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CURVATURE AND GEOMETRIC EQUATIONS IN SMOOTH METRIC MEASURE SPACES

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TESE DE DOUTORAMENTO

Curvature and geometric equations in smooth metric measure spaces

Diego Mojón Álvarez

PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

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It’s a dangerous business, Frodo, going out your door. You step onto the road, and if you don’t keep your feet, there’s no knowing where you might be swept off to.

J. R. R. TOLKIEN

Contents

Abstract

A semi-Riemannian manifold is endowed with a density function, modifying the Riemannian volume element and giving rise to a smooth metric measure space. These spaces appear naturally in many topics in Mathematics, and their study combines ideas from Geometry, Topology and Analysis.

In this thesis, we tackle the study of geometric equations in smooth metric measure spaces. The results presented can be broadly divided into two lines. The first one concerns a natural generalization of Einstein manifolds for Riemannian smooth metric measure spaces called weighted Einstein manifolds. We classify manifolds with this property which additionally satisfy a harmonicity condition on a weighted analogue of the Weyl tensor. We also translate a classical problem of Riemannian geometry to this setting and classify weighted Einstein manifolds admitting another such structure in their conformal class.

The second line concerns the derivation of vacuum weighted Einstein field equations for Lorentzian smooth metric measure spaces. We present both a variational approach and an alternative perspective based on the characterizing properties of the usual Einstein tensor. We classify solutions under conditions on the density and the geometry of the underlying manifold. Among others, we study solutions whose density has lightlike gradient, and also general solutions which have harmonic curvature. Special families of Kundt spacetimes such as Brinkmann waves and *pp*-waves play a distinguished role as solutions with specific features.

Resumo en galego

Un problema central en xeometría semi-riemanniana é a resolución de ecuacións xeométricas en variedades. Estas ecuacións adoitan estar compostas por unha combinación de operadores diferenciais e tensores relacionados coa curvatura do espazo semi-riemanniano subxacente. Algúns exemplos clásicos relevantes e moi relacionados coa temática desta tese son as ecuacións que definen as variedades de Einstein, de grande importancia en relatividade xeral, a ecuación conforme Einstein estudada por primeira vez por Brinkmann [11], ou as distintas xeneralizacións da ecuación de Obata [84, 96, 120]. En coordenadas locais, toman a forma dun sistema de ecuacións diferenciais en derivadas parciais (EDPs) involucrando as compoñentes da métrica e, a miúdo, outras funcións cun certo significado xeométrico.

O estudo deste tipo de ecuacións xeométricas segue unha dobre vertente. Por unha banda, para unha variedade dada, búscase determinar a existencia dunha solución de certa ecuación xeométrica, así como as características da mesma. A cotío isto involucra achar unha métrica e outras funcións relacionadas coa ecuación. Por outra banda, cabe preguntarse ata que punto a existencia de solucións para unha certa ecuación xeométrica restrinxe a xeometría da variedade subxacente. Habitualmente, o obxectivo final é unha combinación de ambas, é dicir, dada unha ecuación xeométrica nunha variedade semi-riemanniana arbitraria, pretendemos entender tanto as xeometrías que admiten solucións como as formas destas solucións. Non obstante, a miúdo estas ecuacións alcanzan un nivel de complexidade que imposibilita a súa resolución en toda a súa xeneralidade. Faise necesaria, por tanto, a imposición de certas hipóteses con significado xeométrico que reduzan o problema a outro máis manexable (algún tipo de simetría, restricións na curvatura...).

Esta tese céntrase no estudo dunha serie de ecuacións xeométricas nun contexto moi relevante matematicamente, pero menos coñecido que o formalismo semi-riemanniano usual: as variedades con densidade. A continuación, presentamos estes obxectos, explicamos algúns dos seus aspectos chave, e resumimos algúns dos resultados principais.

Variedades con densidade

O Capítulo 1 desta tese está centrado na presentación de nocións básicas de xeometría semi-riemanniana e, máis concretamente, de variedades con densidade. De xeito xeral, unha variedade con densidade ou *smooth metric measure space* (que abreviamos como SMMS) é unha variedade semi-riemanniana (M, g) de dimensión $n \geq 3$ dotada dunha medida diferenciable que é distinta, en xeral, da medida riemanniana usual. Esta noción formalízase introducindo unha función diferenciable $f \in C^{\infty}(M)$ chamada función de densidade ou simplemente densidade, a cal proporciona o elemento de volume ponderado $e^{-f}\mathrm{dvol}_g$, onde dvol_g é o elemento de volume riemanniano. Por tanto, a maneira máis sinxela de definir un SMMS é como a tripla (M^n, g, f). O caso no que f é constante denomínase trivial, dado que o estudo de SMMSs triviais redúcese ao caso semi-riemanniano usual. Por tanto, como norma xeral, nesta tese consideraremos SMMSs non triviais. Nocións relacionadas coas variedades con densidade aparecen de xeito natural en numerosos problemas matemáticos, dende a resolución da conxectura de Poincaré [100] á construción de variedades Einstein con estrutura de produto deformado [77], pasando polas desigualdades de Gagliardo-Nirenberg-Sobolev (GNS) [36, 38], o problema isoperimétrico [75, 108] ou teorías modificadas da gravidade [109, 117], entre outros. Nos últimos anos, especialmente en base aos traballos de Case [34–41], estableceuse unha definición máis xeral de SMMS en signatura riemanniana. No formalismo de Case, un SMMS é unha quíntupla (M^n, g, f, m, μ), onde (M, g) é unha variedade riemanniana e $f \in C^\infty(M)$ é a densidade, estando os parámetros $m \in \mathbb{R}^+$ e $\mu \in \mathbb{R}$ relacionados co produto deformado formal

$$M^n \times_{e^{-\frac{f}{m}}} F^m(\mu) = \left(M^n \times F^m, g \oplus e^{-\frac{2f}{m}} q(\mu)\right), \tag{0.1}$$

onde $F(\mu) = (F, q(\mu))$ é un espazo forma m-dimensional de curvatura seccional constante μ. Para poder estudar ecuacións xeométricas neste contexto ponderado, necesitamos un xeito de identificar dous SMMSs. Así, dicimos que $(M_1^n, g_1, f_1, m_1, \mu_1)$ e $(M_2^n, g_2, f_2, m_2, \mu_2)$ son isométricos se existe unha isometría $\psi : (M_1, g_1) \to (M_2, g_2)$ (no sentido riemanniano) tal que $f_1 = f_2 \circ \psi$, $m_1 = m_2$ e $\mu_1 = \mu_2$. Os obxectos xeométricos a estudar neste formalismo son os invariantes ponderados, funcións no espazo Met$(M) \times C^\infty(M)$ de estruturas métrica-densidade nunha certa variedade M que son invariantes respecto da acción natural do grupo de difeomorfismos $\mathrm{Diff}(M)$ (nótese que adicionalmente poden depender dos parámetros m e μ). Así, pódense definir escalares, funcionais ou tensores ponderados, entre outros obxectos. Moitos destes son restricións de invariantes riemannianos do produto deformado formal (0.1) á base deste. En efecto, o elemento de volume ponderado é a restrición do elemento de volume do produto deformado a M. De forma similar, pódense construír análogos ponderados para os tensores de Ricci, Weyl etc. a partir dos tensores relacionados coa curvatura do produto deformado.

Obxecto	**Usual**	**Ponderado**
Curvatura	R	R
Tensor de Ricci	ρ	$\rho_f^m = \rho + \mathrm{Hes}_f - \frac{1}{m} df \otimes df$
Curvatura escalar	τ	$\tau_f^m = \tau + 2\Delta f - \frac{m+1}{m}\|\nabla f\|^2$ $+m(m-1)\mu\, e^{2f/m}$
Tensor de Schouten	$P = \frac{1}{n-2}(\rho - Jg)$ $J = \frac{\tau}{2(n-1)}$	$P_f^m = \frac{1}{m+n-2}(\rho_f^m - J_f^m g)$ $J_f^m = \frac{1}{2(m+n-1)}\tau_f^m$
Tensor de Weyl	$W = R - P \owedge g$	$W_f^m = R - P_f^m \owedge g$

Táboa 1: Comparativa entre tensores relacionados coa curvatura usuais e os seus análogos ponderados. O símbolo $\owedge$ denota o produto de Kulkarni-Nomizu.

Estes tensores, baseados nos introducidos por Case en [34,41], formarán os piares básicos do noso estudo. Así, τ_f^m é a curvatura escalar do produto deformado formal (0.1), ρ_f^m é a restrición do tensor de Ricci do produto deformado a vectores tanxentes á base etc. De entre todos estes tensores ponderados, o máis estudado é precisamente o tensor de Ricci de Bakry-Émery ρ_f^m. Este tensor xa aparecera en relación a procesos de difusión [1], pero tamén dá lugar ás estruturas coñecidas como solitóns de Ricci gradientes ($\rho_f^\infty = \rho + \mathrm{Hes}_f = \lambda g$ para algún $\lambda \in \mathbb{R}$) e variedades quasi-Einstein (QE, $\rho_f^m = \lambda g$). As primeiras están relacionadas co fluxo de Ricci [69] e a proba da conxectura de Poincaré por parte de Perelman [100]; mentres que os segundos aparecen na construción de produtos warped Einstein [77], que teñen interese, por exemplo, en relatividade xeral [80]. Por estas razóns, ámbalas estruturas foron intensamente estudadas nos últimos vinte anos, baixo distintas hipóteses sobre o tipo de variedade subxacente, tanto en signatura riemanniana coma noutras (especialmente lorentziana e neutra). Algúns dos numerosos traballos nesta liña son [12,15,31,32,59,93] para resultados sobre solitóns e [13,14,19,43,45,71] para variedades QE. Así, as expresións dos solitóns de Ricci gradientes e variedades QE son exemplos de dúas ecuacións xeométricas de grande interese en variedades con densidade. Nesta tese non consideramos o caso formal $m = \infty$, centrándonos só no caso $m \in \mathbb{R}^+$, polo que non aparecen os solitóns. Non obstante, as variedades QE si xogan un papel importante en certos resultados de clasificación.

O tensor de Ricci de Bakry-Émery tamén aparece en signatura riemanniana, relacionado coa ecuación dun fluído estático perfecto [79], e en signatura lorentziana, en relación a teoremas de descomposición e singularidades [42, 118] e teorías escalar-tensor da gravidade [117]. Outros invariantes ponderados relevantes, aínda que non entran dentro do ámbito desta tese, son as constantes de Yamabe ponderadas [36, 38], relacionadas coas desigualdades GNS de Del Pino-Dolbeaut [53], e as σ_k-curvaturas ponderadas [34, 35].

Variedades Einstein ponderadas

Nunha variedade semi-riemanniana, a condición de ser Einstein pode enunciarse de maneira equivalente en termos do tensor de Ricci ρ ou do tensor de Schouten P. En efecto, $\rho = \lambda g$ para algún $\lambda \in \mathbb{R}$ implica que P tamén é un múltiplo da métrica, e viceversa. Porén, no caso ponderado isto non é así. En efecto, $\rho_f^m = \lambda g$ proporciona a noción de variedade quasi-Einstein xa comentada, mentres que $P_f^m = \lambda g$ dá lugar ás variedades Einstein ponderadas (ou WE, do inglés *weighted Einstein*). Este concepto, de feito, é máis xeral ca o das variedades QE, xa que para toda variedade WE (M^n, g, f, m, μ) existe unha constante κ (chamada escala) tal que o escalar de Schouten ponderado J_f^m satisfai

$$J_f^m = (m+n)\lambda - m\kappa e^{\frac{f}{m}}.$$

Nótese que, se $\kappa = 0$ ou o SMMS é trivial, entón a variedade é QE. Ademais, no caso non trivial, se (M^n, g, f, m) é QE, existe un $\mu \in \mathbb{R}$ tal que (M^n, g, f, m, μ) é WE con escala cero [34].

As variedades Einstein ponderadas non son só unha xeneralización das variedades quasi-Einstein, xa que para $\mu = 0$ aparecen de forma similar ás variedades Einstein usuais como puntos críticos do funcional curvatura escalar ponderada total [36], o cal está relacionado coa resolución do problema de Yamabe ponderado. Por estas razóns, son a estrutura tipo Einstein máis natural a estudar nas variedades con densidade, e son estruturas puramente ponderadas, no sentido de que non se herdan directamente de invariantes riemannianos do produto deformado formal (0.1). Ademais, a súa maior xeneralidade fai que non estean tan entendidas coma as variedades QE, pero presentan suficiente rixidez como para obter resultados de clasificación baixo hipóteses de curvatura razoables. Finalmente, unha propiedade esencial que probamos na Sección 1.6 é que as variedades WE son (real) analíticas en coordenadas harmónicas, o que nos permite determinar propiedades xeométricas a partir de valores nun conxunto aberto.

As variedades Einstein ponderadas máis simples constrúense sobre os tres espazos forma: a esfera $\mathbb{S}^n$, o espazo hiperbólico $\mathbb{H}^n$ e o espazo euclidiano $\mathbb{E}^n$. En efecto, definindo funcións de densidade e valores do parámetro μ axeitados nestas variedades, podemos construír exemplos de variedades Einstein ponderadas (con $P_f^m = \lambda g$), completas e de curvatura seccional constante 2λ (escollendo a métrica correspondente sobre os espazos forma), que realizan calquera valor de m e λ (ver [34, 40] para algúns primeiros exemplos, e a Sección 1.5 para a construción completa). Dadas as súas propiedades, estes espazos son chave nos resultados de clasificación de variedades Einstein ponderadas completas nesta tese. Porén, non toda variedade WE é Einstein, de feito existen estruturas máis complexas que aparecen nos nosos resultados.

Na seguinte táboa preséntanse de maneira esquemática os espazos forma ponderados, onde c.s.c é unha abreviatura de curvatura seccional constante. A métrica dos espazos forma pódese escribir utilizando a súa descomposición como produtos warped $I \times_\varphi \mathbb{S}^{n-1}$, sendo I un intervalo aberto e t unha coordenada que o parametriza por lonxitude de arco.

	Variedade subxacente	**Parámetros do SMMS**
$\lambda > 0$	n-esfera de c.s.c 2λ	$f_m(t) = -m\log\left(A + B\cos\left(t\sqrt{2\lambda}\right)\right)$ $\mu = 2\lambda(B^2 - A^2)$ (ou $m = 1$), $\kappa = 2\lambda A$
$\lambda < 0$	n-espazo hiperbólico de c.s.c 2λ	$f_m(t) = -m\log\left(A + B\cosh\left(t\sqrt{-2\lambda}\right)\right)$ $\mu = 2\lambda(B^2 - A^2)$ (ou $m = 1$), $\kappa = 2\lambda A$
$\lambda = 0$	n-espazo euclidiano	$f_m(t) = -m\log\left(A + Bt^2\right)$ $\mu = -4AB$ (ou $m = 1$), $\kappa = 2B$

Táboa 2: Espazos forma ponderados contruídos prescribindo funcións de densidade e parámetros axeitados aos espazos forma usuais. Chamamos a estes SMMSs n-esfera m-ponderada, n-espazo euclidiano m-ponderado e n-espazo euclidiano m-ponderado, respectivamente. As constantes A e B deben tomarse de xeito que f_m estea ben definida en toda a variedade subxacente.

Parte I. A xeometría das variedades Einstein ponderadas

Nos Capítulos 2 e 3, os cales forman a Parte I desta tese, buscamos aumentar o coñecemento sobre as variedades Einstein ponderadas en signatura riemanniana mediante resultados de clasificación (salvo isometría de SMMSs). Dada a complexidade da ecuación xeométrica $P_f^m = \lambda g$, centrámonos en dous problemas máis concretos que explicamos a continuación.

Variedades WE baixo condicións de harmonicidade. Unha restrición habitual na análise de estruturas como as variedades quasi-Einstein é a da harmonicidade do tensor de Weyl, $\operatorname{div} W = 0$, onde div denota a diverxencia. Esta condición aparece naturalmente en discusións sobre variedades conformemente Einstein, é dicir, variedades (M, g) que admiten localmente un cambio conforme $\widehat{g} = e^{-2\sigma} g$ tal que a variedade resultante é Einstein (ver [83]). A tradución natural desta condición ao contexto dos SMMSs é a anulación da diverxencia ponderada do tensor de Weyl ponderado:

$$0 = \mathrm{div}_f \, W_f^m = \mathrm{div} \, W_f^m - \iota_{\nabla f} W_f^m = \mathrm{div} \, W_f^m - W_f^m(\nabla f, -, -, -).$$

Así, no Capítulo 2 analizamos variedades WE baixo esta condición de harmonicidade ponderada. Nótese que unha variedade WE con $P_f^m = \lambda g$ e $W_f^m = 0$ ten curvatura seccional constante 2λ, pero este non é o caso para a condición $\mathrm{div}_f \, W_f^m = 0$, que é máis débil. De feito, esta condición non implica o carácter Einstein da variedade subxacente.

Exemplo 2.4. Sexa un SMMS da forma $(I \times_\varphi N, g, f, \frac{1}{2}, 0)$, onde $I \times_\varphi N$ é un produto deformado dun intervalo aberto $I \subset \mathbb{R}^+$ e unha variedade Ricci-chá N. Prescribindo as funcións de deformación e densidade

$$\varphi(t) = A(Bt)^{\frac{1}{n-1}}, \quad f(t) = -\log(Bt),$$

onde t é a coordenada natural de $\mathbb{R}^+$ e $A, B \in \mathbb{R}^+$, séguese que esta variedade non é Einstein. En efecto, a curvatura escalar da variedade subxacente, $\tau = \frac{(n-2)}{(n-1)t^2}$, non é constante. Porén, o SMMS si ten tensor de Weyl harmónico ponderado.

Este exemplo non Einstein acaba sendo crucial no resultado de clasificación local principal deste capítulo. Para obtelo, utilizamos o feito de que as hipersuperficies de nivel da densidade f arredor de puntos regulares resultan ser esféricas, o que permite a descomposición local de (M, g) como produto deformado [104]. Grazas a isto, as EDPs na ecuación $P_f^m = \lambda g$ redúcense a un sistema sobredeterminado de EDOs, o que permite a súa resolución completa.

Teorema 2.14. *Sexa (M^n, g, f, m, μ) un SMMS tal que $P_f^m = \lambda g$ e $\mathrm{div}_f \, W_f^m = 0$. Entón, para cada punto regular p de f, existe unha isometría riemanniana entre unha veciñanza $\mathcal{U}$ de p e un produto deformado $I \times_\varphi N$, onde $I \subset \mathbb{R}$ é un intervalo aberto, N é unha variedade Einstein $(n-1)$-dimensional, e ∇f é tanxente a I. Ademais, satisfaise unha das seguintes condicións:*

1. *$I \times_\varphi N$ é Einstein con $\rho = 2(n-1)\lambda g$.*
2. *$(\mathcal{U}, g\big|_U, f\big|_U, m, \mu)$ é isométrico a $(I \times_\varphi N, g, f, \frac{1}{2}, 0)$, con este último dado polo Exemplo 2.4.*

O caso Einstein está menos restrinxido ca o non Einstein, e analizámolo en profundidade na Sección 2.2. Ademais, impoñendo a condición de completude, e utilizando o carácter analítico das variedades WE e a relación da ecuación $P_f^m = \lambda g$ coa ecuación de Obata xeneralizada (ver [120]), probamos un resultado de clasificación global máis estrito para SMMSs completos, que non realizan o Exemplo 2.4 anterior.

Teorema 2.23. *Sexa (M^n, g, f, m, μ) un SMMS completo tal que $P_f^m = \lambda g$ (con escala κ) e $\mathrm{div}_f \, W_f^m = 0$. Entón, (M^n, g, f, m, μ) é isométrico a un dos seguintes espazos:*

1. *Un espazo forma ponderado.*

2. *Un produto deformado Einstein $\mathbb{R} \times_\varphi N$, onde N é unha variedade Ricci-chá completa. Neste caso, existe unha coordenada t que parametriza $\mathbb{R}$ por lonxitude de arco, e tal que as funcións de deformación e densidade toman as formas*

$$\varphi(t) = Ae^{t\sqrt{-2\lambda}}, \qquad f(t) = -m \log\left(\tfrac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}}\right),$$

para algúns $B \geq 0$ e $\kappa \leq 0$. Ademais, $m = 1$ ou $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$.

Por tanto, temos os exemplos completos esperados dados polos espazos forma ponderados, pero tamén espazos que non teñen curvatura seccional constante e non son necesariamente simplemente conexos, construídos a partir do Teorema 2.23 (2).

Variedades WE na mesma clase conforme. No contexto ponderado, ideas e conceptos de xeometría conforme entran de maneira natural en discusións sobre variedades QE [37, 39–41] e problemas analíticos relacionados coas desigualdades de Gagliardo-Nirenberg-Sobolev [34–36, 38]. Para definir unha transformación conforme dun SMMS, hai que ter en conta o cambio da densidade e, de forma similar á construción dos invariantes ponderados relacionados coa curvatura, podemos definir nocións conformes a partir do produto deformado formal $M^n \times_{e^{-f/m}} F^m(\mu)$ dado en (0.1).

Dicimos que dous SMMSs (M, g, f, m, μ) e $(M, \widehat{g}, \widehat{f}, m, \mu)$ son conformemente equivalentes se existe unha función diferenciable $\phi \in C^\infty(M)$ tal que $\widehat{g} = e^{-2\phi/m} g$ e $\widehat{f} = f + \phi$. Isto significa que os correspondentes produtos deformados formais son conformemente equivalentes no sentido riemanniano.

A determinación das variedades conformemente Einstein é un problema clásico en xeometría semi-riemanniana (ver [83] para unha explicación detallada). Un problema relacionado, tamén clásico, pero máis manexable, é o da clasificación de variedades Einstein que admiten outra estrutura Einstein na súa clase conforme. Os primeiros resultados neste sentido déronse en [10] (ver tamén [82, 122]). Ademais, a non unicidade de variedades Einstein ponderadas nunha mesma clase conforme de SMMSs foi considerada por Case en [34, 36] en relación ao problema de Yamabe ponderado e á busca de desigualdades GNS de orde superior. Porén, os seus traballos só proporcionan resultados parciais. Así, no Capítulo 3, resolvemos completamente esta cuestión, dando unha clasificación dos SMMSs que admiten varias estruturas WE na mesma clase conforme (descartando homotecias, é dicir, transformacións conformes con factor conforme $u = e^{\phi/m}$ constante).

Primeiro, damos un resultado de rixidez local, para o que utilizamos unha descomposición en forma de produto deformado similar á do apartado anterior, coa diferenza de que neste caso utilizamos hipersuperficies de nivel do factor conforme e non da densidade.

Teorema 3.4. *Sexa (M^n, g, f, m, μ) un SMMS Einstein ponderado con $P_f^m = \lambda g$, tal que existe un SMMS conformemente equivalente $(M^n, \widehat{g}, \widehat{f}, m, \mu)$ que é Einstein ponderado con $\widehat{P}_{\widehat{f}}^m = \widehat{\lambda}\widehat{g}$. Entón, nunha veciñanza de cada punto regular do factor conforme u, (M, g) descompón coma* *un produto deformado $I \times_\varphi N$, onde $I \subset \mathbb{R}$ é un intervalo aberto e ∇u é tanxente a I. Ademais, satisfaise unha das seguintes condicións:*

1. *(M, g) e $(M, \widehat{g})$ son Einstein con $\rho = 2(n-1)\lambda g$ e $\widehat{\rho} = 2(n-1)\widehat{\lambda} g$, e a densidade ten a forma $f = -m \log(\varphi v_N + \alpha)$, onde v_N é unha función definida en N e α é unha función definida en I.*

   Ademais, a fibra (N, g^N) é Einstein e existen constantes ξ, ν determinadas por v e u tales que $\mathrm{Hes}^N_{v_N} = (\xi - (\nu^2 - 4\lambda\widehat{\lambda})v_N)g^N$.

2. *(M^n, g, f, m) e $(M^n, \widehat{g}, \widehat{f}, m)$ son quasi-Einstein con $\rho^m_f = 2(m+n-1)\lambda g$ e $\widehat{\rho}^m_{\widehat{f}} = 2(m+n-1)\widehat{\lambda}\widehat{g}$, e a densidade f descompón como $f = -m\log(\varphi) + f_N$ onde f_N é unha función en N.*

   Ademais, a fibra (N, g^N, f_N, m) tamén é quasi-Einstein.

De novo, a condición de completude resulta nunha clasificación máis estrita, pero con certa flexibilidade.

Teorema 3.10. *Sexa (M^n, g, f, m, μ) un SMMS completo tal que $P^m_f = \lambda g$, con escala κ, e tal que existe un SMMS Einstein ponderado conformemente equivalente a el. Entón, (M, g, f, m, μ) é isométrico a un dos seguintes SMMSs:*

1. *Un espazo forma ponderado.*
2. *Un produto deformado $\mathbb{R} \times_\varphi N$, con N completa, e tal que $\varphi(t) = Ae^{t\sqrt{-2\lambda}}$, onde t parametriza $\mathbb{R}$ por lonxitude de arco. Ademais, $\lambda < 0$ e satisfaise unha das seguintes condicións:*

   (a) (M, g) é Einstein e (N, g^N) é Ricci-chá. A función de densidade toma a forma

   $$f(t) = -m \log\left(\frac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}}\right),$$

   para algúns $B \geq 0$ e $\kappa \leq 0$. Ademais, $m = 1$ ou $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$.

   (b) (M, g, f, m) é quasi-Einstein e f descompón como $f = -m\log\varphi + f_N$. Ademais, (N, g^N, f_N, m) é quasi-Einstein con $(\rho^m_{f_N})^N = 0$.

Nótese que nos apartados (1) e (2.a) do Teorema 3.10 aparecen exactamente as variedades WE completas con $\operatorname{div}_f W^m_f = 0$ do Teorema 2.23, as cales son Einstein e están relacionadas coa versión clásica (non ponderada) deste problema (ver [82]). Non obstante, tamén aparecen as construcións do Teorema 3.10 (2.b), que son QE pero non Einstein.

Como consecuencia do Teorema 3.10, mostramos que no caso compacto o SMMS é necesariamente unha esfera ponderada (Corolario 3.12), xeneralizando un resultado parcial dado por Case en [36]. Para finalizar esta primeira parte da tese, na Sección 3.3 realizamos unha pequena discusión sobre a relación entre as variedades WE e a condición de ser Bach-chá (no sentido ponderado), centrándonos en como os SMMSs estudados nos Capítulos 2 e 3 aparecen de forma natural ao considerar esta condición.

Parte II. Ecuacións de campo de Einstein ponderadas en variedades Lorentzianas

Os Capítulos 4–7 constitúen a Parte II desta tese, e abandonan o contexto riemanniano para centrarse na signatura lorentziana, de grande interese por ser a base de moitas teorías métricas da gravidade. A máis coñecida, e precursora de numerosas teorías máis modernas, é a relatividade xeral, onde as ecuacións que codifican a relación entre materia e xeometría veñen dadas polo tensor de Einstein (con constante cosmolóxica) $G = \rho - \frac{\tau}{2} g + \Lambda g$. As correspondentes ecuacións de Einstein de baleiro obtéñense mediante a condición $G = 0$ (ver [114] para unha recompilación detallada de solucións exactas das ecuacións de campo en relatividade xeral). Nocións de variedades con densidade, sobre todo relacionadas co tensor de Ricci de Bakry-Émery, aparecen en teorías modificadas da gravidade coma as teorías escalar-tensor [109, 117].

Así, o propósito do Capítulo 4 é obter unhas ecuacións de campo axeitadas para variedades con densidade, de maneira que codifiquen información tanto sobre a xeometría da variedade subxacente como sobre a densidade. Para acadar este obxectivo, utilizamos unha estratexia dobre. Por unha banda, tomando $m = 1$ (de forma que μ non desempeña un papel na definición dos tensores ponderados) e $h = e^{-f}$, para un SMMS lorentziano (M, g, h), establecemos o problema variacional consistente en atopar os puntos críticos do funcional de Einstein-Hilbert ponderado

$$\mathcal{S} : (g, h) \mapsto \mathcal{S}_{(g,h)} = \int_M \tau h \, \mathrm{dvol}_g,$$

para variacións da estrutura métrica-medida da forma

$$g[t] = g + t\bar{g}, \qquad h[t] = h + t\bar{h}, \qquad \mathrm{dV}_h[t] = h[t] dvol_{g[t]},$$

suxeitas á restrición $\frac{d}{dt}\big|_{t=0} \mathrm{dV}_h[t] = 0$ (as variacións deixan o elemento de volume ponderado invariante). O cálculo desta variación acaba por reducirse ao do adxunto formal da linearización da curvatura escalar, o cal aparece recorrentemente na literatura [3, 5, 8, 62, 63, 87]. Así, os puntos críticos están caracterizados pola ecuación

$$G^h = h\rho - \mathrm{Hes}_h + \Delta h g = 0,$$

onde G^h, ao que chamamos tensor de Einstein ponderado, vén dado polo citado adxunto formal. As ecuacións determinadas pola igualdade $G^h = 0$, por tanto, son as ecuacións de Einstein ponderadas de baleiro (a partir de aquí, simplemente ecuacións de campo) sen constante cosmolóxica. Estas ecuacións xa apareceran en signatura riemanniana, onde as solucións se coñecen como espazos estáticos de baleiro debido á súa relación (mediante unha redución dimensional) coa ecuación dun fluído estático perfecto nun espazo-tempo lorentziano [80]. A versión riemanniana desta ecuación espertou grande interese, cos traballos fundacionais [79, 84] e outros máis modernos como [78, 105, 112]. Porén, as solucións lorentzianas non foron exploradas, o que motiva as aportacións da tese neste sentido.

Unha segunda estratexia para obter as ecuacións de campo consiste en considerar as catro propiedades que caracterizan o tensor de Einstein [89]: simetría, diverxencia cero, dependencia da métrica e as súas dúas primeiras derivadas, e linearidade nas segundas derivadas da métrica. Así, a partir do tensor de Ricci de Bakry-Émery $\rho_f^1 = \rho - h^{-1}\operatorname{Hes}_h$, construímos

$$G^h = h\rho - \operatorname{Hes}_h + (\Delta h + \Lambda)g, \quad \text{onde } \Lambda \in \mathbb{R},$$

como un tensor simétrico, dependente da métrica, a densidade e as súas dúas primeiras derivadas, linear nas segundas derivadas da métrica e a densidade e de diverxencia nula cando a variedade ten curvatura escalar constante. A igualdade $G^h = 0$ proporciona as ecuacións de campo con constante cosmolóxica. Nótese que, se $G^h = 0$, entón a curvatura escalar da variedade é constante (ver Lema 4.5), polo que as solucións ponderadas comparten esta propiedade coas variedades Einstein usuais.

Unha vez establecidas as ecuacións de campo, o resto da tese céntrase na análise das súas solucións mediante resultados de clasificación locais (salvo isometría de SMMSs) baixo hipóteses xeométricas naturais. No Capítulo 5, estudamos solucións isotrópicas, isto é, SMMSs (M, g, h) tales que $G^h = 0$ e $g(\nabla h, \nabla h) = 0$. Estas condicións levan a que a variedade subxacente presente unha distribución nula distinguida, e acaban forzando nela unha estrutura de espazo-tempo de Kundt ou algunha das súas subfamilias (ondas de Brinkmann, pr-waves...). Os espazo-tempos de Kundt son variedades lorentzianas de grande importancia en matemáticas (entre outros motivos, pola estrutura da súa holonomía [86]) e física (por exemplo, porque modelizan certos tipos de ondas gravitacionais [57, 95, 101, 114]). Así, a literatura sobre o tema é moi extensa, sendo [7, 9, 46, 48–50, 103] só algúns exemplos adicionais.

Na Sección 5.1, unha análise das ecuacións de campo mostra que toda solución isotrópica satisfai $\Delta h = \Lambda = 0$. Os principais resultados acadados no estudo da ecuación $h\rho = \operatorname{Hes}_h$ resultante recóllense no seguinte teorema, que constitúe o principal resultado do Capítulo 5.

Teorema 5.4. *Sexa (M^n, g, h) unha solución isotrópica das ecuacións de campo de Einstein ponderadas. Entón, satisfaise unha das seguintes posibilidades:*

1. *(M, g) é unha onda de Brinkmann Ricci-chá e $\operatorname{Hes}_h = 0$.*
2. *O operador de Ricci é nilpotente en 2 pasos e (M, g) é unha onda de Brinkmann.*
3. *O operador de Ricci é nilpotente en 3 pasos e (M, g) é un espazo-tempo de Kundt.*

Ademais deste resultado en dimensión arbitraria, na Sección 5.2 proporcionamos unha clasificación local completa de solucións isotrópicas tridimensionais. Nótese que, polo Teorema 5.4, todos os autovalores do operador de Ricci, Ric, dunha solución isotrópica anúlanse. Non obstante, isto non implica que a solución sexa Ricci-chá. En efecto, en signatura riemanniana, o operador de Ricci diagonaliza en calquera punto nunha base ortonormal axeitada (isto é consecuencia do seu carácter autoadxunto). Polo contrario, en signatura lorentziana os operadores autoadxuntos poden presentar distintas formas normais de Jordan, incluíndo unha con autovalores complexos non reais.

No Capítulo 6, estas formas de Jordan desempeñan un papel esencial. Nel estudamos solucións xerais (M, g, h) das ecuacións de campo sen constante cosmolóxica baixo condicións no tensor de Weyl usual, W. Debido a que as características das solucións varían notablemente dependendo do carácter causal de ∇h, traballamos localmente e supoñemos que este carácter non cambia na variedade. Así, temos solucións isotrópicas ($g(\nabla h, \nabla h) = 0$) e non isotrópicas ($g(\nabla h, \nabla h) \neq 0$).

Primeiro, na Sección 6.1, proporcionamos unha clasificación de solucións localmente conformemente chás ($W = 0$) no Teorema 6.3. Neste caso, para solucións non-isotrópicas Ric diagonaliza e estas solucións son similares ás súas análogas riemannianas [79]. Por tanto, aumentamos a flexibilidade considerando a condición $\operatorname{div} W = 0$ (que resulta ser equivalente a $\operatorname{div} R = 0$, é dicir, a curvatura é harmónica) para solucións de dimensión catro. Esta situación foi considerada en signatura riemanniana en [78]. No noso contexto aparecen distintas formas de Jordan para o operador de Ricci con xeometrías moi diferentes, polo que para poder analizalas impoñemos que a forma de Jordan sexa constante en M. Nas Seccións 6.2–6.4 estudamos cada unha das posibles formas, dando resultados de clasificación para solucións isotrópicas e non isotrópicas. Este estudo culmina coa seguinte clasificación local.

Teorema 6.36. *Sexa (M, g, h) unha solución de dimensión $n = 4$ das ecuacións de campo de Einstein ponderadas (sen constante cosmolóxica) tal que (M, g) ten curvatura harmónica e non é localmente conformemente chá. Asumamos ademais que a forma normal de Jordan do operador de Ricci,* Ric*, é constante en M. Entón, os autovalores de* Ric *son reais e satisfaise unha das seguintes condicións:*

1. Ric *diagonaliza en (M, g) e $g(\nabla h, \nabla h) \neq 0$. Ademais, existe un conxunto aberto e denso M_{Ric} de M onde (M, g) é localmente isométrica a:*

   (a) Un produto directo $I_2 \times \tilde{M}$, onde $\tilde{M} = I_1 \times_\xi N$ é unha solución tridimensional construída sobre un produto deformado con $\tilde{\tau} = 0$ e N unha superficie de curvatura de Gauss constante.

   (b) Un produto directo $N_1 \times N_2$ de dúas superficies de curvatura de Gauss constante $\frac{\kappa}{2}$ e κ, respectivamente.

2. *(M, g) é un espazo-tempo de Kundt e, dependendo do carácter causal de ∇h, dáse unha das seguintes opcións:*

   (a) Se $g(\nabla h, \nabla h) = 0$, entón Ric *é nilpotente e ∇h determina a distribución nula paralela distinguida. Ademais, se* Ric *é identicamente nulo ou nilpotente en 2 pasos, a variedade subxacente é unha pp-wave.*

   (b) Se $g(\nabla h, \nabla h) \neq 0$, entón ∇h é espacial e o campo de vectores nulo distinguido é ortogonal a ∇h.

As solucións diagonalizables teñen propiedades similares ás que aparecen en [78], pero as non diagonalizables destacan o carácter lorentziano da variedade subxacente ao estar realizadas en espazo-tempos de Kundt. Isto está garantido para solucións isotrópicas polo Teorema 5.4, pero tamén resultan ser chave no caso non-isotrópico, como se aprecia en Teorema 6.36 (2.b). Ademais, non existen solucións con autovalores do operador de Ricci non reais, pero este feito, lonxe de ser evidente, requiriu para a súa proba de ferramentas alxébricas coma o cálculo de bases de Gröbner (mediante o uso de software específico).

Para finalizar a discusión da tese, no Capítulo 7 damos algúns exemplos notables de solucións das ecuacións de campo de Einstein ponderadas. Dado que o Teorema 6.36 mostra a importancia das solucións realizadas en espazo-tempos de Kundt, pero a súa métrica é demasiado complicada como para ser analizada en toda a súa xeneralidade, restrinximos o problema analizando ondas de radiación pura (ou pr-waves, do inglés *pure radiation waves* [86]). Esta é unha familia de espazo-tempos de Kundt con relevancia física, e a métrica dunha pr-wave en dimensión catro pódese escribir en coordenadas locais (u, v, x, y) como

$$g = 2dudv + F(u, v, x, y)dv^2 + dx^2 + dy^2.$$

Primeiro, proporcionamos unha clasificación local completa deste tipo de solucións.

Teorema 7.2. *Sexa (M, g, h) unha solución de dimensión $n = 4$ non chá das ecuacións de campo de Einstein ponderadas, realizada nunha pr-wave. Entón, ∇h é nulo ou espacial, e ademais:*

1. *Se ∇h é nulo, entón (M, g) é unha pp-wave. A función que determina a métrica nas coordenadas (u, v, x, y) satisfai $\partial_u F = 0$ e $\partial_x^2 F + \partial_y^2 F = \gamma(v)$ e a densidade $h = h(v)$ satisfai a EDO $2h'' + h\gamma = 0$.*

2. *Se ∇h é espacial, entón Ric é nilpotente e, adicionalmente:*

   (a) Se Ric é nilpotente en 2 pasos, entón (M, g) é unha pp-wave. Ademais, nas coordenadas (u, v, x, y) temos $\partial_u F = 0$ e $h(v, x, y) = h_0(v) + (x + Ay)h_x$, con $A \in \mathbb{R}$ e $h_x \neq 0$, satisfacendo

   $$0 = -2G^h_{vv} = 2h_0'' + h_x(\partial_x F + A\partial_y F) + h(\partial_x^2 F + \partial_y^2 F).$$

   (b) Se Ric é nilpotente en 3 pasos, entón existen coordenadas (u, v, x, y) tales que a densidade ten a forma $h(v, x, y) = h_0(v) + (x + Ay)h_x(v)$, con $A \in \mathbb{R}$ e $h_x' \neq 0$, e a función que determina a métrica é da forma

   $$F(u, v, x, y) = F_0(v, x, y) + u\left(\frac{2h_x'(v)\log(h(v, x, y))}{h_x(v)} + \alpha(v)\right)$$

   con

   $$\begin{aligned} 0 &= -2h_x G^h_{vv} \\ &= 2h_x'(h_0' + (x + Ay)h_x')\log(h_0 + (x + Ay)h_x) \\ &\quad + h_x^2(\partial_x F_0 + A\partial_y F_0 + (x + Ay)(\partial_x^2 F_0 + \partial_y^2 F_0)) \\ &\quad + h_x(\alpha(h_0' + (x + Ay)h_x') + 2h_0'' + 2(x + Ay)h_x'') \\ &\quad + h_x h_0(\partial_x^2 F_0 + \partial_y^2 F_0). \end{aligned}$$

Como consecuencia deste teorema, e enlazando co estudo realizado no Capítulo 6, obtéñense resultados máis estritos impoñendo que a curvatura sexa harmónica (Corolario 7.5), mostrando que toda solución non isotrópica é unha onda plana. O capítulo remata cunha pequena discusión de solucións xeodesicamente completas realizadas en ondas planas en $\mathbb{R}^4$.

Introduction

The study of differential equations is ubiquitous in Mathematics. In a considerable number of cases, these equations are closely related to Differential Geometry, because they codify relevant geometric properties and conditions. Many equations of great physical significance are studied within the general framework of smooth manifolds as well. These *geometric* equations usually involve a combination of differential operators and tensors related to the curvature of the underlying semi-Riemannian manifold. Some classical examples which are intimately related to the topics in this thesis are the equations that define Einstein manifolds, which are key in General Relativity, the conformal Einstein equation introduced by Brinkmann [11], or the different generalizations of the Obata equation [84, 96, 120]. In local coordinates, geometric equations take the form of a (generically overdetermined) system of partial differential equations (PDEs) involving the components of the metric and, often, additional functions of geometric significance.

The study of these kinds of geometric equations follows a two-pronged approach. On the one hand, on a given manifold, we seek to determine the existence of solutions of a certain equation, as well as their characteristics. Frequently, this involves finding a suitable metric, as well as other functions related to the geometric equation itself. On the other hand, we might wonder to what extent the existence of solutions of a certain geometric equation restricts the admissible geometries of the underlying manifold. Usually, the final goal is a combination of the two, meaning that given a geometric equation on a generic semi-Riemannian manifold, we aim to understand both the geometries admitting solutions and the form of the solutions themselves. Nevertheless, these equations often reach a level of complexity that makes them impossible to solve in full generality. Thus, it becomes necessary to impose suitable geometric conditions that reduce the problem to a manageable state (some sort of symmetry, curvature restrictions...).

This thesis focuses on the study of a series of geometric equations in a highly mathematically relevant context which is, however, less well-known than the standard semi-Riemannian formalism: smooth metric measure spaces (also known as manifolds with density).

Thus, this dissertation starts with an preliminary chapter (Chapter 1) devoted to the introduction of the necessary notions of semi-Riemannian and weighted geometry in smooth metric measure spaces. Broadly speaking, a smooth metric measure space (SMMSs for short) is a semi-Riemannian manifold (M, g) (we only consider manifolds of dimension $n \geq 3$) endowed with a smooth measure which is, in general, different from its Riemannian measure. This can be formalized via the inclusion of a smooth density function $f \in C^\infty(M)$ which gives rise to the weighted volume element $e^{-f}\mathrm{dvol}_g$, where dvol_g is the standard Riemannian volume element. Whenever f is constant, the SMMS is said to be trivial and its analysis boils down to standard semi-Riemannian techniques. Thus, as a general rule, we will consider non-trivial SMMSs.

Notions of manifolds with density appear naturally in many mathematical and physical problems, such as the resolution of the Poincaré conjecture by Perelman [100], the construction of Einstein warped products [77], analytic topics related to Gagliardo-Nirenberg-Sobolev (GNS) inequalities [36, 38], the isoperimetric problem [75, 108] and modified theories of gravity [109, 117], among many others.

Although the simplest way to define a SMMS is as the triple (M, g, f), in the past fifteen years, mostly thanks to works by Case [34–41], a more general definition has been introduced, specially for certain problems in Riemannian signature. Within this formalism, a SMMS is a 5-tuple (M^n, g, f, m, μ), where $m \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ are parameters of geometric significance. The objects of study in this setting are weighted invariants, i.e., functions in the space $\text{Met}(M) \times C^\infty(M)$ of metric-measure structures on M which are invariant under the action of the diffeomorphism group $\text{Diff}(M)$ (note that they can also depend on m and μ). We now present some weighted invariants of particular interest to this work. The most well-known one is the Bakry-Émery Ricci tensor

$$\rho_f^m = \rho + \text{Hes}_f - \frac{1}{m} df \otimes df,$$

where ρ is the Ricci tensor and Hes_f is the Hessian of f. We will also use the weighted scalar curvature

$$\tau_f^m = \tau + 2\Delta f - \frac{m+1}{m} \|\nabla f\|^2 + m(m-1)\mu\, e^{\frac{2}{m} f},$$

where τ is the standard scalar curvature of (M, g). The weighted Schouten tensor and its associated scalar then take the form

$$P_f^m = \frac{1}{m+n-2}(\rho_f^m - J_f^m g), \qquad J_f^m = \frac{1}{2(m+n-1)} \tau_f^m,$$

and finally, the weighted Weyl tensor is given by

$$W_f^m = R - P_f^m \owedge g,$$

where $\owedge$ represents the Kulkarni-Nomizu product.

The use of the tensor ρ_f^m introduced in [1] is widespread in Mathematics and the literature related to it is extensive. For example, it is relevant in the study of the Ricci flow [100], the construction of warped product Einstein manifolds [71, 77], and it gives rise to gradient Ricci solitons (i.e., solutions to the equation $\rho_f^\infty = \rho + \text{Hes}_f = \lambda g$ for some $\lambda \in \mathbb{R}$) and quasi-Einstein (QE) manifolds (solutions to $\rho_f^m = \lambda g$), which are two structures of interest, with the latter being highly related to this work (see [12, 15, 31, 43, 45] for some results on gradient Ricci solitons and QE manifolds). Much less is known, however, about the influence of P_f^m and $W_f^m = R - P_f^m \owedge g$ on SMMSs.

In order to study such weighted geometric equations, we need a way to identify SMMSs. To this end, we say that two SMMSs

$$(M_1^n, g_1, f_1, m_1, \mu_1) \quad \text{and} \quad (M_2^n, g_2, f_2, m_2, \mu_2)$$

are isometric if there exists a Riemannian isometry $\psi : (M_1, g_1) \to (M_2, g_2)$ such that $f_1 = f_2 \circ \psi$, and the parameters satisfy $m_1 = m_2$ and $\mu_1 = \mu_2$.

A natural question in this context is how to formulate a notion of "Einstein manifold in the weighted sense". In the unweighted setting, the Einstein condition can be stated equivalently using the Ricci tensor ($\rho = \lambda g$ for some $\lambda \in \mathbb{R}$) or the usual Schouten tensor ($P = \lambda g$ for some $\lambda \in \mathbb{R}$). However, these conditions are not equivalent in the weighted setting. Indeed, $\rho_f^m = \lambda g$ gives rise to QE manifolds, while $P_f^m = \lambda g$ gives rise to weighted Einstein (WE) manifolds. This notion turns out to be more general than that of QE manifolds, since for any SMMS (M^n, g, f, m, μ) satisfying $P_f^m = \lambda g$, there exist a constant κ (called the scale) such that the weighted Schouten scalar is given by $J_f^m = (m + n)\lambda - m\kappa e^{f/m}$. Notice that if $\kappa = 0$ or the SMMS is trivial, then we have a QE manifold. Moreover, in the non-trivial case, if (M^n, g, f, m) is QE, then there exists a value of $\mu \in \mathbb{R}$ such that (M^n, g, f, m, μ) is WE with vanishing scale [34].

The fact that WE manifolds generalize QE manifolds makes them a prime candidate for being the weighted analogues of Einstein manifolds. Moreover, for $\mu = 0$, they also arise as critical points of a suitable total weighted scalar curvature functional [36] related to the weighted Yamabe problem, similarly to how Einstein manifolds arise from variations of the Einstein-Hilbert functional. This fact further motivates the interest of studying weighted Einstein structures on SMMSs.

The simplest examples of WE manifolds are built on the three model spaces: the sphere $\mathbb{S}^n$, the hyperbolic space $\mathbb{H}^n$ and the Euclidean space $\mathbb{E}^n$, through the prescription of suitable density functions and values of the parameter μ. Indeed, by choosing the appropriate metric on these space forms, we construct complete WE manifolds with $P_f^m = \lambda g$ and constant sectional curvature 2λ, for any value of m and λ. First examples of such structures were introduced by Case (see [34, 40]). In Section 1.6, we present a comprehensive list of families of WE manifolds realized on space forms for all values of μ that generalize those already existing in literature. We call these the weighted space forms, and they play a key role in several of our classification results. Nevertheless, not every WE manifold is Einstein, and more complex structures also appear.

In Part I of this thesis, which is comprised of Chapters 2 and 3, we aim to further our understanding of weighted Einstein manifolds in Riemannian signature via classification results up to isometry of SMMSs. Given the complexity of the WE equation $P_f^m = \lambda g$, we focus on two specific problems which we present below. A key trait of WE manifolds which we will make extensive use of throughout Part I is their real analyticity in harmonic coordinates, which we prove in Section 1.6. This property allows us to determine global geometric properties from values of tensors in a given open set.

