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TESE DE DOUTORAMENTO

Conformal structures and solitons in pseudo-Riemannian geometry

María Ferreiro Subrido

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“Sobrio no tengo la sabiduría”

– Sebastian Pena

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Abstract

This thesis is divided into three distinct parts, each of which explores different aspects of mathematical structures. The first part focuses on the investigation of locally conformally flat structures on four-dimensional manifolds. More precisely, the study delves into the local conformal flatness of Kähler, para-Kähler, and null-Kähler manifolds, and provides a complete geometric description of four-dimensional para-Kähler Lie algebras.

Moving on to the second part, the research focuses on the analysis of solitons associated to both the Bach and Ricci flows. This part offers a complete classification of four-dimensional left-invariant Lorentzian Ricci solitons and Riemannian algebraic Bach and Ricci solitons in dimension four.

Finally, the third part covers the description of four-dimensional homogeneous Riemannian manifolds that have half-harmonic Weyl curvature tensor and those homogenous manifolds which admit more than one homogeneous structure in dimension free.

Introduction

Let (M, g) be a pseudo-Riemannian manifold and denote by $[g]$ the conformal class of all the pseudo-Riemannian metrics which are conformal to g. A basic problem in conformal Riemannian geometry is the existence of distinguished structures – such as Einstein metrics, Kähler or para-Kähler structures, Ricci solitons, etc. – in the same conformal class of a given metric. In this Ph.D. thesis we examine this sort of problems and obtain some classification results under certain conditions.

The existence of flat metrics in the conformal class of a given metric is a very restrictive property. From a local perspective, this property is characterized by the vanishing of the Weyl curvature tensor in dimension four, which ensures the existence of locally defined functions $\sigma\colon \mathcal{U} \subset M \to \mathbb{R}$ such that $(\mathcal{U}, \overline{g} = e^{2\sigma} g)$ is flat. Granted that there exist bountiful examples of locally conformally flat metrics, the particular example of the n-dimensional sphere $\mathbb{S}^n$ shows that local conformal flatness cannot always be extended to a global condition.

Our first objective has been to completely understand the local structure of four-dimensional Kähler and para-Kähler metrics that have at least one flat representative in their conformal classes. It was already known that in dimension greater than four the existence of such representatives is strictly restricted to flat manifolds, but the four-dimensional situation still remained an open problem. Patterson proved the existence of non-flat four-dimensional examples when the associated Ricci operators are two-step nilpotent in the case of split-signature Kähler metrics (see [119]). In Chapter 2 in this thesis we complete the description of these metrics by finding the existence of examples with complex Ricci operators.

Theorem 2.2 *Any indecomposable locally conformally flat Kähler surface (M, g, J_+) is locally isometric to the cotangent bundle $(T^*\Sigma, g)$ of a Riemannian surface (Σ, g_Σ) of constant curvature with a metric given by*

(i) $g = g_{\nabla^\Sigma}$ *if the Gaussian curvature is non-zero, or*

(ii) $g = \iota J_\Sigma \circ \iota \operatorname{Id} + g_{\nabla^\Sigma}$ *if the Gaussian curvature vanishes,*

where ∇^Σ is the Levi-Civita connection of (Σ, g_Σ) and J_Σ is the Kähler structure on Σ associated to the Riemannian volume form. Furthermore, the complex structure J_+ on $T^*\Sigma$ is determined by the symplectic form $\Omega_+ = -d\iota J_\Sigma$.

A somehow analogous result describes the locally conformally flat para-Kähler surfaces.

Theorem 2.1 *Any indecomposable locally conformally flat para-Kähler surface (M, g, J_-) is locally isometric to the cotangent bundle $T^*\Sigma$ of a flat affine surface (Σ, D) with a para-complex* *structure determined by* $J|_{\ker \pi_*} = \mathrm{Id}$*, where* π *denotes the canonical projection from the cotangent bundle, and the metric* g *is given by* $g = \iota T \circ \iota \,\mathrm{Id} + g_D$ *where*

(i) T *is a parallel nilpotent* $(1,1)$*-tensor field on* (Σ, D)*, or*

(ii) T *is a parallel* $(1,1)$*-tensor field on* (Σ, D) *satisfying* $T^2 = -\kappa^2 \,\mathrm{Id}$.

Moreover, in both cases the para-Kähler two-form $\Omega_-(X,Y) = g(J_-X, Y)$ *is the canonical symplectic form of the cotangent bundle.*

Even though the local structures correspond to modified Riemannian extensions of affine surfaces in both situations, the affine connections on the base surfaces are essentially different. In the Kähler case, the extended connection is the Levi-Civita connection of an affine surface of constant Gaussian curvature, whereas in the para-Kähler situation the extended connection is flat, which makes it less restrictive than the complex situation. The Ricci operators associated to the metrics in Theorem 2.1-*(ii)* and Theorem 2.2-*(ii)* determine self-dual parallel complex structures on the manifolds, which induce anti-Kähler structures (see [21]) whose associated metrics have the same Levi-Civita connection than the original metrics.

Since the metrics in Theorems 2.1 and 2.2 are locally symmetric, we pondered the possibility of their realization as left-invariant metrics on Lie groups. Ovando had already determined all the – both Riemannian and split-signature – Kähler structures on four-dimensional Lie groups in her work [116]. Nevertheless, the complete description of the four-dimensional para-Kähler Lie groups still remained an open problem, despite the number of previous attempts to understand it [31, 32, 101]. Para-Kähler structures are determined by the symplectic Lie algebras that admit a decomposition $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{L}'$ as a direct sum of Lagrangian subalgebras, so the possibilities for the existence of such structures are more flexible than in the Kählerian case and, consequently, they are plentiful.

In Chapter 3 we give a complete description of the left-invariant para-Kähler metrics on four-dimensional Lie groups, which allows us to describe all the locally conformally flat para-Kähler metrics in terms of these structures (see Corollary 3.2). We give our description in terms of the curvature of the left-invariant metrics. In Theorem 3.1 we determine all the classes of symmetric para-Kähler Lie groups. The non-symmetric situation splits into the semi-symmetric case – that occurs only when the corresponding Ricci tensors vanish – and the non-semi-symmetric case – which corresponds to four families among which we found some of the 3-symmetric spaces – as stated in Theorems 3.6 and 3.7. As a consequence of our study, we obtained an alternative description of the hypersymplectic Lie groups previously obtained by Andrada [5].

The existence of Einstein metrics in the conformal class of a given metric implies the existence of an underlying C-space structure, i.e., the existence of a vector field X on the manifold such that

$$\operatorname{div} W + \iota_X W = 0.$$

When the vector field X is the gradient of a function, this equation is equivalent to the existence of a conformal metric $\overline{g}$ for which $\overline{\operatorname{div} W} = 0$, since the Weyl curvature of an Einstein manifold is harmonic – i.e., divergence-free. On four-dimensional oriented manifolds, the decomposition of the Weyl curvature tensor into its self-dual and anti-self-dual components $W = W^+ + W^-$, which is conformally invariant, makes the condition $\operatorname{div} W = 0$ become $\operatorname{div} W^+ + \operatorname{div} W^- = 0$. In this way, a four-dimensional oriented manifold has half-harmonic Weyl curvature if either one of the two summands in the previous equation vanishes. It is important to emphasize that this condition is not conformally invariant, since

$$\overline{\operatorname{div} W}^+ = \operatorname{div} W^+ - \iota_{\nabla\sigma} W^+$$

for a conformal metric $\overline{g} = e^{2\sigma} g$. Four-dimensional homogeneous Riemannian manifolds have harmonic Weyl curvature tensor if and only if they are symmetric [123]. In Chapter 6 we give the complete description of the four-dimensional homogeneous spaces that have half-harmonic Weyl curvature, as stated in the main result of the chapter.

Theorem 6.1 *Let (M, g) be a four-dimensional locally homogeneous Riemannian manifold with half-harmonic Weyl curvature tensor. Then it is symmetric or locally homothetic to one of the following semi-direct extensions of the Heisenberg group.*

(i) The left-invariant metric on $H^3 \rtimes \mathbb{R}$ determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = -2e_3\,,$$

(ii) or the left-invariant metric on $H^3 \rtimes \mathbb{R}$ determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \tfrac{1}{2}e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = -\tfrac{1}{2}e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{h}_3 \rtimes \mathfrak{r}$.

It is important to emphasize that the Weyl curvature tensor of a locally symmetric manifold is parallel and so it is divergence-free. Besides, even though orientation does not play a relevant role in Theorem 6.1, it does when additional structures on the manifold are considered. A Kähler surface satisfies $\operatorname{div} W^- = 0$ if and only if it is weakly Bochner-flat, in which case it is locally symmetric when its scalar curvature is constant (see [89]). Furthermore, since the self-dual Weyl curvature of an oriented four-dimensional Kähler metric takes the form

$$W^+ = \tfrac{\tau}{12} \operatorname{diag}[2, -1, -1],$$

then $\tau \operatorname{div}_1 W^+ + \iota_{\nabla\tau} W^+ = 0$ and so $\operatorname{div}_1 W^+ = 0$ if the scalar curvature is a non-zero constant. If the scalar curvature is not constant, then it determines a conformal metric whose Weyl curvature tensor is half-harmonic [56, 87]. The only non-symmetric homogeneous Kähler metric is the 3-symmetric space, which is necessarily half-harmonic, and it corresponds to Theorem 6.1-*(ii)*.

The Lie groups given in Theorem 6.1 correspond to homogeneous manifolds that admit self-dual homogeneous structures. Generalizing Cartan's characterization of symmetric spaces as those whose curvature tensors are parallel, Ambrose and Singer [3] considered homogeneous structures on a Riemannian manifold (M, g) as tensor fields of type $(1, 2)$ for which the connection $\widetilde{\nabla} = \nabla - T$ makes the metric g, its associated curvature tensor R and the tensor field T parallel. In this way, a complete and simply connected Riemannian manifold is homogeneous if it admits a homogeneous structure.

Sekigawa proved in [126] that any three-dimensional simply connected homogeneous Riemannian manifold is either symmetric or isometric to a Lie group endowed with a left-invariant metric $(G, \langle\cdot,\cdot\rangle)$. Lie groups are trivially homogeneous, just considering the action of G on itself determined by the left translations. Therefore, all of them have a natural homogeneous structure, called the *canonical homogeneous structure* of the Riemannian Lie group.

Given the fact that a homogeneous manifold may admit different presentations as a homogeneous space, it may as well admit more than one homogeneous structure. Therefore, we decided to try and determine all the three-dimensional Riemannian homogeneous spaces that admit more than one homogeneous structure, and saw that in the non-symmetric situation this occurs only when their isometry groups are four-dimensional.

Theorem 7.1 *A non-symmetric simply connected three-dimensional Riemannian Lie group admits a homogeneous structure different from the canonical one if and only if it admits a naturally reductive homogeneous structure. Moreover, in such a case, it admits exactly a one-parameter family of homogeneous structures.*

In Theorems 7.2 and 7.3 we also determine all the possible homogeneous structures on three-dimensional Riemannian Lie groups.

Since the four-dimensional homogeneous manifolds that have half-harmonic Weyl curvature admit self-dual homogeneous structures, we started the study of such structures in Section 7.3. Even though there already are classification results under certain additional conditions [109], the complete description of four-dimensional self-dual homogeneous structures seems to be an arduous problem that we only sketch in the mentioned section.

The Weyl curvature tensor of any locally conformally flat manifold vanishes and so the functional

$$g \mapsto \int_M \|W_g\|^2 dvol_g$$

reaches a minimum on every such manifold. This functional is conformally invariant in the four-dimensional situation and its gradient is determined by the Bach tensor

$$\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \tfrac{1}{2} W[\rho].$$

Therefore, Bach-flat four-dimensional manifolds are a natural generalization of locally conformally flat manifolds. The fact that four-dimensional both conformally Einstein and (anti-)self-dual manifolds are also Bach-flat places more importance on the study of such manifolds. The Bach flow

$$\tfrac{\partial}{\partial t} g_t = \mathfrak{B}_{g_t} + \tfrac{1}{12}(\Delta \tau_{g_t}) g_t$$

has been intensively studied recently with the intention of improving the behaviour of a given metric in terms of its Bach tensor and obtaining Bach-flat metrics at the limit.

The existence of solitons associated to this flow has been studied in the context of product homogeneous manifolds and under the additional hypothesis of the soliton being a gradient. In Chapter 5 we tackle the study of such solitons from the algebraic point of view. A Riemannian Lie group $(G, \langle\cdot,\cdot\rangle)$ is an algebraic Bach soliton if and only if

$$\mathfrak{D} = \widehat{\mathfrak{B}} - \mu \mathrm{Id}$$

is a derivation of the Lie algebra of G, in which case it gives rise to a Bach soliton or, equivalently, to a self-similar solution $g_t = \sigma(t)\psi_t^*\langle\cdot,\cdot\rangle$, where ψ_t is one-parameter group of automorphisms of G.

Since every Einstein metric is Bach-flat, one might expect Ricci solitons to correspond to Bach solitons. This is actually the situation at the algebraic level, although there are algebraic Bach solitons which are not Ricci solitons. The following result provides a complete description of four-dimensional Riemannian algebraic Bach solitons.

Theorem 5.8 *A four-dimensional simply connected Riemannian Lie group is an algebraic Bach soliton if and only if it is Bach-flat, an algebraic Ricci soliton or homothetic to one of the following Lie groups.*

(i) The product Lie group $SU(2)\times\mathbb{R}$ with the product left-invariant metric determined by

$$[e_1,e_2]=4e_3,\quad [e_1,e_3]=-4e_2,\quad [e_2,e_3]=e_1.$$

(ii) The semi-direct extension $H^3\rtimes\mathbb{R}$ with the left-invariant metric determined by

$$[e_1,e_2]=e_3,\quad [e_1,e_4]=ae_1,\quad [e_2,e_4]=\tfrac{1}{a}e_2,\quad [e_3,e_4]=\tfrac{a^2+1}{a}\,e_3,$$

for $a\in(0,1)$.

Here $\{e_1,e_2,e_3,e_4\}$ denotes an orthonormal basis of the corresponding Lie algebra.

The Bach soliton obtained from Assertion *(i)* is a gradient soliton on $\mathbb{S}^3\times\mathbb{R}$, where the metric on $\mathbb{S}^3$ is not the round metric, but a Berger one. On the other hand, the family of metrics in Assertion *(ii)* does not give rise to gradient solitons.

The four-dimensional algebraic Ricci solitons were determined – up to isomorphisms – by Lauret in [98]. In Theorem 5.4 we give a description of such solitons up to homotheties, which makes the description shorter and more manageable. The techniques developed in Chapter 5 are based on the analysis of general algebraic T-solitons for a geometric flow $\partial_t g_t = T_{g_t}$ given by an isometrically invariant symmetric, divergence-free $(0,2)$-tensor field. This general approach led to unexpected simplifications which enable us to consider more complicated geometric flows.

Homogeneous Ricci solitons are critical for some curvature quadratic functional

$$g\mapsto\int_M\left\{\|\rho\|^2+t\tau^2\right\}dvol_g$$

with zero energy in dimensions three and four (see [23]) and Riemannian signature. These solitons are algebraic or gradient solitons. In the same way, homogeneous gradient or algebraic Lorentzian Ricci solitons are critical for some curvature quadratic functional with zero energy in dimension three. Nevertheless, and in sharp contrast with the Riemannian situation, the Lorentzian signature allows the existence of left-invariant Ricci solitons on Lie groups. Such solitons had already been classified in the three-dimensional situation [24], and we tackle the four-dimensional problem in Chapter 4 of this thesis. The situation is far more complicated than the three-dimensional one and we obtained the following result.

Theorem 4.2 *A non-symmetric four-dimensional Lorentzian Lie group which is not a pp-wave is a non-trivial left-invariant Ricci soliton if and only if it is homothetic to one of the following:*

(i) $G_\alpha = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_1, e_4] = \alpha e_1,\ [e_2, e_4] = \varepsilon\left(1-\tfrac{\alpha^2}{2}\right)^{\frac{1}{2}} e_2 - e_3,\ [e_3, e_4] = e_2 + \varepsilon\left(1-\tfrac{\alpha^2}{2}\right)^{\frac{1}{2}} e_3,$$

where the parameter satisfies $0 \le \alpha \le \sqrt{2}$ *and* $\{e_1, e_2, e_3, e_4\}$ *is an orthonormal basis with timelike* e_3. *If* $\alpha = 0$, *then* $\varepsilon = 1$. *If* $0 < \alpha < \sqrt{2}$, *then* $\varepsilon^2 = 1$. *In the latter case,* $\alpha \neq \frac{2\sqrt{3}}{3}$ *when* $\varepsilon = -1$.

(ii) $G_\alpha = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_4] = \alpha u_1, \quad [u_2, u_4] = -\alpha u_2 + u_3, \quad [u_3, u_4] = u_1, \qquad \alpha > 0,$$

where $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis for which the non-zero inner products are* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$.

(iii) $G = E(1,1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_2, e_4] = -[e_1, e_2] = e_2, \quad [e_1, e_3] = [e_3, e_4] = \tfrac{1}{2}[e_1, e_4] = e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ *is an orthonormal basis with timelike* e_3.

(iv) $G_{\alpha\beta} = E(1,1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_2] = u_1, \qquad [u_1, u_4] = -2\alpha(\alpha\beta + 1)u_1, \quad [u_2, u_3] = u_3,$$
$$[u_2, u_4] = \beta u_1, \quad [u_3, u_4] = \alpha u_3,$$

where $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis for which the non-zero inner products are* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$, *and the parameters* $\alpha > 0$ *and* $\alpha\beta \notin \{-2, -1, -\frac{1}{2}\}$.

The metrics in Assertions *(i)* and *(iii)* have complex Ricci curvatures, while those of the remaining two are real. Besides, the metrics in Assertions *(i)*, *(ii)* and *(iv)* are critical for some curvature quadratic functionals, whereas the metrics in Theorem 4.2-*(iii)* never are.

The case of left-invariant Ricci solitons which are pp-wave and plane wave Lie groups are described in Theorems 4.9 and 4.11, respectively.

The outline of this thesis

In Chapter 1 the reader can find the preliminary concepts that will be necessary for the complete understanding of the contents of this thesis, which is divided in three distinguished parts.

Part I is devoted to the study of locally conformally flat structures. In particular, in Chapter 2 we give a complete description of four-dimensional locally conformally flat Kähler, para-Kähler and null-Kähler structures. In Chapter 3 we study the realization of the para-Kähler families obtained in the previous chapter as left-invariant metrics on Lie groups and give a complete geometric description of the four-dimensional para-Kähler Lie algebras.

In Part II we study the solitons associated to two particular geometric flows: the Bach flow and the Ricci flow. In Chapter 4 we give a complete description of the four-dimensional left-invariant Lorentzian Ricci solitons and Chapter 5 is devoted to the study of the algebraic situation in Riemannian signature. In this chapter we introduce a general technique to describe four-dimensional Riemannian algebraic solitons associated to the flow determined by a generic symmetric and divergence-free $(0, 2)$-tensor field and use it to describe all the algebraic Bach and Ricci solitons in dimension four.

Part III covers Chapters 6 and 7 of this thesis. In Chapter 6 we give a complete description of the four-dimensional homogeneous Riemannian manifolds that have half-harmonic Weyl curvature. In Chapter 7 we determine all the non-symmetric three-dimensional Riemannian homogeneous spaces that admit more than one homogeneous structure. Besides, motivated by the fact that the metrics obtained in Chapter 6 admit self-dual homogeneous structures, we began to study these sort of structures in dimension four and the reader can find a sketch of this problem in Section 7.3. This remains an open problem for which we have obtained only partial results so far.

Chapter 1
Preliminaries

In this chapter we will introduce the concepts and notations that will be necessary for the complete understanding of this thesis. We will omit most of the proofs and remit the reader to different bibliographic references for more details.

1.1 Pseudo-Riemannian geometry

1.1.1 Pseudo-Riemannian manifolds

An n-dimensional *pseudo-Riemannian manifold* is a pair (M, g) where M is a smooth manifold and g is a metric tensor, i.e., a symmetric and non-degenerate $(0, 2)$-tensor field on M. The signature of the metric g is the pair $(n - \nu, \nu)$ such that $n - \nu$ and ν are the number of negative and positive eigenvalues of its associated matrix, respectively. Let (M, g) be an n-dimensional pseudo-Riemannian manifold. It is said to be ***Riemannian*** if its signature is $(0, n)$ and *Lorentzian* if its signature is $(1, n - 1)$. Moreover, if M is even-dimensional and the signature of g is $\left(\frac{n}{2}, \frac{n}{2}\right)$ then the manifold has neutral (or split) signature.

We will denote by TM and T^*M the tangent and cotangent bundles of the manifold M and by $\mathfrak{X}(M)$ the space of vector fields which are tangent to M. We will use capital letters to denote vector fields and small letters to denote tangent vectors at a given point. Given a non-zero vector $v \in T_pM$ tangent to M at a point $p \in M$, it is said to be *timelike* if $g(v, v) < 0$, *spacelike* if $g(v, v) > 0$ and *null* or *lightlike* if $g(v, v) = 0$.

For any pseudo-Riemannian manifold (M, g) there exists a unique adapted linear connection ∇ which is torsion-free and parallel, i.e., such that

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \quad \text{and} \quad \nabla g = 0.$$

This connection is known as the *Levi-Civita connection* of the pseudo-Riemannian manifold and it is characterized by the Koszul formula

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) = {} & Xg\left(Y, Z\right) + Yg\left(X, Z\right) - Zg\left(X, Y\right) \\ & - g\left(X, [Y, Z]\right) - g\left(Y, [X, Z]\right) + g\left(Z, [X, Y]\right), \end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $[\cdot, \cdot]$ denotes the Lie bracket. The Levi-Civita connection can also be described by means of the *Christoffel symbols*. Let $(x^1, \dots, x^n)$ be local coordinates on M. The Christoffel symbols of the *first kind* are given by

$$\Gamma_{ij\ell} = \frac{1}{2}\left(\frac{\partial g_{\ell j}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\right)$$

and so the Christoffel symbols of the *second kind* are

$$\Gamma_{ij}{}^{k} = g^{k\ell}\Gamma_{ij\ell},$$

being (g^{ij}) the inverse matrix of (g_{ij}). Therefore the Levi-Civita connection can be written in coordinates as

$$\nabla_{\partial_{x^i}}\partial_{x^j} = \Gamma_{ij}{}^{k}\partial_{x^k},$$

where $\partial_{x^i} := \frac{\partial}{\partial x^i}$ denote the locally defined coordinate vector fields.

1.1.2 The curvature tensor

In terms of the Levi-Civita connection we can define the *curvature operator*, or curvature tensor of type $(1,3)$ by the convention

$$R\,(X,Y)\,Z = \nabla_{[X,Y]}Z - [\nabla_X,\nabla_Y]\,Z.$$

Considering local coordinates $(x^1,\dots,x^n)$ on M, the components of the curvature operator are given by $R\,(\partial_{x^i},\partial_{x^j})\,\partial_{x^k} = R_{ijk}{}^{\ell}\partial_{x^\ell}$. We can obtain the *curvature tensor* of type $(0,4)$ by lowering indices in the previous expression

$$R\,(X,Y,Z,V) = g\,(R(X,Y)Z,V)\,,$$

so that its components are given by $R_{ijk\ell} = g_{\ell r}R_{ijk}{}^{r}$. Moreover, the curvature tensor has the following algebraic properties

$$\begin{aligned}
&(i) \quad R(X,Y,Z,V) = -R(Y,X,Z,V) = -R(X,Y,V,Z),\\
&(ii) \quad R(X,Y,Z,V) + R(Y,Z,X,V) + R(Z,X,Y,V) = 0,\\
&(iii) \quad R(X,Y,Z,V) = R(Z,V,X,Y),
\end{aligned} \tag{1.1}$$

and the differential identity

$$(iv) \quad (\nabla_X R)\,(Y,Z,U,V) + (\nabla_Y R)\,(Z,X,U,V) + (\nabla_Z R)\,(X,Y,U,V) = 0.$$

Identities (ii) and (iv) are known as the first and second *Bianchi identities*, respectively.

A $(0,4)$-tensor field $A\colon \mathcal{V}\times\mathcal{V}\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ on a vector space $\mathcal{V}$ satisfying identities (1.1) is said to be an *algebraic curvature tensor*.

The *sectional curvature* of a given Riemannian manifold (M,g) is the real function K define on the Grassmannian of 2-planes by

$$K(\Pi) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where $\Pi = \operatorname{span}\{X,Y\}$ is a two-dimensional subspace of T_pM. In the pseudo-Riemannian case we must consider the restriction to the Grassmannian of non-degenerate planes, i.e., those planes such that

$$g\,(X,X)\,g\,(Y,Y) - g\,(X,Y)^2 \neq 0.$$

If $K(\Pi)$ is independent of the plane Π, then the curvature tensor is given by

$$R(X, Y, Z, V) = KR^0(X, Y, Z, V),$$

where R^0 is the *standard algebraic curvature tensor* given by

$$R^0(X, Y, Z, V) = g(X, Z)g(Y, V) - g(X, V)g(Y, Z). \tag{1.2}$$

In dimension greater than two, if M is connected then the second Bianchi identity implies that if K is pointwise constant, then it is necessarily a global constant. A pseudo-Riemannian manifold has constant sectional curvature K if and only if its curvature tensor can be written as

$$R(X, Y)Z = K\{g(X, Z)Y - g(Y, Z)X\},$$

in which case the manifold is locally isometric to a pseudo-sphere $\mathbb{S}^n_\nu$ (when $K > 0$), to a pseudo-Euclidean space $\mathbb{E}^n_\nu$ (when $K = 0$) or to a pseudo-hyperbolic space $\mathbb{H}^n_\nu$ (when $K < 0$). We refer to O'Neil's book [112] for more details on this topic.

We will denote by ρ the *Ricci tensor*, which is defined as the second trace of the curvature operator,

$$\rho(X, Y) = \mathrm{tr}\,(Z \mapsto R(X, Z)\,Y)$$

and its associated $(1, 1)$-tensor field, known as the *Ricci operator*, is characterized by

$$g\,(\mathrm{Ric}(X), Y) = \rho(X, Y).$$

The *scalar curvature* is defined as the trace of the Ricci operator $\tau = \mathrm{tr}(\mathrm{Ric})$. It follows from the curvature identities (1.1) that the Ricci tensor is symmetric and, equivalently, the Ricci operator is self-adjoint. The Ricci tensor and the scalar curvature can be expressed in coordinates as

$$\rho_{ij} = g^{r\ell}R_{irj\ell}, \quad \tau = g^{ij}\rho_{ij}.$$

Any two-dimensional pseudo-Riemannian manifold satisfies $\rho = \frac{\tau}{2}g$. A pseudo-Riemannian manifold of dimension greater than two is said to be an *Einstein space* if its Ricci tensor is a constant multiple of the metric, $\rho = \lambda g$. Tracing on the previous expression one sees that

$$\rho = \frac{\tau}{n}g, \tag{1.3}$$

and if M is connected, then the second Bianchi identity leads to the constancy of τ. In dimension three, satisfying the Einstein condition (1.3) is equivalent to having constant sectional curvature, while in dimension four there exist Einstein metrics which are not of constant sectional curvature. The four-dimensional case then appears as the first non-trivial case for consideration.

1.1.3 The Weyl tensor

The *Schouten tensor* of an algebraic curvature tensor A on an n-dimensional inner product vector space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is the symmetric $(0,2)$-tensor field defined as

$$\mathfrak{S}_A = \frac{1}{n-2}\left(\rho_A - \frac{\tau_A}{2(n-1)}\langle\cdot,\cdot\rangle\right),$$

where ρ_A and τ_A are the Ricci tensor and the scalar curvature associated to the algebraic curvature tensor A.

Let D and B be two symmetric bilinear forms on a vector space $\mathcal{V}$. Their *Kulkarni-Nomizu* product $D \odot B$ is the $(0,4)$-tensor field on $\mathcal{V}$ defined as

$$\begin{aligned}(D \odot B)(x,y,z,v) = {} & D(x,z)B(y,v) + D(y,v)B(x,z) \\ & - D(x,v)B(y,z) - D(y,z)B(x,v),\end{aligned}$$

for $x,y,z,v \in \mathcal{V}$. It is easy to check that $D \odot B$ is an algebraic curvature tensor on $(\mathcal{V}, \langle\cdot,\cdot\rangle)$. For example, the standard curvature tensor $R^0 = \frac{1}{2}\langle\cdot,\cdot\rangle \odot \langle\cdot,\cdot\rangle$.

The *Weyl curvature tensor* arises from the Kulkarni-Nomizu product of the Schouten tensor and the metric tensor as $W_A = A - \mathfrak{S}_A \odot \langle\cdot,\cdot\rangle$. Therefore, the Weyl curvature tensor of a pseudo-Riemannian manifold (M,g) is defined as

$$W = R - \mathfrak{S} \odot g,$$

which can be written at each point $p \in M$ as

$$\begin{aligned}W(x,y,z,v) = {} & R(x,y,z,v) + \tfrac{\tau}{(n-1)(n-2)}\left\{g(x,z)g(y,v) - g(x,v)g(y,z)\right\} \\ & - \tfrac{1}{n-2}\left\{\rho(x,z)g(y,v) - \rho(x,v)g(x,z) + \rho(y,v)g(x,z) - \rho(y,z)g(x,v)\right\},\end{aligned}$$

for all $x,y,z,v \in T_pM$. An important property of the Weyl curvature tensor is that it is trace-free and in dimension three it vanishes identically.

Local conformal flatness

A pseudo-Riemannian manifold (M,g) is said to be *locally conformally flat* if for every point $p \in M$ there exists an open neighbourhood $\mathcal{U}$ of p and a smooth function $\sigma\colon \mathcal{U} \to \mathbb{R}$ such that the metric $\bar{g} = e^{2\sigma}g$ is flat.

The vanishing of the Weyl curvature tensor characterizes the locally conformally flat spaces in dimension greater than three. In the three-dimensional case, local conformal flatness is characterized by the total symmetry of the covariant derivative of the Schouten tensor, which means that it must be such that $(\nabla_X\mathfrak{S})(Y,Z) = (\nabla_Y\mathfrak{S})(X,Z)$ (see [99]).

In local differential geometry, the most important invariant of a conformal structure is given by the conformal Weyl curvature tensor of type $(0,4)$, which satisfies $\overline{W} = e^{2\sigma}W$ for any two conformally related metrics $\bar{g} = e^{2\sigma}g$.

1.1.4 Curvature decomposition

Consider for now an n-dimensional real vector space $\mathcal{V}$ with basis $\{e_1, \ldots, e_n\}$ and let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathcal{V}$. A *bivector* on $\mathcal{V}$ is an element of the form $\sum_{i,j=1}^{n} a_{ij} e_i \wedge e_j$, where $a_{ij} \in \mathbb{R}$. The set of all bivectors is known as the bivector space and it is usually denoted by $\Lambda^2\mathcal{V}$. The bivector space has the following properties:

(i) $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0$ for all $i, j \in \{1, \ldots, n\}$.

(ii) The set
$$\{e_1 \wedge e_2, \ldots, e_1 \wedge e_n, e_2 \wedge e_3, \ldots, e_2 \wedge e_n, \ldots, e_{n-1} \wedge e_n\}$$
is a basis of $\Lambda^2\mathcal{V}$.

Therefore $\Lambda^2\mathcal{V}$ is an $\frac{n(n-1)}{2}$-dimensional vector space. We define the *wedge product* of two elements $x = x^i e_i$ and $y = y^j e_j$ in $\mathcal{V}$ by

$$x \wedge y = x^i e_i \wedge y^j e_j = \sum_{i<j} \left(x^i y^j - x^j y^i\right) e_i \wedge e_j \in \Lambda^2\mathcal{V}.$$

The inner product on $\mathcal{V}$ naturally extends to an inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on the bivector space as [95]
$$\langle\!\langle x \wedge y, z \wedge v \rangle\!\rangle = \langle x, z\rangle\langle y, v\rangle - \langle x, v\rangle\langle y, z\rangle \tag{1.4}$$
and if $\{e_i \colon i = 1, \ldots, n\}$ is a $\langle \cdot, \cdot \rangle$-orthonormal basis of $\mathcal{V}$, then $\{e_i \wedge e_j \colon i < j\}$ is an orthonormal basis of $\Lambda^2\mathcal{V}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Given an algebraic curvature tensor A on $(\mathcal{V}, \langle \cdot, \cdot \rangle)$, it induces a unique self-adjoint endomorphism $\mathcal{A} \colon \Lambda^2\mathcal{V} \to \Lambda^2\mathcal{V}$ determined by

$$\langle\!\langle \mathcal{A}(x \wedge y), z \wedge v \rangle\!\rangle = A(x, y, z, v) \quad \text{for all} \quad x, y, z, v \in \mathcal{V}.$$

The converse is not necessarily true, since a given self-adjoint endomorphism of $\Lambda^2\mathcal{V}$ may not satisfy the first Bianchi identity, but there exists a bijective correspondence between the set of algebraic curvature tensors on $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ and the set of self-adjoint endomorphisms of $(\Lambda^2\mathcal{V}, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$ satisfying

$$\langle\!\langle \mathcal{A}(x \wedge y), z \wedge v \rangle\!\rangle + \langle\!\langle \mathcal{A}(y \wedge z), x \wedge v \rangle\!\rangle + \langle\!\langle \mathcal{A}(z \wedge x), y \wedge v \rangle\!\rangle = 0.$$

In particular, the standard curvature tensor R^0 corresponds to the endomorphism $\mathcal{R}^0 = \mathrm{Id}_{\Lambda^2}$.

The following result provides a decomposition of all the algebraic curvature tensors which, in turn, motivates the tensors introduced above.

Theorem 1.1 ([95])**.** *Any algebraic curvature tensor A on an inner product vector space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ decomposes as*
$$A = \mathfrak{U}_A + \mathfrak{Z}_A + W_A,$$

where

$$\mathfrak{U}_A = \frac{\tau_A}{2n(n-1)}\langle\cdot,\cdot\rangle \odot \langle\cdot,\cdot\rangle, \qquad \mathfrak{Z}_A = \frac{1}{n-2}\left(\rho_A - \tfrac{\tau_A}{n}\langle\cdot,\cdot\rangle\right) \odot \langle\cdot,\cdot\rangle,$$
$$W_A = A - \mathfrak{U}_A - \mathfrak{Z}_A = A - \mathfrak{S}_A \odot \langle\cdot,\cdot\rangle,$$

being ρ_A, τ_A and $\mathfrak{S}_A$ the Ricci tensor, the scalar curvature and the Schouten tensor of the algebraic curvature tensor A, respectively.

The components $\mathfrak{U}_A$, $\mathfrak{Z}_A$ and W_A in Theorem 1.1 correspond to the following orthogonal components:

- $\mathfrak{U}_A$ is the orthogonal projection on the space of algebraic curvature tensors of constant sectional curvature.
- The vanishing of $\mathfrak{Z}_A$ corresponds to Einstein algebraic curvature tensors.
- In dimension greater than three, the vanishing of the component W_A represents locally conformally flat algebraic curvature tensors.

1.1.5 Self-duality and anti-self-duality

The curvature decomposition given in Theorem 1.1 can be expressed in a more simple manner in low-dimensional cases. In dimension two every algebraic curvature tensor takes the form $A = \frac{\tau_A}{2}\langle\cdot,\cdot\rangle \odot \langle\cdot,\cdot\rangle$, while in dimension three every algebraic curvature tensor is determined by its associated Schouten tensor as $A = \mathfrak{S}_A \odot \langle\cdot,\cdot\rangle$. The situation in dimension four is more complicated, but the properties of the Hodge-star operator allow us to refine the curvature decomposition given above.

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of a four-dimensional inner product space that we will denote by $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ and $\{e^1, e^2, e^3, e^4\}$ be its dual basis. Consider the volume form on $\mathcal{V}$, $vol = e^1 \wedge e^2 \wedge e^3 \wedge e^4$. The *Hodge-star operator* $\star\colon \Lambda^2\mathcal{V} \to \Lambda^2\mathcal{V}$ is defined as

$$\alpha \wedge \star\beta = \langle\!\langle \alpha, \beta \rangle\!\rangle vol$$

for all $\alpha, \beta \in \Lambda^2\mathcal{V}$.

The properties of the Hodge-star operator depend on the signature of the inner product $\langle\cdot,\cdot\rangle$ under consideration. In this way, in Lorentzian signature the Hodge-star operator $\star$ defines a complex structure ($\star^2 = -\mathrm{Id}_{\Lambda^2\mathcal{V}}$), while both in Riemannian and neutral signatures it defines a para-complex structure ($\star^2 = \mathrm{Id}_{\Lambda^2\mathcal{V}}$). In this case, the Hodge-star operator induces a decomposition of the space of bivectors $\Lambda^2\mathcal{V} = \Lambda^2_+\mathcal{V} \oplus \Lambda^2_-\mathcal{V}$, where $\Lambda^2_+\mathcal{V}$ and $\Lambda^2_-\mathcal{V}$ denote the spaces of self-dual and anti-self-dual bivectors, respectively,

$$\Lambda^2_+\mathcal{V} = \left\{\alpha \in \Lambda^2\mathcal{V}\colon\ \star\alpha = \alpha\right\} \quad \text{and} \quad \Lambda^2_-\mathcal{V} = \left\{\alpha \in \Lambda^2\mathcal{V}\colon\ \star\alpha = -\alpha\right\},$$

which are the eigenspaces associated to the ± 1 eigenvalues of the Hodge-star operator.

For the orthonormal basis considered above and denoting $\varepsilon_i = \langle e_i, e_i \rangle$, the self-dual and anti-self-dual subspaces are spanned by $\{E_1^\pm, E_2^\pm, E_3^\pm\}$, where

$$\begin{aligned} E_1^\pm &= \tfrac{1}{\sqrt{2}} \left(e^1 \wedge e^2 \pm \varepsilon_3 \varepsilon_4 e^3 \wedge e^4\right), \\ E_2^\pm &= \tfrac{1}{\sqrt{2}} \left(e^1 \wedge e^3 \mp \varepsilon_2 \varepsilon_4 e^2 \wedge e^4\right), \\ E_3^\pm &= \tfrac{1}{\sqrt{2}} \left(e^1 \wedge e^4 \pm \varepsilon_2 \varepsilon_3 e^2 \wedge e^3\right). \end{aligned}$$

The metric induced on the bivector space by the inner product on $\mathcal{V}$, given by (1.4), is Riemannian when $\langle \cdot, \cdot \rangle$ is positive definite and has signature $(++----)$ when $\langle \cdot, \cdot \rangle$ has neutral signature $(2,2)$. In the latter case, the signature of the restriction of the metric to the subspaces $\Lambda^2_\pm \mathcal{V}$ is $(+--)$ and $\{E_1^\pm, E_2^\pm, E_3^\pm\}$ is an orthonormal basis in which $E_1^\pm$ is spacelike and $E_2^\pm$ and $E_3^\pm$ are lightlike.

Now consider an algebraic curvature tensor A on $\mathcal{V}$ as an endomorphism of the bivector space. The decomposition $\Lambda^2 \mathcal{V} = \Lambda^2_+ \mathcal{V} \oplus \Lambda^2_- \mathcal{V}$ induces a decomposition of the Weyl curvature tensor as $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-$, where $\mathcal{W}^\pm$ denote the restriction of the Weyl curvature tensor acting on the space of bivectors to the self-dual and anti-self-dual subspaces, respectively. In dimension four the curvature decomposition given in Theorem 1.1 can be refined as

$$\mathcal{A} \equiv \tfrac{\tau}{12} \mathrm{Id}_{\Lambda^2 \mathcal{V}} + \rho_0 + \mathcal{W}^+ + \mathcal{W}^- \colon \Lambda^2 \mathcal{V} \to \Lambda^2 \mathcal{V},$$

where ρ_0 denotes the trace-less Ricci tensor. The Weyl curvature tensor is said to be *self-dual* if $\mathcal{W}^- = 0$ and *anti-self-dual* if $\mathcal{W}^+ = 0$.

1.1.6 Differential operators

Let (M, g) be an n-dimensional pseudo-Riemannian manifold and let $f \colon M \to \mathbb{R}$ be a differentiable function. The *gradient of* f is the vector field $\nabla f = \sharp df$, which is determined by the gradient operator $\nabla \colon C^\infty(M) \to \mathfrak{X}(M)$ as

$$g(\nabla f, X) = X(f) \quad \text{for all} \quad X \in \mathfrak{X}(M).$$

The gradient can be expressed in a system of local coordinates $(x^1, \dots, x^n)$ on M as

$$\nabla f = \sum_{i,k=1}^{n} g^{ik} \frac{\partial f}{\partial x^k} \partial_{x^i}.$$

The *Hessian operator of* f is defined as the endomorphism $\mathrm{h}_f \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ that is given by the second covariant derivative

$$\mathrm{h}_f(X) = \nabla_X \nabla f.$$

Now, the *Hessian tensor* is the symmetric tensor field of type $(0, 2)$ defined as

$$\mathrm{Hes}_f(X, Y) = g(\mathrm{h}_f(X), Y) = g(\nabla_X \nabla f, Y) = XY(f) - (\nabla_X Y) f.$$

In terms of a system of local coordinates on M, the Hessian tensor can be expressed as

$$\mathrm{Hes}_f(\partial_{x^i}, \partial_{x^j}) = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}{}^k \frac{\partial f}{\partial x^k}.$$

Note that, since the Levi-Civita connection is trace-free and adapted to the metric, the Hessian operator is self-adjoint and so the corresponding $(0,2)$-tensor field is symmetric. The Hessian operator also allows us to define the *Laplacian of a function* f as

$$\Delta f = \mathrm{tr}(\mathrm{h}_f).$$

Considering the extension of the Levi-Civita connection as a derivation of tensor fields on the manifold, the *Laplacian of a* $(0,k)$*-tensor field* T is defined as

$$\Delta T(X_1, \ldots, X_k) = \mathrm{tr}\left(\nabla^2 T\right)(X_1, \ldots, X_k) = \sum_{i=1}^{n} \left(\nabla^2_{E_i E_i} T\right)(X_1, \ldots, X_k),$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame.

The *divergence* of a vector field X is defined as the function $\mathrm{div} X = \mathrm{tr}(\nabla X)$. Considering a local orthonormal frame $\{E_1, \ldots, E_n\}$, then

$$\mathrm{div} X = \sum_{i=1}^{n} \varepsilon_i g(\nabla_{E_i} X, E_i),$$

where $\varepsilon_i = g(E_i, E_i)$. In general, if T is $(0,s)$-tensor field, its r*-divergence* is defined as the $(0, s-1)$-tensor field given by

$$(\mathrm{div}_r T)(X_1, \ldots, X_{s-1}) = \sum_{i=1}^{n} \varepsilon_i (\nabla_{E_i} T)(X_1, \ldots, X_{r-1}, E_i, X_{r+1}, \ldots, X_{s-1}),$$

for all $X_1, \ldots, X_{s-1} \in \mathfrak{X}(M)$. Since the r-divergence of the tensor field T is given by the r-th trace of ∇T, its definition does not depend on the choice of the local frame.

1.2 Curvature functionals

Let (M, g) be an n-dimensional pseudo-Riemannian manifold and consider a local orthonormal frame $\{E_i\}$. A *curvature scalar invariant* is a polynomial that involves the components of the curvature tensor and its covariant derivatives that does not depend on the choice of the local frame.

- The space of curvature scalar invariants of order one of a pseudo-Riemannian manifold has dimension one and is generated by the scalar curvature τ.
- The space of curvature scalar invariants of order two of a pseudo-Riemannian manifold has dimension four and is generated by $\{\tau^2, \|\rho\|^2, \|R\|^2, \Delta\tau\}$.

We refer the reader to [17, 80] for more information.

1.2.1 The Hilbert-Einstein functional

Given a compact n-dimensional pseudo-Riemannian manifold (M, g), the associated *Hilbert-Einstein functional* is given by

$$S_{HE}\colon g \mapsto \int_M \tau \, dvol_g.$$

If we consider variations of the form $g[t] = g + th$, where h is a symmetric $(0, 2)$-tensor field, we will say that the metric g is critical for this variation if

$$\tfrac{d}{dt}\big|_{t=0} S_{HE}(g[t]) = 0.$$

This expression can be written as

$$\tfrac{d}{dt}\big|_{t=0} S_{HE}(g[t]) = \int_M \langle \nabla S_{HE}, h\rangle dvol_g$$

for some symmetric $(0, 2)$-tensor field ∇S_{HE}, which is called the *gradient of the functional* S_{HE}. The gradient of the Hilbert-Einstein functional is given by

$$\nabla S_{HE} = -\rho + \tfrac{\tau}{n} g.$$

Given that the Hilbert-Einstein functional is not homothetically invariant, we need to consider its restriction to the space of metrics of constant volume on a manifold M. Bearing in mind that all the metrics of the variation $g[t]$ have constant volume as long as the tensor field h is orthogonal to the metric g, the corresponding Euler-Lagrange equations are determined by

$$\nabla S_{HE} = \lambda g.$$

Tracing both sides of this identity, it follows that

$$\lambda = \tfrac{1}{n}\mathrm{tr}\,(\nabla S_{HE}) = \tfrac{n-2}{2n}\tau,$$

and so

$$\rho - \tfrac{\tau}{n} g = 0. \tag{1.5}$$

As a consequence, the metrics which are critical for the Hilbert-Einstein functional – when restricted to variations of constant volume – are precisely the Einstein metrics, which are those whose scalar curvature is “better” distributed across the manifold.

1.2.2 Quadratic functionals

If we now consider the space of curvature scalar invariants of order two, a curvature quadratic functional is given by

$$g \mapsto \int_M \left\{ a\tau^2 + b\|\rho\|^2 + c\|R\|^2 + d\Delta\tau \right\} dvol_g,$$

for some $a, b, c, d \in \mathbb{R}$. If we assume that M is a compact manifold without boundary, then

$$\begin{aligned} g \mapsto & \textstyle\int_M \{a\tau^2 + b\|\rho\|^2 + c\|R\|^2 + d\Delta\tau\}\, dvol_g \\ & = \textstyle\int_M \{a\tau^2 + b\|\rho\|^2 + c\|R\|^2\}\, dvol_g, \end{aligned}$$

and so every quadratic functional can be expressed in terms of the three functionals

$$\mathcal{S}\colon g \mapsto \int_M \tau^2 dvol_g, \quad \mathcal{T}\colon g \mapsto \int_M \|\rho\|^2 dvol_g, \quad \mathcal{R}\colon g \mapsto \int_M \|R\|^2 dvol_g,$$

which are what we will refer to as *curvature quadratic functionals*. The gradients of these three functionals are (see [16])

$$\begin{aligned} \nabla\mathcal{S} &= 2\nabla^2\tau - 2\Delta\tau g - \tau\left(2\rho - \tfrac{1}{2}g\right), \\ \nabla\mathcal{T} &= -\Delta\rho + \nabla^2\tau - \tfrac{1}{2}\Delta\tau g - 2R[\rho] + \tfrac{1}{2}\|\rho\|^2 g, \\ \nabla\mathcal{R} &= -4\Delta\rho + 2\nabla^2\tau - 2\check{R} + \tfrac{1}{2}\|R\|^2 g - 4R[\rho] + 4\check{\rho}. \end{aligned}$$

The curvature tensor R of a three-dimensional pseudo-Riemannian manifold is completely determined by its Ricci tensor, which means that the three curvature scalar invariants satisfy the identity

$$\|R\|^2 = 2\|\rho\|^2 - \tfrac{1}{2}\tau^2.$$

Consequently, the functional determined by the L^2-norm of the curvature tensor can be expressed as a linear combination of the other two as

$$\mathcal{R} = 2\mathcal{T} - \tfrac{1}{2}\mathcal{S}$$

in the three-dimensional case. In the four-dimensional setting, the Chern-Gauss-Bonnet Theorem gives the Euler characteristic of a compact pseudo-Riemannian manifold with no boundary in terms of the three curvature scalar invariants as

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left\{\|R\|^2 - 4\|\rho\|^2 + \tau^2\right\} dvol_g.$$

Therefore, the four-dimensional curvature functional $\mathcal{R}$ is determined by

$$\mathcal{R} = t\pi^2\chi(M) + 4\mathcal{T} - \mathcal{S},$$

and, since the Euler characteristic of a manifold is a topological invariant, the critical points of $\mathcal{R}$ correspond to the critical points of $4\left(\mathcal{T} - \frac{1}{4}\mathcal{S}\right)$. Therefore, the functionals $\mathcal{R}$ and $\mathcal{T} - \frac{1}{4}\mathcal{S}$ are equivalent. This shows that it is enough to study the functionals

$$\mathcal{S}\colon g \mapsto \int_M \tau^2 dvol_g, \quad \mathcal{F}_t\colon g \mapsto \int_M \left\{\|\rho\|^2 + t\tau^2\right\} dvol_g,$$

for all $t \in \mathbb{R}$. The gradients of the functionals $\mathcal{F}_t$ are given by

$$\nabla\mathcal{F}_t = -\Delta\rho + (1+2t)\nabla^2\tau - \tfrac{1+4t}{2}\Delta\tau g - 2R[\rho] + \tfrac{1}{2}\|\rho\|^2 g + \tfrac{1}{2}t\tau^2 g - 2t\tau\rho.$$

Proceeding in the same way as we did for the Hilbert-Einstein functional, when we restrict our study to the space of metrics of constant volume, we obtain that the Euler-Lagrange equations corresponding to these functionals are given by

$$\nabla^2\tau - \tfrac{1}{n}\Delta\tau g - \tau\left(\rho - \tfrac{\tau}{n}g\right) = 0,$$
$$\Delta\rho - (1+2t)\nabla^2\tau + \tfrac{2}{n}t\Delta\tau g + 2\left(R[\rho] - \tfrac{1}{n}\|\rho\|^2 g\right) + 2t\tau\left(\rho - \tfrac{\tau}{n}g\right) = 0.$$

Note that Einstein metrics are critical for the functionals $\mathcal{S}$ and $\mathcal{F}_t$ for any value of t. Consequently, Einstein metrics are critical for all the curvature quadratic functionals in dimensions three and four, but this is no longer true in higher dimensions.

The L^2-norm of the Weyl curvature tensor: the Bach tensor

In dimension four, the functional given by

$$\mathcal{W}\colon g \mapsto \mathcal{W}(g) = \int_M \|W_g\|^2 dvol_g,$$

where W_g denotes the Weyl curvature tensor associated to g, quantifies the deflection of a Riemannian metric g from being locally conformally flat. A remarkable property of this functional is that it is conformally invariant in dimension four. Indeed, if $\bar{g} = e^{2\sigma}g$, then

$$\begin{aligned}\|\overline{W}\|^2 dvol_{\bar{g}} &= \overline{W}_{ijk\ell}\overline{W}^{ijk\ell} dvol_{\bar{g}} \\ &= e^{2\sigma}W_{ijk\ell}e^{2\sigma}e^{-8\sigma}W^{ijk\ell}e^{4\sigma}dvol_g = \|W\|^2 dvol_g.\end{aligned}$$

Furthermore, it follows from the Chern-Gauss-Bonnet Theorem that

$$\mathcal{W}(g) = 32\pi^2\chi(M) + 2\mathcal{F}_{-1/3},$$

and so this functional is equivalent to $\mathcal{F}_{-1/3}$. The conformal invariance of $\mathcal{W}$ shows that four-dimensional conformally Einstein metrics are $\mathcal{F}_{-1/3}$-critical. Such metrics are equivalently characterized by the vanishing of their Bach tensor (see [10]), which is defined as the $(0,2)$-tensor field given by

$$\mathfrak{B} = \operatorname{div}_2\operatorname{div}_4 W + \tfrac{1}{2}W[\rho],$$

where $W[\rho]_{ij} = W_{ijk\ell}\rho^{k\ell}$.

In addition to conformally Einstein metrics, half conformally flat metrics are also Bach-flat in dimension four. Recall that the Hirzebruch signature formula allows us to express the Hirzebruch signature as

$$\tau[M] = \frac{1}{12\pi^2}\int_M \left\{\|W^+\|^2 - \|W^-\|^2\right\} dvol_g,$$

and so the functional $\mathcal{W}$ can be written as

$$\begin{aligned}\mathcal{W}(g) = \textstyle\int_M \|W\|^2 dvol_g &= \textstyle\int_M \left\{\|W^+\|^2 + \|W^-\|^2\right\} dvol_g \\ &= \pm 12\pi^2\tau[M] + 2\textstyle\int_M \|W^{\mp}\|^2 dvol_g,\end{aligned}$$

which shows that half conformally flat metrics are critical for the functional $\mathcal{W}$, and therefore Bach-flat. Besides, since the functional $\mathcal{W}$ is equivalent to the one given by the L^2-norm of the self-dual Weyl curvature tensor W^+, if g is a Kähler metric (see Part I), whose self-dual conformal operator W^+ is diagonalizable and takes the form $W^+ = \frac{\tau}{12}\,\mathrm{diag}[2,-1,-1]$, then

$$\mathcal{W}(g) = -12\pi^2\tau[M] + 2\int_M \|W^+\|^2 dvol_g = -12\pi^2\tau[M] + \frac{1}{12}\int_M \tau^2 dvol_g.$$

Consequently, the Calabi functional – which is the restriction of $\mathcal{S}$ to metrics in the same Kähler class – is equivalent to $\mathcal{F}_{-1/3}$ when restricted to such variations (see [29, 127]).

1.3 Affine and projective geometry

An *affine manifold* is a pair (M, D) where M is a smooth manifold and D is an affine torsion-free connection on M. The Ricci tensor associated to D is defined as

$${}^D\rho(X,Y) := \mathrm{tr}\left(Z \to {}^DR(X,Z)Y\right),$$

and, since it need not be symmetric in general, we introduce the symmetrization ${}^D\rho_s$ and skew-symmetrization ${}^D\rho_{sk}$ as

$$\begin{aligned} {}^D\rho_s(X,Y) &:= \tfrac{1}{2}\left\{{}^D\rho(X,Y) + {}^D\rho(Y,X)\right\}, \\ {}^D\rho_{sk}(X,Y) &:= \tfrac{1}{2}\left\{{}^D\rho(X,Y) - {}^D\rho(Y,X)\right\}. \end{aligned} \tag{1.6}$$

An affine manifold is *flat* if its associated curvature tensor ${}^DR$ vanishes. In this case, there exist local coordinates where the Christoffel symbols are zero.

A *projective structure* on an affine manifold (M, D) is an equivalence class $[D]$ of affine connections on TM sharing the same unparametrized geodesics. Two affine connections D and $\widetilde{D}$ are projectively equivalent if and only if there exists a one-form $\omega = \omega_i dx^i$ on M such that

$$\widetilde{\Gamma}_{ij}{}^k = \Gamma_{ij}{}^k + \delta_i{}^k\omega_j + \delta_j{}^k\omega_i,$$

where $\Gamma_{ij}{}^k$ and $\widetilde{\Gamma}_{ij}{}^k$ denote the Christoffel symbols of the affine connections D and $\widetilde{D}$, respectively, and $\delta_i{}^k$ denotes the Kronecker delta. According to this, an affine manifold (M, D) is said to be *projectively flat* if there exists a flat representative in the class $[D]$, i.e., if there exists a one-form ω on M such that

$$\Gamma_{ij}{}^k = -\left(\delta_i{}^k\omega_j + \delta_j{}^k\omega_i\right). \tag{1.7}$$

Furthermore, if there exists a real-valued function f locally defined on M such that $\omega = df$, then (M, D) is said to be *locally strongly projectively flat*. Two-dimensional projectively flat affine manifolds are characterized as follows (see [110]).

Theorem 1.2. *Let (Σ, D) be an affine surface. Then (Σ, D) is projectively flat if and only if ${}^D\rho$ and $D\,{}^D\rho$ are totally symmetric.*

An affine manifold is *curvature recurrent* (respectively, *Ricci recurrent*) if $D^D R = \xi \otimes {}^D R$ (respectively, $D^D \rho = \xi \otimes {}^D \rho$) for some one-form ξ on M and (M, D) is *locally symmetric* if $D^D R = 0$. Since the curvature tensor of any affine surface is determined by the associated Ricci tensor as

$${}^D R(X, Y)Z = {}^D \rho(X, Z)Y - {}^D \rho(Y, Z)X,$$

curvature recurrence and Ricci recurrence are equivalent in the two-dimensional case.

1.3.1 Walker structures

Let (M, g) be a pseudo-Riemannian manifold and consider a distribution $\mathcal{D}$ of the tangent space. $\mathcal{D}$ is parallel if $\nabla_X Y \in \mathcal{D}$ for every smooth vector field X and every $Y \in \mathcal{D}$, and degenerate if $g|_{\mathcal{D}}$ vanishes.

It is well-known that the existence of a parallel distribution on a Riemannian manifold induces a local de Rham decomposition as a product. This property is also true in the pseudo-Riemannian setting as long as the parallel distribution is non-degenerate. The situation in which the distribution is degenerate was studied by Walker [132], who gave a canonical expression for this kind of metrics. This is why pseudo-Riemannian manifolds admitting a parallel degenerate distribution are called *Walker manifolds*.

Walker metrics give rise to many strictly pseudo-Riemannian situations, such as degenerate homogeneous pseudo-Riemannian structures, strictly conformally symmetric manifolds, conformally flat metrics with two-step nilpotent Ricci operator, Einstein hypersurfaces in manifolds with constant sectional curvature and nilpotent shape operator, para-Kähler structures and so on.

In the most general situation, the existence of adapted Walker coordinates is given by the following result (see [132]).

Theorem 1.3. *Let (M, g) be an n-dimensional Walker manifold and let $\mathcal{D}$ be an r-dimensional, parallel degenerate distribution. There exists a local system of adapted coordinates*

$$\left(x^1, \dots, x^{n-r}, x^{n-r+1}, \dots, x^n\right)$$

on M with respect to which the metric tensor takes the form

$$(g_{ij}) = \begin{pmatrix} B & H & \mathrm{Id}_r \\ {}^tH & A & 0 \\ \mathrm{Id}_r & 0 & 0 \end{pmatrix},$$

where Id_r is the identity matrix of order r and A, B and H are matrices whose components are functions of the adapted coordinates satisfying the following conditions:

1. *A is an $(n-2r) \times (n-2r)$ matrix and B is an $r \times r$ matrix.*

2. *H is an $r \times (n-2r)$ matrix and ${}^tH$ denotes its transpose matrix.*

3. *A and H are independent of the last $n-r$ coordinates.*

Moreover, the distribution $\mathcal{D}$ is spanned by the last $n-r$ coordinate vector fields.

The canonical form given in the previous theorem turns out to be simpler when the manifold has even dimension $n=2m$ and the Walker distribution is of maximal dimension. In this case, there exist local Walker coordinates

$$\left(x^1,\dots,x^m,x_{1'},\dots,x_{m'}\right)$$

in which the metric tensor takes the form

$$g=\begin{pmatrix} B & \mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix}, \tag{1.8}$$

where B is a matrix whose components are functions of $(x^1,\dots,x^m,x_{1'},\dots,x_{m'})$. This kind of metrics will play an important role in the work developed in this thesis. Their non-zero Christoffel symbols and curvature operator are given by the following results (see [41]).

Lemma 1.4. *Let $(M,g,\mathcal{D})$ be a Walker manifold of dimension $n=2m$, where $\dim\mathcal{D}=m$. Then its Christoffel symbols are given, up to the corresponding symmetries, by*

$$\begin{aligned}
\Gamma_{ij}^{k} &= -\tfrac{1}{2}\partial_{k'}g_{ij},\\
\Gamma_{i'j}^{k'} &= \tfrac{1}{2}\partial_{i'}g_{jk},\\
\Gamma_{ij}^{k'} &= \tfrac{1}{2}\left(-\partial_k g_{ij}+\partial_j g_{ik}+\partial_i g_{jk}+g_{ks}\partial_{s'}g_{ij}\right),
\end{aligned}$$

where $1\le s\le m$.

Lemma 1.5. *Let (M,g) be a Walker manifold of dimension $n=2m$ with m-dimensional parallel degenerate distribution $\mathcal{D}$. The non-zero components of its curvature tensor of type $(1,3)$ are given, up to the corresponding symmetries, by*

$$\begin{aligned}
R_{ijk}^{h} &= -\tfrac{1}{2}\left(\partial_j\partial_{h'}g_{ik}-\partial_i\partial_{h'}g_{jk}\right)-\tfrac{1}{4}\left(\partial_{s'}g_{jk}\partial_{h'}g_{is}-\partial_{s'}g_{ik}\partial_{h'}g_{js}\right),\\
R_{ijk}^{h'} &= -\tfrac{1}{2}\left(\partial_i\partial_k g_{jh}-\partial_i\partial_h g_{jk}+\partial_j\partial_h g_{ik}-\partial_j\partial_k g_{ih}\right)\\
&\quad -\tfrac{1}{4}\left\{\partial_{s'}g_{jk}\left(\partial_h g_{is}-\partial_s g_{ih}-\partial_i g_{sh}-g_{ht}\partial_{t'}g_{is}\right)\right.\\
&\qquad -\partial_{s'}g_{ik}\left(\partial_h g_{js}-\partial_s g_{jh}-\partial_j g_{sh}-g_{ht}\partial_{t'}g_{js}\right)\\
&\qquad -\partial_{s'}g_{ih}\left(\partial_s g_{jk}-\partial_k g_{js}-\partial_j g_{ks}-g_{st}\partial_{t'}g_{jk}\right)\\
&\qquad +\partial_{s'}g_{jh}\left(\partial_s g_{ik}-\partial_k g_{is}-\partial_i g_{ks}-g_{st}\partial_{t'}g_{ik}\right)\\
&\qquad \left.+2\partial_i\left(g_{hs}\partial_{s'}g_{jk}\right)-2\partial_j\left(g_{hs}\partial_{s'}g_{ik}\right)\right\},\\
R_{ij'k}^{h} &= -\tfrac{1}{2}\partial_{j'}\partial_{h'}g_{ik},\\
R_{ij'k}^{h'} &= -\tfrac{1}{2}\left(\partial_h\partial_{j'}g_{ik}-\partial_k\partial_{j'}g_{ih}\right)\\
&\quad -\tfrac{1}{4}\left(\partial_{s'}g_{ik}\partial_{j'}g_{sh}+\partial_{s'}g_{ih}\partial_{j'}g_{sk}-2\partial_{j'}(g_{hs}\partial_{s'}g_{ik})\right),\\
R_{ijk'}^{h'} &= -\tfrac{1}{2}\left(\partial_i\partial_{k'}g_{jh}-\partial_j\partial_{k'}g_{ih}\right)-\tfrac{1}{4}\left(\partial_{k'}g_{js}\partial_{s'}g_{ih}-\partial_{k'}g_{is}\partial_{s'}g_{jh}\right),\\
R_{ij'k'}^{h'} &= \tfrac{1}{2}\partial_{j'}\partial_{k'}g_{ih},
\end{aligned}$$

where $1\le s\le m$ and $1\le t\le m$.

In general, the fact that $\mathcal{D}$ is parallel implies that the curvature tensor of any Walker manifold is such that

$$R(\mathcal{D}, \mathcal{D}^\perp, \cdot, \cdot) = 0, \quad R(\mathcal{D}, \mathcal{D}, \cdot, \cdot) = 0, \quad \text{and} \quad R(\mathcal{D}^\perp, \mathcal{D}^\perp, \mathcal{D}, \cdot) = 0.$$

1.3.2 Metrics on the cotangent bundle

A particular class of Walker metrics are those known as Riemannian extensions. The interest of these metrics resides in the fact that they allow the translation of problems in affine geometry to problems in pseudo-Riemannian geometry and vice versa.

Let T^*M denote the cotangent bundle of an n-dimensional smooth manifold and consider the natural projection from the cotangent bundle onto the base manifold $\pi\colon T^*M \to M$. A point $\tilde{p} \in T^*M$ is of the form $\tilde{p} = (p, \omega)$, where $p := \pi(\tilde{p}) \in M$ and $\omega \in T_p^*M$. We will give some basic notions about the geometry of the cotangent bundle before introducing Riemannian extensions.

Let $\tilde{p} = (p, \omega)$ be a point in T^*M and consider local coordinates $(x^1, \dots, x^n)$ on a neighbourhood $\mathcal{U}$ of the point $p \in M$. In this neighbourhood we can write

$$\omega = x_{i'} dx^i,$$

which allows us to define a system of local coordinates on $\widetilde{\mathcal{U}} := \pi^{-1}(\mathcal{U}) \subset T^*M$ as

$$\left(x^1, \dots, x^n, x_{1'}, \dots, x_{n'}\right).$$

In terms of these local coordinates, the canonical symplectic structure of the cotangent bundle is given by

$$\Omega := d\omega = dx_{i'} \wedge dx^i. \tag{1.9}$$

Let us consider a vector field X on M. Its evaluation map ιX is the differentiable map on the cotangent bundle given by

$$\iota X(p, \omega) = \omega\left(X_p\right).$$

If we write $X = X^i \partial_{x^i}$, where $X^i = dx^i(X)$, then $\iota X\left(x^i, x_{i'}\right) = x_{i'} X^i$.

Vectors fields on T^*M are determined by their action on evaluation maps of vector fields on M. In this way, two vector fields $\widetilde{Y}$ and $\widetilde{Z}$ on M are the same if and only if $\widetilde{Y}(\iota X) = \widetilde{Z}(\iota X)$ for every vector field X on M. Bearing this in mind, the complete lift of a vector field X on M is a vector field on T^*M characterized by the identity

$$X^C(\iota Z) = \iota[X, Z],$$

for every vector field Z on M. The tangent space to the cotangent bundle of a smooth manifold is generated by the complete lifts of all smooth vector fields on M. Complete lifts allow us to characterize tensor fields of type $(0, s)$ in the sense that two tensor fields $\tilde{S}$ and $\tilde{T}$ are the same if and only if

$$\widetilde{T}\left(X_1^C, \dots, X_s^C\right) = \widetilde{S}\left(X_1^C, \dots, X_s^C\right)$$

for any vector fields $X_1, \ldots, X_s$ on M. Knowing this, it is easy to see that the two-form (1.9) does not depend on the system of local coordinates in consideration and is characterized by the identity

$$\omega\left(X^C, Y^C\right) = \iota\,[X, Y]\,.$$

Now, if T is a $(1,1)$-tensor field on M, it is an endomorphism of the tangent bundle TM, so one can define a one-form ιT on the cotangent bundle characterized by the identity

$$\iota T\left(X^C\right) = \iota\,(TX)\,,$$

which takes the form $\iota T = x_{k'} T_i{}^k dx^i$ with respect to the coordinates induced on T^*M.

Riemannian extensions

The construction of this kind of metrics defines a Walker metric on the cotangent bundle of an affine n-manifold (M, D), where D denotes a torsion-free connection on M. T^*M can be equipped with a pseudo-Riemannian metric of neutral signature (n, n) given by the identity

$$g_D\left(X^C, Y^C\right) = -\iota\,(D_X Y + D_Y X)\,.$$

This metric is called a *Riemannian extension* (see [120]) and in terms of the system of local coordinates induced on T^*M takes the form

$$g_D = \begin{pmatrix} -2x_{k'}\Gamma_{ij}{}^k & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}, \quad i, j = 1, \ldots, n, \quad i' = i + n, \tag{1.10}$$

where $\Gamma_{ij}{}^k$ are the Christoffel symbols of the affine connection D with respect to the coordinates (x^i) on M.

These metrics are particular cases of Walker metrics for which the parallel, degenerate distributions have maximal dimension and are given by $\mathcal{D} = \ker \pi_*$. As we have already mentioned, Riemannian extensions provide a link between affine geometry and Riemannian geometry so that some properties of the affine connection D can be studied through the properties of g_D. For instance, D is projectively flat if and only if g_D is locally conformally flat.

Deformed Riemannian extensions

These metrics are a slight generalization of Riemannian extensions involving an additional $(0,2)$-tensor field on M that we will call Φ. Deformed Riemannian extensions are again pseudo-Riemannian metrics of neutral signature (n, n) on the cotangent bundle of an affine manifold and are given by

$$\begin{aligned} g_{D,\Phi}\left(X^C, Y^C\right) &= g_D + \pi^*\Phi \\ &= -\iota\,(D_X Y + D_Y X) + \pi^*\Phi. \end{aligned}$$

Deformed Riemannian extensions can be expressed in terms of the coordinates induced on the cotangent bundle as

$$g_{D,\Phi} = \begin{pmatrix} -2x_{k'}\Gamma_{ij}{}^k + \Phi_{ij} & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}, \quad i, j = 1, \ldots, n, \quad i' = i + n.$$

A criterium to characterize deformed Riemannian extensions amongst Walker manifolds was shown in [1], where it is stated that if $(M, g, \mathcal{D})$ is a Walker manifold then g is a deformed Riemannian extension of an affine connection if and only if the curvature operator associated to g is such that

$$R(\cdot, \mathcal{D})\,\mathcal{D} = 0.$$

Moreover, the scalar curvature of every deformed Riemannian extension vanishes and its Ricci operator is nilpotent, so a deformed Riemannian extension will be Einstein if and only if it is Ricci-flat.

Modified Riemannian extensions

Going further in the generalization of Riemannian extensions we find a new kind of metrics known as modified Riemannian extensions. These metrics involve two new elements S and T which are $(1,1)$-tensor fields on the affine manifold M. Modified Riemannian extensions are split-signature metrics on the cotangent bundle of M as well and are defined by

$$\begin{aligned} g_{D,\Phi,T,S} &= \iota T \circ \iota S + g_{D,\Phi} \\ &= \iota T \circ \iota S + g_D + \pi^*\Phi, \end{aligned}$$

where $\circ$ denotes the symmetric product of one-forms. This kind of metrics can be expressed in terms of the local coordinates induced on the cotangent bundle as

$$g_{D,\Phi,T,S} = \begin{pmatrix} \frac{1}{2}x_{r'}x_{s'}\left(T_i{}^r S_j{}^s + T_j{}^r S_i{}^s\right) - 2x_{k'}\Gamma_{ij}{}^k + \Phi_{ij} & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix},$$

where $i, j = 1, \ldots, n$ and $i' = i + n$. In the particular case where $T = c\,\mathrm{Id}$ and $S = \mathrm{Id}$, the metric is denoted by $g_{D,\Phi,c}$ and it is a Walker metric for which the tensor B_{ij} in (1.8) is a quadratic function of the fibre coordinates $(x_{i'})$. The Walker distribution is given by $\mathcal{D} = \ker \pi_*$ and its scalar curvature is a multiple (depending on the dimension of the manifold) of the parameter c.

Modified Riemannian extensions can be characterized amongst the Walker metrics in terms of the covariant derivative of their curvature by

$$\nabla_{\mathcal{D}} R(\cdot, \mathcal{D})\,\mathcal{D} = 0$$

(see [1]). As a consequence, an even-dimensional Walker manifold admitting a parallel, degenerate distribution of maximal dimension is locally symmetric if and only if it is a suitable modified Riemannian extension. In addition to all this, modified Riemannian extensions turn out to be a source of examples of Einstein metrics. In fact, the modified Riemannian extension $g_{D,\Phi,c}$, with $c \neq 0$, is Einstein if and only if $\Phi = \frac{4}{c(n-1)}\,{}^D\rho_s$.

If (M, D) is flat, its cotangent bundle is not only Einstein but also para-Kähler with constant para-holomorphic sectional curvature, when equipped with a modified Riemannian extension [41]. We will see this in detail in Part I.

1.4 Lie groups

A *Lie group* is a smooth manifold endowed with a group structure such that the operation in the group $\sigma\colon G \times G \to G$ and the inversion are differentiable. A group homomorphism that is differentiable as a map between two smooth manifolds is called a *Lie group homomorphism*. A *real Lie algebra* is a vector space $\mathfrak{g}$ endowed with a skew-symmetric bilinear operator

$$[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

for all $x, y, z \in \mathfrak{g}$. A *Lie algebra homomorphism* is a linear map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ that preserves the Lie brackets, i.e., such that $\varphi[x,y]_{\mathfrak{g}} = [\varphi(x), \varphi(y)]_{\mathfrak{h}}$ for all $x, y \in \mathfrak{g}$. Given a Lie algebra $\mathfrak{g}$, a vector subspace $\mathfrak{h}$ is a *Lie subalgebra* of $\mathfrak{g}$ if $\mathfrak{h}$ is closed for the Lie brackets, i.e., if $[x,y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$. An *ideal* $\mathfrak{a}$ of $\mathfrak{g}$ is a Lie subalgebra satisfying the stronger condition that $[x,a] \in \mathfrak{a}$ for all $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$. A particular example of an ideal is the *derived subalgebra* of a given Lie algebra $\mathfrak{g}$, which is the subspace $\mathfrak{g}' = \operatorname{span}\{[x,y]\colon x, y \in \mathfrak{g}\}$ and is also denoted by $[\mathfrak{g},\mathfrak{g}]$. A Lie algebra such that the sequence of subalgebras given by

$$\mathfrak{g} \geq [\mathfrak{g},\mathfrak{g}] \geq [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] \geq [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]] \geq \dots$$

terminates in the zero subalgebra is said to be *nilpotent*. The Lie algebras whose derived subalgebras are nilpotent are called *solvable*, and those which have no non-zero solvable ideals are said to be *semi-simple*.

Given a Lie group G, a vector field X on G is said to be *left-invariant* if for all $g, h \in G$,

$$(L_g)_{*h} X_h = X_{L_g(h)},$$

where $L_g\colon h \in G \to gh \in G$ denotes the left translations on G.

The Lie brackets induce a Lie algebra structure in the space of left-invariant vector fields on a Lie group G, which is called the *Lie algebra of* G and is denoted by $Lie(G) = \mathfrak{g}$. The Lie algebra of a Lie group is isomorphic to the tangent space at the identity element of G. Besides, any Lie algebra of finite dimension $\mathfrak{g}$ is isomorphic to the Lie algebra of a unique connected and simply connected Lie group – up to isomorphisms. This allows us to identify each simply connected Lie group with its Lie algebra.

In the situation where the Lie group is also a pseudo-Riemannian manifold, the need to study metrics for which L_g is an isometry – known as *left-invariant metrics* – arises naturally. Giving a left-invariant metric on a Lie group G is equivalent to giving an inner product on its Lie algebra $\mathfrak{g}$. The invariance of the metric allows us to obtain general expressions for the Levi-Civita connection, curvature, etc. of a pseudo-Riemannian Lie group just by knowing how the brackets and the metric behave.

Semi-direct products

Given two Lie algebras $(\mathfrak{h}_1, [\cdot,\cdot]_1)$ and $(\mathfrak{h}_2, [\cdot,\cdot]_2)$, if we consider the Lie algebra $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ given by the direct sum of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ as vector spaces, we can take the Lie brackets $[v, v'] = [v, v']_1$, $[w, w'] = [w, w']_2$ and $[v, w] = 0$ for all $v, v' \in \mathfrak{h}_1$ and $w, w' \in \mathfrak{h}_2$. Then $(\mathfrak{g}, [\cdot,\cdot])$ is the direct sum Lie algebra.

Let $\mathfrak{g}, \mathfrak{h}_1$ and $\mathfrak{h}_2$ be a Lie algebra, an ideal of $\mathfrak{g}$ and a Lie subalgebra of $\mathfrak{g}$, respectively, such that $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. If $w \in \mathfrak{h}_2$, and it follows from the Jacobi identity that $\mathfrak{D} = \mathrm{ad}(w) = [w, \cdot]$ is a derivation of the Lie algebra $\mathfrak{h}_1$, i.e.,

$$\mathfrak{D}[u, v] = [\mathfrak{D}u, v] + [u, \mathfrak{D}v]$$

for all $u, v \in \mathfrak{h}_1$. In this way, $\mathrm{ad}\colon \mathfrak{h}_2 \to \mathrm{Der}(\mathfrak{h}_1)$ provides a Lie algebra homomorphism between $\mathfrak{h}_2$ and the Lie algebra of the derivations of $\mathfrak{h}_1$, where the Lie brackets are given by the commutator. In fact, this homomorphism determines all the Lie brackets between the elements of $\mathfrak{h}_2$ and $\mathfrak{h}_1$, since $[w, v] = \mathrm{ad}(w)v$ for all $v \in \mathfrak{h}_1$ and $w \in \mathfrak{h}_2$.

The construction given above can be extended to an arbitrary Lie algebra homomorphism $\varphi\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$. It can be checked that there exists a unique Lie algebra structure in the vector space $\mathfrak{g} \oplus \mathfrak{h}$ such that

$$[v, v'] = [v, v']_{\mathfrak{g}}, \quad [w, w'] = [w, w']_{\mathfrak{h}}, \quad [w, v] = \varphi(w)v,$$

for all $v, v' \in \mathfrak{g}$ and $w, w' \in \mathfrak{h}$. With these brackets, $\mathfrak{g} \oplus \mathfrak{h}$ is known as the *semi-direct product* of $\mathfrak{g}$ and $\mathfrak{h}$ via φ and is denoted by either $\mathfrak{h} \ltimes_\varphi \mathfrak{g}$ or $\mathfrak{g} \rtimes_\varphi \mathfrak{h}$.

At a Lie group level, given the product of two Lie groups $G \times H$, its Lie algebra is given by $Lie(G \times H) = \mathfrak{g} \oplus \mathfrak{h}$ were $\mathfrak{g}$ and $\mathfrak{h}$ are ideals of their direct sum. Conversely, given a Lie group G and two subgroups H_1 and H_2 such that $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, then $G = H_1 \times H_2$ as Lie groups.

In the same way as in the Lie algebra situation, this notion can be extended to semi-direct products of Lie groups. Given two Lie groups G and H, G acts on H by homomorphisms if there exists a differentiable action $\sigma\colon G \times H \to H$ such that $\sigma(g, \cdot) \in \mathrm{Aut}(H)$ for all $g \in G$ – equivalently, there exists a Lie group homomorphism $\overline{\sigma}\colon G \to \mathrm{Aut}(H)$. The multiplication in $G \times H$ given by

$$(g_1, h_1) \cdot (g_2, h_2) = \left(g_1\, g_2, \sigma(g_2^{-1}, h_1)h_2\right), \quad \text{where} \quad g_1, g_2 \in G,\ h_1, h_2 \in H,$$

induces a Lie group structure in $G \times H$, known as the *semi-direct product* of G and H via σ and denoted by either $G \ltimes_\sigma H$ or $H \rtimes_\sigma G$. When the action σ is trivial, the structure corresponds to a direct product of Lie groups.

The Lie algebra of a semi-direct product of Lie groups $H \rtimes_\sigma G$ is the semi-direct product of their Lie algebras via the derivative of $\overline{\sigma}(g, \cdot)$, $\mathfrak{h} \rtimes_{d\overline{\sigma}} \mathfrak{g}$. Conversely, given a semi-direct product of Lie algebras $\mathfrak{h} \rtimes_\varphi \mathfrak{g}$, let G and H be the connected and simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. There is a unique action by homomorphisms Φ from G on H such that $d\Phi = \varphi$ and $H \rtimes_\varphi G$ is the unique connected and simply connected Lie group with Lie algebra $\mathfrak{h} \rtimes_\varphi \mathfrak{g}$. This gives a correspondence between semi-direct products of Lie groups and semi-direct products of Lie algebras.

1.4.1 Three-dimensional Lie groups

In dimension three we can make use of the Euclidean cross product operation to simplify the computations related to the Lie brackets and, in fact, classify the unimodular Lie groups (see [104]).

Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a Lie algebra endowed with a non-degenerate inner product and consider $\times\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ the only cross product such that $\langle e_i\times e_j, e_k\rangle = \det(e_i, e_j, e_k)$ for any orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$. We can define a linear map $L\colon \mathfrak{g}\to\mathfrak{g}$ such that

$$L(e_1\times e_2) = [e_1, e_2], \quad L(e_2\times e_3) = [e_2, e_3], \quad L(e_1\times e_3) = [e_1, e_3]$$

with respect to an orthonormal basis of the Lie algebra. Therefore, L is such that $L(u\times v) = [u, v]$ for all vectors $u, v\in\mathfrak{g}$ and so it provides us with all the Lie brackets of the Lie algebra. L is known as *structure operator* of $\mathfrak{g}$.

Unimodular Lie groups are characterized in terms of their adjoint automorphism as those connected Lie groups G such that $\mathrm{ad}(x)\colon \mathfrak{g}\to\mathfrak{g}$ is trace-free for all vectors $x\in\mathfrak{g} = Lie(G)$ (see [104]). Consequently, a Lie algebra $\mathfrak{g}$ such that $\mathrm{tr}\,\mathrm{ad}(x) = 0$ for all $x\in\mathfrak{g}$ is said to be a *unimodular Lie algebra*.

Both in the Riemannian and the Lorentzian situations, writing $L(e_i) = c_i{}^j e_j$ and tracing $\mathrm{ad}(e_i)$ in terms of the $c_i{}^j$'s one can see that G is unimodular if and only if L is self-adjoint. Therefore, it is possible to study unimodular Lie algebras through the study of these self-adjoint operators.

Riemannian unimodular Lie groups

Considering a three-dimensional Riemannian Lie algebra $\mathfrak{g}$, one has that

$$e_1\times e_2 = e_3, \quad e_3\times e_1 = e_2, \quad e_2\times e_3 = e_1,$$

and so

$$L(e_1) = [e_2, e_3], \quad L(e_2) = [e_3, e_1], \quad L(e_3) = [e_1, e_2].$$

If $\mathfrak{g}$ is a unimodular Lie algebra, then L is diagonalizable with respect to an orthonormal basis, i.e., if λ_1, λ_2 and λ_3 are the eigenvalues of L, there exists an orthonormal basis of eigenvectors with respect to which

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.$$

Unimodular Lie groups can therefore be described in terms of the signs of the eigenvalues of L as (see [104])

Signs of the eigenvalues λ_1, λ_2, λ_3	Associated Lie group
$+\ +\ +$	$SU(2)$
$+\ +\ -$	$\widetilde{SL}(2,\mathbb{R})$
$+\ +\ 0$	$\widetilde{E}(2)$
$+\ -\ 0$	$E(1,1)$
$+\ 0\ 0$	H^3
$0\ 0\ 0$	$\mathbb{R}^3$

The descriptions of the Lie groups that appear in the table are the following.

- $SU(2)$ is the group of unitary 2×2 matrices of determinant 1. Its Lie algebra $\mathfrak{su}(2)$ is the algebra of skew-Hermitian 2×2 matrices with no trace.
- $\widetilde{SL}(2,\mathbb{R})$ is the universal covering of the group of the 2×2 real matrices of determinant 1. Its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is the algebra of 2×2 real matrices with no trace.
- $\widetilde{E}(2)$ is the universal covering of the group of rigid motions of the Euclidean plane. Its Lie algebra $\mathfrak{e}(2)$, known as the *Euclidean algebra*, consists of the semi-direct product $\mathfrak{r}^2 \rtimes \mathfrak{r}$ determined by an endomorphisms of $\mathfrak{r}^2$ with imaginary eigenvalues.
- $E(1,1)$ is the group of rigid motions of the Minkowskian plane. Its Lie algebra $\mathfrak{e}(1,1)$, known as the *Poincaré algebra*, is the semi-direct product $\mathfrak{r}^2 \rtimes \mathfrak{r}$ given by an endomorphism of $\mathfrak{r}^2$ with real and not-equal eigenvalues.
- H^3 is the Heisenberg group of order three, which consists of the 3×3 matrices of the form

  $$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad a, b, c \in \mathbb{R}.$$

  Its Lie algebra $\mathfrak{h}^3$ is the algebra of upper triangular matrices with zero diagonal.
- $\mathbb{R}^3$ is the Abelian group, whose Lie algebra is the Abelian algebra $\mathfrak{r}^3$.

Remark 1.6. With the intention of making the reading of this thesis easier, and since Chapter 4 is the only chapter in which we work with Lorentzian signature, the description of Lorentzian unimodular Lie groups will be given there. The reader can find it in Sections 4.1, 4.2 and 4.3.

Riemannian non-unimodular Lie groups

If a three-dimensional Riemannian Lie group is not unimodular, then there exists a basis of its Lie algebra with respect to which the Lie brackets take the form (see [104])

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \qquad [e_1, e_3] = \gamma e_2 + \delta e_3,$$

with $[e_2, e_3] = 0$, and so that the matrix associated to $\mathrm{ad}(e_1)$

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

has trace $\alpha + \delta = 2$ and $\alpha\gamma + \beta\delta = 0$. If we exclude the case where $A = \mathrm{Id}$ – in which case the group corresponds to a unimodular one –, the determinant $D = \alpha\delta - \beta\gamma$ provides a complete isomorphism invariant for this Lie algebra.

1.4.2 Four-dimensional Lie groups

Any real Lie algebra can be written as the semi-direct product of its radical – its maximal solvable ideal – and a semi-simple subalgebra known as the *Levi factor* (see [90]). Therefore, the classification of four-dimensional Lie algebras can be reduced to the study of low-dimensional semi-simple Lie algebras, solvable Lie algebras and semi-direct products of semi-simple and solvable Lie algebras. Semi-simple Lie algebras further decompose into direct sums of simple subalgebras – non Abelian Lie algebras without any non-zero proper ideals – which are orthogonal with respect to the Killing-Cartan form. Solvable Lie algebras have been classified in low dimensions – up to six – and we refer the reader to [6] for a description of four-dimensional solvable Lie algebras.

Simply connected four-dimension Lie groups are either isomorphic to the direct extensions $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ or solvable Lie groups, which correspond to the semi-direct extensions of the Poincaré, Euclidean, Abelian and Heisenberg groups. Since solvable Lie groups play a special role in the study of left-invariant symplectic structures, we will outline some results on the classification of their corresponding Lie algebras.

Solvable Lie groups

Consider a four-dimensional Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{v}$ of codimension one in $\mathfrak{g}$. Taking $e_0 \in \mathfrak{g} \setminus \mathfrak{v}$, we can write $\mathfrak{g} = \mathbb{R} \ltimes_\varphi \mathfrak{v}$, where $\varphi\colon \mathbb{R}e_0 \to \mathrm{Der}\mathfrak{v}$ is a linear map such that $\varphi(e_0) = \mathrm{ad}(e_0)$.

The following result, which can be found in [6, Proposition 1.3] (see [55] for a different proof), proves that any four-dimensional solvable real Lie algebra is a semi-direct product of $\mathbb{R}$ and a three-dimensional unimodular Lie algebra. Therefore, the classification of four-dimensional solvable Lie algebras is reduced to the study of the derivations of three-dimensional unimodular Lie algebras.

Proposition 1.7. *Let $\mathfrak{g}$ be a four-dimensional solvable real Lie algebra. Then*

$$\mathfrak{g} \cong \mathbb{R} \ltimes_\varphi \mathfrak{v},$$

where $\mathfrak{v}$ is isomorphic to either $\mathfrak{r}^3$, $\mathfrak{h}_3$, $\mathfrak{e}(1,1)$ or $\mathfrak{e}(2)$.

Proof. Consider the Lie algebra homomorphism given by

$$\begin{array}{rcl} \chi\colon \mathfrak{g} & \to & \mathbb{R} \\ x & \mapsto & \chi(x) = \mathrm{tr}(\mathrm{ad}(x)). \end{array}$$

Its kernel $\mathfrak{u} = \ker\chi$, known as the *unimodular kernel*, is an ideal of the Lie algebra $\mathfrak{g}$ containing the derived subalgebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$.

If $\mathfrak{g}$ is not unimodular, then its unimodular kernel $\mathfrak{u} = \ker\chi$ is three-dimensional, and therefore it is isomorphic to either $\mathbb{R}^3$, $\mathfrak{h}_3$, $\mathfrak{e}(1,1)$ or $\mathfrak{e}(2)$ and the result follows by taking $\mathfrak{v} = \mathfrak{u}$.

If $\mathfrak{g}$ is unimodular, its derived subalgebra $\mathfrak{g}'$ is nilpotent and its dimension is lower than or equal to three. Therefore, $\mathfrak{g}'$ is isomorphic to either $\{0\}$, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathfrak{h}_3$.

- If $\mathfrak{g}' \cong \mathbb{R}^3$ or $\mathfrak{g}' \cong \mathfrak{h}_3$, the result immediately follows by taking $\mathfrak{v} = \mathfrak{g}'$.
- If $\mathfrak{g} \cong \{0\}$, $\mathfrak{g}$ is Abelian and so $\mathfrak{v} = \mathbb{R}^3$ is an ideal of $\mathfrak{g}$.
- If $\mathfrak{g}' \cong \mathbb{R}$, take $\mathfrak{g}' = \mathbb{R}e_3$ and there exist $e_1, e_2 \in \mathfrak{g}$ such that $[e_1, e_2] = e_3$ and, since $\mathfrak{g}$ is unimodular, the set $\{e_1, e_2, e_3\}$ is linearly independent, and so it generates an ideal which is isomorphic to $\mathfrak{h}_3$.
- If $\mathfrak{g}' \cong \mathbb{R}^2$, there are two different possibilities:

  Case (i): There exists $x \notin \mathfrak{g}'$ such that $\mathrm{ad}(x)|_{\mathfrak{g}'}$ is non-singular.

  Case (ii): $\mathrm{ad}(x)|_{\mathfrak{g}'}$ is singular for all $x \in \mathfrak{g}$.

  Making use of the real Jordan normal form of the corresponding complex transformation in both cases we obtain that

  $$\chi(x) = \mathrm{tr}(\mathrm{ad}(x)) = \lambda_1 + \lambda_2 \quad \text{with} \quad \lambda_i \in \mathbb{C},\ i = 1, 2.$$

  In Case (i), there is a basis of $\mathfrak{g}'$ such that the action of x is given – up to a non-zero multiple – by

  $$\text{(i.a)} \quad \mathrm{ad}(x)|_{\mathfrak{g}'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \text{(i.b)} \quad \mathrm{ad}(x)|_{\mathfrak{g}'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

  where Case (i.b) corresponds to the complex eigenvalues $\pm i$. Thus, $\mathbb{R}x \oplus \mathfrak{g}'$ is an ideal of $\mathfrak{g}$ that is isomorphic to $\mathfrak{e}(1, 1)$ – in Case (i.a) – and $\mathfrak{e}(2)$ – in Case (i.b).

  In Case (ii), since one of λ_1 and λ_2 vanishes, the unimodular condition implies that both of them have to. Therefore, for any fixed $x \notin \mathfrak{g}'$ there is a basis of $\mathfrak{g}'$ with respect to which the action of $\mathrm{ad}(x)|_{\mathfrak{g}'}$ takes one of the following forms.

  $$\text{(ii.a)} \quad \mathrm{ad}(x)|_{\mathfrak{g}'} = 0 \quad \text{or} \quad \text{(ii.b)} \quad \mathrm{ad}(x)|_{\mathfrak{g}'} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

  As a consequence, $\mathbb{R}x \oplus \mathfrak{g}'$ is an ideal of $\mathfrak{g}$ that is isomorphic to $\mathbb{R}^3$ in Case (ii.a) and to $\mathfrak{h}_3$ in Case (ii.b). This completes the proof.

□

Remark 1.8 (The Poincaré Lie algebra $\mathfrak{e}(1, 1)$). The Poincaré Lie group can be equivalently described as

(i) the semi-direct product $\mathbb{R}^2 \rtimes_{\psi_1} \mathbb{R}e_1$ with

$$\psi_1 = \mathrm{ad}(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

in which case its non-zero Lie brackets are given by

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3,$$

(ii) or the semi-direct product $\mathbb{R}^2 \rtimes_{\psi_2} \mathbb{R}e_3$ with

$$\psi_2 = \mathrm{ad}(e_3) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

in which case its non-zero Lie brackets are given by

$$[e_3, e_1] = -e_2, \quad [e_3, e_2] = -e_1.$$

Let us consider the description of the Poincaré group given by (i). The derivations of this Lie algebra are

$$\mathrm{Der}(\mathfrak{e}(1,1)) = \left\{ \varphi = \begin{pmatrix} 0 & 0 & 0 \\ c & a & 0 \\ d & 0 & b \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

and so every semi-direct extension is of the form $E(1,1) \rtimes_\varphi \mathbb{R}e_0$, where φ belongs to $\mathrm{Der}(\mathfrak{e}(1,1))$.

If we now rescale the vector e_0 and take $\hat{e}_0 = e_0 - ae_1 + ce_2 - de_3$, and consider the basis of $E(1,1) \rtimes_\varphi \mathbb{R}\hat{e}_0$ given by $\{e_1, e_2, e_3, \hat{e}_0\}$, then

$$[\hat{e}_0, e_1] = 0, \quad [\hat{e}_0, e_2] = 0, \quad [\hat{e}_0, e_3] = (a+b)e_3,$$

so there are two different possibilities depending on whether or not $a + b = 0$.

If $a + b = 0$, then $E(1,1) \rtimes_\varphi \mathbb{R}\hat{e}_0$ is unimodular and its Lie brackets are

$$[\hat{e}_0, e_i] = 0, \quad [e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3.$$

This corresponds to the product Lie group $E(1,1) \times \mathbb{R}\hat{e}_0$.

If $a + b \neq 0$, then $E(1,1) \rtimes_\varphi \mathbb{R}e_0$ is non-unimodular and its non-zero Lie brackets are

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3, \quad [e_3, \hat{e}_0] = \lambda e_3, \quad (\lambda \neq 0).$$

If we now consider the Lie group homomorphism Φ given by

$$\Phi(e_1) = -\tfrac{1}{\lambda}\hat{e}_0, \quad \Phi(e_2) = e_3, \quad \Phi(e_3) = e_1 - \tfrac{1}{\lambda}\hat{e}_0, \quad \Phi(\hat{e}_0) = \lambda e_2$$

and take the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, $\mathbf{v}_i = \Phi(e_i)$, of $E(1,1) \rtimes_\varphi \mathbb{R}\hat{e}_0$, then the Lie brackets are

$$[\mathbf{v}_1, \mathbf{v}_2] = \mathbf{v}_2, \qquad [\mathbf{v}_3, \mathbf{v}_4] = \mathbf{v}_4$$

and this Lie algebra corresponds to $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, where $\mathfrak{aff}(\mathbb{R})$ denotes the real affine Lie algebra.

The discussion in the previous remark can be summarized as follows.

Lemma 1.9. *Let $G = E(1,1) \rtimes \mathbb{R}$ be a semi-direct extension of the Poincaré Lie group. Then*

(i) G is unimodular if and only if it is isomorphic to the product $E(1,1) \times \mathbb{R}$, and

(ii) G is non-unimodular if and only if it is isomorphic to $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$.

Remark 1.10 (The Euclidean Lie algebra $\mathfrak{e}(2)$). The Euclidean Lie group can be described as the semi-direct product $\mathbb{R}^2 \rtimes_\psi \mathbb{R}$ given by

$$\psi = \mathrm{ad}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so its non-zero Lie brackets are given by

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = -e_2.$$

The derivations of this Lie algebra are

$$\mathrm{Der}(\mathfrak{e}(2)) = \left\{ \varphi = \begin{pmatrix} 0 & 0 & 0 \\ c & a & -b \\ d & b & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

and so every semi-direct extension is of the form $\widetilde{E}(2) \rtimes_\varphi \mathbb{R}e_0$, where $\varphi \in \mathrm{Der}(\mathfrak{e}(2))$. We can rescale e_0 by $\hat{e}_0 = e_0 - be_1 + de_2 - ce_3$ and consider the basis of $\widetilde{E}(2) \times \mathbb{R}\hat{e}_0$ given by $\{e_1, e_2, e_3, \hat{e}_0\}$ so that

$$[\hat{e}_0, e_1] = 0, \quad [\hat{e}_0, e_2] = ae_2, \quad [\hat{e}_0, e_3] = ae_3.$$

This gives rise to two different possibilities depending on whether or not $a = 0$.

If $a = 0$, then $\widetilde{E}(2) \rtimes_\varphi \mathbb{R}\hat{e}_0$ is unimodular and its Lie brackets are

$$[\hat{e}_0, e_i] = 0, \quad [e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2.$$

This corresponds to the product Lie group $\widetilde{E}(2) \times \mathbb{R}\hat{e}_0$.

If $a \neq 0$, then the semi-direct product $\widetilde{E}(2) \rtimes_\varphi \mathbb{R}\hat{e}_0$ is non-unimodular and its Lie algebra is isomorphic to

$$[\hat{e}_0, e_2] = e_2, \quad [\hat{e}_0, e_3] = e_3, \quad [e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2,$$

which corresponds to the complex affine Lie algebra $\mathfrak{aff}(\mathbb{C})$.

The discussion above can be summarized as follows.

Lemma 1.11. *Let $G = \widetilde{E}(2) \rtimes \mathbb{R}$ be a semi-direct extension of the Euclidean Lie group. Then*

(i) G is unimodular if and only if it is isomorphic to the product $\widetilde{E}(2) \times \mathbb{R}$, and

(ii) G is non-unimodular if and only if it is isomorphic to $\mathfrak{aff}(\mathbb{C})$.

The left-invariant metrics on each four-dimensional solvable Riemannian Lie group can be described in terms of an orthonormal basis as follows.

