsf{Sp}_{r,r}/\mathsf{Sp}_r \times \mathsf{Sp}_r$	$\mathsf{Sp}_{r-1,r}/\mathsf{Sp}_{r-1}\mathsf{Sp}_r$	$4r$	$r \geq 3$
$\mathsf{SL}_{r+1}(\mathbb{R})/\mathsf{SO}_{r+1}$	$\mathbb{R} \times \mathsf{SL}_r(\mathbb{R})/\mathsf{SO}_r$	r	$r \geq 2$
$\mathsf{SU}^*_6/\mathsf{Sp}_3$	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	6	
$\mathsf{SU}^*_8/\mathsf{Sp}_4$	$\mathsf{Sp}_{2,2}/\mathsf{Sp}_2\mathsf{Sp}_2$	11	
$\mathsf{SU}^*_{2r+2}/\mathsf{Sp}_{r+1}$	$\mathbb{R} \times \mathsf{SU}^*_{2r}/\mathsf{Sp}_r$	$4r$	$r \geq 4$
$\mathsf{Sp}_r(\mathbb{R})/\mathsf{U}_r$	$\mathsf{Sp}_1(\mathbb{R})/\mathsf{U}_1 \times \mathsf{Sp}_{r-1}(\mathbb{R})/\mathsf{U}_{r-1}$	$2r-2$	$r \geq 3$
$\mathsf{SO}^*_{4r}/\mathsf{U}_{2r}$	$\mathsf{SO}^*_{4r-2}/\mathsf{U}_{2r-1}$	$4r-2$	$r \geq 3$
$\mathsf{SO}^*_{4r+2}/\mathsf{U}_{2r+1}$	$\mathsf{SO}^*_{4r}/\mathsf{U}_{2r}$	$4r$	$r \geq 2$
$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	3	
$\mathsf{SL}_4(\mathbb{C})/\mathsf{SU}_4$	$\mathsf{Sp}_2(\mathbb{C})/\mathsf{Sp}_2$	5	
$\mathsf{SL}_{r+1}(\mathbb{C})/\mathsf{SU}_{r+1}$	$\mathbb{R} \times \mathsf{SL}_r(\mathbb{C})/\mathsf{SU}_r$	$2r$	$r \geq 4$
$\mathsf{SO}_{2r+1}(\mathbb{C})/\mathsf{SO}_{2r+1}$	$\mathsf{SO}_{2r}(\mathbb{C})/\mathsf{SO}_{2r}$	$2r$	$r \geq 2$
$\mathsf{Sp}_r(\mathbb{C})/\mathsf{Sp}_r$	$\mathsf{Sp}_1(\mathbb{C})/\mathsf{Sp}_1 \times \mathsf{Sp}_{r-1}(\mathbb{C})/\mathsf{Sp}_{r-1}$	$4r-4$	$r \geq 3$
$\mathsf{SO}_{2r}(\mathbb{C})/\mathsf{SO}_{2r}$	$\mathsf{SO}_{2r-1}(\mathbb{C})/\mathsf{SO}_{2r-1}$	$2r-1$	$r \geq 4$

Table 7.4: Examples of totally geodesic submanifolds Σ in classical symmetric spaces M with $\mathrm{codim}(\Sigma) = i(M)$.

Among the pairs of spaces given in the table, there are some infinite series and also some isolated examples in low dimensions. For the infinite series, the assertion of the theorem follows in all cases from Lemma 7.7.2 and the multiplicative property of the Dynkin index.

We will treat the remaining isolated examples individually. First note that the Dynkin index of $\mathfrak{sp}_2(\mathbb{C})$ in $\mathfrak{sp}_{2,2}$ is $(1,1)$, as follows from Lemma 7.7.2 since the corresponding complexifications are $\mathfrak{sp}_2(\mathbb{C}) \oplus \mathfrak{sp}_2(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$. The Dynkin index of $\mathfrak{sl}_3(\mathbb{C})$ in $\mathfrak{su}^*_6$ is $(1,1)$, as the complexifications are $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sl}_6(\mathbb{C})$. The subalgebra $\mathfrak{sl}_3(\mathbb{R})$ is a real form of $\mathfrak{sl}_3(\mathbb{C})$ and therefore has Dynkin index one. The subalgebra $\mathfrak{sp}_2(\mathbb{C})$ of $\mathfrak{sl}_4(\mathbb{C})$ has Dynkin index one by Lemma 7.7.2, since it corresponds to the subalgebra $\mathfrak{so}_5(\mathbb{C}) \subset \mathfrak{so}_6(\mathbb{C})$ under the isomorphism $\mathfrak{so}_6(\mathbb{C}) \simeq \mathfrak{sl}_4(\mathbb{C})$. Finally, it remains to determine the Dynkin index of the subalgebra $\mathfrak{sp}_{2,2}$ in $\mathfrak{su}^*_8$. Since the Dynkin indices of $\mathfrak{sp}_2(\mathbb{C}) \subset \mathfrak{sp}_4(\mathbb{C})$ and $\mathfrak{sl}_4(\mathbb{C}) \subset \mathfrak{sl}_8(\mathbb{C})$ are one by Lemma 7.7.2, and we know from the previous case that the

Dynkin index of $\mathfrak{sp}_2(\mathbb{C}) \subset \mathfrak{sl}_4(\mathbb{C})$ is one, using the multiplicative property of the Dynkin index, we conclude from the following diagram

$$\begin{array}{ccc} \mathfrak{sp}_2(\mathbb{C}) & \xrightarrow{1} & \mathfrak{sl}_4(\mathbb{C}) \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\ \mathfrak{sp}_4(\mathbb{C}) & \longrightarrow & \mathfrak{sl}_8(\mathbb{C}) \end{array}$$

that the Dynkin index of $\mathfrak{sp}_4(\mathbb{C})$ in $\mathfrak{sl}_8(\mathbb{C})$ is also one. □

Table 7.5: Maximal totally geodesic submanifolds of symmetric spaces of G_2-type.

M	Σ	Dynkin index	Reflective?	$\dim\Sigma$
$\mathsf{G}_2^2/\mathsf{SO}_4$	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	$(3,1)$	Yes	4
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	1	No	5
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_2)$	1	No	4
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	28	No	2
$\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2\times\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2$	$(3,1)$	Yes	6
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2$	28	No	3
	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	1	No	8
	$\mathsf{G}_2^2/\mathsf{SO}_4$	1	Yes	8

Table 7.6: Maximal totally geodesic submanifolds of symmetric spaces of F_4-type.

M	Σ	Dynkin index	Reflective?	$\dim\Sigma$
$\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1$	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3\times\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	$(1,2)$	No	10
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_2)\times\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_2)$	$(1,2)$	No	8
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{Sp}_3(\mathbb{R})/\mathsf{U}_3$	$(1,1)$	Yes	14
	$\mathsf{Sp}_{1,2}/\mathsf{Sp}_1\times\mathsf{Sp}_2$	1	Yes	8
	$\mathsf{SO}^0_{4,5}/\mathsf{SO}_4\times\mathsf{SO}_5$	1	Yes	20
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	156	No	2
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{G}_2^2/\mathsf{SO}_4$	$(8,1)$	No	10
$\mathsf{F}_4^{-20}/\mathsf{Spin}_9$	$\mathsf{SO}^0_{1,8}/\mathsf{SO}_8$	1	Yes	8
	$\mathsf{Sp}_{1,2}/\mathsf{Sp}_1\times\mathsf{Sp}_2$	1	Yes	8
$\mathsf{F}_4(\mathbb{C})/\mathsf{F}_4$	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2\times\mathsf{Sp}_3(\mathbb{C})/\mathsf{Sp}_3$	$(1,1)$	Yes	24
	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3\times\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	$(1,2)$	No	16
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2$	156	No	3
	$\mathsf{SO}_9(\mathbb{C})/\mathsf{SO}_9$	1	Yes	36
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2\times\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$	$(8,1)$	No	17
	$\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1$	1	Yes	28
	$\mathsf{F}_4^{-20}/\mathsf{Spin}_9$	1	Yes	16

Table 7.7: Maximal totally geodesic submanifolds of symmetric spaces of E_6-type.

M	Σ	Dynkin index	Reflective?	$\dim \Sigma$
$\mathsf{E}_6^6/\mathsf{Sp}_4$	$(\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3)^3$	$(1,1,1)$	No	15
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_2) \times \mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	$(1,1,1)$	No	12
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 \times \mathsf{SL}_6(\mathbb{R})/\mathsf{SO}_6$	$(1,1)$	Yes	22
	$\mathsf{SU}_6^*/\mathsf{Sp}_3$	1	Yes	14
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3 \times \mathsf{G}_2^2/\mathsf{SO}_4$	$(2,1)$	No	13
	$\mathsf{Sp}_{2,2}/\mathsf{Sp}_2 \times \mathsf{Sp}_2$	1	Yes	16
	$\mathsf{Sp}_4(\mathbb{R})/\mathsf{U}_4$	1	Yes	20
	$\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1$	1	Yes	28
	$\mathbb{R} \times \mathsf{SO}_{5,5}^0/\mathsf{SO}_5 \times \mathsf{SO}_5$	1	Yes	26
$\mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1$	$\mathsf{SO}_{4,6}^0/\mathsf{SO}_4 \times \mathsf{SO}_6$	1	Yes	24
	$\mathsf{SO}_{10}^*/\mathsf{U}_5$	1	Yes	20
	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3 \times \mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	$(1,1,1)$	No	13
	$(\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_2))^3$	$(1,1,1)$	No	12
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 \times \mathsf{SU}_{3,3}/\mathsf{S}(\mathsf{U}_3 \times \mathsf{U}_3)$	$(1,1)$	Yes	20
	$\mathsf{SU}_{2,4}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_4)$	1	Yes	16
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_2)$	9	No	4
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	9	No	5
	$\mathsf{G}_2^2/\mathsf{SO}_4$	3	No	8
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_2) \times \mathsf{G}_2^2/\mathsf{SO}_4$	$(2,1)$	No	12
	$\mathsf{Sp}_{1,3}/\mathsf{Sp}_1 \times \mathsf{Sp}_3$	1	Yes	12
	$\mathsf{Sp}_4(\mathbb{R})/\mathsf{U}_4$	1	Yes	20
	$\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1$	1	Yes	28
$\mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1$	$\mathsf{SO}_{10}^*/\mathsf{U}_5$	1	Yes	20
	$\mathsf{SO}_{2,8}^0/\mathsf{SO}_2 \times \mathsf{SO}_8$	1	Yes	16
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 \times \mathsf{SU}_{1,5}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_5)$	$(1,1)$	Yes	12
	$\mathsf{Sp}_{2,2}/\mathsf{Sp}_2 \times \mathsf{Sp}_2$	1	Yes	16
	$\mathsf{F}_4^{-20}/\mathsf{Spin}_9$	1	Yes	16
	$\mathsf{SU}_{2,4}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_4)$	1	Yes	16
$\mathsf{E}_6^{-26}/\mathsf{F}_4$	$\mathbb{R} \times \mathsf{SO}_{1,9}^0/\mathsf{SO}_9$	1	Yes	10
	$\mathsf{SU}_6^*/\mathsf{Sp}_3$	1	Yes	14
	$\mathsf{Sp}_{1,3}/\mathsf{Sp}_1 \times \mathsf{Sp}_3$	1	Yes	12
	$\mathsf{F}_4^{-20}/\mathsf{Spin}_9$	1	Yes	16
$\mathsf{E}_6(\mathbb{C})/\mathsf{E}_6$	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	9	No	8
	$\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$	3	No	14
	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3 \times \mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$	$(2,1)$	No	22
	$\mathsf{Sp}_4(\mathbb{C})/\mathsf{Sp}_4$	1	Yes	36
	$\mathsf{F}_4(\mathbb{C})/\mathsf{F}_4$	1	Yes	52
	$\mathbb{R} \times \mathsf{SO}_{10}(\mathbb{C})/\mathsf{SO}_{10}$	1	Yes	46
	$(\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3)^3$	$(1,1,1)$	No	24
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2 \times \mathsf{SL}_6(\mathbb{C})/\mathsf{SU}_6$	$(1,1)$	Yes	38
	$\mathsf{E}_6^6/\mathsf{Sp}_4$	1	Yes	42
	$\mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1$	1	Yes	40
	$\mathsf{E}_6^{-26}/\mathsf{F}_4$	1	Yes	26
	$\mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1$	1	Yes	32

Table 7.8: Maximal totally geodesic submanifolds of symmetric spaces of E_7-type.

M	Σ	Dynkin index	Reflective?	$\dim\Sigma$
$\mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1$	$\mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1$	1	Yes	40
	$\mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1$	1	Yes	32
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{SO}_{12}^*/\mathsf{U}_6$	$(1,1)$	Yes	32
	$\mathsf{SO}_{4,8}^0/\mathsf{SO}_4\times\mathsf{SO}_8$	1	Yes	32
	$\mathsf{SU}_{4,4}/\mathsf{S}(\mathsf{U}_4\times\mathsf{U}_4)$	1	Yes	32
	$\mathsf{SU}_{2,6}/\mathsf{S}(\mathsf{U}_2\times\mathsf{U}_6)$	1	Yes	24
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3\times\mathsf{SU}_6^*/\mathsf{Sp}_3$	$(1,1)$	No	19
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_2)\times\mathsf{SU}_{2,4}/\mathsf{S}(\mathsf{U}_2\times\mathsf{U}_4)$	$(1,1)$	No	20
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_2)$	21	No	4
	$\mathsf{Sp}_{1,2}/\mathsf{Sp}_1\times\mathsf{Sp}_2\times\mathsf{G}_2^2/\mathsf{SO}_4$	$(1,1)$	No	16
$\mathsf{E}_7^{-25}/\mathsf{E}_6\mathsf{U}_1$	$\mathbb{R}\times\mathsf{E}_6^{-26}/\mathsf{F}_4$	1	Yes	27
	$\mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1$	1	Yes	32
	$\mathsf{SU}_8^*/\mathsf{Sp}_4$	1	Yes	27
	$\mathsf{SU}_{2,6}/\mathsf{S}(\mathsf{U}_2\times\mathsf{U}_6)$	1	Yes	24
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_2)\times\mathsf{SU}_{1,5}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_5)$	$(1,1)$	No	14
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{F}_4^{-20}/\mathsf{Spin}_9$	$(3,1)$	No	18
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{SO}_{2,10}^0/\mathsf{SO}_2\times\mathsf{SO}_{10}$	$(1,1)$	Yes	22
	$\mathsf{SO}_{12}^*/\mathsf{U}_6$	1	Yes	30
$\mathsf{E}_7^7/\mathsf{SU}_8$	$\mathbb{R}\times\mathsf{E}_6^6/\mathsf{Sp}_4$	1	Yes	43
	$\mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1$	1	Yes	40
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{SO}_{6,6}^0/\mathsf{SO}_6\times\mathsf{SO}_6$	$(1,1)$	Yes	38
	$\mathsf{SO}_{12}^*/\mathsf{U}_6$	1	Yes	30
	$\mathsf{SL}_8(\mathbb{R})/\mathsf{SO}_8$	1	Yes	35
	$\mathsf{SU}_{4,4}/\mathsf{S}(\mathsf{U}_4\times\mathsf{U}_4)$	1	Yes	32
	$\mathsf{SU}_8^*/\mathsf{Sp}_4$	1	Yes	27
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3\times\mathsf{SL}_6(\mathbb{R})/\mathsf{SO}_6$	$(1,1)$	No	25
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1\times\mathsf{U}_2)\times\mathsf{SU}_{3,3}/\mathsf{S}(\mathsf{U}_3\times\mathsf{U}_3)$	$(1,1)$	No	22
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	$231, 399$	No	2
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	21	No	5
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	$(15,24)$	No	4
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{G}_2^2/\mathsf{SO}_4$	$(7,2)$	No	10
	$\mathsf{Sp}_3(\mathbb{R})/\mathsf{U}_3\times\mathsf{G}_2^2/\mathsf{SO}_4$	$(1,1)$	No	20
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2\times\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1$	$(3,1)$	No	30
$\mathsf{E}_7(\mathbb{C})/\mathsf{E}_7$	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2$	$231, 399$	No	3
	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	21	No	8
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2\times\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2$	$(15,24)$	No	6
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2\times\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$	$(7,2)$	No	17
	$\mathsf{Sp}_3(\mathbb{C})/\mathsf{Sp}_3\times\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$	$(1,1)$	No	35
	$\mathbb{R}\times\mathsf{E}_6(\mathbb{C})/\mathsf{E}_6$	1	Yes	79
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2\times\mathsf{SO}_{12}(\mathbb{C})/\mathsf{SO}_{12}$	$(1,1)$	Yes	69
	$\mathsf{SL}_8(\mathbb{C})/\mathsf{SU}_8$	1	Yes	63
	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3\times\mathsf{SL}_6(\mathbb{C})/\mathsf{SU}_6$	$(1,1)$	No	43
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2\times\mathsf{F}_4(\mathbb{C})/\mathsf{F}_4$	$(3,1)$	No	55
	$\mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1$	1	Yes	64
	$\mathsf{E}_7^7/\mathsf{SU}_8$	1	Yes	70
	$\mathsf{E}_7^{-25}/\mathsf{E}_6\mathsf{U}_1$	1	Yes	54

Table 7.9: Maximal totally geodesic submanifolds of symmetric spaces of E_8-type.