In the analysis of Einstein-type structures such as quasi-Einstein manifolds, a condition that often appears in literature is the harmonicity of the standard Weyl tensor, $\operatorname{div} W = 0$, where div denotes the usual divergence (see [19, 44]). This condition arises naturally in discussions on conformally Einstein manifolds, i.e., manifolds (M, g) admitting local conformal transformations $\widehat{g} = e^{-2\sigma} g$ such that the resulting manifold is Einstein (see, for example, [83]). The natural translation of $\operatorname{div} W = 0$ to the weighted setting is the vanishing of the weighted divergence of the weighted Weyl tensor:

$$0 = \operatorname{div}_f W_f^m = \operatorname{div} W_f^m - \iota_{\nabla f} W_f^m = \operatorname{div} W_f^m - W_f^m(\nabla f, -, -, -).$$

Thus, in Chapter 2, we analyze weighted Einstein manifolds under this weighted harmonicity condition. Notice that a WE manifold with $P_f^m = \lambda g$ and $W_f^m = 0$ has constant sectional curvature 2λ, but this is not the case for the weaker condition $\operatorname{div}_f W_f^m = 0$. In fact, this condition does not imply the Einstein character of the underlying manifold, although the non-Einstein case is heavily restricted (see Example 2.4).

In order to obtain the main local classification result in Chapter 2, we use the fact that the level hypersurfaces of the density function f turn out to be spherical, which allows for a local decomposition of the manifold as a warped product around regular points of f [104]. Due to this fact, the PDEs corresponding to the equation $P_f^m = \lambda g$ are reduced to an overdetermined system of ODEs, which we then solve.

Theorem 2.14. *Let (M^n, g, f, m, μ) be a SMMS such that $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$. Then, for each regular point p of f, there exists a Riemannian isometry between a neighborhood $\mathcal{U}$ of p and a warped product $I \times_\varphi N$, where $I \subset \mathbb{R}$ is an open interval, N is an $(n-1)$-dimensional Einstein manifold, and ∇f is tangent to I. Moreover, one of the following conditions holds:*

1. *$I \times_\varphi N$ is Einstein with $\rho = 2(n-1)\lambda g$.*
2. *$(\mathcal{U}, g\big|_U, f\big|_U, m, \mu)$ is isometric to $(I \times_\varphi N, g, f, \frac{1}{2}, 0)$ as given in Example 2.4.*

The Einstein case is less restricted than its non-Einstein counterpart, and we analyze it thoroughly in Section 2.2. Moreover, by imposing completeness of the underlying manifold and taking advantage of the analyticity of WE manifolds and the relationship between the WE equation $P_f^m = \lambda g$ and the generalized Obata equation (see [120]), we prove a stronger global classification result for complete SMMSs, where it is shown that the non-Einstein case is not admissible.

Theorem 2.23. *Let (M^n, g, f, m, μ) be a complete SMMS such that $P_f^m = \lambda g$ (with scale κ) and $\operatorname{div}_f W_f^m = 0$. Then, (M^n, g, f, m, μ) is isometric to one of the following spaces:*

1. *A weighted space form as described in Examples 1.12, 1.13 and 1.14.*
2. *An Einstein warped product $\mathbb{R} \times_\varphi N$, where N is a Ricci-flat complete manifold. In this case, there is a coordinate t parameterizing $\mathbb{R}$ by arc length such that the warping and density functions take the forms*

$$\varphi(t) = Ae^{t\sqrt{-2\lambda}}, \qquad f(t) = -m \log\left(\tfrac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}}\right),$$

for some $B \geq 0$ and $\kappa \leq 0$. Moreover, $m = 1$ or $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$.

We see how the weighted space forms (which have $W_f^m = 0$) appear, but by choosing appropriate Ricci-flat fibers N in Theorem 2.23 (2), we also get spaces which do not have constant sectional curvature or which are not simply connected.

The second problem regarding Riemannian SMMSs that we tackle comes from the fact that ideas of conformal geometry turn up naturally in discussions of SMMSs. Two examples of this are QE manifolds [37, 39–41] and analytic problems related to Gagliardo-Nirenberg-Sobolev inequalities [34–36, 38]. Hence, weighted conformal geometry deserves further attention, and in Chapter 3 we turn towards the weighted version of a classical problem: The classification of Einstein manifolds admitting more than one Einstein representative in their conformal class (see [10, 82, 122]).

In our weighted context, we say that two SMMSs (M, g, f, m, μ) and $(M, \widehat{g}, \widehat{f}, m, \mu)$ are conformally equivalent if there exists a smooth function $\phi \in C^\infty(M)$ such that $\widehat{g} = e^{-2\phi/m} g$ and $\widehat{f} = f + \phi$ (i.e., the corresponding formal warped products are conformally equivalent in the Riemannian sense).

Thus, in Chapter 3 we look for WE manifolds admitting another WE representative in their weighted conformal class (excluding homotheties, i.e., rescalings of the metric by a constant factor). Some results in this regard were given by Case [34,36] in relation to the Yamabe problem and the search for sharp higher order GNS inequalities. However, these were only partial results, and in this thesis we complete the classification in full generality. Firstly, we give a local rigidity result by leveraging a warped product decomposition similar to the one in Chapter 2, with the caveat that now the fibers correspond to level hypersurfaces of the conformal factor $u = e^{\phi/m}$ instead of the density.

Theorem 3.4. *Let (M^n, g, f, m, μ) be a weighted Einstein SMMS, with $P^m_f = \lambda g$, such that there exists a conformally equivalent SMMS $(M^n, \widehat{g}, \widehat{f}, m, \mu)$ which is weighted Einstein with $\widehat{P}^m_{\widehat{f}} = \widehat{\lambda}\widehat{g}$. Then, on a neighborhood of each regular point of the conformal factor u, M decomposes as a warped product $I \times_\varphi N$, where $I \subset \mathbb{R}$ is an open interval and ∇u is tangent to I. Furthermore, one of the following holds:*

1. *(M, g) and $(M, \widehat{g})$ are Einstein with $\rho = 2(n-1)\lambda g$ and $\widehat{\rho} = 2(n-1)\widehat{\lambda} g$, and the density takes the form $f = -m \log(\varphi v_N + \alpha)$, where v_N is a function on N and α is a function on I.*

   Moreover, the fiber (N, g^N) is Einstein and there exist constants ξ, ν determined by v and u such that $\mathrm{Hes}^N_{v_N} = (\xi - (\nu^2 - 4\lambda\widehat{\lambda})v_N)g^N$.

2. *(M^n, g, f, m) and $(M^n, \widehat{g}, \widehat{f}, m)$ are quasi-Einstein with $\rho^m_f = 2(m+n-1)\lambda g$ and $\widehat{\rho}^m_{\widehat{f}} = 2(m+n-1)\widehat{\lambda}\widehat{g}$, and the density f splits as $f = -m\log(\varphi) + f_N$ where f_N is a function on N.*

   Moreover, the fiber (N, g^N, f_N, m) is quasi-Einstein too.

Then, we achieve a stronger global rigidity result by imposing, once again, the condition of completeness of our SMMSs of interest.

Theorem 3.10. *Let (M^n, g, f, m, μ) be a complete SMMS such that $P^m_f = \lambda g$, with scale κ, and such that there exists a conformally equivalent weighted Einstein SMMS. Then, (M, g, f, m, μ) is isometric to one of the following SMMSs:*

1. *A weighted space form as described in Examples 1.12, 1.13 and 1.14.*
2. *A warped product $\mathbb{R} \times_\varphi N$, with N complete, and such that $\varphi(t) = Ae^{t\sqrt{-2\lambda}}$, where t parameterizes $\mathbb{R}$ by arc length. Moreover, $\lambda < 0$ and one of the following holds:*
   (a) *(M, g) is Einstein and (N, g^N) is Ricci-flat. The density function has the form*
   $$f(t) = -m \log\left(\frac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}}\right),$$
   for some $B \geq 0$ and $\kappa \leq 0$. Moreover, $m = 1$ or $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$.
   (b) *(M, g, f, m) is quasi-Einstein, f splits as $f = -m \log \varphi + f_N$, and (N, g^N, f_N, m) is also quasi-Einstein with $(\rho^m_{f_N})^N = 0$.*

Note that the SMMSs in items (1) and (2.a) of Theorem 3.10 are exactly the WE manifolds with $\operatorname{div}_f W^m_f = 0$ in Theorem 2.23, all of which are Einstein and related to the classical unweighted version of this problem [82]. Nevertheless, we also get the constructions in Theorem 3.10 (2.b), which are QE but not Einstein.

As a consequence of Theorem 3.10, we show that in the compact case the SMMS is necessarily an m-weighted n-sphere (Corollary 3.12), thus generalizing the partial result given by Case in [36]. We also build an example of the new structures arising from Theorem 3.10 (2.b). Finally, we end Part I of the thesis by briefly discussing, in Section 3.3, the relationship between WE manifolds and weighted Bach-flatness, by focusing on how the SMMSs presented in Chapters 2 and 3 arise naturally when considering this condition.

Part II of this dissertation, which includes Chapters 4–7, abandons the Riemannian setting to focus instead on Lorentzian smooth metric measure spaces. Lorentzian manifolds are highly significant in Physics because they are the basis of many metric theories of gravity, with General Relativity being the foundational one. In this framework, the field equations codifying the relationship between matter and geometry are given by the Einstein tensor (with cosmological constant) $G = \rho - \frac{\tau}{2}g + \Lambda g$. The corresponding vacuum field equations, which correspond to the usual Einstein condition, are obtained by setting $G = 0$ (see [114] for a detailed review of exact solutions of the Einstein field equations in General Relativity). Notions of smooth metric measure spaces, specially ones concerning the Bakry-Émery Ricci tensor, appear in modified theories of gravity such as scalar-tensor theories [109, 117].

Thus, the purpose of Chapter 4 is to obtain suitable field equations for SMMSs, so that they still contain standard semi-Riemannian geometric information, along with information on the density function. We achieve this goal through two different approaches: On the one hand, taking $m = 1$ so that the parameter μ does not play a role, and $h = e^{-f}$, for a Lorentzian SMMS (M, g, h) we define the weighted Einstein-Hilbert functional

$$\mathcal{S} : (g, h) \mapsto \mathcal{S}_{(g,h)} = \int_M \tau h \, \mathrm{dvol}_g.$$

We then look for the critical points of this functional for variations of the metric-measure structure of the form

$$g[t] = g + t\bar{g}, \qquad h[t] = h + t\bar{h}, \qquad \mathrm{dV}_h[t] = h[t] dvol_{g[t]},$$

such that the weighted volume element remains invariant, i.e., $\frac{d}{dt}\big|_{t=0}\mathrm{dV}_h[t] = 0$. The analysis of this variation amounts to the computation of the formal adjoint of the linearization of the scalar curvature. This object appears in a recurrent manner in literature [3, 5, 8, 62, 63, 87]. It follows that critical points are characterized by the equation

$$G^h = h\rho - \mathrm{Hes}_h + \Delta h g = 0,$$

where G^h, which we call the weighted Einstein tensor, is given by the aforementioned formal adjoint. The equations determined by $G^h = 0$ are thus called the vacuum weighted Einstein field equations (henceforth, weighted Einstein field equations or just field equations). These equations had already been studied in Riemannian signature due to their formal relation (via dimensional reduction) to the usual field equations of a perfect fluid in a Lorentzian spacetime [80]. The Riemannian version of this equation has drawn significant interest, with seminal works like [79, 84] and more modern ones like [78, 105, 112]. Nevertheless, the Lorentzian solutions explored in this thesis constitute a new contribution which had not been analyzed in literature.

On the other hand, the second approach consists on considering the four properties that characterize the usual Einstein tensor: symmetry, vanishing divergence, dependence on the metric and its first two derivatives, and linearity on the second derivatives of the metric [89]. Hence, from the Bakry-Émery Ricci tensor, we build

$$G^h = h\rho - \mathrm{Hes}_h + (\Delta h + \Lambda) g, \quad \text{where } \Lambda \in \mathbb{R},$$

as a symmetric tensor whose divergence vanishes if the scalar curvature of the underlying manifold is constant. Moreover, G^h depends on the metric g, the density h and their first two derivatives, and it is linear in the second derivatives of both g and h. Then, $G^h = 0$ gives the field equations with cosmological constant. Note that, if $G^h = 0$, then the scalar curvature of the underlying manifold is constant (see Lemma 4.5), so solutions of the weighted Einstein field equations share this property with standard Einstein manifolds.

Once the field equations have been established, the focus of the remainder of the thesis shifts toward the analysis of their solutions via local classification results (up to isometry of SMMSs). We start in Chapter 5 by studying isotropic solutions, i.e., SMMSs (M, g, h) such that $G^h = 0$ and $g(\nabla h, \nabla h) = 0$. It turns out that these conditions imply that $\operatorname{span} \nabla h$ is a distinguished lightlike distribution, forcing the underlying manifold to be a general Kundt spacetime or a more specific family of such spaces (Brinkmann waves, pr-waves...). Kundt spacetimes are important in Mathematics, in part due to their holonomy structure (see [86]) and in Physics, among other reasons, due to their links to gravitational waves (see [57, 95, 101, 114])). Literature on the topic is thus extensive, with [7, 9, 46, 48–50, 103] being only some additional examples.

In Section 5.1, we show that any isotropic solution of the field equations satisfies $\Lambda = \Delta h = 0$. Thus, the equations reduce to $h\rho = \mathrm{Hes}_h$, which we further study in order to prove the main rigidity result of Chapter 5.

Theorem 5.4. *Let (M^n, g, h) be an isotropic solution of the weighted Einstein field equations. Then, one of the following possibilities holds:*

1. (M, g) *is a Ricci-flat Brinkmann wave and* $\mathrm{Hes}_h = 0$.

2. *The Ricci operator is* 2*-step nilpotent and* (M, g) *is a Brinkmann wave.*

3. *The Ricci operator is* 3*-step nilpotent and* (M, g) *is a Kundt spacetime.*

In addition to this, we also provide a full classification of 3-dimensional isotropic solutions in Section 5.2, where admissible solutions are described explicitly in local coordinates. Notice that, by Theorem 5.4, the Ricci eigenvalues of any isotropic solution vanish, although this does not mean that it is Ricci-flat. Recall that, in Riemannian signature, as a consequence of its self-adjoint character, the Ricci operator Ric diagonalizes at every point $p \in M$ for an appropriate orthonormal basis of T_pM. However, in Lorentzian signature, self-adjoint operators might present several distinct Jordan normal forms.

These different normal forms play an essential role in Chapter 6, where we study general solutions (M, g, h) under conditions on the standard Weyl tensor W. Since the features and analysis of solutions are quite different depending on the causal character of ∇h, we work locally and assume that it does not change, giving rise to isotropic solutions (∇h lightlike) and non-isotropic ones (∇h timelike or spacelike). A first result is given in Section 6.1 for locally conformally flat solutions (Theorem 6.3). In this case, the Ricci operator diagonalizes for non-isotropic solutions, and these solutions are similar to their Riemannian analogues [79]. In order to find purely Lorentzian non-isotropic solutions, the bulk of the chapter centers around the weaker condition of harmonic curvature ($\operatorname{div} R = 0$) on 4-dimensional SMMSs, which was previously considered for Riemannian manifolds in [78]. Since each possible Jordan normal form for Ric requires a different approach, and our analysis is local, we assume the constancy of this normal form and provide, throughout Sections 6.2–6.4, rigidity results for each one. This culminates in the following main classification result.

Theorem 6.36. *Let* (M, g, h) *be a* 4*-dimensional solution of the weighted Einstein field equations (without cosmological constant) such that* (M, g) *has harmonic curvature tensor (not locally conformally flat). Assume that the Jordan normal form of the Ricci operator* Ric *is constant in* M*. Then, the eigenvalues of* Ric *are real and one of the following is satisfied:*

1. Ric *diagonalizes on* (M, g) *and* $g(\nabla h, \nabla h) \neq 0$*. Furthermore, there exists an open dense subset* M_{Ric} *of* M *where* (M, g) *is locally isometric to:*

   (a) A direct product $I_2 \times \tilde{M}$*, where* $\tilde{M} = I_1 \times_\xi N$ *is a warped product* 3*-dimensional solution with* $\tilde{\tau} = 0$ *and* N *a surface of constant Gauss curvature.*

   (b) A direct product $N_1 \times N_2$ *of two surfaces of constant Gauss curvature* $\frac{\kappa}{2}$ *and* κ*, respectively.*

2. (M, g) *is a Kundt spacetime and, depending on the causal character of* ∇h*, one of the following applies:*

   (a) If $g(\nabla h, \nabla h) = 0$*, then* Ric *is nilpotent and* ∇h *determines the lightlike parallel line field. Moreover, if* Ric *vanishes or is 2-step nilpotent, the underlying manifold is a pp-wave.*

(b) If $g(\nabla h, \nabla h) \neq 0$, then ∇h is spacelike and the distinguished lightlike vector field is orthogonal to ∇h.

Diagonalizable solutions present similar features to those of their Riemannian analogues in [78], but non-diagonalizable ones are realized on Kundt spacetimes (even in the non-isotropic case), which highlights their purely Lorentzian character. Moreover, we emphasize that there are no solutions with complex, non-real eigenvalues, which is possible for a general self-adjoint operator in a Lorentzian space. The nonexistence of solutions of this kind is proved by employing tools from computational algebra (Gröbner bases).

The contributions of this dissertation end in Chapter 7, where we present some examples of solutions of geometric interest. Since Kundt spacetimes are key in Theorem 6.36, but their metric is too complicated to tackle in full generality, we focus on the problem of classifying all 4-dimensional solutions realized on the physically significant family of pure radiation waves (pr-waves for short, see [86] for details). This classification includes the explicit description of solutions in local coordinates and is given in Theorem 7.2. We link this to Theorem 6.36 by obtaining stronger results for solutions on pr-waves with harmonic curvature (Corollary 7.5). Non-isotropic solutions in this context turn out to be realized on plane waves, and we end the chapter by presenting a brief discussion on geodesically complete solutions on plane waves in $\mathbb{R}^4$.

Chapter 1
Preliminaries

This chapter is devoted to the introduction of the basic concepts, notations and conventions that will be necessary for a complete understanding of this thesis. Some motivational material is also included, as well as new results that are considered essential knowledge for the development of subsequent chapters. We will omit the proofs of most known results and direct the reader to several references for further details. For a more comprehensive introduction to Riemannian and semi-Riemannian geometry, we refer to the well-known books by O'Neill [97], Kühnel [81] and Lee [85].

1.1 Semi-Riemannian geometry

Let M^n be a (connected) n-dimensional (smooth) manifold (without boundary), with $n \geq 3$. A *metric tensor* (or simply a *metric*) g on M^n is a (smooth) symmetric and non-degenerate (0,2)-tensor field on M. The pair (M^n, g) (or just (M, g) if the dimension is known) is called a *semi-Riemannian manifold*. The *signature* of a metric g is the pair $(n - \nu, \nu)$, where $n - \nu$ and ν refer to the number of negative and positive eigenvalues of the associated matrix of g, respectively. If g has signature $(0, n)$ (i.e., if it is positive definite), the manifold (M, g) is said to be *Riemannian*. Similarly, if the signature is $(1, n-1)$, then (M, g) is said to be *Lorentzian*. Equivalently, the signature can be defined as the dimension of the maximal subspace where the metric is negative definite, so Riemannian and Lorentzian manifolds have signatures 0 and 1 respectively. Thoughout this thesis, we will also use the word *spacetime* to refer to a Lorentzian manifold, meaning that we will not take into account the notion of time orientation that is sometimes imposed in the definition of spacetimes.

Let T_pM denote the tangent space to M at $p \in M$, TM the tangent bundle of M, and $\mathfrak{X}(M)$ the space of (smooth) tangent vector fields to M, this is, the space of smooth sections of TM. A non-zero vector $v \in T_pM$ is called *timelike* if $g_p(v, v) < 0$, *lightlike* or *null* if $g_p(v, v) = 0$, and *spacelike* if $g_p(v, v) > 0$. Likewise, a vector field $X \in \mathfrak{X}(M)$ is timelike, lightlike or spacelike if X_p presents these behaviors for all $p \in M$. Note that, for Riemannian manifolds, all non-zero vectors are spacelike.

For any semi-Riemannian manifold (M, g), the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X), \end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$, determines the *Levi-Civita connection* ∇, which is the unique linear connection such that

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \quad \text{and} \quad \nabla g = 0$$

for all $X, Y \in \mathfrak{X}(M)$, where $[-,-]$ denotes the Lie bracket. The first condition means that ∇ is torsion-free, while the second one is often referred to as metric compatibility, since it means that the metric is parallel. When working with local coordinates $(x_1, \ldots, x_n)$ on M, we denote the corresponding coordinate vector fields by ∂_{x_i}, for $i = 1, \ldots, n$. Then, the connection is also characterized by the *Christoffel symbols* determined by the Koszul formula. The Christoffel symbols of the *first kind* are given by the expression

$$\Gamma_{ijl} = \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x^l}\right),$$

while the Christoffel symbols of the *second kind* are

$$\Gamma_{ij}{}^k = g^{kl}\Gamma_{ijl},$$

where (g^{ij}) is the inverse matrix of (g_{ij}), whose entries are $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$, and we have used Einstein notation to signify a sum over the repeated index l. Hence, covariant derivatives of coordinate vectors are given by $\nabla_{\partial_{x_i}}\partial_{x_j} = \Gamma_{ij}{}^k\partial_{x_k}$.

The connection gives rise to the concept of *geodesics*, i.e., curves $\gamma : I \subset \mathbb{R} \to M$, where I is an open interval, such that $\nabla_{\gamma'}\gamma' = 0$. Geodesics are relevant in Mathematics and Physics, for example, due to their local length-minimizing properties and the fact that free-falling particles move along geodesics in General Relativity. For the purposes of this dissertation, however, we will not be discussing geodesic curves, but manifolds such that all maximal geodesics (those that cannot be extended) are defined in $\mathbb{R}$. These manifolds are said to be *geodesically complete*, and this notion interacts with the metric in an interesting way. Indeed, for Riemannian manifolds, due to the Hopf-Rinow Theorem, this is equivalent to completeness as a metric space (with the distance function given by the metric), but this equivalence fails for indefinite signatures. Thus, in Part I of this thesis, which focuses on Riemannian manifolds, we will refer to geodesically complete manifolds merely as *complete*.

The presence of a metric allows for a process known as the *raising* (or *lowering*) of indices, which transforms a tensor T of type (r, s) into one or type $(r+1, s-1)$ or $(r-1, s+1)$, respectively. Indeed, if the components of T are given in the coordinates $(x_1, \ldots, x_n)$ as

$$T = T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\,\partial_{x_{i_1}} \otimes \cdots \otimes \partial_{x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}$$

then a *metrically equivalent* $(r+1, s-1)$-tensor is, for example

$$\begin{aligned} Q_T &= g^{j_1 k} T^{i_1\cdots i_r}{}_{k j_2\cdots j_s}\,\partial_{x_{i_1}} \otimes \cdots \otimes \partial_{x_{i_r}} \otimes \partial_{x_{j_1}} \otimes dx_{j_2} \otimes \cdots \otimes dx_{j_s} \\ &= T^{i_1\cdots i_r j_1}{}_{j_2\cdots j_s}\,\partial_{x_{i_1}} \otimes \cdots \otimes \partial_{x_{i_r}} \otimes \partial_{x_{j_1}} \otimes dx_{j_2} \otimes \cdots \otimes dx_{j_s} \end{aligned}$$

where we have raised the first covariant index. Henceforth, when using index notation, we will disregard the coordinates and describe a tensor only by its components.

A case of particular relevance for this work is that of symmetric $(0, 2)$-tensors. If T_{ij} is one such tensor, then raising either index gives rise to the metrically equivalent $(1, 1)$-tensor $(Q_T)^i{}_j = g^{ik}T_{kj}$ (which can be regarded as an endomorphism $Q_T : \mathfrak{X}(M) \to \mathfrak{X}(M)$). Both tensors are related by

$$T(X, Y) = g(Q_T(X), Y), \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

1.1.1 Differential operators

In this section, we present some useful differential operators associated with a semi-Riemannian manifold (M^n, g). Let $C^\infty(M)$ be the space of smooth functions $f : M \to \mathbb{R}$.

Gradient: Let $f \in C^\infty(M)$. The *gradient* of f, denoted by ∇f, is the vector field which satisfies

$$g(\nabla f, X) = X(f), \quad \text{for all } X \in \mathfrak{X}(M).$$

Points $p \in M$ where ∇f vanishes are called *critical*, while those where $(\nabla f)_p \neq 0$ are *regular*. Note that the gradient can be considered as the metrically equivalent tensor to the 1-form df.

Hessian and Laplacian: The *Hessian operator* of a smooth function f is the $(1,1)$-tensor given by the second covariant derivative of f:

$$\mathrm{hes}_f(X) = \nabla_X \nabla f.$$

Its metrically equivalent $(0,2)$-tensor, the *Hessian tensor* of f, is thus defined as

$$\mathrm{Hes}_f(X,Y) = g(\mathrm{hes}_f(X), Y) = g(\nabla_X \nabla f, Y) = X(Y(f)) - (\nabla_X Y)(f),$$

where the right-hand side follows from the compatibility of the Levi-Civita connection with the metric. Moreover, because the Levi-Civita connection is torsion-free,

$$X(Y(f)) - Y(X(f)) = [X,Y](f) = (\nabla_X Y)(f) - (\nabla_Y X)(f).$$

Hence, X and Y can be reversed in the formulas above. This means that Hes_f is symmetric, and that hes_f is self-adjoint, i.e., $g(\mathrm{hes}_f(X), Y) = g(X, \mathrm{hes}_f(Y))$.

Taking the trace of the Hessian operator yields the *Laplacian* of the function f:

$$\Delta f = \mathrm{tr}(X \mapsto \mathrm{hes}_f(X)) = (\mathrm{hes}_f)^i{}_i\,.$$

Divergence: Consider a local orthonormal frame $\{E_1, \ldots, E_n\}$, which means that $g(E_i, E_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = \pm 1$ and δ_{ij} is the Kronecker delta. Let T be a $(0,s)$-tensor. Then, the r-*divergence* of T is given by

$$\mathrm{div}_r\, T(X_1, \ldots, X_{s-1}) = \sum_{i=1}^n \varepsilon_i (\nabla_{E_i} T)(X_1, \ldots, X_{r-1}, E_i, X_r, \ldots, X_{s-1}),$$

i.e., it is the (metric) trace of ∇T over its first and $(r+1)$-th arguments. Hence, it does not depend on the choice of local frame. Henceforth, we will write $\mathrm{div}_1 = \mathrm{div}$ and refer to the 1-divergence simply as the divergence.

This definition is motivated by the fact that, if M is compact and orientable, then by Stokes' theorem div is the negative of the formal adjoint of the covariant derivative of tensors. This means that for any two tensors T, K of types $(0,s)$ and $(0, s-1)$ respectively,

$$\int_M \langle T, \nabla K \rangle \mathrm{dvol}_g = - \int_M \langle \mathrm{div}\, T, K \rangle \mathrm{dvol}_g$$

where $\langle -, - \rangle$ denotes the product of tensors given by the metric g, $\langle T, \nabla K \rangle = T^{i_1 \cdots i_s} \nabla_{i_1} K_{i_2 \cdots i_s}$; and $\mathrm{dvol}_g = \sqrt{|g|}\, dx_1 \wedge \cdots \wedge dx_n$ is the usual Riemannian volume element, with $|g| = \varepsilon \det(g_{ij})$, where $\varepsilon = \pm 1$ depending on the signature of the metric. We say that T is *harmonic if* $\mathrm{div}\, T = 0$.

1.1.2 Tensors related to curvature

In terms of the Levi-Civita connection, we define the $(1,3)$-*curvature tensor* by the convention

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z.$$

The curvature tensor of type $(0,4)$ is then given by

$$R(X,Y,Z,U) = g(R(X,Y)Z,U).$$

The $(0,4)$-curvature tensor has the following symmetries:

$$\begin{aligned}
&\text{(i)} \quad R(X,Y,Z,U) = -R(Y,X,Z,U) = -R(X,Y,U,Z),\\
&\text{(ii)} \quad R(X,Y,Z,U) = R(Z,U,X,Y),\\
&\text{(iii)} \quad R(X,Y,Z,U) + R(Y,Z,X,U) + R(Z,X,Y,U) = 0.
\end{aligned}$$

Item (iii) is known as the *first Bianchi identity*. Additionally, it satisfies the differential property known as the *second Bianchi identity*:

$$\text{(iv)} \quad (\nabla_X R)(Y,Z,U,V) + (\nabla_Y R)(Z,X,U,V) + (\nabla_Z R)(X,Y,U,V) = 0.$$

On a semi-Riemannian manifold (M,g), the *standard curvature tensor* is

$$R_0(X,Y,Z,U) = \frac{1}{2}(g \owedge g)(X,Y,Z,U) = g(X,Z)g(Y,U) - g(Y,Z)g(X,U),$$

where $\owedge$ denotes the *Kulkarni-Nomizu product*, which acts on two symmetric $(0,2)$-tensors T and S as

$$\begin{aligned}
(T \owedge S)(X,Y,Z,U) &= T(X,Z)S(Y,U) + T(Y,U)S(X,Z)\\
&\quad -T(X,U)S(Y,Z) - T(Y,Z)S(X,U).
\end{aligned}$$

A tangent plane $\Pi = \mathrm{span}\{x,y\} \subset T_pM$ is called non-degenerate if $R_0(x,y,x,y) \neq 0$, in which case we define its *sectional curvature* as

$$K(\Pi) = \frac{R(x,y,x,y)}{R_0(x,y,x,y)}.$$

If at each $p \in M$, $K(\Pi) = c$ for some constant c, for all non-degenerate tangent planes $\Pi \subset T_pM$, then the curvature tensor takes the form

$$R(X,Y,Z,U) = cR_0(X,Y,Z,U)$$

and it is said that (M,g) has *constant sectional curvature* c. Note that we do not need to impose that the constant c be the same for all $p \in M$, since this is a consequence of Schur's Lemma. If R vanishes, we say that the manifold is *flat*. Geodesically complete manifolds of constant sectional curvature are called *space forms*. The universal cover of a space form is isometric to a pseudosphere $\mathbb{S}^n_\nu(r)$, to a pseudo-hyperbolic space $\mathbb{H}^n_\nu(r)$, or to the indefinite flat space $\mathbb{R}^n_\nu$, depending on the signature of its metric and the sign of its sectional curvature. In Riemannian signature, we get standard spheres, hyperbolic spaces and Euclidean spaces, and this becomes the well-known Killing-Hopf Theorem. As a result of this fact, the three simply connected space forms are sometimes referred to as the model spaces.

Another important geometric object is the *Ricci tensor*, a $(0,2)$-tensor given by the contraction

$$\rho(X,Y) = \mathrm{tr}(Z \mapsto R(X,Z)Y).$$

Thus, in index notation, $\rho_{ij} = R_{ikj}{}^k$, and for an orthonormal frame $\{E_1,\dots,E_n\}$,

$$\rho(X,Y) = \sum_{i=1}^n \varepsilon_i R(X,E_i,Y,E_i).$$

Due to the symmetries of the curvature tensor, the Ricci tensor is symmetric. Thus, its metrically equivalent $(1,1)$-tensor is self-adjoint. We refer to this tensor as the *Ricci operator*, and denote it by Ric, so that $\rho(X,Y) = g(\mathrm{Ric}(X),Y) = g(X,\mathrm{Ric}(Y))$. If $\rho = 0$, we say that (M,g) is *Ricci-flat*, while if $\rho = \lambda g$ for some $\lambda \in \mathbb{R}$, we say that (M,g) is *Einstein*. This is a condition of great importance, not only from the point of view of Geometry, but also in Physics, in particular in metric theories of gravitation such as General Relativity (see Section 1.4 for further details on Einstein metrics).

The contraction of the Ricci tensor yields the *scalar curvature*,

$$\tau = \mathrm{tr}(X \mapsto \mathrm{Ric}(X)) = \mathrm{Ric}^i{}_i\,.$$

From the Ricci tensor, we also define the *Schouten tensor* P and the *Schouten scalar* J, which is its trace:

$$P = \frac{1}{n-2}\left(\rho - \frac{\tau}{2(n-1)}g\right), \quad J = \mathrm{tr}\,P = \frac{\tau}{2(n-1)}. \tag{1.1}$$

The *Weyl tensor* of type $(0,4)$ is now given by

$$W = R - P \owedge g = R - \frac{1}{n-2}\left(\rho \owedge g - \frac{\tau}{n-1}R_0\right).$$

Note that, if (M,g) is Einstein with $\rho = \lambda g$, then contraction yields $\tau = n\lambda$, and consequently $R = \frac{\tau}{n(n-1)}R_0 + W$. Thus, any Einstein manifold with $W = 0$ has constant sectional curvature. As we will see shortly, the Weyl tensor is related to the conformal properties of the underlying manifold. Moreover, for any 3-dimensional manifold, the Weyl tensor vanishes, so the curvature is completely determined by the Ricci tensor. In particular, any Einstein manifold of dimension three has constant sectional curvature (so any Ricci-flat 3-dimensional manifold is flat).

Finally, another important object is the *Cotton tensor*, given by the skew-symmetrization of ∇P:

$$dP(X,Y,Z) = (n-2)\left\{(\nabla_Y P)(X,Z) - (\nabla_Z P)(X,Y)\right\}. \tag{1.2}$$

Conformal transformations, local conformal flatness and the Weyl tensor

Let (M, g) and $(N, \widehat{g})$ be two semi-Riemannian manifolds and $\Phi : M \to N$ a diffeomorphism between them. We say that Φ is a *conformal map* if there exists a positive function $u \in C^\infty(M)$ such that the pullback $\Phi^*\widehat{g}$ satisfies

$$(\Phi^*\widehat{g})_p(x, y) = \widehat{g}_{\Phi(p)}(\Phi_*(p)x, \Phi_*(p)y) = (u(p))^{-2} g_p(x, y),$$

where $\Phi_*(p)$ is the pushforward at p, for all $p \in M$ and $x, y \in T_pM$. We then say that (M, g) and $(N, \widehat{g})$ are *conformally equivalent* (in the Riemannian sense).

If u is a constant, we say that Φ is a homothety (it rescales the metric), while if $u = 1$, it is a *Riemannian isometry* (since it preserves the metric). In these cases, we say that (M, g) and $(N, \widehat{g})$ are *homothetic* or *isometric* (in the Riemannian sense), respectively.

Thus, we say that a manifold (M, g) is *locally conformally flat* if for every $p \in M$ there exists a neighborhood of p which is conformally equivalent to a flat manifold. If (M, g) is globally conformally equivalent to a flat manifold, then it is *conformally flat*. This condition is related to the Weyl tensor through the following result.

Lemma 1.1. *Let (M, g) and $(N, \widehat{g})$ be two conformally equivalent manifolds with $\widehat{g} = u^{-2}g$, and W and $\widehat{W}$ their Weyl tensors. Then, $\widehat{W} = u^{-2}W$. In particular, if a semi-Riemannian manifold is locally conformally flat, then $W = 0$.*

For a manifold with $n \geq 4$, the Cotton tensor dP vanishes whenever W does. This follows from the fact that they are related via the differential identity

$$\operatorname{div} W = \frac{n-3}{n-2} dP. \tag{1.3}$$

However, when $n = 3$, this is not the case. Indeed, the Weyl tensor always vanishes for $n = 3$, but the Cotton tensor does not in general. Furthermore, local conformal flatness is characterized in terms of W and dP, depending on the dimension of the manifold.

Theorem 1.2. *Let (M, g) be an n-dimensional manifold. Then,*

- *If $n \geq 4$, (M, g) is locally conformally flat if and only if the Weyl tensor vanishes in M.*
- *If $n = 3$, (M, g) is locally conformally flat if and only if the Cotton tensor vanishes in M.*

Some differential identities related to curvature

Throughout this thesis, we will make use of some identities involving curvature-related objects and differential operators. Here we present some of them. Firstly, from the second Bianchi identity, it follows that the divergence of the Riemann curvature tensor is given by

$$\operatorname{div} R(X, Y, Z) = (\nabla_Y \rho)(X, Z) - (\nabla_Z \rho)(X, Y).$$

Thus, the curvature is harmonic if and only if the skew-symmetrization of $\nabla\rho$ vanishes. This is also known as the *Codazzi* condition. Now, recalling the relation (1.3) between the Weyl and Cotton tensors, we see that if $n > 3$, W is harmonic if and only if the Schouten tensor is Codazzi.

Moreover, contracting and taking the divergence on the second Bianchi identity yields the *contracted Bianchi identity*:

$$\operatorname{div}\rho = \frac{1}{2} d\tau.$$

Finally, the Hessian tensor of a function can also be related to curvature through the *Bochner formula*

$$\operatorname{div}\operatorname{Hes}_f = d(\Delta f) + \rho(\nabla f, -),$$

and the following expression involving the skew-symmetrization of $\nabla \operatorname{Hes}_f$:

$$(\nabla_Z \operatorname{Hes}_f)(X,Y) - (\nabla_Y \operatorname{Hes}_f)(X,Z) = R(\nabla h, X, Y, Z).$$

1.1.3 Self-adjoint operators in Lorentzian spaces

Many of the tensor fields that play a role in the context of this work are symmetric $(0,2)$-tensors, such as the Ricci and Hessian tensors. By raising an index, from a $(0,2)$-tensor T one can always build the associated linear operator Q_T, which is thus self adjoint, meaning

$$T(X,Y) = g(Q_T(X),Y) = g(X,Q_T(Y)), \qquad \text{for all } X,Y \in \mathfrak{X}(N).$$

In Riemannian signature, this means that, at each point $p \in M$, we can always find an orthonormal basis $\mathcal{B} = \{e_1,\dots,e_n\}$ of T_pM such that each e_i is an eigenvector of Q_T (moreover, this can be extended to a local orthonormal frame comprised of eigenvectors). Thus, in the basis $\mathcal{B}$, the associated matrix of Q_T becomes diagonal. However, this is not always true in other signatures. In particular, the structure of self-adjoint operators in Lorentzian spaces plays a key role in Part II of this thesis.

A self-adjoint linear operator Q on a Lorentzian vector space V has an associated matrix of exactly one of the following four types (see [97]):

On the one hand, relative to an orthonormal basis $\mathcal{B}_1 = \{e_1,\dots,e_n\}$, where $g(e_1,e_1) = -1$, $g(e_i,e_i) = 1$ for $i \geq 2$,

$$Q = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & \lambda_n \end{pmatrix} \quad \text{or} \quad Q = \left(\begin{array}{cc|ccc} a & b & & & \\ -b & a & & & \\ \hline & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{array}\right)$$

with $b \neq 0$. Notice that the eigenvalues are $\{\lambda_1,\dots,\lambda_n\}$ in the first case, whereas in the second one they are $\{a \pm bi, \lambda_1,\dots,\lambda_{n-2}\}$. Following standard terminology, we refer to the diagonal case as *Type I.a* and to the case with two complex (non-real) eigenvalues as *Type I.b*.

Alternatively, relative to a pseudo-orthonormal basis of the form $\mathcal{B}_2 = \{u, v, e_1, \dots, e_{n-2}\}$, where the only non-vanishing terms of the metric are given by $g(e_i, e_i) = 1$ and $g(u, v) = 1$, there are two more possible forms:

$$Q = \left(\begin{array}{cc|ccc} \alpha & 0 & & & \\ \varepsilon & \alpha & & & \\ \hline & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{array}\right) \quad \text{or} \quad Q = \left(\begin{array}{ccc|ccc} \alpha & 0 & 1 & & & \\ 0 & \alpha & 0 & & & \\ 0 & 1 & \alpha & & & \\ \hline & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-3} \end{array}\right),$$

where $\varepsilon = \pm 1$, whose eigenvalues are $\{\alpha, \lambda_1, \dots, \lambda_{n-2}\}$ and $\{\alpha, \lambda_1, \dots, \lambda_{n-3}\}$, respectively. We call these forms *Type II* and *Type III*, respectively. Notice that, in any case, the non-diagonal part of the matrix arises due to the indefinite metric, and the remaining diagonal part is associated to spacelike eigenvectors.

The metrics under consideration in Part II of this thesis are Lorentzian. Thus, since for such a manifold, for every $p \in M$, the tangent space T_pM is a Lorentzian vector space, it follows that the Ricci operator Ric takes exactly one of these four forms at each point of M. This fact will be crucial for the classification results presented in Chapters 5 and 6.

1.2 Warped products

A simple geometric structure is that of a direct product manifold, this is, a semi-Riemannian manifold $\mathcal{M} = (M, g)$ which decomposes as $\mathcal{M} = (B \times F, g^B \oplus g^F)$, where (B, g^B) and (F, g^F) are also semi-Riemannian manifolds. If the metric on the second factor is modified by a positive function $\varphi \in C^\infty(M)$, the resulting manifold M endowed with the metric $g = g^B \oplus \varphi^2 g^F$ is called a *twisted product*, and φ is the *twisting function*.

A more specific and highly relevant example occurs when φ is defined only on B. This gives rise to a structure $(M, g = g^B \oplus \varphi^2 g^F) = B \times_\varphi F$ known as a *warped product*. The manifolds B and F are called the *base* and the *fiber* of the product, respectively, and φ is referred to as the *warping function*. This construction was introduced in [4] as a means to study manifolds with negative curvature, and it is found in multiple contexts in Mathematics and Physics, such as the construction of static spacetimes or Einstein manifolds with certain properties. The latter is closely related to the concept of quasi-Einstein manifolds (see Section 1.4).

Throughout this thesis, we will work with warped products with 1-dimensional base of the form $I \times_\varphi N$, where I is an open interval in $\mathbb{R}$. This structure allows for the components of the curvature and the Levi-Civita connection to be described in terms of the curvature and connection of the fiber and the warping function (see [97]). Indeed, let t be a local coordinate parameterizing I by arc length, and let X, Y, Z be lifts to $I \times_\varphi N$ of vector fields in N. Then, the Levi-Civita connection of the warped product takes the form

$$\nabla_{\partial_t}\partial_t = 0, \quad \nabla_X \partial_t = \nabla_{\partial_t} X = \frac{\varphi'}{\varphi} X, \quad \nabla_X Y = -\varepsilon \frac{\varphi'}{\varphi} g(X, Y)\partial_t + \nabla^N_X Y, \tag{1.4}$$

where ∇^N is the Levi-Civita connection of (N, g^N) and $\varepsilon = g(\partial_t, \partial_t) = \pm 1$. For the curvature tensor R, the only components that may not vanish (modulo symmetries) are the following:

$$\begin{aligned} R(X,\partial_t)\partial_t &= \tfrac{\varphi''}{\varphi}X, \qquad R(\partial_t, X)Y = \varepsilon\tfrac{\varphi''}{\varphi}g(X,Y)\partial_t, \\ R(X,Y)Z &= R^N(X,Y)Z - \varepsilon\tfrac{(\varphi')^2}{\varphi^2}(g(X,Z)Y - g(Y,Z)X). \end{aligned} \tag{1.5}$$

Contracting these expressions, the components of the Ricci tensor follow:

$$\begin{aligned} \rho(\partial_t,\partial_t) &= -(n-1)\tfrac{\varphi''}{\varphi}, \qquad \rho(\partial_t, X) = 0, \\ \rho(X,Y) &= \rho^N(X,Y) - \varepsilon\left(\tfrac{\varphi''}{\varphi} + (n-2)\tfrac{(\varphi')^2}{\varphi^2}\right)g(X,Y), \end{aligned} \tag{1.6}$$

and further contraction yields the scalar curvature

$$\tau = \frac{\varepsilon}{\varphi^2}\left(\varepsilon\tau^N - 2(n-1)\varphi\varphi'' - (n-1)(n-2)(\varphi')^2\right), \tag{1.7}$$

where R^N, ρ^N and τ^N are the curvature tensor, Ricci tensor and scalar curvature of (N, g^N), respectively.

Notice that, for this structure, the canonical foliation $\mathfrak{L}_I$ is totally geodesic (this follows from $\nabla_{\partial_t}\partial_t = 0$), and the canonical foliation $\mathfrak{L}_N$ is umbilical, since the second fundamental form associated to $\mathfrak{L}_N$ is given by $II(X,Y) = -\varepsilon\frac{\varphi'}{\varphi}g(X,Y)\partial_t$. Hence, the normal curvature vector of each leaf of $\mathfrak{L}_N$, which is the vector H such that $II(X,Y) = g(X,Y)H$, is $H = -\varepsilon\frac{\varphi'}{\varphi}\partial_t$. Since for umbilical submanifolds the mean curvature vector coincides with the normal curvature vector, and $\nabla_X\partial_t$ is orthogonal to ∂_t for all X tangent to N, it follows that the mean curvature vector is parallel in the normal bundle, i.e., $\nabla^\perp H = 0$, where $\nabla^\perp$ is the normal connection. Thus, $\mathfrak{L}_N$ is indeed spherical. It turns out that these properties characterize general warped products, with twisted products being characterized by a weaker condition.

Theorem 1.3 [72, 104]. *Let g be a metric defined on a product manifold $B \times F$, such that the canonical foliations $\mathfrak{L}_B$ and $\mathfrak{L}_F$ intersect orthogonally. Then,*

- *If $\mathfrak{L}_B$ is totally geodesic and $\mathfrak{L}_F$ is totally umbilical, then the manifold splits locally as a twisted product.*
- *If $\mathfrak{L}_B$ is totally geodesic and $\mathfrak{L}_F$ is spherical, then the manifold splits locally as a warped product.*

We will point out the following useful alternative characterization of twisted products which can be reduced to warped products:

Theorem 1.4 [61]. *Let $B \times_{\tilde\varphi} F$ be a twisted product of the manifolds (B, g^B) and (F, g^F), with $\dim F > 1$. Then, $\rho(X,V) = 0$ for all X, V lifts of vector fields in B and F respectively if and only if $B \times_{\tilde\varphi} F$ can be expressed as a warped product $B \times_\varphi F$ of (B, g^B) and $(F, \widehat{g}^F)$, where $\widehat{g}^F$ is a conformal metric to g^F.*

Geodesic completeness of Riemannian warped products is an essential notion for the global results that we will present in Chapters 2 and 3. As it turns out, due to the Hopf-Rinow Theorem, this property is inferred from the completeness of the base and the fiber.

Theorem 1.5 [97]. *If (B, g^B) and (F, g^F) are complete Riemannian manifolds, then the warped product $B \times_\varphi F$ is complete for every warping function φ.*

Thus, in the case of 1-dimensional fiber, if $I = \mathbb{R}$ (with positive definite metric) and (N, g^N) is a complete Riemannian manifold, then $I \times_\varphi N$ is complete, irrespective of the form of the warping function $\varphi \in C^\infty(\mathbb{R})$. However, this characterization does not hold for manifolds with indefinite metrics, even if both B and F have definite metrics (see [97]). Some partial results for the geodesic completeness of warped products in the semi-Riemannian setting can be found in [107], but even for warped products with Lorentzian 1-dimensional base and Riemannian fiber the characterization of geodesic completeness is no easy task [110].

As a final note on warped products, we will point out that the Weyl tensor of a warped product depends strongly on the geometry of the fiber, particularly in the case where the base is 1-dimensional, as the following result shows.

Theorem 1.6 [21, 66]. *Let $I \times_\varphi N$ be a semi-Riemannian warped product with 1-dimensional base. Then,*

- *$I \times_\varphi N$ is locally conformally flat if and only if (N, g^N) is a manifold of constant sectional curvature.*
- *$I \times_\varphi N$ has harmonic Weyl tensor if and only if (N, g^N) is Einstein.*

Warped products can be generalized by adding additional fibers (F_1, g_1), ..., (F_k, g_k), with their corresponding warping functions $\varphi_1, \ldots, \varphi_k$ defined on B, giving rise to a *multiply warped product* $B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ with the metric given by $g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k$. The expressions for the components of the Levi-Civita connection and the Ricci tensor for these structures are computed, for example, in [55].

1.3 Spacetimes with a distinguished lightlike vector field

Spacetimes characterized by the existence of a distinguished lightlike vector field have attracted the attention of both mathematicians and physicists for a long time, and some of these families play a pivotal role in Part II of this thesis. In this section, we recall definitions and basic facts about these spacetimes.

Kundt spacetimes

Kundt spacetimes are Lorentzian manifolds of dimension $n \geq 3$ with a lightlike geodesic vector field V which is recurrent in its orthogonal complement (see, for example, [7]). This means that there exists a differential 1-form ω such that

$$\nabla_V V = 0 \quad \text{and} \quad \nabla_X V = \omega(X) V \quad \text{for all } X \in V^\perp. \tag{1.8}$$

Alternatively, Kundt spacetimes can be defined using optical scalars. For any lightlike vector field V, the *optical scalars* of *expansion*, *shear* and *twist* are given, respectively, by

$$\begin{aligned} \theta &= \tfrac{1}{n-2}\nabla_i V^i, \\ \sigma^2 &= (\nabla^i V^j)\nabla_{(i}V_{j)} - (n-2)\theta^2, \\ \omega^2 &= (\nabla^i V^j)\nabla_{[i}V_{j]}, \end{aligned} \tag{1.9}$$

where parentheses denote symmetrization and brackets denote skew-symmetrization when placed in the subindices. Kundt spacetimes are then characterized by a lightlike geodesic vector field with zero optical scalars, which means that it is expansion-free, shear-free and twist-free (see [46, 48, 103]).

