Isaac Carcacía Campos

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MÁSTER EN MATEMÁTICAS

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Homotopy in small categories

Isaac Carcacía Campos

Xullo, 2022

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Contents

Resumen

En este trabajo abordaremos distintas aproximaciones topológicas a la teoría de categorías. Entre ellas destacan el espacio clasificante y la distancia homotópica entre functores como una adaptación al contexto de las categorías pequeñas del concepto topológico de distancia homotópica entre aplicaciones continuas. Esta noción puede entenderse como una generalización de la recientemente estudiada categoría-LS y la complejidad categórica, dos importantes invariantes por equivalencias de homotopía entre categorías pequeñas. Además veremos cómo la distancia homotópica puede ser generalizada a la distancia homotópica *higher* y cómo se comporta en relación a las fibraciones.

Abstract

In this work we explore topological approximations to category theory. More precisely, we will study the classifying space and the homotopic distance between functors as an adaptation of the topological concept of homotopic distance between continuous maps to the context of small categories. This notion can be viewed as a generalization of the recently studied LS-category and categorical complexity, two important invariants by homotopic equivalence between small categories. Moreover we will see how homotopic distance can be generalized to the higher homotopic distance and how it relates to fibrations.

Introduction

The *topological complexity* $\mathrm{TC}(X)$ of a topological space X was introduced in 2003 by Farber [8] in order to approach the motion planning problem in robotics. Moreover, topological complexity is an homotopy invariant that also has a deep relation with another topological invariant, namely the *LS-Category* of a topological space, that it was already know in 1934 [15] and had been used in different context such as Calculus of variants (numbers of critical points of a function) or Riemann Geometry (existence of closed geodesics). Both invariants had been studied a lot but are pretty difficult to obtain, so various scholars have developed approximations and variations of them such as the strong category, the topological complexity of a work-map [20], the category of a fibration [23] and much more.

Recently, Macías-Mosquera [18] introduced the homotopic distance between two continuous maps, that have as particular cases both the LS-Category and the topological complexity. Concretely, $\mathrm{cat}(X) = \mathrm{D}(i_1, i_2)$ the distance between the two inclusions $i_j\colon X \to X \times X$ such that $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$ where x_0 is any fix point in X, whereas $\mathrm{TC}(X) = \mathrm{D}(p_1, p_2)$, the distance between the two projections $p_j\colon X \times X \to X$.

Our work is framed in a different context in which the topological spaces, continuous maps and homotopies are in some sense replaced by similar notions in category theory, namely categories, functors and natural transformations. More precisely, we will work with small categories, i.e categories such that objects and morphism between objects are in a set. There is a lot of interesting examples of small categories as we will see in the following pages.

For a small category $\mathfrak{C}$ is possible to obtain a technically useful definition of homotopy and cover by subcategories, so we can easily extend some relevant notions that appears in the topological setting. Moreover, we can define the *classifying space* $\mathrm{B}\mathfrak{C}$ (a *CW-Complex*) of a category. In this way we can compare the invariants of $\mathfrak{C}$ and $\mathrm{B}\mathfrak{C}$.

To a large degree this change of setting is also motivated by the fact that in the recent years there was a considerable amount of work aimed to extend topological notions to categorical settings. Examples of this trend are the works of Tanaka in which he introduced a theory of Euler Calculus ([27] and [28]) and the development of LS-Category for small categories [25], the definition of the Euler Characteristic in the context of small categories by Leinster [3] and finally the construction of homotopic distance between functors by Macías-Mosquera [19].

Nonetheless both articles by Tanaka and Macías-Mosquera did not offer a proper enunciation nor a proof of some results that anyone will expect under the assumption of some relevant analogies between the topological and the categorical notions. This work aims to complete their papers and have a more deep understanding of their results by the careful study of some relevant examples. In particular we will study the notion of fibration that Tanaka study in the case of the LS-Category [25] under the more broad notion of homotopic distance.

The organization of this TFM is as follows:

In the first chapter we will introduce some *topological* notions in the categorical setting. More precisely we will see how the classifying space is a natural way to associate to each category a CW-Complex. Later we will see how both connectedness and homotopy are well defined notions when we work with categories. We will close the chapter with the introduction o a family of examples, namely posets.

The next chapter is an introduction to fibrations, cofibrations and bifibrations, some notions that are a categorical counterpart of fibrations in Topology. After we define them we will show how they really are analogous to fibrations, that is we will prove that they have the homotopy lifting property and how and when we can talk about the fiber in a fibration.

In the last chapter we will define the homotopic distance between functors that Macías-Mosquera [19] introduced. More precisely we we will first define what is a geometrical cover of a small category and then we will finally arrive at the notion of homotopic distance between two functors. Next we will define the LS-Category and the categorical complexity to then prove that they are particulars cases of homotopic distance. Once we have shown that both notions are particular cases we will study some interesting properties such as compositions inequalities and the fact that the distance is invariant under homotopy equivalence. Finally we will expand the notion of homotopic distance in two different ways taking as guide two results in the topological setting plus the Varadarajan inequality that Tanaka prove for the LS-category for small categories ([4], [18] and [25]). First we will apply the homotopic distance in the context of bifibrations in order two show a inequality of homotopic distance between two bifibrations and how this inequality allow us to obtain two known inequalities. Next we will conclude defining a generalization of homotopic distance (higher homotopic distance) to then finish this work proving that it generalizes the notion of higher categorical complexity.

Chapter 1

Relations between categories and topology

The aim of this chapter is to show in various domains the close relation between topological notions and category theory. We begin by describing how one can associate to every category a topological space that is called the classifying space of the category [7]. This construction is, as we will show, a natural one that uses common techniques in topology and category theory ([16] and [22]).

Later we will explore how we can introduce the notions of connectedness and homotopy in the context of small categories. Connectedness is a more familiar notion [16] introduced by using a family of interval categories that have the same role as the unit interval in topology. These interval categories will also provide us with a way of defining homotopy between functors and related notions. However this will only be one option since homotopy can be introduced using two apparently different ways, first using the same interval categories and then with a sequence of functors and natural transformations ([12] and [13]).

Finally we will study posets since they admit both topological [1] and category points of view, so they are the perfect candidate to study how this notions relate to each other.

1.1 Nerve and Classifying space

Mathematical history show us how one of the most important ways of understanding mathematical objects relies on assigning to them other objects. For example is commonly know that in order to understand topological spaces algebraic invariants such as the fundamental group are essential tools. In a similar way now we will introduce a *natural way* to associate to each category a topological space. However in order to do that we must introduce some notions. First we will define what are simplicial categories.

1.1.1 Simplicial categories

Definition 1.1. We define the category Δ as the following one:

- The objects are posets of the form $[n] = \{0 < 1 < 2 < \cdots < n\}$ with $n \in \mathbb{N}$.
- The morfism are order preserving maps between posets.

Notation 1.2. We will assume that $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{N}^* = \{1, 2, ...\}$.

Definition 1.3. If we choose some $n \in \mathbb{N}^*$ we define for each i and $j \in \{0, \cdots, n\}$ the morfisms:

- *face* $d^i\colon [n-1] \to [n]$ defined as

$$d^i(x) = \begin{cases} x & if \quad x < i \\ \\ x+1 & if \quad i \leq x \end{cases}$$

  In ordinary language we can say that this morfism *skips*.

- *degeneracy* $s^j\colon [n+1] \to [n]$ in Δ

$$s^j(x) = \begin{cases} x & if \quad x \leq i \\ \\ x-1 & if \quad i < x \end{cases}$$

  In ordinary language we can say that this morfism *repeats* j or that *pass* j *twice*.

Example 1.4. If we have a low number as 2 we can easily write all face and degeneracy morphisms. In this case there are only three face maps: $d^0\colon [1] \to [2]$ that is defined as the one that takes 0 to 1 and 1 to 2 (we can see this as the morphism that takes everything *one step to te right*), $d^1\colon [1] \to [2]$ which satisfied that takes 0 to 0 and 1 to 2 and finally $d^2\colon [1] \to [2]$ that takes 0 to 0 and 1 to 1 and we can see as the *natural inmersion* of [1] into [2]. Analogously it can easily be checked that there only three degeneracy maps s^0, s^1 and s^2.

Now that we have presented this category, we can define the following notions that are the key point in the process of building the nerve of a category.

Definition 1.5 (Simplicial object). A simplicial object in a category $\mathfrak{C}$ is a contravariant functor $X\colon \Delta^{op} \to \mathfrak{C}$.

Notation 1.6. If we have a simplicial object $X\colon \Delta^{op} \to \mathfrak{C}$ and $n \in \mathbb{N}$ we denote by X_n the object $X([n])$.

Definition 1.7 (Simplicial map). A simplicial map in a category $\mathfrak{C}$ is a natural transformation between two simplicial objects.

Definition 1.8 (Simplicial category). Let $\mathfrak{C}$ be a category, we denote by $s\mathfrak{C}$ the category whose objects are simplicial objects in $\mathfrak{C}$ and whose morphisms are simplicial maps.

These notions are well known when the category sitting in the right-side is the category **Set** of Sets with applications between sets as morphisms. In this case the notion has proven its usefulness and these objects are commonly referred as *simplicial*. Moreover, the justification of this construction can be obtained by the Yoneda lemma. In order to show this we define the followings simplicial sets:

Definition 1.9 (Standard n-simplex). For all $n \in \mathbb{N}$ we define the standard n-simplex $\Delta[n]$ as the simplicial set $\mathrm{Hom}(-,[n])\colon \Delta \to \mathbf{Set}$, the functor that takes $[k]$ to the set of morphism of the form $[k] \to [n]$.

Proposition 1.10. *For all simplicial sets $X\colon \Delta \to \boldsymbol{Set}$ there is a natural bijection between $X_n = X([n])$ and simplicial maps $\Delta[n] \to X$.*

Proof. It follows directly from Yoneda lemma. □

Example 1.11. If we have $n = 0$ it is easy to see that for any natural number $k \in \mathbb{N}$ the set $\Delta[0]([k]) = \mathrm{Hom}([k],[0])$ has only one element: the constant map that maps each element of $[k]$ into 0.

A more insightful example but still not hard to see is the following one.

Example 1.12. For each natural number $k \in \mathbb{N}$ we have that $\Delta[1]([k]) = \mathrm{Hom}([k],[1])$ has $k+2$ elements. In order to prove it is sufficient to see that it follows from the fact that all morphism that interest us are in either one of these options:

- A morphism $\bar{m}$ with $m \in [k]$ defined as:

$$\bar{m}(x) = \begin{cases} 0 \quad if \quad x \leq k \\ \\ 1 \quad if \quad k < x \end{cases}$$

- The constant map to 1.

Remark 1.13. Let $X\colon \Delta^{\mathrm{Op}} \to \mathbf{Set}$ be a simplicial set, the morphism $X(d^i)$ and $X(s^j)$ are usually denoted by d^i and s^j. They are also called face and degeneracy map.

1.1.2 Nerve and classifiying space

Now that we know what are simplicial objects and in particular simplicial sets we can define the nerve of a category. As we will see intuitively the nerve of a category $\mathfrak{C}$ is nothing more that the simplicial set of *chains*, i.e. sequence of composable morphism in $\mathfrak{C}$.

Definition 1.14. The n-chain category $\mathcal{I}_n$ is the category generated by the following diagram:

$$0 \longrightarrow 1 \longrightarrow ... \longrightarrow n$$

Definition 1.15. A n-chain $\mathcal{C}$ in a small category $\mathfrak{C}$ is a functor $\mathcal{C}\colon \mathcal{I}_n \to \mathfrak{C}$. We denote by X_i the object $\mathcal{C}(i)$.

Definition 1.16 (Nerve of a category). We define the nerve of the category $\mathfrak{C}$ as the simplicial set $N\mathfrak{C}\colon \Delta^{op} \to \mathbf{Set}$ which satisfies that $N\mathfrak{C}_n$ is the set of n-chains in $\mathfrak{C}$.

Remark 1.17. If we apply the face morphism d^i to an n-chain $\mathcal{C}$:

$$X_0 \longrightarrow ... \longrightarrow X_{i-1} \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow ... \longrightarrow X_n$$

we delete the object $\mathcal{C}(i)$ by composing the morphisms $X_{i-1} \to X_i \to X_{i+1}$ if $n \neq i \neq 0$:

$$X_0 \longrightarrow ... \longrightarrow X_{i-1} \longrightarrow X_{i+1} \longrightarrow ... \longrightarrow X_n \ .$$

In other cases we simply delete the first or the last morphism

$$X_1 \longrightarrow ... \longrightarrow X_n \qquad\qquad X_0 \longrightarrow ... \longrightarrow X_{n-1} \ .$$

Whereas if we apply the degeneracy s^i morphism to $\mathcal{C}$ we obtain the $n+1$-chain that has all composable morphism in $\mathcal{C}$ with an extra element, the identity morphism between $X_i \to X_i$:

$$X_1 \longrightarrow ... \longrightarrow X_i \xrightarrow{\mathrm{id}_{X_i}} X_i \longrightarrow ... \longrightarrow X_n$$

We have shown that for each category we can build a simplicial set associated to it, but our goal, as we know, is to associate to each category a topological space. As the reader can guess the plan now is to use the nerve of a category as the intermediary step in an our journey. So the next step will be to construct a topological space from a simplicial set. In order to do that we need the notions of topological simplex and the geometric realization of a simplicial set.

Definition 1.18 (Topological simplex). For each natural number $n \in \mathbb{N}$ we define the topological n-simplex as the following subset of $\mathbb{R}^{n+1}$:

$$\Delta^n := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : \text{for each}\, i\, t_i \geq 0\, \text{and} \sum_{i=0}^{n} t_i = 1\}.$$

Definition 1.19. Let n be a natural number we define the followings continuous maps:

- $\delta^i\colon \Delta^n \to \Delta^{n+1}$ $\delta^i(t_0, ..., t_{i-1}, t_i, t_{i+1}, ...t_n) = (t_0, ..., t_i, 0, t_{i+1}, ..., t_n)$.
- $\sigma^i\colon \Delta^{n+1} \to \Delta^n$ $\sigma^i(t_0, ..., t_{i-1}, t_i, t_{i+1}, ..., t_{n+1}) = (t_0, ..., t_{i-1}, t_i + t_{i+1}, ..., t_{n+1})$.

Definition 1.20 (Geometric realization). Let X be a simplicial set, we define the geometric realization of X and we denoted by $|X|$ the topological space:

$$|X| := \sqcup_{n\geq 0} X_n \times \Delta^n/\sim.$$

where X_n has the discrete topology, Δ^n has the usual topology of a real subset and $\sim$ verifies:

1. $(d^i(x), \bar{t}) \sim (x, \delta^i(\bar{t}))$ with $x \in X_n$ and $\bar{t} \in \Delta^{n-1}$.
2. $(s^i(x), \bar{t}) \sim (x, \sigma^i(\bar{t}))$ with $x \in X_n$ and $\bar{t} \in \Delta^{n+1}$.

With all the previous definition we finally reach our destination. We must only bring together all the ingredients.

Notation 1.21. We will denote by $\mathrm{K}X$ the topological space $\sqcup_{n\geq 0} X_n \times \Delta^n$ where X_n has the discrete topology and Δ^n has the usual topology.

Definition 1.22 (Classifying space). For each category $\mathfrak{C}$ we define its classifying space as $\mathrm{B}\mathfrak{C} = |\mathrm{N}\mathfrak{C}|$.

1.1.3 Examples and interesting results

As usual in order to show the usefulness of a definition we must apply it, i.e. we must show the intuition behind the idea using relevant examples. But if we want to do it directly we face a problem. The strategy of looking into every chain in a category lead us to complications because every category have identity maps that can create chains of every length. So in order to properly calculate some examples we must begin by getting rid of that trivial chains.

Definition 1.23. If $\mathcal{C}$ is a chain in a category $\mathfrak{C}$ that contains an identity we say that it is a trivial chain.

Proposition 1.24. *Let $\mathfrak{C}$ be a small category, then $\mathrm{B}\mathfrak{C} = |\hat{\mathrm{N}}\mathfrak{C}|$ where $\hat{\mathrm{N}}\mathfrak{C}$ is the simplicial set of non trivial chains of $\mathfrak{C}$.*

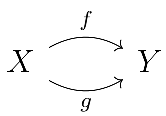
Proof. Suppose that $\mathcal{C}$ is a trivial n-chain that have an identity in the i-position. For every point $(\mathcal{C}, t)$ with $t = (t_0, ..., t_{i-1}, t_i, t_{i+1}, ...t_n)$ in the connected subspace $X = \{\mathcal{C}\} \times \Delta^n$, we know that

$$(\mathcal{C}, t) = (s^i(d^i(\mathcal{C})), t) \simeq (d^i(\mathcal{C}), \sigma^i(t)) = (d^i(\mathcal{C}), (t_0, ..., i_{i-1}, t_{i+1}, ...t_n)).$$

So we have proved that $X \simeq d^i\{\mathcal{C}\} \times \Delta^{n-1}$. If $d^i\{\mathcal{C}\}$ is trivial we repeat the argument until we have a non trivial chain. $\square$

Now we can see some examples.

Example 1.25. Let $\mathfrak{C}$ be the category generated by the following diagram:



We claim that $\mathrm{B}\mathfrak{C} \simeq \mathbb{S}^1$. The only non trivial chains are the constant functors X, Y,$f: X \to Y$ and $g\colon X \to Y$. So the classifiying space will be the quotient of two segments in which we identify the initial and finals points, which is homeomorphic to $\mathbb{S}^1$.

Example 1.26. Let $\mathcal{I}_2$ be the 2-chain category:

$$0 \xrightarrow{f} 1 \xrightarrow{g} 2$$

then $\mathrm{B}\mathcal{I}_2 \simeq \Delta^2$. The only 2-chain is $\mathcal{C} = \{g, f\}$, whereas the 1-chains are f, g and $g \circ f$. The gluing that we make is to identify the chains $f \times \Delta^1$, $g \times \Delta^1$ and $g \circ f \times \Delta^1$ with subsets in $\{\mathcal{C}\} \times \Delta^2$. For example for every point $(\mathcal{C}, t)$ in $\{\mathcal{C}\} \times \Delta^2$ we have:

1. If $t = (t_0, t_1, 0)$ then $(\mathcal{C}, t) = (\mathcal{C}, \delta^2 \circ \sigma^2 t) \simeq (d^2\mathcal{C}, \sigma^2 t) = (f, (t_0, t_1))$.
2. If $t = (t_0, 0, t_2)$ then $(\mathcal{C}, t) = (\mathcal{C}, \delta^2 \circ \sigma^2 t) \simeq (d^1\mathcal{C}, \sigma^1 t) = (g \circ f, (t_0, t_2))$.
3. If $t = (0, t_1, t_2)$ then $(\mathcal{C}, t) = (\mathcal{C}, \delta^2 \circ \sigma^2 t) \simeq (d^0\mathcal{C}, \sigma^0 t) = (g, (t_1, t_2))$.

In general it can be proved that $\mathrm{B}\mathcal{I}_n \simeq \Delta^n$.

Example 1.27. Let $\mathfrak{C}$ be the following category:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

where $f \circ g = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$, i.e. $\mathfrak{C}$ is a groupoid. It can be proved that $\mathrm{B}\mathfrak{C} = S^\infty$ ([7]).

Example 1.28. Let $\mathfrak{C}$ be the discrete abelian group $\mathbb{Z}_2$ considered as a category, that is $\mathfrak{C}$ is the category generated by the following diagram:

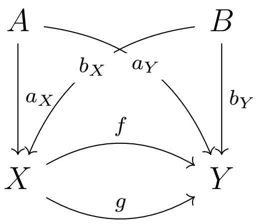
where $f \circ f = \mathrm{id}_\bullet$. It can be shown that $\mathrm{B}\mathfrak{C} = \mathbb{RP}^\infty$.

Example 1.29. Let $\mathfrak{C}$ be a commutative square, i.e. $\mathfrak{C}$ is the category generated by the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{h} & D \\ {\scriptstyle f}\downarrow & \searrow^{l\circ h}_{g\circ f} & \downarrow{\scriptstyle l} \\ B & \xrightarrow[g]{} & D \end{array}$$

In $\mathrm{B}\mathfrak{C}$ there are only two cell of dimension 2 and both have the side label as $g \circ f$ or $l \circ h$ by the commutativity of the diagram. As we know if we have two triangles with one side in common the full picture is something homeomorphic to a square and we can conclude that $\mathrm{B}\mathfrak{C} \simeq [0, 1]^2$.