M	Σ	Dynkin index	Reflective?	$\dim \Sigma$
$\mathsf{E}_8^8/\mathsf{SO}_{16}$	$(\mathsf{SU}_{1,4}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_4))^2$	$(1,1)$	No	16
	$(\mathsf{SU}_{2,3}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_3))^2$	$(1,1)$	No	24
	$\mathsf{SL}_5(\mathbb{R})/\mathsf{SO}_5 \times \mathsf{SL}_5(\mathbb{R})/\mathsf{SO}_5$	$(1,1)$	No	28
	$\mathsf{SO}^0_{8,8}/\mathsf{SO}_8 \times \mathsf{SO}_8$	1	Yes	64
	$\mathsf{SO}^*_{16}/\mathsf{U}_8$	1	Yes	56
	$\mathsf{SU}_{1,8}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_8)$	1	No	16
	$\mathsf{SU}_{4,5}/\mathsf{S}(\mathsf{U}_4 \times \mathsf{U}_5)$	1	No	40
	$\mathsf{SL}_9(\mathbb{R})/\mathsf{SO}_9$	1	No	44
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3 \times \mathsf{E}_6^6/\mathsf{Sp}_4$	$(1,1)$	No	47
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_2) \times \mathsf{E}_6^2/\mathsf{SU}_6\mathsf{Sp}_1$	$(1,1)$	No	44
	$\mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1$	1	Yes	64
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 \times \mathsf{E}_7^7/\mathsf{SU}_8$	$(1,1)$	Yes	72
	$\mathsf{SO}^0_{2,3}/\mathsf{SO}_2 \times \mathsf{SO}_3$	12	No	6
	$\mathsf{SO}^0_{1,4}/\mathsf{SO}_4$	12	No	4
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	$520, 760, 1240$	No	2
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 \times \mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_2)$	$(16,6)$	No	6
	$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2 \times \mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3$	$(16,6)$	No	7
	$\mathsf{F}_4^4/\mathsf{Sp}_3\mathsf{Sp}_1 \times \mathsf{G}_2^2/\mathsf{SO}_4$	$(1,1)$	No	36
	$\mathsf{G}_2(\mathbb{C})/\mathsf{G}_2 \times \mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	$(1,1,8)$	No	16
$\mathsf{E}_8^{-24}/\mathsf{E}_7\mathsf{Sp}_1$	$\mathsf{SU}_{1,4}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_4) \times \mathsf{SU}_{2,3}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_3)$	$(1,1)$	No	20
	$\mathsf{SO}^0_{4,12}/\mathsf{SO}_4 \times \mathsf{SO}_{12}$	1	Yes	48
	$\mathsf{SO}^*_{16}/\mathsf{U}_8$	1	Yes	56
	$\mathsf{SU}_{3,6}/\mathsf{S}(\mathsf{U}_3 \times \mathsf{U}_6)$	1	No	36
	$\mathsf{SU}_{2,7}/\mathsf{S}(\mathsf{U}_2 \times \mathsf{U}_7)$	1	No	28
	$\mathsf{SU}_{1,2}/\mathsf{S}(\mathsf{U}_1 \times \mathsf{U}_2) \times \mathsf{E}_6^{-14}/\mathsf{Spin}_{10}\mathsf{U}_1$	$(1,1)$	No	36
	$\mathsf{SL}_3(\mathbb{R})/\mathsf{SO}_3 \times \mathsf{E}_6^{-26}/\mathsf{F}_4$	$(1,1)$	No	31
	$\mathsf{E}_7^{-5}/\mathsf{SO}_{12}\mathsf{Sp}_1$	1	Yes	64
	$\mathsf{E}_7^{-25}/\mathsf{E}_6\mathsf{U}_1 \times \mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$	$(1,1)$	Yes	56
	$\mathsf{F}_4^{-20}/\mathsf{Spin}_9 \times \mathsf{G}_2^2/\mathsf{SO}_4$	$(1,1)$	No	24
$\mathsf{E}_8(\mathbb{C})/\mathsf{E}_8$	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2$	$520, 760, 1240$	No	3
	$\mathsf{SO}_5(\mathbb{C})/\mathsf{SO}_5$	12	No	10
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2 \times \mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3$	$(16,6)$	No	11
	$\mathsf{F}_4(\mathbb{C})/\mathsf{F}_4 \times \mathsf{G}_2(\mathbb{C})/\mathsf{G}_2$	$(1,1)$	No	66
	$\mathsf{SO}_{16}(\mathbb{C})/\mathsf{SO}_{16}$	1	Yes	120
	$\mathsf{SL}_9(\mathbb{C})/\mathsf{SU}_9$	1	No	80
	$\mathsf{SL}_5(\mathbb{C})/\mathsf{SU}_5 \times \mathsf{SL}_5(\mathbb{C})/\mathsf{SU}_5$	$(1,1)$	No	48
	$\mathsf{SL}_3(\mathbb{C})/\mathsf{SU}_3 \times \mathsf{E}_6(\mathbb{C})/\mathsf{E}_6$	$(1,1)$	No	86
	$\mathsf{SL}_2(\mathbb{C})/\mathsf{SU}_2 \times \mathsf{E}_7(\mathbb{C})/\mathsf{E}_7$	$(1,1)$	Yes	136
	$\mathsf{E}_8^8/\mathsf{SO}_{16}$	1	Yes	128
	$\mathsf{E}_8^8/\mathsf{E}_7\mathsf{Sp}_1$	1	Yes	112

Chapter 8
Hopf fibrations and totally geodesic submanifolds

In this chapter, we give a classification of totally geodesic submanifolds in Hopf-Berger spheres, which constitute a special family of homogeneous spaces diffeomorphic to spheres.

It is well-known that any homogeneous metric on the even-dimensional sphere S^{2n} is isometric to a round metric, while homogeneous metrics on odd-dimensional spheres S^{2n+1} of dimension bigger than three are homothetic to a metric lying in one of the following families, see [192]:

(1) a 1-parameter family of U_n-invariant metrics on S^{2n+1},

(2) a 1-parameter family of $\mathsf{Sp}_1\mathsf{Sp}_n$-invariant metrics on S^{4n+3},

(3) a 1-parameter family of Spin_9-invariant metrics on S^{15},

(4) a 3-parameter family of Sp_n-invariant metrics on S^{4n+3}.

It turns out that metrics in (2) lie in (4). Moreover, metrics in (1), (2) and (3) can be obtained by rescaling the round metric of the total space of a Hopf fibration in the direction of the fibers. We recall that Hopf fibrations are as follows:

$$\begin{aligned}
&\mathsf{S}^1 \to \mathsf{S}^{2n+1} \to \mathbb{C}\mathsf{P}^n,\\
&\mathsf{S}^3 \to \mathsf{S}^{4n+3} \to \mathbb{H}\mathsf{P}^n,\\
&\mathsf{S}^7 \to \mathsf{S}^{15} \to \mathsf{S}^8,
\end{aligned}$$

where $n \geq 1$. We will refer to these fibrations as the complex, quaternionic, or octonionic Hopf fibrations, respectively.

Let us equip the base and the total spaces of the Hopf fibrations with the corresponding symmetric metrics of diameter $\pi/2$, and sectional curvature equal to 1, respectively. Under these conditions Hopf fibrations become Riemannian submersions with totally geodesic fibers. These fibrations constitute the fundamental examples of Riemannian submersions with a round sphere as the total space, as the following theorem implies. Let $\pi\colon \mathsf{S}^n \to M$ be a Riemannian submersion from a round sphere with connected fibers of positive dimension. Then, Gromoll, Grove and Wilking [87, 183] proved that $\pi\colon \mathsf{S}^n \to M$ is metrically congruent to a Hopf fibration.

Let us consider, for each $\tau > 0$, the total space of the complex, quaternionic, or octonionic Hopf fibration endowed with the Riemannian metric obtained from rescaling the metric tensor of the round sphere by a factor τ in the vertical directions. Such homogeneous spaces are denoted by $\mathsf{S}^{2n+1}_{\mathbb{C},\tau}$, $\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$, and $\mathsf{S}^{15}_{\mathbb{O},\tau}$, depending on whether the Hopf fibration considered is the complex, the quaternionic, or the octonionic one, respectively. We refer to such spaces as the *Hopf-Berger spheres*. Every Hopf-Berger sphere with $\tau \neq 1$ is homothetic to a geodesic sphere of a rank one symmetric space $\bar{M}$, see [192]. The simply connected symmetric spaces of rank one are: S^n, $\mathbb{C}\mathsf{P}^n, \mathbb{H}\mathsf{P}^n, \mathbb{O}\mathsf{P}^2$; and their non-compact duals: $\mathbb{R}\mathsf{H}^n, \mathbb{C}\mathsf{H}^n, \mathbb{H}\mathsf{H}^n, \mathbb{O}\mathsf{H}^2$. We consider the metrics of these symmetric spaces rescaled in such a way that the minimal absolute value of their sectional curvatures is equal to 1. Notice that two geodesic spheres of $\bar{M}$ of the same radius are congruent. Furthermore, it can be proved that every geodesic sphere $\mathsf{S}(r)$ of radius r in a symmetric space of rank one $\bar{M}$ can be obtained by performing a homothety (cf. [33]) of ratio α to $\mathsf{S}^n_{\mathbb{F},\tau}$, with $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $\tau \in (0, +\infty)$, where the values of r, α and τ are related by:

$$\begin{cases} r = \arccos(\sqrt{\tau}), \quad \alpha = \sqrt{1-\tau}, \quad \text{for } 0 < \tau < 1, \text{when } \bar{M} \text{ is compact}, \\ r = \operatorname{arccosh}(\sqrt{\tau}), \quad \alpha = \sqrt{\tau - 1}, \quad \text{for } 1 < \tau < +\infty, \text{when } \bar{M} \text{ is not compact}. \end{cases} \tag{8.1}$$

As we pointed out in Chapter 5, the problem of classifying totally geodesic submanifolds in a given Riemannian manifold is a classical topic in the field of submanifold geometry. In the setting of Riemannian symmetric spaces, we recall that this problem has been extensively studied. However, despite all the efforts toward a general classification of totally geodesic submanifolds in symmetric spaces, we only have classifications for symmetric spaces of rank one [187], symmetric spaces of rank two [48, 49, 107, 108, 109], and special classes of totally geodesic submanifolds such as reflective ones [123, 124, 125] or non-semisimple maximal ones [20], apart from the classifications derived in Chapters 6 and 7 of this thesis.

In the context of symmetric spaces, since the curvature tensor of a symmetric space is parallel under the Levi-Civita connection and can be expressed by means of an easy formula in terms of Lie brackets, the problem of classifying totally geodesic submanifolds turns out to be equivalent to classifying subspaces of the tangent spaces that satisfy the harmless-looking but extremely complicated condition of being a Lie triple system, see Section §5.3.

In the more general setting of Riemannian homogeneous spaces, the classification of totally geodesic submanifolds is a harder problem for a number of reasons. Firstly, in the setting of symmetric spaces, in order to classify totally geodesic submanifolds, we have to characterize those subspaces of the tangent space that are invariant under the curvature tensor, or equivalently, that satisfy the "simple" property of being Lie triple systems. However, in the setting of homogeneous spaces, we need to look for subspaces of the tangent space that are not only invariant by the curvature tensor, which has a much more involved expression, see Equation (1.4), but also by all its covariant derivatives, see Theorem 5.2.5.

Secondly, although complete totally geodesic submanifolds are intrinsically homogeneous (see Proposition 5.1.6), they are not necessarily extrinsically homogeneous, i.e. orbits of a subgroup of the isometry group of the ambient space, see for instance [105]. This suggests the necessity of developing new tools to study totally geodesic submanifolds in homogeneous spaces.

Moreover, the utilization of totally geodesic submanifolds has been of capital importance when studying homogeneous spaces. For example, in the investigation of pinching constants for homogeneous spaces of positive curvature, see [157], they have played a crucial role since every critical point for the sectional curvature of a totally geodesic submanifold is a critical point for the sectional curvature of the ambient space. Indeed, the setting of homogeneous spaces of positive curvature turns out to be an interesting context where to study totally geodesic submanifolds since every totally geodesic submanifold of dimension $d \geq 2$ is again a homogeneous space of positive curvature.

In this chapter, we classify totally geodesic submanifolds in Hopf-Berger spheres with $\tau \geq 1/2$. Hopf-Berger spheres provide some of the easiest examples of non-symmetric homogeneous metrics with positive sectional curvature. Indeed, they have positive sectional curvature if and only if $\tau \in (0, 4/3)$, see [180].

Recall that, on the one hand, the round sphere S^n can be seen as a geodesic sphere of $\mathbb{R}^{n+1}$ of radius 1. On the other hand, every complete totally geodesic submanifold of S^n is the intersection of a complete totally geodesic submanifold of $\mathbb{R}^{n+1}$ passing through the origin (a linear subspace) with S^n. The following theorem generalizes this well-known geometric fact to the setting of 2-point homogeneous spaces and provides the classification of totally geodesic submanifolds in Hopf-Berger spheres.

Theorem A. *Let $\mathsf{S}^n_{\mathbb{F},\tau}$, with $\tau \neq 1$ and $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, be a Hopf-Berger sphere, and $\bar{M}$ the symmetric space of rank one where $\mathsf{S}^n_{\mathbb{F},\tau}$ is realized as a geodesic sphere. Then, if $\tau \geq 1/2$, the following statements are equivalent:*

i) Σ is a complete totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$ of dimension $d \geq 2$.

ii) Σ is the intersection of $\mathsf{S}^n_{\mathbb{F},\tau}$ (regarded as a geodesic sphere of $\bar{M}$) with a complete totally geodesic submanifold M of $\bar{M}$ of dimension $d' \geq 3$ and containing the center of the geodesic sphere.

Every pair $(\Sigma, \mathsf{S}^n_{\mathbb{F},\tau})$ satisfying the above equivalent conditions is listed in Table 8.1. Conversely, each pair listed in Table 8.1 corresponds to a congruence class of totally geodesic submanifolds of $\mathsf{S}^n_{\mathbb{F},\tau}$.

Remark 8.0.1. Notice that for $\tau = 1$ the sphere $\mathsf{S}^n_{\mathbb{F},\tau}$ is round. If $\tau \neq 1$, we have to assume that Σ has dimension $d \geq 2$, since there always are non-closed geodesics in $\mathsf{S}^n_{\mathbb{F},\tau}$, see Lemma 8.4.10, and a non-closed geodesic in $\mathsf{S}^n_{\mathbb{F},\tau}$ cannot arise as the intersection with a totally geodesic submanifold of $\bar{M}$.

Let us denote by S^n_κ the round sphere with constant sectional curvature equal to $\kappa > 0$. As a consequence of Theorem A and Remark 8.3.4, maximal totally geodesic submanifolds

$\mathsf{S}^{2n+1}_{\mathbb{C},\tau}$		
	$\mathsf{S}^{2k+1}_{\mathbb{C},\tau}$	$1 \leq k \leq n-1$
	S^{k}_{1}	$2 \leq k \leq n$
$\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$		
	$\mathsf{S}^{4k+3}_{\mathbb{H},\tau}$	$1 \leq k \leq n-1$
	$\mathsf{S}^{2k+1}_{\mathbb{C},\tau}$	$1 \leq k \leq n$
	S^{k}_{1}	$2 \leq k \leq n$
	$\mathsf{S}^{k}_{1/\tau}$	$2 \leq k \leq 3$
$\mathsf{S}^{15}_{\mathbb{O},\tau}$		
	$\mathsf{S}^{7}_{\mathbb{H},\tau}$	
	$\mathsf{S}^{3}_{\mathbb{C},\tau}$	
	$\mathsf{S}^{k}_{1/\tau}$	$2 \leq k \leq 7$

Table 8.1: Totally geodesic submanifolds of dimension $d \geq 2$ in Hopf-Berger spheres with $\tau \geq 1/2$, up to congruence.

of $\mathsf{S}^3_{\mathbb{C},\tau}$ are geodesics, and for the other Hopf-Berger spheres with $\tau \geq 1/2$ they are:

$$\begin{aligned}
\mathsf{S}^{2n+1}_{\mathbb{C},\tau}:&\quad \mathsf{S}^{2n-1}_{\mathbb{C},\tau}, \quad \mathsf{S}^{n}_{1}, \quad \text{where } n \geq 2,\\
\mathsf{S}^{4n+3}_{\mathbb{H},\tau}:&\quad \mathsf{S}^{4n-1}_{\mathbb{H},\tau}, \quad \mathsf{S}^{2n+1}_{\mathbb{C},\tau}, \quad \text{where } n \geq 2,\\
\mathsf{S}^{7}_{\mathbb{H},\tau}:&\quad \mathsf{S}^{3}_{\mathbb{C},\tau}, \quad \mathsf{S}^{3}_{1/\tau},\\
\mathsf{S}^{15}_{\mathbb{O},\tau}:&\quad \mathsf{S}^{7}_{\mathbb{H},\tau}, \quad \mathsf{S}^{7}_{1/\tau}.
\end{aligned}$$

Furthermore, every totally geodesic submanifold of a Hopf-Berger sphere with $\tau \geq 1/2$ is extrinsically homogeneous, see Remark 8.3.3.

As we have already seen, Hopf-Berger spheres maintain a close relationship with symmetric spaces of rank one, but they fail to be symmetric. A way to measure the extend to which a homogeneous space fails to be symmetric is by employing the index of symmetry, see [25] or [146]. Recall that a homogeneous space $M = \mathsf{G}/\mathsf{K}$ is symmetric if and only if for every point $p \in M$ and every $v \in T_pM$, there is a Killing vector field X on M with $X_p = v$ and $(\nabla X)_p = 0$. For every $p \in M$, the symmetric subspace of T_pM is defined as

$$\mathfrak{s}_p := \{X_p \in T_pM : \ X \in \mathfrak{K}(M) \text{ and } (\nabla X)_p = 0\},$$

where $\mathfrak{K}(M)$ denotes the set of Killing vector fields on M. It turns out that the symmetric subspaces of $M = \mathsf{G}/\mathsf{K}$ form a G-invariant distribution on M called the *distribution of symmetry* of M. This distribution is integrable and its leaves are symmetric spaces that are totally geodesically embedded in M. The index of symmetry of a homogeneous manifold M is the rank of its distribution of symmetry. We denote it by $\mathrm{ind}_S(M)$. In this chapter, we also compute the index of symmetry of Hopf-Berger spheres.

Theorem B. *Let* $\mathsf{S}^n_{\mathbb{F},\tau}$*,* $\tau \neq 1$*, be a Hopf-Berger sphere. Then, its index of symmetry is given by*

$$\operatorname{ind}_S(\mathsf{S}^n_{\mathbb{F},\tau}) = \begin{cases} 0 & \text{if } \mathbb{F} = \mathbb{H} \text{ or } \mathbb{O} \text{ and } \tau \neq 1/2, \\ 1 & \text{if } \mathbb{F} = \mathbb{C}, \\ 3 & \text{if } \mathbb{F} = \mathbb{H} \text{ and } \tau = 1/2, \\ 7 & \text{if } \mathbb{F} = \mathbb{O} \text{ and } \tau = 1/2. \end{cases}$$

This chapter is organized as follows. In Section §8.1, some tools that will be used in later sections are developed. Of particular relevance for the chapter is Lemma 8.1.3, which states that the totally geodesic property is preserved under rescalings in the direction of the fibers of the metric of the total space of a Riemannian submersion for a certain class of totally geodesic submanifolds (these submanifolds will be called well-positioned in Subsection §8.4.1). The purpose of Section §8.2 is to recall some well-known facts about totally geodesic submanifolds in rank one symmetric spaces.

In Section §8.3 we introduce the reductive decomposition of Hopf-Berger spheres that we use in this chapter. This allows us to compute the curvature tensor and to provide examples of totally geodesic submanifolds in Subsections §8.3.1 and §8.3.2.