Kundt spacetimes are interesting from both a geometric and a physical point of view. Due to this fact, literature devoted to these spacetimes is vast, with [7, 9, 46, 48, 103, 114] being just a few examples. We refer to [9] for relations with supersymmetric solutions of supergravity theories and their role in string theory. They also play a central role in the study of spaces with *vanishing scalar invariants* (VSI) (respectively, *constant scalar invariants* (CSI)) i.e., spaces such that all polynomial scalar invariants constructed from the curvature tensor and its covariant derivatives are zero (respectively, constant). Indeed, in dimension three, every CSI spacetime is either Kundt or locally homogeneous [49]. Moreover, in arbitrary dimension, every VSI spacetime is Kundt [50].

For an n-dimensional Kundt spacetime, the metric can be written in appropriate local coordinates $(u, v, x_1, \ldots, x_{n-2})$ as

$$g = dv\left(2du + F(u,v,x)dv + \sum_{i=1}^{n-2} 2K_{x_i}(u,v,x)dx_i\right) + \sum_{i,j=1}^{n-2} g_{ij}(v,x)dx_i dx_j, \tag{1.10}$$

where F, K_{x_i} and g_{ij} are functions of the specified coordinates, and ∂_u is lightlike and has vanishing optical scalars. In dimension four, coordinates in (1.10) can be further specialized so that $g_{ij} = P(v,x)\delta_{ij}$ for some function P (see [114]).

In dimension three, the geometry of Kundt spacetimes is more rigid than in higher dimensions. Indeed, the presence of an expansion-free lightlike geodesic vector field guarantees that the shear and twist vanish as well, so the spacetime is Kundt [46]. In this case, the expression (1.10) can be normalized so that $g_{11} = 1$. Thus, the metric can be written in local coordinates (u, v, x) as

$$g(u,v,x) = dv(2du + F(u,v,x)dv + 2K(u,v,x)dx) + dx^2. \tag{1.11}$$

Brinkmann waves

A more specific situation appears when on a Kundt spacetime the distinguished lightlike geodesic vector field V is recurrent, i.e. $\nabla_X V = \omega(X)\otimes V$, for a 1-form ω, which means that the line field $\mathrm{span}\{V\}$ is parallel. A spacetime admitting a parallel lightlike line field is said to be a *Brinkmann wave*. Due to their holonomy structure, these spacetimes highlight some fundamental differences between Riemannian and Lorentzian geometry. Indeed, let (M, g) be a semi-Riemannian manifold and $p \in M$. If $E \subset T_pM$ is an invariant subspace for the holonomy action, then $E^\perp$ is also invariant. Hence, every holonomy invariant subspace defines, via parallel translation, two parallel distributions D and $D^\perp$ (i.e., $\nabla D \subset D$ and $\nabla D^\perp \subset D^\perp$) in TM. The holonomy group acts *irreducibly* if it does not admit any invariant subspace E which is proper ($1 \leq \dim E \leq n-1$). In this case, we say that (M, g) is *irreducible*.

If E is invariant and, additionally, it is non-degenerate, then we say that the holonomy acts *decomposably* and that (M, g) is *decomposable*. In this case, (M, g) splits locally as a product $M = M_1 \times M_2$ with metric $g = g_1 \oplus g_2$, where (M_i, g_i) are semi-Riemannian manifolds which are integral manifolds of the distributions D and $D^\perp$. This decomposition is global if the manifold is assumed to be simply connected and geodesically complete, as shown by the de Rham-Wu splitting theorem [52, 119]. If there is no non-degenerate proper subspace that is invariant under the holonomy action, we say that the holonomy acts *indecomposably*, or that (M, g) is *indecomposable*. Hence, it follows that every reducible Riemannian manifold is decomposable.

However, if the metric g is semi-Riemannian and E is a degenerate subspace ($E \cap E^\perp \neq \{0\}$) which is invariant under the holonomy action, then there is a totally degenerate distribution $D \cap D^\perp$ with totally geodesic leaves. Thus, the holonomy group may act indecomposably but not irreducibly, meaning that the de Rham-Wu splitting theorem does not apply. This is the case if the holonomy action admits a totally lightlike (i.e., spanned by lightlike vectors) invariant subspace, but no non-degenerate invariant subspaces. Brinkmann waves illustrate this phenomenon in Lorentzian geometry.

Local coordinates given for Kundt spacetimes in (1.10) are further specialized for Brinkmann waves. Indeed, we can write the metric of a Brinkmann wave as

$$g = dv\left(2du + F(u,v,x)dv + \sum_{i=1}^{n-2} 2K_{x_i}(v,x)dx_i\right) + \sum_{i,j=1}^{n-2} g_{ij}(v,x)dx_i dx_j.$$

Notice that, in contrast to (1.10), now the functions K_{x_i} do not depend on the coordinate u. This guarantees that ∂_u is recurrent (in addition to being lightlike), so it plays the role of the distinguished vector field V. Moreover, if ∂_u can rescaled to a parallel vector field, then $\partial_u F = 0$ (see, for example, [86]).

In particular, the coordinates of a 3-dimensional Brinkmann wave can be manipulated so that the metric takes the form

$$g(u,v,x) = dv\,(2du + F(u,v,x)dv) + dx^2. \tag{1.12}$$

Pure radiation waves, pp-waves and plane waves

Among Brinkmann waves, there are special families of interest that are obtained by imposing some conditions on the curvature. Following terminology in [86], *pure radiation waves* (*pr*-waves for short), are Brinkmann waves whose curvature tensor satisfies $R(V^\perp, V^\perp, -, -) = 0$.

In this case, the Brinkmann wave metric reduces to a much simpler form:

$$g = 2dudv + F(u, v, x)dv^2 + \sum_{i,j=1}^{n-2} dx_i^2. \tag{1.13}$$

Whenever $\partial_u = V$ is parallel, the pr-wave is said to be a *plane-fronted wave with parallel propagation*, or pp-wave for short. For pp-waves, F can be taken to satisfy $\partial_u F = 0$. Note that a Brinkmann wave with parallel lightlike vector field V is a pp-wave if and only if $R(V^\perp, V^\perp, -, -) = 0$. Moreover, it was shown in [86] that a pr-wave is a pp-wave if and only if it is Ricci-isotropic, i.e., $g(\mathrm{Ric}(X), \mathrm{Ric}(X)) = 0$ for all $X \in \mathfrak{X}(M)$. For Brinkmann waves, this is equivalent to the condition that $\mathrm{Ric}(X) = 0$ for all $X \in V^\perp$.

Finally, a pp-wave with transversally parallel curvature tensor (i.e., such that $\nabla_{V^\perp} R = 0$) is called a *plane wave*. In local coordinates, the metric of a plane wave is given by the pr-wave metric (1.13) where

$$F(u, v, x) = \sum_{i,j=1}^{n} a_{ij}(v) x_i x_j$$

and the coefficients a_{ij} are smooth functions of v. Plane waves defined on $\mathbb{R}^n$ are examples of geodesically complete Lorentzian manifolds [30]. Note that, if the a_{ij} are constants, these metrics correspond to *Cahen-Wallach symmetric spaces* [27].

Among plane waves metrics in dimension three, given by (1.13) with $F(v, x) = \alpha(v)x^2$, there are two families that are locally homogeneous [65]. The first one is precisely the family of Cahen-Wallach symmetric spaces $\mathcal{CW}_\varepsilon$, with $F(v, x) = \varepsilon x^2$. The second one is the family $\mathcal{P}_c$, defined by $F(v, x) = -\beta(v)x^2$ with $\beta' = c\beta^{3/2}$ for a constant c and $\beta > 0$.

Moreover, it was shown recently in [70] that compact locally homogeneous plane waves of any dimension are quotients of a homogeneous plane wave by a discrete subgroup of its isometry group.

Pure radiation waves, pp-waves and plane waves are examples of the famed family of *gravitational waves*, which are predicted as special solutions of the Einstein field equations in General Relativity. The detection of such deformations of spacetime by experimental means in recent years has sparked a renewed interest in the topic (see [98, 114] for further details and [57, 95, 101] for some classic examples of gravitational waves).

1.4 Notable geometric equations

As its title suggests, this thesis is centered around the analysis of several geometric equations in the context of smooth metric measure spaces. The study of geometric equations on manifolds is a storied field, with classic examples like Einstein manifolds in General Relativity, the conformal Einstein equation introduced by Brinkmann [11] (see also [83]) or the different generalizations of the Obata equation [84, 96, 120], to cite just a few examples closely related to the topics of this work.

Usually, these equations feature a combination of curvature-related tensors and differential operators. Thus, when working in coordinates, they take the form of a system of PDEs involving the components of the metric and, often, additional functions with a certain geometric meaning. In this section, we go over some notable geometric equations which appear throughout the thesis.

Einstein metrics

Einstein manifolds, which we have already mentioned, arise as critical points of the Einstein-Hilbert functional, also known as the total scalar curvature functional. This is a key functional in the theory of General Relativity (see, for example, [3, 114]) and it is given for a compact and orientable semi-Riemannian manifold (M, g) as the volume integral

$$\mathcal{S}_{HE} : g \mapsto \int_M \tau \, \mathrm{dvol}_g .$$

We say that the metric g is *critical* under variations of the form $g[t] = g + t\bar{g}$, where $\bar{g}$ is a symmetric $(0,2)$-tensor, if the variational derivative $\delta(\mathcal{S}_{HE}) = \frac{d}{dt}\big|_{t=0}\mathcal{S}_{HE}(g[t])$ vanishes. This variation takes the form (see, for example, [3])

$$\delta(\mathcal{S}_{HE}) = \int_M \left\langle \rho - \frac{\tau}{2} g, \bar{g} \right\rangle \mathrm{dvol}_g . \tag{1.14}$$

The tensor $G = \rho - \frac{\tau}{2} g$ is often referred to as the *gradient* of the functional $\mathcal{S}_{HE}$. Due to its importance in General Relativity, it is also known as the *Einstein tensor*. Since for critical metrics $\delta(\mathcal{S}_{HE})$ vanishes for all variations $\bar{g}$, it follows from the expression above that G must also vanish. However, in this case, taking the trace of G shows that $\frac{2-n}{2}\tau = 0$, so $\tau = 0$ (recall that we are taking $n \geq 3$). Thus, critical metrics for $\mathcal{S}_{HE}$ are Ricci-flat.

In order to arrive at the standard definition of Einstein metrics, and since the Einstein-Hilbert functional is not invariant under homotheties, one considers the restriction of the variational problem to variations that keep the volume of the manifold, i.e., the integral $\int_M \mathrm{dvol}_g$, constant. These variations turn out to be exactly those that are orthogonal to the metric tensor, meaning $\langle g, \bar{g} \rangle = 0$. Thus, from (1.14), it follows that critical metrics for the restricted variational problem are metrics whose Einstein tensor is a multiple of g at each point of the manifold. This is equivalent to the condition

$$\rho = \lambda g, \quad \text{for some } \lambda \in C^\infty(M).$$

By taking traces in this equation, it follows that $\lambda = \frac{\tau}{n}$. Then, taking divergences and using the contracted Bianchi identity yields $\frac{1}{2} d\tau = d\lambda = \frac{1}{n} d\tau$. Since $n \neq 2$, λ and the scalar curvature τ are constants. This result is known as Schur's Lemma and it motivates the definition of Einstein manifolds as those with $\rho = \lambda g$ for some $\lambda \in \mathbb{R}$, as given in Section 1.1.2.

In dimension two, the situation is slightly different. In this case, the Gauss-Bonnet Theorem shows that $\mathcal{S}_{HE}$ is always a multiple of the Euler characteristic, so all metrics are critical and $\rho = \frac{\tau}{2} g$ for all 2-dimensional semi-Riemannian manifolds (M, g). However, this does not imply the constancy of τ.

In Physics, especially within the framework of General Relativity in Lorentzian manifolds, it is common to write the condition $\rho = \lambda g$ in terms of the Einstein tensor G as

$$G + \Lambda g = \rho - \frac{\tau}{2} g + \Lambda g = 0, \quad \text{for some } \Lambda \in \mathbb{R},$$

giving rise to the *Einstein field equations*. Einstein manifolds model the behavior of spacetimes in the absence of matter, so they are also referred to as *vacuum solutions* to the field equations, while Λ is known as the *cosmological constant*, due to its influence in the rate of expansion of the Universe. Moreover, the divergence of this equation vanishes, which is interpreted as a law of conservation of energy. When matter is introduced, this is usually taken into account via the introduction of a *stress-energy* tensor T which models the type of matter that populates the spacetime. The Einstein field equations then become $G + \Lambda g = T$ (see [114] for details on different vacuum and non-vacuum solutions). Throughout the rest of the thesis, we will often include the term Λg in the Einstein tensor itself, and refer to it as the Einstein tensor with cosmological constant.

Notice that, although this characterization of Einstein metrics as critical points of a certain functional requires the underlying manifold to be closed (compact without boundary), the same process works for general manifolds (with or without boundary), as long as the problem is restricted to variations with compact support which vanish on a neighborhood of the boundary of the support. Furthermore, the tensorial equations resulting from this variational approach make sense for any semi-Riemannian manifold, so "being Einstein" is not an exclusive property of closed (or orientable) manifolds.

An important property of Riemannian Einstein metrics is their (real) analyticity in harmonic coordinates (that is, coordinates $(x_1, \ldots, x_n)$ such that $\Delta x_i = 0$ for all $i = 1, \ldots n$), which always exist locally (see [3, K.45]). This means that there exists an atlas, contained in the maximal smooth atlas of the manifold, such that the coordinates for each chart are harmonic and analytic, and such that the transition functions are analytic as well. This property of Einstein metrics comes from standard results in elliptic PDEs (see, for example, [3, 5.26], and Theorem 1.15 for a related result in a more general context). In particular, this means that any analytic function of the components of the metric is determined by its value on an open set. For example, an immediate consequence is that if two Riemannian Einstein metrics coincide as tensors fields on an open set, then they are equal on the whole manifold.

Notice that, according to the decomposition $R = P \owedge g + W$ of the curvature tensor in Section 1.1.2, an Einstein manifold is locally conformally flat if and only if it has constant sectional curvature $\frac{\tau}{n(n-1)}$. However, since the scalar curvature τ is constant for all Einstein manifolds, the Schouten tensor P is Codazzi and the Cotton tensor dP vanishes, even if $W \neq 0$. Moreover, by (1.3), we have that $\operatorname{div} W$ is a multiple of dP, therefore Einstein manifolds have harmonic Weyl tensor. This is not the case for other Einstein-type structures, so the condition $\operatorname{div} W = 0$ is often introduced as a means to study them (see the section below for examples of this).

Gradient Ricci solitons and quasi-Einstein manifolds

Ricci solitons arise, in a certain sense, as self-similar solutions to the Ricci flow, given by the evolution equation $\frac{\partial}{\partial t} g_t = -2\rho_{g_t}$, where g_t is a 1-parameter family of semi-Riemannian metrics on a manifold M. This flow was introduced by Hamilton [69], as a tool intended to be utilized in the resolution of the Poincaré Conjecture. For an initial Einstein metric $g = g_t\big|_{t=0}$, as the flow evolves, g_t remains invariant modulo homotheties. In general, if in addition to homotheties we allow an initial metric g to change via diffeomorphisms, the solution g_t is said to be *self-*

similar, i.e., there exists a positive function $\sigma(t)$ and a 1-parameter family of diffeomorphisms $\varphi_t \in \mathrm{Diff}(M)$ such that $g_t = \sigma(t)\varphi_t^* g$. There exists a correspondence between self-similar solutions and vector fields $X \in \mathfrak{X}(M)$ such that

$$\rho + \mathcal{L}_X g = \lambda g, \qquad \text{for some } \lambda \in \mathbb{R},$$

where $\mathcal{L}$ denotes the Lie derivative. Thus, a triple (M, g, X) satisfying the geometric equation above is called a *Ricci soliton*. A more specific example of this structure is given by *gradient Ricci solitons*, i.e., triples (M, g, f) where $f \in C^\infty(M)$ and $(M, g, \nabla f)$ is a Ricci soliton as portrayed above. In this case, the Ricci soliton equation becomes

$$\rho + \mathrm{Hes}_f = \lambda g.$$

These structures appear, for example, in Perelman's work on the Ricci flow [100], which eventually provided the first proof of the Poincaré Conjecture.

A well-known generalization of the gradient Ricci soliton equation is the quasi-Einstein equation:

$$\rho + \mathrm{Hes}_f - \alpha df \otimes df = \lambda g, \qquad \text{where } \lambda, \alpha \in \mathbb{R}.$$

Manifolds satisfying this equation for some f, α and λ are said to be *quasi-Einstein (QE)*. Note that gradient Ricci solitons are QE manifolds with $\alpha = 0$. Structures of this type appear in numerous contexts in Mathematics and Physics, from the construction of Einstein warped products (see, for example, [77]) to modified theories of gravity (see [117]). A particularly relevant context in which the QE equation plays a role is the construction of manifolds which can be conformally transformed into an Einstein manifold. Indeed, the Ricci tensor transforms under the conformal change $\widehat{g} = e^{-\frac{2}{n}f} g$ as (see, for example, [83])

$$\widehat{\rho} = \rho + \mathrm{Hes}_f + \frac{1}{n} df \otimes df + \frac{1}{n}(\Delta f - \|\nabla f\|^2) g.$$

Hence, $\widehat{g}$ is Einstein if and only if the underlying manifold is QE for the density f and the parameter $\alpha = -\frac{1}{n}$ (this is often called the *conformal Einstein equation*). Consequently, this value of α is distinguished and the behavior of solutions is different than that of solutions for other values of α (see [14, 19, 44] for some examples of this fact).

Due to the fact that smooth metric measure spaces are characterized by the presence of a density $f \in C^\infty(M)$, it comes as no surprise that geometric objects such as Ricci solitons and QE manifolds are intimately related to them. We refer to Section 1.5 below for further motivation for QE manifolds from the point of view of smooth metric measure spaces.

Given their importance, both Ricci solitons and QE manifolds have drawn considerable interest over the past twenty years, especially in Riemannian signature. Thus, literature on the topic is extensive and we refer, for example, to [32, 59, 93], the survey [31] and references therein for results on gradient Ricci solitons; and to [43, 45, 71] for QE manifolds. In Lorentzian signature, results can also be found in works such as [12, 15] for solitons and [14, 19] for QE manifolds. See also [13] for results for solitons in signature $(2, 2)$.

In a similar way to that of Einstein manifolds, Riemannian Ricci solitons and QE manifolds are shown to be real analytic in harmonic coordinates [71], meaning that both the metric and the function f are. Thus, both of them are determined by their values on an open set.

Notice that, for Einstein manifolds with $n \geq 3$, the constancy of the λ term in the equation $\rho = \lambda g$ is guaranteed by the contracted Bianchi identity. This is no longer the case for quasi-Einstein manifolds, so it makes sense to consider a further generalization known as *generalized quasi-Einstein (GQE) manifolds* [44], i.e., manifolds (M, g) satisfying

$$\rho + \operatorname{Hes}_f - \alpha df \otimes df = \lambda g, \qquad \text{for some } f, \lambda \in C^\infty(M), \alpha \in \mathbb{R}.$$

GQE manifolds generalize a number of geometric conditions and therefore have been extensively analyzed in literature. For example, unlike Einstein manifolds, QE and GQE manifolds do not have harmonic Weyl tensor in general, and classification results under conditions of this type have been found in both Riemannian [44] and Lorentzian [19] signatures. When $\alpha = 0$, the resulting structure is known as an *almost gradient Ricci soliton*, and such manifolds have also been studied (see, for example, [102], where they were introduced).

The generalized Obata equation

In his seminal work [96], Obata studied the geometric equation

$$\operatorname{Hes}_u + cug = 0, \qquad \text{for some } c \in \mathbb{R}^+,$$

for a smooth function u on a Riemannian manifold (M, g), and showed that the only complete manifold (up to isometry) admitting a solution is the standard round sphere of radius $\frac{1}{\sqrt{c}}$. Since then, this equation has been referred to as the *Obata equation*, and several different generalizations of it have appeared in literature (see, for example, [84, 120]), related to a number of relevant geometric problems. Among them, we highlight conformal transformations between Einstein spaces [10, 82] and the Lichnerowicz–Obata theorem for the first eigenvalue of the Laplacian on compact Einstein manifolds [96]. A version of this equation also plays a central role in Part I of this thesis.

A natural generalization of the Obata equations is as follows. Given a semi-Riemannian manifold (M, g), we say that a function $u \in C^\infty(M)$ is a solution of the *generalized Obata equation* if it satisfies

$$\operatorname{Hes}_u + \gamma(u) g = 0, \quad \text{for some smooth function } \gamma. \tag{1.15}$$

If γ is linear in u, the function u is said to be *concircular*, in which case this equation is related to conformal transformations between Einstein spaces [82].

Notice that solutions to this equation are necessarily solutions of the local Möbius equation $\operatorname{Hes}_h = \frac{\Delta h}{n} g$ (see [99, 121]), which is also related to conformal changes of Einstein metrics that are Einstein as well. We refer to [83] for a survey of this topic in semi-Riemannian geometry. Also, the local Möbius equation was applied to give the warped product structure of a Schwarzschild space-time in [60].

The existence of a global solution of (1.15) on a Riemannian manifold (M, g) has considerable implications on the geometry of said manifold. Indeed, by the following construction, which was presented in detail in [120] (although many aspects of it were outlined in the older work [82]), we can build manifolds that admit a solution with at least one critical point for a suitable function γ.

Let γ be a smooth function defined on an interval $I = (a, b)$, $[a, b)$, $(a, b]$ or $[a, b]$ (with a, b possibly infinite), and assume that there exists $\eta \in I$ such that $\gamma(\eta) \neq 0$. Let ω be the unique maximally extended solution of the initial value problem

$$\omega'' + \gamma(\omega) = 0, \quad \omega(0) = \eta, \quad \omega'(0) = 0.$$

Define T as the (possibly infinite) supremum of t such that ω is defined on $[0, t]$ and $\omega' \neq 0$ in $(0, t]$, and consider the following warped metric on $(0, T) \times \mathbb{S}^{n-1}$:

$$g = dt^2 + \gamma(\eta)^{-2}(\omega')^2 g_{\mathbb{S}^{n-1}},$$

which extends smoothly through $t = 0$ to the Euclidean open ball of radius T, $\mathcal{B}_T(0)$. Then, if $T = \infty$, then $\mathcal{B}_T(0) = \mathbb{R}^n$. On the other hand, if T is finite, g extends smoothly to $\mathbb{S}^n$, where $\mathbb{S}^n \setminus \{p, -p\}$ is identified with $(0, T) \times \mathbb{S}^{n-1}$. These complete extensions are denoted by M^η_γ. Moreover, taking $u = \omega(t)$ on $(0, T) \times \mathbb{S}^{n-1}$ guarantees that u extends smoothly to M^η_γ. Note that $u(0) = \eta$, that u has a critical point at $t = 0$ and another one at $t = T$ when T is finite; and that u satisfies the generalized Obata equation (1.15) on M^η_γ.

As it turns out, this process not only provides a way to build complete manifolds admitting solutions, but these constructions are the only ones (up to isometry) with solutions with critical points. Moreover, solutions without critical points are also restricted, as the following theorem shows.

Theorem 1.7. *[120, Theorem 4.6] Let (M^n, g) be a complete Riemannian manifold admitting a non-constant smooth solution u of the generalized Obata equation* (1.15) *for a smooth function γ. Then,*

1. *If u has critical points (at most, it can have two), then (M, g) is isometric to a suitable M^η_γ.*
2. *If u does not have critical points, (M, g) is isometric to a warped product $\mathbb{R} \times_\varphi N$, where N is complete and u is defined on the base $\mathbb{R}$.*

In Chapters 2 and 3, we will make use of Theorem 1.7 in order to both determine the geometry of complete solutions, and to give explicit expressions for the density in local coordinates.

1.5 Smooth metric measure spaces

Smooth metric measure spaces (SMMSs for short) are the main focus of this thesis. Roughly speaking, a *manifold with density* or *smooth metric measure space* is a semi-Riemannian manifold (M, g) endowed with a smooth measure which is, in general, different from the usual Riemannian measure. This can be formalized by introducing a smooth function $f \in C^\infty(M)$, called *density function* or simply *density*, such that the aforementioned smooth measure is given by the *weighted volume element* $d\nu = e^{-f}\mathrm{dvol}_g$, Thus, a first definition for a SMMS is a triple $(M, g, e^{-f}\mathrm{dvol}_g)$. Alternatively, we can consider the density as a positive function $h = e^{-f}$ and make use of the simpler notation (M, g, h). This is the notation that we will use for Part II of this dissertation. When the density function f is constant, we say that the corresponding SMMS is *trivial*, since in this case the scope of the analysis is reduced to standard semi-Riemannian geometry. Henceforth, unless specifically stated, we will assume that all SMMSs are non-trivial. We refer to Morgan's book [92] for a good introduction to geometric measure theory, including basic notions of the geometry of smooth metric measure spaces.

When using this definition, we say that two SMMSs (M_1^n, g_1, h_1), (M_2^n, g_2, h_2) are *isometric* if there exists a Riemannian isometry $\psi : (M_1, g_1) \to (M_2, g_2)$ preserving the density, i.e., such that $h_1 = h_2 \circ \psi$.

When presented with the notion of SMMSs, a question which arises naturally is that of the influence of the density on the geometric features of the underlying semi-Riemannian manifold. The problem in this sense is twofold: First, one might wonder how to define appropriate geometric objects which retain geometric meaning while incorporating information on the density. From this perspective, *weighted invariants* are defined as functions on the space of metric-measure structures

$$\mathfrak{M}(M) = \mathrm{Met}(M) \times C^\infty(M),$$

where $\mathrm{Met}(M)$ is the space of metrics (of whatever signature we are considering); which are invariant with respect to the action of the diffeomorphism group $\mathrm{Diff}(M)$. With this, we can define *weighted functionals*, i.e., maps $\mathcal{S} : \mathfrak{M} \to \mathbb{R}$ such that $\mathcal{S}(\varphi^* g, \varphi^* f) = \mathcal{S}(g, f)$ for every $\varphi \in \mathrm{Diff}(M)$ and every $(g, f) \in \mathfrak{M}$. Another class consists of *weighted scalars*, namely maps $I : \mathfrak{M} \to C^\infty(M)$ such that $I(\varphi^* g, \varphi^* f) = \varphi^* I(g, f)$. Similarly, we can define *weighted tensors*, and so on.

Secondly, as a result of these weighted invariants involving both the metric and the density, it is natural to question how the geometry of the manifolds themselves reflects on the admissible forms of the density, and vice versa.

Among the different weighted local invariants of SMMSs, the most well-known one is perhaps the m-Bakry-Émery Ricci tensor

$$\rho_f^m = \rho + \mathrm{Hes}_f - \frac{1}{m} df \otimes df, \tag{1.16}$$

on whose study rests much of the literature on SMMSs. The earliest version of this object in literature is the ∞-Bakry-Émery Ricci tensor, where we formally set $m = \infty$ so that

$$\rho_f^\infty = \rho + \mathrm{Hes}_f$$

(see [88] and references therein for some geometric properties of this tensor). The tensor ρ_f^∞ was introduced in relation to diffusion processes [1], but it also gives rise to gradient Ricci solitons when $\rho_f^\infty = \lambda g$ for some $\lambda \in \mathbb{R}$ (see Section 1.4 for details), which is relevant for the proof of the Poincaré conjecture by Perelman [100]. Indeed, shrinking gradient Ricci solitons arise as critical points related to the $\mathcal{W}$-functional that Perelman uses in his results. This is in turn related to the isoperimetric problem through the Gaussian logarithmic Sobolev inequality (see also [75, 108] for some results on the isoperimetric problem in manifolds with density).

Nevertheless, in this dissertation, we do not consider the case $m = \infty$. Indeed, we will restrict our study to positive values of m, as explained below. In this case, the m-Bakry-Émery Ricci tensor has also proved to be significant in numerous contexts. One example is quasi-Einstein structures ($\rho_f^m = \lambda g$, see Section 1.4), which along with gradient Ricci solitons can be formulated in a useful uniform manner using the formalism of SMMSs. In another vein, works such as [2, 74, 116] consider bounded Bakry-Émery Ricci tensors and some curvature-dimension inequalities to extend spectral gap theorems or to obtain topological restrictions. This tensor also appears in Riemannian signature linked to the study of the static perfect fluid Einstein equation [79] and in Lorentzian signature in relation to splitting and singularity theorems [42, 118] and scalar-tensor gravity theories [117].

As we will see shortly, most of the weighted invariants of interest to this dissertation are constructed from the Bakry-Émery Ricci tensor, so it will be a central aspect of our discussion.

SMMSs and their relation to GNS inequalities

In recent years, a new framework has been introduced, mostly through the works of Case (see [34–41]), which generalizes the concept of SMMSs explained above. Although this framework makes sense in any signature, the motivation behind it stems from problems in Riemannian signature, so this is the formalism that we will use in Part I of this thesis, where we focus on results for Riemannian SMMSs.

Sobolev-type inequalities are a quintessential example of how analytic inequalities play an important role in geometric analysis. For example, the classical Sobolev inequality states that there is a constant C_S such that

$$\left(\int_{\mathbb{R}^n} w^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_S \left(\int_{\mathbb{R}^n} \|\nabla w\|^2\right), \qquad \text{for all } w \text{ in the Sobolev space } L_1^2(\mathbb{R}^n),$$

where $\|\nabla w\|^2 = g(\nabla w, \nabla w)$. On the other hand, the Yamabe problem (the question of when one can find a metric of constant scalar curvature in a given conformal class on a compact manifold) is equivalent to finding a smooth function w which realizes the *Yamabe constant*

$$\sigma_1(g) = \inf \left\{ \frac{\int \left(\|\nabla w\|^2 + \frac{n-2}{4(n-1)}\tau w^2\right) \mathrm{dvol}_g}{\left(\int w^{\frac{2n}{n-2}} \mathrm{dvol}_g\right)^{\frac{n-2}{n}}} \;:\; 0 \neq w \in L_1^2(M) \right\}.$$

Its standard resolution requires knowing the best (or *sharp*) value of the constant C_S in the Sobolev inequality on $\mathbb{R}^n$, which turns out to be the inverse of the Yamabe constant on the sphere (see [111] and references therein for details).

For other types of Gagliardo-Nirenberg-Sobolev inequalities, the sharp constants are not known. However, they have been computed by Del Pino and Dolbeaut [53] for the following special cases, which are useful for studying some fast diffusion equations:

$$\Lambda_{m,n}\left(\int_{\mathbb{R}^n} w^{\frac{2(m+n)}{m+n-2}}\right)^{\frac{2m+n-2}{n}} \leq \left(\int_{\mathbb{R}^n} \|\nabla w\|^2\right)\left(\int_{\mathbb{R}^n} w^{\frac{2(m+n-1)}{m+n-2}}\right)^{\frac{2m}{n}},$$

for $m \in [0, \infty)$, $\Lambda_{m,n}$ the known sharp constant and suitable functions w. Note that the value $m = 0$ recovers the classical Sobolev inequality. Case proved [38] that a similar connection exists between Yamabe-type constants and these inequalities by defining, using the formalism of SMMSs, a series of conformally invariant (in an appropriate weighted sense which we will define shortly) quantities known as *weighted Yamabe constants*. These invariants coincide with the sharp Del Pino-Dolbeaut constants when working in the Euclidean space, making explicit the connection between weighted conformal geometry and GNS inequalities. Moreover, Case also describes and partially solves in [36] a weighted Yamabe-type problem, and shows how the weighted Yamabe constants interpolate between the usual Yamabe constant (for $m = 0$) and another Riemannian invariant of importance, known as the ν-entropy (for $m = \infty$), which arises in Perelman's study of the Ricci flow [100] as the infimum of the $\mathcal{W}$-functional (which we have already mentioned) over the density and a certain scale parameter. Moreover, the ν-entropy realizes the sharp constant for the Gaussian logarithmic Sobolev inequality in Euclidean space (see [36]).

Definition of SMMSs and weighted tensors

Throughout Part I of this thesis, we will use the following definition for smooth metric measure spaces. This is the most general way to define SMMSs within Case's formalism.

Definition 1.8. A *smooth metric measure space* is a five-tuple (M^n, g, f, m, μ), where (M^n, g) is an n-dimensional Riemannian manifold (recall that we consider $n \geq 3$), $f \in C^\infty(M)$ is the density, $m \in \mathbb{R}^+$ is a dimensional parameter and $\mu \in \mathbb{R}$ is an auxiliary curvature parameter.

Notice that, in contrast to the definition at the start of this section, we include the parameter m from the m-Bakry-Émery Ricci tensor and an additional parameter μ. Both of them have relevant geometric meanings, which we will explain shortly. The theory of SMMSs developed in the aforementioned previous works usually assumes that m is non-negative (indeed, in the Del Pino-Dolbeaut inequalities, this is also the case). Moreover, by convention, weighted invariants are defined in such a way that they coincide with their Riemannian counterparts when $m = 0$. Hence, we assume $m \in \mathbb{R}^+$. However, the results in this thesis formally extend to negative values $m \in \mathbb{R}^- \backslash \{1-n, 2-n\}$ (values $m = 1-n, 2-n$ are not admissible, see expression (1.19) below and results thereafter).

When working with this definition, in order to identify SMMSs, we need to fix the values of the parameters m and μ. Thus, we say that two SMMSs

$$(M_1^n, g_1, f_1, m_1, \mu_1) \quad \text{and} \quad (M_2^n, g_2, f_2, m_2, \mu_2)$$

are *isometric* if there exists a Riemannian isometry $\psi : (M_1, g_1) \to (M_2, g_2)$ such that $f_1 = f_2 \circ \psi$, $m_1 = m_2$ and $\mu_1 = \mu_2$. Consequently, weighted invariants may now depend on m and μ, along with the metric and the density (for further details, see [34]).

Besides the weighted Yamabe constants, some of the weighted invariants defined within this formalism are weighted analogues of σ_k-curvatures, which were introduced in [35] and further analyzed in [34]. Tractor calculus has also been introduced into the study of SMMSs and quasi-Einstein spaces [39, 41]. Nevertheless, for the purposes of this thesis, we will be interested in weighted invariants related to curvature and some natural geometric equations that arise from them. These are based, except for the placement of some constants, on those defined in [34, 41].

For a SMMS (M, g, f, m, μ) the positive density is now defined as $v = e^{-\frac{f}{m}}$, so that the weighted volume element becomes $d\nu = v^m \mathrm{dvol}_g$. From this point of view, the dimensional parameter m indicates that we wish to consider $(M, g, d\nu)$ as an $(m+n)$-dimensional metric-measure space, meaning that we define curvature-related weighted invariants using the formal warped product

$$M^n \times_v F^m(\mu) = (M^n \times F^m, g \oplus v^2 q(\mu)), \tag{1.17}$$

where $F(\mu) = (F, q(\mu))$ is an m-dimensional space form of constant sectional curvature μ. This gives further justification for the assumption $m \in \mathbb{R}^+$. With this, many weighted invariants can be regarded as restrictions of Riemannian invariants of (1.17) to M. Indeed, $d\nu$ is the restriction of the Riemannian volume element of the warped product to its base, and the m-Bakry-Émery Ricci tensor (1.16) is the restriction of the Ricci tensor of the product to vector fields tangent to M. Similarly, the *weighted scalar curvature* is

$$\tau_f^m = \tau + 2\Delta f - \frac{m+1}{m}\|\nabla f\|^2 + m(m-1)\mu\, e^{\frac{2}{m}f}, \tag{1.18}$$

which is the scalar curvature of (1.17), considered as a function on M. Recently, the linearization of the weighted scalar curvature (for $\mu = 0$) has been used to propose a weighted analogue of Riemannian vacuum static spaces [73].

Following this pattern, the *weighted Schouten tensor* and *weighted Schouten scalar* are given, respectively, by

$$P_f^m = \frac{1}{m+n-2}(\rho_f^m - J_f^m g), \qquad J_f^m = \frac{1}{2(m+n-1)}\tau_f^m. \tag{1.19}$$

Although τ_f^m is regarded as a weighted analogue of the usual scalar curvature τ, it is not the trace of the Bakry-Émery Ricci tensor (1.16). Moreover, J_f^m is not the trace of P_f^m, as opposed to the usual Schouten tensor and scalar (1.1). The difference between these two quantities is denoted by $Y_f^m = J_f^m - \operatorname{tr} P_f^m$ and will play a role in Chapter 2.

From the Schouten tensor, we get the *weighted Weyl tensor*:

$$W_f^m = R - P_f^m \owedge g. \tag{1.20}$$

As we will show, this tensor is key when discussing conformal aspects of SMMSs, much like the Weyl tensor in the context of semi-Riemannian manifolds. Finally, we will need the *weighted Cotton tensor*, given by the skew-symmetrization of ∇P_f^m:

$$dP_f^m(X, Y, Z) = (\nabla_Y P_f^m)(X, Z) - (\nabla_Z P_f^m)(X, Y). \tag{1.21}$$

The relations between the main weighted and unweighted tensors can be summarized through the following table:

Object	**Usual**	**Weighted**
Curvature tensor	R	R
Ricci tensor	ρ	$\rho_f^m = \rho + \mathrm{Hes}_f - \frac{1}{m} df \otimes df$
Scalar curvature	τ	$\tau_f^m = \tau + 2\Delta f - \frac{m+1}{m}\|\nabla f\|^2$ $+m(m-1)\mu\, e^{2f/m}$
Schouten tensor	$P = \frac{1}{n-2}(\rho - Jg)$ $J = \frac{\tau}{2(n-1)}$	$P_f^m = \frac{1}{m+n-2}(\rho_f^m - J_f^m g)$ $J_f^m = \frac{1}{2(m+n-1)}\tau_f^m$
Weyl tensor	$W = R - P \owedge g$	$W_f^m = R - P_f^m \owedge g$

Table 1.1: Comparative between standard curvature-related tensors and their weighted analogues.

Note that the role of the auxiliary curvature parameter μ is made explicit in the definition of the weighted scalar curvature. However, if $m = 1$, the value of μ becomes irrelevant. Indeed, in this case, the fiber of the warped product (1.17) is 1-dimensional, so it has no sectional curvature. Thus, we adapt notation and refer to SMMSs with $m = 1$ by the quadruple $(M, g, f, 1)$.

Like in the case of curvature-related tensors, this pattern of generalization of Riemannian geometric objects to the weighted setting also applies to some differential operators. Indeed, a natural one to consider is the *weighted divergence*

$$\mathrm{div}_f T = \mathrm{div}\, T - \iota_{\nabla f} T = \mathrm{div}\, T - T(\nabla f, \cdots), \tag{1.22}$$

which is the negative of the formal adjoint of the covariant derivative of tensors with respect to the weighted measure $e^{-f}\mathrm{dvol}_g$. The *weighted Laplacian* Δ_f on functions is thus the formally self-adjoint operator given by

$$\Delta_f \varphi = \Delta\varphi - g(\nabla f, \nabla\varphi)$$

for any $\varphi \in C^\infty(M)$.

Weighted conformal classes

As previously pointed out, ideas of conformal geometry naturally enter into the study of SMMSs, be it for quasi-Einstein manifolds [37, 39–41], the weighted Yamabe problem [36], or the properties of weighted σ_k-curvatures [34, 35]. Hence, in this section, we explain how conformal geometry is naturally introduced into the weighted setting, and show some conformal properties of the curvature-related weighted tensors of interest to this thesis.

Two SMMSs (M, g, f, m, μ) and $(N, \widehat{g}, \widehat{f}, m, \mu)$ are *conformally equivalent* (in the weighted sense) if there exists a Riemannian conformal map $\Phi : M \to N$, with $\Phi^*\widehat{g} = e^{-2\phi/m} g$ for some function $\phi \in C^\infty(M)$, and such that $\widehat{f} \circ \Phi = f + \phi$. In order to simplify notation, and since Φ is a diffeomorphism, we say that the SMMSs (M, g, f, m, μ) and $(M, \widehat{g}, \widehat{f}, m, \mu)$ are conformally equivalent if there exists a smooth function $\phi \in C^\infty(M)$ such that $\widehat{g} = e^{-2\phi/m} g$ and $\widehat{f} = f + \phi$.

This definition is motivated by standard notions of conformal geometry on the formal warped product (1.17). Indeed, consider the warped products

$$(M, g) \times_v F^m(\mu) \quad \text{and} \quad (M, \widehat{g}) \times_{\widehat{v}} F^m(\mu), \tag{1.23}$$

where $v = e^{-\frac{f}{m}}$ and $\widehat{v} = e^{-\frac{\widehat{f}}{m}}$, and let $u = e^{\phi/m}$. Then, we can rephrase the definition saying that two SMMSs are conformally equivalent if there exists a positive $u \in C^\infty(M)$ such that $\widehat{g} = u^{-2} g$ and $\widehat{v} = u^{-1} v$. As a consequence, the products in (1.23) are conformally equivalent in the Riemannian sense, with the conformal factor u depending only on the base M. These definitions can also be stated in local terms if instead of a global function ϕ (equivalently, u), there is a function defined only on a neighborhood of each point.

A weighted invariant $T = T(g, v)$ is said to be *conformally invariant* of weight s if for any conformal factor u,

$$T(u^{-2} g, u^{-1} v) = u^{-s} T(g, v).$$

For example, the weighted volume element of the conformally transformed manifold is $d\widehat{\nu} = u^{-m} v^m \mathrm{dvol}_{\widehat{g}} = u^{-(m+n)} v^m \mathrm{dvol}_g$, so it is conformally invariant of weight $m + n$. Moreover, it transforms like the volume element of an $(m + n)$-dimensional manifold would under the conformal transformation $\widehat{g} = u^{-2} g$, which is consistent with the interpretation of SMMSs as bases of the formal warped products (1.23).

Once a notion of conformal class for SMMSs has been established, a natural question is how to define weighted local conformal flatness. To this end, notice that direct products of the form $F^n(-\mu) \times F^m(\mu)$ are locally conformally flat, and those of the form $F^n(c) \times \mathbb{R}$ are locally conformally flat for any c [124]. Thus, locally conformally flat SMMSs are those whose associated warped product is in the conformal class of one of these structures (in the Riemannian sense), taking into account that the conformal change is only defined on the base. This motivates the following definition:

Definition 1.9. A SMMS (M^n, g, f, m, μ) with $m \neq 1$ is *locally conformally flat* if it is locally conformally equivalent to $(F^n, q(-\mu), 0, m, \mu)$. If $m = 1$, $(M^n, g, f, 1)$ is *locally conformally flat* if it is locally conformally equivalent to $(F^n, q(c), 0, 1)$ for some sectional curvature c.

Like in the unweighted context, the weighted Weyl tensor W_f^m is intimately related to local conformal flatness. Indeed, let (M, g, f, m, μ) and $(M, \widehat{g}, \widehat{f}, m, \mu)$ be two conformally equivalent smooth metric measure spaces with $\widehat{g} = u^{-2} g$ and $\widehat{v} = u^{-1} v$. Then,

$$\widehat{W}_{\widehat{f}}^m = u^{-2} W_f^m,$$

so we see that the weighted Weyl tensor is a weighted conformal invariant (with weight 2), like its analogue in the usual setting. Hence, the curvature tensor is completely controlled by the weighted Schouten tensor within a weighted conformal class. Moreover, a SMMS with $n \geq 3$ and $m + n \neq 3$ (which are satisfied for all the SMMSs that we will consider) is locally conformally flat in the weighted sense if and only if $W_f^m = 0$, in which case the weighted Cotton tensor

dP_f^m also vanishes. These properties, like many other weighted conditions, stem from their unweighted analogue in the formal warped product (1.17), when one considers conformal changes defined only on the base. See [34, 41] for the derivation of these formulas, other transformations of curvature-related weighted invariants, and more details on conformal aspects of SMMSs.

1.6 Weighted Einstein manifolds

In Section 1.4, we presented Einstein manifolds as those that satisfy $\rho = \lambda g$ for some $\lambda \in \mathbb{R}$, explained how they arise through a variational approach, and touched on their importance in both Mathematics and Physics. Thus, it makes sense to wonder how one should define an appropriate Einstein-type structure in the aforementioned formalism for smooth metric measure spaces.

In this context, quasi-Einstein manifolds (those with $\rho_f^m = \lambda g$) arise as a strong candidate, due to the Bakry-Émery Ricci tensor serving as the weighted analogue of the standard Ricci tensor. However, note that, in the standard setting, the Einstein condition can be formulated in terms of the Ricci tensor or, equivalently, in terms of the Schouten tensor, by saying a manifold is Einstein if $P = \lambda g$ for some $\lambda \in \mathbb{R}$. These two conditions are not equivalent for the weighted analogues of these two tensors, so they lead to two distinct geometric notions. The first one is, as previously mentioned, that of quasi-Einstein manifolds. The second one, which is the focus of Part I of this thesis, is the following.

Definition 1.10 [34]. A smooth metric measure space (M^n, g, f, m, μ) is *weighted Einstein* if its weighted Schouten tensor satisfies $P_f^m = \lambda g$ for some $\lambda \in \mathbb{R}$.

For simplicity, since the SMMS structure is implied in our discussion, we will often refer to weighted Einstein SMMSs simply as *weighted Einstein (WE) manifolds*. More explicitly, for such a SMMS, the weighted Schouten tensor satisfies

$$P_f^m = \frac{1}{m+n-2}(\rho_f^m - J_f^m g) = \lambda g, \qquad \text{for some } \lambda \in \mathbb{R}.$$

From this formula, it is clear that WE manifolds are particular cases of generalized quasi-Einstein manifolds (those with $\rho_f^m = \alpha g$ for some function $\alpha \in C^\infty(M)$). Indeed, we have that a WE manifold is GQE with

$$\alpha = (m+n-2)\lambda + J_f^m. \tag{1.24}$$

The motivation for saying that the notion of weighted Einstein manifolds is the appropriate generalization of Einstein-type structures to SMMSs, instead of QE or GQE manifolds, is twofold:

Variational properties: On the one hand, WE manifolds with $\mu = 0$ are critical points of the *total weighted scalar curvature functional*

$$\mathcal{W}_m[\kappa] : (g, f) \mapsto \int_M (\tau_f^m + 2m(m+n-2)\kappa(e^{\frac{f}{m}} - 1))\, e^{-f} \mathrm{dvol}_g,$$

under simultaneous variations of the metric and the density that preserve the total weighted volume $\int_M e^{-f}\mathrm{dvol}_g$. Indeed, for a fixed κ, a metric-measure structure (g, f) is critical if and only if there exists $\lambda \in \mathbb{R}$ such that

$$P_f^m = \lambda g \quad \text{and} \quad J_f^m = (m+n)\lambda - m\kappa e^{\frac{f}{m}} \tag{1.25}$$

(see [36]). The total weighted scalar curvature functional is an important piece in the discussion of the weighted Yamabe problem. Moreover, weighted Einstein manifolds also arise among the critical points of functionals related to the weighted σ_k-curvatures, which are expected to lead to sharp fully non-linear Gagliardo-Nirenberg-Sobolev inequalities (see [34]).

Relationship with QE manifolds: The constant κ that appears in (1.25) plays an essential role in our study of weighted Einstein manifolds. As it turns out, such a constant exists for every such manifold, not only in the case $\mu = 0$, but for every $\mu \in \mathbb{R}$. Moreover, it encodes the relationship between non-trivial quasi-Einstein and weighted Einstein manifolds in a remarkably simple way (note that a trivial WE manifold is trivial QE with $\rho_f^m = \rho = (2(m+n-1)\lambda - m\kappa e^{f/m})g$. These facts are summarized in the following lemma.

Lemma 1.11 [34]**.** *Let (M^n, g, f, m, μ) be a non-trivial SMMS such that $P_f^m = \lambda g$ for some $\lambda \in \mathbb{R}$. Then, there is a unique constant $\kappa \in \mathbb{R}$ (called scale) such that*

$$J_f^m = (m+n)\lambda - m\kappa e^{\frac{f}{m}}.$$

Moreover, if (M, g, f, m, μ) is weighted Einstein with $\kappa = 0$, then (M, g, f, m) is quasi-Einstein. Also, if a quadruple (M, g, f, m) is quasi-Einstein, there exists an appropriate μ for which (M, g, f, m, μ) is weighted Einstein with $\kappa = 0$.