Example 1.30. Let $\mathfrak{C}$ be the category generated by the following conmutative diagram:

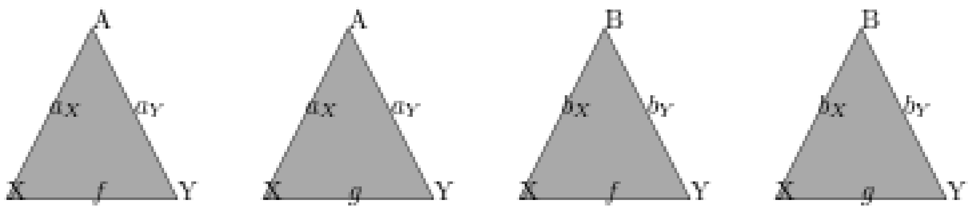


We claim that $\mathrm{B}\mathfrak{C} = \mathbb{S}^2$. In order to prove our claim is enough to consider that the non-degenerated chains of length 2 are:

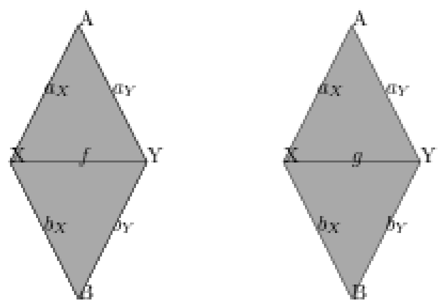
$$A \xrightarrow{a_X} X \xrightarrow{f} Y \qquad\qquad B \xrightarrow{b_X} X \xrightarrow{f} Y$$

$$A \xrightarrow{a_X} X \xrightarrow{g} Y \qquad\qquad B \xrightarrow{b_Y} X \xrightarrow{g} Y$$

and they have associated the following 2-simplices (recall that $f \circ a_x = a_Y = g \circ a_x$ and $f \circ b_x = b_Y = g \circ b_x$:



So if we glue together keeping in mind the labels:



Finally we can see that if we glue the boundaries we get $\mathrm{B}\mathfrak{C} \simeq \mathbb{S}^2$.

1.2 Properties of the classifying space

The classifying space of a category is not only a way to associate to each small category with a topological space, but it is also a natural way of doing it. The character of being natural means that it is compatible with many other notions such as the product (in some cases), suspension or, more importantly the structure of small categories as a category **cat** in which objects are categories and morphism are functors between them.

1.2.1 Functorial character of B

Lemma 1.31. *Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor between two small categories. There is a continuous map $\mathrm{B}F\colon \mathrm{B}\mathfrak{C} \to \mathrm{B}\mathfrak{D}$ induced by the functor F.*

Proof. Take any chain $\mathcal{C}$ in $\mathfrak{C}$ it is easy to see that it induces a chain in $F(\mathcal{C})$ in $\mathfrak{D}$ by the composition, i.e. $F(\mathcal{C}) = F \circ \mathcal{C}\colon \mathcal{I}_m \to \mathfrak{D}$. Then we have a continuous map $\mathrm{K}F\colon \mathrm{K}\mathfrak{C} \to \mathrm{K}\mathfrak{D}$ that takes a pair $(\mathcal{C}, t)$ to $(F(\mathcal{C}), t)$.

We only have to prove that $\mathrm{K}F$ is compatible with the equivalence relations that give us the classifying spaces.

1. $(\mathcal{C}, t) \sim (s^i(\mathcal{C}), \delta^i(t))$, so we must prove that $(F(\mathcal{C}), t) \sim (Fs^i(\mathcal{C}), \delta^i(t))$. But the truth of this follows from the fact functors respect identities, so $Fs^i(\mathcal{C}) = s^iF(\mathcal{C})$.

2. If t has a 0 in the i with entry then $(\mathcal{C}, t) \sim (d^i\mathcal{C}, \sigma^i(t)$ since $t = \delta^i(\sigma^i(t))$. So we must prove that in that case $(F(\mathcal{C}), t) \sim (F(d^i\mathcal{C}), \sigma^i(t)$. This can be proved analogously to the previous result since $d^iF(\mathcal{C}) = Fd^i(\mathcal{C})$ since functors preserve compositions.

□

Remark 1.32. There is also a functor $\mathrm{N}\colon$ **cat** $\to$ **SimSet** that takes ever category to $\mathrm{N}\mathfrak{C}$ its nerve category.

Proposition 1.33. *There is a functor* $\mathrm{B}\colon$ ***cat*** $\to$ ***TOP***.

Proof. We must only prove that B behaves well with identities and compositions.

1. $\mathrm{B}(\mathrm{id}_{\mathfrak{C}}) = \mathrm{id}_{\mathrm{B}\mathfrak{C}}$ since the induced continuous map originates from the map $\mathrm{K}\mathrm{id}_{\mathfrak{C}}$ that takes every pair $(\mathcal{C}, t)$ to itself.

2. For every pair of functors $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{B}$ we have that $\mathrm{B}(G \circ F) = \mathrm{B}G \circ \mathrm{B}F$. This follows from the fact that $B(G \circ F)$ is induced by $\mathrm{K}(G \circ F)$ that takes every pair $(\mathcal{C}, t)$ to $(G \circ F(\mathcal{C}), t)$.

□

Definition 1.34. Let $F, G\colon \mathfrak{C} \to \mathfrak{D}$ be two functor between two small categories. We say that F and G are weak homotopy if $\mathrm{B}F \simeq \mathrm{B}G$.

Definition 1.35. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor between two small categories. We say that F is a weak homotopy equivalence if there is a functor $G\colon \mathfrak{D} \to \mathfrak{C}$ such that $\mathrm{B}G \circ \mathrm{B}F \simeq \mathrm{id}_{\mathrm{B}\mathfrak{C}}$ and $\mathrm{B}G \circ \mathrm{B}G \simeq \mathrm{id}_{\mathrm{B}\mathfrak{D}}$.

1.2.2 Product

In order to properly state the facts and conditions that we want we need to introduce some auxiliary notions from general and algebraic topology (see for example [11]).

Definition 1.36. Let X be a topological space and $\{X_\alpha\}_{\alpha\in\mathrm{I}}$ be a collection of subsets of X such that $\cup_{\alpha\in\mathrm{I}}X_\alpha = X$. We say that X is generated by $\{X_\alpha\}_{\alpha\in\mathrm{I}}$ if for every subset $A \subset X$ we have that A is closed if and only if $X_\alpha \cap A$ is closed $\forall\alpha \in \mathrm{I}$.

Example 1.37. The topological space $\mathbb{I} = [0,1]$ is compactly generated. Indeed, take any closed subspace $F \subset \mathbb{I}$, then $K \cap F$ is closed for every K. Alternatively if $F \cap K$ is closed for every K then it is also closed in $\mathbb{I}$ since $\mathbb{I}$ is also closed.

Definition 1.38. Let X be a topological space we say that X is a compactly generated space if it is generated by its compact subspaces. We say that a compactly generated space is a *k-space*.

Definition 1.39. Let X and Y be two simplicial sets, we define the product $X \times Y : \Delta^{op} \to$ **set** as $(X \times Y)(n) = X(n) \times Y(n)$.

Proposition 1.40. *Let X and Y be two simplicial sets. There is a natural bijection* $\mathrm{B}(X) \times B(Y) \to \mathrm{B}(X \times Y)$ *that is a homemorphism if the right product is formed in the category of k-spaces.*

Proof. See chapter III of [10]. □

Proposition 1.41. *The product $X \times Y$ of a compactly generated Hausdorff space and is locally compact space is compactly generated.*

Proof. See proposition A.15 of [11]. □

Corollary 1.42. *$X \times \mathbb{I}$ is compactly generated if X is a CW-complex.*

1.2.3 Suspension

The Example 1.30 can be generalized but in order to do that we have to introduce some notions and a lemma.

Definition 1.43 (Suspension). Let $\mathfrak{C}$ be a small category. We define the categorical suspension or the suspension of $\mathfrak{C}$ as the category $\mathrm{S}(\mathfrak{C})$ with:

1. Objects from the disjoint union of objects of $\mathfrak{C}$ and $\{A, B\}$.

2. All the morphism in $\mathfrak{C}$ plus two unique morphism $A \to C$ and $B \to C$ for every object C in $\mathfrak{C}$.

Definition 1.44. Let X be a topological space. We define the topological suspension or the suspension of X as the topological space $\mathrm{S}(X)$:

$$\mathrm{S}(X) = X \times I/\sim$$

where I is the real interval $[-1, 1]$ and $\sim$ is the relation that identifies the followings points:

1. $(x, 1) \sim (y, 1)$.
2. $(x, -1) \sim (y, -1)$.

Lemma 1.45. *Let $\mathfrak{C}$ be a small category then*

$$\mathrm{S}(\mathrm{B}(\mathfrak{C})) \simeq \mathrm{S}(\sqcup_{n\geq 0}\mathfrak{C}_n \times \Delta^n)/\sim$$

where $\sim$ is the equivalence relation generated by

1. $[(\mathcal{C}, t), t'] \sim [(\hat{\mathcal{C}}, \hat{t}), t']$ *if* $[\mathcal{C}, t] = [\hat{\mathcal{C}}, \hat{t}]$ *in* $\mathrm{B}(\mathfrak{C})$.
2. $[(\mathcal{C}, t), 1] \sim [(\hat{\mathcal{C}}, \hat{t}), 1]$.
3. $[(\mathcal{C}, t), 1] \sim [(\hat{\mathcal{C}}, \hat{t}), 1]$.

Proof. Take $g\colon \mathrm{B}(\mathfrak{C})\times I \to Sc(\sqcup_{n\geq 0}\mathfrak{C}_n \times \Delta^n) \sim$ the continuous map defined as $g([\mathcal{C}, t], t') = [[(\mathcal{C}, t), t']]$. It is easy to show that it is well defined and behaves well with the relation that defines $\mathrm{S}(\mathrm{B}(\mathfrak{C}))$. Indeed, suppose that $[\mathcal{C}, t] = [\hat{\mathcal{C}}, \hat{t}]$ then it follows from the definition that $[(\mathcal{C}, t), t'] \sim [(\hat{\mathcal{C}}, \hat{t}), t']$. Moreover, if $([\mathcal{C}, t], \pm 1) \simeq [\hat{\mathcal{C}}, \hat{t}], \pm 1)$ then $g([\mathcal{C}, t], \pm 1) = [[(\mathcal{C}, t), \pm 1]] \simeq [[(\hat{\mathcal{C}}, \hat{t}), 1]] = g([\mathcal{C}, t], \pm 1)$. □

Proposition 1.46. *For every small category $\mathfrak{C}$ with $B(\mathfrak{C})$ a finite CW-complex we have that* $\mathrm{B}(\mathrm{S}(\mathfrak{C})) \simeq \mathrm{S}(\mathrm{B}(\mathfrak{C}))$.

Proof. In order to prove this proposition it is enough to define a homeomorphism
$f\colon \mathrm{B}(\mathrm{S}(\mathfrak{C})) \to \mathrm{S}(\mathrm{B}(\mathfrak{C}))$. The strategy we will follow can be summarised in the following diagram:

$$\begin{array}{ccc} \mathrm{KS}\mathfrak{C} & \xrightarrow{\bar{f}} & \mathrm{SK}\mathfrak{C} \\ \downarrow & & \downarrow \\ \mathrm{BS}\mathfrak{C} & \xrightarrow{f} & \mathrm{SB}\mathfrak{C} \end{array}$$

First we begin by constructing a continuous maps $\bar{f}\colon \mathrm{KS}\mathfrak{C} \to \mathrm{SK}\mathfrak{C}$. Take an arbitrary chain in the first space, i.e. a point $(\mathcal{C}, t)$ with $\mathcal{C}$ a chain of length n and $t = (t_0, ..., t_n) \in \Delta^n$. By the definition of $\mathrm{S}(\mathfrak{C})$ the chain $\mathcal{C}$ has one of the following forms:

1. $$C_1 \longrightarrow C_2 \longrightarrow ... \longrightarrow C_n$$

   where every C_i is in $\mathfrak{C}$.

2.
$$X \longrightarrow C_1 \longrightarrow C_2 \longrightarrow ... \longrightarrow C_n$$
where X is either A or B and C_i is in $\mathfrak{C}$.

If $\mathcal{C}$ has the first form we define $f(\mathcal{C}, t) = [(\mathcal{C}, t), 0]$ whereas if $\mathcal{C}$ begins with either A or B:

$$\bar{f}(\mathcal{C}, t) = \begin{cases} \left[(d^0\mathcal{C}, \sigma^0(t)), t_0\right] & if \quad X = A \\ \left[(d^0\mathcal{C}, \sigma^0(t)), -t_0\right] & if \quad X = B \end{cases}$$

It is easy to prove that $\bar{f}$ is continuous and surjective. Now we will see that $\bar{f}$ induces a continuous map f since $\bar{f}$ is compatible with the equivalence relations. Indeed, suppose we take an arbitrary element $(\mathcal{C}, t)$ then:

1. If t has a 0 in the i with entry then $(\mathcal{C}, t) \sim (d^i\mathcal{C}, \sigma^i(t)$ since $t = \delta^i(\sigma^i(t))$. Now in order to see that f is well defined we must prove that $\bar{f}((\mathcal{C}, t) \sim \bar{f}((d^i\mathcal{C}, \sigma^i(t))$. However, since
$$\bar{f}(\mathcal{C}, t) = \begin{cases} [(\mathcal{C}, t), 0] & if \quad \mathcal{C} \,\text{is in}\, \mathfrak{C} \\ \left[(d^0\mathcal{C}, \sigma^0(t)), t_0\right] & if \quad X = A \\ \left[(d^0\mathcal{C}, \sigma^0(t)), -t_0\right] & if \quad X = B \end{cases}$$
and if we denote the entries of $\sigma^i(t) = (\bar{t}_0, ..., \bar{t}_{n-1})$ then
$$\bar{f}(d^i\mathcal{C}, \sigma^i(t)) = \begin{cases} [(d^i\mathcal{C}, \sigma^i(t), 0] & if \quad d^i\mathcal{C} \,\text{is in}\, \mathfrak{C} \\ \left[(d^0d^i\mathcal{C}, \sigma^0\delta^i(t)), \bar{t}_0\right] & if \quad X = A \\ \left[(d^0d^i\mathcal{C}, \sigma^0\delta^i(t)), -\bar{t}_0\right] & if \quad X = B \end{cases}$$
In any case we have that $\bar{f}((\mathcal{C}, t) \sim \bar{f}((d^i\mathcal{C}, \sigma^i(t))$. Indeed:
   (a) If $\mathcal{C}$ is in $\mathfrak{C}$ then obviously $[(\mathcal{C}, t), 0] \sim [(d^i\mathcal{C}, \sigma^i(t), 0]$.
   (b) If $\mathcal{C}$ begins with $X = A$ we have that
      - If $i \neq 0$ then $t_0 = \bar{t}_0$ and
      $$\left[(d^0\mathcal{C}, \sigma^0(t)), t_0\right] \sim \left[(d^{i-1}d^0\mathcal{C}, \sigma^{i-1}\sigma^0(t)), t_0\right] = \left[(d^0d^i\mathcal{C}, \sigma^0\sigma^i(t)), t_0\right].$$
      - If $i = 0$ then $d^0\mathcal{C}$ lies in $\mathfrak{C}$ and we have that
      $$[(\mathcal{C}, t), t_0 = 0] \sim [d^0\mathcal{C}, \sigma^0(t)), 0].$$
   (c) If $\mathcal{C}$ begins with $X = B$ we can repeat the previous argument changing t_0 by $-t_0$.
2. $(\mathcal{C}, t) \sim (s^i(\mathcal{C}), \delta^i(t))$. This case can be proved in a similar way.

Since $\mathrm{B}(\mathrm{S}(\mathfrak{C}))$ is compact, $\mathrm{S}(\mathrm{B}(\mathfrak{C}))$ is Hausdorff and f is a continuous bijection we have that f is a homeomorphism. □

Proposition 1.47. *For every* $n \in \{1, 2, ...\}$ *we have that* $\mathrm{S}^n(\mathbb{S}^0) = \mathrm{S}(\mathrm{S}(...(\mathbb{S}^0)...)) \simeq \mathbb{S}^n$.

Corollary 1.48. *For every* $n \in \{1, 2, ...\}$ *we have that* $\mathrm{B}(\mathrm{S}^n(D_2) = \mathrm{B}(\mathrm{S}(\mathrm{S}(...(D_2)...))) \simeq \mathbb{S}^n$ *where* D_2 *is the discrete category with two objects.*

Definition 1.49. There is a functor $\mathrm{S}\colon$ **cat** $\to$ **cat** who takes every small category $\mathfrak{C}$ to $\mathrm{S}(\mathfrak{C})$ and every functor $F\colon \mathfrak{C} \to \mathfrak{D}$ to $\mathrm{S}(f)\colon \mathrm{S}(\mathfrak{C}) \to \mathrm{S}(\mathfrak{D})$ defined as:

- $\mathrm{S}(F)(C) = F(C)$ if C is in $\mathfrak{C}$.
- $\mathrm{S}(F)(A) = A$ and $\mathrm{S}(F)(B) = B$.

1.3 Connectedness in Categories

Definition 1.50. We define the interval category of length m and we denoted by $\mathbb{I}_m$ the following *zig-zag category*:

$$0 \to 1 \leftarrow 2... \to (\leftarrow)m.$$

Definition 1.51 (Path). Let $\mathfrak{C}$ be a small category. A path in $\mathfrak{C}$ is a functor $F\colon \mathbb{I}_m \to \mathfrak{C}$. Furthermore, we denote the objects $F(0)$ and $F(m)$ by the initial and final object of the path respectively.

Definition 1.52. We say that a small category $\mathfrak{C}$ is connected if for every pair of objects X and Y in $\mathfrak{C}$ there is a path $F : \mathbb{I}_m \to \mathfrak{C}$ such that X and Y are the initial and final object of F are X and Y.

Remark 1.53. For every path $F\colon \mathbb{I}_m \to \mathfrak{C}$ in $\mathfrak{C}$ we can define the inverse path $F^{-1}\colon \mathbb{I}_m \to \mathfrak{C}$ as the functor that takes a number i to the object $F(m-i)$. Furthermore if we have two paths $F\colon \mathbb{I}_m \to \mathfrak{C}$ and $G\colon \mathbb{I}_n \to \mathfrak{C}$ such that $F(m) = G(0)$ we can define the concatenation path $G * F\colon \mathbb{I}_{m+n} \to \mathfrak{C}$ if m is even or $G * F\colon \mathbb{I}_{m+n+1} \to \mathfrak{C}$ if m is odd. Indeed, we only have to construct $G * F$ as the functor:

- If m is even, $G * F(i) = F(i)$ if $i \leq m$ and $G(i)$ if $m \leq i$.
- If m is odd, $G * F(i) = F(i)$ if $i \leq m$, $G * F(m+1) = G(m) = F(m)$ (connected by the identity morphism to $G * F(i)$) and $G(i+1)$ iF $m + 1 \leq i$.

Definition 1.54 (Connected component). Let $\mathfrak{C}$ be a small category. A subcategory $\mathfrak{U}$ of $\mathfrak{C}$ is a connected component if

1. $\mathfrak{U}$ is connected.
2. Is maximal in the sense that there are no connected subcategories $\mathfrak{V}$ of $\mathfrak{C}$ such that $\mathfrak{U}$ is a proper subcategory of $\mathfrak{V}$.

Proposition 1.55. *Every small category* $\mathfrak{C}$ *admits an unique decomposition into connected components.*

Proof. It follows immediately from the fact that the relation between object *there is a path between them* is an equivalence relation. □

Remark 1.56. This fact is important since justifies us in assuming without loss of generality that we work with a connected category since we can always restrict ourselves to the connected components.

1.4 Homotopies

In order to understand the notions of homotopic distance and related notions such as the LS-category we have to define homotopies in the context of small categories. We follow the ideas of Lee in [12] and [13]. This notions is also compatible with the classifying space in the sense that if two functors F and G are homotopic then $\mathrm{B}F$ and $\mathrm{B}G$ are homotopic.