Section §8.4 is devoted to the classification of totally geodesic submanifolds of Hopf-Berger spheres. In Subsection §8.4.1 we define and study well-positioned totally geodesic submanifolds of $\mathsf{S}^n_{\mathbb{F},\tau}$. It is important to highlight Proposition 8.4.2, which asserts that every complete well-positioned totally geodesic submanifold is characterized as the intersection of $\mathsf{S}^n_{\mathbb{F},\tau}$ regarded as a geodesic sphere of $\bar{M}$ with a totally geodesic submanifold M of a rank one symmetric space $\bar{M}$, with M containing the center of the geodesic sphere $\mathsf{S}^n_{\mathbb{F},\tau}$. Moreover, by performing certain involved computations, we prove that every totally geodesic surface of $\mathsf{S}^n_{\mathbb{F},\tau}$, $\tau \geq 1/2$, is well-positioned. In Subsection §8.4.3, we prove that every totally geodesic submanifold of dimension $d \geq 2$ in a Hopf-Berger sphere with $\tau \geq 1/2$ is well-positioned, thus obtaining Theorem A. The idea is to show that every not well-positioned totally geodesic submanifold contains a not well-positioned totally geodesic surface, see Proposition 8.4.11, which gives a contradiction with the fact that every totally geodesic surface in $\mathsf{S}^n_{\mathbb{F},\tau}$ is well-positioned. Some ingredients used in this section are: the polarity of the isotropy representation of $\mathsf{S}^n_{\mathbb{F},\tau}$, certain algebraic properties of bounded Killing vector fields, and the fact that a homogeneous space all whose geodesics are closed and of the same length is a symmetric space of rank one and compact type.

Finally, in Section §8.5, we compute the index of Hopf-Berger spheres by noticing that every symmetry of a Hopf-Berger sphere (see the beginning of Section §8.5 for the definition) is an isometry, and by using the techniques to compute this invariant developed in [146].

8.1 Totally geodesic submanifolds, geodesic spheres and Riemannian submersions

Let $(\bar{M}, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold. We recall that we denote by $\bar{\nabla}$ its Levi-Civita connection, by $\bar{R}$ its curvature tensor defined by the convention $\bar{R}(X,Y) = [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}$, and by $\operatorname{inj}(p)$ its injectivity radius at $p \in \bar{M}$.

Let M be an immersed submanifold of $\bar{M}$. We denote its induced Levi-Civita connection by ∇ and its second fundamental form by II. Recall that a connected submanifold M is totally geodesic if every geodesic of M is also a geodesic of $\bar{M}$. Equivalently, a connected submanifold M of $\bar{M}$ is totally geodesic if and only if $II = 0$. Moreover, every totally geodesic submanifold M of $\bar{M}$ can be extended to a complete totally geodesic submanifold of $\bar{M}$, see Theorem 5.2.5. Given a complete totally geodesic submanifold M of $\bar{M}$ passing through $p \in \bar{M}$, we have $\overline{\exp}_p(V) = M$, for some linear subspace V of $T_p\bar{M}$, where $\overline{\exp}_p$ denotes the Riemannian exponential map of $\bar{M}$ at $p \in \bar{M}$.

In what follows, we give an overview of some well-known facts about the extrinsic geometry of geodesic spheres in Riemannian manifolds. Let $v(s)$, $s \in (-\varepsilon, \varepsilon)$, be a smooth curve in the unit tangent sphere $\mathsf{S}(T_p\bar{M})$ and let us consider the geodesic variation $\Gamma(s,t) := \gamma_{v(s)}(t) = \overline{\exp}_p(tv(s))$. Let us denote by $\frac{D}{\partial t}$ and $\frac{D}{\partial s}$ the partial covariant derivatives. Then, if $0 < t < \operatorname{inj}(p)$, the field along $\gamma_{v(s)}(t)$ given by $\dot{\gamma}_{v(s)}(t) = \frac{\partial}{\partial t}\gamma_{v(s)}(t)$ is the outer unit normal at $\gamma_{v(s)}(t)$ to the geodesic sphere $\mathsf{S}_t(p) := \overline{\exp}_p(t\mathsf{S}(T_pM))$ of radius $t > 0$. Since $\|\dot{\gamma}_{v(s)}(t)\| = 1$, we have that $\frac{D}{\partial s}_{|s=0}\dot{\gamma}_{v(s)}(t)$ and $\dot{\gamma}_{v(s)}(t)$ are orthogonal. Thus,

$$\frac{D}{\partial s}_{|s=0}\dot{\gamma}_{v(s)}(t) = -\mathcal{S}^t J(t),$$

where $\mathcal{S}^t$ denotes the shape operator of the geodesic sphere $\mathsf{S}_t(p)$ in the outer direction and $J(t) = \frac{\partial}{\partial s}_{|s=0}\Gamma(s,t)$ is a Jacobi field along $\gamma_{v(0)}(t)$ with initial conditions $J(0) = 0$ and $\frac{D}{\partial t}_{|t=0}J(t) = \dot{v}(0)$. Now using that $\frac{D}{\partial s}_{|s=0}\dot{\gamma}_{v(s)}(t) = \frac{D}{\partial t}J(t)$, we obtain the following expression for the shape operator of the geodesic sphere

$$\mathcal{S}^t J(t) = -\frac{D}{\partial t}J(t).$$

Lemma 8.1.1. *Let $\bar{M}$ be a Riemannian manifold and let Σ be an embedded totally geodesic submanifold of $\bar{M}$. Let M be an embedded submanifold of $\bar{M}$ satisfying the following conditions:*

i) $\Sigma \cap M$ is non-empty.

ii) $T_q\Sigma$ contains the normal space $\nu_q M$ of M for every $q \in \Sigma \cap M$.

Then, any connected component of $\Sigma \cap M$ is an embedded totally geodesic submanifold of M and $T_q(\Sigma \cap M)$ is invariant under the shape operator $\mathcal{S}_\xi$ of M, for every $\xi \in \nu_q M$ and $q \in \Sigma \cap M$.

Proof. By *i)* and *ii)*, Σ and M are transverse at every $q \in \Sigma \cap M$. Then, it follows that $\Sigma \cap M$ is an embedded submanifold of $\bar{M}$. Moreover, $T_q(\Sigma \cap M) = (T_q\Sigma) \cap (T_qM)$ and $T_q\Sigma = T_q(\Sigma \cap M) \oplus \nu_q M$, for every $q \in \Sigma \cap M$.

We may assume that $\dim(\Sigma \cap M) > 0$. Let X, Y be tangent fields of $\Sigma \cap M$ defined around q. Thus $(\bar{\nabla}_X Y)_q \in T_q\Sigma = T_q(\Sigma \cap M) \oplus \nu_q M$. Hence, the tangent projection of $(\bar{\nabla}_X Y)_q$ to M belongs to $T_q(\Sigma \cap M)$. This shows that $\Sigma \cap M$ is a totally geodesic submanifold of M. Let ξ be a unit normal vector field of M defined around $q \in \Sigma \cap M$ and $v \in T_q(\Sigma \cap M)$. Since Σ is a totally geodesic submanifold of $\bar{M}$, $(\bar{\nabla}_v \xi)_q \in T_q\Sigma = T_q(\Sigma \cap M) \oplus \nu_q M$. Consequently, the projection of $(\bar{\nabla}_v \xi)_q$ to T_qM belongs to $T_q(\Sigma \cap M)$ and this shows that $\mathcal{S}_{\xi_q} T_q(\Sigma \cap M) \subset T_q(\Sigma \cap M)$. □

As a consequence of Lemma 8.1.1 and the previous discussion, we obtain the following result.

Corollary 8.1.2. *Let $\bar{M}$ be a Riemannian manifold, $p \in \bar{M}$ and $\mathsf{S}_t(p)$ the geodesic sphere of radius $t \in (0, \mathrm{inj}(p))$. Let $V \subset T_p\bar{M}$ be a vector subspace and consider the set $V_r := \{v \in V : \|v\| < r\}$, where $r \in (t, \mathrm{inj}(p))$. Assume that $\Sigma = \overline{\exp}_p(V_r)$ is a totally geodesic submanifold of $\bar{M}$. Then, $\mathsf{S}_t(p) \cap \Sigma$ is a totally geodesic submanifold of $\mathsf{S}_t(p)$ whose tangent space is invariant under the shape operator $\mathcal{S}^t$ of $\mathsf{S}_t(p)$.*

The following lemma shows that there is a certain class of totally geodesic submanifolds of the total space of a given Riemannian submersion that remain totally geodesic when we rescale the metric on the vertical distribution.

Lemma 8.1.3. *Let $\pi\colon M \to B$ be a Riemannian submersion with totally geodesic fibers and denote by $\mathcal{H}$ and $\mathcal{V}$ the horizontal and vertical distributions, respectively. Consider the family of Riemannian metrics $\langle\cdot,\cdot\rangle^\tau$ of M obtained by rescaling the metric tensor $\langle\cdot,\cdot\rangle$ of M by a factor $\tau > 0$ in the vertical distribution. Moreover, let Σ be a totally geodesic submanifold of M, with respect to the original metric, such that*

$$T_q\Sigma = W_1(q) \oplus W_2(q), \quad \text{for all } q \in \Sigma,$$

where $W_1(q) \subset \mathcal{H}_q$ and $W_2(q) \subset \mathcal{V}_q$. Then, Σ is a totally geodesic submanifold of M with respect to any metric $\langle\cdot,\cdot\rangle^\tau$.

Proof. Let us assume that B has dimension m and M has dimension $m + n$. By the continuity of the vertical and horizontal projections, we have that $q \in \Sigma \mapsto \dim(W_i(q))$ is an upper semicontinuous function for each $i \in \{1, 2\}$. Thus, since $T_q\Sigma = W_1(q) \oplus W_2(q)$ for every $q \in \Sigma$, we deduce that $\dim(W_i(q))$ is constant for every $i \in \{1, 2\}$. From now on, we will assume that $r = \dim(W_1(q))$ and $s = \dim(W_2(q))$ for every $q \in \Sigma$.

We work locally, since the property of being totally geodesic is local. From the assumptions, it follows that $\pi(\Sigma)$ is a submanifold of B of dimension r, since π has constant rank m. Moreover, $\pi(\Sigma)$ is totally geodesic in B. Indeed, let γ be a geodesic of B with initial velocity tangent to $\pi(\Sigma)$. Observe that horizontal geodesics do not depend on τ. Then, γ lifts to a horizontal geodesic $\widetilde{\gamma}$ in M. Now, since Σ is totally geodesic in $(M, \langle\cdot,\cdot\rangle)$, the geodesic $\widetilde{\gamma}$ is contained in Σ and γ is contained in $\pi(\Sigma)$, thus proving that $\pi(\Sigma)$ is totally geodesic in B.

Let $(\check{E}_1, \cdots, \check{E}_r, \cdots, \check{E}_m)$ be an orthonormal frame around $\pi(q)$ where the first r fields are tangent to $\pi(\Sigma)$ when restricted to $\pi(\Sigma)$. Since the fibers of π are totally geodesic, we have that $\Sigma \cap \pi^{-1}(\pi(q))$ is a totally geodesic submanifold around q. Let $(E_1, \cdots, E_m)$ be the horizontal lift of this frame around q. Notice that $E_1, \ldots, E_r$ restricted to Σ around q are tangent to Σ. Let $(\xi_1, \cdots, \xi_n)$ be an orthonormal vertical frame, with respect to $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle^1$, such that the first s fields are tangent to $\Sigma \cap \pi^{-1}(\pi(q))$ when restricted to $\Sigma \cap \pi^{-1}(\pi(q))$.

Notice that $\langle H_1 + U_1, H_2 + U_2 \rangle^\tau = \langle H_1, H_2 \rangle^1 + \tau \langle U_1, U_2 \rangle^1 = \langle H_1, H_2 \rangle + \tau \langle U_1, U_2 \rangle$, where H_i is a horizontal vector field and U_i is a vertical vector field, $i \in \{1, 2\}$. Then, the frame $(E_1, \cdots E_m, \xi_1, \cdots, \xi_n)$ is orthogonal with respect to any $\langle \cdot, \cdot \rangle^\tau$ and each one of its elements has constant length. Let ∇^τ be the Levi-Civita connection associated with $\langle \cdot, \cdot \rangle^\tau$ and denote by ∇ the Levi-Civita connection of M when $\tau = 1$. By the Koszul formula, we get

$$2\langle \nabla^\tau_X Y, Z \rangle^\tau = \langle [X, Y], Z \rangle^\tau - \langle [X, Z], Y \rangle^\tau - \langle [Y, Z], X \rangle^\tau$$

for all $X, Y, Z \in \{E_1, \cdots, E_m, \xi_1, \cdots, \xi_n\}$.

Let $X, Y \in \{E_1, \cdots, E_r, \xi_1, \cdots, \xi_s\}$ and $Z \in \{E_{r+1}, \cdots, E_m, \xi_{s+1}, \cdots, \xi_n\}$. Since Σ is totally geodesic when $\tau = 1$, we have $\langle \nabla_X Y, Z \rangle_q = 0$. This combined with the fact that the vertical distribution is integrable, and the bracket of a projectable vector field with a vertical vector field is vertical, implies that $\langle \nabla^\tau_X Y, Z \rangle^\tau_q = 0$. This proves that Σ is a totally geodesic submanifold of $(M, \langle \cdot, \cdot \rangle^\tau)$ for every $\tau > 0$. □

Notice that homogeneous spaces are real analytic, see [35, Lemma 1.1]. Let $M = \mathsf{G}/\mathsf{K}$ be a Riemannian homogeneous space. See Subsection §1.3.1 for a quick introduction to homogeneous spaces. Since every G-invariant tensor is parallel under the canonical connection ∇^c, we have $\nabla^c R = 0$ and $\nabla^c D = 0$, where R and D denote the curvature tensor and the difference tensor of $M = \mathsf{G}/\mathsf{K}$, respectively. Thus, by Equations (1.2) and (1.3), the covariant derivatives of the curvature tensor can be expressed in terms of D and R (see Remark 1.3.1). Hence, every subspace $\mathfrak{p}_\Sigma$ of $\mathfrak{p}$ invariant under D and R is invariant under every covariant derivative of R, which implies by Theorem 5.2.5 that $\mathfrak{p}_\Sigma$ is the tangent space of some totally geodesic submanifold of M and $\Sigma = \exp_o \mathfrak{p}_\Sigma$ is a complete totally geodesic submanifold of M, where $\exp_o$ denotes the Riemannian exponential map of M at o. Hence, we obtain the following result.

Lemma 8.1.4. *Let $M = \mathsf{G}/\mathsf{K}$ be a Riemannian homogeneous space with base point $o \in M$ and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{p}_\Sigma$ be a subspace of $\mathfrak{p}$ invariant under D and R. Then, there is a complete totally geodesic submanifold Σ of M such that $T_o\Sigma = \mathfrak{p}_\Sigma$.*

We remark that Lemma 8.1.4 provides a sufficient condition to obtain totally geodesic submanifolds in homogeneous spaces in terms of a linear algebraic property. However, this condition does not need to be necessary.

$\mathbb{C}\mathsf{P}^n$		
	$\mathbb{C}\mathsf{P}^k$	$2 \leq k \leq n-1$
	$\mathbb{R}\mathsf{P}^k$	$2 \leq k \leq n$
	S^k_4	$1 \leq k \leq 2$
$\mathbb{H}\mathsf{P}^n$		
	$\mathbb{H}\mathsf{P}^k$	$2 \leq k \leq n-1$
	$\mathbb{C}\mathsf{P}^k$	$2 \leq k \leq n$
	$\mathbb{R}\mathsf{P}^k$	$2 \leq k \leq n$
	S^k_4	$1 \leq k \leq 4$
$\mathbb{O}\mathsf{P}^2$		
	$\mathbb{H}\mathsf{P}^2$	
	$\mathbb{C}\mathsf{P}^2$	
	$\mathbb{R}\mathsf{P}^2$	
	S^k_4	$1 \leq k \leq 8$

$\mathbb{C}\mathsf{H}^n$		
	$\mathbb{C}\mathsf{H}^k$	$2 \leq k \leq n-1$
	$\mathbb{R}\mathsf{H}^k$	$2 \leq k \leq n$
	$\mathbb{R}\mathsf{H}^k(4)$	$1 \leq k \leq 2$
$\mathbb{H}\mathsf{H}^n$		
	$\mathbb{H}\mathsf{H}^k$	$2 \leq k \leq n-1$
	$\mathbb{C}\mathsf{H}^k$	$2 \leq k \leq n$
	$\mathbb{R}\mathsf{H}^k$	$2 \leq k \leq n$
	$\mathbb{R}\mathsf{H}^k(4)$	$1 \leq k \leq 4$
$\mathbb{O}\mathsf{H}^2$		
	$\mathbb{H}\mathsf{H}^2$	
	$\mathbb{C}\mathsf{H}^2$	
	$\mathbb{R}\mathsf{H}^2$	
	$\mathbb{R}\mathsf{H}^k(4)$	$1 \leq k \leq 8$

Figure 8.1: Totally geodesic submanifolds in symmetric spaces of rank one and non-constant sectional curvature, up to congruence.

8.2 Rank one symmetric spaces and their totally geodesic submanifolds

In this section we discuss some facts that will be useful later about totally geodesic submanifolds in symmetric spaces of rank one.

Let $\bar{M} = \bar{\mathsf{G}}/\bar{\mathsf{K}}$ be a connected Riemannian symmetric space, where $\bar{\mathsf{G}}$ is up to some finite quotient equal to $\mathrm{Isom}^0(\bar{M})$, the connected component of the identity of the isometry group of $\bar{M}$, and $\bar{\mathsf{K}}$ is the isotropy at some fixed point $p \in \bar{M}$. Let s_q be the geodesic reflection of $\bar{M}$ at $q \in \bar{M}$. It turns out that a submanifold Σ of $\bar{M}$ is a complete totally geodesic submanifold of $\bar{M}$ if and only if $s_q\Sigma = \Sigma$ for every $q \in \Sigma$. Recall that the rank of a symmetric space is defined as the dimension of a maximal flat totally geodesic submanifold.

The complete Riemannian manifolds with non-zero constant sectional curvature are: real hyperbolic spaces, round spheres and real projective spaces. These are symmetric spaces of rank one. Furthermore, the remaining symmetric spaces of rank one are hyperbolic and projective spaces over $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. The totally geodesic submanifolds in symmetric spaces of rank one were classified by Wolf in [187], see Figure 8.1.

Let $\kappa \in (0, \infty)$. We denote by S^n_κ, $\mathbb{R}\mathsf{P}^n(\kappa)$ and $\mathbb{R}\mathsf{H}^n(\kappa)$ the round sphere of sectional curvature κ, the real projective space of sectional curvature κ and the real hyperbolic space with sectional curvature $-\kappa$, respectively. In the case that $\kappa = 1$ or $\kappa = -1$, we write S^n, $\mathbb{R}\mathsf{P}^n$, or $\mathbb{R}\mathsf{H}^n$, respectively.

Now we assume that $\bar{M} = \bar{\mathsf{G}}/\bar{\mathsf{K}}$ has rank one and that the minimal absolute value of its sectional curvature is equal to 1. One has that $\bar{\mathsf{K}}$, via the isotropy representation, acts transitively on $\mathsf{S}(T_p\bar{M})$, the unit sphere of $T_p\bar{M}$. Moreover, $\bar{\mathsf{K}}$ acts transitively on the geodesic sphere $\mathsf{S}_t(p)$ of radius t centered at $p \in \bar{M}$, for each $t \in (0, \operatorname{inj}(p))$. This implies that geodesic spheres are $\bar{\mathsf{K}}$-homogeneous spaces. Recall that the Jacobi operator $\bar{R}_v$ associated with $v \in T_p\bar{M}$ is defined by $\bar{R}_v(X) = \bar{R}(X, v, v)$ for every $X \in T_p\bar{M}$. In this case, the Jacobi operators $\{\bar{R}_v : v \in T_p\bar{M},\ \|v\| = t\}$ are all conjugate by elements of $\bar{\mathsf{K}}$.