From this result, it follows that weighted Einstein manifolds generalize quasi-Einstein manifolds, while being in turn more rigid than generalized quasi-Einstein manifolds. Moreover, the fact that they arise naturally within this framework through a variational approach means that they are well-suited for weighted geometric conditions. Since they are a more general notion, the geometric properties of WE manifolds are less well understood than those of QE manifolds (see Section 1.4 and references therein for some results on the latter). Besides, they inhabit a middle ground where they present enough rigidity to obtain classification results under some natural weighted assumptions, but not so much that their study is reduced to known problems. Finally, we point out that the WE condition is a purely weighted notion, in the sense that it is not inherited directly from unweighted conditions such as the Einstein character of the formal warped product (1.17) (see Example 2.1 and Remark 2.3 for an example of this behavior). Indeed, this leads to the notion of QE manifolds instead.

These factors, as well as the importance of WE manifolds in various analytic problems in SMMSs, motivate the need for the research of such structures, which will be the focus of Part I of this thesis.

The weighted space forms

The simplest way to build examples of weighted Einstein manifolds is by taking the three model spaces (spheres, hyperbolic spaces and Euclidean spaces), and prescribing on them appropriate densities and values for the parameters m and μ. This process, in the case of quasi-Einstein manifolds, gives rise to the positive and negative elliptic m-Gaussians [40], which are built on the upper hemisphere of $\mathbb{S}^n$ and on $\mathbb{H}^n$ respectively. In [34], a WE manifold with positive scale (thus, not quasi-Einstein), is constructed for $\mu = 0$ on the punctured sphere (see Example 1.12 below). Nevertheless, the literature on the topic lacked a comprehensive list of families of WE manifolds realized on space forms for all values of μ. Aiming to fill this gap, and due to the fact that they play an essential role in the classification results in Chapters 2 and 3, we presented in [24] the following constructions, which we called *weighted space forms*.

Example 1.12 (m-weighted n-sphere $(\mathbb{S}^n(2\lambda), g_{\mathbb{S}}^{2\lambda}, f_m, m, \mu)$)**.** Let $(\mathbb{S}^n(2\lambda), g_{\mathbb{S}}^{2\lambda})$ be the n-sphere of constant sectional curvature $2\lambda > 0$ (equivalently, of radius $1/\sqrt{2\lambda}$), with the standard round metric

$$g_{\mathbb{S}}^{2\lambda} = dt^2 + (2\lambda)^{-1}\sin^2(t\sqrt{2\lambda})g_{\mathbb{S}^{n-1}}, \quad t \in \left(0, \tfrac{\pi}{\sqrt{2\lambda}}\right),$$

where t denotes the geodesic distance from the pole N of the sphere and $\mathbb{S}^{n-1}$ is the $(n-1)$-sphere of radius 1. This metric extends smoothly to the poles N and $-N$. Take the positive density $v(t) = A + B\cos(\sqrt{2\lambda}t)$ for $A \in \mathbb{R}^+$, $B \in \mathbb{R}$ such that $A > |B|$ and define, by continuity, $v(N) = A + B$ and $v(-N) = A - B$. For $m \neq 1$, fix $\mu = 2\lambda(B^2 - A^2)$. Then, the SMMSs $(\mathbb{S}^n(2\lambda), g_{\mathbb{S}}^{2\lambda}, f_m, m, \mu)$ and $(\mathbb{S}^n(2\lambda), g_{\mathbb{S}}^{2\lambda}, f_1, 1)$, where $f_m = -m\log v$, are weighted Einstein with $P_f^m = \lambda g$ and scale $\kappa = 2\lambda A > 0$. Hence, these weighted spheres are only quasi-Einstein in the trivial case, where $\rho_f^m = \rho = 2(n-1)\lambda g$.

Note that, by removing the condition $A > |B|$, we also get incomplete examples defined on the open set of points where $v > 0$. We present the two notable examples which where mentioned above (cf. [34]):

1. $A = B = 1$ on the punctured sphere $\mathbb{S}^n(2\lambda) - \{-N\}$ gives the *standard m-weighted n-sphere* of curvature 2λ.
2. $A = 0$ and $B = 1$ on the upper hemisphere $\mathbb{S}^n_+(2\lambda)$ gives the *positive elliptic m-Gaussian*, which is a quasi-Einstein manifold since its scale vanishes (indeed, $J_f^m = \lambda(m+n)$ in this case).

Example 1.13 (m-weighted n-Euclidean space $(\mathbb{R}^n, g_{\mathbb{E}}, f_m, m, \mu)$)**.** Let $(\mathbb{R}^n, g_{\mathbb{E}})$ be the standard Euclidean space, whose metric can be written as a warped product as

$$g_{\mathbb{E}} = dt^2 + t^2 g_{\mathbb{S}^{n-1}}, \quad t \in (0, \infty),$$

extending smoothly to $t = 0$. Consider the positive density $v(t) = A + Bt^2$ with $A \in \mathbb{R}^+$, $B \in [0, \infty)$. For $m \neq 1$, set the parameter $\mu = -4AB$. The SMMSs $(\mathbb{R}^n, g_{\mathbb{E}}, f_m, m, \mu)$ and $(\mathbb{R}^n, g_{\mathbb{E}}, f_1, 1)$ are weighted Einstein with $P_f^m = 0$ and scale $\kappa = 2B \geq 0$, so they are quasi-Einstein only when they are trivial.

Example 1.14 (m-weighted n-hyperbolic space $(\mathbb{H}^n(2\lambda), g_{\mathbb{H}}^{2\lambda}, f_m, m, \mu)$)**.** Let $(\mathbb{H}^n(2\lambda), g_{\mathbb{H}}^{2\lambda})$ be the n-hyperbolic space of constant sectional curvature $2\lambda < 0$, with the metric

$$g_{\mathbb{H}}^{2\lambda} = dt^2 + (-2\lambda)^{-1}\sinh^2(t\sqrt{-2\lambda})g_{\mathbb{S}^{n-1}}, \quad t \in (0, \infty),$$

extending smoothly to $t = 0$. Take the positive density $v(t) = A + B\cosh(\sqrt{-2\lambda}t)$, with $B \in [0,\infty)$, $A \in \mathbb{R}$ such that $A > -B$. Moreover, for $m \neq 1$, fix $\mu = 2\lambda(B^2 - A^2)$. Then, the SMMSs $(\mathbb{H}^n(2\lambda), g_{\mathbb{H}}^{2\lambda}, f_m, m, \mu)$ and $(\mathbb{H}^n(2\lambda), g_{\mathbb{H}}^{2\lambda}, f_1, 1)$ are weighted Einstein with $P_f^m = \lambda g$ and scale $\kappa = 2\lambda A$. Note that the scale can have any sign, depending on the value of A and, in contrast to the two previous models, taking $A = 0$ results in a family of non-trivial quasi-Einstein manifolds.

Thus, for any values of λ and m, we can build a complete weighted Einstein manifold by considering the corresponding weighted space form. Hence, it is not surprising that these families are key in global classification results of WE manifolds (see Theorems 2.23 and 3.10)

Analyticity of weighted Einstein manifolds

In order to achieve the classification results contained in Part I of this thesis, we will need to make use of the analytic properties of weighted Einstein manifolds, namely of their (real) analyticity in harmonic coordinates. For Einstein manifolds, Ricci solitons and quasi-Einstein metrics, this was already known (see Section 1.4 and references therein). However, since WE manifolds are a strict generalization of QE manifolds, a new result is needed which proves the real analyticity, in harmonic coordinates, of both the metric and the density function, using similar techniques in the context of SMMSs. This is indeed the purpose of the following theorem, which is contained in [24].

Theorem 1.15. *Let (M^n, g, f, m, μ) be a weighted Einstein SMMS. Then, both g and f are real analytic in harmonic coordinates on M.*

Proof. Assume $P_f^m = \lambda g$ and take traces in this equation, using (1.18) and (1.19), to obtain

$$(2m+n-2)\tau + 2(m-1)\Delta f + \frac{(m-1)(n-2)}{m}\|\nabla f\|^2 - nm(m-1)\mu e^{\frac{2}{m}f} = 2(m+n-1)(m+n-2)n\lambda.$$

Note that, if $m = 1$, this becomes $\tau = 2n(n-1)\lambda$, so τ is constant. If $m \neq 1$, we can write

$$\begin{aligned}\Delta f \;=\;& \tfrac{1}{2(m-1)}\left(-(2m+n-2)\tau + 2(m+n-1)(m+n-2)n\lambda\right)\\ &-\tfrac{n-2}{2m}\|\nabla f\|^2 + \tfrac{mn}{2}\mu e^{\frac{2}{m}f}.\end{aligned} \tag{1.26}$$

Furthermore, let $\kappa \in \mathbb{R}$ be the scale of (M^n, g, f, m, μ), so that $J_f^m = (m+n)\lambda - m\kappa e^{\frac{f}{m}}$ (see Lemma 1.11). Solving this equation for τ yields

$$\begin{aligned}\tau + 2\Delta f \;=\;& 2(m+n-1)\left((m+n)\lambda - m\kappa e^{\frac{f}{m}}\right)\\ &+\tfrac{m+1}{m}\|\nabla f\|^2 - m(m-1)\mu\, e^{\frac{2}{m}f}.\end{aligned} \tag{1.27}$$

If $m = 1$, since τ is constant, equation (1.27) becomes $\Delta f + \text{l.o.t} = 0$, where l.o.t. stands for lower order terms involving the metric and the density function. If $m \neq 1$, we can use (1.27) to write $\tau + 2\Delta f + \text{l.o.t} = 0$, while, by (1.26), we have $\Delta f + \frac{2m+n-2}{2(m-1)}\tau + \text{l.o.t} = 0$. Combining both equations, we have

$$\frac{m+n-1}{m-1}\Delta f + \text{l.o.t} = 0.$$

Since $n \geq 3$ and $m \in \mathbb{R}^+ - \{1\}$, we can write $\Delta f + \text{l.o.t} = 0$. Moreover, from (1.24), we can write the weighted Einstein equation as $\rho_f^m = ((m+n-2)\lambda + J_f^m)g$. Using Lemma 1.11 again, this expression takes the form

$$\rho + \text{Hes}_f - \frac{1}{m} df \otimes df = (2(m+n-1)\lambda - m\kappa e^{\frac{f}{m}})g$$

so, for any $m \in \mathbb{R}^+$, we end up with

$$\begin{aligned} \rho + \text{Hes}_f + \text{l.o.t} &= 0, \\ \Delta f + \text{l.o.t} &= 0. \end{aligned}$$

In harmonic coordinates $(x_1, \ldots, x_n)$ (recall that this means that $\Delta x_i = 0$ for all $i = 1, \ldots, n$), which always exist locally [3, K.45], a direct computation shows that these geometric equations become a quasi-linear second-order system of PDEs:

$$\begin{aligned} -\tfrac{1}{2} g^{rs} \tfrac{\partial^2 g_{ij}}{\partial x^r \partial x^s} + \tfrac{\partial^2 f}{\partial x_i \partial x_j} + \text{l.o.t} &= 0, \\ g^{rs} \tfrac{\partial^2 f}{\partial x^r \partial x^s} + \text{l.o.t} &= 0. \end{aligned}$$

Let $S^2(T^*M)$ be the space of symmetric $(0,2)$-tensor fields on M. Then, the principal symbol associated to this quasi-linear system is the linear map $\sigma_\xi : S^2(T^*M) \oplus C^\infty(M) \to S^2(T^*M) \oplus C^\infty(M)$ given by

$$(h, \omega) \mapsto \sigma_\xi(h, \omega) = \left(-\frac{1}{2}\|\xi\|^2 h + \omega \xi \otimes \xi, \|\xi\|^2 \omega \right).$$

If $\sigma_\xi(h, \omega) = 0$ and $\xi \neq 0$, then it follows that $\omega = 0$. In this case, h must also vanish. Thus, σ_ξ is an automorphism of $S^2(T^*M) \oplus C^\infty(M)$, which means that the quasi-linear system is elliptic. Moreover, the whole system is of the form $F(g, f, \partial g, \partial f, \partial^2 g, \partial^2 f) = 0$, where F is real analytic. From regularity results for quasi-linear elliptic PDEs, it follows that both the metric and the density function are real analytic in harmonic coordinates (see [3, J.41]). □

The real analyticity of weighted Einstein manifolds proved in Theorem 1.15 has a number of immediate and highly relevant geometric implications. Firstly, this means that if the metrics (respectively, densities) of two weighted Einstein structures on the same manifold coincide on an open subset, then they are equal on the whole manifold. Moreover, for any non-trivial WE manifold (M, g, f, m, μ), since f is non-constant and analytic in harmonic coordinates, the set $\tilde{M} = \{p \in M | (\nabla f)_p \neq 0\}$ of regular points of f is open and dense in M.

Part I

The geometry of weighted Einstein manifolds

Throughout Section 1.5, we went over several key aspects about smooth metric measure spaces, like their relation to important problems in geometric analysis and the kinds of weighted invariants that are of interest when working in a weighted context. Specifically for the Riemannian setting, we explained how SMMSs are defined as five-tuples (M, g, f, m, μ), where f is the density function and m and μ are parameters with certain geometric interpretations.

In particular, in Section 1.6 we presented a natural analogue of Einstein manifolds, weighted Einstein manifolds, coming from a relevant variational problem, and generalizing an important geometric notion such as quasi-Einstein manifolds. These are SMMSs whose weighted Schouten tensor (1.19) satisfies

$$P_f^m = \frac{1}{m+n-2}(\rho_f^m - J_f^m g) = \lambda g, \quad \text{for some } \lambda \in \mathbb{R}.$$

Motivated by the importance of these structures for the study of SMMSs, Part I of this dissertation is devoted to the analysis of the weighted Einstein equation under several weighted geometric conditions that arise naturally for manifods with density. In Chapter 2, we translate the condition of the harmonicity of the Weyl tensor (see [19,44] for results for quasi-Einstein manifolds under this assumption) to the weighted setting and study WE manifolds with $\operatorname{div}_f W_f^m = 0$. On the other hand, in Chapter 3 we tackle a problem from weighted conformal geometry (see [34,36]): The classification of WE manifolds which admit another WE structure in their conformal class. This translates a classical problem in Riemannian Geometry due to Brinkmann [10] to the weighted setting, and it entails a thorough study of a generalized Obata equation (2.22) involving the conformal factor u which relates both WE structures.

In both chapters, we start by performing a local study of the geometry of the SMMSs of interest. Then, we impose the condition of completeness in order to obtain global classification results. In Chapter 2, we assume all SMMSs are non-trivial (f non-constant), while in Chapter 3, we impose non-constancy on the conformal factor u. Thus, the analyticity of WE manifolds, which is guaranteed by Theorem 1.15 (see also Lemma 3.1), plays a key role by ensuring that the set of regular points of f or u is dense in the underlying manifold. As it turns out, these geometric conditions mean that the weighted analogues of the space forms that we defined in Section 1.5 naturally play an essential role. However, more involved structures are also admissible, many of which are not built on Einstein manifolds and arise as purely weighted objects without unweighted counterparts.

Chapter 2
Weighted Einstein manifolds with weighted harmonic Weyl tensor

In this chapter, we characterize the geometric structure of weighted Einstein manifolds with weighted harmonic Weyl tensor. The results in this chapter are contained in the article [24].

When discussing Einstein-type structures on semi-Riemannian manifolds, such as gradient Ricci solitons or quasi-Einstein manifolds, conditions on the Weyl tensor are often used as a means to obtain classification results. This is a very natural process, given that the Einstein-type equation provides information on the part of the curvature controlled by the Ricci tensor, so restrictions on the Weyl tensor usually provide additional information. The most natural one is local conformal flatness ($W = 0$). However, this is quite restrictive, so the weaker assumption $\operatorname{div} W = 0$ is often made instead in order to gain flexibility while meaningfully reducing the complexity of the problem. Indeed, this harmonicity condition has been used to study, for example, Ricci solitons [59, 93] and generalized QE manifolds [19, 44].

The equation $\operatorname{div} W = 0$ arises naturally in discussions of spaces which are conformally Einstein, i.e., semi-Riemannian manifolds (M, g) which locally admit a conformal transformation $\widehat{g} = e^{-2\sigma} g$ such that the resulting manifold is Einstein (see Section 1.4 for comments on the relation between a certain quasi-Einstein equation and this conformal problem). This is because the divergence of the standard Weyl tensor transforms under this conformal change as $\widehat{\operatorname{div}}\widehat{W} = \operatorname{div} W + (3-n)\iota_{\nabla\sigma} W$. Additionally, the Einstein condition implies the harmonicity of the Weyl tensor, since the Schouten tensor (1.1) of an Einstein manifold is Codazzi. Thus, if $\widehat{g}$ is Einstein, then $\operatorname{div} W + (3-n)\iota_{\nabla\sigma} W = 0$.

Therefore, a natural question when studying weighted Einstein (Riemannian) smooth metric measure spaces is how analogous conditions on the weighted Weyl tensor W_f^m given by (1.20) affect their geometry. Recall that the WE equation is the Einstein-type equation $P_f^m = \lambda g$ for some $\lambda \in \mathbb{R}$, where P_f^m is the weighted Schouten tensor (1.19). If we impose local conformal flatness in the weighted sense, i.e., $W_f^m = 0$, it follows that

$$R = \lambda g \owedge g,$$

that is, the underlying manifold has constant sectional curvature 2λ. This greatly limits the amount of admissible geometries for these SMMSs. Hence, we translate the harmonicity condition $\operatorname{div} W = 0$ to the weighted setting by considering the weighted divergence (1.22) and imposing the weighted harmonicity of the weighted Weyl tensor:

$$0 = \operatorname{div}_f W_f^m = \operatorname{div} W_f^m - \iota_{\nabla f} W_f^m = \operatorname{div} W_f^m - W_f^m(\nabla f, -, -, -).$$

For simplicity, we say that a SMMS such that $\operatorname{div}_f W_f^m = 0$ has *weighted harmonic Weyl tensor*. The purpose of this chapter is to study the geometric structure of weighted Einstein manifolds under this condition.

An important detail to point out is that this is not a direct translation of an unweighted geometric problem. Indeed, in contrast to the usual setting, for smooth metric measure spaces, the condition $\operatorname{div}_f W_f^m = 0$ does not follow from $P_f^m = \lambda g$, for any value of the dimensional parameter m. The following examples illustrate this fact appropriately.

Example 2.1. Let $m \in \mathbb{R}^+ - \{\frac{1}{2}\}$, and let $(\mathbb{R}^+ \times \mathbb{R}^3, g)$ be the 4-dimensional Riemannian manifold with local coordinates $(x_1, \ldots, x_4)$ and metric given by the only non-vanishing components $g(\partial_{x_i}, \partial_{x_i}) = x_1^{2m}$ for $i = 1, \ldots, 4$. The Ricci tensor is determined by the following non-vanishing components

$$\rho\left(\partial_{x_1}, \partial_{x_1}\right) = \frac{3m}{x_1^2}, \quad \rho\left(\partial_{x_i}, \partial_{x_i}\right) = \frac{m - 2m^2}{x_1^2} \text{ for } i > 1.$$

For the density function $f_m(x_1) = -2m(m+1)\log(x_1)$, and the parameter $\mu = 0$, the Bakry-Émery Ricci tensor is diagonal with $\rho_f^m\left(\partial_{x_i}, \partial_{x_i}\right) = \frac{m-4m^2-2m^3}{x_1^2}$ for every $i = 1, \ldots, n$. Now, direct computations show that

$$J_f^m = -m\left(2m^2 + 4m - 1\right) x_1^{-2(m+1)}$$

and that the corresponding SMMS is weighted Einstein with $P_f^m = 0$. However, its weighted Weyl tensor is not weighted harmonic. Indeed, the non-vanishing components of $\operatorname{div}_f W_f^m$ are $\operatorname{div}_f W_f^m(\partial_{x_i}, \partial_{x_1}, \partial_{x_i}) = \frac{2m(2m^2+m-1)}{x_1^3}$, $i > 1$ (up to symmetries).

The following SMMS in dimension 3 provides an example of weighted Einstein manifold with $\operatorname{div}_f W_f^m \neq 0$ for the remaining value $m = \frac{1}{2}$.

Example 2.2. Let $(\mathbb{R}^+ \times \mathbb{R}^2, g)$ be the 3-dimensional Riemannian manifold with the metric given by the only non-vanishing components $g(\partial_{x_i}, \partial_{x_i}) = x_1^{\frac{2}{3}(3-\sqrt{6})}$ for $i = 1, 2, 3$. The Ricci tensor is given by

$$\rho\left(\partial_{x_1}, \partial_{x_1}\right) = -\frac{2\left(\sqrt{6} - 3\right)}{3x_1^2}, \quad \rho\left(\partial_{x_i}, \partial_{x_i}\right) = \frac{\sqrt{6} - 2}{3x_1^2}.$$

For the density function $f(x_1) = -\sqrt{\frac{2}{3}}\log(x_1)$, and the parameters $m = \frac{1}{2}$ and $\mu = 0$, this manifold satisfies $\rho_f^{1/2} = 0$, hence it is quasi-Einstein, and $P_f^{1/2} = 0$. However, its weighted Weyl tensor is not harmonic in the weighted sense. The non-vanishing components of $\operatorname{div}_f W_f^{1/2}$ are $\operatorname{div}_f W_f^{1/2}(\partial_{x_i}, \partial_{x_1}, \partial_{x_i}) = \frac{4(\sqrt{6}-3)}{9x_1^3}$, $i > 1$ (up to symmetries).

Remark 2.3. Although several geometric conditions on SMMSs have a counterpart on formal warped products of the form (1.17), the ones that we are considering in this chapter do not in general. In fact, fix $m = 3$ in Example 2.1 and take the warped product $\mathbb{R}^+ \times \mathbb{R}^3 \times_v \mathbb{R}^3$, where $v(x_1) = x_1^8$. This warped product is not Einstein but has harmonic Weyl tensor. Thus, Example 2.1 further illustrates that a weighted Einstein SMMS does not give rise to an Einstein warped product (1.17) and that a warped product (1.17) with harmonic Weyl tensor does not induce a SMMS with weighted harmonic Weyl tensor.

Additionally, the weighted harmonicity condition on the weighted Weyl tensor does not induce, in general, a warped product (1.17) with harmonic Weyl tensor (cf. Remark 2.20). Indeed, for $\mu \neq 0$, consider the SMMS $(\mathbb{R}^+ \times_\varphi \mathbb{R}^3, f, 2, \mu)$, where $\varphi(t) = t^{\frac{1}{3}}$ and $f(t) = -\log(t)$ (hence $v(t) = t^{\frac{1}{2}}$). This SMMS has weighted harmonic Weyl tensor, but it is not weighted Einstein for any $\lambda \in \mathbb{R}$. However, the warped product $\mathbb{R}^+ \times_\varphi \mathbb{R}^3 \times_v F^2(\mu)$ does not have harmonic Weyl tensor.

Outline of the chapter

This chapter is broadly divided into three sections. First, we analyze the local structure of weighted Einstein manifolds (M, g, f, m, μ) with weighted harmonic Weyl tensor without further assumptions (Section 2.1) and prove the main local rigidity result (Theorem 2.14) which guarantees the splitting of the underlying manifold as a warped product $I \times_\varphi N$ around regular points of f. This result gives two cases (Einstein and non-Einstein) which are analyzed separately.

Thus, in Section 2.2, we focus on the Einstein case, which is less rigid than its non-Einstein counterpart. We describe both the warping and density functions φ and f, as well as the Einstein constant of the fiber N and the value of the parameter μ, to obtain Theorem 2.16, completing the local classification result around regular points of f. Moreover, since f is real analytic in harmonic coordinates by Theorem 1.15, the set of its regular points is dense in M. Therefore, Theorems 2.14 and Theorem 2.16 determine the local geometric features of an open dense subset of these SMMSs. In low dimensions, stronger rigidity results are provided in Corollary 2.18.

Finally, in Section 2.3, making use of the aforementioned analyticity of WE manifolds, we prove Theorem 2.23, a global rigidity result which states that there exist only four families of complete weighted Einstein SMMSs with weighted harmonic Weyl tensor (the three weighted space forms and an additional family of warped products).

2.1 Local structure

The objective in this section is to study the local geometric structure of weighted Einstein manifolds with weighted harmonic Weyl tensor, culminating with the proof of Theorem 2.14. The following family of examples will play a key role in the non-Einstein case of this result.

Example 2.4. Take a SMMS of the form $(I \times_\varphi N, g, f, \frac{1}{2}, 0)$, where $I \times_\varphi N$ is a warped product of an open interval $I \subset \mathbb{R}^+$ and a Ricci-flat manifold N. Now, set the warping and density functions

$$\varphi(t) = A(Bt)^{\frac{1}{n-1}}, \quad f(t) = -\log(Bt),$$

where t is the natural coordinate in $\mathbb{R}^+$ and $A, B \in \mathbb{R}^+$. The Ricci tensor of $I \times_\varphi N$ is given by

$$\rho(\partial_t, \partial_t) = \frac{(n-2)}{(n-1)t^2}, \quad \rho(\partial_t, X) = 0, \quad \rho(X, Y) = 0,$$

for any $X, Y \in \mathfrak{X}(N)$. Moreover, the Bakry-Émery Ricci tensor is given by

$$\begin{aligned} \rho_f^{1/2}(\partial_t, \partial_t) &= -\tfrac{1}{(n-1)t^2}, \\ \rho_f^{1/2}(\partial_t, X) &= 0, \\ \rho_f^{1/2}(X, Y) &= -\tfrac{A^2(Bt)^{\frac{2}{n-1}}}{(n-1)t^2} g^N(X, Y), \end{aligned}$$

for any $X, Y \in \mathfrak{X}(N)$. The weighted Schouten scalar is $J_f^{1/2} = -\frac{1}{(n-1)t^2}$. Hence, these SMMSs satisfy $P_f^{1/2} = 0$ and $\operatorname{div}_f W_f^{1/2} = 0$, so they are weighted Einstein and have weighted harmonic Weyl tensor. However, the scalar curvature of the underlying manifold is non-constant, $\tau = \frac{(n-2)}{(n-1)t^2}$, therefore they are not Einstein. In particular, if N is the usual flat Euclidean space $\mathbb{R}^{n-1}$ with coordinates $(x_1, \ldots, x_{n-1})$, the weighted Einstein tensor $W_f^{1/2}$ presents the following non-zero components (up to symmetries):

$$\begin{aligned} W_f^{1/2}(\partial_t, \partial_{x_i}, \partial_t, \partial_{x_i}) &= \tfrac{(n-2)\varphi(t)^2}{(n-1)^2 t^2}, \\ W_f^{1/2}(\partial_{x_i}, \partial_{x_j}, \partial_{x_i}, \partial_{x_j}) &= -\tfrac{\varphi(t)^4}{(n-1)^2 t^2}, \quad i \neq j. \end{aligned}$$

Note that $(I \times_\varphi N, g)$ is an incomplete manifold, and it cannot be isometrically embedded (as an open set) in any complete manifold (see Lemma 2.21).

Remark 2.5. Notice that the value $m = \frac{1}{2}$ is special in the family of SMMSs given by Example 2.1. Indeed, the SMMS $(\mathbb{R}^+ \times \mathbb{R}^3, g, f_{1/2}, \frac{1}{2}, 0)$ corresponds to Example 2.4 for $t = \frac{2}{3}(x_1)^{3/2}$, $A = 1$ and $B = \frac{3}{2}$. The weighted Schouten tensor and the weighted divergence of its weighted Weyl tensor vanish ($P_f^{1/2} = 0$ and $\operatorname{div}_f W_f^{1/2} = 0$), but the weighted Weyl tensor itself does not. Its non-zero components are $W_{f_{1/2}}^{1/2}(\partial_{x_i}, \partial_{x_1}, \partial_{x_i}, \partial_{x_1}) = \frac{1}{2x_1}$ for $i \neq 1$, and $W_{f_{1/2}}^{1/2}(\partial_{x_i}, \partial_{x_j}, \partial_{x_i}, \partial_{x_j}) = -\frac{1}{4x_1}$ for $1 < i < j$ (up to symmetries).

In order to prove the local classification result, we begin by computing some geometric formulas and proving the local splitting of these SMMSs as warped products with Einstein fiber. Afterwards, we find necessary and sufficient conditions, in terms of an overdetermined system of ODEs, for a SMMS to satisfy both the weighted Einstein and the weighted harmonicity conditions. Solving this system yields the theorem.

The following lemma, which was adapted from the derivation in [41] to present the form given in [34], provides a key identity involving the trace of the weighted Cotton tensor (1.21) and the scalar $Y_f^m = J_f^m - \operatorname{tr} P_f^m$.

Lemma 2.6 [41]. *Let (M, g, f, m, μ) be a SMMS. Then,*

$$\operatorname{tr}(dP_f^m) = -\left(\iota_{\nabla f} P_f^m + dY_f^m - \frac{1}{m} Y_f^m df\right),$$

where the trace is taken over the first and third arguments of dP_f^m.

Proof. Consider the change of variable $v = e^{-f/m}$. With this, the Bakry-Émery Ricci tensor (1.16) becomes

$$\rho_f^m = \rho - mv^{-1}\operatorname{Hes}_v,$$

and the weighted scalar curvature (1.18) is

$$\tau_f^m = \tau - 2mv^{-1}\Delta v + m(m-1)v^{-2}(\mu - \|\nabla v\|^2). \tag{2.1}$$

On the one hand, let $A = -\frac{1}{n}(\Delta v + Jv)$, where $J = \frac{\tau}{2(n-1)}$ is the usual Schouten scalar. On the other hand, let $B = \mu - 2Av - \|\nabla v\|^2$. Then, the weighted scalar curvature can be written as

$$\begin{aligned} \tau_f^m &= \tfrac{m+n-1}{n-1}\tau + 2mnv^{-1}A + m(m-1)v^{-2}(\mu - \|\nabla v\|^2) \\ &= \tfrac{m+n-1}{n-1}\tau + 2m(m+n-1)v^{-1}A + m(m-1)v^{-2}B. \end{aligned}$$

Inserting his expression into the weighted Schouten tensor (1.19) yields

$$\begin{aligned} P_f^m &= \tfrac{1}{m+n-2}(\rho - mv^{-1}\operatorname{Hes}_v - J_f^m g) \\ &= \tfrac{1}{m+n-2}\left(\rho - Jg - mv^{-1}\operatorname{Hes}_v - mv^{-1}\left(A + \tfrac{m-1}{2(m+n-1)}v^{-1}B\right)g\right) \\ &= \tfrac{n-2}{m+n-2}P - \tfrac{mv^{-1}}{m+n-2}\left(\operatorname{Hes}_v + Ag + \tfrac{m-1}{2(m+n-1)}v^{-1}Bg\right), \end{aligned} \tag{2.2}$$

where $P = \frac{1}{n-2}(\rho - Jg)$ is the usual Schouten tensor. The next step is to take this expression and compute the weighted Cotton tensor (1.21), which is the skew-symmetrization of ∇P_f^m. To that end, note that

$$(\nabla_Y \operatorname{Hes}_h)(X, Z) - (\nabla_Z \operatorname{Hes}_h)(X, Y) = R(\nabla h, X, Z, Y) = -\iota_{\nabla h} R(X, Y, Z),$$

and for any function $F \in C^\infty(M)$,

$$\begin{aligned} (\nabla_Y Fg)(X, Z) - (\nabla_Z Fg)(X, Y) &= dF(Y)g(X, Z) - dF(Z)g(X, Y) \\ &= -g \wedge dF(X, Y, Z). \end{aligned}$$

where $\wedge$ acts on a 1-form ω and a tensor T of type $(0, k)$ as

$$T \wedge \omega(\ldots, Y, Z) = \omega(Z)T(\ldots, Y) - \omega(Y)T(\ldots, Z).$$

Thus, it follows from (2.2) and the definitions of the Cotton (1.2) and weighted Cotton (1.21) tensors that

$$\begin{aligned} dP_f^m &= \tfrac{1}{m+n-2}dP - \tfrac{mv^{-2}}{m+n-2}\left(\operatorname{Hes}_v + Ag\right)\wedge dv \\ &\quad + \tfrac{mv^{-1}}{m+n-2}\left(\iota_{\nabla v}R + g\wedge dA + vg\wedge d\left(\tfrac{m-1}{2(m+n-1)}v^{-2}B\right)\right) \\ &= \tfrac{1}{m+n-2}dP + \tfrac{mv^{-1}}{m+n-2}\Big(\iota_{\nabla v}R + g\wedge dA \\ &\quad - v^{-1}(\operatorname{Hes}_v + Ag)\wedge dv + vg\wedge d\left(\tfrac{m-1}{2(m+n-1)}v^{-2}B\right)\Big). \end{aligned} \tag{2.3}$$

Moreover, since $R = P \owedge g + W$, we have

$$\begin{aligned} (P \owedge g)(\nabla v, X, Y, Z) &= P(\nabla v, Y)g(X,Z) + P(X,Z)g(\nabla v, Y) \\ &\quad - P(\nabla v, Z)g(X,Y) - P(X,Y)g(\nabla v, Z) \\ &= -(P\wedge dv + g\wedge \iota_{\nabla v}P)(X,Y,Z), \end{aligned}$$

so (2.3) becomes

$$\begin{aligned} dP_f^m &= \tfrac{1}{m+n-2}dP + \tfrac{mv^{-1}}{m+n-2}\Big(\iota_{\nabla v}W + g\wedge(dA - \iota_{\nabla v}P) \\ &\quad - v^{-1}(vP + \operatorname{Hes}_v + Ag)\wedge dv + vg\wedge d\left(\tfrac{m-1}{2(m+n-1)}v^{-2}B\right)\Big). \end{aligned} \tag{2.4}$$

Now, we can take the trace of this expression. To that end, note that if ω is a 1-form and T is a tensor of type $(0,2)$, in an orthonormal frame $\{E_1,\dots,E_n\}$,

$$\begin{aligned} \operatorname{tr}(T\wedge\omega)(X) &= \textstyle\sum_{i=1}^n T\wedge\omega(E_i, X, E_i) \\ &= \textstyle\sum_{i=1}^n T(E_i, X)\omega(E_i) - \sum_{i=1}^n T(E_i,E_i)\omega(X) \\ &= T(X,\omega^\#) - \omega(X)\operatorname{tr}T, \end{aligned}$$

where $\omega^\#$ is the metrically equivalent vector of the 1-form ω and we have taken the trace over the first and third arguments of $T\wedge\omega$. In particular, $\operatorname{tr}(g\wedge\omega) = -(n-1)\omega$. Moreover, the Cotton and Weyl tensors are traceless, and by the definition of A, $\operatorname{tr}(vP + \operatorname{Hes}_v + Ag) = 0$. Thus, the trace of equation (2.4) becomes

$$\begin{aligned} \operatorname{tr} dP_f^m = \tfrac{mv^{-1}}{m+n-2}\Big(&(n-2)\iota_{\nabla v}P - (n-1)dA - v^{-1}\iota_{\nabla v}(\operatorname{Hes}_v + Ag) \\ &- \tfrac{(m-1)(n-1)}{2(m+n-1)}vd\left(v^{-2}B\right)\Big). \end{aligned}$$

Using (2.2), we write this equation in terms of the weighted Schouten tensor:

$$\begin{aligned} \operatorname{tr} dP_f^m &= mv^{-1}\iota_{\nabla v}P_f^m + \tfrac{mv^{-1}}{m+n-2}\Big((m-1)v^{-1}(\iota_{\nabla v}\operatorname{Hes}_v + Adv) \\ &\quad - (n-1)dA + \tfrac{m(m-1)}{2(m+n-1)}v^{-2}Bdv - \tfrac{(m-1)(n-1)}{2(m+n-1)}vd\left(v^{-2}B\right)\Big). \end{aligned}$$

We also have $dB = -d(2vA + \|\nabla v\|^2) = -2(Adv + vdA + \iota_{\nabla v}\operatorname{Hes}_v)$, so

$$(m-1)v^{-1}(\iota_{\nabla v}\operatorname{Hes}_v + Adv) = -(m-1)\left(dA + \frac{1}{2}v^{-1}dB\right).$$

Substituting this value into the equation above and grouping the terms involving B into a single differential, we arrive at the following expression for the trace of the weighted Cotton tensor in terms of the weighted Schouten tensor P_f^m and the auxiliary functions A and B:

$$\operatorname{tr} dP_f^m = mv^{-1}\left(\iota_{\nabla v}P_f^m - dA - \frac{(m+2n-2)(m-1)}{2(m+n-1)(m+n-2)}d(v^{-1}B)\right). \tag{2.5}$$

Now, we need to compute $Y_f^m = J_f^m - \operatorname{tr} P_f^m$ and its differential. To that end, note that, from (1.19),

$$\operatorname{tr} P_f^m = \frac{1}{m+n-2}(\tau - mv^{-1}\Delta v - nJ_f^m),$$

so using (2.1) we obtain

$$\begin{aligned} Y_f^m &= \tfrac{(m+2n-2)J_f^m - (\tau - mv^{-1}\Delta v)}{m+n-2} \\ &= -\tfrac{mv^{-1}}{2(m+n-1)(m+n-2)}\big(\tau v + 2(n-1)\Delta v \\ &\qquad -(m-1)(m+2n-2)v^{-1}(\mu - \|\nabla v\|^2)\big) \\ &= mv^{-1}\left(A + \tfrac{(m+2n-2)(m-1)}{2(m+n-1)(m+n-2)}v^{-1}B\right). \end{aligned}$$

Therefore,

$$dY_f^m = -Y_f^m v^{-1}dv + mv^{-1}\left(dA + \frac{(m+2n-2)(m-1)}{2(m+n-1)(m+n-2)}d(v^{-1}B)\right),$$

and equation (2.5) becomes

$$\begin{aligned} \operatorname{tr} dP_f^m &= mv^{-1}\iota_{\nabla v}P_f^m - dY_f^m - Y_f^m v^{-1}dv \\ &= -\left(\iota_{\nabla f}P_f^m + dY_f^m - \tfrac{1}{m}Y_f^m df\right), \end{aligned}$$

where we have used the fact that, since $v = e^{-f/m}$, we have $df = -mv^{-1}dv$. □

The following lemma gives an additional key formula for the weighted divergence of the weighted Weyl tensor W_f^m on a SMMS which is weighted Einstein.

Lemma 2.7. *Let (M^n, g, f, m, μ) be a SMMS such that $P_f^m = \lambda g$. Then, the following equation is satisfied for all $X, Y, Z \in \mathfrak{X}(M)$:*

$$\begin{aligned} \operatorname{div}_f W_f^m(X,Y,Z) &= \left(\tfrac{1}{m}Y_f^m + \lambda\right)\{df(Y)g(X,Z) - df(Z)g(X,Y)\} \\ &\quad -\tfrac{1}{m}\{df(Y)\operatorname{Hes}_f(X,Z) - df(Z)\operatorname{Hes}_f(X,Y)\}. \end{aligned} \tag{2.6}$$

Proof. Since the SMMS satisfies $P_f^m = \lambda g$, its weighted Schouten tensor P_f^m is Codazzi, so the weighted Cotton tensor (1.21) vanishes. Thus, the expression given by Lemma 2.6 transforms into $dJ_f^m = dY_f^m = \left(\frac{1}{m}Y_f^m - \lambda\right) df$. We calculate the covariant derivative of the Bakry-Émery Ricci tensor,

$$\begin{aligned}(\nabla_Y \rho_f^m)(X,Z) &= (\nabla_Y \rho)(X,Z) + g(\nabla_Y \nabla_Z \nabla f, X) - g(\nabla_{\nabla_Y Z}\nabla f, X) \\ &\quad -\tfrac{1}{m}\left\{df(X)\operatorname{Hes}_f(Y,Z) + df(Z)\operatorname{Hes}_f(X,Y)\right\}.\end{aligned}$$

Furthermore, the weighted Einstein equation reads $\rho_f^m = \{(m+2n-2)\lambda + Y_f^m\}g$, so we also have $(\nabla_Y \rho_f^m)(X,Z) = \left(\frac{1}{m}Y_f^m - \lambda\right) df(Y) g(X,Z)$. We now take the difference $(\nabla_Y \rho_f^m)(X,Z) - (\nabla_Z \rho_f^m)(X,Y)$ to find (cf. [19])

$$\begin{aligned}R(\nabla f, X, Y, Z) &= \left(\tfrac{1}{m}Y_f^m - \lambda\right)\left\{df(Z)g(X,Y) - df(Y)g(X,Z)\right\} \\ &\quad +(\nabla_Y \rho)(X,Z) - (\nabla_Z \rho)(X,Y) \\ &\quad +\tfrac{1}{m}\left\{df(Y)\operatorname{Hes}_f(X,Z) - df(Z)\operatorname{Hes}_f(X,Y)\right\}.\end{aligned} \tag{2.7}$$

Finally, since $P_f^m = \lambda g$, we have $W_f^m = R - \lambda g \owedge g$. Hence,

$$\begin{aligned}\operatorname{div}_f W_f^m(X,Y,Z) &= \operatorname{div} W_f^m(X,Y,Z) - \iota_{\nabla f} W_f^m(X,Y,Z) \\ &= \operatorname{div} R(X,Y,Z) - R(\nabla f, X, Y, Z) \\ &\quad +2\lambda\{df(Y)g(X,Z) - df(Z)g(X,Y)\}.\end{aligned} \tag{2.8}$$

Since $\operatorname{div} R(X,Y,Z) = (\nabla_Y \rho)(X,Z) - (\nabla_Z \rho)(X,Y)$, a combination of (2.7) and (2.8) yields equation (2.6). □

We are now ready to prove a first rigidity result, concerning the warped product structure of these SMMSs around regular points of f.

Lemma 2.8. *Let (M^n, g, f, m, μ) be a SMMS with $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$. Let $p \in M$ be a regular point of f. Then, there exists a Riemannian isometry between a neighborhood $\mathcal{U}$ of p in M and a warped product $I \times_\varphi N$, where $I \subset \mathbb{R}$ is an open interval, N is an $(n-1)$-dimensional Einstein manifold, and ∇f is tangent to I.*

Proof. Since p is a regular point of f, $\nabla f \neq 0$ in a neighborhood of p. Thus, consider an orthonormal frame $\mathcal{B} = \{E_1, \ldots, E_n\}$ around p, where $E_1 = \nabla f / \|\nabla f\|$. Since W_f^m is weighted harmonic, the left-hand side of equation (2.6) vanishes. Consequently, we can take $X = Z = E_1$ and $Y = E_i$, $i \neq 1$, to find

$$\operatorname{Hes}_f(E_1, E_i) = 0, \quad i \neq 1, \tag{2.9}$$

which shows that the integral submanifold of ∇f is totally geodesic. Furthermore, taking $X = E_i$, $Y = E_j$, $Z = E_1$, with $i, j \neq 1$, equation (2.6) yields

$$\operatorname{Hes}_f(E_i, E_j) = (Y_f^m + m\lambda)\delta_{ij}, \quad i, j \neq 1. \tag{2.10}$$

It follows that the level hypersurfaces of f around p are totally umbilical. Consequently, (M, g) splits in a neighborhood $\mathcal{U}$ of p as a twisted product $I \times_{\tilde{\varphi}} N$, where $I \subset \mathbb{R}$ is an open interval, for some twisting function $\tilde{\varphi}$ on $I\times N$ (see Theorem 1.3). Moreover, from (2.10), the mean curvature vector field of each leaf of the fiber is $H = -\frac{Y_f^m+m\lambda}{\|\nabla f\|}E_1$, which is parallel in the normal bundle ($\nabla^\perp H = 0$, where $\nabla^\perp$ is the normal connection). Indeed, for $i \neq 1$,

$$\begin{aligned}\nabla^\perp_{E_i} H &= -g\left(\nabla_{E_i}\tfrac{Y_f^m+m\lambda}{\|\nabla f\|}E_1, E_1\right)E_1 \\ &= -E_i\left(\tfrac{Y_f^m+m\lambda}{\|\nabla f\|}\right)E_1 - \tfrac{Y_f^m+m\lambda}{\|\nabla f\|^2}\operatorname{Hes}_f(E_1,E_i)E_1 = 0,\end{aligned}$$

where we have used (2.9) and the fact that $dY_f^m = \left(\frac{1}{m}Y_f^m - \lambda\right)df$ by Lemma 2.6. Therefore, the leaves of the fiber are spherical and, by Theorem 1.3, the twisted product reduces to a warped product $I\times_\varphi N$ for some function φ on I. Alternatively, since $\rho_f^m = \{(m+2n-2)\lambda + Y_f^m\}g$ and Hes_f diagonalizes in the frame $\mathcal{B}$, so does the Ricci tensor. Hence, the vanishing condition on mixed terms for ρ given in Theorem 1.4 is satisfied and also implies the reduction of the twisted product to the warped product.

Now we show that N is Einstein as follows. Let t be a coordinate parameterizing I by arc length, and consider the local orthonormal frame $\{\partial_t, E_2,\dots,E_n\}$. Note that $E_2,\dots,E_n$ are tangent to N. Thus, from the weighted Einstein condition and (2.10), we get that

$$\begin{aligned}\rho(E_i,E_j) &= \rho_f^m(E_i,E_j) - \operatorname{Hes}_f(E_i,E_j) \\ &= \{(m+2n-2)\lambda + Y_f^m\}\delta_{ij} - (Y_f^m + m\lambda)\delta_{ij} \\ &= 2(n-1)\lambda\delta_{ij}\end{aligned}$$

for $i,j = 2,\dots,n$. Moreover, consider the basis $\{\bar{E}_i = \varphi E_i\}_{i=2,\dots,n}$ which is orthonormal on N. From the expression of the Ricci tensor of a warped product (1.6), we have

$$\begin{aligned}\rho^N(\bar{E}_i,\bar{E}_j) &= \rho(\bar{E}_i,\bar{E}_j) + g(\bar{E}_i,\bar{E}_j)\left(\tfrac{\varphi''}{\varphi} + (n-2)\tfrac{(\varphi')^2}{\varphi^2}\right) \\ &= \varphi^2\left(2(n-1)\lambda + \tfrac{\varphi''}{\varphi} + (n-2)\tfrac{(\varphi')^2}{\varphi^2}\right)\delta_{ij}.\end{aligned} \tag{2.11}$$

Since $\rho^N(\bar{E}_i,\bar{E}_j)$ is a function defined on the fiber, it does not depend on t, which is a coordinate of the base. Hence, $\rho^N = \beta g^N$ for some $\beta\in\mathbb{R}$ and N is Einstein. □

Remark 2.9. Note that the local splitting given by Lemma 2.8 is reminiscent of the result found by Catino [44], in the unweighted Riemannian setting, for generalized quasi-Einstein manifolds with harmonic Weyl tensor such that $\iota_{\nabla f}W$=0. However, we do not require both summands in $\operatorname{div}_f W_f^m$ to vanish, but merely that they cancel out (see Example 2.4).

Remark 2.10. Warped products of the form $I\times_\varphi N$ with N Einstein have harmonic Weyl tensor (see Theorem 1.6 and also [3, 16.26(i)]). Hence, it follows from the local structure described in Lemma 2.8 and the density of regular points of f that the underlying manifold of a SMMS which is weighted Einstein and has weighted harmonic Weyl tensor is not necessarily Einstein, but does have harmonic Weyl tensor in the unweighted sense.

By Lemma 2.8, whenever we are working locally around any regular point of f, we can assume without loss of generality that our SMMSs are built on a warped product of the form $I \times_\varphi N$ with the density function defined on I. Such a warped product is not weighted Einstein with weighted harmonic Weyl tensor in general. Indeed, equation (2.11) imposes a constraint on the warping function. The next result provides necessary and sufficient conditions that identify these SMMSs in terms of an overdetermined system of ODEs.