Definition 1.57 (**Homotopy between functors**)**.** Let $F, G\colon \mathfrak{C} \to \mathfrak{D}$ be two functors between two small categories. We say that a functor $H\colon \mathfrak{C} \times \mathbb{I}_m \to \mathfrak{D}$ is a homotopy between F and G if

- $H_0\colon \mathfrak{C} \to \mathfrak{D}$ equals to F.
- $H_m\colon \mathfrak{C} \to \mathfrak{D}$ equals to G.

If there is a homotopy between two functors F and G we say that they are homotopic and we denote it by $F \cong G$

Proposition 1.58. *The relation being homotopic between functors is an equivalence relation.*

Proposition 1.59. *Let $F, G\colon \mathfrak{C} \to \mathfrak{D}$ be two functors between two small categories. The two following statements are equivalent:*

1. *There is a homotopy $H\colon \mathfrak{C} \times \mathbb{I}_m \to \mathfrak{D}$ between F and G.*
2. *There is a sequence of functors $\{H_0, ..., H_m\}$ such that $H_0 = F$, $H_m = G$ and there are natural transformations $\alpha_i\colon H_k \implies H_{k+1}$ is k is even or $\alpha\colon H_k \implies H_{k-1}$ if k is odd.*

Now we will introduce some lemmas that will serve us in latter task. First we will show how to relate connectedness and constant functors. Recall that every constant functors $F\colon \mathfrak{C} \to \mathfrak{D}$ is obviously associated to a the object Y in $\mathfrak{D}$ such that $F(X) = Y$ for every X in $\mathfrak{C}$. From now on we will denote a constant functor by the object associated to it.

Lemma 1.60. *Two constant functors $A, B\colon \mathfrak{C} \to \mathfrak{D}$ are homotopic if and only if there is a path between them.*

Proof. Assume there is a path $\alpha\colon \mathcal{I}_m \to \mathfrak{D}$ between A and B then we have a homotopy $H\colon \mathfrak{C} \times \mathcal{I}_m \to \mathfrak{D}$ defined as:

$$H(X, i) = I(i)$$

fo every object X in $\mathfrak{C}$.

Conversely, if we have a homotopy $H\colon \mathfrak{C} \times \mathcal{I}_m \to \mathfrak{D}$ such that $H_0 = A$ and $H_m = B$ we can construct a path α usign any object of $\mathfrak{C}$. Indeed, for every object X we can define a path as $\alpha(i) = H(X, i)$. □

Corollary 1.61. *In a connected category $\mathfrak{C}$ every pair of objects induce two homotopic constant functor.*

Lemma 1.62. *If there is a natural transformation $\alpha\colon F \implies G$ between two functors $F, G\colon \mathfrak{C} \to \mathfrak{D}$, then $\mathrm{B}F \cong \mathrm{B}G$.*

Proof. It follows from the Corollary 1.42 and the fact that a natural transformation α is equivalent to a functor $H\colon \mathfrak{C} \times \mathbb{I}_1 \to \mathfrak{D}$. Indeed, since $\mathrm{B}(\mathfrak{C} \times \mathbb{I}_1) \simeq \mathrm{B}(\mathfrak{C}) \times \mathrm{B}(\mathbb{I}_1) \simeq \mathrm{B}(\mathfrak{C}) \times \mathbb{I}$ then we have $\mathrm{B}(H)\colon B(\mathfrak{C}) \times \mathbb{I} \to \mathrm{B}(\mathfrak{D})$, i.e. a homotopy between $\mathrm{B}(F)$ and $\mathrm{B}(G)$. □

Proposition 1.63. *If two functors F and G are homotopic then they are weak homotopic.*

Proof. It follows directly from the previous lemma. □

Definition 1.64 (Homotopy equivalence). Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor between two small categories. We say that F is a homotopy equivalence if there is another functor r $G\colon \mathfrak{D} \to \mathfrak{C}$ such that:

- $G \circ F \cong \mathrm{id}_{\mathfrak{C}}$.
- $F \circ G \cong \mathrm{id}_{\mathfrak{D}}$.

Furthermore if two categories $\mathfrak{C}$ and $\mathfrak{D}$ have a homotopy equivalence we say that they are homotopy equivalent and we denote it by $\mathfrak{C} \cong \mathfrak{D}$-

Remark 1.65. If the functor $F\colon \mathfrak{C} \to \mathfrak{D}$ satisfies that there is another functor $G\colon \mathfrak{D} \to \mathfrak{C}$ such that $G \circ F \cong \mathrm{id}_{\mathfrak{C}}$ then we say that G is a left homotopy inverse. If G satisfies $F \circ G \cong \mathrm{id}_{\mathfrak{D}}$ we call it a right homotopy inverse. Finally if G fulfills both condition we simply say that G is a homotopy inverse of F.

Proposition 1.66 (Homotopy and compositions). *Let $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{B}$ be three small categories such that there are functors $F, G\colon \mathfrak{C} \to \mathfrak{D}$ and $M\colon \mathfrak{D} \to \mathfrak{B}$ satisfying that $F \cong G$. Then $M \circ F \cong M \circ G$.*

Similarly, let $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{B}$ be three small categories such that there are three functors $F, G\colon \mathfrak{C} \to \mathfrak{D}$ and $N\colon \mathfrak{B} \to \mathfrak{C}$ satisfying that $F \cong G$. Then $F \circ N \cong G \circ N$.

Proof. Take $H\colon \mathfrak{C} \times \mathbb{I}_m \to \mathfrak{D}$ a homotopy between F and G. If we have a functor $M\colon \mathfrak{D} \to \mathfrak{B}$ then we can take the following functor $K\colon \mathfrak{C} \times \mathbb{I}_m \to \mathfrak{B}$:

- $K(X,i) = M((H(X,i)))$.
- $K(f,g) = M \circ (H(f,g))$.

That it is well defined and it also is the homotopy that we were looking for since $K_0 = M \circ H_0 = M \circ F$ and $K_m = M \circ H_m = M \circ G$. Alternatively if we have a functor $N \colon \mathfrak{B} \to \mathfrak{C}$ then there is a functor $L \colon \mathfrak{B} \times \mathbb{I}_m \to \mathfrak{D}$:

- $L(X,i) = H(N(X),i)$.
- $L(f,g) = H(N(f),g)$.

such that it is is a homotopy between $F \circ N$ and $G \circ N$ since $L_0(X) = H(N(X),0) = F \circ N(X)$ and $L_m(X) = H(N(X),m) = G \circ N(X)$ □

Proposition 1.67. *The relation between categories being homotopy equivalent is an equivalence relation.*

1.4.1 Contractible categories

Definition 1.68 (Contractible). We say that a category $\mathfrak{C}$ is contractible if $\mathfrak{C} \cong \bullet$ where $\bullet$ is the trivial category with only one object and only one morphism the identity.

Remark 1.69. Recall that every object induces a constant functor and vice versa so we will often denote any constant functor $F \colon \mathfrak{C} \to \mathfrak{D}$ as X where X is the image of every object in $\mathfrak{C}$.

Proposition 1.70. *A category $\mathfrak{C}$ is contractible if there is a homotopy between the identity and a constant functor.*

Proposition 1.71. *If a category $\mathfrak{C}$ has a initial or final object then it is contractible.*

Proof. We prove it assuming $\mathfrak{C}$ has an initial object since the other case is completely analogous. Suppose I is a initial object in $\mathfrak{C}$, then $\mathrm{id}_{\mathfrak{C}}$ is homotopic to the constant functor I by the following natural transformation:

$$\begin{array}{ccc} I & \xrightarrow{i_X} & X \\ \big\downarrow{\scriptstyle \mathrm{id}_I} & & \big\downarrow{\scriptstyle f} \\ I & \xrightarrow{i_Y} & Y \end{array}$$

Where $f \colon X \to Y$ is any arrow in $\mathfrak{C}$ and i_X and i_Y are the unique morphism from I to X and Y respectively. The diagram is obviously commutative since there is only one morphism from I to any object in $\mathfrak{C}$. □

Proposition 1.72. *A category that admits finite products is contractible.*

Proof. Suppose $\mathfrak{C}$ is a category with finite products. For every object C in $\mathfrak{C}$ we can define the functor $C \times -\colon \mathfrak{C} \to \mathfrak{C}$ that takes any object D to $C \times D$ and every morphism $f\colon D \to D'$ to $\mathrm{id}_C \times f\colon C \times D \to C \times D'$. On the one hand, there is a natural transformation $\alpha\colon C \times - \implies \mathrm{id}_{\mathfrak{C}}$ defined as:

$$\begin{array}{ccc} C \times D & \xrightarrow{\quad p_2 \quad} & D \\ \Big\downarrow{\scriptstyle \mathrm{id}_C \times f} & & \Big\downarrow{\scriptstyle f} \\ C \times D' & \xrightarrow{\quad p_2 \quad} & D' \end{array}$$

where p_2 is the second projection.

On the other, there is also a natural transformation $\beta\colon C \times - \implies C$ between $C \times -$ and the constant functor C defined as:

$$\begin{array}{ccc} C \times D & \xrightarrow{\quad p_1 \quad} & C \\ \Big\downarrow{\scriptstyle \mathrm{id}_C \times f} & & \Big\downarrow{\scriptstyle \mathrm{id}_C} \\ C \times D' & \xrightarrow{\quad p_1 \quad} & C \end{array}$$

□

1.5 Posets

In this section we will show some interesting results in the study of Posets. First we will see how we can introduce in a natural way the language of category theory and topology in order to study them.

Definition 1.73. Let $(P, \leq)$ be a poset, we can view P as a category whose objects are the elements of P and such that there is an arrow $X \to Y$ just in case $X \geq Y$.

Remark 1.74. It is easy to show that we have a category and that this follows from the fact that $\leq$ is a partial order relation. Indeed, on the one hand by reflexivity we have the identity morphism from every object. On the other hand by transitive we have the composition property.

Proposition 1.75. *A functor between two posets $F\colon P \to Q$ is an order preserving map.*

Proof. If $X \geq Y$ then there is a morphism $f\colon X \to Y$ and applying the functor F we get $F(f)\colon F(X) \to F(Y)$, so $F(X) \geq F(Y)$. □

Proposition 1.76. *A natural transformation α between two functors $F, G\colon P \to Q$ between posets is equivalent to the fact that $F(X) \geq G(X) \forall X \in P$.*

Proof. Recall that a natural transformation α gives for every object X in P a morphism $\alpha(X)\colon F(X) \to G(X)$. For the reciprocal case we just have to consider that to get the following commutative diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha(X)} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha(Y)} & G(Y) \end{array}$$

is enough to have the morphism $\alpha(X)$ and $\alpha(Y)$ since the commutative follows from the fact that we have only one morphism between two non isomorphic objects. ☐

Corollary 1.77. *Let $F, G\colon P \to Q$ be two functors between posets, F is homotopic to G if and only if there is a fence $F_0 = F, F_1, ...F_n = G$ of functors between then such that $F_0 \geq F_1 \leq F_2 \geq ...(\leq) \geq F_n$.*

Definition 1.78. Let P be a poset. For every object $X \in P$ we define the two posets:

- $U_X = \{Z \in P |\ Z \leq X\}$.
- $F_X = \{Z \in P |\ X \leq Z\}$.

Proposition 1.79. *U_X and F_X are contractible.*

Proof. It follows immediately from the Proposition 1.71. ☐

Definition 1.80 (Beat-point). Let P be a poset, a beat point is an element X in P such that it satisfies any of the followings claims:

- The subposet $\widehat{U_X} = \{Z \in P |\ Z < X\}$ has a unique maximum. If that is the case we say that X is a up beat point.
- The subset $\widehat{F_X} = \{Z \in P |\ Z > X\}$ has a unique minimum. If that is the case we say that X is a down beat point.

Definition 1.81 (Minimal poset). A minimal poset P is a Poset without beat points.

Proposition 1.82. *Let P be a poset whit a beat point X. There is a homotopy equivalence in the categorical sense between P and Q, the poset obtained by deleting the element X.*

Proof. The homotopy equivalence between P and Q is induced by the inclusio functor $i\colon Q \to P$ and the functor $F\colon P \to Q$ defined as the identity for every object that is not X and that takes X to $\bar{X}$, the maximum of $\widehat{U_X}$ if X is an up beat point or the minimum of $\widehat{F_X}$ if X is a down beat point. Now we will prove that there are homotopies between $(F \circ i)$ and id_Q and between $(i \circ F)$ and id_P. The former is trivial since $F \circ i = \mathrm{id}_Q$.

Moreover, there is a natural transformation, and by definition a homotopy equivalence, between $(i \circ F)$ and id_P . Indeed, on the one hand $(i \circ F)(Z) = Z$ for every $Z \neq X$, Furthermore $(i \circ F)(X) \geq X$ if X is a up beat point and $(i \circ F)(X) \leq X$ if X is a down beat point. Thus we have that $(i \circ F)(Y) \leq Y$ or $(i \circ F)(Y) \geq Y$ and so we can conclude that $(i \circ F) \cong \mathrm{id}_P$. ☐

Definition 1.83 (Core of a Poset). Let P and P_0 be two posets, we claim that P_0 is the core of P if $P \cong P_0$ and P_0 is minimal.

Definition 1.84. Let P and Q be two posets. A functor $F\colon P \to Q$ is homeomorphism if there is another functor $F^{-1}\colon Q \to P$ such that $F^{-1} \circ F = \mathrm{id}_P$ y $F \circ F^{-1} = \mathrm{id}_Q$.

If there is a homeomorphism between two posets we say that they are homeomorphic.

Proposition 1.85. *Let P be a minimal a finite poset. If we have a endofunctor $F\colon P \to P$ such that $F \cong \mathrm{id}_P$, then $F = \mathrm{id}_P$.*

Proof. Take the fence $F_0 = F \geq F_1 \leq ...(\leq) \geq F_n = \mathrm{id}_P$. Suppose that every functors $F_i \neq F_n$ for every $i \in \{0, ..., n-1\}$ and take $G = F_{n-1}$. Assume that $G \leq \mathrm{id}_P$. The set $A = \{Y \in P | G(Y) \neq Y\}$ it is not empty since $G \neq \mathrm{id}_P$. Moreover, we can take X the minimum of A. By hypothesis $F(X) < X$ and for every object $Y < X$ we have that $F(Y) = Y \leq F(X)$, so X is an up beat-point and we have contradicted the fact that P is minimal. If we assume that $G \geq \mathrm{id}_P$ the same contradiction appears taking X to be the maximum of A and proving that then X is a down beat point. In order to avoid the contradiction we must assume that $G = F_{n-1} = \mathrm{id}_P$. Repeating the argument in the fence $F_0 = F \geq F_1 \leq ...(\leq) \geq F_{n-1} = \mathrm{id}_P$ we can conclude that $F = \mathrm{id}_P$. □

Corollary 1.86. *Let P and Q be two minimal and finite posets. If P and Q have the same homotoype type then they are homeomorphic.*

Proof. Take $F\colon P \to Q$ and $G\colon Q \to P$ functors such that $G \circ F \cong \mathrm{id}_P$ and $F \circ G \cong \mathrm{id}_Q$. By the previous theorems $G \circ F = \mathrm{id}_P$ and $F \circ G = \mathrm{id}_Q$. □

Corollary 1.87. *Let P be a finite poset. If P_0 are P_0' two cores of P then they are homeomorphic.*

Proof. Since $P \cong P_0$ and $P \cong P_0'$ it follows that $P_0 \cong P_0'$. Now since P_0 and P_0' are minimal posets such that $P_0 \cong P_0'$ by the previous result there is a homeomorphism between them. □

Proposition 1.88 (Classification Theorem). *Let P and Q be two finite posets. The two followings claims are equivalent:*

- *P and Q have the same homotopy type.*
- *P and Q have the same core (up to homeomorphism).*

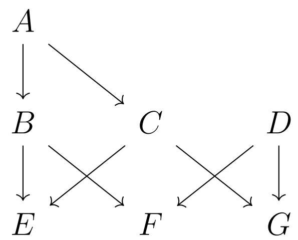
Proof.

- $\Longrightarrow$ Let P_0 a be a core of P and let Q_0 be a core of Q. By the definition of core of a poset we have that $P_0 \cong P$ and $Q_0 \cong Q$. Since $P \cong Q$ and $\cong$ is a equivalence relation we can conclude that $P_0 \cong Q_0$. Now, by the Corollary 1.87 they are homeomorphic.
- $\Longleftarrow$ Since $P_0 \cong P$, $Q_0 \cong Q$ and $Q_0 \cong P_0$ then $P \cong Q$.

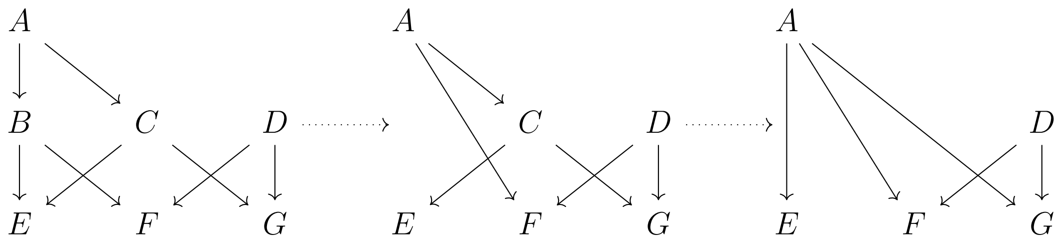
□

Now we will exemplify how to get the core of a Poset.

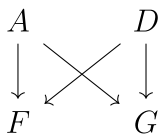
Example 1.89. Let P be the following Poset:



We will obtain its core by removing every beat point until we get a minimal poset. First we delete B since $\widehat{F_B} = \{A\}$. Then we remove C applying the same reasoning since $\widehat{F_C} = \{A\}$. Finally we remove F. In the diagram we have that:



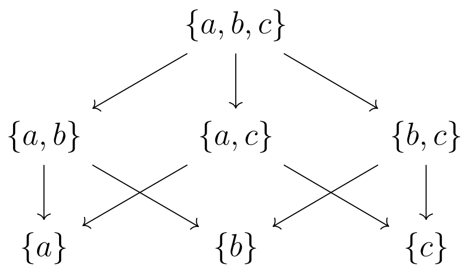
And



Posets not only have interesting criteria for the homotopy type, they also have interesting properties regarding the classifying space. The most direct one is that the classifying space of a poset is always a simplicial complex since for every two points there is only a morphism between them, hence there is at most only one n-simplices with the same set of vertices. We will use this fact in order to associated to each poset another poset. In order to do that we must introduce the face poset associated to a finite simplicial complex.

Definition 1.90 (Face poset). Let K be a simplicial complex we define the face poset of K as $\mathcal{X}(K)$ as the poset of chains of K ordered by inclusions.

Example 1.91. Suppose you have the 2-simplex, if you denote the 0-simplices as $\{a\}$,$\{b\}$ and $\{c\}$ the face poset is:



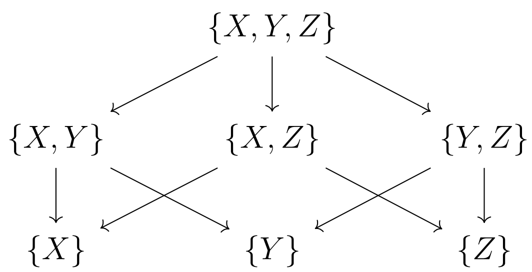
Definition 1.92 (**Barycentric subdivision**)**.** Let P be a poset we define its barycentric subdivision $\mathrm{sd}(P)$ as $\mathcal{X}(\mathrm{B}(P))$.

Remark 1.93. This can be also be defined if we take an acyclic category, i.e a category such that if whenever there is a morphism between two different objects X and Y there is no other morphism in the other direction and there is only one morphism between one object and itself namely the identity.

Example 1.94. Let P be the following poset:

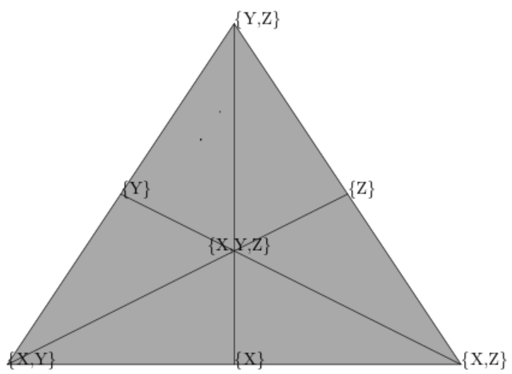
$$X \longrightarrow Y \longrightarrow Z$$

We know by Example 1.26 that its classifying space is Δ^2. So we can easy check that the barycentric subdivision of P is:



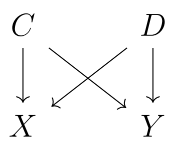
that is, the same as the previous example.