Let us assume that $\bar{M}$ does not have constant sectional curvature. It is well known that, for any non-zero $v \in T_p\bar{M}$, the Jacobi operator $\bar{R}_v$ restricted to $T_p\bar{M} \ominus \mathbb{R}v$ has only two different eigenvalues λ_1 and λ_2. Let $v \in T_p\bar{M}$ be of unit length. Then, $\lambda_1 = \pm 4,\ \lambda_2 = \pm 1$, where the signs are positive if $\bar{M}$ is of the compact type, and negative if $\bar{M}$ is of non-compact type. Observe that $\bar{R}_v$ has exactly one more eigenvalue that is equal to zero and has associated eigenspace $\mathbb{R}v$. Let $\mathbb{V}_i(v)$ be the eigenspace of $\bar{R}_v$ associated with λ_i for each $i \in \{1, 2\}$. When $\bar{M}$ is of compact type, $\operatorname{inj}(\bar{M}) = \pi/2$. However, if $\bar{M}$ is of non-compact type, then it follows by the Cartan-Hadamard Theorem that $\operatorname{inj}(\bar{M}) = +\infty$.

Fix $p \in \bar{M}$. For each $i \in \{1, 2\}$ we define $\mathbb{V}_i^t(v)$ to be the parallel transport of $\mathbb{V}_i(v)$ along the unit speed geodesic $\gamma_v\colon [0, \operatorname{inj}(\bar{M})) \to \bar{M}$ with initial conditions $p \in \bar{M}$ and $v \in T_p\bar{M}$. The equation $\bar{\nabla}\bar{R} = 0$ implies that $\mathbb{V}_i^t(v)$ is the eigenspace of $\bar{R}_{\cdot, \dot{\gamma}_v(t)}\dot{\gamma}_v(t)$ associated with λ_i, for each $i \in \{1, 2\}$. Moreover, $\mathbb{V}_i^t(v) \perp \dot{\gamma}_v(t)$ and then we have the following orthogonal decomposition

$$T_{\gamma_v(t)}\mathsf{S}_t(p) = \mathbb{V}_1^t(v) \oplus \mathbb{V}_2^t(v).$$

Let $\bar{\mathsf{K}}_v$ be the isotropy of $\bar{\mathsf{K}}$ at v. Since $\bar{\mathsf{K}}$ preserves the curvature tensor $\bar{R}$ of $\bar{M}$ at p, then $\bar{\mathsf{K}}_v$ commutes with the Jacobi operator $\bar{R}_v$ and therefore leaves $\mathbb{V}_i(v)$ invariant. Notice that $\bar{\mathsf{K}}_v$ coincides with the isotropy $\bar{\mathsf{K}}_{\gamma_v(t)}$ of the action of $\bar{\mathsf{K}}$ on the geodesic sphere $\mathsf{S}_t(p)$ at $\gamma_v(t)$ for every $t \in (0, \operatorname{inj}(\bar{M}))$. Moreover, $\bar{\mathsf{K}}_{\gamma_v(t)}$ leaves invariant $\mathbb{V}_i^t(v)$ since $\bar{\mathsf{K}}_{\gamma_v(t)}$ and $\bar{R}_{\cdot, \dot{\gamma}_v(t)}\dot{\gamma}_v(t)$ commute. Hence, for any fixed $t \in (0, \operatorname{inj}(\bar{M}))$, the subspace $\mathbb{V}_i^t(v)$ extends to a $\bar{\mathsf{K}}$-invariant distribution $\mathcal{D}_i^t$ on the geodesic sphere $\mathsf{S}_t(p)$. Thus,

$$T\mathsf{S}_t(p) = \mathcal{D}_1^t \oplus \mathcal{D}_2^t.$$

The distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ coincide with the vertical and horizontal distributions, respectively, defined by the Hopf fibrations.

Remark 8.2.1. Using Jacobi field theory, one can prove that $\mathbb{V}_1^t(v)$ and $\mathbb{V}_2^t(v)$ are the eigenspaces of the shape operator $\mathcal{S}^t$ of $\mathsf{S}_t(p)$ in $\bar{M}$ with corresponding eigenvalues β_1^t and β_2^t, respectively. These eigenvalues are given by the following expressions (see e.g. [33, p. 16]).

$$\beta_1^t = \begin{cases} -2\cot(2t) & \text{if } \bar{M} \text{ is of the compact type,} \\ -2\coth(2t) & \text{if } \bar{M} \text{ is of the non-compact type.} \end{cases} \tag{8.2}$$

$$\beta_2^t = \begin{cases} -\cot(t) & \text{if } \bar{M} \text{ is of the compact type,} \\ -\coth(t) & \text{if } \bar{M} \text{ is of the non-compact type.} \end{cases} \tag{8.3}$$

Notice that $\beta_1^t \neq \beta_2^t$ for every $t \in (0, \operatorname{inj}(\bar{M}))$. Moreover, since the connected group $\bar{\mathsf{K}}$ acts isometrically on $\mathsf{S}_t(p)$, the shape operator $\mathcal{S}^t$ is a $\bar{\mathsf{K}}$-invariant tensor. Then, $\mathcal{D}_i^t$ is an $\mathcal{S}^t$-invariant distribution of $\mathsf{S}_t(p)$ and $\mathcal{S}^t_{|\mathcal{D}_i^t} = \beta_i^t \operatorname{Id}_{|\mathcal{D}_i^t}$, for each $i \in \{1, 2\}$.

Remark 8.2.2. If $\bar{M}$ is of compact type, it is well known that $\mathbb{V}_1(v) \oplus \mathbb{R}v \subset T_p\bar{M}$ is the tangent space of a totally geodesic sphere of $\bar{M}$ isometric to S^{m+1}_4, which we denote by $\mathsf{S}^{m+1}(v)$, where $m = \mathrm{rk}(\mathcal{D}^t_1)$. This is a so-called Helgason sphere, see [89] and [144].

This implies that $\mathcal{D}^t_1$ is an integrable distribution of $\mathsf{S}_t(p)$ whose integral submanifold through $\gamma_w(t) = \overline{\exp}(tw)$, where $w \in T_p\bar{M}$ is of unit length, is the intersection of $\mathsf{S}^{m+1}(w)$ with $\mathsf{S}_t(p)$. Moreover, by Corollary 8.1.2, $\mathcal{D}^t_1$ is an autoparallel distribution of $\mathsf{S}_t(p)$. The same is true when $\bar{M}$ is of non-compact type. But in this case $\mathbb{V}_1(v) \oplus \mathbb{R}v$ is the tangent space at p of a totally geodesic hyperbolic space of $\bar{M}$ that is isometric to $\mathbb{R}\mathsf{H}^{m+1}(4)$.

We finish this section with a remark that will be useful in Section §8.4.

Remark 8.2.3. Let $\bar{M} = \bar{\mathsf{G}}/\bar{\mathsf{K}}$ be a simply connected compact symmetric space with non-constant sectional curvature and minimal sectional curvature 1. Recall that $\mathrm{inj}(\bar{M})$ is $\pi/2$. Let $p \in \bar{M}$ and let M be a complete totally geodesic submanifold of $\bar{M}$ that contains p. Then, M is a rank one symmetric space of compact type with $\mathrm{inj}(M) = \pi/2$ or a closed geodesic of length π.

Let $B^E_{\pi/2}(0)$ be the Euclidean open ball of $T_p\bar{M}$ of radius $\pi/2$ centered at the origin. Hence, $\overline{\exp}_p\colon B^E_{\pi/2}(0) \to B_{\pi/2}(p)$ is a diffeomorphism, where $B_{\pi/2}(p)$ is the open ball of $\bar{M}$ with center p and radius $\pi/2$. Now, since $\mathrm{inj}(M) = \pi/2$, the points x of M that are in the complement of $\overline{\exp}_p(B^E_{\pi/2}(0) \cap T_pM)$ must be at a distance of $\pi/2$ from p in $\bar{M}$. Indeed, if $\gamma\colon [0, \pi/2] \to M$ is a minimizing unit speed geodesic in M from p to x, we have that γ restricted to $[0, \pi/2)$ is a minimizing geodesic of $\bar{M}$, since M is totally geodesic. Thus, the distance in $\bar{M}$ from p to x is $\pi/2$. This implies that $M \cap B_{\pi/2}(p) = \overline{\exp}_p(B^E_{\pi/2}(0) \cap T_pM)$ and then $M \cap \mathsf{S}_t(p) = \overline{\exp}_p(B^E_{\pi/2}(0) \cap T_pM) \cap \mathsf{S}_t(p)$, where $t \in (0, \pi/2)$.

8.3 Reductive decomposition of Hopf-Berger spheres

In this section we describe a reductive decomposition for each Hopf-Berger sphere.

Recall that $\{\mathsf{S}^n_{\mathbb{F},\tau}\}_{\tau>0}$ denotes the family of complex, quaternionic or octonionic Hopf-Berger spheres of dimension n, according to the value of $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. These are geodesic orbit spaces, see [175]. This means that every geodesic of $\mathsf{S}^n_{\mathbb{F},\tau}$ is the orbit of a 1-parameter subgroup of $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})$. Furthermore, according to [191], $\mathsf{S}^n_{\mathbb{F},\tau}$ admits, for each $\tau > 0$, a naturally reductive decomposition when $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. However, $\mathsf{S}^{15}_{\mathbb{O},\tau}$ does not admit a naturally reductive decomposition for most $\tau \in (0, +\infty)$, see [192].

The complex and quaternionic Hopf fibrations are invariant under the action of U_{n+1} and $\mathsf{Sp}_{n+1}\mathsf{Sp}_1$, respectively. The complex and quaternionic Hopf-Berger spheres can be expressed as $\mathsf{S}^{2n+1}_{\mathbb{C},\tau} = \mathsf{U}_{n+1}/\mathsf{U}_n$ and $\mathsf{S}^{4n+3}_{\mathbb{H},\tau} = \mathsf{Sp}_{n+1}/\mathsf{Sp}_n$. Now we will describe the reductive decomposition for $\mathsf{S}^n_{\mathbb{F},\tau} = \mathsf{G}/\mathsf{K}$, for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, where the Lie groups G and K are as above. We refer the reader to Section §1.3 for more details on homogeneous spaces and reductive decompositions. Let $\mathfrak{g}$ be the Lie algebra of G for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. We denote by $\mathrm{Im}(x)$ the imaginary part of an element $x \in \mathbb{F}$ and by A^* the conjugate transpose of a matrix A with entries in $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. Then, we consider the following subspaces of $\mathfrak{g}$:

$$\mathfrak{k} = \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & 0 \end{array} \right), \quad \mathfrak{p}_1 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \operatorname{Im}(x) \end{array} \right), \quad \mathfrak{p}_2 = \left(\begin{array}{c|c} 0 & v \\ \hline -v^* & 0 \end{array} \right),$$

where $x \in \mathbb{F}$, $v \in \mathbb{F}^n$ and Z belongs to $\mathfrak{u}_n$ or $\mathfrak{sp}_n$, when $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$, respectively. Observe that $\mathfrak{k}$ is the Lie algebra of K and $[\mathfrak{k}, \mathfrak{p}_1] = 0$. On the one hand, $\mathfrak{p}_1$ is spanned by $\{X_l\}_{l=1}^{\dim_{\mathbb{R}} \mathbb{F}-1}$, where each X_l is the matrix filled with zeroes except the last entry, which is equal to one of the imaginary units i, j or k, for X_1, X_2 or X_3, respectively. On the other hand, $\mathfrak{p}_2$ is identified with $\mathbb{F}^n$, and thus, we can consider the set $\{Y_j\}_{j=1}^n$ of $\mathfrak{p}_2$ where Y_j is identified with the j-th element of the canonical basis of $\mathbb{F}^n$ for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. Furthermore, the adjoint representation of G restricted to K acts on $\mathfrak{p}_2$ as the standard representation of U_n or Sp_n, depending on whether $\mathbb{F}$ is equal to $\mathbb{C}$ or $\mathbb{H}$.

Now let $\mathbb{F} = \mathbb{O}$. The octonionic Hopf fibration is invariant under the action of Spin_9 and $\mathsf{S}^{15}_{\mathbb{O},\tau} = \mathsf{Spin}_9/\mathsf{Spin}_7$. For positive integers i, j, such that $i < j < 9$, let $E_{ij} \in \mathfrak{spin}_9$ be the matrix filled with zeroes except for the entries (i, j) and (j, i) which are equal to 1 and -1, respectively. We define the following elements in $\mathfrak{g} = \mathfrak{spin}_9$.

$$\begin{aligned}
X_1 &= (E_{15} + E_{26} + E_{37} + E_{48}), & X_2 &= (E_{17} + E_{28} - E_{35} - E_{46}), \\
X_3 &= (E_{13} - E_{24} - E_{57} + E_{68}), & X_4 &= (E_{16} - E_{25} - E_{38} + E_{47}), \\
X_5 &= (E_{18} - E_{27} + E_{36} - E_{45}), & X_6 &= (E_{12} + E_{34} - E_{56} - E_{78}), \\
X_7 &= (E_{14} + E_{23} - E_{58} - E_{67}), & Y_1 &= 2E_{19}, \\
Y_2 &= 2E_{29}, & Y_3 &= 2E_{39}, \\
Y_4 &= 2E_{49}, & Y_5 &= 2E_{59}, \\
Y_6 &= 2E_{69}, & Y_7 &= 2E_{79}, \\
Y_8 &= 2E_{89}.
\end{aligned}$$

Then, we consider $\mathfrak{p}_1$ and $\mathfrak{p}_2$ as the subspaces of $\mathfrak{g}$ spanned by $\{X_i\}_{i=1}^7$ and $\{Y_i\}_{i=1}^8$, respectively. We define $\mathfrak{k}$ as the orthogonal complement of $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ with respect to the Killing form of $\mathfrak{g} = \mathfrak{spin}_9$. It turns out that $\mathfrak{k} \cong \mathfrak{spin}_7$ is the Lie algebra of the isotropy $\mathsf{K} = \mathsf{Spin}_7$ for the action of $\mathsf{G} = \mathsf{Spin}_9$ in $\mathsf{S}^{15}_{\mathbb{O},\tau}$ (cf. [180, p. 476]). Moreover, $\mathfrak{k}$ acts on $\mathfrak{p}_1$ as the standard representation of $\mathfrak{so}_7 = \mathfrak{spin}_7$ and on $\mathfrak{p}_2$ as the irreducible spin representation of dimension eight.

To sum up, for each Hopf-Berger sphere $\mathsf{S}^n_{\mathbb{F},\tau}$, it can be checked that $[\mathfrak{k}, \mathfrak{p}_1 \oplus \mathfrak{p}_2] \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Thus, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{p} := \mathfrak{p}_1 \oplus \mathfrak{p}_2$, defines a reductive decomposition for $\mathsf{S}^n_{\mathbb{F},\tau} = \mathsf{G}/\mathsf{K}$.

Notice that for each $X_i \in \mathfrak{p}_1$, the map $J \colon \mathfrak{p}_2 \to \mathfrak{p}_2$ given by $J_{X_i} Y := [Y, X_i]$ defines a complex structure in $\mathfrak{p}_2$. In the following, we simply write J_i to denote J_{X_i}, for each $i \in \{1, \ldots, \dim \mathbb{F} - 1\}$. Let $m := \dim \mathfrak{p}_2 / \dim \mathbb{F}$. We consider the unique K-invariant inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{p}$ such that $\{X_i/\sqrt{\tau}, Y_j, J_i Y_j\}_{i,j}^{\dim \mathbb{F}-1, m}$ and $\{X_i/(2\sqrt{\tau}), Y_j\}_{i,j}^{\dim \mathbb{F}-1, 8}$ are orthonormal bases for $\mathfrak{p}$ when $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$ and $\mathbb{F} = \mathbb{O}$, respectively. Thus, the tangent space of $\mathsf{S}^n_{\mathbb{F},\tau}$ at a base point o is identified with $\mathfrak{p}$ as inner product spaces. Furthermore, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ can be identified with the vertical and horizontal spaces at $o \in \mathsf{S}^n_{\mathbb{F},\tau}$ defined by the corresponding Hopf fibration, respectively.

It can be checked that J_i is a skew-symmetric endomorphism of $\mathfrak{p}_2$ with respect to the inner product $\langle \cdot, \cdot \rangle$, hence J_i is an orthogonal complex structure for each subindex $i \in \{1, \dots, \dim \mathbb{F} - 1\}$. On the one hand, if $\mathbb{F} = \mathbb{H}$, we have the relations $J_j J_i = J_k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. On the other hand, if $\mathbb{F} = \mathbb{O}$, the relations among the complex structures $\{J_i\}_{i=1}^7$ are given by the labeling of the Fano plane indicated in Figure 8.2. The Fano plane has 7 points and 7 lines. Each line I contains exactly three points, and each of these triples has a cyclic ordering shown by the arrows in such a way that if (i, j, k) are cyclically ordered, and in this case we have $J_j J_{i|\mathbb{H}_I Y_1} = J_{k|\mathbb{H}_I Y_1}$, where $\mathbb{H}_I Y_1 := \operatorname{span}\{Y_1, J_l Y_1\}_{l=1}^3$.

Remark 8.3.1. Let $v_1 \in \mathfrak{p}_1$ and $v_2 \in \mathfrak{p}_2$ be non-zero elements of lengths r and s, respectively. Moreover, denote by $\mathsf{S}_1(r)$ the sphere of radius r of $\mathfrak{p}_1$, and by $\mathsf{S}_2(s)$ the sphere of radius s of $\mathfrak{p}_2$. If $\mathbb{F} = \mathbb{O}$, then K acts polarly on $\mathfrak{p}$ and its orbits are given by

$$\mathsf{K} \cdot (v_1 + v_2) = \mathsf{S}_1(r) \times \mathsf{S}_2(s).$$

When $\mathbb{F} = \mathbb{H}$, we can extend the group $\mathsf{G} = \mathsf{Sp}_{n+1}$ acting transitively on $\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$ to the Lie group $\check{\mathsf{G}} = \mathsf{Sp}_{n+1}\mathsf{Sp}_1$ which also acts transitively and isometrically on $\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$. The isotropy of the action of $\check{\mathsf{G}}$ at $o \in \mathsf{S}^{4n+3}_{\mathbb{H},\tau}$ is $\check{\mathsf{K}} = \mathsf{Sp}_n\mathsf{Sp}_1$. Then $\check{\mathsf{K}}$ acts polarly on $\mathfrak{p}$ and its orbits are given by

$$\check{\mathsf{K}} \cdot (v_1 + v_2) = \mathsf{S}_1(r) \times \mathsf{S}_2(s).$$

Now, let $\mathbb{F} = \mathbb{C}$ and consider $\check{\mathsf{K}}$, the disconnected Lie group generated by K and σ, where σ is the isometry lying in the isotropy of $\mathsf{S}^{2n-1}_{\mathbb{C},\tau}$ induced by the standard conjugation in $\mathbb{C}^n$. Then, $\check{\mathsf{K}}$ and K act polarly in $\mathfrak{p}$ and their orbits are given by

$$\mathsf{K} \cdot (v_1 + v_2) = \{v_1\} \times \mathsf{S}_2(s), \quad \check{\mathsf{K}} \cdot (v_1 + v_2) = \mathsf{S}_1(r) \times \mathsf{S}_2(s).$$

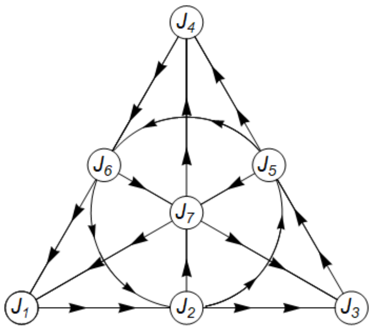

Figure 8.2: Fano plane with the appropiate labeling.