Left-invariant Riemannian metrics on $H^3 \rtimes \mathbb{R}$

Let $\mathfrak{g} = \mathfrak{h}^3 \rtimes \mathbb{R}$ be the semi-direct extension of the Heisenberg algebra $\mathfrak{h}^3$. Let $\langle \cdot, \cdot \rangle$ be an inner product in $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle_3$ be its restriction to $\mathfrak{h}^3$. It follows from Milnor's work [104] that there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{h}^3$ such that

$$[\mathbf{v}_1, \mathbf{v}_2] = \gamma \mathbf{v}_3, \quad [\mathbf{v}_1, \mathbf{v}_3] = 0, \quad [\mathbf{v}_2, \mathbf{v}_3] = 0, \quad \gamma \neq 0. \tag{1.11}$$

The algebra of derivations of $\mathfrak{h}^3$ with respect to a rotated basis that we will also denote by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is given by

$$\mathrm{Der}(\mathfrak{h}^3) = \left\{ \begin{pmatrix} \tilde{a} & \tilde{c} & 0 \\ -\tilde{c} & \tilde{d} & 0 \\ \tilde{h} & \tilde{f} & \tilde{a} + \tilde{d} \end{pmatrix} : \tilde{a}, \tilde{c}, \tilde{d}, \tilde{h}, \tilde{f} \in \mathbb{R} \right\}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ with $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ given by Equation (1.11), and $\mathfrak{g} = \mathfrak{h}^3 \oplus \mathbb{R}\mathbf{v}_4$. Since $\mathbb{R}\mathbf{v}_4$ need not be orthogonal to $\mathfrak{h}^3$, we set $\tilde{k}_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$, and consider $\hat{e}_4 = \mathbf{v}_4 - \sum_i \tilde{k}_i \mathbf{v}_i$. If we now normalize it we get an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that

$$\begin{aligned} &[e_1, e_2] = \gamma e_3, && [e_4, e_1] = \tfrac{1}{R}\{\tilde{a} e_1 - \tilde{c} e_2 + (\tilde{h} + \tilde{k}_2 \gamma) e_3\}, \\ &[e_4, e_3] = \tfrac{1}{R}(\tilde{a} + \tilde{d}) e_3, && [e_4, e_2] = \tfrac{1}{R}\{\tilde{c} e_1 + \tilde{d} e_2 + (\tilde{f} - \tilde{k}_1 \gamma) e_3\}, \end{aligned} \tag{1.12}$$

where $R > 0$. In order to simplify the expressions we define

$$a = -\tfrac{\tilde{a}}{R}, \quad c = -\tfrac{\tilde{c}}{R}, \quad d = -\tfrac{\tilde{d}}{R}, \quad h = -\tfrac{\tilde{h}}{R}, \quad f = -\tfrac{\tilde{f}}{R},$$

$$k_1 = -\tfrac{\tilde{k}_1}{R}, \quad k_2 = -\tfrac{\tilde{k}_2}{R}$$

and use the notation $F = f - k_1 \gamma$ and $H = h + k_2 \gamma$. Now the Lie brackets in Equation (1.12) become

$$\begin{aligned} &[e_1, e_2] = \gamma e_3, && [e_1, e_4] = a e_1 - c e_2 + H e_3, \\ &[e_3, e_4] = (a + d) e_3, && [e_2, e_4] = c e_1 + d e_2 + F e_3. \end{aligned} \tag{1.13}$$

Left-invariant Riemannian metrics on $E(1,1) \rtimes \mathbb{R}$ and $\widetilde{E}(2) \rtimes \mathbb{R}$

Let $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathbb{R}$ be the semi-direct extension of the three-dimensional Lie algebra $\mathfrak{g}_3$, being $\mathfrak{g}_3$ either $\mathfrak{e}(1,1)$ or $\mathfrak{e}(2)$. Let $\langle \cdot, \cdot \rangle$ be an inner product in $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle_3$ denote its restriction to $\mathfrak{g}_3$. According to Milnor's work [104], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \quad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \quad [\mathbf{v}_1, \mathbf{v}_2] = 0, \tag{1.14}$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 \neq 0$. The associated Lie groups correspond to $E(2)$ whenever $\lambda_1\lambda_2 > 0$ and $E(1,1)$ whenever $\lambda_1\lambda_2 < 0$. Moreover, the algebra of derivations of $\mathfrak{g}_3$ is given by

$$\mathrm{Der}(\mathfrak{g}_3) = \left\{ \begin{pmatrix} \tilde{b} & \tilde{a} & \tilde{c} \\ -\frac{\lambda_2}{\lambda_1}\tilde{a} & \tilde{b} & \tilde{d} \\ 0 & 0 & 0 \end{pmatrix} : \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R} \right\}.$$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ for which $\mathrm{ad}(\mathbf{v}_4)$ is determined by a derivation as above. After a normalization, like in the previous section, we get an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for which the non-zero Lie brackets are given by

$$\begin{aligned} &[e_2, e_3] = \lambda_1 e_1, \qquad\qquad [e_3, e_1] = \lambda_2 e_2, \\ &[e_4, e_1] = \tfrac{1}{R}\{\tilde{b} e_1 - \lambda_2(\tfrac{\tilde{a}}{\lambda_1} + \tilde{k}_3) e_2\}, \\ &[e_4, e_2] = \tfrac{1}{R}\{(\tilde{a} + \tilde{k}_3\lambda_1) e_1 + \tilde{b} e_2\}, \\ &[e_4, e_3] = \tfrac{1}{R}\{(\tilde{c} - \tilde{k}_2\lambda_1) e_1 + (\tilde{d} + \tilde{k}_1\lambda_2) e_2\}, \end{aligned} \tag{1.15}$$

where $R > 0$. In order to simplify the notation, we define

$$a = -\tfrac{\tilde{a}}{R},\ b = -\tfrac{\tilde{b}}{R},\ c = -\tfrac{\tilde{c}}{R},\ d = -\tfrac{\tilde{d}}{R},\ k_1 = -\tfrac{\tilde{k}_1}{R},\ k_2 = -\tfrac{\tilde{k}_2}{R},\ k_3 = -\tfrac{\tilde{k}_3}{R}$$

and set $A = \frac{a}{\lambda_1} + k_3$, $C = c - k_2\lambda_1$ and $D = d + k_1\lambda_2$. Now the Lie brackets given in Equation (1.15) become

$$\begin{aligned} &[e_1, e_3] = -\lambda_2 e_2, && [e_2, e_3] = \lambda_1 e_1, \\ &[e_1, e_4] = b e_1 - A\lambda_2 e_2, && [e_2, e_4] = A\lambda_1 e_1 + b e_2, \\ &[e_3, e_4] = C e_1 + D e_2. \end{aligned} \tag{1.16}$$

Left-invariant Riemannian metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$

Let $\mathfrak{g} = \mathfrak{r}^3 \rtimes \mathbb{R}$ be a semi-direct extension of the Abelian Lie algebra $\mathfrak{r}^3$. Let $\langle\cdot,\cdot\rangle$ be a inner product on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_3$ be its restriction to $\mathfrak{r}^3$. The algebra of all the derivations of $\mathfrak{r}^3$ is $\mathfrak{gl}(3,\mathbb{R})$ and there exists a $\langle\cdot,\cdot\rangle_3$-orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{r}^3$ where a derivation decomposes as a sum of a diagonal matrix and a skew-symmetric matrix. Therefore, the algebra of derivations of $\mathfrak{r}^3$ is given by

$$\mathrm{Der}\left(\mathfrak{r}^3\right) = \left\{ \begin{pmatrix} \tilde{a} & -\tilde{b} & -\tilde{c} \\ \tilde{b} & \tilde{f} & -\tilde{h} \\ \tilde{c} & \tilde{h} & \tilde{p} \end{pmatrix} : \tilde{a}, \tilde{b}, \tilde{c}, \tilde{f}, \tilde{h}, \tilde{p} \in \mathbb{R} \right\}.$$

Now, the corresponding semi-direct product $\mathfrak{g} = \mathfrak{r}^3 \rtimes \mathbb{R}$ is given by

$$\begin{aligned} &[\mathbf{v}_1, \mathbf{v}_2] = 0, && [\mathbf{v}_4, \mathbf{v}_1] = \tilde{a}\mathbf{v}_1 + \tilde{b}\mathbf{v}_2 + \tilde{c}\mathbf{v}_3, \\ &[\mathbf{v}_1, \mathbf{v}_3] = 0, && [\mathbf{v}_4, \mathbf{v}_2] = -\tilde{b}\mathbf{v}_1 + \tilde{f}\mathbf{v}_2 + \tilde{h}\mathbf{v}_3, \\ &[\mathbf{v}_2, \mathbf{v}_3] = 0, && [\mathbf{v}_4, \mathbf{v}_3] = -\tilde{c}\mathbf{v}_1 - \tilde{h}\mathbf{v}_2 + \tilde{p}\mathbf{v}_4, \end{aligned}$$

with respect to some basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ so that $\mathfrak{g} = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \oplus \mathbb{R}\mathbf{v}_4$. Since $\mathbb{R}\mathbf{v}_4$ need not be orthogonal to $\mathfrak{r}^3$, we can consider $k_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for all $i = 1, 2, 3$, and define $\hat{e}_4 = \mathbf{v}_4 - \sum_i k_i \mathbf{v}_i$. If we normalize it, we obtain an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ so that

$$\begin{aligned} &[e_4, e_1] = \tfrac{1}{R}(\tilde{a}e_1 + \tilde{b}e_2 + \tilde{c}e_3), \qquad [e_4, e_2] = \tfrac{1}{R}(-\tilde{b}e_1 + \tilde{f}e_2 + \tilde{h}e_3), \\ &[e_4, e_3] = \tfrac{1}{R}(-\tilde{c}e_1 - \tilde{h}e_2 + \tilde{p}e_3), \quad R > 0. \end{aligned} \tag{1.17}$$

In order to simplify the notation, we define

$$a = \tfrac{\tilde{a}}{R}, \quad b = \tfrac{\tilde{b}}{R}, \quad c = \tfrac{\tilde{c}}{R}, \quad f = \tfrac{\tilde{f}}{R}, \quad h = \tfrac{\tilde{h}}{R}, \quad p = \tfrac{\tilde{p}}{R}.$$

Now the Lie brackets given in Equation (1.17) become

$$\begin{aligned} &[e_1, e_4] = ae_1 + be_2 + ce_3, \qquad [e_2, e_4] = -be_1 + fe_2 + he_3, \\ &[e_3, e_4] = -ce_1 - he_2 + pe_3. \end{aligned} \tag{1.18}$$

Non-solvable Lie groups

Four-dimensional non-solvable Lie groups are isomorphic to either one of the direct products $SU(2) \times \mathbb{R}$ and $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ and the left-invariant metrics on them can be described in terms of an orthonormal basis as follows.

Left-invariant Riemannian metrics on $SU(2) \times \mathbb{R}$ and $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$

Let $\mathfrak{g} = \mathfrak{g}_3 \times \mathbb{R}$ be a direct extension of the unimodular Lie algebra $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle_3$ denote its restriction to $\mathfrak{g}_3$. Following Milnor's work [104], there exists an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathfrak{g}_3$ such that

$$[\mathbf{v}_2, \mathbf{v}_3] = \lambda_1 \mathbf{v}_1, \quad [\mathbf{v}_3, \mathbf{v}_1] = \lambda_2 \mathbf{v}_2, \quad [\mathbf{v}_1, \mathbf{v}_2] = \lambda_3 \mathbf{v}_3, \tag{1.19}$$

where λ_1, λ_2, $\lambda_3 \in \mathbb{R}$ and $\lambda_1\lambda_2\lambda_3 \neq 0$. Moreover, the associated Lie group corresponds to $SU(2)$ if λ_1, λ_2 and λ_3 have the same sign, and to $SL(2, \mathbb{R})$ otherwise.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis of $\mathfrak{g}$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by Equation (1.19) and $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}\mathbf{v}_4$. Since $\mathbb{R}\mathbf{v}_4$ need not be orthogonal to $\mathfrak{g}_3$, we consider $\tilde{k}_i = \langle \mathbf{v}_i, \mathbf{v}_4 \rangle$, for $i = 1, 2, 3$. Let $\hat{e}_4 = \mathbf{v}_4 - \sum_i \tilde{k}_i \mathbf{v}_i$ and normalize it to get an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}$ such that

$$\begin{aligned} &[e_1, e_2] = \lambda_3 e_3, && [e_2, e_3] = \lambda_1 e_1, \\ &[e_3, e_1] = \lambda_2 e_2, && [e_1, e_4] = \tfrac{1}{R}(\tilde{k}_3\lambda_2 e_2 - \tilde{k}_2\lambda_3 e_3), \\ &[e_2, e_4] = \tfrac{1}{R}(\tilde{k}_1\lambda_3 e_3 - \tilde{k}_3\lambda_1 e_1), && [e_3, e_4] = \tfrac{1}{R}(\tilde{k}_2\lambda_1 e_1 - \tilde{k}_1\lambda_2 e_2), \end{aligned} \tag{1.20}$$

where $R > 0$. In order to simplify the expressions, we define $k_i = \frac{\tilde{k}_i}{R}$, so the Lie brackets now take the form

$$\begin{aligned} &[e_1, e_2] = \lambda_3 e_3, \qquad [e_1, e_3] = -\lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1, \\ &[e_1, e_4] = k_3\lambda_2 e_2 - k_2\lambda_3 e_3, \qquad [e_2, e_4] = k_1\lambda_3 e_3 - k_3\lambda_1 e_1, \\ &[e_3, e_4] = k_2\lambda_1 e_1 - k_1\lambda_2 e_2. \end{aligned} \tag{1.21}$$

1.5 Homogeneous spaces

Roughly speaking, in pseudo-Riemannian geometry, homogeneity means that the geometry of a manifold is the same at each of its points. What this means is that for any two points in the manifold, there exists an isometry sending one to the other. At the same time, in affine geometry, homogeneity means that for any two points there exists an affine transformation sending one point to the other. It is important to be aware of the fact that a pseudo-Riemannian manifold may be affine homogeneous for its Levi-Civita connection but not necessarily homogeneous (see [92]).

Riemannian homogeneous spaces

A connected Riemannian manifold (M, g) is said to be *homogeneous* if its isometry group acts transitively on M, i.e, if for any two points $p, q \in M$ there exists an isometry φ of (M, g) such that $\varphi(p) = q$. In this situation, the connected component of the identity of the isometry group also acts transitively on M. This definition of homogeneity is equivalent to the existence of a connected Lie group G and a smooth map

$$\begin{array}{rcl} G \times M & \longrightarrow & M \\ (q, p) & \longmapsto & L_q(p) = q\,p \end{array}$$

such that

(i) L_q is an isometry of (M, g).

(ii) $L_{q_1} L_{q_2} = L_{q_1 q_2}$.

(iii) For any $p_1, p_2 \in M$, there exists an element $q_1 \in G$ such that $L_{q_1}(p_1) = p_2$.

If G acts effectively on M, i.e., if L_q is the identity transformation of M if and only if q is the identity element $e \in G$, we can always replace G by the quotient group G/K, where K is the kernel of the map $q \in G \mapsto L_q \in \mathrm{Isom}(M)$. Therefore, if G is a connected Lie group that acts on (M, g) as a transitive and effective group of isometries, then G can be identified with a Lie subgroup of the isometry group of (M, g).

Let $p \in M$ and $H = \{q \in G \colon q\,p = p\}$ be the isotropy group of p. Then M is diffeomorphic to the quotient G/H and we have the canonical projection

$$\pi \colon G \longrightarrow G/H.$$

This gives a fibre bundle over M with structure group H, where the subgroup H is closed, but not necessarily connected. A Riemannian metric $\langle \cdot, \cdot \rangle$ on G/H is said to be G-invariant if the action

$$t_q \colon sH \in G/H \longmapsto t_q(sH) = q\,sH$$

is an isometry for all $q \in G$. In this case $(G/H, \langle \cdot, \cdot \rangle)$ is called a *Riemannian homogeneous space*. (M, g) is *locally homogeneous* if for each two points $p, q \in M$ there exist neighbourhoods $\mathcal{U}$ of p and $\mathcal{V}$ of q and a local isometry $\varphi\colon \mathcal{U} \to \mathcal{V}$ such that $\varphi(p) = q$.

Simply connected homogeneous Riemannian manifolds of dimension two are symmetric. Three-dimensional simply connected homogeneous Riemannian manifolds are either symmetric spaces or Lie groups endowed with left-invariant Riemannian metrics (see [126], [103] for a modern presentation and [30] for an extension to the three-dimensional Lorentzian case). Bérard-Bergery showed in [15] that the same result holds true in the four-dimensional Riemannian situation.

Theorem 1.12. *Let (M, g) be a four-dimensional simply connected Riemannian homogeneous manifold. Then, it is either symmetric or isometric to a Lie group endowed with a left-invariant metric.*

Let (M, g) be a connected n-dimensional Riemannian manifold. What is more, consider $M = G/H$, where G is a subgroup of the group of isometries of M acting transitively and effectively on M, and H is the isotropy group of a point $p \in M$. If we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively, then $M = G/H$ is said to be *reductive* if there exists a vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$g = \mathfrak{h} \oplus \mathfrak{m},$$

and $\mathfrak{m}$ is the $\mathrm{ad}(H)$-invariant subspace of $\mathfrak{g}$.

1.6 A note on Gröbner bases

Gröbner bases were introduced by Bruno Buchberger around the year 1960 and they have proven themselves to be extraordinarily useful in many different mathematical contexts. The aim of this section is to provide the reader with some basic knowledge on these algebraic objects, since we will be making use of them in different sections throughout this thesis as a tool to solve large system of polynomial equations.

1.6.1 Monomial order and ideals

Given a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the exponents $\alpha = (\alpha_1, \ldots, \alpha_n)$ are elements of $\mathbb{Z}^n_{\geq 0}$, which establishes a one-to-one correspondence between the monomials in the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ and $\mathbb{Z}^n_{\geq 0}$. A *monomial order* on $\mathbb{R}[x_1, \ldots, x_n]$ is a relation $>$ on $\mathbb{Z}^n_{\geq 0}$ or, equivalently, on the set of monomials x^α, satisfying the following properties.

1. $>$ is a total order on $\mathbb{Z}^n_{\geq 0}$.
2. If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.
3. $>$ is a well-order on $\mathbb{Z}^n_{\geq 0}$.

There are many different monomial orders, but we will be most interested in the following three:

- *Lexicographical order*: $\alpha >_{lex} \beta$ if the leftmost non-zero entry in the vector $\alpha - \beta \in \mathbb{Z}^n$ is positive.
- *Graded lexicographical order*: $\alpha >_{grlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ with $\alpha >_{lex} \beta$.
- *Graded reverse lexicographical order*: $\alpha >_{grevlex} \beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost non-zero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.

The lexicographical order corresponds to the alphabetical order and a variable dominates any monomial involving only smaller variables, regardless of its total degree. If we want to take into account the total degrees of the monomials so that the monomials of higher degree are the greatest, we can use the graded lexicographical order.

Let $\mathfrak{P} = \sum_\alpha a_\alpha x^\alpha$ be a non-zero polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ and let $>$ be a monomial order. The *multidegree* of $\mathfrak{P}$ is the maximum (with respect to the monomial order $>$) $\alpha \in \mathbb{Z}^n_{\geq 0}$ so that $a_\alpha \neq 0$. The corresponding monomial is called the *leading term* $LT(\mathfrak{P}) = a_\alpha x^\alpha$. A *monomial ideal* in $\mathbb{R}[x_1, \ldots, x_n]$ is a polynomial ideal that can be generated by monomials. A polynomial $\mathfrak{P}$ belongs to a monomial ideal $\mathcal{I}$ if and only if all of its terms are elements of $\mathcal{I}$. We denote by $LT(\mathcal{I})$ the set of leading terms of the non-zero elements of $\mathcal{I}$, i.e.,

$$LT(\mathcal{I}) = \{cx^\alpha \colon \exists \mathfrak{P} \in \mathcal{I} \setminus \{0\} \text{ s.t. } LT(\mathfrak{P}) = cx^\alpha\},$$

and $\langle LT(\mathcal{I})\rangle$ denotes the ideal generated by the elements of $LT(\mathcal{I})$. Notice that if $\mathfrak{P}_i \in \mathcal{I}$, for $i = 1, \ldots, k$, then $LT(\mathfrak{P}_i) \in LT(\mathcal{I}) \subset \langle LT(\mathcal{I})\rangle$ and so

$$\langle LT(\mathfrak{P}_1), \ldots, LT(\mathfrak{P}_k)\rangle \subset \langle LT(\mathcal{I})\rangle.$$

However, if $\mathcal{I} = \langle \mathfrak{P}_1, \ldots, \mathfrak{P}_k\rangle$, $\langle LT(\mathcal{I})\rangle$ might be strictly larger than $\langle LT(\mathfrak{P}_1), \ldots, LT(\mathfrak{P}_k)\rangle$. Consider, for instance, the ideal $\mathcal{I} = \langle \mathfrak{P}_1, \mathfrak{P}_2\rangle$ where

$$\mathfrak{P}_1 = y^3 - 2xy, \qquad \mathfrak{P}_2 = xy^2 - 2x^2 + y,$$

and fix the graded lexicographical order for monomials. The polynomial

$$y\mathfrak{P}_2 - x\mathfrak{P}_1 = y^2$$

belongs to $\mathcal{I}$ and $y^2 = LT(x^2) \in \langle LT(\mathcal{I})\rangle$, but

$$y^2 \notin \langle LT(\mathfrak{P}_1), LT(\mathfrak{P}_2)\rangle = \langle y^3, xy^2\rangle.$$

Theorem 1.13 (Hilbert Basis Theorem)**.** *Every ideal $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ has a finite generating set.*

The analogue result for monomial ideals is called Dickson's Lemma. The Hilbert Basis Theorem guarantees that any non-zero ideal $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_n]$ admits a Gröbner basis.

Definition 1.14. A finite subset $\mathcal{G} = \{g_1, \dots g_\nu\}$ of an ideal $\mathcal{I}$ with a fixed monomial order such that

$$\langle LT(g_1), \dots, LT(g_\nu) \rangle = \langle LT(\mathcal{I}) \rangle$$

is said to be a *Gröbner basis* (or a *Gröbner-Shirshov basis*) with respect to the given monomial order.

Gröbner bases provide us with a powerful tool to find quite simple algorithmic solutions to various algebraic problems. For example:

The Ideal Membership Problem

The remainder of the division algorithm applied to a polynomial $\mathfrak{P}$ divided by the elements of a Gröbner basis $\mathcal{G}$ of an ideal $\mathcal{I}$ is zero if and only if $\mathfrak{P}$ belongs to $\mathcal{I}$, and this property does not necessarily hold if $\mathcal{G}$ is not a Gröbner basis.

Solving large systems of polynomial equations

Let $\{\mathfrak{P}_i\}$ be a set of polynomials in $\mathbb{R}[x_1, \dots, x_n]$ and consider the system of polynomial equations given by $\{\mathfrak{P}_i = 0\}$. If the system in consideration is simple, it is an elementary problem to find all the common roots, but if the number of unknowns, equations and their degree increase, then finding all the solutions might become quite an unmanageable problem. What can one do to make the task easier?

If two sets of polynomials generate the same ideal, the corresponding zero sets must be identical. The theory of Gröbner bases provides a well-known strategy to solve rather large polynomial systems obtaining "better" polynomials that belong to the ideal generated by the initial polynomial system.

We refer to [53] for more information regarding the theory of Gröbner bases.

1.6.2 Buchberger's algorithm

Buchberger's algorithm is the oldest algorithm ever introduced for the computation of Gröbner bases. It was devised by Bruno Buchberger at the same time that he introduced Gröbner bases. A crude version of this algorithm to find a Gröbner basis of an ideal of polynomials $\mathcal{I}$ proceeds as follows:

Input: A set of polynomials P that generate $\mathcal{I}$.
Output: A Gröbner basis $\mathcal{G}$ for $\mathcal{I}$.

(1) Set $\mathcal{G} = P$.

(2) If $p_i, p_j \in P$ and we denote by f_i, f_j the coefficients of their leading terms with respect to a given monomial ordering, respectively, set $a_{ij} = \operatorname{lcm}\{f_i, f_j\}$.

(3) Define $g_{ij} = \frac{a_{ij}}{f_i} p_i - \frac{a_{ij}}{f_j} p_j$. Note that the leading terms here will cancel. The polynomials p_i and p_j are called a *critical pair*.

(4) Reduce g_{ij} as much as possible using the multivariate division algorithm with respect to $\mathcal{G}$. If the result is non-zero, then add g_{ij} to $\mathcal{G}$.

(5) Repeat (2)–(4) until all possible pairs have been considered, including those involving the new polynomials added to $\mathcal{G}$ in step (4).

There are many ways in which the algorithm above can be improved and, in fact, has been improved (see, for instance, [64, 65]).

Part I

Locally conformally flat structures

In this part we will devote ourselves to the study of Kähler, para-Kähler and null-Kähler structures. In Chapter 2 we will study the geometry of locally conformally flat four-dimensional structures and in Chapter 3 we will give the classification of four-dimensional para-Kähler Lie groups. Before we start, we will briefly outline some basic notions and notation on these structures.

Kähler structures

If a real $2n$-dimensional manifold M admits a globally defined $(1,1)$-tensor field J such that

$$J^2 = -\mathrm{Id},$$

then (M, J) is said to be an *almost complex manifold* and J is an *almost complex structure* on M. The almost complex structure is *integrable* if it corresponds to the underlying structure of a complex manifold – this is, a smooth manifold admitting a holomorphic atlas. Newlander and Nirenberg showed in [108] that the integrability of an almost complex structure on a manifold M is equivalent to the vanishing of its associated Nijenhuis tensor

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]. \tag{I.1}$$

A pseudo-Riemannian metric g on (M, J) is an *almost Hermitian metric* if the almost complex structure J is an isometry in each tangent space, i.e.,

$$g(JX, JY) = g(X, Y) \quad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

Now the triple (M, g, J) is called an *almost Hermitian manifold*. In particular, if J is integrable, then (M, g, J) is a *Hermitian manifold*.

There always exists a non-degenerate two-form associated to any almost Hermitian structure (g, J) which is called the *Kähler form* and given by

$$\Omega(X, Y) = g(JX, Y).$$

This two-form is covariantly constant and so it is closed. The covariant derivative of an almost complex structure, its Nijenhuis tensor and its associated Kähler two-form are related by

$$2g((\nabla_X J)Y, Z) + 3d\Omega(X, Y, Z) - 3d\Omega(X, JY, JZ) - g(JX, N_J(Y, Z)) = 0.$$

An (almost) Hermitian manifold (M, g, J) whose associated Kähler two-form is closed is said to be an (*almost*) *Kähler* manifold. In other words, Kähler manifolds are characterized by the parallelizability of their complex structure, $\nabla J = 0$, and their curvature tensor satisfies

$$R(X, Y, Z, W) = R(JX, JY, Z, W).$$

As a consequence, any Kähler manifold of constant sectional curvature is necessarily flat. The restriction of the sectional curvature to non-degenerate planes Π which are invariant under the action of the complex structure J is called the *holomorphic sectional curvature* and given by

$$H(\Pi) = \frac{R(X, JX, X, JX)}{g(X, X)^2}.$$

Notice that the holomorphic sectional curvature determines the curvature tensor of Kähler manifolds. What is more, a Kähler manifold has constant holomorphic sectional curvature c if and only if its curvature tensor decomposes as

$$R = \tfrac{c}{4}\left(R^0 + R^J\right),$$

where R^0 is the standard algebraic curvature tensor given in (1.2) and

$$R^J(X,Y)Z = g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ.$$

Such a Kähler manifold is locally isometric either to the complex space $\mathbb{C}^{2n}_\nu$ if $c = 0$, or to the complex projective space $\mathbb{CP}^{2n}_\nu$ if $c > 0$, or to the complex hyperbolic space $\mathbb{CH}^{2n}_\nu$ if $c < 0$ (see [13]).

An almost Hermitian manifold (M, g, J) is said to be *locally conformally Kähler* (resp. *locally conformally symplectic*) if there exists a local conformal deformation $\bar{g} = e^{2\sigma} g$ so that $(M, \bar{g}, J)$ is a Kähler (resp. symplectic) manifold. Some characterizations for these kinds of manifolds have been given in [59, 131] as follows.

- (M, g, J) is locally conformally symplectic if and only if its associated two-form satisfies $d\Omega = \theta \wedge \Omega$ for some other one-form θ such that $d\theta = 0$.
- (M, g, J) is locally conformally Kähler if and only if it is locally conformally symplectic and J is integrable.

Assume that (M, g, J) is a four-dimensional Kähler manifold oriented so that the associated Kähler form is self-dual, i.e., $\Omega \in \Lambda^2_+$. Then its self-dual Weyl curvature operator satisfies

$$\mathcal{W}^+ = \tfrac{\tau}{12}\mathrm{diag}[2, -1, -1].$$

Therefore, the self-dual part of the Weyl curvature tensor of any locally conformally Kähler metric has two equal eigenvalues and a distinguished one. Derdziński proved in [56] the following converse to this statement.

Theorem 1.15. *Let (M, g) be an oriented four-dimensional Riemannian Einstein manifold such that $\mathcal{W}^+$ has at most two different eigenvalues at each point. Then the metric $\bar{g} = (24\|\mathcal{W}^+\|^2)^{\frac{1}{3}} g$ is Kähler on the open set where $\mathcal{W}^+ \neq 0$.*

Para-Kähler structures

A $(1,1)$-tensor field J on a $2n$-dimensional manifold M is said to be an *almost product structure* if $J^2 = \mathrm{Id}$. In this case, the pair (M, J) is called an *almost product manifold*. If the eigenspaces associated to the eigenvalues ± 1 of J have the same rank, the almost product manifold (M, J) is said to be an *almost para-complex manifold* and J is an *almost para-complex structure* on M.

A manifold endowed with an almost para-complex structure J and a metric tensor g for which J is an anti-isometry at each point, i.e., such that

$$g(JX, JY) = -g(X, Y),$$

is called an *almost para-Hermitian manifold*. Any almost para-Hermitian manifold has an associated almost symplectic form, i.e., a non-degenerate two-form, which is given by

$$\Omega(X, Y) = g(JX, Y),$$

for any vector fields X, Y on M. The pair (M, Ω) where Ω is an almost symplectic form is called an *almost symplectic manifold*. If $L \subset M$ is an n-dimensional submanifold of a $2n$-dimensional symplectic manifold such that $\Omega|_L = 0$, then L is a *Lagrangian submanifold* of M. An almost symplectic manifold is an almost para-Hermitian manifold if its tangent bundle decomposes as a Whitney sum of Lagrangian subbundles. Indeed, in that case, $TM = L_1 \oplus L_2$ and the $(1,1)$-tensor field defined as $J = \pi_{L_1} - \pi_{L_2}$, where π_{L_i} are the projections from TM on L_i, determines an almost para-complex structure on M and the metric tensor is determined by the para-complex structure and the two-form Ω as $g(X, Y) = \Omega(JX, Y)$.

A *para-Kähler manifold* is a symplectic manifold that is diffeomorphic to a product of two Lagrangian submanifolds. For such manifolds, there is a relation between Ω and the covariant derivative and integrability of J given by the equation

$$2g((\nabla_X J)Y, Z) + 3d\Omega(X, Y, Z) + 3d\Omega(X, JY, JZ) + g(JX, N_J(Y, Z)) = 0,$$

where N_J is de Nijenhuis tensor associated to the almost para-complex structure J given as in (I.1). This relation allows us to characterize para-Kähler manifolds by the parallelism of J. Therefore, a triple (M, g, J) is a *para-Kähler manifold* if and only if

$$J^2 = \mathrm{Id}, \quad g(JX, JY) = -g(X, Y), \quad \text{and} \quad \nabla J = 0.$$

In this case, the curvature tensor satisfies that

$$R(X, Y, Z, W) = -R(JX, JY, Z, W).$$

The restriction of the sectional curvature to non-degenerate planes Π that are invariant under the action of the para-complex structure J is called the *para-holomorphic sectional curvature* and given by

$$H(\Pi) = -\frac{R(X, JX, X, JX)}{g(X, X)^2}.$$

As it happened in the Kähler case, the para-holomorphic sectional curvature determines the curvature tensor of a para-Kähler manifold and it is constant if and only if it can be written as

$$R = \tfrac{c}{4}(R^0 - R^J),$$

where R^0 is the standard algebraic curvature tensor given in (1.2) and

$$R^J(X, Y)Z = g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ.$$

A para-Kähler manifold of constant para-holomorphic sectional curvature is locally isometric (or maybe anti-isometric) to $\mathbb{R}^{2n}$ if $c = 0$, or to the para-complex projective space $\mathbb{P}^m(\mathbb{B})$ if $c \neq 0$ (see [74]), and in the latter case, it is isometric to the cotangent bundle of a flat affine manifold equipped with a suitable Riemannian extension [41].

The Bochner curvature tensor was introduced by S. Bochner in [20]. It is formally defined as an analogue to the Weyl curvature tensor, so that the curvature of a Bochner-flat manifold is completely determined by its Ricci tensor. The *Bochner curvature tensor* of a $2n$-dimensional para-Kähler manifold is defined as

$$B(X,Y)Z = R(X,Y)Z + \tfrac{\tau}{(2n+2)(2n+4)} R^0(X,Y)Z - \tfrac{1}{2(n+2)} R^1(X,Y)Z$$

for all vector fields X, Y, Z on the manifold, where R^0 is the standard algebraic curvature tensor given in (1.2) and

$$\begin{aligned} R^1(X,Y)Z = {} & g(X,Z)\mathrm{Ric}(Y) - g(Y,Z)\mathrm{Ric}(X) + g(X,JZ)\mathrm{Ric}(JY) \\ & - g(Y,JZ)\mathrm{Ric}(JX) + 2g(X,JY)\mathrm{Ric}(JZ) + \rho(X,Z)Y \\ & - \rho(Y,Z)X + \rho(X,JZ)JY - \rho(Y,JZ)JX + 2\rho(X,JY)JZ. \end{aligned}$$

A para-Kähler manifold is said to be *Bochner-flat* if its Bochner tensor vanishes identically. Even though the condition of being Bochner-flat is somehow analogous to that of being locally conformally flat, it is more restrictive. Besides, the anti-self-dual curvature tensor of an oriented para-Kähler manifold is completely determined by its Bochner tensor, so $W^- = 0$ if and only if the manifold is Bochner-flat [27].

Null-Kähler structures

A $(1,1)$-tensor field J on a $4n$-dimensional manifold M is said to be an *almost tangent structure* if $J^2 = 0$. If, in addition, $\mathrm{rank}(J) = 2n$, J is said to be a *null structure* on M.

A metric tensor g on M is said to be *null-Hermitian* if $g(JX,Y) = g(X,JY)$ for all vector fields X, Y on M, where J is a null structure on M. This implies that $g(X,JX) = 0$. The signature of a null-Hermitian metric on M is neutral $(2n,2n)$ and to each null-Hermitian metric there is an associated two-form defined by $\Omega(X,Y) = g(JX,Y)$. The kernel of a null structure is an integrable distribution of the tangent bundle and this integrability condition is equivalent to the vanishing of the associated Nijenhuis tensor given by (I.1).

A null-Hermitian manifold (M,g,J) is said to be *null-Kähler* if $\nabla J = 0$, where ∇ denotes de Levi-Civita connection of g. The fundamental two-form given by $\Omega(X,Y) = g(JX,Y)$ is covariantly-constant and so it is closed. Besides, it satisfies $\Omega^{\wedge n} = \Omega \wedge \cdots \wedge \Omega \neq 0$ and $\Omega^{\wedge(n+1)} = 0$, in contrast with the Kähler and para-Kähler conditions, where $\Omega^{\wedge 2n} \neq 0$.

We refer to Dunajski's works [60,61] for more information about null-Kähler structures.

Notation

In what follows, (M, g, J_ε) with

$$J_\varepsilon^2 = -\varepsilon \mathrm{Id}, \quad g(J_\varepsilon X, Y) + g(X, J_\varepsilon Y) = 0, \quad \nabla J_\varepsilon = 0,$$

will denote a Kähler, a para-Kähler or a null-Kähler manifold for $\varepsilon = 1$, $\varepsilon = -1$, or $\varepsilon = 0$, respectively.

Chapter 2
Locally conformally flat four-dimensional structures

The contents of this chapter regarding locally conformally flat Kähler and para-Kähler surfaces are contained in the work [68].

2.1 Locally conformally flat four-dimensional Kähler and para-Kähler structures

Let (M, g, J_ε) be a (para-)Kähler manifold, where $J_\varepsilon^2 = -\varepsilon \operatorname{Id}$ is the (para-)complex structure satisfying $g(J_\varepsilon X, J_\varepsilon Y) = \varepsilon g(X,Y)$ for $\varepsilon = \pm 1$. Since the (para-)complex structure is parallel, the curvature identity $R(X,Y) \cdot J_\varepsilon = J_\varepsilon \cdot R(X,Y)$, which strictly restricts the curvature tensor, holds. A consequence of this is that the Ricci tensor of a $2n$-dimensional locally conformally flat (para-)Kähler manifold satisfies

$$(2n-4)\,\rho(X,Y) = -\frac{\tau}{2n-1} g(X,Y)$$

and the manifold is flat if n is greater than or equal to three. In addition, the Ricci tensor of a locally conformally flat (para-)Kähler manifold is parallel an so (M, g) is locally symmetric in dimension four [111, 129]. Tanno proved in [129] that locally conformally flat positive definite four-dimensional Kähler manifolds are either flat or a product of two surfaces of constant opposite curvature. This result does not cover all the possibilities in the pseudo-Riemannian case of split signature, as Patterson pointed out in [119]. Motivated by this and some recent interest in locally conformally flat (para-)Kähler surfaces [2,4,82], in this chapter we will give the complete classification of locally conformally flat (para-)Kähler surfaces showing the existence of two additional possibilities. The result that completes the classification in the para-Kähler setting is given by Theorem 2.1, which is the main result in this chapter.

Theorem 2.1. *Any indecomposable locally conformally flat para-Kähler surface (M, g, J_-) is locally isometric to the cotangent bundle $T^*\Sigma$ of a flat affine surface (Σ, D) with a para-complex structure determined by $J|_{\ker \pi_*} = \operatorname{Id}$, where π denotes the canonical projection from the cotangent bundle, and the metric g is given by $g = \iota T \circ \iota \operatorname{Id} + g_D$ where*

(i) T is a parallel nilpotent $(1,1)$-tensor field on (Σ, D), or

(ii) T is a parallel $(1,1)$-tensor field on (Σ, D) satisfying $T^2 = -\kappa^2 \operatorname{Id}$.

Moreover, in both cases the para-Kähler two-form $\Omega_-(X,Y) = g(J_-X,Y)$ *is the canonical symplectic form of the cotangent bundle.*

Recall that a pseudo-Riemannian manifold is indecomposable if it does not admit a non-degenerate subspace that is invariant under the action of its holonomy group. Besides, the holonomy group may act indecomposably without acting irreducibly.

Para-Kähler surfaces in Case *(i)* in Theorem 2.1 had already been reported by Patterson in [119], while Case *(ii)* seems to be missing in previous works.

Considering the Ricci operator in Case *(ii)*, $\frac{1}{\kappa}\mathrm{Ric}$ defines a self-adjoint complex structure that is parallel, and so it is a Riemannian complex structure. Since the Ricci operator and the para-complex structure J_- commute with each other, one has that $J_+ = \frac{1}{\kappa}\mathrm{Ric}\,J_-$ is a complex structure so that (g, J_+) is a locally conformally flat Kähler structure. Therefore, the result that completes the classification in the Kähler setting is given by Theorem 2.2.

Theorem 2.2. *Any indecomposable locally conformally flat Kähler surface* (M, g, J_+) *is locally isometric to the cotangent bundle* $(T^*\Sigma, g)$ *of a Riemannian surface* (Σ, g_Σ) *of constant curvature with a metric given by*

(i) $g = g_{\nabla^\Sigma}$ *if the Gaussian curvature is non-zero, or*

(ii) $g = \iota J_\Sigma \circ \iota\,\mathrm{Id} + g_{\nabla^\Sigma}$ *if the Gaussian curvature vanishes,*

where ∇^Σ *is the Levi-Civita connection of* (Σ, g_Σ) *and* J_Σ *is the Kähler structure on* Σ *associated to the Riemannian volume form. Furthermore, the complex structure* J_+ *on* $T^*\Sigma$ *is determined by the symplectic form* $\Omega_+ = -d\iota J_\Sigma$.

Remark 2.3. In Chapter 3 we will see that it is possible to give a description of the structures in Theorems 2.1 and 2.2 in terms of left-invariant metrics on Lie groups.

The metric tensors in *(ii)* in both Theorem 2.1 and Theorem 2.2 are the same, but their associated symplectic structures and underlying geometries are different. The metrics in Case *(i)* in both theorems above correspond to different curvature models that will be described in Section 2.1.1.

The Kähler metrics in Theorem 2.2-*(i)* are locally isometric (up to reversing the metric) to those studied by Guilfoyle and Klingenberg on the space of oriented affine lines in $\mathbb{R}^3$ by means of the minitwistor correspondence in [82]. Analogously, the metrics in Theorem 2.1-*(i)* are locally isometric (up to reversing the metric) to the space of oriented spacelike or timelike lines in $\mathbb{R}^3_1$ (see [2,4]). The metrics corresponding to Assertion *(ii)* in Theorem 2.1 and Theorem 2.2 are locally isometric (up to reversing the metric) to the non-Einstein para-Kähler and Kähler metrics in the space of spacelike and timelike oriented geodesics of the de Sitter space constructed by Anciaux in [4]. Besides, the non-locally conformally flat Kähler-Einstein metrics on the de Sitter space constructed in [4] correspond to those in Remark 2.13.

2.1.1 Curvature models

In this section we will work at a purely algebraic level to describe the curvature models found in our description of locally conformally flat Kähler and para-Kähler surfaces. Bearing this in mind, let $(\mathcal{V}, \langle\cdot,\cdot\rangle, J_\pm)$ be a (para-)Hermitian inner product space. Recall that an algebraic curvature tensor is a multilinear map

$$A : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$$

satisfying the identities (1.1). If the corresponding Weyl curvature tensor vanishes, then the algebraic curvature tensor A is determined by the associated Ricci tensor and a straightforward calculation shows that the Ricci operator of a locally conformally flat (para-)Kähler surface is either diagonalizable or (following the discussion in [112]) its Jordan normal form corresponds to one of the following.

1. The Ricci operator has two 2×2 Jordan blocks. At each point there is a basis $\{u_1, v_1, u_2, v_2\}$ of the tangent space so that the Ricci operator and the non-zero inner products are given by

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \langle u_i, v_i\rangle = \varepsilon_i, \quad \varepsilon_i^2 = 1 \quad (i = 1, 2).$$

Besides, if the associated Weyl curvature vanishes, there are two different possibilities as $\varepsilon_1\varepsilon_2 = \pm 1$ (up to reversing the metric).

1.a. If $\varepsilon_1\varepsilon_2 = 1$, there is a unique (up to sign) Ricci-commuting Hermitian structure $(\langle\cdot,\cdot\rangle, J_+)$ given by

$$J_+ u_1 = -u_2, \quad J_+ v_1 = -v_2$$

and there are no Ricci-commuting para-Hermitian structures.

1.b. If $\varepsilon_1\varepsilon_2 = -1$, there is a unique (up to sign) Ricci-commuting para-Hermitian structure $(\langle\cdot,\cdot\rangle, J_-)$ given by

$$J_- u_1 = v_2, \quad J_- v_1 = u_2$$

and there are no Ricci-commuting Hermitian structures.

2. The Ricci operator is complex diagonalizable with imaginary eigenvalues $\pm i\kappa$. At each point there is a basis $\{u_1, v_1, u_2, v_2\}$ of the tangent space so that the Ricci operator and the non-zero inner products are given by

$$\mathrm{Ric} = \begin{pmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa \\ 0 & 0 & -\kappa & 0 \end{pmatrix}, \qquad \langle u_i, u_i\rangle = 1 = -\langle v_i, v_i\rangle, \quad (i = 1, 2).$$

One may assume that $\kappa > 0$ so that there are a unique (up to reversing the metric) Ricci-commuting para-Hermitian structure $(\langle\cdot,\cdot\rangle, J_-)$ given by

$$J_- u_1 = v_2, \quad J_- v_1 = -u_2$$

and a unique (up to reversing the metric) Ricci-commuting Hermitian structure $(\langle\cdot,\cdot\rangle, J_+)$ such that

$$J_+ u_1 = u_2, \quad J_+ v_1 = -v_2.$$

Regarding the discussion above, we introduce the following locally conformally flat algebraic curvature models $(\mathcal{V}, \langle\cdot,\cdot\rangle, A)$ given by $A = \frac{1}{2}\langle\cdot,\cdot\rangle \odot \rho_A$, where ρ_A denotes the Ricci tensors corresponding to the Ricci operators above and $\odot$ is the Kulkarni–Nomizu's product.

$(\mathfrak{M}^+)$: $(\mathcal{V}, \langle\cdot,\cdot\rangle, A)$ given by

$$A_{1413} = A_{3231} = \frac{1}{2}$$

with respect to a basis $\{u_1, u_2, u_3, u_4\}$ for which the non-zero inner products are

$$\langle u_1, u_2\rangle = 1 = \langle u_3, u_4\rangle.$$

$(\mathfrak{M}^-)$: $(\mathcal{V}, \langle\cdot,\cdot\rangle, A)$ given by

$$A_{1413} = A_{3231} = -\frac{1}{2}$$

with respect to a basis $\{u_1, u_2, u_3, u_4\}$ for which the non-zero inner products are

$$\langle u_1, u_2\rangle = 1 = -\langle u_3, u_4\rangle.$$

$(\mathfrak{N}_\kappa)$: $(\mathcal{V}, \langle\cdot,\cdot\rangle, A)$ given by

$$A_{1413} = A_{1442} = A_{3224} = A_{3231} = \frac{\kappa}{2}$$

with respect to a basis $\{u_1, u_2, u_3, u_4\}$ where u_1 and u_3 are spacelike vectors and u_2 and u_4 are timelike vectors. Notice that, even though the curvature models $(\mathfrak{N}_\kappa)$ are not isometric, they are all homothetic to the curvature model $(\mathfrak{N}_1)$.

We will see that the curvature tensors of the locally conformally flat para-Kähler manifolds in Assertions *(i)* and *(ii)* in Theorem 2.1 are modelled on $(\mathfrak{M}^-)$ and $(\mathfrak{N}_k)$, respectively. The curvature tensors of the locally conformally flat Kähler manifolds in Assertions *(i)* and *(ii)* in Theorem 2.2 are modelled on $(\mathfrak{M}^+)$ and $(\mathfrak{N}_k)$, respectively.

2.1.2 Self-dual Walker manifolds

Let $(M, g, \mathcal{D})$ be a four-dimensional Walker manifold, i.e., a pseudo-Riemannian manifold of split signature (M, g) admitting a parallel, degenerate plane field $\mathcal{D}$ of maximal dimension. As we have already mentioned in Section 1.3.1, there exist local coordinates $(x^1, x^2, x_{1'}, x_{2'})$ so that the Walker distribution is $\mathcal{D} = \text{span}\left\{\partial_{x_{1'}}, \partial_{x_{2'}}\right\}$ and the metric takes the form

$$g = dx^i \otimes dx_{i'} + dx_{i'} \otimes dx^i + g_{ij}\left(x^1, x^2, x_{1'}, x_{2'}\right) dx^i \otimes dx^j. \qquad (2.1)$$

The existence of a two-dimensional degenerate distribution $\mathcal{D}$ on a split-signature four-dimensional manifold (M, g) naturally induces an orientation. Let $\{u, v\}$ be a basis of $\mathcal{D}_p$ for $p \in M$, and denote by u^* and v^* their corresponding dual forms. The Hodge-star operator satisfies $\star(u^* \wedge v^*) = \pm(u^* \wedge v^*)$ and so, any four-dimensional Walker manifold is naturally oriented by the self-duality of $u^* \wedge v^*$. Considering local coordinates as in (2.1), the Walker orientation determined by

$$\star(dx_{1'} \wedge dx_{2'}) = dx_{1'} \wedge dx_{2'}$$

corresponds to the volume element $vol_g = dx^1 \wedge dx^2 \wedge dx_{1'} \wedge dx_{2'}$.

Self-dual Walker manifolds have been described by Calviño-Louzao, García-Río, Gilkey and Vázquez-Lorenzo as follows.

Theorem 2.4. ([41], Theorem 7.1) *A four-dimensional Walker manifold is self-dual if and only if it is locally isometric to the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) with metric*

$$g = \iota X(\iota \operatorname{Id} \circ \iota \operatorname{Id}) + \iota T \circ \iota \operatorname{Id} + g_D + \pi^*\Phi \tag{2.2}$$

where X is a vector field on Σ and T and Φ are a $(1,1)$-tensor field and a symmetric $(0,2)$-tensor field on Σ, respectively.

A special case of this result describes the local structure of para-complex space forms as follows.

Theorem 2.5. ([41], Theorem 2.2) *A para-Kähler surface of non-zero constant para-holomorphic sectional curvature c is locally isometric to the cotangent bundle of a flat affine surface equipped with the modified Riemannian extension $g = c\,\iota \operatorname{Id} \circ \iota \operatorname{Id} + g_D$.*

Consider $\theta_{(p,\omega)} = \pi^*\omega_p = x_{\ell'} dx^\ell$ the tautological one-form of $T^*\Sigma$ and let

$$\Omega = d\theta = dx_{\ell'} \wedge dx^\ell$$

be the canonical symplectic form of $T^*\Sigma$. Given the modified Riemannian extension

$$g = c\,\iota \operatorname{Id} \circ \iota \operatorname{Id} + g_D,$$

one naturally has a para-complex structure J_- determined by $\Omega(X, Y) = g(J_- X, Y)$, whose components are

$$J_-\partial_{x_{i'}} = \partial_{x_{i'}}, \quad J_-\partial_{x^i} = -\partial_{x^i} + c x_{i'} x_{j'} \partial_{x_{j'}},$$

and the Walker distribution $\mathcal{D} = \ker \pi_*$ corresponds to the eigenspace $\mathcal{D}_+ = \ker(J_- - \operatorname{Id})$.

2.1.3 Locally symmetric self-dual Walker surfaces

Given that locally conformally flat Kähler surfaces with non-diagonalizable Ricci operator and locally conformally flat para-Kähler surfaces are locally symmetric, we study first which self-dual Walker surfaces in Theorem 2.4 are locally symmetric.

Let ${}^D\rho$ denote the Ricci tensor of (Σ, D) and decompose it as ${}^D\rho = {}^D\rho_s + {}^D\rho_{sk}$, where ${}^D\rho_s$ and ${}^D\rho_{sk}$ denote the symmetric and the skew-symmetric parts of ${}^D\rho$, respectively.

Lemma 2.6. *Let (M,g) be a locally symmetric self-dual Walker manifold. Then the Riemannian extension g in Theorem 2.4 satisfies $g = \iota T \circ \iota \operatorname{Id} + g_D + \pi^*\Phi$ for some parallel $(1,1)$-tensor field T on (Σ, D).*

Furthermore, if the scalar curvature is not zero, then (M,g) is locally isometric to a para-complex space form as in Theorem 2.5.

Proof. The scalar curvature of any Riemannian extension in Theorem 2.4 is given by

$$\tau = 3 \operatorname{tr} T + 12\iota X.$$

Since the scalar curvature is constant, then $X = 0$ and $\operatorname{tr} T = \kappa$ for some $\kappa \in \mathbb{R}$. Given the fact that any self-dual Walker manifold is locally isometric to a Riemannian extension given by Theorem 2.4, the covariant derivatives of the curvature operator are polynomials on the fibre coordinates $(x_{1'}, x_{2'})$.

$$\begin{aligned}
(\nabla_{\partial_{x^1}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) &= -\tfrac{1}{8}\kappa\,(T_1{}^2)^2\, x_{2'}^3 + \text{ other terms},\\
(\nabla_{\partial_{x^2}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) &= \tfrac{1}{4}\kappa\,(T_2{}^1)^2\, x_{1'}^3 + \text{ other terms},\\
(\nabla_{\partial_{x^2}} R)(\partial_{x_{2'}}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) &= -\tfrac{\kappa}{8}\,(\kappa - 2T_2{}^2)\, x_{1'} + \text{ other terms}.
\end{aligned}$$

Assume in the first place that the scalar curvature $\tau = 3\kappa \neq 0$. If $\nabla R = 0$ then the previous expressions show that the tensor field T must be a scalar multiple of the identity, $T = c\operatorname{Id}$. Further calculations now show that

$$\begin{aligned}
(\nabla_{\partial_{x^1}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) &= c\{2{}^D\rho_s(\partial_{x^1}, \partial_{x^1}) - \tfrac{1}{2}c\Phi_{11}\}x_{2'} + \text{ other terms},\\
(\nabla_{\partial_{x^2}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) &= \tfrac{3}{2}c\{2{}^D\rho_s(\partial_{x^1}, \partial_{x^2}) - \tfrac{1}{2}c\Phi_{12}\}x_{2'} + \text{ other terms},\\
(\nabla_{\partial_{x^2}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x_{1'}}) &= \tfrac{1}{2}c^2\{2{}^D\rho_s(\partial_{x^2}, \partial_{x^2}) - \tfrac{1}{2}c\Phi_{22}\}x_{1'}^3 + \text{ other terms},
\end{aligned}$$

from where it follows that the symmetric $(0,2)$-tensor field $\Phi = \frac{4}{c}{}^D\rho_s$. In this situation, the Ricci operator $\operatorname{Ric} = \frac{3c}{2}\operatorname{Id}$ and, according to [41, Theorem 2.1], the corresponding metric is Einstein. Furthermore, one has that for any unit vector field X, the Jacobi operators $R_X = R(\,\cdot\,, X)X$ have eigenvalues $\{0, c, \frac{1}{4}c, \frac{1}{4}c\}$ and the eigenspace associated to c is timelike. Consequently, (M,g) is locally a para-complex space form (see [77]), thus being locally isometric to a modified Riemannian extension given by Theorem 2.5.

Now assume that the scalar curvature $\tau = 0$. Let DT denote the covariant derivative of the $(1,1)$-tensor field T with respect to the affine connection D. We set

$$DT = DT_{j;i}{}^k dx^i \otimes dx^j \otimes \partial_{x^k},$$

where $DT_{j;i}{}^k = \partial_{x^i} T_j{}^k + T_j{}^{\ell\, D}\Gamma_{i\ell}{}^k$. Since the scalar curvature vanishes, $\operatorname{tr} T = 0$. Hence $T_1{}^1 = -T_2{}^2$ and thus $DT_{1;1}{}^1 = -DT_{2;1}{}^2$ and $DT_{1;2}{}^1 = -DT_{2;2}{}^2$. A straightforward calculation now shows that

$$
\begin{aligned}
(\nabla_{\partial_{x^1}} R)(\partial_{x_{2'}}, \partial_{x^1}, \partial_{x^1}, \partial_{x^1}) &= \tfrac{1}{2} DT_{1;1}{}^2 \,, \\
(\nabla_{\partial_{x^1}} R)(\partial_{x_{2'}}, \partial_{x^2}, \partial_{x^2}, \partial_{x^1}) &= \tfrac{1}{2} DT_{2;1}{}^1 \,, \\
(\nabla_{\partial_{x^2}} R)(\partial_{x_{2'}}, \partial_{x^2}, \partial_{x^1}, \partial_{x^2}) &= \tfrac{1}{2} DT_{1;2}{}^2 \,, \\
(\nabla_{\partial_{x_{1'}}} R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^2}, \partial_{x^1}) &= -\tfrac{1}{2} DT_{2;2}{}^1 \,, \\
(\nabla_{\partial_{x_{1'}}} R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x^1}) &= DT_{2;2}{}^2 + \tfrac{1}{2} DT_{2;1}{}^1 \,, \\
(\nabla_{\partial_{x_{2'}}} R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^2}, \partial_{x^2}) &= DT_{2;1}{}^2 - \tfrac{1}{2} DT_{1;2}{}^2 \,,
\end{aligned}
$$

from where it follows that the trace-free $(1,1)$-tensor field T is D-parallel. □

The existence of a parallel $(1,1)$-tensor field on an affine surface (Σ, D) was considered in [40] showing that (besides the trivial case where $T = 0$) a parallel trace-free $(1,1)$-tensor field corresponds to one of the following:

(a) An *affine para-Kähler structure* ($\det T = -k^2 < 0$), which in suitable adapted coordinates becomes $T = k(\partial_{x^1} \otimes dx^1 - \partial_{x^2} \otimes dx^2)$.

(b) An *affine nilpotent Kähler structure* ($T^2 = 0$), which in suitable adapted coordinates becomes $T = k\partial_{x^1} \otimes dx^2$.

(c) An *affine Kähler structure* ($\det T = k^2 > 0$), which in suitable adapted coordinates becomes $T = k(\partial_{x^2} \otimes dx^1 - \partial_{x^1} \otimes dx^2)$.

Each of the three possibilities above gives rise to different geometric structures which are locally conformally flat. We study each case separately in what follows.

Locally symmetric self-dual Walker surfaces given by an affine para-Kähler structure

We will see that this case leads to product manifolds.

Lemma 2.7. *Let $(T^*\Sigma, g)$ be a locally symmetric self-dual Walker manifold determined by an affine para-Kähler structure T on (Σ, D). Then $(T^*\Sigma, g)$ is locally conformally flat and locally isometric to a product of two Lorentzian surfaces of constant opposite Gaussian curvature.*

Proof. Let (Σ, D) be an affine surface and choose local coordinates (x^1, x^2) so that the parallel tensor field T is locally given by $T = k(\partial_{x^1} \otimes dx^1 - \partial_{x^2} \otimes dx^2)$. Then T is parallel if and only if the Christoffel symbols are such that

$$ {}^D\Gamma_{11}{}^2 = {}^D\Gamma_{12}{}^1 = {}^D\Gamma_{12}{}^2 = {}^D\Gamma_{22}{}^1 = 0. $$

We refer to [40] for more information on the matter. Let $(x^1, x^2, x_{1'}, x_{2'})$ be the induced coordinates on $T^*\Sigma$. Then the symmetric and skew-symmetric Ricci tensors of (Σ, D) are given by

$$
\begin{aligned}
{}^D\rho_s &= -(\partial_{x^1} {}^D\Gamma_{22}{}^2 + \partial_{x^2} {}^D\Gamma_{11}{}^1) dx^1 \circ dx^2 \,, \\
{}^D\rho_{sk} &= \tfrac{1}{2}(\partial_{x^1} {}^D\Gamma_{22}{}^2 - \partial_{x^2} {}^D\Gamma_{11}{}^1) dx^2 \wedge dx^1 \,.
\end{aligned}
$$

A straightforward calculation now shows that the Ricci operator of $(T^*\Sigma, g)$, when expressed on the coordinate basis, satisfies

$$\mathrm{Ric} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & k\Phi_{12} + 2{}^D\rho_s(\partial_{x^1}, \partial_{x^2}) & k & 0 \\ -k\Phi_{12} + 2{}^D\rho_s(\partial_{x^1}, \partial_{x^2}) & 0 & 0 & -k \end{pmatrix},$$

from where it follows that the Ricci curvatures are $\pm k$. Moreover, a straightforward calculation shows that $(\mathrm{Ric} - k\,\mathrm{Id})(\mathrm{Ric} + k\,\mathrm{Id}) = 0$, so the Ricci operator is diagonalizable with respect to an orthonormal basis. If it is parallel, then the manifold is locally isometric to a product of two Lorentzian surfaces of constant opposite Gaussian curvature, thus being locally conformally flat. □

Remark 2.8. Let (Σ, D, T) be an affine para-Kähler surface. Let (x^1, x^2) be local coordinates on Σ so that the tensor field T expresses locally as in (a) above. Then there exists a locally defined deformation tensor Φ so that $g = \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi$ is locally symmetric. A straightforward calculation shows that

$$(\nabla_{\partial_{x^2}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x_{1'}}) = -k^2\{k\Phi_{12} + 2{}^D\rho_{sk}(\partial_{x^1}, \partial_{x^2})\}x_{1'}^2 x_{2'} + \text{other terms.}$$

Setting $\Phi_{12} = -\frac{2}{k}{}^D\rho_{sk}(\partial_{x^1}, \partial_{x^2})$, one has

$$\begin{aligned}(\nabla_{\partial_{x^1}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x_{1'}}) &= \tfrac{k}{2}\left\{k\partial_{x^2}\Phi_{11} - 2(D_{\partial_{x^1}}{}^D\rho)(\partial_{x^2}, \partial_{x^1})\right\} x_{1'}^2 \\ &\quad + \text{other terms},\\ (\nabla_{\partial_{x^2}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x_{1'}}) &= -\tfrac{k}{2}\left\{k\partial_{x^1}\Phi_{22} + 2(D_{\partial_{x^2}}{}^D\rho)(\partial_{x^1}, \partial_{x^2})\right\} x_{1'}^2 \\ &\quad + \text{other terms},\end{aligned}$$

which determines Φ_{11} and Φ_{22}. A straightforward calculation now shows that the covariant derivative ∇R vanishes when choosing the deformation tensor Φ as above.

Locally symmetric self-dual Walker surfaces given by an affine nilpotent Kähler structure

The case given by an affine nilpotent Kähler structure (Σ, D, T), where $\det T = 0$ and $T \neq 0$, gives rise to a single curvature model.

Lemma 2.9. *Let $(T^*\Sigma, g)$ be a locally symmetric self-dual Walker manifold determined by an affine nilpotent Kähler structure T on (Σ, D). Then $(T^*\Sigma, g)$ is locally conformally flat modelled on $(\mathfrak{M}^-)$, and locally isometric to a modified Riemannian extension $g = \iota T \circ \iota\,\mathrm{Id} + g_D$ where (Σ, D) is a flat affine surface.*

Proof. Let (x^1, x^2) be local coordinates on Σ in which the tensor field T is locally expressed as $T = k\,\partial_{x^1} \otimes dx^2$. Then it follows from [40] that the Christoffel symbols satisfy

$${}^D\Gamma_{11}{}^1 = {}^D\Gamma_{11}{}^2 = {}^D\Gamma_{12}{}^2 = 0 \quad \text{and} \quad {}^D\Gamma_{12}{}^1 = {}^D\Gamma_{22}{}^2.$$

The symmetric and skew-symmetric components of the Ricci tensor of (Σ, D) are given by

$$^D\rho_s = (\partial_{x^1}{}^D\Gamma_{22}{}^1 - \partial_{x^2}{}^D\Gamma_{22}{}^2)dx^2 \otimes dx^2\,, \qquad {}^D\rho_{sk} = \partial_{x^1}{}^D\Gamma_{22}{}^2 dx^2 \wedge dx^1\,.$$

Let $(x^1, x^2, x_{1'}, x_{2'})$ be the induced local coordinates on $T^*\Sigma$. A straightforward calculation shows that the component of the Weyl tensor

$$\begin{aligned} W(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x_{1'}}) &= \tfrac{1}{k}\partial_{x_{1'}}(\nabla_{\partial_{x^1}}R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^2}, \partial_{x^1}) \\ &= \tfrac{k}{4}\Phi_{11} - {}^D\rho_{sk}(\partial_{x^1}, \partial_{x^2})\,. \end{aligned}$$

Assuming that $(T^*\Sigma, g)$ is locally symmetric, one has $\Phi_{11} = \frac{4}{k}{}^D\rho_{sk}(\partial_{x^1}, \partial_{x^2})$. Now, given the expression of Φ_{11} one gets that the only non-zero component of the Weyl tensor (up to the usual symmetries) is

$$\begin{aligned} W(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x^2}) = \; & \tfrac{1}{4k}x_{1'}\partial_{x_{1'}}\partial_{x_{1'}}(\nabla_{\partial_{x^2}}R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^2}, \partial_{x_{1'}}) \\ & -\tfrac{1}{k^2}\partial_{x_{1'}}\partial_{x_{1'}}\partial_{x^1}(\nabla_{\partial_{x^2}}R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^2}, \partial_{x_{2'}}) \\ & +\tfrac{1}{4k^2}\partial_{x_{1'}}\partial_{x_{1'}}\partial_{x^2}(\nabla_{\partial_{x^2}}R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^2}, \partial_{x_{1'}}) \\ & -\tfrac{1}{2k^2}{}^D\Gamma_{22}{}^2\partial_{x_{1'}}\partial_{x_{1'}}(\nabla_{\partial_{x^2}}R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^2}, \partial_{x_{1'}})\,. \end{aligned}$$

This shows that if $(T^*\Sigma, g)$ is locally symmetric, then it is locally conformally flat. Moreover, since $\Phi_{11} = \frac{4}{k}{}^D\rho_{sk}(\partial_{x^1}, \partial_{x^2})$ it is easy to see that

$$\begin{aligned} (\nabla_{\partial_{x^1}}R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^2}, \partial_{x^1}) = \; & \tfrac{k}{2}\partial_{x^1}\Phi_{12} - (D_{\partial_{x^2}}{}^D\rho)(\partial_{x^1}, \partial_{x^2}) \\ & + \partial_{x^1}{}^D\rho_s(\partial_{x^2}, \partial_{x^2}) - \partial_{x^2}{}^D\rho_{sk}(\partial_{x^1}, \partial_{x^2}). \end{aligned}$$

Therefore, $\Phi_{12} = \frac{2}{k}\left\{\left({}^D\Gamma_{22}{}^2\right)^2 - \partial_{x^2}{}^D\Gamma_{22}{}^2 - \partial_{x^1}{}^D\Gamma_{22}{}^1\right\} + \phi_{12}(x^2)$ for some function $\phi_{12}(x^2)$. Using the expressions obtained for Φ_{11} and Φ_{12} one has

$$(\nabla_{\partial_{x^2}}R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^2}, \partial_{x^1}) = \partial_{x^1}\left(\tfrac{k}{2}\Phi_{22} + 2\partial_{x^2}{}^D\Gamma_{22}{}^1 - 2{}^D\Gamma_{22}{}^{2D}\Gamma_{22}{}^1\right) - \tfrac{k}{2}\phi'_{12}(x^2),$$

which gives $\Phi_{22} = \frac{4}{k}({}^D\Gamma_{22}{}^{2D}\Gamma_{22}{}^1 - \partial_{x^2}{}^D\Gamma_{22}{}^1) + x^1\phi'_{12}(x^2) + \phi_{22}(x^2)$ for some function $\phi_{22}(x^2)$.

A straightforward calculation shows that the metric $g = \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi$, where the tensor Φ is given by the expressions above, is locally symmetric (thus being locally conformally flat). Besides, the Ricci operator takes the form

$$\mathrm{Ric} = \begin{pmatrix} 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2{}^D\rho_{sk}(\partial_{x^2}, \partial_{x^1}) & 0 & 0 \\ 2{}^D\rho_{sk}(\partial_{x^1}, \partial_{x^2}) & 2{}^D\rho_s(\partial_{x^2}, \partial_{x^2}) & k & 0 \end{pmatrix}.$$

Hence, the Ricci operator is two-step nilpotent and a straightforward calculation shows that its Jordan normal form corresponds to that discussed in Section 2.1.1-(1.b) with $\varepsilon_1\varepsilon_2 = -1$. Therefore, the curvature tensor is determined by the 0-model $(\mathfrak{M}^-)$.

Now let (Σ, D) be a flat affine surface and let T be an affine nilpotent Kähler structure. Setting $\Phi = 0$, a direct calculation using the expressions above shows that the Riemannian extension $g = \iota T \circ \iota \operatorname{Id} + g_D$ is locally symmetric, thus completing the proof. □

Remark 2.10. Let (Σ, D, T) be an affine nilpotent Kähler surface. Let (x^1, x^2) be local coordinates on Σ so that the tensor field T expresses locally as $T = k\,\partial_{x^1} \otimes dx^2$. Then there exists a locally defined tensor field Φ given as in the proof of Lemma 2.9 so that the cotangent bundle $(T^*\Sigma, g = \iota T \circ \iota \operatorname{Id} + g_D + \pi^*\Phi)$ is locally symmetric.

Locally symmetric self-dual Walker surfaces given by an affine Kähler structure

We will see that this situation corresponds to the 0-model $(\mathfrak{N}_k)$.

Lemma 2.11. *Let $(T^*\Sigma, g)$ be a locally symmetric self-dual Walker manifold determined by an affine Kähler structure T on (Σ, D). Then $(T^*\Sigma, g)$ is locally conformally flat modelled on $(\mathfrak{N}_k)$ and locally isometric to the Riemannian extension $g = \iota T \circ \iota \operatorname{Id} + g_D$, where (Σ, D) is a flat affine surface.*

Proof. Let (x^1, x^2) be local coordinates on Σ so that the tensor field T locally takes the form $T = k\,(\partial_{x^2} \otimes dx^1 - \partial_{x^1} \otimes dx^2)$. Then it follows from the work in [40] that the Christoffel symbols are such that

$$ {}^D\Gamma_{11}{}^1 = {}^D\Gamma_{12}{}^2 = -{}^D\Gamma_{22}{}^1 \quad \text{and} \quad {}^D\Gamma_{12}{}^1 = -{}^D\Gamma_{11}{}^2 = {}^D\Gamma_{22}{}^2. $$

Besides, the symmetric and skew-symmetric parts of the Ricci tensor are given by

$$ \begin{aligned} {}^D\rho_s &= (\partial_{x^1}{}^D\Gamma_{22}{}^1 - \partial_{x^2}{}^D\Gamma_{22}{}^2)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2), \\ {}^D\rho_{sk} &= (\partial_{x^2}{}^D\Gamma_{22}{}^1 + \partial_{x^1}{}^D\Gamma_{22}{}^2)(dx^2 \otimes dx^1 - dx^1 \otimes dx^2). \end{aligned} $$

Now, since

$$ (\nabla_{\partial_{x^1}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x_{1'}}) = \tfrac{k^2}{8}\left\{k(\Phi_{11} + \Phi_{22}) - 4\,{}^D\rho_{sk}(\partial_{x^2}, \partial_{x^1})\right\} x_{1'}^3 + \text{other terms}, $$

assuming that $(T^*\Sigma, g)$ is locally symmetric we set $\Phi_{11} = -\Phi_{22} + \frac{4}{k}{}^D\rho_{sk}(\partial_{x^2}, \partial_{x^1})$. A long but straightforward calculation now shows that the only non-zero component (up to the usual symmetries) of the Weyl tensor is

$$ \begin{aligned} W(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x^2}) &= \tfrac{1}{k} x_{1'} \partial_{x_{1'}} \partial_{x_{2'}} (\nabla_{\partial_{x^1}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^2}, \partial_{x_{1'}}) \\ &\quad + \tfrac{1}{k} x_{2'} \partial_{x_{2'}} \partial_{x_{2'}} (\nabla_{\partial_{x^2}} R)(\partial_{x^2}, \partial_{x^1}, \partial_{x^1}, \partial_{x_{1'}}) \\ &\quad - \tfrac{1}{3k^2} \partial_{x^2} \partial_{x_{1'}} \partial_{x_{2'}} (\nabla_{\partial_{x^1}} R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x_{2'}}) \\ &\quad + \tfrac{2}{3k^2}\,{}^D\Gamma_{22}{}^2 \partial_{x_{1'}} \partial_{x_{2'}} (\nabla_{\partial_{x^1}} R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x_{2'}}) \\ &\quad - \tfrac{1}{k^2} \partial_{x^1} \partial_{x_{1'}} \partial_{x_{2'}} (\nabla_{\partial_{x^2}} R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x_{2'}}) \\ &\quad - \tfrac{2}{k^2}\,{}^D\Gamma_{22}{}^1 \partial_{x_{1'}} \partial_{x_{2'}} (\nabla_{\partial_{x^2}} R)(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x_{2'}})\,. \end{aligned} $$

Therefore, the local symmetry of $(T^*\Sigma, g)$ implies local conformal flatness. Moreover, the Ricci operator of $(T^*\Sigma, g)$ is given by

$$\mathrm{Ric} = \begin{pmatrix} 0 & -k & 0 & 0 \\ k & 0 & 0 & 0 \\ 2^D\rho_s(\partial_{x^1}, \partial_{x^1}) & 2^D\rho_{sk}(\partial_{x^2}, \partial_{x^1}) & 0 & k \\ 2^D\rho_{sk}(\partial_{x^1}, \partial_{x^2}) & 2^D\rho_s(\partial_{x^1}, \partial_{x^1}) & -k & 0 \end{pmatrix}.$$

Hence, the Ricci curvatures are $\pm ik$ and $\mathrm{Ric}^2 = -k^2 \,\mathrm{Id}$, which shows that the Ricci operator is complex diagonalizable. Since the curvature tensor is completely determined by the Ricci operator, it corresponds to the 0-model $(\mathfrak{N}_k)$.

To conclude, let (Σ, D) be a flat affine surface and T be an affine Kähler structure. A direct calculation shows that the Riemannian extension $g = \iota T \circ \iota \,\mathrm{Id} + g_D$ is locally symmetric, thus completing the proof. □

Remark 2.12. Let (Σ, D, T) be an affine Kähler surface. Then there exists a suitable locally defined deformation tensor Φ so that $(T^*\Sigma, g = \iota T \circ \iota \,\mathrm{Id} + g_D + \pi^*\Phi)$ is locally symmetric. A straightforward calculation as in the proof of Lemma 2.11 gives that

$$\Phi_{11} = -\Phi_{22} + \frac{4}{k} {}^D\rho_{sk}(\partial_{x^2}, \partial_{x^1})$$

and the coefficients Φ_{12} and Φ_{22} satisfy

$$\begin{aligned} \partial_{x^1}\Phi_{22} - \partial_{x^2}\Phi_{12} &= \tfrac{2}{k}\left\{(D\partial_{x^1}{}^D\rho_{sk})(\partial_{x^1}, \partial_{x^2}) + (D\partial_{x^2}{}^D\rho_s)(\partial_{x^1}, \partial_{x^1})\right. \\ &\qquad \left. +\partial_{x^1}{}^D\rho_{sk}(\partial_{x^2}, \partial_{x^1})\right\}, \\ \partial_{x^2}\Phi_{22} + \partial_{x^1}\Phi_{12} &= \tfrac{2}{k}\left\{(D\partial_{x^2}{}^D\rho_{sk})(\partial_{x^2}, \partial_{x^1}) + (D\partial_{x^1}{}^D\rho_s)(\partial_{x^1}, \partial_{x^1})\right. \\ &\qquad \left. +\partial_{x^2}{}^D\rho_{sk}(\partial_{x^2}, \partial_{x^1})\right\}. \end{aligned}$$

The system of equations above is just the inhomogeneous Cauchy-Riemann equation $\partial_{\bar z}\phi = f$ for $\phi = \Phi_{22} + i\Phi_{12}$, where the function $f = f_1 + if_2$ is given by the right hand side of the expressions above, which admit local solutions for any affine Kähler surface (Σ, D) (see [106]).

Remark 2.13. Let (M, g) be a locally symmetric Walker metric as in Lemma 2.11. Then the Ricci operator defines a complex structure $S = \frac{1}{k}\,\mathrm{Ric}$ which is self-adjoint and parallel (hence, a Riemannian complex structure). Furthermore, the twin metric

$$\hat g(X, Y) = g(\frac{1}{k}\,\mathrm{Ric}\, X, Y) = \frac{1}{k}\rho(X, Y)$$

is locally symmetric and Einstein with scalar curvature $\hat\tau = 4k$, which is a special case of the main Theorem in [22].

Locally symmetric self-dual Walker surfaces with $T = 0$

In contrast with the previous cases, the class of locally symmetric Riemannian extensions of the form $g = g_D + \pi^*\Phi$ is much larger and the underlying structure is not necessarily locally conformally flat.

Example 2.14. Let (Σ, D) be the flat plane and let Φ be the symmetric $(0,2)$-tensor field

$$\Phi = x^1x^2(dx^1 \otimes dx^2 + dx^2 \otimes dx^1).$$

Then $(T^*\Sigma, g_D + \pi^*\Phi)$ is a Ricci-flat locally symmetric manifold which is not locally conformally flat. The curvature tensor is determined by the only non-zero component $R(\partial_{x^1}, \partial_{x^2}, \partial_{x^1}, \partial_{x^2}) = 1$ (up to the usual symmetries).

It follows from the work in [1] that if $(T^*\Sigma, g = g_D + \pi^*\Phi)$ is locally symmetric, then so is (Σ, D). Locally symmetric affine surfaces were described by Opozda in [113], where it is stated that the affine connection corresponds to one of the following:

(i) The flat affine surface modelled on $\mathbb{R}^2$ with ${}^D\Gamma_{ij}{}^k = 0$.

(ii) The Levi-Civita connection of the hyperbolic plane $\mathbb{H}^2 = \mathbb{R}^+ \times \mathbb{R}$ given by

$${}^D\Gamma_{11}{}^1 = {}^D\Gamma_{12}{}^2 = -{}^D\Gamma_{22}{}^1 = -\frac{1}{x^1}.$$

(iii) The Levi-Civita connection of the Lorentzian hyperbolic plane

$${}^D\Gamma_{11}{}^1 = {}^D\Gamma_{12}{}^2 = {}^D\Gamma_{22}{}^1 = -\frac{1}{x^1}.$$

(iv) The Levi-Civita connection of the standard sphere.

(v) One of the two non-metrizable affine connections modelled on $\mathbb{R}^2$ whose non-zero Christoffel symbols are ${}^D\Gamma_{11}{}^1 = 1$ and ${}^D\Gamma_{22}{}^1 = \pm 1$.

The Ricci operator of $(T^*\Sigma, g = g_D + \pi^*\Phi)$ vanishes in Case (i) and it is two-step nilpotent otherwise. Besides, Ric has rank two in Cases (ii), (iii) and (iv), while it has rank one in Case (v).

In order to describe the locally symmetric Riemannian extensions which are locally conformally flat, we introduce the following algebraic curvature model:

$(\mathfrak{P})$: $(V, \langle\cdot,\cdot\rangle, \mathcal{A})$ given by

$$\mathcal{A}_{1313} = \mathcal{A}_{1441} = \frac{1}{2}$$

with respect to a basis $\{u_1, u_2, u_3, u_4\}$ where the non-zero inner products are

$$\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1 = -\langle u_4, u_4\rangle.$$

If $(T^*\Sigma, g = g_D + \pi^*\Phi)$ is locally conformally flat, then there exist coordinates on (Σ, D) so that one may assume $\Phi = 0$. This situation is summarized as follows:

Lemma 2.15. *Let $(T^*\Sigma, g = g_D + \pi^*\Phi)$ be a locally symmetric and locally conformally flat Riemannian extension. Then it is either flat or locally isometric to a Riemannian extension $(T^*\Sigma, g_D)$ where (Σ, D) corresponds to one of the following affine connections.*

(i) The Levi-Civita connection of a Riemannian surface of constant curvature, in which case the curvature tensor of $(T^*\Sigma, g_D)$ is modelled on $(\mathfrak{M}^+)$.

(ii) The Levi-Civita connection of a Lorentzian surface of constant curvature, in which case the curvature tensor of $(T^*\Sigma, g_D)$ is modelled on $(\mathfrak{M}^-)$.

(iii) One of the two non-metrizable affine connections modelled on $\mathbb{R}^2$ whose Christoffel symbols are ${}^D\Gamma_{11}{}^1 = 1$ and ${}^D\Gamma_{22}{}^1 = \pm 1$. In this case, the curvature tensor of $(T^*\Sigma, g_D)$ is modelled on $(\mathfrak{P})$.

2.1.4 Locally conformally flat para-Kähler surfaces

Let (M, g, J_-) be a para-Kähler surface. Since $J_-^2 = \mathrm{Id}$ and $J_-^* g = -g$, then $\mathcal{D}_\pm = \ker(J_- \mp \mathrm{Id})$ are totally degenerate. Moreover, the fact that $\nabla J_- = 0$ implies that $\mathcal{D}_\pm$ are parallel, so (M, g, J_-) is a Walker manifold with respect to both distributions $\mathcal{D}_\pm$. We set the parallel distribution $\mathcal{D} = \mathcal{D}_+$ so that $J_- |_{\mathcal{D}} = \mathrm{Id}$ and

$$g = dx^i \otimes dx_{i'} + dx_{i'} \otimes dx^i + g_{ij}(x^1, x^2, x_{1'}, x_{2'}) dx^i \otimes dx^j , \tag{2.3}$$

with respect to the Walker coordinates $(x^1, x^2, x_{1'}, x_{2'})$. The locally defined almost para-Hermitian structures satisfying $J_- |_{\mathcal{D}} = \mathrm{Id}$ are parametrized by a real-valued function $f(x^1, x^2, x_{1'}, x_{2'})$ and are given by

$$\begin{aligned} J_-^f \partial_{x^1} &= -\partial_{x^1} + g_{11}\partial_{x_{1'}} + f\partial_{x_{2'}}, & J_-^f \partial_{x_{1'}} &= \partial_{x_{1'}}, \\ J_-^f \partial_{x^2} &= -\partial_{x^2} + (2g_{12} - f)\partial_{x_{1'}} + g_{22}\partial_{x_{2'}}, & J_-^f \partial_{x_{2'}} &= \partial_{x_{2'}} . \end{aligned} \tag{2.4}$$

Let (g, J_-^f) be an almost para-Hermitian structure determined by Equations (2.3)–(2.4). Then the para-Kähler form $\Omega_f(X, Y) = g(J_-^f X, Y)$ is given by

$$\Omega_f = (f - g_{12}) dx^1 \wedge dx^2 + dx_{1'} \wedge dx^1 + dx_{2'} \wedge dx^2$$

and consequently

$$d\Omega_f = \partial_{x_{1'}} (f - g_{12})\, dx_{1'} \wedge dx^1 \wedge dx^2 + \partial_{x_{2'}} (f - g_{12})\, dx_{2'} \wedge dx^1 \wedge dx^2 .$$

Therefore, $d\Omega_f = 0$ if and only if

$$f(x^1, x^2, x_{1'}, x_{2'}) = g_{12}(x^1, x^2, x_{1'}, x_{2'}) + h(x^1, x^2)$$

for some function $h(x^1, x^2)$ and the almost para-complex structure becomes

$$\begin{aligned} J_-^h \partial_{x^1} &= -\partial_{x^1} + g_{11}\partial_{x_{1'}} + (g_{12} + h)\partial_{x_{2'}}, & J_-^h \partial_{x_{1'}} &= \partial_{x_{1'}}, \\ J_-^h \partial_{x^2} &= -\partial_{x^2} + (g_{12} - h)\partial_{x_{1'}} + g_{22}\partial_{x_{2'}}, & J_-^h \partial_{x_{2'}} &= \partial_{x_{2'}} . \end{aligned} \tag{2.5}$$

Let (g, J_-^h) be an almost para-Hermitian structure determined by (2.3) and (2.5). The para-Kähler two-form is given by

$$\Omega_h = h\, dx^1 \wedge dx^2 + dx_{1'} \wedge dx^1 + dx_{2'} \wedge dx^2.$$

Notice that the para-Kähler orientation and the Walker orientation are opposite. Indeed, the para-Kähler two-form Ω_h is anti-self-dual for the para-Kähler orientation determined by the para-complex structure J_-^h, but it is self-dual for the Walker orientation.

Meaning to describe all the anti-self-dual para-Kähler surfaces of constant scalar curvature we consider the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) with the metric $g = \iota T \circ \iota \operatorname{Id} + g_D + \pi^*\Phi$ as discussed in Section 2.1.3 and set the para-complex structure satisfying the condition $J_- |_{\ker \pi_*} = \operatorname{Id}$. The almost para-Hermitian structures (g, J_-^h) defined by (2.3) and (2.5) are not para-Kähler in general. In order to express the components of ∇J_-^h on $T^*\Sigma$ we use the notation $(\nabla_{\partial_{x^\alpha}} J_-^h)\partial_{x^\beta} = (\nabla J_-^h)_{\beta;\alpha}{}^\gamma \partial_{x^\gamma}$ and $(D\partial_{x^i}\Phi)(\partial_{x^j}, \partial_{x^k}) = D\Phi_{jk;i}$ to represent the covariant derivative of the symmetric $(0,2)$-tensor field Φ on Σ. In this notation, the components of the covariant derivative of the para-complex structures J_-^h are given by the following result.

Lemma 2.16. *Let $(T^*\Sigma, g)$ be a locally symmetric self-dual Walker structure on the cotangent bundle of an affine surface (Σ, D). Let J_-^h be a (locally defined) almost para-complex structure determined by $J_-^h |_{\ker \pi_*} = \operatorname{Id}$ so that (g, J_-^h) is an almost para-Hermitian structure locally given by* (2.5)*. Then the non-zero components of ∇J_-^h are determined by*

$$\begin{aligned}
2\left(\nabla J_-^h\right)_{1;1}{}^{2'} &= \tfrac{1}{4}x_{1'}^2 x_{2'}\left\{(T_2{}^2)^2 - (T_1{}^1)^2\right\} - \tfrac{1}{2}x_{2'}\left\{8\,{}^D\rho(\partial_{x^1}, \partial_{x^1}) - 4hT_1{}^2\right\} \\
&\quad + \tfrac{1}{2}x_{1'}\left\{8\,{}^D\rho(\partial_{x^2}, \partial_{x^1}) + 5hT_1{}^1 + hT_2{}^2\right. \\
&\qquad\qquad \left. + 2(T_2{}^1\Phi_{11} - T_1{}^2\Phi_{22} + (T_2{}^2 - T_1{}^1)\Phi_{12})\right\} \\
&\quad + 2\left\{\partial_1 h - h({}^D\Gamma_{11}{}^1 + {}^D\Gamma_{12}{}^2) + D\Phi_{11;2} - D\Phi_{12;1}\right\},
\end{aligned}$$

$$\begin{aligned}
2(\nabla J_-^h)_{1;2}{}^{2'} &= \tfrac{1}{4}x_{1'} x_{2'}^2\left\{(T_2{}^2)^2 - (T_1{}^1)^2\right\} + \tfrac{1}{2}x_{1'}\left\{8\,{}^D\rho(\partial_{x^2}, \partial_{x^2}) + 4hT_2{}^1\right\} \\
&\quad - \tfrac{1}{2}x_{2'}\left\{8\,{}^D\rho(\partial_{x^1}, \partial_{x^2}) - hT_1{}^1 - 5hT_2{}^2\right. \\
&\qquad\qquad \left. - 2(T_2{}^1\Phi_{11} - T_1{}^2\Phi_{22} + (T_2{}^2 - T_1{}^1)\Phi_{12})\right\} \\
&\quad + 2\left\{\partial_2 h - h({}^D\Gamma_{22}{}^2 + {}^D\Gamma_{12}{}^1) + D\Phi_{12;2} - D\Phi_{22;1}\right\},
\end{aligned}$$

where T is a trace-free parallel $(1,1)$-tensor field on (Σ, D) and Φ is a symmetric $(0,2)$-tensor field on Σ.

Theorem 2.1 follows immediately from the following result describing the local structure of anti-self-dual para-Kähler surfaces with constant scalar curvature.

Theorem 2.17. *Let (M, g, J_-) be an anti-self-dual para-Kähler surface with constant scalar curvature. Then it is locally isometric to a Riemannian extension of the form $(T^*\Sigma, \tilde{g} = \iota T \circ \iota \operatorname{Id} + g_D)$ with para-complex structure determined by $J_- |_{\ker \pi_*} = \operatorname{Id}$, where T is a parallel $(1,1)$-tensor field on a flat affine surface (Σ, D) satisfying one of the following conditions.*

(i) $T = c\,\mathrm{Id}$ *and* (M, g, J_-) *has constant para-holomorphic sectional curvature* c.

(ii) $T = 0$ *and* (M, g, J_-) *is flat.*

(iii) $T^2 = k^2\,\mathrm{Id}$ *and* (M, g, J_-) *is locally isometric to a product of two Lorentzian surfaces of constant opposite curvature.*

(iv) $T^2 = 0$ *and* (M, g, J_-) *is modelled on* $(\mathfrak{M}^-)$.

(v) $T^2 = -k^2\,\mathrm{Id}$ *and* (M, g, J_-) *is modelled on* $(\mathfrak{N}_k)$.

In all the cases above the para-Kähler two-form is the canonical symplectic two-form of $T^*\Sigma$.

Proof. Let (M, g, J_-) be an anti-self-dual para-Kähler surface. Then there exists a Walker structure $(M, g, \mathcal{D})$ so that (M, g) is self-dual with respect to the Walker orientation and (M, g, J_-) is locally isometric to the cotangent bundle of an affine surface (Σ, D) with para-complex structure determined by $J_- \,|_{\ker \pi_*} = \mathrm{Id}$ and metric tensor $g = \iota X(\iota\,\mathrm{Id} \circ \iota\,\mathrm{Id}) + \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi$.

A para-Kähler surface is anti-self-dual if and only if its Bochner tensor vanishes (see [27]), and it is locally symmetric if and only if the scalar curvature is constant. Assertion *(i)* corresponds to the case when its scalar curvature is non-zero and Lemma 2.6 shows that the $(1, 1)$-tensor field T is parallel in this case.

Anti-self-dual para-Kähler surfaces whose scalar curvature is zero are locally conformally flat and locally symmetric. Therefore, the underlying structure is induced by an affine para-Kähler structure, an affine nilpotent Kähler structure, or an affine Kähler structure as discussed in Lemma 2.7, Lemma 2.9, and Lemma 2.11, respectively.

Let (Σ, D, T) be an affine surface equipped with a parallel trace-free $(1, 1)$-tensor field T. It follows from Lemma 2.16 that, if the almost para-complex structure J_-^h determined by $J_-^h \,|_{\ker \pi_*} = \mathrm{Id}$ is parallel, then it is uniquely determined. If (Σ, D, T) is an affine para-Kähler surface, then the coefficients of $x_{1'}$ and $x_{2'}$ in Lemma 2.16 show that $h = -\frac{2}{k}{}^D\rho_s(\partial_{x^1}, \partial_{x^2})$ for a deformation tensor field Φ given as in Remark 2.8. If (Σ, D, T) is an affine nilpotent Kähler surface, then Lemma 2.16 shows that $h = -\frac{2}{k}{}^D\rho(\partial_{x^2}, \partial_{x^2})$ and if (Σ, D, T) is an affine Kähler surface, then $h = \frac{2}{k}{}^D\rho(\partial_{x^2}, \partial_{x^2})$.

Moreover, a straightforward calculation shows that for any (Σ, D, T) there is an appropriate deformation tensor field Φ so that $(T^*\Sigma, \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi, J_-^h)$ is para-Kähler, where Φ is given as in Remark 2.8, Remark 2.10 and Remark 2.12. In all these cases, the cotangent bundle $(T^*\Sigma, \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi, J_-^h)$ is locally isometric to the Riemannian extension $\iota T \circ \iota\,\mathrm{Id} + g_D$ of a flat affine surface (Σ, D, T) that is affine para-Kähler, affine nilpotent Kähler or affine Kähler, and the two-form of the corresponding locally conformally flat para-Kähler manifold is the canonical symplectic form of $T^*\Sigma$, from where Assertions *(iii)*, *(iv)* and *(v)* follow.

Finally, we consider the case $T = 0$ corresponding to Assertion *(ii)*. Setting $T = 0$ in

Lemma 2.16, one has that the non-zero components of ∇J_-^h are given by

$$\begin{aligned}(\nabla J_-^h)_{1;1}{}^{2'} &= 2x_{1'}\rho_{21}^D - 2x_{2'}\rho_{11}^D + \{\partial_1 h - h({}^D\Gamma_{11}{}^1 + {}^D\Gamma_{12}{}^2) \\ &\qquad + D\Phi_{11;2} - D\Phi_{12;1}\}\,, \\ (\nabla J_-^h)_{1;2}{}^{2'} &= 2x_{1'}\rho_{22}^D - 2x_{2'}\rho_{12}^D + \{\partial_2 h - h({}^D\Gamma_{22}{}^2 + {}^D\Gamma_{12}{}^1) \\ &\qquad + D\Phi_{12;2} - D\Phi_{22;1}\}\,.\end{aligned}$$

It now follows from the coefficients of the terms of degree one above that if $\nabla J_-^h = 0$, the Ricci tensor ${}^D\rho$ vanishes and (Σ, D) is flat. Since the Ricci tensor of $g_D + \pi^*\Phi$ is determined by the symmetric part of ${}^D\rho$ one has that $(T^*\Sigma, \tilde{g} = g_D + \pi^*\Phi)$ is Ricci-flat. Therefore, if $(T^*\Sigma, \tilde{g} = g_D + \pi^*\Phi)$ is para-Kähler, then it must be Ricci-flat and thus flat – since it is locally conformally flat.

Observe that in all the cases above the para-complex structure J_-^h is uniquely determined because $h = 0$ if the base surface is flat and $T \neq 0$ (which follows in all cases from the expressions in Lemma 2.16). Moreover, the corresponding para-Kähler form is again the canonical symplectic two-form of the cotangent bundle. □

Remark 2.18. The Ricci operator of any metric in Assertion *(v)* of Theorem 2.17 satisfies $\mathrm{Ric}^2 = -k^2\,\mathrm{Id}$ and, since the para-complex structure J_- commute with the Ricci operator, defining $J_+ = \frac{1}{k}\,\mathrm{Ric}\cdot J_-$ one has that (g, J_+) is a locally conformally flat indefinite Kähler structure.

2.2 Locally symmetric Kähler surfaces

Let (M, g, J_+) be a locally symmetric four-dimensional Kähler manifold. Then its Ricci operator is parallel. In the diagonalizable case the metric is Einstein or locally isometric to a product of two surfaces of constant curvature. The non-diagonalizability of the Ricci operator leads to a Walker structure and therefore to the situation in Section 2.1.3.

Lemma 2.19. *A Kähler surface (M, g, J_+) with parallel and non-diagonalizable Ricci operator is a Walker manifold.*

Proof. Since the Ricci operator Ric commutes with the complex structure J_+, then the trace-less Ricci operator $\mathrm{Ric}^0 = \mathrm{Ric} - \frac{\tau}{4}\,\mathrm{Id}$ is either two-step nilpotent or complex diagonalizable.

If Ric^0 is two-step nilpotent, since $\ker \mathrm{Ric}^0$ is J_+-invariant, then it must be two-dimensional and parallel thus determining a totally degenerate parallel distribution, which shows that (M, g) is a Walker manifold.

If Ric^0 is complex diagonalizable with eigenvalues $\pm ik$, then $\frac{1}{k}\,\mathrm{Ric}^0$ is a self-adjoint complex structure so that $(M, g, \frac{1}{k}\,\mathrm{Ric}^0)$ is complex Riemannian and the $(1,1)$-tensor field $J_- = \frac{1}{k}\,\mathrm{Ric}^0 \cdot J_+$ determines a para-Kähler structure (g, J_-) so that (M, g, J_-) is a Walker manifold whose parallel distribution is $J_\pm$-invariant. □

The proof of Theorem 2.2. Assertion *(ii)* in Theorem 2.2 follows immediately from Remark 2.18, which shows that any affine Kähler structure T satisfying $T^2 = -k^2 \operatorname{Id}$ on a flat affine surface (Σ, D) induces a Kähler structure (g, J_+) on $T^*\Sigma$ with

$$g = \iota T \circ \iota \operatorname{Id} + g_D$$

and $J_+ = -\frac{1}{k} \operatorname{Ric} \cdot J_-$, where J_- is the para-Kähler structure determined by

$$J_- \,|_{\ker \pi_*} = \operatorname{Id}.$$

Moreover, let g_Σ be a flat Riemannian metric on Σ with Levi-Civita connection D. A straightforward calculation shows that the corresponding Kähler two-form is given by $\Omega_+ = -d\iota J_\Sigma$, where J_Σ is the complex structure on (Σ, g_Σ) induced by the volume element of the flat metric g_Σ. Choosing local adapted coordinates (x^1, x^2) on Σ so that the metric tensor in given by $g_\Sigma = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$, one has that the complex structure J_+ on $T^*\Sigma$ is characterized by

$$J_+ \partial_{x_{1'}} = \partial_{x_{2'}}\,, \qquad J_+ \partial_{x_{2'}} = -\partial_{x_{1'}}\,,$$

thus being a proper complex structure in the sense of [102] whose corresponding Kähler form is $\Omega_+ = dx^1 \wedge dx_{2'} - dx^2 \wedge dx_{1'}$.

We will construct a locally conformally flat Kähler surface showing the geometric realizability of the model $(\mathfrak{M}^+)$ and thus proving Assertion *(i)* in Theorem 2.2. Let (Σ, g_Σ) be a Riemannian surface of non-zero constant Gaussian curvature and let D be its Levi-Civita connection. The Riemannian extension $(T^*\Sigma, g_D)$ is a locally symmetric four-dimensional manifold with curvature tensor modelled on $(\mathfrak{M}^+)$. Let ω_Σ be the Riemannian volume form of (Σ, g_Σ) and let J_Σ be the complex structure $g_\Sigma(J_\Sigma X, Y) = \omega_\Sigma(X, Y)$. Then $\Omega_+ = -d\iota J_\Sigma$ is a symplectic structure on $(T^*\Sigma, g = g_D)$ which induces a Kähler structure with complex structure $g_D(J_+\xi, \eta) = \Omega_+(\xi, \eta)$ for all vector fields ξ and η on $T^*\Sigma$.

Let (x^1, x^2) be a system of local coordinates on Σ so that the metric takes the form

$$g_\Sigma = \Psi(x^1, x^2)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) \quad \text{and} \quad J_\Sigma \partial_{x^1} = \partial_{x^2},\; J_\Sigma \partial_{x^2} = -\partial_{x^1}.$$

Then the complex structure J_+ is determined, with respect to the induced system of local coordinates $(x^1, x^2, x_{1'}, x_{2'})$, by

$$J_+ \partial_{x_{1'}} = \partial_{x_{2'}}\,, \qquad J_+ \partial_{x_{2'}} = -\partial_{x_{1'}}\,.$$

It corresponds to a proper Kähler structure whose corresponding Kähler form is given by

$$\Omega_+ = dx^1 \wedge dx_{2'} - dx^2 \wedge dx_{1'}$$

. □

Remark 2.20. Let $\Sigma = \mathbb{H}^2_1$ be the Lorentzian hyperbolic plane and let $(T^*\Sigma, g_D)$ be the Riemannian extension of its Levi-Civita connection. Let J_Σ be the para-complex structure on (Σ, g_Σ) determined by the Lorentzian volume form and $\Omega_- = -d\iota J_\Sigma$. Then it determines a para-Kähler structure (g_D, J_-) by $g_D(J_-\xi, \eta) = \Omega_-(\xi, \eta)$ which is locally conformally flat with curvature tensor modelled on $(\mathfrak{M}^-)$, and thus locally isometric to the one in Theorem 2.1-*(i)*.

Furthermore, let (x^1, x^2) be a system of local coordinates on Σ so that the metric tensor $g_\Sigma = \frac{1}{(x^1)^2}(dx^1 \otimes dx^1 - dx^2 \otimes dx^2)$ and $J_\Sigma \partial_{x^1} = \partial_{x^2}$, $J_\Sigma \partial_{x^2} = \partial_{x^1}$. Let $(x^1, x^2, x_{1'}, x_{2'})$ be the induced coordinates on $T^*\Sigma$. It now follows that the para-complex structure J_- is given by

$$J_-\partial_{x_{1'}} = -\partial_{x_{2'}}\,, \quad J_-\partial_{x^1} = \partial_{x^2} - 2\tfrac{x_{2'}}{x^1}\partial_{x_{1'}} - 2\tfrac{x_{1'}}{x^1}\partial_{x_{2'}}\,,$$
$$J_-\partial_{x_{2'}} = -\partial_{x_{1'}}\,, \quad J_-\partial_{x^2} = \partial_{x^1} - 2\tfrac{x_{1'}}{x^1}\partial_{x_{1'}} - 2\tfrac{x_{2'}}{x^1}\partial_{x_{2'}}\,,$$

with the corresponding Kähler two-form $\Omega_- = -d\iota J_\Sigma = dx^1 \wedge dx_{2'} + dx^2 \wedge dx_{1'}$.

2.3 Locally conformally flat null-Kähler structures

In this section we will focus on those $(1,1)$-tensor fields that are two-step nilpotent, i.e., such that $J^2 = 0$. Such a tensor field is commonly said to be an *almost tangent structure* on M. The study of this kind of structures started with the works [51] and [52] around 1960 and they were further investigated during that decade in works such as [63] and [86]. If $\operatorname{rank}(J) = n$, J is said to be a tangent structure or, as we will call it in what follows, a *null-Kähler structure* (see [60, 61]).

Lemma 2.21. *A null-Kähler surface* (M, g, J_0) *is a Walker manifold.*

Proof. $\mathcal{D} = \ker J = \operatorname{Im} J$ is a Walker distribution on (M, g, J_0). Indeed, since $g(J_0X, Y) + g(X, J_0Y) = 0$, then

$$0 = g(J_0X, J_0Y) + g(X, J_0^2Y) = g(J_0X, JY),$$

so $\mathcal{D}$ is degenerate. Besides, since J_0 is parallel with respect to the Levi-Civita connection of g,

$$\nabla_Z J_0 X = J_0 \nabla_Z X$$

for all vector field Z on M and all $X \in \mathcal{D}$, which shows that $\mathcal{D}$ is also parallel. □

As a consequence of Lemma 2.21, we can consider the induced Walker coordinates on M, with respect to which the metric is given by (2.1) and the Walker distribution is $\mathcal{D} = \operatorname{span}\{\partial_{x_{1'}}, \partial_{x_{2'}}\}$. Since $\mathcal{D} = \operatorname{Im}(J_0)$ we can write $J_0\partial_{x^1} = a\partial_{x_{1'}} + b\partial_{x_{2'}}$ and $J_0\partial_{x^2} = c\partial_{x_{1'}} + d\partial_{x_{2'}}$ for some real-valued smooth functions a, b, c and d on M. Imposing the condition for J_0 to be anti-self-adjoint we see that

$$0 = g(J_0\partial_{x^1}, \partial_{x^1}) + g(\partial_{x^1}, J_0\partial_{x^1}) = 2g(J_0\partial_{x^1}, \partial_{x^1}) = 2g(a\partial_{x_{1'}} + b\partial_{x_{2'}}, \partial_{x1}) = 2a$$

and, analogously

$$2d = 0, \qquad c = -b.$$

Hence, in Walker coordinates, the null-Kähler structure takes the form

$$J_0 = \lambda(x^1, x^2, x_{1'}, x_{2'}) \left(dx_{2'} \otimes \frac{\partial}{\partial x^1} - dx_{1'} \otimes \frac{\partial}{\partial x^2} \right).$$

In this situation the associated two-form given by $\Omega(X,Y) = g(JX,Y)$ takes the form $\Omega = \lambda(x^1, x^2, x_{1'}, x_{2'})dx^1 \wedge dx^2$ and a standard calculation shows that it is closed if and only if $\lambda(x^1, x^2, x_{1'}, x_{2'}) = \lambda(x^1, x^2)$.

Theorem 2.22. *Let (M, g, J) be a locally conformally flat null-Kähler structure. Then it is locally isometric to the cotangent bundle $T^*\Sigma$ of a strongly projectively flat affine surface (Σ, D) endowed with the Riemannian extension $g = g_D$.*

Proof. It is known that the scalar curvature of a four-dimensional locally conformally flat Walker manifold $\tau = 3\,\mathrm{tr}(T) + 12\iota X$ is necessarily zero, so the vector field X must vanish identically and the tensor field T must be trace-free. In this situation, the non-zero components of the covariant derivative ∇J_0 are given by

$$\begin{aligned}(\nabla_{\partial_{x^1}} J_0)\partial_{x^1} &= T_1{}^1\lambda x_{1'} + T_1{}^2\lambda x_{2'} - \lambda(\Gamma_{11}{}^1 + \Gamma_{12}{}^2) + \partial_{x^1}\lambda \\ &\qquad\qquad = -(\nabla_{\partial_{x^2}} J_0)\partial_{x^1} \\ (\nabla_{\partial_{x^1}} J_0)\partial_{x^2} &= T_2{}^1\lambda x_{1'} - T_1{}^1\lambda x_{2'} - \lambda(\Gamma_{12}{}^1 + \Gamma_{22}{}^2) + \partial_{x^2}\lambda \\ &\qquad\qquad = -(\nabla_{\partial_{x^2}} J_0)\partial_{x^2}\end{aligned} \tag{2.6}$$

which are polynomials on the fibre coordinates. After analysing the conditions under which the expressions above vanish we easily deduce that the $(1,1)$-tensor field T must be zero. This implies that the metric must be a deformed Riemannian extension $g = g_D + \pi^*\Phi$, which, in these coordinates, takes the form

$$g = \{-2\Gamma_{ij}{}^k y_k + \Phi_{ij}\}dx^i \otimes dx^j + dx^i \otimes dx_{i'} + dx_{i'} \otimes dx^i. \tag{2.7}$$

If we now consider $\bar{x}^i = x^i$ and $\bar{x}_{i'} = x_{i'} + \eta_i$ for some real-valued functions η_i on the base surface, and write $\partial_{\bar{x}^i} = a_i{}^k\partial_{x^k} + b_i{}^k\partial_{x_{k'}}$ and $\partial_{\bar{x}_{i'}} = c_i{}^k\partial_{x^k} + d_i{}^k\partial_{x_{k'}}$, we have

$$\begin{aligned}&0 = d\bar{x}^j(\partial_{\bar{x}_{i'}}) = dx^j(c_i{}^k\partial_{x^k} + d_i{}^k\partial_{x_{k'}}) = c_i{}^j, \\ &\delta_i^j = d\bar{x}_{j'}(\partial_{\bar{x}_{i'}}) = dx_{j'}(d_i{}^k\partial_{x_{k'}}) = d_i{}^j, \\ &\delta_i^j = d\bar{x}^j(\partial_{\bar{x}^i}) = dx^j(a_i{}^k\partial_{x^k}) = a_i{}^j, \\ &0 = d\bar{x}_{j'}(\partial_{\bar{x}_i}) = (dx_{j'} + \partial_{x^\ell}\eta_j dx^\ell)(a_i{}^k\partial_{x^k} + b_i{}^k\partial_{x_{k'}}) = \partial_{x^i}\eta_j + b_i{}^j.\end{aligned}$$

so the coordinate vector fields are given by $\partial_{\bar{x}^i} = \partial_{x^i} - \partial_{x^i}\eta_k\partial_{y_k}$ and $\partial_{\bar{x}_{i'}} = \partial_{x_{i'}}$ for some real-valued functions η_i, $i = 1, 2$, on the base surface. It is easy to see that in these coordinates the metric takes the form

$$g = \{-2\Gamma_{ij}{}^k x_{k'} + 2\Gamma_{ij}{}^k\eta_k - \partial_{x^i}\eta_j - \partial_{x^j}\eta_i + \Phi_{ij}\}dx^i \otimes dx^j + dx^i \otimes dx_{i'} + dx_{i'} \otimes dx^i.$$

Afifi showed in [1] that it is possible to find two smooth functions η_1 and η_2 on Σ such that

$$\Phi_{ij} = -2\Gamma_{ij}{}^k\eta_k + \partial_{x^i}\eta_j + \partial_{x^j}\eta_i,$$

or, equivalently, a one-form η on Σ such that

$$\Phi(\partial_{x^i}, \partial_{x^j}) = (\nabla_{\partial_{x^i}}\eta)\partial_{x^j} + (\nabla_{\partial_{x^j}}\eta)\partial_{x^i}. \tag{2.8}$$

Therefore, the $(0,2)$-tensor field Φ can be transformed away from Expression (2.7).