Moreover if we now apply the classifiying space functor to the poset $\mathrm{sd}(P)$ we get the following topological space:

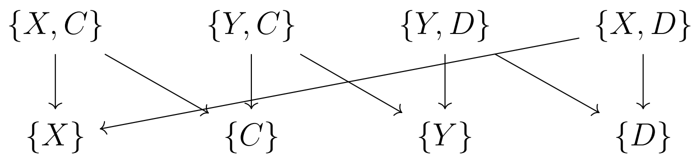


As we can notice by this example the name *barycentric subdivision* was well chosen.

Example 1.95. Let P be the following poset



The barycentric subdivision of P is



Chapter 2

Fibration

In this chapter we will continue the study of topological notions in the context of category theory by examining an analogous notion to Hurewicz fibration. More precisely we want to define the notions of fibered and opfibered categories introduced by Grothendieck in a Bourbaki Seminar in $1959 - 1960$ and that are well explain in chapter 12 of [2], in [24] and in the final part of the article [25]. In order to do that we will define two special classes of arrow associated to a functor $P\colon \mathfrak{E} \to \mathfrak{B}$, namely cartesian and opcartesian arrows. Then we will use them to define when the functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a fibration or a opfibration. Moreover we will see that our definition behaves in a similar fashion as the usual fibrations in topology. This means that we will check that our definition has the homotopy lifting property and that we can talk about **the** fiber of a fibration (up to homotopy equivalence) if the base category $\mathfrak{B}$ is connected.

2.1 Cartesian morphism and fibrations

2.1.1 Cartesian arrows

First we begin by defining the auxiliary notion of cartesian arrow associated to a functor $P\colon \mathfrak{E} \to \mathfrak{B}$. Recall that for every category $\mathfrak{E}$ and any pair of objects e_1 and e_2 we will denote by $\mathfrak{E}(e_1, e_2)$ the morphism between them.

Definition 2.1 (**Cartesian morphism**). Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a functor between two small categories and let $\phi\colon e_1 \to e_2$ be a morphism in $\mathfrak{E}$. The morphism ϕ is P-cartesian if for every morphism $\beta \in \mathfrak{E}(e, e_2)$ and every morphism $\bar{\alpha} \in \mathfrak{B}(Pe, Pe_1)$ such that $P\phi \circ \bar{\alpha} = P\beta$, there exists a unique arrow $\alpha \in \mathfrak{E}(e, e_1)$ such that $\phi \circ \alpha = \beta$ and $P\alpha = \bar{\alpha}$.

Remark 2.2. The previous definition is better understand with the following diagrams in mind:

$$\begin{array}{ccc} & & e_1 \\ & \overset{\alpha}{\nearrow} & \downarrow \phi \\ e & \xrightarrow{\beta} & e_2 \end{array} \qquad\qquad \begin{array}{ccc} & & P(e_1) \\ & \overset{\bar{\alpha}}{\nearrow} & \downarrow P(\phi) \\ P(e) & \xrightarrow{P(\beta)} & P(e_1) \end{array}$$

Remark 2.3. If the context is clear we will simply say that a morphism is *cartesian* without mentioning the functor.

Remark 2.4. The intuition behind the definition of a cartesian a is the possibility of *lifting a morphism* in the base category to $\mathfrak{E}$ in a rational manner. That property is the key to show the most important property of fibration, that is the *homotopy lifting property*.

Before we define fibrations we will show a more *categorical* definition of cartesian morphisms that will also explain why we choose that name. Recall that a functor $P\colon \mathfrak{E} \to \mathfrak{B}$ induces an application between $\bar{P}\colon \mathfrak{E}(e_1, e_2) \to \mathfrak{B}(P(e_1), P(e_2))$. Moreover any morphism $\phi\colon e_1 \to e_2$ induces an application $\phi \circ -\colon \mathfrak{E}(e, e_1) \to \mathfrak{E}(e, e_2)$ by composition. If we take all together we can easily prove that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{E}(e, e_1) & \xrightarrow{\Phi\circ -} & \mathfrak{E}(e, e_2) \\ \downarrow{\scriptstyle \bar{P}} & & \downarrow{\scriptstyle \bar{P}} \\ \mathfrak{B}(P(e), P(e_1)) & \xrightarrow{P(\phi)\circ -} & \mathfrak{B}(P(e), P(e_2)) \end{array}$$

since P is a functor and therefore $P(\Phi \circ \Psi) = P(\Phi) \circ P(\Phi)$.

Now we can formulate the characterization:

Proposition 2.5. *Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a functor. A morphism $\phi\colon e_1 \to e_2$ is P-cartesian $\iff$ for every object e in $\mathfrak{E}$ the pullback $\mathcal{P}$ of the following diagram*

$$\begin{array}{ccc} & & \mathfrak{E}(e, e_2) \\ & & \downarrow{\scriptstyle \bar{P}} \\ \mathfrak{B}(P(e), P(e_1)) & \xrightarrow{P(\phi)\circ -} & \mathfrak{B}(P(e), P(e_2)) \end{array}$$

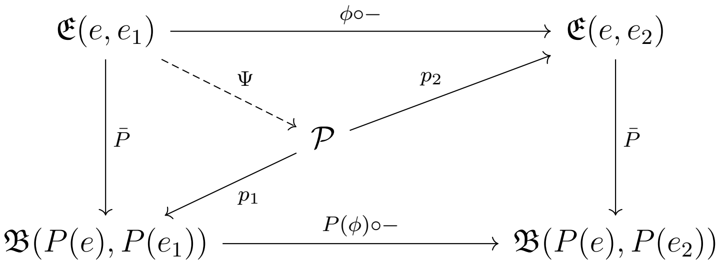
is $\mathfrak{E}(e, e_1)$.

Proof. Recall that we work under the assumption that the categories are locally small so the previous diagram lies in the category **SET** of sets and applications. Moreover in **SET** there are pullbacks and $\mathcal{P}$ is defined as:

$$\mathcal{P} = \{(\bar{\alpha}, \beta) \in \mathfrak{B}(P(e), P(e_1)) \times \mathfrak{E}(e, e_2) \mid P(\phi) \circ \bar{\alpha} = P(\beta)\}.$$

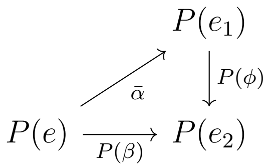
1. $\implies$ Suppose ϕ is P-cartesian, by the commutative of the diagram previously defined and the universal property of Pullback we know that there is a unique morphism

$\Psi\colon \mathfrak{E}(e, e_1) \to \mathcal{P}$ such that the following diagram commutes:

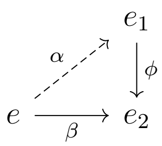


Now we must proof that Ψ is an isomorphism but as we know we are working in the category of sets and applications so a isomorphism is nothing more than a bijective application.

Take any pair $(\bar{\alpha}, \beta) \in \mathcal{P}$ we have the following commutative diagram:

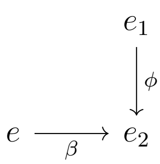


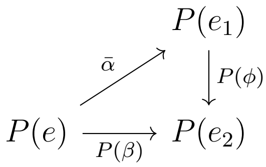
So using the fact that ϕ is cartesian there is a unique morphism α in $\mathfrak{E}(e, e_1)$ such that makes the following diagram commute



It is easy to see that α is the preimage of $(\bar{\alpha}, \beta)$ since $\bar{P}(\alpha) = P(\alpha) = \bar{\alpha}$ and $\phi\circ\alpha = \beta$ by the commutative of the previous diagram. Moreover α is unique. Hence Ψ is a isomorphism in **SET**.

2. $\Longleftarrow$ Suppose that we have the following diagrams:





By the previous consideration we can define the lift of $\bar{\alpha}$ as the preimage of $(\bar{\alpha}, \beta)$ in $\mathcal{P}$ by the isomorphism with $\mathfrak{E}(e, e_1)$.

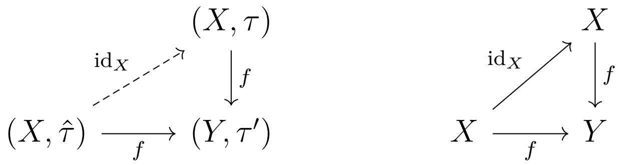
□

Examples

Now we will see some interesting cases of a cartesian morphims.

Example 2.6. Let $U\colon \mathbf{TOP} \to \mathbf{SET}$ be the forgetful functor, i.e. the functor that takes every topological space (X, τ) to the set X and every continuous map $f\colon (X,\tau) \to (Y,\tau')$ to the application $f\colon X \to Y$. It is easy to see that a continuous map $f\colon (X,\tau) \to (Y,\tau')$ is U-cartesian if and only if τ has the initial topology, that is, τ is the smallest (coarsest) topology making f continuous or equivalently a subset $V \subset X$ is open if and only if there is $W \in \tau'$ such that $V = f^{-1}(W)$. Indeed, take any continuous map $g\colon Z \to Y$ and a application $h\colon Z \to X$ such that $f \circ h = g$, we must show that h is continuous. Take any open set $V \subset X$ by the definition of the topology τ there is an open set $W \subset Y$ such that $V = f^{-1}(W)$. Now if we take the preimage $h^{-1}(V) = h^{-1}(f^{-1}(W)) = (f \circ h)^{-1}(W) = g^{-1}(W)$ that is open by the continuity of g.

Conversely assume that f is U-cartesian and take the topological space $(X, \hat{\tau})$ where $\hat{\tau}$ is the smallest topology making f continuous. We have the following diagrams:



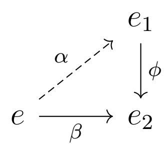
So $\mathrm{id}_X\colon (X,\hat{\tau}) \to (X,\tau)$ is continuous. Now using the previous fact that the coarsest topology is U-cartesian we can prove that $\mathrm{id}_X\colon (X,\tau) \to (X,\hat{\tau})$ is also continuous hence $\tau = \hat{\tau}$.

Example 2.7. Let $p\colon \mathfrak{E} \to \bullet$ be a functor between a category $\mathfrak{C}$ and $\bullet$, the category with only one object and the identity morphism. A morphism $\phi\colon e_1 \to e_2$ is P-cartesian if and only if ϕ is an isomorphism. Suppose ϕ is P-cartesian. On the one hand we have that for the morphism id_{e_2} there is a unique morphism ψ such that

Hence $\phi \circ \psi = \mathrm{id}_{e_2}$.

On the other hand since $\phi \circ \psi = \mathrm{id}_{e_2}$ we can conclude that $\phi = \mathrm{id}_{e_2} \circ \phi = (\phi \circ \psi) \circ \phi = \phi \circ (\psi \circ \phi)$, so the following diagram is commutative and a lift of $\mathrm{id}_\bullet$:

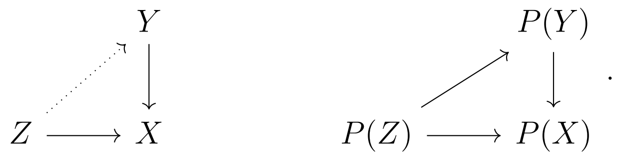
But we know that id_{e_1} has the same property so $\mathrm{id}_{e_1} = \psi \circ \phi$. Alternatively if ϕ is a isomorphism we have that for every object e and morphism $\beta\colon e \to e_2$ there is only one morphism $\alpha \in \mathfrak{E}(e, e_1)$ such that the following diagram is commutative:



namely $\alpha = \psi \circ \beta$ where ψ is the inverse of ϕ.

Example 2.8. Every isomorphism is a cartesian morphism for every functor. This follows from a straightforward repetition of the last argument in the previous example.

Example 2.9. Let $F\colon P \to Q$ be a functor between posets, that is an order preserving application. As we saw in the previous chapter there is a morphism $X \to Y$ in a poset if and only if $X \geq Y$. Now we can see that a pair $X \geq Y$ is F-cartesian if and only if it verifies the following condition: $\forall Z \in P$ if $Z \geq X$ and $P(Z) \geq P(Y)$ then $Z \geq Y$:



For a study of cartesian morphism and fibrations in the setting of posets we refer the reader to [5].

2.1.2 Fibrations

Now that we know what is a cartesian morphism we are prepared to define fibrations as the functors that can *lift* morphism in the base with a fixed element in the preimage to a cartesian morphism.

Definition 2.10 (Fibrations). We say that a functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a fibration if for any arrow $\bar{\phi} \in \mathfrak{B}(b_1, b_2)$ and any object $e_2 \in \mathfrak{E}$ such that $P(e_2) = b_2$, there exists a cartesian arrow $\phi \in \mathfrak{E}(e_1, e_2)$ such that $P(\phi) = \bar{\phi}$.

Remark 2.11. The arrow ϕ is called a cartesian lift of $\bar{\phi}$ with codomain e_2. We say **a** cartesian lift and not **the** cartesian lift because it is not necessary that ϕ is unique but as we will see every two possible lifts are closely related.

Remark 2.12. Not every fibration is surjective, not even essentially surjective. For example take the constant functor $P\colon \mathcal{I}_0 \to \mathcal{I}_1$ that takes 0 to 0. It is easy to see that P is a fibration since identities are always cartesian and the only morphism in $\mathcal{I}_2(b_1, b_2)$ with b_2 having a preimage are morphism of the form $\mathcal{I}_2(0,0) = \{\mathrm{id}_0\}$.

What we always have is the fact that if an object b has a preimage in $\mathfrak{E}$ then every object b' such that there is a morphism $\hat{\phi}\colon b' \to b$ has also a preimage. Indeed, since P is a fibration and there is object e such that $P(e) = b$ then there is a morphism $\phi\colon e' \to e$ such that $P(\phi) = \bar{\phi}$ and therefore $P(e') = b'$.

Now that we have defined what is a fibration we must introduce some related notions.

Definition 2.13 (Fiber). Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a functor, we define the fiber of an object b of $\mathfrak{B}$ and we denote by $\mathfrak{E}_b$ the category $P^{-1}(X)$, i.e. the category with objects e in $\mathfrak{E}$ such that $P(e) = b$ and morphism f that verifies $P(f) = \mathrm{id}_B$. These arrows are called vertical arrows.

Remark 2.14. In the definition we do not use the notion of fibration, P is an arbitrary functor.

Remark 2.15. The fiber of the object b is the pullback of the following diagram in **Cat**:

$$\begin{array}{ccc} & & \mathfrak{E} \\ & & \big\downarrow{\scriptstyle P} \\ \bullet & \xrightarrow{\;b\;} & \mathfrak{B} \end{array}$$

where b is the constant functor to b.

The last thing we mention in the last definition, the vertical arrows, have some interest since they are the key to understand how we can relate two different lifts of a morphism $\bar{\phi}$ in $\mathfrak{B}$. To be more precise, the cartesian lift ϕ of $\bar{\phi}$ is unique up to a vertical arrow. Indeed, take any pair of lifts $\phi\colon e_1 \to e_2$ and $\phi\colon e_1' \to e_2$ of a morphism $\bar{\phi}\colon b_1 \to b_2$. Using the fact that ϕ is cartesian there is only one morphism ν such that we have the following commutative diagram:

$$\begin{array}{ccc} & & e_1 \\ & {\scriptstyle \nu}\nearrow & \big\downarrow{\scriptstyle \phi} \\ e_1' & \xrightarrow{\;\phi'\;} & e_2 \end{array} \qquad \begin{array}{ccc} & & b_1 \\ & {\scriptstyle \mathrm{id}_{b_1}}\nearrow & \big\downarrow{\scriptstyle \bar{\phi}} \\ b_1 & \xrightarrow{\;\bar{\phi}\;} & b_2 \end{array} .$$

Since ν satisfies the property that $P(\nu) = \mathrm{id}_{b_1}$, ν is a vertical arrow.

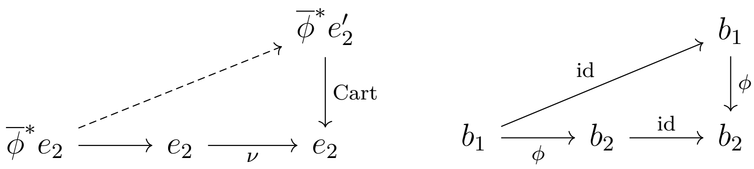
Now we must explore another important fact regarding the fiber of a fibration $P\colon \mathfrak{E} \to \mathfrak{B}$. Suppose we choose in every possible morphism $\bar{\phi}$ in $\mathfrak{B}$ that admits a lift one particular lift that we will denote by:

$$\mathrm{Cart}(\overline{\phi}, e_2)\colon \bar{\phi}^* e_2 \to e_2.$$

This particular choice defines a functor $\bar{\phi}^*\colon \mathfrak{E}_{b_2} \to \mathfrak{E}_{b_1}$, where the image of a vertical arrow $\nu \in \mathfrak{E}_{b_2}$ is given by the unique arrow $\overline{\phi}^* \nu$ making the following diagram commute:

$$\begin{array}{ccc} \overline{\phi}^* e_2 & \xrightarrow{\mathrm{Cart}(\overline{\phi}, e_2)} & e_2 \\ {\scriptstyle \overline{\phi}^* \nu}\big\downarrow & & \big\downarrow{\scriptstyle \nu} \\ \overline{\phi}^* e_2' & \xrightarrow{\mathrm{Cart}(\overline{\phi}, e_2')} & e_2'. \end{array}$$

It corresponds to the diagram



The functoriality of $\overline{\phi}^*$ follows again from unicity. It is called the *pullback functor*.

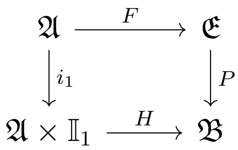
We usually work in contexts where we want to have some way of choosing a particular lift, so we need the following notion:

Definition 2.16. A cleavage for a fibration $P\colon \mathfrak{E} \to \mathfrak{B}$ is a way of choosing for any arrow $\bar{\phi} \in \mathfrak{B}(b_1, b_2)$ and any object $e_2 \in \mathfrak{E}$ such that $P(e_2) = b_2$ a cartesian arrow $\phi \in \mathfrak{E}(e_1, e_2)$ such that $P(\phi) = \bar{\phi}$, that is a particular lift of ϕ.

From now on we will work under the assumption that the categories are sufficiently small and they admit a cleavage. Moreover if we have a morphism $\bar{\phi} \in \mathfrak{B}(b_1, b_2)$ and any object $e_2 \in \mathfrak{E}$ such that $P(e_2) = b_2$ we will denote the lift of that morphism as $\mathrm{Cart}(\phi, e_2)$ and the preimage of b_1 will be $\bar{\phi}^* e_2$.

Taking the notation in account we can proof a interesting result that show us the usefulness of fibrations and how they are related to homotopy theory.

Proposition 2.17. *The functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a fibration if and only if for every category $\mathfrak{A}$ the functor P has the right lifting property, i.e. if we have the following commutative diagram:*



where i_1 is the functor that takes and object a to $(a, 1)$, then there is a unique functor $G\colon \mathfrak{A} \times \mathbb{I}_1 \to \mathfrak{E}$ that makes the whole diagram commutative.

Proof. Suppose that P is a fibration. In order to make everything work properly we must define G such that $G_1 = F$, the problem now is how to define G_0 in order to have a functor. To do that we have to use the definition of fibered category. As we know if we have an arrow in $\mathfrak{B}$ and we have an object in the fiber of the target we can *lift* the arrow into a cartesian morphism. In particular if we have the arrow $H(\mathrm{id}_a, s)\colon H(a, 0) \to H(a, 1)$ where $s\colon 0 \to 1$ in $\mathbb{I}_1$ and the object $F(a) = G(a, 1)$, there is a cartesian morphism

$$\mathrm{Cart}(H(\mathrm{id}_a, s), H(a, 1))\colon H(\mathrm{id}_a, s)^* F(a) \to F(a).$$

If we define G such that $G(a, 0) = H(\mathrm{id}_a, s)^* F(a)$ and $G(\mathrm{id}_a, s) = \mathrm{Cart}(H(\mathrm{id}_a, s), H(a, 1))$ we have a candidate to the functor.