Lemma 2.11. *Let $(I \times_\varphi N, g, f, m, \mu)$ be an n-dimensional warped product SMMS where I is an open interval, ∇f is tangent to I, and such that $\rho^N = \beta g^N$ for some $\beta \in \mathbb{R}$. Then $P_f^m = \lambda g$, for $\lambda \in \mathbb{R}$, and $\operatorname{div}_f W_f^m = 0$ if and only if the following system of ODEs is satisfied:*

$$0 = \beta - \varphi''\varphi - (n-2)(\varphi')^2 - 2(n-1)\lambda\varphi^2, \tag{2.12}$$

$$0 = f'' - (n-1)\frac{\varphi''}{\varphi} - \frac{1}{m}(f')^2 - \frac{\varphi' f'}{\varphi} - 2(n-1)\lambda, \tag{2.13}$$

$$0 = \frac{\varphi' f'}{\varphi} + (n-m)\lambda - J_f^m, \tag{2.14}$$

where the weighted Schouten scalar J_f^m is given by

$$\begin{aligned} 2(m+n-1)J_f^m &= (n-1)\tfrac{\beta-(n-2)(\varphi')^2}{\varphi^2} + 2(n-1)\tfrac{\varphi' f' - \varphi''}{\varphi} \\ &\quad + 2f'' - \tfrac{1+m}{m}(f')^2 + m(m-1)e^{2f/m}\mu. \end{aligned} \tag{2.15}$$

Proof. Let t be a local coordinate parameterizing I by arc length. We work in the local orthonormal frame $\mathcal{B} = \{\partial_t, E_2, \dots, E_n\}$. Using the warped product expressions for the connection (1.4) and the Ricci tensor (1.6), and (2.11), the Bakry-Émery Ricci tensor (1.16) takes the form

$$\begin{aligned} \rho_f^m(\partial_t, \partial_t) &= -(n-1)\tfrac{\varphi''}{\varphi} + f'' - \tfrac{1}{m}(f')^2, \\ \rho_f^m(\partial_t, E_i) &= 0, \\ \rho_f^m(E_i, E_j) &= \left(\tfrac{\beta}{\varphi^2} - \tfrac{\varphi''}{\varphi} - (n-2)\tfrac{(\varphi')^2}{\varphi^2} + \tfrac{\varphi' f'}{\varphi}\right)\delta_{ij}. \end{aligned}$$

Thus, the fact that $(I \times_\varphi N, g, f, m, \mu)$ is weighted Einstein, using (1.24) to express it as $\rho_f^m = ((m+n-2)\lambda + J_f^m)g$, is equivalent to the following two equations:

$$-(n-1)\frac{\varphi''}{\varphi} + f'' - \frac{1}{m}(f')^2 = (m+n-2)\lambda + J_f^m, \tag{2.16}$$

$$\frac{\beta}{\varphi^2} - \frac{\varphi''}{\varphi} - (n-2)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi' f'}{\varphi} = (m+n-2)\lambda + J_f^m. \tag{2.17}$$

On the one hand, a direct calculation on the warped product yields

$$\operatorname{Hes}_f(E_i, E_i) = \frac{\varphi' f'}{\varphi}.$$

On the other hand, if $\operatorname{div}_f W_f^m = 0$, then equation (2.10) is also satisfied, giving $\operatorname{Hes}_f(E_i, E_i) = (m-n)\lambda + J_f^m$, where we have used that $Y_f^m = J_f^m - n\lambda$ on a weighted Einstein SMMS. Hence, $\frac{\varphi' f'}{\varphi} = (m-n)\lambda + J_f^m$, which is equation (2.14). Now, using (2.14) to substitute the term J_f^m in (2.17) and (2.16) yields, respectively, (2.12) and (2.13). The form of the weighted Schouten scalar J_f^m follows from a direct computation of the weighted scalar curvature (1.18), using the expression (1.7) of the scalar curvature of a warped product $I \times_\varphi N$.

Conversely, if (2.12)-(2.14) are satisfied, then (2.16) and (2.17) hold and the equation $P_f^m = \lambda g$ is also satisfied. Thus, we only need to check that they are sufficient conditions for the weighted harmonicity condition $\operatorname{div}_f W_f^m = 0$. To that end, we use the expression given by (2.6) for $\operatorname{div}_f W_f^m$, which applies to any weighted Einstein manifold. By the symmetries of this tensor, we only need to analyze the following terms:

$$\begin{aligned}
\operatorname{div}_f W_f^m(\partial_t, E_i, E_j) &= 0, \\
\operatorname{div}_f W_f^m(E_i, E_j, E_k) &= 0, \\
\operatorname{div}_f W_f^m(\partial_t, E_i, \partial_t) &= \tfrac{f'}{m} \operatorname{Hes}_f(\partial_t, E_i) = 0, \\
\operatorname{div}_f W_f^m(E_i, \partial_t, E_j) &= \left(\tfrac{1}{m} Y_f^m + \lambda\right) f'\delta_{ij} - \tfrac{1}{m} f' \operatorname{Hes}_f(E_i, E_j) \\
&= \tfrac{1}{m}\left(J_f^m + (m-n)\lambda - \tfrac{\varphi' f'}{\varphi}\right) f'\delta_{ij} \overset{(2.14)}{=} 0.
\end{aligned}$$

Hence, equations (2.12)-(2.14) are sufficient for the warped product $I \times_\varphi N$ to be a weighted Einstein manifold with weighted harmonic Weyl tensor. □

Remark 2.12. Consider a warped product SMMS $(I \times_\varphi N, g, f, m, \mu)$ satisfying (2.12)–(2.14) as in Lemma 2.11. Then, the Ricci tensor is readily determined using equations (2.12) and (1.6):

$$\rho(\partial_t, \partial_t) = -(n-1)\frac{\varphi''}{\varphi}, \quad \rho(\partial_t, X) = 0, \quad \rho(X, Y) = 2(n-1)\lambda g(X, Y),$$

for any $X, Y \in \mathfrak{X}(N)$. Therefore, the underlying manifold is Einstein if and only if $\varphi'' = -2\lambda\varphi$. As we will shortly show, this is one of two cases which are allowed for this kind of manifold, with the geometry of the non-Einstein case being very heavily restricted.

Lemma 2.13. *Let $(I \times_\varphi N, g, f, m, \mu)$ be an n-dimensional warped product SMMS where I is an open interval, ∇f is tangent to I, and such that $\rho^N = \beta g^N$ for some $\beta \in \mathbb{R}$. Let t be a local coordinate parameterizing I by arc length. If $P_f^m = \lambda g$ for some $\lambda \in \mathbb{R}$ and $\operatorname{div}_f W_f^m = 0$, then either $I \times_\varphi N$ is Einstein, or $\varphi(t) = Ae^{-\frac{f(t)}{n-1}}$, for some $A \in \mathbb{R}^+$.*

Proof. We adopt the notation in Lemma 2.11 and keep working in a local orthonormal frame $\mathcal{B} = \{\partial_t, E_2, \dots, E_n\}$. The weighted Einstein condition and the harmonicity of the weighted Weyl tensor guarantee that

$$0 = \operatorname{div}_f W_f^m(E_i, \partial_t, E_i) = \operatorname{div} R(E_i, \partial_t, E_i) - R(\nabla f, E_i, \partial_t, E_i) + 2\lambda f'. \tag{2.18}$$

We will use this to obtain an additional ODE. Firstly, consider the divergence of the Riemann curvature tensor, given by $\operatorname{div} R(X, Y, Z) = (\nabla_Y \rho)(X, Z) - (\nabla_Z \rho)(X, Y)$. On the one hand,

$(\nabla_{\partial_t}\rho)(E_i,E_i)=\partial_t(\rho(E_i,E_i))-2\rho(\nabla_{\partial_t}E_i,E_i)$. But $\rho(E_i,E_i)=2(n-1)\lambda$ (see Remark 2.12), so $\partial_t(\rho(E_i,E_i))=0$. Moreover, $g(\nabla_{\partial_t}E_i,E_i)=\frac{1}{2}\partial_t(g(E_i,E_i))=0$, so $\nabla_{\partial_t}E_i\perp E_i$. Since $\mathcal{B}$ is a frame of eigenvectors for the Ricci operator (see the proof of Lemma 2.8), we have $\rho(\nabla_{\partial_t}E_i,E_i)=0$. On the other hand, we use the expression of the connection for a warped product (1.4) to compute

$$\begin{aligned}(\nabla_{E_i}\rho)(\partial_t,E_i) &= -\rho(\nabla_{E_i}\partial_t,E_i)-\rho(\partial_t,\nabla_{E_i}E_i)\\ &= \tfrac{\varphi'}{\varphi}\left(\rho(\partial_t,\partial_t)-\rho(E_i,E_i)\right),\end{aligned}$$

so

$$\operatorname{div}R(E_i,\partial_t,E_i)=\frac{\varphi'}{\varphi}(\rho(E_i,E_i)-\rho(\partial_t,\partial_t))=(n-1)\frac{\varphi'(\varphi''+2\lambda\varphi)}{\varphi^2}.$$

Additionally, the curvature term $R(\nabla f,E_i,\partial_t,E_i)$ takes the form

$$R(\nabla f,E_i,\partial_t,E_i)=-\frac{\operatorname{Hes}_\varphi(\partial_t,\nabla f)}{\varphi}=-\frac{\varphi''f'}{\varphi}$$

by equation (1.5). With this, equation (2.18) becomes

$$\frac{1}{\varphi^2}(\varphi''+2\lambda\varphi)(\varphi f'+(n-1)\varphi')=0.$$

Thus, on a suitable open set, $\varphi''+2\lambda\varphi=0$, or $\varphi f'+(n-1)\varphi'=0$. In the first case, the underlying manifold is Einstein (see Remark 2.12). In the second one, we solve the ODE to get $\varphi(t)=Ae^{-\frac{f(t)}{n-1}}$, for $A\in\mathbb{R}^+$. Moreover, by the real analyticity of the metric (see Theorem 1.15), if $I\times_\varphi N$ is Einstein in an open set, then it is Einstein everywhere. Thus, the result extends to the whole product $I\times_\varphi N$. □

Lemma 2.13 reduces our study to only two possibilities. We will discuss the Einstein case in detail in Section 2.2, but for now we give the main local rigidity result of this chapter.

Theorem 2.14. *Let (M^n,g,f,m,μ) be a SMMS such that $P_f^m=\lambda g$ and $\operatorname{div}_f W_f^m=0$. Then, for each regular point p of f, there exists a Riemannian isometry between a neighborhood $\mathcal{U}$ of p and a warped product $I\times_\varphi N$, where $I\subset\mathbb{R}$ is an open interval, N is an $(n-1)$-dimensional Einstein manifold, and ∇f is tangent to I. Moreover, one of the following conditions holds:*

1. *$I\times_\varphi N$ is Einstein with $\rho=2(n-1)\lambda g$.*
2. *$(\mathcal{U},g\big|_U,f\big|_U,m,\mu)$ is isometric to $(I\times_\varphi N,g,f,\frac{1}{2},0)$ as given in Example 2.4.*

Proof. Let (M^n,g,f,m,μ) be a SMMS such that $P_f^m=\lambda g$ and $\operatorname{div}_f W_f^m=0$. By Lemma 2.8, around every regular point of f there exists a Riemannian isometry between a neighborhood $\mathcal{U}$ and a warped product of the form $I\times_\varphi N$, where $I\subset\mathbb{R}$ is an open interval, ∇f is tangent to I, and $\rho^N=\beta g^N$ for some $\beta\in\mathbb{R}$. Using Lemma 2.13, we have that either $I\times_\varphi N$ is Einstein and Theorem 2.14 (1) holds, or the warping and density functions are related by $\varphi(t)=Ae^{-\frac{f(t)}{n-1}}$ for some $A \in \mathbb{R}^+$, where t is a coordinate parameterizing I by arc length. Assume that the latter is satisfied. Then, the necessary and sufficient conditions given by Lemma 2.11 take on a simpler form. Indeed, equation (2.12) becomes

$$0 = \beta - \frac{A^2}{n-1} e^{\frac{-2f}{n-1}} (2(n-1)^2\lambda + (f')^2 - f''), \tag{2.19}$$

while equation (2.13) turns into

$$(f')^2 = 2m(f'' - (n-1)\lambda). \tag{2.20}$$

Now, taking the derivative of (2.20) yields $f^{(3)} = \frac{f'f''}{m}$. Substituting this expression into the derivative of (2.19) and using (2.20), we have

$$0 = \frac{A^2 e^{-2f/(n-1)} f'}{m(n-1)^2} (4m(m-n+1)(n-1)\lambda - (4m^2 - 2mn + n - 1)f''). \tag{2.21}$$

Note that the factor $4m^2 - 2mn + n - 1$ vanishes if and only if $m = \frac{1}{2}$ or $m = \frac{1}{2}(n-1)$. This results in three cases we need to analyze separately.

$m \notin \{\frac{1}{2}, \frac{1}{2}(n-1)\}$: Let $B = \frac{4m(m-n+1)(n-1)\lambda}{4m^2-2mn+n-1}$. Then, from (2.21) it follows that $f(t) = \frac{B}{2}t^2 + Ct + D$, where $C, D \in \mathbb{R}$. Hence, we have $0 = f^{(3)} = \frac{f'f''}{m}$, but $f' \neq 0$, so $f'' = B$ must vanish. Since $m > 0$ and $n \geq 3$, $B = 0$ if and only if $\lambda = 0$ or $m = n-1$.

If $\lambda = 0$, from (2.20), we have that $f' = 0$, which is not possible.

If $m = n-1$, then from (2.20) we deduce $C^2 = -2(n-1)^2\lambda$. With this, it follows from the expression of φ that $\frac{\varphi''}{\varphi} = -2\lambda$, and the manifold is Einstein (see Remark 2.12).

$m = \frac{1}{2}(n-1)$: From (2.21), it follows that $\lambda = 0$. Solving (2.20), and through a suitable change of the coordinate t (preserving the parameterization by arc length), if needed, we find $f(t) = -(n-1)\log(Et)$, where $E \in \mathbb{R}^+$. Hence, $\varphi(t) = AEt$, so $\varphi'' = 0$ and the manifold is Einstein (indeed, Ricci-flat).

$m = \frac{1}{2}$: From (2.21), it follows that $(2n-3)\lambda = 0$ and, since $n \geq 3$, we get that $\lambda = 0$. Now, we solve (2.20) (translating t if needed) to find that $f(t) = -\log(Bt)$, where $B \in \mathbb{R}^+$, and hence $\varphi(t) = A(Bt)^{\frac{1}{n-1}}$. Then, (2.19) reduces to $\beta = 0$, so the fiber N must be Ricci-flat. Finally, a direct computation shows that the condition given by (2.14) reduces to $0 = \frac{\mu}{4(2n-1)(Bt)^4}$, so $\mu = 0$. Hence, a SMMS of the form $(I \times_\varphi N, g, f, \frac{1}{2}, 0)$ satisfies equations (2.12), (2.13) and (2.14) and, moreover, is the only solution whose underlying manifold is not Einstein. This corresponds to Theorem 2.14 (2). □

Notice that we could eliminate the logarithm in the density function in Theorem 2.14 by the change of variable $v = e^{-f/m}$. Recall that the choice of v as a density is related to the interpretation of weighted objects as their corresponding standard Riemannian counterparts on certain formal warped products (we refer to Section 1.5 and Remark 2.20 for details).

Remark 2.15. Note that, by the real analyticity of the metric (see Theorem 1.15), if (M, g) is Einstein in an open set, then it is Einstein in the whole connected component containing it. Thus, since we are dealing with connected manifolds, the Einstein behavior around a single regular point is enough to infer that the whole manifold is Einstein with $\rho = 2(n-1)\lambda g$, even if the parameters m, λ and μ were to coincide with those acceptable in the non-Einstein case given by Example 2.4.

2.2 The Einstein case

Let (M^n, g, f, m, μ) be a SMMS with $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$. By Theorem 2.14, around any regular point p of f, (M, g) is isometric to a warped product $I \times_\varphi N$, where N is Einstein. We have already shown that the non-Einstein case is heavily restricted, with only one allowed value for the parameters λ, m and μ and for the Einstein constant of the fiber β, with the warping and density functions also fixed up to integration constants. However, the next result shows that if the total space $I \times_\varphi N$ is Einstein (which implies that the whole manifold M is Einstein by analyticity, since M is connected), its geometry is more flexible, allowing for solutions to the necessary and sufficient equations (2.12)-(2.14) for different combinations of parameters and functions. The value $m = 1$ is exceptional and is excluded in the statement, although this case also follows with an extra degree of freedom (see Remark 2.17 below).

Theorem 2.16. *Let (M^n, g, f, m, μ) be a SMMS with (M, g) Einstein and such that $P_f^m = \lambda g$ (with scale κ) and $\operatorname{div}_f W_f^m = 0$, with $m \neq 1$. Then, for each regular point p of f, there exists a Riemannian isometry between a neighborhood $\mathcal{U}$ of p and a warped product $I \times_\varphi N$, where $I \subset \mathbb{R}$ is an open interval, $\rho^N = \beta g^N$, and f, φ, β and μ take the following forms (t is a coordinate parameterizing I by arc length):*

1. *If $\lambda > 0$, then*

$$\begin{aligned}\varphi(t) &= a\cos(t\sqrt{2\lambda}) + b\sin(t\sqrt{2\lambda}),\\ f(t) &= -m\log\left(\tfrac{\kappa}{2\lambda} - bc\cos(t\sqrt{2\lambda}) + ac\sin(t\sqrt{2\lambda})\right),\end{aligned}$$

and

$$\beta = 2(a^2+b^2)(n-2)\lambda, \qquad \mu = 2(a^2+b^2)c^2\lambda - \frac{\kappa^2}{2\lambda}.$$

2. *If $\lambda = 0$, then*

$$\begin{aligned}\varphi(t) &= a(\kappa t + b),\\ f(t) &= -m\log\left(\tfrac{\kappa}{2}t^2 + bt + d\right),\end{aligned}$$

and

$$\beta = a^2\kappa^2(n-2), \qquad \mu = b^2 - 2d\kappa.$$

3. *If* $\lambda < 0$, *then*

$$\begin{aligned}\varphi(t) &= ae^{t\sqrt{-2\lambda}} + be^{-t\sqrt{-2\lambda}},\\ f(t) &= -m\log\left(\tfrac{\kappa}{2\lambda} + ace^{t\sqrt{-2\lambda}} - bce^{-t\sqrt{-2\lambda}}\right),\end{aligned}$$

and

$$\beta = 8ab(n-2)\lambda, \qquad \mu = -8abc^2\lambda - \frac{\kappa^2}{2\lambda}.$$

The constants a, b, c, d *and the scale* κ *are such that* φ *is positive and* f *is well-defined on* I.

Proof. Take a regular point p of f, and the local splitting as a warped product $I \times_\varphi N$ given by Theorem 2.14 around p. The weighted Einstein condition and the harmonicity of the weighted Weyl tensor guarantee that the necessary and sufficient equations (2.12)-(2.14) are satisfied. Moreover, since the underlying manifold is Einstein, the ODE $\varphi'' + 2\lambda\varphi = 0$ must also be satisfied (see Remark 2.12). Solving this equation, we can fix the different forms of φ, depending on the sign of λ. We analyze each case separately, describing the case $\lambda > 0$ in detail and giving a schematic proof of the cases $\lambda \leq 0$, which are analogous.

For $\lambda > 0$, $\varphi(t) = a\cos(\sqrt{2\lambda}t) + b\sin(\sqrt{2\lambda}t)$, where the constants a, b are given by the data of the corresponding initial value problem. Substituting φ into equation (2.12) yields

$$0 = \beta - 2(a^2+b^2)(n-2)\lambda,$$

from where $\beta = 2(a^2+b^2)(n-2)\lambda$. Now, consider the usual change of variable $v = e^{-f/m}$. Then, equation (2.13) imposes that

$$\frac{m(\varphi'v' - \varphi v'')}{\varphi v} = 0.$$

Since $v' \neq 0$ (recall that we are only considering non-trivial SMMSs), it follows that $c\sqrt{2\lambda}\,\varphi = v'$ and so $v(t) = d - bc\cos(t\sqrt{2\lambda}) + ac\sin(t\sqrt{2\lambda})$ for some constants c, d.

Lastly, in order to fix the value of μ, we substitute the known values of β, φ and f into the last of the necessary and sufficient equations, which is (2.14), where we compute J_f^m using the expression (2.15):

$$0 = \frac{m(m-1)(2\lambda c^2\,(a^2+b^2) - 2\lambda d^2 - \mu)}{2(m+n-1)v^2}.$$

From this expression we get that $\mu = 2\lambda((a^2+b^2)c^2 - d^2)$. Finally, from the scale equation $J_f^m = (m+n)\lambda - m\kappa v^{-1}$ provided by Lemma 1.11, we have $m(\kappa - 2\lambda d)v^{-1} = 0$, so that the scale κ of this weighted Einstein SMMS satisfies $\kappa = 2\lambda d$ and d can be substituted by $\frac{\kappa}{2\lambda}$ in the previous expressions. This concludes the case $\lambda > 0$.

For $\lambda < 0$, $\varphi'' + 2\lambda\varphi = 0$ yields $\varphi(t) = ae^{t\sqrt{-2\lambda}} + be^{-t\sqrt{-2\lambda}}$. Equation (2.12) then transforms into $0 = 8ab(n-2)\lambda - \beta$. Moreover, via the expression $\varphi'v' - \varphi v'' = 0$ provided by equation (2.13), we have $v(t) = d + ace^{t\sqrt{-2\lambda}} - bce^{-t\sqrt{-2\lambda}}$. Substituting these into (2.14), we get

$$0 = \frac{m(m-1)(2\lambda(4abc^2 + d^2) + \mu)}{2(m+n-1)v^2},$$

from where $\mu = -2\lambda(4abc^2 + d^2)$. Finally, the scale equation yields $\kappa = 2\lambda d$ once again, which concludes the case $\lambda < 0$.

For $\lambda = 0$, $\varphi'' + 2\lambda\varphi = 0$ yields $\varphi(t) = \tilde{a}t + \tilde{b}$. Equation (2.12) then transforms into $0 = \tilde{a}^2(n-2) - \beta$ and since $\varphi'v' - \varphi v'' = 0$ by (2.13), we can write $v(t) = \left(\frac{\tilde{a}}{2}t^2 + \tilde{b}t\right)\tilde{c} + d$ with $\tilde{c} \neq 0$. Substituting these into (2.14), we get

$$0 = \frac{2m(m-1)(2\tilde{a}\tilde{c}d - \tilde{b}^2\tilde{c}^2 + \mu)}{(m+n-1)v^2},$$

so $\mu = \tilde{b}^2\tilde{c}^2 - 2\tilde{a}\tilde{c}d$. Finally, the scale equation yields $\kappa = \tilde{a}\tilde{c}$. Now, renaming the constants so that $\tilde{b}\tilde{c} = b$ and $a = \tilde{c}^{-1}$ yields the result for $\lambda = 0$. □

Remark 2.17. In the case $m = 1$, Theorem 2.16 still holds for the same values of φ, f and β. The only difference with the case $m \neq 1$ is that, since the auxiliary curvature parameter μ does not appear in the definition of any weighted tensors when $m = 1$ (see Section 1.5), equations (2.12)-(2.14) are satisfied for arbitrary values of μ.

Recall that, for Einstein manifolds, the curvature tensor decomposes as $R = \frac{\tau}{2n(n-1)} g \owedge g + W$. If (M^n, g, f, m, μ) is an Einstein SMMS with $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$, then $\rho = 2(n-1)\lambda g$ and $\tau = 2n(n-1)\lambda$ (see Remark 2.12), which implies that $W = W_f^m = R - \lambda g \owedge g$, i.e. the weighted and unweighted Weyl tensors become equal. Consequently, the following three conditions are equivalent in this context:

1. M has constant sectional curvature.
2. M is locally conformally flat in the usual sense.
3. M is locally conformally flat in the weighted sense.

Moreover, by Theorem 1.6, a warped product $I \times_\varphi N$ is locally conformally flat if and only if N has constant sectional curvature. Then, the following rigidity result in low dimensions follows immediately.

Corollary 2.18. *Let (M^n, g, f, m, μ) be an Einstein SMMS with $n = 3$ or 4. If $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$, then (M, g) has constant sectional curvature 2λ.*

Proof. By Theorem 2.14, the open dense set $\tilde{M} \subset M$ of regular points of f is locally isometric to a warped product $I \times_\varphi N$ with the fiber N a 2 or 3-dimensional Einstein manifold. Hence, N has constant sectional curvature and $\tilde{M}$ is locally conformally flat (see Theorem 1.6). By the smoothness of the Weyl and Cotton tensors, it follows that M is locally conformally flat. □

Nevertheless, for $n \geq 5$, Corollary 2.18 no longer holds, and there exist Einstein SMMSs which are weighted Einstein and have weighted harmonic Weyl tensor, but are not locally conformally flat. In order to build an example, it suffices to consider a warped product $I \times_\varphi N$ with N Einstein but not locally conformally flat. The following construction illustrates this fact.

Example 2.19. Let (M, g) be the warped product $I \times_\varphi N$, where N is the Riemannian product $S_1 \times S_2$ of two surfaces of constant Gauss curvature β. Thus, N is Einstein with $\rho^N = \beta g^N$. Choose local coordinates (x_1, x_2) and (x_3, x_4), respectively, for S_1 and S_2 and consider the metric of the warped product given by the non-vanishing components

$$g(\partial_t, \partial_t) = 1, \qquad g(\partial_{x_1}, \partial_{x_1}) = g(\partial_{x_2}, \partial_{x_2}) = \frac{\varphi(t)^2}{\left(1+\frac{\beta}{4}(x_1^2+x_2^2)\right)^2},$$
$$g(\partial_{x_3}, \partial_{x_3}) = g(\partial_{x_4}, \partial_{x_4}) = \frac{\varphi(t)^2}{\left(1+\frac{\beta}{4}(x_3^2+x_4^2)\right)^2}.$$

Now, for $\lambda \in \mathbb{R}$, fix $\varphi(t)$, $f(t)$, β and μ as in Theorem 2.16 (in agreement with the sign of λ), choosing constants such that $\beta \neq 0$. The SMMS defined by $(I \times_\varphi N, g, f, m, \mu)$ is Einstein, and satisfies $P_m^f = \lambda g$ and $\operatorname{div}_f W_f^m = 0$, but is not of constant sectional curvature. Indeed, up to symmetries, the nonzero components of the usual Weyl tensor (hence also of the weighted Weyl tensor) are

$$\begin{aligned}
W(\partial_{x_1}, \partial_{x_2}, \partial_{x_1}, \partial_{x_2}) &= \frac{512\,\beta\,\varphi(t)^2}{3\left(4+\beta(x_1^2+x_2^2)\right)^4},\\
W(\partial_{x_3}, \partial_{x_4}, \partial_{x_3}, \partial_{x_4}) &= \frac{512\,\beta\,\varphi(t)^2}{3\left(4+\beta(x_3^2+x_4^2)\right)^4},\\
W(\partial_{x_i}, \partial_{x_j}, \partial_{x_i}, \partial_{x_j}) &= -\frac{256\,\beta\,\varphi(t)^2}{3\left(4+\beta(x_1^2+x_2^2)\right)^2\left(4+\beta(x_3^2+x_4^2)\right)^2}, \quad i = 1, 2,\ j = 3, 4.
\end{aligned}$$

Remark 2.20. Let us consider a SMMS (M^n, g, f, m, μ) which is Einstein, with $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$, and adopt the notation in Theorem 2.16. Note that, if we make the change of variable $v = e^{-f/m}$ in Theorem 2.16, we find that the density and warping functions satisfy $v'(t) = A^{-1}\varphi(t)$, where $A \neq 0$ is an integration constant fixed by the initial data. Thus, the warped product $I \times_{Av'} N$ is Einstein and the conformal metric $v^{-2}g$ is also Einstein, since v is a solution of the local Möbius equation $\operatorname{Hes}_v = \frac{\Delta v}{n} g$ (see [83] and Section 1.4 for details). A classical result by Brinkmann [10] states that warped products in Theorem 2.16 are characteristic of Einstein metrics that are conformally transformed into Einstein metrics.

Moreover, these warped products have harmonic Weyl tensor in the usual sense (see Remark 2.10). In fact, we have that $W_f^m = W$, so the harmonicity condition $\operatorname{div}_f W_f^m = 0$ can be reformulated in terms of W as the condition $\iota_{\nabla f} W = 0$. Also, recall that the divergence of the Weyl tensor is modified by a conformal change $\tilde{g} = e^{-2f} g$ as $\widehat{\operatorname{div}}\widehat{W} = \operatorname{div} W + (3-n)\iota_{\nabla f} W$ (see [83]). Hence, given that W is harmonic and $\iota_{\nabla f} W = 0$, it follows that $\widehat{\operatorname{div}}\widehat{W} = 0$, so we can rephrase the role of the density function by stating that *it defines a conformal change of the metric that preserves the harmonicity of the Weyl tensor of M.*

In terms of the geometric interpretation of weighted objects as Riemannian invariants of the formal warped product (1.17), we have that its auxiliary manifold becomes a multiply warped product of the form $I \times_{Av'} N \times_v F^m(\mu)$. Multiply warped product metrics have been considered in different contexts to obtain examples of manifolds with some curvature features; see, for example, [20, 55] for studies of these metrics related to the Einstein condition, local conformal flatness and negative curvature. Similarly to how quasi-Einstein manifolds can be used as bases to find Einstein warped products (see [77]), we can use weighted Einstein manifolds with weighted harmonic Weyl tensor in order to find multiply warped products satisfying certain geometric properties. For example, these multiply warped products have harmonic Weyl tensor. Indeed, notice that a conformal change of the form $g \oplus v^2 g^F \mapsto \frac{1}{v^2} g \oplus g^F$ transforms the warped product into a direct product of two Einstein manifolds, so this product manifold has harmonic Weyl tensor. Since v only depends on t and $\iota_{\partial_t} W = 0$, the inverse of the previous conformal change preserves the vanishing of the divergence of the Weyl tensor. This harmonicity can also be proved directly on the multiply warped product by checking the conditions in [66].

Furthermore, although the multiply warped products $I \times_{Av'} N \times_v F^m(\mu)$ are not Einstein in general, we can make them so by choosing appropriate constants of integration, namely by taking $\kappa = 0$, with the Einstein constant being $\tilde{\lambda} = 2(m+n-1)\lambda$. Recall that, by Lemma 1.11, taking the scale κ to be zero makes the resulting SMMS a quasi-Einstein manifold.

2.3 Global results

Now we turn our attention to global questions and study obstructions to the existence of complete weighted Einstein manifolds with weighted harmonic Weyl tensor. In related contexts, several authors have given results for complete and simply connected quasi-Einstein manifolds (see, for example, [71, Theorem 1.2]). However, in this weighted setting, we will show that taking advantage of a relation between the weighted Einstein and generalized Obata equations (see Section 1.4), we can disregard simple connectedness and prove the main global rigidity result of this chapter (Theorem 2.23) by imposing only the completeness assumption.

The following two lemmas highlight some properties of both the Einstein and non-Einstein cases which will be key in our proof of Theorem 2.23.

Lemma 2.21. *Let $(I \times_\varphi N, g, f, \frac{1}{2}, 0)$ be a SMMS given as in Example 2.4. Then, $(I \times_\varphi N, g)$ is incomplete, and cannot be isometrically embedded as an open set in any complete manifold.*

Proof. The Ricci tensor of $I \times_\varphi N$ has only one non-zero component:

$$\rho(\partial_t, \partial_t) = \frac{(n-2)}{(n-1)t^2}.$$

Moreover, $\alpha(t) = t$ is a geodesic, since $\alpha'(t) = \partial_t$ and $\nabla_{\alpha'(t)}\alpha'(t) = \nabla_{\partial_t}\partial_t = 0$. Notice that $\rho(\alpha'(t), \alpha'(t)) = \frac{(n-2)}{(n-1)t^2}$ on I. Suppose $(I \times_\varphi N, g)$ can be isometrically embedded into a complete manifold. Then $\lim_{t\to 0^+} \rho(\alpha'(t), \alpha'(t))$ exists, contradicting the formula $\rho(\alpha'(t), \alpha'(t)) = \frac{(n-2)}{(n-1)t^2}$. □

Lemma 2.22. *Let (M^n, g, f, m, μ) be a SMMS with (M, g) Einstein and such that $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$. Then, the function $v = e^{-\frac{f}{m}}$ is a solution in M of the generalized Obata equation*

$$\operatorname{Hes}_v + \gamma(v) g = 0, \tag{2.22}$$

with $\gamma(v) = 2\lambda v - \kappa$, where $\kappa \in \mathbb{R}$ is the scale of (M^n, g, f, m, μ).

Proof. Since (M, g) is Einstein, by Theorem 2.14 we know that $\rho = 2(n-1)\lambda g$. Now, by the change of variable $v = e^{-\frac{f}{m}}$, we have $\operatorname{Hes}_f - \frac{1}{m} df \otimes df = -\frac{m}{v} \operatorname{Hes}_v$. Using the scale equation $J_f^m = (m+n)\lambda - m\kappa e^{\frac{f}{m}}$ (see Lemma 1.11), the weighted Einstein equation $P_f^m = \lambda g$ reads

$$\begin{aligned} \lambda g &= \tfrac{1}{m+n-2} \left\{ \rho + \operatorname{Hes}_f - \tfrac{1}{m} df \otimes df - J_f^m \right\} \\ &= \tfrac{1}{m+n-2} \left(-\tfrac{m}{v} \operatorname{Hes}_v + ((n-m-2)\lambda + \tfrac{m}{v}\kappa) g \right), \end{aligned}$$

from where

$$-m \operatorname{Hes}_v + (m\kappa - 2m\lambda v) g = 0,$$

and the result follows. □

Now, using the previous results, we are ready to prove the global result characterizing complete Einstein SMMSs with weighted harmonic Weyl tensor.

Theorem 2.23. *Let (M^n, g, f, m, μ) be a complete SMMS such that $P_f^m = \lambda g$ (with scale κ) and $\operatorname{div}_f W_f^m = 0$. Then, (M^n, g, f, m, μ) is isometric to one of the following spaces:*

1. *A weighted space form as described in Examples 1.12, 1.13 and 1.14.*
2. *An Einstein warped product $\mathbb{R} \times_\varphi N$, where N is a Ricci-flat complete manifold. In this case, there is a coordinate t parameterizing $\mathbb{R}$ by arc length such that the warping and density functions take the forms*

$$\varphi(t) = A e^{t\sqrt{-2\lambda}}, \qquad f(t) = -m \log\left(\tfrac{\kappa}{2\lambda} + B e^{t\sqrt{-2\lambda}} \right),$$

for some $B \geq 0$ and $\kappa \leq 0$. Moreover, $m = 1$ or $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$.

Proof. Let (M^n, g, f, m, μ) be a complete SMMS with $P_f^m = \lambda g$ and $\operatorname{div}_f W_f^m = 0$. By Lemma 2.14, around regular points of f, (M, g) is either Einstein or given by Example 2.4. If (M, g) is Einstein around any regular point of f, then it is Einstein everywhere by analyticity (see Remark 2.15). On the other hand, if (M, g) is not Einstein, Lemma 2.21 guarantees that Example 2.4 cannot be complete. Thus, we assume that (M, g) is Einstein henceforth. Then, by Lemma 2.22, $v = e^{-f/m}$ satisfies the generalized Obata equation (2.22) with $\gamma(v) = 2\lambda v - \kappa$.

Firstly, assume that v has critical points. Then, by Theorem 1.7, (M, g) is isometric to a complete extension of the warped metric

$$g = dt^2 + \varphi(t)^2 g_{\mathbb{S}^{n-1}}, \quad t \in (0, T),$$

with $\varphi(t) = \frac{(v'(t))}{(2\lambda v(0) - \kappa)}$, where

$$v'' + 2\lambda v - \kappa = 0, \quad v(0) = \xi > 0, \quad v'(0) = 0.$$

If $\lambda > 0$, it follows that $v(t) = \frac{\kappa}{2\lambda} + \frac{(2\xi\lambda - \kappa)\cos(\sqrt{2\lambda} t)}{2\lambda}$, with $t \in (0, \frac{\pi}{\sqrt{2\lambda}})$. The warping function is $\varphi(t) = \frac{\sin(\sqrt{2\lambda} t)}{\sqrt{2\lambda}}$. Hence, (M, g) is isometric to a sphere of constant sectional curvature 2λ. By imposing the weighted Einstein equation, we obtain that $\mu = 2\xi(\xi\lambda - \kappa)$ or $m = 1$ and, therefore, (M, g, f, m, μ) and $(M, g, f, 1)$ are isometric to the m-weighted n-sphere as in Example 1.12. On the other hand, if $\lambda = 0$, $\kappa \neq 0$, then $v(t) = \xi + \frac{\kappa}{2}t^2$ with $t \in (0, \infty)$; the warping function is $\varphi(t) = t$ and $\mu = -2\xi\kappa$, so (M, g, f, m, μ) is isometric to the m-weighted n-Euclidean space as in Example 1.13. Finally, if $\lambda < 0$, we have $v(t) = \frac{\kappa}{2\lambda} + \frac{(2\xi\lambda-\kappa)\cosh\left(\sqrt{-2\lambda}t\right)}{2\lambda}$, with $t \in (0, \infty)$. The warping function is given by $\varphi(t) = \frac{\sinh\left(\sqrt{-2\lambda}t\right)}{\sqrt{-2\lambda}}$. Similarly to the first case, the weighted Einstein equation yields $\mu = 2\xi(\xi\lambda - \kappa)$ and (M, g, f, m, μ) is isometric to the m-weighted n-hyperbolic space as in Example 1.14. These three cases constitute Theorem 2.23 (1).

Lastly, consider all remaining cases, where it is assumed that v has no critical points. Then, by Theorem 1.7, (M, g) splits globally as a warped product $\mathbb{R} \times_\varphi N$ where N is complete. Thus, the forms of the warping and density functions given by Theorem 2.16 for the Einstein case can be taken to be global. For $\lambda \geq 0$, these density functions either present critical points (so they correspond to a local description of one of the previous examples) or are such that $v = e^{-f/m}$ turns nonpositive for some values of a coordinate t parameterizing $\mathbb{R}$ by arc length (so they result in incomplete manifolds). Hence, let us focus on the case $\lambda < 0$. The form of the warping function is $\varphi(t) = ae^{t\sqrt{-2\lambda}} + be^{-t\sqrt{-2\lambda}}$ as in Theorem 2.16 (3). For φ to stay positive for all $t \in \mathbb{R}$, a and b must be nonnegative. Note that this also prevents $v(t) = \frac{\kappa}{2\lambda} + ace^{t\sqrt{-2\lambda}} - bce^{-t\sqrt{-2\lambda}}$ from presenting critical points. In addition, v must remain positive for all $t \in \mathbb{R}$. Assume first that $a, b > 0$, then v turns nonpositive for large enough values of t if $c < 0$, and for small enough values of t if $c > 0$, so this case is not admissible. Hence, either $a > 0$ and $b = 0$, or $a = 0$ and $b > 0$. Notice that a reparametrization of the form $t \to -t$ together with a change $c \to -c$ interchange a and b, so we can assume $b = 0$ and $v(t) = \frac{\kappa}{2\lambda} + ace^{t\sqrt{-2\lambda}}$. Thus, v remains positive if and only if $c > 0$ and $\frac{\kappa}{2\lambda} \geq 0$ (hence $\kappa \leq 0$).

Now, let $A = a$ and $B = ac$. It follows from Theorem 2.16 (3) that $\beta = 0$ (hence N is Ricci-flat),

$$\varphi(t) = Ae^{\sqrt{-2\lambda}\,t}, \qquad f(t) = -m \log\left(\frac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}}\right),$$

with $A, B \in \mathbb{R}^+$ and, moreover, either $m = 1$ (see Remark 2.17) or $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$. This is the remaining case, Theorem 2.23 (2). □

Notice that, if $\lambda \geq 0$, the weighted space forms in Examples 1.12 and 1.13 are the only complete SMMSs which are weighted Einstein and satisfy $\operatorname{div}_f W_f^m = 0$. In contrast, if $\lambda < 0$, we have both m-weighted n-hyperbolic space (Example 1.14) and the warped products in Theorem 2.23 (2). If the dimension is $n \leq 4$, both of these latter examples have an underlying manifold of negative constant sectional curvature (see Corollary 2.18). Nevertheless, these two SMMSs are not isometric, indeed the density function has one critical point in the m-weighted n-hyperbolic space, but has no critical points in Theorem 2.23 (2). For $n \geq 5$, any complete Ricci-flat (non-flat) manifold N gives rise to a complete SMMS via the construction in Theorem 2.23 (2). Moreover, the underlying Riemannian manifold does not have constant sectional curvature. Examples of complete Ricci-flat, non-flat manifolds in Riemannian signature are the Eguchi-Hanson metric on the cotangent bundle of the 2-sphere $T^*\mathbb{S}^2$ (see [56]); and Calabi-Yau manifolds, which are notable examples of complete, Ricci-flat Kähler manifolds (see [123]).

Chapter 3
Weighted Einstein manifolds in the same conformal class

As we pointed out in our introduction of (Riemannian) smooth metric measure spaces in Section 1.5, ideas related to conformal geometry naturally come up, and the definitions of local conformal class and local conformal flatness in the weighted sense are built from their Riemannian counterparts in the formal warped product (1.17). Thus, one might wonder to what extent one can generalize problems of standard conformal geometry to the context of SMMSs, what the similarities and differences between both settings are, and what the consequences of the corresponding weighted results are on the underlying Riemannian spaces.

A question of great significance in Riemannian geometry is whether a certain geometric property is satisfied by some manifold in a given conformal class. Examples of this are the conformal Einstein equation and the Yamabe problem (see Sections 1.4 and 1.5 for details, respectively). Related to the former, the matter of finding which manifolds admit more than one Einstein metric in a conformal class is a classical one. First contributions on it date back to Brinkmann [10], and literature on the topic is extensive. See, for example, [82, 122], where the former includes a detailed review of the subject.

This question has a seamless generalization to SMMSs. Indeed, since weighted Einstein manifolds are the natural Einstein-type structure for SMMSs, one starts by considering the conformal weighted Einstein equation. For two conformally equivalent SMMSs (M^n, g, f, m, μ) and $(M^n, \widehat{g}, \widehat{f}, m, \mu)$, where $\widehat{g} = e^{-\frac{2\phi}{m}} g$ and $\widehat{f} = f + \phi$, taking $u = e^{\frac{\phi}{m}}$ and using the transformation formulae in [34], one has

$$\widehat{P}^m_{\widehat{f}} = P^m_f + u^{-1}\operatorname{Hes}_u - \frac{1}{2}u^{-2}\|\nabla u\|^2 g. \tag{3.1}$$

Thus, the problem of finding conformal classes which admit weighted Einstein representatives entails finding a Riemannian metric g, a density function f and a conformal function u defined on M so that (3.1) is satisfied for $\widehat{P}^m_{\widehat{f}} = \widehat{\lambda} u^{-2} g$ and some constant $\widehat{\lambda}$. This translates into finding a solution of the system of PDEs

$$P^m_f + u^{-1}\operatorname{Hes}_u = u^{-2}\left(\widehat{\lambda} + \frac{1}{2}\|\nabla u\|^2\right) g \tag{3.2}$$

for a constant $\widehat{\lambda}$, which turns out to be an unmanageable problem in general. However, the complexity of the system can be reduced by considering the question of non-uniqueness, this is, to find weighted Einstein SMMSs that admit another weighted SMMS in their conformal class (excluding homotheties, i.e., rescalings of the metric by a constant factor).

Moreover, this is not just an *ad-hoc* extension of the classical unweighted problem, but one that is of use both in the weighted Yamabe problem [36] and in the search for sharp fully non-linear Sobolev inequalities based on the weighted σ_k-curvatures [34]. Indeed, first results in this regard were given in [36] for $\mu = 0$. On the one hand, it was shown that if M is compact then the SMMS is conformal to the standard sphere $(S^n, g, 0, 1, 0)$. On the other hand, if the manifold is complete and the weighted Schouten tensor (1.19) vanishes, then it is isometric to the Euclidean space with a particular family of possible density functions. Additionally, some partial results were given in [34] for specific families of SMMSs. In this chapter, we generalize these results and give a complete answer to this problem, determining all conformal classes which admit more than one weighted Einstein representative. These results are contained in the work [18].

We will show that, in fact, some of the solutions we obtain are built on Einstein manifolds which conformally transform into another Einstein manifold. Thus, we recover part of the known results from the context where the density is constant, although in our case there is, additionally, a change of density. We emphasize, however, that there are also weighted Einstein solutions whose underlying manifold is not Einstein (see Theorems 3.4 and 3.10 below).

Outline of the chapter

Much like in Chapter 2, this analysis is divided into two distinct types of results: local and global. On the one hand, in Section 3.1 we discuss the local geometric structure of manifolds which admit two weighted Einstein structures in the same weighted conformal class. From the transformation formula (3.1), we prove the splitting of these SMMSs as warped products with one-dimensional base. Further study of the transformation formula and the resulting weighted Einstein equations allows us to prove the main local classification result (Theorem 3.4). Additionally, we provide some remarks and examples of SMMSs that give more insight into the types of geometries that arise for the different items of the theorem.

On the other hand, in Section 3.2 we prove a global classification result (Theorem 3.10) for SMMSs whose underlying manifold (M, g) is complete. In this context, the admissible geometries with two weighted Einstein representatives of the same weighted conformal class are either weighted space forms (see definitions in Section 1.5) or special families of warped products. Then, we perform an analysis of the compact case, extending a result in [36] and showing that if the weighted Einstein manifolds are compact then they are necessarily a sphere (Corollary 3.12). We end the chapter with a brief note on WE weighted Bach-flat manifolds (Section 3.3), which further highlights the relation between the SMMSs in Chapters 2 and 3.

Computations in this chapter are more easily performed by considering the change of variable $v = e^{-\frac{f}{m}}$ (see Section 1.5 for motivation for this alternative definition of the density), so that a conformal change transforms the metric and the density as $\widehat{g} = u^{-2}g$ and $\widehat{v} = u^{-1}v$, respectively. It is also worth noting that, since conformal transformations can transform non-trivial manifolds into trivial ones and vice versa, it is useful to include the trivial case in our analysis.

3.1 Conformally weighted Einstein SMMSs: local study

Before diving into the geometric features of the SMMSs of interest in this chapter, we discuss the analytic properties of the conformal factor. Thus, suppose that for two locally conformally equivalent SMMSs with $\widehat{g} = u^{-2}g$ and $\widehat{v} = u^{-1}v$, we have $P_f^m = \lambda g$ and $\widehat{P}_{\widehat{f}}^m = \widehat{\lambda}\widehat{g}$. As shown in Theorem 1.15, an important property of weighted Einstein manifolds is that both the metric and the density function are real analytic in harmonic coordinates. As a consequence, we show that the conformal factor u relating both of them is also analytic.

Lemma 3.1. *If u is a solution of* (3.2) *on a weighted Einstein SMMS (M, g, f, m, μ), then u is (real) analytic in harmonic coordinates on M.*

Proof. From the fact that (M, g, f, m, μ) satisfies $P_f^m = \lambda g$ for some constant λ, it follows that (3.2) becomes

$$\mathrm{Hes}_u = u\left(\widehat{\lambda}u^{-2} - \lambda + \frac{1}{2}u^{-2}\|\nabla u\|^2\right) g, \tag{3.3}$$

or, taking the trace of this equation,

$$\Delta u + \mathrm{l.o.t} = 0,$$

where l.o.t. stands for lower order terms. In harmonic coordinates, this geometric equation becomes the quasi-linear second-order PDE $g^{rs}\frac{\partial^2 u}{\partial x^r \partial x^s} + \mathrm{l.o.t} = 0$, which is elliptic due to the fact that the metric g is positive definite (recall that we are working in the Riemannian setting).

Moreover, since WE metrics are real analytic in harmonic coordinates (see Theorem 1.15), the equation is of the form $F(u, \partial u, \partial^2 u) = 0$, with F real analytic. It follows that the conformal factor u is real analytic in harmonic coordinates (see, for example, [3, J.41]). □

As a result of the previous lemma, the set of regular points of u is open and dense in M. We will use this fact in subsequent arguments.

Now, we begin the analysis of the transformation formula (3.2) from a local point of view without further assumptions. We start by proving that the conformal factor satisfies a generalized Obata equation (compare with the proof of [36, Proposition 9.5]). This is the same equation satisfied by a conformal factor transforming an Einstein metric into another one (cf. [10, 82] and see Section 1.4 for further details) and it provides some information on the structure of the underlying manifold, which decomposes as a warped product.

Lemma 3.2. *Let (M^n, g, v, m, μ) be a SMMS such that $P_f^m = \lambda g$ and u a non-constant solution of* (3.2). *Then, the function u is a solution of the generalized Obata equation* (2.22)

$$\mathrm{Hes}_u + \gamma(u) g = 0, \tag{3.4}$$

with $\gamma(u) = 2\lambda u - \nu$, for some constant $\nu \in \mathbb{R}$. Moreover, around any regular point of u, (M, g) is locally isometric to a warped product $I \times_\varphi N$, where $I \subset \mathbb{R}$ is an open interval, ∇u is tangent to I, and $\varphi(t) = \pm u'(t)$, where t is a local coordinate parameterizing I by arc length.