At this point, the only conditions we have left for J_0 to be parallel are

$$\lambda({\Gamma_{11}}^1 + {\Gamma_{12}}^2) + \partial_{x^1}\lambda = 0, \quad \lambda({\Gamma_{12}}^1 + {\Gamma_{22}}^2) + \partial_{x^2}\lambda = 0, \tag{2.9}$$

and the compatibility condition for this system of partial differential equations reduces to

$$\lambda \left\{ \partial_{x^2}({\Gamma_{12}}^2 + {\Gamma_{11}}^1) - \partial_{x^1}({\Gamma_{12}}^1 + {\Gamma_{22}}^2) \right\} = 0. \tag{2.10}$$

Besides, it is known [62, Sec. 34] that if a Riemannian extension of the form $g = g_D + \pi^*\Phi$ is locally conformally flat, then the base affine surface (Σ, D) must be projectively flat. This means that there exists a one-form ω on Σ such that

$$ {\Gamma_{ij}}^k = -(\omega_i {\delta_j}^k + \omega_j {\delta_i}^k). $$

This implies that the compatibility condition (2.10) can be written as

$$-3\partial_{x^2}\omega(\partial_{x^1}) + 3\partial_{x^1}\omega(\partial_{x^2}) = 0, \tag{2.11}$$

which means that the one-form ω must be closed and therefore there exists a real-valued function f locally defined on Σ such that $\omega = df$. Consequently, (Σ, D) must be locally strongly projectively flat. Under these conditions the metric is locally conformally flat and null-Kähler.

Conversely, if (Σ, D) is a strongly projectively flat affine surface, then there exist a one-form ω and a real-valued function f such that $\omega = df$ and the compatibility condition (2.11) automatically holds. □

Chapter 3
Four-dimensional Kähler and para-Kähler Lie groups

In this chapter we will describe all the left-invariant para-Kähler structures on four-dimensional Lie groups and analyse their geometry, thus completing the analysis previously carried out in [31, 32, 101]. The geometry of the left-invariant Kähler structures obtained by Ovando in [116] will also be clarified. The results in this chapter are contained in the work [70].

3.1 Summary of results

A para-Kähler Lie algebra is a triple $(\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ so that

$$J^2 = \mathrm{Id}, \quad \langle Jx, Jy\rangle = -\langle x, y\rangle, \quad \nabla J = 0,$$

for all vectors $x, y \in \mathfrak{g}$. The associated two-form $\Omega(x, y) = \langle Jx, y\rangle$ is non-degenerate and closed, so $(\mathfrak{g}, \Omega)$ is a symplectic Lie algebra satisfying

$$\Omega(Jx, Jy) = -\Omega(x, y).$$

The eigenspaces $\mathfrak{L} = \ker(J - \mathrm{Id})$ and $\mathfrak{L}' = \ker(J + \mathrm{Id})$ are Lagrangian subalgebras and

$$\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{L}'$$

is a Lagrangian decomposition of $(\mathfrak{g}, \Omega)$.

For a fixed symplectic structure Ω on a Lie algebra $\mathfrak{g}$ we will describe all the para-Kähler structures $(J, \langle\cdot,\cdot\rangle, \Omega)$, up to isometric automorphisms preserving the symplectic structure and modulo reversing the metric – in both cases the corresponding Lagrangian decomposition $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{L}'$ is preserved. The different Lagrangian decompositions, which are of much interest in the para-Kähler setting, will also be explicitly described in each case.

The geometry of four-dimensional para-Kähler Lie groups naturally splits into the symmetric and the non-symmetric cases. The latter splits into the semi-symmetric and the non-semi-symmetric situations. The symmetric case can be summarized as follows.

Theorem 3.1. *Let $(G, \langle\cdot,\cdot\rangle, J)$ be a non-flat locally symmetric four-dimensional para-Kähler Lie group. Then, there are two distinct situations.*

(i) If the Ricci operator is diagonalizable, then one of the following holds:

(i.a) The Ricci operator vanishes and the anti-self-dual Weyl curvature operator is two-step nilpotent.

(i.b) The para-holomorphic sectional curvature is a non-zero constant.

(i.c) The metric is Einstein with non-zero scalar curvature, and the self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues.

(i.d) The manifold is locally a product of two surfaces of constant Gaussian curvature. The self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues.

(ii) If the Ricci operator is non-diagonalizable, then one of the following holds:

(ii.a) The Ricci operator has complex eigenvalues. The self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues.

(ii.b) The Ricci operator is two-step nilpotent and the anti-self-dual Weyl curvature operator either vanishes or is two-step nilpotent.

The structures in Assertion *(i.a)*, which do not have a Kählerian counterpart, are realized on $\mathfrak{r}_{4,-1}$, $\mathfrak{r}_{4,-1,-1}$ and $\mathfrak{d}_{4,1}$, and they all correspond to symmetric Osserman manifolds – which are four-dimensional Einstein manifolds that are either self-dual or anti-self-dual – with non-diagonalizable Jacobi operators [77].

A para-Kähler manifold (M, g, J) is said to be *opposite para-Kähler* if there exists a para-Kähler structure (J', g) so that $JJ' = J'J$. In the four-dimensional setting, the corresponding para-Kähler forms Ω and Ω' induce opposite orientations and $Q = JJ'$ defines a parallel product structure on the manifold. In this situation, (M, g) locally splits as a product of two oriented surfaces $M = N_1 \times N_2$ so that $J = J_1 \oplus J_2$ and $J' = J_1 \oplus -J_2$, where J_i is the para-complex structure induced by the volume form on N_i, for $i = 1, 2$. Conversely, a four-dimensional product of two oriented Lorentzian surfaces naturally inherits a para-Kähler and opposite para-Kähler structure. This is the case of the structures corresponding to Assertion *(i.d)* in Theorem 3.1.

As an immediate consequence of Theorem 3.1 and the results of Chapter 2 we conclude that locally conformally flat left-invariant para-Kähler structures can be modelled as follows.

Corollary 3.2. *Let (M, g, J) be a locally conformally flat four-dimensional para-Kähler manifold. Then, it is either flat or locally isometric to one of the following para-Kähler Lie groups.*

(i) The symplectic Lie algebra $(\mathfrak{r}'_2, \Omega)$ determined by $\Omega = e^{14} + e^{23}$, with the metric

$$\langle \cdot, \cdot \rangle = \tfrac{2}{\kappa}\, e^1 \circ e^2 - 2\kappa\, e^3 \circ e^4 .$$

(ii) The symplectic Lie algebra $(\mathfrak{d}_{4,1}, \Omega)$ determined by $\Omega = e^{12} - e^{34}$, with the metric

$$\langle \cdot, \cdot \rangle = 2(e^1 \circ e^3 + e^2 \circ e^4).$$

(iii) The symplectic Lie algebra $(\mathfrak{r}_2\mathfrak{r}_2, \Omega)$ *determined by* $\Omega = e^{12} + e^{34}$, *with the metric*

$$\langle\cdot,\cdot\rangle = \tfrac{1}{\kappa}(e^1 \circ e^1 - e^3 \circ e^3) - \kappa(e^2 \circ e^2 - e^4 \circ e^4).$$

Remark 3.3. Four-dimensional self-dual para-Kähler manifolds are either para-complex space forms – which are realizable as Lie groups and are the only ones that are strictly self-dual – or locally conformally flat, and so they are covered by the previous corollary (see [67]).

A pseudo-Riemannian manifold (M, g) is *semi-symmetric* if its curvature tensor satisfies $R(X,Y)\cdot R = 0$ for all vector fields X, Y on M, where $R(X,Y)$ is acting on the curvature tensor R as a derivation. The special significance of the semi-symmetry condition lies in the fact that if a curvature tensor is semi-symmetric, then it is pointwise the curvature tensor of a symmetric space. However, the model symmetric space may change from point to point and there are examples of semi-symmetric manifolds which are not even locally homogeneous.

Another generalization of symmetric spaces is given by manifolds of *recurrent curvature*, which are those that admit a one-form ξ such that the covariant derivative of the curvature tensor satisfies $\nabla R = \xi \otimes R$. As we mentioned in Section 1.1.4, the curvature tensor induces a unique self-adjoint endomorphism $\mathcal{R}$ of the space of two-forms defined by

$$\langle\!\langle \mathcal{R}(e^i \wedge e^j), e^k \wedge e^\ell \rangle\!\rangle = R(e_i, e_j, e_k, e_\ell).$$

A recurrent manifold is said to be *special* if there exist two two-forms $\alpha \wedge \beta$ and $\gamma \wedge \delta$ so that the curvature tensor acting on the space of two-forms is completely described by

$$\mathcal{R}(\alpha \wedge \beta) = \pm \gamma \wedge \delta.$$

Remark 3.4. Special recurrent manifolds are simply harmonic spaces, whose local structure was given in [125]. There it is shown that there exist local coordinates (x^1, x^2, x^3, x^4) with respect to which their metrics take the form

$$g = \Psi(x^1, x^2) dx^1 \circ dx^1 + 2dx^1 \circ dx^4 + 2dx^2 \circ dx^3, \tag{3.1}$$

where $\Psi(x^1, x^2)$ is an arbitrary function with non-constant $\partial_2\partial_2\Psi$ – otherwise, g would be locally symmetric. A standard calculation shows that special recurrent manifolds are semi-symmetric, since their curvature tensors correspond to those of the locally symmetric metrics given by $\Psi(x^1, x^2) = \pm(x^2)^2$. Moreover, given that

$$\begin{aligned} J_K &= \partial_{x^2} \otimes dx^1 + \partial_{x^3} \otimes dx^4 - \partial_{x^1} \otimes dx^2 - \partial_{x^4} \otimes dx^3 + \tfrac{1}{2}\Psi(\partial_{x^3} \otimes dx^1 + \partial_{x^4} \otimes dx^2), \\ J_{pK} &= \partial_{x^2} \otimes dx^2 + \partial_{x^4} \otimes dx^4 - \partial_{x^1} \otimes dx^1 - \partial_{x^3} \otimes dx^3 + \Psi \partial_{x^4} \otimes dx^1 \end{aligned}$$

are anti-commuting Kähler and para-Kähler structures with associated symplectic forms

$$\omega_{J_K} = \frac{1}{2}\Psi\, dx^1 \wedge dx^2 + dx^1 \wedge dx^3 - dx^2 \wedge dx^4 \quad \text{and} \quad \omega_{J_{pK}} = dx^1 \wedge dx^4 - dx^2 \wedge dx^3,$$

the metrics given by Equation (3.1) are (locally) hypersymplectic. Therefore, any special recurrent manifold admits a locally defined Kähler structure (g, J_K) and two locally defined para-Kähler structures (g, J_{pK}) and $(g, J_K J_{pK})$. Hypersymplectic structures on four-dimensional Lie algebras were classified by Andrada (see [5]), who showed that, besides the Abelian algebra $\mathfrak{r}^4$, only $\mathfrak{rh}_3$, $\mathfrak{r}_{4,-1,-1}$ or $\mathfrak{d}_{4,2}$ admit this kind of structures. The non-flat cases will be described in Section 3.11 and Section 3.17.1.

Remark 3.5. In addition to having a hypersymplectic structure, any special recurrent manifold is locally para-Kähler and opposite almost para-Kähler. Indeed, there exists an almost para-complex structure J'_{pK}, given by

$$J'_{pK} = \partial_{x^1}\otimes dx^1 + \partial_{x^2}\otimes dx^2 - \partial_{x^3}\otimes dx^3 - \partial_{x^4}\otimes dx^4 - \Psi\partial_{x^4}\otimes dx^1 \,,$$

that commutes with J_{pK} and (g, J'_{pK}) determines a locally defined almost para-Hermitian structure with closed fundamental form

$$\Omega_{J'_{pK}} = -dx^1 \wedge dx^4 - dx^2 \wedge dx^3.$$

Para-Kähler and opposite almost para-Kähler structures on non-flat Lie groups were considered in [45], where it is shown that their corresponding Ricci operators either vanish or are diagonalizable with two-dimensional kernels. In sharp contrast with the Kähler situation [7,44], there are many para-Kähler Lie groups which admit an opposite almost para-Kähler structure, even in the symmetric case. The Lie algebras $\mathfrak{rr}_{3,-1}$, $\mathfrak{r}_{4,-1,\beta}$ and $\mathfrak{r}_{4,-1,-1}$ admit Ricci-flat para-Kähler structures with associated opposite almost para-Kähler structures, and so do $\mathfrak{r}_{4,0}$, $\mathfrak{r}_{4,-1}$ and $\mathfrak{r}_2\mathfrak{r}_2$. The Lie algebra $\mathfrak{d}_{4,2}$ admits para-Kähler structures that are not semi-symmetric and have associated opposite almost para-Kähler structure.

The geometry of para-Kähler Lie groups is quite rigid in the non-symmetric case, where there are two essentially different possibilities depending on whether the curvature tensor is semi-symmetric or not. The semi-symmetric case can be summarized as follows.

Theorem 3.6. *Let $(G, \langle\cdot,\cdot\rangle, J)$ be a four-dimensional para-Kähler Lie group that is not locally symmetric. Then, its curvature tensor is semi-symmetric if and only if its associated Ricci operator vanishes. Moreover, all these structures are recurrent and harmonic.*

Although all the structures covered by Theorem 3.6 have analogous local descriptions as in Remark 3.4, they are not necessarily isometric, since their associated recurrence one-forms may have different causalities. Theorem 3.6 also holds true in the Kähler situation (cf. Theorem 3.16).

Finally, the class of non-symmetric para-Kähler Lie groups whose curvature tensors are not semi-symmetric is described as follows.

Theorem 3.7. *Let $(G, \langle\cdot,\cdot\rangle, J)$ be a four-dimensional para-Kähler Lie group whose curvature tensor is not semi-symmetric. Then, one of the following statements holds:*

(i) The Ricci operator has a single eigenvalue which is a double root of its minimal polynomial. The anti-self-dual Weyl curvature operator is three-step nilpotent.

(ii) The Ricci operator is diagonalizable with two-dimensional kernel. Moreover,

(ii.a) the self-dual and anti-self-dual Weyl curvature operators have the same eigenvalues and W^- has a double root of its minimal polynomial, or

(ii.b) the self-dual and anti-self-dual Weyl curvature operators have opposite eigenvalues and both operators $W^\pm$ are diagonalizable.

The only geometry in Theorem 3.7 with a Kähler counterpart corresponds to Assertion *(ii.b)*, as shown in Theorem 3.16.

3.2 Symplectic Lie algebras

Four-dimensional symplectic Lie algebras are necessarily solvable (see [50]) and they were classified by Ovando in [115]. We follow the notation in Ovando's works to denote the different four-dimensional symplectic Lie algebras, which correspond to certain semi-direct extensions of the Abelian algebra $\mathfrak{r}^3$, the Euclidean algebra $\mathfrak{e}(2)$, the Poincaré algebra $\mathfrak{e}(1,1)$ and the Heisenberg algebra $\mathfrak{h}^3$ admitting symplectic structures.

The non-zero Lie brackets on each Lie algebra are given in terms of a basis $\{e_1, e_2, e_3, e_4\}$ of the corresponding algebras and e^i will denote the dual of e_i. In order to simplify the notation, in what follows we will write e^{ij} to denote $e^i \wedge e^j$.

Solvable Lie algebra	Symplectic form
$\mathfrak{r}_2\mathfrak{r}_2\colon\ [e_1,e_2]=e_2,\,[e_3,e_4]=e_4$	$\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{34}e^{34}$ $\alpha_{12}\alpha_{34}\neq 0$
$\mathfrak{r}\mathfrak{h}_3\colon\ [e_1,e_2]=e_3$	$\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{14}e^{14}+\alpha_{23}e^{23}+\alpha_{24}e^{24}$ $\alpha_{14}\alpha_{23}-\alpha_{13}\alpha_{24}\neq 0$
$\mathfrak{r}\mathfrak{r}_{3,0}\colon\ [e_1,e_2]=e_2$	$\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{14}e^{14}+\alpha_{34}e^{34}$ $\alpha_{12}\alpha_{34}\neq 0$
$\mathfrak{r}\mathfrak{r}'_{3,0}\colon\ [e_1,e_2]=-e_3,\ [e_1,e_3]=e_2$	$\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{14}e^{1}4+\alpha_{23}e^{23}$ $\alpha_{14}\alpha_{23}\neq 0$
$\mathfrak{r}\mathfrak{r}_{3,-1}\colon\ [e_1,e_2]=e_2,\,[e_1,e_3]=-e_3$	$\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{14}e^{14}+\alpha_{23}e^{23}$ $\alpha_{14}\alpha_{23}\neq 0$
$\mathfrak{r}'_2\colon$ $[e_1,e_3]=e_3,\quad [e_1,e_4]=e_4,$ $[e_2,e_3]=e_4,\quad [e_2,e_4]=-e_3$	$\omega=\alpha_{12}e^{12}+\alpha_{13}\left(e^{13}-e^{24}\right)+\alpha_{14}\left(e^{14}+e^{23}\right)$ $\alpha_{13}^2+\alpha_{14}^2\neq 0.$
$\mathfrak{n}_4\colon\ [e_1,e_4]=-e_2,\ [e_2,e_4]=-e_3$	$\omega=\alpha_{12}e^{12}+\alpha_{14}e^{14}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ $\alpha_{12}\alpha_{34}\neq 0$
$\mathfrak{r}_{4,0}\colon\ [e_1,e_4]=-e_1,\,[e_3,e_4]=-e_2$	$\omega=\alpha_{14}e^{14}+\alpha_{23}e^{23}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ $\alpha_{14}\alpha_{23}\neq 0$
$\mathfrak{r}_{4,-1}\colon$ $[e_1,e_4]=-e_1,\ [e_2,e_4]=e_2$ $[e_3,e_4]=e_3-e_2$	$\omega=\alpha_{13}e^{13}+\alpha_{14}e^{14}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ $\alpha_{13}\alpha_{24}\neq 0$

Solvable Lie algebra	Symplectic form
$\mathfrak{r}_{4,-1,\beta}$: $[e_1,e_4]=-e_1,\ [e_2,e_4]=e_2,$ $[e_3,e_4]=-\beta e_3,\ -1<\beta<0$	$\omega=\alpha_{12}e^{12}+\alpha_{14}e^{14}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ $\alpha_{12}\alpha_{34}\neq 0$
$\mathfrak{r}_{4,-1,-1}$: $[e_1,e_4]=-e_1,\ [e_2,e_4]=e_2,$ $[e_3,e_4]=e_3$	$\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{14}e^{14}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ $\alpha_{13}\alpha_{24}-\alpha_{12}\alpha_{34}\neq 0$
$\mathfrak{r}_{4,-\alpha,\alpha}$: $[e_1,e_4]=-e_1,\ [e_2,e_4]=\alpha e_2,$ $[e_3,e_4]=-\alpha e_3,\ 0<\alpha<1$	$\omega=\alpha_{14}e^{14}+\alpha_{23}e^{23}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ $\alpha_{14}\alpha_{23}\neq 0$
$\mathfrak{r}'_{4,0,\delta}$: $[e_1,e_4]=-e_1,\ [e_2,e_4]=\delta e_3,$ $[e_3,e_4]=-\delta e_2,\ \delta>0$	$\omega=\alpha_{14}e^{14}+\alpha_{23}e^{23}+\alpha_{24}e^{24}+\alpha_{34}e^{34}$ $\alpha_{14}\alpha_{23}\neq 0$
$\mathfrak{h}_4$: $[e_1,e_2]=e_3,\ [e_1,e_4]=-\frac{1}{2}e_1$ $[e_3,e_4]=-e_3,$ $[e_2,e_4]=-e_1-\frac{1}{2}e_2$	$\omega=\alpha_{12}\left(e^{12}-e^{34}\right)+\alpha_{14}e^{14}+\alpha_{24}e^{24}$ $\alpha_{12}\neq 0$
$\mathfrak{d}_{4,1}$: $[e_1,e_2]=e_3,\ [e_1,e_4]=-e_1,$ $[e_3,e_4]=-e_3$	$\omega=\alpha_{12}\left(e^{12}-e^{34}\right)+\alpha_{14}e^{14}+\alpha_{24}e^{24}$ $\alpha_{12}\neq 0$
$\mathfrak{d}_{4,\frac{1}{2}}$: $[e_1,e_2]=e_3,\ [e_1,e_4]=-\frac{1}{2}e_1$ $[e_2,e_4]=-\frac{1}{2}e_2,\ [e_3,e_4]=-e_3$	$\omega=\alpha_{12}\left(e^{12}-e^{34}\right)+\alpha_{14}e^{14}+\alpha_{24}e^{24}$ $\alpha_{12}\neq 0$
$\mathfrak{d}_{4,\lambda}$: $[e_1,e_2]=e_3,\ [e_1,e_4]=-\lambda e_1,$ $[e_2,e_4]=(\lambda-1)e_2,$ $[e_3,e_4]=-e_3,$ $\lambda>\frac{1}{2},\ \lambda\neq 1,2$	$\omega=\alpha_{12}\left(e^{12}-e^{34}\right)+\alpha_{14}e^{14}+\alpha_{24}e^{24}$ $\alpha_{12}\neq 0$
$\mathfrak{d}_{4,2}$: $[e_1,e_2]=e_3,\quad [e_1,e_4]=-2e_1,$ $[e_2,e_4]=e_2,\quad [e_3,e_4]=-e_3$	$\omega=\alpha_{12}\left(e^{12}-e^{34}\right)+\alpha_{14}e^{14}+\alpha_{23}e^{23}+\alpha_{24}e^{24}$ $\alpha_{12}^2-\alpha_{14}\alpha_{23}\neq 0$
$\mathfrak{d}'_{4,\delta}$: $[e_1,e_2]=e_3,\ [e_1,e_4]=e_2-\frac{\delta}{2}e_1,$ $[e_2,e_4]=-e_1-\frac{\delta}{2}e_2,$ $[e_3,e_4]=-\delta e_3,\ \delta>0$	$\omega=\alpha_{12}\left(e^{12}-\delta e^{34}\right)+\alpha_{14}e^{14}+\alpha_{24}e^{24}$ $\alpha_{12}\neq 0$

Table 3.1: Four-dimensional symplectic Lie algebras as given in [115]. We have highlighted those which do not admit a para-Kähler structure.

For each symplectic Lie algebra $(\mathfrak{g},\omega)$ in Table 3.1 we will determine all the compatible para-Kähler structures. To do so, we start with an arbitrary $(1,1)$-tensor field $J=(a_{ij})$ and determine all the conditions on its components that are necessary for it to satisfy

(1) $J^2=\mathrm{Id}$,

(2) $\omega(Jx,Jy)=-\omega(x,y)$, and

(3) the integrability condition given by

$$N_J(x,y)=[Jx,Jy]-J[Jx,y]-J[x,Jy]+[x,y]=0,$$

so that the metric $\langle x,y\rangle=\omega(Jx,y)$ gives a para-Kähler metric. Note that the conditions (1)–(3) determine a system of polynomial equations on the parameters $\{a_{ij}\}$ which we will need to solve explicitly in order to describe all the possible para-Kähler structures in each case.

First of all, we point out that not all of the symplectic Lie algebras $(\mathfrak{g},\omega)$ in Table 3.1 admit a para-Kähler structure. Indeed, even though the Lie algebras

$$\mathfrak{rr}'_{3,0}: \quad [e_1,e_2]=-e_3,\ [e_1,e_3]=e_2,$$

$$\mathfrak{n}_4: \quad [e_1,e_4]=-e_2,\ [e_2,e_4]=-e_3,$$

$$\mathfrak{r}'_{4,0,\delta}: \quad [e_1,e_4]=-e_1,\ [e_2,e_4]=\delta e_3,\ [e_3,e_4]=-\delta e_2,$$

$$\mathfrak{d}'_{4,\delta}: \quad [e_1,e_2]=e_3,\ [e_1,e_4]=e_2-\tfrac{\delta}{2}e_1,\ [e_2,e_4]=-e_1-\tfrac{\delta}{2}e_2,\ [e_3,e_4]=-\delta e_3,$$

where $\delta>0$, admit symplectic structures, a straightforward calculation shows that none of them admits a para-Kähler structure.

Remark 3.8. According to [6], the four Lie algebras listed above correspond to the unimodular Lie group given by the trivial extension $\mathbb{R}\times\widetilde{E}(2)$ (see Lemma 1.11-*(i)*), the semi-direct extensions $\mathbb{R}^3\rtimes_{\varphi_i}\mathbb{R}e_4$ given by

$$\varphi_1=\mathrm{ad}(e_4)\colon \quad \varphi(e_1)=e_2,\quad \varphi(e_2)=e_3,\quad \varphi(e_3)=0,$$

$$\varphi_2=\mathrm{ad}(e_4)\colon \quad \varphi(e_1)=e_1,\quad \varphi(e_2)=-\delta e_3,\quad \varphi(e_3)=\delta e_2,$$

and the semi-direct extension $H^3\rtimes_{\varphi}\mathbb{R}e_4$ given by

$$\varphi=\mathrm{ad}(e_4)\colon \quad \varphi(e_1)=\tfrac{\delta}{2}e_1-e_2,\quad \varphi(e_2)=e_1+\tfrac{\delta}{2}e_2,\quad \varphi(e_3)=\delta e_3,$$

respectively.

Any other symplectic Lie algebra in Ovando's classification [115] does admit at least one para-Kähler structure. We will study all of them separately in the subsequent sections. For the sake of simplicity, we will omit the details of most of the calculations and outline the different para-Kähler structures and their curvatures. Our description of para-Kähler Lie algebras will be given up to symplectomorphical equivalence, i.e., up to Lie algebra automorphisms that preserve the symplectic structure.

3.3 Para-Kähler structures on $\mathfrak{r}_{4,0}$

Let $\{e_1,e_2,e_3,e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,0}$ determined by

$$[e_1,e_4]=-e_1,\quad [e_3,e_4]=-e_2,$$

which, as it was shown in [6], is equivalent to the Lie algebra of the semi-direct extension $\mathbb{R}e_4\ltimes_{\varphi}\mathbb{R}^3$ where

$$\varphi=\mathrm{ad}(e_4)\colon \quad \varphi(e_1)=e_1,\quad \varphi(e_2)=0,\quad \varphi(e_3)=-e_2.$$

The symplectic structures on $\mathfrak{r}_{4,0}$ are given by (see [115])

$$\omega=\alpha_{14}e^{14}+\alpha_{23}e^{23}+\alpha_{24}e^{24}+\alpha_{34}e^{34},\qquad \alpha_{14}\alpha_{23}\neq 0.$$

Making use of the automorphisms of $\mathfrak{r}_{4,0}$, which are of the form

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & z_{23} & z_{24} \\ 0 & 0 & z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \;\; z_{11}z_{22} \neq 0,$$

it is easy to see that any symplectic form on $\mathfrak{r}_{4,0}$ is symplectomorphically equivalent to

$$\Omega_\varepsilon = e^{14} + \varepsilon e^{23}, \quad \varepsilon^2 = 1.$$

Now, the automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,0}, \Omega_\varepsilon)$ are those determined by Φ above satisfying the conditions

$$\Phi_1 \colon z_{11} = 1, \quad z_{24} = 0, \quad z_{34} = 0, \quad z_{22} = 1,$$
$$\Phi_2 \colon z_{11} = 1, \quad z_{24} = 0, \quad z_{34} = 0, \quad z_{22} = -1.$$

Bearing this in mind, we determine the conditions that are necessary for the compatibility of the para-complex structure with the symplectic structures and the integrability of the para-complex structure (i.e., the integrability of the eigenspaces $\ker(J \mp \mathrm{Id})$ corresponding to the eigenvalues ± 1 of J, which is equivalent to the vanishing of the Nijenhuis tensor).

The compatibility condition $\Omega_\varepsilon(J\cdot, J\cdot) = -\Omega_\varepsilon(\cdot,\cdot)$ determines the following system of polynomial equation on the components $\{a_{ij}\}$

$$\varepsilon a_{31} - a_{42} = 0, \qquad -(\varepsilon a_{21} + a_{43}) = 0, \qquad -\varepsilon(a_{22} + a_{33}) = 0,$$
$$-(a_{11} + a_{44}) = 0, \qquad -(a_{12} + \varepsilon a_{34}) = 0, \qquad -a_{13} + \varepsilon a_{24} = 0,$$

and so the para-complex structure J must be such that

$$Je_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4, \quad Je_3 = \varepsilon a_{24}e_1 + a_{23}e_2 - a_{22}e_3 - \varepsilon a_{21}e_4,$$
$$Je_2 = -\varepsilon a_{34}e_1 + a_{22}e_2 + a_{32}e_3 + \varepsilon a_{31}e_4, \; Je_4 = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 - a_{11}e_4.$$

Now, a straightforward calculation shows that the non-zero components of the Nijenhuis tensor are determined by

$$N_J(e_2, e_1) = \varepsilon a_{34}a_{41}e_1 - (a_{32}a_{41} + \varepsilon a_{31}(a_{21} - a_{31}))\, e_2 - \varepsilon a_{31}^2 e_3 - \varepsilon a_{31}a_{41}e_4,$$
$$N_J(e_3, e_1) = -\varepsilon a_{41}(a_{24} + a_{34})e_1 + (2a_{22}a_{41} + \varepsilon a_{21}(a_{21} - a_{31}))\, e_2 + (a_{32}a_{41} + \varepsilon a_{21}a_{31})e_3 + \varepsilon a_{41}(a_{21} + a_{31})e_4,$$
$$N_J(e_3, e_2) = -\,(a_{24}a_{31} - (a_{21} - a_{31})a_{34})\, e_1 + \varepsilon(2a_{22}a_{31} - a_{21}a_{32})e_2 + \varepsilon a_{31}a_{32}e_3 + a_{31}^2 e_4,$$
$$N_J(e_4, e_1) = (1 - a_{11}^2 - a_{14}a_{41} + \varepsilon a_{31}a_{34})\, e_1 - (a_{31}(a_{11} + a_{22}) + a_{34}a_{41})\, e_2 - a_{31}a_{32}e_3 - \varepsilon a_{31}^2 e_4,$$

$$\begin{aligned} N_J(e_4,e_2) &= -\varepsilon\,(a_{14}a_{31} - (2a_{11}+a_{32})a_{34})\,e_1 \\ &\quad - (a_{32}(a_{11}+a_{22}) - \varepsilon(a_{21}-a_{31})a_{34})\,e_2 + (\varepsilon a_{31}a_{34} - a_{32}^2)e_3 + \varepsilon(a_{34}a_{41} - a_{31}a_{32})e_4, \\ N_J(e_4,e_3) &= \varepsilon\,(a_{14}a_{21} - a_{22}a_{34} - a_{11}(2a_{24}+a_{34}))\,e_1 \\ &\quad + (1 + a_{22}(2a_{11}+a_{22}) + \varepsilon a_{21}(a_{34}-a_{24}))\,e_2 + (a_{32}(a_{11}+a_{22}) - \varepsilon a_{24}a_{31})\,e_3 \\ &\quad + \varepsilon\,(a_{31}(a_{11}+a_{22}) - a_{24}a_{41})\,e_4, \end{aligned}$$

which again determine a system of polynomial equations on $\{a_{ij}\}$ that needs to be solved in order to obtain the integrability conditions on J. It immediately follows from the expressions of $N_J(e_2,e_1)$ and $N_J(e_4,e_2)$ that both a_{31} and a_{32} must vanish. In this situation

$$N_J(e_3,e_2) = a_{21}a_{34}e_1$$

and so there are two different possibilities depending on whether $a_{34}=0$ or not. A straightforward calculation shows that there can be no para-Kähler structures with $a_{34}\neq 0$. Therefore, we can assume that $a_{34}=0$. Then the condition $J^2=\mathrm{Id}$ gives the following system of polynomial equations

$$a_{11}^2 + a_{14}a_{41} = 0, \qquad a_{21}(a_{11}+a_{22}) + a_{24}a_{41} = 0,$$

$$a_{22}^2 - 1 = 0, \qquad a_{14}a_{21} + a_{24}(a_{22}-a_{11}) = 0,$$

from where it follows that $a_{22}=\varepsilon_2$, with $\varepsilon_2^2=1$. At this point,

$$N_J(e_3,e_1) = -\varepsilon a_{24}a_{41}e_1 + (2\varepsilon_2 a_{41} + \varepsilon a_{21}^2)e_2 + \varepsilon a_{21}a_{41}e_4,$$

which implies that there are again two different possibilities depending on whether$a_{41}=0$ or not. A straightforward calculation shows that the condition $a_{41}\neq 0$ is incompatible with the integrability of the para-complex structure. Consequently, we assume that $a_{41}=0$. Now,

$$N_J(e_3,e_1) = \varepsilon a_{21}^2 e_2, \qquad N_J(e_4,e_1) = \left(1 - a_{11}^2\right)e_1,$$

so $a_{21}=0$ and $a_{11}=\varepsilon_1$, for $\varepsilon_1^2=1$. At this point, the only non-zero component of the Nijenhuis tensor is

$$N_J(e_4,e_3) = -2\varepsilon_1\varepsilon a_{24}e_1 + 2(1+\varepsilon_1\varepsilon_2)e_2.$$

Therefore, $a_{24}=0$ and $\varepsilon_2=-\varepsilon_1$, and the condition $J^2=\mathrm{Id}$ automatically holds. What we have just obtained is the para-Kähler structures $(J,\langle\cdot,\cdot\rangle)$ on $\mathfrak{r}_{4,0}$ given by

$$\begin{cases} Je_1 = \varepsilon_1 e_1,\ Je_2 = -\varepsilon_1 e_2,\ Je_3 = a_{23}e_2 + \varepsilon_1 e_3,\ Je_4 = a_{14}e_1 - \varepsilon_1 e_4, \\ \langle\cdot,\cdot\rangle = 2(-\varepsilon_1 e^1\circ e^4 - \varepsilon_1\varepsilon\, e^2\circ e^3) - \varepsilon a_{23}e_3\circ e_3 - a_{14}e_4\circ e_4. \end{cases}$$

It is easy to see that all these structures are symplectomorphically equivalent to

$$(J,\langle\cdot,\cdot\rangle): \begin{cases} Je_1 = -e_1, \quad Je_2 = e_2, \quad Je_3 = -e_3, \quad Je_4 = e_4, \\ \langle\cdot,\cdot\rangle = 2(e^1\circ e^4 - \varepsilon\, e^2\circ e^3) \end{cases}$$

through the symplectomorphism determined by Φ_1 satisfying $z_{14} = -\frac{\varepsilon_1}{2}a_{14}$ and $z_{23} = \frac{\varepsilon_1}{2}a_{23}$. These structures correspond to the decomposition of $(\mathfrak{r}_{4,0}, \Omega_\varepsilon)$ as a direct sum of Lagrangian subalgebras as

$$\mathfrak{r}_{4,0} = \mathfrak{L} \oplus \mathfrak{L}' = \mathrm{span}\{e_2, e_4\} \oplus \mathrm{span}\{e_1, e_3\},$$

where $\mathcal{L}$ and $\mathcal{L}'$ are the eigenspaces associated to the eigenvalues ± 1 of the para-complex structure J, respectively.

The structures $(\mathfrak{r}_{4,0}, J, \langle\cdot,\cdot\rangle)$ are Ricci-flat and a straightforward calculation shows that their curvature tensors satisfy $\nabla R = 2e^4 \otimes R$, so they are recurrent with recurrence one-form $\xi = 2e^4$. Besides, the corresponding curvature tensors acting on the space of two-forms are given by

$$\mathcal{R}(e^3 \wedge e^4) = -e^1 \wedge e^2 = \varepsilon R(e_3, e_4, e_3, e_4) e^1 \wedge e^2.$$

Therefore, the para-Kähler structures $(\mathfrak{r}_{4,0}, J, \langle\cdot,\cdot\rangle)$ are special recurrent and, consequently, simply harmonic, so they are locally modelled on (3.1) and their curvature tensors are semi-symmetric. These structures are thus covered by Theorem 3.6.

Direct calculations similar to the ones described above show that the structures $(J, \langle\cdot,\cdot\rangle)$ admit opposite almost para-Kähler structures compatible with the opposite symplectic forms $\Omega' = e^{14} - \varepsilon e^{23} - \mu e^{34}$, with $\mu \in \mathbb{R}$.

3.4 Para-Kähler structures on $\mathfrak{r}_{4,-1}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-1}$ determined by

$$[e_1, e_4] = -e_1, \qquad [e_2, e_4] = e_2, \qquad [e_3, e_4] = e_3 - e_2,$$

which, according to [6], corresponds to the Lie algebra of the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi \mathbb{R}^3$ where

$$\varphi = \mathrm{ad}(e_4): \quad \varphi(e_1) = e_1, \quad \varphi(e_2) = -e_2, \quad \varphi(e_3) = e_3 - e_2.$$

Any symplectic form on $\mathfrak{r}_{4,-1}$ is of the form

$$\omega = \alpha_{13}e^{13} + \alpha_{14}e^{14} + \alpha_{24}e^{24} + \alpha_{34}e^{34}, \qquad \alpha_{13}\alpha_{24} \neq 0,$$

and the automorphisms of this Lie algebra are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & z_{23} & z_{24} \\ 0 & 0 & z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11}z_{22} \neq 0.$$

Now, it is not difficult to check that any symplectic structure is symplectomorphically equivalent to $\Omega = e^{13} + e^{24}$. Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,-1}, \Omega)$ are given by Φ above with $z_{11} = 1$, $z_{22} = 1$, $z_{34} = 0$ and $z_{23} = z_{14}$. Now, a straightforward calculation as in the previous section shows that any para-Kähler structure on $\mathfrak{r}_{4,-1}$ is equivalent to

$$(J, \langle\cdot,\cdot\rangle) : \begin{cases} Je_1 = -e_1, \quad Je_2 = e_2, \quad Je_3 = -\kappa e_1 + e_3, \quad Je_4 = -e_4, \\ \langle\cdot,\cdot\rangle = 2(e^1 \circ e^3 - e^2 \circ e^4) + \kappa\, e^3 \circ e^3, \quad \kappa \in \mathbb{R}, \end{cases}$$

which correspond to the Lagrangian decomposition

$$\mathfrak{r}_{4,-1} = \mathfrak{L} \oplus \mathfrak{L}' = \operatorname{span}\{e_2, e_3 - \tfrac{\kappa}{2} e_1\} \oplus \operatorname{span}\{e_1, e_4\}.$$

The structures above are flat if $\kappa = 0$. Otherwise, they are Ricci-flat, locally symmetric – thus covered by Theorem 3.1 – and the underlying structures are locally modelled on (3.1) with

$$\Psi(x^1, x^2) = \pm(x^2)^2.$$

Furthermore, the structures above admit opposite almost para-Kähler structures which are compatible with the opposite symplectic form $\Omega' = e^{13} - e^{24} + \mu e^{34}$ with $\mu \in \mathbb{R}$.

3.5 Para-Kähler structures on $\mathfrak{r}_2\mathfrak{r}_2$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ determined by

$$[e_1, e_2] = e_2, \qquad [e_3, e_4] = e_4,$$

which, according to [6], corresponds to the Lie algebra of the non-unimodular semi-direct extension $E(1,1) \rtimes \mathbb{R}$ isomorphic to $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$ as in Lemma 1.9.

We make use of Lie algebra automorphisms to simplify the final expressions of the structures. The automorphisms of this Lie algebra are given by

$$\Phi_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & z_{23} & z_{24} \\ 1 & 0 & 0 & 0 \\ z_{41} & z_{42} & 0 & 0 \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & z_{43} & z_{44} \end{pmatrix}.$$
$$z_{24} z_{42} \neq 0 \qquad\qquad z_{22} z_{44} \neq 0$$

Any symplectic structure on $\mathfrak{r}_2\mathfrak{r}_2$ is given by

$$\omega = \alpha_{12} e^{12} + \alpha_{13} e^{13} + \alpha_{34} e^{34}, \qquad \alpha_{12}\alpha_{34} \neq 0,$$

and Ovando showed in [115] that these $(\mathfrak{r}_2\mathfrak{r}_2, \omega)$ are symplectomorphically equivalent to

$$\Omega_\lambda = e^{12} + e^{34} + \lambda e^{13} \quad \text{with} \quad \lambda \geq 0.$$

It is easy to check that the symplectic form Ω_λ is preserved by any automorphism Φ_2 with $z_{22} = z_{44} = 1$ (in the special case where $\lambda = 0$, the automorphisms Φ_1 with $z_{24} = z_{42} = 1$ also preserve the symplectic form Ω_0). Considering the action of the symplectomorphisms Φ_2 that preserve Ω_λ we are allowed to restrict our study to the para-complex structures $J = (a_{ij})$ satisfying one of the following conditions.

Case 1. $a_{12} = 0$ and $a_{14} = 0$.
Case 2. $a_{12} = 0$ and $a_{14} \neq 0$, in which case one may also assume $a_{13} = 0$.
Case 3. $a_{12} \neq 0$ and $a_{14} = 0$, in which case one may also assume $a_{11} = 0$.
Case 4. $a_{12} \neq 0$ and $a_{14} \neq 0$, in which case one may also assume $a_{11} = a_{13} = 0$.

In each case we determine the necessary conditions for the compatibility of the para-complex structure with the symplectic structures and the integrability of the latter. To do so, we distinguish the two possibilities corresponding to the symplectic structures Ω_λ (with $\lambda \neq 0$) and Ω_0, since they give rise to different geometries.

3.5.1 Para-Kähler structures on $(\mathfrak{r}_2\mathfrak{r}_2, \Omega_\lambda)$ with $\lambda > 0$

A straightforward calculation reveals that the Cases 2, 3 and 4 above are not compatible with the existence of para-Kähler structures. Therefore we focus on the case $a_{12} = a_{14} = 0$. Long but straightforward calculations show that the para-Kähler structures $(J, \langle\cdot,\cdot\rangle)$ which are compatible with Ω_λ are symplectomorphically equivalent to one of the following three families of para-Kähler structures

$$(J_{11}, \langle\cdot,\cdot\rangle_{11}) : \begin{cases} J_{11}e_1 = -e_1, \quad J_{11}e_2 = e_2, \quad J_{11}e_3 = e_3, \quad J_{11}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{11} = 2(e^1 \circ e^2 + \lambda e^1 \circ e^3 - e^3 \circ e^4), \end{cases}$$

$$(J_{12}, \langle\cdot,\cdot\rangle_{12}) : \begin{cases} J_{12}e_1 = -e_1 + 2\lambda e_2 - 2e_3, \;\; J_{12}e_2 = e_2, \\ J_{12}e_3 = e_3, \;\; J_{12}e_4 = 2e_2 - e_4, \\ \langle\cdot,\cdot\rangle_{12} = 2(e^1 \circ e^2 + \lambda e^1 \circ e^3 + 2e^1 \circ e^4 - e^3 \circ e^4), \end{cases}$$

$$(J_{13}, \langle\cdot,\cdot\rangle_{13}) : \begin{cases} J_{13}e_1 = -e_1, \;\; J_{13}e_2 = e_2 - 2e_4, \\ J_{13}e_3 = 2e_1 + e_3 + 2\lambda e_4, \;\; J_{13}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{13} = 2(e^1 \circ e^2 + \lambda e^1 \circ e^3 - 2e^2 \circ e^3 - e^3 \circ e^4). \end{cases}$$

The structures above correspond to the decompositions $\mathfrak{r}_2\mathfrak{r}_2 = \mathfrak{L}_{1i} \oplus \mathfrak{L}'_{1i}$ as direct sums of Lagrangian subalgebras given by

$$\begin{aligned} \mathfrak{r}_2\mathfrak{r}_2 &= \mathfrak{L}_{11} \oplus \mathfrak{L}'_{11} = \operatorname{span}\{e_2, e_3\} \oplus \operatorname{span}\{e_1, e_4\} \\ &= \mathfrak{L}_{12} \oplus \mathfrak{L}'_{12} = \operatorname{span}\{e_2, e_3\} \oplus \operatorname{span}\{e_4 - e_2, e_1 + e_3 - \lambda e_2\} \\ &= \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \operatorname{span}\{e_4 - e_2, e_1 + e_3 + \lambda e_2\} \oplus \operatorname{span}\{e_1, e_4\}. \end{aligned}$$

Moreover, a straightforward calculation shows that all the structures above are non-flat and Ricci-flat with recurrent curvature, i.e., their curvature tensors satisfy

$$\nabla_{1i} R_{1i} = \xi_{1i} \otimes R_{1i},$$

and their recurrence one-forms ξ_{1i} are given by

$$\xi_{11} = 2(e^1 + e^3), \quad \xi_{12} = 2e^3, \quad \xi_{13} = 2e^1.$$

Furthermore, the corresponding curvature operators $\mathcal{R}_{1i} : \Lambda^2 \to \Lambda^2$ acting on the space of two-forms are given by

$$\mathcal{R}_{1i}(e^1 \wedge e^2) = -\lambda e^2 \wedge e^4 = R_{1i}(e_1, e_3, e_3, e_1) e^2 \wedge e^4, \quad \text{for } i = 1, 2, 3,$$

from where it follows that all the para-Kähler structures above are special recurrent and thus locally modelled on (3.1). Consequently, their curvature tensors are semi-symmetric and they are covered by Theorem 3.6.

Finally, the structures $(J_{11}, \langle\cdot,\cdot\rangle_{11})$ admit opposite almost para-Kähler structures compatible with the opposite symplectic forms $\Omega'_{11} = e^{12} - e^{34} + (\mu - \lambda)e^{13}$ with $\mu \in \mathbb{R}$, while the structures $(J_{12}, \langle\cdot,\cdot\rangle_{12})$ and $(J_{13}, \langle\cdot,\cdot\rangle_{13})$ do not admit opposite almost para-Kähler structures.

3.5.2 Para-Kähler structures on $(\mathfrak{r}_2\mathfrak{r}_2, \Omega_0)$

Proceeding as in the previous case, it is straightforward to check that no para-Kähler structures exist in Case 2 while the other three cases give rise to three essentially different geometries.

Para-Kähler structures of constant para-holomorphic sectional curvature

Assuming that $a_{12} \neq 0$ and $a_{14} \neq 0$ as in Case 4, any para-Kähler structure is equivalent to the structures $(J_{21}, \langle\cdot,\cdot\rangle_{21})$, which are given by

$$\begin{cases} J_{21}e_1 = e_3 - \frac{1}{\kappa}e_2,\ J_{21}e_2 = -\kappa(e_1 + e_3),\ J_{21}e_4 = e_4 - e_2 - \kappa(e_1 + e_3), \\ J_{21}e_3 = -e_3, \\ \langle\cdot,\cdot\rangle_{21} = \kappa(e^2 \circ e^2 + e^4 \circ e^4 + 2e^2 \circ e^4) - 2e^1 \circ e^4 + 2e^3 \circ e^4 - \frac{1}{\kappa}e^1 \circ e^1, \end{cases}$$

where $\kappa \neq 0$. These correspond to the Lagrangian decompositions

$$\mathfrak{r}_2\mathfrak{r}_2 = \mathfrak{L}_{21} \oplus \mathfrak{L}'_{21} = \operatorname{span}\{e_4 - e_2, e_1 + e_3 - \tfrac{1}{\kappa}e_2\} \oplus \operatorname{span}\{e_3, e_2 + \kappa e_1\}.$$

Besides, one can easily check that their para-holomorphic sectional curvatures are constantly $H_{21} = \kappa$, so these structures correspond to Theorem 3.1-*(i.b)*.

Flat para-Kähler structures

Assuming that $a_{12}=0$ and $a_{14}=0$ as in Case 1, if $a_{34}=0$, then there are three inequivalent flat para-Kähler structures given by

$$(J_{22},\langle\cdot,\cdot\rangle_{22}):\begin{cases} J_{22}e_1=-e_1,\ J_{22}e_2=e_2-2e_4,\ J_{22}e_3=2e_1+e_3,\\ J_{22}e_4=-e_4,\\ \langle\cdot,\cdot\rangle_{22}=2(e^1\circ e^2-2e^2\circ e^3-e^3\circ e^4), \end{cases}$$

$$(J_{23},\langle\cdot,\cdot\rangle_{23}):\begin{cases} J_{23}e_1=-e_1,\ \ J_{23}e_2=e_2,\ \ J_{23}e_3=\varepsilon e_3,\ \ J_{23}e_4=-\varepsilon e_4,\\ \langle\cdot,\cdot\rangle_{23}=2(e^1\circ e^2-\varepsilon e^3\circ e^4),\quad \varepsilon=\pm1, \end{cases}$$

$$(J_{24},\langle\cdot,\cdot\rangle_{24}):\begin{cases} J_{24}e_1=-e_1-2e_3,\ \ J_{24}e_2=e_2,\ \ J_{24}e_3=e_3,\\ J_{24}e_4=2e_2-e_4,\\ \langle\cdot,\cdot\rangle_{24}=2(e^1\circ e^2+2e^1\circ e^4-e^3\circ e^4), \end{cases}$$

which correspond to the Lagrangian decompositions $\mathfrak{r}_2\mathfrak{r}_2=\mathfrak{L}_{2i}\oplus\mathfrak{L}'_{2i}$ given by

$$\begin{aligned} \mathfrak{r}_2\mathfrak{r}_2&=\mathfrak{L}_{22}\oplus\mathfrak{L}'_{22}=\operatorname{span}\{e_1+e_3,e_4-e_2\}\oplus\operatorname{span}\{e_1,e_4\}\\ &=\mathfrak{L}_{23}\oplus\mathfrak{L}'_{23}=\operatorname{span}\{e_2,e_3\}\oplus\operatorname{span}\{e_1,e_4\} &&(\varepsilon=1)\\ &=\mathfrak{L}_{23}\oplus\mathfrak{L}'_{23}=\operatorname{span}\{e_2,e_4\}\oplus\operatorname{span}\{e_1,e_3\} &&(\varepsilon=-1)\\ &=\mathfrak{L}_{24}\oplus\mathfrak{L}'_{24}=\operatorname{span}\{e_2,e_3\}\oplus\operatorname{span}\{e_1+e_3,e_4-e_2\}. \end{aligned}$$

Para-Kähler and opposite para-Kähler structures

The remaining possibilities corresponding to Case 1 with $a_{34}\neq0$ and Case 3 give rise to two families of para-Kähler structures determined by

$$(J_{25},\langle\cdot,\cdot\rangle_{25}):\begin{cases} J_{25}e_1=-\frac{1}{\kappa_1}e_2,\ \ J_{25}e_2=-\kappa_1e_1,\ \ J_{25}e_3=-e_3,\ \ J_{25}e_4=e_4,\\ \langle\cdot,\cdot\rangle_{25}=-\frac{1}{\kappa_1}e^1\circ e^1+\kappa_1e^2\circ e^2+2e^3\circ e^4, \end{cases}$$

$$(J_{26},\langle\cdot,\cdot\rangle_{26}):\begin{cases} J_{26}e_1=-\frac{1}{\kappa_1}e_2,\ J_{26}e_2=-\kappa_1e_1,\ J_{26}e_3=-\frac{1}{\kappa_2}e_4,\\ J_{26}e_4=-\kappa_2e_3,\\ \langle\cdot,\cdot\rangle_{26}=-\frac{1}{\kappa_1}e^1\circ e^1+\kappa_1e^2\circ e^2-\frac{1}{\kappa_2}e^3\circ e^3+\kappa_2e^4\circ e^4, \end{cases}$$

where $\kappa_1\kappa_2 \neq 0$. The corresponding Lagrangian decompositions are given by

$$\begin{aligned}\mathfrak{r}_2\mathfrak{r}_2 &= \mathfrak{L}_{25} \oplus \mathfrak{L}'_{25} = \operatorname{span}\{e_4, e_2 - \kappa_1 e_1\} \oplus \operatorname{span}\{e_3, e_2 + \kappa_1 e_1\} \\ &= \mathfrak{L}_{26} \oplus \mathfrak{L}'_{26} = \operatorname{span}\{e_4 - \kappa_2 e_3, e_2 - \kappa_1 e_1\} \oplus \operatorname{span}\{e_4 + \kappa_2 e_3, e_2 + \kappa_1 e_1\}.\end{aligned}$$

Now, it is straightforward to see that both metrics are locally symmetric, so they are covered by Theorem 3.1-*(i.d)*. Their Ricci operators are diagonalizable with respect to the basis $\{e_1, e_2, e_3, e_4\}$ with Ricci curvatures

$$\operatorname{Ric}_{25} = \operatorname{diag}[\kappa_1, \kappa_1, 0, 0] \quad \text{and} \quad \operatorname{Ric}_{26} = \operatorname{diag}[\kappa_1, \kappa_1, \kappa_2, \kappa_2].$$

Therefore, the underlying pseudo-Riemannian manifolds split locally as a product of two Lorentzian surfaces of constant Gaussian curvature, namely, $N_1(\kappa_1) \times \mathbb{L}^2$ or $N_1(\kappa_1) \times N_2(\kappa_2)$, where $\mathbb{L}^2$ denotes the Minkowskian plane. Thus they also admit opposite para-Kähler structures. Furthermore, the metrics $\langle \cdot, \cdot \rangle_{26}$ are Einstein if $\kappa_1 = \kappa_2$ and locally conformally flat if $\kappa_1 = -\kappa_2$, in which case they correspond to the metrics in Corollary 3.2-*(iii)*.

3.6 Para-Kähler structures on $\mathfrak{rh}_3$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{rh}_3$ determined by

$$[e_1, e_2] = e_3,$$

which corresponds to the Lie algebra of either $H^3 \times \mathbb{R}$ or the semi-direct extension $\mathbb{R}e_1 \ltimes_\varphi \mathbb{R}^3$ given by

$$\varphi = \operatorname{ad}(e_1): \quad \varphi(e_2) = e_3, \quad \varphi(e_3) = 0, \quad \varphi(e_4) = 0.$$

Symplectic forms on $\mathfrak{rh}_3$ are given by

$$\begin{gathered}\omega = \alpha_{12}e^{12} + \alpha_{13}e^{13} + \alpha_{14}e^{14} + \alpha_{23}e^{23} + \alpha_{24}e^{24}, \\ \alpha_{14}\alpha_{23} - \alpha_{13}\alpha_{24} \neq 0,\end{gathered}$$

as shown in [115]. All these symplectic structures $(\mathfrak{rh}_3, \omega)$ are symplectomorphically equivalent to $\Omega = e^{14} + e^{23}$ through an automorphism of the form

$$\Phi = \begin{pmatrix} z_{11} & z_{12} & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ z_{31} & z_{32} & z_{11}z_{22} - z_{12}z_{21} & z_{34} \\ z_{41} & z_{42} & 0 & z_{44} \end{pmatrix}, \quad \text{with} \quad (z_{11}z_{22} - z_{12}z_{21})z_{44} \neq 0.$$

The ones that preserve the symplectic structure $(\mathfrak{rh}_3, \Omega)$ determined by $z_{11}z_{22}^2 = 1$, $z_{12}z_{22} = -z_{34}$, $z_{21} = 0$, $z_{42} = z_{22}^3 z_{31} - z_{22}z_{34}z_{41}$ and $z_{44} = z_{22}^2$. A long but straightforward calculation shows that any para-Kähler structure on $\mathfrak{rh}_3$ must be flat and equivalent to one of the following two para-Kähler structures

$$(J_1, \langle\cdot,\cdot\rangle_1) : \begin{cases} J_1e_1 = e_1, \quad J_1e_2 = -e_2, \quad J_1e_3 = e_3, \quad J_1e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_1 = 2(-e^1 \circ e^4 + e^2 \circ e^3), \end{cases}$$

$$(J_2, \langle\cdot,\cdot\rangle_2) : \begin{cases} J_2e_1 = -e_2, \quad J_2e_2 = -e_1, \quad J_2e_3 = e_4, \quad J_2e_4 = e_3, \\ \langle\cdot,\cdot\rangle_2 = 2(e^1 \circ e^3 + e^2 \circ e^4). \end{cases}$$

In each case, $\mathfrak{rh}_3$ decomposes as a direct sum of Lagrangian subalgebras as

$$\begin{aligned} \mathfrak{rh}_3 &= \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \mathrm{span}\{e_1, e_3\} \oplus \mathrm{span}\{e_2, e_4\} \\ &= \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \mathrm{span}\{e_3 + e_4, e_2 - e_1\} \oplus \mathrm{span}\{e_1 + e_2, e_4 - e_3\}. \end{aligned}$$

3.7 Para-Kähler structures on $\mathfrak{rr}_{3,0}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{rr}_{3,0}$ determined by

$$[e_1, e_2] = e_2,$$

which corresponds to the Lie algebra of $E(1,1) \times \mathbb{R}$ or to the semi-direct extension $\mathbb{R}e_1 \ltimes_\varphi \mathbb{R}^3$ given by

$$\varphi = \mathrm{ad}(e_1)\colon \quad \varphi(e_2) = e_2, \quad \varphi(e_3) = 0, \quad \varphi(e_4) = 0.$$

The symplectic structures on $\mathfrak{rr}_{3,0}$ are given by

$$\omega = \alpha_{12}e^{12} + \alpha_{13}e^{13} + \alpha_{14}e^{14} + \alpha_{34}e^{34},$$
$$\alpha_{12}\alpha_{34} \neq 0,$$

as shown in [115]. All these symplectic structures $(\mathfrak{rr}_{3,0}, \omega)$ are symplectomorphically equivalent to $\Omega = e^{12} + e^{34}$ through an automorphism of the form

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & 0 & 0 \\ z_{31} & 0 & z_{33} & z_{34} \\ z_{41} & 0 & z_{43} & z_{44} \end{pmatrix}, \quad \text{with} \quad (z_{33}z_{44} - z_{34}z_{43})z_{22} \neq 0.$$

Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{rr}_{3,0}, \Omega)$ are the ones determined by Φ above satisfying

$$\Phi_1 : z_{22} = 1, \quad z_{31} = 0, \quad z_{41} = 0, \quad z_{44}z_{33} = 1 + z_{34}z_{43} \quad \text{with } z_{33} \neq 0,$$

$$\Phi_2 : z_{22} = 1, \quad z_{31} = 0, \quad z_{41} = 0, \quad z_{33} = 0, \quad z_{43}z_{34} = -1.$$

Proceeding as in the previous cases, a straightforward calculation shows that any para-Kähler structure on $(\mathfrak{rr}_{3,0}, \Omega)$ is equivalent to one of the following two families

$$(J_1,\ \langle\cdot,\cdot\rangle_1):\begin{cases} J_1e_1=\varepsilon\, e_1, \quad J_1e_2=-\varepsilon\, e_2, \quad J_1e_3=e_3, \quad J_1e_4=-e_4,\\ \langle\cdot,\cdot\rangle_1=-2(\varepsilon\, e^1\circ e^2+e^3\circ e^4), \quad \varepsilon=\pm 1,\end{cases}$$

$$(J_2,\ \langle\cdot,\cdot\rangle_2):\begin{cases} J_2e_1=\frac{1}{\kappa}e_2, \quad J_2e_2=\kappa e_1, \quad J_2e_3=e_3, \quad J_2e_4=-e_4,\\ \langle\cdot,\cdot\rangle_2=\frac{1}{\kappa}e^1\circ e^1-\kappa e^2\circ e^2-2e^3\circ e^4, \quad \kappa\neq 0.\end{cases}$$

These correspond to the Lagrangian decompositions

$$\begin{aligned}\mathfrak{rr}_{3,0}&=\mathfrak{L}_1\oplus\mathfrak{L}_1'=\operatorname{span}\{e_1,e_3\}\oplus\operatorname{span}\{e_2,e_4\} && (\varepsilon=1)\\ &=\mathfrak{L}_1\oplus\mathfrak{L}_1'=\operatorname{span}\{e_2,e_3\}\oplus\operatorname{span}\{e_1,e_4\} && (\varepsilon=-1)\\ &=\mathfrak{L}_2\oplus\mathfrak{L}_2'=\operatorname{span}\{e_2+\kappa e_1,e_3\}\oplus\operatorname{span}\{e_2-\kappa e_1,e_4\}.\end{aligned}$$

Both structures $(J_1,\langle\cdot,\cdot\rangle_1)$, with $\varepsilon=\pm1$, are flat, while all the structures $(J_2,\langle\cdot,\cdot\rangle_2)$ are locally symmetric with diagonalizable Ricci operator

$$\mathrm{Ric}_2=-\operatorname{diag}[\kappa,\kappa,0,0].$$

Thus, the underlying pseudo-Riemannian manifolds split locally as products of the form $N\times\mathbb{L}^2$, where N is a Lorentzian surface of constant Gaussian curvature $K_N=-\kappa$ and $\mathbb{L}^2$. Therefore, $(J_2,\langle\cdot,\cdot\rangle_2)$ are para-Kähler and opposite para-Kähler structures, and correspond to the metrics in Theorem 3.1-*(i.d)*

3.8 Para-Kähler structures on $\mathfrak{rr}_{3,-1}$

Let $\{e_1,e_2,e_3,e_4\}$ be a basis of the Lie algebra $\mathfrak{rr}_{3,-1}$ determined by

$$[e_1,e_2]=e_2, \qquad [e_1,e_3]=-e_3.$$

which corresponds to a unimodular semi-direct extension of $E(1,1)$, given by the Lie algebra of $\mathbb{R}\times E(1,1)$ as in Lemma 1.9, and can also be seen as the semi-direct extension $\mathbb{R}e_1\ltimes_\varphi\mathbb{R}^3$ given by

$$\varphi=\operatorname{ad}(e_1):\quad \varphi(e_2)=e_2, \quad \varphi(e_3)=-e_3, \quad \varphi(e_4)=0.$$

The symplectic structures on $\mathfrak{rr}_{3,-1}$ are given by

$$\omega=\alpha_{12}e^{12}+\alpha_{13}e^{13}+\alpha_{14}e^{14}+\alpha_{23}e^{23}, \qquad \alpha_{14}\alpha_{23}\neq 0,$$

as shown in [115]. All these symplectic structures $(\mathfrak{rr}_{3,-1},\omega)$ are symplectomorphically equivalent to $\Omega=e^{14}+e^{23}$ through an automorphism of the form

$$\Phi_1=\begin{pmatrix}1&0&0&0\\ z_{21}&z_{22}&0&0\\ z_{31}&0&z_{33}&0\\ z_{41}&0&0&z_{44}\end{pmatrix}\quad\text{or}\quad\Phi_2=\begin{pmatrix}-1&0&0&0\\ z_{21}&0&z_{23}&0\\ z_{31}&z_{32}&0&0\\ z_{41}&0&0&z_{44}\end{pmatrix},$$

with $z_{22}z_{33}z_{44}\neq 0$ in the first case and $z_{23}z_{32}z_{44}\neq 0$ in the second one. Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{rr}_{3,-1},\Omega)$ are given by

$$\Phi_1: z_{21}=0,\quad z_{31}=0,\quad z_{33}z_{22}=1,\quad z_{44}=1,$$

$$\Phi_2: z_{21}=0,\quad z_{31}=0,\quad z_{32}z_{23}=-1,\quad z_{44}=-1.$$

Proceeding as in the previous sections we see that para-Kähler structure on $(\mathfrak{rr}_{3,-1},\Omega)$ corresponds to one of the following possibilities.

Flat para-Kähler structures on $\mathfrak{rr}_{3,-1}$

Assuming that $a_{14}\neq 0$, the corresponding para-Kähler structures are all flat and equivalent to

$$(J_1,\langle\cdot,\cdot\rangle_1):\begin{cases}J_1e_1=\frac{1}{\kappa}e_4,\quad J_1e_2=-e_2,\quad J_1e_3=e_3,\quad J_1e_4=\kappa e_1,\\ \langle\cdot,\cdot\rangle_1=\frac{1}{\kappa}e^1\circ e^1+2e^2\circ e^3-\kappa e^4\circ e^4,\qquad \kappa\neq 0.\end{cases}$$

These structures correspond to the decompositions

$$\mathfrak{rr}_{3,-1}=\mathfrak{L}_1\oplus\mathfrak{L}_1'=\operatorname{span}\{e_4+\kappa e_1,e_3\}\oplus\operatorname{span}\{e_4-\kappa e_1,e_2\}$$

of $(\mathfrak{rr}_{3,-1},\Omega)$ as direct sums of Lagrangian subalgebras.

Ricci-flat para-Kähler structures on $\mathfrak{rr}_{3,-1}$

If the component of the para-complex structure $a_{14}=0$, any para-Kähler structure is equivalent to

$$(J_2,\langle\cdot,\cdot\rangle_2):\begin{cases}J_2e_1=-e_1,\quad J_2e_2=-e_2,\quad J_2e_3=e_3-\kappa e_2,\quad J_2e_4=e_4,\\ \langle\cdot,\cdot\rangle_2=2(e^1\circ e^4+e^2\circ e^3)+\kappa e^3\circ e^3,\quad \kappa=0,\pm 1.\end{cases}$$

The symplectic Lie algebra $(\mathfrak{rr}_{3,-1},\Omega)$ decomposes as a direct sum of Lagrangian subalgebras as $\mathfrak{rr}_{3,-1}=\mathfrak{L}_2\oplus\mathfrak{L}_2'=\operatorname{span}\{e_4,e_3-\frac{\kappa}{2}e_2\}\oplus\operatorname{span}\{e_1,e_2\}$.

The para-Kähler structures $(J_2,\langle\cdot,\cdot\rangle_2)$ are flat if $\kappa=0$ and Ricci-flat otherwise, so they are covered by Theorem 3.1-*(i.a)*. Moreover, if $\kappa\neq 0$, then the metric has recurrent curvature with recurrence one-form $\xi = 2e^1$ (i.e., $\nabla_2 R_2 = 2e^1 \otimes R_2$), and the corresponding curvature operator acting on the space of two-forms is given by

$$\mathcal{R}_2(e^1 \wedge e^3) = 2\,\kappa\, e^2 \wedge e^4 = R_2(e_1, e_3, e_3, e_1) e^2 \wedge e^4.$$

Therefore, they are special recurrent and simply harmonic manifolds modelled on (3.1). Hence, their curvature tensor is semi-symmetric. Furthermore, the structures given by $(J_2, \langle\cdot,\cdot\rangle_2)$ with $\kappa \neq 0$ admit opposite almost para-Kähler structures compatible with the opposite symplectic form $\Omega_2' = e^{14} - e^{23} + \mu e^{13}$ with $\mu \in \mathbb{R}$.

3.9 Para-Kähler structures on $\mathfrak{r}_2'$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the symplectic Lie algebra $\mathfrak{r}_2'$ determined by

$$[e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_2, e_3] = e_4, \quad [e_2, e_4] = -e_3,$$

which, according to [6], corresponds to the non-unimodular semi-direct extension $\widetilde{E}(2) \rtimes \mathbb{R}$ isomorphic to $\mathfrak{aff}(\mathbb{C}) \times \mathfrak{aff}(\mathbb{C})$ as in Lemma 1.11.

It was shown in [115] that the symplectic forms on $\mathfrak{r}_2'$ are given by

$$\omega = \alpha_{12} e^{12} + \alpha_{13}(e^{13} - e^{24}) + \alpha_{14}(e^{14} + e^{23}), \qquad \alpha_{13}^2 + \alpha_{14}^2 \neq 0.$$

The automorphisms of $\mathfrak{r}_2'$ are of the form

$$\Phi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z_{31} & z_{32} & z_{33} & z_{34} \\ -z_{32} & z_{31} & -z_{34} & z_{33} \end{pmatrix} \quad \text{or} \quad \Phi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{32} & -z_{31} & z_{34} & -z_{33} \end{pmatrix},$$

with $z_{33}^2 + z_{34}^2 \neq 0$, from where it follows that any symplectic form is symplectomorphically equivalent to $\Omega_\lambda = \lambda e^{12} + e^{14} + e^{23}$, for some $\lambda \in \mathbb{R}$. We emphasize that even though there exist Kähler structures with associated symplectic form Ω_λ for $\lambda \neq 0$ (see [116]), a straightforward calculation shows that Ω_λ does not support any para-Kähler structure for $\lambda \neq 0$. On the other hand, there exist para-Kähler structures whose associated symplectic structure is $\Omega_0 = e^{14} + e^{23}$. In order to describe these structures, we make use of the automorphisms preserving the symplectic structure $(\mathfrak{r}_2', \Omega_0)$, which are given by Φ_1 (resp., Φ_2) above with $z_{33} = 1$ and $z_{34} = 0$ (resp., $z_{33} = -1$ and $z_{34} = 0$). A long but straightforward calculation shows that any para-Kähler structure on $(\mathfrak{r}_2', \Omega_0)$ corresponds to one of the different situations given by $a_{13} \neq 0$ or $a_{13} = 0$. While the Ricci operator is diagonalizable in the case $a_{13} = 0$, this does not happen when $a_{13} \neq 0$. If $a_{13} = 0$, there are three essentially different cases depending on whether the component of the para-complex structure $a_{14} = 0$ or $a_{14} \neq 0$.

Para-Kähler structures on $\mathfrak{r}'_2$ with complex Ricci operator

Assuming the component of the para-complex structure $a_{13} \neq 0$, then para-Kähler structures are equivalent to $(J_1, \langle\cdot,\cdot\rangle_1)$ given by

$$\begin{cases} J_1 e_1 = \widehat{\beta} e_3 + \widehat{\alpha} e_4,\ J_1 e_2 = -\widehat{\alpha} e_3 + \widehat{\beta} e_4,\ J_1 e_3 = \beta e_1 - \alpha e_2,\ J_1 e_4 = \alpha e_1 + \beta e_2, \\ \langle\cdot,\cdot\rangle_1 = \widehat{\alpha}(e^1 \circ e^1 - e^2 \circ e^2) + \alpha(e^3 \circ e^3 - e^4 \circ e^4) + 2(\widehat{\beta} e^1 \circ e^2 - \beta e^3 \circ e^4), \end{cases}$$

where $\widehat{\beta} = \frac{\beta}{\beta^2+\alpha^2}$ and $\widehat{\alpha} = \frac{\alpha}{\beta^2+\alpha^2}$, $\alpha, \beta \in \mathbb{R}$, and $\beta \neq 0$. In this situation, $\mathfrak{r}'_2$ decomposes into direct sums of Lagrangian subalgebras $\mathfrak{r}'_2 = \mathfrak{L}_1 \oplus \mathfrak{L}'_1$ where

$$\mathfrak{L}_1 = \operatorname{span}\{e_4 + \alpha e_1 + \beta e_2, e_3 + \beta e_1 - \alpha e_2\},$$
$$\mathfrak{L}'_1 = \operatorname{span}\{e_4 - \alpha e_1 - \beta e_2, e_3 - \beta e_1 + \alpha e_2\}.$$

The Ricci operators associated to the metrics above are complex diagonalizable with eigenvalues $-2(\alpha \pm \beta\sqrt{-1})$ and the underlying pseudo-Riemannian structures are locally symmetric. Furthermore, their self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues

$$W^{\pm} = \operatorname{diag}\left[-\tfrac{4}{3}\alpha, \tfrac{2}{3}\alpha, \tfrac{2}{3}\alpha\right].$$

Therefore, they correspond to the metrics in Theorem 3.1-*(ii.a)* and the metrics are locally conformally flat if and only if $\alpha = 0$, as in Corollary 3.2-*(i)*.

Remark 3.9. The Ricci tensor ρ_1 defines another para-Kähler structure $(\mathfrak{r}'_2, J_1, \rho_1)$ with the same Levi-Civita connection as $(\mathfrak{r}'_2, J_1, \langle\cdot,\cdot\rangle_1)$. Straightforward calculations show that $(\mathfrak{r}'_2, J_1, \rho_1)$ is an Einstein manifold of non-constant para-holomorphic sectional curvature so, after a normalization of the associated symplectic structure, it corresponds to one of the para-Kähler structures $(J_4, \langle\cdot,\cdot\rangle_4)$ described below.

Flat para-Kähler structures on $\mathfrak{r}'_2$

If $a_{13} = 0$ and $a_{14} = 0$, proceeding as in the previous sections, it is easy to see that any para-Kähler structure is equivalent to the flat structure given by

$$(J_2, \langle\cdot,\cdot\rangle_2) : \begin{cases} J_2 e_1 = e_1, \quad J_2 e_2 = e_2, \quad J_2 e_3 = -e_3, \quad J_2 e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_2 = -2(e^1 \circ e^4 + e^2 \circ e^3), \end{cases}$$

so that $\mathfrak{r}'_2 = \mathfrak{L}_2 \oplus \mathfrak{L}'_2 = \operatorname{span}\{e_1, e_2\} \oplus \operatorname{span}\{e_3, e_4\}$.

Einstein para-Kähler structures on $\mathfrak{r}_2'$ of non-zero curvature

Assume that $a_{13} = 0$ and $a_{14} \neq 0$. Then any para-Kähler structure is equivalent to another one in one of the following two families.

$$(J_3, \langle\cdot,\cdot\rangle_3) : \begin{cases} J_3e_1 = -\frac{1}{\kappa}e_4, \quad J_3e_2 = \frac{2}{\kappa}e_3 - e_2, \quad J_3e_3 = e_3, \quad J_3e_4 = -\kappa e_1, \\ \langle\cdot,\cdot\rangle_3 = -\frac{1}{\kappa}(e^1 \circ e^1 - 2e^2 \circ e^2) + 2e^2 \circ e^3 + \kappa e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

$$(J_4, \langle\cdot,\cdot\rangle_4) : \begin{cases} J_4e_1 = -\frac{1}{\kappa}e_4, \quad J_4e_2 = \frac{1}{\kappa}e_3, \quad J_4e_3 = \kappa e_2, \quad J_4e_4 = -\kappa e_1, \\ \langle\cdot,\cdot\rangle_4 = -\frac{1}{\kappa}\left(e^1 \circ e^1 - e^2 \circ e^2\right) - \kappa(e^3 \circ e^3 - e^4 \circ e^4), \quad \kappa \neq 0. \end{cases}$$

These give decompositions of the symplectic Lie algebra $(\mathfrak{r}_2', \Omega_0)$ as direct sums of Lagrangian subalgebras as

$$\begin{aligned} \mathfrak{r}_2' = \mathfrak{L}_3 \oplus \mathfrak{L}_3' &= \operatorname{span}\{e_3, e_4 - \kappa e_1\} \oplus \operatorname{span}\{e_3 - \kappa e_2, e_4 + \kappa e_1\} \\ = \mathfrak{L}_4 \oplus \mathfrak{L}_4' &= \operatorname{span}\{e_3 + \kappa e_2, e_4 - \kappa e_1\} \oplus \operatorname{span}\{e_4 + \kappa e_1, e_3 - \kappa e_2\}. \end{aligned}$$

It is easy to see that the structures $(J_3, \langle\cdot,\cdot\rangle_3)$ have non-zero constant para-holomorphic sectional curvature $H_3 = \kappa$ – thus corresponding to those in Theorem 3.1-*(i.b)* – while the structures $(J_4, \langle\cdot,\cdot\rangle_4)$ are Einstein with $\operatorname{Ric}_4 = 2\kappa \operatorname{Id}(\neq 0)$ and their self-dual and anti-self-dual Weyl curvature operators are diagonalizable with the same eigenvalues – which implies that the para-holomorphic sectional curvature is not constant and they are covered by Theorem 3.1-*(i.c)*.

A straightforward calculation shows that the structures $(J_4, \langle\cdot,\cdot\rangle_4)$ have a compatible anti-Kähler structure (see [21]) defined by a complex structure $\mathfrak{J}$ given by $\mathfrak{J}e_1 = e_2$, $\mathfrak{J}e_3 = e_4$, which commutes with the para-complex structure J_4. Hence, the structure $(\langle\cdot,\cdot\rangle^*, J_4)$ is also a locally symmetric para-Kähler structure, where $\langle X, Y\rangle^* = \langle \mathfrak{J}X, Y\rangle_4$ is the twin metric of $\langle\cdot,\cdot\rangle_4$. A straightforward calculation shows that the Ricci operator Ric^* has complex eigenvalues $\pm 2\kappa\sqrt{-1}$, so this metric corresponds to a metric in Section 3.9 (up to renormalization of the corresponding symplectic structure).

3.10 Para-Kähler structures on $\mathfrak{r}_{4,-1,\beta}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-1,\beta}$ determined by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -\beta e_3,$$

with $-1 < \beta < 0$. This Lie algebra corresponds to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi \mathbb{R}^3$ given by

$$\varphi = \operatorname{ad}(e_4)\colon \quad \varphi(e_1) = e_1, \quad \varphi(e_2) = -e_2, \quad \varphi(e_3) = \beta e_3.$$

Any symplectic form on this Lie algebra must take the form

$$\omega = \alpha_{12}e^{12} + \alpha_{14}e^{14} + \alpha_{24}e^{24} + \alpha_{34}e^{34}, \qquad \alpha_{12}\alpha_{34} \neq 0,$$

as shown in [115]. These symplectic forms are symplectomorphically equivalent to $\Omega = e^{12}+e^{34}$ through an automorphisms of $\mathfrak{r}_{4,-1,\beta}$, which are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ 0 & 0 & z_{33} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11}z_{22}z_{33} \neq 0.$$

The ones that preserve the symplectic structure $(\mathfrak{r}_{4,-1,\beta}, \Omega)$ are those which satisfy $z_{14} = z_{24} = 0$, $z_{33} = 1$ and $z_{22}z_{11} = 1$. The para-Kähler structures on $(\mathfrak{r}_{4,-1,\beta}, \Omega)$ split into two different classes depending on the value of the coefficient a_{43} of the para-complex structure as follows.

Ricci-flat para-Kähler structures on $\mathfrak{r}_{4,-1,\beta}$

Assuming that $a_{43} = 0$, the corresponding para-Kähler structures are equivalent to one of the following two families.

$$(J_1, \langle\cdot,\cdot\rangle_1) : \begin{cases} J_1e_1 = -e_1 + \kappa\, e_2, \quad J_1e_2 = e_2, \quad J_1e_3 = -e_3, \quad J_1e_4 = e_4, \\ \langle\cdot,\cdot\rangle_1 = \kappa\, e^1 \circ e^1 + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

$$(J_2, \langle\cdot,\cdot\rangle_2) : \begin{cases} J_2e_1 = -e_1, \quad J_2e_2 = -\kappa\, e_1 + e_2, \quad J_2e_3 = e_3, \quad J_2e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_2 = \kappa\, e^2 \circ e^2 + 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1. \end{cases}$$

These two families of para-Kähler structures induce the Lagrangian decompositions

$$\begin{aligned} \mathfrak{r}_{4,-1,\beta} &= \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_3, e_2 - \tfrac{2}{\kappa}e_1\} \quad (\kappa \neq 0) \\ &= \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_1, e_3\} \quad (\kappa = 0) \\ &= \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_3, e_2 - \tfrac{\kappa}{2}e_1\} \oplus \operatorname{span}\{e_1, e_4\}. \end{aligned}$$

The structures above are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and recurrent with recurrence one-forms $\xi_1 = 2(\beta-1)e^4$ and $\xi_2 = 2(\beta+1)e^4$, respectively, and their curvature operators acting on the space of two-forms are given by

$$\mathcal{R}_1(e^1 \wedge e^4) = \kappa(\beta - 2)e^2 \wedge e^3 = R_1(e_1, e_4, e_1, e_4)e^2 \wedge e^3,$$

$$\mathcal{R}_2(e^2 \wedge e^4) = \kappa(\beta + 2)e^1 \wedge e^3 = R_2(e_2, e_4, e_4, e_2)e^1 \wedge e^3.$$

Therefore, the metrics above are simply harmonic and special recurrent, so they are modelled on (3.1). Hence, their curvature tensors are semi-symmetric and so they are covered by Theorem 3.6. Finally, both $(J_1, \langle\cdot,\cdot\rangle_1)$ and $(J_2, \langle\cdot,\cdot\rangle_2)$ admit opposite almost para-Kähler structures that are compatible with the opposite symplectic forms $\Omega_1' = e^{12} - e^{34} + \mu e^{14}$ and $\Omega_2' = e^{12} - e^{34} + \mu e^{24}$, respectively, where $\mu \in \mathbb{R}$.

Para-Kähler and opposite para-Kähler structures on $\mathfrak{r}_{4,-1,\beta}$

If $a_{43} \neq 0$, then any para-Kähler structure is equivalent to

$$(J_3, \langle\cdot,\cdot\rangle_3) : \begin{cases} J_3e_1 = -e_1, \quad J_3e_2 = e_2, \quad J_3e_3 = \kappa e_4, \quad J_3e_4 = \frac{1}{\kappa}e_3, \\ \langle\cdot,\cdot\rangle_3 = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{r}_{4,-1,\beta} = \mathfrak{L}_3 \oplus \mathfrak{L}_3' = \operatorname{span}\{e_2, e_4 + \frac{1}{\kappa}e_3\} \oplus \operatorname{span}\{e_1, e_4 - \frac{1}{\kappa}e_3\}$.

The metrics above are locally symmetric with diagonalizable Ricci operator

$$\operatorname{Ric}_3 = \operatorname{diag}[0, 0, \kappa\beta^2, \kappa\beta^2].$$

Therefore, the underlying manifolds are locally isometric to a product $\mathbb{L}^2 \times N$ of the Minkowskian plane and a Lorentzian surface of constant sectional curvature

$$K_N = \kappa\beta^2.$$

Consequently, they are para-Kähler and opposite para-Kähler and they correspond to the metrics in Theorem 3.1-*(i.d)*.

3.11 Para-Kähler structures on $\mathfrak{r}_{4,-1,-1}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-1,-1}$ determined by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3.$$

This Lie algebra corresponds to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi \mathbb{R}^3$ given by

$$\varphi = \operatorname{ad}(e_4)\colon \quad \varphi(e_1) = e_1, \quad \varphi(e_2) = -e_2, \quad \varphi(e_3) = -e_3.$$

Any symplectic structure on $\mathfrak{r}_{4,-1,-1}$ is of the form

$$\omega = \alpha_{12}e^{12} + \alpha_{13}e^{13} + \alpha_{14}e^{14} + \alpha_{24}e^{24} + \alpha_{34}e^{34}, \quad \alpha_{13}\alpha_{24} - \alpha_{12}\alpha_{34} \neq 0,$$

and all of them are symplectomorphically equivalent to $\Omega = e^{12} + e^{34}$ through an automorphism of $\mathfrak{r}_{4,-1,-1}$, which are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & z_{23} & z_{24} \\ 0 & z_{32} & z_{33} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11}(z_{22}z_{33} - z_{23}z_{32}) \neq 0.$$

Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,-1,-1}, \Omega)$ are the ones above with $z_{23} = z_{24} = 0$, $z_{33} = 1$, $z_{32} = z_{14}z_{22}$ and $z_{22}z_{11} = 1$. The associated para-Kähler structures split into two cases depending on whether $a_{43} \neq 0$ or $a_{43} = 0$ as follows.

Para-Kähler and opposite para-Kähler structures on $\mathfrak{r}_{4,-1,-1}$

Straightforward calculations as in the previous sections show that any para-Kähler structure with $a_{43} \neq 0$ is equivalent to

$$(J_1, \langle\cdot,\cdot\rangle_1) : \begin{cases} J_1 e_1 = -e_1, \quad J_1 e_2 = e_2, \quad J_1 e_3 = \kappa e_4\,, \quad J_1 e_4 = \frac{1}{\kappa} e_3, \\ \langle\cdot,\cdot\rangle_1 = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa} e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{r}_{4,-1,-1} = \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{\frac{1}{\kappa}e_3 + e_4, e_2\} \oplus \operatorname{span}\{e_1, e_4 - \frac{1}{\kappa}e_3\}$.

It is not difficult to check that these structures are locally symmetric and their Ricci operators are diagonalizable

$$\mathrm{Ric}_1 = \mathrm{diag}[0, 0, \kappa, \kappa].$$

Therefore, the underlying pseudo-Riemannian structures split locally as a product $\mathbb{L}^2 \times N$ of a Lorentzian surface N of constant Gaussian curvature $K_N = \kappa$ and the Minkowskian plane. As a consequence, they also admit opposite para-Kähler structures. These structures correspond to those in Theorem 3.1-*(i.d)*.

Ricci-flat para-Kähler structures on $\mathfrak{r}_{4,-1,-1}$

Assuming that $a_{43} = 0$, the para-Kähler structures on $(\mathfrak{r}_{4,-1,-1}, \Omega)$ are equivalent to one of the following families, which correspond to different geometric situations.

$$(J_2, \langle\cdot,\cdot\rangle_2) : \begin{cases} J_2 e_1 = -e_1, \quad J_2 e_2 = -\kappa e_1 + e_2, \quad J_2 e_3 = e_3, \quad J_2 e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_2 = \kappa e^2 \circ e^2 + 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

so that $\mathfrak{r}_{4,-1,-1} = \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_3, e_2 - \frac{\kappa}{2}e_1\} \oplus \operatorname{span}\{e_4, e_1\}$,

$$(J_3, \langle\cdot,\cdot\rangle_3) : \begin{cases} J_3 e_1 = -e_1 + \kappa e_2, \quad J_3 e_2 = e_2, \quad J_3 e_3 = -e_3, \quad J_3 e_4 = e_4, \\ \langle\cdot,\cdot\rangle_3 = \kappa e^1 \circ e^1 + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

so that the Lie algebra decomposes as

$$\begin{aligned} \mathfrak{r}_{4,-1,-1} = \mathfrak{L}_3 \oplus \mathfrak{L}_3' &= \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_3, e_2 - \tfrac{2}{\kappa}e_1\} \quad (\kappa \neq 0) \\ &= \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_3, e_1\} \qquad (\kappa = 0), \end{aligned}$$

or

$$(J_4, \langle\cdot,\cdot\rangle_4) : \begin{cases} J_4 e_1 = \kappa e_2 + e_4, \quad J_4 e_2 = e_3, \quad J_4 e_3 = e_2, \quad J_4 e_4 = e_1 - \kappa e_3, \\ \langle\cdot,\cdot\rangle_4 = \kappa(e^1 \circ e^1 + e^4 \circ e^4) + 2(e^1 \circ e^3 - e^2 \circ e^4), \quad \kappa \in \mathbb{R}, \end{cases}$$

so that $(\mathfrak{r}_{4,-1,-1}, \Omega)$ decomposes as direct sums of Lagrangian subalgebras

$$\mathfrak{r}_{4,-1,-1} = \mathfrak{L}_4 \oplus \mathfrak{L}_4' = \operatorname{span}\{e_2 + e_3, e_1 + e_4 + \kappa e_2\} \oplus \operatorname{span}\{e_3 - e_2, e_4 - e_1 + \kappa e_2\}.$$

The structures $(J_2, \langle\cdot,\cdot\rangle_2)$ are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and locally symmetric, thus locally modelled on (3.1) with $\Psi(x^1, x^2) = \pm(x^2)^2$. Moreover, these structures admit opposite almost para-Kähler structures compatible with the opposite symplectic two-forms $\Omega_2' = e^{12} - e^{34} + \mu e^{24}$ with $\mu \in \mathbb{R}$. These structures correspond to Theorem 3.1-*(i.a)*.

The structures $(J_3, \langle\cdot,\cdot\rangle_3)$ are flat if $\kappa = 0$ and Ricci-flat otherwise. They are special recurrent with recurrence one-form $\xi_3 = -4e^4$ and curvature operator determined by

$$\mathcal{R}_3(e^1 \wedge e^4) = -3\kappa e^2 \wedge e^3 = R_3(e_1, e_4, e_1, e_4) e^2 \wedge e^3.$$

Therefore, these structures are simply harmonic and locally modelled on (3.1), and their curvature tensor is semi-symmetric. Furthermore, they admit opposite almost para-Kähler structures compatible with the opposite symplectic two-forms

$$\Omega_3' = e^{12} - e^{34} + \mu e^{14}, \quad \mu \in \mathbb{R}.$$

These structures also have an associated one-parameter family of hypersymplectic structures $(J_3, J_\delta, \langle\cdot,\cdot\rangle_3)$ given by the Kähler structures

$$J_\delta e_1 = -\tfrac{\kappa\delta}{2} e_3 - \tfrac{1}{\delta} e_4, \quad J_\delta e_2 = -\delta e_3, \quad J_\delta e_3 = \tfrac{1}{\delta} e_2, \quad J_\delta e_4 = \delta e_1 - \tfrac{\kappa\delta}{2} e_2,$$

so that $J_\delta J_3 = -J_3 J_\delta$ for any $\delta \neq 0$ (see Remark 3.4 and [5] for more information).

Finally, the structures $(J_4, \langle\cdot,\cdot\rangle_4)$ are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and special recurrent with recurrence one-form $\xi_4 = -4e^4$ and curvature operators given by

$$\mathcal{R}_4(e^1 \wedge e^4) = -3\kappa e^2 \wedge e^3 = R_4(e_1, e_4, e_1, e_4) e^2 \wedge e^3.$$

Therefore, they are simply harmonic and locally modelled on (3.1), so their curvature tensors are semi-symmetric. Moreover, these structures admit opposite almost para-Kähler structures compatible with the opposite symplectic two-forms

$$\Omega_4' = e^{13} + e^{24} + \mu e^{14}, \quad \mu \in \mathbb{R}.$$

The structures $(J_3, \langle\cdot,\cdot\rangle_3)$ and $(J_4, \langle\cdot,\cdot\rangle_4)$ are covered by Theorem 3.6.

3.12 Para-Kähler structures on $\mathfrak{r}_{4,-\alpha,\alpha}$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{r}_{4,-\alpha,\alpha}$ determined by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = \alpha e_2, \quad [e_3, e_4] = -\alpha e_3,$$

where $0 < \alpha < 1$. According to [6], this Lie algebra corresponds to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi \mathbb{R}^3$ given by

$$\varphi = \operatorname{ad}(e_4)\colon \quad \varphi(e_1) = e_1, \quad \varphi(e_2) = -\alpha e_2, \quad \varphi(e_3) = \alpha e_3.$$

The symplectic forms on $\mathfrak{r}_{4,-\alpha,\alpha}$ are given by

$$\omega = \alpha_{14}e^{14} + \alpha_{23}e^{23} + \alpha_{24}e^{24} + \alpha_{34}e^{34}, \quad \alpha_{14}\alpha_{23} \neq 0,$$

and all of them are symplectomorphically equivalent to $\Omega = e^{14} + e^{23}$ through a Lie algebra automorphism of the form

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ 0 & 0 & z_{33} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad z_{11}z_{22}z_{33} = 0.$$

Moreover, the automorphisms preserving the symplectic structure $(\mathfrak{r}_{4,-\alpha,\alpha}, \Omega)$ are given by Φ above with $z_{11} = 1$, $z_{24} = z_{34} = 0$ and $z_{33}z_{22} = 1$. The different classes of para-Kähler structures on this Lie algebra arise from the cases $a_{41} \neq 0$ and $a_{41} = 0$, which we study separately in what follows.

Para-Kähler and opposite para-Kähler structures on $\mathfrak{r}_{4,-\alpha,\alpha}$

Straightforward calculations as in the previous sections show that any para-Kähler structure with $a_{41} \neq 0$ is equivalent to

$$(J_1, \langle\cdot,\cdot\rangle_1) : \begin{cases} J_1e_1 = \kappa e_4, \quad J_1e_2 = -e_2, \quad J_1e_3 = e_3, \quad J_1e_4 = \frac{1}{\kappa}e_1, \\ \langle\cdot,\cdot\rangle_1 = \kappa e^1 \circ e^1 - \frac{1}{\kappa}e^4 \circ e^4 + 2e^2 \circ e^3, \quad \kappa \neq 0, \end{cases}$$

which induces the decompositions

$$\mathfrak{r}_{4,-\alpha,\alpha} = \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \operatorname{span}\{e_3, e_4 + \tfrac{1}{\kappa}e_1\} \oplus \operatorname{span}\{e_2, e_4 - \tfrac{1}{\kappa}e_1\}$$

of $\mathfrak{r}_{4,-\alpha,\alpha}$ as direct sums of Lagrangian subalgebras. These structures are locally symmetric and their Ricci operators are diagonalizable

$$\operatorname{Ric}_1 = \operatorname{diag}[0, 0, \kappa, \kappa].$$

Therefore, the underlying manifolds split locally as products $\mathbb{L}^2 \times N$ of a Lorentzian surface N of constant Gaussian curvature $K_N = \kappa$ and the Minkowskian plane. Consequently, they also admit opposite para-Kähler structures and are covered by Theorem 3.1-*(i.d)*.

Ricci-flat para-Kähler structures

Assuming that $a_{41} = 0$, the corresponding para-Kähler structures are equivalent to one of the following two families of para-Kähler structures.

$$(J_2, \langle\cdot,\cdot\rangle_2) : \begin{cases} J_2e_1 = -e_1, \quad J_2e_2 = -e_2 + \kappa e_3, \quad J_2e_3 = e_3, \quad J_2e_4 = e_4, \\ \langle\cdot,\cdot\rangle_2 = \kappa e^2 \circ e^2 + 2(e^1 \circ e^4 + e^2 \circ e^3), \quad \kappa = 0, \pm 1, \end{cases}$$

which induces the Lagrangian decompositions

$$\mathfrak{r}_{4,-\alpha,\alpha} = \mathfrak{L}_2 \oplus \mathfrak{L}'_2 = \operatorname{span}\{e_3, e_4\} \oplus \operatorname{span}\{e_1, e_3 - \tfrac{2}{\kappa} e_2\} \quad (\kappa \neq 0)$$
$$= \operatorname{span}\{e_3, e_4\} \oplus \operatorname{span}\{e_1, e_2\} \qquad (\kappa = 0),$$

or

$$(J_3, \langle\cdot,\cdot\rangle_3) : \begin{cases} J_3 e_1 = e_1, \quad J_3 e_2 = -e_2, \quad J_3 e_3 = -\kappa e_2 + e_3, \quad J_3 e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_3 = \kappa e^3 \circ e^3 + 2(e^2 \circ e^3 - e^1 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

so that $\mathfrak{r}_{4,-\alpha,\alpha} = \mathfrak{L}_3 \oplus \mathfrak{L}'_3 = \operatorname{span}\{e_1, e_3 - \frac{\kappa}{2} e_2\} \oplus \operatorname{span}\{e_2, e_4\}$.

The structures $(J_2, \langle\cdot,\cdot\rangle_2)$ are flat if $\kappa = 0$, while structures $(J_3, \langle\cdot,\cdot\rangle_3)$ are flat if either $\kappa = 0$ or $\alpha = \frac{1}{2}$. Otherwise, all the structures above are Ricci-flat and special recurrent with recurrence one-forms given by $\xi_2 = 2(1+\alpha)e^4$ and $\xi_3 = 2(1-\alpha)e^4$, respectively, and curvature operators determined by

$$\mathcal{R}_2(e^2 \wedge e^4) = \alpha(1+2\alpha)\kappa e^1 \wedge e^3 = R_2(e_2, e_4, e_4, e_2) e^1 \wedge e^3,$$
$$\mathcal{R}_3(e^3 \wedge e^4) = \alpha(1-2\alpha)\kappa e^1 \wedge e^2 = R_3(e_3, e_4, e_3, e_4) e^1 \wedge e^2.$$

Therefore, they are simply harmonic and locally modelled on (3.1), so their curvature tensor is semi-symmetric. Furthermore, these structures admit opposite almost para-Kähler structures compatible with the opposite symplectic forms given by

$$\Omega'_2 = e^{14} - e^{23} + \mu e^{24}, \quad \Omega'_3 = e^{14} - e^{23} + \mu e^{34},$$

where $\mu \in \mathbb{R}$. All these structures are covered by Theorem 3.6.

3.13 Para-Kähler structures on $\mathfrak{h}_4$

Let $\mathfrak{h}_4$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$, so that

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\tfrac{1}{2} e_1, \quad [e_2, e_4] = -e_1 - \tfrac{1}{2} e_2, \quad [e_3, e_4] = -e_3.$$

According to [6], this Lie algebra corresponds to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi H^3$ given by

$$\varphi = \operatorname{ad}(e_4) \colon \quad \varphi(e_1) = \tfrac{1}{2} e_1, \quad \varphi(e_2) = e_1 + \tfrac{1}{2} e_2, \quad \varphi(e_3) = e_3.$$

The symplectic structures on $\mathfrak{h}_4$, which are given by

$$\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14} e^{14} + \alpha_{24} e^{24}, \quad \alpha_{12} \neq 0,$$

are equivalent to $\Omega_\varepsilon = \varepsilon(e^{12} - e^{34})$, $\varepsilon = \pm 1$, through a Lie algebra automorphism

$$\Phi = \begin{pmatrix} z_{22} & z_{12} & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ 2 z_{22} z_{24} & 2(z_{12} + 2 z_{22}) z_{24} - 2 z_{14} z_{22} & z_{22}^2 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{22} \neq 0.$$

Moreover, the Lie algebra automorphisms preserving the symplectic structure $(\mathfrak{h}_4, \Omega_\varepsilon)$ are given by Φ above with $z_{24} = z_{14} = 0$ and $z_{22} = \pm 1$. Now, it is easy to see that any para-Kähler structure on $(\mathfrak{h}_4, \Omega_\varepsilon)$ is equivalent to

$$(J, \langle\cdot,\cdot\rangle) : \begin{cases} Je_1 = -\varepsilon\, e_1, \quad Je_2 = \varepsilon\, e_2, \quad Je_3 = \varepsilon\, e_3, \quad Je_4 = -\varepsilon\, e_4, \\ \langle\cdot,\cdot\rangle = 2(e^1 \circ e^2 + e^3 \circ e^4), \qquad \varepsilon = \pm 1, \end{cases}$$

which correspond to the Lagrangian decomposition

$$\mathfrak{h}_4 = \mathfrak{L} \oplus \mathfrak{L}' = \operatorname{span}\{e_2, e_3\} \oplus \operatorname{span}\{e_1, e_4\}.$$

A straightforward calculation shows that the structures $(J, \langle\cdot,\cdot\rangle)$ are Ricci-flat with recurrent curvature. The recurrence one-form is given by $\xi = e^4$ and their curvature operator is determined by

$$\mathcal{R}(e^2 \wedge e^4) = -e^1 \wedge e^3 = R(e_2, e_4, e_2, e_4)e^1 \wedge e^3.$$

As a consequence, $(\mathfrak{h}_4, J, \langle\cdot,\cdot\rangle)$ are simply harmonic manifolds with special recurrent curvature and locally modelled on (3.1) whose curvature tensor is semi-symmetric and so they are covered by Theorem 3.6. A straightforward calculation shows that the structures $(\mathfrak{h}_4, J, \langle\cdot,\cdot\rangle)$ do not admit any opposite almost para-Kähler structures.

Remark 3.10. These structures, together with the structures $(J_{13}, \langle\cdot,\cdot\rangle_{13})$ on $(\mathfrak{d}_{4,1}, \Omega_1)$ given in Section 3.14.1, $(J_{1i}, \langle\cdot,\cdot\rangle_{1i})$, $i = 2, 3$, on $(\mathfrak{d}_{4,2}, \Omega_1)$ given in Section 3.17.1, and $(J_{32}, \langle\cdot,\cdot\rangle_{32})$ on $(\mathfrak{d}_{4,2}, \Omega_3)$ given in Section 3.17.3, are the only Ricci-flat para-Kähler structures that do not admit any left-invariant opposite almost para-Kähler structures.