For every morphism $\psi\colon a_1 \to a_2$ in $\mathfrak{A}$ we have the following commutative diagram in $\mathfrak{B}$:

$$\begin{array}{ccc} H(a_1,0) & \xrightarrow{H(\mathrm{id}_{a_1},s)} & H(a_1,1) \\ \Big\downarrow{\scriptstyle H(\psi,\mathrm{id}_0)} & & \Big\downarrow{\scriptstyle H(\psi,\mathrm{id}_1)} \\ H(a_2,0) & \xrightarrow{H(\mathrm{id}_{a_2},s)} & H(a_2,1) \end{array}$$

That induces the following diagram in $\mathfrak{E}$:

$$\begin{array}{ccc} G(a_1,0) & \xrightarrow{G(\mathrm{id}_{a_1},s)} & G(a_1,1) = F(a_1) \\ & & \Big\downarrow{\scriptstyle G(\psi,\mathrm{id}_1)=F(\psi)} \\ G(a_2,0) & \xrightarrow{G(\mathrm{id}_{a_2},s)} & G(a_2,1) = F(a_2) \end{array}$$

But as we know the morphism $G(\mathrm{id}_{a_2}, s)$ is a cartesian morphism so there is an unique morphism $G(\psi, 0)\colon G(a_1, 0) \to G(a_2, 0)$.

For the converse statement, let $\bar{\phi}\colon b_1 \to b_2$ in $\mathfrak{B}$ be an arrow and consider the diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{G} & \mathfrak{E} \\ {\scriptstyle i_1}\Big\downarrow & \overset{\tilde{H}}{\nearrow} & \Big\downarrow{\scriptstyle P} \\ \bullet \times \mathbb{I}_1 & \xrightarrow[H]{} & \mathfrak{B} \end{array}$$

where $G(\bullet) = e_2$ and $H(\mathrm{id}_\bullet \times s) = \bar{\phi}$. Then the map $\phi = \tilde{H}(\mathrm{id}_\bullet \times s)$ is cartesian and verifies that $P\phi = \bar{\phi}$. This ends the proof. □

The problem that we face is that, as we will immediately see, there is no such property if the diagram is defined with $i_0 : \mathfrak{A} \to \mathfrak{A} \times \mathbb{I}_1$ that takes a to $(a, 0)$. This fact means that we can not lift a homotopy because we can only lift some direction in the natural transformations. In order to be able to do that we must expand the notion of fibration using duality as we will see in the next section.

Examples

Now we must explore some applications of the notion of fibrations and some interesting facts. But first we will show a easy fibration in which we can not lift a homotopy so we can have a strong case for the introduction of opfibrations.

Example 2.18. As we have seen the constant functor $P\colon \mathcal{I}_0 \to \mathcal{I}_1$ that goes to 0 is a fibration. Moreover is easy to see that for every category $\mathfrak{A}$ the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{0} & \mathbb{I}_0 \\ \Big\downarrow{\scriptstyle i_0} & & \Big\downarrow{\scriptstyle P} \\ \mathfrak{A} \times \mathbb{I}_1 & \xrightarrow{p_2} & \mathbb{I}_1 \end{array}$$

But there is no functor $G\colon \mathfrak{A} \times \mathbb{I}_1 \to \mathbb{I}_0$ such that:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{0} & \mathbb{I}_0 \\ \downarrow{\scriptstyle i_0} & \overset{G}{\nearrow} & \downarrow{\scriptstyle P} \\ \mathfrak{A} \times \mathbb{I}_1 & \xrightarrow{p_2} & \mathbb{I}_1 \end{array}$$

is a commutative square. First recall that there is only one possible functor $G\colon \mathfrak{A} \times \mathbb{I}_1 \to \mathbb{I}_0$ namely the constant functor 0. But this functor can not commute since we get different results if we take an object $(a, 1)$ in $\mathfrak{A} \times \mathbb{I}_1$:

$$\begin{array}{ccc} & & 0 \\ & \overset{G}{\nearrow} & \downarrow{\scriptstyle P} \\ (a,1) & & 0 \end{array} \qquad (a,1) \xrightarrow{p_2} 1 \quad .$$

Example 2.19. Recall that every group can be viewed from a categorical point of view as a category with only one object in which morphisms are the elements of the group, the identity morphism is the identity and the composition of two morphism is given by the operation. Moreover a functor $f\colon G \to H$ between two groups is nothing more than a group homomorphism since the image of the only object of G is trivially the only object of H, $F(e_G) = e_H$ and $F(gh) = F(g \circ h) = F(g) \circ f(h) = f(g)F(h)$.

Taking this in account we can easily see that a functor $f\colon G \to H$ between two groups is nothing more than a epimorphism in groups, i.e. a surjective homomorphism of groups. Indeed, every arrow is cartesian since in a group all the elements have an inverse and therefore every arrow is an isomorphism. Also since we have only one object in H in order to have a fibration every morphism has to have a lift, hence f must be surjective. Additionally it is easy to see that the fiber of the only element in H is precisely the kernel of the homomorphism since it is constituted by the arrows that go to the identity.

Example 2.20. Let $\mathfrak{C}$ be a category, there is a category $\mathfrak{C}^{\mathcal{I}_1}$ of functors $F\colon \mathcal{I}_1 \to \mathfrak{C}$ and natural transformatins between them. It is easy to see that a functor $f\colon \mathcal{I}_1 \to \mathfrak{C}$ is identifiable with a morphism $f\colon f(0) \to f(1)$ whereas a natural transformation $\alpha\colon f \implies g$ can be viewed as a commutative square:

$$\begin{array}{ccc} f(0) & \xrightarrow{\alpha(0)} & g(0) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ f(1) & \xrightarrow{\alpha(1)} & g(1) \end{array}$$

Moreover there is a functor $\mathrm{Cod}\colon \mathfrak{C}^{\mathcal{I}_1} \to \mathfrak{C}$ that takes every morphism $F\colon X \to Y$ to the codomain and every commutative square to the morphims between the codomains, i.e the morphism $\alpha(1)$ in the previous diagrams. This functors is a fibration if and only if $\mathfrak{C}$ has enough pullbacks. Indeed, suppose that you fix a morphism $f\colon X \to Y$ in $\mathfrak{C}$ and then you choose any object $g\colon Z \to Y$ (i.e an object such that $\mathrm{Cod}(g) = Y$, if we want a morphism that lift f by g then that morphism must be a commutative square α with the following form:

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Z \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

since $\mathrm{Cod}(\alpha) = f$ and $\alpha\colon p_1 \to g$. Moreover α is cartesian if and only if P is the pullback. Take any commutative triangle in $\mathfrak{C}$ of the form:

$$\begin{array}{ccc} & & X \\ & \overset{l}{\nearrow} & \Big\downarrow{\scriptstyle f} \\ B & \xrightarrow[h]{} & Y \end{array} .$$

Now suppose that you have β a commutative square such that $\mathrm{Cod}(\beta) = h$ and that goes to $g\colon Z \to Y$ i.e a commutative square of the form:

$$\begin{array}{ccc} A & \xrightarrow{n} & Z \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle g} \\ B & \xrightarrow[h]{} & Y \end{array} .$$

We must prove that there is only one commutative square γ such that:

$$\begin{array}{ccc} & & p_1\colon P \to X \\ & \overset{\gamma}{\nearrow} & \Big\downarrow{\scriptstyle \alpha} \\ m\colon A \to B & \xrightarrow{\beta} & g\colon Z \to Y \end{array}$$

if and only if P is the pullback. But as we know γ must satisfy that $\alpha \circ \gamma = \beta$ and $\mathrm{Cod}(\gamma) = l$, hence γ must be a commutative square of the following form:

$$\begin{array}{ccc} A & \xrightarrow{s} & P \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle p_1} \\ B & \xrightarrow[l]{} & X \end{array} .$$

This fact implies that γ depends exclusively on the morphism $s\colon A \to P$ but there is only one morphism that makes the following diagram commutative (i.e $\alpha \circ \gamma = \beta$):

$$\begin{array}{ccccc} A & \xrightarrow{s} & P & \xrightarrow{p_2} & Z \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle g} \\ B & \xrightarrow[l]{} & X & \xrightarrow[f]{} & Y \end{array}$$

if and only if P is a pullback.

Example 2.21. Take a category $\mathfrak{C}$ with an object X such that $\mathfrak{B}(Y, X) = \emptyset$ for every object $Y \neq X$ and $\mathfrak{B}(X, X) = \{\mathrm{id}_X\}$. The functor $\mathcal{I}_0 \to \mathfrak{C}$ such that $P(0) = X$ is a fibration.

2.2 Opcartesian morphism and opfibrations

In the categorical setting we usually have some kind of duality using the notion of the dual category $\mathfrak{C}^{op}$, i.e. the category with the same objects as $\mathfrak{C}$ but arrows that goes backwards, that is there is an arrow $f\colon Y \to X$ in $\mathfrak{C}^{op}$ for every arrow $f\colon X \to Y$ in $\mathfrak{C}$. Moreover, we have the notion of a contravariant functor between $\mathfrak{C}$ and $\mathfrak{D}$ as a functor $F\colon \mathfrak{C}^{op} \to \mathfrak{D}$. Using this well known categorical tools we will define opcartesian morphism and opfibrations. But first we must spend some time with some observations.

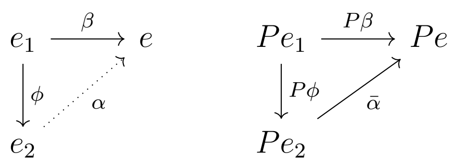
As the reader could have already guessed we have change the usual prefix *co-* to *op-* to denote duality in the case of cartesian arrows and fibrations. This decision was made in order to avoid confusing these notions with the *cofibrations* that appears in topology [21].

Since the propositions, examples and definitions in this section are the dual of the previous one we will spare only some few words dealing with them. We hope the reader will be able to understand everything properly despite this and also save some time reading almost the same.

2.2.1 Opcartesian morphism

Definition 2.22 (Opcartesian morphism). Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a functor between two categories and let $\phi\colon e_1 \to e_2$ be a morphism in $\mathfrak{E}$. The morphism ϕ is P-opcartesian if the morphism ϕ of $\mathfrak{E}^{op}$ is P^{op} cartesian.

Remark 2.23. Explicitly, this means that a morphism is opcartesian if for every morphism $\beta \in \mathfrak{E}(e_1, e)$ and every morphism $\bar{\alpha} \in \mathfrak{B}(Pe_1, Pe)$ such that $P\phi \circ \bar{\alpha} = P\beta$, there exists a unique arrow $\alpha \in \mathfrak{E}(e_2, e_2)$ such that $\alpha \circ \phi = \beta$ and $P\alpha = \bar{\alpha}$. This can be illustrated by the following diagrams:



Again we can characterize this definition in a more *categorical way.*

Proposition 2.24. *Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a functor between two categories. A morphism $\phi\colon e_1 \to e_2$ in $\mathfrak{E}$ is P-opcartesian if and only if the following commutative square is a* *pullback square:*

$$\begin{array}{ccc} \mathfrak{E}(e_2, e) & \xrightarrow{-\circ\phi} & \mathfrak{E}(e_1, e) \\ \downarrow P & & \downarrow P \\ \mathfrak{B}(Pe_2, P_e) & \xrightarrow{-\circ P\phi} & \mathfrak{B}(Pe_1, Pe_2) \end{array}$$

Examples

Example 2.25. Let $U\colon \mathbf{Top} \longrightarrow \mathbf{Set}$ be the forgetfull functor. A continuous map $f\colon (X,\tau) \to (Y,\tau')$ is U-opcartesian if and oly if (Y,τ') has the quotient topology, i.e $V \in \tau'$ if and only if $f^{-1}(V) \in \tau$.

Example 2.26. Let $p\colon \mathfrak{E} \to \bullet$ be a functor between a category $\mathfrak{C}$ and $\bullet$. A morphism $\phi\colon e_1 \to e_2$ is P-opcartesian if and only if ϕ is an isomorphism.

Example 2.27. Every isomorphism is opcartesian for every functor.

2.2.2 Opfibrations

Definition 2.28 (Opfibrations). We say that a functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a opfibration if for any arrow $\bar{\phi} \in \mathfrak{B}(b_1, b_2)$ and any object $e_1 \in \mathfrak{E}$ such that $P(e_1) = b_1$, there exists a cartesian arrow $\phi \in \mathfrak{E}(e_1, e_2)$ such that $P(\phi) = \bar{\phi}$.

Remark 2.29. The arrow ϕ is called a opcartesian lift of $\bar{\phi}$ with codomain e_1. The opcartesian lift is unique up to vertical arrows.

Remark 2.30. Not every opfibration is surjective, not even essentially surjective. For example take the constant functor $P\colon \mathcal{I}_0 \to \mathcal{I}_1$ that takes 0 to 1.

What we always have is the fact that if an object b has a preimage in $\mathfrak{E}$ then every object b' such that there is a morphism $\hat{\phi}\colon b \to b'$ has also a preimage. Indeed, since P is a opfibration and there is object e such that $P(e) = b$ then there is a morphism $phi\colon e \to e'$ such that $P(\phi) = \bar{\phi}$ and therefore $P(e') = b'$.

Definition 2.31. A opcleavage for a opfibration $P\colon \mathfrak{E} \to \mathfrak{B}$ is a way of choosing for any arrow $\bar{\phi} \in \mathfrak{B}(b_1, b_2)$ and any object $e_1 \in \mathfrak{E}$ such that $P(e_1) = b_1$ an opcartesian arrow $\phi \in \mathfrak{E}(e_1, e_2)$ such that $P(\phi) = \bar{\phi}$, that is a particular lift of ϕ.

Remark 2.32. If we have a morphism $\bar{\phi} \in \mathfrak{B}(b_1, b_2)$ and any object $b_2 \in \mathfrak{E}$ such that $P(e_1) = b_1$ we will denote the lift of that morphism as $\mathrm{opCart}(\bar{\phi}, e_1)$ and the preimage of b_2 will be $\bar{\phi}^* e_1$, i.e. $\mathrm{opCart}(\bar{\phi}, e_1)\colon e_1 \to \bar{\phi}^* e_1$.

Remark 2.33. Again, this also defines a functor between the fibers of every to objects b_1 and b_2 in B such that there is an arrow $\bar{\phi}\colon b1 \to b_2$. The difference is that in opfibrations the functor between the opfibrations acts in the same direction as $\bar{\phi}$, i.e. $\bar{\phi}_*\colon \mathfrak{E}_{b_1} \to \mathfrak{E}_{b_2}$. This is called the *pushforward fuctor*.

Whereas fibrations satisfy the right lifting property they do not behave well with the left lifting property, opfibration behave in the opposite way, that is the have the left lifting property but not the right one.

Proposition 2.34. *The functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a opfibration if and only if for every category $\mathfrak{A}$ the functor P has the left lifting property, i.e. if we have the following commutative diagram:*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{F} & \mathfrak{E} \\ \big\downarrow{\scriptstyle i_0} & & \big\downarrow{\scriptstyle P} \\ \mathfrak{A}\times\mathbb{I}_1 & \xrightarrow{H} & \mathfrak{B} \end{array}$$

where i_0 is the functor that takes and object a to $(a, 0)$, then there is a unique functor $G\colon \mathfrak{A} \times \mathbb{I}_1 \to \mathfrak{E}$ that makes the whole diagram commutative.

Examples

First we will see that opfibrations fail to satisfy the right lifting property.

Example 2.35. The constant functor $P\colon \mathcal{I}_0 \to \mathcal{I}_1$ that goes to 1 is a opfibration. Moreover is easy to see that for every category $\mathfrak{A}$ the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{0} & \mathbb{I}_0 \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle P} \\ \mathfrak{A}\times\mathbb{I}_1 & \xrightarrow{p_2} & \mathbb{I}_1 \end{array}$$

But there is no functor $G\colon \mathfrak{A} \times \mathbb{I}_1 \to \mathbb{I}_0$ such that:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{0} & \mathbb{I}_0 \\ \big\downarrow{\scriptstyle i_1} & \nearrow{\scriptstyle G} & \big\downarrow{\scriptstyle P} \\ \mathfrak{A}\times\mathbb{I}_1 & \xrightarrow{p_2} & \mathbb{I}_1 \end{array}$$

is a commutative square since there is only the constant functor and

$$\begin{array}{ccc} & & 0 \\ & \nearrow{\scriptstyle G} & \big\downarrow{\scriptstyle P} \\ (a,0) & & 1 \end{array} \qquad (a,0) \xrightarrow{p_2} 0 \quad .$$

Example 2.36. Let $f\colon G \to H$ be a functor between two groups regarded as categories. The functor f is a fibration if and only if f is surjective.

Example 2.37. Let $\mathrm{Dom}\colon \mathfrak{C}^{\mathcal{I}_1} \to \mathfrak{C}$ be the functor that takes every morphism $F\colon X \to Y$ to the domain (i.e X) and every commutative square to the morphims between the domains. This functors is a opfibration if and only if $\mathfrak{C}$ has enough push-outs.

Example 2.38. Take a category $\mathfrak{C}$ with an object X such that $\mathfrak{B}(X, Y) = \emptyset$ for every object $Y \neq X$ and $\mathfrak{B}(X, X) = \{\mathrm{id}_X\}$. The functor $\mathcal{I}_0 \to \mathfrak{C}$ such that $P(0) = X$ is a opfibration.

2.3 Bifibrations

In previous sections we have study both fibrations and opfibrations, now we must combine both notions.

Definition 2.39 (Bifibration). We say that a functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is bifibration if it is both a fibration and a opfibration.

With this definition we finally arrive at something that really works like fibrations in topology. But before we prove this properties we will state the following fact.

Proposition 2.40. *Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a bifibration with $\mathfrak{B}$ a connected category. Then P is surjective in the objects and in the maps.*

Proof. This follows from the remarks that we made after the definitions of fibrations and opfibrations. □

Now we will see some examples that will allow us to understand more deeply this notion.

2.3.1 Examples

First we will use the previous examples.

Example 2.41. Take the constant functors $0\colon \mathcal{I}_0 \to \mathcal{I}_1$ and $1\colon \mathcal{I}_0 \to \mathcal{I}_1$. The first is a fibration but not a opfibration whereas the second one is an opfibration but not a fibration.

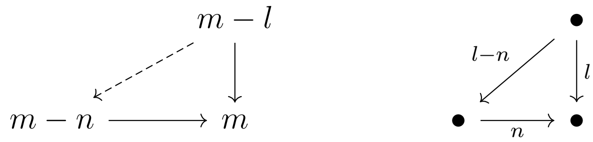
Example 2.42. A functor between groups is a bifibration if and only if is surjective.

Example 2.43 (Natural numbers). Let $\mathfrak{E}$ be the category with objects the integers $\mathbb{Z}$ and a unique morphism between two object $n \to m$ if and only if $n \leq m$.

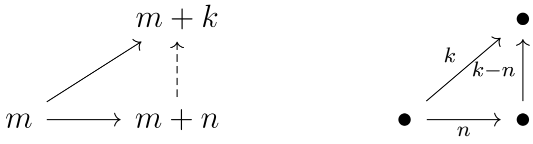
$$... \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow ...$$

And let $\mathfrak{B}$ be the monoid of natural numbers as a category.

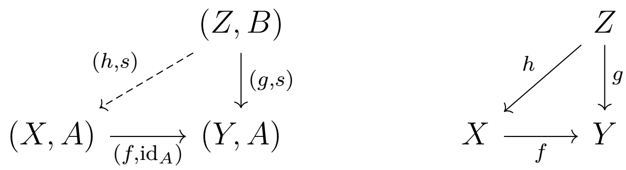
It is easy to show that the functor $P(n) = \bullet$ and $P(n \to m) = m - n$ is a fibration. Let consider $n\colon \bullet \to \bullet$ a morphism in $\mathfrak{B}$, m any object in $\mathfrak{E}$ and morphism $m - l \to m$ such that there is a commutative triangle in $\mathfrak{B}$ associate to it we have that



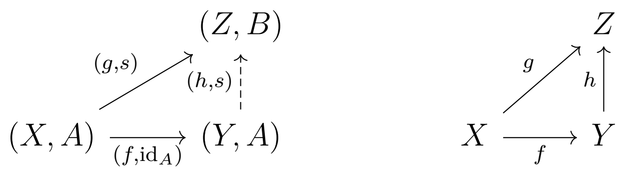
Furthermore, P is a opfibration by a similar argument using the following diagrams:



Example 2.44. Take a base category $\mathfrak{B}$ and another category $\mathfrak{C}$ then $P_1 \colon \mathfrak{B} \times \mathfrak{C} \to \mathfrak{C}$ is a bifibration where P_1 is the first projection. In order to prove it we only must take in account that:



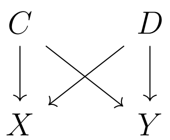
and



Example 2.45 (The 1-sphere). Let $\mathfrak{E}$ be the category with objects the natural numbers and ziz-zag morphism:

$$\ldots \longleftarrow -2 \longrightarrow -1 \longleftarrow 0 \longrightarrow 1 \longleftarrow 2 \longrightarrow \ldots$$

And let $\mathfrak{B}$ be the following category



It is easy to show that the functor

$$P(n) = \begin{cases} C & if \quad N = 0 \,\mathrm{mod} 4 \\ Y & if \quad N = 1 \,\mathrm{mod} 4 \\ D & if \quad N = 2 \,\mathrm{mod} 4 \\ X & if \quad N = 3 \,\mathrm{mod} 4 \end{cases}$$

is a bifibration.