Proof. Since $P_f^m = \lambda g$, the Hessian of u is given by (3.3). Thus, it follows that the level hypersurfaces of u around regular points are totally umbilical. Moreover, letting $E_1 = \nabla u/\|\nabla u\|$, by (3.3) we have

$$g(\nabla_X Y, E_1) = -\frac{1}{\|\nabla u\|}\operatorname{Hes}_u(X,Y) = Hg(X,Y) \quad \text{for } X, Y \in \nabla u^\perp,$$

where $H = -\frac{u}{\|\nabla u\|}\left(\widehat{\lambda}u^{-2} - \lambda + \frac{1}{2}u^{-2}\|\nabla u\|^2\right)$. Now, note that for a vector field $X \in \nabla u^\perp$, by (3.3),

$$2\|\nabla u\| X(\|\nabla u\|) = X(\|\nabla u\|^2) = 2\operatorname{Hes}_u(\nabla u, X) = -\|\nabla u\| H g(\nabla u, X) = 0,$$

and thus $X(H) = 0$. Consequently, the mean curvature vector field HE_1 is parallel in the normal bundle $\operatorname{span}\{\nabla u\}$. Indeed, once again by (3.3),

$$\nabla_X^\perp (HE_1) = g(\nabla_X(HE_1), E_1)E_1 = \frac{H}{\|\nabla u\|}\operatorname{Hes}_u(E_1, X) = 0,$$

so the level hypersurfaces of u are spherical. Therefore, as a consequence of Theorem 1.3, in a neighborhood of each regular point, (M, g) splits as a warped product $I \times_\varphi N$, for some function φ defined on I.

Let t be a coordinate parameterizing I by arc length. Then, evaluating equation (3.3) in (∂_t, ∂_t) yields

$$u^{-2}\left(\lambda u^2 - \widehat{\lambda} - \frac{1}{2}(u')^2 + uu''\right) = 0. \tag{3.5}$$

Now, we compute

$$((u')^2 u^{-1} + 2\lambda u + 2\widehat{\lambda}u^{-1})' = u^{-2}(2u''u - (u')^2 - 2\widehat{\lambda} + 2\lambda u^2)u' = 0$$

where the middle expression vanishes by (3.5). Hence $(u')^2 u^{-1} + 2\lambda u + 2\widehat{\lambda}u^{-1} = \nu$ for some constant $\nu \in \mathbb{R}$, which yields $(u')^2 = -2\lambda u^2 + 2\nu u - 2\widehat{\lambda}$. Substituting the value of $(u')^2$ into (3.3) we obtain $\operatorname{Hes}_u + (2\lambda u - \nu)g = 0$, which is the generalized Obata equation (3.4) around regular points of u. Since the set of regular points of u is open and dense in M by Lemma 3.1, by smoothness, equation (3.4) extends to M.

Moreover, for any unitary vector field $X \in \partial_t^\perp$, we use the warped product decomposition and the formula (1.4) for its connection to compute $\operatorname{Hes}_u(X, X) = \frac{u'\varphi'}{\varphi}$. Hence, (3.3) yields

$$\frac{u'\varphi'}{u\varphi} = \frac{\widehat{\lambda}}{u^2} - \lambda + \frac{(u')^2}{2u^2},$$

whereas by (3.5) we have

$$\frac{u''}{u} = \frac{\widehat{\lambda}}{u^2} - \lambda + \frac{(u')^2}{2u^2}.$$

From these two expressions it follows that $u'\varphi' - u''\varphi = 0$, and then $\varphi = Ku'$ for some $K \in \mathbb{R}$ such that $\varphi > 0$ in I (recall that u is non-constant). Rescaling the metric in N, we can assume $K = 1$ if $u' > 0$ and $K = -1$ if $u' < 0$. □

Notice that, as pointed out in [82, Proposition 4], conformal changes given by solutions to equation (3.4) preserve the constancy of the scalar curvature, which is a crucial fact in the study of conformal transformations preserving the Einstein character. However, recall that weighted Einstein manifolds do not have constant scalar curvature in general (see, for example, Example 2.4), so this fact does not always apply in this context.

Another essential difference between the usual setting and conformal changes that transform a WE manifold into another is the transformation of the density. As a next step, we analyze the form of the density function assuming that the manifold decomposes as a warped product according to Lemma 3.2.

Lemma 3.3. *Let* (M^n, g, f, m, μ)*, where* $(M, g) = I \times_\varphi N$ *and* $P_f^m = \lambda g$*, admit a non-constant solution of* (3.2)*. Then* $v = e^{-\frac{f}{m}}$ *splits as*

$$v = \varphi(t) v_N(x_1, \dots, x_{n-1}) + \alpha(t)$$

where t *parameterizes* I *by arc length and* $x_1, \dots x_{n-1}$ *are coordinates of the fiber* N*.*

Proof. By Lemma 3.2, we have that $u''+2\lambda u-\nu = 0$ and $\varphi = \pm u'$. Hence $\frac{\varphi''}{\varphi} = -2\lambda$. Moreover, since $(M, g) = I \times_\varphi N$, the Ricci tensor takes the form portrayed in (1.6). In particular, we have $\rho(\partial_t, X) = 0$ for $X \in \partial_t^\perp$ and t parameterizing I by arc length. Now, from the definition of the Schouten tensor (1.19), and using the scale equation $J_f^m = (m + n)\lambda - m\kappa e^{\frac{f}{m}}$ given by Lemma 1.11, the weighted Einstein equation $P_f^m = \lambda g$ can be written as

$$\rho + \operatorname{Hes}_f - \frac{1}{m} df \otimes df = \left(2(m + n - 1)\lambda - m\kappa e^{\frac{f}{m}}\right) g. \tag{3.6}$$

Under the change of variable $v = e^{-\frac{f}{m}}$, equation (3.6) takes the form

$$\rho - mv^{-1} \operatorname{Hes}_v = (2(m + n - 1)\lambda - m\kappa v^{-1})g. \tag{3.7}$$

Thus, taking X the lift of a vector field in N we have $-mv^{-1} \operatorname{Hes}_v(\partial_t, X) = 0$, and computing this Hessian using the expressions for the Levi-Civita connection of a warped product (1.4) yields

$$0 = \operatorname{Hes}_v(\partial_t, X) = \partial_t X(v) - \frac{\varphi'}{\varphi} X(v).$$

Therefore, locally either $X(v) = 0$ or $\frac{\partial_t X(v)}{X(v)} = \frac{\varphi'}{\varphi}$. In the latter case we have

$$\partial_t \log(X(v)) = (\log \varphi)',$$

and integrating with respect to t we obtain

$$\log X(v) = \log \varphi + \bar{v}_N^X(\overrightarrow{x}),$$

where $\overrightarrow{x} = (x_1, \dots, x_{n-1})$ are coordinates on N. Hence, in either case, $X(v) = \varphi(t) v_N^X(\overrightarrow{x})$ where v_N^X does not depend on t, but may depend on the choice of X. Since φ depends only on t

and there is no confusion, we omit this dependence to simplify notation henceforth. Set notation $v_N^{\partial_{x_i}} = v_N^i$ and take $X = \partial_{x_1}$, so $\partial_{x_1}(v) = \varphi\, v_N^1(\overrightarrow{x})$. Integrating this expression with respect to x_1 yields

$$v = \varphi\, \tilde{v}_N^1(\overrightarrow{x}) + \alpha_1(t, x_2, \dots, x_{n-1}) \tag{3.8}$$

for some function $\tilde{v}_N^1(\overrightarrow{x})$ on N and some function α_1 which does not depend on x_1. Now, we work with this expression to see that the form in (3.8) can be rearranged so that α_1 does not depend on x_2. We differentiate this expression with respect to ∂_{x_2} to find

$$\varphi\, v_N^2(\overrightarrow{x}) = \partial_{x_2}(v) = \varphi\, \partial_{x_2}\tilde{v}_N^1(\overrightarrow{x}) + \partial_{x_2}\alpha_1(t, x_2, \dots, x_{n-1}).$$

Hence,

$$v_N^2(\overrightarrow{x}) - \partial_{x_2}\tilde{v}_N^1(\overrightarrow{x}) = \varphi^{-1}\, \partial_{x_2}\alpha_1(t, x_2, \dots, x_{n-1}).$$

Since the left-hand side of this equation does not depend on t, differentiating with respect to t gives

$$0 = \varphi^{-2}(\varphi\, \partial_t\partial_{x_2}\alpha_1 - \varphi'\partial_{x_2}\alpha_1),$$

from where it follows that either α_1 does not depend on x_2, and thus we attain our objective, or $\frac{\partial_t\partial_{x_2}\alpha_1}{\partial_{x_2}\alpha_1} = \frac{\varphi'}{\varphi}$. Assuming that the latter holds, we integrate with respect to t to find

$$\partial_{x_2}\alpha_1(t, x_2, \dots, x_{n-1}) = \varphi\, \gamma_1(x_2, \dots, x_{n-1})$$

for some function γ_1 on N. Hence, we write

$$\begin{aligned} \alpha_1(t, x_2, \dots, x_{n-1}) &= \varphi \textstyle\int \gamma_1(x_2, \dots, x_{n-1})dx_2 + \alpha_2(t, x_3, \dots, x_{n-1}) \\ &= \varphi\, \tilde{\gamma}_1(x_2, \dots, x_{n-1}) + \alpha_2(t, x_3, \dots, x_{n-1}) \end{aligned}$$

for some $\tilde{\gamma}_1$ defined on N and α_2 not depending on x_1 and x_2. Substituting this value of α_1 into (3.8), we have

$$\begin{aligned} v &= \varphi\, \tilde{v}_N^1(\overrightarrow{x}) + \alpha_1(t, x_2, \dots, x_{n-1}) \\ &= \varphi\, (\tilde{v}_N^1(\overrightarrow{x}) + \tilde{\gamma}_1(x_2, \dots, x_{n-1})) + \alpha_2(t, x_3, \dots, x_{n-1}) \\ &= \varphi\, \tilde{v}_N^2(\overrightarrow{x}) + \alpha_2(t, x_3, \dots, x_{n-1}), \end{aligned}$$

where $\tilde{v}_N^2(\overrightarrow{x}) = \tilde{v}_N^1(\overrightarrow{x}) + \tilde{\gamma}_1(x_2, \dots, x_{n-1})$. Thus, we get that v is of the form of (3.8) with α not depending on x_1 and x_2. Now, using $\partial_{x_i}(v) = v_N^i(\overrightarrow{x})\varphi(t)$ for $i = 3, \dots, n-1$, we repeat the process outlined above, eliminating the dependence on the x_i variable from the corresponding α_i function. After a number of iterations, v becomes

$$v = \varphi\, v_N(x_1, \dots, x_{n-1}) + \alpha(t)$$

for a function v_N defined on the fiber and a function α defined on the base. □

For the form of the density function given by Lemma 3.3, the remaining components of the weighted Einstein equation (3.7) provide additional information on the density and the geometry of the underlying manifold, allowing us to prove the main local classification result.

Theorem 3.4. *Let (M^n, g, f, m, μ) be a weighted Einstein SMMS, with $P^m_f = \lambda g$, such that there exists a conformally equivalent SMMS $(M^n, \widehat{g}, \widehat{f}, m, \mu)$ which is weighted Einstein with $\widehat{P}^m_{\widehat{f}} = \widehat{\lambda}\widehat{g}$. Then, on a neighborhood of each regular point of the conformal factor u, M decomposes as a warped product $I \times_\varphi N$, where $I \subset \mathbb{R}$ is an open interval and ∇u is tangent to I. Furthermore, one of the following holds:*

1. *(M, g) and $(M, \widehat{g})$ are Einstein with $\rho = 2(n-1)\lambda g$ and $\widehat{\rho} = 2(n-1)\widehat{\lambda}\widehat{g}$, and the density takes the form $f = -m\log(\varphi v_N + \alpha)$, where v_N is a function on N and α is a function on I.*

   Moreover, the fiber (N, g^N) is Einstein and there exist constants ξ, ν determined by v and u such that $\operatorname{Hes}^N_{v_N} = (\xi - (\nu^2 - 4\lambda\widehat{\lambda})v_N)g^N$.

2. *(M^n, g, f, m) and $(M^n, \widehat{g}, \widehat{f}, m)$ are quasi-Einstein with $\rho^m_f = 2(m+n-1)\lambda g$ and $\widehat{\rho}^m_{\widehat{f}} = 2(m+n-1)\widehat{\lambda}\widehat{g}$, and the density f splits as $f = -m\log(\varphi) + f_N$ where f_N is a function on N.*

   Moreover, the fiber (N, g^N, f_N, m) is quasi-Einstein too.

Proof. Let (M^n, g, f, m, μ) be a SMMS such that $P^m_f = \lambda g$ and such that there exists a SMMS $(M^n, \widehat{g}, \widehat{f}, m, \mu)$, with $\widehat{g} = u^{-2}g$, $\widehat{v} = u^{-1}v$ (where $v = e^{-\frac{f}{m}}$ and u is non-constant) and such that $\widehat{P}^m_{\widehat{f}} = \widehat{\lambda}\widehat{g}$. By Lemma 3.2, (M, g) splits locally around the regular points of the conformal factor u as a warped product $I \times_\varphi N$, where $\varphi'' = -2\lambda\varphi$. Moreover, by Lemma 3.3, the density takes the form $v = \varphi v_N + \alpha$, where v_N is defined on N and α is defined on I.

From the expression (1.6) for the Ricci tensor of a warped product, we have $\rho(\partial_t, \partial_t) = -(n-1)\frac{\varphi''}{\varphi} = 2(n-1)\lambda$, so the weighted Einstein equation (3.7) yields

$$\rho^m_f(\partial_t, \partial_t) = 2(n-1)\lambda - mv^{-1}\operatorname{Hes}_v(\partial_t, \partial_t) = 2(m+n-1)\lambda - m\kappa v^{-1}.$$

Hence $\operatorname{Hes}_v(\partial_t, \partial_t) = \partial^2_t v = -2\lambda v + \kappa$. Since $v = \varphi v_N + \alpha$, and knowing that $\varphi'' = -2\lambda\varphi$, this implies $\alpha'' = -2\lambda\alpha + \kappa$. Now, consider the following decomposition:

$$\operatorname{Hes}_v = v_N \operatorname{Hes}_\varphi + \varphi \operatorname{Hes}_{v_N} + dv_N \otimes d\varphi + d\varphi \otimes dv_N + \operatorname{Hes}_\alpha.$$

Take two lifts X, Y of vector fields in N. Since φ and α depend only on t, we have

$$\operatorname{Hes}_v(X, Y) = v_N(\varphi')^2\varphi g^N(X, Y) + \varphi \operatorname{Hes}_{v_N}(X, Y) + \alpha'\varphi'\varphi g^N(X, Y).$$

Using this expression and (1.6), equation (3.7) for X, Y reads

$$\begin{aligned} \rho^m_f(X, Y) &= \rho^N(X, Y) - (\varphi''\varphi + (n-2)(\varphi')^2)\, g^N(X, Y) \\ &\quad - mv^{-1}(v_N(\varphi')^2\varphi + \alpha'\varphi'\varphi)g^N(X, Y) \\ &\quad - mv^{-1}\varphi \operatorname{Hes}_{v_N}(X, Y) \\ &= (2(m+n-1)\lambda - m\kappa v^{-1})\varphi^2 g^N(X, Y). \end{aligned} \tag{3.9}$$

Let X be a unit eigenvector of the Ricci operator Ric^N, and let

$$\rho^N(X,X) = g^N(\mathrm{Ric}^N(X),X) = r(X)$$

be its associated eigenvalue. Also, denote $\mathrm{Hes}_{v_N}(X,X) = h(X)$ and notice that, by (3.9), X is also an eigenvector of the Hessian operator. In order to simplify notation, we omit the X dependence in the following calculations unless explicitly needed. From the formula

$$\begin{aligned} h = \mathrm{Hes}_{v_N}(X,X) = g(\nabla_X \nabla v_N, X) &= \varphi^2 g^N(\nabla_X(\varphi^{-2}\nabla^N v_N), X) \\ &= g^N(\nabla^N_X \nabla^N v_N, X) = \mathrm{Hes}^N_{v_N}(X,X), \end{aligned}$$

it follows that h, as well as r, is also a function defined on the fiber N. Taking $Y = X$ in (3.9) and multiplying by v, we obtain the following equation:

$$\begin{aligned} 0 &= vr - v\left(\varphi''\varphi + (n-2)(\varphi')^2\right) - m(v_N(\varphi')^2\varphi + \alpha'\varphi'\varphi) \\ &\quad -m\varphi h - (2(m+n-1)\lambda v - m\kappa)\varphi^2, \end{aligned} \tag{3.10}$$

where $v = \varphi v_N + \alpha$, φ and α depend on t, and the remaining functions depend only on the coordinates $x_1, \ldots, x_{n-1}$ of the fiber N. Now, differentiating (3.10) with respect to any x_i and substituting $\varphi'' = -2\lambda\varphi$ yields

$$\alpha\partial_{x_i} r + \varphi\left(\partial_{x_i}(v_N r) - m\partial_{x_i} h - (m+n-2)(2\lambda\varphi^2 + (\varphi')^2)\partial_{x_i} v_N\right) = 0.$$

Note that the expression $2\lambda\varphi^2 + (\varphi')^2$ is constant. Indeed,

$$(2\lambda\varphi^2 + (\varphi')^2)' = 2\varphi'(2\lambda\varphi + \varphi'') = 0.$$

Moreover, since $\varphi = \pm u'$, $u'' = -2\lambda u + \nu$ and $(u')^2 = -2\lambda u^2 + 2\nu u - 2\widehat{\lambda}$ by Lemma 3.2, we can write

$$2\lambda\varphi^2 + (\varphi')^2 = \nu^2 - 4\lambda\widehat{\lambda}. \tag{3.11}$$

Thus, the equation above becomes

$$\alpha\partial_{x_i} r + \varphi\left(\partial_{x_i}(v_N r) - m\partial_{x_i} h - (m+n-2)(\nu^2 - 4\lambda\widehat{\lambda})\partial_{x_i} v_N\right) = 0.$$

On a suitable open set, it follows that either $\partial_{x_i} r = 0$ or

$$\alpha\varphi^{-1} = -\frac{\partial_{x_i}(v_N r) - m\partial_{x_i} h - (m+n-2)(\nu^2 - 4\lambda\widehat{\lambda})\partial_{x_i} v_N}{\partial_{x_i} r} = A$$

for some constant A, since the left-hand side is a function of t and the expression in the middle is defined on N. Moreover, if we take $\tilde{v}_N = v_N + A$, we have $v = \varphi\tilde{v}_N$, so this can be considered as a solution with $\alpha = 0$. Hence, we conclude that $\partial_{x_i} r(X) = 0$ for all $i = 1, \ldots, n-1$ and all eigenvectors X (name it Case 1), or v splits as $v = \varphi\tilde{v}_N$ (name it Case 2). We analyze the two possibilities separately.

Case 1. Assume that $\partial_{x_i} r(X) = 0$ for all $i = 1, \ldots, n-1$ and all eigenvectors X on an open set. Then, we have

$$0 = -m\partial_{x_i} h + (r - (m+n-2)(\nu^2 - 4\lambda\widehat{\lambda}))\partial_{x_i} v_N,$$

which can be integrated and solved for h to find

$$h = m^{-1}(r - (m+n-2)(\nu^2 - 4\lambda\widehat{\lambda}))v_n - \xi \tag{3.12}$$

for some constant ξ. Substituting this value of h into (3.10), and using $\varphi'' = -2\lambda\varphi$ and (3.11) yields

$$0 = \alpha(r - (n-2)(\nu^2 - 4\lambda\widehat{\lambda}) - 2m\lambda\varphi^2) + m\varphi(\xi + \kappa\varphi - \alpha'\varphi'). \tag{3.13}$$

Notice that, since $\alpha = 0$ leads to Case 2, we can further assume $\alpha \neq 0$ in this instance. Dividing by φ and differentiating with respect to t, we simplify using that $\varphi'' = -2\lambda\varphi$ and $\alpha'' = -2\lambda\alpha + \kappa$ to obtain

$$0 = \frac{(r - (n-2)(\nu^2 - 4\lambda\widehat{\lambda}))(\varphi\alpha' - \alpha\varphi')}{\varphi^2},$$

so, locally, either $\varphi\alpha' - \alpha\varphi'$ or $r = (n-2)(\nu^2 - 4\lambda\widehat{\lambda})$. The first case implies $\alpha = C\varphi$ for some $C \in \mathbb{R}$, so it can be reformulated as $\alpha = 0$ again, thus fitting into Case 2 below.

Assume $r = (n-2)(\nu^2 - 4\lambda\widehat{\lambda})$. Since $r = \rho^N(X,X)$ for an arbitrary Ricci eigenvector, it follows that

$$\rho^N = (n-2)(\nu^2 - 4\lambda\widehat{\lambda})g^N = (n-2)\left(2\lambda + \frac{(\varphi')^2}{\varphi^2}\right)g \tag{3.14}$$

and the fiber (N, g^N) is Einstein. Additionally, by (1.6), we have

$$\rho(X,Y) = \rho^N(X,Y) - \left(\frac{\varphi''}{\varphi} + (n-2)\frac{(\varphi')^2}{\varphi^2}\right)g(X,Y) = 2(n-1)\lambda g(X,Y),$$

so the Ricci tensor satisfies $\rho = 2(n-1)\lambda g$, i.e., (M,g) is Einstein on an open set. Moreover, (3.13) becomes

$$m\varphi(\xi + (\kappa - 2\lambda\alpha)\varphi - \alpha'\varphi') = 0, \tag{3.15}$$

from where $\xi = \alpha'\varphi' - (\kappa - 2\lambda\alpha)\varphi$, which is indeed a constant due to the equations $\varphi'' = -2\lambda\varphi$ and $\alpha'' = -2\lambda\alpha + \kappa$. Furthermore, note that this shows that ξ does not depend on the choice of the eigenvalue X. Equation (3.12) now yields that $h(X) = -(\xi + (\nu^2 - 4\lambda\widehat{\lambda})v_N)$ for every eigenvector X, so v_N satisfies the generalized Obata equation

$$\mathrm{Hes}^N_{v_N} = -(\xi + (\nu^2 - 4\lambda\widehat{\lambda})v_N)g^N. \tag{3.16}$$

Moreover, the form of the conformal factor u given by Lemma 3.2 around its regular points yields $\widehat{\rho} = 2(n-1)\widehat{\lambda}\widehat{g}$ (see Remark 3.6 for details). This corresponds to Theorem 3.4 (1) on a suitable open set.

Case 2. Consider now solutions with $\alpha = 0$, i.e., those whose density is $v = \varphi v_N$. From the fact that $\alpha'' = -2\lambda\alpha + \kappa$, it follows that $\kappa = 0$, so (M, g, f, m) is quasi-Einstein. Moreover, from (3.6) we obtain that $\rho^m_f = 2(m + n - 1)\lambda g$. With this, equation (3.9) takes the form

$$\begin{aligned}\left(\rho^N - m v_N{}^{-1} \operatorname{Hes}_{v_N}\right)(X, Y) &= (m + n - 2)(2\lambda\varphi^2 + (\varphi')^2) g^N(X, Y) \\ &= (m + n - 2)(\nu^2 - 4\lambda\widehat{\lambda}) g^N(X, Y)\end{aligned} \tag{3.17}$$

where we have also used equation (3.11). The left-hand side in (3.17) is the Bakry-Émery Ricci tensor $\rho^m_{f_N}$ on $(N, g^N, f_N = -m \log v_N, m)$, so the fiber is also quasi-Einstein. Now the form of u given by Lemma 3.2 leads to $\widehat{\rho}^m_{\widehat{f}} = 2(m + n - 1)\widehat{\lambda}\widehat{g}$ (see Remark 3.6 for details), so Theorem 3.4 (2) follows on a suitable open set.

Finally, since by Theorem 1.15 weighted Einstein manifolds and their densities are real analytic in harmonic coordinates, and (M, g) and $(M, \widehat{g})$ are Einstein (respectively, quasi-Einstein) on an open set, they are Einstein (respectively, quasi-Einstein) everywhere. □

Remark 3.5. Although we are considering manifolds of dimension $n \geq 3$, Lemma 3.2 also applies to the 2-dimensional case. In this context, the lemma implies that, around regular points of the conformal factor u, (M^2, g) splits as a warped product of two open intervals $I_1 \times_\varphi I_2$, where the warping function satisfies $\varphi'' = -2\lambda\varphi$. A direct computation shows that the sectional curvature is constant (indeed, it is 2λ), and this property is extended by smoothness to (M, g). Thus, the two-dimensional setting becomes a reduced version of the Einstein case in Theorem 3.4. This is also the case for the global result for complete manifolds (see Theorem 3.10).

3.1.1 Remarks and examples

In this section, we will provide some additional information on the weighted Einstein SMMSs that appear in Theorem 3.4. We will also construct some examples of such SMMSs and explicit conformal factors that transform them into conformally equivalent weighted Einstein manifolds.

Notice that, as a consequence of Theorem 3.4, some of the manifolds of interest to this work have constant scalar curvature. Indeed, they are Einstein (Item (1)). However, this does not hold in general for manifolds in Item (2), which do have constant weighted scalar curvature (since they are quasi-Einstein), but not constant scalar curvature. In both cases, the corresponding constancy is preserved by the conformal transformation, as the following remark explains.

Remark 3.6. The conformal factor in Theorem 3.4 transforms Einstein manifolds into Einstein manifolds (Item (1)) and quasi-Einstein manifolds into quasi-Einstein manifolds (Item (2)). The first case corresponds to conformal transformations that have been already described in the literature (see [82]), so only the second case needs to be checked. Nevertheless, in order to be self-contained and justify the precise description in Theorem 3.4, we include the details of both transformations as follows.

The underlying manifolds of SMMSs described in Theorem 3.4 (1) are Einstein with $\rho = 2(n - 1)\lambda g$. In this case, the conformally transformed manifold $(M, \widehat{g})$ where $\widehat{g} = u^{-2} g$ is also

Einstein with $\widehat{\rho} = 2(n-1)\widehat{\lambda}\widehat{g}$. Indeed, by the transformation formula for the Ricci tensor (see, for example, [83]),

$$\widehat{\rho} = \rho + u^{-2}((n-2)u \operatorname{Hes}_u + (u\Delta u)g - (n-1)\|\nabla u\|^2 g),$$

the conformal factor is a concircular function ($\operatorname{Hes}_u = (\nu - 2\lambda u)g$), and since $(u')^2 = -2\lambda u^2 + 2\nu u - 2\widehat{\lambda}$ by Lemma 3.2, the value of $\widehat{\rho}$ follows.

For SMMSs in Theorem 3.4 (2), (M, g, f, m) is quasi-Einstein with $\rho_f^m = 2(m+n-1)\lambda g$. We check that the conformal transformation preserves the quasi-Einstein character as follows. The Bakry-Émery Ricci tensor transforms under a weighted conformal change $\widehat{g} = u^{-2}g$, $\widehat{v} = u^{-1}v$ as (see [40])

$$\begin{aligned} \widehat{\rho}_{\widehat{f}}^m &= \rho_f^m + (m+n-2)u^{-1} \operatorname{Hes}_u \\ &\quad + (u^{-1}(\Delta u - g(\nabla f, \nabla u)) - (m+n-1)u^{-2}\|\nabla u\|^2)g. \end{aligned}$$

Moreover, by Lemmas 3.2 and 3.3, and the proof of Theorem 3.4, we have $\varphi = \pm u'$, $u'' = \nu - 2\lambda u$ and $f = -m\log(\varphi v_N)$. Hence, $g(\nabla f, \nabla u) = -mu'' = -m(\nu - 2\lambda u)$ and substituting these values into the expression above yields $\widehat{\rho}_{\widehat{f}}^m = 2(m+n-1)\widehat{\lambda}\widehat{g}$, so the transformed manifold is quasi-Einstein. Hence, weighted conformal transformations of manifolds in Theorem 3.4 (1) stay in that family, and the same is true for Theorem 3.4 (2).

Remark 3.7. For non-trivial manifolds in Theorem 3.4 (1) (the Einstein case), a warped product decomposition similar to that given by Lemma 3.2 arises with respect to the density v. Indeed, since the underlying manifold (M, g) is Einstein with $\rho = 2(n-1)\lambda g$, the weighted Einstein equation (3.7) reads

$$\operatorname{Hes}_v = -(2\lambda v - \kappa)g, \tag{3.18}$$

(compare with the proof of Lemma 2.22). Hence, the arguments from Lemma 3.2 can be mimicked to split M around regular points of v, with the fibers being level hypersurfaces of v. This approach has the advantage of v depending only on a coordinate of the base, but in general this will no longer apply to u.

Moreover, for the same manifolds in Theorem 3.4 (1) and the warped product splitting $I \times_\varphi N$ given in the theorem (i.e. with the fibers being level hypersurfaces of u), (3.16) is again an Obata-type equation holding on the fiber. Hence, it induces an additional splitting of the form $N = I_2 \times_{\varphi_2} N_2$ around regular points of v_N, where the fibers are level hypersurfaces of v_N. This can be used to extract further information about the geometry of N, using known properties of the solutions of the generalized Obata equation (see Theorem 1.7, and also [120] for further details).

Remark 3.8. For manifolds in Theorem 3.4 (1), since $\rho = 2(n-1)\lambda g$, the curvature tensor satisfies $R = \lambda g \owedge g + W$. Additionally, since $P_f^m = \lambda g$, the weighted Weyl tensor (1.20) takes the form $W_f^m = W = R - \lambda g \owedge g$.

The warped product splitting $I \times_\varphi N$ with N Einstein guarantees that the Weyl tensor is harmonic (see Theorem 1.6). Also, if we consider the alternative warped product splitting discussed in Remark 3.7 around regular points of the density v, it is straightforward to check that

$\iota_{\nabla f} W_f^m = 0$. Hence, it follows that the weighted divergence $\operatorname{div}_f W_f^m = \operatorname{div} W_f^m - \iota_{\nabla f} W_f^m$ vanishes too. In the non-trivial case, it follows that the manifolds in Theorem 3.4 (1) are thus exactly those that arise in Theorem 2.16 (the local result for the Einstein case with $\operatorname{div}_f W_f^m = 0$). In particular, weighted space forms are included in this family, but other kinds of SMMSs with $W_f^m \neq 0$ also appear (see Example 2.19).

Consequently, we can use the local warped product decomposition from Theorem 2.16, where (M, g) is locally isometric to $I \times_\varphi N$, where N is Einstein with $\rho^N = \beta g^N$. Let t be a coordinate parameterizing I by arc length. In general, as pointed out in Remark 3.7, the density v and the conformal factor u do not necessarily depend on the same variables. However, in the case where $u = u(t)$ and $v = v(t)$, we can use the aforementioned theorem, along with Lemma 3.2, to describe these SMMSs very explicitly. Indeed, the functions v, φ and u and the parameters β and μ take the following forms (the value of μ being irrelevant if $m = 1$):

$\lambda > 0$	$\varphi(t) = a\cos(t\sqrt{2\lambda}) + b\sin(t\sqrt{2\lambda})$	$\beta = 2(a^2+b^2)(n-2)\lambda$
	$v(t) = \frac{\kappa}{2\lambda} - bc\cos(t\sqrt{2\lambda}) + ac\sin(t\sqrt{2\lambda})$	$\mu = 2(a^2+b^2)c^2\lambda - \frac{\kappa^2}{2\lambda}$
	$u(t) = \frac{\nu}{2\lambda} \mp \frac{b}{\sqrt{2\lambda}}\cos(t\sqrt{2\lambda}) \pm \frac{a}{\sqrt{2\lambda}}\sin(t\sqrt{2\lambda})$	$\widehat{\lambda} = \frac{\nu^2}{4\lambda} - \frac{a^2+b^2}{2}$
$\lambda = 0$	$\varphi(t) = a\kappa t + b$	$\beta = a^2\kappa^2(n-2)$
	$v(t) = \frac{\kappa}{2}t^2 + ct + d$ where $b = ac$ if $\kappa \neq 0$	$\mu = c^2 - 2d\kappa$
	$u(t) = \frac{\nu t^2}{2} \pm bt + l \qquad (\nu = \pm a\kappa)$	$\widehat{\lambda} = -\frac{1}{2}b^2 + l\nu$
$\lambda < 0$	$\varphi(t) = ae^{t\sqrt{-2\lambda}} + be^{-t\sqrt{-2\lambda}}$	$\beta = 8ab(n-2)\lambda$
	$v(t) = \frac{\kappa}{2\lambda} + ace^{t\sqrt{-2\lambda}} - bce^{-t\sqrt{-2\lambda}}$	$\mu = -8abc^2\lambda - \frac{\kappa^2}{2\lambda}$
	$u(t) = \frac{\nu}{2\lambda} \pm \frac{a}{\sqrt{-2\lambda}}e^{t\sqrt{-2\lambda}} \mp \frac{b}{\sqrt{-2\lambda}}e^{-t\sqrt{-2\lambda}}$	$\widehat{\lambda} = \frac{\nu^2}{4\lambda} - 2ab$

Table 3.1: Local descriptions of the density v, warping function φ and conformal factor u in the warped product decomposition $I \times_\varphi N$ of SMMSs in Theorem 3.4 (1) when u and v are defined on I (cf. Theorem 2.16); and corresponding values of the parameters β, μ and $\widehat{\lambda}$.

The constants a, b, c, d and the scale κ are such that φ, v and u are positive on I. Notice that the density and the conformal factor present qualitatively the same behavior, up to some constant translation and rescaling. This allows for conformal changes between trivial and non-trivial SMMSs by suitably adjusting the constants involved (cf. Remark 3.15).

Moreover, observe that in all three cases the value of $\widehat{\lambda}$ after the conformal change can be any real number. This contrasts with the behavior of global conformal changes (see Theorem 3.10 and Corollary 3.12).

Although many relatively simple weighted Einstein structures are realized on Einstein manifolds (such as the weighted space forms), more involved structures are admissible as well, like those that fall into Theorem 3.4 (2), whose underlying manifold is quasi-Einstein. The following is an example constructed from a quasi-Einstein manifold with $\rho_f^m = 0$ that was built in [113, Example 2].

Example 3.9. For any value of $m \neq 1$, consider the direct product $(N, g_0) = (\tilde{N}_2^{n-2} \times \mathbb{R}, g^{\tilde{N}} \oplus ds^2)$, where $\tilde{N}$ is an Einstein manifold with positive Einstein constant ξ, i.e. $\rho^{\tilde{N}} = \xi g^{\tilde{N}}$. Define the function $\sigma : C^\infty(N) \to \mathbb{R}$ by $\sigma = \frac{m\sqrt{\xi}}{\sqrt{m+n-3}}\pi_{\mathbb{R}}$, where $\pi_{\mathbb{R}}$ denotes the projection on $\mathbb{R}$, and take the conformal metric $g^N = e^{-2\frac{\sigma}{m}} g_0$. Then, (N, g^N, σ, m) is a quasi-Einstein manifold with $(\rho_\sigma^m)^N = 0$ (see [113]).

Next, consider the direct product $(M^n, g) = (I \times N, dt^2 \oplus g^N)$, where $I \subset \mathbb{R}$ is an open interval, and set $f = \sigma$. Taking $\mu = \frac{\xi}{m-1}$ makes (M, g, f, m, μ) a weighted Einstein manifold with $\lambda = 0$ (and it is also quasi-Einstein with $\rho_f^m = 0$). Moreover, let $u = at + b$ for constants $a \neq 0, b$ such that $u > 0$ in I. Take the weighted conformal change given by $\widehat{g} = u^{-2} g$ and $\widehat{f} = f + m\log(u)$. Then, $(M, \widehat{g}, \widehat{f}, m, \mu)$ is weighted Einstein with $\widehat{\lambda} = -\frac{a^2}{2} < 0$, and also quasi-Einstein with $\widehat{\rho}_{\widehat{f}}^m = -(m+n-1)a^2\widehat{g}$. Note that neither (M, g) nor $(M, \widehat{g})$ are Einstein. Indeed, $\tau = \frac{m(n-2)}{m+n-3}\xi e^{2\frac{f}{m}}$ and $\widehat{\tau} = u^2\tau - n(n-1)a^2$ are non-constant.

3.2 The complete case

A key point in our analysis of SMMSs with two weighted Einstein structures in the same weighted conformal class is that the conformal factor satisfies the generalized Obata equation (3.4), but the results stated in the previous section have mostly been centered around the local features of the geometry of the SMMSs of interest. In this section, we focus on SMMSs with complete underlying manifold to prove the main global classification result of the chapter, taking into account that the function γ in (3.4) takes the form $\gamma(u) = 2\lambda u - \nu$ given in Lemma 3.2.

Theorem 3.10. *Let (M^n, g, f, m, μ) be a complete SMMS such that $P_f^m = \lambda g$, with scale κ, and such that there exists a conformally equivalent weighted Einstein SMMS. Then, (M, g, f, m, μ) is isometric to one of the following SMMSs:*

1. *A weighted space form as described in Examples 1.12, 1.13 and 1.14.*
2. *A warped product $\mathbb{R} \times_\varphi N$, with N complete, and such that $\varphi(t) = Ae^{t\sqrt{-2\lambda}}$, where t parameterizes $\mathbb{R}$ by arc length. Moreover, $\lambda < 0$ and one of the following holds:*
   (a) *(M, g) is Einstein and (N, g^N) is Ricci-flat. The density function has the form*
   $$f(t) = -m\log\left(\frac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}}\right),$$
   for some $B \geq 0$ and $\kappa \leq 0$. Moreover, $m = 1$ or $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$.
   (b) *(M, g, f, m) is quasi-Einstein, f splits as $f = -m\log\varphi + f_N$, and (N, g^N, f_N, m) is also quasi-Einstein with $(\rho_{f_N}^m)^N = 0$.*

Proof. Let (M^n, g, f, m, μ) be a complete SMMS with $P_f^m = \lambda g$, such that there exists a SMMS $(M^n, \widehat{g}, \widehat{f}, m, \mu)$, with $\widehat{P}_{\widehat{f}}^m = \widehat{\lambda}\widehat{g}$, related by a non-constant conformal factor u, i.e. $\widehat{g} = u^{-2}g$ and $\widehat{v} = u^{-1}v$. By Lemma 3.2 we know that the (globally defined) conformal factor u satisfies the generalized Obata equation $\mathrm{Hes}_u = (\nu - 2\lambda u)g$, which allows us to apply Theorem 1.7. There are two possibilities that we analyze separately. First, consider the case where u has critical points. Then, (M, g) is isometric to

$$g = dt^2 + \varphi(t)^2 g_{\mathbb{S}^{n-1}}, \quad t \in (0, T),$$

with $\varphi(t) = \frac{u'(t)}{(2\lambda u(0) - \nu)}$, where

$$u'' + 2\lambda u - \nu = 0, \quad u(0) = \xi > 0, \quad u'(0) = 0.$$

If $\lambda > 0$, it follows that $u(t) = \frac{1}{2\lambda}\left(\nu + (2\xi\lambda - \nu)\cos(t\sqrt{2\lambda})\right)$, with $t \in (0, \frac{\pi}{\sqrt{2\lambda}})$. The warping function is $\varphi(t) = \frac{1}{\sqrt{2\lambda}} \sin\left(t\sqrt{2\lambda}\right)$. Hence, (M, g) is isometric to a sphere of constant sectional curvature 2λ, which only admits solutions in the form of the m-weighted n-spheres as described in Example 1.12. For $\lambda = 0$ and $\lambda < 0$ analogous processes lead to the m-weighted n-Euclidean spaces of Example 1.13 and to the m-weighted n-hyperbolic spaces of Example 1.14. Locally, these spaces look like the SMMSs in Theorem 3.4 (1). Moreover, by the argument in Remark 3.8, they have weighted harmonic Weyl tensor, so the classification in Theorem 2.23 also implies this conclusion in the non-trivial case. These are the SMMSs in Theorem 3.10 (1).

Now, we consider the other possibility in Theorem 1.7, this is, the case where u has no critical points, which guarantees that (M, g) is isometric to a warped product of the form $\mathbb{R} \times_\varphi N$ where ∇u is tangent to $\mathbb{R}$.

Then, we use the fact that the functions v, φ and u are globally defined. The warping function φ and the conformal factor u satisfy the ODEs $u'' + 2\lambda u - \nu = 0$ and $\varphi = \pm u'$ given in Lemma 3.2, and the only global solutions such that u has no critical points and both u and φ remain positive on $\mathbb{R}$ are (after an inversion of the sign of t if necessary) of the form $\varphi = Ae^{t\sqrt{-2\lambda}}$ and $u(t) = \frac{\nu}{2\lambda} + \frac{A}{\sqrt{-2\lambda}} e^{t\sqrt{-2\lambda}}$ for $\lambda < 0$ and some $A > 0$, $\nu \leq 0$ (compare with the proof of Theorem 2.23). Now, the expression of φ yields, from (3.11), $\nu^2 - 4\lambda\widehat{\lambda} = 2\lambda\varphi^2 + (\varphi')^2 = 0$. Thus, $\widehat{\lambda} = \frac{\nu^2}{4\lambda} \leq 0$, so $u(t) = \sqrt{\frac{\widehat{\lambda}}{\lambda}} + \frac{A}{\sqrt{-2\lambda}} e^{t\sqrt{-2\lambda}}$.

By Lemma 3.3, the density v takes the form $v = \varphi v_N + \alpha$. We distinguish between the cases where $\alpha \neq 0$ and $\alpha = 0$.

Case 1. Assume $\alpha \neq 0$. First, observe that the SMMS falls into Theorem 3.4 (1), so g and $\tilde{g}$ are Einstein. From (3.14), since $2\lambda\varphi^2 + (\varphi')^2 = 0$, we obtain $\rho^N = 0$, i.e. the fiber N is Ricci-flat.

From $\alpha'' = -2\lambda\alpha + \kappa$, it follows that $\alpha(t) = \frac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}} + Ce^{-t\sqrt{-2\lambda}}$ for some $B, C \geq 0$ such that $v > 0$. Moreover, for this form of α, equation (3.15) reduces to $\xi = 4AC\lambda$, so the generalized Obata equation (3.16) for the fiber takes the form $\mathrm{Hes}_{v_N} = -4AC\lambda g^N$. Hence, either v_N is constant and $C = 0$; or $C > 0$.

- If v_N is constant and $C = 0$, redefining the constant B if necessary, we can take $v = \alpha(t) = \frac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}}$. In order for v to stay positive in $\mathbb{R}$, it follows that $\kappa \leq 0$. Moreover, a straightforward calculation from the weighted Einstein equation $P_f^m = \lambda g$ yields that $m = 1$ or $\mu = -\frac{\kappa^2}{2\lambda} \geq 0$. This corresponds to Theorem 3.10 (2.a).

- If $C > 0$, from equation $\mathrm{Hes}_{v_N} = -4AC\lambda g^N$, it follows that N is isometric to $\mathbb{R}^{n-1}$ and $v_N(r) = -2AC\lambda r^2 + D$, where r is the radial coordinate around some point of $\mathbb{R}^{n-1}$ (see [120, Theorem 6.3]). The manifold $\mathbb{R} \times_\varphi \mathbb{R}^{n-1}$ is isometric to the hyperbolic space. In order to see that this case corresponds to Example 1.14, we keep analyzing the density function. Since $A, C > 0$ and $\lambda < 0$, it follows that v remains positive for all values of r if and only if $D \geq 0$. Redefining the constant B, we can write $v = \frac{\kappa}{2\lambda} + Be^{t\sqrt{-2\lambda}} + Ce^{-t\sqrt{-2\lambda}} - 2A^2C\lambda r^2 e^{t\sqrt{-2\lambda}}$, so v necessarily has a critical point. Now, applying the splitting given in Remark 3.7 and knowing that v has critical points, by Theorem 1.7, we conclude that this case corresponds to Example 1.14 (the m-weighted n-hyperbolic space) so it falls into Theorem 3.10 (1). Additionally, with this expression for v, one checks that these SMMSs satisfy $W = W_f^m = 0$, so the conclusion also follows from the classification in Theorem 2.23.

Case 2. Assume $\alpha = 0$. Notice that, in this case, the SMMS falls into Theorem 3.4 (2). Then, equation (3.17) guarantees that $(N, g^N, f_N = -m \log v_N, m)$ is quasi-Einstein with $(\rho^m_{f_N})^N = 0$. This yields the quasi-Einstein manifolds in Theorem 3.10 (2.b). □

The underlying manifolds (M, g) for the SMMSs in Theorem 3.10 (2.a) and the weighted space forms in Theorem 3.10 (1) are Einstein with $\rho = 2(n-1)\lambda g$. Hence, they fall into Theorem 3.4 (1). Furthermore, these complete manifolds are precisely those that admit a non-homothetic conformal change into another Einstein manifold (see [82, Theorem 27]). Consequently, for trivial SMMSs, i.e., those with constant density function, we recover this classical result. Strikingly, the non-trivial SMMSs in these families correspond exactly with those which are weighted Einstein and have harmonic weighted Weyl tensor, as shown in Theorem 2.23.

Note that, much like in Theorem 2.23 (3.b), the warped product construction $I \times_\varphi N$ in Theorem 3.10 (2.a) can be performed for any complete Ricci-flat fiber N, without changing the parameters of the resulting SMMS $(I \times_\varphi N, g, f, m, \mu)$. The most obvious choice is the flat Euclidean manifold $\mathbb{R}^{n-1}$, but if the fiber is of dimension $n \geq 4$, then N can be taken to be complete and Ricci-flat, but not flat. For example, two types of manifolds which fit this description and were already mentioned in Chapter 2 are the cotangent bundle of the 2-sphere $T^*\mathbb{S}^2$ with the Eguchi-Hanson metric (see [56]), and Calabi-Yau manifolds (see [123]).

As a result of the discussion above, it follows that the genuinely new metric structures in Theorem 3.10 are those given in Theorem 3.10 (2.b). In order to build an example of this kind, it suffices to take any non-trivial complete quasi-Einstein manifold with $\rho_f^m = 0$ as a fiber. The following one illustrates this fact, using a complete fiber related to the construction of the *generalized Schwarzschild metric* (see [3, Example 9.118a] and [71] for more details).

Example 3.11. For $m > 1$, consider the two-dimensional manifold $N = \mathbb{R}^2$ endowed with the warped product metric $g^N = dx^2 + (\omega'(x))^2 d\theta^2$, where ω is the unique solution on $[0, \infty)$ of the problem

$$(\omega')^2 = 1 - \omega^{1-m}, \quad \omega(0) = 1, \quad \omega \geq 0.$$

For example, if $m = 3$, then $\omega(x) = \sqrt{1 + x^2}$. This metric extends smoothly across $x = 0$, and the resulting manifold $(N, g^N, -m\log(\omega), m)$ is complete and quasi-Einstein (with quasi-Einstein constant 0), but ρ^N is not, in general, a constant multiple of the metric (indeed, for

$m=3$, $\rho^N=\frac{3}{(1+x^2)^2}g^N$). Thus, for $(\mathbb{R}^3, g=dt^2+\varphi^2 g^N)$, $\lambda<0$, $\varphi(t)=e^{\sqrt{-2\lambda}t}$ and $f(t,x,\theta)=-m\left(\sqrt{-2\lambda}\,t+\log(\omega(x))\right)$, the SMMS $(\mathbb{R}^3,g,f,m,\mu)$ is quasi-Einstein with $\rho_f^m=2(m+2)\lambda g$ and thus, weighted Einstein for an appropriate μ (for $m=3$, the value is $\mu=1$). Moreover, for $\widehat{\lambda}<0$ and the conformal factor $u(t)=\sqrt{\frac{\widehat{\lambda}}{\lambda}}+\frac{1}{\sqrt{-2\lambda}}e^{\sqrt{-2\lambda}t}$, the conformally transformed SMMS $(\mathbb{R}^3,u^{-2}g,f+m\log(u),m,\mu)$ is also quasi-Einstein with $\widehat{\rho}^m_{\widehat{f}}=2(m+2)\widehat{\lambda}\widehat{g}$, hence weighted Einstein with $\widehat{P}^m_{\widehat{f}}=\widehat{\lambda}\widehat{g}$.

Theorem 3.10 has very strong consequences in the case of compact SMMSs. Indeed, the following rigidity result generalizes the one given in [36] for $\mu=0$. Moreover, from the point of view that smooth metric measure spaces generalize manifolds with constant density, we can say that it also extends [82, Corollary 23] to the weighted setting, in the sense that the existence of a solution of the generalized Obata equation forces the underlying manifold to be conformally equivalent to a sphere.