3.14 Para-Kähler structures on $\mathfrak{d}_{4,1}$

Let $\mathfrak{d}_{4,1}$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ so that

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -e_1, \quad [e_3, e_4] = -e_3.$$

According to [6], this Lie algebra corresponds to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi H^3$ given by

$$\varphi = \operatorname{ad}(e_4) : \quad \varphi(e_1) = e_1, \quad \varphi(e_2) = 0, \quad \varphi(e_3) = e_3.$$

The symplectic structures on $\mathfrak{d}_{4,1}$ are given by

$$\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14}e^{14} + \alpha_{24}e^{24}, \quad \alpha_{12} \neq 0.$$

All these two-forms are symplectomorphycally equivalent to either $\Omega_1 = e^{12} - e^{34}$ or $\Omega_2 = e^{12} - e^{34} + e^{24}$ through an automorphisms of $\mathfrak{d}_{4,1}$, which are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & 0 \\ z_{31} & -z_{14}z_{22} & z_{11}z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad z_{11}z_{22} \neq 0.$$

Besides, the automorphisms Φ_i that preserve each symplectic structure $(\mathfrak{d}_{4,1}, \Omega_i)$, $i = 1, 2$, are determined by the conditions

$$\Phi_1 : z_{31} = 0, \quad z_{22}z_{11} = 1 \qquad \text{and} \qquad \Phi_2 : z_{31} = 0, \quad z_{22} = 1, \quad z_{11} = 1.$$

We study the two symplectic structures separately.

3.14.1 Para-Kähler structures on $(\mathfrak{d}_{4,1}, \Omega_1)$

There are three distinct situations that give rise to different geometries.

Para-Kähler structures of non-zero constant para-holomorphic sectional curvature

Assume that $a_{43} \neq 0$. Then necessarily $a_{23} = 0$ and a straightforward calculation shows that any para-Kähler structure is equivalent to

$$(J_{11}, \langle\cdot,\cdot\rangle_{11}) : \begin{cases} J_{11}e_1 = -e_1, \quad J_{11}e_2 = e_2, \quad J_{11}e_3 = -\kappa e_4, \quad J_{11}e_4 = -\frac{1}{\kappa}e_3, \\ \langle\cdot,\cdot\rangle_{11} = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{d}_{4,1}$ splits as direct sums of Lagrangian subalgebras as

$$\mathfrak{d}_{4,1} = \mathfrak{L}_{11} \oplus \mathfrak{L}'_{11} = \operatorname{span}\{e_2, e_4 - \frac{1}{\kappa}e_3\} \oplus \operatorname{span}\{e_1, e_4 + \frac{1}{\kappa}e_3\}.$$

A straightforward calculation shows that the structures $(J_{11}, \langle\cdot,\cdot\rangle_{11})$ have constant para-holomorphic sectional curvature $H_{11} = \kappa$, thus corresponding to those given by Theorem 3.1-*(i.b)*.

Locally symmetric Ricci-flat para-Kähler structures

Set $a_{43} = 0$, $a_{23} = 0$. Then, any para-Kähler structure is equivalent to one of the following two families.

$$(J_{12}, \langle\cdot,\cdot\rangle_{12}) : \begin{cases} J_{12}e_1 = -e_1,\ J_{12}e_2 = -\kappa\, e_1 + e_2,\ J_{12}e_3 = e_3,\ J_{12}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{12} = \kappa\, e^2 \circ e^2 + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

$$(J_{13}, \langle\cdot,\cdot\rangle_{13}) : \begin{cases} J_{13}e_1 = e_1 + \kappa\, e_2,\ J_{13}e_2 = -e_2,\ J_{13}e_3 = e_3,\ J_{13}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{13} = \kappa\, e^1 \circ e^1 - 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1. \end{cases}$$

These structures induce the Lagrangian decompositions

$$\begin{aligned} \mathfrak{d}_{4,1} &= \mathfrak{L}_{12} \oplus \mathfrak{L}'_{12} = \operatorname{span}\{e_3, e_2 - \tfrac{\kappa}{2}e_1\} \oplus \operatorname{span}\{e_1, e_4\} \\ &= \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \operatorname{span}\{e_3, e_2 + \tfrac{2}{\kappa}e_1\} \oplus \operatorname{span}\{e_2, e_4\} \quad (\kappa \neq 0) \\ &= \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \operatorname{span}\{e_1, e_3\} \oplus \operatorname{span}\{e_2, e_4\} \qquad (\kappa = 0). \end{aligned}$$

The structures $(J_{12}, \langle\cdot,\cdot\rangle_{12})$ are all flat, while the structures $(J_{13}, \langle\cdot,\cdot\rangle_{13})$ are flat if and only if $\kappa = 0$. Otherwise, they are locally symmetric and Ricci-flat and so they correspond to those in Theorem 3.1-*(i)*. As a consequence, their underlying pseudo-Riemannian structures are modelled on (3.1) with $\Psi(x^1, x^2) = \pm(x^2)^2$.

Finally, $(\mathfrak{d}_{4,1}, J_{13}, \langle\cdot,\cdot\rangle_{13})$ do not admit any invariant opposite almost para-Kähler structures (see Remark 3.10).

Locally symmetric para-Kähler structures with nilpotent Ricci operator

Setting $a_{43} = 0$ and $a_{23} \neq 0$, it is non difficult to see that any para-Kähler structure is equivalent to

$$(J_{14}, \langle\cdot,\cdot\rangle_{14}) : \begin{cases} J_{14}e_1 = \kappa e_2 - e_4,\ J_{14}e_2 = e_3,\ J_{14}e_3 = e_2,\ J_{14}e_4 = -e_1 + \kappa e_3, \\ \langle\cdot,\cdot\rangle_{14} = \kappa(e^1 \circ e^1 + e^4 \circ e^4) + 2(e^1 \circ e^3 + e^2 \circ e^4), \quad \kappa \in \mathbb{R}, \end{cases}$$

which correspond to the Lagrangian decompositions of $\mathfrak{d}_{4,1}$

$$\mathfrak{L}_{14} \oplus \mathfrak{L}'_{14} = \mathrm{span}\{e_2 + e_3, e_4 - e_1 - \kappa e_2\} \oplus \mathrm{span}\{e_3 - e_2, e_4 + e_1 - \kappa e_2\}.$$

The structures $(J_{14}, \langle\cdot,\cdot\rangle_{14})$ are locally symmetric and have two-step nilpotent Ricci operators. Furthermore, the metrics $\langle\cdot,\cdot\rangle_{14}$ are locally conformally flat if and only if $\kappa = 0$. In any other case, their anti-self-dual Weyl curvature operators are two-step nilpotent, and so this structures correspond to those in Theorem 3.1-*(ii.b)*.

3.14.2 Para-Kähler structures on $(\mathfrak{d}_{4,1}, \Omega_2)$

Direct calculations show that all the para-Kähler structures are flat in this case and they are equivalent to the structures

$$(J_{21}, \langle\cdot,\cdot\rangle_{21}) : \begin{cases} J_{21}e_1 = -e_1,\ J_{21}e_2 = -\kappa e_1 + e_2,\ J_{21}e_3 = e_3,\ J_{21}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{21} = \kappa e^2 \circ e^2 + 2(e^1 \circ e^2 - e^2 \circ e^4 + e^3 \circ e^4), \quad \kappa \in \mathbb{R}, \end{cases}$$

which correspond to the Lagrangian decompositions

$$\mathfrak{d}_{4,1} = \mathfrak{L}_{21} \oplus \mathfrak{L}'_{21} = \mathrm{span}\{e_3, e_2 - \tfrac{\kappa}{2}e_1\} \oplus \mathrm{span}\{e_1, e_4\}.$$

3.15 Para-Kähler structures on $\mathfrak{d}_{4,\frac{1}{2}}$

Let $\mathfrak{d}_{4,\frac{1}{2}}$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ so that

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\tfrac{1}{2}e_1, \quad [e_2, e_4] = -\tfrac{1}{2}e_2, \quad [e_3, e_4] = -e_3,$$

whose automorphisms correspond to

$$\Phi = \begin{pmatrix} z_{11} & z_{12} & 0 & z_{14} \\ z_{21} & z_{22} & 0 & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad \begin{aligned} z_{31} &= 2(z_{11}z_{24} - z_{14}z_{21}), \\ z_{32} &= 2(z_{12}z_{24} - z_{14}z_{22}), \\ z_{33} &= z_{11}z_{22} - z_{12}z_{21} \neq 0. \end{aligned}$$

According to [6], this Lie algebra corresponds to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi H^3$ given by

$$\varphi = \mathrm{ad}(e_4)\colon \quad \varphi(e_1) = \tfrac{1}{2}e_1, \quad \varphi(e_2) = \tfrac{1}{2}e_2, \quad \varphi(e_3) = e_3.$$

All the symplectic forms on $\mathfrak{d}_{4,\frac{1}{2}}$ are given by

$$\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14}e^{14} + \alpha_{24}e^{24}, \quad \alpha_{12} \neq 0,$$

so they are symplectomorphically equivalent to $\Omega = e^{12} - e^{34}$ (see [115]). To give a description of the para-Kähler structures on this Lie algebra, we make use of the Lie algebra automorphisms that preserve Ω, i.e., those which satisfy

$$z_{11}z_{22} - z_{12}z_{21} = 1, \quad z_{14}z_{21} - z_{11}z_{24} = 0, \quad z_{14}z_{22} - z_{12}z_{24} = 0.$$

Now, the description of para-Kähler structures depends on whether or not $a_{43} = 0$.

Para-Kähler structures of non-zero constant para-holomorphic sectional curvature

If $a_{43} \neq 0$, then any para-Kähler structures is equivalent to

$$(J_1, \langle\cdot,\cdot\rangle_1) : \begin{cases} J_1e_1 = -e_1, \quad J_1e_2 = e_2, \quad J_1e_3 = -\kappa e_4, \quad J_1e_4 = -\frac{1}{\kappa}e_3, \\ \langle\cdot,\cdot\rangle_1 = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

which corresponds to the Lagrangian decompositions

$$\mathfrak{d}_{4,\frac{1}{2}} = \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \mathrm{span}\{e_2, e_4 - \tfrac{1}{\kappa}e_3\} \oplus \mathrm{span}\{e_1, e_4 + \tfrac{1}{\kappa}e_3\}.$$

The para-holomorphic sectional curvatures of these structures are constant $H_1 = \kappa$, so they correspond to those given in Theorem 3.1-*(i.b)*

Flat para-Kähler structures

If $a_{43} = 0$, then the corresponding para-Kähler structures are flat and equivalent to

$$(J_2, \langle\cdot,\cdot\rangle_2) : \begin{cases} J_2e_1 = e_1, \quad J_2e_2 = -e_2, \quad J_2e_3 = e_3, \quad J_2e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_2 = 2(-e^1 \circ e^2 + e^3 \circ e^4), \end{cases}$$

so that $\mathfrak{d}_{4,\frac{1}{2}} = \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \mathrm{span}\{e_3, e_1\} \oplus \mathrm{span}\{e_4, e_2\}$.

3.16 Para-Kähler structures on $\mathfrak{d}_{4,\lambda}$

Let $\mathfrak{d}_{4,\lambda}$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ so that

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\lambda e_1, \quad [e_2, e_4] = (\lambda - 1)e_2, \quad [e_3, e_4] = -e_3,$$

for $\lambda > \frac{1}{2}$, $\lambda \neq 1, 2$. This Lie algebra is equivalent to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi H^3$ given by

$$\varphi = \mathrm{ad}(e_4): \quad \varphi(e_1) = \lambda e_1, \quad \varphi(e_2) = (1-\lambda)e_2, \quad \varphi(e_3) = e_3.$$

The automorphisms of the Lie algebra are given by

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ \frac{z_{11}z_{24}}{1-\lambda} & -\frac{z_{22}z_{14}}{\lambda} & z_{11}z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11}z_{22} \neq 0.$$

Thus, any symplectic form on $\mathfrak{d}_{4,\lambda}$, which are

$$\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14}e^{14} + \alpha_{24}e^{24}, \quad \alpha_{12} \neq 0,$$

is symplectomorphically equivalent to $\Omega = e^{12} - e^{34}$ (see [115]). Now, the Lie algebra automorphisms preserving Ω are given by Φ with $z_{24} = z_{14} = 0$ and $z_{22}z_{11} = 1$.

The description of the para-Kähler structures depends on whether the component a_{43} vanishes or not.

Para-Kähler structures of non-zero constant para-holomorphic sectional curvature

If $a_{43} \neq 0$, the para-Kähler structures on $(\mathfrak{d}_{4,\lambda}, \Omega)$ are equivalent to

$$(J_1, \langle\cdot,\cdot\rangle_1): \begin{cases} J_1e_1 = -e_1, \quad J_1e_2 = e_2, \quad J_1e_3 = -\kappa e_4, \quad J_1e_4 = -\frac{1}{\kappa}e_3, \\ \langle\cdot,\cdot\rangle_1 = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0. \end{cases}$$

These structures induce the Lagrangian decompositions

$$\mathfrak{d}_{4,\lambda} = \mathfrak{L}_1 \oplus \mathfrak{L}_1' = \mathrm{span}\{e_2, e_4 - \tfrac{1}{\kappa}e_3\} \oplus \mathrm{span}\{e_1, e_4 + \tfrac{1}{\kappa}e_3\}.$$

A straightforward calculation shows that their para-holomorphic sectional curvatures are constant $H_1 = \kappa$, so these metrics correspond to those given in Theorem 3.1-*(i.b)*.

Ricci-flat para-Kähler structures

If $a_{43} = 0$, then the resulting para-Kähler structures are Ricci-flat equivalent to one of the families

$$(J_2, \langle\cdot,\cdot\rangle_2) : \begin{cases} J_2e_1 = -e_1, \quad J_2e_2 = -\kappa\, e_1 + e_2, \quad J_2e_3 = e_3, \quad J_2e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_2 = 2(e^1 \circ e^2 + e^3 \circ e^4) + \kappa\, e^2 \circ e^2, \quad \kappa = 0, \pm 1, \end{cases}$$

$$(J_3, \langle\cdot,\cdot\rangle_3) : \begin{cases} J_3e_1 = e_1 + \kappa\, e_2, \quad J_3e_2 = -e_2, \quad J_3e_3 = e_3, \quad J_3e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_3 = \kappa\, e^1 \circ e^1 - 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1. \end{cases}$$

These structures induce the Lagrangian decompositions

$$\begin{aligned} \mathfrak{d}_{4,\lambda} &= \mathfrak{L}_2 \oplus \mathfrak{L}_2' = \operatorname{span}\{e_3, e_2 - \tfrac{\kappa}{2}e_1\} \oplus \operatorname{span}\{e_1, e_4\} \\ &= \mathfrak{L}_3 \oplus \mathfrak{L}_3' = \operatorname{span}\{e_3, e_2 + \tfrac{2}{\kappa}e_1\} \oplus \operatorname{span}\{e_4, e_2\} \quad (\kappa \neq 0) \\ &= \mathfrak{L}_3 \oplus \mathfrak{L}_3' = \operatorname{span}\{e_3, e_1\} \oplus \operatorname{span}\{e_4, e_2\} \qquad (\kappa = 0). \end{aligned}$$

Both structures above are flat if and only if $\kappa = 0$. Otherwise, they have recurrent curvature with recurrence one-forms $\xi_2 = 2\lambda e^4$ and $\xi_3 = 2(1-\lambda)e^4$, respectively. Moreover, their curvature operators acting on the space of two-forms are given by

$$\begin{aligned} \mathcal{R}_2(e^2 \wedge e^4) &= -\kappa(2\lambda^2 - 3\lambda + 1)e^1 \wedge e^3 = R_2(e_2, e_4, e_2, e_4)e^1 \wedge e^3, \\ \mathcal{R}_3(e^1 \wedge e^4) &= \kappa\lambda(2\lambda - 1)e^2 \wedge e^3 = R_3(e_1, e_4, e_4, e_1)e^2 \wedge e^3, \end{aligned}$$

respectively. Consequently, the corresponding manifolds are simply harmonic with special recurrent curvature tensor, thus locally isometric to a metric given by (3.1) and their curvature tensors are semi-symmetric. Furthermore, neither $(J_2, \langle\cdot,\cdot\rangle_2)$ nor $(J_3, \langle\cdot,\cdot\rangle_3)$ admit any opposite almost para-Kähler structure. These structures are thus covered by Theorem 3.6.

3.17 Para-Kähler structures on $\mathfrak{d}_{4,2}$

Let $\mathfrak{d}_{4,2}$ be the Lie algebra generated by $\{e_1, e_2, e_3, e_4\}$ with Lie brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -e_3,$$

which, according to [6], corresponds to the semi-direct extension $\mathbb{R}e_4 \ltimes_\varphi H^3$ given by

$$\varphi = \operatorname{ad}(e_4)\colon \quad \varphi(e_1) = 2e_1, \quad \varphi(e_2) = -e_2, \quad \varphi(e_3) = e_3.$$

The symplectic forms on $\mathfrak{d}_{4,2}$ are of the given by

$$\omega = \alpha_{12}(e^{12} - e^{34}) + \alpha_{14}e^{14} + \alpha_{23}e^{23} + \alpha_{24}e^{24}, \quad \alpha_{12}^2 - \alpha_{14}\alpha_{23} \neq 0.$$

Any such form is symplectomorphically equivalent to

$$\Omega_1 = e^{12} - e^{34}, \quad \Omega_2 = e^{14} + e^{23}, \quad \text{or} \quad \Omega_3 = e^{14} - e^{23},$$

through a Lie algebra automorphism given by (see [115])

$$\Phi = \begin{pmatrix} z_{11} & 0 & 0 & z_{14} \\ 0 & z_{22} & 0 & z_{24} \\ -z_{11}z_{24} & -\frac{z_{14}z_{22}}{2} & z_{11}z_{22} & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad z_{11}z_{22} \neq 0. \tag{3.2}$$

The geometric situation is different for each Ω_i, so we consider the three possibilities separately.

3.17.1 Para-Kähler structures on $(\mathfrak{d}_{4,2}, \Omega_1)$

The Lie algebra automorphisms preserving the symplectic structure $(\mathfrak{d}_{4,2}, \Omega_1)$ with $\Omega_1 = e^{12} - e^{34}$ are the ones given by Φ above with $z_{14} = z_{24} = 0$ and $z_{22}z_{11} = 1$. Now, we consider the cases $a_{43} \neq 0$ and $a_{43} = 0$, which give rise to different geometries as follows.

Para-Kähler structures of non-zero constant para-holomorphic sectional curvature

If $a_{43} \neq 0$, then any para-Kähler structure is equivalent to

$$(J_{11}, \langle\cdot,\cdot\rangle_{11}) : \begin{cases} J_{11}e_1 = -e_1,\ J_{11}e_2 = e_2,\ J_{11}e_3 = -\kappa e_4,\ J_{11}e_4 = -\frac{1}{\kappa}e_3, \\ \langle\cdot,\cdot\rangle_{11} = 2e^1 \circ e^2 + \kappa e^3 \circ e^3 - \frac{1}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that $\mathfrak{d}_{4,2} = \mathfrak{L}_{11} \oplus \mathfrak{L}'_{11} = \operatorname{span}\{e_2, e_4 - \frac{1}{\kappa}e_3\} \oplus \operatorname{span}\{e_1, e_4 + \frac{1}{\kappa}e_3\}$.

A straightforward calculation shows that all these structures have constant para-holomorphic sectional curvature $H_{11} = \kappa$, and so they correspond to those in Theorem 3.1-*(i.b)*.

Ricci-flat para-Kähler structures

Assuming that $a_{43} = 0$, the corresponding para-Kähler structures on $(\mathfrak{d}_{4,2}, \Omega_1)$ are equivalent either to

$$(J_{12}, \langle\cdot,\cdot\rangle_{12}) : \begin{cases} J_{12}e_1 = -e_1 + \kappa e_2,\ J_{12}e_2 = e_2,\ J_{12}e_3 = -e_3,\ J_{12}e_4 = e_4, \\ \langle\cdot,\cdot\rangle_{12} = \kappa e^1 \circ e^1 + 2(e^1 \circ e^2 - e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

which induce the Lagrangian decompositions

$$\begin{aligned} \mathfrak{d}_{4,2} = \mathfrak{L}_{12} \oplus \mathfrak{L}'_{12} &= \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_3, e_2 - \tfrac{2}{\kappa}e_1\} \qquad (\kappa \neq 0) \\ &= \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_1, e_3\} \qquad (\kappa = 0), \end{aligned}$$

or to

$$(J_{13}, \langle\cdot,\cdot\rangle_{13}) : \begin{cases} J_{13}e_1 = -e_1,\ J_{13}e_2 = -\kappa e_1 + e_2,\ J_{13}e_3 = e_3,\ J_{13}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{13} = \kappa e^2 \circ e^2 + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa = 0, \pm 1, \end{cases}$$

where $\mathfrak{d}_{4,2} = \mathfrak{L}_{13} \oplus \mathfrak{L}'_{13} = \operatorname{span}\{e_2 - \frac{\kappa}{2}e_1, e_3\} \oplus \operatorname{span}\{e_1, e_4\}$.

The structures $(J_{12}, \langle\cdot,\cdot\rangle_{12})$ and $(J_{13}, \langle\cdot,\cdot\rangle_{13})$ are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and recurrent, with recurrence one-forms $\xi_{12} = -2e^4$ and $\xi_{13} = 4e^4$, respectively. Their corresponding curvature operators are given by

$$\mathcal{R}_{12}(e^1 \wedge e^4) = 6\kappa e^2 \wedge e^3 = R_{12}(e_1, e_4, e_4, e_1)e^2 \wedge e^3,$$
$$\mathcal{R}_{13}(e^2 \wedge e^4) = -3\kappa e^1 \wedge e^3 = R_{13}(e_2, e_4, e_2, e_4)e^1 \wedge e^3.$$

Therefore, these two families are simply harmonic and special recurrent, locally modelled by (3.1). Consequently, their curvature tensors are semi-symmetric and so they are covered by Theorem 3.6. None of these structures admits any opposite almost para-Kähler structures (see Remark 3.10).

Besides, the structures $(J_{13}, \langle\cdot,\cdot\rangle_{13})$ have an associated one-parameter family of hypersymplectic structures $(J_{13}, J_\delta, \langle\cdot,\cdot\rangle_{13})$ which are given by the Kähler structures

$$J_\delta e_1 = \tfrac{1}{\delta}e_3, \quad J_\delta e_2 = \tfrac{\kappa}{2\delta}e_3 + \delta e_4, \quad J_\delta e_3 = -\delta e_1, \quad J_\delta e_4 = \tfrac{1}{\delta}(\tfrac{\kappa}{2}e_1 - e_2),$$

so that $J_\delta J_{13} = -J_{13}J_\delta$, for any $\delta \neq 0$ (see Remark 3.4 and [5] for information).

3.17.2 Para-Kähler structures on $(\mathfrak{d}_{4,2}, \Omega_2)$

The Lie algebra automorphisms that preserve the symplectic structure $(\mathfrak{d}_{4,2}, \Omega_2)$, where $\Omega_2 = e^{14} + e^{23}$, are the ones given by Φ in (3.2) with $z_{24} = z_{34} = 0$, $z_{11} = 1$ and $z_{22} = \pm 1$. Any para-Kähler structure on this Lie algebra satisfies $a_{21} = 0$ and different situations may occur depending on whether $a_{23} = 0$ or $a_{23} \neq 0$. We study each case separately.

Flat para-Kähler structures

Assuming that $a_{23} = 0$ and $a_{41} = 0$, straightforward calculations show that any para-Kähler structure is equivalent to the flat structure

$$(J_{21}, \langle\cdot,\cdot\rangle_{21}) : \begin{cases} J_{21}e_1 = e_1, \quad J_{21}e_2 = -e_2, \quad J_{21}e_3 = e_3, \quad J_{21}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{21} = 2(e^2 \circ e^3 - e^1 \circ e^4), \end{cases}$$

so that $\mathfrak{d}_{4,2} = \mathfrak{L}_{21} \oplus \mathfrak{L}'_{21} = \operatorname{span}\{e_1, e_3\} \oplus \operatorname{span}\{e_2, e_4\}$.

Para-Kähler structures which are not semi-symmetric

Assuming that $a_{23} = 0$ and $a_{41} \neq 0$, straightforward calculations as in the previous sections show that the para-Kähler structures on this Lie algebra are equivalent to

$$(J_{22}, \langle\cdot,\cdot\rangle_{22}) : \begin{cases} J_{22}e_1 = e_1 + \kappa e_4,\ J_{22}e_2 = -e_2,\ J_{22}e_3 = e_3,\ J_{22}e_4 = -e_4, \\ \langle\cdot,\cdot\rangle_{22} = \kappa e^1 \circ e^1 + 2(e^2 \circ e^3 - e^1 \circ e^4), \quad \kappa \neq 0, \end{cases}$$

which induce the Lagrangian decomposition

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{22} \oplus \mathfrak{L}'_{22} = \operatorname{span}\{e_3, e_4 + \tfrac{2}{\kappa}e_1\} \oplus \operatorname{span}\{e_2, e_4\}.$$

The Ricci operators of these structures are diagonalizable

$$\mathrm{Ric}_{22} = \mathrm{diag}[0, 0, 4\kappa, 4\kappa],$$

and their self-dual and anti-self-dual Weyl curvature operators have the same eigenvalues. Moreover, the anti-self-dual Weyl curvature operator W^- has a double root of its minimal polynomial. Besides, their curvature tensor is not semi-symmetric, so it does not correspond to the curvature of any symmetric space. These structures correspond to those given in Theorem 3.7-*(ii.a)*.

Assuming that $a_{23} \neq 0$, the para-Kähler structures are equivalent to

$$(J_{23}, \langle\cdot,\cdot\rangle_{23}) : \begin{cases} J_{23}e_1 = \tfrac{\kappa}{2}e_4,\ \ J_{23}e_2 = -\tfrac{1}{\kappa}e_3,\ \ J_{23}e_3 = -\kappa e_2,\ \ J_{23}e_4 = \tfrac{2}{\kappa}e_1, \\ \langle\cdot,\cdot\rangle_{23} = \tfrac{\kappa}{2}e^1 \circ e^1 - \tfrac{1}{\kappa}e^2 \circ e^2 + \kappa e^3 \circ e^3 - \tfrac{2}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

so that

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{23} \oplus \mathfrak{L}'_{23} = \operatorname{span}\{e_3 - \kappa e_2, e_4 + \tfrac{2}{\kappa}e_1\} \oplus \operatorname{span}\{e_3 + \kappa e_2, e_4 - \tfrac{2}{\kappa}e_1\}.$$

The Ricci operators of these structures are diagonalizable

$$\mathrm{Ric}_{23} = \mathrm{diag}[0, 0, 3\kappa, 3\kappa],$$

and their self-dual and anti-self-dual Weyl curvature operators are diagonalizable with opposite eigenvalues. Furthermore, their curvature tensor is not semi-symmetric. These structures correspond to those given in Theorem 3.7-*(ii.b)*.

Remark 3.11. Let Q_{2i} be the almost product structures associated to the Ricci operators Ric_{2i}, for $i = 2, 3$, so that $Q_{2i} = -\mathrm{Id}$ on $\ker \mathrm{Ric}_{2i}$ and $Q_{2i} = \mathrm{Id}$ on the orthogonal distribution corresponding to eigenspace of the non-zero Ricci curvature. These give opposite almost para-complex structures defined by

$$J'_{2i} = J_{2i}Q_{2i}, \quad i = 2, 3.$$

A straightforward calculation now shows that J'_{2i} are opposite almost para-Kähler structures commuting with J_{2i} that have associated symplectic forms

$$\Omega'_{2i}(x, y) = \langle J'_{2i}x, y\rangle_{2i} = \Omega_3, \quad i = 2, 3.$$

3.17.3 Para-Kähler structures on $(\mathfrak{d}_{4,2}, \Omega_3)$

The symplectic structure $(\mathfrak{d}_{4,2}, \Omega_3)$ given by $\Omega_3 = e^{14} - e^{23}$ is preserved by the automorphisms Φ in (3.2) with $z_{24} = z_{34} = 0$, $z_{11} = 1$ and $z_{22} = \pm 1$. Considering the action of the automorphisms which preserve the symplectic structure, the different possibilities arise from the following cases.

Case 1. $a_{21} = a_{23} = a_{41} = 0$, $a_{22} \neq 0$, in which case one may assume $a_{32} = 0$.
Case 2. $a_{21} = a_{23} = a_{41} = a_{22} = 0$.
Case 3. $a_{21} = a_{23} = 0$, $a_{41} \neq 0$, in which case one may assume $a_{11} = 0$.
Case 4. $a_{21} = 0$, $a_{23} \neq 0$, in which case one may assume $a_{22} = 0$.
Case 5. $a_{21} \neq 0$, in which case one may assume $a_{13} = 0$.

We will study all these situations separately.

Flat para-Kähler structures

Assuming that $a_{21} = a_{23} = a_{41} = 0$ and $a_{22} \neq 0$ as in Case 1, if $a_{31} = 0$ then the corresponding para-Kähler structures are equivalent to the flat para-Kähler structure given by

$$(J_{31}, \langle\cdot,\cdot\rangle_{31}) : \begin{cases} J_{31}e_1 = -e_1, \quad J_{31}e_2 = e_2, \quad J_{31}e_3 = -e_3, \quad J_{31}e_4 = e_4, \\ \langle\cdot,\cdot\rangle_{31} = 2(e^1 \circ e^4 + e^2 \circ e^3), \end{cases}$$

which induces the Lagrangian decomposition

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{31} \oplus \mathfrak{L}'_{31} = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_1, e_3\}.$$

The case where $a_{31} \neq 0$ will be considered bellow, since it gives rise to a different geometric situation.

Ricci-flat para-Kähler structures

Assuming that $a_{21} = a_{23} = a_{41} = a_{22} = 0$ as in Case 2, the corresponding para-Kähler structures are equivalent to the structures $(J_{32}, \langle\cdot,\cdot\rangle_{32})$ given by

$$\begin{cases} J_{32}e_1 = -e_3, \;\; J_{32}e_2 = -\kappa e_3 + e_4, \;\; J_{32}e_3 = -e_1, \;\; J_{32}e_4 = -\kappa e_1 + e_2, \\ \langle\cdot,\cdot\rangle_{32} = \kappa(e^2 \circ e^2 + e^4 \circ e^4) + 2(e^1 \circ e^2 + e^3 \circ e^4), \quad \kappa \in \mathbb{R}, \end{cases}$$

which correspond to the Lagrangian decomposition of $\mathfrak{d}_{4,2}$ as

$$\mathfrak{L}_{32} \oplus \mathfrak{L}'_{32} = \operatorname{span}\{e_3 - e_1, e_4 + e_2 - \kappa e_1\} \oplus \operatorname{span}\{e_3 + e_1, e_4 - e_2 + \kappa e_1\}.$$

These structures are flat if $\kappa = 0$. Otherwise, they are Ricci-flat and special recurrent with recurrence one-forms $\xi_{32} = 4e^4$ and curvature operators

$$\mathcal{R}_{32}(e^2 \wedge e^4) = -3\kappa e^1 \wedge e^3 = R_{32}(e_2, e_4, e_2, e_4) e^1 \wedge e^3.$$

Consequently, the underlying structures are locally modelled on (3.1) and thus their curvature tensor is semi-symmetric and these structures are covered by Theorem 3.6. Finally, these structures do not admit opposite almost para-Kähler structures (see Remark 3.10).

Para-Kähler structures with diagonalizable Ricci operator which are not semi-symmetric

Assuming that $a_{21} = a_{23} = 0$ and $a_{41} \neq 0$ as in Case 3, the para-Kähler structures are equivalent to

$$(J_{33}, \langle\cdot,\cdot\rangle_{33}) : \begin{cases} J_{33}e_1 = -e_1 + \kappa e_4,\ J_{33}e_2 = e_2,\ J_{33}e_3 = -e_3,\ J_{33}e_4 = e_4, \\ \langle\cdot,\cdot\rangle_{33} = \kappa e^1 \circ e^1 + 2(e^1 \circ e^4 + e^2 \circ e^3), \quad \kappa \neq 0, \end{cases}$$

so that

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{33} \oplus \mathfrak{L}'_{33} = \operatorname{span}\{e_2, e_4\} \oplus \operatorname{span}\{e_3, e_4 - \tfrac{2}{\kappa}e_1\}.$$

Their Ricci operators are diagonalizable with eigenvalues $\{4\kappa, 4\kappa, 0, 0\}$ and their self-dual and anti-self-dual Weyl curvature operators have the same eigenvalues. Moreover, W^- has a double root of its minimal polynomial, and the curvature tensors are not semi-symmetric. These structures correspond to those in Theorem 3.7-*(ii.a)*.

The assumption that $a_{21} = 0$ and $a_{23} \neq 0$ as in Case 4 leads to para-Kähler structures equivalent to

$$(J_{34}, \langle\cdot,\cdot\rangle_{34}) : \begin{cases} J_{34}e_1 = \tfrac{\kappa}{2}e_4,\ \ J_{34}e_2 = -\tfrac{1}{\kappa}e_3,\ \ J_{34}e_3 = -\kappa e_2,\ \ J_{34}e_4 = \tfrac{2}{\kappa}e_1, \\ \langle\cdot,\cdot\rangle_{34} = \tfrac{\kappa}{2}e^1 \circ e^1 + \tfrac{1}{\kappa}e^2 \circ e^2 - \kappa e^3 \circ e^3 - \tfrac{2}{\kappa}e^4 \circ e^4, \quad \kappa \neq 0, \end{cases}$$

which correspond to the Lagrangian decompositions

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{34} \oplus \mathfrak{L}'_{34} = \operatorname{span}\{e_4 + \tfrac{2}{\kappa}e_1, e_3 - \kappa e_2\} \oplus \operatorname{span}\{e_4 - \tfrac{2}{\kappa}e_1, e_3 + \kappa e_2\}.$$

Their Ricci operators are diagonalizable with eigenvalues $\{3\kappa, 3\kappa, 0, 0\}$ and their self-dual and the anti-self-dual Weyl curvature operators are diagonalizable with opposite eigenvalues. Moreover, the curvature tensor is not semi-symmetric and these structures correspond to those in Theorem 3.7-*(ii.b)*.

Remark 3.12. Let Q_{3i} be the almost product structures associated to the Ricci operators Ric_{3i}, for $i = 2, 3$, so that $Q_{3i} = -\operatorname{Id}$ on $\ker \operatorname{Ric}_{3i}$ and $Q_{3i} = \operatorname{Id}$ on the orthogonal distribution corresponding to the eigenspace of the non-zero Ricci curvature. These determine opposite almost para-complex structures defined by

$$J'_{3i} = J_{3i}Q_{3i}, \quad i = 2, 3.$$

A straightforward calculation now shows that J'_{3i} are opposite almost para-Kähler structures commuting with J_{3i} that have associated symplectic forms

$$\Omega'_{3i}(x, y) = \langle J'_{3i}x, y\rangle_{3i} = \Omega_2, \quad i = 3, 4.$$

Para-Kähler structures which are not semi-symmetric with non-diagonalizable Ricci operator

Assuming that $a_{21} = a_{23} = a_{41} = 0$ and $a_{22} \neq 0$ as in Case 1, if $a_{31} \neq 0$, the para-Kähler structures on this Lie algebra are equivalent to the structures $(J_{35}, \langle\cdot,\cdot\rangle_{35})$ given by

$$\begin{cases} J_{35}e_1 = -e_1 - 2e_3,\ J_{35}e_2 = 2e_4 - e_2 - \kappa e_3,\ J_{35}e_3 = e_3,\ J_{35}e_4 = e_4, \\ \langle\cdot,\cdot\rangle_{35} = \kappa e^2 \circ e^2 + 2(e^1 \circ e^4 + 2e^1 \circ e^2 - e^2 \circ e^3), \quad \kappa \in \mathbb{R}, \end{cases}$$

which induce the Lagrangian decompositions

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{35} \oplus \mathfrak{L}'_{35} = \mathrm{span}\{e_3, e_4\} \oplus \mathrm{span}\{e_4 - e_2 + \tfrac{\kappa}{2}e_1, e_3 + e_1\}.$$

Their Ricci operators are two-step nilpotent and their curvature tensors are not semi-symmetric. Moreover, their anti-self-dual Weyl curvature operator is three-step nilpotent and these structures correspond to those in Theorem 3.7-*(i)*.

Finally, if $a_{21} \neq 0$ as in Case 5, then the para-Kähler structures are equivalent to the structures $(J_{36}, \langle\cdot,\cdot\rangle_{36})$ given by

$$\begin{cases} J_{36}e_1 = -e_1 + \kappa(e_2 + e_4),\ J_{36}e_2 = e_2,\ J_{36}e_3 = \kappa(e_2 + e_4) - e_3,\ J_{36}e_4 = e_4, \\ \langle\cdot,\cdot\rangle_{36} = \kappa(e^1 \circ e^1 + e^3 \circ e^3 + 2e^1 \circ e^3) + 2(e^1 \circ e^4 + e^2 \circ e^3), \quad \kappa \neq 0, \end{cases}$$

so that

$$\mathfrak{d}_{4,2} = \mathfrak{L}_{36} \oplus \mathfrak{L}'_{36} = \mathrm{span}\{e_2, e_4\} \oplus \mathrm{span}\{e_2 - \tfrac{2}{\kappa}e_1 + e_4, e_3 - e_1\}.$$

The Ricci operators of these structures have a single eigenvalue $\frac{3}{2}\kappa \neq 0$ which is a double root of their minimal polynomials, and their anti-self-dual Weyl curvature operators are three-step nilpotent. Besides, their curvature tensors are not semi-symmetric and these structures correspond to those in Theorem 3.7-*(i)*. Since their anti-self-dual Weyl curvature operators are three-step nilpotent the structures above do not admit any opposite almost para-Kähler structure commuting with the Ricci operator [45].

3.18 Kähler Lie algebras

Kähler Lie algebras were classified by Ovando in [116] and the geometry of the corresponding structures, similar though it may be to that of para-Kähler Lie algebras, is more rigid, allowing less possibilities. The symmetric case is essentially the same, but there are no left-invariant locally symmetric Ricci-flat Kählerian structures in contrast with Theorem 3.1-(1.a).

Theorem 3.13. *Let $(G, \langle\cdot,\cdot\rangle, J)$ be a non-flat locally symmetric four-dimensional indefinite Kähler Lie group. Then, it corresponds to one of the following situations.*

(1) Its Ricci operator is diagonalizable and one of the following holds:

(1.a) Its holomorphic sectional curvature is a non-zero constant.

(1.b) Its metric is Einstein with non-zero scalar curvature.

(1.c) The underlying manifold is locally a product of two surfaces of constant Gaussian curvature.

(2) Its Ricci operator is non-diagonalizable and one of the following holds:

(2.a) Its Ricci operator has complex eigenvalues.

(2.b) Its Ricci operator is two-step nilpotent.

Following Ovando's classification, the structures in Assertion *(1.a)* correspond to the Kähler structures on the Lie groups determined by $\mathfrak{d}_{4,\frac{1}{2}}$ and $\mathfrak{d}'_{4,\delta}$, where $\delta > 0$.

The Kähler structures in Assertion *(1.b)* correspond to the metrics

$$\langle\cdot,\cdot\rangle = a(e^1 \circ e^1 - e^2 \circ e^2 + e^3 \circ e^3 - e^4 \circ e^4)$$

on $\mathfrak{r}'_2$ with symplectic form $\Omega = a(e^{13} - e^{24})$.

The Kähler structures which admit an opposite Kähler structure as in Assertion *(1.c)* correspond to the Kähler structures on $\mathfrak{rr}_{3,0}$, $\mathfrak{r}'_{4,0,\delta}$ for $\delta > 0$, and the structures on $\mathfrak{r}_2\mathfrak{r}_2$ given by the metrics

$$\langle\cdot,\cdot\rangle = a(e^1 \circ e^1 + e^2 \circ e^2) + b(e^3 \circ e^3 + e^4 \circ e^4)$$

with the symplectic form $\Omega = ae^{12} + be^{34}$, for $ab < 0$.

The Kähler structures in Assertion *(2.a)* correspond to the metrics

$$\langle\cdot,\cdot\rangle = a(e^1 \circ e^1 - e^2 \circ e^2 + e^3 \circ e^3 - e^4 \circ e^4) + 2b(e^1 \circ e^2 + e^3 \circ e^4)$$

on $\mathfrak{r}'_2$ with the symplectic form $\Omega = a(e^{13} - e^{24}) + b(e^{14} + e^{23})$, for $b \neq 0$.

Assertion *(2.b)* corresponds to the Kähler structures on $\mathfrak{d}_{4,1}$.

Remark 3.14. The metrics corresponding to Assertions *(1.b)* and *(2.a)* are linked by anti-Kähler structures, so that they have the same Levi-Civita connection as in the para-Kähler case.

Moreover, amongst the structures above there are locally conformally flat Kähler Lie groups as stated in the following result.

Corollary 3.15. *Let (M, g, J) be a locally conformally flat four-dimensional indefinite Kähler manifold. Then, it is flat or it is locally isometric to the Kähler Lie group determined by one of the following:*

(1) The Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ with the metrics $\langle\cdot,\cdot\rangle = a(e^1 \circ e^1 + e^2 \circ e^2 - e^3 \circ e^3 - e^4 \circ e^4)$ and the symplectic structure $\Omega = a(e^{12} - e^{34})$.

(2) The Lie algebra $\mathfrak{r}'_2$ with the metrics $\langle\cdot,\cdot\rangle = 2b(e^1 \circ e^2 + e^3 \circ e^4)$ and the symplectic structure $\Omega = b(e^{14} + e^{23})$.

(3) The Lie algebra $\mathfrak{d}_{4,1}$ with the metrics $\langle\cdot,\cdot\rangle = 2a(e^2 \circ e^4 - e^1 \circ e^3)$ and the symplectic structure $\Omega = a(e^{12} - e^{34})$.

Finally, the non-symmetric case is significantly simpler than its para-Kähler counterpart since the existence of Kähler and opposite almost Kähler structures is much more rigid than the corresponding para-Kähler analogue (see [44, 72]).

Theorem 3.16. *Let $(G, \langle\cdot,\cdot\rangle, \Omega)$ be a non-symmetric four-dimensional indefinite Kähler Lie group. Then, one of the following holds.*

(1) *$(G, \langle\cdot,\cdot\rangle)$ is semi-symmetric if and only if its Ricci operator vanishes, in which case its curvature tensor is special recurrent and the metric is simply harmonic.*

(2) *$(G, \langle\cdot,\cdot\rangle)$ is not semi-symmetric if and only if it corresponds to the 3-symmetric space determined by the Kähler metrics*

$$\langle\cdot,\cdot\rangle = a\left(\tfrac{1}{2}e^1 \circ e^1 + 2e^4 \circ e^4\right) + b(e^2 \circ e^2 + e^3 \circ e^3)$$

on $\mathfrak{d}_{4,2}$ with the symplectic form $\Omega = ae^{14} + be^{23}$, for $ab < 0$.

Assertion *(1)* corresponds to the metrics

$$\langle\cdot,\cdot\rangle = -c(e^1 \circ e^1 + e^2 \circ e^2) - 2a(e^1 \circ e^4 + e^2 \circ e^3) + 2b(e^1 \circ e^3 - e^2 \circ e^4)$$

on $\mathfrak{r}'_2$ with the symplectic form $\Omega = ce^{12} + a(e^{13} - e^{24}) + b(e^{14} + e^{23})$, where $c(a^2 + b^2) \neq 0$, the metrics

$$\langle\cdot,\cdot\rangle = -c(e^1 \circ e^1 + e^4 \circ e^4) + 2b(e^1 \circ e^2 - e^3 \circ e^4) - 2a(e^1 \circ e^3 + e^2 \circ e^4)$$

on $\mathfrak{r}_{4,-1,-1}$ with symplectic the form $\Omega = a(e^{12}+e^{34})+b(e^{13}-e^{24})+ce^{14}$, where $c(a^2+b^2) \neq 0$, and the metrics

$$\langle\cdot,\cdot\rangle = b(e^2 \circ e^2 + e^4 \circ e^4) + 2a(e^1 \circ e^2 + e^3 \circ e^4)$$

on $\mathfrak{d}_{4,2}$ with the symplectic form $\Omega = be^{24} + a(e^{14} + e^{23})$, for $a \neq 0$.

Part II

Solitons associated to geometric flows

In this part, we will devote ourselves to the study of solitons associated to two particular geometric flows: the Ricci flow and the Bach flow. In Chapter 4 we will give a complete description of four-dimensional Lorentzian left-invariant Ricci solitons and in Chapter 5 we will introduce a general technique to approach the classification of algebraic solitons and give a complete classification of four-dimensional Riemannian both Ricci and Bach solitons. But before we start, we will briefly introduce the two geometric flows that are the subject of our study and their corresponding solitons.

The Ricci flow: Ricci solitons

The Ricci flow was introduced by Hamilton in [83] with the intention to solve the Poincaré conjecture, which declares that any three-dimensional closed and simply connected manifold is homeomorphic to $\mathbb{S}^3$.

The Ricci flow is given by the evolution equation

$$\tfrac{\partial}{\partial t} g_t = -2\,\rho_{g_t}, \tag{II.1}$$

where g_t is a one-parameter family of pseudo-Riemannian metrics on a manifold M. For any given differentiable metric g_0 on a closed manifold M, there exists a unique solution g_t, with $t \in [0,\varepsilon)$ for some $\varepsilon > 0$, to the Ricci flow equation such that $g_t|_{t=0} = g_0$.

The first examples of solutions to the Ricci flow are given by Einstein metrics, which provide solutions of the form

$$g_t = (1-2\mu t) g_0, \quad \text{where} \quad \begin{cases} t \in \left(-\infty, \frac{1}{2\mu}\right) & \text{if} \quad \mu > 0, \\ t \in \left(\frac{1}{2\mu}, \infty\right) & \text{if} \quad \mu < 0, \\ t \in (-\infty, \infty) & \text{if} \quad \mu = 0, \end{cases}$$

for an initial Einstein metric g_0 such that $\rho_{g_0} = \mu g_0$. In any of the three cases given by the different values of μ, g_0 remains invariant modulo homotheties. If we allow the initial metric to change not only by homotheties but also by diffeomorphisms, a solution g_t to the Ricci flow is said to be ***self-similar*** if there exists a positive function $\sigma(t)$ and a one-parameter group of diffeomorphisms $\psi_t \colon M \to M$ such that

$$g_t = \sigma(t) \psi_t^* g_0. \tag{II.2}$$

Remark 3.17. Assume that Equation (II.2) determines a solution to the Ricci flow and differentiate it to obtain

$$\tfrac{\partial}{\partial t} g_t = -2\rho_{g_t} = \tfrac{d\sigma}{dt}(t) \psi_t^* g_0 + \sigma(t) \psi_t^* \left(\mathcal{L}_X g_0\right), \tag{II.3}$$

where $\mathcal{L}$ denotes the Lie derivative and X is the time-dependent vector field given by

$$X\left(\psi_t(p)\right) = \tfrac{d}{dt}(\psi_t(p))$$

for any $p \in M$. Now, since $\rho_{g_t} = \psi_t^* \rho g_0$, we can actually drop the pull-back in Equation (II.3) and so

$$-2\rho_{g_0} = \tfrac{d\sigma}{dt}(t) g_0 + \mathcal{L}_{\tilde{X}} g_0, \tag{II.4}$$

where $\tilde{X}(t) = \sigma(t)X(t)$. If we now set

$$\mu = -\tfrac{1}{2}\tfrac{d\sigma}{dt}\Big|_{t=0} \quad \text{and} \quad X_0 = \tilde{X}(0),$$

Equation (II.4) becomes

$$-2\rho_{g_0} = -2\mu g_0 + 2\mathcal{L}_{X_0} g_0$$

at $t = 0$. This proves that for any self-similar solution to the Ricci flow, there exists a vector field X on M such that

$$\mathcal{L}_X g + \rho = \mu g.$$

Conversely, let X be a complete vector field on a pseudo-Riemannian manifold (M, g) and denote by $\psi_t \colon M \to M$ the family of diffeomorphisms generated by X according to

$$\tfrac{\partial}{\partial t}\psi_t(p) = \tfrac{1}{1-2\mu t} X(\psi_t(p)) \quad \text{and} \quad \psi(0) = \mathrm{Id}_M,$$

which is defined for all $t \in (-\infty, \frac{1}{2\mu})$ if $\mu > 0$ and for all $t \in (\frac{1}{2\mu}, \infty)$ if $\mu < 0$. If we now consider the one-parameter family of metrics

$$g_t = (1 - 2\mu t)\psi_t^* g,$$

then

$$\begin{aligned} \tfrac{\partial}{\partial t} g_t &= -2\mu\psi_t^* g + (1 - 2\mu)\psi_t^* \mathcal{L}_{\frac{1}{1-2\mu t}X} g \\ &= \psi_t^* \left(-2\mu g + \mathcal{L}_{X(\psi_t(p))} g\right). \end{aligned}$$

Now, if $\mathcal{L}_X g + \rho = \mu g$, then

$$\tfrac{\partial}{\partial t} g_t = \psi_t^*(-2\rho) = -2\psi_t^*\rho = -2\rho(\psi_t^* g) = -2\rho(g_t),$$

and so g_t is a solution to the Ricci flow.

The remark above shows that there exists a correspondence between self-similar solutions to the Ricci flow and what are known as Ricci solitons.

Ricci solitons

A triple (M, g, X) where (M, g) is a pseudo-Riemannian manifold and X is a vector field on M satisfying the differential equation

$$\mathcal{L}_X g + \rho = \mu g, \quad \mu \in \mathbb{R}, \tag{II.5}$$

is called a *Ricci soliton*. These metrics not only generalize Einstein metrics but also are self-similar solutions to the Ricci flow and conversely, thus corresponding to geometric fixed points of the flow (modulo homotheties and diffeomorphisms). A Ricci soliton is said to be *expanding*, *steady*, or *shrinking* if the soliton constant $\mu < 0$, $\mu = 0$ or $\mu > 0$, respectively. Besides, if the soliton vector field X is the gradient of some potential function $f\colon M \to \mathbb{R}$, then the soliton is said to be a *gradient Ricci soliton*. In this case $\mathcal{L}_X g_0 = 2\mathrm{Hes}_{g_0}(f)$ and so

$$\mathrm{Hes}_{g_0}(f) + \rho_{g_0} = \mu g_0. \tag{II.6}$$

A Ricci soliton is said to be *trivial* if the corresponding pseudo-Riemannian metric is Einstein, in which case one may solve Equation (II.5) just by setting $X = 0$. It immediately follows from (II.5) that two Ricci soliton vector fields X_1 and X_2 on a given manifold (M, g) differ in a homothetic vector field $\xi = X_1 - X_2$. While the existence of homothetic vector fields is a very rigid condition in the positive definite case, Lorentzian manifolds may admit homothetic vector fields without being flat. Moreover, the Ricci soliton equation (II.5) is invariant by homotheties in the sense that (M, g, X) is a Ricci soliton with soliton constant μ if and only if $(M, \kappa g, \frac{1}{\kappa}X)$ is a Ricci soliton with soliton constant $\frac{\mu}{\kappa}$ for any $\kappa > 0$. Therefore, we develop our study modulo homotheties. We refer to [49] for more information.

The scalar curvature of a Ricci soliton satisfies

$$2\Delta\tau = \mu\tau - \|\rho\|^2 + \langle X, \nabla\tau\rangle$$

(see for example [47]). Therefore, homogeneous steady Ricci solitons are necessarily Ricci flat, and therefore flat. As a consequence of local homogeneity, a shrinking Ricci soliton is necessarily a gradient (see [105]).

A gradient Ricci soliton (M, g, f) is *rigid* if the manifold (M, g) splits isometrically as a product $N \times \mathbb{R}^k$, where the non-Euclidean factor (N, g_N) is Einstein and the potential function is given by the norm of the projection on the Euclidean factor, $f = \frac{\mu}{2}\|\pi_{\mathbb{R}^k}\|^2$. Besides, the soliton constant is given by $\rho_N = \mu g_N$. It was shown in [121] that homogeneous gradient Ricci solitons are necessarily rigid, and Arroyo and Lafuente showed in [9] that any four-dimensional homogeneous expanding Ricci soliton is homothetic to an algebraic Ricci soliton. Hence, any four-dimensional homogeneous Ricci soliton is either Einstein, or an algebraic Ricci soliton or homothetic to a product $\mathbb{S}^2 \times \mathbb{R}^2$.

Algebraic Ricci solitons

A metric Lie group $(G, \langle\cdot,\cdot\rangle)$ is an *algebraic Ricci soliton* if its Ricci operator satisfies

$$\mathrm{Ric} = \mu\,\mathrm{Id} + \mathfrak{D}$$

for some derivation $\mathfrak{D}$ of the corresponding Lie algebra (see [97]). Algebraic Ricci solitons are critical points of the scalar curvature for an appropriately restricted family of metrics [97], and they are critical for a quadratic curvature functional with zero energy in dimensions three and four [23]. On the other hand, Ricci solitons on Lie groups with left-invariant soliton vector field are not necessarily critical for any quadratic curvature functional, thus being of a different nature.

If a simply connected pseudo-Riemannian Lie group $(G, \langle\cdot,\cdot\rangle)$ is an algebraic Ricci soliton, then it is a Ricci soliton (see [133]). Indeed, let $\{\varphi_t : G \to G\}$ be the one-parameter group of automorphisms of G determined by

$$d(\varphi_t)_e = \operatorname{Exp}(\tfrac{t}{2}\mathfrak{D}),$$

where $\mathfrak{D} = \operatorname{Ric} - \mu \operatorname{Id}$ is the derivation of the Lie algebra determining the algebraic Ricci soliton. Define a vector field X on G as the infinitesimal generator of $\{\varphi_t\}$, i.e.,

$$X(p) = \left.\frac{d}{dt}\right|_{t=0} \varphi_t(p)$$

for any $p \in G$. Then

$$\begin{aligned}
(\mathcal{L}_X \langle\cdot,\cdot\rangle)(e_i, e_j) &= \tfrac{d}{dt}(\varphi_t^* \langle\cdot,\cdot\rangle)(e_i, e_j) = \tfrac{1}{2}\left\{\langle \mathfrak{D}e_i, e_j\rangle + \langle e_i, \mathfrak{D}e_j\rangle\right\} \\
&= \tfrac{1}{2}\left\{\langle \operatorname{Ric}(e_i), e_j\rangle + \langle e_i, \operatorname{Ric}(e_j)\rangle\right\} - \mu\langle e_i, e_j\rangle \\
&= (\rho - \mu\langle\cdot,\cdot\rangle)(e_i, e_j),
\end{aligned}$$

from where it follows that $\mathcal{L}_X\langle\cdot,\cdot\rangle - \rho = -\mu\langle\cdot,\cdot\rangle$. Replacing X by $-X$ one gets the Ricci soliton equation (II.5).

The Bach flow: Bach solitons

The Bach tensor $\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \frac{1}{2} W[\rho]$ is conformally invariant, trace-free and divergence-free in dimension four.

The *Bach flow* is the differential equation

$$\tfrac{\partial}{\partial t} g_t = \mathfrak{B}_{g_t} + \tfrac{1}{12}(\Delta \tau_{g_t}) g_t \tag{II.7}$$

where g_t is a one-parameter family of pseudo-Riemannian metrics on a manifold M. For any given differentiable metric g_0 on a closed manifold there exists a unique solution g_t with $t \in [0, \varepsilon)$ for some $\varepsilon > 0$, to the Bach flow equation such that $g_t|_{t=0} = g_0$ (see [11, 12]).

Given a homothetic transformation $\bar{g} = \lambda g$ of a pseudo-Riemannian metric on a manifold M, the corresponding Bach tensor rescales as $\overline{\mathfrak{B}} = \lambda^{-1}\mathfrak{B}$ and so the right-hand side in (II.7) is homogeneous of degree $d = -1$ under homotheties. Therefore, there is a one-to-one correspondence between self-similar solutions of the Bach flow and *Bach solitons*, i.e., (M, g, X, μ) such that

$$\mathcal{L}_X g + \left(\mathfrak{B} + \tfrac{1}{2}\Delta\tau g\right) = \mu g,$$

just like in the Ricci flow case (see [133]).

In the homogeneous case – or, in a more general situation, if the scalar curvature is constant –, Equation (II.7) becomes

$$\tfrac{\partial}{\partial t} g_t = \mathfrak{B}_{g_t}, \tag{II.8}$$

and the corresponding Bach solitons are determined by triples (M, g, X) so that

$$\mathcal{L}_X g + \mathfrak{B} = \mu g, \tag{II.9}$$

where X is a vector field on M and $\mu \in \mathbb{R}$. Following the terminology we introduced when we talked about Ricci solitons, a Bach soliton is said to be expanding, steady or shrinking if $\mu < 0$, $\mu = 0$ or $\mu > 0$, respectively. Moreover, the soliton is said to be a *gradient* Bach soliton if the vector field is the gradient of a potential function f, $X = \frac{1}{2}\nabla f$.

We now assume that $\dim M = 4$, as it is the situation that concerns our study of the Bach flow. Since the Bach tensor is trace-free, it is easy to see that

$$\mu = \tfrac{1}{2} \operatorname{div} X$$

just by tracing (II.9).

Let X be a vector field on a Riemannian manifold (M, g) and let Ψ be a symmetric $(0, 2)$-tensor field on M. Then

$$\langle \mathcal{L}_X g, \Psi \rangle = 2 \operatorname{div}(\iota_X \Psi) + 2(\operatorname{div} \Psi)(X).$$

Applying the identity above to a Bach soliton ($\mathcal{L}_X g + \mathfrak{B} = \mu g$) and setting $\Psi = \mathfrak{B}$, one has that

$$\begin{aligned} 0 &= \langle \mathcal{L}_X g + \mathfrak{B} - \mu g, \mathfrak{B} \rangle = \langle \mathcal{L}_X g, \mathfrak{B} \rangle + \|\mathfrak{B}\|^2 - \mu \langle g, \mathfrak{B} \rangle \\ &= 2 \operatorname{div}(\iota_X \mathfrak{B}) + 2(\operatorname{div} \mathfrak{B})(X) + \|\mathfrak{B}\|^2 - \mu \operatorname{tr}_g \mathfrak{B} \\ &= 2 \operatorname{div}(\iota_X \mathfrak{B}) + \|\mathfrak{B}\|^2, \end{aligned}$$

from where it follows that compact Bach solitons are necessarily Bach-flat – even if the scalar curvature is not constant, just replacing the constant μ by the function $\hat{\mu} = \mu - \frac{1}{2}\Delta\tau$ and proceeding as above (see also [81]).

Just like gradient Ricci solitons, gradient Bach solitons are rigid in the homogeneous case. This follows immediately from the following general result obtained by Petersen and Wylie in [122], just by applying it to $\tilde{q} = \mu g - \mathfrak{B}$.

Theorem 3.18. *Let (M, g) be a homogeneous manifold and let $\tilde{q}$ be a symmetric $(0, 2)$-tensor field which is divergence-free and invariant by isometries. If there is a non-constant function f satisfying the equation $\operatorname{Hes}_f = \tilde{q}$, then (M, g) splits as a product $N \times \mathbb{R}^k$ and f is a function on the Euclidean factor.*

Griffin showed in [81] that homogeneous gradient Bach solitons either are Bach-flat or split as a product $N \times \mathbb{R}^k$, where the potential function depends only on the Euclidean factor and (N, g_N) is a homogeneous manifold. Moreover, homogeneous steady gradient Bach solitons are necessarily Bach-flat. The existence of gradient Bach solitons on those products were considered in [81], where it is shown that

- Non-Bach-flat homogeneous gradient shrinking Bach solitons reduce to the products $\mathbb{S}^2 \times \mathbb{R}^2$ and $\mathbb{H}^2 \times \mathbb{R}^2$.

- The only non-Bach-flat homogeneous expanding Bach soliton on a product $N^3 \times \mathbb{R}$ corresponds to $\mathbb{S}^3 \times \mathbb{R}$, where the metric on $\mathbb{S}^3$ is not the round metric, but a Berger one.

The non-gradient case is more subtle. For instance, the existence of non-Bach-flat homogeneous steady Bach solitons is still an open question. We provide a partial answer in Chapter 5, where we show that steady algebraic Bach solitons are necessarily Bach-flat.

Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group equipped with a left-invariant Riemannian metric. We say that $(G, \langle\cdot,\cdot\rangle)$ is an *algebraic Bach soliton* if $\mathfrak{D} = \widehat{\mathfrak{B}} - \mu \mathrm{Id}$ is a derivation of the Lie algebra $\mathfrak{g}$ of G for some $\mu \in \mathbb{R}$, where $\widehat{\mathfrak{B}}$ is the $(1,1)$-tensor field associated to the Bach tensor.

As in the Ricci flow case, if a simply connected Riemannian Lie group $(G, \langle\cdot,\cdot\rangle)$ is an algebraic Bach soliton, then it is a Bach soliton and thus a self-similar solution of the homogeneous Bach flow (see [133]). Indeed, let $\{\varphi_t : G \to G\}$ be the one-parameter family of automorphisms of G determined by

$$d(\varphi_t)_e = \mathrm{Exp}(\tfrac{t}{2}\mathfrak{D}),$$

where $\mathfrak{D} = \widehat{\mathfrak{B}} - \mu\,\mathrm{Id}$ is the derivation of the Lie algebra determining the algebraic Bach soliton. Define a vector field X on G as the infinitesimal generator of $\{\varphi_t\}$, i.e., $X(p) = \frac{d}{dt}\big|_{t=0}\varphi_t(p)$ for any $p \in G$. Then one has that

$$\begin{aligned}
(\mathcal{L}_X\langle\cdot,\cdot\rangle)(e_i,e_j) &= \tfrac{d}{dt}(\varphi_t^*\langle\cdot,\cdot\rangle)(e_i,e_j) = \tfrac{1}{2}\left\{\langle\mathfrak{D}e_i,e_j\rangle + \langle e_i,\mathfrak{D}e_j\rangle\right\} \\
&= \tfrac{1}{2}\left\{\langle\widehat{\mathfrak{B}}e_i,e_j\rangle + \langle e_i,\widehat{\mathfrak{B}}e_j\rangle\right\} - \mu\langle e_i,e_j\rangle \\
&= (\mathfrak{B} - \mu\langle\cdot,\cdot\rangle)(e_i,e_j),
\end{aligned}$$

from where it follows that $\mathcal{L}_X\langle\cdot,\cdot\rangle - \mathfrak{B} = -\mu\langle\cdot,\cdot\rangle$. Replacing X by $-X$ one gets the homogeneous Bach soliton equation (II.9). Therefore, algebraic Bach solitons correspond to self-similar solutions to the homogeneous Bach flow (II.8) of the form $g_t = \sigma(t)\psi_t^*\langle\cdot,\cdot\rangle$, where $\sigma(t)$ is a positive real function and ψ_t is a one-parameter family of automorphisms of $(G, \langle\cdot,\cdot\rangle)$.

In Chapter 5 we determine all the algebraic Bach solitons, showing that they either are algebraic Ricci solitons or correspond to the above-mentioned gradient shrinking Bach soliton on $\mathbb{S}^3 \times \mathbb{R}$, or to a one-parameter family of left-invariant semi-direct extensions of the Heisenberg algebra (cf. Theorem 5.8). Due to the difficulty in manipulating the Bach tensor we approach the problem in an indirect way, considering the existence of general abstract algebraic T-solitons on each four-dimensional Lie group. The study of the existence of algebraic solitons associated to a generic $(0,2)$-tensor field T provides us with some necessary conditions for the existence of such general T-solitons which turn out to be good enough to simplify the analysis of the algebraic Bach solitons.

Chapter 4

Ricci solitons on four-dimensional Lorentzian Lie groups

Non-trivial homogeneous Riemannian Ricci solitons are rigid or expanding. Moreover the expanding ones are necessarily algebraic in dimension four [9]. None of these four-dimensional Ricci solitons can be realized as a left-invariant vector field on a Lie group. Indeed it was shown in [58] that non-trivial left-invariant Ricci solitons do not exist on Riemannian unimodular Lie groups and there are no three-dimensional left-invariant Ricci solitons on Riemannian Lie groups. A straightforward calculation shows that no four-dimensional homogeneous Riemannian Ricci soliton may be determined by a left-invariant vector field unless they are trivial.

In sharp contrast, the Lorentzian signature does support such solitons (see [24]). In spite of this general discrepancy, there does exist a certain correspondence between Riemannian and Lorentzian algebraic Ricci solitons in the nilpotent case (see [135] for more information about this relation).

In this chapter we will give a complete classification of left-invariant Ricci solitons on four-dimensional Lorentzian Lie groups. After reviewing left-invariant Einstein metrics and plane waves, we will recall the situation in dimension three, which is much simpler than the four-dimensional one. The main result of this chapter, which is Theorem 4.2, provides a complete description, modulo homotheties, of non-trivial left-invariant Ricci solitons which are neither symmetric nor pp-waves. The symmetric case is treated in Remark 4.5 and the pp-wave Lie groups are considered in Section 4.4. The results in this chapter are contained in the work [69].

Einstein metrics on Lorentzian four-dimensional Lie groups

While four-dimensional homogeneous Einstein metrics are locally symmetric in the Riemannian setting [88], the Lorentzian signature allows other possibilities. Left-invariant Einstein metrics on four-dimensional Lorentzian Lie groups were studied in [36] and a different approach shows that left-invariant Einstein metrics split into three categories: symmetric spaces, plane waves and left-invariant metrics which do not correspond to any of these.

Indecomposable locally symmetric Lorentzian spaces either are irreducible (and therefore of constant sectional curvature), or they correspond to Cahen-Wallach symmetric spaces [28], which are a special class of plane waves. Four-dimensional products $N^3 \times \mathbb{R}$ are Einstein if and only if they are flat and so the only decomposable four-dimensional Einstein Lorentzian symmetric spaces of non-constant sectional curvature are products $M_1(c)\times M_2(c)$ of two surfaces with the same constant sectional curvature. The other possibilities are covered by the following result (see [114]).

Theorem 4.1. *Let $(G, \langle \cdot, \cdot \rangle)$ be a four-dimensional Lie group with a left-invariant Einstein Lorentzian metric which is neither locally symmetric nor a plane wave. Then, it is locally homothetic to the Lie group determined by one of the following:*

(i) The Ricci-flat semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[e_1, e_4] = -2e_1, \quad [e_2, e_4] = e_2 + \sqrt{3}e_3, \quad [e_3, e_4] = -\sqrt{3}e_2 + e_3, \quad \textit{or}$$

(ii) the semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[u_1, u_4] = -u_1 + \delta u_2, \quad [u_2, u_4] = 5u_2, \quad [u_3, u_4] = 2u_3, \quad \delta \neq 0, \quad \textit{or}$$

(iii) the semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[u_1, u_4] = 4u_1, \quad [u_2, u_4] = -2u_2 + \delta u_3, \quad [u_3, u_4] = \delta u_1 + u_3, \quad \delta \neq 0,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis with e_3 timelike, and $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = \langle u_4, u_4 \rangle = 1$.

The curvature operator $\mathcal{R} : \Lambda^2 \to \Lambda^2$ of the metrics corresponding to Assertion *(i)* has real and complex eigenvalues, and $\|\nabla R\|^2 \neq 0$. The metrics corresponding to Assertion *(ii)* have scalar curvature $\tau = -48$ and their Weyl curvature operator is two-step nilpotent. Moreover, they are locally isometric to the only non-reductive homogeneous space which is Einstein but not of constant sectional curvature [35, 66]. The metrics corresponding to Assertion *(iii)* have scalar curvature $\tau = -12$ and their Weyl curvature operator is three-step nilpotent.

Homogeneous pp-waves and plane waves

Let $(M, g, \mathcal{U})$ be a Brinkmann wave, i.e., a Lorentzian manifold admitting a parallel degenerate line field $\mathcal{U}$. $(M, g, \mathcal{U})$ is said to be a *pp-wave* if the parallel line field is locally generated by a parallel null vector field and (M, g) is transversally flat, i.e., its curvature tensor satisfies $R(X, Y) = 0$ for all $X, Y \in \mathcal{U}^\perp$. In such case there exist local coordinates (u, v, x^1, x^2) so that

$$g = du \circ dv + H(v, x^1, x^2) dv \circ dv + dx^1 \circ dx^1 + dx^2 \circ dx^2 \,.$$

Leistner showed in [100] that a Brinkmann wave $(M, g, \mathcal{U})$ is a pp-wave if and only if it is transversally flat and Ricci isotropic, i.e., $g(\operatorname{Ric} X, \operatorname{Ric} X) = 0$ for any vector field X on M.

A pp-wave is said to be a *plane wave* if the covariant derivative of the curvature tensor satisfies $\nabla_X R = 0$ for all $X \in \mathcal{U}^\perp$. In this case the local coordinates above can be specialized so that $H(v, x^1, x^2) = a_{ij}(v) x^i x^j$. The Ricci operator of any pp-wave is two-step nilpotent and the metric is Ricci-flat if $\Delta_x H = 0$, where $\Delta_x = \partial_{x^1 x^1} + \partial_{x^2 x^2}$ is the spacelike Laplacian. It was shown in [78] that locally homogeneous Ricci-flat pp-waves are plane waves in the four-dimensional case. Homogeneous steady Ricci solitons on pp-waves which are not plane waves are given in Section 4.4, where we show that the result in [78] does not extend to Ricci solitons.

Homogeneous plane waves of dimension four are described in terms of a 2×2 skew-symmetric matrix F and a 2×2 symmetric matrix A_0 so that the defining function $H(v, x^1, x^2)$ takes the form $H = \vec{x}^T A(v)\vec{x}$, where the matrix $A(v)$ is given by (see [19])

$$A(v) = e^{vF} A_0 e^{-vF}, \quad \text{or} \quad A(v) = \frac{1}{(v+b)^2} e^{\log(v+b)F} A_0 e^{-\log(v+b)F}.$$

Furthermore, the plane wave metric is Ricci-flat if and only if A_0 is trace-free.

The existence of Ricci solitons on plane waves was investigated in [25] where it is shown that any plane wave is a steady gradient Ricci soliton. Due to the existence of homothetic vector fields, one also has the existence of expanding and shrinking Ricci solitons on some special classes of plane waves. In any case, the soliton vector field needs not be left-invariant for a plane wave Lie group, and hence the existence of left-invariant Ricci solitons on plane wave Lie groups will be considered in Section 4.4.

Left-invariant Ricci solitons on three-dimensional Lorentzian Lie groups

Non-trivial three-dimensional left-invariant Ricci solitons are either non-symmetric pp-waves or locally isometric to a left-invariant metric on $G = O(1, 2)$, the universal cover of $SL(2, \mathbb{R})$ or the non-unimodular semi-direct extension $\mathbb{R}^2 \rtimes \mathbb{R}$, respectively given by the Lorentzian Lie algebras

$$\begin{array}{llll} (i) & [u_1, u_2] = \lambda u_3, & [u_1, u_3] = -\lambda u_1 \mp u_2, & [u_2, u_3] = \lambda u_2, \quad \lambda \neq 0, \\ (ii) & [u_1, u_2] = u_1 + \lambda u_3, & [u_1, u_3] = -\lambda u_1, & [u_2, u_3] = \lambda u_2 + u_3, \quad \lambda \neq 0, \\ (iii) & [e_1, e_3] = e_1 - e_2, & [e_2, e_3] = e_1 + e_2 & \end{array}$$

where $\{u_1, u_2, u_3\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1$, and $\{e_1, e_2, e_3\}$ is an orthonormal basis with timelike e_1.

The three-dimensional Lorentzian Lie groups corresponding to cases (i) and (ii) have a single Ricci curvature which is either a double or a triple root of the minimal polynomial of the Ricci operator (see [24]). Moreover, the Lie group corresponding to (iii), which was omitted in [24], has complex Ricci curvatures $-2 \pm 2i$.

There are two different possibilities for three-dimensional left-invariant pp-waves which are Ricci solitons: a locally conformally flat plane wave (which is locally isometric to a $\mathcal{P}_c$–space), or a pp-wave that is locally isometric to a $\mathcal{N}_b$–space. We recommend to consult [76] for a classification of homogeneous pp-waves in dimension three, definitions of $\mathcal{P}_c$ and $\mathcal{N}_b$–spaces and more details.

Left-invariant Ricci solitons on four-dimensional Lorentzian Lie groups

As in the Einstein case, the four-dimensional situation is more complicated than the corresponding three-dimensional one. We consider the case of left-invariant Ricci solitons on pp-wave Lie groups separately in Section 4.4. The remaining possibilities are given the main result of this chapter as follows.

Theorem 4.2. *A non-symmetric four-dimensional Lorentzian Lie group which is not a pp-wave is a non-trivial left-invariant Ricci soliton if and only if it is homothetic to one of the following:*

(i) $G_\alpha = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_1, e_4] = \alpha e_1,\ [e_2, e_4] = \varepsilon\left(1-\frac{\alpha^2}{2}\right)^{\frac{1}{2}} e_2 - e_3,\ [e_3, e_4] = e_2 + \varepsilon\left(1-\frac{\alpha^2}{2}\right)^{\frac{1}{2}} e_3,$$

where the parameter satisfies $0 \le \alpha \le \sqrt{2}$ *and* $\{e_1, e_2, e_3, e_4\}$ *is an orthonormal basis with timelike* e_3*. If* $\alpha = 0$*, then* $\varepsilon = 1$*. If* $0 < \alpha < \sqrt{2}$*, then* $\varepsilon^2 = 1$*. In the latter case,* $\alpha \neq \frac{2}{\sqrt{3}}$ *when* $\varepsilon = -1$*.*

(ii) $G_\alpha = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_4] = \alpha u_1, \quad [u_2, u_4] = -\alpha u_2 + u_3, \quad [u_3, u_4] = u_1, \qquad \alpha > 0,$$

where $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis for which the non-zero inner products are* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$*.*

(iii) $G = E(1,1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_2, e_4] = -[e_1, e_2] = e_2, \quad [e_1, e_3] = [e_3, e_4] = \tfrac{1}{2}[e_1, e_4] = e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ *is an orthonormal basis with timelike* e_3*.*

(iv) $G_{\alpha\beta} = E(1,1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$\begin{aligned} &[u_1, u_2] = u_1, \quad [u_1, u_4] = -2\alpha(\alpha\beta+1)u_1, \quad [u_2, u_3] = u_3,\\ &[u_2, u_4] = \beta u_1, \quad [u_3, u_4] = \alpha u_3, \end{aligned}$$

where $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis for which the non-zero inner products are* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$*, and the parameters* $\alpha > 0$ *and* $\alpha\beta \notin \{-2, -1, -\frac{1}{2}\}$*.*

Remark 4.3. The left-invariant Ricci solitons corresponding to G_α in Assertion *(i)* are steady and their left-invariant soliton vector field is defined by $X = X_1 e_1 + e_4$ if the parameter $\alpha = 0$, and by $X = \frac{1}{2}\left(\alpha + \varepsilon\sqrt{4-2\alpha^2}\right) e_4$ otherwise. Moreover, the Ricci operator has eigenvalues

$$\begin{aligned} &\xi_1 = 0, \qquad \xi_2 = -\alpha\left(\alpha + \varepsilon\,(4-2\alpha^2)^{\frac{1}{2}}\right),\\ &\xi_3 = \alpha^2 - 2 - \varepsilon\alpha\left(1-\frac{\alpha^2}{2}\right)^{\frac{1}{2}} + \left(\alpha^2 - 4 - 2\varepsilon\alpha\,(4-2\alpha^2)^{\frac{1}{2}}\right)^{\frac{1}{2}},\\ &\xi_4 = \alpha^2 - 2 - \varepsilon\alpha\left(1-\frac{\alpha^2}{2}\right)^{\frac{1}{2}} - \left(\alpha^2 - 4 - 2\varepsilon\alpha\,(4-2\alpha^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, the Ricci curvatures are $\{0, 0, -2 \pm 2i\}$ if $\alpha = 0$, $\{0, \lambda, \alpha \pm \beta i\}$ with $\lambda\alpha\beta \neq 0$ if $0 < \alpha < \sqrt{2}$, and $\{0, -2, \pm\sqrt{2}\, i\}$ if $\alpha = \sqrt{2}$.

The left-invariant Ricci solitons corresponding to G_α in Assertion *(ii)* are steady and their left-invariant soliton vector field is defined by $X = X_1 u_1 - X_1 \alpha u_3 - \frac{1}{2}\alpha u_4$. Moreover, their Ricci operator is three-step nilpotent.

The left-invariant Ricci solitons corresponding to Assertion *(iii)* are steady and their left-invariant soliton vector field is defined by $X = -\frac{1}{2}e_1 + \frac{3}{2}e_4$. Furthermore, their Ricci operator has eigenvalues $\{0, -2, -2 \pm \sqrt{6}\, i\}$.

The left-invariant Ricci solitons corresponding to $G_{\alpha\beta}$ in Assertion *(iv)* are expanding with $\mu = -(2(\alpha\beta + 1)^2 + 1)\alpha^2$ and their left-invariant soliton vector field is defined by $X = X_1 u_1 + X_2 u_2 + X_4 u_4$, where

$$\begin{aligned} X_1 &= \tfrac{1}{2(2\alpha\beta+1)}(\alpha\beta + 2)(2(\alpha\beta + 1)\alpha\beta - 1), \\ X_2 &= \tfrac{1}{2\alpha\beta+1}(\alpha\beta + 2)(2(\alpha\beta + 2)\alpha\beta + 3)\alpha^2, \\ X_4 &= \tfrac{1}{2\alpha\beta+1}(\alpha\beta + 2)^2\, \alpha\,. \end{aligned}$$

Moreover, their Ricci operator is diagonalizable with non-zero real eigenvalues

$$\begin{aligned} &\xi_1 = \xi_2 = -(2\alpha\beta + 1)(\alpha\beta + 1)\alpha^2, \\ &\xi_3 = (2\alpha\beta + 1)\alpha^2, \qquad \xi_4 = -(2(\alpha\beta + 2)\alpha\beta + 3)\alpha^2\,. \end{aligned}$$

Remark 4.4. Let $(G_1, \langle\cdot,\cdot\rangle_1)$ and $(G_2, \langle\cdot,\cdot\rangle_2)$ be two Lorentzian Lie groups with non-zero scalar curvatures. If $(G_1, \langle\cdot,\cdot\rangle_1)$ and $(G_2, \langle\cdot,\cdot\rangle_2)$ are homothetic, then

$$\tau_1^{-2}\|R_1\|^2 = \tau_2^{-2}\|R_2\|^2 \quad \text{and} \quad \tau_1^{-2}\|W_1\|^2 = \tau_2^{-2}\|W_2\|^2,$$

where R_i and W_i denote the curvature tensor and the Weyl conformal curvature tensor for $i = 1, 2$, respectively. We use the quadratic scalar curvature invariants to show that left-invariant metrics in different assertions in Theorem 4.2 correspond to distinct homothetic classes. It also follows that different values of the parameter in Assertion *(i)* determine distinct homothetic classes. Metrics in Assertion *(iv)* with different $\alpha\beta$ correspond to distinct homothetic classes.

Remark 4.5. Locally symmetric Lorentzian spaces which are neither of constant sectional curvature nor a Cahen-Wallach symmetric space split as a product [28]. Besides, left-invariant symmetric Ricci solitons which are neither Einstein nor plane waves are locally isometric to $\mathbb{L}^2 \times N(c)$, where $N(c)$ is a surface of constant curvature, and correspond to one of the following Lie groups:

- $G_{\alpha\beta}$ in Assertion *(iv)* of Theorem 4.2 for $\alpha\beta = -1$, as discussed in Section 4.1.4.
- The Lie group $H^3 \rtimes \mathbb{R}$ determined by the Lie algebra

$$\begin{aligned} &[u_1, u_2] = \lambda_1 u_1, &&[u_1, u_4] = -\tfrac{\gamma_3\lambda_1^2}{\gamma_4} u_1, \\ &[u_2, u_4] = \gamma_3\lambda_1^2 u_3, &&[u_3, u_4] = \gamma_4\lambda_1 u_3, \end{aligned}$$

where $\lambda_1\gamma_4 \neq 0$ and $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis such that $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$. This is a expanding Ricci soliton with $\mu = -\lambda_1^2$ and left-invariant soliton vector field $X = -\frac{\gamma_3\lambda_1^2}{\gamma_4^2}u_2 + \frac{\gamma_3^2\lambda_1^3}{2\gamma_4^3}u_3 - \frac{\lambda_1}{\gamma_4}u_4$, as discussed in Section 4.3.2.

Remark 4.6. The Bach tensor of a four-dimensional manifold is defined by

$$\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \frac{1}{2}W[\rho].$$

Four-dimensional Bach-flat metrics are conformally invariant and Bach-flatness is a necessary condition for being conformally Einstein. The left-invariant metrics in Theorem 4.2 are Bach-flat if and only if they correspond to Assertion *(iv)* when the parameters satisfy $\alpha\beta = -\frac{5}{4}$. In this case, the vector field $X = \frac{3}{2}u_1 - \frac{3\alpha}{2}u_4$ is locally a gradient and satisfies $\operatorname{div}_4 W + \frac{1}{2}W(\cdot,\cdot,\cdot,X) = 0$. A straightforward calculation shows that the Weyl operator acting on the space of two-forms has non-zero eigenvalues and thus the metric is weakly-generic. Therefore, it is conformally Einstein (see [94] for more information).

Remark 4.7. The metrics in Theorem 4.2-*(i)* are critical for the functional given by the L^2-norm of the Ricci tensor or, equivalently $\mathcal{F}_0$-critical with zero energy. These metrics also are steady algebraic Ricci solitons, which do not exist in the Riemannian situation.

The scalar curvature of the metrics in Theorem 4.2-*(ii)* is zero, so they are $\mathcal{S}$-critical, but they are not critical for any other curvature quadratic functional and nor are they algebraic Ricci solitons.

The metrics in Theorem 4.2-*(iii)* are never critical for any curvature quadratic functional and they are not algebraic Ricci solitons either.

The metrics in Theorem 4.2-*(iv)* are $\mathcal{F}_t$-critical with zero energy for

$$t = -\frac{2\alpha^2\beta^2 + 4\alpha\beta + 3}{6\alpha^2\beta^2 + 8\alpha\beta + 4},$$

and they are never algebraic Ricci solitons.

Left-invariant metrics and Gröbner bases

Connected and simply connected four-dimensional Lie groups are either products $SU(2) \times \mathbb{R}$, $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$, or one of the solvable semi-direct extensions of three-dimensional unimodular Lie groups $\widetilde{E}(2) \rtimes \mathbb{R}$, $E(1,1) \rtimes \mathbb{R}$, $H^3 \rtimes \mathbb{R}$ or $\mathbb{R}^3 \rtimes \mathbb{R}$, where $\widetilde{E}(2)$, $E(1,1)$, H^3 and $\mathbb{R}^3$ denote the Euclidean, the Poincaré, the Heisenberg and the Abelian three-dimensional Lie algebras, respectively. Since our purpose is to investigate left-invariant Ricci solitons, we work at the purely algebraic level, and therefore we restrict to the corresponding Lie algebras. Left-invariant Riemannian metrics are described, using the work of Milnor [104], in terms of the corresponding derivations on the three-dimensional unimodular Lie subalgebras. The Lorentzian situation is more subtle due to the fact that the restriction of the metric to the three-dimensional subalgebras $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{e}(2)$, $\mathfrak{e}(1,1)$, $\mathfrak{h}$ or $\mathfrak{r}^3$ may be a positive definite, Lorentzian or degenerate inner product. We follow [33] and consider separately the three possibilities above.

Let $(G, \langle\cdot,\cdot\rangle)$ be a four-dimensional Lorentzian Lie group and let X be a left-invariant vector field on G. Then $(G, \langle\cdot,\cdot\rangle, X, \mu)$ is a left-invariant Ricci soliton if and only if the symmetric tensor field $\frac{1}{2}\mathfrak{P} = \mathcal{L}_X\langle\cdot,\cdot\rangle + \rho - \mu\langle\cdot,\cdot\rangle$ vanishes identically. It is now immediate, since the vector field X is left-invariant, that the condition $\mathfrak{P} = 0$ equals to a system of polynomial equations on the structure constants which needs to be solved in order to obtain a complete classification. The theory of Gröbner bases provides a well-known strategy to solve rather large polynomial systems obtaining "better" polynomials that belong to the ideal generated by the initial polynomial system. We make use of Gröbner bases to show non-existence results in some cases. We refer to Section 1.6 and [53] for more information on Gröbner bases.

4.1 Extensions of Lorentzian Lie groups

Let $(G, \langle\cdot,\cdot\rangle)$ be a four-dimensional Lorentzian Lie group of the form $G_3 \rtimes \mathbb{R}$ so that the restriction of the metric to the three-dimensional subalgebra $\mathfrak{g}_3$ is Lorentzian. Three-dimensional unimodular Lie algebras are completely described by a Milnor-type frame associated to the self-dual structure tensor L given by

$$L(X \times Y) = [X, Y],$$

where "$\times$" denotes the vector-cross product $\langle X \times Y, Z\rangle = \det(X, Y, Z)$. The self-duality of L ensures the existence of an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}_3$ that diagonalizes the structure tensor in the positive definite case [104]. If the inner product is of Lorentzian signature, then L may have non-trivial Jordan normal form as follows (see, for example [112]).

Ia. L is real diagonalizable. Therefore, there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where we assume e_3 to be timelike, so that $L(e_i) = \lambda_i e_i$.

Ib. L has complex eigenvalues. Consequently, there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where we assume e_3 to be timelike, so that

$$L = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}, \quad \beta \neq 0\,.$$

II. L has a double root of its minimal polynomial. Then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ so that

$$L = \begin{pmatrix} \lambda_1 & 0 & 0 \\ \varepsilon & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad \varepsilon = \pm 1, \quad \text{where} \quad \langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1\,.$$

III. L has a triple root of its minimal polynomial. Then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ so that

$$L = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \quad \text{where} \quad \langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = 1\,.$$

In what follows, we set $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ and L denotes the structure operator of the unimodular subalgebra $\mathfrak{g}_3$. We follow the work of Rahmani [124] to describe Lorentzian left-invariant metrics on $\mathfrak{g}_3$ and analyse the existence of left-invariant Ricci solitons on each of the possibilities above. It follows that all the left-invariant metrics in Theorem 4.2 are realized as extensions of unimodular Lorentzian Lie groups.

4.1.1 The structure operator L is diagonalizable

There exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with timelike e_3, where $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \operatorname{span}\{e_4\}$, so that

$$[e_1, e_2] = -\lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \quad \underset{(i=1,2,3)}{[e_i, e_4]} = \sum_{j=1}^{3} {\alpha_i}^j e_j,$$

for certain ${\alpha_i}^j \in \mathbb{R}$ depending on the eigenvalues λ_i. The Jacobi identity leads to the following different possibilities.

Structure operator with non-zero eigenvalues on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$

Assume $\lambda_1\lambda_2\lambda_3 \neq 0$. The left-invariant metrics on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$ are described by the corresponding Lie algebra structure

$$\begin{aligned} &[e_1, e_2] = -\lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, && [e_1, e_4] = \gamma_1\lambda_2 e_2 + \gamma_2\lambda_3 e_3, \\ &[e_2, e_3] = \lambda_1 e_1, \quad\;\; [e_2, e_4] = -\gamma_1\lambda_1 e_1 + \gamma_3\lambda_3 e_3, && [e_3, e_4] = \gamma_2\lambda_1 e_1 + \gamma_3\lambda_2 e_2, \end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. A straightforward calculation shows that a left-invariant vector field $X = \sum_\ell X_\ell e_\ell$ is a Ricci soliton if and only if the tensor field $\frac{1}{2}\mathfrak{P} = \mathcal{L}_X\langle\cdot,\cdot\rangle + \rho - \mu\langle\cdot,\cdot\rangle$ vanishes identically. Equivalently $\{\mathfrak{P}_{ij} = 0\}$, where the polynomials $\mathfrak{P}_{ij}$

are given by

$$\mathfrak{P}_{11} = (\gamma_1^2 - \gamma_2^2 - 1)\lambda_1^2 - (\gamma_1^2 - 1)\lambda_2^2 + (\gamma_2^2 + 1)\lambda_3^2 - 2\lambda_2\lambda_3 - 2\mu,$$

$$\mathfrak{P}_{12} = \gamma_2\gamma_3(\lambda_3^2 - \lambda_1\lambda_2) - 2(X_4\gamma_1 - X_3)(\lambda_1 - \lambda_2),$$

$$\mathfrak{P}_{13} = -\gamma_1\gamma_3(\lambda_2^2 - \lambda_1\lambda_3) + 2(X_4\gamma_2 - X_2)(\lambda_1 - \lambda_3),$$

$$\mathfrak{P}_{14} = \gamma_3(\lambda_2 - \lambda_3)^2 + 2(X_2\gamma_1 - X_3\gamma_2)\lambda_1,$$

$$\mathfrak{P}_{22} = -(\gamma_1^2 - 1)\lambda_1^2 + (\gamma_1^2 - \gamma_3^2 - 1)\lambda_2^2 + (\gamma_3^2 + 1)\lambda_3^2 - 2\lambda_1\lambda_3 - 2\mu,$$

$$\mathfrak{P}_{23} = \gamma_1\gamma_2(\lambda_1^2 - \lambda_2\lambda_3) + 2(X_4\gamma_3 + X_1)(\lambda_2 - \lambda_3),$$

$$\mathfrak{P}_{24} = -\gamma_2(\lambda_1 - \lambda_3)^2 - 2(X_1\gamma_1 + X_3\gamma_3)\lambda_2,$$

$$\mathfrak{P}_{33} = -(\gamma_2^2 + 1)\lambda_1^2 - (\gamma_3^2 + 1)\lambda_2^2 + (\gamma_2^2 + \gamma_3^2 + 1)\lambda_3^2 + 2\lambda_1\lambda_2 + 2\mu,$$

$$\mathfrak{P}_{34} = \gamma_1(\lambda_1 - \lambda_2)^2 + 2(X_1\gamma_2 + X_2\gamma_3)\lambda_3,$$

$$\mathfrak{P}_{44} = -\gamma_1^2(\lambda_1 - \lambda_2)^2 + \gamma_2^2(\lambda_1 - \lambda_3)^2 + \gamma_3^2(\lambda_2 - \lambda_3)^2 - 2\mu\,.$$

Since $\lambda_1\lambda_2\lambda_3 \neq 0$, we may assume $\lambda_1 = 1$ in our computations, which means we will be working with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_1}e_i$. Now, let $\mathcal{I} \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \lambda_2, \lambda_3, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded reverse lexicographical order and we see that the polynomials

$$\mathbf{g}_1 = \mu^2 \quad \text{and} \quad \mathbf{g}_2 = 4\lambda_2\lambda_3 + 3(\lambda_2 + \lambda_3 + 1)\mu$$

belong to $\mathcal{G}$. Since $\lambda_2\lambda_3 \neq 0$, there are no left-invariant Ricci solitons in this case.

Structure operator with a zero eigenvalue on $\tilde{E}(2) \rtimes \mathbb{R}$ or $E(1,1) \rtimes \mathbb{R}$

We distinguish two possibilities depending on the causality of $\ker L$. If $\ker L$ is spacelike then either $\lambda_1 = 0$ or $\lambda_2 = 0$, while if $\ker L$ is timelike then $\lambda_3 = 0$. Next we will show that left-invariant Ricci solitons exist only in the flat case.

Structure operator L with spacelike kernel

Without loss of generality, we can assume $\lambda_1 = 0$ and $\lambda_2\lambda_3 \neq 0$. Left-invariant metrics in this case are given by

$$[e_1, e_2] = -\lambda_3 e_3, \qquad [e_1, e_3] = -\lambda_2 e_2, \qquad [e_1, e_4] = \gamma_1 e_2 + \gamma_2 e_3,$$

$$[e_2, e_4] = \gamma_3 e_2 + \gamma_4\lambda_3 e_3, \quad [e_3, e_4] = \gamma_4\lambda_2 e_2 + \gamma_3 e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. We focus on the following components of the tensor field $\mathfrak{P}$:

$$\mathfrak{P}_{11} = (\lambda_3 - \lambda_2)^2 - \gamma_1^2 + \gamma_2^2 - 2\mu,$$
$$\mathfrak{P}_{14} = \gamma_4(\lambda_3 - \lambda_2)^2,$$
$$\mathfrak{P}_{22} = -(\gamma_4^2 + 1)(\lambda_2^2 - \lambda_3^2) + \gamma_1^2 - 4(\gamma_3 - X_4)\gamma_3 - 2\mu,$$
$$\mathfrak{P}_{33} = -(\gamma_4^2 + 1)(\lambda_2^2 - \lambda_3^2) + \gamma_2^2 + 4(\gamma_3 - X_4)\gamma_3 + 2\mu,$$
$$\mathfrak{P}_{44} = \gamma_4^2(\lambda_3 - \lambda_2)^2 - \gamma_1^2 + \gamma_2^2 - 4\gamma_3^2 - 2\mu\,.$$

It is easy to check that

$$\mathfrak{P}_{11} + \gamma_4\mathfrak{P}_{14} - \mathfrak{P}_{44} = (\lambda_2 - \lambda_3)^2 + 4\gamma_3^2.$$

Therefore, $\lambda_3 = \lambda_2$ and $\gamma_3 = 0$. Now, we have

$$\mathfrak{P}_{22} + \mathfrak{P}_{33} = \gamma_1^2 + \gamma_2^2,$$

which implies $\gamma_1 = \gamma_2 = 0$, so the metric is flat.

Structure operator L with timelike kernel

If $\lambda_3 = 0$ and $\lambda_1\lambda_2 \neq 0$ then left-invariant metrics are described by

$$[e_1, e_3] = -\lambda_2 e_2, \quad [e_1, e_4] = \gamma_1 e_1 + \gamma_2\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1,$$
$$[e_2, e_4] = -\gamma_2\lambda_1 e_1 + \gamma_1 e_2, \quad [e_3, e_4] = \gamma_3 e_1 + \gamma_4 e_2,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. In this situation, a straightforward calculation leads to the following components of the tensor field $\mathfrak{P}$:

$$\mathfrak{P}_{11} = (\gamma_2^2 - 1)(\lambda_1^2 - \lambda_2^2) - \gamma_3^2 - 4(\gamma_1 - X_4)\gamma_1 - 2\mu,$$
$$\mathfrak{P}_{34} = \gamma_2(\lambda_1 - \lambda_2)^2,$$
$$\mathfrak{P}_{33} = -(\lambda_1 - \lambda_2)^2 - \gamma_3^2 - \gamma_4^2 + 2\mu,$$
$$\mathfrak{P}_{44} = -\gamma_2^2(\lambda_1 - \lambda_2)^2 - 4\gamma_1^2 + \gamma_3^2 + \gamma_4^2 - 2\mu\,.$$

It is easy to see that

$$\mathfrak{P}_{33} + \gamma_2\mathfrak{P}_{34} + \mathfrak{P}_{44} = -(\lambda_1 - \lambda_2)^2 - 4\gamma_1^2,$$

so $\lambda_2 = \lambda_1$ and $\gamma_1 = 0$. Now,

$$\mathfrak{P}_{11} + \mathfrak{P}_{33} = -2\gamma_3^2 - \gamma_4^2,$$

which implies that $\gamma_3 = \gamma_4 = 0$ and the metric is flat as in the previous case.

Structure operator of rank one: metrics on $H^3 \rtimes \mathbb{R}$

We will consider the cases when the restriction of the metric to $\ker L$ is positive definite ($\lambda_3 \neq 0$) or Lorentzian ($\lambda_3 = 0$) separately, and make use of Gröbner bases to show non-existence of left-invariant Ricci solitons in both cases.

Structure operator L with positive definite kernel

Setting $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$ left-invariant metrics are described by

$$[e_1, e_2] = -\lambda_3 e_3, \qquad [e_1, e_4] = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3,$$
$$[e_2, e_4] = \gamma_4 e_1 + \gamma_5 e_2 + \gamma_6 e_3, \qquad [e_3, e_4] = (\gamma_1 + \gamma_5) e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. Now, a vector $X \in \mathfrak{h} \rtimes \mathbb{R}$ determines a left-invariant Ricci soliton if and only if the system of polynomial equations given by $\{\mathfrak{P}_{ij} = 0\}$ is satisfied, where

$$\mathfrak{P}_{11} = \lambda_3^2 - 4\gamma_1^2 - \gamma_2^2 + \gamma_3^2 + \gamma_4^2 - 4\gamma_1\gamma_5 + 4X_4\gamma_1 - 2\mu,$$
$$\mathfrak{P}_{12} = -\gamma_1\gamma_2 - 3\gamma_1\gamma_4 - 3\gamma_2\gamma_5 + \gamma_3\gamma_6 - \gamma_4\gamma_5 + 2X_4(\gamma_2 + \gamma_4),$$
$$\mathfrak{P}_{13} = 2X_2\lambda_3 + 2\gamma_1\gamma_3 + 3\gamma_3\gamma_5 - \gamma_4\gamma_6 - 2X_4\gamma_3,$$
$$\mathfrak{P}_{14} = \gamma_6\lambda_3 - 2X_1\gamma_1 - 2X_2\gamma_4,$$
$$\mathfrak{P}_{22} = \lambda_3^2 + \gamma_2^2 - \gamma_4^2 - 4\gamma_5^2 + \gamma_6^2 - 4\gamma_1\gamma_5 + 4X_4\gamma_5 - 2\mu,$$
$$\mathfrak{P}_{23} = -2X_1\lambda_3 + 3\gamma_1\gamma_6 - \gamma_2\gamma_3 + 2\gamma_5\gamma_6 - 2X_4\gamma_6,$$
$$\mathfrak{P}_{24} = -\gamma_3\lambda_3 - 2X_1\gamma_2 - 2X_2\gamma_5,$$
$$\mathfrak{P}_{33} = \lambda_3^2 + 4\gamma_1^2 + \gamma_3^2 + 4\gamma_5^2 + \gamma_6^2 + 8\gamma_1\gamma_5 - 4X_4(\gamma_1 + \gamma_5) + 2\mu,$$
$$\mathfrak{P}_{34} = 2\{X_3(\gamma_1 + \gamma_5) + X_1\gamma_3 + X_2\gamma_6\},$$
$$\mathfrak{P}_{44} = -4\gamma_1^2 - \gamma_2^2 + \gamma_3^2 - \gamma_4^2 - 4\gamma_5^2 + \gamma_6^2 - 4\gamma_1\gamma_5 - 2\gamma_2\gamma_4 - 2\mu\,.$$

Since $\lambda_3 \neq 0$, we can assume $\lambda_3 = 1$ and work with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_3} e_i$. Let $\mathcal{I}_1 \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order and obtain that the polynomials

$$\mathbf{g}_{11} = X_3(2617344X_4^8 + 13139712X_4^6 + 18557248X_4^4 + 7213356X_4^2 + 61803),$$
$$\mathbf{g}_{12} = X_4(83755008X_4^{14} + 776429568X_4^{12} + 2689679360X_4^{10} + 4517104000X_4^8$$
$$+ 4237066048X_4^6 + 2362718304X_4^4 + 591574590X_4^2 + 5006043)$$

belong to $\mathcal{G}_1$. Thus, $X_3 = X_4 = 0$. We compute a second Gröbner basis $\mathcal{G}_2$ of the ideal generated by $\mathcal{G}_1 \cup \{X_3, X_4\} \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ with respect to the lexicographical order, obtaining that the polynomial

$$\mathbf{g}_{21} = X_1^2 + X_2^2$$

belongs to $\mathcal{G}_2$, which shows that $X = 0$ and Ricci solitons are Einstein metrics, which do not exist in this case.

Structure operator L with Lorentzian kernel

In this case, $\lambda_3 = 0$ and we can assume without loss of generality that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Therefore, left-invariant metrics are described by

$$[e_1, e_3] = -\lambda_2 e_2, \quad [e_1, e_4] = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3,$$
$$[e_2, e_4] = \gamma_4 e_2, \qquad [e_3, e_4] = \gamma_5 e_1 + \gamma_6 e_2 - (\gamma_1 - \gamma_4) e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. Proceeding as in the previous case, the polynomials $\mathfrak{P}_{ij}$ are given by

$$\mathfrak{P}_{11} = \lambda_2^2 - \gamma_2^2 + \gamma_3^2 - \gamma_5^2 - 4\gamma_1\gamma_4 + 4\gamma_1 X_4 - 2\mu,$$
$$\mathfrak{P}_{12} = -2X_3\lambda_2 + \gamma_1\gamma_2 - 3\gamma_2\gamma_4 - \gamma_5\gamma_6 + 2X_4\gamma_2,$$
$$\mathfrak{P}_{13} = -2\gamma_1(\gamma_3 + \gamma_5) - \gamma_2\gamma_6 + 3\gamma_3\gamma_4 - \gamma_4\gamma_5 - 2X_4(\gamma_3 - \gamma_5),$$
$$\mathfrak{P}_{14} = \gamma_6\lambda_2 - 2(X_1\gamma_1 + X_3\gamma_5),$$
$$\mathfrak{P}_{22} = -\lambda_2^2 + \gamma_2^2 - 4\gamma_4^2 - \gamma_6^2 + 4X_4\gamma_4 - 2\mu,$$
$$\mathfrak{P}_{23} = 2X_1\lambda_2 - \gamma_1\gamma_6 - \gamma_2\gamma_3 - 2\gamma_4\gamma_6 + 2X_4\gamma_6,$$
$$\mathfrak{P}_{24} = -2\left(X_1\gamma_2 + X_2\gamma_4 + X_3\gamma_6\right),$$
$$\mathfrak{P}_{33} = -\lambda_2^2 + \gamma_3^2 + 4\gamma_4^2 - \gamma_5^2 - \gamma_6^2 - 4\gamma_1\gamma_4 + 4X_4(\gamma_1 - \gamma_4) + 2\mu,$$
$$\mathfrak{P}_{34} = \gamma_2\lambda_2 - 2X_3(\gamma_1 - \gamma_4) + 2X_1\gamma_3,$$
$$\mathfrak{P}_{44} = -4\gamma_1^2 - \gamma_2^2 + \gamma_3^2 - 4\gamma_4^2 + \gamma_5^2 + \gamma_6^2 + 4\gamma_1\gamma_4 - 2\gamma_3\gamma_5 - 2\mu\,.$$

Since $\lambda_2 \neq 0$, we assume $\lambda_2 = 1$ and work with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_2} e_i$. Let $\mathcal{I}_1 \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order and see that the polynomials

$$\mathbf{g}_{11} = X_2(2617344X_4^8 + 13139712X_4^6 + 18557248X_4^4 + 7213356X_4^2 + 61803),$$
$$\mathbf{g}_{12} = X_4(83755008X_4^{14} + 776429568X_4^{12} + 2689679360X_4^{10} + 4517104000X_4^8 + 4237066048X_4^6 + 2362718304X_4^4 + 591574590X_4^2 + 5006043)$$

belong to $\mathcal{G}_1$. Therefore, $X_2 = X_4 = 0$.