8.3.1 The curvature tensor of Hopf-Berger spheres

Now we compute the curvature tensor R and the difference tensor D of $\mathsf{S}^n_{\mathbb{F},\tau}$ when $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. Notice that D restricted to $\mathfrak{p}_1$ is zero and satisfies $D_X Y = -D_Y X$ for every

$X, Y \in \mathfrak{p}_2$. Let $Y, Z, W \in \mathfrak{p}_2$ spanning a totally real subspace of $\mathfrak{p}_2$ and let (i, j, k) be a cyclic permutation of $(1, 2, 3)$. Then, we have

$$\begin{aligned} &D_Y X_i = \tau J_i Y, \quad D_{X_i} Y = (\tau - 1) J_i Y, \quad D_Y J_i Y = -X_i, \quad D_{J_i Y} J_j Y = X_k, \\ &D_Y Z = 0. \end{aligned} \tag{8.4}$$

By using Equation (1.4), we have for a cyclic permutation (i, j, k) of $(1, 2, 3)$,

$$\begin{aligned} R(X_i, X_j)Y &= 2\tau(1-\tau)J_k Y, & R(X_i, X_j)X_i &= -X_j \\ R(X_i, Y)X_i &= -\tau^2 Y, & R(X_i, Y)J_j Y &= (1-\tau)X_k, \\ R(X_i, Y)X_j &= \tau(1-\tau)J_k Y, & R(X_i, Y)Y &= \tau X_i, \\ R(Y, J_i Y)Y &= (-4+3\tau)J_i Y, & R(Y, J_i Y)Z &= 2(-1+\tau)J_i Z, \\ R(Y, J_i Y)X_j &= 2(1-\tau)X_k, & R(Y, J_i Y)J_i Y &= (4-3\tau)Y, \\ R(Y, Z)Y &= -Z, & R(Y, Z)J_i Y &= (-1+\tau)J_i Z \end{aligned} \tag{8.5}$$

$$\begin{aligned} R(X_i, X_j)X_k &= R(X_i, Y)J_i Y = R(X_i, Y)Z = R(Y, J_i Y)X_i = R(Y, J_i Y)J_j Y \\ &= R(Y, Z)X_i = R(Y, Z)W = 0. \end{aligned} \tag{8.6}$$

Let us consider $R_X \colon \mathfrak{p} \to \mathfrak{p}$, the Jacobi operator associated with a vector $X \in \mathfrak{p}$, which is given by $R_X(Y) = R(Y, X)X$.

Let $X \in \mathfrak{p}_1$ be of unit length. By Equations (8.5) and (8.6), we deduce that R_X leaves $\mathfrak{p}_1$ and $\mathfrak{p}_2$ invariant. Moreover, $R_{X|\mathfrak{p}_2} = \tau \operatorname{Id}_{\mathfrak{p}_2}$, and the eigenvalues of $R_{X|\mathfrak{p}_1}$ are $1/\tau$ and 0, with multiplicity $\dim \mathbb{F} - 2$ and 1, respectively. Let $Y \in \mathfrak{p}_2$ be a unit vector. By Equations (8.5) and (8.6), we deduce that R_Y leaves $\mathfrak{p}_1$ and $\mathfrak{p}_2$ invariant. Moreover, $R_{Y|\mathfrak{p}_1} = \tau \operatorname{Id}_{\mathfrak{p}_1}$, and the eigenvalues of $R_{Y|\mathfrak{p}_2}$ are $4 - 3\tau$, 1 and 0 of multiplicities $\dim \mathbb{F} - 1$, $n - 2\dim \mathbb{F} + 1$ and 1, respectively.

Now we compute the curvature tensor R and the difference tensor D of $\mathsf{S}^{15}_{\mathbb{O},\tau}$. We have again that D restricted to $\mathfrak{p}_1$ is zero and satisfies $D_X Y = -D_Y X$ for every $X, Y \in \mathfrak{p}_2$. Let (i, j, k) be an ordered triple contained in a line of the Fano plane, see Figure 8.2. Then, we have

$$\begin{aligned} &D_{Y_1} X_i = 2\tau J_i Y_1, \quad D_{X_i} Y_1 = (2\tau - 1) J_i Y_1, \quad D_{Y_1} J_i Y_1 = -X_i/2, \\ &D_{J_i Y_1} J_j Y_1 = -X_k/2. \end{aligned} \tag{8.7}$$

Moreover, by using Equation (1.4), we get

$$\begin{aligned} R(X_i, X_j)X_i &= -4X_j, & R(X_i, X_j)Y_1 &= 8\tau(1-\tau)J_k Y_1, \\ R(X_i, Y_1)Y_1 &= \tau X_i, & R(X_i, Y_1)X_j &= 4\tau(1-\tau)J_k Y_1, \\ R(X_i, Y_1)X_i &= -4\tau^2 Y_1, & R(X_i, Y_1)J_j Y_1 &= (1-\tau)X_k, \\ R(Y_1, J_i Y_1)X_j &= 2(1-\tau)X_k, & R(Y_1, J_i Y_1)Y_1 &= (-4+3\tau)J_i Y_1, \\ R(Y_1, J_i Y_1)J_i Y_1 &= (4-3\tau)Y_1. \end{aligned} \tag{8.8}$$

$$R(X_i, X_j)X_k = R(X_i, Y_1)J_i Y_1 = R(Y_1, J_i Y_1)X_i = R(Y_1, J_i Y_1)J_j Y_1 = 0. \tag{8.9}$$

Let $X \in \mathfrak{p}_1$ be of unit length. By Equations (8.8) and (8.9), we deduce that R_X leaves $\mathfrak{p}_1$ and $\mathfrak{p}_2$ invariant. Moreover, $R_{X|\mathfrak{p}_2} = \tau \operatorname{Id}_{\mathfrak{p}_2}$ and the eigenvalues of $R_{X|\mathfrak{p}_1}$ are $1/\tau$ and 0, with multiplicity 6 and 1, respectively. Let $Y \in \mathfrak{p}_2$ be a unit vector. By Equations (8.8) and (8.9), we have that, R_Y leaves $\mathfrak{p}_1$ and $\mathfrak{p}_2$ invariant. Moreover, $R_{Y|\mathfrak{p}_1} = \tau \operatorname{Id}_{\mathfrak{p}_1}$ and the eigenvalues of $R_{Y|\mathfrak{p}_2}$ are $4 - 3\tau$ and 0 with multiplicities 7 and 1, respectively.

In Table 8.2 we summarize the pairs (λ, m_λ) of eigenvalues and multiplicities for the Jacobi operator.

	R_X	R_Y
$\mathfrak{p}_1$	$(0,1)$ $(1/\tau, \dim(\mathbb{F}) - 2)$	$(\tau, \dim(\mathbb{F}) - 1)$
$\mathfrak{p}_2$	$(\tau, n - \dim(\mathbb{F}) + 1)$	$(0,1)$ $(4 - 3\tau, \dim(\mathbb{F}) - 1)$ $(1, n - 2\dim(\mathbb{F}) + 1)$

Table 8.2: Eigenvalues and multiplicities of the Jacobi operator of $\mathsf{S}^n_{\mathbb{F},\tau}$ associated with unit vectors X and Y in $\mathfrak{p}_1$ and $\mathfrak{p}_2$, respectively.

8.3.2 Examples of totally geodesic submanifolds in Hopf-Berger spheres

In what follows, we provide examples of totally geodesic submanifolds in Hopf-Berger spheres.

Lemma 8.3.2. *Let Σ be a totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$, with $\tau \neq 1$. Let $\mathfrak{p}_\Sigma \subset \mathfrak{p}$ be identified with the tangent space of Σ at $o \in \mathsf{S}^n_{\mathbb{F},\tau}$. Then, if $\mathfrak{p}_\Sigma \cap \mathfrak{p}_i \neq 0$ for some $i \in \{1,2\}$, we have $\mathfrak{p}_\Sigma = (\mathfrak{p}_\Sigma \cap \mathfrak{p}_1) \oplus (\mathfrak{p}_\Sigma \cap \mathfrak{p}_2)$.*

Proof. Let us identify $\mathfrak{p}_\Sigma \subset \mathfrak{p}$ with the tangent space at $o \in \mathsf{S}^n_{\mathbb{F},\tau}$ of a totally geodesic submanifold Σ in $\mathsf{S}^n_{\mathbb{F},\tau}$. If $X \in \mathfrak{p}_\Sigma \cap \mathfrak{p}_1$ is a non-zero vector, we have $R_X \mathfrak{p}_\Sigma \subset \mathfrak{p}_\Sigma$. The sets of eigenvalues of $R_{X|\mathfrak{p}_1}$ and $R_{X|\mathfrak{p}_2}$ have non-trivial intersection if and only if $\tau = 1$, see Table 8.2. If $Y \in \mathfrak{p}_\Sigma \cap \mathfrak{p}_2$ is a non-zero vector, we have $R_Y \mathfrak{p}_\Sigma \subset \mathfrak{p}_\Sigma$. The sets of eigenvalues of $R_{Y|\mathfrak{p}_1}$ and $R_{Y|\mathfrak{p}_2}$ have non-trivial intersection if and only if $\tau = 1$, see Table 8.2. Hence, if there is some non-zero vector $X \in \mathfrak{p}_\Sigma \cap \mathfrak{p}_i$, for some $i \in \{1,2\}$, we have $\mathfrak{p}_\Sigma = (\mathfrak{p}_\Sigma \cap \mathfrak{p}_1) \oplus (\mathfrak{p}_\Sigma \cap \mathfrak{p}_2)$ when $\tau \neq 1$. □

Remark 8.3.3. Now we will discuss some examples of totally geodesic submanifolds of $\mathsf{S}^n_{\mathbb{F},\tau}$. Consider the following subspaces $(1)_{\mathbb{F}}$ and $(2)_{\mathbb{F}}$ of $\mathfrak{p}$, where $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

$(1)_{\mathbb{C}}$ $\mathfrak{p}_\Sigma = V$, where V is a totally real subspace of $\mathfrak{p}_2$.

$(2)_{\mathbb{C}}$ $\mathfrak{p}_\Sigma = \mathfrak{p}_1 \oplus V$, where $V \subset \mathfrak{p}_2$ is totally complex.

$(1)_{\mathbb{H}}$ $\mathfrak{p}_\Sigma = \mathbb{R}X \oplus V$, where $X \in \mathfrak{p}_1$ and V is a totally complex subspace of $\mathfrak{p}_2$ with respect to J_X, see Section §4.1.

$(2)_{\mathbb{H}}$ $\mathfrak{p}_\Sigma = \mathfrak{p}_1 \oplus V$, where V is invariant under J_X for every $X \in \mathfrak{p}_1$.

$(1)_{\mathbb{O}}$ $\mathfrak{p}_\Sigma = \operatorname{span}\{X_i\}_{i\in I} \oplus \mathbb{H}_I Y_1$, where I is a line of the Fano plane (see Figure 8.2).

$(2)_{\mathbb{O}}$ $\mathfrak{p}_\Sigma = \mathfrak{p}_1$.

It is clear from Equations (8.4-8.9) that these subspaces are invariant under the difference and curvature tensors. Hence, by Lemma 8.1.4, they induce totally geodesic submanifolds in $\mathsf{S}^n_{\mathbb{F},\tau}$. Additionally, it can be checked that these subspaces are invariant under U, see Subsection §1.3.1 for the definition of U.

On the one hand, if we consider Example $(1)_{\mathbb{F}}$, for each $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, we have that $\mathfrak{k}_\Sigma := [\mathfrak{p}_\Sigma, \mathfrak{p}_\Sigma]_{\mathfrak{k}}$ is isomorphic to $\mathfrak{so}_k$, $\mathfrak{u}_k$, or $\mathfrak{sp}_1 \oplus \mathfrak{sp}_1$, where k is the real or complex dimension of V, respectively. The Lie algebra generated by $\mathfrak{p}_\Sigma$ is $\mathfrak{g}_\Sigma = \mathfrak{k}_\Sigma \oplus \mathfrak{p}_\Sigma$, which is isomorphic to $\mathfrak{so}_{k+1}$, $\mathfrak{u}_{k+1}$ or $\mathfrak{sp}_2 \oplus \mathfrak{sp}_1$, respectively. On the other hand, if we consider Example $(2)_{\mathbb{F}}$, we have that $\mathfrak{k}_\Sigma$ is isomorphic to $\mathfrak{u}_k$, $\mathfrak{sp}_k$, or $\mathfrak{so}_7$, where k is the complex or quaternionic dimension of V, respectively, and the Lie algebra generated by $\mathfrak{p}_\Sigma$ is $\mathfrak{g}_\Sigma = \mathfrak{k}_\Sigma \oplus \mathfrak{p}_\Sigma$, which is isomorphic to $\mathfrak{u}_{k+1}$, $\mathfrak{sp}_{k+1}$, or $\mathfrak{so}_8$. Let G_Σ be the connected subgroup of G with Lie algebra $\mathfrak{g}_\Sigma$. Observe that $\mathfrak{p}_\Sigma$ is identified with $T_o(\mathsf{G}_\Sigma \cdot o)$, and the connected component of the isotropy of G_Σ at $o \in \mathsf{S}^n_{\mathbb{F},\tau}$ is K_Σ, which is the connected Lie subgroup of G with Lie algebra $\mathfrak{k}_\Sigma$. Now, using Equations (1.2) and (1.3) and the fact that $\mathfrak{p}_\Sigma$ is invariant under D and U, we have that the second fundamental form of $\Sigma = \mathsf{G}_\Sigma \cdot o$ vanishes at o. Thus, since Σ is an orbit, Σ is a totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$, and $\Sigma = \exp_o(\mathfrak{p}_\Sigma)$ by the uniqueness of complete totally geodesic submanifolds. Hence, these totally geodesic submanifolds are extrinsically homogeneous submanifolds.

Moreover, by using the Gauss equation, one can compute the curvature of these examples using Equations (8.4), (8.5), (8.6) and (8.7), (8.8), (8.9), and it turns out that Example $(1)_{\mathbb{F}}$ is isometric, for each $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, to S^k_1, $\mathsf{S}^{2k+1}_{\mathbb{C},\tau}$, or $\mathsf{S}^7_{\mathbb{H},\tau}$, respectively. Furthermore, Example $(2)_{\mathbb{F}}$ is isometric, for each $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, to $\mathsf{S}^{2k+1}_{\mathbb{C},\tau}$, $\mathsf{S}^{4k+3}_{\mathbb{H},\tau}$, or $\mathsf{S}^7_{1/\tau}$, respectively.

Remark 8.3.4. Notice that for $\tau \neq 1$ and $n \geq 1$, we have the totally geodesic inclusions

$$\begin{aligned} \mathsf{S}^n_1 \subset\ & \mathsf{S}^{2n+1}_{\mathbb{C},\tau} \subset \mathsf{S}^{4n+3}_{\mathbb{H},\tau}, \\ \mathsf{S}^3_{\mathbb{C},\tau} \subset\ & \mathsf{S}^7_{\mathbb{H},\tau} \quad \subset \mathsf{S}^{15}_{\mathbb{O},\tau}, \end{aligned}$$

for the totally geodesic submanifolds constructed in Remark 8.3.3.

8.4 Totally geodesic submanifolds of Hopf-Berger spheres

In this section, we will carry out the main efforts toward the classification of totally geodesic submanifolds of Hopf-Berger spheres.

8.4.1 Well-positioned totally geodesic submanifolds of Hopf-Berger spheres

The goal of this subsection is to give a characterization of well-positioned totally geodesic submanifolds in Hopf-Berger spheres.

Motivated by Lemma 8.1.3, we say that a totally geodesic submanifold Σ of a Hopf-Berger sphere $\mathsf{S}^n_{\mathbb{F},\tau}$ is *well-positioned* if $T_p\Sigma = (T_p\Sigma \cap \mathcal{H}_p) \oplus (T_p\Sigma \cap \mathcal{V}_p)$ for every $p \in M$, where $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and vertical distributions of the corresponding Hopf fibration, respectively.

Remark 8.4.1. Let $\bar{M}$ be a symmetric space of rank one with minimal absolute value of its sectional curvature equal to one, and consider for each $t \in (0, \operatorname{inj}(p))$ the natural map $h^t \colon \mathsf{S}(T_p\bar{M}) \to \mathsf{S}_t(p)$ given by $h^t(q) := \overline{\exp}_p(tq) = \gamma_q(t)$, where $\mathsf{S}_t(p)$ is endowed with the Riemannian metric induced by $\bar{M}$. We compute the pull-back metric $\langle \cdot, \cdot \rangle^t$ induced by h^t on $\mathsf{S}(T_p\bar{M})$ at a point $q \in \mathsf{S}(T_p\bar{M})$.