There are no non trivial commutative triangles in $\mathfrak{B}$ so in order to prove that $P \colon \mathfrak{E} \to \mathfrak{B}$ is a bifibration is enough to show that for every every morphism $f \colon A \to B$ in $\mathfrak{B}$ and every objects n and m such that $P(n) = A$ and $P(m) = B$ there is a morphism that goes to f. Since $n \to n \pm 1$ and $n \pm 1 \to n$ satisfy the previous conditions we can conclude that $P \colon \mathfrak{E} \to \mathfrak{B}$ is a bifibration.

Example 2.46. Let $\mathfrak{E}$ be the groupoid:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

and $\mathfrak{B}$ be $\mathbb{Z}_2$ considered as a category:

$$\overset{h}{\circlearrowleft} \atop \bullet \qquad .$$

The functor $P\colon \mathfrak{E} \to \mathfrak{B}$ defined as $F(X) = \bullet = F(Y)$ and $F(f) = h = F(g)$ is a bifibration.

2.3.2 Properties of bifibrations

As we said the previous definition really generalizes the notion of topological fibration to a different realm. Now we must show that it really behaves as expected, i.e. similarly to fibrations in topology. In order to do that we begin showing how a bifibration has the lifting property.

Taking in account the propositions 2.17 and 2.34 we can prove directly the lifting property.

Theorem 2.47 (Homotopy lifting property). *Let $H\colon \mathfrak{A} \times \mathcal{I}_m \to \mathfrak{B}$ be a homotopy, $P\colon \mathfrak{E} \to \mathfrak{B}$ be a bifibration and $F\colon \mathfrak{A} \to \mathfrak{E}$ a functor such that $H_0 = P \circ F$. There is a unique homotopy $G\colon \mathfrak{A} \times \mathbb{I}_m \to \mathfrak{E}$ that makes the following diagram commutative:*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{F} & \mathfrak{E} \\ \downarrow{\scriptstyle i_0} & \overset{G}{\nearrow} & \downarrow{\scriptstyle P} \\ \mathfrak{A} \times \mathcal{I}_m & \xrightarrow{H} & \mathfrak{B} \end{array}$$

where i_0 is the functor that takes and object e to $(e, 0)$

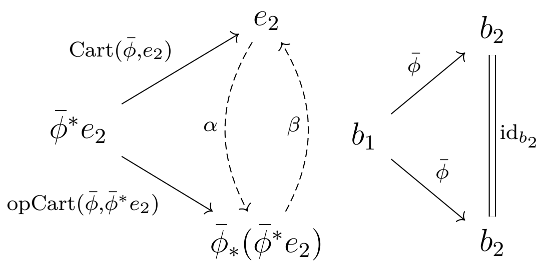
As we know by the Example 2.44 bifibrations are generalizations of the product and as in the topological settings they have a *horizontal* and *vertical* part. The horizontal one is like the base category and the vertical one is the one we called *the fiber* that we define in Definition 2.13.

As we have defined the fiber depends on the base object that we choose, but in reality that it is not the case as we will immediately see, at least if the base category is connected.

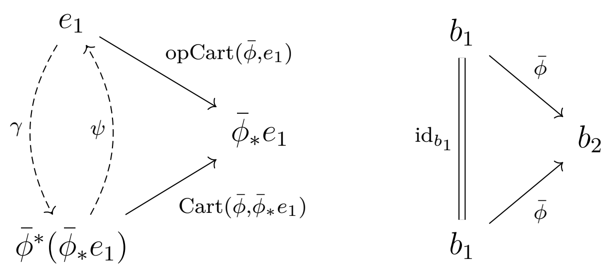
Theorem 2.48 (Equivalence of fibers). *Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a bifibration. If b_1 and b_2 are two objects in $\mathfrak{E}$ and there is a morphism $\bar{p}hi\colon e_1 \to e_2$ then there is a categorical equivalence between $\mathfrak{E}_{e_1}$ and $\mathfrak{E}_{e_2}$.*

Proof. In order to prove the equivalence we must define two functors $F\colon \mathfrak{E}_{b_1} \to \mathfrak{E}_{b_2}$ and $G\colon \mathfrak{E}_{b_2} \to \mathfrak{E}_{b_1}$ such that $G \circ F \cong \mathrm{id}_{\mathfrak{E}_{b_1}}$ and $F \circ G \cong \mathrm{id}_{\mathfrak{E}_{b_2}}$. We simply define F and G using the pushforward functor and the pullback functor. More precisely we define F as $\bar{\phi}_*\colon \mathfrak{E}_{b_1} \to \mathfrak{E}_{b_2}$ that takes any object e_1 in $\mathfrak{E}_{b_1}$ to $\bar{\phi}_* e_1$ and we take G to be the functor $\bar{\phi}^*\colon \mathfrak{E}_{b_2} \to \mathfrak{E}_{b_1}$ that takes any object e_2 in $\mathfrak{E}_{b_2}$ to $\bar{\phi}^* e_2$.

Now in order to show that F and G induces a categorical equivalence is enough to show that for every object e_2 in $\mathfrak{E}_{b_2}$ we have the following two diagrams in $\mathfrak{E}$ and $\mathfrak{B}$:



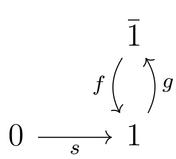
where α and β exist because $\bar{f}$ and $\hat{f}$ are a cartesian and opcartesian morphism (respectively) and their compositions are the identities. Thus, we have a natural isomorphism between $\mathrm{id}_{\mathfrak{E}_{e_2}}$ and $F \circ G$. Analogously we have the other natural isomorphism using the followings diagrams:



□

Corollary 2.49. *Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a bifibration. If b_1 and b_2 are two objects in $\mathfrak{B}$ and there is a path $F\colon \mathbb{I}_m \to \mathfrak{B}$ between them, then $\mathfrak{E}_{b_1}$ and $\mathfrak{E}_{b_2}$ are homotopy equivalent.*

Example 2.50. Let $\mathfrak{E}$ be the category generated by the following diagram:



where $f \circ g = \mathrm{id}_1$ and $g \circ f = \mathrm{id}_{\bar{1}}$. Let $\mathfrak{B} = \mathbb{I}_1$ the category with only one arrow $0 \xrightarrow{s} 1$.

Take the functor $P\colon \mathfrak{E} \to \mathfrak{B}$ defined as $P(0) = 0$, $P(1) = P(\bar{1}) = 1$, $P(s) = s$ and $P(f) = P(g) = \mathrm{id}_1$. It is a bifibration because the arrows id_0, id_1, $\mathrm{id}_{\bar{1}}$, s and $g \circ s$ are cartesian and opcartesian. The fiber over 0 is the discrete category with one object 0 and the fiber over 1 is the following category:

$$\begin{array}{c} \bar{1} \\ f \downarrow \uparrow g \\ 1 \end{array} .$$

It is trivial to prove that $\mathfrak{E}_0$ and $\mathfrak{E}_1$ are different but this does not contradict our result since $\mathfrak{E}_1$ is contractible by the natural transformation α between the identity and the constant functor $\bar{1}$ given by $\alpha(1) = g$, $\alpha(\bar{1}) = \mathrm{id}_{\bar{1}}$. In fact, we have

$$\begin{array}{ccc} 1 & \xrightarrow{\alpha(1)=g} & \bar{1} \\ \downarrow g & & \downarrow \mathrm{id}_{\bar{1}} \\ \bar{1} & \xrightarrow{\alpha(\bar{1})=\mathrm{id}_{\bar{1}}} & \bar{1} \end{array} \qquad \begin{array}{ccc} \bar{1} & \xrightarrow{\alpha(\bar{1})=\mathrm{id}_{\bar{1}}} & \bar{1} \\ \downarrow f & & \downarrow \mathrm{id}_{\bar{1}} \\ 1 & \xrightarrow{\alpha(1)=g} & \bar{1} \end{array}$$

Chapter 3

Homotopic distance

The notion of homotopic distance was first introduced in [18] and then expanded in [17] as a tool to measure how far two continuous map are from being homotopic. Moreover, the homotopic distance is a generalization of the LS-category and the topological complexity of a topological space. Both notions are related to important results and applications such as Morse-Theory [6], robotics [9] and much more.

As we know in Category Theory we have analogous notions to the ones that appears in Topology so homotopic distance was extended to the categorical setting where it also measures how far are two functors from being homotopic [19]. Similarly, the homotopic distance generalizes both the LS-Category of a category, which was studied in [25], and the categorical complexity, which was introduced in [26].

In this chapter we will follow the previous steps. So we will first introduce the notion of homotopic distance between two functors. Next we will define the LS-Category and the categorical complexity to then prove that they are particulars cases of homotopic distance. Once we have shown that both notions are particular cases we will study some interesting properties such as compositions inequalities and the invariance of the distance by homotopic equivalence that give us interesting inequalities.

Finally we will expand the notion of homotopic distance in two different ways taking as guide two results in the topological setting plus the Varadarajan inequality that Tanaka prove for the LS-category for small categories ([4], [18] and [25]). First we will apply the homotopic distance in the context of bifibrations in order two show a inequality of homotopic distance between two bifibrations and how this inequality allow us to obtain two known inequalities. Next we will conclude defining a generalization of homotopic distance (higher homotopic distance) to then finish this work proving that it generalizes the notion of higher categorical complexity.

3.1 Homotopic distance

The idea behind homotopic distance between two functors $F, G\colon \mathfrak{C} \to \mathfrak{D}$ is to try to obtain a cover of a $\mathfrak{C}$ such that the functors F and G are homotopic when we restrict them to the members of such cover, that is if U is the cover then $F|_U \cong G|_U$. The first problem in order to formalise the idea is what properties must the element of the cover satisfied. In topology, as usual, we want the cover to be an open cover, but in Categories there are no such thing. They are various attempts to use different notions to solve this problem. In this work we will follow [25], so we will use geometric covers, i.e. covers such that they are *compatible* with the nerve and the classifying space. More precisely, we define them as:

Definition 3.1 (**Geometric cover**)**.** Let $\mathfrak{C}$ be a small category, a set$\{U_i\}_{i\in I}$ of subcategories of $\mathfrak{C}$ is a geometric cover of $\mathfrak{C}$ if for every $\mathcal{C}$ in $\mathfrak{C}$ there is $i \in I$ such that $\mathcal{C}$ lies in U_i in the sense that every object and morphism in the image category lies in U_i.

Remark 3.2. Recall that the image of a category is not always a subcategory. For example,if we take the two following categories:

$$A \xrightarrow{f} B$$

$$C \xrightarrow{g} D \qquad\qquad X \xrightarrow{m} Y \xrightarrow{n} Z$$

and take the functor F that takes f to m and g to n, then the image of the functor is not a subcategory since $n \circ m$ is not in it but both m and n belong to it.

Example 3.3. Take $\mathfrak{C}$ the category generated by the following diagram:

$$\begin{array}{ccc} & & A \\ & \swarrow & \downarrow \\ Z \longleftarrow & X & Y \end{array}$$

Now take the two following covers:

$$\begin{array}{ccccc} & & & & \\ X & & & & A \\ \downarrow & & & \swarrow & \downarrow \\ Z & & X & & Y \end{array} \qquad\qquad \begin{array}{ccc} A & & \\ \downarrow & & \\ X & & A \\ \downarrow & & \downarrow \\ Z & & Y \end{array}$$

The left one is a geometric but the right cover is not since the chain $A \to X \to Z$ does not belong to any subcategory of the cover.

With the definition of geometric cover now we can finally define the homotopic distance between two functors.

Definition 3.4 (**Homotopic distance between functors**)**.** Let $\mathfrak{C}$ and $\mathfrak{D}$ be two small categories and let $F, G\colon \mathfrak{C} \to \mathfrak{D}$ be two functors. The homotopic distance $\mathrm{cD}(F,G)$ between F and G is the least positive integer $n \in \mathbb{N}$ such that there is geometric cover $\{U_0, ..., U_n\}$ such that $F|_{U_i} \cong G|_{U_i}$ for every $0 \leq i \leq n$. If there is no such cover we define $\mathrm{cD}(F,G)$ as ∞.

Definition 3.5. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two small categories and let $F, G\colon \mathfrak{C} \to \mathfrak{D}$ be two functors such that $\mathrm{cD}(F, G) = n$ and $\{U_0, ..., U_n\}$ is a geometric cover such that $F|_{U_i} \cong G|_{U_i}$ for every $0 \leq i \leq n$. We say that $\{U_0, ..., U_n\}$ is a geometric cover by homotopy domains between F and G.

Remark 3.6. If the context is clear we will only say that $\{U_0, ..., U_n\}$ is a geometric cover by homotopy domains.

Definition 3.7 (**Weak categorical homotopic distance between functors**)**.** Let $\mathfrak{C}$ and $\mathfrak{D}$ be two small categories and let $F, G\colon \mathfrak{C} \to \mathfrak{D}$ be two functors. The weak categorical homotopic distance $\mathrm{wcD}(F, G)$ between F and G is the least positive integer $n \in \mathbb{N}$ such that there is geometric cover $\{U_0, ..., U_n\}$ such that $\mathrm{B}F|_{U_i} \cong \mathrm{B}G|_{U_i}$ for every $0 \leq i \leq n$. If there is no such cover we define $w\mathrm{cD}(F, G)$ as ∞.

Proposition 3.8. *Let $F, G\colon \mathfrak{C} \to \mathfrak{D}$ be two functors:*

1. $\mathrm{cD}(F, G) = \mathrm{cD}(G, F)$.

2. $\mathrm{cD}(G, F) = 0$ *if and only if* $F \cong G$.

3. *If* $F \cong \hat{F}$ *and* $G \cong \hat{G}$ *then* $\mathrm{cD}(F, G) = \mathrm{cD}(\hat{F}, \hat{G})$.

4. *For every geometric cover $\{U_0, ..., U_n\}$ of $\mathfrak{C}$ we have that*

$$\mathrm{cD}(F, G) \leq \sum_{i=0}^{n} \mathrm{cD}(F|_{U_i}, G|_{U_i}) + n$$

Proof. 1. If $\{U_0, ..., U_n\}$ is a geometric cover of $\mathfrak{C}$ such that for all $0 \leq i \leq n$ we have $F|_{U_i} \cong G|_{U_i}$ then it follows from the fact that being homotopic is an equivalence relation that $G|_{U_i} \cong F|_{U_i}$. So $\mathrm{cD}(G, F) \leq cD(F, G)$ and by a similar argument $\mathrm{cD}(F, G) \leq cD(G, F)$.

2. If $\mathrm{cD}(F, G) = 0$ the geometric cover $\{U\}$ such that $F|_U = G_U$ have only one subcategory of $\mathfrak{C}$. So we only have to prove that U must equal $\mathfrak{C}$. But this fact is trivial because every object is a 0-chain and every morphism is a 1-chain..

3. We only have to prove that if $F \cong \hat{F}$ then every subcategory U of $\mathfrak{C}$ satisfies that $F|_U \cong \hat{F}|_U$. But this fat follows from the existence of a homotopy $H_U\colon U \times \mathbb{I} \to \mathfrak{D}$ that we define from the homotopy $H\colon \mathfrak{C} \times \mathbb{I} \to \mathfrak{D}$ betwen F and $\hat{F}$ by restricting it using $H_U(X, i) = H(X, i)$ and $H_U(f, g) = H(f, g)$.

   Now we can prove the total result. If $U = \{U_0, ..., U_n\}$ is a geometric cover by homotopy domains then U is also a geometric cover by homotopy domains between $\hat{F}$ and $\hat{G}$. As we know $F|_{U_i} \cong \hat{F}|_{U_i}$, $G|_{U_i} \cong \hat{G}|_{U_i}$ and $F|_{U_i} \cong G|_{U_i}$, so it follows that $\hat{F}|_{U_i} \cong \hat{G}|_{U_i}$.

4. Let $\{U_0, ..., U_n\}$ be a geometric cover of $\mathfrak{C}$ and let $\mathrm{cD}(F_{U_i}, G_{U_i}) = m_i$ for every $0 \leq i \leq n$. For every $0 \leq i \leq n$ we have $\{U_i^0, ..., U_i^{m_i}\}$, a geometric cover by homtopy domains between $F|_{U_i}$ and $F|_{U_i}$. It is easy to prove that

$$\{U_0^0, ..., U_0^{m_1}, ..., U_i^0, ..., U_i^{m_i}, ..., U_n^0, ..., U_n^{m_n}\}$$

is a geometric cover by $\sum_{i=0}^{n} m_i + n$ subcategoriess such that every U_i^j satisfies $F|_{U_i^j} \cong G|_{U_i^j}$.

□

Proposition 3.9. *Let $F, G \colon \mathfrak{C} \to \mathfrak{D}$ be two funtors between small categories. Then*

$$\mathrm{wcd}(F, G) \leq \mathrm{cD}(F, G)$$

.

Proof. It follows from Proposition 1.63. □

3.2 LS-Category and Categorical Complexity

In this section we will show how LS-category and categorical complexity are both particular cases of homotopic distance. This will show how homotopic distance is an useful tool that can show us how these two notions are related.

3.2.1 LS-Category

The LS-category is a well studied topological invariant with numerous applications and relation with interesting topics in various domains [6]. The categorical counterpart was well study in [25]. Both notions define how far is a topological space and a category, respectively, from being contractible.

Definition 3.10. A subcategory $\mathfrak{U}$ of a small category $\mathfrak{C}$ is called 0-categorical if the inclusion functor $i \colon \mathfrak{U} \to \mathfrak{C}$ is homotopic to a constant functor.

Definition 3.11 (Geometric cover)**.** A geometric cover $\{U_0, ..., U_m\}$ of a small category $\mathfrak{C}$ is called categorical if ever U_i is 0-categorical.

Definition 3.12 (LS-Category). Let $\mathfrak{C}$ be a small category. We defined the normalized Lusternik-Schnirelmann category of $\mathfrak{C}$ and we will call $\mathrm{ccat}(\mathfrak{C})$ to the least $n \in \mathbb{N}$ such that there is a categorical cover of $\mathfrak{C}$. If there is no such cover we defined $\mathrm{ccat}(\mathfrak{C})$ as ∞.

Proposition 3.13. *For every small and connected category $\mathfrak{C}$ we have that*

$$\mathrm{ccat}(\mathfrak{C}) = \mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, \bullet)$$

where $\bullet$ is a constant functor.

Proof. Let $\{U_0, ..., U_n\}$ be a geometric cover satisfying that $n+1 = \mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, \bullet)$ and $i_{U_i} \cong \bullet$ for every $i \in \{0, ..., n\}$. It is evident that $\mathrm{id}|_{U_i} = i_{U_i} \cong \bullet$ so $\{U_0, ..., U_n\}$ is a homotopy domain for $\mathrm{id}_{\mathfrak{C}}$ and $\bullet$. So we have proved that $\mathrm{ccat}(\mathfrak{C}) \geq \mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, \bullet)$. Alternatively if $\{U_0, ..., U_n\}$ is a homotopy domain for $\mathrm{id}_{\mathfrak{C}}$ and $\bullet$ it follows that $id|_{U_i} = i_{U_i} \cong \bullet$ and $\mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, \bullet) \geq \mathrm{ccat}(\mathfrak{C})$. □

Corollary 3.14. *If $\mathfrak{C}$ is a small category the following claims are equivalent:*

1. *$\mathfrak{C}$ is contractible.*
2. $\mathrm{ccat}(\mathfrak{C}) = 0$.