We compute a second Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\mathcal{G}_1 \cup \{X_2, X_4\}$ in $\mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ with respect to the lexicographical order and see that the polynomial

$$\mathbf{g}_{21} = \gamma_4^2 + 1$$

belongs to $\mathcal{G}_2$, which shows that there are no left-invariant Ricci solitons in this case.

Structure operator with zero eigenvalues: metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$

Since $\lambda_1 = \lambda_2 = \lambda_3 = 0$, any linear map $D : \mathfrak{r}^3 \to \mathfrak{r}^3$ is a derivation. In order to simplify the structure constants, we proceed as follows. Let

$$\Phi(x, y) = \langle Dx, y \rangle$$

be the bilinear form associated to $D(\,\cdot\,) = [\,\cdot\,, e_4]$, and let

$$\Phi_s = \frac{1}{2}(\Phi + {}^t\Phi) \quad \text{and} \quad \Phi_a = \frac{1}{2}(\Phi - {}^t\Phi)$$

be the symmetric and skew-symmetric parts of Φ, respectively. We denote the corresponding self-adjoint and anti-self-adjoint endomorphisms by D_{sad} and D_{asad}, which are defined by

$$\Phi_s(x, y) = \langle D_{sad}x, y \rangle \quad \text{and} \quad \Phi_a(x, y) = \langle D_{asad}x, y \rangle,$$

respectively. We analyse the different Jordan normal forms of D_{sad} separately.

The self-adjoint part of the derivation D_{sad} is diagonalizable

In this case, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}^3$ with timelike e_3, so that

$$D_{sad} = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 \\ -\gamma_1 & 0 & \gamma_3 \\ \gamma_2 & \gamma_3 & 0 \end{pmatrix}.$$

Therefore left-invariant metrics are described by

$$\begin{aligned} &[e_1, e_4] = \eta_1 e_1 - \gamma_1 e_2 + \gamma_2 e_3, \quad [e_2, e_4] = \gamma_1 e_1 + \eta_2 e_2 + \gamma_3 e_3, \\ &[e_3, e_4] = \gamma_2 e_1 + \gamma_3 e_2 + \eta_3 e_3, \end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{r}^3 \rtimes \mathbb{R}$ with timelike e_3. After a straightforward calculation, we obtain the following polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$:

$$\begin{aligned} \tilde{\mathfrak{P}}_{11} &= -\eta_1(\eta_1 + \eta_2 + \eta_3 - 2X_4) - \mu, \\ \tilde{\mathfrak{P}}_{22} &= -\eta_2(\eta_1 + \eta_2 + \eta_3 - 2X_4) - \mu, \\ \tilde{\mathfrak{P}}_{33} &= \eta_3(\eta_1 + \eta_2 + \eta_3 - 2X_4) + \mu, \\ \tilde{\mathfrak{P}}_{44} &= -\eta_1^2 - \eta_2^2 - \eta_3^2 - \mu\,. \end{aligned}$$

Considering the combinations

$$\eta_2\tilde{\mathfrak{P}}_{11} - \eta_1\tilde{\mathfrak{P}}_{22} = (\eta_1 - \eta_2)\mu,$$
$$\eta_3\tilde{\mathfrak{P}}_{11} + \eta_1\tilde{\mathfrak{P}}_{33} = (\eta_1 - \eta_3)\mu,$$

together with the expression of $\tilde{\mathfrak{P}}_{44}$, we see that $\eta_1 = \eta_2 = \eta_3 = \kappa$. Now, a standard calculation shows that the corresponding left-invariant metric has constant sectional curvature $-\kappa^2$.

The self-adjoint part of the derivation D_{sad} has complex eigenvalues

If the self-dual part of the derivation, D_{sad}, has complex eigenvalues then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}^3$ with timelike e_3, so that

$$D_{sad} = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \delta & \nu \\ 0 & -\nu & \delta \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 \\ -\gamma_1 & 0 & \gamma_3 \\ \gamma_2 & \gamma_3 & 0 \end{pmatrix},$$

where $\nu \neq 0$. The corresponding left-invariant metrics are described by

$$[e_1, e_4] = \eta e_1 - \gamma_1 e_2 + \gamma_2 e_3, \qquad [e_2, e_4] = \gamma_1 e_1 + \delta e_2 + (\gamma_3 - \nu)e_3,$$
$$[e_3, e_4] = \gamma_2 e_1 + (\gamma_3 + \nu)e_2 + \delta e_3,$$

and a standard calculation shows that the polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\tilde{\mathfrak{P}}_{11} = -\eta^2 - 2(\delta - X_4)\eta - \mu, \qquad \tilde{\mathfrak{P}}_{12} = \gamma_1(\delta - \eta) - \gamma_2\nu,$$
$$\tilde{\mathfrak{P}}_{13} = \gamma_2(\delta - \eta) + \gamma_1\nu, \qquad \tilde{\mathfrak{P}}_{14} = -X_1\eta - X_2\gamma_1 - X_3\gamma_2,$$
$$\tilde{\mathfrak{P}}_{22} = -2\delta^2 - (\eta - 2X_4)\delta - 2\gamma_3\nu - \mu, \quad \tilde{\mathfrak{P}}_{23} = -(2\delta + \eta - 2X_4)\nu,$$
$$\tilde{\mathfrak{P}}_{24} = X_1\gamma_1 - X_2\delta - X_3(\nu + \gamma_3), \qquad \tilde{\mathfrak{P}}_{33} = 2\delta^2 + (\eta - 2X_4)\delta - 2\gamma_3\nu + \mu,$$
$$\tilde{\mathfrak{P}}_{34} = X_1\gamma_2 - X_2(\nu - \gamma_3) + X_3\delta, \qquad \tilde{\mathfrak{P}}_{44} = -2\delta^2 - \eta^2 + 2\nu^2 - \mu\,.$$

Since $\nu \neq 0$, we can rescale the basis to work with the homothetic metric determined by $\hat{e}_i = \frac{1}{\nu}e_i$ and assume that $\nu = 1$ for the rest of our calculations. If we now consider the linear combinations

$$\gamma_2\tilde{\mathfrak{P}}_{12} - \gamma_1\tilde{\mathfrak{P}}_{13} = -\gamma_1^2 - \gamma_2^2 \quad \text{and} \quad \tilde{\mathfrak{P}}_{22} + \tilde{\mathfrak{P}}_{33} = -4\gamma_3,$$

it follows that $\gamma_1 = \gamma_2 = \gamma_3 = 0$. Now, the combinations

$$\tilde{\mathfrak{P}}_{23} = -2\delta - \eta + 2X_4,$$
$$\tilde{\mathfrak{P}}_{24} - \delta\,\tilde{\mathfrak{P}}_{34} = -X_3(\delta^2 + 1),$$
$$\delta\,\tilde{\mathfrak{P}}_{24} + \tilde{\mathfrak{P}}_{34} = -X_2(\delta^2 + 1)$$

lead to $X_2 = X_3 = 0$ and $X_4 = \delta + \frac{1}{2}\eta$. Consequently, the system of polynomial equations $\{\tilde{\mathfrak{P}}_{ij} = 0\}$ reduces to

$$\tilde{\mathfrak{P}}_{11} = \tilde{\mathfrak{P}}_{22} = -\tilde{\mathfrak{P}}_{33} = -\mu = 0,$$
$$\tilde{\mathfrak{P}}_{14} = -X_1\eta = 0,$$
$$\tilde{\mathfrak{P}}_{44} = -2\delta^2 - \eta^2 - \mu + 2 = 0,$$

which shows that $X_1\eta = 0$ and the left-invariant metric given by

$$[e_1, e_4] = \eta e_1, \quad [e_2, e_4] = \varepsilon\sqrt{1 - \tfrac{1}{2}\eta^2}\, e_2 - e_3, \quad [e_3, e_4] = e_2 + \varepsilon\sqrt{1 - \tfrac{1}{2}\eta^2}\, e_3,$$

with $-\sqrt{2} \leq \eta \leq \sqrt{2}$ and $\varepsilon^2 = 1$, is a left-invariant steady Ricci soliton which corresponds to Assertion *(i)* in Theorem 4.2. Moreover, the left-invariant soliton vector field is given by $X = X_1 e_1 + \varepsilon\, e_4$ if $\eta = 0$, and $X = \frac{1}{2}\left(\eta + \varepsilon\sqrt{4 - 2\eta^2}\right) e_4$ if $\eta \neq 0$, as stated in Remark 4.3.

Note that $(e_1, e_2, e_3, e_4) \mapsto (e_1, e_2, -e_3, -e_4)$ defines an isometry interchanging (η, ε) and $(-\eta, -\varepsilon)$. Therefore, we can assume $0 \leq \eta \leq \sqrt{2}$. In particular, if $\eta = 0$, the same isometry interchanges $\varepsilon = 1$ and $\varepsilon = -1$. A straightforward calculation shows that the metrics described above are never symmetric and they are Einstein if and only if $\eta = -\frac{2\varepsilon}{\sqrt{3}}$, which corresponds to Assertion *(i)* in Theorem 4.1.

The self-adjoint part of the derivation D_{sad} has a double root

In this case, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathfrak{r}^3$ such that $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1$ and

$$D_{sad} = \begin{pmatrix} \eta_1 & 0 & 0 \\ \varepsilon & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} \gamma_1 & 0 & \gamma_2 \\ 0 & -\gamma_1 & \gamma_3 \\ -\gamma_3 & -\gamma_2 & 0 \end{pmatrix},$$

where $\varepsilon^2 = 1$. In this situation, the corresponding left-invariant metrics are described by

$$[u_1, u_4] = (\eta_1 + \gamma_1)u_1 + \varepsilon u_2 - \gamma_3 u_3, \quad [u_2, u_4] = (\eta_1 - \gamma_1)u_2 - \gamma_2 u_3,$$
$$[u_3, u_4] = \gamma_2 u_1 + \gamma_3 u_2 + \eta_2 u_3,$$

where $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis for which the non-zero inner products are $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$. We consider the following polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$:

$$\tilde{\mathfrak{P}}_{11} = -\varepsilon(2\eta_1 + \eta_2 + 2\gamma_1 - 2X_4), \qquad \tilde{\mathfrak{P}}_{12} = -\eta_1(2\eta_1 + \eta_2) + 2X_4\eta_1 - \mu,$$
$$\tilde{\mathfrak{P}}_{13} = -\gamma_3(\eta_1 - \eta_2) - \varepsilon\gamma_2, \qquad \tilde{\mathfrak{P}}_{23} = -\gamma_2(\eta_1 - \eta_2),$$
$$\tilde{\mathfrak{P}}_{33} = -\eta_2(2\eta_1 + \eta_2) + 2X_4\eta_2 - \mu, \qquad \tilde{\mathfrak{P}}_{44} = -2\eta_1^2 - \eta_2^2 - \mu\,.$$

It is easy to check that

$$\gamma_2\tilde{\mathfrak{P}}_{13} - \gamma_3\tilde{\mathfrak{P}}_{23} = -\varepsilon\gamma_2^2,$$
$$\eta_2\tilde{\mathfrak{P}}_{12} - \eta_1\tilde{\mathfrak{P}}_{33} = (\eta_1 - \eta_2)\mu,$$
$$\varepsilon\eta_1\tilde{\mathfrak{P}}_{11} - \tilde{\mathfrak{P}}_{33} + \tilde{\mathfrak{P}}_{44} = -\eta_1(4\eta_1 - \eta_2 + 2\gamma_1) + 2X_4(\eta_1 - \eta_2),$$

and it now follows from the expression of $\tilde{\mathfrak{P}}_{44}$ above that $\gamma_2 = 0$, $\eta_2 = \eta_1$ and $\eta_1(3\eta_1 + 2\gamma_1) = 0$.

If $3\eta_1 + 2\gamma_1 = 0$, the resulting left-invariant metric is Einstein and it corresponds to Assertion *(ii)* in Theorem 4.1. Finally, if $\eta_1 = \gamma_2 = \eta_2 = 0$ and $\gamma_1 \neq 0$, then the left-invariant metric corresponds to

$$[u_1, u_4] = \gamma_1 u_1 + \varepsilon u_2 - \gamma_3 u_3, \quad [u_2, u_4] = -\gamma_1 u_2, \quad [u_3, u_4] = \gamma_3 u_2, \tag{4.1}$$

and u_2 is a recurrent null vector. Furthermore, a straightforward calculation shows that the curvature tensor is transversally flat (i.e., $R(Y, Z) = 0$ for all $Y, Z \in u_2^\perp$) and the Ricci operator is isotropic ($\rho_{11} = -2\varepsilon\gamma_1$ is the only non-zero component of the Ricci tensor). Therefore, the underlying structure is that of a pp-wave which is neither symmetric nor locally conformally flat.

The self-adjoint part of the derivation D_{sad} has a triple root

Let $\{u_1, u_2, u_3\}$ be a pseudo-orthonormal basis of $\mathfrak{r}^3$ with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1$, so that

$$D_{sad} = \begin{pmatrix} \eta & 0 & 1 \\ 0 & \eta & 0 \\ 0 & 1 & \eta \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} \gamma_1 & 0 & \gamma_2 \\ 0 & -\gamma_1 & \gamma_3 \\ -\gamma_3 & -\gamma_2 & 0 \end{pmatrix}.$$

The corresponding left-invariant metrics are given by

$$[u_1, u_4] = (\eta + \gamma_1)u_1 - \gamma_3 u_3, \qquad [u_2, u_4] = (\eta - \gamma_1)u_2 - (\gamma_2 - 1)u_3,$$
$$[u_3, u_4] = (\gamma_2 + 1)u_1 + \gamma_3 u_2 + \eta u_3,$$

where $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis of of $\mathfrak{r}^3 \rtimes \mathbb{R}$ for which the non-zero inner products are $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$. A straightforward calculation shows that the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\tilde{\mathfrak{P}}_{12} = -3\eta^2 + 2X_4\eta + \gamma_3 - \mu, \qquad \tilde{\mathfrak{P}}_{14} = -X_2(\eta - \gamma_1) - X_3\gamma_3,$$
$$\tilde{\mathfrak{P}}_{22} = 2\gamma_2, \qquad \tilde{\mathfrak{P}}_{23} = -3\eta + \gamma_1 + 2X_4,$$
$$\tilde{\mathfrak{P}}_{24} = -X_1(\eta + \gamma_1) - X_3(\gamma_2 + 1), \qquad \tilde{\mathfrak{P}}_{33} = -3\eta^2 + 2X_4\eta - 2\gamma_3 - \mu,$$
$$\tilde{\mathfrak{P}}_{34} = -X_3\eta + X_2(\gamma_2 - 1) + X_1\gamma_3, \qquad \tilde{\mathfrak{P}}_{44} = -3\eta^2 - \mu\,.$$

It follows from the combinations

$$\tilde{\mathfrak{P}}_{22} = 2\gamma_2,$$
$$\tilde{\mathfrak{P}}_{12} - \tilde{\mathfrak{P}}_{33} = 3\gamma_3,$$
$$\tilde{\mathfrak{P}}_{12} - \eta\tilde{\mathfrak{P}}_{23} - \tilde{\mathfrak{P}}_{44} = \eta(3\eta - \gamma_1) + \gamma_3$$

that $\gamma_2 = \gamma_3 = 0$ and $\eta(3\eta - \gamma_1) = 0$.

Now, if $3\eta - \gamma_1 = 0$, the corresponding left-invariant metric is Einstein, and it corresponds to Assertion *(iii)* in Theorem 4.1 if $\gamma_1 = 3\eta \neq 0$. The case where $\eta = \gamma_1 = 0$ corresponds to a Ricci-flat plane wave.

If $\eta = 0$ and $\gamma_1 \neq 0$, then a straightforward calculation shows that the corresponding left-invariant metrics, which are given by

$$[u_1, u_4] = \gamma_1 u_1, \quad [u_2, u_4] = -\gamma_1 u_2 + u_3, \quad [u_3, u_4] = u_1,$$

are neither Einstein nor symmetric. Moreover, the system of polynomial equations $\{\tilde{\mathfrak{P}}_{ij} = 0\}$ reduces to

$$\tilde{\mathfrak{P}}_{12} = \tilde{\mathfrak{P}}_{33} = \tilde{\mathfrak{P}}_{44} = -\mu = 0, \qquad \tilde{\mathfrak{P}}_{24} = -X_1\gamma_1 - X_3 = 0,$$
$$\tilde{\mathfrak{P}}_{14} = X_2\gamma_1 = 0, \qquad \tilde{\mathfrak{P}}_{34} = -X_2 = 0,$$
$$\tilde{\mathfrak{P}}_{23} = \gamma_1 + 2X_4 = 0,$$

and it defines a left-invariant steady Ricci soliton with left-invariant soliton vector field $X = X_1 u_1 - X_1\gamma_1 u_3 - \frac{1}{2}\gamma_1 u_4$.

Finally, notice that $(u_1, u_2, u_3, u_4) \mapsto (-u_1, -u_2, u_3, -u_4)$ defines an isometry interchanging γ_1 and $-\gamma_1$. Therefore, we can restrict the parameter γ_1 to $\gamma_1 > 0$ without losing generality. Setting $\alpha = \gamma_1$, this case corresponds to Assertion *(ii)* in Theorem 4.2.

4.1.2 The structure operator L has complex eigenvalues

If the structure operator L is of type **Ib**, there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with timelike e_3, where $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \operatorname{span}\{e_4\}$, so that

$$[e_1, e_2] = -\beta e_2 - \alpha e_3, \;\; [e_1, e_3] = -\alpha e_2 + \beta e_3, \;\; [e_2, e_3] = \lambda e_1, \;\; \underset{(i=1,2,3)}{[e_i, e_4]} = \sum_{j=1}^{3} \alpha_i^j e_j,$$

for certain $\alpha_i^j \in \mathbb{R}$ and $\beta \neq 0$. Next, we consider the cases where the real eigenvalue $\lambda = 0$ and $\lambda \neq 0$ separately.

Case of zero real eigenvalue: metrics on $E(1,1) \rtimes \mathbb{R}$

If $\lambda = 0$, then the corresponding metrics are given by

$$[e_1, e_2] = -\beta e_2 - \alpha e_3, \qquad [e_1, e_3] = -\alpha e_2 + \beta e_3, \qquad [e_1, e_4] = \gamma_1 e_2 + \gamma_2 e_3,$$
$$[e_2, e_4] = 2\gamma_3\beta e_2 + (\gamma_3 - \gamma_4)\alpha e_3, \qquad [e_3, e_4] = (\gamma_3 - \gamma_4)\alpha e_2 + 2\gamma_4\beta e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{e}(1,1) \rtimes \mathfrak{r}$ with timelike e_3. A straightforward calculation shows that the polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{aligned}
\mathfrak{P}_{11} &= -4\beta^2 - \gamma_1^2 + \gamma_2^2 - 2\mu,\\
\mathfrak{P}_{12} &= (\gamma_2(\gamma_3 - \gamma_4) - 2X_3)\alpha - 2(\gamma_1(2\gamma_3 + \gamma_4) + X_2)\beta + 2X_4\gamma_1,\\
\mathfrak{P}_{13} &= -(\gamma_1(\gamma_3 - \gamma_4) - 2X_2)\alpha + 2(\gamma_2(\gamma_3 + 2\gamma_4) - X_3)\beta - 2X_4\gamma_2,\\
\mathfrak{P}_{14} &= -4(\gamma_3 - \gamma_4)\beta^2,\\
\mathfrak{P}_{22} &= -8\gamma_3(\gamma_3 + \gamma_4)\beta^2 + 4(2X_4\gamma_3 + X_1)\beta + \gamma_1^2 - 2\mu,\\
\mathfrak{P}_{23} &= -4((\gamma_3 - \gamma_4)^2 + 1)\alpha\beta - \gamma_1\gamma_2,\\
\mathfrak{P}_{24} &= -(\gamma_2 + 2X_3(\gamma_3 - \gamma_4))\alpha + (\gamma_1 - 4X_2\gamma_3)\beta - 2X_1\gamma_1,\\
\mathfrak{P}_{33} &= 8(\gamma_3 + \gamma_4)\gamma_4\beta^2 - 4(2X_4\gamma_4 - X_1)\beta + \gamma_2^2 + 2\mu,\\
\mathfrak{P}_{34} &= (\gamma_1 + 2X_2(\gamma_3 - \gamma_4))\alpha + (\gamma_2 + 4X_3\gamma_4)\beta + 2X_1\gamma_2\\
\mathfrak{P}_{44} &= -8(\gamma_3^2 + \gamma_4^2)\beta^2 - \gamma_1^2 + \gamma_2^2 - 2\mu\,.
\end{aligned}$$

Since $\beta \neq 0$, we can assume that $\beta = 1$ and work with the homothetic metric determined by $\hat{e}_i = \frac{1}{\beta}e_i$. Using the expressions for $\mathfrak{P}_{14}$, $\mathfrak{P}_{11}$, $\mathfrak{P}_{23}$ and $\mathfrak{P}_{44}$ given above, together with the combination

$$\mathfrak{P}_{22} + \mathfrak{P}_{33} = \gamma_1^2 + \gamma_2^2 - 8(\gamma_3^2 - \gamma_4^2 - X_4(\gamma_3 - \gamma_4) - X_1),$$

we see that

$$\gamma_4 = \gamma_3, \quad \mu = -\tfrac{1}{2}(\gamma_1^2 - \gamma_2^2 + 4), \quad \alpha = -\tfrac{1}{4}\gamma_1\gamma_2, \quad \gamma_3 = \tfrac{\varepsilon_1}{2}, \quad X_1 = -\tfrac{1}{8}(\gamma_1^2 + \gamma_2^2),$$

where $\varepsilon_1^2 = 1$. Now, it is easy to check that

$$\varepsilon_1\mathfrak{P}_{12} - \mathfrak{P}_{24} - \tfrac{1}{4}\gamma_1\gamma_2\mathfrak{P}_{34} + \tfrac{1}{2}\gamma_1\mathfrak{P}_{33} = \tfrac{1}{16}\gamma_1(\gamma_2^2 - 8)(\gamma_2^2 + 2\gamma_1^2 + 8),$$
$$\varepsilon_1\mathfrak{P}_{13} - \tfrac{1}{4}\gamma_1\gamma_2\mathfrak{P}_{24} + \mathfrak{P}_{34} - \tfrac{1}{2}\gamma_2\mathfrak{P}_{33} = -\tfrac{1}{16}\gamma_2(\gamma_1^2 + 8)(2\gamma_2^2 + \gamma_1^2 - 8),$$

from where it follows that $\gamma_1 = 0$ and $\gamma_2 \in \{-2, 0, 2\}$. A standard calculation shows that the corresponding left-invariant metric, which is given by

$$[e_1, e_2] = -e_2, \qquad [e_1, e_3] = e_3, \qquad [e_1, e_4] = \gamma_2 e_3,$$
$$[e_2, e_4] = \varepsilon_1 e_2, \qquad [e_3, e_4] = \varepsilon_1 e_3,$$

is Einstein (and locally isometric to a product of two surfaces with the same constant curvature) if and only if $\gamma_2 = 0$. If $\gamma_2 \neq 0$, we take $\gamma_2 = 2\varepsilon_2$, with $\varepsilon_2^2 = 1$, and the system of polynomial equations $\{\mathfrak{P}_{ij} = 0\}$ is now given by

$$\mathfrak{P}_{12} = -2X_2 = 0, \qquad \mathfrak{P}_{24} = -2\varepsilon_1 X_2 = 0,$$
$$\mathfrak{P}_{13} = -2(X_3 + 2\varepsilon_2 X_4) + 6\varepsilon_1\varepsilon_2 = 0, \qquad \mathfrak{P}_{33} = -4\varepsilon_1 X_4 + 6 = 0,$$
$$\mathfrak{P}_{22} = 4\varepsilon_1 X_4 - 6 = 0, \qquad \mathfrak{P}_{34} = 2\varepsilon_1 X_3 = 0,$$

which shows that $X_2 = X_3 = 0$, $X_4 = \frac{3\varepsilon_1}{2}$, and the left-invariant metric given by

$$[e_1, e_2] = -e_2, \qquad [e_1, e_3] = e_3, \qquad [e_1, e_4] = 2\varepsilon_2 e_3,$$
$$[e_2, e_4] = \varepsilon_1 e_2, \qquad [e_3, e_4] = \varepsilon_1 e_3,$$

is a steady Ricci soliton with left-invariant soliton vector field $X = -\frac{1}{2}e_1 + \frac{3\varepsilon_1}{2}e_4$.

Notice that $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_2, -e_3, -e_4)$ is an isometry interchanging $\varepsilon_1 = 1$ and $\varepsilon_1 = -1$, and $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_2, -e_3, e_4)$ defines an isometry which interchanges $\varepsilon_2 = 1$ and $\varepsilon_2 = -1$. Therefore, we can set $\varepsilon_1 = \varepsilon_2 = 1$, which leads to Assertion *(iii)* in Theorem 4.2.

Case of non-zero real eigenvalue: metrics on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$

If $\lambda \neq 0$ then the corresponding left-invariant metrics are given by

$$[e_1, e_2] = -\beta e_2 - \alpha e_3, \qquad [e_1, e_4] = (\alpha^2 + \beta^2)(\gamma_1 e_2 + \gamma_2 e_3)$$
$$[e_1, e_3] = -\alpha e_2 + \beta e_3, \qquad [e_2, e_4] = -(\gamma_1\alpha - \gamma_2\beta)\lambda e_1 + \gamma_3\beta e_2 + \gamma_3\alpha e_3,$$
$$[e_2, e_3] = \lambda e_1, \qquad [e_3, e_4] = (\gamma_2\alpha + \gamma_1\beta)\lambda e_1 + \gamma_3\alpha e_2 - \gamma_3\beta e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathfrak{r}$ with timelike e_3. A straightforward calculation shows that the polynomials $\mathfrak{P}_{ij}$ are given by

$$\mathfrak{P}_{11} = -\left((\alpha^2+\beta^2)^2 - (\alpha^2-\beta^2)\lambda^2\right)(\gamma_1^2 - \gamma_2^2) - 4\alpha\beta\lambda^2\gamma_1\gamma_2 - 4\beta^2 - \lambda^2 - 2\mu,$$
$$\begin{aligned}\mathfrak{P}_{12} = {} & (2X_4(\alpha^2+\beta^2-\alpha\lambda) - (\alpha^2+\beta^2+2\alpha\lambda)\beta\gamma_3)\gamma_1 \\ & + \left(\left((\alpha^2+\beta^2)\,\alpha - (\alpha^2-\beta^2)\lambda\right)\gamma_3 + 2X_4\beta\lambda\right)\gamma_2 - 2(X_3(\alpha-\lambda) + X_2\beta),\end{aligned}$$
$$\mathfrak{P}_{13} = -\left(\left((\alpha^2+\beta^2)\alpha - (\alpha^2-\beta^2)\lambda\right)\gamma_3 - 2X_4\beta\lambda\right)\gamma_1$$

$$-\left(2X_4(\alpha^2+\beta^2-\alpha\lambda)+(\alpha^2+\beta^2+2\alpha\lambda)\beta\gamma_3\right)\gamma_2+2(X_2(\alpha-\lambda)-X_3\beta),$$

$$\mathfrak{P}_{14}=2(X_2\alpha-X_3\beta)\lambda\gamma_1-2(X_3\alpha+X_2\beta)\lambda\gamma_2-4\beta^2\gamma_3,$$

$$\mathfrak{P}_{22}=\left((\alpha^2+\beta^2)^2-\alpha^2\lambda^2\right)\gamma_1^2-\beta^2\lambda^2\gamma_2^2+2\alpha\beta\lambda^2\gamma_1\gamma_2+4X_4\beta\gamma_3+4X_1\beta-(2\alpha-\lambda)\lambda-2\mu,$$

$$\mathfrak{P}_{23}=\alpha\beta\left(\lambda^2(\gamma_1^2-\gamma_2^2)-4\gamma_3^2\right)-\left((\alpha^2+\beta^2)^2-(\alpha^2-\beta^2)\lambda^2\right)\gamma_1\gamma_2-2(2\alpha-\lambda)\beta,$$

$$\mathfrak{P}_{24}=\left((\alpha^2+\beta^2)(\beta-2X_1)-\beta\lambda^2\right)\gamma_1-\left((\alpha^2+\beta^2)(\alpha-2\lambda)+\alpha\lambda^2\right)\gamma_2-2(X_3\alpha+X_2\beta)\gamma_3,$$

$$\mathfrak{P}_{33}=-\beta^2\lambda^2\gamma_1^2+\left((\alpha^2+\beta^2)^2-\alpha^2\lambda^2\right)\gamma_2^2-2\alpha\beta\lambda^2\gamma_1\gamma_2+4X_4\beta\gamma_3+4X_1\beta+(2\alpha-\lambda)\lambda+2\mu,$$

$$\mathfrak{P}_{34}=\left((\alpha^2+\beta^2)(\alpha-2\lambda)+\alpha\lambda^2\right)\gamma_1+\left((\alpha^2+\beta^2)(2X_1+\beta)-\beta\lambda^2\right)\gamma_2+(2X_2\alpha-2X_3\beta)\gamma_3,$$

$$\mathfrak{P}_{44}=-\left(\alpha^2+\beta^2-(\alpha+\beta)\lambda\right)\left(\alpha^2-\alpha\lambda+(\beta+\lambda)\beta\right)(\gamma_1^2-\gamma_2^2)-4\beta^2\gamma_3^2$$
$$-4\left(\alpha^2+\beta^2-\alpha\lambda\right)\beta\lambda\gamma_1\gamma_2-2\mu\,.$$

In this case, we make use of Gröbner bases again but, since it is extraordinarily difficult to obtain such a basis using the above polynomials $\mathfrak{P}_{ij}$, we need to reduce the number of variables. Let us focus on the expressions of $\mathfrak{P}_{11}$, $\mathfrak{P}_{22}$ and the linear combinations

$$\beta\,\mathfrak{P}_{12}-(\alpha-\lambda)\mathfrak{P}_{13}\quad\text{and}\quad(\alpha-\lambda)\mathfrak{P}_{12}+\beta\,\mathfrak{P}_{13}.$$

They allow us to clear μ, X_1, X_2 and X_3, respectively, as

$$\mu=-\tfrac{1}{2}((\alpha^2+\beta^2)^2-(\alpha^2-\beta^2)\lambda^2)(\gamma_1^2-\gamma_2^2)-2\alpha\beta\lambda^2\gamma_1\gamma_2-2\beta^2-\tfrac{1}{2}\lambda^2,$$

$$X_1=-\tfrac{1}{4\beta}\left((\alpha^2+\beta^2)^2-\alpha^2\lambda^2\right)\gamma_1^2+\tfrac{1}{4}\beta\lambda^2\gamma_2^2-\tfrac{1}{2}\alpha\lambda^2\gamma_1\gamma_2-X_4\gamma_3$$
$$-\tfrac{1}{4\beta}((\lambda-2\alpha)\lambda-2\mu),$$

$$X_2=\left(\tfrac{1}{2}\left(\alpha^2-\beta^2-\tfrac{4\alpha\beta^2\lambda}{(\alpha-\lambda)^2+\beta^2}\right)\gamma_3+X_4\beta\right)\gamma_1+\left(\tfrac{\alpha\beta(\alpha^2+\beta^2-\lambda^2)}{(\alpha-\lambda)^2+\beta^2}\gamma_3+X_4\alpha\right)\gamma_2,$$

$$X_3=-\left(\tfrac{\alpha\beta(\alpha^2+\beta^2-\lambda^2)}{(\alpha-\lambda)^2+\beta^2}\gamma_3-X_4\alpha\right)\gamma_1+\left(\tfrac{1}{2}\left(\alpha^2-\beta^2-\tfrac{4\alpha\beta^2\lambda}{(\alpha-\lambda)^2+\beta^2}\right)\gamma_3-X_4\beta\right)\gamma_2\,.$$

Consequently, we can eliminate these variables from the polynomials $\mathfrak{P}_{ij}$ and, as a consequence, X_4 is also eliminated. Let us denote by $\mathfrak{Q}_{ij}$ the expressions obtained from the polynomials $\mathfrak{P}_{ij}$ after substituting μ, X_1, X_2 and X_3. These expressions are not polynomials, since they contain fractional expressions with variables in the denominators, but we can easily avoid this inconvenience by considering $\mathfrak{Q}'_{ij}$, which are given by

$$\mathfrak{Q}'_{14}=((\alpha-\lambda)^2+\beta^2)\mathfrak{Q}_{14},\qquad\mathfrak{Q}'_{23}=\mathfrak{Q}_{23},$$

$$\mathfrak{Q}'_{24}=2((\alpha-\lambda)^2+\beta^2)\beta\,\mathfrak{Q}_{24},\qquad\mathfrak{Q}'_{33}=\mathfrak{Q}_{33},$$

$$\mathfrak{Q}'_{34}=2((\alpha-\lambda)^2+\beta^2)\beta\,\mathfrak{Q}_{34},\qquad\mathfrak{Q}'_{44}=\mathfrak{Q}_{44},$$

and the remaining ones are zero. This way, $\mathfrak{Q}'_{ij}\in\mathbb{R}[\gamma_1,\gamma_2,\gamma_3,\lambda,\alpha,\beta]$. Now, let $\mathcal{I}\subset\mathbb{R}[\gamma_1,\gamma_2,\gamma_3,\lambda,\alpha,\beta]$ be the ideal generated by the polynomials $\mathfrak{Q}'_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$

with respect to the lexicographical order and see that the polynomial

$$\mathbf{g} = (\alpha^2 + \beta^2)^2 \beta^2$$

belongs to $\mathcal{G}$. Since $\beta \neq 0$, there are no left-invariant Ricci solitons in this case.

4.1.3 The structure operator L has a double root of its minimal polynomial

If the structure operator is of type **II** then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with $\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = \langle u_4, u_4 \rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$, so that

$$[u_1, u_2] = \lambda_2 u_3, \quad [u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2, \quad [u_2, u_3] = \lambda_1 u_2, \quad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\alpha_i^j \in \mathbb{R}$ and $\varepsilon^2 = 1$. Next, depending on the eigenvalues λ_i, we are led to the following different possibilities.

Case $\lambda_i = 0$: metrics on $H^3 \rtimes \mathbb{R}$

If $\lambda_1 = \lambda_2 = 0$ then the corresponding metrics are determined by

$$\begin{aligned} &[u_1, u_3] = -\varepsilon u_2, && [u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3, \\ &[u_2, u_4] = \gamma_4 u_2, && [u_3, u_4] = \gamma_5 u_1 + \gamma_6 u_2 - (\gamma_1 - \gamma_4) u_3, \end{aligned}$$

and the polynomials $\mathfrak{P}_{ij}$ are

$$\begin{aligned} \mathfrak{P}_{12} &= -2\gamma_4^2 - 2\gamma_1\gamma_4 + \gamma_5\gamma_6 + 2X_4\gamma_1 + 2X_4\gamma_4 - 2\mu, \\ \mathfrak{P}_{14} &= -2X_1\gamma_2 - 2X_2\gamma_4 - 2X_3\gamma_6 - \varepsilon\gamma_5, \\ \mathfrak{P}_{22} &= \gamma_5^2, \\ \mathfrak{P}_{24} &= -2(X_1\gamma_1 + X_3\gamma_5), \\ \mathfrak{P}_{33} &= -2(2\gamma_4^2 - 2\gamma_1\gamma_4 + \gamma_5\gamma_6 + 2X_4\gamma_1 - 2X_4\gamma_4 + \mu), \\ \mathfrak{P}_{34} &= 2(X_3\gamma_1 - X_1\gamma_3 - X_3\gamma_4), \\ \mathfrak{P}_{44} &= -3\gamma_1^2 - 3\gamma_4^2 + 2\gamma_1\gamma_4 - 2\gamma_3\gamma_5 - 2\gamma_5\gamma_6 - 2\mu\,. \end{aligned}$$

Therefore γ_5 must vanish. Considering the linear combination

$$0 = 2(\gamma_1 - \gamma_4)\mathfrak{P}_{12} + (\gamma_1 + \gamma_4)\mathfrak{P}_{33} = 2(\gamma_4 - 3\gamma_1)\mu$$

we see that there are two different possibilities depending on whether $\mu = 0$ or $\gamma_4 = 3\gamma_1$. If $\mu = 0$ then

$$\mathfrak{P}_{44} = -2\gamma_1^2 - 2\gamma_4^2 - (\gamma_1 - \gamma_4)^2,$$

and if $\gamma_4 = 3\gamma_1$, it is easy to check that

$$\begin{aligned} &\mathfrak{P}_{24} = -2X_1\gamma_1, \qquad \gamma_1\mathfrak{P}_{34} - \gamma_3\mathfrak{P}_{24} = -4X_3\gamma_1^2, \\ &2\gamma_1^2\mathfrak{P}_{14} - (2\gamma_1\gamma_2 - \gamma_3\gamma_6)\mathfrak{P}_{24} - \gamma_1\gamma_6\mathfrak{P}_{34} = -12X_2\gamma_1^3, \\ &\mathfrak{P}_{12} - \mathfrak{P}_{44} = 8X_4\gamma_1\,. \end{aligned}$$

In any case, $\gamma_1 = \gamma_4 = 0$, and the left-invariant metrics are given by

$$[u_1, u_3] = -\varepsilon u_2, \quad [u_1, u_4] = \gamma_2 u_2 + \gamma_3 u_3, \quad [u_3, u_4] = \gamma_6 u_2\,. \tag{4.2}$$

A straightforward calculation shows that u_2 is parallel and the curvature tensor satisfies $R(Y, Z) = 0$ and $\nabla_Y R = 0$ for all $Y, Z \in u_2^\perp$. Thus, the underlying structure is a plane wave.

Case $\lambda_1 = 0$, $\lambda_2 \neq 0$: metrics on $\tilde{E}(2) \rtimes \mathbb{R}$ or $E(1, 1) \rtimes \mathbb{R}$

In this case the left-invariant metrics are described by

$$\begin{aligned} &[u_1, u_2] = \lambda_2 u_3, \qquad [u_2, u_4] = \gamma_3 u_2 + \gamma_4\lambda_2 u_3, \\ &[u_1, u_3] = -\varepsilon u_2, \qquad [u_3, u_4] = -\varepsilon\gamma_4 u_2 + \gamma_3 u_3, \\ &[u_1, u_4] = \gamma_1 u_2 + \gamma_2 u_3, \end{aligned}$$

and a straightforward calculation leads to the polynomials

$$\begin{aligned} \mathfrak{P}_{12} &= \lambda_2^2 - \gamma_2\gamma_4\lambda_2 - 2(\gamma_3^2 - X_4\gamma_3 + \mu), \\ \mathfrak{P}_{24} &= -\gamma_4\lambda_2^2, \\ \mathfrak{P}_{44} &= 2(\gamma_4\varepsilon - \gamma_2)\gamma_4\lambda_2 - 3\gamma_3^2 - 2\mu, \\ \mathfrak{P}_{33} &= 2\gamma_2\gamma_4\lambda_2 - \lambda_2^2 - 2(2\gamma_3^2 - 2X_4\gamma_3 + \mu)\,. \end{aligned}$$

It now follows that

$$2\,\mathfrak{P}_{12} - \tfrac{2(\gamma_2 + \varepsilon\gamma_4)}{\lambda_2}\mathfrak{P}_{24} - \mathfrak{P}_{33} - \mathfrak{P}_{44} = 3(\lambda_2^2 + \gamma_3^2)$$

and, since $\lambda_2 \neq 0$, there are no left-invariant Ricci solitons in this case.

Case $\lambda_1 \neq 0$, $\lambda_2 = 0$: metrics on $E(1,1) \rtimes \mathbb{R}$

If $\lambda_1 \neq 0$ and $\lambda_2 = 0$ then

$$[u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2, \qquad [u_2, u_4] = -(2\varepsilon\gamma_2\lambda_1 - \gamma_1)u_2,$$
$$[u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2, \qquad [u_3, u_4] = \gamma_3 u_1 + \gamma_4 u_2,$$
$$[u_2, u_3] = \lambda_1 u_2 .$$

Straightforward calculations show that the non-zero polynomials $\mathfrak{P}_{ij}$ are given by

$$\mathfrak{P}_{11} = -4\varepsilon\lambda_1 + \gamma_4^2 - 4\gamma_1\gamma_2 + 4X_4\gamma_2 - 4\varepsilon X_3,$$
$$\mathfrak{P}_{12} = -4\gamma_2^2\lambda_1^2 + 4\varepsilon(2\gamma_1 - X_4)\gamma_2\lambda_1 - 4\gamma_1^2 + \gamma_3\gamma_4 + 4X_4\gamma_1 - 2\mu,$$
$$\mathfrak{P}_{13} = (2\varepsilon\gamma_2\gamma_4 - 2X_2)\lambda_1 - 3\gamma_1\gamma_4 - \gamma_2\gamma_3 + 2X_4\gamma_4 + 2\varepsilon X_1,$$
$$\mathfrak{P}_{14} = (4\varepsilon X_2\gamma_2 - \gamma_4)\lambda_1 - 2X_2\gamma_1 - 2X_1\gamma_2 - \varepsilon\gamma_3 - 2X_3\gamma_4,$$
$$\mathfrak{P}_{22} = \gamma_3^2,$$
$$\mathfrak{P}_{23} = 2(2\varepsilon\gamma_2\gamma_3 + X_1)\lambda_1 - 3\gamma_1\gamma_3 + 2X_4\gamma_3,$$
$$\mathfrak{P}_{24} = \gamma_3\lambda_1 - 2X_1\gamma_1 - 2X_3\gamma_3,$$
$$\mathfrak{P}_{33} = -2(\gamma_3\gamma_4 + \mu)\,,$$
$$\mathfrak{P}_{44} = -4\gamma_2^2\lambda_1^2 + 8\varepsilon\gamma_1\gamma_2\lambda_1 - 4\gamma_1^2 - 2\gamma_3\gamma_4 - 2\mu\,.$$

We immediately see that γ_3 must be zero, so that

$$\mathfrak{P}_{23} = 2X_1\lambda_1 \quad \text{and} \quad \mathfrak{P}_{33} = -2\mu,$$

and so $X_1 = \mu = 0$. Now,

$$\mathfrak{P}_{44} = -4(\varepsilon\gamma_2\lambda_1 - \gamma_1)^2,$$

which implies that $\gamma_1 = \varepsilon\gamma_2\lambda_1$ and now the polynomial

$$\mathfrak{P}_{13} = -(\varepsilon\gamma_2\gamma_4 + 2X_2)\lambda_1 + 2X_4\gamma_4,$$

from where we get $X_2 = -\frac{\varepsilon}{2}\gamma_2\gamma_4 + X_4\frac{\gamma_4}{\lambda_1}$. At this point, the left-invariant metric is given by

$$[u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2, \quad [u_1, u_4] = \varepsilon\gamma_2\lambda_1 u_1 + \gamma_2 u_2, \quad [u_2, u_3] = \lambda_1 u_2,$$
$$[u_2, u_4] = -\varepsilon\gamma_2\lambda_1 u_2, \qquad [u_3, u_4] = \gamma_4 u_2,$$

and the system of polynomial equations $\{\mathfrak{P}_{ij} = 0\}$ is now given by

$$\mathfrak{P}_{11} = -4\varepsilon(\gamma_2^2 + 1)\lambda_1 + \gamma_4^2 + 4X_4\gamma_2 - 4\varepsilon X_3 = 0,$$

$$\mathfrak{P}_{14} = -\gamma_4\{(\gamma_2^2 + 1)\lambda_1 + 2(X_3 - \varepsilon X_4\gamma_2)\} = 0\,.$$

Let us set

$$v_1 = u_1, \qquad v_2 = \tfrac{1}{2}u_2, \qquad v_3 = \varepsilon\gamma_2 u_3 + u_4, \qquad v_4 = u_3.$$

A straightforward calculation shows that $[v_i, v_j] = 0$ for all $i, j \in \{1, 2, 3\}$ and $[v_4, v_i] \in \operatorname{span}\{v_1, v_2, v_3\}$. Therefore, every left-invariant metric above is isometric to some left-invariant metric on $\mathbb{R}^3 \rtimes \mathbb{R}$ as discussed in Section 4.1.1.

Case of non-zero eigenvalues: metrics on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$

In this case one has the metric expressed in terms of the Lie brackets

$$[u_1, u_2] = \lambda_2 u_3, \quad [u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2, \quad [u_2, u_3] = \lambda_1 u_2,$$

$$[u_1, u_4] = \lambda_1\gamma_1 u_1 + \varepsilon\gamma_1 u_2 + \gamma_2\lambda_2 u_3, \qquad [u_2, u_4] = -\gamma_1\lambda_1 u_2 + \gamma_3\lambda_2 u_3,$$

$$[u_3, u_4] = -\gamma_3\lambda_1 u_1 - (\gamma_2\lambda_1 + \varepsilon\gamma_3)u_2,$$

and a straightforward calculation shows that the polynomials $\mathfrak{P}_{ij}$ are given by

$$\mathfrak{P}_{11} = \gamma_2^2(\lambda_1^2 - \lambda_2^2) - 2\varepsilon(2\gamma_1^2 - \gamma_2\gamma_3 + 2)\lambda_1 + 2\varepsilon\lambda_2 + \gamma_3^2 + 4\varepsilon X_4\gamma_1 - 4\varepsilon X_3,$$

$$\mathfrak{P}_{12} = \gamma_2\gamma_3\lambda_1^2 - (\gamma_2\gamma_3 - 1)\lambda_2^2 - 2\lambda_1\lambda_2 + \varepsilon\gamma_3^2\lambda_1 - 2\mu,$$

$$\begin{aligned}\mathfrak{P}_{13} = {}& \gamma_1\gamma_2(\lambda_1^2 - \lambda_1\lambda_2) + 2(\varepsilon\gamma_1\gamma_3 - X_4\gamma_2 - X_2)\lambda_1 + (\varepsilon\gamma_1\gamma_3 + 2X_4\gamma_2 + 2X_2)\lambda_2 \\ & - 2\varepsilon\gamma_3 X_4 + 2\varepsilon X_1,\end{aligned}$$

$$\mathfrak{P}_{14} = \gamma_2(\lambda_1 - \lambda_2)^2 + 2(X_2\gamma_1 + X_3\gamma_2 + \varepsilon\gamma_3)\lambda_1 - 2\varepsilon\gamma_3\lambda_2 - 2\varepsilon X_1\gamma_1 + 2\varepsilon X_3\gamma_3,$$

$$\mathfrak{P}_{22} = \gamma_3^2(\lambda_1^2 - \lambda_2^2),$$

$$\mathfrak{P}_{23} = -\gamma_1\gamma_3(\lambda_1^2 - \lambda_1\lambda_2) - 2(X_4\gamma_3 - X_1)(\lambda_1 - \lambda_2),$$

$$\mathfrak{P}_{24} = -\gamma_3(\lambda_1^2 + \lambda_2^2) + 2\gamma_3\lambda_1\lambda_2 - 2(X_1\gamma_1 - X_3\gamma_3)\lambda_1,$$

$$\mathfrak{P}_{33} = -2\gamma_2\gamma_3\lambda_1^2 + (2\gamma_2\gamma_3 - 1)\lambda_2^2 - 2\varepsilon\gamma_3^2\lambda_1 - 2\mu,$$

$$\mathfrak{P}_{34} = -2(X_1\gamma_2 + X_2\gamma_3)\lambda_2,$$

$$\mathfrak{P}_{44} = -2\gamma_2\gamma_3(\lambda_1 - \lambda_2)^2 - 2\varepsilon\gamma_3^2(\lambda_1 - \lambda_2) - 2\mu\,.$$

Let $\mathcal{I} \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \varepsilon, \lambda_1, \lambda_2, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded reverse lexicographical order and obtain that the polynomial $\mathfrak{g} = \lambda_2^3$ belongs to $\mathcal{G}$. Since $\lambda_2 \neq 0$, there are no left-invariant Ricci solitons in this case.

4.1.4 The structure operator L has a triple root of its minimal polynomial

If the structure operator L is of type **III**, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with $\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = \langle u_4, u_4 \rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$, so that

$$[u_1, u_2] = u_1 + \lambda u_3, \quad [u_1, u_3] = -\lambda u_1, \quad [u_2, u_3] = \lambda u_2 + u_3, \quad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\alpha_i^j \in \mathbb{R}$. In what follows, we will consider the cases $\lambda = 0$ and $\lambda \neq 0$ separately.

Case $\lambda = 0$: metrics on $E(1,1) \rtimes \mathbb{R}$

If $\lambda = 0$, then

$$[u_1, u_2] = u_1, \qquad [u_1, u_4] = \gamma_1 u_1, \qquad [u_2, u_3] = u_3,$$
$$[u_2, u_4] = \gamma_2 u_1 + \gamma_3 u_3, \qquad [u_3, u_4] = \gamma_4 u_3,$$

and a straightforward calculation shows that the non-zero polynomials $\mathfrak{P}_{ij}$ are given by

$$\mathfrak{P}_{12} = -\gamma_1^2 - \gamma_1\gamma_4 + 2X_4\gamma_1 + 2X_2 - 2\mu, \qquad \mathfrak{P}_{23} = -2\gamma_3\gamma_4 + 2X_4\gamma_3 + 2X_3,$$
$$\mathfrak{P}_{22} = -\gamma_3^2 - 2\gamma_2\gamma_4 + 4X_4\gamma_2 - 4X_1 - 4, \qquad \mathfrak{P}_{34} = -2(X_2\gamma_3 + X_3\gamma_4),$$
$$\mathfrak{P}_{24} = -(2X_1 + 1)\gamma_1 - 2X_2\gamma_2 + 2\gamma_4, \qquad \mathfrak{P}_{44} = -\gamma_1^2 - 2\gamma_4^2 - 2\mu,$$
$$\mathfrak{P}_{33} = -2(\gamma_4^2 + \gamma_1\gamma_4 - 2X_4\gamma_4 + 2X_2 + \mu)\,.$$

From the expressions of $\mathfrak{P}_{22}$, $\mathfrak{P}_{23}$ and $\mathfrak{P}_{44}$ we obtain

$$X_1 = -\tfrac{1}{4}\gamma_3^2 - \tfrac{1}{2}(\gamma_4 - 2X_4)\gamma_2 - 1, \quad X_3 = (\gamma_4 - X_4)\gamma_3, \quad \mu = -\tfrac{1}{2}\gamma_1^2 - \gamma_4^2.$$

Therefore,

$$\mathfrak{P}_{12} = 2\gamma_4^2 - (\gamma_4 - 2X_4)\gamma_1 + 2X_2$$

and

$$X_2 = -\gamma_4^2 + \tfrac{1}{2}(\gamma_4 - 2X_4)\gamma_1.$$

Now,

$$\mathfrak{P}_{24} = \tfrac{1}{2}(\gamma_3^2 + 2)\gamma_1 + 2(\gamma_2\gamma_4 + 1)\gamma_4,$$

so

$$\gamma_1 = -\frac{4(\gamma_2\gamma_4+1)\gamma_4}{\gamma_3^2+2}.$$

At this point, the system of polynomial equations $\{\mathfrak{P}_{ij}=0\}$ reduces to

$$\mathfrak{P}_{33} = \tfrac{4}{(\gamma_3^2+2)^2}\left\{(\gamma_3^2+2\gamma_2\gamma_4+4)^2\gamma_4 + X_4(\gamma_3^2+2)(\gamma_3^2-4\gamma_2\gamma_4-2)\right\}\gamma_4 = 0,$$

$$\mathfrak{P}_{34} = \tfrac{2}{\gamma_3^2+2}\left\{2(\gamma_2\gamma_4+1)\gamma_4 + (\gamma_3^2-4\gamma_2\gamma_4-2)X_4\right\}\gamma_3\gamma_4 = 0\,.$$

It is easy to check that

$$\tfrac{\gamma_3}{2}\mathfrak{P}_{33} - \mathfrak{P}_{34} = \tfrac{1}{2(\gamma_3^2+2)^2}\left\{3\gamma_3^4 + 12\gamma_3^2 + (\gamma_3^2+4\gamma_2\gamma_4+6)^2 + 12\right\}\gamma_3\gamma_4^2,$$

which implies that $\gamma_3\gamma_4$=0.

If $\gamma_4 = 0$ (which implies $\gamma_1 = 0$), the left-invariant metrics are given by

$$[u_1,u_2] = u_1, \qquad [u_2,u_3] = u_3, \qquad [u_2,u_4] = \gamma_2 u_1 + \gamma_3 u_3, \tag{4.3}$$

and a standard calculation shows that u_1 is a recurrent null vector. Moreover, the only non-zero component of the Ricci tensor

$$\rho_{22} = -2 - \tfrac{1}{2}\gamma_3^2$$

shows that the Ricci operator is isotropic, $R(Y,Z) = 0$, and $\nabla_Y R = 0$ for all $Y, Z \in u_1^\perp$. Consequently, the underlying structure corresponds to a plane wave.

If $\gamma_4 \neq 0$, then $\gamma_3 = 0$ and

$$\mathfrak{P}_{33} = 4\left\{(\gamma_2\gamma_4+2)^2\gamma_4 - X_4(2\gamma_2\gamma_4+1)\right\}\gamma_4.$$

Notice that if $2\gamma_2\gamma_4+1=0$ then $\mathfrak{P}_{33}\neq 0$. Hence the left-invariant metric is given by

$$[u_1,u_2] = u_1, \qquad [u_1,u_4] = -2(\gamma_2\gamma_4+1)\gamma_4 u_1, \qquad [u_2,u_3] = u_3,$$

$$[u_2,u_4] = \gamma_2 u_1, \qquad [u_3,u_4] = \gamma_4 u_3,$$

and it is an expanding left-invariant Ricci soliton with $\mu = -(2(\gamma_2\gamma_4+1)^2+1)\gamma_4^2$ and left-invariant soliton vector field $X = X_1u_1 + X_2u_2 + X_4u_4$, where

$$X_1 = \tfrac{1}{2(2\gamma_2\gamma_4+1)}(\gamma_2\gamma_4+2)(2(\gamma_2\gamma_4+1)\gamma_2\gamma_4-1),$$

$$X_2 = \tfrac{1}{2\gamma_2\gamma_4+1}(\gamma_2\gamma_4+2)(2(\gamma_2\gamma_4+2)\gamma_2\gamma_4+3)\gamma_4^2,$$

$$X_4 = \tfrac{1}{2\gamma_2\gamma_4+1}(\gamma_2\gamma_4+2)^2\,\gamma_4\,.$$

A straightforward calculation shows that the metric above is symmetric if and only if $(\gamma_2\gamma_4+1)(\gamma_2\gamma_4+2)=0$. Moreover, it is Einstein if and only if $\gamma_2\gamma_4+2=0$, in which case the sectional curvature is constant. Otherwise, if $\gamma_2\gamma_4+1=0$, the metric is locally a product $\mathbb{L}^2\times N(c)$, where $\mathbb{L}^2$ is the Minkowskian plane and $N(c)$ is a surface of constant curvature c. Finally, note that $(u_1,u_2,u_3,u_4)\mapsto(u_1,u_2,u_3,-u_4)$ defines an isometry interchanging (γ_4,γ_2) and $(-\gamma_4,-\gamma_2)$, which allows us to restrict the parameter γ_4 to $\gamma_4>0$ without losing generality. Setting $\alpha=\gamma_4$ and $\beta=\gamma_2$, this case corresponds to Assertion *(iv)* in Theorem 4.2 and Remark 4.5.

Case $\lambda \neq 0$: metrics on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$

If $\lambda \neq 0$, then

$$[u_1, u_2] = u_1 + \lambda u_3, \qquad [u_1, u_3] = -\lambda u_1,$$
$$[u_1, u_4] = \gamma_1 \lambda u_1 + \gamma_2 \lambda^2 u_3, \qquad [u_2, u_3] = \lambda u_2 + u_3,$$
$$[u_3, u_4] = -\gamma_3 \lambda u_1 - \gamma_2 \lambda^2 u_2 - \gamma_2 \lambda u_3,$$
$$[u_2, u_4] = \gamma_3 u_1 - (\gamma_1 - \gamma_2)\lambda u_2 - (\gamma_1 - \gamma_2 - \gamma_3 \lambda) u_3,$$

and we see that

$$\mathfrak{P}_{12} = -(\gamma_2^2 - \gamma_1\gamma_2 + 1)\lambda^2 + 2X_4\gamma_2\lambda + 2X_2 - 2\mu, \qquad \mathfrak{P}_{13} = 3\gamma_2^2\lambda^3,$$
$$\mathfrak{P}_{33} = (2\gamma_2^2 - 2\gamma_1\gamma_2 - 1)\lambda^2 - 4X_4\gamma_2\lambda - 4X_2 - 2\mu, \qquad \mathfrak{P}_{44} = -3\gamma_2^2\lambda^2 - 2\mu\,.$$

One easily checks that

$$2\,\mathfrak{P}_{12} - \tfrac{3}{\lambda}\,\mathfrak{P}_{13} + \mathfrak{P}_{33} - 3\,\mathfrak{P}_{44} = -3\lambda^2.$$

Since $\lambda \neq 0$, there are no left-invariant Ricci solitons in this case.

4.2 Extensions of Riemannian Lie groups

In this section we analyse the left-invariant Lorentzian metrics which are extensions of three-dimensional unimodular Riemannian Lie groups. In particular, we show that any left-invariant Ricci soliton in this setting is trivial.

Lemma 4.8. *A four-dimensional Lie group $G = G_3 \rtimes \mathbb{R}$ equipped with a left-invariant Lorentzian metric whose restriction to G_3 is Riemannian is a left-invariant Ricci soliton if and only if it is a space of non-negative constant sectional curvature.*

Let $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ and let L be the structure operator of $\mathfrak{g}_3$. Since L is self-adjoint and diagonalizable, there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$, with timelike e_4, where $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \operatorname{span}\{e_4\}$, so that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \quad \underset{(i=1,2,3)}{[e_i, e_4]} = \sum_{j=1}^{3} \alpha_i^j e_j,$$

for certain $\alpha_i^j \in \mathbb{R}$. Next, depending on the eigenvalues λ_i and imposing the Jacobi identity, we are led to the following different possibilities.

4.2.1 Case of non-zero eigenvalues: $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

If $\lambda_1\lambda_2\lambda_3 \neq 0$, the left-invariant Lorentzian metrics are described by

$$
\begin{aligned}
&[e_1,e_2] = \lambda_3 e_3, \quad [e_1,e_3] = -\lambda_2 e_2, \qquad\qquad [e_1,e_4] = \gamma_1\lambda_2 e_2 + \gamma_2\lambda_3 e_3,\\
&[e_2,e_3] = \lambda_1 e_1, \quad [e_2,e_4] = -\gamma_1\lambda_1 e_1 + \gamma_3\lambda_3 e_3, \quad [e_3,e_4] = -\gamma_2\lambda_1 e_1 - \gamma_3\lambda_2 e_2,
\end{aligned}
$$

and, proceeding as in Section 4.1.1, a straightforward calculation shows that there are no left-invariant Ricci solitons in this case.

4.2.2 Case of a single null eigenvalue: $\tilde{E}(2) \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$

Without loss of generality, we assume $\lambda_3 = 0$ and $\lambda_1\lambda_2 \neq 0$. The Lorentzian left-invariant metrics on $\tilde{E}(2) \rtimes \mathbb{R}$ or $E(1,1) \rtimes \mathbb{R}$ are given by

$$
\begin{aligned}
&[e_1,e_3] = -\lambda_2 e_2, \qquad\qquad [e_1,e_4] = \gamma_1 e_1 + \gamma_2\lambda_2 e_2, \quad [e_2,e_3] = \lambda_1 e_1,\\
&[e_2,e_4] = -\gamma_2\lambda_1 e_1 + \gamma_1 e_2, \quad [e_3,e_4] = \gamma_3 e_1 + \gamma_4 e_2\,.
\end{aligned}
$$

Proceeding as in Section 4.1.1, we see that the existence of left-invariant Ricci solitons leads to the conditions $\lambda_2 = \lambda_1$, $\gamma_1 = \gamma_3 = \gamma_4 = 0$, which determine flat metrics on $\tilde{E}(2) \rtimes \mathbb{R}$.

4.2.3 Structure operator of rank one: $H^3 \rtimes \mathbb{R}$

Set $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$ so the left-invariant Lorentzian metrics can be expressed as

$$
\begin{aligned}
&[e_1,e_2] = \lambda_3 e_3, \qquad\qquad\qquad [e_1,e_4] = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3,\\
&[e_2,e_4] = \gamma_4 e_1 + \gamma_5 e_2 + \gamma_6 e_3, \quad [e_3,e_4] = (\gamma_1 + \gamma_5) e_3\,.
\end{aligned}
$$

A straightforward calculation as in Section 4.1.1 shows that there are no left-invariant Ricci solitons in this case.

4.2.4 Case of zero eigenvalues: $\mathbb{R}^3 \rtimes \mathbb{R}$

Proceeding as in Section 4.1.1, the left-invariant metrics are described by

$$
\begin{aligned}
&[e_1,e_4] = \eta_1 e_1 - \gamma_1 e_2 - \gamma_2 e_3, \quad [e_2,e_4] = \gamma_1 e_1 + \eta_2 e_2 - \gamma_3 e_3,\\
&[e_3,e_4] = \gamma_2 e_1 + \gamma_3 e_2 + \eta_3 e_3\,.
\end{aligned}
$$

Analogous calculations to those in Section 4.1.1 show that $\mathbb{R}^3 \rtimes \mathbb{R}$ is a left-invariant Ricci soliton if and only if $\eta_1 = \eta_2 = \eta_3 = \kappa$, in which case the sectional curvature is constantly κ^2.

4.3 Extensions of degenerate Lie groups

In this section we will study left-invariant Lorentzian metrics which are extensions of three-dimensional unimodular Lie groups with a degenerate metric. We will show that the underlying structure of any non-Einstein soliton is either a plane wave (obtained in Section 4.3.1 and Section 4.3.2) or a symmetric product $\mathbb{L}^2 \times N(c)$ (studied in Section 4.3.2). While the products of the form $\mathbb{L}^2 \times N(c)$, discussed in Section 4.3.2, are left-invariant Ricci solitons, the case of plane waves is more complicated and will be analysed in Section 4.4.

Let $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ be a four-dimensional Lie algebra with a Lorentzian inner product $\langle \cdot, \cdot \rangle$ which is degenerate when restricted to $\mathfrak{g}_3$. Let $\mathfrak{g}_3' = [\mathfrak{g}_3, \mathfrak{g}_3]$ be the derived subalgebra of $\mathfrak{g}_3$. We consider the different cases given by $\dim \mathfrak{g}_3' \in \{0, 1, 2, 3\}$ separately.

4.3.1 $\dim \mathfrak{g}_3' = 0$: left-invariant metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$

In this case the Lie algebra $\mathfrak{g}_3$ is Abelian. There exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \operatorname{span}\{u_4\}$ with $\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle = \langle u_3, u_4 \rangle = 1$, so that

$$[u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3, \qquad [u_2, u_4] = \gamma_4 u_1 + \gamma_5 u_2 + \gamma_6 u_3,$$
$$[u_3, u_4] = \gamma_7 u_1 + \gamma_8 u_2 + \gamma_9 u_3,$$

for $\gamma_i \in \mathbb{R}$. A straightforward calculation leads to the polynomials

$$\mathfrak{P}_{11} = -\gamma_7^2 + 4X_4\gamma_1 - 2\mu, \qquad \mathfrak{P}_{13} = 2X_4\gamma_7,$$
$$\mathfrak{P}_{34} = \gamma_7^2 + \gamma_8^2 + 2X_4\gamma_9 - 2\mu, \qquad \mathfrak{P}_{23} = 2X_4\gamma_8\,.$$

It follows from the expressions of $\mathfrak{P}_{13}$ and $\mathfrak{P}_{23}$, together with the combination

$$\mathfrak{P}_{11} - \mathfrak{P}_{34} = -2\gamma_7^2 - \gamma_8^2 + 2X_4(2\gamma_1 - \gamma_9),$$

that $\gamma_7 = \gamma_8 = 0$. Therefore, the left-invariant metric is given by

$$\begin{aligned} &[u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3, \qquad [u_3, u_4] = \gamma_9 u_3, \\ &[u_2, u_4] = \gamma_4 u_1 + \gamma_5 u_2 + \gamma_6 u_3\,. \end{aligned} \tag{4.4}$$

It is not difficult to check that u_3 is a recurrent null vector such that $R(Y, Z) = 0$ and $\nabla_Y R = 0$ for all $Y, Z \in u_3^\perp$. Moreover, the only non-zero component of the Ricci tensor is

$$\rho_{44} = -\gamma_1^2 - \tfrac{1}{2}(\gamma_2 + \gamma_4)^2 - \gamma_5^2 + (\gamma_1 + \gamma_5)\gamma_9,$$

which shows that the Ricci operator is isotropic and the underlying structure is a plane wave.

4.3.2 $\dim \mathfrak{g}_3' = 1$: left-invariant metrics on $H^3 \rtimes \mathbb{R}$

Since the restriction of the metric to $\mathfrak{g}_3$ has signature $(+,+,0)$, then $\mathfrak{g}_3' = \operatorname{span}\{v\}$ can be a null or a spacelike subspace. We analyse these two possibilities separately.

$\mathfrak{g}_3' = \operatorname{span}\{v\}$ is a null subspace

In this case, setting $u_3 = v$, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$, so that

$$[u_1, u_2] = \lambda_1 u_3, \quad [u_1, u_3] = \lambda_2 u_3, \quad [u_2, u_3] = \lambda_3 u_3, \quad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\alpha_i^j \in \mathbb{R}$ and where at least one of λ_1, λ_2 and λ_3 is non-zero. We are led to the following different possibilities depending on the values of the λ_i's.

Case $\lambda_2 = \lambda_3 = 0$

If $\lambda_2 = \lambda_3 = 0$, then necessarily $\lambda_1 \neq 0$ and

$$\begin{aligned} &[u_1, u_2] = \lambda_1 u_3, && [u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3, \\ &[u_2, u_4] = \gamma_4 u_1 + \gamma_5 u_2 + \gamma_6 u_3, && [u_3, u_4] = (\gamma_1 + \gamma_5) u_3\,. \end{aligned} \tag{4.5}$$

A standard calculation shows that u_3 is a recurrent vector field and the curvature tensor satisfies $R(Y, Z) = 0$ and $\nabla_Y R = 0$ for all $Y, Z \in u_3^\perp$. The only non-zero component of the Ricci tensor is

$$\rho_{44} = \tfrac{1}{2}\left\{\lambda_1^2 + 4\gamma_1\gamma_5 - (\gamma_2 + \gamma_4)^2\right\},$$

which shows that the Ricci operator is isotropic. Therefore, the underlying structure is a plane wave.

Case $\lambda_2 = 0$**,** $\lambda_3 \neq 0$

In this case one has

$$[u_1, u_2] = \lambda_1 u_3, \qquad [u_1, u_4] = \gamma_1\lambda_3 u_1 + (\gamma_1 - \gamma_2)\lambda_1 u_3, \quad [u_2, u_3] = \lambda_3 u_3,$$

$$[u_2, u_4] = \gamma_3 u_1 + \gamma_4 u_3, \quad [u_3, u_4] = \gamma_2 \lambda_3 u_3,$$

and the non-zero polynomials $\mathfrak{P}_{ij}$ are given by

$$\mathfrak{P}_{11} = 4X_4\gamma_1\lambda_3 - 2\mu,$$
$$\mathfrak{P}_{12} = 2X_4\gamma_3,$$
$$\mathfrak{P}_{14} = \lambda_1\lambda_3 + 2(X_4(\gamma_1 - \gamma_2) + X_2)\lambda_1 - 2X_1\gamma_1\lambda_3 - 2X_2\gamma_3,$$
$$\mathfrak{P}_{22} = -\lambda_3^2 - 2\mu,$$
$$\mathfrak{P}_{24} = -\gamma_1\lambda_3^2 - 2X_1\lambda_1 + 2X_3\lambda_3 + 2X_4\gamma_4,$$
$$\mathfrak{P}_{34} = (2X_4\gamma_2 - 2X_2)\lambda_3 - \lambda_3^2 - 2\mu,$$
$$\mathfrak{P}_{44} = \lambda_1^2 - 2\gamma_1(\gamma_1 - \gamma_2)\lambda_3^2 - 4X_1(\gamma_1 - \gamma_2)\lambda_1 - 4X_3\gamma_2\lambda_3 - 4X_2\gamma_4 - \gamma_3^2\,.$$

It is easy to check that

$$\mathfrak{P}_{11} - \mathfrak{P}_{22} = (\lambda_3 + 4X_4\gamma_1)\lambda_3,$$

which implies that $X_4 \neq 0$ (since $\lambda_3 = 0$). Now, this combination, together with the expressions of $\mathfrak{P}_{12}$ and $\mathfrak{P}_{22}$, lead to

$$\gamma_3 = 0, \quad \gamma_1 = -\tfrac{1}{4X_4}\lambda_3, \quad \mu = -\tfrac{1}{2}\lambda_3^2,$$

and a direct calculation shows that

$$2\lambda_1\mathfrak{P}_{14} - 2\gamma_2\lambda_3\mathfrak{P}_{24} + \tfrac{2}{\lambda_3}(\lambda_1^2 - \gamma_3\lambda_1 + \gamma_4\lambda_3)\mathfrak{P}_{34} - \lambda_3\mathfrak{P}_{44} = \tfrac{1}{8X_4^2}\lambda_3^5\,.$$

Since $\lambda_3 \neq 0$, there are no left-invariant Ricci solitons in this case.

Case $\lambda_2 \neq 0$

If $\lambda_2 \neq 0$, then the Lie algebra structure is given by

$$[u_1, u_2] = \lambda_1 u_3, \quad [u_1, u_3] = \lambda_2 u_3, \quad [u_2, u_3] = \lambda_3 u_3, \quad [u_3, u_4] = \gamma_4\lambda_2 u_3,$$
$$[u_1, u_4] = -\gamma_1\lambda_2\lambda_3 u_1 + \gamma_1\lambda_2^2 u_2 + \gamma_2\lambda_2 u_3,$$
$$[u_2, u_4] = -\gamma_3\lambda_3 u_1 + \gamma_3\lambda_2 u_2 + (\gamma_1\lambda_1\lambda_3 - (\gamma_3 - \gamma_4)\lambda_1 + \gamma_2\lambda_3)u_3,$$

and the non-zero polynomials $\mathfrak{P}_{ij}$ are

$$\begin{aligned}
\mathfrak{P}_{11} &= -\lambda_2^2 - 4X_4\gamma_1\lambda_2\lambda_3 - 2\mu, \\
\mathfrak{P}_{12} &= 2X_4\gamma_1\lambda_2^2 - \lambda_2\lambda_3 - 2X_4\gamma_3\lambda_3, \\
\mathfrak{P}_{14} &= \gamma_1\lambda_2^2\lambda_3 - \gamma_3\lambda_2^2 + \lambda_1\lambda_3 + 2X_2(\lambda_1 + \gamma_3\lambda_3) + 2(X_4\gamma_2 + X_3 + X_1\gamma_1\lambda_3)\lambda_2, \\
\mathfrak{P}_{22} &= -\lambda_3^2 + 4X_4\gamma_3\lambda_2 - 2\mu, \\
\mathfrak{P}_{24} &= \gamma_1\lambda_2\lambda_3^2 - 2X_1\gamma_1\lambda_2^2 - \lambda_1\lambda_2 + 2X_4\gamma_1\lambda_1\lambda_3 - \gamma_3\lambda_2\lambda_3 \\
&\quad - 2(X_4(\gamma_3 - \gamma_4) + X_1)\lambda_1 - 2X_2\gamma_3\lambda_2 + 2(X_4\gamma_2 + X_3)\lambda_3, \\
\mathfrak{P}_{34} &= -\lambda_2^2 - \lambda_3^2 + 2(X_4\gamma_4 - X_1)\lambda_2 - 2X_2\lambda_3 - 2\mu, \\
\mathfrak{P}_{44} &= -\gamma_1^2\lambda_2^4 - 2\gamma_1^2\lambda_2^2\lambda_3^2 + 2\gamma_1(\gamma_3 - \gamma_4)\lambda_2^2\lambda_3 + \lambda_1^2 - 2\gamma_3(\gamma_3 - \gamma_4)\lambda_2^2 - \gamma_3^2\lambda_3^2 \\
&\quad - 4X_2\gamma_1\lambda_1\lambda_3 + 4X_2(\gamma_3 - \gamma_4)\lambda_1 - 4(X_1\gamma_2 + X_3\gamma_4)\lambda_2 - 4X_2\gamma_2\lambda_3\,.
\end{aligned}$$

Since $\lambda_2 \neq 0$, if we consider the expressions

$$\begin{aligned}
\mathfrak{P}_{11} - \mathfrak{P}_{22} &= -\lambda_2^2 + \lambda_3^2 - 4X_4(\gamma_1\lambda_3 + \gamma_3)\lambda_2, \\
\mathfrak{P}_{12} &= -\lambda_2\lambda_3 + 2X_4(\gamma_1\lambda_2^2 - \gamma_3\lambda_3),
\end{aligned}$$

then $X_4 \neq 0$. Now, from the expressions of $\mathfrak{P}_{12}$, $\mathfrak{P}_{11}$ and $\mathfrak{P}_{22}$ we obtain

$$\gamma_1 = \frac{(\lambda_2 + 2X_4\gamma_3)\lambda_3}{2X_4\lambda_2^2}, \quad \mu = -\frac{\lambda_2^3 + 2(\lambda_2 + 2X_4\gamma_3)\lambda_3^2}{2\lambda_2}, \quad \gamma_3 = -\frac{\lambda_2}{4X_4},$$

and a direct calculation shows that

$$\begin{aligned}
&2(\gamma_4\lambda_2^2 - \lambda_1\lambda_3)\mathfrak{P}_{14} + 2(\lambda_1 + \gamma_4\lambda_3)\lambda_2\mathfrak{P}_{24} + (\lambda_2^2 + \lambda_3^2)\mathfrak{P}_{44} \\
&\qquad -2\left(\lambda_1^2 + (\gamma_1\lambda_1 + \gamma_2)\left(\lambda_2^2 + \lambda_3^2\right) + \gamma_4\lambda_1\lambda_3\right)\mathfrak{P}_{34} = -\frac{(\lambda_2^2 + \lambda_3^2)^3}{8X_4^2}\,.
\end{aligned}$$

Since $\lambda_2 \neq 0$, there are no left-invariant Ricci solitons in this case.

$\mathfrak{g}_3' = \operatorname{span}\{v\}$ is a spacelike subspace

Considering $u_1 = \frac{v}{\|v\|}$, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$, so that

$$[u_1, u_2] = \lambda_1 u_1, \quad [u_1, u_3] = \lambda_2 u_1, \quad [u_2, u_3] = \lambda_3 u_1, \quad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\alpha_i^j \in \mathbb{R}$ and where at least one of λ_1, λ_2 and λ_3 is non-zero. Depending on the values of the λ_i's, we are led to the following different possibilities.

Case $\lambda_1 = \lambda_2 = 0$

Since $\lambda_3 \neq 0$ one has the Lie algebra structure

$$[u_1, u_4] = \gamma_1 u_1, \qquad [u_2, u_3] = \lambda_3 u_1,$$
$$[u_2, u_4] = \gamma_2 u_1 + \gamma_3 u_2 + \gamma_4 u_3, \qquad [u_3, u_4] = \gamma_5 u_1 + \gamma_6 u_2 + (\gamma_1 - \gamma_3) u_3 \,.$$

A direct calculation shows $\mathfrak{P}_{33} = -\lambda_3^2$. Therefore, there are no left-invariant Ricci solitons in this case.

Case $\lambda_1 = 0$***,*** $\lambda_2 \neq 0$

In this case, the left-invariant metrics are described as

$$[u_1, u_3] = \lambda_2 u_1, \qquad [u_1, u_4] = \gamma_1 \lambda_2 u_1, \qquad [u_2, u_3] = \lambda_3 u_1,$$
$$[u_2, u_4] = (\gamma_1 - \gamma_2)\lambda_3 u_1 + \gamma_2 \lambda_2 u_2, \quad [u_3, u_4] = \gamma_3 u_1 + \gamma_4 u_2 \,.$$

It now follows from $\mathfrak{P}_{33} = -2\lambda_2^2 - \lambda_3^2$ that there are no left-invariant Ricci solitons in this case.

Case $\lambda_1 \neq 0$

If $\lambda_1 \neq 0$, then the Lie algebra structure takes the form

$$[u_1, u_2] = \lambda_1 u_1, \quad [u_1, u_3] = \lambda_2 u_1, \quad [u_1, u_4] = \gamma_1 \lambda_1 u_1, \quad [u_2, u_3] = \lambda_3 u_1,$$
$$[u_2, u_4] = \lambda_1 \gamma_2 u_1 - \gamma_3 \lambda_1 \lambda_2 u_2 + \gamma_3 \lambda_1^2 u_3,$$
$$[u_3, u_4] = -(\gamma_3 \lambda_2 \lambda_3 - \gamma_2 \lambda_2 + (\gamma_1 - \gamma_4)\lambda_3) u_1 - \gamma_4 \lambda_2 u_2 + \gamma_4 \lambda_1 u_3,$$

and the polynomials $\mathfrak{P}_{ij}$ are

$$\begin{aligned}
\mathfrak{P}_{11} = &- \gamma_3^2 \lambda_2^2 \lambda_3^2 + 2\gamma_3 \lambda_1 \lambda_2^2 + 2\gamma_2 \gamma_3 \lambda_2^2 \lambda_3 - 2(\gamma_1 - \gamma_4)\gamma_3 \lambda_2 \lambda_3^2 - 2\lambda_1^2 - \gamma_2^2 \lambda_2^2 \\
&- (\gamma_1 - \gamma_4)^2 \lambda_3^2 - 2(2\gamma_1 + \gamma_4)\lambda_1 \lambda_2 + 2\gamma_2 \lambda_1 \lambda_3 + 2(\gamma_1 - \gamma_4)\gamma_2 \lambda_2 \lambda_3 \\
&+ 4(X_4 \gamma_1 + X_2)\lambda_1 + 4X_3 \lambda_2 - 2\mu,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{12} = &-\gamma_3 \gamma_4 \lambda_2^2 \lambda_3 + \gamma_2 \gamma_4 \lambda_2^2 - 2\gamma_2 \lambda_1 \lambda_2 - (2\gamma_1 + \gamma_4)\lambda_1 \lambda_3 - (\gamma_1 - \gamma_4)\gamma_4 \lambda_2 \lambda_3 \\
&+ 2(X_4 \gamma_2 - X_1)\lambda_1 + 2X_3 \lambda_3,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{13} = &\, 2\gamma_3 \lambda_2^2 \lambda_3 - 2\gamma_2 \lambda_2^2 + 2\lambda_1 \lambda_3 + 2(\gamma_1 - \gamma_4 - X_4 \gamma_3)\lambda_2 \lambda_3 + (2X_4 \gamma_2 - 2X_1)\lambda_2 \\
&- 2(X_4(\gamma_1 - \gamma_4) + X_2)\lambda_3,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{14} = &\, \gamma_3^2 \lambda_1 \lambda_2^2 \lambda_3 + \gamma_3 \lambda_1^2 \lambda_3 - \gamma_2 \gamma_3 \lambda_1 \lambda_2^2 - (\gamma_1 + \gamma_4)\gamma_3 \lambda_1 \lambda_2 \lambda_3 + 2\gamma_2 \lambda_1^2 + 2\gamma_1 \gamma_2 \lambda_1 \lambda_2 \\
&- 2\gamma_1(\gamma_1 - \gamma_4)\lambda_1 \lambda_3 + 2X_3 \gamma_3 \lambda_2 \lambda_3 - 2(X_1 \gamma_1 + X_2 \gamma_2)\lambda_1 - 2X_3 \gamma_2 \lambda_2
\end{aligned}$$

$$
\begin{aligned}
&+ 2X_3(\gamma_1-\gamma_4)\lambda_3,\\
\mathfrak{P}_{22} &= 2\gamma_3\lambda_1\lambda_2^2 - 2\lambda_1^2 - \gamma_4^2\lambda_2^2 - 4X_4\gamma_3\lambda_1\lambda_2 - 2\gamma_2\lambda_1\lambda_3 - 2\mu,\\
\mathfrak{P}_{23} &= \gamma_3\lambda_2\lambda_3^2 + \gamma_4\lambda_2^2 + (\gamma_1-\gamma_4)\lambda_3^2 - 2\lambda_1\lambda_2 - \gamma_2\lambda_2\lambda_3 - 2X_4\gamma_4\lambda_2,\\
\mathfrak{P}_{24} &= -2\gamma_3\lambda_1^2\lambda_2 + 2\gamma_3\gamma_4\lambda_1\lambda_2^2 - \gamma_2\gamma_3\lambda_1\lambda_2\lambda_3 - 2(\gamma_1 - X_4\gamma_3)\lambda_1^2 + (\gamma_2^2 - \gamma_1\gamma_4 + 2X_2\gamma_3)\lambda_1\lambda_2\\
&\quad - (\gamma_1-\gamma_4)\gamma_2\lambda_1\lambda_3 + 2X_3\gamma_4\lambda_2,\\
\mathfrak{P}_{33} &= -2\lambda_2^2 - \lambda_3^2,\\
\mathfrak{P}_{34} &= \gamma_3^2\lambda_2^2\lambda_3^2 - 2\gamma_2\gamma_3\lambda_2^2\lambda_3 + 2(\gamma_1-\gamma_4)\gamma_3\lambda_2\lambda_3^2 + (\gamma_2^2+\gamma_4^2)\lambda_2^2 + (\gamma_1-\gamma_4)^2\lambda_3^2\\
&\quad - (2\gamma_1+\gamma_4)\lambda_1\lambda_2 - \gamma_2\lambda_1\lambda_3 - 2(\gamma_1-\gamma_4)\gamma_2\lambda_2\lambda_3 + 2X_4\gamma_4\lambda_1 - 2\mu,\\
\mathfrak{P}_{44} &= -2\gamma_3^2\lambda_1^2\lambda_2^2 + 2\gamma_3\lambda_1^3 - (2\gamma_1^2 + \gamma_2^2 - 2\gamma_1\gamma_4 + 4X_2\gamma_3)\lambda_1^2 - 4X_3\gamma_4\lambda_1\,.
\end{aligned}
$$

It follows from the expression of $\mathfrak{P}_{33}$ that both λ_2 and λ_3 must be zero. Now,

$$\mathfrak{P}_{22} - \mathfrak{P}_{34} = -2(\lambda_1 + X_4\gamma_4)\lambda_1$$

implies that $X_4 \neq 0$ and $\gamma_4 \neq 0$. At this point, the expressions

$$
\begin{aligned}
\gamma_2\mathfrak{P}_{11} - 2\gamma_1\mathfrak{P}_{12} + 2\,\mathfrak{P}_{14} - \gamma_2\mathfrak{P}_{22} &= 4\gamma_2\lambda_1^2,\\
\gamma_1\mathfrak{P}_{22} - \mathfrak{P}_{24} - \gamma_1\mathfrak{P}_{34} &= -2X_4(\gamma_3\lambda_1 + \gamma_1\gamma_4)\lambda_1,
\end{aligned}
$$

lead to $\gamma_2 = 0$, and $\gamma_1 = -\frac{\gamma_3\lambda_1}{\gamma_4}$. Finally, a standard calculation shows that the corresponding left-invariant metric, which given by

$$[u_1,u_2] = \lambda_1 u_1, \quad [u_1,u_4] = -\tfrac{\gamma_3\lambda_1^2}{\gamma_4}u_1, \quad [u_2,u_4] = \gamma_3\lambda_1^2 u_3, \quad [u_3,u_4] = \gamma_4\lambda_1 u_3,$$

is symmetric and locally isometric to a product $\mathbb{L}^2 \times N(c)$, where N is a surface of constant curvature c. Furthermore, it is a expanding Ricci soliton with $\mu = -\lambda_1^2$ and left-invariant soliton vector field

$$X = -\frac{\gamma_3\lambda_1^2}{\gamma_4^2}u_2 + \frac{\gamma_3^2\lambda_1^3}{2\gamma_4^3}u_3 - \frac{\lambda_1}{\gamma_4}u_4.$$

4.3.3 $\dim \mathfrak{g}_3' = 2$: left-invariant metrics on $\tilde{E}(2) \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$

Let $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ be the derived subalgebra of $\mathfrak{g}$. We can assume that $\mathfrak{g}' = \mathfrak{g}_3$ without losing generality. Indeed, if $\dim \mathfrak{g}' < 3$, then there exist two linearly independent vectors $x_1, x_2 \in \mathfrak{g}$ acting as derivations on $\mathfrak{g}$. Since $\mathfrak{g}$ is Lorentzian, we can choose a non-null vector $y \in \operatorname{span}\{x_1,x_2\}$ so that $\mathfrak{g} = \mathfrak{h} \rtimes \operatorname{span}\{y\}$, where the restriction of the metric to the three-dimensional subalgebra $\mathfrak{h}$ is non-degenerate. Thus, $\mathfrak{g}$ corresponds to one of the cases already studied in Sections 4.1 and 4.2.

Let $\mathfrak{g}_3' = \operatorname{span}\{w_1,w_2\}$ with $w_i = v_i + \xi_i u_3$, where v_i is a spacelike vector field and u_3 is null and orthogonal to v_1 and v_2.

If $\{v_1, v_2\}$ are linearly independent, which means that $\mathfrak{g}_3'$ is a spacelike subspace, we choose an orthonormal basis $\{u_1, u_2\}$ for $\operatorname{span}\{v_1, v_2\}$ that can be completed to a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ such that $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$. In this situation, the left-invariant metrics are of the form

$$[u_1, u_2] = \gamma_1 u_1 + \gamma_2 u_2, \qquad [u_1, u_3] = \gamma_3 u_1 + \gamma_4 u_2,$$

$$[u_2, u_3] = \gamma_5 u_1 + \gamma_6 u_2, \qquad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain γ_i, $\alpha_i^j \in \mathbb{R}$.

If $\{v_1, v_2\}$ are linearly dependent, i.e., the restriction of the metric to $\mathfrak{g}_3'$ is degenerate, then $\{u_1 = \frac{v_1}{\|v_1\|}, u_3\}$ is a basis of $\mathfrak{g}_3'$ that can be completed to a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ such that $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$. In this case, the left-invariant metrics are expressed as

$$[u_1, u_2] = \gamma_1 u_1 + \gamma_2 u_3, \qquad [u_1, u_3] = \gamma_3 u_1 + \gamma_4 u_3,$$

$$[u_2, u_3] = \gamma_5 u_1 + \gamma_6 u_3, \qquad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain γ_i, $\alpha_i^j \in \mathbb{R}$.

In both cases above, a straightforward calculation shows that the Jacobi identity is not satisfied since $\dim \mathfrak{g}_3' = 2$ and $\dim \mathfrak{g}' = 3$. Therefore, there are no left-invariant Ricci solitons in this case.

4.3.4 $\dim \mathfrak{g}_3' = 3$: left-invariant metrics on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

In this case, $\mathfrak{g}_3' = \mathfrak{g}_3$ and we consider the pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \operatorname{span}\{u_4\}$ with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle$ and $\operatorname{ad}_{u_3} : \mathfrak{g}_3 \to \mathfrak{g}_3$. Since $\mathfrak{g}_3' = \mathfrak{g}_3$, ad_{u_3} must be of rank 2 and, apart from 0, it must have either two real eigenvalues or two conjugate complex eigenvalues. Moreover, writing

$$u_3 = [x_1, x_2], \qquad x_1, x_2 \in \mathfrak{g}_3,$$

we have

$$\operatorname{ad}_{u_3} = \operatorname{ad}_{x_1} \circ \operatorname{ad}_{x_2} - \operatorname{ad}_{x_2} \circ \operatorname{ad}_{x_1},$$

which implies $\operatorname{tr}(\operatorname{ad}_{u_3}) = 0$. Consequently, there are two different possibilities, none of them supporting left-invariant Ricci solitons.

ad_{u_3} **has real eigenvalues** $\{0, \lambda, -\lambda\}$**, with** $\lambda \neq 0$

Let v_1 and v_2 be unit eigenvectors, i.e., $[v_1, u_3] = \lambda v_1$ and $[v_2, u_3] = -\lambda v_2$. The Jacobi identity implies $[v_1, v_2] \in \mathrm{span}\{u_3\}$. Thus, rescaling u_3 if necessary, we obtain a basis $\{v_1, v_2, v_3, v_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = \langle v_3, v_4\rangle = 1$ and $\langle v_1, v_2\rangle = \kappa \neq \pm 1$, where $\mathfrak{g}_3 = \mathrm{span}\{v_1, v_2, v_3\}$ and $\mathfrak{r} = \mathrm{span}\{v_4\}$. In this situation

$$[v_1, v_2] = v_3, \qquad [v_1, v_3] = \lambda v_1, \qquad [v_1, v_4] = \gamma_1 v_1 + \gamma_2 v_3,$$
$$[v_2, v_3] = -\lambda v_2, \qquad [v_2, v_4] = -\gamma_1 v_2 + \gamma_3 v_3, \qquad [v_3, v_4] = \gamma_3\lambda v_1 + \gamma_2 \lambda v_2\,.$$

We compute $\mathfrak{P}_{33} = \frac{4\lambda^2}{\kappa^2-1}$ and, since $\lambda \neq 0$, there are no left-invariant Ricci solitons in this case.

ad_{u_3} **has complex eigenvalues** $\{0, i\beta, -i\beta\}$**, with** $\beta \neq 0$

Let v_1 and v_2 be unit vectors so that $[v_1, u_3] = \beta v_2$ and $[v_2, u_3] = -\beta v_1$. The Jacobi identity implies $[v_1, v_2] \in \mathrm{span}\{u_3\}$. Thus, rescaling u_3 if necessary, we obtain a basis $\{v_1, v_2, v_3, v_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = \langle v_3, v_4\rangle = 1$ and $\langle v_1, v_2\rangle = \kappa \neq \pm 1$, where $\mathfrak{g}_3 = \mathrm{span}\{v_1, v_2, v_3\}$ and $\mathfrak{r} = \mathrm{span}\{v_4\}$. In this case

$$[v_1, v_2] = v_3, \qquad [v_1, v_3] = \beta v_2, \qquad [v_1, v_4] = \gamma_1 v_2 + \gamma_2 v_3,$$
$$[v_2, v_3] = -\beta v_1, \qquad [v_2, v_4] = -\gamma_1 v_1 + \gamma_3 v_3, \qquad [v_3, v_4] = \gamma_2\beta v_1 + \gamma_3 \beta v_2\,.$$

We will consider the polynomials $\tilde{\mathfrak{P}}_{ij} = (\kappa^2 - 1)\mathfrak{P}_{ij}$ given by

$$\begin{aligned}
\tilde{\mathfrak{P}}_{11} &= -(\kappa^2-1)\beta^2(\gamma_2+\kappa\gamma_3)^2 - 4(2\kappa\beta - X_4(\kappa^2-1))\kappa\gamma_1 \\
&\quad + 2(2X_3\kappa^3 - \kappa^2 - 2X_3\kappa + 1)\beta - 2(\kappa^2-1)\mu, \\
\tilde{\mathfrak{P}}_{12} &= -(\kappa^2-1)\kappa\beta^2(\gamma_2^2+\gamma_3^2) - (\kappa^4-1)\beta^2\gamma_2\gamma_3 - 8\kappa\beta\gamma_1 - 2(\kappa^2-1)\kappa\mu, \\
\tilde{\mathfrak{P}}_{22} &= -(\kappa^2-1)\beta^2(\kappa\gamma_2+\gamma_3)^2 - 4(2\kappa\beta + X_4(\kappa^2-1))\kappa\gamma_1 \\
&\quad - 2(2X_3\kappa^3 + \kappa^2 - 2\kappa X_3 - 1)\beta - 2(\kappa^2-1)\mu, \\
\tilde{\mathfrak{P}}_{33} &= 4\beta^2\kappa^2, \\
\tilde{\mathfrak{P}}_{44} &= 4\kappa^2\gamma_1^2 - 2(\kappa^2-1)\beta(\gamma_2^2+\gamma_3^2) - 4(\kappa^2-1)(X_1\gamma_2 + X_2\gamma_3) - 1\,.
\end{aligned}$$

Since $\beta \neq 0$, the expression of $\tilde{\mathfrak{P}}_{33}$ shows that $\kappa = 0$. It is now easy to check that

$$\gamma_3\tilde{\mathfrak{P}}_{11} - \gamma_2\tilde{\mathfrak{P}}_{12} - \gamma_3\tilde{\mathfrak{P}}_{22} = -\beta^2\gamma_3^3, \qquad \gamma_2\tilde{\mathfrak{P}}_{11} + \gamma_3\tilde{\mathfrak{P}}_{12} - \gamma_2\tilde{\mathfrak{P}}_{22} = \beta^2\gamma_2^3\,.$$

Therefore, $\gamma_2 = \gamma_3 = 0$ and so $\tilde{\mathfrak{P}}_{44} = -1$, which shows that there are no left-invariant Ricci solitons in this case.

4.4 Left-invariant Ricci solitons on pp-wave Lie groups

Based on the analysis carried out in the previous sections, left-invariant Ricci solitons on pp-wave Lie groups split naturally into two distinct possibilities depending on whether they are plane waves or not. The case of pp-wave Lie groups which are not plane waves can be summarized as follows

Theorem 4.9. *A four-dimensional Lorentzian pp-wave Lie group that is not a plane wave is a non-trivial left-invariant Ricci soliton if and only it is homothetic to the Lie group* $G = \mathbb{R}^3 \rtimes \mathbb{R}$ *endowed with a left-invariant metric given by the two-parameter family of Lie algebras*

$$[u_1, u_4] = \gamma_1 u_1 + \varepsilon u_2, \quad [u_2, u_4] = -\gamma_1 u_2,$$

where $\gamma_1 \neq 0$*,* $\varepsilon = \pm 1$*, and* $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis such that* $\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = \langle u_4, u_4 \rangle = 1$.

Proof. The Lorentzian Lie groups above are extensions of unimodular Lorentzian Lie groups and have been discussed in Section 4.1.1. Since the sectional curvature is independent of the structure constant γ_3, we set $\gamma_3 = 0$ in Equation (4.1) and work at the homothetic level (see [91,96]). Now, the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ reduce to

$$\tilde{\mathfrak{P}}_{11} = 2\varepsilon(X_4 - \gamma_1), \qquad \tilde{\mathfrak{P}}_{12} = \tilde{\mathfrak{P}}_{33} = \tilde{\mathfrak{P}}_{44} = -\mu,$$
$$\tilde{\mathfrak{P}}_{14} = X_2\gamma_1 - \varepsilon X_1, \qquad \tilde{\mathfrak{P}}_{24} = -X_1\gamma_1,$$

so we have a left-invariant steady Ricci soliton with left-invariant soliton vector field $X = X_3 u_3 + \gamma_1 u_4$, for any $\gamma_1 \neq 0$.

□

Remark 4.10. Globke and Leistner proved in [78] that four-dimensional Ricci-flat homogeneous pp-waves are plane waves. The examples in Theorem 4.9 show that the result above cannot be extended to steady Ricci soliton pp-waves. Moreover, the pp-wave Lie groups in Theorem 4.9 are conformal C-spaces, but not conformally Einstein (see [26, 79] for more information).

Theorem 4.11. *A four-dimensional Lorentzian plane wave Lie group is a non-trivial left-invariant Ricci soliton if and only it is homothetic to one of the following Lie groups.*

(i) $G = H^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_3] = u_2, \quad [u_1, u_4] = \gamma_3 u_3,$$

where $\gamma_3 \neq 0$ *and* $\{u_1, u_2, u_3, u_4\}$ *is pseudo-orthonormal such that*

$$\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = \langle u_4, u_4 \rangle = 1.$$

(ii) $G = E(1,1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_2] = u_1, \quad [u_2, u_3] = u_3, \quad [u_2, u_4] = \gamma_3 u_3,$$

and $\{u_1, u_2, u_3, u_4\}$ *is pseudo-orthonormal with*

$$\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = \langle u_4, u_4 \rangle = 1.$$

(iii) $G = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2, \quad [u_2, u_4] = \gamma_4 u_1 + \gamma_5 u_2, \quad [u_3, u_4] = u_3,$$

where $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis with*

$$\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle = \langle u_3, u_4 \rangle = 1.$$

(iv) $G = H^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$\begin{aligned} [u_1, u_2] &= \lambda_1 u_3, & [u_1, u_4] &= \gamma_1 u_1 + \gamma_2 u_2, \\ [u_2, u_4] &= \gamma_4 u_1 + \gamma_5 u_2, & [u_3, u_4] &= (\gamma_1 + \gamma_5) u_3, \end{aligned}$$

where $\gamma_1 + \gamma_5 \neq 0$*, and* $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis with*

$$\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle = \langle u_3, u_4 \rangle = 1.$$

Proof. The Lie groups in Assertions *(i)* and *(ii)* are Lorentzian extensions of unimodular Lorentzian Lie groups. Assertion *(i)* was considered in Section 4.1.3 and a straightforward calculation shows that the curvature tensor does not involve the structure constants ε and γ_2, so we can take $\varepsilon = -1$ and $\gamma_2 = 0$ in Equation (4.2) (see [91, 96]). Now, the only non-zero component of the Ricci tensor is

$$\rho_{11} = -\tfrac{1}{2}(\gamma_3^2 - \gamma_6^2)$$

and the non-zero polynomials $\mathfrak{P}_{ij}$ are

$$\begin{aligned} &\mathfrak{P}_{11} = -\tfrac{1}{2}\{\gamma_3^2 - \gamma_6^2 - 4X_3\}, && \mathfrak{P}_{12} = \mathfrak{P}_{33} = \mathfrak{P}_{44} = -\mu, \\ &\mathfrak{P}_{13} = X_4(\gamma_3 + \gamma_6) - X_1, && \mathfrak{P}_{14} = -X_3\gamma_6, \qquad \mathfrak{P}_{34} = -X_1\gamma_3 . \end{aligned}$$

Now the metric is a left-invariant steady Ricci soliton if and only if it is Ricci-flat ($\gamma_3^2 = \gamma_6^2$) or, otherwise, $\gamma_6 = 0$ and $\gamma_3 \neq 0$. In this latter case, the left-invariant soliton vector field is given by

$$X = X_2 u_2 + \tfrac{1}{4}\gamma_3^2 u_3.$$

Assertion *(ii)* was treated in Section 4.1.4 and since the curvature tensor does not depend on the structure constant γ_2, we can eliminate it in Equation (4.3) remaining in the same homothetic class, according to the work [96] (see also [91]). The non-zero polynomials $\mathfrak{P}_{ij}$ now reduce to

$$\mathfrak{P}_{12} = X_2 - \mu, \qquad \mathfrak{P}_{22} = -\tfrac{1}{2}\gamma_3^2 - 2(X_1 + 1), \qquad \mathfrak{P}_{23} = X_4\gamma_3 + X_3,$$

$$\mathfrak{P}_{33} = -2X_2 - \mu, \qquad \mathfrak{P}_{34} = -X_2\gamma_3, \qquad \mathfrak{P}_{44} = -\mu,$$

and we obtain a left-invariant steady Ricci soliton with left-invariant soliton vector field

$$X = -(\tfrac{1}{4}\gamma_3^2 + 1)u_1 - X_4\gamma_3 u_3 + X_4 u_4.$$

The plane wave Lie groups in Assertion *(iii)* are Lorentzian extensions of unimodular degenerate Lie groups and correspond to those studied in Section 4.3.1. First of all, observe that proceeding as in the previous cases the constants γ_3 and γ_6 can be removed from Equation (4.4) and we would still be working in the same homothetic class. A straightforward calculation shows that the Ricci tensor vanishes if and only if

$$\rho_{44} = -\gamma_1^2 - \frac{1}{2}(\gamma_2 + \gamma_4)^2 - \gamma_5^2 + (\gamma_1 + \gamma_5)\gamma_9 = 0,$$

and the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\tilde{\mathfrak{P}}_{11} = 2X_4\gamma_1 - \mu, \qquad \tilde{\mathfrak{P}}_{14} = -X_1\gamma_1 - X_2\gamma_4, \qquad \tilde{\mathfrak{P}}_{12} = X_4(\gamma_2 + \gamma_4),$$

$$\tilde{\mathfrak{P}}_{34} = X_4\gamma_9 - \mu, \qquad \tilde{\mathfrak{P}}_{22} = 2X_4\gamma_5 - \mu, \qquad \tilde{\mathfrak{P}}_{24} = -X_1\gamma_2 - X_2\gamma_5,$$

$$\tilde{\mathfrak{P}}_{44} = \rho_{44} - 2X_3\gamma_9\,.$$

We consider the cases $\gamma_9 \neq 0$ and $\gamma_9 = 0$ separately. Assuming $\gamma_9 \neq 0$, we can take $\gamma_9 = 1$ without losing generality. If $\gamma_2 = -\gamma_4$ and $\gamma_1 = \gamma_5 = \frac{1}{2}$, then

$$X = \frac{1}{2}\rho_{44}u_3 + \mu u_4$$

defines a locally conformally flat expanding, steady or shrinking left-invariant Ricci soliton. Otherwise, if $\gamma_2 \neq -\gamma_4$, or $\gamma_1 \neq \frac{1}{2}$, or $\gamma_5 \neq \frac{1}{2}$, then

$$X = \frac{1}{2}\rho_{44}u_3$$

determines a steady left-invariant Ricci soliton. Finally, if $\gamma_9 = 0$, then G is a left-invariant Ricci soliton if and only if it is Ricci-flat.