Corollary 3.12. *Let (M^n,g,f,m,μ) be a non-trivial compact weighted Einstein SMMS. If there exists a non-constant conformal factor such that the transformed manifold is weighted Einstein, then (M,g,f,m,μ) is an m-weighted n-sphere (which is conformally equivalent to a standard sphere with vanishing density).*

Proof. Out of the admissible geometries for complete SMMSs given by Theorem 3.10, the only compact ones are the weighted spheres in Example 1.12. Hence, $\lambda>0$ and (M^n,g,f,m,μ) is globally isometric to such a sphere, so $v(t)=A+B\cos(\sqrt{2\lambda}t)$ for some constants $A\in\mathbb{R}^+$, $B\in\mathbb{R}$ such that $A>|B|$. Now, we can take the conformal factor $u=v$, so that $\widehat{f}=f+m\log u=0$. Then, it is straightforward to prove that the conformally transformed SMMS $(M,u^{-2}g,0,m,\mu)$ has constant sectional curvature $\widehat{\lambda}=(A^2-B^2)\lambda>0$, so it is isometric to the sphere $(\mathbb{S}^n,g_{\mathbb{S}}^{2\widehat{\lambda}},0,m,\mu)$, which is a trivial weighted Einstein manifold with $\widehat{P}^m_{\widehat{f}}=\widehat{\lambda}g_{\mathbb{S}}^{2\widehat{\lambda}}$. □

Remark 3.13. Note that, according to Example 1.12, the curvature parameter μ for the weighted spheres is given by $\mu=2\lambda(B^2-A^2)$. Thus, the condition $A>|B|$ guarantees that the only weighted spheres with $\mu=0$ are those with $m=1$, where μ does not play a role. Thus, this recovers the result in [36].

Example 3.14. Due to the fact that the density v satisfies (3.18), the only weighted Einstein structures on standard spheres are the m-weighted n-spheres portrayed in Example 1.12. However, since the symmetry of the sphere allows for the poles to be any two antipodal points, the conformal factor does not necessarily vary in the same direction as the density, and it can be used to rotate it and modify its radius while maintaining its weighted Einstein character. For example, let $\lambda>0$ and consider the sphere $(\mathbb{S}^n,g_{\mathbb{S}}^{2\lambda})$, whose metric can be written as

$$g_{\mathbb{S}}^{2\lambda}=dt^2+(2\lambda)^{-1}\sin^2(\sqrt{2\lambda}t)(d\theta^2+\sin^2(\theta)g_{\mathbb{S}^{n-2}}),\;\; t\in\left(0,\tfrac{\pi}{\sqrt{2\lambda}}\right),\;\;\theta\in(0,\pi)\,.$$

The warping function $\varphi(t)=\frac{\sin(t\sqrt{2\lambda})}{\sqrt{2\lambda}}$ is induced by the conformal transformation given by $u(t)=\frac{\nu}{2\lambda}-\frac{\cos(t\sqrt{2\lambda})}{2\lambda}$, with $\nu>1$, since $\varphi=u'$. Now, instead of the density given in Example 1.12, consider $v(t,\theta)=A\varphi(t)\cos(\theta)+B\cos(t\sqrt{2\lambda})+\frac{\kappa}{2\lambda}$ (where, A,B,κ are such that v

is always positive). The corresponding SMMS $(\mathbb{S}^n, g_{\mathbb{S}}^{2\lambda}, f, m, \mu)$ is weighted Einstein with scale κ and $P_f^m = \lambda g$ for $\mu = A^2 + 2\lambda B^2 - \frac{\kappa^2}{2\lambda}$ or $m = 1$. Notice that v is of the form given in Lemma 3.3 with $\alpha(t) = \frac{\kappa}{2\lambda} + B\cos(t\sqrt{2\lambda})$, so that $\xi = \alpha'\varphi' - (\kappa - 2\lambda\alpha)\varphi = 0$, where ξ is the constant in the generalized Obata equation (3.16).

Moreover, the transformed manifold $(\mathbb{S}^n, \widehat{g}, \widehat{f}, m, \mu)$, is weighted Einstein with $\widehat{P}_{\widehat{f}}^m = \widehat{\lambda}\widehat{g}$ and $\widehat{\lambda} = \frac{\nu^2-1}{4\lambda} > 0$. In this case, $v_N(\theta) = A\cos(\theta)$ is indeed a solution of the generalized Obata equation (3.16) on $\mathbb{S}^{n-1}$ for $\xi = 0$ and $\nu^2 - 4\lambda\widehat{\lambda} = 1$, as stated in Theorem 3.4 (1), and the metric $\widehat{g}$ has constant sectional curvature $2\widehat{\lambda}$, so $(\mathbb{S}^n, \widehat{g}, \widehat{f}, m, \mu)$ is a weighted sphere.

Remark 3.15. The proof of Corollary 3.12 relies on the fact that we can take the conformal factor u to be equal to the density function v so that $\widehat{f} = f + m\log u = 0$. This is possible for most of the other Einstein manifolds in Theorem 3.10. However, the sign of the weighted Einstein constant is not necessarily preserved under such transformations when $\lambda \leq 0$, in constrast to the case $\lambda > 0$.

$\underline{\lambda = 0}$: For the m-weighted n-Euclidean space as in Example 1.13 and the conformal change $u(t) = v(t) = A + Bt^2$, $(M^n, \widehat{g} = u^{-2}g_{\mathbb{E}}, 0, m, \mu)$ is isometric to the punctured sphere $(\mathbb{S}^n \setminus \{N\}, g_{\mathbb{S}}^{2\widehat{\lambda}}, 0, m, \mu)$ with $\widehat{\lambda} = 2AB > 0$, which is weighted Einstein with $\widehat{P}_{\widehat{f}}^m = \widehat{\lambda}g_{\mathbb{S}}^{2\widehat{\lambda}}$. Indeed, note that $\widehat{g} = u^{-2}g$ is essentially the change of the metric given by stereographic projection.

$\underline{\lambda < 0}$: For the m-weighted n-hyperbolic space of constant sectional curvature 2λ as in Example 1.14 and the conformal change given by $u(t) = v(t) = A + B\cosh(\sqrt{2\lambda}t)$, $(M^n, \widehat{g} = u^{-2}g_{\mathbb{H}}^{2\lambda}, 0, m, \mu)$ is weighted Einstein with $\widehat{\lambda} = (A^2 - B^2)\lambda$, and it has constant sectional curvature $2\widehat{\lambda}$. In this case, $\widehat{\lambda}$ can be positive, zero or negative depending on the values of A and B (since $A > -B$), so $(M, \widehat{g})$ can be isometric to a punctured sphere, a Euclidean space or a hyperbolic space.

On the other hand, for a warped product $\mathbb{R} \times_\varphi N$ as in Theorem 3.10 (2.a), where $u(t) = v(t) = \frac{\kappa}{2\lambda} + \frac{A}{\sqrt{-2\lambda}}e^{t\sqrt{-2\lambda}}$, N is a Ricci-flat complete manifold and $\varphi(t) = Ae^{t\sqrt{-2\lambda}}$, we have that for $\widehat{\lambda} = \frac{\kappa^2}{4\lambda} \leq 0$, the transformed SMMS $(M, \widehat{g}, 0, m, \mu)$ satisfies $\widehat{P}_{\widehat{f}}^m = \widehat{\lambda}\widehat{g}$.

In contrast to the previous items, in the cases in Theorem 3.10 (2.b) where v_N is non-constant, the conformal factor is defined on the base of the warped product $I \times_\varphi N$, while the density has a non-constant component in N, so we cannot take $u = v$. Nevertheless, in Theorem 3.10 (2.b) we can take $u = \varphi$, in which case the transformed density $\widehat{f}$ will be constant on the base of the product.

3.3 A note on weighted Bach-flat WE manifolds

As a final note before moving on to Part II of the dissertation, we point out the interesting interplay between the SMMSs in Chapters 2 and 3 and the weighted analogue of the Bach tensor B, which is related to the conformal properties of semi-Riemannian manifolds. In particular, the Bach-flat condition (i.e., $B = 0$) is useful in discussions around the conformal Einstein equation introduced by Brinkmann [11]. Firstly, let T be a $(0, 4)$-tensor and S a $(0, 2)$-tensor, and denote by $T(-,-)-$ and $S(-)$ the $(1,3)$ and $(1,1)$-tensors such that $g(T(X,Y)Z,U) = T(X,Y,Z,U)$ and $g(S(X),Y) = S(X,Y)$ for all $X,Y,Z,U \in \mathfrak{X}(M)$. Then, we define the following contraction:

$$T[S](X,Y) = \operatorname{tr}(Z \mapsto T(X,S(Z))Y).$$

Let $\{E_1,\dots,E_n\}$ be a local orthonormal frame, then

$$T[S](X,Y) = \sum_{i,j=1}^{n} T(X,E_i,Y,E_j)S(E_i,E_j).$$

or, in index notation, $T[S]_{ij} = T_{ikjl}S^{kl}$. Thus, the trace of the weighted Weyl tensor over its second and fourth arguments, which coincides with its trace over the first and third arguments due to its symmetries, is $\operatorname{tr} W_f^m = W_f^m[g]$. With this, the usual Bach tensor is is written as $B = \operatorname{div}_2 dP + W[\rho] = 0$ (up to constant scaling). For SMMSs, the *weighted Bach tensor* takes a similar form:

$$B_f^m = (\operatorname{div}_f)_2 dP_f^m + \frac{1}{m} df \otimes \operatorname{tr} dP_f^m + W_f^m \left[P_f^m - \frac{Y_f^m}{m} g \right], \tag{3.19}$$

where

$$(\operatorname{div}_f)_2 dP_f^m(X,Y) = \sum_{i=1}^{n} (\nabla_{E_i} dP_f^m)(X,E_i,Y) - dP_f^m(X,\nabla f,Y),$$

$Y_f^m = J_f^m - \operatorname{tr} P_f^m$ and $\operatorname{tr} dP_f^m$ is taken over its first and third arguments. This definition comes from the application of tractor calculus to SMMSs [39, 41] and, although it plays an important role in variational results regarding the weighted σ_k-curvatures (see [34]), its influence over the geometry of SMMS is more nebulous than that of its unweighted counterpart. Thus, one might wonder to what extent, if at all, the relation between some significant families of manifolds and the Bach-flat condition extends to their weighted counterparts. For example, since $A_f^m = 0$ implies $dP_f^m = 0$ (see [34, 41] and Section 1.5 for details), it follows that locally conformally flat SMMSs are weighted Bach-flat, thus mirroring this characteristic of locally conformally flat manifolds in the standard setting.

However, this analogous behavior does not extend to weighted Einstein manifolds. In the unweighted setting, Einstein manifolds of any dimension, as well as 4-dimensional conformally Einstein manifolds, are Bach-flat (see, for example, [83]). In contrast, weighted Einstein manifolds do not necessarily have vanishing weighted Bach tensor. Nevertheless, we will see that the less general quasi-Einstein manifolds do present this property. In fact, QE manifolds and the WE manifolds with $\operatorname{div}_f W_f^m = 0$ discussed in the Einstein case of Chapter 2 turn out to be the only examples of weighted Bach-flat WE manifolds. In order to prove this, we start by recalling a useful computation from [41], whose derivation we adapt to our notation.

Lemma 3.16 [41]. *Let (M,g,f,m,μ) be a SMMS. Then,*

$$\operatorname{tr} W_f^m = mP_f^m + mv^{-1}\operatorname{Hes}_v + Y_f^m g.$$

Proof. From the definition of the weighted Weyl tensor (1.20), we have

$$W_f^m = R - P_f^m \owedge g = W - (P_f^m - P) \owedge g.$$

Now, notice that for any symmetric $(0,2)$-tensor T, the trace over the first and third arguments of the Kulkarni-Nomizu product $T \owedge g$ is $\operatorname{tr}(T \owedge g) = (n-2)T + (\operatorname{tr} T)g$. Thus, since the usual Weyl tensor is traceless,

$$\operatorname{tr} W_f^m = -(n-2)(P_f^m - P) - (J_f^m - J)g + Y_f^m g.$$

Moreover, from the formulas in Lemma 2.6, it follows that $J_f^m - J = mv^{-1}*$, and

$$P_f^m - P = -\frac{m}{m+n-2}P - \frac{mv^{-1}}{m+n-2}\left(\operatorname{Hes}_v + *\, g\right),$$

for a scalar $*$ whose value is irrelevant to our argument. Therefore,

$$\begin{aligned}\operatorname{tr} W_f^m &= \tfrac{m(n-2)}{m+n-2}P + \tfrac{mv^{-1}(n-2)}{m+n-2}\left(\operatorname{Hes}_v + *\, g\right) - mv^{-1} * g + Y_f^m g \\ &= mP_f^m + mv^{-1}\operatorname{Hes}_v + Y_f^m g.\end{aligned}$$

□

Our claim then follows from the study of the form of (3.19) for a WE manifold.

Lemma 3.17. *Let (M^n, g, f, m, μ) be a non-trivial weighted Einstein SMMS with $P_f^m = \lambda g$ and scale κ. Then, it is weighted Bach-flat if and only if it is quasi-Einstein (i.e., $\kappa = 0$) or an Einstein SMMS with $\operatorname{div}_f W_f^m = 0$ as in Theorem 2.16.*

Proof. For any WE manifold, we have $dP_f^m = 0$. Moreover, since $P_f^m = \lambda g$ and the scale is κ, the weighted Schouten scalar takes the form $J_f^m = (m+n)\lambda - m\kappa v^{-1}$. With this, the weighted Bach tensor (3.19) reduces to

$$B_f^m(X,Y) = W_f^m\left[\lambda g - \frac{J_f^m - n\lambda}{m} g\right] = \kappa v^{-1}\operatorname{tr} W_f^m. \tag{3.20}$$

Hence, a weighted Einstein manifold is weighted Bach-flat if and only if it is quasi-Einstein ($\kappa = 0$) or it satisfies $\operatorname{tr} W_f^m = 0$. In the case with non-vanishing scale, by Lemma 2.7, we have

$$\begin{aligned}\operatorname{div}_f W_f^m &= -\left(\tfrac{1}{m}Y_f^m + \lambda\right) g \wedge df + \tfrac{1}{m}\operatorname{Hes}_f \wedge df \\ &= \left((Y_f^m + m\lambda)g + mv^{-1}\operatorname{Hes}_v\right) \wedge v^{-1}dv \\ &= \operatorname{tr} W_f^m \wedge v^{-1}dv\end{aligned}$$

On the other hand, since $W_f^m = R - \lambda g \owedge g$ by the weighted Einstein condition, we have $\operatorname{tr} W_f^m = \rho - 2(n-1)\lambda g$. Thus, $\operatorname{tr} W_f^m = 0$ if and only if $\rho = 2(n-1)\lambda g$. Moreover, by the equation above, these SMMSs have $\operatorname{div}_f W_f^m = 0$. The SMMSs satisfiying these conditions are described in Theorem 2.16. □

From Lemma 3.17, we see that the condition $\operatorname{div}_f W_f^m = 0$ does not guarantee the weighted Bach-flatness of a weighted Einstein manifold. Indeed, the key non-Einstein SMMS portrayed in Example 2.4 has $B_f^m \neq 0$, since its weighted Schouten scalar is $J_f^{1/2} = -\frac{1}{(n-1)t^2} = -m\kappa v^{-1} \neq 0$ (recall that $\lambda = 0$ for this SMMS), which implies $\kappa \neq 0$. This contrasts with the situation in the problem of non-uniqueness of WE structures in the same conformal class, given that all of the SMMSs presented throughout Chapter 3 are weighted Bach-flat, as shown by the following result.

Corollary 3.18. *Let (M^n, g, f, m, μ) be a non-trivial weighted Einstein SMMS with $P_f^m = \lambda g$ and scale κ. If (M^n, g, f, m, μ) admits another WE representative in its conformal class, then it is weighted Bach-flat.*

Proof. By Theorem 3.4, weighted Einstein manifolds (including trivial ones) which admit another WE representative in their conformal class are either quasi-Einstein with $\rho_f^m = 2(m+n-1)\lambda$ (so $\kappa = 0$) or Einstein with $\rho = 2(n-1)\lambda g$ (in which case $\operatorname{tr} W_f^m$ and $\operatorname{div}_f W_f^m$ vanish). From equation (3.20), it follows that (M^n, g, f, m, μ) is weighted Bach-flat in either case. □

Part II

The weighted Einstein field equations

Up until this point, this thesis has been mostly centered around results concerning smooth metric measure spaces in Riemannian signature, using definitions coined by Case and motivated by problems in geometric analysis for definite metrics (see Section 1.5). Nevertheless, in principle, SMMSs can present metrics of any signature, since the introduction of a density function also makes sense for indefinite metrics.

In this context, Lorentzian manifolds with density, which we will also refer to as *smooth metric measure spacetimes*, are particularly relevant due to their potential applications to modified theories of gravity. An example of this are scalar–tensor gravitational theories (such as dilaton gravity and Brans-Dicke theory), in particular when the Jordan frame replaces the Einstein frame to be used as conformal gauge. For instance, in this context, in the Brans-Dicke family of theories the density function is taken as a scalar field non-minimally coupled to the metric tensor in the Einstein frame [117].

Similarly, comparison geometry results for standard Lorentzian manifolds have also been extended to SMMSs by Case [42] and Woolgar and Wylie [118], who stated new versions of the Hawking-Penrose singularity theorem and the timelike splitting theorem in terms of the Bakry-Émery Ricci tensor (1.16). Moreover, Rupert and Woolgar [109] explored the extension of analogues of theorems from black holes in General Relativity by imposing energy conditions on this tensor and the density function.

This part of the thesis revolves around the definition and study of an analogue, suitable for manifolds with density, of the usual Einstein tensor (with cosmological constant), which is

$$G = \rho - \frac{\tau}{2} g + \Lambda g. \tag{II.1}$$

The name of the cosmological constant Λ is a reference to its influence on the accelerated expansion of the Universe. This tensor plays an essential role in General Relativity, and it arises both from the variation of the Einstein-Hilbert functional (see Section 1.4) and as the only symmetric $(0, 2)$-tensor which is divergence-free, dependent only on the metric and its first two derivatives, and which is linear on the second derivatives of the metric [89]. Thus, in Chapter 4, we define a weighted Einstein tensor that includes information on the density function through two different avenues: Firstly, we utilize a variational approach from a weighted Einstein-Hilbert functional, and secondly, we consider the translation of the characterizing properties of G to the weighted setting in a very natural way. As a result of this process, we define our weighted analogue of the Einstein tensor (with cosmological constant) as

$$G^h = h\rho - \operatorname{Hes}_h + (\Delta h + \Lambda) g$$

for a positive density function $h \in C^\infty(M)$. Manifolds admitting a density such that G^h vanishes are said to be solutions of the (vacuum) weighted Einstein field equations.

Thus, Chapters 5 and 6 are devoted to the study of the local geometric structure of manifolds with vanishing weighted Einstein tensor under different geometric conditions. In the first of these two chapters, we study isotropic solutions, meaning those whose density h has a lightlike gradient ∇h. In Chapter 6, we analyze solutions under several conditions on the Weyl tensor, focusing on solutions with harmonic Weyl tensor ($\operatorname{div} W = 0$). Finally, in Chapter 7 we present some notable examples, including a full classification of 4-dimensional solutions realized on pure radiation waves. Solutions within this family of manifolds illustrate many of the features described in previous chapters, especially in the isotropic case.

Throughout this part of the thesis, since we are concerned with local geometric features of solutions, all SMMSs are assumed to be non-trivial and we work around regular points of the density function ($\nabla h \neq 0$).

Chapter 4
Derivation of the field equations

Throughout the rest of this dissertation, we work in a Lorentzian smooth metric measure space of the form $(M, g, h\,\mathrm{dvol}_g)$, where h is a positive density function and $\mathrm{dvol}_g = \sqrt{|g|}\,dx_1 \wedge \cdots \wedge dx_n$ is the usual Riemannian volume element, with $|g| = -\det(g_{ij})$. Since dvol_g is determined by the metric, we will make use of the simpler notation (M, g, h) (see Section 1.5 for more details on smooth metric measure spaces).

This chapter is devoted to the derivation of weighted Einstein field equations that take into account the presence of the density h in an appropriate manner. These equations amount to the vanishing of the tensor

$$G^h = h\rho - \mathrm{Hes}_h + (\Delta h + \Lambda)g,$$

which generalizes the usual Einstein tensor with cosmological constant (II.1) (indeed, the vanishing of G^h reduces to the Einstein condition when h is constant). Firstly, we employ a variational approach by looking for suitably constrained critical points of a modified Einstein-Hilbert functional, which yields the equation $G^h = 0$ (with $\Lambda = 0$). Secondly, we translate the characterizing properties of the standard Einstein tensor to te weighted setting to obtain G^h. These derivations are included in the works [22, 25]. Recall that, for the rest of the thesis, we are only interested in non-trivial SMMSs, so we assume that the density h is non-constant on any open subset of M.

4.1 Variational problem with constraints

In this section, we use a modified version of the Einstein-Hilbert functional to obtain the weighted Einstein tensor. To this end, consider the space $\mathcal{M}$ of Lorentzian metric-measure structures on M, i.e. $\mathcal{M} = \mathrm{Lor}(M) \times \mathcal{C}^\infty(M;\mathbb{R}^+)$, where $\mathrm{Lor}(M)$ is the space of Lorentzian metrics on M and $\mathcal{C}^\infty(M;\mathbb{R}^+)$ is the space of positive smooth functions on M, which will be acting as densities. In this section, we will be considering integrals with respect to the weighted volume element $\mathrm{dV}_h = h\,\mathrm{dvol}_g$. Thus, exclusively for the purpose of developing the following variations, assume that the density function has compact support, so that all integrands do.

Now, define the *weighted Einstein-Hilbert functional* as the functional $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ given by

$$\mathcal{S} : (g, h) \mapsto \mathcal{S}_{(g,h)} = \int_M \tau h\,\mathrm{dvol}_g,$$

which generalizes the usual Einstein-Hilbert functional, since for $h \equiv 1$, we recover it exactly. Note that this functional shows a dependence on the metric through its scalar curvature and a direct dependence on the density function through the weighted volume. For simplicity, we assume the dependence on the metric and we write $\mathcal{S}_{(g,h)} = \mathcal{S}_h$ in order to highlight the difference between the weighted Einstein-Hilbert functional and its unweighted analogue. Since we are assuming that h is non-constant, the behavior of $\mathcal{S}_h$ is quite different from its unweighted counterpart.

A defining feature of a SMMS (M, g, h) as a weighted geometric object is its weighted volume element dV_h. Hence, it is natural to pose the variational problem of finding the critical points of $\mathcal{S}_h$, constrained to variations of the metric-measure structure of (M, g, h) which keep the weighted volume element invariant at each point of the manifold. By a variation of the metric-measure structure, we mean a simultaneous variation of both the metric and the density function, with the same variation parameter t:

$$g[t] = g + t\bar{g}, \qquad h[t] = h + t\bar{h}, \qquad \mathrm{dV}_h[t] = h[t]dvol_{g[t]}, \tag{4.1}$$

where $\bar{g}$ is a symmetric $(0, 2)$-tensor and $\bar{h} \in C^\infty(M)$. In order to maintain the compact support of the integrands, we assume that any variations of h have compact support as well. Since we are working with manifolds without boundary, we do not need to worry about boundary terms. Otherwise, we would need to impose the conditions that the variations and their first derivatives vanish on the boundary (see, for example, [106]).

For a variation of the metric-measure structure (4.1), through the well-known expression $\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\,\mathrm{tr}\,\bar{g}$ (where we use δ to denote the variation), the invariance of the weighted volume element reads

$$\begin{aligned} \delta(\mathrm{dV}_h) = \tfrac{d}{dt}\big|_{t=0}\mathrm{dV}_h[t] &= \bar{h}\,\mathrm{dvol}_g + h\tfrac{d}{dt}\big|_{t=0}\mathrm{dvol}_{g[t]} \\ &= \left(\bar{h} + \tfrac{1}{2}h\,\mathrm{tr}\,\bar{g}\right)\,\mathrm{dvol}_g = 0. \end{aligned} \tag{4.2}$$

Also due to this invariance, we compute the variation of the functional,

$$\delta\mathcal{S}_h = \frac{d}{dt}\mathcal{S}_h\bigg|_{t=0} = \int_M \frac{d\tau}{dt}\bigg|_{t=0}\mathrm{dV}_h + \int_M \tau\,\delta(\mathrm{dV}_h) = \int_M \frac{d\tau}{dt}\bigg|_{t=0} h\,\mathrm{dvol}_g.$$

Hence, we need the linearization $\delta\tau(\bar{g}) = \frac{d\tau}{dt}\big|_{t=0}$ of the scalar curvature τ. This is a known computation (see Remark 4.1 below), but, for the sake of self-containment, we compute the variation explicitly.

Using index notation, let $\delta g^{ij} = \frac{dg[t]^{ij}}{dt}\Big|_{t=0}$, and notice that $\delta g^{ij} \neq g^{ik}g^{jl}\bar{g}_{kl}$. Indeed, since $g^{ik}g_{kj} = \delta^i{}_j$ (the identity matrix), we have $\delta(g^{ik}g_{kj}) = 0$ and so $g^{ik}\bar{g}_{kj} = -\delta g^{ik}g_{kj}$. From this, it follows that $\delta g^{ij} = -g^{ik}g^{jl}\bar{g}_{kl}$. Now, through a slight abuse of notation, take $\delta g^{ij} = \bar{g}^{ij}$, so that for any $(0, 2)$-tensor T, the formula $T_{ij}\bar{g}^{ij} = -\langle T, \bar{g}\rangle$ is satisfied.

Now, since $\tau = \rho^i{}_i = \rho_{ij}g^{ij} = \langle\rho, g\rangle$, we can write

$$\delta\tau(\bar{g}) = \frac{d\tau}{dt}\bigg|_{t=0} = \rho_{ij}\bar{g}^{ij} + g^{ij}\delta\rho_{ij} = -\langle\rho, \bar{g}\rangle + \langle\delta\rho, g\rangle.$$

Inserting this into the variation of $\mathcal{S}_h$, we arrive at a variational expression that must hold at critical points

$$\delta\mathcal{S}_h = \int_M (-\langle\rho, \bar{g}\rangle + \langle\delta\rho, g\rangle)h\,\mathrm{dvol}_g = 0.$$

Since the first summand already explicitly features the variation of the metric, we need to focus on the second term of this expression. The process to obtain this variation is laid out, for example, in [106], and some of the formulas used are explained in [67, Appendix A], so we retrace the most important steps here. Firstly, use the Palatini identity for the variation of the Ricci tensor in terms of the variation of the Christoffel symbols:

$$\delta\rho_{ij} = \nabla_k(\delta\Gamma_{ij}{}^k) - \nabla_j(\delta\Gamma_{ik}{}^k),$$

so that we have

$$\int_M \langle\delta\rho, g\rangle h \,\mathrm{dvol}_g = \int_M g^{ij}(\nabla_k(\delta\Gamma_{ij}{}^k) - \nabla_j(\delta\Gamma_{ik}{}^k))h \,\mathrm{dvol}_g. \tag{4.3}$$

Now, we can rewrite the two terms in the integral as

$$\begin{aligned} hg^{ij}\nabla_k(\Gamma_{ij}{}^k) &= \nabla_k(hg^{ij}\delta\Gamma_{ij}{}^k) - (\nabla_k h)g^{ij}\delta\Gamma_{ij}{}^k, \\ hg^{ij}\nabla_j(\delta\Gamma_{ik}{}^k) &= \nabla_j(hg^{ij}\delta\Gamma_{ik}{}^k) - (\nabla_j h)g^{ij}\delta\Gamma_{ik}{}^k. \end{aligned}$$

The first terms on the right-hand side of these equations are divergences, whose integral over M vanishes by virtue of Stokes' theorem. Moreover, interchanging the dummy indices j and k in the second expression, the remaining terms can be written in terms of the variation of the metric as follows [67, Appendix A]:

$$(\nabla_k h)(g^{ik}\delta\Gamma_{ij}{}^j - g^{ij}\delta\Gamma_{ij}{}^k) = (\nabla_k h)(\nabla_i(\bar{g}^{ik}) - g_{ij}\nabla^k(\bar{g}^{ij})).$$

Thus, the term (4.3) of the variation takes the form

$$\int_M (\nabla_k h)(\nabla_i(\bar{g}^{ik}) - g_{ij}\nabla^k(\bar{g}^{ij}))\mathrm{dvol}_g.$$

Like in the previous step, we now decompose (using the fact that $\nabla g = 0$) the two terms of the integral in a way that includes a divergence term which can be ignored as we integrate over M:

$$\begin{aligned} (\nabla_k h)\nabla_i(\bar{g}^{ik}) &= \nabla_i((\nabla_k h)\bar{g}^{ik}) - (\nabla_i\nabla_k h)\bar{g}^{ik}, \\ (\nabla_k h)g_{ij}\nabla^k(\bar{g}^{ij}) &= \nabla^k((\nabla_k h)g_{ij}(\bar{g}^{ij})) - (\nabla^k\nabla_k h)g_{ij}\bar{g}^{ij}. \end{aligned}$$

Hence, (4.3) finally reads

$$\int_M (g_{ij}\nabla^k\nabla_k h - \nabla_i\nabla_j h)\bar{g}^{ij}\mathrm{dvol}_g = \int_M \langle \mathrm{Hes}_h - \Delta h g, \bar{g}\rangle \mathrm{dvol}_g,$$

and the complete variation of the action is

$$\delta\mathcal{S}_h = \int_M \langle -h\rho + \mathrm{Hes}_h - \Delta h g, \bar{g}\rangle \mathrm{dvol}_g. \tag{4.4}$$

Note that we have the freedom to choose any variation of the metric $\bar{g}$, since there always exists a variation of the density, given by (4.2) as $\bar{h} = -\frac{1}{2}h \operatorname{tr}\bar{g}$, which preserves the weighted volume element $h\,\mathrm{dvol}_g$. Hence, the left-hand side of the product $\langle -h\rho + \mathrm{Hes}_h - \Delta h g, \bar{g}\rangle$ must vanish for all critical points of $\mathcal{S}_h$, restricted to variations of the metric-measure structure preserving the weighted volume element. Thus, we define the *weighted Einstein tensor* of the SMMS (M, g, h) as

$$G^h = h\rho - \mathrm{Hes}_h + \Delta h g, \tag{4.5}$$

and it follows that the critical points in our constrained variational problem satisfy the equation

$$G^h = h\rho - \mathrm{Hes}_h + \Delta h g = 0. \tag{4.6}$$

By analogy with the vacuum Einstein field equations of General Relativity, we call this the *(vacuum) weighted Einstein field equations*. Henceforth, we will refer to them as the weighted Einstein field equations or simply as the field equations, when the weighted context is clear. A *solution* of the field equations is a SMMS (M, g, h) which satisfies $G^h = 0$.

Remark 4.1. The linearization of τ was already computed in [5, 87], in works on General Relativity (see also [3, 8, 62, 63] for discussions in Riemannian signature), as the operator acting on symmetric $(0, 2)$-tensors by

$$\delta\tau(\bar{g}) = \left.\frac{d\tau}{dt}\right|_{t=0} = -\Delta\,\mathrm{tr}(\bar{g}) + \mathrm{div}\,\mathrm{div}\,\bar{g} - \langle \rho, \bar{g}\rangle.$$

Indeed, we can obtain this formula by using the formal adjoint of $\delta\tau$ with respect to the L^2-inner product on (M, g):

$$\int_M \delta\tau(\bar{g}) h\,\mathrm{dvol}_g = \int_M \langle \delta\tau^*(h), \bar{g}\rangle \mathrm{dvol}_g.$$

By the discussion above and the expression (4.4), it follows that

$$\delta\tau^*(h) = -h\rho + \mathrm{Hes}_h - \Delta h g,$$

and, since $\mathrm{Hes}_h = \nabla dh$, div is the negative of the formal adjoint of ∇, and Δ is formally self-adjoint,

$$\int_M \langle -h\rho + \mathrm{Hes}_h - \Delta h g, \bar{g}\rangle \mathrm{dvol}_g = \int (-\langle \rho, \bar{g}\rangle + \mathrm{div}\,\mathrm{div}\,\bar{g} - \Delta(\langle g, \bar{g}\rangle))\, h\,\mathrm{dvol}_g.$$

Hence, the value of $\delta\tau(\bar{g})$ follows. From this point of view, the search for solutions to the vacuum weighted Einstein field equations starts by determining what kinds of manifolds satisfy $\ker(\Gamma) \neq 0$, where $\Gamma : \mathcal{M} \to S^2(T^*M)$ is the map that takes the metric-measure structure (g, h) to the symmetric $(0, 2)$-tensor $\delta\tau^*(h) = -h\rho + \mathrm{Hes}_h - \Delta h g$. Then, computing $\ker(\Gamma)$ gives the explicit solutions on a given manifold.

Besides the discussion in this thesis, equation (4.6) and formally similar variants of it appear in several contexts as very natural second order differential equations with geometric interest. For example,considering the space of manifolds with constant scalar curvature, critical metrics for the volume functional admit non-trivial solutions for the equation $\delta\tau^*(h) = \kappa g$ for κ constant. This analysis was localized to the case where the metric deformation is supported on the closure of a bounded domain in [51, 90], defining the V-static spaces.

Remark 4.2. Geometrically, (4.6) presents a notable relationship with certain families of warped products. Indeed, the Ricci tensor of a warped product of the form $N \times_\varphi I$, where N is n-dimensional and $I \subset \mathbb{R}$ is a real interval, is given by [97]:

$$\begin{aligned} \rho(X,Y) &= \rho^N(X,Y) - \tfrac{1}{\varphi}\,\mathrm{Hes}_\varphi(X,Y), \\ \rho(X,\partial_t) &= 0, \\ \rho(\partial_t,\partial_t) &= -\varphi(\Delta\varphi)\, g^I(\partial_t,\partial_t), \end{aligned}$$

where X, Y are vector fields tangent to N, t is a coordinate parameterizing I by arc length, and ρ^N is the Ricci tensor of N. Necessary and sufficient conditions for a warped product $N \times_\varphi I$ to be Einstein follow:

$$\rho^N - \frac{1}{\varphi}\,\mathrm{Hes}^N_\varphi = \lambda g^N, \quad -\Delta\varphi = \lambda\varphi,$$

where λ is constant. By replacing λ in the expression on the left-hand side, one gets $\varphi\rho^N - \mathrm{Hes}_\varphi + \Delta\varphi g^N = 0$, which corresponds to equation (4.6). Thus, for any Einstein warped product $N \times_\varphi I$, the smooth metric measure space (N, g^N, φ) is a solution of the vacuum weighted Einstein field equation (4.6). When the vector field ∂_t is timelike, this process gives rise to Riemannian vacuum static spaces.

Remark 4.2 shows how a static vacuum solution of the usual Einstein field equations is constructed from a Riemannian manifold with a density satisfying (4.6). When the stress-energy tensor of a perfect fluid is introduced, an analogous process gives rise to the formally similar equation

$$h\rho - \mathrm{Hes}_h = \frac{1}{n}\,(h\tau - \Delta h)\, g,$$

where τ and h are related to the pressure and energy density of the higher dimensional static Lorentzian spacetime (see [80]).

Due to its physical significance and its relation to the linearization of the scalar curvature, equation (4.6) has drawn significant attention in the Riemannian setting. The works of Kobayashi and Lafontaine [79, 84] laid the groundwork for a systematic study of its Riemannian solutions (see also [78, 105, 112]). In this thesis, we present new results concerning solutions in Lorentzian signature, which had not been explored previously.

It is also noteworthy that, recently, Ho and Shin studied the kernel of the formal adjoint for the linearization of the weighted scalar curvature (1.18) with $\mu = 0$ to propose a weighted analogue of vacuum static spaces in the Riemannian setting. They also studied the locally conformally flat weighted case, and considered some stability and prescribed weighted scalar curvature problems associated to closed SMMSs [73].

Remark 4.3. Another example of the geometric significance of equations formally related to (4.6) is the following: In [3], it was shown that an n-dimensional compact Riemannian manifold which is critical for the Einstein-Hilbert functional, restricted to the space of metrics with constant scalar curvature and unit volume, satisfies the *Critical Point Equation (CPE):*

$$(f+1)\rho - \mathrm{Hes}_f + \left(\Delta f - \frac{\tau}{n}\right) g = 0,$$

for a certain function f. Besse conjectured in [3] that the only critical compact Riemannian manifolds are standard spheres. Since then, a number of papers have provided positive results under some extra assumptions (see, for example, [76, 94]). Due to their formal resemblance, solutions of the CPE equation in the Lorentzian setting share some geometric features with solutions of the weighted Einstein field equations (see Remark 5.13).

Remark 4.4. Metric theories of gravitation often derive their field equations from a modified version of the Einstein-Hilbert functional, either by taking a function of the scalar curvature, as in $f(R)$-gravity [26], or by incorporating scalar fields which play similar roles to our density, like in Jordan-Brans-Dicke theory [91].

In a similar manner, a way to generalize the weighted field equations (4.6) is to include a density-dependent *potential function* $V(h)$ and a density-independent *matter Lagrangian* $\mathcal{L}_m$, by analogy with physicists' interpretation of the stress-energy tensor in General Relativity. Thus, the functional $\mathcal{S}_h$ becomes

$$\mathcal{S}_{h,V,\mathcal{L}_m} = \int_M (\tau h - 2V(h) + \mathcal{L}_m)\,\mathrm{dvol}_g,$$

and the variational problem with constraints leads to the following equation:

$$h\rho - \mathrm{Hes}_h + (\Delta h + V - hV')g = T,$$

where $V' = \frac{dV}{dh}$ and T is such that $\langle T, \bar{g}\rangle = -\frac{1}{\sqrt{|g|}}\delta(\sqrt{|g|}\mathcal{L}_m)$. The potential terms come from the constraint (4.2) and the variation

$$\delta(2V\mathrm{dvol}_g) = 2V'(h)\bar{h}\,\mathrm{dvol}_g + V(h)\mathrm{tr}\,\bar{g}\,\mathrm{dvol}_g = (V - hV')\mathrm{tr}\,\bar{g}\,\mathrm{dvol}_g.$$

(cf. [91]). The stress-energy tensor T is physically interpreted as the one which models the matter content of the spacetime, so $T = 0$ indicates a vacuum. If, additionally, the potential V is constant, the equation reduces to $h\rho - \mathrm{Hes}_h + (\Delta h + V)g = 0$. In this case, V plays the role of a cosmological constant. As we will see shortly, the non-variational derivation of the weighted field equations naturally yields such a constant. However, the variational approach requires an *ad-hoc*, although standard, modification of the weighted functional $\mathcal{S}_h$ of the form

$$\mathcal{S}_{h,\Lambda} = \int_M (\tau h - 2\Lambda)\,\mathrm{dvol}_g.$$

4.2 Derivation through the characterizing properties of G

As mentioned in the introduction to Part II, the Einstein tensor (II.1) on a spacetime (M, g) is characterized (up to multiplicative constants) by the following four properties (see [89]):

1. Symmetry.
2. Vanishing divergence.

3. Dependence on the metric tensor g and its first two derivatives.

4. Linearity in the second derivatives of g.

Our objective in this section is to define a tensor on a smooth metric measure space that suitably generalizes the Einstein tensor while also satisfying analogous characterizing properties, with the caveat that now they must include information on the density function h. Thus, we expect our tensor to present the following properties:

1. Symmetry.

2. Vanishing divergence.

3. Dependence on the metric tensor g, the density function h and their first two derivatives.

4. Linearity in the second derivatives of g and h.

Since the Bakry-Émery Ricci tensor (1.16) is the natural Ricci-type tensor in smooth metric measure spaces, we will use it as our key building block. Moreover, we will see that the weighted Einstein tensor defined through this process essentially mimics the one obtained through a variational approach in Section 4.1.

Now, motivated by the computations in the previous section and the relationship between the value of the dimensional parameter m and vacuum static spaces (see Remark 4.2), and given that the auxiliary curvature parameter μ does not play a role in this setting (see Section 1.5 and recall that the value $m = 1$ makes μ irrelevant), we take $m = 1$ and $h = e^{-f}$ to rewrite (1.16) as

$$\rho^h = \rho - \frac{\operatorname{Hes}_h}{h}.$$

Since a suitable generalization of the Einstein tensor must depend on the metric tensor (property 3 above), we allow a summand which is a multiple of g. Thus, we consider a tensor of the form $\rho^h + \lambda g$, where $\lambda \in C^\infty(M)$. Motivated by the linearity in the second derivatives (property 4), we perform a linearization of this tensor, resulting in

$$G^h = h\rho - \operatorname{Hes}_h + \lambda h g.$$

Einstein manifolds have constant scalar curvature and we will show (see Lemma 4.5 below) that the weighted analogue that we are going to define also has this property. Hence, we compute the divergence of G^h in the case where τ is constant:

$$\begin{aligned} \operatorname{div}(G^h) &= \operatorname{div}(h\rho) - \operatorname{div}\operatorname{Hes}_h + \operatorname{div}(\lambda h g) \\ &= h \operatorname{div}\rho + \iota_{\nabla h}\rho - d\Delta h - \iota_{\nabla h}\rho + d(\lambda h) \\ &= \tfrac{1}{2} h\, d\tau - d\Delta h + d(\lambda h) \\ &= d(\lambda h - \Delta h), \end{aligned}$$

where ι denotes the interior product, $\iota_X \rho = \rho(X, \cdot)$, and we have used the contracted Bianchi identity $\operatorname{div}\rho = \frac{1}{2} d\tau$ and the Bochner formula $\operatorname{div}\operatorname{Hes}_h = d\Delta h + \iota_{\nabla h}\rho$ (see Section 1.1.2). Thus, for G^h to be divergence-free if τ is constant, we get that $\lambda h = \Delta h + \Lambda$, where $\Lambda \in \mathbb{R}$ plays the role of a cosmological constant. Consequently, we define a *weighted Einstein tensor* on a smooth metric measure space (M, g, h) by

$$G^h = h\rho - \operatorname{Hes}_h + (\Delta h + \Lambda)g, \tag{4.7}$$

as a $(0, 2)$-tensor satisfying properties 1, 3 and 4 above. Property 2 (vanishing divergence) is not satisfied in general, but it is when the scalar curvature is constant (see Lemma 4.5 below). Notice that, defined in this way (i.e., including the cosmological constant), G^h is indeed a generalization of the Einstein tensor (II.1), since the vanishing of G^h is equivalent to the underlying manifold being Einstein if h is constant. The associated (vacuum) field equations are thus

$$G^h = h\rho - \operatorname{Hes}_h + (\Delta h + \Lambda)g = 0. \tag{4.8}$$

The definition of G^h in (4.7) only differs from (4.5), the one obtained through variational means, by the appearance of a possibly non-zero cosmological constant, and this difference can be overcome through a small modification of the weighted Einstein-Hilbert functional as in Remark 4.4. In the following chapter, we will prove that solutions of (4.8) with ∇h lightlike do not admit $\Lambda \neq 0$, so (4.6) and (4.8) are equivalent in that context. Moreover, there are families of manifolds that only admit solutions with $\Lambda = 0$ (see Theorem 5.6).

As a final note in this chapter, we point out that solutions of the field equations (4.8) have constant scalar curvature, which is also a property presented by the vacuum solutions of the usual Einstein equations (i.e., standard Einstein manifolds). The following lemma, which was proved in [8,63], and was applied in [25] to the Lorentzian setting of smooth metric measure spacetimes, further cements (4.8) as a weighted analogue of the usual Einstein condition by proving that this property is conserved for solutions with non-trivial densities.

Lemma 4.5. *If (M, g, h) is a solution of the vacuum weighted Einstein field equations, then its scalar curvature τ is constant.*

Proof. We take the divergence of equation (4.8) to see, using the Bochner formula and the contracted Bianchi identity, that $0 = h \operatorname{div} \rho + \iota_{\nabla h}\rho - \operatorname{div} \operatorname{Hes}_h + d\Delta h = \frac{1}{2} h\, d\tau$. Hence, since $h \neq 0$ in every open subset, we conclude that τ is constant. □

Chapter 5
Isotropic solutions

Once a weighted analogue of the Einstein field equations has been determined, in the form of (4.6) or (4.8), our interest shifts towards the analysis of solutions, i.e, smooth metric measure spacetimes (M, g, h) such that the weighted field equations are satisfied in M. Given the level of complexity that this task reaches in the usual setting, it comes as no surprise that its weighted version

$$0 = h\rho - \mathrm{Hes}_h + (\Delta h + \Lambda) g$$

is also too unwieldy to consider in its full generality.

Nevertheless, due to the fact that we are working in Lorentzian signature, the causal character of ∇h crucially influences the geometry of solutions. Thus, since we only consider solutions with $\nabla h \neq 0$, we can split the problem of solving the field equations into two more narrow cases by defining *isotropic solutions* as those with ∇h lightlike, and *non-isotropic* ones as those with ∇h timelike or spacelike. In each case, the approach in treating an equation like (4.8) is different, as are often distinct the features of the resulting solutions. Although the causal character of ∇h can, in principle, change within the manifold, since our analysis in this part of the thesis is local, we will restrict our work to open sets where this causal character remains constant.

Due to the formal similarity between our equations and those of vacuum static spaces in Riemannian signature, some solutions in the non-isotropic case present similar geometric features to those of their Riemannian counterparts (see Chapter 6). However, in this chapter we focus on the purely Lorentzian context given by isotropic solutions. The results in this chapter are contained in the article [25].

Outline of the chapter

Firstly, in Section 5.1 we consider isotropic solutions of (4.8) of arbitrary dimension $n \geq 3$. We will see that, in general, they are realized on Kundt spacetimes and, in certain cases, on Brinkmann waves. Moreover, the scalar curvature vanishes and the Ricci operator is nilpotent. These results are summarized in Theorem 5.4, which is the main result in arbitrary dimension in this chapter.

For solutions in dimension three, the geometry of the underlying spacetime is more rigid than in higher dimensions, so stronger results can be achieved, and the metric and density can be more explicitly described in local coordinates. Indeed, in Section 5.2 we restrict the context to dimension three to classify solutions on pp-waves, provide some illustrative examples, and prove a complete classification result for 3-dimensional isotropic solutions (Theorem 5.11). Finally, in Section 5.3 we provide some remarks on 4-dimensional spacetimes: we prove that 4-dimensional Ricci-flat isotropic solutions are pp-waves; show that the classification result in three dimensions does not extend to four dimensions by giving an appropriate example; and build Ricci-flat 4-dimensional warped products from the 3-dimensional solutions given in Section 5.2.

5.1 The field equations in arbitrary dimension

Let (M, g, h) be a smooth metric measure spacetime of dimension n. Taking traces in the field equations, we have

$$0 = h\tau + (n-1)\Delta h + n\Lambda, \tag{5.1}$$

so Δh can be given in terms of h, τ and Λ as $\Delta h = -\frac{h\tau + n\Lambda}{n-1}$. The following result shows that, for isotropic solutions, ∇h is geodesic and an eigenvector of the Ricci operator.

Lemma 5.1. *Let* (M, g, h) *be an isotropic solution. Then,*

$$\nabla_{\nabla h}\nabla h = 0 \quad \textit{and} \quad \mathrm{Ric}(\nabla h) = \frac{h\tau + \Lambda}{(n-1)h}\nabla h.$$

Proof. Since $g(\nabla h, \nabla h) = 0$, we have

$$0 = (\nabla_X g)(\nabla h, \nabla h) = -2\,\mathrm{Hes}_h(\nabla h, X) \text{ for all vector fields } X.$$

Hence $\mathrm{hes}_h(\nabla h) = \nabla_{\nabla h}\nabla h = 0$ and, from equation (4.8),

$$\mathrm{Ric}(\nabla h) = -\frac{\Delta h + \Lambda}{h}\nabla h = \frac{h\tau + \Lambda}{(n-1)h}\nabla h. \qquad \square$$

Let $\alpha = \frac{h\tau+\Lambda}{(n-1)h}$ be the eigenvalue of Ric associated to ∇h. Since ∇h is lightlike and $\mathrm{Ric}(\nabla h) = \alpha\nabla h$, the Ricci operator has real eigenvalues. Moreover, since the Ricci operator is self-adjoint, from the discussion in Section 1.1.3 on self-adjoint operators in Lorentzian spaces, there exists a pseudo-orthonormal basis

$$\mathcal{B} = \{\nabla h, U, X_1, \ldots, X_{n-2}\} \text{ such that } g(\nabla h, U) = g(X_i, X_i) = 1,$$

(other terms of g being zero) and such that the Ricci operator takes the form

$$\mathrm{Ric} = \left(\begin{array}{ccc|ccc} \alpha & \nu & \mu & & & \\ 0 & \alpha & 0 & & 0 & \\ 0 & \mu & \beta_1 & & & \\ \hline & & & \beta_1 & & \\ & 0 & & & \ddots & \\ & & & & & \beta_{n-2} \end{array}\right) \tag{5.2}$$

In other words, we have $\mathrm{Ric}(\nabla h) = \alpha\nabla h$, $\mathrm{Ric}(U) = \nu\nabla h + \alpha U + \mu X_1$, $\mathrm{Ric}(X_1) = \mu\nabla h + \beta_1 X_1$ and $\mathrm{Ric}(X_i) = \beta_i X_i$ if $i \neq 1$.

In the next lemma we show that the Ricci operator is indeed nilpotent and, moreover, the constant Λ and the Laplacian of h vanish.