Let $w \in T_q\mathsf{S}(T_p\bar{M})$ and let $c(s)$ be a curve in $\mathsf{S}(T_p\bar{M})$ with initial conditions $c(0) = q$ and $\dot{c}(0) = w$. Then $h^t(c(s)) = \gamma_{c(s)}(t)$ is a variation of radial geodesics starting at p. Then

$$h^t_*(\dot{c}(0)) = \frac{\partial}{\partial s}_{|s=0} \gamma_{c(s)}(t) = J_w(t),$$

where $J_w(t)$ is the Jacobi field along $\gamma_q(t)$ with initial conditions $J(0) = 0$ and $J'_w(0) = w$. Hence,

$$\langle w, w \rangle^t = \langle J_w(t), J_w(t) \rangle.$$

There are two complementary distributions $\mathcal{F}_1$ and $\mathcal{F}_2 = \mathcal{F}_1^\perp$ (with respect to $\langle \cdot, \cdot \rangle^1$) on $\mathsf{S}(T_p\bar{M})$ defined by the eigenspaces associated with the eigenvalues $\lambda_1 = \pm 4$ and $\lambda_2 = \pm 1$ of the Jacobi operator $\bar{R}_{\cdot,q}q$ of $\bar{M}$ at p (the plus sign is for the compact case and the minus sign for the non-compact case). One has that $(h^t)_*(\mathcal{F}_i) = \mathcal{D}^t_i$, for each $i \in \{1, 2\}$ (see Section §8.2 for the definition of $\mathcal{D}^t_i$). Let $\widetilde{w}_i$ be the parallel vector field along γ_q with initial condition $\widetilde{w}_i(0) = w_i$. If $w_i \in \mathcal{F}_i(q)$, then $J_{w_i}(t) = \frac{1}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i}\, t)\tilde{w}_i(t)$ if $\bar{M}$ is of compact type, or $J_{w_i}(t) = \frac{1}{\sqrt{-\lambda_i}} \sinh(\sqrt{-\lambda_i}\, t)\tilde{w}_i(t)$ if $\bar{M}$ is of non-compact type. In addition to that, $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle^t = 0$ and

$$\begin{cases} \langle w_1, w_1 \rangle^t = \frac{1}{4} \sin^2(2t) \|w_1\|^2, \\ \langle w_2, w_2 \rangle^t = \sin^2(t) \|w_2\|^2, \end{cases}$$

if $\bar{M}$ is of compact type, or

$$\begin{cases} \langle w_1, w_1 \rangle^t = \frac{1}{4} \sinh^2(2t) \|w_1\|^2, \\ \langle w_2, w_2 \rangle^t = \sinh^2(t) \|w_2\|^2, \end{cases}$$

if $\bar{M}$ is of non-compact type. Recall that $\mathsf{S}^n_{\mathbb{F},1}$ is a round sphere of constant sectional curvature equal to 1. Thus, $\mathsf{S}^n_{\mathbb{F},\tau}$ is homothetic to the geodesic sphere $\mathsf{S}_t(p)$ via a homothety of ratio $\alpha = \sin(t)$ or $\alpha = \sinh(t)$, depending on whether $\bar{M}$ is of compact or non-compact type, respectively. Notice that this proves that the radius t of the geodesic sphere and τ are related by $t = \arccos(\sqrt{\tau})$ in the compact setting, as we pointed out in Equation (8.1), since $\tau = \frac{\sin^2(2t)}{4\sin^2(t)} = \cos^2(t)$; the relation $t = \operatorname{arccosh}(\sqrt{\tau})$ in the non-compact setting is obtained analogously.

Proposition 8.4.2. *Let Σ be a totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$, $\tau \neq 1$, and let $\bar{M}$ be the corresponding rank one symmetric space such that $\mathsf{S}^n_{\mathbb{F},\tau}$ arises as a geodesic sphere of $\bar{M}$ centered at $p \in \bar{M}$. Then, the following statements are equivalent:*

i) Σ is a well-positioned totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$.

ii) $T_q\Sigma$ has a vertical or a horizontal non-zero vector for every $q \in \Sigma$.

iii) $T_q\Sigma$ is invariant under the shape operator of $\mathsf{S}^n_{\mathbb{F},\tau}$ in $\bar{M}$ for every $q \in \Sigma$.

iv) Σ is the intersection of a totally geodesic submanifold M of $\bar{M}$ containing $p \in \bar{M}$ with $\mathsf{S}^n_{\mathbb{F},\tau}$ regarded as a geodesic sphere of $\bar{M}$.

Proof. First of all, let us identify $\mathsf{S}^n_{\mathbb{F},\tau}$ with the geodesic sphere $\mathsf{S}_t(p)$ of the appropiate symmetric space $\bar{M}$ of rank one centered at $p \in \bar{M}$ and $t = \arccos(\sqrt{\tau})$, in the compact case; or $t = \operatorname{arccosh}(\sqrt{\tau})$, in the non-compact case, see Equation (8.1).

Now, observe that *i)* and *ii)* are equivalent by Lemma 8.3.2. Moreover, let us recall that *i)* and *iii)* are equivalent since the only eigenvalue of the shape operator $\mathcal{S}^t$ of $\mathsf{S}_t(p)$ restricted to the horizontal distribution is different to the only eigenvalue of the shape operator restricted to the vertical distribution, for every $t \in (0, \operatorname{inj}(p))$, see Equations (8.2), (8.3) and the paragraph below them.

Now, by Corollary 8.1.2 and Remark 8.2.3 the intersection of a totally geodesic submanifold of $\bar{M}$ that contains p with $\mathsf{S}_t(p)$ is totally geodesic in $\mathsf{S}_t(p)$ and its tangent space is invariant under $\mathcal{S}^t$. Hence, *iv)* implies *iii)*.

Let us prove that *iii)* implies *iv)*. Let $0 < s, t < \operatorname{inj}(p)$ and consider the map $f_s : \mathsf{S}_t(p) \to \mathsf{S}_s(p)$ given by $f_s(\overline{\exp}_p(tv)) = \overline{\exp}_p(sv)$, for all $v \in T_pM$ of unit length. Then the pull-back by f_s of the Riemannian metric of $\mathsf{S}_s(p)$, if M is of the compact type, is given by modifying the metric on $\mathsf{S}_t(p)$ by the factor $\frac{\sin^2(2s)}{\sin^2(2t)}$ on the distribution $\mathcal{D}^t_1$ and by the factor $\frac{\sin^2(s)}{\sin^2(t)}$ on the distribution $\mathcal{D}^t_2$ (in the non-compact case the trigonometric functions have to be replaced by the corresponding hyperbolic functions). By Remark 8.2.1, if we rescale this pull-back metric by a factor of $\frac{\sin^2(t)}{\sin^2(s)}$, we are under the assumptions of Lemma 8.1.3. This shows that Σ is also totally geodesic with respect to the pull-back metric for every $s \in (0, \operatorname{inj}(p))$. Or equivalently, $f_s(\Sigma)$ is a totally geodesic submanifold of $\mathsf{S}_s(p)$. Furthermore, using that $T_q\Sigma$ is invariant under the shape operator for every q and that the integral curves of the outer unit normal vector field to the geodesic spheres are geodesics, it is standard to show that $\widehat{\Sigma} := \bigcup_{0<s<\operatorname{inj}(p)} f_s(\Sigma)$ is a totally geodesic submanifold of the open ball $B_{\operatorname{inj}(p)}(p)$ which contains a piece of the radial geodesic $\gamma_v(t) = \overline{\exp}_p(tv)$. Since a totally geodesic submanifold of a symmetric space extends to a complete totally geodesic submanifold, the complete extension $\widetilde{\Sigma}$ of $\widehat{\Sigma}$ contains p. By making use of Remark 8.2.3, we have $\Sigma = \widetilde{\Sigma} \cap \mathsf{S}_t(p)$, and it follows that *iii)* implies *iv)*. □

Notice that if $\dim(\Sigma) \geq \dim \mathbb{F}$, then $T_p\Sigma$ has a horizontal non-zero vector for every $p \in \Sigma$. Thus, Proposition 8.4.2 implies the following:

Corollary 8.4.3. *Let* Σ *be a totally geodesic submanifold of* $\mathsf{S}^n_{\mathbb{F},\tau}$. *If* $\dim \Sigma \geq \dim \mathbb{F}$, *then* Σ *is well-positioned.*

Remark 8.4.4. Notice that the intersection of a complete totally geodesic submanifold M of $\bar{M}$ passing through p with a geodesic sphere $\mathsf{S}_t(p)$ of $\bar{M}$ is a geodesic sphere of M of radius $t > 0$. Indeed, let us denote by $\mathsf{S}'_t(p)$ the geodesic sphere of M of radius $t > 0$ centered at p. Let us denote by d_M and $d_{\bar{M}}$ the distances on M and $\bar{M}$ induced by the respective Riemannian metrics. Then,

$$\mathsf{S}_t(p) \cap M = \{q \in M : d_{\bar{M}}(p,q) = t\} = \{q \in M : d_M(p,q) = t\} = \mathsf{S}'_t(p),$$

where we have used that M is a complete totally geodesic submanifold of $\bar{M}$.

Thus, as a consequence of Proposition 8.4.2 and Corollary 8.4.3, we obtain the classification of totally geodesic submanifolds of $\mathsf{S}^n_{\mathbb{C},\tau}$, for every $\tau \in (0,\infty)$. The classification when $\tau < 1$ and $\mathbb{F} = \mathbb{C}$ was already obtained in [177].

8.4.2 Totally geodesic surfaces of Hopf-Berger spheres

The aim of this subsection is to prove that every totally geodesic surface of a Hopf-Berger sphere is well-positioned when $\tau \geq 1/2$.

Lemma 8.4.5. *Let* V *be a curvature invariant* 2*-plane in* $\mathsf{S}^{15}_{\mathbb{O},\tau}$ *with* $\tau \neq 1$. *Then,* V *is contained in the tangent space of a totally geodesic* $\mathsf{S}^7_{\mathbb{H},\tau}$ *in* $\mathsf{S}^{15}_{\mathbb{O},\tau}$.

Proof. First of all, using the isotropy of $\mathsf{S}^{15}_{\mathbb{O},\tau}$ (see [180]), we can assume that the 2-plane V is spanned by $X, Y \in \mathfrak{p}$, where

$$\begin{aligned} X &= a_1 X_1/(2\sqrt{\tau}) + a_2 X_2/(2\sqrt{\tau}) + a_3 X_3/(2\sqrt{\tau}) + a_4 Y_1, \\ Y &= b_1 X_1/(2\sqrt{\tau}) + b_2 X_2/(2\sqrt{\tau}) + b_3 X_3/(2\sqrt{\tau}) + b_4 X_4/(2\sqrt{\tau}) + b_5 Y_1 + b_6 J_1 Y_1, \end{aligned}$$

where we are using the notation of Section §8.3. Let us proceed by contradiction. Then, we can assume that $b_4 \neq 0$, since otherwise, V is contained in the tangent space of some totally geodesic submanifold $\mathsf{S}^7_{\mathbb{H},\tau}$ of $\mathsf{S}^{15}_{\mathbb{O},\tau}$, see Remark 8.3.3. Now,

$$\begin{aligned} 0 &= \frac{1}{2\sqrt{\tau}} \langle R(X,Y)Y, X_6 \rangle = -3 a_4 b_4 b_6 (-1+\tau), \\ 0 &= \langle R(X,Y)Y, Y_6 \rangle = 3 a_1 b_4 b_6 (-1+\tau), \end{aligned} \tag{8.10}$$

since V is curvature invariant. Thus we either have that $b_6 = 0$ or $a_1 = a_4 = 0$.

Let us assume that $b_6 = 0$. Thus,

$$\begin{aligned}
0 &= \langle R(X,Y)Y, Y_8\rangle = 3a_3b_4b_5(-1+\tau),\\
0 &= \langle R(X,Y)Y, Y_4\rangle = 3a_2b_4b_5(-1+\tau),\\
0 &= \langle R(X,Y)Y, Y_2\rangle = -3a_1b_4b_5(-1+\tau),
\end{aligned}$$

since V is curvature invariant. Hence, $a_1 = a_2 = a_3 = 0$ or $b_5 = 0$. However, in the first case, V is contained in the tangent space of some totally geodesic submanifold $\mathsf{S}^7_{\mathbb{H},\tau}$ of $\mathsf{S}^{15}_{\mathbb{O},\tau}$. Thus, $b_5 = 0$. Observe that we can assume that $a_4 \neq 0$, since otherwise V is contained in the tangent space of some totally geodesic submanifold $\mathsf{S}^7_{\mathbb{H},\tau}$ of $\mathsf{S}^{15}_{\mathbb{O},\tau}$. Then,

$$\begin{aligned}
0 &= \langle R(Y,X)X, Y_8\rangle = -3a_3a_4b_4(-1+\tau),\\
0 &= \langle R(Y,X)X, Y_4\rangle = -3a_2a_4b_4(-1+\tau),\\
0 &= \langle R(Y,X)X, Y_2\rangle = 3a_1a_4b_4(-1+\tau),
\end{aligned}$$

since V is curvature invariant. Hence, $a_1 = a_2 = a_3 = 0$ and V is contained in the tangent space of some totally geodesic submanifold $\mathsf{S}^7_{\mathbb{H},\tau}$ of $\mathsf{S}^{15}_{\mathbb{O},\tau}$.

Let us assume that $b_6 \neq 0$. Then, by Equation (8.10), $a_1 = a_4 = 0$. Moreover,

$$\begin{aligned}
0 &= \langle R(X,Y)Y, Y_8\rangle = 3b_4(a_3b_5 + a_2b_6)(-1+\tau),\\
0 &= \langle R(X,Y)Y, Y_4\rangle = 3b_4(a_2b_5 - a_3b_6)(-1+\tau),
\end{aligned}$$

since V is curvature invariant. Hence, $a_2 = a_3 = 0$, implying that V is contained in the tangent space of some totally geodesic submanifold $\mathsf{S}^7_{\mathbb{H},\tau}$ of $\mathsf{S}^{15}_{\mathbb{O},\tau}$. □

Remark 8.4.6. Let V be a 2-plane that is a minimum or a maximum for the sectional curvature of $\mathsf{S}^{15}_{\mathbb{O},\tau}$ at $o \in \mathsf{S}^{15}_{\mathbb{O},\tau}$. Hence, by [178, Theorem 4.1], V is curvature invariant. Thus, by Lemma 8.4.5, V is contained in the tangent space of a totally geodesic $\mathsf{S}^7_{\mathbb{H},\tau}$ in $\mathsf{S}^{15}_{\mathbb{O},\tau}$. Consequently, $\mathsf{S}^{15}_{\mathbb{O},\tau}$ has the same pinching as $\mathsf{S}^7_{\mathbb{H},\tau}$.

Lemma 8.4.7. *Let Σ be a totally geodesic surface in $\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$ with $\tau \neq 1$. Then, if $\tau \geq 1/2$, we have that Σ is well-positioned.*

Proof. Let Σ be a totally geodesic submanifold of dimension two in $\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$ and $\mathfrak{p}_\Sigma$ its tangent space at $o \in \Sigma$ identified with a subspace of $\mathfrak{p}$. Using the isotropy of $\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$ (see [180]), we can assume that $\mathfrak{p}_\Sigma$ is spanned by the basis $\{X, Y\}$ given by

$$\begin{aligned}
X &= a_1X_1/\sqrt{\tau} + a_2X_2/\sqrt{\tau} + a_3X_3/\sqrt{\tau} + a_4Y_1,\\
Y &= b_1X_1/\sqrt{\tau} + b_2X_2/\sqrt{\tau} + b_3X_3/\sqrt{\tau} + b_4Y_1 + b_5J_1Y_1 + b_6Y_2,
\end{aligned}$$

where $a_i, b_j \in \mathbb{R}$ for $i \in \{1,\dots,4\}$ and $j \in \{1,\dots,6\}$.

Let us suppose that $\mathfrak{p}_\Sigma$ does not contain horizontal or vertical vectors and let us derive a contradiction. Then, $a_4 \neq 0$. Moreover, we can assume without loss of generality that $b_4 = 0$. If not, $\{X, Y - b_4/a_4X\}$ is also a basis for $\mathfrak{p}_\Sigma$ where the second vector projects trivially over Y_1.

Since $\mathfrak{p}_\Sigma$ is curvature invariant, and assuming $n \geq 2$, we have:

$$\begin{aligned} 0 &= \langle R(X,Y)Y, J_1Y_2\rangle = 3(a_3b_2 - a_2b_3 + a_4b_5)(-1+\tau)b_6, \\ 0 &= \langle R(X,Y)Y, J_2Y_2\rangle = 3(a_3b_1 - a_1b_3)(1-\tau)b_6, \\ 0 &= \langle R(X,Y)Y, J_3Y_2\rangle = 3(a_2b_1 - a_1b_2)(-1+\tau)b_6. \end{aligned} \tag{8.11}$$

By Proposition 5.1.6 every totally geodesic submanifold of a homogeneous space is intrinsically homogeneous and every homogeneous space of dimension two has constant sectional curvature. Then, every covariant derivative of the curvature tensor R restricted to $\mathfrak{p}_\Sigma$ vanishes. Thus,

$$0 = \langle (\nabla_X R)(Y,X)X, Y_1\rangle = -4a_1a_4^2b_5\sqrt{\tau}(-1+\tau).$$

This implies that $a_1 = 0$ or $b_5 = 0$. In the latter case, $b_6 \neq 0$, since otherwise there would be a vertical vector in $\mathfrak{p}_\Sigma$. Thus, $n \geq 2$ and by Equation (8.11)

$$a_3b_2 - a_2b_3 = a_3b_1 - a_1b_3 = a_2b_1 - a_1b_2 = 0.$$

Hence, (a_1, a_2, a_3) and (b_1, b_2, b_3) are proportional and there would be a horizontal vector in $\mathfrak{p}_\Sigma$.

Now assume that $b_5 \neq 0$ and thus $a_1 = 0$. Then,

$$\begin{aligned} 0 &= \langle R(X,Y)Y, J_2Y_1\rangle = 3a_2b_1b_5(1-\tau), \\ 0 &= \langle R(X,Y)Y, J_3Y_1\rangle = 3a_3b_1b_5(1-\tau). \end{aligned}$$

Thus, $b_1 = 0$ or $a_2 = a_3 = 0$, but the latter cannot happen because it would imply the existence of a horizontal vector. Hence, $b_1 = 0$. Moreover, we have

$$0 = \langle (\nabla_X R)(Y,X)Y, X_1\rangle = 4a_4(a_2b_2 + a_3b_3)b_5(-1+\tau)(-1+2\tau).$$

This yields $\tau = 1/2$ or $a_2b_2 + a_3b_3 = 0$. Let us assume that $\tau = 1/2$. Then,

$$\begin{aligned} 0 &= \langle (\nabla_X R)(Y,X)X, J_2Y_1\rangle = -\sqrt{2}a_4^3b_2, \\ 0 &= \langle (\nabla_X R)(Y,X)X, J_3Y_1\rangle = -\sqrt{2}a_4^3b_3. \end{aligned}$$

Hence, $b_2 = b_3 = 0$ and we obtain a contradiction due to the existence of a horizontal vector in $\mathfrak{p}_\Sigma$. Now, let us assume that $a_2b_2 + a_3b_3 = 0$. Then, there exist $r, s \in \mathbb{R}$ and $\varphi \in [0, 2\pi]$ such that

$$a_2 = r\cos(\varphi), \quad a_3 = r\sin(\varphi), \quad b_2 = -s\sin(\varphi), \quad b_3 = s\cos(\varphi).$$

Thus, we have

$$0 = \langle (\nabla_X R)(Y,X)X, X_1\rangle = 4a_4b_5(-1+\tau)(a_4^2\tau + r^2(-1+2\tau)).$$

Hence, since $b_5 \neq 0$ and $\tau \geq 1/2$, we get a contradiction. □

Proposition 8.4.8. *Let Σ be a totally geodesic surface in $\mathsf{S}^n_{\mathbb{F},\tau}$ with $\tau \neq 1$. Then, if $\mathbb{F} = \mathbb{C}$, Σ is well-positioned. Moreover, if $\tau \geq 1/2$ and $\mathbb{F} \in \{\mathbb{H}, \mathbb{O}\}$, Σ is well-positioned.*

Proof. Let us assume that $\mathbb{F} = \mathbb{C}$. Then, for dimensional reasons, $\mathfrak{p}_\Sigma$ has a vertical or horizontal vector. Hence, the conclusion follows from Proposition 8.4.2.