The plane wave Lie groups in Assertion *(iv)* are Lorentzian extensions of unimodular degenerate Lie groups and correspond to those in Section 4.3.2. We proceed as in the previous case and eliminate the structure constants γ_3 and γ_6, so that the Ricci tensor vanishes if and only if

$$\rho_{44} = \frac{1}{2}\left\{\lambda_1^2 + 4\gamma_1\gamma_5 - (\gamma_2 + \gamma_4)^2\right\} = 0,$$

and the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\begin{aligned} \tilde{\mathfrak{P}}_{11} &= 2X_4\gamma_1 - \mu, & \tilde{\mathfrak{P}}_{12} &= X_4(\gamma_2 + \gamma_4), \\ \tilde{\mathfrak{P}}_{14} &= -X_1\gamma_1 + X_2(\lambda_1 - \gamma_4), & \tilde{\mathfrak{P}}_{22} &= 2X_4\gamma_5 - \mu, \\ \tilde{\mathfrak{P}}_{24} &= -X_1(\lambda_1 + \gamma_2) - X_2\gamma_5, & \tilde{\mathfrak{P}}_{34} &= X_4(\gamma_1 + \gamma_5) - \mu, \\ \tilde{\mathfrak{P}}_{44} &= \rho_{44} - 2X_3(\gamma_1 + \gamma_5)\,. \end{aligned}$$

If $\gamma_1 + \gamma_5 = 0$, left-invariant Ricci solitons only exist in the Ricci-flat case. Assuming that $\gamma_1 + \gamma_5 \neq 0$, there are two distinct possibilities.

If $\gamma_2 + \gamma_4 = 0$ and $\gamma_1 - \gamma_5 = 0$, then

$$X = \frac{1}{2(\gamma_1 + \gamma_5)}\rho_{44}u_3 + \frac{1}{\gamma_1 + \gamma_5}\mu u_4$$

determines a locally conformally flat expanding, steady or shrinking left-invariant Ricci soliton. Otherwise, if $\gamma_2 + \gamma_4 \neq 0$ or $\gamma_1 - \gamma_5 \neq 0$, then

$$X = \frac{1}{2(\gamma_1 + \gamma_5)}\rho_{44}u_3$$

defines a steady left-invariant Ricci soliton. □

Remark 4.12. The plane wave Lie groups in Theorem 4.11 have vanishing Cotton tensor, and thus they are conformally Einstein [26]. The plane wave Lie groups corresponding to Assertion *(iii)* and Assertion *(iv)* which admit expanding, steady and shrinking left-invariant Ricci solitons are locally conformally flat.

Chapter 5
Algebraic Ricci and Bach solitons on four-dimensional Riemannian Lie groups

In this chapter we will give a complete description of the four-dimensional Riemannian algebraic Bach and Ricci solitons. In an endeavour to simplify the calculations, we will first work with a generic $(0,2)$-tensor field T and study the conditions it must satisfy in order to determine an algebraic soliton for its associated flow. Imposing these conditions on specific tensor fields significantly eases the problem of determining all the algebraic solitons associated to the corresponding geometric flows. The results in this chapter are partially contained in the work [71].

5.1 Algebraic T-solitons

Let T be a left-invariant symmetric $(0,2)$-tensor field on a Lie group $(G,\langle\cdot,\cdot\rangle)$ that is endowed with a left-invariant Riemannian metric and denote by $\widehat{T}$ its associated $(1,1)$-tensor field. Throughout this chapter, we will assume that T is divergence-free and isometrically invariant. $(G,\langle\cdot,\cdot\rangle)$ is said to be an *algebraic T-soliton* if

$$\widehat{T} = \mu\,\mathrm{Id} + \mathfrak{D}$$

for some derivation $\mathfrak{D}$ of the corresponding Lie algebra $\mathfrak{g}$ and some $\mu \in \mathbb{R}$, and it will be expanding, steady or shrinking if $\mu < 0$, $\mu = 0$, or $\mu > 0$, respectively.

If $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{g}$, we will use the notation $T_{ij} = T(e_i, e_j)$ and, since T is symmetric, when we need to write it in matrix form we will omit the entries below the diagonal and, for the sake of simplicity, write '$*$' instead.

The endomorphism $\mathfrak{D} = \widehat{T} - \mu\,\mathrm{Id}$ is a derivation of the corresponding Lie algebra $\mathfrak{g}$ if it satisfies the condition

$$\mathfrak{D}[e_i, e_j] - [\mathfrak{D}e_i, e_j] - [e_i, \mathfrak{D}e_j] = 0, \quad i, j = 1, \ldots, 4,$$

which, when expressed with respect to the basis $\{e_1, e_2, e_3, e_4\}$, is equivalent to

$$\mathfrak{P}_{ijk} = \mathfrak{D}_\ell{}^k c_{ij}{}^\ell - \mathfrak{D}_i{}^\ell c_{\ell j}{}^k - \mathfrak{D}_j{}^\ell c_{i\ell}{}^k = 0,$$

where $\mathfrak{D}_s{}^r = \widehat{T}_s{}^r - \mu\delta_s{}^r$ and the structure constants $c_{ij}{}^\ell$ are given by $[e_i, e_j] = c_{ij}{}^\ell e_\ell$. Note that $T_{rs} = T(e_r, e_s) = \sum_{\alpha=1}^4 \widehat{T}_\alpha{}^r \langle e_\alpha, e_s\rangle$ and therefore $\widehat{T}_s{}^r = T_{rs}$ if the basis $\{e_1, e_2, e_3, e_4\}$ is orthonormal.

We say that an algebraic T-soliton is *trivial* if $T = \kappa \langle \cdot, \cdot \rangle$ for some real constant κ (or equivalently, $\widehat{T}$ and $\mathfrak{D}$ are multiples of the identity). Note that trivial algebraic Ricci solitons correspond to Einstein spaces, while trivial algebraic Bach solitons are precisely the Bach-flat manifolds.

Remark 5.1. A Riemannian manifold (M, g) is a T-*soliton* if there is a vector field X on M so that

$$\mathcal{L}_X g + T = \lambda g$$

for some $\lambda \in \mathbb{R}$, and it is expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively. If the vector field X is a gradient of a real-valued function, $X = \frac{1}{2}\nabla f$, then the equation above becomes

$$\operatorname{Hes}(f) + T = \lambda g.$$

In such a case we say that (M, g, T, f) is a *gradient T-soliton* and refer to f as the potential function of the T-soliton. Given a geometric evolution equation

$$\frac{\partial}{\partial t} g_t = T_{g_t}$$

associated to an isometrically invariant symmetric $(0, 2)$-tensor field T, a solution that evolves by scaling and diffeomorphisms is said to be a *self-similar solution*. These solutions are of the form

$$g_t = \sigma(t)\psi_t^* g,$$

where ψ_t is a one-parameter family of diffeomorphisms of M. Any self-similar solution gives rise to a T-soliton and the converse is true if the tensor field T is homogeneous of degree d, i.e., $\overline{T} = \kappa^d T$ for any homothetic transformation $\overline{g} = \kappa g$ (see [133]). Since the Ricci and Bach tensors are homothetically invariant, self-similar solutions are equivalent to solitons for both the Ricci and Bach flows.

If a simply connected Riemannian Lie group $(G, \langle \cdot, \cdot \rangle)$ is an algebraic T-soliton, then it is a T-soliton (see [133]). Indeed, let $\{\varphi_t : G \to G\}$ be the one-parameter group of automorphisms of G determined by

$$d(\varphi_t)_e = \operatorname{Exp}(\tfrac{t}{2}\mathfrak{D}),$$

where $\mathfrak{D} = \widehat{T} - \mu \operatorname{Id}$ is the derivation of the Lie algebra determining the algebraic T-soliton. Define a vector field X on G as the infinitesimal generator of $\{\varphi_t\}$, i.e., $X(p) = \frac{d}{dt}\big|_{t=0} \varphi_t(p)$ for any $p \in G$. Then

$$\begin{aligned}
(\mathcal{L}_X \langle \cdot, \cdot \rangle)(e_i, e_j) &= \tfrac{d}{dt}(\varphi_t^* \langle \cdot, \cdot \rangle)(e_i, e_j) = \tfrac{1}{2}\left\{\langle \mathfrak{D} e_i, e_j \rangle + \langle e_i, \mathfrak{D} e_j \rangle\right\} \\
&= \tfrac{1}{2}\left\{\langle \widehat{T} e_i, e_j \rangle + \langle e_i, \widehat{T} e_j \rangle\right\} - \mu \langle e_i, e_j \rangle \\
&= (T - \mu \langle \cdot, \cdot \rangle)(e_i, e_j),
\end{aligned}$$

from where it follows that $\mathcal{L}_X \langle \cdot, \cdot \rangle - T = -\mu \langle \cdot, \cdot \rangle$. Replacing X by $-X$ one gets the T-soliton equation above with $\lambda = \mu$.

Remark 5.2. If a Riemannian manifold (M, g) admits two distinct T-solitons, i.e., vector fields X_i so that $\mathcal{L}_{X_i}g+T = \lambda_i g$, $i = 1, 2$, then one has that $\xi = X_1-X_2$ satisfies $\mathcal{L}_\xi g = (\lambda_1-\lambda_2)g$ and so ξ is a homothetic vector field. What is more, if we assume that (M, g) is homogeneous, then either the manifold is flat or ξ is a Killing vector field. This shows that, if they exist, T-solitons are unique (up to Killing vector fields) in the homogeneous category.

Remark 5.3. Petersen and Wylie showed in [122] that if (M, g) is a homogeneous manifold and $\widetilde{T}$ is a divergence-free, symmetric, and isometrically invariant tensor field of type $(0, 2)$, then any non-constant function satisfying $\mathrm{Hes}(f) = \widetilde{T}$ induces a splitting of the manifold as a product $N \times \mathbb{R}^k$, and f is a function on the Euclidean factor.

For a given T-flow, considering the tensor field $\widetilde{T} = \lambda g - T$, it is a direct consequence of the previous result that any homogeneous gradient T-soliton (such that $\mathrm{Hes}(f) + T = \lambda g$) splits as a product if the tensor field T is divergence-free. The result above is no longer true in the T-soliton is not a gradient, which is the case of the solitons constructed in this chapter.

5.1.1 Four-dimensional algebraic Ricci solitons

First of all note that the Ricci tensor is invariant by homotheties. Equivalently, the Ricci operator, $\mathrm{Ric}_i{}^k = \rho_{ij}g^{jk}$, is homogeneous of degree $d = -1$. Therefore, there is a one-to-one correspondence between self-similar solutions of the Ricci flow and Ricci solitons. Algebraic Ricci solitons correspond to self-similar solutions which are invariant up to homotheties and automorphisms of the group.

Four-dimensional homogeneous Einstein manifolds were described by Jensen, who showed in [88] that they are necessarily symmetric and, in the simply connected case, homothetic to a real space form ($\mathbb{S}^4$, $\mathbb{R}^4$ or $\mathbb{H}^4$), to a complex space form ($\mathbb{C}P^2$ or $\mathbb{C}H^2$), or to a product $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{H}^2 \times \mathbb{H}^2$.

Recall that, even though any four-dimensional homogeneous expanding Ricci soliton is homothetic to an algebraic Ricci soliton, algebraic Ricci solitons may be shrinking, in which case they are rigid gradient Ricci solitons (see Theorem 5.4-*(v)*). Four-dimensional algebraic Ricci solitons were described by Lauret in [98].

One of our aims in this chapter is to understand the Riemannian geometry of algebraic Ricci solitons. To do so, we work modulo homotheties also considering non-isomorphic homotheties in order to give a shorter description. Lauret's classification follows as a special situation of the analysis carried out in this chapter.

Theorem 5.4. *A four-dimensional simply connected Riemannian Lie group is an algebraic Ricci soliton if and only if it is Einstein or either of the following conditions holds.*

(a) The Riemannian Lie group is not locally symmetric and it is homothetic to one of the following:

(i) The semi-direct extension $\mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_2, e_4] = e_2 + e_3, \quad [e_3, e_4] = -e_2 - e_3.$$

(ii) The semi-direct extension $\mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_1, e_4] = \tfrac{\sqrt{2}}{2}(e_2 + e_3), \quad [e_2, e_4] = -\tfrac{\sqrt{2}}{2}e_1 + e_2, \quad [e_3, e_4] = -\tfrac{\sqrt{2}}{2}e_1 - e_3.$$

(iii) The semi-direct extension $\mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = f e_2, \quad [e_3, e_4] = p e_3,$$

where

$$(f, p) \in \{(f, p) \in \mathbb{R}^2 \colon -1 \le f \le p \le 1\} \setminus \{(-1, p) \in \mathbb{R}^2 \colon -1 \le p < 0\}$$

and $(f, p) \notin \{(0, 0), (0, 1), (1, 1)\}$*.*

(iv) The semi-direct extension $H^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = a e_1, \quad [e_2, e_4] = d e_2, \quad [e_3, e_4] = (a + d) e_3,$$

where $a \in \left[-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$*. For a fixed* a*, the parameter* d *is given by the only positive solution of* $4(a^2 + d^2 + ad) - 3 = 0$*.*

(b) The Riemannian Lie group is locally symmetric and it is homothetic to one of the following:

(v) The product Lie group $SU(2) \times \mathbb{R}$ *with the product metric determined by*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1,$$

which is homothetic to $\mathbb{S}^3 \times \mathbb{R}$*.*

(vi) The semi-direct extension $\mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_3, e_4] = e_3,$$

which is homothetic to the product $\mathbb{H}^2 \times \mathbb{R}^2$*.*

(vii) The semi-direct extension $\mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_2,$$

which is homothetic to the product $\mathbb{H}^3 \times \mathbb{R}$*.*

In all the cases above $\{e_1, e_2, e_3, e_4\}$ *denotes an orthonormal basis.*

Remark 5.5. It follows from the result above that a simply connected four-dimensional homogeneous Ricci soliton is either a rigid gradient Ricci soliton or homothetic to a metric *(i)–(iv)* in Theorem 5.4.

Observe that the assumption of simply connectedness is relevant. Indeed, the metric in *(i)* above is isomorphically homothetic to the left-invariant product metric on the product $H^3 \times \mathbb{R}$. It is clear that $H^3 \times \mathbb{R}$ admits discrete subgroups Γ so that the quotient nilmanifold $H^3 \times \mathbb{R}/\Gamma$ is a compact (but not simply connected) nilmanifold. Although the left-invariant metric descends to the quotient (and thus $H^3 \times \mathbb{R}$ and the corresponding nilmanifold are locally isometric), the Ricci soliton cannot pass to the quotient, since compact expanding Ricci solitons are necessarily Einstein.

Remark 5.6. All the algebraic Ricci solitons in the previous theorem are also algebraic Bach solitons. Moreover, while the algebraic Ricci solitons are expanding in all the cases but *(v)*, the algebraic Bach solitons may be shrinking, expanding or steady (the latter only in the Bach-flat situation). We analyse their structure case by case in what follows.

(i) A straightforward calculation shows that the Ricci and Bach operators in the orthonormal basis $\{e_k\}$ take the forms

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad \widehat{\mathfrak{B}} = -\tfrac{8}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

In this situation,

$$\mathfrak{D} = \mathrm{Ric} + 6\mathrm{Id} \quad \text{and} \quad \mathfrak{D} = \widehat{\mathfrak{B}} - \tfrac{88}{3}\mathrm{Id}$$

are derivations of the Lie algebra, from where it follows that $(G, \langle\cdot,\cdot\rangle)$ is an expanding algebraic Ricci soliton ($\mu = -6$) and a shrinking algebraic Bach soliton ($\mu = \frac{88}{3}$).

The metric in Theorem 5.4-*(i)* is isometric to the product metric on $H^3 \times \mathbb{R}$, which is a non-gradient Ricci soliton. Moreover, it follows from the work in [81] that the corresponding Bach soliton is not a gradient either.

(ii) The Ricci and Bach operators of the left-invariant metric in Theorem 5.4-*(ii)* are given by

$$\mathrm{Ric} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{2} \end{pmatrix}, \quad \widehat{\mathfrak{B}} = \tfrac{7}{6} \begin{pmatrix} -1 & \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 & 0 \\ -\sqrt{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Therefore,

$$\mathfrak{D} = \mathrm{Ric} + 3\mathrm{Id} \quad \text{and} \quad \mathfrak{D} = \widehat{\mathfrak{B}} - \tfrac{35}{6}\mathrm{Id}$$

are derivations of the Lie algebra, which shows that $(G, \langle\cdot,\cdot\rangle)$ is an expanding algebraic Ricci soliton and a shrinking algebraic Bach soliton. Although the Ricci operator has a zero eigenvalue with associated eigenvector given by

$$e_2 + e_3,$$

the corresponding eigenspace is not parallel and thus the associated Ricci and Bach solitons are not gradient solitons.

(iii) The Ricci operators of the metrics in Theorem 5.4-*(iii)* are given by

$$\mathrm{Ric} = -\operatorname{diag}\left[f+p+1, f(f+p+1), p(f+p+1), f^2+p^2+1\right],$$

from where it follows that $\mathfrak{D} = \mathrm{Ric} + (f^2+p^2+1)\mathrm{Id}$ are derivations of the Lie algebras determining expanding algebraic Ricci solitons.

Moreover, the Bach operators are given by $\widehat{\mathfrak{B}} = \frac{1}{6}\operatorname{diag}[\mathfrak{B}_{11}, \mathfrak{B}_{22}, \mathfrak{B}_{33}, 6\mu]$, where

$$\begin{aligned}
\mathfrak{B}_{11} &= -\left(f^2-2(p+1)f+(p-1)^2\right)\left(f^2+(p-1)f+(p-1)p-1\right),\\
\mathfrak{B}_{22} &= \left(f^2-2(p+1)f+(p-1)^2\right)\left(f^2+(p+1)f-(p+1)p-1\right),\\
\mathfrak{B}_{33} &= -\left(f^2-2(p+1)f+(p-1)^2\right)\left(f^2-(p-1)f-(p+1)p+1\right),\\
\mu &= \tfrac{1}{6}\left(f^2-2(p+1)f+(p-1)^2\right)\left(f^2-(p+1)f+(p-1)p+1\right).
\end{aligned}$$

Thus, $\mathfrak{D} = \widehat{B} - \mu\mathrm{Id}$ are derivations of the corresponding Lie algebras and hence they are algebraic Bach solitons, which may be expanding, steady or shrinking depending on the values of the parameters (f,p). They are steady algebraic Bach solitons if and only if $f = (\sqrt{p}-1)^2$, with $\frac{1}{4} \leq p < 1$, in which case the corresponding metrics are Bach-flat (see Case *(ii)* in Remark 5.7).

(iv) The Ricci and Bach operators of the left-invariant metrics in Theorem 5.4-*(iv)* are given by

$$\mathrm{Ric} = \operatorname{diag}\left[2(d^2-1), -2d(a+d)-\tfrac{1}{2}, -2ad-1, -\tfrac{3}{2}\right],$$

$$\widehat{\mathfrak{B}} = \tfrac{1}{24}\operatorname{diag}[\mathfrak{B}_{11}, -\mathfrak{B}_{11}, -(4ad-1)(8ad+1), (4ad-1)(8ad+1)],$$

where $\mathfrak{B}_{11} = -32(d^4-d^2-ad^3)-20ad-3$. In this situation,

$$\mathfrak{D} = \mathrm{Ric} + \tfrac{3}{2}\mathrm{Id}, \quad \mathfrak{D} = \widehat{\mathfrak{B}} - \tfrac{1}{24}(4ad-1)(8ad+1)\mathrm{Id}$$

are derivations of the Lie algebra, from where it follows that these metrics are expanding algebraic Ricci solitons and also algebraic Bach solitons, which may be expanding, steady or shrinking depending on the values of the parameters (a,d). They are steady algebraic Bach solitons if and only if

$$(4ad-1)(8ad+1) = 0,$$

in which case they are Bach-flat and correspond to Case *(i)* in Remark 5.7.

(v) The metric in Theorem 5.4-*(v)* is homothetic to the product metric on $\mathbb{S}^3 \times \mathbb{R}$. Therefore, it is locally conformally flat and thus Bach-flat, so it is a steady algebraic Bach soliton. Moreover, its associated Ricci operator is diagonal, $\mathrm{Ric} = \frac{1}{2}\operatorname{diag}[1,1,1,0]$, and $\mathfrak{D} = \mathrm{Ric} - \frac{1}{2}\mathrm{Id}$, which shows that it is a shrinking Ricci soliton that is a gradient.

(vi) The product $\mathbb{H}^2 \times \mathbb{R}^2$ is an expanding rigid gradient Ricci soliton which is also algebraic since the Ricci operator is diagonal $\mathrm{Ric} = \mathrm{diag}[0, 0, -1, -1]$ and $\mathfrak{D} = \mathrm{Ric} + \mathrm{Id}$. Moreover, it is a shrinking algebraic Bach soliton determined by $\mathfrak{D} = \widehat{\mathfrak{B}} - \frac{1}{6}\mathrm{Id}$, where the Bach operator is given by $\widehat{\mathfrak{B}} = \frac{1}{6}\,\mathrm{diag}[-1, -1, 1, 1]$. The Bach soliton associated to the algebraic Bach soliton coincides with the gradient Bach soliton given by Ho in [85] up to some Killing vector field.

(vii) The metric in Theorem 5.4-*(vii)* is homothetic to the product metric on $\mathbb{H}^3 \times \mathbb{R}$, thus being locally conformally flat. It is a steady algebraic Bach soliton and an expanding rigid gradient Ricci soliton with $\mathrm{Ric} = -2\,\mathrm{diag}[1, 1, 0, 1]$ and $\mathfrak{D} = \mathrm{Ric} + 2\mathrm{Id}$.

The proof of Theorem 5.4 is obtained as the result of the general study of the algebraic Ricci solitons on four-dimensional Lie groups.

5.1.2 Four-dimensional algebraic Bach solitons

The Bach tensor $\mathfrak{B}$ is conformally invariant in dimension four and thus it is also homothetically invariant. Besides, the Bach operator $\widehat{\mathfrak{B}}$ is homogeneous of degree $d = -1$ and thus there is a one-to-one correspondence between self-similar solutions to the Bach flow and Bach solitons.

Remark 5.7. Locally conformally flat manifolds are trivially Bach-flat. In the homogeneous situation Takagi showed that they are also symmetric [128], and thus they either correspond to real space forms, or are homothetic to a product $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{H}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{H}^2$.

In contrast with the Einstein and the locally conformally flat situations, there are non-symmetric Bach-flat homogeneous four-manifolds, which have already been described in [39] as Riemannian Lie groups that are homothetic to those given by

(i) The semi-direct product $H^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[e_1, e_2] = e_3, \qquad [e_1, e_4] = -\tfrac{1}{4}\sqrt{7-3\sqrt{5}}\, e_1,$$
$$[e_2, e_4] = \tfrac{1}{4}\sqrt{7+3\sqrt{5}}\, e_2, \qquad [e_3, e_4] = \tfrac{\sqrt{5}}{2\sqrt{2}}\, e_3.$$

(ii) The semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = (\sqrt{p} - 1)^2\, e_2, \quad [e_3, e_4] = p\, e_3, \qquad \tfrac{1}{4} \leq p < 1\,.$$

(iii) The semi-direct product $E(1, 1) \rtimes \mathbb{R}$ with Lie algebra given by

$$[e_1, e_3] = (2-\sqrt{3})\, e_2, \quad [e_1, e_4] = \sqrt{6-3\sqrt{3}}\, e_1,$$
$$[e_2, e_3] = e_1, \qquad [e_2, e_4] = \sqrt{6-3\sqrt{3}}\, e_2.$$

(iv) The semi-direct product $H^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = 2e_3.$$

Note that, while the left-invariant metrics (i) and (ii) above are algebraic Ricci solitons, the metrics corresponding to (iii)–(iv) are not isometric to any Ricci soliton. Moreover, the left-invariant metric determined in (iv) is anti-self-dual, and its Weyl conformal tensor is half-harmonic (see Theorem 6.1 in this thesis).

Proceeding as in Remark 5.3, Griffin [81] showed that homogeneous gradient Bach solitons either are Bach-flat or split as a product $N \times \mathbb{R}^k$, where the potential function depends only on the Euclidean factor, and (N, g_N) is a homogeneous manifold. Moreover, homogeneous steady gradient Bach solitons are necessarily Bach-flat. Gradient Bach solitons on products $N \times \mathbb{R}$ which are not Bach-flat reduce to the case $\mathbb{S}^3 \times \mathbb{R}$, where the metric on $\mathbb{S}^3$ is not the round metric, but a Berger one [81].

Moving from the gradient to the algebraic situation, the next result shows that algebraic Bach solitons are steady if and only if they are Bach-flat and, moreover, all but two of them also are algebraic Ricci solitons.

Theorem 5.8. *A four-dimensional simply connected Riemannian Lie group is an algebraic Bach soliton if and only if it is Bach-flat, an algebraic Ricci soliton or homothetic to one of the following Lie groups.*

(i) The product Lie group $SU(2) \times \mathbb{R}$ with the product left-invariant metric determined by

$$[e_1, e_2] = 4e_3, \quad [e_1, e_3] = -4e_2, \quad [e_2, e_3] = e_1.$$

(ii) The semi-direct extension $H^3 \rtimes \mathbb{R}$ with the left-invariant metric determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1, \quad [e_2, e_4] = \tfrac{1}{a}e_2, \quad [e_3, e_4] = \tfrac{a^2+1}{a}\, e_3,$$

for $a \in (0, 1)$.

Here $\{e_1, e_2, e_3, e_4\}$ denotes an orthonormal basis of the corresponding Lie algebra.

Remark 5.9. The left-invariant metric in Theorem 5.8-*(i)* is the product metric of the Berger sphere determined by the left-invariant metric on $SU(2)$ given by

$$[e_1, e_2] = 4e_3, \quad [e_1, e_3] = -4e_2, \quad [e_2, e_3] = e_1,$$

and the real line. Moreover, it follows from [81, 84] that the induced Bach soliton is a gradient soliton. The Berger sphere above is the only non-symmetric homogeneous three-dimensional manifold which is critical for the curvature quadratic functional

$$\mathcal{F}_{-1/3}(g) = \int_M \left\{ \|\rho\|^2 - \tfrac{1}{3}\tau^2 \right\} dvol_g.$$

The associated Bach operator is diagonal of the form $\widehat{\mathfrak{B}} = \frac{1}{2}\operatorname{diag}[1, 1, 1, -3]$ and the algebraic Bach soliton is shrinking, with $\mu = \frac{1}{2}$.

The left-invariant metrics in *(ii)* have diagonal Bach operator of the form

$$\widehat{\mathfrak{B}} = \tfrac{1}{2a^2}\left[-3(a^4-1), 3(a^4-1), (a^2-1)^2, -(a^2-1)^2\right]$$

and $\mathfrak{D} = \widehat{\mathfrak{B}} + \frac{(a^2-1)^2}{2a^2}\mathrm{Id}$ is a derivation determining an expanding algebraic Bach soliton. Moreover, the corresponding Ricci operator is diagonal,

$$\mathrm{Ric} = -\operatorname{diag}\left[2a^2 + \tfrac{5}{2}, \tfrac{5a^2+4}{2a^2}, \tfrac{4a^4+7a^2+4}{2a^2}, \tfrac{2(a^4+a^2+1)}{a^2}\right],$$

and therefore its eigenvalues are always non-zero. This shows that the metrics cannot split as products $N \times \mathbb{R}^k$ and thus the corresponding solitons are not gradient.

Remark 5.10. Gradient Bach solitons on products $N^2 \times \mathbb{R}^2$ were considered by Ho in [85], where the existence of non-trivial gradient Bach solitons on $\mathbb{S}^2 \times \mathbb{R}^2$ and $\mathbb{H}^2 \times \mathbb{R}^2$ was shown. While the soliton on $\mathbb{S}^2 \times \mathbb{R}^2$ cannot be algebraic, the gradient Bach soliton on $\mathbb{H}^2 \times \mathbb{R}^2$ corresponds to the one in Theorem 5.4-*(vi)*.

Remark 5.11. The algebraic Ricci solitons in Theorem 5.4 are critical metrics for some curvature quadratic functional

$$\mathcal{F}_t : g \mapsto \mathcal{F}_t(g) = \int_M \{\|\rho\|^2 + t\tau^2\} dvol_g$$

with zero energy, i.e., $\|\rho\|^2 + t\tau^2 = 0$ (see [23]). This is no longer true for algebraic Bach solitons corresponding to Cases *(i)* and *(ii)* in Theorem 5.8, which are not critical for any curvature quadratic functional.

Remark 5.12. The two-loop renormalization group flow – RG2 flow for short – is a perturbation of the Ricci flow $\partial_t g_t = -2\rho_t$, which is mathematically described by

$$\partial_t g_t = -2RG[\Upsilon]_t,$$

where $RG[\Upsilon]$ is the symmetric $(0,2)$-tensor field given by

$$RG[\Upsilon] = \rho + \tfrac{\Upsilon}{4}\check{R},$$

where Υ denotes the coupling constant, and $\check{R}$ is the symmetric $(0,2)$-tensor field given by

$$\check{R}_{ij} = R_{iabc}R_j{}^{abc}.$$

Let G be a Lie group with left-invariant metric $\langle\cdot,\cdot\rangle$ and let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ denote the corresponding Lie algebra. An RG2 *algebraic soliton* corresponds to a derivation of the Lie algebra $\mathfrak{g}$ given by

$$\mathfrak{D} = \widehat{RG}[\Upsilon] - \mu\,\mathrm{Id},$$

where $\widehat{RG}[\Upsilon]$ is the $(1,1)$-tensor field metrically equivalent to $RG[\Upsilon]$ and $\mu \in \mathbb{R}$. RG2 algebraic solitons give rise to RG2 solitons as in the Ricci flow and the Bach flow cases (see [97, 133]).

The metric in Theorem 5.8-*(i)* is an algebraic soliton for the RG2 flow. Indeed,

$$\mathfrak{D} = \widehat{RG}\left[\tfrac{-4}{7}\right] - \tfrac{13}{28}\,\mathrm{Id}$$

is a derivation of the Lie algebra, where $RG[\frac{-4}{7}] = \rho - \frac{1}{7}\check{R}$.

The metrics in Theorem 5.8-*(ii)* are algebraic RG2 solitons as well. Indeed,

$$\mathfrak{D} = \widehat{RG}[\Upsilon] - \mu\,\mathrm{Id}$$

is a derivation of the Lie algebra for $\mu = -\frac{20(a^4+a^2+1)}{4a^4+15a^2+4}$ and $\Upsilon = \frac{8a^2}{4a^4+15a^2+4}$.

The proof of Theorem 5.8 is obtained as the result of the general study of the algebraic Bach solitons on four-dimensional Lie groups.

Remark 5.13. Four-dimensional homogeneous expanding Ricci solitons are homothetic to algebraic solitons [9]. It is an open question whether the analogous situation holds true for homogeneous Bach solitons.

5.2 Algebraic solitons on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

The left-invariant metrics on the product Lie groups $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ have already been described in Section 1.4.2. We recall that in this case there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$ and $\mathfrak{su}(2) \times \mathbb{R}$ such that

$$\begin{aligned} &[e_1,e_2] = \lambda_3 e_3, \qquad [e_1,e_3] = -\lambda_2 e_2, \qquad [e_2,e_3] = \lambda_1 e_1, \\ &[e_1,e_4] = k_3\lambda_2 e_2 - k_2\lambda_3 e_3, \qquad [e_2,e_4] = k_1\lambda_3 e_3 - k_3\lambda_1 e_1, \\ &[e_3,e_4] = k_2\lambda_1 e_1 - k_1\lambda_2 e_2, \end{aligned} \tag{5.1}$$

where $\lambda_1\lambda_2\lambda_3 \neq 0$. The associated Lie group corresponds to $SU(2) \times \mathbb{R}$ if λ_1, λ_2 and λ_3 have the same sign, and to $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ otherwise.

Remark 5.14. $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ is never locally symmetric while $SU(2) \times \mathbb{R}$ is locally symmetric if and only if $\lambda_1 = \lambda_2 = \lambda_3$, in which case it is homothetic to $\mathbb{S}^3 \times \mathbb{R}$.

5.2.1 Algebraic T-solitons on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

The existence of algebraic T-solitons on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ is a very restrictive condition which essentially reduces to the fact that $\widehat{T}$ decomposes according to the product and the restriction of $\widehat{T}$ to the semi-simple Lie algebra is a multiple of the identity.

Theorem 5.15. $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ *and* $SU(2) \times \mathbb{R}$ *are non-trivial algebraic* T*-solitons with soliton constant* μ *if and only if*

$$\widehat{T} = \mathrm{diag}[\mu,\mu,\mu,T_{44}], \quad T_{44} \neq \mu,$$

and the left-invariant metrics correspond to the product metric, i.e.,

$$k_1 = k_2 = k_3 = 0.$$

Proof. First of all, note that the vanishing of the divergence of T leads to

$$\begin{aligned}
&(\lambda_2-\lambda_3)T_{23}-k_3\lambda_2T_{24}+k_2\lambda_3T_{34}=0,\\
&(\lambda_1-\lambda_3)T_{13}-k_3\lambda_1T_{14}+k_1\lambda_3T_{34}=0,\\
&(\lambda_1-\lambda_2)T_{12}-k_2\lambda_1T_{14}+k_1\lambda_2T_{24}=0,\\
&k_3(\lambda_1-\lambda_2)T_{12}-k_2(\lambda_1-\lambda_3)T_{13}+k_1(\lambda_2-\lambda_3)T_{23}=0.
\end{aligned}\tag{5.2}$$

To determine the conditions for $\mathfrak{D}=\widehat{T}-\mu\,\mathrm{Id}$ to be a derivation we will need to solve a system of twenty four (up to duplicities) polynomial equations on the soliton constant μ, the structure constants (5.1) and the components T_{ij}, given by the vanishing of the polynomials

$$\begin{aligned}
\mathfrak{P}_{211}&=-(\lambda_1+\lambda_3)T_{13}+k_3\lambda_1T_{14},\\
\mathfrak{P}_{212}&=-(\lambda_2+\lambda_3)T_{23}+k_3\lambda_2T_{24},\\
\mathfrak{P}_{213}&=\lambda_3(T_{11}+T_{22}-T_{33}-k_1T_{14}-k_2T_{24}-\mu),\\
\mathfrak{P}_{214}&=-\lambda_3T_{34},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{311}&=(\lambda_1+\lambda_2)T_{12}-k_2\lambda_1T_{14},\\
\mathfrak{P}_{312}&=-\lambda_2(T_{11}-T_{22}+T_{33}-k_1T_{14}-k_3T_{34}-\mu),\\
\mathfrak{P}_{313}&=(\lambda_2+\lambda_3)T_{23}-k_2\lambda_3T_{34},\\
\mathfrak{P}_{314}&=\lambda_2T_{24},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{321}&=-\lambda_1(T_{11}-T_{22}-T_{33}+k_2T_{24}+k_3T_{34}+\mu),\\
\mathfrak{P}_{322}&=-(\lambda_1+\lambda_2)T_{12}+k_1\lambda_2T_{24},\\
\mathfrak{P}_{323}&=-(\lambda_1+\lambda_3)T_{13}+k_1\lambda_3T_{34},\\
\mathfrak{P}_{324}&=-\lambda_1T_{14},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{411}&=-k_3(\lambda_1+\lambda_2)T_{12}+k_2(\lambda_1+\lambda_3)T_{13},\\
\mathfrak{P}_{412}&=k_3\lambda_2(T_{11}-T_{22}+T_{44}-\mu)-k_1\lambda_2T_{13}+k_2\lambda_3T_{23}-\lambda_2T_{34},\\
\mathfrak{P}_{413}&=-k_2\lambda_3(T_{11}-T_{33}+T_{44}-\mu)+k_1\lambda_3T_{12}-k_3\lambda_2T_{23}+\lambda_3T_{24},\\
\mathfrak{P}_{414}&=-k_3\lambda_2T_{24}+k_2\lambda_3T_{34},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{421}&=k_3\lambda_1(T_{11}-T_{22}-T_{44}+\mu)-k_1\lambda_3T_{13}+k_2\lambda_1T_{23}+\lambda_1T_{34},\\
\mathfrak{P}_{422}&=k_3(\lambda_1+\lambda_2)T_{12}-k_1(\lambda_2+\lambda_3)T_{23},\\
\mathfrak{P}_{423}&=k_1\lambda_3(T_{22}-T_{33}+T_{44}-\mu)-k_2\lambda_3T_{12}+k_3\lambda_1T_{13}-\lambda_3T_{14},\\
\mathfrak{P}_{424}&=k_3\lambda_1T_{14}-k_1\lambda_3T_{34},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{431}&=-k_2\lambda_1(T_{11}-T_{33}-T_{44}+\mu)+k_1\lambda_2T_{12}-k_3\lambda_1T_{23}-\lambda_1T_{24},\\
\mathfrak{P}_{432}&=k_1\lambda_2(T_{22}-T_{33}-T_{44}+\mu)-k_2\lambda_1T_{12}+k_3\lambda_2T_{13}+\lambda_2T_{14},\\
\mathfrak{P}_{433}&=-k_2(\lambda_1+\lambda_3)T_{13}+k_1(\lambda_2+\lambda_3)T_{23},\\
\mathfrak{P}_{434}&=-k_2\lambda_1T_{14}+k_1\lambda_2T_{24}.
\end{aligned}$$

We start by considering the polynomials

$$\mathfrak{P}_{324} = -\lambda_1 T_{14}, \quad \mathfrak{P}_{314} = \lambda_2 T_{24}, \quad \mathfrak{P}_{214} = -\lambda_3 T_{34},$$

$$\mathfrak{P}_{321} = -\lambda_1(T_{11} - T_{22} - T_{33} + k_2 T_{24} + k_3 T_{34} + \mu),$$

$$\mathfrak{P}_{312} = -\lambda_2(T_{11} - T_{22} + T_{33} - k_1 T_{14} - k_3 T_{34} - \mu),$$

$$\mathfrak{P}_{213} = \lambda_3(T_{11} + T_{22} - T_{33} - k_1 T_{14} - k_2 T_{24} - \mu),$$

from where it follows that

$$T_{14} = T_{24} = T_{34} = 0, \quad T_{11} = T_{22} = T_{33} = \mu. \tag{5.3}$$

Under the conditions above we obtain

$$\mathfrak{P}_{311} = (\lambda_1 + \lambda_2)T_{12}, \quad \mathfrak{P}_{211} = -(\lambda_1 + \lambda_3)T_{13}, \quad \mathfrak{P}_{212} = -(\lambda_2 + \lambda_3)T_{23},$$

and now the conditions necessary for the vanishing of the divergence of T given by Equation (5.2) lead to

$$(\lambda_1 - \lambda_2)T_{12} = 0, \quad (\lambda_1 - \lambda_3)T_{13} = 0, \quad (\lambda_2 - \lambda_3)T_{23} = 0.$$

Since $\lambda_1\lambda_2\lambda_3 \neq 0$, these equations lead to

$$T_{12} = T_{13} = T_{23} = 0. \tag{5.4}$$

Equations (5.3) and (5.4) imply that $\widehat{T} = \operatorname{diag}[\mu, \mu, \mu, T_{44}]$. Moreover, T is divergence-free (see Equation (5.2)) and the system $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$k_1(T_{44} - \mu) = 0, \quad k_2(T_{44} - \mu) = 0, \quad k_3(T_{44} - \mu) = 0,$$

which complete the proof. □

5.2.2 Algebraic Ricci solitons on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

A straightforward calculation shows that the components ρ_{ij} of the Ricci tensor of both $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ are determined by

$$\begin{aligned}
2\rho_{11} &= (\lambda_1^2 - \lambda_3^2)k_2^2 + (\lambda_1^2 - \lambda_2^2)k_3^2 + (\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3),\\
2\rho_{12} &= (\lambda_3^2 - \lambda_1\lambda_2)k_1 k_2,\\
2\rho_{13} &= (\lambda_2^2 - \lambda_1\lambda_3)k_1 k_3,\\
2\rho_{14} &= (\lambda_2 - \lambda_3)^2 k_1,\\
2\rho_{22} &= (\lambda_2^2 - \lambda_3^2)k_1^2 - (\lambda_1^2 - \lambda_2^2)k_3^2 - (\lambda_1 - \lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3),\\
2\rho_{23} &= (\lambda_1^2 - \lambda_2\lambda_3)k_2 k_3,\\
2\rho_{24} &= (\lambda_1 - \lambda_3)^2 k_2,\\
2\rho_{33} &= (\lambda_3^2 - \lambda_2^2)k_1^2 - (\lambda_1^2 - \lambda_3^2)k_2^2 - (\lambda_1 - \lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3),
\end{aligned}$$

$$2\rho_{34} = (\lambda_1 - \lambda_2)^2 k_3,$$
$$2\rho_{44} = -(\lambda_2 - \lambda_3)^2 k_1^2 - (\lambda_1 - \lambda_3)^2 k_2^2 - (\lambda_1 - \lambda_2)^2 k_3^2.$$

In the locally symmetric case, which can only occur on $SU(2) \times \mathbb{R}$, since

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \neq 0$$

(see Remark 5.14) a direct calculation shows that the Ricci operator is given by

$$\mathrm{Ric} = \tfrac{1}{2}\lambda^2 \,\mathrm{diag}[1, 1, 1, 0].$$

Thus, by Theorem 5.15, we obtain a non-trivial algebraic Ricci soliton for $k_1 = k_2 = k_3 = 0$ and $\mu = \frac{1}{2}\lambda^2$. Note that we can take $\lambda = 1$ remaining in the same homothetic class and hence the left-invariant metric is isomorphically homothetic to the one given by

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1,$$

which is non-Einstein and corresponds to Assertion *(v)* in Theorem 5.4.

Next we analyse non-trivial algebraic Ricci solitons in the non-symmetric case. Note that λ_1, λ_2, λ_3 must be not equal. Also, according to Theorem 5.15, the possible non-zero components of the Ricci tensor must reduce to ρ_{ii}, satisfying

$$\rho_{11} = \rho_{22} = \rho_{33} = \mu,$$

and moreover $k_1 = k_2 = k_3 = 0$. Setting $k_1 = k_2 = k_3 = 0$ the non-zero components of the Ricci tensor are given by

$$\begin{aligned}
\rho_{11} &= \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3),\\
\rho_{22} &= -\tfrac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3),\\
\rho_{33} &= -\tfrac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3).
\end{aligned}$$

Since λ_1, λ_2 and λ_3 are not equal, $\rho_{11} = \rho_{22} = \rho_{33}$ never holds. Hence, there does not exist any non-symmetric algebraic Ricci soliton on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$.

5.2.3 Algebraic Bach solitons on $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

A long but straightforward calculation shows that the components $\mathfrak{B}_{ij}$ of the Bach tensors of $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ are determined by the following polynomials on the structure constants.

$$\begin{aligned}
24\mathfrak{B}_{11} = &-4(\lambda_2 - \lambda_3)^2(\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3)k_1^4\\
&- 4(5\lambda_1^4 - 3\lambda_3^4 - \lambda_1\lambda_3(3\lambda_1^2 - \lambda_3^2))k_2^4\\
&- 4(5\lambda_1^4 - 3\lambda_2^4 - \lambda_1\lambda_2(3\lambda_1^2 - \lambda_2^2))k_3^4\\
&+ (8\lambda_3^4 + 4\lambda_3^3(\lambda_1 - \lambda_2) - 24\lambda_1^2\lambda_2^2 + 3\lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 + 2\lambda_1\lambda_2\lambda_3(6\lambda_1 + 2\lambda_2 - \lambda_3))k_1^2k_2^2
\end{aligned}$$

$$
\begin{aligned}
&+ (8\lambda_2^4 + 4\lambda_2^3(\lambda_1 - \lambda_3) + 3\lambda_1^2\lambda_2^2 - 24\lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 + 2\lambda_1\lambda_2\lambda_3(6\lambda_1 - \lambda_2 + 2\lambda_3))k_1^2k_3^2 \\
&- (40\lambda_1^4 - 12\lambda_1^3(\lambda_2 + \lambda_3) - \lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - 24\lambda_2^2\lambda_3^2 + 2\lambda_1\lambda_2\lambda_3(\lambda_1 + 2\lambda_2 + 2\lambda_3))k_2^2k_3^2 \\
&+ (\lambda_2 - \lambda_3)^2(3\lambda_1^2 + 8\lambda_2^2 + 8\lambda_3^2 + 4\lambda_1\lambda_2 + 4\lambda_1\lambda_3 + 8\lambda_2\lambda_3)k_1^2 \\
&- (\lambda_1 - \lambda_3)(40\lambda_1^3 + 24\lambda_3^3 - 4\lambda_1^2(3\lambda_2 - 4\lambda_3) - \lambda_2^2(\lambda_1 + 3\lambda_3) \\
&\qquad\qquad + 4\lambda_3^2(4\lambda_1 - 3\lambda_2) - 8\lambda_1\lambda_2\lambda_3)k_2^2 \\
&- (\lambda_1 - \lambda_2)(40\lambda_1^3 + 24\lambda_2^3 + 4\lambda_1^2(4\lambda_2 - 3\lambda_3) + 4\lambda_2^2(4\lambda_1 - 3\lambda_3) \\
&\qquad\qquad - \lambda_3^2(\lambda_1 + 3\lambda_2) - 8\lambda_1\lambda_2\lambda_3)k_3^2 \\
&- 4(5\lambda_1^4 - 3\lambda_2^4 - 3\lambda_3^4 - 3\lambda_1^3(\lambda_2 + \lambda_3) + \lambda_2^3(\lambda_1 + 3\lambda_3) + \lambda_3^3(\lambda_1 + 3\lambda_2) \\
&\qquad\qquad + \lambda_1\lambda_2\lambda_3(\lambda_1 - \lambda_2 - \lambda_3)),
\end{aligned}
$$

$$
\begin{aligned}
12\mathfrak{B}_{12} = -k_1k_2\big(&(8\lambda_3^4 - 8\lambda_1\lambda_2^3 - 4\lambda_2\lambda_3^3 - \lambda_2^2\lambda_3^2 + \lambda_1\lambda_2\lambda_3(4\lambda_2 + \lambda_3))k_1^2 \\
&+ (8\lambda_3^4 - 8\lambda_1^3\lambda_2 - 4\lambda_1\lambda_3^3 - \lambda_1^2\lambda_3^2 + \lambda_1\lambda_2\lambda_3(4\lambda_1 + \lambda_3))k_2^2 \\
&+ (\lambda_3^2 - \lambda_1\lambda_2)(8\lambda_1^2 + 8\lambda_2^2 + 5\lambda_1\lambda_2)k_3^2 \\
&+ 8\lambda_3^4 - 8\lambda_1^3\lambda_2 - 8\lambda_1\lambda_2^3 - 4\lambda_3^3(\lambda_1 + \lambda_2) - 5\lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 \\
&+\lambda_1\lambda_2\lambda_3(10\lambda_1 + 10\lambda_2 + 3\lambda_3)\big),
\end{aligned}
$$

$$
\begin{aligned}
12\mathfrak{B}_{13} = -k_1k_3\big(&(8\lambda_2^4 - 4\lambda_2^3\lambda_3 - 8\lambda_1\lambda_3^3 - \lambda_2^2\lambda_3^2 + \lambda_1\lambda_2\lambda_3(\lambda_2 + 4\lambda_3))k_1^2 \\
&+ (\lambda_2^2 - \lambda_1\lambda_3)(8\lambda_1^2 + 8\lambda_3^2 + 5\lambda_1\lambda_3)k_2^2 \\
&+ (8\lambda_2^4 - 8\lambda_1^3\lambda_3 - 4\lambda_1\lambda_2^3 - \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2\lambda_3(4\lambda_1 + \lambda_2))k_3^2 \\
&+ 8\lambda_2^4 - 8\lambda_1^3\lambda_3 - 4\lambda_2^3(\lambda_1 + \lambda_3) - 8\lambda_1\lambda_3^3 - \lambda_1^2\lambda_2^2 - 5\lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 \\
&+\lambda_1\lambda_2\lambda_3(10\lambda_1 + 3\lambda_2 + 10\lambda_3)\big),
\end{aligned}
$$

$$
\begin{aligned}
12\mathfrak{B}_{14} = -k_1\big(&8(\lambda_2 - \lambda_3)^2(\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3)k_1^2 \\
&+ (8\lambda_3^4 - 4\lambda_3^3(\lambda_1 + \lambda_2) + 8\lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3(2\lambda_1 + 2\lambda_2 - \lambda_3))k_2^2 \\
&+ (8\lambda_2^4 - 4\lambda_2^3(\lambda_1 + \lambda_3) - \lambda_1^2\lambda_2^2 + 8\lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3(2\lambda_1 - \lambda_2 + 2\lambda_3))k_3^2 \\
&-(\lambda_2 - \lambda_3)^2(\lambda_1^2 - 8\lambda_2^2 - 8\lambda_3^2 + 4\lambda_1\lambda_2 + 4\lambda_1\lambda_3 - 8\lambda_2\lambda_3)\big),
\end{aligned}
$$

$$
\begin{aligned}
24\mathfrak{B}_{22} = &-4(5\lambda_2^4 - 3\lambda_3^4 - \lambda_2\lambda_3(3\lambda_2^2 - \lambda_3^2))k_1^4 \\
&- 4(\lambda_1 - \lambda_3)^2(\lambda_1^2 + \lambda_3^2 + \lambda_1\lambda_3)k_2^4 \\
&+ 4(3\lambda_1^4 - 5\lambda_2^4 - \lambda_1\lambda_2(\lambda_1^2 - 3\lambda_2^2))k_3^4 \\
&+ (8\lambda_3^4 - 4\lambda_3^3(\lambda_1 - \lambda_2) - 24\lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 + 3\lambda_2^2\lambda_3^2 + 2\lambda_1\lambda_2\lambda_3(2\lambda_1 + 6\lambda_2 - \lambda_3))k_1^2k_2^2 \\
&+ (-40\lambda_2^4 + 12\lambda_2^3(\lambda_1 + \lambda_3) + \lambda_1^2\lambda_2^2 + 24\lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3(2\lambda_1 + \lambda_2 + 2\lambda_3))k_1^2k_3^2 \\
&+ (8\lambda_1^4 + 4\lambda_1^3(\lambda_2 - \lambda_3) + 3\lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - 24\lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3(\lambda_1 - 6\lambda_2 - 2\lambda_3))k_2^2k_3^2 \\
&- (\lambda_2 - \lambda_3)(40\lambda_2^3 + 24\lambda_3^3 - \lambda_1^2(\lambda_2 + 3\lambda_3) - 4\lambda_2^2(3\lambda_1 - 4\lambda_3) \\
&\qquad\qquad - 4\lambda_3^2(3\lambda_1 - 4\lambda_2) - 8\lambda_1\lambda_2\lambda_3)k_1^2 \\
&+ (\lambda_1 - \lambda_3)^2(8\lambda_1^2 + 3\lambda_2^2 + 8\lambda_3^2 + 4\lambda_1\lambda_2 + 8\lambda_1\lambda_3 + 4\lambda_2\lambda_3)k_2^2 \\
&+ (\lambda_1 - \lambda_2)(24\lambda_1^3 + 40\lambda_2^3 + 4\lambda_1^2(4\lambda_2 - 3\lambda_3) + 4\lambda_2^2(4\lambda_1 - 3\lambda_3) \\
&\qquad\qquad - \lambda_3^2(3\lambda_1 + \lambda_2) - 8\lambda_1\lambda_2\lambda_3)k_3^2 \\
&+ 4(3\lambda_1^4 - 5\lambda_2^4 + 3\lambda_3^4 - \lambda_1^3(\lambda_2 + 3\lambda_3) + 3\lambda_2^3(\lambda_1 + \lambda_3) - \lambda_3^3(3\lambda_1 + \lambda_2) \\
&\qquad\qquad + \lambda_1\lambda_2\lambda_3(\lambda_1 - \lambda_2 + \lambda_3)),
\end{aligned}
$$

$$\begin{aligned}
12\mathfrak{B}_{23} = -k_2k_3\big(&(\lambda_1^2-\lambda_2\lambda_3)(8\lambda_2^2+8\lambda_3^2+5\lambda_2\lambda_3)k_1^2\\
&+(8\lambda_1^4-4\lambda_1^3\lambda_3-8\lambda_2\lambda_3^3-\lambda_1^2\lambda_3^2+\lambda_1\lambda_2\lambda_3(\lambda_1+4\lambda_3))k_2^2\\
&+(8\lambda_1^4-4\lambda_1^3\lambda_2-8\lambda_2^3\lambda_3-\lambda_1^2\lambda_2^2+\lambda_1\lambda_2\lambda_3(\lambda_1+4\lambda_2))k_3^2\\
&+8\lambda_1^4-4\lambda_1^3(\lambda_2+\lambda_3)-8\lambda_2^3\lambda_3-8\lambda_2\lambda_3^3-\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2-5\lambda_2^2\lambda_3^2\\
&+\lambda_1\lambda_2\lambda_3(3\lambda_1+10\lambda_2+10\lambda_3)\big),
\end{aligned}$$

$$\begin{aligned}
12\mathfrak{B}_{24} = -k_2\big(&(8\lambda_3^4-4\lambda_3^3(\lambda_1+\lambda_2)+8\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2-\lambda_2^2\lambda_3^2\\
&\qquad-2\lambda_1\lambda_2\lambda_3(2\lambda_1+2\lambda_2-\lambda_3))k_1^2\\
&+8(\lambda_1-\lambda_3)^2(\lambda_1^2+\lambda_3^2+\lambda_1\lambda_3)k_2^2\\
&+(8\lambda_1^4-4\lambda_1^3(\lambda_2+\lambda_3)-\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2+8\lambda_2^2\lambda_3^2+2\lambda_1\lambda_2\lambda_3(\lambda_1-2\lambda_2-2\lambda_3))k_3^2\\
&+(\lambda_1-\lambda_3)^2(8\lambda_1^2-\lambda_2^2+8\lambda_3^2-4\lambda_1\lambda_2+8\lambda_1\lambda_3-4\lambda_2\lambda_3)\big),
\end{aligned}$$

$$\begin{aligned}
12\mathfrak{B}_{34} = -k_3\big(&(8\lambda_2^4-4\lambda_2^3(\lambda_1+\lambda_3)-\lambda_1^2\lambda_2^2+8\lambda_1^2\lambda_3^2-\lambda_2^2\lambda_3^2\\
&\qquad-2\lambda_1\lambda_2\lambda_3(2\lambda_1-\lambda_2+2\lambda_3))k_1^2\\
&+(8\lambda_1^4-4\lambda_1^3(\lambda_2+\lambda_3)-\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2+8\lambda_2^2\lambda_3^2+2\lambda_1\lambda_2\lambda_3(\lambda_1-2\lambda_2-2\lambda_3))k_2^2\\
&+8(\lambda_1-\lambda_2)^2(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)k_3^2\\
&+(\lambda_1-\lambda_2)^2(8\lambda_1^2+8\lambda_2^2-\lambda_3^2+8\lambda_1\lambda_2-4\lambda_1\lambda_3-4\lambda_2\lambda_3)\big).
\end{aligned}$$

$$\begin{aligned}
24\mathfrak{B}_{33} = \;&4(3\lambda_2^4-5\lambda_3^4-\lambda_2\lambda_3(\lambda_2^2-3\lambda_3^2))k_1^4\\
&+4(3\lambda_1^4-5\lambda_3^4-\lambda_1\lambda_3(\lambda_1^2-3\lambda_3^2))k_2^4\\
&-4(\lambda_1-\lambda_2)^2(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)k_3^4\\
&-(40\lambda_3^4-12\lambda_3^3(\lambda_1+\lambda_2)-24\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_3^2-\lambda_2^2\lambda_3^2+2\lambda_1\lambda_2\lambda_3(2\lambda_1+2\lambda_2+\lambda_3))k_1^2k_2^2\\
&+(8\lambda_2^4-4\lambda_2^3(\lambda_1-\lambda_3)-\lambda_1^2\lambda_2^2-24\lambda_1^2\lambda_3^2+3\lambda_2^2\lambda_3^2+2\lambda_1\lambda_2\lambda_3(2\lambda_1-\lambda_2+6\lambda_3))k_1^2k_3^2\\
&+(8\lambda_1^4-4\lambda_1^3(\lambda_2-\lambda_3)-\lambda_1^2\lambda_2^2+3\lambda_1^2\lambda_3^2-24\lambda_2^2\lambda_3^2-2\lambda_1\lambda_2\lambda_3(\lambda_1-2\lambda_2-6\lambda_3))k_2^2k_3^2\\
&+(\lambda_2-\lambda_3)(24\lambda_2^3+40\lambda_3^3-\lambda_1^2(3\lambda_2+\lambda_3)-4\lambda_2^2(3\lambda_1-4\lambda_3)\\
&\qquad-4\lambda_3^2(3\lambda_1-4\lambda_2)-8\lambda_1\lambda_2\lambda_3)k_1^2\\
&+(\lambda_1-\lambda_3)(24\lambda_1^3+40\lambda_3^3-4\lambda_1^2(3\lambda_2-4\lambda_3)-\lambda_2^2(3\lambda_1+\lambda_3)\\
&\qquad+4(4\lambda_1-3\lambda_2)\lambda_3^2-8\lambda_1\lambda_2\lambda_3)k_2^2\\
&+(\lambda_1-\lambda_2)^2(8\lambda_1^2+8\lambda_2^2+3\lambda_3^2+8\lambda_1\lambda_2+4\lambda_1\lambda_3+4\lambda_2\lambda_3)k_3^2\\
&+4(3\lambda_1^4+3\lambda_2^4-5\lambda_3^4-\lambda_1^3(3\lambda_2+\lambda_3)-\lambda_2^3(3\lambda_1+\lambda_3)+3\lambda_3^3(\lambda_1+\lambda_2)\\
&\qquad+\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2-\lambda_3)),
\end{aligned}$$

Recall that $\mathfrak{B}_{44}=-\mathfrak{B}_{11}-\mathfrak{B}_{22}-\mathfrak{B}_{33}$ since the Bach tensor is trace-free. If the space is locally symmetric we have $\lambda_1=\lambda_2=\lambda_3$ (see Remark 5.14) and a direct calculation shows it is locally conformally flat and thus Bach-flat.

In the non-symmetric case – where $\lambda_1,\lambda_2,\lambda_3$ are not equal –, we make use of Theorem 5.15 to analyse the existence of non-trivial algebraic Bach solitons. Thus, the only possible non-zero components of the Bach tensor must correspond to $\mathfrak{B}_{ii}$, with

$$\mathfrak{B}_{11}=\mathfrak{B}_{22}=\mathfrak{B}_{33}=\mu,$$

and moreover, $k_1=k_2=k_3=0$. In what follows we take an isomorphic homothety to assume

$\lambda_1 = 1$. A direct computation shows that $\mathfrak{B}_{ij} = 0$ for $i \neq j$ and, moreover,

$$\begin{aligned} 0 &= (\lambda_2 - 1)(\mathfrak{B}_{22} - \mathfrak{B}_{33}) + (\lambda_2 - \lambda_3)(\mathfrak{B}_{11} - \mathfrak{B}_{22}) \\ &= -\tfrac{1}{3}(\lambda_2 - \lambda_3)(\lambda_2 - 1)(\lambda_3 - 1) \\ &\quad \times \left((\lambda_2 + 1)^2 + (\lambda_3 + 1)^2 + (\lambda_2 + \lambda_3)^2 + 3(\lambda_2^2 + \lambda_3^2 + 1)\right). \end{aligned}$$

Since the last factor above does not vanish, then either $\lambda_2 = \lambda_3$, or $\lambda_2 = 1$, or $\lambda_3 = 1$.

If $\lambda_2 = \lambda_3$, then

$$\mathfrak{B}_{11} = -\tfrac{1}{6}(\lambda_2 - 1)(\lambda_2 - 5),$$

$$\mathfrak{B}_{22} = \mathfrak{B}_{33} = \tfrac{1}{6}(\lambda_2 - 1)(\lambda_2 - 3),$$

so the condition $\mathfrak{B}_{11} = \mathfrak{B}_{22}$ implies that $\lambda_2 = 4$. Thus,

$$\widehat{B} = \tfrac{1}{2}\,\mathrm{diag}[1, 1, 1, -3], \quad \mu = \tfrac{1}{2},$$

and the left-invariant metric is given by

$$[e_1, e_2] = 4e_3, \quad [e_1, e_3] = -4e_2, \quad [e_2, e_3] = e_1.$$

This space, which is not a Ricci soliton, corresponds to Assertion *(i)* in Theorem 5.8.

If $\lambda_2 = 1$, we proceed as in the previous case and obtain that $\lambda_3 = \frac{1}{4}$, so the left-invariant metric is given by

$$[e_1, e_2] = \tfrac{1}{4}e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1.$$

Now, the transformation $(e_1, e_2, e_3, e_4) \mapsto 4(e_3, e_2, -e_1, e_4)$ shows that this case is isomorphically homothetic to the previous one.

Finally, if $\lambda_3 = 1$, proceeding exactly as in the previous cases we get $\lambda_2 = \frac{1}{4}$. Hence, the left-invariant metric is given by

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -\tfrac{1}{4}e_2, \quad [e_2, e_3] = e_1,$$

and the transformation $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_3, e_2, e_4)$ shows that this case is isomorphically isometric to the previous one.

5.3 Algebraic solitons on $\mathbb{R}^3 \rtimes \mathbb{R}$

The left-invariant metrics on semi-direct extensions of the Abelian Lie algebra have already been described in Section 1.4.2. In this case there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that

$$\begin{aligned} &[e_1, e_4] = ae_1 + be_2 + ce_3, \qquad [e_2, e_4] = -be_1 + fe_2 + he_3, \\ &[e_3, e_4] = -ce_1 - he_2 + pe_3. \end{aligned} \tag{5.5}$$

Remark 5.16. $\mathbb{R}^3 \rtimes \mathbb{R}$ is locally symmetric if and only if it is Einstein ($a = f = p$) in which case it is homothetic to $\mathbb{H}^4$. Otherwise, it is isomorphically isometric to (5.5) with $a = f = c = h = 0$, $p \neq 0$ (thus being homothetic to $\mathbb{H}^2 \times \mathbb{R}^2$), or $a = f \neq 0$, $p = c = h = 0$ (thus being homothetic to $\mathbb{H}^3 \times \mathbb{R}$).

Note that in the Abelian case any metric is flat and any tensor field T is an algebraic T-soliton. Hence, we exclude this case throughout this section and assume that at least one of the structure constants in (5.5) is non-zero.

Remark 5.17. The isomorphic isometry determined by

$$(e_1, e_2, e_3, e_4) \mapsto (e_2, e_1, e_3, e_4)$$

implies that $(a, f, p, b, c, h) \sim (f, a, p, -b, h, c)$. Analogously,

$$(e_1, e_2, e_3, e_4) \mapsto (e_3, e_2, e_1, e_4)$$

gives $(a, f, p, b, c, h) \sim (p, f, a, -h, -c, -b)$ and

$$(e_1, e_2, e_3, e_4) \mapsto (e_1, e_3, e_2, e_4)$$

shows that $(a, f, p, b, c, h) \sim (a, p, f, c, b, -h)$.

5.3.1 Algebraic T-solitons on $\mathbb{R}^3 \rtimes \mathbb{R}$

Recall that we are assuming that the left-invariant tensor field T is divergence-free, so the following conditions hold:

$$\begin{aligned}
&(2a + f + p)T_{14} + bT_{24} + cT_{34} = 0,\\
&bT_{14} - (a + 2f + p)T_{24} - hT_{34} = 0,\\
&cT_{14} + hT_{24} - (a + f + 2p)T_{34} = 0,\\
&aT_{11} + fT_{22} + pT_{33} - (a + f + p)T_{44} = 0.
\end{aligned} \tag{5.6}$$

The conditions for $\mathfrak{D} = \widehat{T} - \mu\,\mathrm{Id}$ to be a derivation are determined by a system of twenty one (up to duplicities) polynomial equations on the soliton constant μ, the structure constants in (5.5) and the components T_{ij}, given by $\{\mathfrak{P}_{ijk} = 0\}$, where

$\mathfrak{P}_{211} = bT_{14} + aT_{24}$,
$\mathfrak{P}_{212} = -fT_{14} + bT_{24}$,
$\mathfrak{P}_{213} = -hT_{14} + cT_{24}$,

$\mathfrak{P}_{311} = cT_{14} + aT_{34}$,
$\mathfrak{P}_{312} = hT_{14} + bT_{34}$,
$\mathfrak{P}_{313} = -pT_{14} + cT_{34}$,

$$
\begin{aligned}
\mathfrak{P}_{321} &= cT_{24} - bT_{34}, \\
\mathfrak{P}_{322} &= hT_{24} + fT_{34}, \\
\mathfrak{P}_{323} &= -pT_{24} + hT_{34}, \\
\mathfrak{P}_{411} &= a(T_{44} - \mu) - 2bT_{12} - 2cT_{13}, \\
\mathfrak{P}_{412} &= b(T_{11} - T_{22} + T_{44} - \mu) - (a - f)T_{12} - hT_{13} - cT_{23}, \\
\mathfrak{P}_{413} &= c(T_{11} - T_{33} + T_{44} - \mu) + hT_{12} - (a - p)T_{13} - bT_{23}, \\
\mathfrak{P}_{414} &= -aT_{14} - bT_{24} - cT_{34}, \\
\mathfrak{P}_{421} &= b(T_{11} - T_{22} - T_{44} + \mu) + (a - f)T_{12} - hT_{13} - cT_{23}, \\
\mathfrak{P}_{422} &= f(T_{44} - \mu) + 2bT_{12} - 2hT_{23}, \\
\mathfrak{P}_{423} &= h(T_{22} - T_{33} + T_{44} - \mu) + cT_{12} + bT_{13} - (f - p)T_{23}, \\
\mathfrak{P}_{424} &= bT_{14} - fT_{24} - hT_{34}, \\
\mathfrak{P}_{431} &= c(T_{11} - T_{33} - T_{44} + \mu) + hT_{12} + (a - p)T_{13} - bT_{23}, \\
\mathfrak{P}_{432} &= h(T_{22} - T_{33} - T_{44} + \mu) + cT_{12} + bT_{13} + (f - p)T_{23}, \\
\mathfrak{P}_{433} &= p(T_{44} - \mu) + 2cT_{13} + 2hT_{23}, \\
\mathfrak{P}_{434} &= cT_{14} + hT_{24} - pT_{34}.
\end{aligned}
$$

In order to determine all the algebraic T-solitons, it is important to analyse the behaviour of the distinguished direction e_4, since it must be an eigenspace of the endomorphism $\widehat{T}$ in most cases.

Lemma 5.18. *Let $\mathbb{R}^3 \rtimes \mathbb{R}$ be an algebraic T-soliton with soliton constant μ. Then the following relations hold.*

(a) With no further assumptions, either e_4 is an eigenvector of $\widehat{T}$, i.e.,

$$T_{14} = T_{24} = T_{34} = 0,$$

or a, f and p are different with $afp = 0$ and $bch = 0$. In the latter case, the original metric is isomorphically isometric to a metric with $a = 0$ and $b = c = 0$.

(b) Assume that a, f and p are different. If e_4 is an eigenvector of $\widehat{T}$ with associated eigenvalue $T_{44} \neq \mu$, then $afp = 0$. In addition, the original metric is isomorphically isometric to a metric with $a = 0$.

Proof. We prove each assertion separately in what follows.

Assertion *(a)*

Let us assume that e_4 is not an eigenvector of $\widehat{T}$, i.e., at least one of the components T_{14}, T_{24} and T_{34} does not vanish. Firstly, we show that a, f and p are necessarily different. If two of them are equal, using Remark 5.17 one may assume $f = a$. If $f = a$, we consider

$$\mathfrak{P}_{211} = bT_{14} + aT_{24}, \quad \mathfrak{P}_{212} = -aT_{14} + bT_{24}.$$

Hence, either $T_{14} = T_{24} = 0$ or $a = b = 0$. If $T_{14} = T_{24} = 0$ then

$$\mathfrak{P}_{311} = aT_{34}, \quad \mathfrak{P}_{434} = -pT_{34}, \quad \mathfrak{P}_{312} = bT_{34}, \quad \mathfrak{P}_{313} = cT_{34}, \quad \mathfrak{P}_{323} = hT_{34},$$

so $T_{34} \neq 0$ implies that the Lie group is Abelian. If some of T_{14} and T_{24} does not vanish, then $a = b = 0$ and

$$\mathfrak{P}_{313} + \mathfrak{P}_{414} = -pT_{14}, \quad \mathfrak{P}_{323} + \mathfrak{P}_{424} = -pT_{24},$$

which together with

$$\mathfrak{P}_{311} = cT_{14}, \quad \mathfrak{P}_{321} = cT_{24}, \quad \mathfrak{P}_{312} = hT_{14}, \quad \mathfrak{P}_{322} = hT_{24},$$

imply that $p = c = h = 0$, so the Lie group turns out to be Abelian again. This shows that a, f and p have to be different.

Secondly, we show that $bch = 0$ and $afp = 0$. The linear combinations

$$\begin{aligned} \mathfrak{P}_{213} - \mathfrak{P}_{312} - \mathfrak{P}_{321} &= -2hT_{14}, \\ \mathfrak{P}_{213} + \mathfrak{P}_{312} + \mathfrak{P}_{321} &= \quad 2cT_{24}, \\ \mathfrak{P}_{213} + \mathfrak{P}_{312} - \mathfrak{P}_{321} &= \quad 2bT_{34}, \end{aligned}$$

clearly lead to $bch = 0$. If $b = 0$, then

$$\mathfrak{P}_{212} = -fT_{14}, \quad \mathfrak{P}_{211} = aT_{24}, \quad f\mathfrak{P}_{311} + c\mathfrak{P}_{212} = afT_{34}.$$

Consequently, $afp = 0$. The same conclusion is obtained if $c = 0$, since

$$\mathfrak{P}_{313} = -pT_{14}, \quad p\mathfrak{P}_{211} + b\mathfrak{P}_{313} = apT_{24}, \quad \mathfrak{P}_{311} = aT_{34},$$

or if $h = 0$, using

$$p\mathfrak{P}_{212} + b\mathfrak{P}_{323} = -fpT_{14}, \quad \mathfrak{P}_{323} = -pT_{24}, \quad \mathfrak{P}_{322} = fT_{34}.$$

Thus, $afp = 0$ and $bch = 0$. Finally, by Remark 5.17 we can take $a = 0$ working, if necessary, with an isomorphically isometric metric. Now, if $b = 0$, we compute

$$\mathfrak{P}_{311} = cT_{14}, \quad \mathfrak{P}_{321} = cT_{24}, \quad \mathfrak{P}_{414} = -cT_{34},$$

so necessarily $c = 0$. If $c = 0$, then $b = 0$ since

$$\mathfrak{P}_{211} = bT_{14}, \quad \mathfrak{P}_{414} = -bT_{24}, \quad \mathfrak{P}_{321} = -bT_{34}.$$

Finally, if $h = 0$ we use

$$\mathfrak{P}_{211} = bT_{14}, \quad \mathfrak{P}_{212} = -fT_{14} + bT_{24}, \quad \mathfrak{P}_{312} = bT_{34},$$

to obtain that $b = 0$ and, therefore, necessarily $c = 0$. This proves Assertion *(a)*.

Assertion *(b)*

Since

$$\mathfrak{P}_{411} + \mathfrak{P}_{422} + \mathfrak{P}_{433} = (a + f + p)(T_{44} - \mu),$$

and we are assuming $T_{44} \neq \mu$, it follows that $a + f + p = 0$. Therefore, we can take $p = -a - f$, and with this condition we obtain

$$\begin{aligned}
\mathfrak{P}_{421} - \mathfrak{P}_{412} &= 2\left((a - f)T_{12} - b(T_{44} - \mu)\right),\\
\mathfrak{P}_{431} - \mathfrak{P}_{413} &= 2\left((2a + f)T_{13} - c(T_{44} - \mu)\right),\\
\mathfrak{P}_{432} - \mathfrak{P}_{423} &= 2\left((a + 2f)T_{23} - h(T_{44} - \mu)\right).
\end{aligned}$$

Since $p = -a - f$ and a, f, p are different, we have $a - f \neq 0$, $2a + f \neq 0$ and $a + 2f \neq 0$, so

$$T_{12} = \tfrac{b}{a-f}(T_{44} - \mu), \quad T_{13} = \tfrac{c}{2a+f}(T_{44} - \mu), \quad T_{23} = \tfrac{h}{a+2f}(T_{44} - \mu).$$

We set $T_{14} = T_{24} = T_{34} = 0$. We need to see that $afp = 0$, i.e., $af(a + f) = 0$. Now, we compute the polynomials

$$\begin{aligned}
\mathfrak{P}_{411} &= \left(a - \tfrac{2b^2}{a-f} - \tfrac{2c^2}{2a+f}\right)(T_{44} - \mu),\\
\mathfrak{P}_{422} &= \left(f - \tfrac{2h^2}{a+2f} + \tfrac{2b^2}{a-f}\right)(T_{44} - \mu),\\
\mathfrak{P}_{433} &= -\left((a + f) - \tfrac{2c^2}{2a+f} - \tfrac{2h^2}{a+2f}\right)(T_{44} - \mu),\\
\mathfrak{P}_{421} &= b(T_{11} - T_{22}) - \tfrac{3ch(a+f)}{(2a+f)(a+2f)}(T_{44} - \mu),\\
\mathfrak{P}_{413} &= c(T_{11} - T_{33}) + \tfrac{3bhf}{(a-f)(a+2f)}(T_{44} - \mu),\\
\mathfrak{P}_{432} &= h(T_{22} - T_{33}) + \tfrac{3bca}{(a-f)(2a+f)}(T_{44} - \mu).
\end{aligned} \tag{5.7}$$

If $bch = 0$, it follows immediately from the expressions above that $af(a + f) = 0$. For instance, assume that $b = 0$. Hence, $\mathfrak{P}_{411} = \mathfrak{P}_{422} = 0$ in Equation (5.7) imply that $ch \neq 0$ or, otherwise, $a = 0$ or $f = 0$. Now, if $ch \neq 0$, $\mathfrak{P}_{421} = 0$ in the same equation leads to $a + f = 0$. Thus, in any case, $af(a + f) = 0$. A similar argument works both for $c = 0$ and $h = 0$.

Next we analyse the case $bch \neq 0$. We assume that $af(a + f) \neq 0$ and argue for a contradiction. At this point, we consider Equation (5.6), which characterizes the vanishing of the divergence of T. A direct calculation shows that such equation reduces to

$$a(T_{11} - T_{33}) + f(T_{22} - T_{33}) = 0. \tag{5.8}$$

Since $ch \neq 0$, we can isolate $T_{11} - T_{33}$ and $T_{22} - T_{33}$ in $\mathfrak{P}_{413} = 0$ and $\mathfrak{P}_{432} = 0$ in Equation (5.7), respectively. Replacing them in Equation (5.8) we get

$$abf(T_{44} - \mu)\left((a + 2f)c^2 + (2a + f)h^2\right) = 0$$

and therefore $(a+2f)c^2 + (2a+f)h^2 = 0$, which leads to

$$c^2 = -\tfrac{(2a+f)h^2}{a+2f}.$$

Finally, using this last expression in $\mathfrak{P}_{433}$ in Equation (5.7) we obtain

$$\mathfrak{P}_{433} = -(a+f)(T_{44}-\mu),$$

which leads to a contradiction since this expression does not vanish.

Hence, we have shown that $afp = 0$. Note that, by Remark 5.17, we can take $a = 0$ working, if necessary, with an isomorphically isometric metric. This proves Assertion *(b)*. ☐

Theorem 5.19. *If $\mathbb{R}^3 \rtimes \mathbb{R}$ is not Einstein, then it is a non-trivial algebraic T-soliton with soliton constant μ if and only if one of the following holds.*

(i) The self-adjoint part of $\operatorname{ad}(e_4)$ has two equal eigenvalues. In this case, the space is isomorphically isometric to a Lie group given by $a = f \neq p$, and

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ * & T_{22} & 0 & 0 \\ * & * & T_{33} & 0 \\ * & * & * & \mu \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle,$$

with

$$bT_{12} = 0, \quad b(T_{11}-T_{22}) = 0, \quad a(T_{11}+T_{22}) + pT_{33} - (2a+p)\mu = 0,$$
$$c(T_{11}-T_{33}) + hT_{12} = 0, \quad h(T_{22}-T_{33}) + cT_{12} = 0.$$

(ii) The self-adjoint part of $\operatorname{ad}(e_4)$ has three different eigenvalues and the tensor field $\widehat{T}$ is diagonal,

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{22}, T_{33}, \mu] \neq \mu \operatorname{Id},$$

with

$$b(T_{11}-T_{22}) = 0, \quad c(T_{11}-T_{33}) = 0, \quad h(T_{22}-T_{33}) = 0,$$
$$aT_{11} + fT_{22} + pT_{33} - (a+f+p)\mu = 0.$$

(iii) The self-adjoint part of $\operatorname{ad}(e_4)$ has three different eigenvalues and the tensor field $\widehat{T}$ is not diagonal. In this case, the space is isomorphically isometric to a Lie group given by one of the following:

(iii.a) $a = 0$, $p = -f \neq 0$, $b = c$, $2b^2 - f^2 + h^2 = 0$, *and*

$$T = \begin{pmatrix} T_{11} & -\frac{b}{f}(T_{44}-\mu) & \frac{b}{f}(T_{44}-\mu) & 0 \\ * & T_{22} & \frac{h}{2f}(T_{44}-\mu) & 0 \\ * & * & T_{22} & 0 \\ * & * & * & T_{44} \end{pmatrix},$$

with

$$T_{44} \neq \mu, \quad b(2f(T_{11}-T_{22}) - 3h(T_{44}-\mu)) = 0.$$

(iii.b) $a = 0$, $p = -f \neq 0$, $b = c = 0$, $h = f$ *and*

$$T = \begin{pmatrix} T_{11} & 0 & 0 & 0 \\ * & T_{22} & \frac{1}{2}(T_{44}-\mu) & T_{24} \\ * & * & T_{22} & -T_{24} \\ * & * & * & T_{44} \end{pmatrix},$$

with $T_{24} \neq 0$.

Proof. We consider the self-adjoint part of $\mathrm{ad}(e_4)$, $\mathrm{diag}[a, f, p]$, and analyse the cases where two of those structure constants are equal or all of them are different separately. Recall that the metric is Einstein if $a = f = p$ (see Remark 5.16), and if two of the structure constants a, f, p are equal, we may assume $a = f \neq p$ just working, if necessary, with an isomorphically isometric metric (see Remark 5.17).

Case 1: $a = f \neq p$**.**

By Lemma 5.18-*(a)*, we know that $T_{14} = T_{24} = T_{34} = 0$. Besides, considering the linear combination

$$\mathfrak{P}_{411} + \mathfrak{P}_{422} + \mathfrak{P}_{433} = (a + f + p)(T_{44} - \mu), \tag{5.9}$$

we must analyse the cases $T_{44} = \mu$ and $a + f + p = 0$ separately.

Case 1.1: $T_{44} = \mu$.

Since

$$\mathfrak{P}_{431} - \mathfrak{P}_{413} = 2(a-p)T_{13}, \quad \mathfrak{P}_{432} - \mathfrak{P}_{423} = 2(a-p)T_{23},$$

we immediately see that

$$T_{13} = T_{23} = 0.$$

Now, a direct calculation shows that the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ is determined by

$$\mathfrak{P}_{422} = 2bT_{12} = 0, \qquad \mathfrak{P}_{421} = b(T_{11} - T_{22}) = 0,$$

$$\mathfrak{P}_{413} = c(T_{11} - T_{33}) + hT_{12} = 0, \quad \mathfrak{P}_{423} = h(T_{22} - T_{33}) + cT_{12} = 0,$$

while the vanishing of the divergence of the tensor T given by Equation (5.6) reduces to

$$a(T_{11} + T_{22}) + pT_{33} - (2a+p)\mu = 0.$$

This proves Assertion *(i)*.

Case 1.2: $T_{44} \neq \mu$ and $a + f + p = 0$.

In this situation we have that $f = a$ and $p = -2a$, with $a \neq 0$. A direct calculation shows that

$$\mathfrak{P}_{421} - \mathfrak{P}_{412} = -2b(T_{44} - \mu),$$

$$\mathfrak{P}_{431} - \mathfrak{P}_{413} = 6aT_{13} - 2c(T_{44} - \mu),$$

$$\mathfrak{P}_{432} - \mathfrak{P}_{423} = 6aT_{23} - 2h(T_{44} - \mu),$$

and therefore $T_{44} \neq \mu$ and $a \neq 0$ imply that

$$b = 0, \quad T_{13} = \tfrac{c}{3a}(T_{44} - \mu), \quad T_{23} = \tfrac{h}{3a}(T_{44} - \mu).$$

Now, the polynomials $\mathfrak{P}_{421}$, $\mathfrak{P}_{411}$ and $\mathfrak{P}_{433}$ reduce to

$$\mathfrak{P}_{421} = -\tfrac{2}{3a}ch(T_{44} - \mu),$$

$$\mathfrak{P}_{411} = \tfrac{1}{3a}(3a^2 - 2c^2)(T_{44} - \mu),$$

$$\mathfrak{P}_{433} = -\tfrac{2}{3a}(3a^2 - c^2 - h^2)(T_{44} - \mu),$$

which imply

$$ch = 0, \quad 3a^2 - 2c^2 = 0, \quad 3a^2 - c^2 - h^2 = 0.$$

Since $a \neq 0$ these equations are incompatible, so we conclude that there are no algebraic T-solitons in this case.

Case 2: $a \neq f \neq p$**,** $a \neq p$**.**

If a, f and p are all different we distinguish two possibilities depending on whether T_{14}, T_{24} and T_{34} are all zero or not. When $T_{14} = T_{24} = T_{34} = 0$ we analyse the two cases given by Equation (5.9), i.e., $T_{44} = \mu$ and $a + f + p = 0$, separately.

Case 2.1: $T_{14} = T_{24} = T_{34} = 0$ and $T_{44} = \mu$.

Since $T_{44} = \mu$, it is not difficult to check that

$$\mathfrak{P}_{421} - \mathfrak{P}_{412} = 2(a - f)T_{12},$$

$$\mathfrak{P}_{431} - \mathfrak{P}_{413} = 2(a - p)T_{13},$$

$$\mathfrak{P}_{432} - \mathfrak{P}_{423} = 2(f - p)T_{23},$$

and so

$$T_{12} = T_{13} = T_{23} = 0.$$

Applying these relations, a direct calculation shows that the tensor $\widehat{T}$ is diagonal,

$$\widehat{T} = \mathrm{diag}[T_{11}, T_{22}, T_{33}, \mu],$$

and the vanishing of the divergence of T (see Equation (5.6)) is given by

$$aT_{11} + fT_{22} + pT_{33} - (a + f + p)\mu = 0,$$

and the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ is now determined by

$$\mathfrak{P}_{412} = b(T_{11} - T_{22}) = 0,$$

$$\mathfrak{P}_{413} = c(T_{11} - T_{33}) = 0,$$

$$\mathfrak{P}_{423} = h(T_{22} - T_{33}) = 0.$$

This proves Assertion *(ii)*.

Case 2.2: $T_{14} = T_{24} = T_{34} = 0$, $T_{44} \neq \mu$ and $a + f + p = 0$.

By Lemma 5.18-*(b)* we can take $a = 0$ working, if necessary, with an isomorphically isometric metric. Hence, $p = -f \neq 0$. Now, we have the following linear combinations

$$\mathfrak{P}_{421} - \mathfrak{P}_{412} = -2\left(fT_{12} + b(T_{44} - \mu)\right),$$

$$\mathfrak{P}_{431} - \mathfrak{P}_{413} = \quad 2\left(fT_{13} - c(T_{44} - \mu)\right),$$

$$\mathfrak{P}_{432} - \mathfrak{P}_{423} = \quad 2\left(2fT_{23} - h(T_{44} - \mu)\right),$$

from where we obtain

$$T_{12} = -\tfrac{b}{f}(T_{44} - \mu), \quad T_{13} = \tfrac{c}{f}(T_{44} - \mu), \quad T_{23} = \tfrac{h}{2f}(T_{44} - \mu).$$

Using these expressions, the vanishing of the divergence of the tensor T given by Equation (5.6) reduces to

$$f(T_{22} - T_{33}) = 0.$$

Besides,

$$\mathfrak{P}_{411} = \tfrac{2}{f}(b^2 - c^2)(T_{44} - \mu).$$

As a consequence

$$T_{22} = T_{33}, \quad c = \varepsilon b, \quad \text{with } \varepsilon^2 = 1,$$

and a direct calculation shows that the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ is given by

$$\mathfrak{P}_{412} = \varepsilon\mathfrak{P}_{413} = \mathfrak{P}_{421} = \varepsilon\mathfrak{P}_{431} = \tfrac{1}{2f} b\left(2f(T_{11} - T_{22}) - 3\varepsilon h(T_{44} - \mu)\right) = 0,$$

$$\mathfrak{P}_{422} = -\mathfrak{P}_{433} = -\tfrac{1}{f}(2b^2 - f^2 + h^2)(T_{44} - \mu) = 0.$$

The associated left-invariant metric is given by

$$[e_1, e_4] = be_2 + \varepsilon be_3,$$

$$[e_2, e_4] = -be_1 + fe_2 + he_3,$$

$$[e_3, e_4] = -\varepsilon be_1 - he_2 - fe_3,$$

and the isometry $e_3 \mapsto -e_3$ interchanges (ε, f, b, h) and $(-\varepsilon, f, b, -h)$, so we can take $\varepsilon = 1$ working with an isomorphically isometric metric if necessary. Thus Assertion *(iii.a)* is obtained.

Case 2.3: Some of T_{14}, T_{24} and T_{34} does not vanish.

In this case, according to Lemma 5.18-*(a)*, we can assume $a = b = c = 0$ and work, if necessary, with an isomorphically isometric metric. In this situation

$$\mathfrak{P}_{212} = -fT_{14}, \quad \mathfrak{P}_{313} = -pT_{14}$$

and, since $f \neq p$, it follows that

$$T_{14} = 0. \tag{5.10}$$

Now, we compute

$$\mathfrak{P}_{322} = hT_{24} + fT_{34}, \quad \mathfrak{P}_{323} = -pT_{24} + hT_{34},$$

$$\mathfrak{P}_{424} = -fT_{24} - hT_{34}, \quad \mathfrak{P}_{434} = hT_{24} - pT_{34}.$$

Since $T_{24} = T_{34} = 0$ is not possible, we have $f^2 = p^2 = h^2$. Moreover, $f \neq p$ and $fp \neq 0$ lead to

$$p = -f \neq 0, \quad h = \varepsilon f, \quad T_{34} = -\varepsilon T_{24} \neq 0, \tag{5.11}$$

where $\varepsilon^2 = 1$.

Using Equations (5.10) and (5.11) the conditions given in Equation (5.6) for the vanishing of the divergence of T reduce to

$$f(T_{22} - T_{33}) = 0,$$

and

$$\mathfrak{P}_{412} = f(T_{12} - \varepsilon T_{13}), \quad \mathfrak{P}_{421} = -f(T_{12} + \varepsilon T_{13}), \quad \mathfrak{P}_{433} = f(2\varepsilon T_{23} - T_{44} + \mu).$$

Therefore,

$$T_{22} = T_{33}, \quad T_{12} = T_{13} = 0, \quad T_{23} = \tfrac{\varepsilon}{2}(T_{44} - \mu), \tag{5.12}$$

and we obtain an algebraic T-soliton with associated left-invariant metric given by

$$[e_2, e_4] = fe_2 + \varepsilon fe_3, \quad [e_3, e_4] = -\varepsilon fe_2 - fe_3.$$

Note that the isometry $e_3 \mapsto -e_3$ interchanges ε and $-\varepsilon$. Thus, working with an isomorphically isometric metric if necessary, we can take $\varepsilon = 1$. Now Equations (5.10), (5.11) and (5.12) lead to the proof of Assertion *(iii.b)*. □

In the previous result we determined the algebraic T-solitons in the non-Einstein case. Even though the Einstein case can usually be handled directly when dealing with particular tensor fields, we include this case in the following result for the sake of completeness.

Theorem 5.20. *If $\mathbb{R}^3 \rtimes \mathbb{R}$ is Einstein, then it is a non-trivial algebraic T-soliton with soliton constant μ if and only if it is isomorphically isometric to a metric with $a = f = p$ and*

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & 0 \\ * & T_{22} & T_{23} & 0 \\ * & * & T_{33} & 0 \\ * & * & * & \mu \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle,$$

with

$$a(T_{11} + T_{22} + T_{33} - 3\mu) = 0, \qquad b(T_{11} - T_{22}) - hT_{13} - cT_{23} = 0,$$

$$bT_{12} + cT_{13} = 0, \qquad c(T_{11} - T_{33}) + hT_{12} - bT_{23} = 0,$$

$$cT_{13} + hT_{23} = 0, \qquad h(T_{22} - T_{33}) + cT_{12} + bT_{13} = 0.$$

Proof. Recall that $\mathbb{R}^3 \rtimes \mathbb{R}$ is Einstein if and only if $a = f = p$ (see Remark 5.16) so, according to Lemma 5.18-*(a)*, $T_{14} = T_{24} = T_{34} = 0$. Moreover, since

$$\mathfrak{P}_{411} + \mathfrak{P}_{422} + \mathfrak{P}_{433} = 3a(T_{44} - \mu),$$

then either $T_{44} = \mu$ or $a = 0$.

If $T_{44} = \mu$, the conditions for the divergence of T to vanish given by Equation (5.6) reduce to

$$a(T_{11} + T_{22} + T_{33} - 3\mu) = 0,$$

and a direct calculation shows that the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ is determined by

$$\mathfrak{P}_{411} = -2(bT_{12} + cT_{13}) = 0,$$

$$\mathfrak{P}_{433} = 2(cT_{13} + hT_{23}) = 0,$$

$$\mathfrak{P}_{412} = b(T_{11} - T_{22}) - hT_{13} - cT_{23} = 0,$$

$$\mathfrak{P}_{413} = c(T_{11} - T_{33}) + hT_{12} - bT_{23} = 0,$$

$$\mathfrak{P}_{423} = h(T_{22} - T_{33}) + cT_{12} + bT_{13} = 0.$$

Assume now that $T_{44} \neq \mu$ and $a = 0$. Since either b, c or h must be non-zero, Remark 5.17 implies that we can assume that $b \neq 0$ (working, if necessary, with an isomorphically isometric metric). Now, a direct calculation shows that

$$\mathfrak{P}_{412} - \mathfrak{P}_{421} = 2b(T_{44} - \mu),$$

which implies that this case cannot occur. This proves the result. □

5.3.2 Algebraic Ricci solitons on $\mathbb{R}^3 \rtimes \mathbb{R}$

Since Einstein metrics are trivially Ricci solitons, we exclude them from our study and part from the conditions in Theorem 5.19. First of all, a direct calculation shows that the Ricci tensor is determined by

$$\rho_{11} = -(a + f + p)a, \qquad \rho_{23} = (f - p)h,$$

$$\rho_{12} = (a - f)b, \qquad \rho_{33} = -(a + f + p)p,$$

$$\rho_{13} = (a - p)c, \qquad \rho_{44} = -a^2 - f^2 - p^2,$$

$$\rho_{22} = -(a + f + p)f.$$

In the locally symmetric case we analyse the two non-Einstein cases given in Remark 5.16. If $a = f = c = h = 0$ and $p \neq 0$, clearly we have

$$\mathrm{Ric} = -p^2 \operatorname{diag}[0, 0, 1, 1],$$

while if $a = f \neq 0$ and $p = c = h = 0$, we get that

$$\mathrm{Ric} = -2a^2 \operatorname{diag}[1, 1, 0, 1].$$

Theorem 5.19-*(i)* guarantees that both are non-trivial algebraic Ricci solitons, and it is immediate to see that their soliton constants are $\mu = -p^2$ and $\mu = -2a^2$, respectively. Setting $p = 1$ in the former case and $a = 1$ in the latter, we remain in the same homothetic classes and the associated left-invariant metrics are given by

$$[e_1, e_4] = be_2, \quad [e_2, e_4] = -be_1, \quad [e_3, e_4] = e_3, \tag{5.13}$$

and

$$[e_1, e_4] = e_1 + be_2, \quad [e_2, e_4] = -be_1 + e_2,$$

respectively. A straightforward calculation shows that the sectional curvatures of the metrics above are independent of the structure constant b. Therefore, these metrics are homothetic – although not isomorphically homothetic – to the corresponding metrics with $b = 0$ (see [96]), which are the metrics in Assertions *(vi)* and *(vii)* in Theorem 5.4.

In the non-symmetric case we analyse every possibility in Theorem 5.19. Note that with no further assumption, $\rho_{14} = \rho_{24} = \rho_{34} = 0$, so Theorem 5.19-*(iii.b)* cannot occur.

Case *(i)*.

We set $a = f \neq p$. Since algebraic Ricci solitons must satisfy the conditions

$$\rho_{13} = (a - p)c = 0 \quad \text{and} \quad \rho_{23} = (a - p)h = 0,$$

then $c = h = 0$. Now note that $\rho_{12} = 0$, so the algebraic Ricci solitons are determined by the vanishing of the polynomials

$$\mathfrak{Q}_1 = b(\rho_{11} - \rho_{22}), \quad \mathfrak{Q}_2 = \rho_{44} - \mu, \quad \mathfrak{Q}_3 = a(\rho_{11} + \rho_{22}) + p\rho_{33} - (2a + p)\mu.$$

A direct calculation leads to

$$\mathfrak{Q}_1 = 0, \quad \mathfrak{Q}_2 = -\mu - 2a^2 - p^2, \quad \mathfrak{Q}_3 = -(2a + p)(\mu + 2a^2 + p^2).$$

Thus, the system $\{\mathfrak{Q}_i = 0\}$ is equivalent to $\mu = -2a^2 - p^2$, which provides an algebraic Ricci soliton with associated left-invariant metric given by

$$[e_1, e_4] = ae_1 + be_2, \quad [e_2, e_4] = -be_1 + ae_2, \quad [e_3, e_4] = pe_3. \tag{5.14}$$

It is not difficult to see that the sectional curvatures of these metrics do not depend on the structure constant b, so the metrics above are homothetic – although not isomorphically homothetic – to the corresponding metrics with $b = 0$ (see [96]). Moreover, the space is locally symmetric if and only if $a = 0$ or $p = 0$. Hence, working in the same homothetic class, we can assume $a = 1$ and $b = 0$, and these metrics are covered by Theorem 5.4-*(iii)* (see the Case *(ii)* for the restrictions on the parameter).

Case *(ii)*.

In this case, $a \neq f \neq p$ and $a \neq p$. The Ricci operator must be diagonal, which occurs if and only if

$$\rho_{12} = (a-f)b = 0, \quad \rho_{13} = (a-p)c = 0 \quad \text{and} \quad \rho_{23} = (f-p)h = 0,$$

i.e., $b = c = h = 0$. Thus, we obtain an algebraic Ricci soliton if the polynomials

$$\mathfrak{Q}_1 = \rho_{44} - \mu, \quad \mathfrak{Q}_2 = a\rho_{11} + f\rho_{22} + p\rho_{33} - (a+f+p)\mu,$$

vanish. A direct calculation shows that

$$\mathfrak{Q}_1 = -\mu - a^2 - f^2 - p^2, \quad \mathfrak{Q}_2 = -(a+f+p)(\mu + a^2 + f^2 + p^2),$$

and $\mu = -a^2 - f^2 - p^2$ provides an algebraic Ricci soliton with associated left-invariant metric given by

$$[e_1, e_4] = ae_1, \quad [e_2, e_4] = fe_2, \quad [e_3, e_4] = pe_3. \tag{5.15}$$

Since $b = c = h = 0$, necessarily some of a, f and p must be non-zero. Working in the same homothetic class, we can assume that $a = 1$ (see Remark 5.17). Now, the isometry

$$e_2 \longmapsto e_3$$

interchanges (f, p) with (p, f), while the isometry

$$(e_1, e_2, e_3, e_4) \longmapsto (e_2, e_1, e_3, -e_4)$$

interchanges $(-1, p)$ with $(-1, -p)$. Besides, if $f \neq 0$, the homothety

$$(e_1, e_2, e_3, e_4) \longmapsto \tfrac{1}{f}(e_2, e_1, e_3, e_4)$$

interchanges (f, p) with $(\frac{1}{f}, \frac{p}{f})$ and, if $p \neq 0$, the homothety

$$(e_1, e_2, e_3, e_4) \longmapsto \tfrac{1}{p}(e_3, e_2, e_1, e_4)$$

interchanges (f, p) with $(\frac{f}{p}, \frac{1}{p})$.

The relations above allow us to show that any metric in this case is homothetic to a metric in Theorem 5.4-*(iii)*. Since $(f, p) \sim (p, f)$, we restrict the domain of (f, p) to $f \leq p$. Now, using the identification $(f, p) \sim (\frac{1}{f}, \frac{p}{f})$, we remove the set

$$\{(f, p) \colon f \leq p, |f| > 1\}$$

since every homothetic class has a representative with $|f| \leq 1$. Now, considering $(f, p) \sim (\frac{f}{p}, \frac{1}{p})$ we can remove the set

$$\{(f, p) \colon f < p, |f| \leq 1, p > 1\}$$

so that the domain of (f, p) reduces to

$$\{(f, p) \colon -1 \leq f \leq p \leq 1\}.$$

Finally, since $(-1,p)\sim(-1,-p)$ we can also eliminate the segment

$$\{(-1,p)\colon p<0\}.$$

The particular points $(0,0)$, $(0,1)$ and $(1,1)$ are also excluded because the space is locally symmetric in those cases. Hence, we conclude that this case corresponds to Assertion *(iii)* in Theorem 5.4.

Case *(iii.a)*.

In this case, the structure constants satisfy $a=0$, $p=-f\neq 0$, $c=b$ and

$$2b^2-f^2+h^2=0.$$

Algebraic Ricci solitons are characterized by the system of polynomial equations $\{\mathfrak{Q}_i=0\}$, where

$$\mathfrak{Q}_1=\rho_{22}-\rho_{33},\quad \mathfrak{Q}_2=\rho_{12}+\tfrac{b}{f}(\rho_{44}-\mu),\quad \mathfrak{Q}_3=\rho_{12}+\rho_{13},$$
$$\mathfrak{Q}_4=\rho_{23}-\tfrac{h}{2f}(\rho_{44}-\mu),\quad \mathfrak{Q}_5=b\left(2f(\rho_{11}-\rho_{22})-3h(\rho_{44}-\mu)\right),$$

with $\rho_{44}\neq\mu$ and $2b^2-f^2+h^2=0$. A direct calculation leads to

$$\mathfrak{Q}_1=0,\quad \mathfrak{Q}_2=-\tfrac{b}{f}(\mu+3f^2),\quad \mathfrak{Q}_3=0,$$
$$\mathfrak{Q}_4=\tfrac{h}{2f}(\mu+6f^2),\quad \mathfrak{Q}_5=3bh(\mu+2f^2),$$

where $\mu\neq-2f^2$ and $2b^2-f^2+h^2=0$. Since $f\neq 0$, the cases $b=h=0$ and $bh\neq 0$ are not possible. Next we analyse the cases $b=0$, $h\neq 0$ and $h=0$, $b\neq 0$ separately.

If $b=0$ and $h\neq 0$, then $h=\varepsilon f$, with $\varepsilon^2=1$, and the system $\{\mathfrak{Q}_i=0\}$ reduces to $\mu=-6f^2$. We take $f=1$ working in the same homothetic class and the left-invariant metric associated to the algebraic Ricci soliton is given by

$$[e_2,e_4]=e_2+\varepsilon e_3,\quad [e_3,e_4]=-\varepsilon e_2-e_3. \tag{5.16}$$

The isometry $e_3\mapsto -e_3$ interchanges ε and $-\varepsilon$. Hence, we can set $\varepsilon=1$ and Assertion *(i)* in Theorem 5.4 follows.