Definition 3.15 (LS-Category of a functor). Let $\mathfrak{C}$ and $\mathfrak{D}$ be two small categories and connected categories and let $F: \mathfrak{C} \to \mathfrak{D}$ be a functor between them. We defined the normalized Lusternik-Schnirelmann category of F as $\mathrm{ccat}(F) = \mathrm{cD}(F, Y)$ where $Y: \mathfrak{C} \to \mathfrak{D}$ is a constant functor.

Remark 3.16. Let $\mathfrak{C}$ be a category and C an object in $\mathfrak{C}$. There are two functors $i_1, i_2: \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ defined as follows:

- For every object X in $\mathfrak{C}$ we have that $i_1(X) = (X, C)$ e $i_2(X) = i_2(C, X)$.
- Let f be a morphism in $\mathfrak{C}$,then $i_1(f) = (f, \mathrm{id}_C)$ e $i_2(f) = (\mathrm{id}_C, f)$.

Now we will characterize the LS-category as a particular case of homotopic distance.

Proposition 3.17 (Characterization of LS-Category). *Let $\mathfrak{C}$ be a small category and C an object in $\mathfrak{C}$. We claim that:*

$$\mathrm{ccat}(\mathfrak{C}) = \mathrm{cD}(i_1, i_2).$$

Proof.
- $\leq$

  Every cover by homotopy domain $\{U_0, ..., U_n\}$ of i_1 and $i_2|_{U_i}$ is a categorical cover. Indeed, let U_i be a homotopy domain in the cover. We know that there is a homotopy $H: U_i \times \mathbb{I}_m \to \mathfrak{C} \times \mathfrak{C}$ such that it makes $i_1|_{U_i} \cong i_2|_{U_i}$. The homotopy $p_1 \circ H: U_i \times \mathbb{I}_m \to \mathfrak{C}$ where p_1 is the first projection makes U_i 0-categorical. In order to prove it it is enough to see that $p_1 \circ i_1|_{U_i} = id|_{U_i}$ and $p_1 \circ i_2|_{U_i} = C$ where C is the constant functor associated to the object C.

- $\geq$

  Let $\{U_0, ..., U_n\}$ be a categorical cover of $\mathfrak{C}$. We must prove that it is also a cover by homotopy domains. Fix a number i in $\{0, ..., n\}$ and show that $i_1|_{U_i} \cong i_2|_{U_i}$. First, we know that there is a homotopy $H: U_i \times \mathbb{I}_m \to \mathfrak{C}$ between i_{U_i} and C a constant functor. Now we can take the following homotopy $K: U_i \times \mathbb{I}_{2m} \to \mathfrak{C} \times \mathfrak{C}$ between i_1 and i_2 defined as follows:

$$K(X, j) = \begin{cases} (H(X, j), C) & if \quad 0 \leq j \leq m \\ (C, H(X, 2m - j) & if \quad m \leq j \leq 2m \end{cases}$$

I order to prove that K is a homotopy between $i_1|_{U_i}$ and $i_2|_{U_i}$ we only have to see that:

$$K(X,0) = (H(X,0),C) = (X,C) = i_1(X),$$
$$K(X,m) = (H(X,m),C) = (C,C) = (C,H(X,j)$$

and

$$K(X,2m) = (C,H(X,2m-2m) = (C,H(X,0)) = (C,X) = i_2(X).$$

□

3.2.2 Categorical complexity

Now it is the turn of categorical complexity. This notions was first introduced by Farber and has a incredible amount of applications in various field [8]. In the categorical setting the most obvious interpretation as a way of stabilising a continuous way of assigning to every two points a path between them it is not so apparent. The correct generalization was also studied by Tanaka in [26]. In order to understand it properly we must define some previous notions.

Definition 3.18 (**Diagonal functor**). Let $\mathfrak{C}$ be a small category, we define the diagonal functor $\Delta\colon \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ as the only functor that takes every object X to $\Delta(X) = (X,X)$ and that takes every morphism f to $\Delta(f) = (f,f)$.

Definition 3.19 (**Farber Subcategory**). We say that a subcategory $\mathfrak{U}$ of a small category $\mathfrak{C} \times \mathfrak{C}$ is Farber if there is a functor $F\colon U \to \mathfrak{C}$ such that $\Delta \circ F \cong i_U$.

Definition 3.20 (**Categorical Complexity**). Let $\mathfrak{C}$ be a small category, The normalized categorical complexity of $\mathfrak{C}$ cTC($\mathfrak{C}$ is the least natural number $n \in \mathbb{N}$ such that there is a geometric cover $\{U_0, ..., U_n\}$ f $\mathfrak{C} \times \mathfrak{C}$ by Farber subcategories. If there is no such that cover we define cTC($\mathfrak{C}$) as ∞.

Proposition 3.21 (**Characterization of categorical complexity**). *Let $\mathfrak{C}$ be a small category category we have that*

$$\mathrm{cTC}(\mathfrak{C}) = \mathrm{cD}(p_1, p_2).$$

Proof.

$\leq$

Let $\{U_0, ..., U_n\}$ be a geometric cover of $\mathfrak{C} \times \mathfrak{C}$ by homotopy domains such that $p_1|_{U_i} \cong p_2|_{U_i}$, we must prove that they are also Farber subcategories. Indeed, for every homotopy domain U_i we have that U_i is a Farber subcategory using as the functor F the first projection p_1. Take $H\colon U_i \times \mathbb{I}_m \to \mathfrak{C}$ the homotopy between $p_1|_{U_i}$ and $p_2|_{U_i}$.The functor $G\colon U_i \times \mathbb{I} \to \mathfrak{C} \times \mathfrak{C}$ defines as $G(X,Y,j) = (X, H(X,Y,j))$ is a homotopy between $\Delta \circ p_1$ and i_{U_i} since

$$G(X,Y,0) = (X, H(X,Y,0)) = (X, p_1(X,Y)) = (X,X)$$

and

$$G(X,Y,m) = (X, H(X,Y,m)) = (X, p_2(X,Y)) = (X,Y).$$

$\geq$

Let $\{U_0, ..., U_n\}$ be a geometric cover of $\mathfrak{C} \times \mathfrak{C}$ by Farber subcategories. Fix some index i in $\{0, ..., n\}$ and see that $p_1|_{U_i} \cong p_2|_{U_i}$. Since U_i is a Farber subcategory there is a functor $F\colon U_i \to \mathfrak{C}$ such that $\Delta \circ F \cong i_{U_i}$ and there is a functor $H\colon U_i \times \mathbb{I}_m \to \mathfrak{C} \times \mathfrak{C}$ such that H verifies that $H_m = \Delta \circ F$ and $H_n = i_{U_i}$. Now we take the functor $K\colon U_i \times \mathbb{I}_{2m} \to \mathfrak{C}$

$$K(X,Y,j) = \begin{cases} p_1 \circ H(X,Y,m-j) & if \quad 0 \leq j \leq m \\ \\ p_2 \circ H(X,Y,j-m) & if \quad m \leq j \leq 2m \end{cases}$$

The functor K is well-defined since:

$$K(X,Y,m) = p_1 \circ H(X,Y,0) = p_1 \circ H((X,Y),0) = \\ p_1 \circ \Delta \circ F(X,Y) = p_1(F(X,Y), F(X,Y) = F(X,Y).$$

and

$$K((X,Y),m) = p_2 \circ H((X,Y),m-m) = p_2 \circ H((X,Y),0) = \\ p_2 \circ \Delta \circ F(X,Y) = p_2(F(X,Y), F(X,Y)) = F(X,Y).$$

Moreover K is a homotopy between p_1 and p_2 since:

$$K(X,Y),0) = p_1 \circ H((X,Y),m) = p_1 \circ H((X,Y),m) \\ = p_1 \circ i_{U_i}(X,Y) = p_1(X,Y),$$

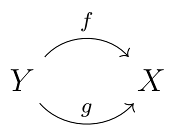
and

$$K(X,Y),2n-m) = p_2 \circ H((X,Y),2m-m) = p_2 \circ H((X,Y),m) \\ = p_2 \circ i_{U_i}(X,Y) = p_2(X,Y).$$

□

3.2.3 Examples

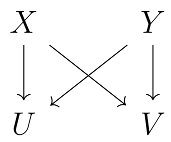
Example 3.22. The category $\mathfrak{C}$ generated by the following diagram



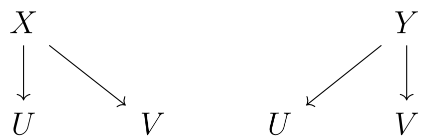
satisfies that $\text{ccat}(\mathfrak{C}) = 1$. First $\text{ccat}(\mathfrak{C}) \neq 0$ since $\mathfrak{C}$ is not contractible. Moreover, $\text{ccat}(\mathfrak{C}) = 1$ since there are two 0-categorical subcategory generated by the following diagrams:

$$X \xrightarrow{f} Y \qquad X \xrightarrow{g} Y \ .$$

Example 3.23. Let $\mathfrak{C}$ be the following Poset



We claim that $\mathrm{ccat}(\mathfrak{C}) = 1$. We can use the proposition 1.88 to show that $\mathfrak{C}$ is not contractible since it has no beat-points. Moreover we can easily show that $\mathrm{ccat}(\mathfrak{C}) = 1$ since they are two 0-categorical subcategories, namely:



They are both 0-categorical since they both have an initial object.

3.3 Homotopic distance and composition

The aim of this chapter is to explore some relations between homotopic distance and compositions. As we will see this relations will be a very useful tool in order to give us some interesting inequalities.

Proposition 3.24 (Inequality of left composition). *Let $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{B}$ be three small categories and let $F, G \colon \mathfrak{C} \to \mathfrak{D}$ and $H \colon \mathfrak{D} \to \mathfrak{B}$ be three functors. We have that:*

$$\mathrm{cD}(H \circ F, H \circ G) \leq \mathrm{cD}(F, G).$$

Proof. Let $\{U_0, ..., U_n\}$ be a geometric cover of $\mathfrak{C}$ by homotopy domains between F and G such that $cD(F, G) = n + 1$. It is easy to show that $\{U_0, ..., U_n\}$ are homotopy domains of $H \circ F$ and $H \circ G$. using the proposition 1.66 since:

$$(H \circ F)|_{U_i} = H \circ (F|_{U_i}) \cong H \circ (G|_{U_i}) = (H \circ F)|_{U_i}$$

. □

Corollary 3.25. *Let $\mathfrak{C}$ be a small and connected category and let $\mathfrak{D}$ be a small category with $F \colon \mathfrak{C} \to \mathfrak{D}$ between them. We claim that:*

$$\mathrm{ccat}(F) \leq \mathrm{ccat}(\mathfrak{C}).$$

Proof. $\mathrm{ccat}(F) = \mathrm{cD}(F, \bullet)$ where $\bullet$ is a constant functor and $\mathrm{ccat}(\mathfrak{C}) = \mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, X)$ and X is another constant functor. Since $F \circ \mathrm{id}_{\mathfrak{C}} = F$ and $F \circ X$ is a constant functor we have by the previous proposition that:

$$\mathrm{cD}(F, F(X)) \leq \mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, X).$$

□

Proposition 3.26 (Inequality of right composition). *Let $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{B}$ be three small categories and let $F, G \colon \mathfrak{C} \to \mathfrak{D}$ and $H \colon \mathfrak{B} \to \mathfrak{C}$ be three functors. We have that:*

$$\mathrm{cD}(H \circ F, H \circ G) \leq \mathrm{cD}(F, G).$$

Proof. Let $\{U_0, ..., U_n\}$ be a geometric cover of $\mathfrak{C}$ by homotopy domains between F and G such that $cD(F, G) = n + 1$. We can take the geometric cover of $\mathfrak{B}$ induced by H $\{V_0, ..., V_n\}$ where V_i is equal to $H^{-1}(U_i)$ the category whose

- objects are objects X in $\mathfrak{B}$ such that $H(X)$ is U_i.
- morphism are morphism f in $\mathfrak{B}$ such that $H(f)$ is in U_i.

For every chain $\mathcal{C}$ in $\mathfrak{B}$ we have that $H(\mathcal{C})$ is a chain in $\mathfrak{C}$ hence $H(\mathcal{C})$ lies inside a subcategory U_i and therefore $\mathcal{C}$ lies inside V_i. Furthermore, the restriction of H to V_i (i.e. the functor $H|_{V_i} \colon V_i \to \mathfrak{C}$) is equal to the composition $i_{U_i} \circ \bar{H}_i$ where $\bar{H}_i \colon V_i \to U_i$ is defined as $\bar{H}_i(X) = H|_{V_i}(X)$ y $\bar{H}_i(f) = H|_{V_i}(f)$ and $i_{U_i} \colon U_i \to \mathfrak{C}$ is the inclusion of U_i in $\mathfrak{C}$. Now we have that:

$$(F \circ H)|_{V_i} = F|_{U_i} \circ \bar{H}_i \cong G|_{V_i} \circ \bar{H}_i = G \circ i_{U_i} \circ \bar{H}_i = G \circ H|_{V_i} = (G \circ H)|_{V_i}$$

Therefore $\{V_0, ..., V_n\}$ is a geometric cover of $\mathfrak{B}$ such that $(F \circ H)|_{V_i} \cong (G \circ H)|_{V_i}$. □

Corollary 3.27. *Let $F \colon \mathfrak{C} \to \mathfrak{D}$ be a functor between a connected small category $\mathfrak{C}$ and a small category $\mathfrak{D}$. We claim that:*

$$\mathrm{ccat}(F) \leq \mathrm{ccat}(\mathfrak{D}).$$

Proof. By the definition of the LS-category for a functor we know that $\mathrm{ccat}(F) = \mathrm{cD}(F, Y)$ where Y is any constant functor while the LS-Category of a category is $\mathrm{ccat}(\mathfrak{D}) = \mathrm{cD}(\mathrm{id}_{\mathfrak{D}}, X)$ when X is a constant functor. Now we use the inequality by composing both $\mathrm{id}_{\mathfrak{C}}$ and X with F in order to obtain the functors $\mathrm{id}_{\mathfrak{C}} \circ F = F$ and $F(X)$ and the following inequality:

$$\mathrm{cD}(F, X) = \mathrm{cD}(id_{\mathfrak{C}} \circ F, F \circ X) \leq \mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, X).$$

□

Definition 3.28. Take U and V two subcategories of a small category $\mathfrak{C}$. We define the category $U \cap V$ as the category whose objects and morphism are the intersection of objects and morphism of U and V.

Proposition 3.29. *Let $\{U_0, ..., U_n\}$ and $\{V_0, ..., V_m\}$ be two geometric covers of $\mathfrak{C}$. Then $\{W_{i,j}\} = \{U_1 \cap V_1, U_1 \cap V_2, ..., U_1 \cap V_j, ..., U_1 \cap V_m, ..., U_i \cap V_1, U_i \cap V_2, ..., U_i \cap V_j, ..., U_i \cap V_m, ..., U_n \cap V_m\}$ is a geometric cover of $\mathfrak{C}$.*

Proof. For every chain $\mathcal{C}$ in $\mathfrak{C}$ we know that there are $i \in \{0, ..., n\}$ and $j \in \{0, ..., m\}$ such that $\mathfrak{C}$ lies in U_i and V_j, so $\mathcal{C}$ lies in $W_{i,j}$. □

Corollary 3.30. *Let* $F, G\colon \mathfrak{C} \to \mathfrak{D}$ *be two functors between small categories. Then*

$$\mathrm{cD}(F, G) + 1 \leq (\mathrm{ccat}(F) + 1)(\mathrm{ccat}(G) + 1).$$

Proof. We suppose that $\mathrm{ccat}(F)$ and $\mathrm{ccat}(G)$ are both finite since if any one is infinite the inequality is obviously true. Take $n = \mathrm{ccat}(F)$, $m = \mathrm{ccat}(F)$, $\{U_0, ..., U_n\}$ a geometric cover by homotopy domains associated to $\mathrm{ccat}(F)$ and $\{V_0, ..., V_m\}$ a geometric cover by homotopy domains associated to $\mathrm{ccat}(G)$.

Now we can obtain the following geometric cover of $\mathfrak{C}$ defined as $\{W_{i,j} \mid 0 \leq i \leq n$ and $0 \leq j \leq m\}$ where every $W_{i,j}$ is the intersection of U_i and V_j. For every i and j we have that $W_{i,j}$ satisfies that $F|_{W_{i,j}} \cong X_0 \cong G|_{W_{i,j}}$. □

Corollary 3.31 (**Relation between LS-category and Categorical complexity**)**.** *Let* $\mathfrak{C}$ *be a small and connected category. Then we have that*

$$\mathrm{ccat}(\mathfrak{C}) \leq \mathrm{cTC}(\mathfrak{C}).$$

Proof. As we know if we fix an object C in $\mathfrak{C}$ we have two *inclusion* functors $i_1, i_2\colon \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$. Also we know that the categorical complexity $\mathrm{cTC}(\mathfrak{C})$ is equal to $\mathrm{cD}(p_1, p_2)$. Now if we apply the previous proposition we have that

$$\mathrm{cD}(p_1 \circ i_1, p_2 \circ i_1) = \mathrm{cD}(\mathrm{id}_{\mathfrak{C}}, C_0) = \mathrm{ccat}(\mathfrak{C}) \leq \mathrm{cD}(p_1, p_2) = \mathrm{cTC}(\mathfrak{C}).$$

□

Corollary 3.32 (**Inequality of categorical complexity**)**.** *Let* $F, G\colon \mathfrak{C} \to \mathfrak{D}$ *be two functors between small categories, The we have that*

$$\mathrm{cD}(F, G) \leq \mathrm{cTC}(\mathfrak{D}).$$

Proof. Using both F and G we can define the functor $(F, G)\colon \mathfrak{C} \to \mathfrak{D} \times \mathfrak{D}$ such that (F, G) is the only functor that satisfies that $p_1 \circ (F, G) = F$ and $p_2 \circ (F, G) = G$. Therefore we have that

$$\mathrm{cD}(F, G) = cD(p_1 \circ (F, G), p_2 \circ (F, G)) \leq \mathrm{cD}(p_1, p_2) = \mathrm{cTC}(\mathfrak{D}).$$

□

3.4 Invariance

In the topological world we know that both LS-category and topological complexity are two homotopy invariants. This fact should incline everyone to establish if in the categorical setting this fact is preserved, i.e if LS-category and categorical complexity are also homotopy invariants. Moreover, this will follows if we can prove that in general homotopic distance is homotopy invariant. This will be the goal of this section.