Let $\mathbb{F} = \mathbb{H}$. Then, the result follows from Lemma 8.4.7.

Finally, let $\mathbb{F} = \mathbb{O}$. By Lemma 8.4.5, every totally geodesic surface of $\mathsf{S}^{15}_{\mathbb{O},\tau}$ can be totally geodesic embedded in a totally geodesic $\mathsf{S}^7_{\mathbb{H},\tau}$ of $\mathsf{S}^{15}_{\mathbb{O},\tau}$, which again by Lemma 8.4.7 proves our result. □

8.4.3 Totally geodesic submanifolds of Hopf-Berger spheres

The purpose of this section is to provide the proof of Theorem A.

Lemma 8.4.9. *Let $\tau \neq 1$ and γ be a geodesic in $\mathsf{S}^n_{\mathbb{F},\tau} = \mathsf{G}/\mathsf{K}$. Then, there exists a totally geodesic submanifold Σ of $\mathsf{S}^n_{\mathbb{F},\tau}$ satisfying that:*

i) Σ *is well-positioned.*

ii) Σ *is isometric to* $\mathsf{S}^3_{\mathbb{C},\tau}$.

iii) $\Sigma = \mathsf{G}_\Sigma/\mathsf{K}_\Sigma$, *where* $\mathsf{G}_\Sigma \subset \mathsf{G}$ *and* $\mathsf{K}_\Sigma \subset \mathsf{K}$ *are connected subgroups.*

iv) K_Σ *acts non-trivially on* Σ *and thus* $\widetilde{\mathsf{K}}_\Sigma \cong \mathsf{U}_1$ *and* $\widetilde{\mathsf{G}}_\Sigma \cong \mathsf{U}_2$, *where* $\widetilde{\mathsf{K}}_\Sigma$ *and* $\widetilde{\mathsf{G}}_\Sigma$ *denote the quotients of the groups* G_Σ *and* K_Σ *by the kernel of their corresponding actions on* Σ, *respectively.*

v) γ *is contained in* Σ.

Proof. Let us consider the presentation of the Hopf-Berger sphere $\mathsf{S}^n_{\mathbb{F},\tau} = \mathsf{G}/\mathsf{K}$ given in Section §8.3 for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. If $\mathbb{F} = \mathbb{H}$, we consider the presentation $\mathsf{S}^{4n+3}_{\mathbb{F},\tau} = \check{\mathsf{G}}/\check{\mathsf{K}}$ given in Remark 8.3.1. For the sake of simplicity, we write K instead of $\check{\mathsf{K}}$ along this proof.

Recall that $\mathsf{K} \cdot (v_0 + v_1)$ is a principal orbit if $v_0 \in \mathfrak{p}_1$ and $v_1 \in \mathfrak{p}_2$ are both non-zero, see Section §8.3. Hence, $\mathsf{K}_{v_0+v_1}$ fixes the normal space $\nu_{v_0+v_1}(\mathsf{K} \cdot (v_0 + v_1))$. Thus, the set of vectors fixed by $\mathsf{K}_{v_0+v_1}$ in $T_o\mathsf{S}^n_{\mathbb{F},\tau}$ has dimension 3. Namely, it is generated by v_0, v_1 and a vector, let us say w_1, in $\mathfrak{p}_2$. Then, the connected component Σ containing o of the set of points fixed by $\mathsf{K}_{v_0+v_1}$ in $\mathsf{S}^n_{\mathbb{F},\tau}$ is a three-dimensional totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$ such that $v_0 \in \mathfrak{p}_1 \cap T_o\Sigma$. Thus, by Lemma 8.3.2, we have that $T_o\Sigma = (T_o\Sigma \cap \mathfrak{p}_1) \oplus (T_o\Sigma \cap \mathfrak{p}_2)$.

Since Σ is a set of fixed points of a set of isometries, by [14, Lemma 9.1.1], $\Sigma = \mathsf{G}_\Sigma/\mathsf{K}_\Sigma$, where $\mathsf{G}_\Sigma \subset \mathsf{G}$ is connected and $\mathsf{K}_\Sigma \subset \mathsf{K}$. Then, $T_q\Sigma = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$ has dimension 1 and is tangent to the vertical subspace $\mathcal{V}_q$ of the Hopf fibration, and $\mathfrak{s}_2$ has dimension 2 and is perpendicular to $\mathcal{V}_q$. This proves *i)*. Hence, by Proposition 8.4.2, Σ is the intersection of a 4-dimensional totally geodesic submanifold M of the appropriate rank one symmetric space $\bar{M}$ in which $\mathsf{S}^n_{\mathbb{F},\tau}$ is identified with a geodesic sphere of $\bar{M}$ of radius $r \in (0, \mathrm{inj}(\bar{M}))$. Then, Σ is isometric to $\mathsf{S}^3_{\mathbb{C},\tau}$ since it can be identified with a geodesic sphere of M of radius r, and then K_Σ must be connected since Σ is simply connected and G_Σ is connected. This proves *ii)* and *iii)*.

Now we prove that K_Σ acts non-trivially. Let Y^* be a Killing field induced by an element Y in the Lie algebra $\mathfrak{k}$ of the isotropy K of $\mathsf{S}^n_{\mathbb{F},\tau}$ at o. Since the projection of $Y^*_{|\Sigma}$ coincides with the restriction to Σ of another Killing field induced by G which is always tangent to Σ (see the proof of [14, Lemma 9.1.1]), we have that K_Σ acts trivially on $T_o\Sigma$ if and only if $Y^*_{|\Sigma}$ is orthogonal to Σ for every $Y \in \mathfrak{k}$. Assume that K_Σ acts trivially and let $v \in T_o\Sigma$. Then the Jacobi field given by $J(t) = Y^*_{\gamma_v(t)}$ must be always perpendicular to $T_{\gamma_v(t)}\Sigma$, where γ_v denotes the geodesic with initial conditions $\gamma_v(0) = o \in \mathsf{S}^n_{\mathbb{F},\tau}$ and $\dot\gamma_v(0) = v \in T_o\mathsf{S}^n_{\mathbb{F},\tau}$. Since $J(0) = 0$, we have that $J'(0) = \nabla_v Y^*$ must be perpendicular to $T_o\Sigma$, where ∇ denotes the Levi-Civita connection of $\mathsf{S}^n_{\mathbb{F},\tau}$. Let ρ be the isotropy representation of K at o. Since $Y \in \mathfrak{k}$, we have $\rho(Y)v = \nabla_v Y^*$, (see for instance [67, §2.1]). Thus, $\rho(Y)v$ is perpendicular to $T_o\Sigma$, for every $v \in T_o\Sigma$ and $Y \in \mathfrak{k}$. Since K acts polarly on $T_o\mathsf{S}^n_{\mathbb{F},\tau}$, we have that $T_o\Sigma$ must be contained in the tangent space of a section. However, the sections of the isotropy representation of $\mathsf{S}^n_{\mathbb{F},\tau}$ have dimension two, yielding a contradiction. Then, K_Σ acts non-trivially, and since Σ is isometric to $\mathsf{S}^3_{\mathbb{C},\tau}$, it follows that *iv)* holds.

Since K acts polarly on $T_o\mathsf{S}^n_{\mathbb{F},\tau}$, for any $w \in T_o\mathsf{S}^n_{\mathbb{F},\tau}$ there exists $k \in \mathsf{K}$ such that $k_{*o}w \in \nu_{v_0+v_1}(\mathsf{K}\cdot(v_0+v_1)) \subset T_o\Sigma$. Then $k \circ \gamma_w = \gamma_{k_{*o}w}$ is a geodesic contained in Σ. Equivalently, γ_w is contained in a totally geodesic $\mathsf{S}^3_{\mathbb{C},\tau}$ in $\mathsf{S}^n_{\mathbb{F},\tau}$. This proves *v)*. □

We define the *slope* of a geodesic γ in $\mathsf{S}^n_{\mathbb{F},\tau}$ to be the quotient between the lengths of the vertical and horizontal projections of the velocity of γ. This quantity is well defined for every geodesic since Hopf-Berger spheres are geodesic orbit spaces, see [175]. Moreover, using the full isotropy representation of $\mathsf{S}^n_{\mathbb{F},\tau}$ described in Remark 8.3.1, it is easy to see that two geodesics are congruent in $\mathsf{S}^n_{\mathbb{F},\tau}$ if and only if they have the same slope.

Lemma 8.4.10. *Let $\mathsf{S}^n_{\mathbb{F},\tau}$, with $\tau \neq 1$, and let γ be a closed geodesic in $\mathsf{S}^n_{\mathbb{F},\tau}$. Then, the set of possible slopes for γ is countable.*

Proof. First of all, by Lemma 8.4.9, we can assume that γ is a closed geodesic of $\mathsf{S}^3_{\mathbb{C},\tau}$ with $\tau \neq 1$. Let us define a naturally reductive decomposition for $\mathsf{S}^3_{\mathbb{C},\tau} = \mathsf{U}_2/\mathsf{U}_1$. Keeping the notation in Section §8.3 applied to $\mathsf{S}^3_{\mathbb{F},\tau}$, we define $\mathfrak{p}_1^\tau$ as the one-dimensional subspace of $\mathfrak{g} = \mathfrak{u}_2$ spanned by the unit vector

$$X^\tau := \frac{1}{\sqrt{\tau}}\left(X_1 + (1-2\tau)Z\right), \quad \text{where } Z = \left(\begin{array}{c|c} i & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{u}_1 \subset \mathfrak{u}_2.$$

Hence, it can be checked that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^\tau$, where $\mathfrak{p}^\tau := \mathfrak{p}_1^\tau \oplus \mathfrak{p}_2$, is a naturally reductive decomposition of $\mathsf{S}^3_{\mathbb{C},\tau}$, for each $\tau \in (0,\infty)$. In particular, every geodesic of $\mathsf{S}^3_{\mathbb{C},\tau}$ is of the form $\operatorname{Exp}(sX)\cdot o$, for some $X \in \mathfrak{p}^\tau$. For our purposes, we can assume that $o = (0,1) \in \mathsf{S}^3_{\mathbb{C},\tau} \subset \mathbb{C}^2$. Let $X := \alpha_1 Y_1 + \alpha_2 J_1 Y_1 + \alpha_3 X^\tau$, where $\alpha_i \in \mathbb{R}$ for every $i \in \{1,2,3\}$ satisfy $\sum_{i=1}^3 \alpha_i^2 = 1$, and consider the geodesic $\gamma_X(s) = \operatorname{Exp}(sX)\cdot o$. We define the quantities

$$P := \sqrt{\alpha_1^2 + \alpha_2^2 + \tau\alpha_3^2}, \qquad Q := \frac{e^{-\frac{is}{\sqrt{\tau}}\left(P\sqrt{\tau}+(\tau-1)\alpha_3\right)}}{2P}.$$

A direct computation shows that

$$\gamma_X(s) = Q((-1+e^{2iPs})(-i\alpha_1+\alpha_2),(1+e^{2iPs})P+(-1+e^{2iPs})\sqrt{\tau}\alpha_3).$$

If X is vertical, $\alpha_3 = \pm 1$, $\alpha_1 = \alpha_2 = 0$, and we have $\gamma_X(s) = (0, e^{\pm\frac{is}{\sqrt{\tau}}})$. If X is horizontal, $\alpha_3 = 0$, $\alpha_1^2+\alpha_2^2 = 1$ and we have

$$\gamma_X(s) = \frac{e^{-is}}{2}((-1+e^{2is})(\alpha_2 - i\alpha_1), 1+e^{2is}).$$

Now, let us assume that X has non-trivial projection onto $\mathfrak{p}_1^\tau$ and $\mathfrak{p}_2$. Then, if γ_X is closed,

$$-1+e^{2iPs} = 0, \quad \text{and} \quad 2PQ = 1, \quad \text{for some } s \in \mathbb{R},\ s \neq 0.$$

Hence, $\alpha_3 = \frac{P\sqrt{\tau}(j+2k)}{j(1-\tau)} = \frac{\sqrt{(1-\alpha_3^2)+\tau\alpha_3^2}\sqrt{\tau}(j+2k)}{j(1-\tau)}$, where $j \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{Z}$. Since the slope of γ_X is given by $\sqrt{\frac{\alpha_3^2}{1-\alpha_3^2}}$, we deduce that the set of possible slopes for γ_X is countable. □

Proposition 8.4.11. *Let $\mathsf{S}^n_{\mathbb{F},\tau}$, with $\tau \neq 1$, and let Σ be a not well-positioned totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$ with dimension greater or equal than 2. Then there exists a 2-dimensional totally geodesic submanifold Σ' of $\mathsf{S}^n_{\mathbb{F},\tau}$ contained in Σ that is also not well-positioned.*

Proof. Let us assume that Σ is a totally geodesic submanifold of $\mathsf{S}^n_{\mathbb{F},\tau}$ that is not well-positioned. We may assume without loss of generality that Σ is complete, see Theorem 5.2.5. Then, there is some $q \in \Sigma$ such that $T_q\Sigma \neq (T_q\Sigma \cap \mathcal{V}_q) \oplus (T_q\Sigma \cap \mathcal{H}_q)$. This is equivalent, by Lemma 8.3.2, to the fact that $T_q\Sigma \cap \mathcal{V}_q = \{0\} = T_q\Sigma \cap \mathcal{H}_q$.

Let us assume that Σ is compact. We prove that every geodesic of Σ is closed. Let $w \in T_q\Sigma$ and let γ_w be a geodesic in $\mathsf{S}^n_{\mathbb{F},\tau}$ that is not closed. Then, by Proposition 8.4.9, there exists a totally geodesic submanifold N of $\mathsf{S}^n_{\mathbb{F},\tau}$, which is the orbit of a subroup G_N of G, isometric to $\mathsf{S}^3_{\mathbb{C},\tau}$ such that $\gamma_w \subset \Sigma \cap N$. Since N is a g.o. space (see [175]), one has that $\gamma_w(t) = \operatorname{Exp}(tz)\cdot q$ for some z in the Lie algebra of $\mathsf{G}_N \subset \mathsf{G}$. Since the rank of the (compact) effectivized presentation group $\widetilde{\mathsf{G}}_N$ is two and the orbit $\operatorname{Exp}(tz)\cdot q = \gamma_w(t)$ is not compact, the closure of the one-parameter group $\operatorname{Exp}(tz)$ in $\widetilde{\mathsf{G}}_N$ is a 2-dimensional torus T^2. Observe that $\mathsf{T}^2\cdot q$ must be 2-dimensional. Otherwise the 1-dimensional compact submanifold $\mathsf{T}^2\cdot q$ would coincide with the image of γ_w and it would be compact. Since $\Sigma\cap N$ must contain $\mathsf{T}^2\cdot q$, we conclude that this intersection is at least 2-dimensional. But N, by construction, contains a two-dimensional horizontal subspace, that must therefore intersect non-trivially with $T_q(\mathsf{T}^2\cdot q)$. Then, $T_q\Sigma$ contains a non-trivial horizontal vector. However, by Lemma 8.3.2, this contradicts our assumptions. Then every geodesic of Σ is closed since Σ is intrinsically homogeneous by Proposition 5.1.6. Moreover, every geodesic of Σ has the same length. Otherwise, there would be two closed geodesics of Σ with different slopes. Notice that these geodesics cannot be vertical, as this would contradict our assumptions by Lemma 8.3.2. However, by using the continuity of the map that assigns to every non-vertical geodesic its slope, and Lemma 8.4.10, we deduce the existence of a non-closed geodesic in Σ. This yields a contradiction. Thus, all the geodesics of Σ have the same length. Since Σ is intrinsically homogeneous, by [31, Theorem 7.55, p. 196], Σ is a symmetric space of compact type and rank one. Hence, Σ admits a totally geodesic surface Σ' that cannot be well-positioned, as follows from the assumption that Σ is not well-positioned and Lemma 8.3.2. This concludes the proof when Σ is compact.

Let us assume that Σ is not compact. Then, the projection of a Killing field of $\mathsf{S}^n_{\mathbb{F},\tau}$ to Σ is a Killing field of Σ (see the proof of Proposition 5.1.6), which is bounded as $\mathsf{S}^n_{\mathbb{F},\tau}$ is compact. Since there is a projected Killing field in any direction tangent to Σ, we conclude that $\mathfrak{b} \cdot q = T_q\Sigma$, where $\mathfrak{b}$ denotes the ideal of bounded intrinsic Killing fields of Σ. Let B be the (transitive) Lie subgroup of isometries of Σ associated with $\mathfrak{b}$. By making use of [133, Theorem 1.3], one has that $\mathsf{B} = \mathsf{HA}$ (almost direct product) where H, if non-trivial, is compact semisimple and A is abelian.

Let us prove that A is closed in $\mathrm{Isom}(\Sigma)$. Let $\bar{\mathfrak{a}}$ be the Lie algebra of the closure of A in $\mathrm{Isom}(\Sigma)$ and $q \in \Sigma$. Then, since $[\mathfrak{b}, \mathfrak{a}] = 0$, we have

$$||X^*_q|| = ||g_* X^*_q|| = ||(\mathrm{Ad}(g)X)^*_{g(q)}|| = ||X^*_{g(q)}||$$

for every $g \in \mathsf{B}$ and $X \in \bar{\mathfrak{a}}$. Hence, X^* is a Killing vector field of constant length and thus bounded since B acts transitively on Σ.

If the normal subgroup H is non-trivial, then the orbits of H in Σ define a B-invariant foliation. Since $\mathsf{S}^n_{\mathbb{F},\tau}$ is a g.o. space, then Σ is a g.o. space with respect to the presentation B/B_q, where B_q is the isotropy of B at q.

Let us prove that $\mathsf{H}\cdot q$ is a totally geodesic submanifold of Σ. Notice that $X \in \mathfrak{b}$ induces a Killing vector field of Σ that projects to a vector field $\bar{X}$ in the quotient $\mathsf{H} \setminus \Sigma$ given by the orbits of the action of H on Σ. Then, if $X_q \in T_q(\mathsf{H} \cdot q)$, we have

$$\pi_{*h(q)} X_{h(q)} = \bar{X}_{\pi(h(q))} = \bar{X}_{\pi(q)} = \pi_{*q} X_q = 0 \quad \text{for every } h \in \mathsf{H},$$

where $\pi \colon \Sigma \to \mathsf{H} \setminus \Sigma$ denotes the quotient map. Then the restriction of X to $\mathsf{H}\cdot q$ is always tangent to $\mathsf{H} \cdot q$. Hence, $\mathrm{Exp}(tX \cdot q) \in \mathsf{H} \cdot q$ for every $t \in \mathbb{R}$. This implies that $\mathsf{H} \cdot q$ is a totally geodesic submanifold of Σ since Σ is g.o. with respect to the presentation B/B_q.