If $h=0$ and $b\neq 0$ then we have $b=\frac{\varepsilon}{\sqrt{2}}f$, with $\varepsilon^2=1$, and the algebraic Ricci solitons are determined by $\mu=-3f^2$. As in the previous case, we can take $f=1$ so that the associated left-invariant metric is given by

$$[e_1,e_4]=\tfrac{\varepsilon}{\sqrt{2}}(e_2+e_3),\quad [e_2,e_4]=-\tfrac{\varepsilon}{\sqrt{2}}e_1+e_2,\quad [e_3,e_4]=-\tfrac{\varepsilon}{\sqrt{2}}e_1-e_3. \tag{5.17}$$

The isometry $(e_1,e_2,e_3,e_4)\mapsto(e_1,-e_2,-e_3,e_4)$ interchanges ε with $-\varepsilon$, so we can set $\varepsilon=1$ and this case corresponds to Assertion *(ii)* in Theorem 5.4.

5.3.3 Algebraic Bach solitons on $\mathbb{R}^3 \rtimes \mathbb{R}$

Einstein metrics are Bach-flat in dimension four and so they are steady Bach solitons. Therefore, we focus on the non-Einstein situation and use Theorem 5.19 to determine all the non-trivial algebraic Bach solitons. A long but straightforward calculation shows that the non-zero components $\mathfrak{B}_{ij}$ of the Bach tensor of $\mathbb{R}^3 \rtimes \mathbb{R}$ are given by

$$\begin{aligned}
6\mathfrak{B}_{11} = {} & a^4 - f^4 - p^4 - a^3 f + 3af^3 - a^3 p + 3ap^3 + f^3 p + fp^3 - 2a^2 f^2 - 2a^2 p^2 \\
& - 7a^2 fp + 3af^2 p + 3afp^2 + 9(b^2 + c^2)a^2 - 3(5b^2 + h^2)f^2 - 3(5c^2 + h^2)p^2 \\
& + 6(b^2 + 2c^2)af + 6(2b^2 + c^2)ap - 6(2b^2 + 2c^2 - h^2)fp + 18bch(f - p),
\end{aligned}$$

$$\begin{aligned}
6\mathfrak{B}_{12} = {} & -b\,(a - f)\,(2a^2 + 2f^2 - p^2 - 16af - 10ap - 10fp) \\
& + 6ch\,(a + f - 2p)\,(a + f + p) - 3b\,((4b^2 + 4c^2 + h^2)a - (4b^2 + c^2 + 4h^2)f \\
& - 3(c^2 - h^2)p)\,,
\end{aligned}$$

$$\begin{aligned}
6\mathfrak{B}_{13} = {} & -c\,(a - p)\,(2a^2 - f^2 + 2p^2 - 10af - 16ap - 10fp) \\
& - 6bh\,(a - 2f + p)\,(a + f + p) - 3c\,((4b^2 + 4c^2 + h^2)a - 3(b^2 - h^2)f \\
& - (b^2 + 4c^2 + 4h^2)p)\,,
\end{aligned}$$

$$\begin{aligned}
6\mathfrak{B}_{22} = {} & -a^4 + f^4 - p^4 + 3a^3 f - af^3 + a^3 p + ap^3 - f^3 p + 3fp^3 - 2a^2 f^2 - 2f^2 p^2 \\
& + 3a^2 fp - 7af^2 p + 3afp^2 - 3(5b^2 + c^2)a^2 + 9(b^2 + h^2)f^2 - 3(c^2 + 5h^2)p^2 \\
& + 6(b^2 + 2h^2)af - 6(2b^2 - c^2 + 2h^2)ap + 6(2b^2 + h^2)fp - 18bch(a - p),
\end{aligned}$$

$$\begin{aligned}
6\mathfrak{B}_{23} = {} & h\,(f - p)\,(a^2 - 2f^2 - 2p^2 + 10af + 10ap + 16fp) \\
& - 6bc\,(2a - f - p)\,(a + f + p) + 3h\,(3(b^2 - c^2)a - (4b^2 + c^2 + 4h^2)f \\
& + (b^2 + 4c^2 + 4h^2)p)\,,
\end{aligned}$$

$$\begin{aligned}
6\mathfrak{B}_{33} = {} & -a^4 - f^4 + p^4 + a^3 f + af^3 + 3a^3 p - ap^3 + 3f^3 p - fp^3 - 2a^2 p^2 - 2f^2 p^2 \\
& + 3a^2 fp + 3af^2 p - 7afp^2 - 3(b^2 + 5c^2)a^2 - 3(b^2 + 5h^2)f^2 + 9(c^2 + h^2)p^2 \\
& + 6(b^2 - 2c^2 - 2h^2)af + 6(c^2 + 2h^2)ap + 6(2c^2 + h^2)fp + 18bch(a - f).
\end{aligned}$$

Recall that the Bach tensor is trace-free, so $\mathfrak{B}_{44} = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33}$. In the locally symmetric case, analysing the cases in Remark 5.16, a direct calculation shows that the only case which is not Bach-flat corresponds to

$$a = f = c = h = 0, \quad p \neq 0$$

and

$$\widehat{\mathfrak{B}} = \tfrac{p^4}{6}\,\mathrm{diag}[-1, -1, 1, 1].$$

This case gives a non-trivial algebraic Bach soliton with soliton constant $\mu = \frac{p^4}{6}$ (see Theorem 5.19-*(i)*) which is also an algebraic Ricci soliton (see Equation (5.13)).

If the space is not locally symmetric, we need to analyse the four cases obtained in Theorem 5.19 separately. Note that, with no further assumption,

$$\mathfrak{B}_{14} = \mathfrak{B}_{24} = \mathfrak{B}_{34} = 0,$$

so Case *(iii.b)* in Theorem 5.19 is not possible.

Case *(i)*.

Take $a = f \neq p$. The conditions that determine an algebraic Bach soliton can be expressed as the vanishing of the polynomials

$$\mathfrak{Q}_1 = \mathfrak{B}_{13}, \quad \mathfrak{Q}_2 = \mathfrak{B}_{23}, \quad \mathfrak{Q}_3 = b\mathfrak{B}_{12}, \quad \mathfrak{Q}_4 = b(\mathfrak{B}_{11} - \mathfrak{B}_{22}),$$
$$\mathfrak{Q}_5 = c(\mathfrak{B}_{11} - \mathfrak{B}_{33}) + h\mathfrak{B}_{12}, \quad \mathfrak{Q}_6 = h(\mathfrak{B}_{22} - \mathfrak{B}_{33}) + c\mathfrak{B}_{12},$$
$$\mathfrak{Q}_7 = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33} - \mu,$$
$$\mathfrak{Q}_8 = a(\mathfrak{B}_{11} + \mathfrak{B}_{22}) + p\mathfrak{B}_{33} - (2a+p)\mu,$$

and a straightforward calculation shows that

$$\mathfrak{Q}_1 = \tfrac{1}{6}(a-p)\left(c(9a^2 - 2p^2 + 26ap) + 6bh(2a+p) - 3c(b^2 + 4c^2 + 4h^2)\right),$$
$$\mathfrak{Q}_2 = \tfrac{1}{6}(a-p)\left(h(9a^2 - 2p^2 + 26ap) - 6bc(2a+p) - 3h(b^2 + 4c^2 + 4h^2)\right),$$
$$\mathfrak{Q}_3 = \tfrac{1}{2}b(a-p)\left(4ch(2a+p) - 3b(c^2 - h^2)\right),$$
$$\mathfrak{Q}_4 = 2b(a-p)\left((c^2 - h^2)(2a+p) + 3bch\right),$$
$$\mathfrak{Q}_5 = \tfrac{1}{6}(a-p)\left(2c(2a+p)(p^2 - 4ap + 12c^2 + 12h^2) + 9bh(c^2 + h^2)\right),$$
$$\mathfrak{Q}_6 = \tfrac{1}{6}(a-p)\left(2h(2a+p)(p^2 - 4ap + 12c^2 + 12h^2) - 9bc(c^2 + h^2)\right),$$
$$\mathfrak{Q}_7 = \tfrac{1}{6}(a-p)^2\left(p^2 - 4ap + 9c^2 + 9h^2\right) - \mu,$$
$$\mathfrak{Q}_8 = (2a+p)\left(\tfrac{1}{6}(a-p)^2(p^2 - 4ap + 9c^2 + 9h^2) - \mu\right).$$

Now, it is easy to see that

$$h\mathfrak{Q}_1 - c\mathfrak{Q}_2 = b(2a+p)(c^2+h^2)(a-p).$$

Next we show that $c = h = 0$ by analysing the cases $p = -2a$ and $b = 0$. If $p = -2a$, which implies $a \neq 0$ since $a \neq p$, we obtain

$$\mathfrak{Q}_1 = -\tfrac{3}{2}ac(17a^2 + b^2 + 4c^2 + 4h^2), \quad \mathfrak{Q}_2 = -\tfrac{3}{2}ah(17a^2 + b^2 + 4c^2 + 4h^2),$$

so $c = h = 0$. If $p \neq -2a$ and $b = 0$ a straightforward calculation shows that

$$\mathfrak{Q}_1 = \tfrac{1}{6}c(a-p)(9a^2 - 2p^2 + 26ap - 12c^2 - 12h^2),$$
$$\mathfrak{Q}_2 = \tfrac{1}{6}h(a-p)(9a^2 - 2p^2 + 26ap - 12c^2 - 12h^2),$$

and

$$\mathfrak{Q}_5 = \tfrac{1}{3}c(a-p)(2a+p)(p^2 - 4ap + 12c^2 + 12h^2),$$
$$\mathfrak{Q}_6 = \tfrac{1}{3}h(a-p)(2a+p)(p^2 - 4ap + 12c^2 + 12h^2),$$

from where it follows that

$$4(2a+p)\mathfrak{Q}_1 + 13\mathfrak{Q}_5 = c(a-p)(2a+p)(6a^2+3p^2+44c^2+44h^2),$$

$$4(2a+p)\mathfrak{Q}_2 + 13\mathfrak{Q}_6 = h(a-p)(2a+p)(6a^2+3p^2+44c^2+44h^2).$$

Hence, again $c = h = 0$. Assuming the vanishing of c and h the system $\{\mathfrak{Q}_i = 0\}$ reduces to

$$\mathfrak{Q}_7 = \tfrac{1}{6}(a-p)^2(p-4a)p - \mu,$$

$$\mathfrak{Q}_8 = (2a+p)(\tfrac{1}{6}(a-p)^2(p-4a)p - \mu),$$

or equivalently,

$$\mu = \tfrac{1}{6}(a-p)^2(p-4a)p.$$

This leads to an algebraic Bach soliton with associated left-invariant metric given by

$$[e_1, e_4] = ae_1 + be_2, \quad [e_2, e_4] = -be_1 + ae_2, \quad [e_3, e_4] = pe_3,$$

which is locally symmetric if and only if $a = 0$ or $p = 0$, and it is Bach-flat if and only if $p = 0$ or $p = 4a$. Note that, in any case, it is also an algebraic Ricci soliton (see Equation (5.14)).

Case *(ii)*.

If $a \neq f \neq p$, $a \neq p$ and the Bach operator is diagonal, the algebraic Bach solitons are determined by the vanishing of the polynomials

$$\begin{aligned}
&\mathfrak{Q}_1 = \mathfrak{B}_{12}, \quad \mathfrak{Q}_2 = \mathfrak{B}_{13}, \quad \mathfrak{Q}_3 = \mathfrak{B}_{23}, \\
&\mathfrak{Q}_4 = b(\mathfrak{B}_{11} - \mathfrak{B}_{22}), \quad \mathfrak{Q}_5 = c(\mathfrak{B}_{11} - \mathfrak{B}_{33}), \\
&\mathfrak{Q}_6 = h(\mathfrak{B}_{22} - \mathfrak{B}_{33}), \quad \mathfrak{Q}_7 = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33} - \mu, \\
&\mathfrak{Q}_8 = a\mathfrak{B}_{11} + f\mathfrak{B}_{22} + p\mathfrak{B}_{33} - (a+f+p)\mu.
\end{aligned} \tag{5.18}$$

Next we split the analysis in several cases, depending on whether bch vanishes or not. According to Remark 5.17, it is enough to consider the case $b = c = h = 0$, the case $b = c = 0$, $h \neq 0$ and the case $ch \neq 0$. We will show that there exist algebraic Bach solitons only in the first case.

Case 1. $b = c = h = 0$.

If $b = c = h = 0$, a direct calculation shows that the Bach operator is diagonal and that Equation (5.18) takes the form

$$\mathfrak{Q}_7 = \tfrac{1}{6}\left(a^2 + (f-p)^2 - 2a(f+p)\right)\left(a^2 + f^2 + p^2 - af - ap - fp\right) - \mu,$$

$$\mathfrak{Q}_8 = (a+f+p)\mathfrak{Q}_7.$$

Therefore, algebraic Bach solitons are determined by

$$\mu = \tfrac{1}{6}\left(a^2 + (f-p)^2 - 2a(f+p)\right)\left(a^2 + f^2 + p^2 - af - ap - fp\right).$$

The associated left-invariant metric is given by

$$[e_1, e_4] = ae_1, \quad [e_2, e_4] = fe_2, \quad [e_3, e_4] = pe_3,$$

so these algebraic Bach solitons are also algebraic Ricci solitons (see Equation (5.15)).

Case 2. $b = c = 0$, $h \neq 0$.

In this case we work with the polynomials $\mathfrak{Q}_3$ and $\mathfrak{Q}_6$ in Equation (5.18) which are given by

$$\mathfrak{Q}_3 = \tfrac{1}{6}h(f-p)\left(a^2 - 2(f-p)^2 + 10a(f+p) + 12fp - 12h^2\right),$$
$$\mathfrak{Q}_6 = \tfrac{1}{3}h(f-p)(a+f+p)\left(a^2 + (f-p)^2 - 2a(f+p) + 12h^2\right).$$

If $a + f + p = 0$ we take $p = -a - f$ and

$$\mathfrak{Q}_3 = -\tfrac{1}{6}h(a+2f)\left(10(a+f)^2 + a^2 + 10f^2 + 12h^2\right) \neq 0,$$

while if $a + f + p \neq 0$ we calculate

$$2(a+f+p)\mathfrak{Q}_3 + 5\mathfrak{Q}_6 = h(f-p)(a+f+p)\left(2a^2 + (f+p)^2 + 16h^2\right) \neq 0.$$

Since none of these two polynomials above vanishes, we conclude that no algebraic Bach soliton may exist in this case.

Case 3. $ch \neq 0$.

Next we show that non-trivial algebraic Bach solitons cannot exist when $ch \neq 0$. To do so, we consider the polynomials

$$\mathfrak{Q}_5 = c(\mathfrak{B}_{11} - \mathfrak{B}_{33}), \quad \mathfrak{Q}_6 = h(\mathfrak{B}_{22} - \mathfrak{B}_{33}),$$
$$\mathfrak{Q}_7 = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33} - \mu,$$
$$\mathfrak{Q}_8 = a\mathfrak{B}_{11} + f\mathfrak{B}_{22} + p\mathfrak{B}_{33} - (a+f+p)\mu,$$

given in Equation (5.18). It is easy to see that

$$\mathfrak{B}_{11} = \mathfrak{B}_{22} = \mathfrak{B}_{33}, \quad \mu = -3\mathfrak{B}_{11}, \quad (a+f+p)(\mathfrak{B}_{11} - \mu) = 0.$$

Hence, $(a+f+p)\mathfrak{B}_{11} = 0$. Since the Bach operator must be diagonal, if $\mathfrak{B}_{11} = 0$ then any possible algebraic Bach soliton would be trivial. Thus, $p = -a - f$ and a direct calculation shows that the polynomials $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$ in Equation (5.18) are given by

$$\begin{aligned}\mathfrak{Q}_2 = -\tfrac{1}{6}c\,((2a+f)(20a^2+11f^2+20af)\\ +3(5b^2+8c^2+5h^2)a - 3(2b^2-4c^2-7h^2)f)\,,\end{aligned}$$
$$\begin{aligned}\mathfrak{Q}_3 = -\tfrac{1}{6}h\,((a+2f)(11a^2+20f^2+20af)\\ -3(2b^2-7c^2-4h^2)a + 3(5b^2+5c^2+8h^2)f)\,,\end{aligned}$$

from where it immediately follows that

$$h\mathfrak{Q}_2 - c\mathfrak{Q}_3 = -\tfrac{1}{12}ch(a-f)\left(47(a+f)^2 + 11(a^2+f^2) + 6(7b^2+c^2+h^2)\right) \neq 0.$$

This implies that non-trivial algebraic Bach solitons cannot exist if $ch \neq 0$.

Case *(iii.a)*.

In this case, the structure constants satisfy $a = 0$, $p = -f \neq 0$, $c = b$ and

$$2b^2 - f^2 + h^2 = 0.$$

In this situation, the algebraic Bach solitons are determined by the system of polynomial equations $\{\mathfrak{Q}_i = 0\}$, where

$$\mathfrak{Q}_1 = \mathfrak{B}_{22} - \mathfrak{B}_{33}, \quad \mathfrak{Q}_2 = \mathfrak{B}_{12} - \tfrac{b}{f}(\mathfrak{B}_{11} + \mathfrak{B}_{22} + \mathfrak{B}_{33} + \mu),$$
$$\mathfrak{Q}_3 = \mathfrak{B}_{12} + \mathfrak{B}_{13}, \quad \mathfrak{Q}_4 = \mathfrak{B}_{23} + \tfrac{h}{2f}(\mathfrak{B}_{11} + \mathfrak{B}_{22} + \mathfrak{B}_{33} + \mu),$$
$$\mathfrak{Q}_5 = b\,((2f+3h)\mathfrak{B}_{11} - (2f-3h)\mathfrak{B}_{22} + 3h\mathfrak{B}_{33} + 3h\mu)\,,$$

with $\mu \neq -(\mathfrak{B}_{11} + \mathfrak{B}_{22} + \mathfrak{B}_{33})$ and $h^2 = f^2 - 2b^2$. A long but direct calculation leads to $\mathfrak{Q}_1 = \mathfrak{Q}_3 = 0$ and

$$\begin{aligned} \mathfrak{Q}_2 &= \tfrac{b}{3f}\left(40f^4 - 45b^2f^2 - 3\mu\right), \\ \mathfrak{Q}_4 &= -\tfrac{h}{6f}\left(88f^4 - 45b^2f^2 - 3\mu\right), \\ \mathfrak{Q}_5 &= -3bh\left(8f^4 - 15b^2f^2 - \mu\right), \end{aligned}$$

where $\mu \neq 8f^4 - 9b^2f^2$ and $h^2 = f^2 - 2b^2$. Note that

$$3fh\mathfrak{Q}_2 + \mathfrak{Q}_5 = 16bhf^4.$$

Hence, since $f \neq 0$, it follows that the cases $b = h = 0$ and $bh \neq 0$ are not possible. In what follows we analyse the cases $b = 0$, $h \neq 0$ and $h = 0$, $b \neq 0$ separately.

If $b = 0$ and $h \neq 0$, then $h = \varepsilon f$, with $\varepsilon^2 = 1$, and the system of polynomial equations $\{\mathfrak{Q}_i = 0\}$ reduces to

$$\mathfrak{Q}_4 = \tfrac{\varepsilon}{6}(3\mu - 88f^4) = 0.$$

Thus, $\mu = \frac{88}{3}f^4$ and the left-invariant metric associated to the algebraic Bach soliton is given by

$$[e_2, e_4] = e_2 + \varepsilon e_3, \quad [e_3, e_4] = -\varepsilon e_2 - e_3,$$

where we have taken $f = 1$ – working in the same homothetic class. Note that these spaces are algebraic Ricci solitons (see Equation (5.16)).

Finally, assume that $h = 0$ and $b \neq 0$. In this case, $b = \frac{\varepsilon}{\sqrt{2}}f$, with $\varepsilon^2 = 1$, and the system $\{\mathfrak{Q}_i = 0\}$ leads to the soliton constant $\mu = \frac{35}{6}f^4$. As we did in the case just above, we can take $f = 1$ and the left-invariant metric associated to the algebraic Bach soliton is given by

$$[e_1, e_4] = \tfrac{\varepsilon}{\sqrt{2}}(e_2 + e_3), \quad [e_2, e_4] = -\tfrac{\varepsilon}{\sqrt{2}}e_1 + e_2, \quad [e_3, e_4] = -\tfrac{\varepsilon}{\sqrt{2}}e_1 - e_3,$$

which also determines an algebraic Ricci soliton (see Equation (5.17)).

5.4 Algebraic solitons on $E(1,1) \rtimes \mathbb{R}$ and $\widetilde{E}(2) \rtimes \mathbb{R}$

We follow the description of the left-invariant metrics on semi-direct extensions of the Poincaré and the Euclidean groups given in Section 1.4.2. Recall that there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that

$$\begin{aligned} &[e_1, e_3] = -\lambda_2 e_2, && [e_2, e_3] = \lambda_1 e_1, \\ &[e_1, e_4] = be_1 - A\lambda_2 e_2, && [e_2, e_4] = A\lambda_1 e_1 + be_2, \\ &[e_3, e_4] = Ce_1 + De_2, \end{aligned} \tag{5.19}$$

where $\lambda_1\lambda_2 \neq 0$. The associated Lie group to $\widetilde{E}(2) \times \mathbb{R}$ if λ_1, λ_2 do not change sign, and corresponds to $E(1,1) \times \mathbb{R}$ otherwise.

Remark 5.21. $\widetilde{E}(2) \rtimes \mathbb{R}$ is locally symmetric if and only if $C = D = 0$ and $\lambda_2 = \lambda_1$ – in which case the metric is flat if $b = 0$, and homothetic to the product $\mathbb{H}^3 \times \mathbb{R}$ otherwise. Analogously, $E(1,1) \rtimes \mathbb{R}$ is locally symmetric if and only if $C = D = 0$, $\lambda_2 = -\lambda_1$ and $b^2 = (A^2+1)\lambda_1^2$ – in which case it is isometric to the product of two surfaces of constant distinct negative curvatures if $A \neq 0$, and homothetic to $\mathbb{H}^2 \times \mathbb{H}^2$ otherwise. We show in Remark 5.24 that these metrics are also realized as left-invariant metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$.

Remark 5.22. The parameters $(\lambda_1, \lambda_2, A, b, C, D)$ in Equation (5.19) can be transformed into $(\lambda_2, \lambda_1, A, b, -D, C)$ through the isometry

$$(e_1, e_2, e_3, e_4) \mapsto (-e_2, e_1, e_3, e_4).$$

Therefore, any left-invariant metric (5.19) with $D = 0$ is isomorphically isometric to a left-invariant metric with $C = 0$.

5.4.1 Algebraic T-solitons on $E(1,1) \rtimes \mathbb{R}$ and $\widetilde{E}(2) \rtimes \mathbb{R}$

Theorem 5.23. *$E(1,1) \rtimes \mathbb{R}$ or $\widetilde{E}(2) \rtimes \mathbb{R}$ are non-trivial algebraic T-solitons with soliton constant μ if and only if*

$$T = \begin{pmatrix} T_{11} & 0 & 0 & 0 \\ * & T_{11} & 0 & 0 \\ * & * & A^2(T_{44}-\mu)+\mu & -A(T_{44}-\mu) \\ * & * & * & T_{44} \end{pmatrix} \neq \mu\langle\cdot,\cdot\rangle,$$

and $b = 0$, with $C = D = 0$ or $T_{11} = A^2(T_{44}-\mu) + T_{44}$.

Remark 5.24. In the case where $b = 0$ and $C = D = 0$, the left-invariant metric associated to an algebraic T-soliton takes the form

$$[e_1, e_3] = -\lambda_2 e_2, \quad [e_1, e_4] = -A\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_2, e_4] = A\lambda_1 e_1.$$

Setting $\zeta = \sqrt{A^2+1}$ and considering the orthonormal basis given by

$$\tilde{e}_1 = \tfrac{1}{\sqrt{2}}(e_1 + e_2), \qquad \tilde{e}_2 = -\tfrac{1}{\sqrt{2}}(e_1 - e_2),$$
$$\tilde{e}_3 = -\tfrac{1}{\zeta}(Ae_3 - e_4), \quad \tilde{e}_4 = \tfrac{1}{\zeta}(e_3 + Ae_4),$$

the corresponding non-zero Lie brackets are now

$$[\tilde{e}_1, \tilde{e}_4] = \tfrac{\zeta}{2}(\lambda_1 - \lambda_2)\tilde{e}_1 - \tfrac{\zeta}{2}(\lambda_1 + \lambda_2)\tilde{e}_2, \quad [\tilde{e}_2, \tilde{e}_4] = \tfrac{\zeta}{2}(\lambda_1 + \lambda_2)\tilde{e}_1 - \tfrac{\zeta}{2}(\lambda_1 - \lambda_2)\tilde{e}_2.$$

Thus, these metrics are isomorphically isometric to metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$ and therefore the corresponding algebraic T-solitons have already been described in Theorems 5.19 and 5.20.

Proof. The vanishing of the divergence of T is given by the system of polynomial equations

$$\begin{aligned}
&3bT_{14} - \lambda_2 T_{23} - A\lambda_2 T_{24} = 0,\\
&\lambda_1 T_{13} + A\lambda_1 T_{14} + 3bT_{24} = 0,\\
&(\lambda_1 - \lambda_2)T_{12} - CT_{14} - DT_{24} - 2bT_{34} = 0,\\
&bT_{11} + bT_{22} - 2bT_{44} + A(\lambda_1 - \lambda_2)T_{12} + CT_{13} + DT_{23} = 0,
\end{aligned} \tag{5.20}$$

while the conditions for $\mathfrak{D} = \widehat{T} - \mu\,\mathrm{Id}$ to be a derivation will be determined by a system of twenty two (up to duplicities) polynomial equations on the soliton constant μ, the structure constants (5.19) and the components T_{ij}, given by $\{\mathfrak{P}_{ijk} = 0\}$, where

$$\begin{aligned}
\mathfrak{P}_{211} &= -\lambda_1(T_{13} + AT_{14}) + bT_{24},\\
\mathfrak{P}_{212} &= -bT_{14} - \lambda_2(T_{23} + AT_{24}),\\
\mathfrak{P}_{311} &= (\lambda_1 + \lambda_2)T_{12} - CT_{14} + bT_{34},\\
\mathfrak{P}_{312} &= -\lambda_2(T_{11} - T_{22} + T_{33} - \mu) - DT_{14} - A\lambda_2 T_{34},\\
\mathfrak{P}_{313} &= \lambda_2 T_{23},\\
\mathfrak{P}_{314} &= \lambda_2 T_{24},\\
\mathfrak{P}_{321} &= -\lambda_1(T_{11} - T_{22} - T_{33} + \mu) - CT_{24} + A\lambda_1 T_{34},\\
\mathfrak{P}_{322} &= -(\lambda_1 + \lambda_2)T_{12} - DT_{24} + bT_{34},\\
\mathfrak{P}_{323} &= -\lambda_1 T_{13},\\
\mathfrak{P}_{324} &= -\lambda_1 T_{14},\\
\mathfrak{P}_{411} &= b(T_{44} - \mu) + A(\lambda_1 + \lambda_2)T_{12} + CT_{13},\\
\mathfrak{P}_{412} &= -A\lambda_2(T_{11} - T_{22} + T_{44} - \mu) + DT_{13} - \lambda_2 T_{34},\\
\mathfrak{P}_{413} &= -bT_{13} + A\lambda_2 T_{23},\\
\mathfrak{P}_{414} &= -bT_{14} + A\lambda_2 T_{24},\\
\mathfrak{P}_{421} &= -A\lambda_1(T_{11} - T_{22} - T_{44} + \mu) + CT_{23} + \lambda_1 T_{34},\\
\mathfrak{P}_{422} &= b(T_{44} - \mu) - A(\lambda_1 + \lambda_2)T_{12} + DT_{23},\\
\mathfrak{P}_{423} &= -A\lambda_1 T_{13} - bT_{23},\\
\mathfrak{P}_{424} &= -A\lambda_1 T_{14} - bT_{24},\\
\mathfrak{P}_{431} &= -C(T_{11} - T_{33} - T_{44} + \mu) - DT_{12} + bT_{13} + \lambda_1(AT_{23} - T_{24}),\\
\mathfrak{P}_{432} &= -D(T_{22} - T_{33} - T_{44} + \mu) - CT_{12} - \lambda_2(AT_{13} - T_{14}) + bT_{23},\\
\mathfrak{P}_{433} &= -CT_{13} - DT_{23},\\
\mathfrak{P}_{434} &= -CT_{14} - DT_{24}.
\end{aligned}$$

Since $\lambda_1\lambda_2 \neq 0$ and

$$\mathfrak{P}_{323} = -\lambda_1 T_{13}, \quad \mathfrak{P}_{324} = -\lambda_1 T_{14}, \quad \mathfrak{P}_{313} = \lambda_2 T_{23}, \quad \mathfrak{P}_{314} = \lambda_2 T_{24},$$
$$\mathfrak{P}_{312} = -\lambda_2(T_{11} - T_{22} + T_{33} - \mu) - DT_{14} - A\lambda_2 T_{34},$$
$$\mathfrak{P}_{321} = -\lambda_1(T_{11} - T_{22} - T_{33} + \mu) - CT_{24} + A\lambda_1 T_{34},$$

it immediately follows that

$$T_{13} = T_{14} = T_{23} = T_{24} = 0, \quad T_{22} = T_{11}, \quad T_{33} = \mu - AT_{34}. \tag{5.21}$$

Using these conditions, we have

$$\mathfrak{P}_{412} = -\lambda_2 \left(T_{34} + A(T_{44} - \mu)\right),$$

which leads to

$$T_{34} = -A(T_{44} - \mu), \quad T_{33} = A^2(T_{44} - \mu) + \mu. \tag{5.22}$$

Now, a direct computation leads to

$$\mathfrak{P}_{311} = \quad (\lambda_1 + \lambda_2)T_{12} - Ab(T_{44} - \mu),$$
$$\mathfrak{P}_{322} = -(\lambda_1 + \lambda_2)T_{12} - Ab(T_{44} - \mu),$$

while the third equation in (5.20) now is

$$(\lambda_1 - \lambda_2)T_{12} + 2Ab(T_{44} - \mu) = 0,$$

which implies that

$$T_{12} = 0. \tag{5.23}$$

At this point, Equations (5.21), (5.22) and (5.23) show that

$$T = \begin{pmatrix} T_{11} & 0 & 0 & 0 \\ * & T_{11} & 0 & 0 \\ * & * & A^2(T_{44} - \mu) + \mu & -A(T_{44} - \mu) \\ * & * & * & T_{44} \end{pmatrix}.$$

The system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\mathfrak{P}_{311} = \mathfrak{P}_{322} = -Ab(T_{44} - \mu) = 0,$$
$$\mathfrak{P}_{411} = \mathfrak{P}_{422} = b(T_{44} - \mu) = 0,$$
$$\mathfrak{P}_{431} = -C\left(T_{11} - A^2(T_{44} - \mu) - T_{44}\right) = 0,$$
$$\mathfrak{P}_{432} = -D\left(T_{11} - A^2(T_{44} - \mu) - T_{44}\right) = 0,$$

and the vanishing conditions for the divergence of T given in Equation (5.20) are

$$Ab(T_{44}-\mu)=0,\quad b(T_{11}-T_{44})=0.$$

Note that if $b\neq 0$ the solution is determined by $T_{11}=T_{44}=\mu$ and the algebraic T-soliton is trivial. Finally, for $b=0$ the solution is given by $\mathfrak{P}_{431}=\mathfrak{P}_{432}=0$ and therefore $C=D=0$ or, otherwise, $T_{11}=A^2(T_{44}-\mu)+T_{44}$, which completes the proof. □

5.4.2 Algebraic Ricci solitons on $E(1,1)\rtimes\mathbb{R}$ and $\widetilde{E}(2)\rtimes\mathbb{R}$

In this case, the components ρ_{ij} of the Ricci tensor are given by

$$\begin{aligned}
2\rho_{11}&=(A^2+1)(\lambda_1^2-\lambda_2^2)-4b^2+C^2, & 2\rho_{13}&=AD\lambda_2-3bC,\\
2\rho_{12}&=-2Ab(\lambda_1-\lambda_2)+CD, & 2\rho_{14}&=-D\lambda_2,\\
2\rho_{22}&=-(A^2+1)(\lambda_1^2-\lambda_2^2)-4b^2+D^2, & 2\rho_{23}&=-AC\lambda_1-3bD,\\
2\rho_{33}&=-(\lambda_1-\lambda_2)^2-C^2-D^2, & 2\rho_{24}&=C\lambda_1,\\
2\rho_{44}&=-A^2(\lambda_1-\lambda_2)^2-4b^2-C^2-D^2, & 2\rho_{34}&=-A(\lambda_1-\lambda_2)^2.
\end{aligned}$$

We study the locally symmetric case in the first place. The possibilities we have in this situation are (see Remark 5.21) the Lie group $\widetilde{E}(2)\rtimes\mathbb{R}$ with $\lambda_2=\lambda_1$ and $C=D=0$, or $E(1,1)\rtimes\mathbb{R}$, with $\lambda_2=-\lambda_1$, $C=D=0$ and $b^2=(A^2+1)\lambda_1^2$. Besides, Theorem 5.23 guarantees that if there exists a non-trivial algebraic Ricci soliton, then necessarily $b=0$, so the latter case is not possible, while in the former case a direct calculation shows that the space is flat due to $b=0$.

In the non-symmetric case, according to Theorem 5.23, non-trivial algebraic Ricci solitons must satisfy $\rho_{14}=\rho_{24}=0$, from where it follows that $C=D=0$. Moreover, b must vanish. Now, Remark 5.24 shows that this case reduces to the previously considered situation of $\mathbb{R}^3\rtimes\mathbb{R}$.

Remark 5.25. Even though we have mentioned that the non-symmetric case reduces to $\mathbb{R}^3\rtimes\mathbb{R}$, we will solve it here for the sake of completeness. We showed that in this situation $b=0$ and $C=D=0$. Therefore, $\rho_{14}=\rho_{24}=0$ and also $\rho_{13}=\rho_{23}=0$, $\rho_{12}=0$. Note that we can assume that $\lambda_1\neq\lambda_2$ (see Remark 5.21) and so the algebraic Ricci solitons are characterized by

$$\rho_{11}=\rho_{22},\quad \rho_{33}=A^2(\rho_{44}-\mu)+\mu,\quad \rho_{34}=-A(\rho_{44}-\mu).$$

If we compute the components of the Ricci tensor, we obtain

$$\begin{aligned}
&\rho_{11}=-\rho_{22}=\tfrac{1}{2}(A^2+1)(\lambda_1^2-\lambda_2^2),\\
&\rho_{33}=-\tfrac{1}{2}(\lambda_1-\lambda_2)^2,\\
&\rho_{34}=A\rho_{33},\quad \rho_{44}=A^2\rho_{33},
\end{aligned}$$

which combined with the previous relations give

$$(A^2+1)(\lambda_1^2-\lambda_2^2)=0,$$
$$(A^2-1)\left(2\mu+(A^2+1)(\lambda_1-\lambda_2)^2\right)=0,$$
$$A\left(2\mu+(A^2+1)(\lambda_1-\lambda_2)^2\right)=0.$$

Therefore $\lambda_2 = -\lambda_1$, $\mu = -2(A^2+1)\lambda_1^2$ and the algebraic Ricci solitons are given by the left-invariant metrics

$$[e_1,e_3]=\lambda_1 e_2, \quad [e_1,e_4]=A\lambda_1 e_2, \quad [e_2,e_3]=\lambda_1 e_1, \quad [e_2,e_4]=A\lambda_1 e_1. \tag{5.24}$$

Finally, setting $\zeta = A^2+1$ and considering the orthogonal basis

$$\tilde{e}_1=\tfrac{1}{\lambda_1\sqrt{2\zeta}}(e_1+e_2), \quad \tilde{e}_2=-\tfrac{1}{\lambda_1\sqrt{2\zeta}}(e_1-e_2),$$
$$\tilde{e}_3=-\tfrac{1}{\lambda_1\zeta}(Ae_3-e_4), \quad \tilde{e}_4=\tfrac{1}{\lambda_1\zeta}(e_3+Ae_4),$$

the non-zero Lie brackets reduce to

$$[\tilde{e}_1,\tilde{e}_4]=\tilde{e}_1, \quad \text{and} \quad [\tilde{e}_2,\tilde{e}_4]=-\tilde{e}_2.$$

Besides, $\langle \tilde{e}_i,\tilde{e}_j\rangle = \frac{1}{\lambda_1^2\zeta}\langle e_i,e_j\rangle$, so these metrics are isomorphically homothetic to the metric in Theorem 5.4-*(iii)* for $f=-1$ and $p=0$.

5.4.3 Algebraic Bach solitons on $E(1,1) \rtimes \mathbb{R}$ and $\widetilde{E}(2) \rtimes \mathbb{R}$

First we compute the components $\mathfrak{B}_{ij}$ of the Bach tensor of $E(1,1) \rtimes \mathbb{R}$ and $\widetilde{E}(2) \rtimes \mathbb{R}$. A long but straightforward calculation shows that they are determined by

$$\begin{aligned}
24\mathfrak{B}_{11} &= -4(A^2+1)^2\left(5\lambda_1^4-3\lambda_2^4-3\lambda_1^3\lambda_2+\lambda_1\lambda_2^3\right)\\
&\quad+(A^2+1)\big((28b^2-40C^2+3D^2)\lambda_1^2-(20b^2-C^2-8D^2)\lambda_2^2\\
&\qquad\qquad-4(2b^2-3C^2-D^2)\lambda_1\lambda_2\big)\\
&\quad-42AbCD(\lambda_1+\lambda_2)+b^2(43C^2+D^2)-4(C^2+D^2)(5C^2+D^2),\\
12\mathfrak{B}_{12} &= 16A(A^2+1)b\left(\lambda_1^3-\lambda_2^3\right)-(A^2+1)CD\left(8\lambda_1^2+8\lambda_2^2+5\lambda_1\lambda_2\right)\\
&\quad+Ab\big((16C^2-5D^2)\lambda_1+(5C^2-16D^2)\lambda_2\big)+CD(21b^2-8C^2-8D^2),\\
12\mathfrak{B}_{13} &= -A(A^2+1)D\left(8\lambda_2^3-\lambda_1^2\lambda_2-4\lambda_1\lambda_2^2\right)\\
&\quad+3bC\left(8(A^2+1)\lambda_1^2-\lambda_2^2-(3A^2+4)\lambda_1\lambda_2\right)\\
&\quad+AD\left(9b^2\lambda_1+4(3b^2-2C^2-2D^2)\lambda_2\right)-3bC(3b^2-8C^2-8D^2),\\
12\mathfrak{B}_{14} &= -\lambda_2\left((A^2+1)D\left(\lambda_1^2-8\lambda_2^2+4\lambda_1\lambda_2\right)+3AbC(\lambda_1+\lambda_2)+D(3b^2-8C^2-8D^2)\right),\\
24\mathfrak{B}_{22} &= 4(A^2+1)^2\left(3\lambda_1^4-5\lambda_2^4-\lambda_1^3\lambda_2+3\lambda_1\lambda_2^3\right)\\
&\quad-(A^2+1)\big((20b^2-8C^2-D^2)\lambda_1^2-(28b^2+3C^2-40D^2)\lambda_2^2\\
&\qquad\qquad+4(2b^2-C^2-3D^2)\lambda_1\lambda_2\big)\\
&\quad+42AbCD(\lambda_1+\lambda_2)+b^2(C^2+43D^2)-4(C^2+D^2)(C^2+5D^2),
\end{aligned}$$

$$\begin{aligned}
12\mathfrak{B}_{23} =& A(A^2+1)C\left(8\lambda_1^3-4\lambda_1^2\lambda_2-\lambda_1\lambda_2^2\right)-3bD\left(\lambda_1^2-8(A^2+1)\lambda_2^2+(3A^2+4)\lambda_1\lambda_2\right)\\
&-AC\left(4(3b^2-2C^2-2D^2)\lambda_1+9b^2\lambda_2\right)-3bD(3b^2-8C^2-8D^2),\\
12\mathfrak{B}_{24} =& -\lambda_1\left((A^2+1)C\left(8\lambda_1^2-\lambda_2^2-4\lambda_1\lambda_2\right)+3AbD(\lambda_1+\lambda_2)-C(3b^2-8C^2-8D^2)\right),\\
24\mathfrak{B}_{33} =& -4(A^2-3)(A^2+1)\left(\lambda_1^4+\lambda_2^4-\lambda_1^3\lambda_2-\lambda_1\lambda_2^3\right)\\
&+((A^2+3)(8C^2-D^2)-12(A^2+1)b^2)\lambda_1^2-((A^2+3)(C^2-8D^2)+12(A^2+1)b^2)\lambda_2^2\\
&-4((A^2+3)(C^2+D^2)-6(A^2+1)b^2)\lambda_1\lambda_2-18AbCD(\lambda_1-\lambda_2)\\
&-3(C^2+D^2)(19b^2-4C^2-4D^2),\\
12\mathfrak{B}_{34} =& 8A(A^2+1)\left(\lambda_1^4+\lambda_2^4-\lambda_1^3\lambda_2-\lambda_1\lambda_2^3\right)\\
&+A\left((8C^2-D^2)\lambda_1^2-(C^2-8D^2)\lambda_2^2-4(C^2+D^2)\lambda_1\lambda_2\right)+9bCD(\lambda_1-\lambda_2).
\end{aligned}$$

Recall that the Bach tensor is trace-free, so $\mathfrak{B}_{44}=-\mathfrak{B}_{11}-\mathfrak{B}_{22}-\mathfrak{B}_{33}$. Next we analyse non-trivial algebraic Bach solitons using Theorem 5.23. Since b has to be zero, in the locally symmetric case, which has already been detailed in Remark 5.21, we only have to deal with the case $\lambda_2=\lambda_1$ and $C=D=0$. A direct calculation shows that the space is flat in this situation, and so it is Bach-flat.

In the non-symmetric case we set $b=0$ and assume that the corresponding Bach tensor is of the form given in Theorem 5.23. Since

$$\mathfrak{B}_{12}=-\tfrac{1}{24}CD\left((A^2+1)(5(\lambda_1+\lambda_2)^2+11(\lambda_1^2+\lambda_2^2))+16C^2+16D^2\right)$$

must vanish, it follows that $CD=0$. According to Remark 5.22, we can assume $C=0$ and work in the isometric class of the initial metric. Moreover, Remark 5.24 shows that the case where $D=0$ reduces to the previously considered situation of $\mathbb{R}^3\rtimes\mathbb{R}$. Hence, we assume that $D\neq 0$. Since

$$\mathfrak{B}_{14}=\tfrac{1}{12}D\lambda_2\left(8D^2-(A^2+1)(\lambda_1^2-8\lambda_2^2+4\lambda_1\lambda_2)\right)$$

must vanish, then

$$D^2=\tfrac{1}{8}(A^2+1)(\lambda_1^2-8\lambda_2^2+4\lambda_1\lambda_2). \tag{5.25}$$

Considering $\mathfrak{B}_{11}-\mathfrak{B}_{22}$, which must vanish, we obtain

$$\tfrac{3}{16}(A^2+1)^2\lambda_1^3(7\lambda_1-4\lambda_2)=0,$$

from where it immediately follows that $\lambda_2=\frac{7}{4}\lambda_1$. This leads to an incompatibility with Equation (5.25), since $\lambda_1^2-8\lambda_2^2+4\lambda_1\lambda_2=-\frac{33}{2}\lambda_1^2<0$, and therefore no algebraic Bach solitons exist if $D\neq 0$.

Remark 5.26. For the sake of completeness, we will solve the non-symmetric case when $D=0$, even though we already know that it reduces to $\mathbb{R}^3\rtimes\mathbb{R}$. If $b=0$ and $C=D=0$, then

$$\mathfrak{B}_{12}=\mathfrak{B}_{13}=\mathfrak{B}_{14}=\mathfrak{B}_{23}=\mathfrak{B}_{24}=0.$$

Moreover, since

$$\begin{aligned}
\mathfrak{B}_{11}&=-\tfrac{1}{6}(A^2+1)^2\left(5\lambda_1^4-3\lambda_2^4-3\lambda_1^3\lambda_2+\lambda_1\lambda_2^3\right),\\
\mathfrak{B}_{22}&=\tfrac{1}{6}(A^2+1)^2\left(3\lambda_1^4-5\lambda_2^4-\lambda_1^3\lambda_2+3\lambda_1\lambda_2^3\right),
\end{aligned}$$

must be equal, then

$$\mathfrak{B}_{11} - \mathfrak{B}_{22} = -\tfrac{2}{3}(A^2+1)^2(\lambda_1^2-\lambda_2^2)\left((\lambda_1 - \tfrac{1}{2}\lambda_2)^2 + \lambda_1^2 + \tfrac{7}{4}\lambda_2^2\right) = 0.$$

Hence, either $\lambda_2 = \lambda_1$ or $\lambda_2 = -\lambda_1$. In the former case ($b = C = D = 0$, $\lambda_2 = \lambda_1$) the space is flat (see Remark 5.21). In the latter case ($b = C = D = 0$, $\lambda_2 = -\lambda_1$) the algebraic Bach solitons are determined by

$$\mathfrak{B}_{33} = A^2(\mathfrak{B}_{44} - \mu) + \mu, \quad \mathfrak{B}_{34} = -A(\mathfrak{B}_{44} - \mu),$$

and a direct calculation shows that

$$\begin{aligned} \mathfrak{B}_{33} - A^2(\mathfrak{B}_{44} - \mu) - \mu &= (A^2-1)\left(\mu - 2(A^2+1)^2\lambda_1^4\right), \\ \mathfrak{B}_{34} + A(\mathfrak{B}_{44} - \mu) &= -A\left(\mu - 2(A^2+1)^2\lambda_1^4\right). \end{aligned}$$

We obtain $\mu = 2(A^2+1)^2\lambda_1^4$ and the associated left-invariant metrics are given by

$$[e_1, e_3] = \lambda_1 e_2, \quad [e_1, e_4] = A\lambda_1 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_2, e_4] = A\lambda_1 e_1,$$

so the algebraic Bach solitons are also the algebraic Ricci soliton (see Equation (5.24)) that have already been covered by the analysis of $\mathbb{R}^3 \rtimes \mathbb{R}$.

5.5 Algebraic solitons on $H^3 \rtimes \mathbb{R}$

We follow the description of the left-invariant metrics on semi-direct extensions of the Heisenberg group given in Section 1.4.2 and consider an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that

$$\begin{aligned} &[e_1, e_2] = \gamma e_3, && [e_1, e_4] = a e_1 - c e_2 + H e_3, \\ &[e_3, e_4] = (a+d) e_3, && [e_2, e_4] = c e_1 + d e_2 + F e_3, \end{aligned} \tag{5.26}$$

where $\gamma \neq 0$.

Remark 5.27. $H^3 \rtimes \mathbb{R}$ is locally symmetric if and only if the structure constants satisfy

$$a = d = \tfrac{\varepsilon}{2}\gamma \quad \text{and} \quad F = H = 0,$$

where $\varepsilon^2 = 1$. The isomorphism determined by $e_4 \mapsto -e_4$ is an orientation reversing isometry so we can set $\varepsilon = 1$ up to a change of orientation. A straightforward calculation shows that the anti-self-dual Weyl curvature operator W^- vanishes an so the underlying manifold is homothetic to the complex hyperbolic plane $\mathbb{CH}^2$ (see [55]). This is the only Einstein metric on $H^3 \rtimes \mathbb{R}$.

Remark 5.28. The isometry $(e_1, e_2, e_3, e_4) \mapsto (-e_2, e_1, e_3, e_4)$ transforms (γ, a, c, d, H, F) in Equation (5.26) into $(\gamma, d, c, a, -F, H)$. Thus, any left-invariant metric (5.26) with $H = 0$ is isomorphically isometric to a left-invariant metric with $F = 0$.

5.5.1 Algebraic T-solitons on $H^3 \rtimes \mathbb{R}$

Theorem 5.29. *$H^3 \rtimes \mathbb{R}$ is a non-trivial algebraic T-soliton with soliton constant μ if and only if one of the following conditions holds.*

(i) $a + d \neq 0$. *In this case, the space is isomorphically isometric to a Riemannian Lie group given by one of the following possibilities.*

(i.a) $a \neq \pm d$, $c = 0$, $F = 0$, *and the tensor field* $\widehat{T}$ *is diagonal,*

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{22}, T_{11} + T_{22} - \mu, \mu] \neq \mu\ \mathrm{Id},$$

with

$$H(T_{22} - \mu) = 0, \quad (2a + d)T_{11} + (a + 2d)T_{22} - 3(a + d)\mu = 0.$$

(i.b) $a = d \neq 0$, $c = 0$, $F = H = 0$ *and*

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ * & 2\mu - T_{11} & 0 & 0 \\ * & * & \mu & 0 \\ * & * & * & \mu \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle.$$

(ii) $a + d = 0$. *In this case, the space is isomorphically isometric to a Riemannian Lie group given by one of the following possibilities.*

(ii.a) $a = -d$, $a^2 \neq c^2$, $F = H = 0$ *and the tensor field* $\widehat{T}$ *is diagonal,*

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{11}, 2T_{11} - \mu, \mu], \quad T_{11} \neq \mu.$$

(ii.b) $a = d = c = 0$ *and*

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & \frac{1}{\gamma}\left(F(T_{22} - T_{33} + T_{44} - \mu) + HT_{12}\right) \\ * & T_{22} & 0 & \frac{-1}{\gamma}\left(H(T_{11} - T_{33} + T_{44} - \mu) + FT_{12}\right) \\ * & * & T_{33} & 0 \\ * & * & * & T_{44} \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle,$$

with

$$\begin{aligned}(H^2 - \gamma^2)T_{11} + (F^2 - \gamma^2)T_{22} - (F^2 + H^2 - \gamma^2)T_{33} \\ + (F^2 + H^2)T_{44} + 2FHT_{12} - (F^2 + H^2 - \gamma^2)\mu = 0.\end{aligned}$$

(ii.c) $a = -d$, $c = a \neq 0$ *and*

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & T_{14} \\ * & T_{11} & 0 & T_{14} \\ * & * & 2T_{11} - \frac{1}{\gamma}(F-H)T_{14} - \mu & 0 \\ * & * & * & \mu - 2T_{12} \end{pmatrix} \neq \mu \langle \cdot, \cdot \rangle$$

where

$$HT_{11} - (F-2H)T_{12} - \tfrac{1}{\gamma}\left(H(F-H) + \gamma^2\right)T_{14} - H\mu = 0,$$

$$FT_{11} + (2F-H)T_{12} - \tfrac{1}{\gamma}\left(F(F-H) - \gamma^2\right)T_{14} - F\mu = 0.$$

Remark 5.30. If $a = c = d = 0$ as in Assertion *(ii.b)* in the previous theorem, then the left-invariant metrics associated to the corresponding algebraic T-solitons are given by

$$[e_1, e_2] = \gamma e_3, \quad [e_1, e_4] = He_3, \quad [e_2, e_4] = Fe_3.$$

Taking $\zeta = \sqrt{F^2 + H^2 + \gamma^2}$ and considering the orthonormal basis

$$\begin{aligned} \tilde{e}_1 &= \tfrac{1}{\zeta}\left(Fe_1 - He_2 + \gamma e_4\right), \\ \tilde{e}_2 &= \tfrac{1}{\sqrt{2(F^2+H^2)}}\left(He_1 + Fe_2 - \sqrt{F^2+H^2}e_3\right), \\ \tilde{e}_3 &= \tfrac{-1}{\sqrt{2(F^2+H^2)}}\left(He_1 + Fe_2 + \sqrt{F^2+H^2}e_3\right), \\ \tilde{e}_4 &= \tfrac{1}{\zeta\sqrt{F^2+H^2}}\left(\gamma Fe_1 - \gamma He_2 - (F^2+H^2)e_4\right), \end{aligned}$$

the only non-zero brackets now correspond to

$$[\tilde{e}_2, \tilde{e}_4] = \tfrac{\zeta}{2}(\tilde{e}_2 + \tilde{e}_3), \quad [\tilde{e}_3, \tilde{e}_4] = -\tfrac{\zeta}{2}(\tilde{e}_2 + \tilde{e}_3).$$

These metrics are isomorphically isometric to metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$ and therefore the corresponding algebraic T-solitons have already been covered by Theorem 5.19.

We will show that the same conclusion holds true when the structure constants $a = -d$ and $a = c \neq 0$ as in Theorem 5.29-*(ii.c)* if, in addition, $H = -F$ and $4a^2 - 2F^2 - \gamma^2 = 0$ (equivalently, the Ricci operator is of rank three). The left-invariant metrics associated to the algebraic T-solitons are given by

$$[e_1, e_2] = \gamma e_3, \quad [e_1, e_4] = ae_1 - ae_2 - Fe_3, \quad [e_2, e_4] = ae_1 - ae_2 + Fe_3, \tag{5.27}$$

and if we consider the orthonormal basis determined by

$$\begin{aligned} \tilde{e}_1 &= \tfrac{1}{2a\sqrt{2}}\left(Fe_1 + Fe_2 + 2ae_3 + \gamma e_4\right), \\ \tilde{e}_2 &= \tfrac{-1}{4a}\left((2a-F)e_1 - (2a+F)e_2 + 2ae_3 - \gamma e_4\right), \\ \tilde{e}_3 &= \tfrac{-1}{4a}\left((2a+F)e_1 - (2a-F)e_2 - 2ae_3 + \gamma e_4\right), \\ \tilde{e}_4 &= \tfrac{1}{2a\sqrt{2}}\left(\gamma e_1 + \gamma e_2 - 2Fe_4\right), \end{aligned}$$

the non-zero brackets correspond to

$$[\tilde{e}_1, \tilde{e}_4] = a(\tilde{e}_2 + \tilde{e}_3), \quad [\tilde{e}_2, \tilde{e}_4] = -a(\tilde{e}_1 - \sqrt{2}\tilde{e}_2), \quad [\tilde{e}_3, \tilde{e}_4] = -a(\tilde{e}_1 + \sqrt{2}\tilde{e}_3).$$

Thus, the above metrics are isomorphically isometric to metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$ and so the corresponding algebraic T-solitons have already been covered by Theorem 5.19.

Proof. Since we are working with a divergence-free tensor field T of type $(0, 2)$, a direct calculation shows that this condition is equivalent to

$$\begin{aligned}
&(3a + 2d)T_{14} + \gamma T_{23} - cT_{24} + HT_{34} = 0,\\
&\gamma T_{13} - cT_{14} - (2a + 3d)T_{24} - FT_{34} = 0,\\
&3(a + d)T_{34} = 0,\\
&aT_{11} + dT_{22} + (a + d)(T_{33} - 2T_{44}) + HT_{13} + FT_{23} = 0.
\end{aligned} \tag{5.28}$$

The conditions for $\mathfrak{D} = \widehat{T} - \mu\,\mathrm{Id}$ to be a derivation can be expressed in terms of a system of twenty two (up to duplicities) polynomial equations on the soliton constant μ, the structure constants (5.26) and the components T_{ij}, given by $\{\mathfrak{P}_{ijk} = 0\}$, where

$$\begin{aligned}
\mathfrak{P}_{211} &= -\gamma T_{13} - cT_{14} + aT_{24},\\
\mathfrak{P}_{212} &= -dT_{14} - \gamma T_{23} - cT_{24},\\
\mathfrak{P}_{213} &= \gamma(T_{11} + T_{22} - T_{33} - \mu) - FT_{14} + HT_{24},\\
\mathfrak{P}_{214} &= -\gamma T_{34},\\
\mathfrak{P}_{311} &= aT_{34},\\
\mathfrak{P}_{312} &= -cT_{34},\\
\mathfrak{P}_{313} &= -(a + d)T_{14} + \gamma T_{23} + HT_{34},\\
\mathfrak{P}_{321} &= cT_{34},\\
\mathfrak{P}_{322} &= dT_{34},\\
\mathfrak{P}_{323} &= -\gamma T_{13} - (a + d)T_{24} + FT_{34},\\
\mathfrak{P}_{411} &= a(T_{44} - \mu) + 2cT_{12} - HT_{13},\\
\mathfrak{P}_{412} &= -c(T_{11} - T_{22} + T_{44} - \mu) - (a - d)T_{12} - HT_{23},\\
\mathfrak{P}_{413} &= H(T_{11} - T_{33} + T_{44} - \mu) + FT_{12} + dT_{13} + cT_{23} + \gamma T_{24},\\
\mathfrak{P}_{414} &= -aT_{14} + cT_{24} - HT_{34},\\
\mathfrak{P}_{421} &= -c(T_{11} - T_{22} - T_{44} + \mu) + (a - d)T_{12} - FT_{13},\\
\mathfrak{P}_{422} &= d(T_{44} - \mu) - 2cT_{12} - FT_{23},\\
\mathfrak{P}_{423} &= F(T_{22} - T_{33} + T_{44} - \mu) + HT_{12} - cT_{13} - \gamma T_{14} + aT_{23},\\
\mathfrak{P}_{424} &= -cT_{14} - dT_{24} - FT_{34},\\
\mathfrak{P}_{431} &= -dT_{13} + cT_{23},\\
\mathfrak{P}_{432} &= -cT_{13} - aT_{23},
\end{aligned}$$

$\mathfrak{P}_{433} = (a+d)(T_{44} - \mu) + HT_{13} + FT_{23},$
$\mathfrak{P}_{434} = -(a+d)T_{34}.$

We start by considering

$$\mathfrak{P}_{214} = -\gamma T_{34},$$

$$\mathfrak{P}_{211} + \mathfrak{P}_{323} - \mathfrak{P}_{424} = -2(\gamma T_{13} - FT_{34}),$$

$$\mathfrak{P}_{212} - \mathfrak{P}_{313} + \mathfrak{P}_{414} = -2(\gamma T_{23} + HT_{34}),$$

which imply that

$$T_{13} = T_{23} = T_{34} = 0, \tag{5.29}$$

and split our analysis differentiating the cases $a + d \neq 0$ and $a + d = 0$.

Case 1: $a + d \neq 0$**.**

Using Equation (5.29), since

$$\mathfrak{P}_{313} = -(a+d)T_{14},$$

$$\mathfrak{P}_{323} = -(a+d)T_{24},$$

$$\mathfrak{P}_{433} = (a+d)(T_{44} - \mu),$$

$$\mathfrak{P}_{213} = \gamma(T_{11} + T_{22} - T_{33} - \mu) - FT_{14} + HT_{24},$$

and $a + d \neq 0$ we obtain

$$T_{14} = T_{24} = 0, \quad T_{44} = \mu, \quad T_{33} = T_{11} + T_{22} - \mu. \tag{5.30}$$

Next we show that the situation is different depending on whether c vanishes or not.

Case 1.1: $c \neq 0$.

In this case Equations (5.29) and (5.30) imply that

$$\mathfrak{P}_{411} = 2cT_{12}, \quad \mathfrak{P}_{421} = -c(T_{11} - T_{22}) + (a-d)T_{12},$$

so the condition $c \neq 0$ leads to

$$T_{12} = 0, \quad T_{22} = T_{11}. \tag{5.31}$$

At this point, Equations (5.29), (5.30) and (5.31) make $\widehat{T}$ diagonal,

$$\widehat{T} = \mathrm{diag}[T_{11}, T_{11}, 2T_{11} - \mu, \mu]$$

and, moreover, the vanishing conditions of the divergence of T given in Equation (5.28) reduce to

$$(a+d)(T_{11} - \mu) = 0.$$

Hence, we conclude that there are no non-trivial algebraic T-soliton in this case.

Case 1.2: $c = 0$.

Using Equation (5.30) together with Equation (5.29) and $c = 0$, it is straightforward to see that

$$T = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ * & T_{22} & 0 & 0 \\ * & * & T_{11} + T_{22} - \mu & 0 \\ * & * & * & \mu \end{pmatrix}.$$

The system $\{\mathfrak{P}_{ijk} = 0\}$ now corresponds to

$$\mathfrak{P}_{412} = -\mathfrak{P}_{421} = -(a-d)T_{12} = 0,$$
$$\mathfrak{P}_{413} = FT_{12} - H(T_{22} - \mu) = 0,$$
$$\mathfrak{P}_{423} = -F(T_{11} - \mu) + HT_{12} = 0,$$

while the vanishing of the divergence of T given by Equation (5.28) reduces to

$$(2a+d)T_{11} + (a+2d)T_{22} - 3(a+d)\mu = 0.$$

If $a - d \neq 0$, then $T_{12} = 0$. Besides, if $FH \neq 0$, then $T_{11} = T_{22} = \mu$ and the tensor field T is trivial. Thus, according to Remark 5.28, we can take $F = 0$ (working with an isomorphically isometric metric if necessary) and Assertion *(i.a)* immediately follows.

If $a = d$, they are both different from zero, since $a + d \neq 0$. Therefore, the last equation above implies that

$$T_{22} = 2\mu - T_{11}$$

while the other equations give

$$FT_{12} + H(T_{11} - \mu) = 0, \quad HT_{12} - F(T_{11} - \mu) = 0.$$

Note that if either F or H is non-zero, then the corresponding algebraic T-soliton is trivial. Therefore, $F = H = 0$ and Assertion *(i.b)* follows.

Case 2: $a + d = 0$**.**

In this situation, we distinguish two cases depending on whether a^2 and c^2 are equal or not.

Case 2.1: $a^2 \neq c^2$.

A direct calculation involving Equation (5.29) and the condition $d = -a$ shows that

$$\begin{aligned}
&\mathfrak{P}_{211} = -cT_{14} + aT_{24}, \\
&\mathfrak{P}_{212} = \quad aT_{14} - cT_{24}, \\
&2a\mathfrak{P}_{411} + c(\mathfrak{P}_{412} - \mathfrak{P}_{421}) = 2(a^2 - c^2)(T_{44} - \mu), \\
&\mathfrak{P}_{213} = \gamma(T_{11} + T_{22} - T_{33} - \mu) - FT_{14} + HT_{24},
\end{aligned}$$

so $a^2 \neq c^2$ implies that

$$T_{14} = T_{24} = 0, \quad T_{44} = \mu, \quad T_{33} = T_{11} + T_{22} - \mu. \tag{5.32}$$

This last equation together with (5.29) lead to

$$\mathfrak{P}_{411} = 2cT_{12}, \quad \mathfrak{P}_{421} = 2aT_{12} - c(T_{11} - T_{22}),$$

while the vanishing of the divergence of T given in Equation (5.28) reduces to

$$a(T_{11} - T_{22}) = 0.$$

Hence, since $a^2 \neq c^2$, it follows that

$$T_{12} = 0, \quad T_{11} = T_{22}. \tag{5.33}$$

Putting together Equations (5.29), (5.32) and (5.33) we obtain that $\widehat{T}$ is diagonal,

$$\widehat{T} = \operatorname{diag}[T_{11}, T_{11}, 2T_{11} - \mu, \mu],$$

and the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\mathfrak{P}_{413} = H(\mu - T_{11}) = 0, \quad \mathfrak{P}_{423} = F(\mu - T_{11}) = 0.$$

Thus, there are non-trivial algebraic T-solitons when $T_{11} \neq \mu$ and $F = H = 0$, so Assertion *(ii.a)* is obtained.

Case 2.2: $a^2 = c^2$.

We set $c = \varepsilon a$, with $\varepsilon^2 = 1$, and assume that the conditions in Equation (5.29) hold. In this case, Equation (5.28), which gives the conditions for T to be divergence-free, transforms into

$$a(T_{11} - T_{22}) = 0, \quad a(\varepsilon T_{14} - T_{24}) = 0.$$

If $a = 0$, then $c = d = 0$ and the tensor T is divergence-free, while the system $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\mathfrak{P}_{423} = F(T_{22} - T_{33} + T_{44} - \mu) + HT_{12} - \gamma T_{14} = 0,$$

$$\mathfrak{P}_{413} = H(T_{11} - T_{33} + T_{44} - \mu) + FT_{12} + \gamma T_{24} = 0,$$

$$\mathfrak{P}_{213} = \gamma(T_{11} + T_{22} - T_{33} - \mu) - FT_{14} + HT_{24} = 0.$$

Clearing T_{14} and T_{24} in the first and second equations above, respectively, Assertion *(ii.b)* is immediately obtained.

If $a \neq 0$, then we compute

$$\mathfrak{P}_{411} = a(2\varepsilon T_{12} + T_{44} - \mu),$$

which together with the conditions for T to be a divergence-free tensor give

$$T_{22} = T_{11}, \quad T_{24} = \varepsilon T_{14}, \quad T_{44} = \mu - 2\varepsilon T_{12}.$$

Thus, the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$ reduces to

$$\mathfrak{P}_{213} = \gamma(2T_{11} - T_{33} - \mu) - (F - \varepsilon H)T_{14} = 0,$$

$$\mathfrak{P}_{413} = H(T_{11} - T_{33}) + (F - 2\varepsilon H)T_{12} + \varepsilon\gamma T_{14} = 0,$$

$$\mathfrak{P}_{423} = F(T_{11} - T_{33}) + (H - 2\varepsilon F)T_{12} - \gamma T_{14} = 0,$$

and from the first equation above we obtain an expression for T_{33}. Note that the corresponding left-invariant metrics are given by

$$[e_1, e_2] = \gamma e_3, \quad [e_1, e_4] = ae_1 - \varepsilon a e_2 + He_3, \quad [e_2, e_4] = \varepsilon a e_1 - ae_2 + Fe_3,$$

and the isometry

$$(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_2, -e_3, e_4)$$

interchanges (ε, a, F, H) and $(-\varepsilon, a, F, -H)$. Thus we can take $\varepsilon = 1$ working, if necessary, with an isomorphically isometric metric and Assertion *(ii.c)* is obtained. □

5.5.2 Algebraic Ricci solitons on $H^3 \rtimes \mathbb{R}$

A straightforward calculation shows that the Ricci tensor is determined by

$$\begin{aligned}
2\rho_{11} &= -4a(a+d) - H^2 - \gamma^2, & 2\rho_{12} &= -2c(a-d) - FH,\\
2\rho_{13} &= -H(2a+3d) + Fc, & 2\rho_{14} &= F\gamma,\\
2\rho_{22} &= -4d(a+d) - F^2 - \gamma^2, & 2\rho_{23} &= -F(3a+2d) - Hc,\\
2\rho_{33} &= -4(a+d)^2 + F^2 + H^2 + \gamma^2, & 2\rho_{24} &= -H\gamma,\\
2\rho_{44} &= -4(a^2+d^2+ad) - F^2 - H^2. & &
\end{aligned}$$

We use Theorem 5.29 to analyse all the possibilities for non-trivial algebraic Ricci solitons. In the locally symmetric case, Remark 5.27 guarantees that $a = d = \pm\frac{1}{2}\gamma$, $F = H = 0$, and a direct calculation shows that

$$\mathrm{Ric} = -\tfrac{3}{2}\gamma_1^2 \mathrm{Id},$$

so the corresponding algebraic Ricci solitons are trivial.

In the non-symmetric case, we study each of the five cases in Theorem 5.29 separately. Note that we can take $\gamma = 1$ and work in the same homothetic class.

Case *(i.a)*.

We take $c = 0$, $F = 0$ and $d \neq \pm a$. Algebraic Ricci solitons must have diagonal Ricci operator, so the expression $\rho_{24} = -\frac{1}{2}H$ implies that $H = 0$. With this last condition, the Ricci operator is diagonal. Now, algebraic Ricci solitons are characterized by the vanishing of the polynomials

$$\mathfrak{Q}_1 = \rho_{33} - \rho_{11} - \rho_{22} + \mu, \quad \mathfrak{Q}_2 = \rho_{44} - \mu,$$

$$\mathfrak{Q}_3 = (2a+d)\rho_{11} + (a+2d)\rho_{22} - 3(a+d)\mu,$$

and a direct calculation shows that

$$\begin{aligned}
\mathfrak{Q}_1 &= \mu + \tfrac{3}{2},\\
\mathfrak{Q}_2 &= -\mu - 2(a^2+d^2+ad),\\
\mathfrak{Q}_3 &= -\tfrac{1}{2}(a+d)\left(6\mu + 8(a^2+d^2+ad) + 3\right).
\end{aligned}$$

Therefore the system $\{\mathfrak{Q}_i = 0\}$ is equivalent to

$$4(a^2+d^2+ad) - 3 = 0, \quad \mu = -\tfrac{3}{2},$$

which determine algebraic Ricci solitons with associated left-invariant metrics given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1, \quad [e_2, e_4] = de_2, \quad [e_3, e_4] = (a+d)e_3. \tag{5.34}$$

Note that the isometry $e_4 \mapsto -e_4$ transforms (a,d) into $(-a,-d)$, while the isometry

$$(e_1,e_2,e_3,e_4) \mapsto (e_2,-e_1,e_3,e_4)$$

transforms (a,d) into (d,a). As a consequence of this together with $d \neq \pm a$, we can assume that $|a| < d$. Besides, the condition $4(a^2+d^2+ad)-3=0$ implies that $a \in (-\frac{\sqrt{3}}{2}, \frac{1}{2})$. This corresponds to Theorem 5.4-*(iv)* for $a \neq -\frac{\sqrt{3}}{2}$.

Case *(i.b)*.

Since $a = d \neq 0$ and $c = F = H = 0$, a direct calculation shows that the Ricci tensor satisfies

$$\rho_{33} = -8a^2 + \tfrac{1}{2} \quad \text{and} \quad \rho_{44} = -6a^2.$$

Since an algebraic Ricci soliton must satisfy $\rho_{33} = \rho_{44} = \mu$, we get $a = \pm\frac{1}{2}$, which implies that the space is locally symmetric (see Remark 5.27).

Case *(ii.a)*.

In this case, $d = -a$, $a^2 \neq c^2$ and $F = H = 0$. A direct calculation shows that $\rho_{11} = \rho_{22} = -\frac{1}{2}$ and, moreover, the Ricci operator is diagonal if and only if $\rho_{12} = 0$, so the algebraic Ricci solitons are determined by the vanishing of the polynomials

$$\mathfrak{Q}_1 = \rho_{12}, \quad \mathfrak{Q}_2 = \rho_{33} - 2\rho_{11} + \mu, \quad \mathfrak{Q}_3 = \rho_{44} - \mu.$$

Computing these expressions we obtain

$$\mathfrak{Q}_1 = -2ac, \quad \mathfrak{Q}_2 = \mu + \tfrac{3}{2}, \quad \mathfrak{Q}_3 = -\mu - 2a^2,$$

which imply that $c = 0$ and $a = \pm\frac{\sqrt{3}}{2}$. With these conditions we obtain an algebraic Ricci soliton with associated left-invariant metric given by Equation (5.34) with

$$a = -d = \pm\tfrac{\sqrt{3}}{2}.$$

As we did in the previous case, we can set $a = -\frac{\sqrt{3}}{2}$, and this metric corresponds to Theorem 5.4-*(iv)* for this value of a.

Case *(ii.b)*.

This case has already been covered by the analysis on $\mathbb{R}^3 \rtimes \mathbb{R}$ (see Remark 5.30).

Case *(ii.c)*.

In this last case $d = -a$ and $c = a \neq 0$. Since

$$\rho_{34} = 0 \quad \text{and} \quad \rho_{13} = -\rho_{23} = \tfrac{1}{2}a(F+H),$$

then we must take $H = -F$ in order for the Ricci tensor to have the matrix form given in Theorem 5.29-*(ii.c)*. Now, an algebraic Ricci soliton is characterized by the vanishing of the polynomials

$$\mathfrak{Q}_1 = \rho_{11} - \rho_{22}, \qquad \mathfrak{Q}_2 = \rho_{33} - 2\rho_{11} + 2F\rho_{14} + \mu,$$
$$\mathfrak{Q}_3 = \rho_{44} + 2\rho_{12} - \mu, \qquad \mathfrak{Q}_4 = \rho_{24} - \rho_{14},$$
$$\mathfrak{Q}_5 = -F\rho_{11} - 3F\rho_{12} + (2F^2-1)\rho_{14} + F\mu,$$

and a direct calculation leads to $\mathfrak{Q}_1 = \mathfrak{Q}_4 = 0$ and

$$\mathfrak{Q}_2 = \mu + 3F^2 + \tfrac{3}{2}, \quad \mathfrak{Q}_3 = -\mu - 6a^2, \quad \mathfrak{Q}_5 = F(\mu + 6a^2). \tag{5.35}$$

Consequently, $4a^2 - 2F^2 - 1 = 0$ and it follows from Remark 5.30 that this case has already been covered by the analysis on $\mathbb{R}^3 \rtimes \mathbb{R}$.

Remark 5.31. For the sake of completeness, we will include here the solution of the two cases covered by the analysis on $\mathbb{R}^3 \rtimes \mathbb{R}$.

Case *(ii.b)* in Theorem 5.29 is given by $a = c = d = 0$. In this case, it is easy to check that $\rho_{13} = \rho_{23} = \rho_{34} = 0$, so the matrix form in Theorem 5.29-*(ii.b)* is satisfied and an algebraic Ricci soliton is determined by the vanishing of the polynomials

$$\mathfrak{Q}_1 = \rho_{14} - F(\rho_{22} - \rho_{33} + \rho_{44} - \mu) - H\rho_{12},$$
$$\mathfrak{Q}_2 = \rho_{24} + H(\rho_{11} - \rho_{33} + \rho_{44} - \mu) + F\rho_{12},$$
$$\begin{aligned}\mathfrak{Q}_3 = {} & (H^2-1)\rho_{11} + (F^2-1)\rho_{22} - (F^2+H^2-1)\rho_{33} \\ & + (F^2+H^2)\rho_{44} + 2FH\rho_{12} - (F^2+H^2-1)\mu.\end{aligned}$$

A direct calculation gives

$$\begin{aligned}\mathfrak{Q}_1 &= \tfrac{1}{2}F(3F^2+3H^2+2\mu+3), \\ \mathfrak{Q}_2 &= -\tfrac{1}{2}H(3F^2+3H^2+2\mu+3), \\ \mathfrak{Q}_3 &= -\tfrac{1}{2}(F^2+H^2-1)(3F^2+3H^2+2\mu+3),\end{aligned}$$

so the system $\{\mathfrak{Q}_i = 0\}$ is equivalent to $\mu = -\frac{3}{2}(F^2+H^2+1)$. The associated left-invariant metrics are given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = He_1, \quad [e_2, e_4] = Fe_3, \tag{5.36}$$

and considering the orthogonal basis

$$\begin{aligned}
\tilde{e}_1 &= \tfrac{-2}{F^2+H^2+1}\left(Fe_1 - He_2 + e_4\right),\\
\tilde{e}_2 &= \tfrac{\sqrt{2}}{\sqrt{(F^2+H^2+1)(F^2+H^2)}}\left(He_1 + Fe_2 - \sqrt{F^2+H^2}e_3\right),\\
\tilde{e}_3 &= \tfrac{-\sqrt{2}}{\sqrt{(F^2+H^2+1)(F^2+H^2)}}\left(He_1 + Fe_2 + \sqrt{F^2+H^2}e_3\right),\\
\tilde{e}_4 &= \tfrac{2}{(F^2+H^2+1)\sqrt{F^2+H^2}}\left(Fe_1 - He_2 - (F^2+H^2)e_4\right),
\end{aligned}$$

the only non-zero brackets are

$$[\tilde{e}_2, \tilde{e}_4] = \tilde{e}_2 + \tilde{e}_3 \quad \text{and} \quad [\tilde{e}_3, \tilde{e}_4] = -\tilde{e}_2 - \tilde{e}_3.$$

Moreover, $\langle \tilde{e}_i, \tilde{e}_j \rangle = \frac{4}{F^2+H^2+1}\langle e_i, e_j \rangle$, which shows that the above metric Lie groups are isomorphically homothetic to the metric in Theorem 5.4-*(i)*.

Finally, in Case *(ii.c)* of Theorem 5.29, using Equation (5.35) we obtain that $4a^2-2F^2-1=0$ and $\mu = -6a^2$ are the conditions that determine the algebraic Ricci solitons in this case. These have associated left-invariant metrics given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1 - ae_2 - Fe_3, \quad [e_2, e_4] = ae_1 - ae_2 + Fe_3. \tag{5.37}$$

Considering the orthogonal basis

$$\begin{aligned}
\tilde{e}_1 &= \tfrac{1}{4a^2}\left(Fe_1 + Fe_2 + 2ae_3 + e_4\right),\\
\tilde{e}_2 &= \tfrac{-1}{4a^2\sqrt{2}}\left((2a-F)e_1 - (2a+F)e_2 + 2ae_3 - e_4\right),\\
\tilde{e}_3 &= \tfrac{-1}{4a^2\sqrt{2}}\left((2a+F)e_1 - (2a-F)e_2 - 2ae_3 + e_4\right),\\
\tilde{e}_4 &= \tfrac{1}{4a^2}\left(e_1 + e_2 - 2Fe_4\right),
\end{aligned}$$

the non-zero brackets correspond to

$$[\tilde{e}_1, \tilde{e}_4] = \tfrac{1}{\sqrt{2}}(\tilde{e}_2 + \tilde{e}_3), \quad [\tilde{e}_2, \tilde{e}_4] = -\tfrac{1}{\sqrt{2}}\tilde{e}_1 + \tilde{e}_2, \quad [\tilde{e}_3, \tilde{e}_4] = -\tfrac{1}{\sqrt{2}}\tilde{e}_1 - \tilde{e}_3,$$

and $\langle \tilde{e}_i, \tilde{e}_j \rangle = \frac{1}{2a^2}\langle e_i, e_j \rangle$. Hence, the metric Lie groups above are isomorphically homothetic to the metric given in Assertion *(ii)* of Theorem 5.4.

5.5.3 Algebraic Bach solitons on $H^3 \rtimes \mathbb{R}$

A long but straightforward calculation shows that the components $\mathfrak{B}_{ij}$ of the Bach tensor of $H^3 \rtimes \mathbb{R}$ are determined by

$$\begin{aligned}
24\mathfrak{B}_{11} = &-16a^3d + 48ad^3 + 84a^2c^2 + 16a^2d^2 - 108c^2d^2 + 24ac^2d\\
&+ (F^2 - 20H^2 - 20\gamma^2)a^2 - 21(F^2 - H^2)c^2 - 3(4F^2 + 19H^2 + 4\gamma^2)d^2\\
&+ 78FHac - 4(22H^2 + 7\gamma^2)ad + 78FHcd\\
&- 4(F^2 + H^2 + \gamma^2)(F^2 - 3H^2 - 3\gamma^2),
\end{aligned}$$

$$\begin{aligned}
12\mathfrak{B}_{12} &= -18a^3c + 24ac^3 - 24c^3d + 18cd^3 - 58a^2cd + 58acd^2 \\
&\quad - 3FH\,(4a^2 - 7c^2 + 4d^2) + (31F^2 - 8H^2 - 2\gamma^2)ac - 53FHad \\
&\quad + (8F^2 - 31H^2 + 2\gamma^2)cd + 8FH(F^2 + H^2 + \gamma^2), \\
12\mathfrak{B}_{13} &= -3Fc^3 - 9Hd^3 + 33Fa^2c + 3Hac^2 - 28Ha^2d - 48Had^2 + 24Hc^2d \\
&\quad - 9Fcd^2 + 53Facd + 8(F^2 + H^2 + \gamma^2)(2Ha - Fc + 3Hd), \\
12\mathfrak{B}_{14} &= \gamma\left(3F\,(a^2 - c^2) - 3Hac + 14Fad + 15Hcd - 8F(F^2 + H^2 + \gamma^2)\right), \\
24\mathfrak{B}_{22} &= 48a^3d - 16ad^3 - 108a^2c^2 + 16a^2d^2 + 84c^2d^2 + 24ac^2d \\
&\quad - 3(19F^2 + 4H^2 + 4\gamma^2)a^2 + 21(F^2 - H^2)c^2 - (20F^2 - H^2 + 20\gamma^2)d^2 \\
&\quad - 78FHac - 4(22F^2 + 7\gamma^2)ad - 78FHcd \\
&\quad + 4(F^2 + H^2 + \gamma^2)(3F^2 - H^2 + 3\gamma^2), \\
12\mathfrak{B}_{23} &= -9Fa^3 + 3Hc^3 + 9Ha^2c + 24Fac^2 - 48Fa^2d - 28Fad^2 + 3Fc^2d \\
&\quad - 33Hcd^2 - 53Hacd + 8(F^2 + H^2 + \gamma^2)(3Fa + Hc + 2Fd), \\
12\mathfrak{B}_{24} &= \gamma\left(3H\,(c^2 - d^2) + 15Fac - 14Had - 3Fcd + 8H(F^2 + H^2 + \gamma^2)\right), \\
24\mathfrak{B}_{33} &= -16a^3d - 16ad^3 - 12a^2c^2 - 48a^2d^2 - 12c^2d^2 + 24ac^2d \\
&\quad + (43F^2 + 28H^2 + 28\gamma^2)a^2 - 9(F^2 + H^2)c^2 + (28F^2 + 43H^2 + 28\gamma^2)d^2 \\
&\quad - 54FHac + (104F^2 + 104H^2 + 44\gamma^2)ad + 54FHcd - 20(F^2 + H^2 + \gamma^2)^2.
\end{aligned}$$

Recall that the Bach tensor is trace-free, so $\mathfrak{B}_{44} = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33}$. We study the existence of non-trivial algebraic Bach solitons using Theorem 5.29. In the locally symmetric case, the underlying manifold is homothetic to the complex hyperbolic plane (see Remark 5.27), and so it is self-dual, and thus Bach-flat.

In the non-symmetric case, we proceed as in the previous section by checking the five cases given in Theorem 5.29 separately. Without loss of generality we can set $\gamma = 1$ remaining in the same homothetic class.

Case *(i.a)*.

In this first case, $c = 0$, $F = 0$ and $d \neq \pm a$. Note that

$$\mathfrak{B}_{12} = \mathfrak{B}_{14} = \mathfrak{B}_{23} = \mathfrak{B}_{34} = 0$$

and recall that $\mathfrak{B}_{44} = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33}$. If we now impose the diagonal form and the conditions in Theorem 5.29-*(i.a)* to the Bach operator we have that an algebraic Bach soliton is determined by the vanishing of the polynomials

$$\mathfrak{Q}_1 = \mathfrak{B}_{13}, \quad \mathfrak{Q}_2 = \mathfrak{B}_{24}, \quad \mathfrak{Q}_3 = -2(\mathfrak{B}_{11} + \mathfrak{B}_{22}),$$

$$\mathfrak{Q}_4 = H(\mathfrak{B}_{11} + 2\mathfrak{B}_{22} + \mathfrak{B}_{33}),$$

$$\mathfrak{Q}_5 = (5a + 4d)\mathfrak{B}_{11} + (4a + 5d)\mathfrak{B}_{22} + 3(a + d)\mathfrak{B}_{33},$$

$$\mathfrak{Q}_6 = \mu + \mathfrak{B}_{11} + \mathfrak{B}_{22} + \mathfrak{B}_{33}.$$

First of all we use $\mathfrak{Q}_2$ and $\mathfrak{Q}_3$ to show that H must be zero. Since

$$\mathfrak{Q}_2 = -\tfrac{1}{12}H(3d^2 + 14ad - 8H^2 - 8)$$

must vanish, if $H \neq 0$ then d cannot be null and $a = \frac{-1}{14d}(3d^2 - 8H^2 - 8)$. With this value for a a long but direct calculation shows that

$$\mathfrak{Q}_3 = \tfrac{1}{223636d^2}\left(d^2 + 2H^2 + 2\right)\left(652H^2(407d^2 + 64H^2 + 128) + (326d^2 - 83)^2 + 34839\right)$$

which leads to a contradiction since $\mathfrak{Q}_3 = 0$ and none of the factors above vanishes. Consequently $H = 0$ and the system $\{\mathfrak{Q}_i = 0\}$ becomes

$$\begin{aligned}
\mathfrak{Q}_3 &= -\tfrac{2}{3}(ad-1)(4(a^2+d^2+ad)-3),\\
\mathfrak{Q}_5 &= \ \ \tfrac{2}{3}(ad-1)(4(a^2+d^2+ad)-3)(a+d),\\
\mathfrak{Q}_6 &= \mu + \tfrac{1}{6}(4ad-1)(a^2+d^2-ad-1),
\end{aligned}$$

or equivalently,

$$\mu = -\tfrac{1}{6}(4ad-1)(a^2+d^2-ad-1), \quad (ad-1)(4(a^2+d^2+ad)-3) = 0.$$

Thus, we obtain algebraic Bach solitons with associated left-invariant metrics given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1, \quad [e_2, e_4] = de_2, \quad [e_3, e_4] = (a+d)e_3.$$

If $4(a^2+d^2+ad)-3 = 0$, the space is an algebraic Ricci soliton (see Equation (5.34)). Otherwise, $ad = 1$, so these structure constants must be non-zero and $d = \frac{1}{a}$. In this case, $\mu = -\frac{(a^2-1)^2}{2a^2}$, the associated left-invariant metrics transform into

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1, \quad [e_2, e_4] = \tfrac{1}{a}e_2, \quad [e_3, e_4] = \tfrac{a^2+1}{a}e_3,$$

and their Bach operators are diagonal, given by

$$\widehat{B} = \tfrac{1}{2a^2}\left[-3(a^4-1), 3(a^4-1), (a^2-1)^2, -(a^2-1)^2\right].$$

The assumption $d \neq \pm a$ implies that $a \neq \pm 1$. Thus, these spaces are neither Bach-flat nor algebraic Ricci solitons. Furthermore, the isometry $e_4 \mapsto -e_4$ interchanges a with $-a$, while $(e_1, e_2, e_3, e_4) \mapsto (e_2, -e_1, e_3, e_4)$ interchanges a with $\frac{1}{a}$, so we can restrict the parameter a to $(0, 1)$. These metrics correspond to Assertion *(ii)* in Theorem 5.8.

Case *(i.b)*.

Since $a = d \neq 0$ and $c = F = H = 0$, a direct calculation shows that the Bach operator is diagonal,

$$\widehat{\mathfrak{B}} = \tfrac{4a^4-5a^2+1}{6}\,\mathrm{diag}[3, 3, -5, -1].$$

Since $\mathfrak{B}_{33}$ and $\mathfrak{B}_{44}$ must be equal, any algebraic Bach soliton is clearly Bach-flat in this case.

Case *(ii.a)*.

The structure constants must satisfy $d = -a$, $a^2 \neq c^2$ and $F = H = 0$. A direct calculation shows that the diagonalizability of the Bach operator depends only on $\mathfrak{B}_{12}$ and the condition $\mathfrak{B}_{11} = \mathfrak{B}_{22}$. Recall that the Bach tensor is trace-free and so $\mathfrak{B}_{44} = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33}$. Hence, the existence of algebraic Bach solitons is determined by the vanishing of the polynomials

$$\mathfrak{Q}_1 = \mathfrak{B}_{12}, \quad \mathfrak{Q}_2 = -3\mathfrak{B}_{11} - \mathfrak{B}_{22}, \quad \mathfrak{Q}_3 = \mu + \mathfrak{B}_{11} + \mathfrak{B}_{22} + \mathfrak{B}_{33},$$

and it is not difficult to see that

$$\begin{aligned} \mathfrak{Q}_1 &= \tfrac{1}{3}ac(20a^2 + 12c^2 - 1), \\ \mathfrak{Q}_2 &= \tfrac{2}{3}(4a^4 + (12c^2 + 1)a^2 - 3), \\ \mathfrak{Q}_3 &= \mu - \tfrac{1}{6}\left(12a^4 + 36a^2c^2 - a^2 - 1\right). \end{aligned}$$

Thus, the system of polynomial equations $\{\mathfrak{Q}_i = 0\}$ is equivalent to $c = 0$, $a = \pm\frac{\sqrt{3}}{2}$ and $\mu = \frac{5}{6}$, which characterize the algebraic Bach solitons whose left-invariant metrics are given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \pm\tfrac{\sqrt{3}}{2}e_1, \quad [e_2, e_4] = \mp\tfrac{\sqrt{3}}{2}e_2,$$

so they are also algebraic Ricci solitons (see Equation (5.34) with $a = -d = \pm\frac{\sqrt{3}}{2}$).

Case *(ii.b)*.

This case reduces to the analysis of $\mathbb{R}^3 \rtimes \mathbb{R}$, according to Remark 5.30.

Case *(ii.c)*.

We take $d = -a$ and $c = a \neq 0$. The Bach tensor must have the matrix form given in Theorem 5.29-*(ii.c)*, and a direct calculation shows that $\mathfrak{B}_{34} = 0$ and

$$\mathfrak{B}_{13} = -\mathfrak{B}_{23} = -\tfrac{2}{3}a(4a^2 + F^2 + H^2 + 1)(H + F).$$

Consequently, $H = -F$. Moreover, since $\mathfrak{B}_{44} = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33}$, an algebraic Bach soliton is characterized by the vanishing of the polynomials

$$\begin{aligned} &\mathfrak{Q}_1 = \mathfrak{B}_{11} - \mathfrak{B}_{22}, && \mathfrak{Q}_2 = \mathfrak{B}_{33} - 2\mathfrak{B}_{11} + 2F\mathfrak{B}_{14} + \mu, \\ &\mathfrak{Q}_3 = \mathfrak{B}_{44} + 2\mathfrak{B}_{12} - \mu, && \mathfrak{Q}_4 = \mathfrak{B}_{24} - \mathfrak{B}_{14}, \\ &\mathfrak{Q}_5 = -F\mathfrak{B}_{11} - 3F\mathfrak{B}_{12} + (2F^2 - 1)\mathfrak{B}_{14} + F\mu. \end{aligned}$$

A direct calculation leads to $\mathfrak{Q}_1 = \mathfrak{Q}_4 = 0$ and

$$\begin{aligned} \mathfrak{Q}_2 &= \tfrac{1}{6}(16a^2 - 22F^2 - 11)(a^2 + 2F^2 + 1) + \mu, \\ \mathfrak{Q}_3 &= \tfrac{1}{6}(11a^2 - 2F^2 - 1)(16a^2 + 2F^2 + 1) - \mu, \\ \mathfrak{Q}_5 &= -F\left(\tfrac{1}{6}(11a^2 - 2F^2 - 1)(16a^2 + 2F^2 + 1) - \mu\right), \end{aligned} \tag{5.38}$$

and thus

$$\mathfrak{Q}_2 + \mathfrak{Q}_3 = 2(4a^2 + 2F^2 + 1)(4a^2 - 2F^2 - 1).$$

We conclude that $4a^2 - 2F^2 - 1 = 0$ and Remark 5.30 shows that the analysis reduces to that in $\mathbb{R}^3 \rtimes \mathbb{R}$.

Remark 5.32. For the sake of completeness, we include here the study of the two cases covered by the analysis carried out on $\mathbb{R}^3 \rtimes \mathbb{R}$.

Considering the situation of Case *(ii.b)* in Theorem 5.29, we take $a = c = d = 0$. For these values, $\mathfrak{B}_{13} = \mathfrak{B}_{23} = \mathfrak{B}_{34} = 0$ and the Bach tensor has the matrix form given in Theorem 5.29-*(ii.b)*. Since $\mathfrak{B}_{44} = -\mathfrak{B}_{11} - \mathfrak{B}_{22} - \mathfrak{B}_{33}$, the vanishing of the polynomials

$$\begin{aligned}
\mathfrak{Q}_1 &= \mathfrak{B}_{14} + F(\mathfrak{B}_{11} + 2\mathfrak{B}_{33} + \mu) - H\mathfrak{B}_{12}, \\
\mathfrak{Q}_2 &= \mathfrak{B}_{24} - H(\mathfrak{B}_{22} + 2\mathfrak{B}_{33} + \mu) + F\mathfrak{B}_{12}, \\
\mathfrak{Q}_3 &= -(F^2 + 1)\mathfrak{B}_{11} - (H^2 + 1)\mathfrak{B}_{22} - (2F^2 + 2H^2 - 1)\mathfrak{B}_{33} \\
&\quad + 2FH\mathfrak{B}_{12} - (F^2 + H^2 - 1)\mu,
\end{aligned}$$

determines the algebraic Bach solitons in this case. A direct calculation shows that

$$\begin{aligned}
\mathfrak{Q}_1 &= -F\left(\tfrac{11}{6}(F^2 + H^2 + 1)^2 - \mu\right), \\
\mathfrak{Q}_2 &= \quad H\left(\tfrac{11}{6}(F^2 + H^2 + 1)^2 - \mu\right), \\
\mathfrak{Q}_3 &= \quad (F^2 + H^2 - 1)(\tfrac{11}{6}(F^2 + H^2 + 1)^2 - \mu),
\end{aligned}$$

and therefore the system $\{\mathfrak{Q}_i = 0\}$ is equivalent to $\mu = \frac{11}{6}(F^2 + H^2 + 1)^2$. The associated left-invariant metrics are given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = He_1, \quad [e_2, e_4] = Fe_3,$$

so they are algebraic Ricci solitons as well (see Equation (5.36)).

Finally, in Case *(ii.c)* of Theorem 5.29 we use Equation (5.38) to obtain that $4a^2 - 2F^2 - 1 = 0$ and $\mu = \frac{70}{3}a^4$ are the conditions which characterize an algebraic Bach soliton. Note that these algebraic Bach solitons, whose associated left-invariant metrics are given by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = ae_1 - ae_2 - Fe_3, \quad [e_2, e_4] = ae_1 - ae_2 + Fe_3,$$

are algebraic Ricci solitons as well (see Equation (5.37)).

Part III

Homogeneous manifolds and homogeneous structures

Jensen proved in [88] a well-known result that states that, in the Riemannian homogeneous setting, Einstein metrics are symmetric, which constitutes a very rigid situation. The Weyl curvature tensor of any Einstein metric is divergence-free – in other words, it is harmonic –, a condition that is equivalent to the fact that the corresponding Schouten tensor is a Codazzi tensor. If the scalar curvature is constant, then the Ricci tensor is also a Codazzi tensor. Podestà and Spiro showed in [123] that the condition $\operatorname{div} W = 0$ implies local symmetry in the locally homogeneous four-dimensional case.

On four-dimensional oriented four-manifolds, the decomposition of the Weyl curvature tensor into its self-dual and anti-self-dual components $W = W^+ + W^-$, which is conformal invariant, makes the harmonicity condition $\operatorname{div} W = 0$ become $\operatorname{div} W^+ + \operatorname{div} W^- = 0$. A four-dimensional manifold is said to have half-harmonic Weyl curvature tensor if either $\operatorname{div} W^+ = 0$ or $\operatorname{div} W^- = 0$. These conditions are natural generalizations of the Einstein condition and have received special attention during the last decade (see [46, 48, 107] and [134]).

In Theorem 6.1 we give the complete description – up to homotheties – of four-dimensional locally homogeneous Riemannian manifolds with half-harmonic Weyl curvature tensor. The manifolds in this result are naturally equipped with a self-dual homogeneous structure which is not necessarily the canonical homogeneous structure of the underlying Lie group. Given the fact that homogeneous manifolds may admit different presentations as homogeneous spaces, they may as well admit more than one homogeneous structure. Therefore we decided to study which Lie groups admit at least two inequivalent homogeneous structures. In Theorem 7.1 we show that a three-dimensional non-symmetric homogeneous space admits at least two inequivalent homogeneous structures if and only if its isometry group is four-dimensional.

Motivated by the results obtained in Theorem 6.1, we started to investigate four-dimensional self-dual homogeneous structures, obtaining some partial classification results. Nevertheless, the whole classification of four-dimensional Riemannian self-dual homogeneous structures remains an open problem.

Chapter 6
Homogeneous four-manifolds with half-harmonic Weyl curvature

In this chapter we will work in the Riemannian homogeneous setting to study what the manifolds that admit half-harmonic Weyl curvature are like. The contents of this chapter are partially contained in the work [37].

6.1 Summary of results

A four-dimensional Riemannian manifold (M, g) has *harmonic Weyl curvature* if its Weyl tensor is divergence free, i.e., $\delta W = \operatorname{div}_1 W = 0$ or, equivalently, if its Cotton tensor vanishes. Even though the Weyl curvature tensor is conformally invariant, the condition $\delta W = 0$ is not. In fact, for any conformal metric $\tilde{g} = e^{2\sigma} g$, one has that $\widetilde{\operatorname{div}_1} \widetilde{W} = \operatorname{div}_1 W - \iota_{\nabla_\sigma} W$. Einstein metrics have harmonic Weyl curvature, which is the reason why the condition $\delta W = 0$ has been investigated in order to extend the geometric properties of Einstein metrics to more general structures. On the other hand, any locally symmetric manifold trivially has harmonic Weyl curvature and it follows from [15, 54] that the converse is also true in the four-dimensional homogeneous setting (see also [123]).

We have already mentioned that the space of two-forms on an oriented four-dimensional Riemannian manifold splits as the direct sum of the spaces of self-dual and anti-self-dual two-forms, $\Lambda^2_\pm(TM)$, under the action of $SO(4)$ and that such decomposition corresponds to the one given by the ± 1 eigenspaces of the Hodge-star operator

$$\star \colon \Lambda^2(TM) \to \Lambda^2(TM),$$

so that the self-dual and anti-self-dual components of any two-form Ω are given by $\Omega_\pm = \frac{1}{2}(\Omega \pm \star\Omega)$. Since the Weyl conformal curvature operator $\mathcal{W} \in \operatorname{End} \Lambda^2(TM)$ and the Hodge-star operator commute, the bundles $\Lambda^2_\pm(TM)$ remain invariant under its action. In what follows, we will denote by $\mathcal{W}^\pm$ the self-dual and anti-self-dual components of the Weyl curvature operator, which are its restrictions to the self-dual and anti-self-dual subspaces of the space of two-forms, respectively, and an oriented Riemannian four-manifold is said to be self-dual (resp. anti-self-dual) if $\mathcal{W}^- = 0$ (resp. $\mathcal{W}^+ = 0$). Given the decomposition of the Weyl curvature operator into its self-dual and anti-self-dual components, its divergence decomposes accordingly and so (M, g) is said to have *half-harmonic Weyl curvature* if either $\delta\mathcal{W}^+ = 0$ or $\delta\mathcal{W}^- = 0$. The conditions $\delta\mathcal{W}^\pm = 0$ have been extensively investigated during the past decade in order to describe different classes of Riemannian manifolds such as Ricci solitons [46, 48, 134] and quasi-Einstein metrics [107].

A first example of metrics that have half-harmonic Weyl curvature can be found in the Kähler setting. The geometry of four-dimensional Kähler manifolds is, to a great extent, codified by their self-dual and anti-self-dual Weyl curvatures. In fact, the anti-self-dual Weyl curvature codifies the Bochner tensor and so the condition $\delta\mathcal{W}^- = 0$ is equivalent to weak Bochner-flatness ($\delta B = 0$) [8, 87]. It was shown in [89] that weakly Bochner-flat Kähler surfaces of constant scalar curvature are locally symmetric (thus having harmonic Weyl curvature). On the other hand, the self-dual component of the Weyl operator is $\mathcal{W}^+ = \frac{\tau}{12}\operatorname{diag}[2, -1, -1]$ and

$$\tau \operatorname{div}_1 W^+ + \iota_{\nabla\tau} W^+ = 0.$$

Therefore, $\delta\mathcal{W}^+ = 0$ if and only if the scalar curvature τ is constant. Moreover, any four-dimensional Kähler surface with $\mathcal{W}^+ \neq 0$ is conformal to a metric with $\delta\mathcal{W}^+ = 0$.

We will show that four-dimensional homogeneous manifolds with half-harmonic Weyl curvature tensor are either symmetric or locally homothetic to the only non-symmetric anti-self-dual homogeneous manifold or to the 3-symmetric space, so non-symmetric homogeneous four-manifolds with half-harmonic Weyl curvature have an underlying complex structure that is either Kähler or locally conformally Kähler. This description is summarised in the main result of this chapter – whose proof is detailed in Section 6.2 – as follows.

Theorem 6.1. *Let (M, g) be a four-dimensional locally homogeneous Riemannian manifold with half-harmonic Weyl curvature tensor. Then it is symmetric or locally homothetic to one of the following semi-direct extensions of the Heisenberg group:*

(i) The left-invariant metric on $H^3 \rtimes \mathbb{R}$ determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = -2e_3\,,$$

(ii) or the left-invariant metric on $H^3 \rtimes \mathbb{R}$ determined by

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = \tfrac{1}{2}e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = -\tfrac{1}{2}e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

Remark 6.2. The metrics in Theorem 6.1-*(i)* are anti-self-dual and correspond to those in [55]. The anti-self-dual Weyl curvature operator has eigenvalues $\{-2, 1, 1\}$ and $\Omega_- = e^1 \wedge e^2 - e^3 \wedge e^4$ is an eigenvector associated to the distinguished eigenvalue -2. A straightforward calculation shows that the associated almost complex structure $J_- e_1 = e_2$, $J_- e_3 = -e_4$ is integrable and $d\Omega_- = -e^4 \wedge \Omega_-$. Therefore, the structure is locally conformally Kähler with respect to the opposite orientation. Furthermore, it was shown in [39] that these metrics are conformally Ricci-flat and, therefore, quasi-Einstein.