Proposition 3.33. *Let $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{B}$ be three small categories and let $F, G \colon \mathfrak{C} \to \mathfrak{D}$ be two functors between them.*

1. *If there is a functor $\alpha \colon \mathfrak{D} \to \mathfrak{B}$ such that α has a left homotopy inverse then:*

$$\mathrm{cD}(F, G) = \mathrm{cD}(\alpha \circ F, \alpha \circ G).$$

2. *If there is a functor $\alpha \colon \mathfrak{D} \to \mathfrak{B}$ such that α has a right homotopy inverse then:*

$$\mathrm{cD}(F, G) = \mathrm{cD}(F \circ \beta, G \circ \beta).$$

Proof. 1. Let $\gamma \colon \mathfrak{B} \to \mathfrak{D}$ be the left homotopy inverse of α. By definition we know that $\gamma \circ \alpha \cong \mathrm{id}_{\mathfrak{D}}$. Therefore $\gamma \circ \alpha \circ F \cong \mathrm{id}_{\mathfrak{D}} \circ F = F$, $\gamma \circ \alpha \circ F \cong \mathrm{id}_{\mathfrak{D}} \circ F = F$ and using the property (3) of the Proposition 3.8 we have that $\mathrm{cD}(F, G) = \mathrm{cD}(\gamma \circ \alpha \circ F, \gamma \circ \alpha \circ G)$. If we recall the proposition 3.24 we can prove that:

$$\mathrm{cD}(\alpha \circ F, \alpha \circ G) \leq \mathrm{cD}(F, G).$$

and

$$\mathrm{cD}(F, G) = \mathrm{cD}(\gamma \circ \alpha \circ F, \gamma \circ \alpha \circ G) \leq \mathrm{cD}(\alpha \circ F, \alpha \circ G).$$

2. Let $\gamma \colon \mathfrak{C} \to \mathfrak{B}$ be the right homotopy inverse of β. By definition we have that $\beta \circ \gamma \cong \mathrm{id}_{\mathfrak{C}}$. Therefore $F \circ \beta \circ \gamma \cong F \circ \mathrm{id}_{\mathfrak{C}} = F$, $G \circ \beta \circ \gamma \cong \circ G \circ \mathrm{id}_{\mathfrak{C}} = G$ and $\mathrm{cD}(F, G) = \mathrm{cD}(F \circ \beta \circ \gamma, G \circ \beta \circ \gamma)$. Using the Proposition 3.26 we have that:

$$\mathrm{cD}(F \circ \beta, G \circ \beta) \leq \mathrm{cD}(F, G).$$

and

$$\mathrm{cD}(F, G) = \mathrm{cD}(F \circ \beta \circ \gamma, G \circ \beta \circ \gamma) \leq \mathrm{cD}(F \circ \beta, G \circ \beta).$$

□

Proposition 3.34. *Let $\mathfrak{C}_1$, $\mathfrak{D}_1$, $\mathfrak{C}_2$ and $\mathfrak{D}_2$ be four small categories such that there are $\alpha \colon \mathfrak{C}_1 \to \mathfrak{C}_2$ y $\beta \colon \mathfrak{D}_1 \to \mathfrak{D}_2$ homotopy equivalence and let $F_1, G_1 \colon \mathfrak{C}_1 \to \mathfrak{D}_1$ and $F_2, G_2 \colon \mathfrak{C}_2 \to \mathfrak{D}_2$ be functors such that they make the following commutative diagram:*

$$\begin{array}{ccc} \mathfrak{C}_1 & \overset{F_1}{\underset{G_1}{\rightrightarrows}} & \mathfrak{D}_1 \\ \downarrow \alpha & & \downarrow \beta \\ \mathfrak{C}_2 & \overset{F_2}{\underset{G_2}{\rightrightarrows}} & \mathfrak{D}_2 \end{array}$$

Then we have that

$$\mathrm{cD}(F_1, G_1) = \mathrm{cD}(F_2, G_2)$$

.

Proof. As we know α is a homotopy equivalence so there is $\alpha' \colon \mathfrak{C}_2 \to \mathfrak{C}_1$ such that $\alpha' \circ \alpha \cong \mathrm{id}_{\mathfrak{C}_1}$ and $\alpha \circ \alpha' \cong \mathrm{id}_{\mathfrak{C}_2}$. Using the previous proposition we have that

$$\mathrm{cD}(F_1, G_1) = \mathrm{cD}(F_1 \circ \alpha', G_1 \circ \alpha')$$

and

$$\mathrm{cD}(F_1 \circ \alpha', G_1 \circ \alpha' = \mathrm{cD}(\beta \circ F_1 \circ \alpha', \beta \circ G_1 \circ \alpha').$$

If we show that $\beta \circ F_1 \circ \alpha' \cong F_2$ and $\beta \circ G_1 \circ \alpha' \cong G_2$ we could conclude the proof. However, since the diagram that appears in the theorem is commutative we know that $\beta \circ F_1 = F_2 \circ \alpha$. Therefore if we apply α' we obtain that

$$\beta \circ F_1 \circ \alpha' = F_2 \circ \alpha \circ \alpha' \cong F_2 \circ \mathrm{id}_{\mathfrak{C}_2} = F_2.$$

□

Corollary 3.35. ccat *and* cTC *are invariants by homotopy equivalence, i.e. if* $\mathfrak{C}$ *and* $\mathfrak{D}$ *are two small categories such that* $\mathfrak{C} \cong \mathfrak{D}$ *then*

$$\mathrm{ccat}(\mathfrak{C}) = \mathrm{ccat}(\mathfrak{D})$$

and

$$\mathrm{cTC}(\mathfrak{C}) = \mathrm{cTC}(\mathfrak{D}).$$

3.5 Fibrations and homotopic distance

In [25] Tanaka generalized to category theory the well known Varadarajan theorem ([6]) that provides an inequality to the LS-category of a topological space using a fibration of that space, precisely using the LS-category of the base space and the LS-category of the fiber. Now we will further explore this using all the tools that we produce in chapter 2 of this work. First we will begin by introducing a functor between two bifibrations.

Definition 3.36 (Functor between bifibrations). Let $P \colon \mathfrak{E} \to \mathfrak{B}$ and $P' \colon \mathfrak{E}' \to \mathfrak{B}'$ be two bifibrations. A functor between the two bifibrations is a pair $(F, \bar{F})$ of functors such that $F \colon \mathfrak{E} \to \mathfrak{E}'$, $\bar{F} \colon \mathfrak{B} \to \mathfrak{B}'$ and $P' \circ F = \bar{F} \circ P$. In other words, F and $\bar{F}$ make the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{F} & \mathfrak{E}' \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P'} \\ \mathfrak{B} & \xrightarrow{\bar{F}} & \mathfrak{B}' \end{array}$$

Proposition 3.37. *Let* $P \colon \mathfrak{E} \to \mathfrak{B}$ *and* $P' \colon \mathfrak{E}' \to \mathfrak{B}'$ *be two bifibrations. A functor* $(F, \bar{F})$ *between the two bifibrations induces and unique functor* $F_b : \mathfrak{E}_b \to \mathfrak{E}'_{b'}$ *for every pair of objects* b *in* $\mathfrak{B}$ *and* $b' = \bar{F}(b)$ *in* $\mathfrak{B}'$.

Proof. It is enough to define $F_e\colon \mathfrak{E}_X \to \mathfrak{E}'_{X'}$ as the restriction of F to $\mathfrak{E}_e$. Indeed, as we know by the definiton of functor betwen bifibrations we have that for every object e in $\mathfrak{E}_b$ we have that $P' \circ F(e) = \bar{F} \circ P(e) = \bar{F}(b) = b'$. So for every e in $\mathfrak{E}_b$ we have that $F(e)$ lies in $\mathfrak{E}'_{b'}$. □

Theorem 3.38 (**Inequality of bifibrations**). *Let $(F, \overline{F})$ and $(G, \overline{G})$ be two functors between the bifibrations $P\colon \mathfrak{E} \to \mathfrak{B}$ and $P'\colon \mathfrak{E}' \to \mathfrak{B}'$. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be path-connected. Let b be an object in $\mathfrak{B}$ such that $F(b) = G(b) = b'$ and let $F_b, G_b\colon \mathfrak{E}_b \to \mathfrak{E}'_{b'}$ be the induced functors between the fibers. Then*

$$\mathrm{cD}(F, G) + 1 \le (\mathrm{cD}(F_b, G_b) + 1) \cdot (\mathrm{ccat}(\mathfrak{B}) + 1).$$

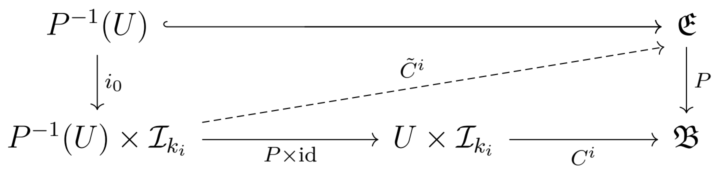
Proof. We assume that $\mathrm{cD}(F_b, G_b)$ and $\mathrm{ccat}\mathfrak{B}$ are both finite, otherwise the result is trivial.

Let $\mathrm{ccat}(\mathfrak{B}) = m$, with $\{U_0, ..., U_m\}$ a categorical cover of $\mathfrak{B}$, and let $\mathrm{cD}(F_b, G_b) = n$, with $\{V_0, ..., V_n\}$ a covering of $\mathfrak{E}_b$ by homotopy domains for F_b and G_b.

For every $i \in \{0, ..., m\}$ we have a homotopy $C^i\colon U_i \times \mathbb{I}_{k_i} \to \mathfrak{B}$ between the inclusion $U_i \hookrightarrow \mathfrak{B}$ and a constant functor $\bullet_i$ for some object $*_i \in \mathfrak{B}$. Since $\mathfrak{B}$ is connected we can assume that all $\bullet_i$ is the same object $*$ for all i (see the proof of Proposition 3.13).

Let $P^{-1}(U_i)$ be the subcategory of $\mathfrak{E}$ whose objects are the objects $e \in \mathfrak{E}$ with $Pe \in U_i$, and whose arrows are the arrows $\alpha \in \mathfrak{E}(e_1, e_2)$ such that $P\alpha$ is an arrow in U_i.

By the homotopy lifting property applied to the following diagram:



we have a homotopy $\tilde{C}^i\colon U_i \times \mathcal{I}_{k_i} \to \mathfrak{E}$ such that $\tilde{C}^i_0$ is the inclusion $P^{-1}(U_i) \hookrightarrow \mathfrak{E}$ and $\tilde{C}^i_{k_i}$ lies inside the fiber $\mathfrak{E}_b$.

For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ we define

$$W_{i,j} = P^{-1}(U_i) \cap (\tilde{C}^i_{k_i})^{-1}(V_j).$$

We claim that $\{W_{i,j}\}_{0 \le i \le m, 0 \le j \le n}$ is a geometric cover of $\mathfrak{E}$ such that each $W_{i,j}$ is a homotopy domain for F and G.

(1) $\{W_{i,j}\}$ is a geometric cover of $\mathfrak{E}$.

Let

$$\mathcal{C}\colon C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_l$$

be a chain in $\mathfrak{E}$. Then we obtain the chain

$$P(\mathcal{C})\colon P(C_1) \longrightarrow P(C_2) \longrightarrow \cdots \longrightarrow P(C_l)$$

in $\mathfrak{B}$. Since $\{U_0, ..., U_m\}$ is a geometric cover of $\mathfrak{B}$, there is some i such that the chain $P(\mathcal{C})$ lies in U_i, so the chain $\mathcal{C}$ lies in $P^{-1}(U_i)$. Moreover, the functor $\tilde{C}^i_{k_i}$ is defined in $\mathcal{C}$, hence we have a new chain $\tilde{C}^i_{k_i}(\mathcal{C})$ that lies in the fiber $\mathfrak{E}_b$. Now, we know that $\{V_j\}$ is a geometric cover of F_b, so $\tilde{C}^i_{k_i}(\mathcal{C})$ lies in some V_j. We conclude that $\mathcal{C}$ is in $W_{i,j}$.

(2) Each $W_{i,j}$ is a homotopy domain for F and G.

For the sake of simplicity we change the notations as follows: $U = U_i$, $\tilde{C} = \tilde{C}^i$, $k = k_i$, $V = V_j$.

Let $K\colon V \times \mathcal{I}_l \to \mathfrak{E}_b$ be a homotopy between $F|_V$ and $G|_V$ and let $\iota\colon \mathfrak{E}_b \hookrightarrow \mathfrak{E}$ be the inclusion of the fiber into the total category $\mathfrak{E}$. The homotopy that we need is the functor

$$H\colon W_{i,j} \times \mathcal{I}_{k+l+k} \to \mathfrak{E}'$$

given by

$$H(c,n) = \begin{cases} F\tilde{C}(c,n) & \text{if} \quad 0 \le n \le k \\ \iota K(\tilde{C}_k c, n-k) & \text{if} \quad k \le n \le k+l \\ G\tilde{C}(c, k+l+k-n) & \text{if} \quad k+l \le n \le k+l+k \end{cases}.$$

It only remains to check that H is well defined and that it is the homotopy that we want:

$$\begin{aligned} H(c,0) =& F\tilde{C}(c,0) = Fc \text{ since } \tilde{C}_0 \text{ is the inclusion,} \\ \iota K(\tilde{C}_k c, 0) =& \iota F(\tilde{C}_k c) = F\tilde{C}(c,k), \\ \iota K(\tilde{C}_k c, l) =& \iota G\tilde{C}_k c = G\tilde{C}(c,k), \\ H(c, k+l+k) =& G\tilde{C}(c,0) = Gc \text{ since } \tilde{C}_0 \text{ is the inclusion.} \end{aligned}$$

This proves that $\mathrm{cD}(F,G) \le m+n$, as stated. □

Corollary 3.39 (Tanaka 2018)**.** *If $P\colon \mathfrak{E} \to \mathfrak{B}$ is a bifibration between small categories, with $\mathfrak{B}$ and* F *connected categories, then*

$$\mathrm{ccat}(\mathfrak{E}) + 1 \le (\mathrm{ccat}(\mathfrak{B}) + 1) \cdot (\mathrm{ccat}(\mathrm{F}) + 1)$$

where F *is the fiber.*

Proof. Take $F = \mathrm{id}_{\mathfrak{E}}$, $G = b$, $\overline{F} = \mathrm{id}_{\mathfrak{B}}$, $\overline{G} = b$. It is easy to see that there is an object such that the constant and the identity are equal, namely the constant. □

Corollary 3.40. *If $P\colon \mathfrak{E} \to \mathfrak{B}$ is a bifibration between small categories, with $\mathfrak{B}$ connected, then*

$$\mathrm{cTC}(\mathfrak{E}) + 1 \le (\mathrm{ccat}(\mathfrak{B} \times \mathfrak{B}) + 1) \cdot (\mathrm{cTC}(\mathrm{F}) + 1)$$

where F *is the fiber.*

Proof. It is easy to see that the functor $P \times P\colon \mathfrak{E} \times \mathfrak{E} \to \mathfrak{B} \times \mathfrak{B}$ is a bifibration since the morphism in the product of two categories are pair of functor between pair of objects. Now if we take this bifibration and the projections the result follow from the theorem. The only problem is how to take the object that has to have the same image, but this can easily be solved since we can use anything in the diagonal. □

Remark 3.41. The latter statement is analogous to the Lemma 7 that appears in [9] in the context of topological spaces.

3.6 Higher homotopic distance

In this final section we will expand some results of [4] to the categorical world, i.e we will define the higher homotopic distance between functors and then we will prove some results concerning it.

Definition 3.42 (**Higher homotopic distance**). Let $\mathfrak{C}$ and $\mathfrak{D}$ be two small categories and let $\{F_i\}_{i=1}^n\colon \mathfrak{C} \to \mathfrak{D}$ be a sequence of funtors between them. The higher homotopic distance $\mathrm{cD}(F_1, ..., F_n)$ is the least positive integer $m \in \mathbb{N}$ such that there is a geometric cover $\{U_0, ..., u_m\}$ that satisfies that $F_i|_{U_k} \cong F_j|_{U_k}$ for every $i, j \in \{1, ..., n\}$ and $0 \le k \le m$. If there is no such cover we define $\mathrm{cD}(F_1, ..., F_n)$ as ∞ .

Notation 3.43. We denote by $\mathfrak{C}^n$ the product of $\mathfrak{C}$ with itself n times, i.e. $\mathfrak{C} \times \mathfrak{C} \times ... \times \mathfrak{C}$.

Definition 3.44 (**Higher n Diagonal functor**). Let $\mathfrak{C}$ be a small category, we define the higher n diagonal functor $\Delta\colon\ \mathfrak{C} \to \mathfrak{C}^n$ as the only functor that takes every object X to $\Delta(X) = (X, ..., X)$ and that takes every morphism f to $\Delta(f) = (f, ..., f)$.

Definition 3.45 (**Highger Farber Subcategory**). We say that a subcategory $\mathfrak{U}$ of a small category $\mathfrak{C}^n$ is Higher n Farber if there is a functor $F\colon U \to \mathfrak{C}$ such that $\Delta \circ F \cong i_U$.

Definition 3.46 (**Higher Categorical Complexety**). Let $\mathfrak{C}$ be a small category. The higher n normalized categorical complexity of $\mathfrak{C}$ that we will denote by $\mathrm{cTC}_n(\mathfrak{C})$ is the least natural number $n \in \mathbb{N}$ such that there is a geometric cover $\{U_0, ..., U_n\}$ of $\mathfrak{C}^n$ by higher Farber subcategories. If there is no such cover we define $\mathrm{cTC}(\mathfrak{C})$ as ∞.

Proposition 3.47 (**Characterization of higher categorical complexity**). *Let $\mathfrak{C}$ be a small category category we have that*

$$\mathrm{cTC}_n(\mathfrak{C}) = cD(p_1, ..., p_n)$$

where $p_i\colon \mathfrak{C}^n \to \mathfrak{C}$ is the i natural projection.

Proof.

$\le$

Let $\{U_0, ..., U_m\}$ be a geometric cover of $\mathfrak{C}^n$ by homotopy domains such that $p_i|_{U_k} \cong p_j|_{U_k}$, we will prove that they also are higher Farber subcategories. Indeed, for every homotopy domain U_k we have that U_k is a Farber subcategory using as the functor F the first projection p_1. For every $i \in \{1, ..., n\}$ we take the homotopy $H^i\colon U_k \times \mathbb{I}_{m_i} \to \mathfrak{C}$ the homotopy between $p_1|_{U_i}$ and $p_i|_{U_i}$ We can normalize all the homotopies taking $m = \max\{m_i\}_{i=0}^n$ and extending $H^i\colon U_k \times \mathbb{I}_{m_i} \to \mathfrak{C}$ by identities if $j \ge m_i$. Now we can define the homotopy $G\colon U_k \times \mathbb{I}_m \to \mathfrak{C}^n$ as

$$G(X_1, X_2, ..., X_n, j) = (X_1, H^2(X_1, ..., X_n, j), ..., H^n(X_1, ..., X_n, j).$$

Finally, we can check that G is a homotopy between $\Delta \circ F$ and i_U since

$$G(X_1, X_2, ..., X_n, 0) = (X_1, H^2(X_1, ..., X_n, 0), ..., H^n(X_1, ..., X_n, 0) = \\ = (X, p_1(X_1, ..., X_n), ..., p_1(X_1, ..., X_n) = (X_1, ..., X_1).$$

and

$$G(X_1, X_2, ..., X_N, m) = G(X_1, H^2(X_1, ..., X_n, m), ..., H^n(X_1, ..., X_n, m)) = \\ = (X_1, p_2(X_1, X_2, ..., X_n), ..., p_n(X_1, ..., X_n)) = (X_1, X_2, ..., X_n).$$

$\geq$

Let $\{U_0, ..., U_m\}$ be a geometric cover of $\mathfrak{C}^n$ by Farber subcategories. Fix some index k in $\{0, ..., n\}$, we will see that $p_i|_{U_i} \cong p_j|_{U_i}$ for every $i, j \in \{1, ..., n\}$. Since U_k is a Farber subcategory there is a functor $F\colon U_k \to \mathfrak{C}$ such that $\Delta \circ F \cong i_{U_i}$, so there is homotopy $H\colon U_k \times \mathbb{I}_m \to \mathfrak{C} \times \mathfrak{C}$ such that H verifies that $H_0 = i_{U_k}$ and $H_m = \Delta \circ F$. Now we take the functor $K\colon U_k \times \mathbb{I}_{2m} \to \mathfrak{C}$

$$K(X_1, ..., X_n, l) = \begin{cases} p_i \circ H(X_1, ..., X_n, m - l) & if \quad 0 \leq l \leq m \\ \\ p_j \circ H(X_1, ..., X_n, l - m) & if \quad m \leq l \leq 2m \end{cases}$$

The functor K is well-defined since:

$$K(X_1, ..., X_n, m) = p_i \circ H(X_1, ..., X_n, m) = \\ p_i \circ \Delta \circ F(X_1, ..., X_n) = p_i(F(X_1, ..., X_n), ..., F(X_1, ..., X_n)) = F(X_1, ..., X_n).$$

and

$$K(X_1, ..., X_n, m) = p_j \circ H((X_1, ..., X_n), m - m) = p_j \circ H((X_1, ..., X_n), 0) = \\ p_j \circ \Delta \circ F(X_1, ..., X_n) = p_j(F(X_1, ..., X_n), ..., F(X_1, ..., X_n)) = F(X_1, ..., X_n).$$

Moreover K is a homotopy between p_i and p_j since:

$$K(X_1, ..., X_n, 0) = p_i \circ H(X_1, ..., X_n, 0) = \\ = p_i \circ i_{U_k}(X_1, ..., X_n) = p_i(X_1, ..., X_n),$$

and

$$K(X_1, ..., X_n, 2m) = p_j \circ H(X_1, ..., X_n, 0) = \\ = p_j \circ i_{U_k}(X_1, ..., X_n) = p_j(X_1, ..., X_n),$$

□

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