We claim that $\mathsf{H} \cdot q$ is a totally geodesic compact submanifold of Σ of dimension at least 2. Indeed, $\mathsf{H} \cdot q$ cannot have dimension zero since a normal subgroup of B cannot be contained in B_q because . If $\mathsf{H} \cdot q$ has dimension one, then H_q has codimension one in H, and then it is a normal subgroup of H since H_q is compact semisimple. This yields a contradiction with the fact that H is semisimple since then H would have a 1-dimensional abelian normal subgroup. Then, since we have already tackled the compact case, we are done. So we may assume that H is trivial. Then $\mathsf{A} \cdot q = \Sigma$, and Σ is flat with dimension at least 2. Then Σ has a totally geodesic flat submanifold Σ' of dimension 2 passing through q that is not well-positioned. This completes the proof. □

We are now in a position to prove Theorem A.

Proof of Theorem A. First of all, notice that every totally geodesic submanifold Σ of dimension $d \geq 2$ of a Hopf-Berger sphere $\mathsf{S}^n_{\mathbb{F},\tau}$, where $\tau \geq 1/2$, is well-positioned. Otherwise, by Proposition 8.4.11, Σ would contain a totally geodesic surface that is not well-positioned, and this cannot happen by Proposition 8.4.8. Let us identify $\mathsf{S}^n_{\mathbb{F},\tau}$ with a geodesic sphere $\mathsf{S}(p)$ centered at some point p of a rank one symmetric space $\bar{M}$. The intersection of a geodesic sphere $\mathsf{S}(p)$ of a symmetric space of rank one $\bar{M}$ with a complete totally geodesic submanifold M passing through $p \in \bar{M}$ is a geodesic sphere of M (see Remark 8.4.4). Hence, the intersection of $\mathsf{S}(p)$ with a complete totally geodesic submanifold M of $\bar{M}$ of dimension $d' \geq 3$ containing p has dimension $d' - 1$. Thus, it follows by Proposition 8.4.2 that *i)* and *ii)* are equivalent.

Furthermore, two totally geodesic submanifolds Σ_1 and Σ_2 of $\mathsf{S}(p)$ of dimension $d \geq 2$ are congruent if and only if they are the intersection of $\mathsf{S}(p)$ with congruent totally geodesic submanifolds M_1 and M_2 of $\bar{M}$, since the isotropy of $\bar{M}$ at p is equal to $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})$ (see Corollary 8.5.2 and the paragraph just above it). Hence, by using the classification of totally geodesic submanifolds (up to congruence) in symmetric spaces of rank one (see Figure 1) we obtain the classification in Table 8.1. □

8.5 The index of symmetry of Hopf-Berger spheres

In this section we compute the index of symmetry of Hopf-Berger spheres. In [84], a *symmetry* of $\mathsf{S}^n_{\mathbb{F},\tau}$ was defined as an isometry of $\mathsf{S}^n_{\mathbb{F},1}$ that maps fibers to fibers of the Hopf fibration corresponding to $\mathsf{S}^n_{\mathbb{F},\tau}$. According to [84] the groups of symmetries for $\mathsf{S}^{2n+1}_{\mathbb{C},\tau}$, $\mathsf{S}^{4n+3}_{\mathbb{H},\tau}$ and $\mathsf{S}^{15}_{\mathbb{O},\tau}$ are

$$\mathsf{U}_{n+1} \rtimes \mathbb{Z}_2, \quad \mathsf{Sp}_{n+1} \times_{\mathbb{Z}_2} \mathsf{Sp}_1, \quad \text{and} \quad \mathsf{Spin}_9, \tag{8.12}$$

respectively.

Notice that every symmetry of $\mathsf{S}^n_{\mathbb{F},\tau}$ is an isometry of $\mathsf{S}^n_{\mathbb{F},\tau}$. The next lemma proves the converse.

Lemma 8.5.1. *Every isometry of $\mathsf{S}^n_{\mathbb{F},\tau}$, where $\tau \neq 1$, is a symmetry.*

Proof. Since the group G of symmetries of $\mathsf{S}^n_{\mathbb{F},\tau}$ acts transitively on $\mathsf{S}^n_{\mathbb{F},\tau}$, it suffices to show that the full isotropy group $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})_p$, at certain $p \in \mathsf{S}^n_{\mathbb{F},\tau}$, leaves the leaf $F(p)$ of the Hopf foliation at p invariant.

Let $(\cdot)^0$ denote the connected component containing the identity of a Lie group. Since $\mathsf{S}^n_{\mathbb{F},\tau}$ is simply connected, the stabilizers of G^0 and $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})^0$ at $p \in \mathsf{S}^n_{\mathbb{F},\tau}$ are connected. Assume that $(\mathsf{G}^0)_p$ is properly contained in $(\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})^0)_p$. Then, either $(\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})^0)_p$ has the same orbits as $(\mathsf{G}^0)_p$, or it is transitive on the unit sphere of $T_p\mathsf{S}^n_{\mathbb{F},\tau}$, since $(\mathsf{G}^0)_p$ acts with codimension 2 on $T_p\mathsf{S}^n_{\mathbb{F},\tau}$. In the first case, $(\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})^0)_p$ leaves invariant $\mathcal{V}_p = T_pF(p)$, and hence also the totally geodesic fiber $F(p)$ of the Hopf fibration. In the second case, $\mathsf{S}^n_{\mathbb{F},\tau}$ is a two-point homogeneous space and therefore $\mathsf{S}^n_{\mathbb{F},\tau}$ is a rank-one symmetric space, yielding a contradiction with the fact that there are non-closed geodesics in $\mathsf{S}^n_{\mathbb{F},\tau}$ (see Lemma 8.4.10). This proves that $(\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})^0)_p$ leaves invariant the leaf $F(p)$ of the Hopf foliation at p.

Now let us prove that the same happens for $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})_p$. The isotropy representation of $(\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})^0)_p$, which is equal to $(\mathsf{G}^0)_p$, has two inequivalent irreducible modules (see Remark 8.3.1), any of which must be preserved by $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})_p$, since $(\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})^0)_p$ is a normal subgroup of $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})_p$. Then, $T_pF(p)$ is invariant under the action $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})_p$, and thus $F(p)$ is invariant by $\mathrm{Isom}(\mathsf{S}^n_{\mathbb{F},\tau})_p$. □

As a consequence of Lemma 8.5.1, since the symmetry group of $\mathsf{S}^n_{\mathbb{F},\tau}$ is isomorphic to the isotropy of the symmetric space of rank one where $\mathsf{S}^n_{\mathbb{F},\tau}$ is embedded as a geodesic sphere (see Equation (8.12)), we obtain the following:

Corollary 8.5.2. *The isometry groups for the Hopf-Berger spheres* $\mathsf{S}^n_{\mathbb{F},\tau}$*, when* $\tau \neq 1$*, are:*

$$\mathrm{Isom}(\mathsf{S}^{2n+1}_{\mathbb{C},\tau}) \cong \mathsf{U}_{n+1} \rtimes \mathbb{Z}_2, \quad \mathrm{Isom}(\mathsf{S}^{4n+3}_{\mathbb{C},\tau}) \cong \mathsf{Sp}_{n+1} \times_{\mathbb{Z}_2} \mathsf{Sp}_1, \quad \mathrm{Isom}(\mathsf{S}^{15}_{\mathbb{O},\tau}) \cong \mathsf{Spin}_9.$$

Proof Theorem B. Let $\mathsf{S}^{2n+1}_{\mathbb{C},\tau} = \mathsf{U}_{n+1}/\mathsf{U}_n$, which is a naturally reductive space for every $\tau > 0$, see [191]. The group G of transvections with respect to the canonical connection ∇^c is a transitive normal connected subgroup of U_{n+1}, see [119, Theorem I.25]. Then, it must be either $\mathsf{G} = \mathsf{U}_{n+1}$ or $\mathsf{G} = \mathsf{SU}_{n+1}$. In any case, the dimension of the subspace of vectors fixed by the isotropy group is one-dimensional. Then, by [146, Theorem B], $\mathrm{ind}_S(\mathsf{S}^{2n+1}_{\mathbb{C},\tau}) = 1$.

Let $\mathsf{S}^{4n+3}_{\mathbb{H},\tau} = \mathsf{Sp}_{n+1}\mathsf{Sp}_1/\mathsf{Sp}_n\mathsf{Sp}_1$, which is a naturally reductive space, see [191]. The group G of transvections with respect to ∇^c, being a transitive normal connected subgroup of $\mathsf{Sp}_{n+1}\mathsf{Sp}_1$, must be either $\mathsf{G} = \mathsf{Sp}_{n+1}\mathsf{Sp}_1$ or $\mathsf{G} = \mathsf{Sp}_{n+1}$. In the first case, the isotropy does not fix any non-zero vector, and thus $\mathrm{ind}_S(\mathsf{S}^{4n+3}_{\mathbb{H},\tau}) = 0$. In the latter case, the space is also naturally reductive with respect to the presentation $\mathsf{Sp}_{n+1}/\mathsf{Sp}_n$. But in this case, the only naturally reductive metric is the normal homogeneous metric, up to rescaling, since Sp_n is simple and there are no isotypical components in the isotropy representation. This normal homogeneous metric corresponds to $\tau = 1/2$ since the tensor U associated with this presentation vanishes if and only if $\tau = 1/2$, see Section §1.3 and Section §8.3. Observe that, in this case, the dimension of the space of vectors fixed by the isotropy Sp_n is 3. Thus, by [146, Theorem B], we have $\mathrm{ind}_S(\mathsf{S}^{4n+3}_{\mathbb{H},\tau}) = 3$ if $\tau = 1/2$; and $\mathrm{ind}_S(\mathsf{S}^{4n+3}_{\mathbb{H},\tau}) = 0$, otherwise.

Let $\mathsf{S}^{15}_{\mathbb{O},\tau} = \mathsf{Spin}_9/\mathsf{Spin}_7$. By Corollary 8.5.2, the isometry group of this space is Spin_9. The symmetry subspace $\mathfrak{s}_o$ at the base point o is a Spin_7-invariant subspace. So, if non-trivial, it must be either $\mathfrak{p}_1$ or $\mathfrak{p}_2$. Since the symmetry distribution is integrable, one has that $\mathfrak{s}_o = \mathfrak{p}_1$ if $\mathfrak{s}_o \neq \{0\}$. Then the leaf of symmetry at o must be $\mathsf{Spin}_8 \cdot o = \mathsf{Spin}_8/\mathsf{Spin}_7$. Observe that $\mathfrak{spin}_8 = \mathfrak{spin}_7 \oplus \mathfrak{p}_1$ is the Cartan decomposition of a leaf of symmetry at o. Let $X \in \mathfrak{spin}_9 \cong \mathfrak{K}(\mathsf{S}^{15}_{\mathbb{O},\tau})$ be a non-zero transvection with respect to the Levi-Civita connection ∇ of $\mathsf{S}^{15}_{\mathbb{O},\tau}$ at o. Then, X restricted to the leaf of symmetry $\mathsf{Spin}_8 \cdot o$ must be an intrinsic transvection of $\mathsf{Spin}_8 \cdot o$ at o. This implies that $X \in \mathfrak{p}_1$. But, using Equation (1.2), we have $(\nabla X)_o = 0$ if and only if $\tau = 1/2$. Hence, $\mathfrak{s}_o = \{0\}$ if $\tau \neq 1/2$, and $\mathfrak{s}_o = \mathfrak{p}_1$ when $\tau = 1/2$. Consequently, $\mathrm{ind}_S(\mathsf{S}^{15}_{\mathbb{O},\tau}) = 0$ for every $\tau \neq 1/2$, and $\mathrm{ind}_S(\mathsf{S}^{15}_{\mathbb{O},1/2}) = 7$. □

Conclusions

The first contribution of this thesis is the construction of an example of a non-isoparametric hypersurface with constant principal curvatures. This is presented in Chapter 2. In particular, we have deduced the following:

- There exist conformally flat Riemannian metrics on compact and non-compact manifolds that admit non-isoparametric hypersurfaces with constant principal curvatures.

The second contribution of this thesis is the classification of cohomogeneity one actions on quaternionic hyperbolic spaces. This is presented in Chapter 4. In particular, we have obtained the following results:

- The classification of protohomogeneous subspaces of quaternionic Euclidean vector spaces $\mathbb{H}^n$.
- The classification of homogeneous hypersurfaces in $\mathbb{H}\mathsf{H}^n$, up to isometric congruence.
- The existence of uncountable families of inhomogeneous isoparametric hypersurfaces with constant principal curvatures in $\mathbb{H}\mathsf{H}^n$ with $n \geq 8$. To our knowledge, these isoparametric families constitute the only such examples known in Riemannian manifolds, apart from the celebrated Ferus, Karcher and Münzner hypersurfaces in spheres and an example in the Cayley hyperbolic plane.

The third contribution of this thesis deals with the classification of totally geodesic submanifolds in products of rank one symmetric spaces. This is presented in Chapter 6. In particular, we have obtained the following:

- A correspondence between semisimple totally geodesic submanifolds in products of rank one symmetric spaces and adapted Young tableaux.

The fourth contribution of this thesis is the construction of infinitely many examples of totally geodesic submanifolds with non-trivial constant Kähler angle in complex Grassmannians. This is presented also in Chapter 6. In particular, this implies the following:

- Every rational number in $[0, 1]$ can be realized as the cosine of the Kähler angle of a totally geodesic submanifold in an irreducible Hermitian symmetric space for a sufficiently large rank.

The fifth contribution of this thesis deals with the classification of maximal totally geodesic submanifolds in exceptional symmetric spaces. This is presented in Chapter 7. In particular, we have achieved the following:

- The classification of maximal totally geodesic submanifolds in exceptional symmetric spaces.
- We have introduced the notion of Dynkin index, which allows us to distinguish the isometry type of totally geodesic embeddings of semisimple symmetric spaces.
- We have proved that every irreducible semisimple symmetric space admits a totally geodesic submanifold realizing the index and whose irreducible factors have Dynkin index equal to one.
- We have listed 168 isometric classes of maximal totally geodesic submanifolds in exceptional symmetric spaces. There are 80 that are reflective and 88 that are not.

The sixth contribution of this thesis deals with the classification of totally geodesic submanifolds in Hopf-Berger spheres. This is presented in Chapter 8. In particular, we have proved the following:

- Every complete totally geodesic submanifold with dimension $d \geq 2$ of a Hopf-Berger sphere $\mathsf{S}^n_{\mathbb{F},\tau}$, $\tau \geq 1/2$, is obtained as the intersection of the Hopf-Berger sphere regarded as a geodesic sphere of the corresponding rank one symmetric space $\bar{M}$ and a totally geodesic submanifold M of $\bar{M}$ passing through the center of $\mathsf{S}^n_{\mathbb{F},\tau}$.
- Two totally geodesic submanifolds of dimension $d \geq 2$ in $\mathsf{S}^n_{\mathbb{F},\tau}$, $\tau \geq 1/2$, are congruent if and only if they are isometric.
- The index of symmetry of $\mathsf{S}^n_{\mathbb{F},\tau}$ with $\tau \neq 1$ is given by:

$$\mathrm{ind}_S(\mathsf{S}^n_{\mathbb{F},\tau}) = \begin{cases} 0 & \text{if } \mathbb{F} = \mathbb{H} \text{ or } \mathbb{O} \text{ and } \tau \neq 1/2, \\ 1 & \text{if } \mathbb{F} = \mathbb{C}, \\ 3 & \text{if } \mathbb{F} = \mathbb{H} \text{ and } \tau = 1/2, \\ 7 & \text{if } \mathbb{F} = \mathbb{O} \text{ and } \tau = 1/2. \end{cases}$$

There are still many open problems and questions in view of the above conclusions. Some of these questions are directly related to the above commented results.

Open problems:

(1) The known classification results and examples of isoparametric hypersurfaces or hypersurfaces with constant principal curvatures occur in spaces with a lot of symmetries. Do isoparametric hypersurfaces or hypersurfaces with constant principal curvatures exist for generic metrics?

(2) The existence of non-isoparametric hypersurfaces with constant principal curvatures in Riemannian manifolds was addressed in Chapter 2. Are there such examples in symmetric spaces?

(3) A problem that seems to be completely out of scope of the current techniques available in the area would be to complete Tables 2.1 and 2.2. The easiest case seems to be the classification of isoparametric hypersurfaces with constant principal curvatures in $\mathbb{H}\mathsf{P}^n$. We conjecture that these hypersurfaces should be open parts of homogeneous hypersurfaces.

(4) Despite all the efforts in Chapter 4, we have not achieved a complete classification of subspaces with constant quaternionic Kähler angle. We conjecture that there are no more examples of non-protohomogeneous subspaces with constant quaternionic Kähler angle than the ones described in Section §4.5. In order to show this, one should prove a version of Corollary 4.3.2 where the hypothesis of protohomogeneity is substituted by the weaker assumption of constant quaternionic Kähler angle.

(5) Continue the classification of cohomogeneity one actions in symmetric spaces of noncompact type and higher rank. This looks like a really hard problem. One should derive new ideas to get a better understanding of the nilpotent construction. A more systematic approach using representation theory and the classification of connected groups acting effectively and transitively on the spheres (see [136]) could give new insights into this problem.

(6) In Section §6.3 it was proved that every rational number in $[0, 1]$ can be realized as the cosine of the Kähler angle of a totally geodesic submanifold in an irreducible Hermitian symmetric space. Are there examples of totally geodesic submanifolds with constant Kähler angle in an irreducible Hermitian symmetric space such that the cosine of its Kähler angle is not rational?

(7) Wolf spaces are symmetric spaces that are quaternionic Kähler. It makes sense to study the quaternionic Kähler angle of totally geodesic submanifolds in Wolf spaces with constant quaternionic Kähler angle. It is worth noticing that totally geodesic submanifolds in Wolf spaces with constant quaternionic Kähler angle $(0, \pi/2, \pi/2)$

were classified in [188]. Is there a totally geodesic submanifold with non-trivial constant quaternionic Kähler angle in a Wolf space? By non-trivial here we mean different from the triples:

$$(0, \varphi, \varphi) \quad \text{or} \quad (\varphi, \pi/2, \pi/2), \quad \text{with } \varphi \in [0, \pi/2].$$

(8) In Chapter 7 we introduce the Dynkin index as an invariant for totally geodesic embeddings of semisimple symmetric spaces in irreducible symmetric spaces. This invariant measures when two totally geodesic embeddings are isometric. Is it possible to define an invariant that determines when two totally geodesic submanifolds are congruent?

(9) Derive sufficient conditions for a homogeneous space to satisfy that all its totally geodesic submanifolds are orbits of some subgroup of the isometry group.

(10) In Chapter 8 we compute the index of symmetry of Hopf-Berger spheres, which provides nice examples of non-symmetric homogeneous spaces of positive curvature. We pose the problem of computing the index of symmetry of homogeneous spaces with positive curvature.

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