Remark 6.3. The self-dual and anti-self-dual Weyl curvature operators of the metrics in Theorem 6.1-*(ii)* have opposite eigenvalues $\{\pm\frac{1}{4}, \mp\frac{1}{2}, \pm\frac{1}{4}\}$. The eigenvectors corresponding to the distinguished eigenvalues of multiplicity one,

$$\Omega_\pm = e^1 \wedge e^3 \mp e^2 \wedge e^4,$$

define a symplectic pair so that the underlying manifold is Kähler and opposite almost Kähler. While Ω_+ determines a Kähler structure $J_+e_1 = e_3$, $J_+e_2 = -e_4$, the opposite almost complex structure $J_-e_1 = e_3$, $J_-e_2 = e_4$ determined by Ω_- is not integrable and the Ricci operator is J_--invariant. Now it follows from [7, Theorem 1] that it corresponds to the unique four-dimensional 3-symmetric space, which is the only homogeneous non-symmetric Kähler surface. Moreover, it is an algebraic Ricci soliton [42], thus being an expanding Ricci soliton.

Remark 6.4. The Bach tensor, $\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \frac{1}{2}W[\rho]$, of the metrics given by Theorem 6.1-(i) vanishes, since they are anti-self-dual. Therefore, these metrics are trivially steady Bach solitons (see [81]). On the contrary, the Bach tensor of the Kähler metrics in Theorem 6.1-*(ii)*,

$$\mathfrak{B} = \operatorname{div}_2 \operatorname{div}_4 W + \frac{1}{2}W[\rho] = \operatorname{div}_2 \operatorname{div}_4 W^+ + \frac{1}{2}W^+[\rho] = \frac{1}{2}W^+[\rho],$$

is determined by the non-zero components

$$\mathfrak{B}(e_1, e_1) = \mathfrak{B}(e_3, e_3) = -\mathfrak{B}(e_2, e_2) = -\mathfrak{B}(e_4, e_4) = -\frac{3}{8}.$$

A straightforward calculation shows that $\mathcal{D} = \hat{\mathfrak{B}} - \frac{3}{8}\operatorname{Id}$ is a derivation of the Lie algebra in Theorem 6.1-(ii), where $\hat{\mathfrak{B}}$ is the $(1,1)$-tensor field associated to the $(0,2)$-Bach tensor $\mathfrak{B}$. Let φ_t be the one-parameter family of automorphisms of the Lie algebra determined by $d\varphi_{t_{|e}} = e^{-\frac{t}{2}\mathcal{D}}$. Proceeding as in Lauret's work [97], the vector field X on $H^3 \rtimes \mathbb{R}$ given by $X = \frac{d}{dt}_{|t=0}\varphi_t$ defines a Bach soliton, i.e., $\mathcal{L}_X\langle\cdot,\cdot\rangle + \mathfrak{B} = \frac{3}{8}\langle\cdot,\cdot\rangle$. Consequently, the 3-symmetric space is both an algebraic Ricci soliton and an algebraic Bach soliton. Moreover, since the metric does not split as a product $\mathbb{R}^k \times N^{4-k}$, neither the Ricci nor the Bach solitons are gradient (see [81, 121]). The soliton above does not correspond to those studied in [81] since the underlying structure is not the product $H^3 \times \mathbb{R}$.

Remark 6.5. The norm of the self-dual Weyl curvature operator of any Kähler surface satisfies $\|\mathcal{W}^+\|^2 = \frac{1}{24}\tau^2$. If the scalar curvature is constant, then it follows from the Weitzenböck formula (see [18])

$$\Delta\|\mathcal{W}^+\|^2 = -\tau\|\mathcal{W}^+\|^2 + 36\det\nolimits_{\Lambda^2_+}\mathcal{W}^+ - 2\|\nabla\mathcal{W}^+\|^2$$

that the self-dual Weyl curvature operator is parallel. Therefore, homogeneous four-manifolds with half-harmonic Weyl curvature are such that $\nabla\mathcal{W}^+ = 0$.

Remark 6.6. The Bach tensor of the anti-self-dual metric in Theorem 6.1-*(i)* vanishes and so it is critical for the functional $\mathcal{F}_t$ with $t = -\frac{1}{3}$. The energy of this functional is strictly negative and this metric is the only one which is Bach-flat with non-zero energy on a semi-direct extension of the Heisenberg group.

The 3-symmetric space in Theorem 6.1-*(ii)* is a Ricci soliton and thus it is critical for the functional $\mathcal{F}_t$ with $t = -\frac{1}{2}$. This is the only metric which is critical with zero energy on a semi-direct extension of the Heisenberg group.

6.2 Half-harmonic Weyl curvature on homogeneous manifolds

Simply connected four-dimensional homogeneous Riemannian manifolds are either symmetric or isometric to a Lie group with a left-invariant metric (see [15]). Any non-symmetric homogeneous metric is thus realized either on the product Lie groups $SU(2) \times \mathbb{R}$ and $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$, or on the semi-direct extensions $E(1,1) \rtimes \mathbb{R}$, $\widetilde{E}(2) \rtimes \mathbb{R}$, $H^3 \rtimes \mathbb{R}$ and $\mathbb{R}^3 \rtimes \mathbb{R}$, where $E(1,1)$, $\widetilde{E}(2)$, H^3 and $\mathbb{R}^3$ denote the Poincaré group, the the universal covering of the Euclidean group, the Heisenberg group and the Abelian group, respectively.

As we have already mentioned, any symmetric four-dimensional homogeneous Riemannian manifold has harmonic Weyl curvature, therefore this situation will not provide examples of strictly half-harmonic Weyl curvature. De Smedt and Salamon proved that if (G, g) is a simply connected Lie group with a left-invariant metric and $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of its Lie algebra, there are two cases in which there exist orientation-reversing isometric automorphisms (see [55, Lemma 2.2]).

1. The Lie algebra $\mathfrak{g}$ has non-zero centre $\mathfrak{z}$ and the isometric automorphism that reverses the orientation of (G, g) is given by
$$(e_1, e_2, e_3, e_4) \longmapsto (-e_1, e_2, e_3, e_4)$$
for an orthonormal basis such that $e_1 \in \mathfrak{z}$. This situation corresponds to the products $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$.

2. G corresponds to the semi-direct extension $\mathbb{R}^3 \rtimes \mathbb{R}$. In this situation, if we fix an orthonormal basis such that $e_2, e_3, e_4 \in \mathfrak{r}^3$, then
$$(e_1, e_2, e_3, e_4) \longmapsto (e_1, -e_2, -e_3, -e_4)$$
determines an orientation-reversing isometric automorphism of G.

As a consequence, the Weyl curvature of these three Lie groups cannot be strictly half-harmonic. Nevertheless, we will address these cases quickly in the following remarks for the sake of completeness.

Remark 6.7. We will use a direct approach to show that the Weyl curvature of $\mathbb{R}^3 \rtimes \mathbb{R}$ cannot be strictly half-harmonic. Let us consider left-invariant metrics on the semi-direct extensions of the Abelian Lie algebra as described in Section 1.4.2. In this case there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that

$$\begin{aligned} &[e_1, e_4] = ae_1 + be_2 + ce_3, \qquad [e_2, e_4] = -be_1 + fe_2 + he_3, \\ &[e_3, e_4] = -ce_1 - he_2 + pe_3. \end{aligned} \tag{6.1}$$

Now, a straightforward calculation shows that the divergences $\operatorname{div}_1 W^{\pm}(e_i, e_j, e_k)$ are determined – up to the corresponding symmetries – by

$$\operatorname{div}_1 W^{\pm}(e_i, e_j, e_k) = \tfrac{1}{4}\mathfrak{P}^{\pm}_{ijk},$$

where $\mathfrak{P}^{\pm}_{ijk}$ are the polynomials on the structure constants in (6.1) given by

$$\mathfrak{P}^{\pm}_{112} = \mp\left(c(a^2 - 2p^2 + af + ap - fp) - bh(a - 2f + p)\right),$$
$$\mathfrak{P}^{\pm}_{113} = \pm\left(b(a^2 - 2f^2 + af + ap - fp) + ch(a + f - 2p)\right),$$
$$\mathfrak{P}^{\pm}_{114} = a^2(f + p) - af^2 - ap^2 - 2(b^2 + c^2)a + 2b^2 f + 2c^2 p,$$
$$\mathfrak{P}^{\pm}_{212} = \mp\left(h(f^2 - 2p^2 + af - ap + fp) - bc(2a - f - p)\right),$$
$$\mathfrak{P}^{\pm}_{213} = \pm\left(a^2 f - (a + p)f^2 + fp^2 - 2b^2 a + 2(b^2 + h^2)f - 2h^2 p\right),$$
$$\mathfrak{P}^{\pm}_{214} = -b(2a^2 - f^2 - af + ap - fp) - ch(a + f - 2p),$$
$$\mathfrak{P}^{\pm}_{312} = \mp\left(a^2 p + f^2 p - (a + f)p^2 - 2c^2 a - 2h^2 f + 2(c^2 + h^2)p\right),$$
$$\mathfrak{P}^{\pm}_{313} = \pm\left(h(2f^2 - p^2 + af - ap - fp) - bc(2a - f - p)\right),$$
$$\mathfrak{P}^{\pm}_{314} = -c(2a^2 - p^2 + af - ap - fp) + bh(a - 2f + p),$$
$$\mathfrak{P}^{\pm}_{412} = -b(a - f)^2,$$
$$\mathfrak{P}^{\pm}_{413} = -c(a - p)^2,$$
$$\mathfrak{P}^{\pm}_{414} = \mp h(f - p)^2.$$

Consequently, the conditions $\delta\mathcal{W}^+ = 0$ and $\delta\mathcal{W}^- = 0$ are equivalent.

Remark 6.8. As in the previous remark, we will make a direct approach to show that the Weyl curvatures of $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ cannot be strictly half-harmonic.

Let us consider left-invariant metrics on the product Lie groups $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$ as described in Section 1.4.2. We recall that in this case there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$ or $\mathfrak{su}(2) \times \mathbb{R}$ such that

$$\begin{aligned} &[e_1, e_2] = \lambda_3 e_3, \qquad [e_1, e_3] = -\lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1, \\ &[e_1, e_4] = k_3\lambda_2 e_2 - k_2\lambda_3 e_3, \qquad [e_2, e_4] = k_1\lambda_3 e_3 - k_3\lambda_1 e_1, \\ &[e_3, e_4] = k_2\lambda_1 e_1 - k_1\lambda_2 e_2, \end{aligned} \tag{6.2}$$

where $\lambda_1\lambda_2\lambda_3 \neq 0$. The associated Lie group corresponds to $SU(2) \times \mathbb{R}$ if λ_1, λ_2, λ_3 do not change sign, and to $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$ otherwise.

Now, a long but straightforward calculation shows that the divergence $\operatorname{div}_1 W^+(e_i, e_j, e_k)$ is determined – up to the corresponding symmetries – by

$$\operatorname{div}_1 W^+(e_i, e_j, e_k) = \tfrac{1}{16}\mathfrak{P}^+_{ijk},$$

where $\mathfrak{P}^+_{ijk}$ are the polynomials on the structure constants in (6.2) given by

$$\begin{aligned} \mathfrak{P}^+_{112} = &-2\left(2\lambda_1^3 - \lambda_3^3 - \lambda_1^2\lambda_3\right)k_2^3 + 2\left(\lambda_3^3 - 2\lambda_1\lambda_2^2 + \lambda_1\lambda_2\lambda_3\right)k_1^2 k_2 \\ &- \left(4\lambda_1^3 - \lambda_1^2(\lambda_2 + \lambda_3) - \lambda_2^2(\lambda_1 + 2\lambda_3) + \lambda_1\lambda_2\lambda_3\right)k_2 k_3^2 \\ &+ \lambda_3\left(\lambda_1^2 - \lambda_2^2 - 3\lambda_1\lambda_2 + 4\lambda_1\lambda_3 - \lambda_2\lambda_3\right)k_1 k_3 \end{aligned}$$

$$
\begin{aligned}
&- \left(4\lambda_1^3 - 2\lambda_3^3 - \lambda_1^2(\lambda_2 + 2\lambda_3) - \lambda_2^2(\lambda_1 - \lambda_3) + \lambda_2\lambda_3^2\right) k_2,\\
\mathfrak{P}^+_{113} &= -2\left(2\lambda_1^3 - \lambda_2^3 - \lambda_1^2\lambda_2\right) k_3^3 + 2\left(\lambda_2^3 - 2\lambda_1\lambda_3^2 + \lambda_1\lambda_2\lambda_3\right) k_1^2 k_3\\
&- \left(4\lambda_1^3 - \lambda_1^2(\lambda_2 + \lambda_3) - \lambda_3^2(\lambda_1 + 2\lambda_2) + \lambda_1\lambda_2\lambda_3\right) k_2^2 k_3\\
&- \lambda_2\left(\lambda_1^2 - \lambda_3^2 + 4\lambda_1\lambda_2 - 3\lambda_1\lambda_3 - \lambda_2\lambda_3\right) k_1 k_2\\
&- \left(4\lambda_1^3 - 2\lambda_2^3 - \lambda_1^2(2\lambda_2 + \lambda_3) + \lambda_2^2\lambda_3 - \lambda_3^2(\lambda_1 - \lambda_2)\right) k_3,\\
\mathfrak{P}^+_{114} &= \left(\lambda_1^2(\lambda_2 - \lambda_3) + \lambda_2^2(4\lambda_1 + 3\lambda_3) - \lambda_3^2(4\lambda_1 + 3\lambda_2)\right) k_1 k_2 k_3\\
&+ 2\left(\lambda_2^3 + \lambda_3^3 - \lambda_2^2\lambda_3 - \lambda_2\lambda_3^2\right) k_1^2\\
&- \left(4\lambda_1^3 - 2\lambda_3^3 - \lambda_1^2(\lambda_2 + 2\lambda_3) + \lambda_2\lambda_3^2\right) k_2^2\\
&- \left(4\lambda_1^3 - 2\lambda_2^3 - \lambda_1^2(2\lambda_2 + \lambda_3) + \lambda_2^2\lambda_3\right) k_3^2\\
&- 2\left(2\lambda_1^3 - \lambda_2^3 - \lambda_3^3 - \lambda_1^2(\lambda_2 + \lambda_3) + \lambda_2^2\lambda_3 + \lambda_2\lambda_3^2\right),\\
\mathfrak{P}^+_{212} &= 2\left(2\lambda_2^3 - \lambda_3^3 - \lambda_2^2\lambda_3\right) k_1^3 - 2\left(\lambda_3^3 - 2\lambda_1^2\lambda_2 + \lambda_1\lambda_2\lambda_3\right) k_1 k_2^2\\
&+ \left(4\lambda_2^3 - \lambda_1^2(\lambda_2 + 2\lambda_3) - \lambda_2^2(\lambda_1 + \lambda_3) + \lambda_1\lambda_2\lambda_3\right) k_1 k_3^2\\
&- \lambda_3\left(\lambda_1^2 - \lambda_2^2 + 3\lambda_1\lambda_2 + \lambda_1\lambda_3 - 4\lambda_2\lambda_3\right) k_2 k_3\\
&+ \left(4\lambda_2^3 - 2\lambda_3^3 - \lambda_1^2(\lambda_2 - \lambda_3) - \lambda_2^2(\lambda_1 + 2\lambda_3) + \lambda_1\lambda_3^2\right) k_1,\\
\mathfrak{P}^+_{213} &= \left(\lambda_1^2(4\lambda_2 + 3\lambda_3) + \lambda_2^2(\lambda_1 - \lambda_3) - \lambda_3^2(3\lambda_1 + 4\lambda_2)\right) k_1 k_2 k_3\\
&+ \left(4\lambda_2^3 - 2\lambda_3^3 - \lambda_2^2(\lambda_1 + 2\lambda_3) + \lambda_1\lambda_3^2\right) k_1^2\\
&- 2\left(\lambda_1^3 + \lambda_3^3 - \lambda_1^2\lambda_3 - \lambda_1\lambda_3^2\right) k_2^2\\
&- \left(2\lambda_1^3 - 4\lambda_2^3 - \lambda_1^2\lambda_3 + \lambda_2^2(2\lambda_1 + \lambda_3)\right) k_3^2\\
&- 2\left(\lambda_1^3 - 2\lambda_2^3 + \lambda_3^3 - \lambda_1^2\lambda_3 + \lambda_2^2(\lambda_1 + \lambda_3) - \lambda_1\lambda_3^2\right),\\
\mathfrak{P}^+_{214} &= 2\left(\lambda_1^3 - 2\lambda_2^3 + \lambda_1\lambda_2^2\right) k_3^3\\
&- \left(4\lambda_2^3 - \lambda_2^2(\lambda_1 + \lambda_3) - \lambda_3^2(2\lambda_1 + \lambda_2) + \lambda_1\lambda_2\lambda_3\right) k_1^2 k_3\\
&+ 2\left(\lambda_1^3 - 2\lambda_2\lambda_3^2 + \lambda_1\lambda_2\lambda_3\right) k_2^2 k_3\\
&+ \lambda_1\left(\lambda_2^2 - \lambda_3^2 + 4\lambda_1\lambda_2 - \lambda_1\lambda_3 - 3\lambda_2\lambda_3\right) k_1 k_2\\
&+ \left(2\lambda_1^3 - 4\lambda_2^3 - \lambda_1^2\lambda_3 + \lambda_2^2(2\lambda_1 + \lambda_3) - \lambda_3^2(\lambda_1 - \lambda_2)\right) k_3,\\
\mathfrak{P}^+_{312} &= \left(\lambda_1^2(3\lambda_2 + 4\lambda_3) - \lambda_2^2(3\lambda_1 + 4\lambda_3) + \lambda_3^2(\lambda_1 - \lambda_2)\right) k_1 k_2 k_3\\
&+ \left(2\lambda_2^3 - 4\lambda_3^3 - \lambda_1\lambda_2^2 + \lambda_3^2(\lambda_1 + 2\lambda_2)\right) k_1^2\\
&+ \left(2\lambda_1^3 - 4\lambda_3^3 - \lambda_1^2\lambda_2 + \lambda_3^2(2\lambda_1 + \lambda_2)\right) k_2^2\\
&+ 2\left(\lambda_1^3 + \lambda_2^3 - \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2\right) k_3^2\\
&+ 2\left(\lambda_1^3 + \lambda_2^3 - 2\lambda_3^3 - \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2 + \lambda_3^2(\lambda_1 + \lambda_2)\right),\\
\mathfrak{P}^+_{313} &= -2\left(\lambda_2^3 - 2\lambda_3^3 + \lambda_2\lambda_3^2\right) k_1^3\\
&+ \left(4\lambda_3^3 - \lambda_1^2(\lambda_3 + 2\lambda_2) - \lambda_3^2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2\lambda_3\right) k_1 k_2^2\\
&- 2\left(\lambda_2^3 - 2\lambda_1^2\lambda_3 + \lambda_1\lambda_2\lambda_3\right) k_1 k_3^2\\
&+ \lambda_2\left(\lambda_1^2 - \lambda_3^2 + \lambda_1\lambda_2 + 3\lambda_1\lambda_3 - 4\lambda_2\lambda_3\right) k_2 k_3\\
&- \left(2\lambda_2^3 - 4\lambda_3^3 - \lambda_1^2(\lambda_2 - \lambda_3) - \lambda_1\lambda_2^2 + \lambda_3^2(\lambda_1 + 2\lambda_2)\right) k_1,\\
\mathfrak{P}^+_{314} &= -2\left(\lambda_1^3 - 2\lambda_3^3 + \lambda_1\lambda_3^2\right) k_2^3\\
&+ \left(4\lambda_3^3 - \lambda_2^2(\lambda_3 + 2\lambda_1) - \lambda_3^2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2\lambda_3\right) k_1^2 k_2\\
&- 2\left(\lambda_1^3 - 2\lambda_2^2\lambda_3 + \lambda_1\lambda_2\lambda_3\right) k_2 k_3^2
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_1 (\lambda_2^2 - \lambda_3^2 + \lambda_1\lambda_2 - 4\lambda_1\lambda_3 + 3\lambda_2\lambda_3)\, k_1k_3 \\
&- (2\lambda_1^3 - 4\lambda_3^3 - \lambda_2\lambda_1^2 - \lambda_2^2(\lambda_1 - \lambda_3) + \lambda_3^2(2\lambda_1 + \lambda_2))\, k_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}^+_{412} = &-2 (\lambda_1^3 + \lambda_2^3 - \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2)\, k_3^3 \\
&- (2\lambda_2^3 - \lambda_2^2(\lambda_1 + \lambda_3) + \lambda_3^2(2\lambda_1 - \lambda_2) - \lambda_1\lambda_2\lambda_3)\, k_1^2k_3 \\
&- (2\lambda_1^3 - \lambda_1^2(\lambda_2 + \lambda_3) - \lambda_3^2(\lambda_1 - 2\lambda_2) - \lambda_1\lambda_2\lambda_3)\, k_2^2k_3 \\
&+ (\lambda_1^2(3\lambda_2 - \lambda_3) - \lambda_2^2(3\lambda_1 - \lambda_3) - \lambda_3^2(\lambda_1 - \lambda_2))\, k_1k_2 \\
&- 2 (\lambda_1^3 + \lambda_2^3 - \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2)\, k_3,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}^+_{413} = &\ 2 (\lambda_1^3 + \lambda_3^3 - \lambda_1^2\lambda_3 - \lambda_1\lambda_3^2)\, k_2^3 \\
&+ (2\lambda_3^3 + \lambda_2^2(2\lambda_1 - \lambda_3) - \lambda_3^2(\lambda_1 + \lambda_2) - \lambda_1\lambda_2\lambda_3)\, k_1^2k_2 \\
&+ (2\lambda_1^3 - \lambda_1^2(\lambda_2 + \lambda_3) - \lambda_2^2(\lambda_1 - 2\lambda_3) - \lambda_1\lambda_2\lambda_3)\, k_2k_3^2 \\
&- (\lambda_1^2(\lambda_2 - 3\lambda_3) + \lambda_2^2(\lambda_1 - \lambda_3) + \lambda_3^2(3\lambda_1 - \lambda_2))\, k_1k_3 \\
&+ 2 (\lambda_1^3 + \lambda_3^3 - \lambda_1^2\lambda_3 - \lambda_1\lambda_3^2)\, k_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{P}^+_{414} = &-2 (\lambda_2^3 + \lambda_3^3 - \lambda_2^2\lambda_3 - \lambda_2\lambda_3^2)\, k_1^3 \\
&- (2\lambda_3^3 + \lambda_1^2(2\lambda_2 - \lambda_3) - \lambda_3^2(\lambda_1 + \lambda_2) - \lambda_1\lambda_2\lambda_3)\, k_1k_2^2 \\
&- (2\lambda_2^3 - \lambda_1^2(\lambda_2 - 2\lambda_3) - \lambda_2^2(\lambda_1 + \lambda_3) - \lambda_1\lambda_2\lambda_3)\, k_1k_3^2 \\
&- (\lambda_1^2(\lambda_2 - \lambda_3) + \lambda_2^2(\lambda_1 - 3\lambda_3) - \lambda_3^2(\lambda_1 - 3\lambda_2))\, k_2k_3 \\
&- 2 (\lambda_2^3 + \lambda_3^3 - \lambda_2^2\lambda_3 - \lambda_2\lambda_3^2)\, k_1.
\end{aligned}
$$

For its part, $\mathrm{div}_1 W^-(e_i, e_j, e_k)$ is given – up to the corresponding symmetries – by

$$
\mathrm{div}_1 W^-(e_i, e_j, e_k) = \tfrac{1}{16}\mathfrak{P}^-_{ijk},
$$

where $\mathfrak{P}^-_{ijk}$ are polynomials on the structure constants in (6.2) that can be expressed in terms of the polynomials $\mathfrak{P}^+_{ijk}$ as follows.

$$
\begin{aligned}
\mathfrak{P}^-_{112} &= -\mathfrak{P}^+_{112} + 2k_1k_3\lambda_3 (\lambda_1^2 - \lambda_2^2 - 3\lambda_1\lambda_2 + 4\lambda_1\lambda_3 - \lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{113} &= -\mathfrak{P}^+_{113} - 2k_1k_2\lambda_2 (\lambda_1^2 - \lambda_3^2 + 4\lambda_1\lambda_2 - 3\lambda_1\lambda_3 - \lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{114} &= -\mathfrak{P}^+_{114} + 2k_1k_2k_3(\lambda_2 - \lambda_3) (\lambda_1^2 + 4\lambda_1\lambda_2 + 4\lambda_1\lambda_3 + 3\lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{212} &= -\mathfrak{P}^+_{212} - 2k_2k_3\lambda_3 (\lambda_1^2 - \lambda_2^2 + 3\lambda_1\lambda_2 + \lambda_1\lambda_3 - 4\lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{213} &= \quad \mathfrak{P}^+_{213} - 2k_1k_2k_3(\lambda_1 - \lambda_3) (\lambda_2^2 + 4\lambda_1\lambda_2 + 3\lambda_1\lambda_3 + 4\lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{214} &= \quad \mathfrak{P}^+_{214} - 2k_1k_2\lambda_1 (\lambda_2^2 - \lambda_3^2 + 4\lambda_1\lambda_2 - \lambda_1\lambda_3 - 3\lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{312} &= \quad \mathfrak{P}^+_{312} - 2k_1k_2k_3(\lambda_1 - \lambda_2) (\lambda_3^2 + 3\lambda_1\lambda_2 + 4\lambda_1\lambda_3 + 4\lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{313} &= -\mathfrak{P}^+_{313} + 2k_2k_3\lambda_2 (\lambda_1^2 - \lambda_3^2 + \lambda_1\lambda_2 + 3\lambda_1\lambda_3 - 4\lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{314} &= \quad \mathfrak{P}^+_{314} + 2k_1k_3\lambda_1 (\lambda_2^2 - \lambda_3^2 + \lambda_1\lambda_2 - 4\lambda_1\lambda_3 + 3\lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{412} &= \quad \mathfrak{P}^+_{412} + 2k_1k_2(\lambda_1 - \lambda_2) (\lambda_3^2 - 3\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\, \lambda_3)\,, \\
\mathfrak{P}^-_{413} &= \quad \mathfrak{P}^+_{413} + 2k_1k_3(\lambda_1 - \lambda_3) (\lambda_2^2 + \lambda_1\lambda_2 - 3\lambda_1\lambda_3 + \lambda_2\lambda_3)\,, \\
\mathfrak{P}^-_{414} &= -\mathfrak{P}^+_{414} - 2k_2k_3(\lambda_2 - \lambda_3) (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 - 3\lambda_2\lambda_3)\,.
\end{aligned}
$$

Since $\lambda_1 \neq 0$, we can consider the orthogonal basis $\hat{e}_i = \frac{1}{\lambda_1} e_i$ and assume that $\lambda_1 = 1$ for the rest of our calculations, just working in the homothetic class of the initial metric. In what follows, we will make use of Gröbner bases (see Section 1.6 and the use we make of them in the following sections for a better understanding). In particular, we consider the polynomial ring $\mathbb{R}[\lambda_2, \lambda_3, k_1, k_2, k_3]$ and fix the graded reverse lexicographical order to compute a Gröbner basis of the ideal generated by the polynomials $\mathfrak{P}^+_{ijk}$. In this way, we obtain that the polynomials

$$\begin{aligned}
\mathbf{g}_1 &= k_1(k_1^2 + k_2^2 + k_3^2 + 1)^2(\lambda_2 - \lambda_3),\\
\mathbf{g}_2 &= (k_2^2 + k_3^2 + 1)(k_1^2 + k_2^2 + k_3^2 + 1)\left(k_1(4\lambda_2 - 5\lambda_3 + 1) + 3k_2k_3(\lambda_3 - 1)\right),\\
\mathbf{g}_3 &= (k_2^2 + k_3^2 + 1)(k_1^2 + k_2^2 + k_3^2 + 1)\left(3k_1(\lambda_2 - \lambda_3) + k_2k_3(\lambda_2 + \lambda_3 - 2)\right),
\end{aligned}$$

belong to the ideal. Note that the expressions for $\mathfrak{P}^-_{ijk}$ given above imply that the space cannot be strictly half-harmonic if any two of the constants k_1, k_2 and k_3 vanish. Hence, excluding that case, the vanishing of the three polynomials $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ clearly leads to $\lambda_1 = \lambda_2 = \lambda_3 = 1$, so again $\mathfrak{P}^-_{ijk}$ coincides with $\mathfrak{P}^+_{ijk}$ – up to a sign in some cases –, so the conditions $\delta\mathcal{W}^+ = 0$ and $\delta\mathcal{W}^- = 0$ are equivalent.

6.2.1 Half-harmonic Weyl curvature on $H^3 \rtimes \mathbb{R}$

We follow the description of the left-invariant metrics on semi-direct extensions of the Heisenberg group given in Section 1.4.2. We fix an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that

$$\begin{aligned}
&[e_1, e_2] = \gamma e_3, &&[e_1, e_4] = a e_1 - c e_2 + H e_3,\\
&[e_3, e_4] = (a + d) e_3, &&[e_2, e_4] = c e_1 + d e_2 + F e_3,
\end{aligned} \tag{6.3}$$

where $\gamma \neq 0$.

For a given orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on the Lie algebra $\mathfrak{g}$, we consider the orientation induced by the volume form $\omega = e^1 \wedge e^2 \wedge e^3 \wedge e^4$ and denote by $\{E_i^\pm\}$ the corresponding orthonormal basis of the spaces of self-dual and anti-self-dual two-forms $\Lambda^2_\pm(\mathfrak{g})$ given by

$$E_1^\pm = \frac{1}{\sqrt{2}}\left(e^{12} \pm e^{34}\right), \quad E_2^\pm = \frac{1}{\sqrt{2}}\left(e^{13} \mp e^{24}\right), \quad E_3^\pm = \frac{1}{\sqrt{2}}\left(e^{14} \pm e^{23}\right),$$

where we are using the notation $e^{ij} = e^i \wedge e^j$ and $\{e^i\}$ is the dual basis of $\{e_i\}$.

The Levi-Civita connection ∇ is determined by (6.3) and the expressions

$$
\begin{aligned}
&\nabla_{e_1} e_1 = -a\, e_4, && \nabla_{e_1} e_2 = \tfrac{1}{2}\gamma\, e_3,\\
&\nabla_{e_1} e_3 = -d\, e_4, && \nabla_{e_1} e_4 = a\, e_1 + \tfrac{1}{2} H\, e_3,\\
&\nabla_{e_2} e_2 = -d\, e_4, && \nabla_{e_2} e_3 = \tfrac{1}{2}\left(\gamma\, e_1 - F\, e_4\right),\\
&\nabla_{e_2} e_4 = d\, e_2 + \tfrac{1}{2} F\, e_3, && \nabla_{e_3} e_3 = -\left(a+d\right) e_4,\\
&\nabla_{e_3} e_4 = \tfrac{1}{2}\left(H\, e_1 + F\, e_2 + 2\left(a+d\right) e_3\right), && \nabla_{e_4} e_4 = 0.
\end{aligned}
\tag{6.4}
$$

The self-dual and anti-self-dual Weyl curvature operators have associated matrices $(\mathcal{W}^{\pm}_{ij})$ whose components $\mathcal{W}^{\pm}_{ij} = \langle \mathcal{W}^{\pm}(E^{\pm}_j), E^{\pm}_i\rangle = \langle \mathcal{W}(E^{\pm}_j), E^{\pm}_i\rangle$ are determined by

$$
\begin{aligned}
&\mathcal{W}^{\pm}_{11} = -\tfrac{1}{6}\left(4ad \pm 3\gamma(a+d) + 2\gamma^2 - F^2 - H^2\right),\\
&\mathcal{W}^{\pm}_{12} = \tfrac{1}{4}\left(Fa + Hc + 2Fd \pm 2\gamma F\right), \qquad \mathcal{W}^{\pm}_{13} = \mp\tfrac{1}{4}\left(2Ha - Fc + Hd \pm 2\gamma H\right),\\
&\mathcal{W}^{\pm}_{22} = \tfrac{1}{6}\left(2ad \pm 3\gamma d + \gamma^2 - 2F^2 + H^2\right), \qquad \mathcal{W}^{\pm}_{23} = \pm\tfrac{1}{2}\left(ac - cd + FH\right).
\end{aligned}
$$

The components of the self-dual and anti-self-dual Weyl curvature tensors are given by

$$
W^{\pm}(e_i, e_j, e_k, e_\ell) = \tfrac{1}{2}\{W(e_i, e_j, e_k, e_\ell) \pm W(e_i, e_j, e_{\bar{k}}, e_{\bar{\ell}})\}
$$

where $e_{\bar{k}}, e_{\bar{\ell}}$ are such that $e^k \wedge e^\ell \wedge e^{\bar{k}} \wedge e^{\bar{\ell}} = e^1 \wedge e^2 \wedge e^3 \wedge e^4$ and $\{e_1, e_2, e_3, e_4\}$ is a positively oriented orthonormal basis. The non-zero components of the self-dual Weyl curvature tensor $W^+_{ijk\ell} = W^+(e_i, e_j, e_k, e_\ell)$ are determined by

$$
\begin{aligned}
&W^+_{1212} = W^+_{1234} = W^+_{3434} = \tfrac{1}{2}\mathcal{W}^+_{11},\\
&W^+_{1313} = -W^+_{1324} = W^+_{2424} = \tfrac{1}{2}\mathcal{W}^+_{22},\\
&W^+_{1213} = -W^+_{1224} = W^+_{1334} = -W^+_{2434} = \tfrac{1}{2}\mathcal{W}^+_{12},\\
&W^+_{1214} = W^+_{1223} = W^+_{1434} = W^+_{2334} = \tfrac{1}{2}\mathcal{W}^+_{13},\\
&W^+_{1314} = W^+_{1323} = -W^+_{1424} = -W^+_{2324} = \tfrac{1}{2}\mathcal{W}^+_{23},\\
&W^+_{1414} = W^+_{1423} = W^+_{2323} = -\tfrac{1}{2}\left(\mathcal{W}^+_{11} + \mathcal{W}^+_{22}\right),
\end{aligned}
$$

up to the corresponding symmetries. The divergence

$$
\operatorname{div}_1 W^+(e_i, e_j, e_k) = \sum_{\alpha=1}^{4}\left(\nabla_{e_\alpha} W^+\right)(e_\alpha, e_i, e_j, e_k)
$$

is determined – up to the corresponding symmetries – by

$$\mathrm{div}_1\, W^+(e_i, e_j, e_k) = \frac{1}{16}\mathfrak{P}_{ijk},$$

where $\mathfrak{P}_{ijk}$ are polynomials on the structure constants in equation (6.3). For instance, using the Levi-Civita connection (6.4) and the expressions for the components $W^+_{ijk\ell}$ above, one obtains

$$\begin{aligned}
\mathfrak{P}_{112} &= 16 \textstyle\sum_{\alpha=1}^{4} \left(\nabla_{e_\alpha} W^+\right)(e_\alpha, e_1, e_1, e_2)\\
&= (4Hd^2 + 8Had - 4Fac - 4\gamma Ha + 2\gamma Fc + 6\gamma Hd - 4\gamma^2 H)\\
&\quad + (4Ha^2 + 2Hd^2 - 4Fac + 2Had + 4\gamma Ha - \gamma Fc + \gamma Hd)\\
&\quad - (2Hc^2 + 4Fac - 4Had + 2Fcd - 4\gamma Ha + 4\gamma Fc - 2\gamma Hd\\
&\qquad\qquad + 2H^3 + 2F^2 H - 2\gamma^2 H)\\
&= 2H\left(2a^2 - c^2 + 3d^2\right) - 8Fac + 14Had - 6FcD + 4\gamma Ha\\
&\qquad - 3\gamma Fc + 9\gamma Hd - 2H\left(F^2 + H^2 + \gamma^2\right).
\end{aligned}$$

It is straightforward to see that the remaining components are given by

$$\begin{aligned}
\mathfrak{P}_{113} &= -4(2a^2c - 3cd^2 + acd) + 2\gamma(ac - dc) - 7FHa + (4F^2 - 3H^2)c - 10FHd,\\
\mathfrak{P}_{114} &= -8(ac^2 + ad^2 - c^2d) - 4\gamma(a^2 + ad) - 2(F^2 - 3H^2 - \gamma^2)a - 7FHc + 9H^2d\\
&\quad + 2\gamma(F^2 + H^2 + \gamma^2),\\
\mathfrak{P}_{212} &= 2F(3a^2 - c^2 + 2d^2) + 6Hac + 14Fad + 8Hcd + 9\gamma Fa + 3\gamma Hc + 4\gamma Fd\\
&\quad - 2F(F^2 + H^2 + \gamma^2),\\
\mathfrak{P}_{213} &= 8(a^2d - ac^2 + c^2d) + 4\gamma(d^2 + ad) - 9F^2a - 7FHc - 2(3F^2 - H^2 + \gamma^2)d\\
&\quad - 2\gamma(F^2 + H^2 + \gamma^2),\\
\mathfrak{P}_{214} &= 4(3a^2c - 2cd^2 - acd) - 2\gamma(ac - cd) + 10FHa + (4H^2 - 3F^2)c + 7FHd,\\
\mathfrak{P}_{312} &= 8(a^2d + ad^2) + 4\gamma(a^2 + d^2 + 2ad) - (7F^2 + 6H^2 + 2\gamma^2)a\\
&\quad - (6F^2 + 7H^2 + 2\gamma^2)d - 4\gamma(F^2 + H^2 + \gamma^2),\\
\mathfrak{P}_{313} &= -2F(2a^2 - c^2 + 2d^2) - 2Hac - 14Fad - 10Hcd - 5\gamma Fa - \gamma Hc - 4\gamma Fd\\
&\quad + 4F(F^2 + H^2 + \gamma^2),\\
\mathfrak{P}_{314} &= 2H(2a^2 - c^2 + 2d^2) - 10Fac + 14Had - 2Fcd + 4\gamma Ha - \gamma Fc + 5\gamma Hd\\
&\quad - 4H(F^2 + H^2 + \gamma^2),\\
\mathfrak{P}_{412} &= 4(a^2c + cd^2 - 2acd) + 3FHa + (F^2 + H^2)c - 3FHd,\\
\mathfrak{P}_{413} &= -2Hd^2 - 2Fac + 4Fcd + 2\gamma Fc - 4\gamma Hd - 2H(F^2 + H^2 + \gamma^2),\\
\mathfrak{P}_{414} &= -2Fa^2 - 2H(2ac - cd) - 4\gamma Fa - 2\gamma Hc - 2F(F^2 + H^2 + \gamma^2).
\end{aligned}$$

Since $\gamma \neq 0$, we can consider the orthogonal basis $\hat{e}_i = \frac{1}{\gamma} e_i$ and work in the homothetic class of the initial metric assuming $\gamma = 1$. The divergence δW^+ vanishes if and only if the structure constants in Equation (6.3) satisfy the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$, where $\mathfrak{P}_{ijk} \in \mathbb{R}[a, c, d, F, H]$. Let $\mathcal{I} \subset \mathbb{R}[a, c, d, F, H]$ be the ideal generated by the polynomials $\mathfrak{P}_{ijk}$. Fixing a monomial order on the polynomial ring $\mathbb{R}[a, c, d, F, H]$, a Gröbner basis of $\mathcal{I}$ is a finite subset $\mathcal{G} = \{\mathrm{g}_1, \ldots, \mathrm{g}_r\}$ such that the leading term of any element of $\mathcal{I}$ is divisible by the leading term of one of the g_i with respect to the given ordering. The Hilbert Basis Theorem guarantees that any non-zero ideal admits a Gröbner basis and Buchberguer's algorithm, among others, provides a constructive strategy to find one such basis (see Section 1.6). The zero-set of $\{\mathfrak{P}_{ijk} = 0\}$ and the roots of $\mathcal{I} = \langle \mathfrak{P}_{ijk} \rangle = \langle \mathcal{G} \rangle$ coincide, and $\mathcal{G}$ can be considered a *good* basis for our purposes if it contains one or more polynomials for which we can determine all the real roots in terms of simple conditions on the structure constants in Equation (6.3).

Computing a Gröbner basis $\mathcal{G}$ of $\mathcal{I} = \langle \mathfrak{P}_{ijk} \rangle$ with respect to the lexicographical order, we get a basis of 29 polynomials, one of which is

$$\begin{aligned} \mathrm{g}_1 = {} & H\left(F^2 + H^2 + 1\right)\left(4H^2 + 1\right)\left(4H^2 + 81\right)\left(64H^2 + 81\right) \\ & \times \left((F^2 - H^2)^2 + F^2 + H^2\right)\left(233317175H^4 + 255147165H^2 + 69861528\right). \end{aligned}$$

Consequently, $H = 0$ and the polynomials $\mathfrak{P}_{414}$ and $\mathfrak{P}_{412}$ are reduced to

$$\begin{aligned} \mathfrak{P}_{414} &= -2F\left((a+1)^2 + F^2\right), \\ \mathfrak{P}_{412} &= c\left(4\,(a-d)^2 + F^2\right). \end{aligned}$$

Therefore, $F = 0$ and there are two different possibilities depending on whether or not $d = a$.

If $d = a$, then $\mathfrak{P}_{114} = -2\,(a+1)\,(2a-1)\,(2a+1)$, from where we are led to the cases $a = -1$ or $a = \pm\frac{1}{2}$.

If $a = \pm\frac{1}{2}$, the metric is Einstein and therefore locally symmetric [88] and so this case does not provide strictly half-harmonic examples.

On the other hand, if $a = -1$, then the associated left-invariant metric

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -e_1 - \kappa e_2, \quad [e_2, e_4] = \kappa e_1 - e_2, \quad [e_3, e_4] = -2e_3$$

is anti-self-dual and not locally symmetric, thus corresponding to that previously obtained by de Smedt and Salamon [55]. A straightforward calculation shows that the sectional curvature of the left-invariant metrics described above does not depend on the real parameter κ. Consequently, the corresponding metric Lie groups are homothetic (see [96]), although not isomorphically homothetic. Assertion (i) in Theorem 6.1 now follows.

If $d \neq a$, then $c = 0$ and

$$\begin{aligned} \mathfrak{P}_{114} &= -2\left(4ad^2 + 2a(a+d) - a - 1\right), \\ \mathfrak{P}_{312} &= 2\left(4ad(a+d) + 2(a+d)^2 - (a+d) - 2\right). \end{aligned}$$

We consider $\mathfrak{P}_{114} + \frac{1}{2}\mathfrak{P}_{312} = (a-d)\,(2a-1)\,(2d-1)$, so either $a = \frac{1}{2}$ (and then $\mathfrak{P}_{114}$ shows that $d = -1$) or $d = \frac{1}{2}$ (in which case $\mathfrak{P}_{114}$ shows that $a = -1$). Now the two cases above are equivalent through the transformation

$$(e_1, e_2, e_3, e_4) \mapsto (e_2, -e_1, e_3, e_4).$$

A straightforward calculation now shows that the resulting left-invariant metric

$$[e_1,e_2]=e_3,\quad [e_1,e_4]=\frac{1}{2}e_1,\quad [e_2,e_4]=-e_2,\quad [e_3,e_4]=-\frac{1}{2}e_3,$$

satisfies $\delta W^+=0$ and $\delta W\neq 0$. This corresponds to Assertion (ii) in Theorem 6.1.

Remark 6.9. In a completely analogous way, left-invariant metrics on $H^3\rtimes\mathbb{R}$ with $\delta W^-=0$ are in correspondence with the ones described above.

6.2.2 Half-harmonic Weyl curvature on $E(1,1)\rtimes\mathbb{R}$ and $\widetilde{E}(2)\rtimes\mathbb{R}$

In this section we will proceed as in the previous one and we will see that there are no left-invariant metrics with strictly half-harmonic Weyl curvature on $E(1,1)\rtimes\mathbb{R}$ and $\widetilde{E}(2)\rtimes\mathbb{R}$.

Next we show that any left-invariant metric with half-harmonic Weyl tensor on $E(1,1)\rtimes\mathbb{R}$ or $\widetilde{E}(2)\rtimes\mathbb{R}$ satisfies $\delta W=0$, thus being necessarily symmetric. We proceed as in the previous section.

We follow the description of the left-invariant metrics on semi-direct extensions of the Poincaré and the Euclidean groups as in Section 1.4.2. We consider an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ such that

$$\begin{aligned}
&[e_1,e_3]=-\lambda_2 e_2, && [e_2,e_3]=\lambda_1 e_1,\\
&[e_1,e_4]=be_1-A\lambda_2 e_2, && [e_2,e_4]=A\lambda_1 e_1+be_2,\\
&[e_3,e_4]=Ce_1+De_2,
\end{aligned}\tag{6.5}$$

where $\lambda_1\lambda_2\neq 0$. The associated Lie group corresponds to $\widetilde{E}(2)\times\mathbb{R}$ if λ_1, λ_2 do not change sign, and corresponds to $E(1,1)\times\mathbb{R}$ otherwise.

The self-dual and anti-self-dual Weyl curvature operators have associated matrices $(\mathcal{W}^\pm)$ whose components $\mathcal{W}^\pm_{ij}=\langle\mathcal{W}^\pm(E^\pm_j),E^\pm_i\rangle=\langle\mathcal{W}(E^\pm_j),E^\pm_i\rangle$ are determined by

$$\begin{aligned}
\mathcal{W}^\pm_{11}&=\frac{1}{6}\left\{(A^2+1)\,(\lambda_1-\lambda_2)^2+2\,(C^2+D^2)\right\},\\
\mathcal{W}^\pm_{12}&=\frac{1}{4}\left\{(2AC\mp D)\,\lambda_1+(AC\mp 2D)\,\lambda_2+bD\right\},\\
\mathcal{W}^\pm_{13}&=\mp\frac{1}{4}\left\{(AD\pm 2C)\,\lambda_1-(2AD\pm C)\,\lambda_2+bC\right\},\\
\mathcal{W}^\pm_{22}&=\frac{1}{6}\left\{(2A^2-1)\,\lambda_1^2-(A^2-2)\,\lambda_2^2-(A^2+1)\,\lambda_1\lambda_2\mp 3b\,(\lambda_1-\lambda_2)-C^2-D^2\right\},\\
\mathcal{W}^\pm_{23}&=\frac{1}{2}\left\{A\,(\lambda_1^2-\lambda_2^2)\pm Ab\,(\lambda_1-\lambda_2)\right\}.
\end{aligned}$$

Proceeding as in the previous section, a long but standard calculation shows that the non-zero components of the divergence $\operatorname{div}_1 W^+$ (up the corresponding symmetries) are given by $\operatorname{div}_1 W^+(e_i,e_j,e_k)=\frac{1}{16}\mathfrak{P}_{ijk}$, where $\mathfrak{P}_{ijk}$ are the following polynomials on the structure constants in Equation (6.5):

$$\begin{aligned}
\mathfrak{P}_{112} &= -(A^2+1)C(4\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2)+b(AD+5C)\lambda_1-b(7AD+3C)\lambda_2\\
&\quad+4C(b^2-C^2-D^2),\\
\mathfrak{P}_{113} &= 2A(A^2+1)(2\lambda_1^3-\lambda_2^3-\lambda_1^2\lambda_2)-4Ab(\lambda_1^2-\lambda_2^2)-(4Ab^2-4AC^2-CD)\lambda_1\\
&\quad+(4Ab^2-AC^2-2AD^2+4CD)\lambda_2+5bCD,\\
\mathfrak{P}_{114} &= -2(A^2+1)(2\lambda_1^3-\lambda_2^3-\lambda_1^2\lambda_2)-2b(3A^2+1)(\lambda_1^2-\lambda_2^2)+(ACD+4b^2-4C^2)\lambda_1\\
&\quad+(4ACD-4b^2+C^2+2D^2)\lambda_2-b(7C^2+2D^2),\\
\mathfrak{P}_{212} &= (A^2+1)D(\lambda_1^2-4\lambda_2^2+\lambda_1\lambda_2)+b(7AC-3D)\lambda_1-b(AC-5D)\lambda_2\\
&\quad+4D(b^2-C^2-D^2),\\
\mathfrak{P}_{213} &= -2(A^2+1)(\lambda_1^3-2\lambda_2^3+\lambda_1\lambda_2^2)-2b(3A^2+1)(\lambda_1^2-\lambda_2^2)\\
&\quad+(4ACD+4b^2-2C^2-D^2)\lambda_1+(ACD-4b^2+4D^2)\lambda_2+b(2C^2+7D^2),\\
\mathfrak{P}_{214} &= -2A(A^2+1)(\lambda_1^3-2\lambda_2^3+\lambda_1\lambda_2^2)+4Ab(\lambda_1^2-\lambda_2^2)\\
&\quad+(4Ab^2-2AC^2-AD^2-4CD)\lambda_1-(4Ab^2-4AD^2+CD)\lambda_2-5bCD,\\
\mathfrak{P}_{312} &= 2(A^2+1)(\lambda_1^3+\lambda_2^3-\lambda_1^2\lambda_2-\lambda_1\lambda_2^2)+(3ACD+2C^2-D^2)\lambda_1\\
&\quad-(3ACD+C^2-2D^2)\lambda_2+9b(C^2+D^2),\\
\mathfrak{P}_{313} &= -D\lambda_1^2+(2A^2D-AC+2D)\lambda_2^2-(AC+D)\lambda_1\lambda_2-b(10AC-3D)\lambda_1\\
&\quad+b(2AC-9D)\lambda_2-2D(3b^2-C^2-D^2),\\
\mathfrak{P}_{314} &= -(2A^2C+AD+2C)\lambda_1^2+C\lambda_2^2-(AD-C)\lambda_1\lambda_2+b(2AD+9C)\lambda_1\\
&\quad-b(10AD+3C)\lambda_2+2C(3b^2-C^2-D^2),\\
\mathfrak{P}_{412} &= 2A(A^2+1)(\lambda_1^3+\lambda_2^3-\lambda_1^2\lambda_2-\lambda_1\lambda_2^2)+(2AC^2-AD^2-3CD)\lambda_1\\
&\quad-(AC^2-2AD^2-3CD)\lambda_2,\\
\mathfrak{P}_{413} &= (2A^2C-AD+2C)\lambda_1^2-A^2C\lambda_2^2-A(AC+D)\lambda_1\lambda_2+b(AD+4C)\lambda_1\\
&\quad-3AbD\lambda_2+2C(b^2+C^2+D^2),\\
\mathfrak{P}_{414} &= -A^2D\lambda_1^2+(2A^2D+AC+2D)\lambda_2^2-A(AD-C)\lambda_1\lambda_2+3AbC\lambda_1\\
&\quad-b(AC-4D)\lambda_2+2D(b^2+C^2+D^2).
\end{aligned}$$

Since $\lambda_1\lambda_2 \neq 0$, we consider the orthonormal basis $\hat{e}_i = \frac{1}{\lambda_1}e_i$ and work in the homothetic class of the initial metric. This way, we can assume that $\lambda_1 = 1$ in what follows. The divergence δW^+ vanishes if and only if the structure constants in equation (6.5) satisfy the system of polynomial equations $\{\mathfrak{P}_{ijk} = 0\}$, where $\mathfrak{P}_{ijk} \in \mathbb{R}[\lambda_2, A, b, C, D]$. Let $\mathcal{I}_1 \subset \mathbb{R}[\lambda_2, A, b, C, D]$ be the ideal generated by the polynomials $\mathfrak{P}_{ijk}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order and we see that one of the 39 polynomials in it is

$$\mathbf{g}_1 = D^5\left(C^2+D^2\right)\left(49D^2+16\right)\left(49D^2+144\right)\left(193600D^4+16560D^2+2187\right).$$

It follows immediately that $D = 0$. Now we compute a Gröbner basis $\mathcal{G}_2$ of the ideal $\mathcal{I}_2$ generated by the polynomials $\mathcal{G}_1 \cup \{D\} \subset \mathbb{R}[\lambda_2, A, b, C, D]$ with respect to the lexicographical order, obtaining that the polynomial

$$\mathbf{g}_2 = C\left(A^2+(b+1)^2+C^2\right)$$

belongs to $\mathcal{G}_2$. Therefore, $C = 0$, and the polynomial $\mathfrak{P}_{312}$ is reduced to

$$\mathfrak{P}_{312} = 2\left(A^2+1\right)(\lambda_2-1)^2(\lambda_2+1)\,.$$

This leads to two different possibilities depending on whether $\lambda_2 = 1$ or $\lambda_2 = -1$.

If $\lambda_2 = 1$, then the left-invariant metric is locally conformally flat. Moreover, it is flat or locally isometric to a product $\mathbb{R}\times N(c)$, where $N(c)$ is a three-dimensional manifold of constant sectional curvature.

If $\lambda_2 = -1$, the polynomial $\mathfrak{P}_{114}$ is reduced to

$$\mathfrak{P}_{114} = 8\left(b^2 - A^2 - 1\right),$$

so $b = \pm\sqrt{A^2+1}$. In this situation, the corresponding left-invariant metric is Einstein (thus locally symmetric [88]) if $A = 0$ and locally isometric to a product $M_1(c_1)\times M_2(c_2)$ of two surfaces of constant curvature $c_1^2 \neq c_2^2$ otherwise.

The Weyl tensor is divergence-free in all the cases above, which shows that there are no non-trivial examples in this case.

Remark 6.10. Proceeding in a completely analogous way, one gets that the metrics with half-harmonic Weyl conformal tensor corresponding to the condition $\delta W^- = 0$ are again the ones described just above.

Chapter 7
Three-dimensional Riemannian homogeneous structures

In this chapter we will devote ourselves to the study of homogeneous structures and give a complete classification of the homogeneous structures on non-symmetric three-dimensional Riemannian Lie groups. We will see that one such group admits a non-canonical homogenous structure if and only if its isometry group is four-dimensional. The results in this chapter are contained in the work [38].

Before we start, we will make a short introduction to the world of homogeneous structures.

7.1 Homogeneous structures

The characterization of Riemannian locally symmetric spaces as those whose curvature is parallel with respect to the Levi-Civita connection was originally given by Cartan. Ambrose and Singer extended this characterization to homogeneous Riemannian manifolds showing that a connected, complete and simply connected n-dimensional Riemannian manifold (M, g) is homogeneous if and only if there exists a $(1,2)$-tensor field T on M such that

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} T = 0, \tag{7.1}$$

where $\widetilde{\nabla}$ is the *Ambrose-Singer connection* given by $\widetilde{\nabla} = \nabla - T$, ∇ is the Levi-Civita connection of the metric g, and R denotes the Riemannian curvature tensor for which we adopt the sign convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ (see [3]).

The tensor field T is said to be a *homogeneous structure* on M. We will also denote by T the associated $(0,3)$-tensor field given by $T(X,Y,Z) = g(T(X,Y),Z)$. Conditions (7.1) were further investigated by Tricerri and Vanhecke in [130], where they considered the space $\mathcal{T}(\mathcal{V})$ of such tensor fields on a vector space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ and decomposed it into three irreducible components under the action of the orthogonal group as $\mathcal{T}(\mathcal{V}) = \mathcal{T}_1(\mathcal{V}) \oplus \mathcal{T}_2(\mathcal{V}) \oplus \mathcal{T}_3(\mathcal{V})$. The subspaces of such decomposition are

$$\begin{aligned}
\mathcal{T}_1(\mathcal{V}) &= \{T \in \mathcal{T}(\mathcal{V}) \colon T(x,y,z) = \langle x,y\rangle\varphi(z) - \langle x,z\rangle\varphi(y)\},\\
\mathcal{T}_2(\mathcal{V}) &= \{T \in \mathcal{T}(\mathcal{V}) \colon c_{12}(T) = 0,\ \sigma_{x,y,z} T(x,y,z) = 0\},\\
\mathcal{T}_3(\mathcal{V}) &= \{T \in \mathcal{T}(\mathcal{V}) \colon T(x,y,z) + T(y,x,z) = 0\},
\end{aligned}$$

where $\varphi \in \mathcal{V}^*$, $\sigma_{x,y,z}$ is the cyclic sum with respect to x, y, z and $c_{12}(T)$ denotes the contraction

$$c_{12}(T)(z) = \sum_i T(e_i, e_i, z)$$

for an arbitrary orthonormal basis $\{e_i\}$ of $\mathcal{V}$. The projections of a homogeneous structure T on each of these subspaces are given by

$$\begin{aligned} p_1(T)(x,y,z) &= \tfrac{1}{n-1}\langle x,y\rangle c_{12}(T)(z) - \tfrac{1}{n-1}\langle x,z\rangle c_{12}(T)(y), \\ p_3(T)(x,y,z) &= \tfrac{1}{3}\sigma_{x,y,z}T(x,y,z), \\ p_2(T)(x,y,z) &= \left(T - p_1(T) - p_3(T)\right)(x,y,z). \end{aligned} \tag{7.2}$$

Homogeneous manifolds admitting a homogeneous structure in one of the eight different classes induced by the decomposition above have been extensively studied in the literature. It was shown in [130] that naturally reductive spaces correspond to non-vanishing homogeneous structures of type $\mathcal{T}_3$ and that a Riemannian manifold admits a non-vanishing structure of type $\mathcal{T}_1$ if and only if it is locally isometric to the real hyperbolic space. The latter also holds true for homogeneous structures of type $\mathcal{T}_1 \oplus \mathcal{T}_3$, $T \notin \mathcal{T}_1$ and $T \notin \mathcal{T}_3$, in dimension greater than three, as shown in [117]. Riemannian manifolds of dimension less than or equal to four that admit a homogeneous structure of type $\mathcal{T}_2$ were described in [93] (see also [34]). Homogeneous structures in the class $\mathcal{T}_1 \oplus \mathcal{T}_2$ in dimension less than or equal to four were described in [73], and those in this class whose fundamental one-form is closed were investigated in [118]. It was shown in [93] that a three-dimensional non-symmetric space admitting a homogeneous structure of type $\mathcal{T}_3$ also admits a $\mathcal{T}_2$-structure.

In dimension two, the decomposition above reduces to $\mathcal{T}(\mathcal{V}) = \mathcal{T}_1(\mathcal{V})$. As a consequence, a surface admits a non-zero homogeneous structure if and only if it is isometric to the hyperbolic plane. Dimension three is particularly relevant to the study of homogeneous spaces. First, it is the lowest possible dimension admitting locally homogeneous metrics which are not locally symmetric and, secondly, any three-dimensional homogeneous manifold is either symmetric or locally isometric to a Lie group endowed with a left-invariant metric [126].

The special case in which (M, g) is a Lie group G equipped with a left-invariant metric $\langle \cdot, \cdot \rangle$ is of special interest for our purposes. Let T^∇ be the *canonical homogeneous structure* defined by

$$2\langle T^\nabla(X,Y), Z\rangle = \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle,$$

for left-invariant vector fields X, Y and Z. Then the corresponding Ambrose-Singer connection $\widetilde{\nabla} = \nabla - T^\nabla$ satisfies $\widetilde{\nabla}_X Y = 0$ for any two left-invariant vector fields. This structure is equivalent to the description $G = G/\{e\}$, which corresponds to the action $G \times G \to G$.

7.2 Riemannian homogeneous structures in dimension three

On the basis of the above outlined, in this section we will clarify the classification of the Riemannian homogeneous structures in dimension three, describing all the possible ones in the non-symmetric case. The following result characterizes the non-symmetric Lie groups admitting more than one homogeneous structure.

Theorem 7.1. *A non-symmetric simply connected three-dimensional Riemannian Lie group admits a homogeneous structure different from the canonical one if and only if it admits a naturally reductive homogeneous structure. Moreover, in such a case, it admits exactly a one-parameter family of homogeneous structures.*

The explicit description of all the homogeneous structures on non-symmetric Lie groups is given in Theorem 7.2 and Theorem 7.3.

Recall that a three-dimensional complete and simply connected manifold is naturally reductive if and only if it admits a non-vanishing homogeneous structure of type $\mathcal{T}_3$. In this case (M, g) is a real space form $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, or it is isometric either to the special unitary group $SU(2)$, or to the universal covering of $\widetilde{SL}(2, \mathbb{R})$ or to the 3-dimensional Heisenberg group H^3, endowed with a suitable left-invariant metric described in terms of the Lie algebras (up to rotations) as

$$\begin{aligned} H^3: &\quad [e_1, e_2] = \lambda e_3, && && \lambda \neq 0, \\ SU(2): &\quad [e_1, e_2] = \mu e_3, \quad [e_2, e_3] = \lambda e_1, \quad [e_3, e_1] = \lambda e_2, && \lambda\,\mu > 0, \\ \widetilde{SL}(2, \mathbb{R}): &\quad [e_1, e_2] = \mu e_3, \quad [e_2, e_3] = \lambda e_1, \quad [e_3, e_1] = \lambda e_2, && \lambda\,\mu < 0, \end{aligned} \tag{7.3}$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis (see [130]).

In this way, (M, g) is naturally reductive if and only if it is isometric to a Lie group endowed with a left-invariant metric whose isometry group is at least four-dimensional.

Theorem 7.1 is thus connected to the following theorem by Meeks and Perez (see [103]): *a simply connected, three-dimensional Lie group with a left-invariant metric $(G_1, \langle\cdot,\cdot\rangle_1)$ is isometric to a second Lie group $(G_2, \langle\cdot,\cdot\rangle_2)$ which is not isomorphic to G_1 if and only if its isometry group has dimension at least four*.

7.2.1 Summary of results

We study the unimodular and non-unimodular cases separately. The unimodular case is dealt with in Section 7.2.2, and the non-unimodular case is considered in Section 7.2.3. We will see that the Riemannian homogeneous structures on a non-symmetric three-dimensional Lie group G equipped with a left-invariant metric are given as follows, from where the proof of Theorem 7.1 is obtained at once.

Unimodular Lie groups

The left-invariant Riemannian metrics $\langle\cdot,\cdot\rangle$ on three-dimensional unimodular Lie groups G were described by Milnor (see [104]) in terms of three structure constants $(\lambda_1, \lambda_2, \lambda_3)$, so that the Lie algebra becomes

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_1, e_3] = -\lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1,$$

for an orthonormal basis $\{e_1, e_2, e_3\}$. It now follows that $(G, \langle\cdot,\cdot\rangle)$ is symmetric if and only if $\lambda_1 = 0$ and $\lambda_2 = \lambda_3$ (up to rotations), in which case it is flat, or $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$, and the sectional curvature is constant and positive. We focus on the non-symmetric situation and we have the following result.

Theorem 7.2. *Let $(G, \langle\cdot,\cdot\rangle)$ be a unimodular Lie group equipped with a non-symmetric left-invariant Riemannian metric. Then there are two mutually excluding cases.*

(i) The three structure constants $\lambda_1, \lambda_2, \lambda_3$ are different and the only homogeneous structure is the canonical one, which is given by

$$\begin{aligned} T^{\nabla} &= -(\lambda_1 - \lambda_2 - \lambda_3)e^1 \otimes (e^2 \wedge e^3) - (\lambda_1 - \lambda_2 + \lambda_3)e^2 \otimes (e^1 \wedge e^3) \\ &\quad +(\lambda_1 + \lambda_2 - \lambda_3)e^3 \otimes (e^1 \wedge e^2). \end{aligned}$$

The canonical homogeneous structure is of type $\mathcal{T}_2$ if $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (see also [73]) and it is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$ otherwise.

(ii) Up to a rotation, the structure constants $\lambda_1 = \lambda_2 \neq \lambda_3$, $\lambda_3 \neq 0$ and there exists a one-parameter family of homogeneous structures

$$T = \lambda_3 e^1 \otimes (e^2 \wedge e^3) - \lambda_3 e^2 \otimes (e^1 \wedge e^3) + 2\kappa e^3 \otimes (e^1 \wedge e^2), \quad \kappa \in \mathbb{R},$$

which corresponds to the canonical structure for $\kappa = \frac{1}{2}(2\lambda_1 - \lambda_3)$. Moreover, it is of type $\mathcal{T}_2$ if $\kappa = -\lambda_3$, of type $\mathcal{T}_3$ if $\kappa = \frac{1}{2}\lambda_3$, and of type $\mathcal{T}_2 \oplus \mathcal{T}_3$ otherwise.

The unimodular Lie groups in Theorem 7.2-*(ii)* correspond to $SU(2)$, $\widetilde{SL}(2,\mathbb{R})$ and H^3 with left-invariant metric as in (7.3), and these include the homogeneous structures on Berger spheres previously considered in [75].

Non-unimodular Lie groups

Non-unimodular Riemannian Lie groups $(G, \langle\cdot,\cdot\rangle)$ are semi-direct extensions $\mathbb{R}^2 \rtimes \mathbb{R}$ of the Abelian group. It was shown in [104] that there exists an orthonormal basis $\{e_1, e_2, e_3\}$ so that

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \qquad [e_1, e_3] = \gamma e_2 + \delta e_3, \qquad [e_2, e_3] = 0,$$

where the trace of the endomorphism determining the semi-direct extension satisfies $\alpha + \delta \neq 0$. Moreover, one may rotate the orthonormal basis $\{e_2, e_3\}$ of the unimodular kernel to assume that their images by the endomorphism are orthogonal, i.e., $\alpha\gamma + \beta\delta = 0$. The Riemannian Lie group $(G, \langle\cdot,\cdot\rangle)$ is symmetric if and only if $\beta = \delta = \gamma = 0$ (up to the isometry $e_2 \mapsto e_3$), and so it is isometric to $\mathbb{R} \times \mathbb{H}^2(-\alpha^2)$, or if $\alpha = \delta \neq 0$ and $\gamma = -\beta$, in which case it is a space of constant sectional curvature $\mathbb{H}^3(-\delta^2)$. In this setting, we have the following result.

Theorem 7.3. *Let $(G, \langle\cdot,\cdot\rangle)$ be a non-unimodular Lie group equipped with a left-invariant non-symmetric Riemannian metric. Then there are two mutually excluding cases.*

(i) If $\delta = \gamma = 0$, $\alpha\beta \neq 0$, then the homogeneous structures are given by one of the following possibilities.

(i.a) The one-parameter family

$$T = \beta e^1 \otimes (e^2 \wedge e^3) - \beta e^2 \otimes (e^1 \wedge e^3) + 2\kappa e^3 \otimes (e^1 \wedge e^2), \quad \kappa \in \mathbb{R}.$$

In this case the homogeneous structure is of type $\mathcal{T}_2$ if $\kappa = -\beta$, of type $\mathcal{T}_3$ if $\kappa = \frac{1}{2}\beta$, and of type $\mathcal{T}_2 \oplus \mathcal{T}_3$ otherwise.

(i.b) The canonical homogeneous structure

$$T^\nabla = \beta e^1 \otimes (e^2 \wedge e^3) - 2\alpha e^2 \otimes (e^1 \wedge e^2) - \beta e^2 \otimes (e^1 \wedge e^3) - \beta e^3 \otimes (e^1 \wedge e^2),$$

which is of the generic type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$.

(ii) If $\delta\alpha \neq 0$, $\beta = -\frac{\alpha\gamma}{\delta}$ and $\alpha \neq \delta$, then the only homogeneous structure is the canonical one, which is given by

$$\begin{aligned} T^\nabla \;=\; & -\tfrac{(\alpha+\delta)\gamma}{\delta} e^1 \otimes (e^2 \wedge e^3) - 2\alpha e^2 \otimes (e^1 \wedge e^2) + \tfrac{(\alpha-\delta)\gamma}{\delta} e^2 \otimes (e^1 \wedge e^3) \\ & + \tfrac{(\alpha-\delta)\gamma}{\delta} e^3 \otimes (e^1 \wedge e^2) - 2\delta e^3 \otimes (e^1 \wedge e^3), \end{aligned}$$

and is of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ if $\gamma = 0$, and $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$ otherwise.

The non-unimodular Lie groups given in Theorem 7.3 are semi-direct extensions $\mathbb{R}^2 \rtimes \mathbb{R}$ of the Abelian Lie group determined by an endomorphism $-\operatorname{ad}(e_1)$. Assertion *(i)* in Theorem 7.3 corresponds to the special situation where $\det \operatorname{ad}(e_1) = 0$, and they are isometric (although not isomorphically isometric) to a left-invariant metric on $\widetilde{SL}(2, \mathbb{R})$ as in (7.3), which corresponds to Theorem 7.2-*(ii)* (cf. [103, 130]). We would like to emphasize that isometries between Riemannian Lie groups need not preserve the Lie group structure, since they are not necessarily realized by group isomorphisms, as evidenced in the above-mentioned situation. On the other hand, the Lie groups in Theorem 7.3-*(ii)* correspond to the generic situation, where one can always specify the orthonormal basis $\{e_2, e_3\}$ so that it is given by eigenvectors of the self-adjoint part of $\operatorname{ad}(e_1)$ (cf. [104]).

Non-symmetric simply connected homogeneous three-dimensional Riemannian manifolds with four-dimensional isometry group are isometric to the unitary group $SU(2)$, the universal cover of $\widetilde{SL}(2, \mathbb{R})$, or the Heisenberg group with the special metrics (7.3). It follows from the description of homogeneous structures in Theorem 7.2 and Theorem 7.3 that (see also [130]) a non-symmetric three-dimensional Riemannian Lie group admits a homogeneous structure different from the canonical one if and only if its isometry group is four-dimensional

Remark 7.4. A more conceptual proof of this last statement can be summarized as follows. For any three-dimensional Lie groups $(G_1, \langle\cdot,\cdot\rangle_1)$ and $(G_2, \langle\cdot,\cdot\rangle_2)$ equipped with a left-invariant Riemannian metric, it follows from Theorems 7.2 and 7.3 that the infinitesimal models associated to their canonical homogeneous structures are isomorphic if and only if the Lie groups $(G_1, \langle\cdot,\cdot\rangle_1)$

and $(G_2, \langle\cdot,\cdot\rangle_2)$ are isomorphically isometric (see [34]). Besides, any non-symmetric homogeneous three-manifold with four-dimensional isometry group admits more than one homogeneous structure. It follows from the works in [103, 126] that a homogeneous three-manifold with three-dimensional isometry group is isometric to a unique Riemannian Lie group, in which case any homogeneous structure is isomorphic to the canonical one.

7.2.2 Homogeneous structures on non-symmetric unimodular Lie groups

Following [104], if $\mathfrak{g}$ is unimodular then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_1, e_3] = -\lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1.$$

Let T be a $(0,3)$-tensor field so that the connection $\widetilde{\nabla} = \nabla - T$ makes the metric tensor parallel, i.e., $T_{xyz} + T_{xzy} = 0$ for x, y, $z \in \mathfrak{g}$. Denoting by $\{e^1, e^2, e^3\}$ the dual basis of $\{e_1, e_2, e_3\}$, then the tensor field T can be written as

$$T = 2\sum_i \sum_{j<k} T_{ijk} e^i \otimes (e^j \wedge e^k).$$

Therefore, the non-zero components of the connection $\widetilde{\nabla} = \nabla - T$ are given by

$$\begin{aligned}
&\widetilde{\nabla}_{112} = -T_{112}, \quad \widetilde{\nabla}_{223} = -T_{223}, \quad \widetilde{\nabla}_{123} = -\tfrac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3) - T_{123}, \\
&\widetilde{\nabla}_{113} = -T_{113}, \quad \widetilde{\nabla}_{313} = -T_{313}, \quad \widetilde{\nabla}_{213} = -\tfrac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3) - T_{213}, \\
&\widetilde{\nabla}_{212} = -T_{212}, \quad \widetilde{\nabla}_{323} = -T_{323}, \quad \widetilde{\nabla}_{312} = \ \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3) - T_{312},
\end{aligned} \tag{7.4}$$

while the $(0,4)$-curvature tensor field is determined by

$$\begin{aligned}
R_{1212} &= \tfrac{1}{4}\left((\lambda_1 - \lambda_2)^2 - 3\lambda_3^2 + 2(\lambda_1 + \lambda_2)\lambda_3\right), \\
R_{1313} &= \tfrac{1}{4}\left((\lambda_1 - \lambda_3)^2 - 3\lambda_2^2 + 2(\lambda_1 + \lambda_3)\lambda_2\right), \\
R_{2323} &= \tfrac{1}{4}\left((\lambda_2 - \lambda_3)^2 - 3\lambda_1^2 + 2(\lambda_2 + \lambda_3)\lambda_1\right).
\end{aligned} \tag{7.5}$$

Let $\mathfrak{R}_{ikj\ell;r} = (\widetilde{\nabla}_{e_r} R)(e_i, e_j, e_k, e_\ell)$. A straightforward calculation involving

Equations (7.4) and (7.5) shows that the condition $\widetilde{\nabla}R = 0$ in Equation (7.1) is given by

$$\begin{aligned}
2\,\mathfrak{R}_{1213;1} &= (\lambda_1-\lambda_2-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1-\lambda_2-\lambda_3+2T_{123}) = 0,\\
\mathfrak{R}_{1213;2} &= (\lambda_1-\lambda_2-\lambda_3)(\lambda_2-\lambda_3)T_{223} = 0,\\
\mathfrak{R}_{1213;3} &= (\lambda_1-\lambda_2-\lambda_3)(\lambda_2-\lambda_3)T_{323} = 0,\\
\mathfrak{R}_{1223;1} &= (\lambda_1-\lambda_2+\lambda_3)(\lambda_1-\lambda_3)T_{113} = 0,\\
2\,\mathfrak{R}_{1223;2} &= (\lambda_1-\lambda_2+\lambda_3)(\lambda_1-\lambda_3)(\lambda_1-\lambda_2+\lambda_3+2T_{213}) = 0,\\
\mathfrak{R}_{1223;3} &= (\lambda_1-\lambda_2+\lambda_3)(\lambda_1-\lambda_3)T_{313} = 0,\\
-\mathfrak{R}_{1323;1} &= (\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2)T_{112} = 0,\\
-\mathfrak{R}_{1323;2} &= (\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2)T_{212} = 0,\\
2\,\mathfrak{R}_{1323;3} &= (\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2)(\lambda_1+\lambda_2-\lambda_3-2T_{312}) = 0.
\end{aligned} \tag{7.6}$$

From here, depending on the eigenvalues λ_i, we are led to the following two possibilities.

Case of three different eigenvalues

In this case, if $\lambda_1-\lambda_2-\lambda_3$, $\lambda_1-\lambda_2+\lambda_3$ and $\lambda_1+\lambda_2-\lambda_3$ do not vanish, then Equations (7.4) and (7.6) clearly imply that $\widetilde{\nabla}_X Y = 0$ for all left-invariant vector fields. Therefore, the only homogeneous structure is the canonical one given by $T_x y = \nabla_x y$, for $x, y \in \mathfrak{g}$, i.e.,

$$\begin{aligned}
T = &-(\lambda_1-\lambda_2-\lambda_3)e^1\otimes(e^2\wedge e^3) - (\lambda_1-\lambda_2+\lambda_3)e^2\otimes(e^1\wedge e^3)\\
&+(\lambda_1+\lambda_2-\lambda_3)e^3\otimes(e^1\wedge e^2).
\end{aligned} \tag{7.7}$$

Next we show that the same holds if any of $\lambda_1-\lambda_2-\lambda_3$, $\lambda_1-\lambda_2+\lambda_3$ and $\lambda_1+\lambda_2-\lambda_3$ vanishes. Suppose that $\lambda_3 = \lambda_1-\lambda_2$ (the other two cases are obtained in a completely analogous way). Then, Equations (7.6) implies

$$T_{112} = T_{113} = T_{212} = T_{313} = 0, \quad T_{213} = -\lambda_1+\lambda_2, \quad T_{312} = \lambda_2. \tag{7.8}$$

Let $\mathfrak{T}_{ijk;r} = (\widetilde{\nabla}_{e_r}T)(e_i, e_j, e_k)$. A straightforward calculation using Equations (7.4) and (7.8) shows that the condition $\widetilde{\nabla}T = 0$ in Equation (7.1) reduces to

$$\begin{aligned}
\mathfrak{T}_{313;r} &= -\mathfrak{T}_{212;r} = (\lambda_1-2\lambda_2)T_{r23} = 0,\\
\mathfrak{T}_{223;r} &= T_{r23}T_{323} = 0,\\
\mathfrak{T}_{323;r} &= -T_{r23}T_{223} = 0,
\end{aligned}$$

or, equivalently, $T_{123} = T_{223} = T_{323} = 0$. Thus, the only homogeneous structure is given by

$$T = -2(\lambda_1-\lambda_2)e^2\otimes(e^1\wedge e^3) + 2\lambda_2 e^3\otimes(e^1\wedge e^2),$$

which corresponds to the structure given by Equation (7.7) for $\lambda_3 = \lambda_1 - \lambda_2$.

Finally, the projections of the homogeneous structure given in Equation (7.7) are obtained by a direct calculation. In particular, $p_1(T) = 0$ and

$$\begin{aligned} p_2(T) = &-\tfrac{2}{3}(2\lambda_1 - \lambda_2 - \lambda_3)e^1 \otimes (e^2 \wedge e^3) - \tfrac{2}{3}(\lambda_1 - 2\lambda_2 + \lambda_3)e^2 \otimes (e^1 \wedge e^3) \\ &+ \tfrac{2}{3}(\lambda_1 + \lambda_2 - 2\lambda_3)e^3 \otimes (e^1 \wedge e^2), \\ p_3(T) = &\ \tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)\left(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) + e^3 \otimes (e^1 \wedge e^2)\right). \end{aligned}$$

Case of two different eigenvalues

In this case, without loss of generality, we can assume $\lambda_1 = \lambda_2 \neq \lambda_3$. Moreover, $\lambda_3 \neq 0$ since the space would be locally symmetric otherwise. Thus, Equation (7.6) implies

$$T_{113} = T_{223} = T_{313} = T_{323} = 0, \quad T_{123} = -T_{213} = \frac{\lambda_3}{2}. \tag{7.9}$$

Let $\mathfrak{T}_{ijk;r} = (\widetilde{\nabla}_{e_r} T)(e_i, e_j, e_k)$. A straightforward calculation using Equations (7.4) and (7.9) shows that the condition $\widetilde{\nabla} T = 0$ in Equation (7.1) reduces to

$$\begin{aligned} \mathfrak{T}_{112;r} &= \ \ T_{r12}T_{212} = 0,\ (r = 1,2), \quad \mathfrak{T}_{112;3} = -\tfrac{1}{2}(2\lambda_1 - \lambda_3 - 2T_{312})T_{212} = 0, \\ \mathfrak{T}_{212;r} &= -T_{r12}T_{112} = 0,\ (r = 1,2), \quad \mathfrak{T}_{212;3} = \ \ \tfrac{1}{2}(2\lambda_1 - \lambda_3 - 2T_{312})T_{112} = 0, \end{aligned}$$

or, equivalently, $T_{112} = T_{212} = 0$. Thus, we obtain a one-parameter family of homogeneous structures given by

$$T = \lambda_3 e^1 \otimes (e^2 \wedge e^3) - \lambda_3 e^2 \otimes (e^1 \wedge e^3) + 2\kappa e^3 \otimes (e^1 \wedge e^2), \quad \kappa \in \mathbb{R}.$$

In the particular case where $\kappa = \frac{1}{2}(2\lambda_1 - \lambda_3)$, it corresponds to the canonical structure. Finally, a direct calculation shows that the projections of these structures are such that $p_1(T) = 0$ and

$$\begin{aligned} p_2(T) &= \tfrac{1}{3}(\lambda_3 - 2\kappa)\left(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) - 2e^3 \otimes (e^1 \wedge e^2)\right), \\ p_3(T) &= \tfrac{2}{3}(\lambda_3 + \kappa)\left(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) + e^3 \otimes (e^1 \wedge e^2)\right). \end{aligned}$$

7.2.3 Homogeneous structures on non-symmetric and non-unimodular Lie groups

If $\mathfrak{g}$ is non-unimodular then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that (see [104])

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \qquad [e_1, e_3] = \gamma e_2 + \delta e_3, \qquad [e_2, e_3] = 0,$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$.

A straightforward calculation shows that a non-unimodular Lie group corresponding to a Lie algebra above is locally symmetric if and only if it is of constant negative sectional curvature (which corresponds to the cases when $\mathrm{ad}(e_1)$ is a multiple of the identity or it has complex eigenvalues), or it is locally isometric to a product $\mathbb{R}\times N(c)$, where $N(c)$ is a surface of constant negative sectional curvature (if $\mathrm{ad}(e_1)$ is of rank-one and $\{e_2,e_3\}$ is an orthonormal basis of eigenvectors).

Using the same notation as in the previous section, the non-zero components of the connection $\widetilde{\nabla}=\nabla-T$ are given by

$$\begin{aligned}
&\widetilde{\nabla}_{112}=-T_{112}, && \widetilde{\nabla}_{223}=-T_{223}, && \widetilde{\nabla}_{123}=\ \tfrac{1}{2}(\beta-\gamma-2T_{123}),\\
&\widetilde{\nabla}_{113}=-T_{113}, && \widetilde{\nabla}_{313}=-\delta-T_{313}, && \widetilde{\nabla}_{213}=-\tfrac{1}{2}(\beta+\gamma+2T_{213}),\\
&\widetilde{\nabla}_{212}=-\alpha-T_{212}, && \widetilde{\nabla}_{323}=-T_{323}, && \widetilde{\nabla}_{312}=-\tfrac{1}{2}(\beta+\gamma+2T_{312}),
\end{aligned}\tag{7.10}$$

while the $(0,4)$-curvature tensor field is determined by

$$\begin{aligned}
R_{1212}&=-\tfrac{1}{4}\left(4\alpha^2+3\beta^2-\gamma^2+2\beta\gamma\right),\\
R_{1313}&=\ \tfrac{1}{4}\left(\beta^2-3\gamma^2-4\delta^2-2\beta\gamma\right),\\
R_{2323}&=\ \tfrac{1}{4}\left(\beta^2+\gamma^2-4\alpha\delta+2\beta\gamma\right).
\end{aligned}\tag{7.11}$$

As in the previous section, set $\mathfrak{R}_{ikj\ell;r}=(\widetilde{\nabla}_{e_r}R)(e_i,e_j,e_k,e_\ell)$. Equations (7.10) and (7.11) imply that the condition $\widetilde{\nabla}R=0$ in Equation (7.1) is given by

$$\begin{aligned}
-2\,\mathfrak{R}_{1213;1}&=(\alpha^2+\beta^2-\gamma^2-\delta^2)(\beta-\gamma-2T_{123})=0,\\
\mathfrak{R}_{1213;2}&=(\alpha^2+\beta^2-\gamma^2-\delta^2)T_{223}=0,\\
\mathfrak{R}_{1213;3}&=(\alpha^2+\beta^2-\gamma^2-\delta^2)T_{323}=0,\\
-\mathfrak{R}_{1223;1}&=(\alpha^2+\beta^2-\alpha\delta+\beta\gamma)T_{113}=0,\\
-2\,\mathfrak{R}_{1223;2}&=(\alpha^2+\beta^2-\alpha\delta+\beta\gamma)(\beta+\gamma+2T_{213})=0,\\
-\mathfrak{R}_{1223;3}&=(\alpha^2+\beta^2-\alpha\delta+\beta\gamma)(\delta+T_{313})=0,\\
\mathfrak{R}_{1323;1}&=(\gamma^2+\delta^2-\alpha\delta+\beta\gamma)T_{112}=0,\\
\mathfrak{R}_{1323;2}&=(\gamma^2+\delta^2-\alpha\delta+\beta\gamma)(\alpha+T_{212})=0,\\
2\,\mathfrak{R}_{1323;3}&=(\gamma^2+\delta^2-\alpha\delta+\beta\gamma)(\beta+\gamma+2T_{312})=0.
\end{aligned}\tag{7.12}$$

Next we analyse the cases $\delta=0$ and $\delta\neq 0$ separately.

Case $\delta=0$

Since $\alpha+\delta\neq 0$ and $\alpha\gamma+\beta\delta=0$, in this case $\gamma=0$ and $\alpha\neq 0$. Moreover, $\beta\neq 0$ because the space is assumed not to be locally symmetric. Thus, Equation (7.12) reduces to

$$T_{113}=T_{223}=T_{313}=T_{323}=0,\quad T_{123}=-T_{213}=\frac{\beta}{2}.\tag{7.13}$$

Following the notation in the previous section, we set $\mathfrak{T}_{ijk;r} = (\widetilde{\nabla}_{e_r} T)(e_i, e_j, e_k)$. A straightforward calculation involving Equations (7.10) and (7.13) shows that the condition $\widetilde{\nabla} T = 0$ in Equation (7.1) reduces to

$$\begin{aligned} \mathfrak{T}_{112;1} &= T_{112}T_{212} = 0, & -\mathfrak{T}_{212;1} &= (T_{112})^2 = 0, \\ \mathfrak{T}_{112;2} &= (\alpha + T_{212})T_{212} = 0, & -\mathfrak{T}_{212;2} &= (\alpha + T_{212})T_{112} = 0, \\ 2\,\mathfrak{T}_{112;3} &= (\beta + 2T_{312})T_{212} = 0, & -2\,\mathfrak{T}_{212;3} &= (\beta + 2T_{312})T_{112} = 0. \end{aligned}$$

Hence, $T_{112} = 0$ and either $T_{212} = 0$ or $T_{212} = -\alpha$, $T_{312} = -\frac{\beta}{2}$.

If $T_{112} = T_{212} = 0$, we obtain a one-parameter family of homogeneous structures given by

$$T = \beta e^1 \otimes (e^2 \wedge e^3) - \beta e^2 \otimes (e^1 \wedge e^3) + 2\kappa e^3 \otimes (e^1 \wedge e^2), \quad \kappa \in \mathbb{R}.$$

The projections of these homogeneous structures are obtained by a direct calculation. In particular, $p_1(T) = 0$ and

$$\begin{aligned} p_2(T) &= \tfrac{1}{3}(\beta - 2\kappa)\left(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) - 2e^3 \otimes (e^1 \wedge e^2)\right), \\ p_3(T) &= \tfrac{2}{3}(\beta + \kappa)\left(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) + e^3 \otimes (e^1 \wedge e^2)\right). \end{aligned}$$

If $T_{112} = 0$ and $T_{212} = -\alpha$, $T_{312} = -\frac{\beta}{2}$, then Equations (7.10) and (7.13) clearly imply that $\widetilde{\nabla}_X Y = 0$ for left-invariant vector fields. Therefore, the only homogeneous structure is the canonical one, given by

$$T = \beta e^1 \otimes (e^2 \wedge e^3) - 2\alpha e^2 \otimes (e^1 \wedge e^2) - \beta e^2 \otimes (e^1 \wedge e^3) - \beta e^3 \otimes (e^1 \wedge e^2).$$

In this case, the projections of this homogeneous structure are given by

$$\begin{aligned} p_1(T) &= -\alpha\left(e^2 \otimes (e^1 \wedge e^2) + e^3 \otimes (e^1 \wedge e^3)\right), \\ p_2(T) &= \tfrac{2}{3}\beta e^1 \otimes (e^2 \wedge e^3) - \alpha e^2 \otimes (e^1 \wedge e^2) - \tfrac{2}{3}\beta e^2 \otimes (e^1 \wedge e^3) \\ &\quad - \tfrac{4}{3}\beta e^3 \otimes (e^1 \wedge e^2) + \alpha e^3 \otimes (e^1 \wedge e^3), \\ p_3(T) &= \tfrac{1}{3}\beta\left(e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) + e^3 \otimes (e^1 \wedge e^2)\right). \end{aligned}$$

Case $\delta \neq 0$

In this case, since $\alpha\gamma + \beta\delta = 0$, we have $\beta = -\frac{\alpha\gamma}{\delta}$. Moreover, the space is locally symmetric if either $\alpha = \delta$ or $\alpha = \gamma = 0$. Now Equation (7.12) becomes

$$\begin{aligned}
2\delta^3\,\mathfrak{R}_{1213;1} &= (\alpha^2-\delta^2)(\gamma^2+\delta^2)((\alpha+\delta)\gamma + 2\delta T_{123}) = 0,\\
\delta^2\,\mathfrak{R}_{1213;2} &= (\alpha^2-\delta^2)(\gamma^2+\delta^2)T_{223} = 0,\\
\delta^2\,\mathfrak{R}_{1213;3} &= (\alpha^2-\delta^2)(\gamma^2+\delta^2)T_{323} = 0,\\
-\delta^2\,\mathfrak{R}_{1223;1} &= \alpha(\alpha-\delta)(\gamma^2+\delta^2)T_{113} = 0,\\
2\delta^3\,\mathfrak{R}_{1223;2} &= \alpha(\alpha-\delta)(\gamma^2+\delta^2)((\alpha-\delta)\gamma - 2\delta T_{213}) = 0,\\
-\delta^2\,\mathfrak{R}_{1223;3} &= \alpha(\alpha-\delta)(\gamma^2+\delta^2)(\delta + T_{313}) = 0,\\
-\delta\,\mathfrak{R}_{1323;1} &= (\alpha-\delta)(\gamma^2+\delta^2)T_{112} = 0,\\
-\delta\,\mathfrak{R}_{1323;2} &= (\alpha-\delta)(\gamma^2+\delta^2)(\alpha + T_{212}) = 0,\\
2\delta^2\,\mathfrak{R}_{1323;3} &= (\alpha-\delta)(\gamma^2+\delta^2)((\alpha-\delta)\gamma - 2\delta T_{312}) = 0.
\end{aligned}$$

Note that $\alpha^2 - \delta^2 \neq 0$, since $\alpha + \delta \neq 0$ and if $\alpha - \delta = 0$ then the space is locally symmetric. Besides, if $\alpha = 0$ then $\beta = 0$ and the metric is isometric to the one in the case corresponding to $\delta = 0$. Hence, the previous equations together with Equation (7.10) imply that $\widetilde{\nabla}_X Y = 0$ for left-invariant vector fields and the only homogeneous structure is the canonical one, which corresponds to

$$\begin{aligned}
T = &-\tfrac{(\alpha+\delta)\gamma}{\delta}e^1\otimes(e^2\wedge e^3) - 2\alpha e^2\otimes(e^1\wedge e^2) + \tfrac{(\alpha-\delta)\gamma}{\delta}e^2\otimes(e^1\wedge e^3)\\
&+ \tfrac{(\alpha-\delta)\gamma}{\delta}e^3\otimes(e^1\wedge e^2) - 2\delta e^3\otimes(e^1\wedge e^3),
\end{aligned}$$

and its projections are given by

$$\begin{aligned}
p_1(T) &= -(\alpha+\delta)\left(e^2\otimes(e^1\wedge e^2) + e^3\otimes(e^1\wedge e^3)\right),\\
p_2(T) &= -\tfrac{2(\alpha+\delta)\gamma}{3\delta}e^1\otimes(e^2\wedge e^3) - (\alpha-\delta)e^2\otimes(e^1\wedge e^2)\\
&\quad + \tfrac{2(\alpha-2\delta)\gamma}{3\delta}e^2\otimes(e^1\wedge e^3)\\
&\quad + \tfrac{2(2\alpha-\delta)\gamma}{3\delta}e^3\otimes(e^1\wedge e^2) + (\alpha-\delta)e^3\otimes(e^1\wedge e^3),\\
p_3(T) &= -\tfrac{(\alpha+\delta)\gamma}{3\delta}\left(e^1\otimes(e^2\wedge e^3) - e^2\otimes(e^1\wedge e^3) + e^3\otimes(e^1\wedge e^2)\right).
\end{aligned}$$

7.3 Self-dual and anti-self-dual homogeneous structures

Dimension four is of special interest for the study of Riemannian manifolds. In this case, the fact that the rotation group $SO(4)$ is not simple gives rise to the concepts of self-duality and anti-self-duality in this context.

Even though $\mathcal{T}_1$ and $\mathcal{T}_3$ are irreducible under the action of $SO(4)$, $\mathcal{T}_2$ splits into two irreducible components and the decomposition of the space of homogeneous structures on a four-dimensional vector space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is now given by

$$\mathcal{T}(\mathcal{V}) = \mathcal{T}_1(\mathcal{V}) \oplus \mathcal{T}_2^+(\mathcal{V}) \oplus \mathcal{T}_2^-(\mathcal{V}) \oplus \mathcal{T}_3(\mathcal{V}),$$

where

$$\mathcal{T}_2^{\pm} = \{T \in \mathcal{T}_2 \colon T(x,\bar{y},\bar{z}) = \pm T(x,y,z)\}$$

for all $x,y,z \in \mathcal{V}$, where $(\bar{y},\bar{z})$ stands for the dual of (y,z) according to the relation $e^k \wedge e^\ell \wedge e^{\bar{k}} \wedge e^{\bar{\ell}} = e^1 \wedge e^2 \wedge e^3 \wedge e^4$ for any positively oriented orthonormal basis $\{e_1,e_2,e_3,e_4\}$. The projections of a homogeneous structure $T \in \mathcal{T}$ on these two subspaces are given by

$$p_2^{\pm}(T)(x,y,z) = \tfrac{1}{2}\{p_2(T)(x,y,z) \pm p_2(T)(x,\bar{y},\bar{z})\},$$

for any $x,y,z \in \mathcal{V}$.

A homogenous structure T on a four-dimensional Riemannian manifold is said to be *self-dual* (resp. *anti-self-dual*) if $p_2^-(T) = 0$ (resp. $p_2^+(T) = 0$).

Self-dual and anti-self-dual homogeneous structures appear naturally in relation to the study of homogeneous manifolds with half-harmonic Weyl curvature.

Proposition 7.5. *Let (M,g) be a simply connected homogeneous four-manifold with half-harmonic Weyl tensor. Then it is symmetric or it admits a self-dual or anti-self-dual homogeneous structure.*

Proof. Let $(G, \langle\cdot,\cdot\rangle)$ be a half-harmonic non-symmetric homogenous four-manifold as in Theorem 6.1-*(i)*, determined by the Lie algebra structure

$$[e_1,e_2] = e_3,\ [e_1,e_4] = -e_1,\ [e_2,e_4] = -e_2,\ [e_3,e_4] = -2e_3.$$

Then, the Riemannian Lie group $(G, \langle\cdot,\cdot\rangle)$ is anti-self-dual and its canonical homogeneous structure is given by

$$\begin{aligned} T &= 2e^1 \otimes (e^1 \wedge e^4) + e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) \\ &\quad + 2e^2 \otimes (e^2 \wedge e^4) - e^3 \otimes (e^1 \wedge e^2) + 4e^3 \otimes (e^3 \wedge e^4) \end{aligned}$$

and a straightforward calculation shows that $p_2^+(T) = 0$, which shows that it is an anti-self-dual homogeneous structure. Moreover, one has that $p_1(T)$ and $p_3(T)$ are non-zero, and thus it is a homogeneous anti-self-dual structure which is not $\mathcal{T}_2$.

The homogeneous manifold in Theorem 6.1-*(ii)* corresponds to the unique four-dimensional 3-symmetric space, and it is isometric to the Lie group determined by the Lie algebra

$$[e_1,e_2] = e_3,\ [e_1,e_4] = \tfrac{1}{2}e_1,\ [e_2,e_4] = -e_2,\ [e_3,e_4] = -\tfrac{1}{2}e_3.$$

Considering the Kähler and opposite almost Kähler structures (J_+, J_-) determined by the Kähler forms

$$\Omega_+ = e^1 \wedge e^2 - e^3 \wedge e^4, \quad \text{and} \quad \Omega_- = e^1 \wedge e^2 + e^3 \wedge e^4,$$

the covariant derivative of the opposite almost Kähler structure J_- determines a homogeneous structure
$$T_{XY} = \frac{1}{2}J_-(\nabla_X J_-)Y$$
which is self-dual [130]. Furthermore, this structure is $\mathcal{T}_2$, and it was shown by Nicolodi in [109] that it is the only self-dual $\mathcal{T}_2$ structure. Furthermore this structure is not the canonical one. □

Remark 7.6. It follows after a long calculation that there are no strictly self-dual or anti-self-dual homogeneous structures on the semi-direct extensions $E(1,1) \rtimes \mathbb{R}$ and $\widetilde{E}(2) \rtimes \mathbb{R}$. On the contrary, semi-direct extensions of the Heisenberg group support such homogeneous structures. Indeed $H_3 \rtimes \mathbb{R}$ admits a self-dual homogeneous structure if and only if it is homothetic to a semi-direct extension with left-invariant metric determined by the following Lie algebras given in terms of an orthonormal basis $\{e_1, e_2, e_3, e_4\}$:

(i) $[e_1, e_2] = e_3,\ [e_1, e_4] = \alpha e_1,\ [e_2, e_4] = \alpha e_2,\ [e_3, e_4] = 2\alpha e_3,\ \alpha \neq 0, \pm\frac{1}{2}$.

In this case,
$$\begin{aligned} T = &-2\alpha e^1 \otimes (e^1 \wedge e^4) + e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) \\ &- 2\alpha e^2 \otimes (e^2 \wedge e^4) - (2\alpha - 1)e^3 \otimes (e^1 \wedge e^2) - 4\alpha e^3 \otimes (e^3 \wedge e^4) \end{aligned}$$
is of type $\mathcal{T}_1 \oplus \mathcal{T}_2^+$ if $\alpha = \frac{3}{2}$, or of type $\mathcal{T}_1 \oplus \mathcal{T}_2^+ \oplus \mathcal{T}_3$ otherwise.

(ii) $[e_1, e_2] = e_3, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = -\frac{1}{2}e_2, \quad [e_3, e_4] = \frac{1}{2}e_3.$

In this case,
$$T = -e^2 \otimes (e^1 \wedge e^3) + e^2 \otimes (e^2 \wedge e^4) - e^3 \otimes (e^1 \wedge e^2) - e^3 \otimes (e^3 \wedge e^4)$$
is of type $\mathcal{T}_2^+$.

There are semi-direct extensions of the Abelian group $\mathbb{R}^3$ which admit self-dual homogeneous structures. Considering the Lie algebra
$$[e_1, e_4] = e_1, \quad [e_2, e_4] = fe_2 + he_3, \quad [e_3, e_4] = -he_2 + fe_3, \quad f \neq 1,$$
where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis, one can check that the homogeneous structures
$$\begin{aligned} T \ = \ &-2e^1 \otimes (e^1 \wedge e^4) + 2(f-1)e^1 \otimes (e^2 \wedge e^3) \\ &- 2fe^2 \otimes (e^2 \wedge e^4) - 2fe^3 \otimes (e^3 \wedge e^4) \end{aligned}$$
are self-dual and never correspond to the canonical homogeneous structure.

Conclusions and open problems

The main achievements of this work can be outlined as follows.

C.1 We completed the classification of four-dimensional locally conformally flat Kähler, para-Kähler and null-Kähler structures.

C.2 We determined all the left-invariant four-dimensional para-Kähler Lie groups up to automorphisms preserving the symplectic structure, showing that all the locally conformally flat structures are realizable as left-invariant Kähler or para-Kähler structures on Lie groups.

C.3 We gave a complete description of all the left-invariant Ricci solitons on four-dimensional Lorentzian Lie groups.

C.4 We determined all the Riemannian algebraic Bach solitons, showing that they are algebraic Ricci solitons or belong to one of two exceptional families.

C.5 We classified the Riemannian homogeneous four-dimensional manifolds with half-harmonic Weyl curvature.

C.6 We showed that a non-symmetric three-dimensional homogeneous manifold admits more than one homogeneous structure if and only if its isometry group has dimension four.

A number of open problems naturally arise as a consequence of our work.

P.1 The complete classification of four-dimensional Bochner-flat para-Kähler structures of non-constant scalar curvature.

It was shown in [67] that any Bochner flat para-Kähler surface of constant scalar curvature is locally a para-complex space form or a locally conformally flat para-Kähler surface as described in Theorem 2.1. The problem of non-constant scalar curvature remains open. Such Bochner flat para-Kähler surfaces are locally isometric to a cotangent bundle with a modified Riemannian extension but a precise parametrization of such structures is still under consideration.

P.2 Algebraic Lorentzian Ricci solitons.

Algebraic Ricci solitons on Lorentzian Lie groups are well-understood in dimension three [14]. One expects to be able to solve the four-dimensional case by following the strategy developed in Chapter 5, although the calculations seem to be much more involved.

P.3 The existence of non-trivial left-invariant Bach solitons.

The existence of homogeneous gradient Bach solitons and algebraic Bach solitons on four-dimensional Riemannian Lie groups was discussed in Chapter 5. While there are no non-trivial Ricci solitons on four-dimensional Lie groups with a left-invariant soliton vector field, it is an open problem whether the same statement holds true in the case of Bach solitons.

P.4 Algebraic T-solitons.

It was shown by Arroyo and Lafuente [9] that any Riemannian expanding homogeneous Ricci soliton is homothetic to an algebraic Ricci soliton in dimension four. Hence, four-dimensional homogeneous Ricci solitons are either symmetric or algebraic.

The situation seems to be much more complicated for other geometric flows where it is not clear whether any non-symmetric soliton is necessarily algebraic. This is the case in the completely solvable case, where isometries are isomorphisms of the group, but the general situation is still an open question.

P.5 The complete classification of four-dimensional Riemannian self-dual homogeneous structures.

Homogeneous four-manifolds with half-harmonic Weyl curvature are equipped with a (not necessarily canonical) self-dual homogeneous structure. Non-symmetric semi-direct extensions of the Heisenberg group H^3, the Euclidean group $\widetilde{E}(2)$, or the Poincaré group $E(1,1)$ do not admit any other self-dual homogeneous structure. As pointed out in Section 7.3, there are however other self-dual homogeneous structures on semi-direct extensions $\mathbb{R}^3 \rtimes \mathbb{R}$. It is an open problem to classify such homogeneous structures and to understand their underlying geometries.

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