Lemma 5.2. *Let* (M, g, h) *be an isotropic solution. Then* Ric *is nilpotent,* $\Delta h = 0$ *and* $\Lambda = 0$*.*

Proof. By Lemma 4.5, the scalar curvature τ is constant. We use the contracted Bianchi identity (see Section 1.1.2) to see that $\operatorname{div} \rho(\nabla h) = \frac{1}{2} d\tau(\nabla h) = 0$. Hence,

$$0 = \operatorname{div} \rho(\nabla h) = (\nabla_{\nabla h}\rho)(U, \nabla h) + (\nabla_U \rho)(\nabla h, \nabla h) + \sum_i (\nabla_{X_i}\rho)(X_i, \nabla h). \tag{5.3}$$

We compute each of these three terms separately. Note that $\alpha = \frac{h\tau + \Lambda}{(n-1)h}$, and since Λ and τ are constants, we have

$$\nabla h(\alpha) = -\frac{\Lambda}{(n-1)h^2}\, g(\nabla h, \nabla h) = 0.$$

Also, since $\nabla_{\nabla h}\nabla h = 0$ and

$$\rho(\nabla_{\nabla h} U, \nabla h) = \alpha g(\nabla_{\nabla h} U, \nabla h) = \alpha\{\nabla h g(U, \nabla h) - g(U, \nabla_{\nabla h}\nabla h)\} = 0,$$

we write

$$(\nabla_{\nabla h}\rho)(U, \nabla h) = \nabla h(\rho(U, \nabla h)) - \rho(\nabla_{\nabla h} U, \nabla h) - \rho(U, \nabla_{\nabla h}\nabla h) = \nabla h(\alpha) = 0.$$

Since $\rho(\nabla h, \nabla h) = 0$, we see that

$$(\nabla_U \rho)(\nabla h, \nabla h) = U(\rho(\nabla h, \nabla h)) - 2\rho(\nabla_U \nabla h, \nabla h) = -2\alpha g(\nabla_{\nabla h}\nabla h, U) = 0.$$

Now, we write the following three formulas:

$$\begin{aligned} \rho(X_i, \nabla h) = \alpha g(X_i, \nabla h) &= 0, \quad \text{for all } i, \\ \textstyle\sum_i \rho(\nabla_{X_i} X_i, \nabla h) = \alpha \sum_i g(\nabla_{X_i} X_i, \nabla h) &= -\alpha \Delta h, \\ \textstyle\sum_i \rho(X_i, \nabla_{X_i}\nabla h) = \sum_i g(X_i, \mathrm{Ric}(\nabla_{X_i}\nabla h)) &= \mathrm{tr}(\mathrm{Ric} \circ \mathrm{hes}_h), \end{aligned}$$

where we have used the fact that the terms

$$\rho(\nabla_U \nabla h, \nabla h), \quad \rho(\nabla_{\nabla h} U, \nabla h) \text{ and } \rho(U, \nabla_{\nabla h}\nabla h)$$

all vanish. We then conclude

$$\begin{aligned} \sum_i (\nabla_{X_i}\rho)(X_i, \nabla h) &= \sum_i \{X_i \rho(X_i, \nabla h) - \rho(\nabla_{X_i} X_i, \nabla h) - \rho(X_i, \nabla_{X_i}\nabla h)\} \\ &= \alpha \Delta h - \mathrm{tr}(\mathrm{Ric} \circ \mathrm{hes}_h). \end{aligned}$$

Hence, from (5.3) it follows that

$$\alpha \Delta h - \mathrm{tr}(\mathrm{Ric} \circ \mathrm{hes}_h) = 0. \tag{5.4}$$

Notice that, by the field equations (4.8), the normal form of hes_h has the same structure as that of Ric. Thus, we set $\mathrm{hes}_h(X_1) = *\nabla h + \gamma_1 X_1$, where the value of $*$ is irrelevant to our argument, and $\mathrm{hes}_h(X_i) = \gamma_i X_i$ for $i \geq 2$. From (4.8) we have

$$0 = G^h(\nabla h, U) = h\alpha + \Delta h + \Lambda,$$
$$0 = G^h(X_i, X_i) = h\beta_i - \gamma_i + \Delta h + \Lambda,$$

so $\gamma_i = h(\beta_i - \alpha)$. Hence, equation (5.4) becomes

$$0 = \alpha \sum_i \gamma_i - \sum_i \beta_i \gamma_i = \sum_i \gamma_i(\alpha - \beta_i) = -\sum_i \frac{\gamma_i^2}{h}.$$

This implies $\gamma_i = 0$ for all i, and therefore $\Delta h = 0$. Moreover, $\beta_i = \alpha$ for all i. Now, from (5.1) we get that $h\tau + n\Lambda = 0$. Since τ and Λ are constant, but h is not, we conclude $\tau = \Lambda = 0$. Furthermore, $\beta_i = \frac{h\tau+\Lambda}{(n-1)h} = 0$ and Ric is nilpotent. □

As a consequence of Lemma 5.2, all isotropic solutions to the field equations (4.8) have vanishing scalar curvature. However, this is not necessarily the case if ∇h is not lightlike. Indeed, in this case, every value of τ is realizable, as the following examples built on two families of Brinkmann waves show.

Example 5.3. We consider $\kappa \neq 0$ and define the following examples:

1. For $\kappa > 0$, let g be a Brinkmann metric defined by (1.12) with
$$F(u,v,x) = \frac{u^2\kappa}{2} + \sigma(v)\left(u + 2\sqrt{\frac{2}{\kappa}}\,\mathrm{arctanh}\left(\tan\left(\frac{x\sqrt{\kappa}}{2\sqrt{2}}\right)\right)\right)$$
for an arbitrary function σ. Then the scalar curvature is $\tau = \kappa$ and the manifold satisfies equation (4.8) for $h(u,v,x) = \cos\left(x\sqrt{\frac{\kappa}{2}}\right)$ and $\Lambda = 0$. Moreover,
$$\nabla h = -\sqrt{\frac{\kappa}{2}}\sin\left(x\sqrt{\frac{\kappa}{2}}\right)\partial_x \text{ and } \|\nabla h\| = \frac{1}{2}\kappa\sin^2\left(x\sqrt{\frac{\kappa}{2}}\right) > 0,$$
so the vector field ∇h is spacelike, since $\nabla h \neq 0$.

2. For $\kappa < 0$, let g be a Brinkmann metric defined by (1.12) with
$$F(u,v,x) = \frac{u^2\kappa}{2} + \sqrt{\frac{2}{-\kappa}}\sigma(v)e^{-\frac{x\sqrt{-\kappa}}{\sqrt{2}}}.$$
Then the scalar curvature is $\tau = \kappa$ and the manifold satisfies equation (4.8) for $h(u,v,x) = e^{\sqrt{\frac{-\kappa}{2}}x}$ and $\Lambda = 0$. Moreover,
$$\nabla h = \sqrt{\frac{-\kappa}{2}}e^{\sqrt{\frac{-\kappa}{2}}x}\partial_x \text{ and } \|\nabla h\| = -\frac{1}{2}\kappa e^{\sqrt{-2\kappa}x} > 0,$$
so the vector field ∇h is globally defined and it is spacelike.

In conclusion, any constant scalar curvature τ is realizable by a solution of the field equations (4.8) with vanishing cosmological constant and a Brinkmann wave as a background metric.

Next, we will continue the analysis of isotropic solutions to the field equations. As a consequence of Lemma 5.2, we have that $\tau = 0$, $\Delta h = 0$ and $\Lambda = 0$, so equation (4.8) reduces to

$$h\rho = \mathrm{Hes}_h\,. \tag{5.5}$$

Notice that this equation is linear in the function h. A more general version of (5.5) called the affine quasi-Einstein equation, was considered in [17] for affine manifolds, so some of the properties of solutions in the affine context extend to the Lorentzian setting. For example, if (M, g) is real analytic then h is also real analytic. Further analysis of (5.5) yields the main rigidity result of this chapter, which holds in arbitrary dimension.

Theorem 5.4. *Let (M^n, g, h) be an isotropic solution of the weighted Einstein field equations* (4.8)*. Then one of the following possibilities holds:*

1. *(M, g) is a Ricci-flat Brinkmann wave and $\mathrm{Hes}_h = 0$.*
2. *The Ricci operator is 2-step nilpotent and (M, g) is a Brinkmann wave.*
3. *The Ricci operator is 3-step nilpotent and (M, g) is a Kundt spacetime.*

Proof. We keep working in the pseudo-orthonormal basis $\mathcal{B}$ where, from (5.2) and as a consequence of Lemma 5.2, the Ricci operator acts as follows:

$$\mathrm{Ric}(\nabla h) = \mathrm{Ric}(X_i) = 0, \text{ for } i = 2, \dots, n-2,$$
$$\mathrm{Ric}(U) = \nu\nabla h + \mu X_1, \qquad \mathrm{Ric}(X_1) = \mu\nabla h.$$

We distinguish three cases: Ric is zero ($\mu = \nu = 0$), Ric is 2-step nilpotent ($\nu \neq 0$ and $\mu = 0$) and Ric is 3-step nilpotent ($\mu \neq 0$).

If the manifold is Ricci-flat, $\mu = \nu = 0$, then equation (5.5) reduces to $\mathrm{Hes}_h = 0$. Hence ∇h is a lightlike parallel vector field, so the manifold is a Ricci-flat Brinkmann wave with distinguished vector field ∇h (see Section 1.3). This corresponds to Theorem 5.4 (1).

If $\nu \neq 0$ and $\mu = 0$, then the Ricci operator and, by (5.5), the Hessian operator are 2-step nilpotent. We have $\nabla_{\nabla h}\nabla h = \nabla_{X_i}\nabla h = 0$ for all $i = 1, \dots, n-2$, while $\nabla_U \nabla h = h\nu\nabla h$, so ∇h is a lightlike recurrent vector field and the manifold is a Brinkmann wave. Theorem 5.4 (2) follows.

If $\mu \neq 0$, then the Ricci and the Hessian operators are 3-step nilpotent. We already know, by Lemma 5.1, that the lightlike vector field ∇h is geodesic. Moreover, since $\nabla_{X_1}\nabla h = h\mu X_1$ and $\nabla_{X_i}\nabla h = 0$ for $i > 1$ the condition (1.8) is satisfied for the 1-form ω given by $\omega(\nabla h) = 0$, $\omega(X_1) = h\mu$ and $\omega(X_i) = 0$ for all $i = 2, \dots, n-2$. It follows that (M, g) is a Kundt spacetime.

Alternatively, we can compute the optical scalars (1.9) for ∇h and see that they vanish. Indeed, because ∇h is a gradient, it is twist-free ($\omega^2 = 0$). Moreover, we check that

$$\theta = \frac{1}{n-2}\nabla_i V^i = \frac{1}{n-2}\Delta h = 0,$$

as a consequence of Lemma 5.2. Since hes_h is nilpotent and $\theta = 0$, ∇h is shear-free as well:

$$\sigma^2 = \| \operatorname{Hes}_h \|^2 - (n-2)\theta^2 = 0,$$

from where we also conclude that (M, g) is a Kundt spacetime. This is the remaining possibility, corresponding to Theorem 5.4 (3). □

We have already pointed out (see Remark 4.2) how SMMSs which satisfy the weighted Einstein field equation with $\Lambda = 0$ are related to Einstein warped products of the form $N \times_h I$. In the isotropic case, as a consequence of the results in Section 5.1, solutions satisfy $\Delta h = 0$ and $\Lambda = 0$. Applying these facts to the formulas in Remark 4.2, we obtain the following consequence.

Corollary 5.5. *A smooth metric measure space (N, g, h) with isotropic density h is a solution to the weighted Einstein field equation* (4.8) *if and only if $N \times_h \mathbb{R}$ is Ricci-flat.*

5.2 The field equations in dimension three

One of the most useful consequences of Theorem 5.4 is that it reduces the study of isotropic solutions to that of different families of Kundt spacetimes. As such, we can make use of the Kundt coordinates and the forms of these metrics discussed in Section 1.3. In particular, Kundt spacetimes in dimension three present a more rigid structure than in higher dimensions, which allows us to explicitly solve the field equations and give expressions for the metric and the isotropic density function h. We begin by studying *pp*-waves and Brinkmann waves, and then we go on to Kundt spacetimes, ending with the statement and proof of the main classification theorem in dimension three (Theorem 5.11).

pp-waves

First, consider 3-dimensional solutions to the field equations with the underlying structure of a *pp*-wave. In this case, we are able to classify non-isotropic solutions as well. Noticeably, there are no solutions with ∇h timelike.

Theorem 5.6. *Let (M, g) be a 3-dimensional pp-wave. If (M, g, h) is a non-flat solution of* (4.8)*, then $\Lambda = 0$ and one of the following possibilities holds:*

1. *∇h is lightlike and (M, g) is a plane wave which in local coordinates can be written as*

$$g(u, v, x) = dv \left(2du - \frac{\alpha''(v)}{\alpha(v)} x^2 dv \right) + dx^2,$$

where $h(u, v, x) = \alpha(v)$ is an arbitrary positive function with $\alpha''(v) \neq 0$.

2. *∇h is spacelike and (M, g) can be written in local coordinates as*

$$g(u, v, x) = dv \left(2du + F(v, x) dv \right) + dx^2,$$

with

$$F(v,x) = \frac{(\gamma_1\alpha(v) + 2\gamma_0(v)\gamma_0''(v))\log(\gamma_0(v)+\gamma_1 x)}{\gamma_1^2} - \frac{2x\gamma_0''(v)}{\gamma_1} + \beta(v),$$

where $h(u,v,x) = \gamma_1 x + \gamma_0(v)$*,* $\gamma_1 \in \mathbb{R}\backslash\{0\}$*, and* γ_0*,* α*,* β *are arbitrary functions such that* $\gamma_1\alpha(v) + 2\gamma_0(v)\gamma_0''(v) \neq 0$ *and* $\gamma_1 x + \gamma_0(v) > 0$*.*

Proof. Since (M,g) is a *pp*-wave, there exist local coordinates so that the metric is given by (1.12) where $F(u,v,x) = F(v,x)$. In order to simplify notation, we denote $G^h(\partial_u,\partial_u) = G^h_{uu}$, $G^h(\partial_u,\partial_x) = G^h_{ux}$, $G^h(\partial_x,\partial_x) = G^h_{xx}$ and so on, and we compute the expression of G^h given by (4.7):

$$G^h_{uu} = -\partial_u^2 h, \quad G^h_{xx} = \Lambda + 2\partial_u\partial_v h - F\partial_u^2 h, \quad G^h_{ux} = -\partial_u\partial_x h,$$
$$G^h_{vv} = F\left(-F\partial_u^2 h + \partial_x^2 h + 2\partial_u\partial_v h + \Lambda\right) + \tfrac{\partial_v F\partial_u h - \partial_x F\partial_x h - 2\partial_v^2 h - h\partial_x^2 F}{2},$$
$$G^h_{vx} = -\partial_v\partial_x h + \tfrac{\partial_x F\partial_u h}{2}, \quad G^h_{uv} = \Lambda + \partial_x^2 h + \partial_u\partial_v h - F\partial_u^2 h.$$

From $G^h_{uu} = G^h_{ux} = 0$ we get that $h(u,v,x) = h_1(v)u + h_0(v,x)$. Now, from $G^h_{xx} = \Lambda + 2h_1'(v) = 0$, we get that $h_1(v) = -\frac{\Lambda}{2}v + k$ for a constant k. From $G^h_{uv} = \Lambda + h_1'(v) + \partial_x^2 h_0(v,x) = 0$, the function h reduces to the form

$$h(u,v,x) = \left(-\frac{\Lambda}{2}v + k\right)u - \frac{\Lambda}{4}x^2 + h_{01}(v)x + h_{00}(v).$$

If we differentiate $G^h_{vx} = -h_{01}'(v) + \frac{1}{4}(2k - v\Lambda)\partial_x F(v,x) = 0$ with respect to x, we obtain $\frac{1}{4}(2k - v\Lambda)\partial_x^2 F(v,x) = 0$. If $\partial_x^2 F(v,x) = 0$ then the manifold is Ricci-flat, and hence flat. Therefore, we conclude that $\Lambda = k = 0$ and $G^h_{vx} = -h_{01}'(v) = 0$, so h_{01} is indeed constant. The function h reduces to $h(u,v,x) = h_{01}x + h_{00}(v)$, with $\nabla h = h_{00}'(v)\partial_u + h_{01}\partial_x$ and $\|\nabla h\|^2 = h_{01}^2$.

We analyze separately the isotropic case (∇h is lightlike: $h_{01} = 0$) and the non-isotropic case (∇h is spacelike: $h_{01} \neq 0$). If $h_{01} = 0$, then the only non-vanishing component of G^h is $G^h_{vv} = -h_{00}''(v) - \frac{1}{2}h_{00}(v)\partial_x^2 F(v,x)$. Setting $G^h_{vv} = 0$ we obtain that $F(v,x)$ is a polynomial of degree two of the form $F(v,x) = -\frac{h_{00}''(v)}{h_{00}(v)}x^2 + F_1(v)x + F_0(v)$ with $h_{00}''(v) \neq 0$, otherwise the manifold is flat. Therefore g is a plane wave and F can be further normalized so that $F(v,x) = -\frac{h_{00}''(v)}{h_{00}(v)}x^2$ (see, for example, [86]). This corresponds to Item (1).

We assume now that ∇h is spacelike, i.e. $h_{01} \neq 0$. There is only one remaining non-zero term of G^h:

$$G^h_{vv} = \frac{1}{2}\left(-\partial_x^2 F(v,x)(h_{00}(v) + h_{01}x) - h_{01}\partial_x F(v,x) - 2h_{00}''(v)\right).$$

We solve $G^h_{vv} = 0$ to obtain the form of F in terms of $\gamma_0(v) = h_{00}(v)$ and $\gamma_1 = h_{01}$ as given in Item (2). Notice that the condition $\gamma_1\alpha(v) + 2\gamma_0(v)\gamma_0''(v) \neq 0$ comes from the assumption that the solution is non-flat, while $\gamma_1 x + \gamma_0(v) > 0$ comes from the positivity of the density function h. □

Remark 5.7. Notice that, as a consequence of Theorem 5.6 (1), for any function $h(v)$ with $h''(v) \neq 0$ there always exists a plane wave (M, g_{pw}) so that (M, g_{pw}, h) is an isotropic solution to the field equations (4.8). Recall from Section 1.3 that, among plane wave metrics, given by expression (1.12) with $F(v,x) = a(v)x^2$, there are two families that are locally homogeneous [65]:

- The family $\mathcal{P}_c$, defined by $F(v,x) = -\beta(v)x^2$ with $\beta' = c\beta^{3/2}$ for a constant c and $\beta > 0$.
- The family of Cahen-Wallach symmetric spaces $\mathcal{CW}_\varepsilon$, defined by $F(v,x) = \varepsilon x^2$.

Since solutions in Theorem 5.6 (1) are of the form $F(v,x) = -\frac{h''(v)}{h(v)}x^2$, we have the following:

- For $c > 0$, metrics in (1.12) with $F(v,x) = -\frac{4}{c^2v^2}x^2$ belong to the family $\mathcal{P}_{-c}$. From the equation $\beta = \frac{h''}{h}$, it follows that for $h = h(v) = a_1 v^{\frac{c-\sqrt{c^2+16}}{2c}} + a_2 v^{\frac{c+\sqrt{c^2+16}}{2c}}$, these metrics are homogeneous solutions to the field equations. They show lightlike singularities and are geodesically incomplete (we refer to [6] for details). Note that F is not defined on $\mathbb{R}^3$.
- For the densities
$$\begin{aligned} h(u,v,x) &= b_1\cos(v\sqrt{\varepsilon}) + b_2\sin(v\sqrt{\varepsilon}), \quad &\text{if } \varepsilon > 0,\\ h(u,v,x) &= b_1 e^{v\sqrt{-\varepsilon}} + b_2 e^{-v\sqrt{-\varepsilon}}, \quad &\text{if } \varepsilon < 0,\end{aligned}$$
Cahen-Wallach spaces $\mathcal{CW}_\varepsilon$ are solutions to the weighted Einstein field equations (4.8) (indeed, $\frac{h''}{h} = -\varepsilon$). Moreover, these metrics are geodesically complete (see [6, 28]). Also, for appropriate choices of $h > 0$ one has $\mathrm{Hes}_h \neq 0$, so there exist global solutions to (4.8).

Brinkmann waves

It was shown in Theorem 5.4 that Brinkmann waves play a role when the Ricci operator is 2-step nilpotent. We now show that all 3-dimensional isotropic solutions in this case are indeed plane waves.

Theorem 5.8. *If* (M, g, h) *is an isotropic solution of* (4.8) *with* (M, g) *being a* 3*-dimensional Brinkmann wave, then* (M, g) *is a plane wave as in Theorem 5.6 (1).*

Proof. Consider local coordinates as in (1.12). By Lemma 5.2, we have $\Lambda = \tau = \Delta h = 0$. The scalar curvature takes the form $\tau = \partial_u^2 F(u,v,x)$, and therefore we obtain $F(u,v,x) = F_1(v,x)u + F_0(v,x)$. With this reduction, the only non-zero component of the square of the Ricci operator is $\mathrm{Ric}^2(\partial_v) = \frac{1}{4}(\partial_x F_1)^2\partial_u$. A direct calculation shows $G^h_{uu} = -\partial_u^2 h(u,v,x)$ and $G^h_{ux} = -\partial_u\partial_x h(u,v,x)$ and, from $G^h_{uu} = G^h_{ux} = 0$, we get that $h(u,v,x) = h_1(v)u + h_0(v,x)$. We differentiate the term $G^h_{xx} = -h_1(v)F_1(v,x) + 2h_1'(v)$ with respect to x to see that $h_1(v)\partial_x F_1(v,x) = 0$. Hence $h_1 = 0$ or $\partial_x F_1(v,x) = 0$.

If $h_1(v) = 0$, then $h(u,v,x) = h_0(v,x)$ and $0 = \|\nabla h\|^2 = (\partial_x h_0(v,x))^2$, so the density function reduces to $h(u,v,x) = h_{00}(v) > 0$. Now, we compute $0 = G^h_{vx} = \frac{1}{2}h_{00}(v)\partial_x F_1(v,x)$ to obtain that in any case $\partial_x F_1(v,x) = 0$. This condition yields $F_1(v,x) = F_1(v)$ and, as a result, the Ricci operator of (M,g) is at most 2-step nilpotent. It now follows that the manifold is a *pp*-wave (see, for example, [86]). Hence, from Theorem 5.6, the result follows. □

Notice that, as a consequence of Theorem 5.8, all 3-dimensional isotropic solutions realized on Brinkmann waves have at most 2-step nilpotent Ricci operator. Thus, any 3-dimensional isotropic solution with 3-step nilpotent Ricci operator is Kundt by Theorem 5.4, but it is not a Brinkmann wave (see Theorem 5.11 (2) for the solutions which present this behavior).

In the cases where ∇h is not lightlike, however, we observe a loss of rigidity in the underlying manifold. Indeed, there exist 3-dimensional non-isotropic solutions which are Brinkmann waves but not pp-waves. The following example illustrates this fact.

Example 5.9. Let (M, g) be a Brinkmann wave with metric given by (1.12) where

$$F(v,x) = \frac{(4uv - x^2)\log(vx) + x^2}{2v^2}.$$

The Ricci operator is given by

$$\mathrm{Ric}(\partial_u) = 0, \;\; \mathrm{Ric}(\partial_v) = \frac{4uv + 2x^2\log(vx) + x^2}{4v^2x^2}\partial_u + \frac{1}{vx}\partial_x, \;\; \mathrm{Ric}(\partial_x) = \frac{1}{vx}\partial_u,$$

so it is 3-step nilpotent and, thus, it is not a pp-wave. A straightforward calculation shows that, for $h(u,v,x) = vx$ and $\Lambda = 0$, (M, g, h) is a solution of equation (4.8). Moreover, $\nabla h = x\partial_u + v\partial_x$, so $\|\nabla h\|^2 = v^2$ and ∇h is spacelike.

Kundt spacetimes

To complete the classification of isotropic solutions, we consider a 3-dimensional Kundt spacetime and work with a metric given in local coordinates as in (1.11).

Lemma 5.10. *Let (M, g) be a 3-dimensional Kundt spacetime with distinguished lightlike vector field V. If $\mathrm{Ric}(V) = 0$ and $\tau = 0$ then either (M, g) is a Brinkmann wave or there exist local coordinates (u, v, x) such that g is of the form given in* (1.11) *with*

$$\begin{aligned} F(u,v,x) &= \tfrac{u^2}{x^2} + \gamma_1(v,x)u + \gamma_0(v,x), \\ K(u,v,x) &= -\tfrac{2u}{x}. \end{aligned} \tag{5.6}$$

Proof. We consider the form of the metric given in (1.11), where $V = \partial_u$. A direct calculation shows that

$$\mathrm{Ric}(V) = \frac{1}{2}\left(\partial_u^2 F - (\partial_u K)^2 + \partial_u\partial_x K - 2K\partial_u^2 K\right)\partial_u + \frac{1}{2}(\partial_u^2 K)\partial_x.$$

Hence, since $\mathrm{Ric}(V) = 0$, we have that $\partial_u^2 K = 0$, so $K(u,v,x) = \omega_1(v,x)u + \omega_0(v,x)$. Now, $\mathrm{Ric}(V) = \frac{1}{2}\left(\partial_u^2 F + \partial_x\omega_1 - \omega_1^2\right)\partial_u$ and $\tau = \partial_u^2 F + 2\partial_x\omega_1 - \frac{3}{2}\omega_1^2$. From these relations we obtain that $2\partial_x\omega_1 - \omega_1^2 = 0$ and, solving this differential equation, we obtain that either $\omega_1 = 0$ or $\omega_1(v,x) = -\frac{2}{x+\varphi(v)}$. Moreover, since $\partial_u^2 F = \omega_1^2 - \partial_x\omega_1 = \partial_x\omega_1$, we get that $F(u,v,x) = \varepsilon\frac{u^2}{(x+\varphi(v))^2} + \gamma_1(v,x)u + \gamma_0(v,x)$, where $\varepsilon = 0$ if $\omega_1 = 0$ and $\varepsilon = 1$ if $\omega_1 = -\frac{2}{x+\varphi(v)}$.

Appropriate changes of coordinates allow us to simplify the form of the functions F and K as follows. We refer to [46] for changes of coordinates of 3-dimensional Kundt spacetimes with functions F and K which are polynomial of degrees 3 and 2, respectively, in the variable u; and to [103] for changes of coordinates in a broader context. Firstly, by setting $(u,v,x) = (\tilde{u},\tilde{v},\tilde{x}+\varphi(\tilde{v}))$ and redefining γ_0 and γ_1, one can write $F(u,v,x) = \varepsilon\frac{u^2}{x^2}+\gamma_1(v,x)u+\gamma_0(v,x)$ and $K(u,v,x) = -\varepsilon\frac{2u}{x}+\omega_0(v,x)$. Moreover, a new change of the form $(u,v,x) = (\tilde{u}+\psi(\tilde{v},\tilde{x}),\tilde{v},\tilde{x})$ for $\psi(\tilde{v},\tilde{x})$ solving the equation $\omega_0+\omega_1\psi+\partial_{\tilde{x}}\psi = 0$ transforms K into either $K=0$, in which case we have a Brinkmann wave, or $K=-\frac{2u}{x}$. □

With this, we are finally ready to prove the main classification result for 3-dimensional isotropic solutions.

Theorem 5.11. *Let (M,g,h) be a non-flat 3-dimensional isotropic solution of the weighted Einstein field equations* (4.8). *Then, the Ricci operator is nilpotent and one of the following holds:*

1. *If* Ric *is* 2*-step nilpotent, then (M,g) is a plane wave and there exist local coordinates (u,v,x) such that*

$$g(u,v,x) = dv\left(2du - \frac{\alpha''(v)}{\alpha(v)}x^2dv\right) + dx^2,$$

   where $h(u,v,x)=\alpha(v)$ is an arbitrary positive function with $\alpha''(v)\neq 0$.

2. *If* Ric *is* 3*-step nilpotent, then (M,g) is a Kundt spacetime and there exist local coordinates (u,v,x) so that $h(u,v,x)=v>0$ and*

$$g(u,v,x) = dv(2du + F(u,v,x)dv + 2K(u,v,x)dx) + dx^2, \tag{5.7}$$

   where

$$\begin{aligned} F(u,v,x) &= \tfrac{u^2}{x^2} + \gamma_1(v,x)u + \gamma_0(v,x), \\ K(u,v,x) &= -\tfrac{2u}{x}, \end{aligned}$$

   with $\gamma_1(v,x) = \alpha_1(v) - \frac{2\log(x)}{v}$ and

$$\begin{aligned} \gamma_0(v,x) &= \tfrac{x^2((\log(x)-2)\log(x)+2)}{v^2} + \tfrac{x^2\alpha_1(v)(1-\log(x))}{v} \\ &\quad + x^2\alpha_2(v) + x\alpha_3(v), \end{aligned}$$

   for arbitrary functions α_1, α_2 and α_3.

Proof. Let (M,g,h) be a 3-dimensional isotropic solution of (4.8). Firstly, we assume that the Ricci operator is 2-step nilpotent in order to prove Theorem 5.11 (1). By Theorem 5.4, (M,g) is a Brinkmann wave where ∇h is a recurrent vector field. Now the result follows from Theorems 5.8 and 5.6 (1).

On the other hand, if the Ricci operator is 3-step nilpotent, by Theorem 5.4, (M,g) is a Kundt spacetime where ∇h is the distinguished lightlike geodesic vector field with vanishing expansion scalar. Hence, there exist coordinates (u,v,x) as in (1.11) with $\nabla h=\partial_u$. For a general function $h(u,v,x)$ we compute

$$\nabla h(u,v,x)=\left(\left(\omega^2-F\right)\partial_u h-\omega\partial_x h+\partial_v h\right)\partial_u+\partial_u h\,\partial_v+\left(\partial_x h-\omega\partial_u h\right)\partial_x$$

to see that $\nabla h=\partial_u$ if and only if $h(u,v,x)=v+\kappa$, where κ is a constant. We normalize the variable v and consider $h(u,v,x)=v$. Now, notice that Lemma 5.10 is satisfied for this solution. Consequently, it is either a Brinkmann wave, in which case it has 2-step nilpotent Ricci operator (see Theorem 5.8) and it is described in Theorem 5.11 (1); or we can consider F and W given by expression (5.6). A direct computation of the tensor G^h shows that the non-zero components, up to symmetries, are

$$\begin{aligned} G^h_{vv} &= -\tfrac{uvx\partial_x\gamma_1(v,x)-vx\partial_x\gamma_0(v,x)+v\gamma_0(v,x)+u}{x^2} \\ &\quad -\tfrac{v\partial_x^2\gamma_0(v,x)+uv\partial_x^2\gamma_1(v,x)+\gamma_1(v,x)}{2}, \\ G^h_{vx} &= \tfrac{1}{2}v\partial_x\gamma_1(v,x)+\tfrac{1}{x}. \end{aligned}$$

From $G^h_{vx}=0$ we get that $\gamma_1(v,x)=\alpha_1(v)-\frac{2\log(x)}{v}$. Finally, the remaining component becomes

$$G^h_{vv}=\left(-\frac{1}{2}\partial_x^2\gamma_0(v,x)+\frac{1}{x}\partial_x\gamma_0(v,x)-\frac{1}{x^2}\gamma_0(v,x)\right)v-\frac{1}{2}\alpha_1(v)+\frac{1}{v}\log(x),$$

which we set to zero to obtain for γ_0 the expression in Theorem 5.11 (2). □

Remark 5.12. Recall that a spacetime is said to have *vanishing scalar invariants* (VSI) (respectively, *constant scalar invariants* (CSI)) if all polynomial scalar invariants constructed from the curvature tensor and its covariant derivatives are zero (respectively, constant).

Three-dimensional locally CSI spacetimes were classified in [49], showing that they are locally homogeneous or a Kundt spacetime. Metrics in Theorem 5.11 (2) are a subclass of VSI Kundt metrics (cf. [50]).

Remark 5.13. In Remark 4.3, we presented the Critical Point Equation (CPE) as

$$(f+1)\rho-\mathrm{Hes}_f+\left(\Delta f-\frac{\tau}{n}\right)g=0,$$

which is geometrically relevant and formally related to the field equations (4.8). Indeed, due to this resemblance, a similar analysis to the one performed in Sections 5.1 and 5.2 leads to classification results for solutions of this equation in the isotropic case if translated to Lorentzian signature. Furthermore, examples of solutions to this equation can be found among Kundt spacetimes and *pp*-waves. Thus, for example, since $\Delta f=\tau=0$ for isotropic solutions, 3-dimensional Cahen-Wallach symmetric spaces ($\mathcal{CW}_\varepsilon$) provide geodesically complete solutions to the CPE, which are not Einstein, for $f(u,v,x)=c_1\cos\left(v\sqrt{\varepsilon}\right)+c_2\sin\left(v\sqrt{\varepsilon}\right)-1$ if $\varepsilon>0$, and for $f(u,v,x)=c_1e^{v\sqrt{-\varepsilon}}+c_2e^{-v\sqrt{-\varepsilon}}-1$ if $\varepsilon<0$ (cf. Remark 5.7).

5.3 Some remarks on four-dimensional spacetimes

In view of Theorem 5.4, if an isotropic solution to equation (4.8) is Ricci-flat, then $\mathrm{Hes}_h = 0$, so ∇h is a parallel lightlike vector field and the underlying spacetime is a Brinkmann wave. The Ricci tensor determines the curvature in dimension three, so Ricci-flat 3-dimensional manifolds are necessarily flat. However, there are 4-dimensional isotropic solutions which are Ricci-flat but not flat. The following result shows that all solutions satisfying these conditions are indeed *pp*-waves.

Theorem 5.14. *Let* (M, g, h) *be a 4-dimensional isotropic Ricci-flat solution of the weighted Einstein field equations* (4.8). *Then* (M, g) *is a pp-wave.*

Proof. If (M, g, h) is an isotropic solution of (4.8) then, from Lemma 5.2, we have $\Delta h = 0$ and $\Lambda = 0$. Since $\rho = 0$, equation (4.8) implies $\mathrm{Hes}_h = 0$. For arbitrary vector fields X, Y, Z we have

$$R(X, Y, Z, \nabla h) = (\nabla_X \mathrm{Hes}_h)(Y, Z) - (\nabla_Y \mathrm{Hes}_h)(X, Z) = 0. \tag{5.8}$$

Let $\mathcal{B} = \{\nabla h, U, X_1, X_2\}$ be a pseudo-orthonormal basis such that $g(\nabla h, U) = g(X_i, X_i) = 1$ for $i = 1, 2$. Hence, $\nabla h^\perp = \mathrm{span}\{\nabla h, X_1, X_2\}$. Due to (5.8), we have that $R(\nabla h, X_i) = 0$. We check that $R(X_1, X_2) = 0$ by computing

$$\begin{aligned} 0 = \rho(X_2, U) &= R(X_2, U, U, \nabla h) + R(X_2, X_1, U, X_1) = R(X_1, X_2, X_1, U), \\ 0 = \rho(X_1, U) &= R(X_1, U, U, \nabla h) + R(X_1, X_2, U, X_2) = -R(X_1, X_2, X_2, U), \\ 0 = \rho(X_1, X_1) &= 2R(X_1, U, X_1, \nabla h) + R(X_1, X_2, X_1, X_2) \\ &= R(X_1, X_2, X_1, X_2). \end{aligned}$$

From this, it follows that (M, g) is a Brinkmann wave with parallel lightlike vector field ∇h such that $R(\nabla h^\perp, \nabla h^\perp) = 0$, so it is a *pp*-wave. □

Remark 5.15. A *pp*-wave of any dimension is given in local coordinates by expression (1.13) with $\partial_u F = 0$. The only possibly non-zero component of its Ricci tensor is $\rho(\partial_v, \partial_v) = -\frac{1}{2}\Delta_x F$, where $\Delta_x = \sum_i \frac{\partial^2}{\partial x_i^2}$ is the Laplacian with respect to the flat spatial metric given by $\sum_{i,j=1}^{n-2} dx_i^2$. Hence, a *pp*-wave is Ricci-flat if and only if $\Delta_x F = 0$. In dimension four, as a consequence of Theorem 5.14, the only Ricci-flat isotropic solutions of the weighted Einstein field equations are *pp*-waves of this type.

On the other hand, setting $h(u, v, x) = v$ in a *pp*-wave of arbitrary dimension, a straightforward calculation shows that $\nabla h = \partial_u$ is lightlike and $\mathrm{Hes}_h = 0$. Thus, any *pp*-wave with $\Delta_x F = 0$ is a Ricci-flat isotropic solution of the weighted Einstein field equations with $h(u, v, x) = v$.

A natural question that arises in view of Theorem 5.11 is whether an analogue of Item (1) holds in higher dimension. The following example shows that, in general, isotropic solutions of equation (4.8) realized on Brinkmann waves are not necessarily *pp*-waves, even if the Ricci operator is 2-step nilpotent.

Example 5.16. We consider local coordinates (u, v, x_1, x_2) and the metric given, up to symmetry, by the following non-vanishing components:

$$g(\partial_u, \partial_v) = 1, \quad g(\partial_{x_i}, \partial_{x_i}) = 1, \quad g(\partial_v, \partial_{x_2}) = x_1 x_2 + v x_2^2,$$
$$g(\partial_v, \partial_v) = (-2vx_2 - x_1 + 2vx_2)u + \tfrac{-2v^2x_1^3x_2 - vx_1^4 + 3vx_1^2x_2^2 + 12vx_1^2x_2 + x_1^3}{6v}.$$

The function $h(u, v, x_1, x_2) = v$ has lightlike gradient vector field $\nabla h = \partial_u$. A direct computation shows that this metric and the function h provide a solution to the field equations (4.8) with $\Lambda = 0$.

The vector field ∇h is recurrent, since $\nabla\nabla h = -\frac{x_1}{2} dv \otimes \nabla h$. Therefore, it is a Brinkmann wave. Moreover, the Ricci tensor has only one non-zero component: $\rho(\partial_v, \partial_v) = -\frac{x_1}{2v}$, so it is 2-step nilpotent.

Notice that $\nabla h^\perp = \operatorname{span}\{\partial_u, \partial_{x_1}, \partial_{x_2}\}$. We check that

$$R(\partial_{x_1}, \partial_{x_2}, \partial_v, \partial_{x_2}) = \frac{1}{2},$$

so $R(\nabla h^\perp, \nabla h^\perp) \neq 0$, which means that the spacetime given by g is not a pp-wave. Consequently, Theorem 5.11 (1) cannot be extended to higher dimension.

It was pointed out in Corollary 5.5 that 3-dimensional isotropic solutions of the weighted Einstein field equations give rise to 4-dimensional warped products which are Ricci-flat. The following are 4-dimensional examples obtained by applying this construction.

Example 5.17. We adopt notation from Theorem 5.6. Let N_1 be the plane wave given in Theorem 5.6 (1), let $h_1(u, v, x) = \alpha(v)$ and let t be the coordinate of $\mathbb{R}$. The 4-dimensional warped product $M_1 = N_1 \times_{h_1} \mathbb{R}$ is Ricci-flat and its Weyl tensor (hence its curvature tensor) is determined, up to symmetries, by the following terms:

$$W(\partial_v, \partial_x, \partial_v, \partial_x) = \frac{\alpha''(v)}{\alpha(v)} \quad \text{and} \quad W(\partial_v, \partial_t, \partial_v, \partial_t) = -\alpha(v)\alpha''(v).$$

Note that M_1 is still a Brinkmann wave with parallel lightlike vector field $V = \partial_u$. Furthermore, it satisfies the curvature conditions $R(V^\perp, V^\perp) = 0$ and $\nabla_{V^\perp} R = 0$, so it is indeed a plane wave.

Let N_2 be the pp-wave given in Theorem 5.6 (2) and $h_2(u, v, x) = \gamma_1 x + \gamma_0(v)$. Then $M_2 = N_2 \times_{h_2} \mathbb{R}$ is a 4-dimensional Ricci-flat warped product. Moreover, the Weyl tensor is determined, up to symmetries, by:

$$\begin{aligned} W(\partial_v, \partial_x, \partial_v, \partial_x) &= \tfrac{\gamma_1\alpha(v) + 2\gamma_0(v)\gamma_0''(v)}{2(\gamma_0(v) + \gamma_1 x)^2}, \\ W(\partial_v, \partial_t, \partial_t, \partial_v) &= \gamma_0(v)\gamma_0''(v) + \tfrac{\gamma_1\alpha(v)}{2}. \end{aligned}$$

As in the previous example, $V = \partial_u$ is still parallel and M_2 satisfies $R(V^\perp, V^\perp) = 0$, thus retaining the pp-wave character of N_2.

We adopt notation from Theorem 5.11 (2). Let N_3 be the Kundt spacetime given by (5.7) and $h_3(u,v,x) = v$. The 4-dimensional warped product $M_3 = N_3 \times_{h_3} \mathbb{R}$ is a Ricci-flat Kundt spacetime and its Weyl tensor is given, up to symmetries, by

$$W(\partial_u,\partial_v,\partial_v,\partial_x) = -\tfrac{1}{vx}, \quad W(\partial_v,\partial_t,\partial_v,\partial_t) = -\tfrac{1}{2}v\alpha_1(v) - \tfrac{uv}{x^2} + \log(x),$$
$$W(\partial_v,\partial_t,\partial_x,\partial_t) = \tfrac{v}{x}, \quad W(\partial_v,\partial_x,\partial_v,\partial_x) = \frac{v\alpha_1(v) - \frac{6uv}{x^2} - 2\log(x)}{2v^2}.$$

Since these examples are Ricci-flat 4-dimensional manifolds, their geometric information is encoded in the Weyl tensor, so it is convenient to analyze their Petrov type (we refer to [68, 114] for details). Since M_1 and M_2 are *pp*-waves, they are of type **N** (one easily checks that $\iota_{\partial_u} W = 0$). The warped product M_3, however, does not satisfy $\iota_X W = 0$ for any vector field X, but $\iota_{\partial_u} W = -\frac{1}{vx}\, dv \otimes (dv \wedge dx)$, therefore it is of type **III** (see [68]). All these examples present a repeated principal lightlike direction spanned by the distinguished lightlike vector field ∂_u. This is a common trait of Ricci-flat Kundt spacetimes, as a consequence of the Goldberg-Sachs theorem (see [114]).

Chapter 6
Solutions with conditions on the Weyl tensor

Throughout the previous chapter, we have seen how the causal character of the gradient of the density function ∇h influences the geometry of solutions of the weighted Einstein field equations, more specifically in the isotropic case. Nevertheless, although Theorem 5.4 guarantees that all isotropic solutions are realized on Kundt spacetimes, this is still a very broad family, and the problem of classifying all solutions in arbitrary dimensions becomes unmanageable. This is even more evident if we allow for solutions to be non-isotropic, due to the fact that, in general, we lose the Kundt structure, so solving the field equations grows increasingly challenging as our control over the metric dwindles.

In this chapter, we analyze solutions of the weighted Einstein field equations with vanishing cosmological constant

$$h\rho - \operatorname{Hes}_h + \Delta h g = 0$$

given by (4.6), both in the isotropic and non-isotropic cases. Since the field equations provide information on the Ricci tensor, we impose some natural geometric conditions on the conformal part of the curvature, i.e., the Weyl tensor. This discussion is local in nature, so we work in open sets where $\nabla h \neq 0$, and given the fundamental differences in the approach needed to analyze isotropic and non-isotropic solutions, we assume that the causal character of ∇h is constant. The results in this chapter are contained in the work [22].

When considering geometric conditions on the Weyl tensor, the strongest possible one is local conformal flatness, that is, $W = 0$. Unsurprisingly, this turns out to be quite restrictive, so the aim of this chapter shifts to the discussion of the less rigid condition of the harmonicity of the Weyl tensor, $\operatorname{div} W = 0$. In this case, we focus on spacetimes in dimension four, due to their physical significance as the basis of models in General Relativity. Recall that, as pointed out in the introduction to Chapter 2, the condition $\operatorname{div} W = 0$ arises naturally when studying conformally Einstein manifolds (see, for example, [83]), and has been used to study other Einstein-type structures such as generalized quasi-Einstein manifolds [19, 44].

Due to the fact that the scalar curvature of any solution of (4.6) is constant by Lemma 4.5, the Cotton tensor (1.2) satisfies

$$dP(X,Y,Z) = \operatorname{div} R(X,Y,Z) = (\nabla_Y \rho)(X,Z) - (\nabla_Z \rho)(X,Y).$$

Thus, for solutions of (4.6) the harmonicity of the Weyl tensor is equivalent to the harmonicity of the curvature tensor, which is in turn equivalent to the Codazzi character of the Ricci tensor. Both local conformal flatness and harmonic curvature are natural geometric conditions that have been studied for vacuum static spaces in Riemannian signature (see [78, 79]). Consequently, parts of our approach and some features of the solutions are similar to those in the aforementioned works.

Nevertheless, the fact that we are working in Lorentzian signature allows for greater flexibility, giving rise to new geometric structures not only in the purely Lorentzian isotropic case, but also for non-isotropic solutions.

Indeed, solutions with harmonic curvature present different Jordan forms of the Ricci operator, which is not possible in the Riemannian context (see Section 1.1.3). Each of these forms requires a different approach, but some geometric features are common to all solutions. Remarkably, it turns out that the underlying spacetimes for these solutions are typical examples that also arise in the study of cosmological models in General Relativity without the presence of a density. In particular, we know from Chapter 5 that Kundt spacetimes play an essential role in the isotropic case, but we see that they are also key in the classification of non-isotropic solutions.

Outline of the chapter

This chapter is organized as follows. In Section 6.1 we consider solutions with vanishing augmented Cotton tensor, which leads to the proof of the main classification result for locally conformally flat solutions (Theorem 6.3). Then, in order to prove Theorem 6.36, which is the main classification result for solutions satisfying $\operatorname{div} W = 0$, each admissible normal form of Ric is tackled in the corresponding section, in both the isotropic and non-isotropic cases. Further details on the geometry and the form of the density function for solutions with different Jordan forms are also provided in each of them.

In particular, in Section 6.2 we consider the diagonalizable case, where we follow some ideas already used in the Riemannian case. In Section 6.3, we prove that the Ricci eigenvalues of solutions with harmonic curvature are necessarily real. This is a long proof that requires a detailed analysis of the geometry of solutions and takes advantage of the use of an algebraic tool (Gröbner bases) on a set of polynomials to show that solutions with non-real eigenvalues do not exist. Finally, in Section 6.4 we study non-diagonalizable solutions with minimal polynomial of degrees two and three.

6.1 Solutions of the field equations

For isotropic solutions, we know from Chapter 5 that the weighted field equations (4.6) are enough to force a Kundt structure on the underlying manifold. Nevertheless, in this case we are implicitly imposing additional restrictions by assuming that ∇h is lightlike. Thus, lacking the rigidity provided by Theorem 5.4, we need to establish some general geometric properties of solutions. We begin by defining a useful auxiliary tensor related to static spaces in Riemannian signature.

To that end, let $J = \frac{\tau}{2(n-1)}$ be the usual Schouten scalar. Taking traces in (4.6), we have $\Delta h = -\frac{h\tau}{n-1} = -2Jh$, so the field equations can also be written as

$$h\,(\rho - 2Jg) = h\left(\rho - \frac{\tau}{n-1}g\right) = \operatorname{Hes}_h, \tag{6.1}$$

and $d\Delta h = -2Jdh$ since the scalar curvature τ is constant by Lemma 4.5. Besides, we have

$$(\nabla_Z \operatorname{Hes}_h)(X,Y) - (\nabla_Y \operatorname{Hes}_h)(X,Z) = R(\nabla h, X, Y, Z),$$

so, using (6.1), we can write

$$\begin{aligned} R(\nabla h, X, Y, Z) &= \nabla_Z(h(\rho - 2Jg))(X,Y) - \nabla_Y(h(\rho - 2Jg))(X,Z) \\ &= (\rho - 2Jg) \wedge dh(X,Y,Z) - h \operatorname{div} R(X,Y,Z) \\ &= ((\rho - 2Jg) \wedge dh - hdP)(X,Y,Z), \end{aligned} \tag{6.2}$$

where, for a $(0,2)$-tensor T and a 1-form ω,

$$T \wedge \omega(X,Y,Z) = T(X,Y)\omega(Z) - T(X,Z)\omega(Y).$$

Given the formal relationship between our field equations and vacuum static spaces, and following terminology in [105], we define the *augmented Cotton tensor*

$$D = hdP + \iota_{\nabla h} W, \tag{6.3}$$

where $\iota_{\nabla h} W(X,Y,Z) = W(\nabla h, X, Y, Z)$. The tensor D is related to the Bach tensor in the direction of ∇h and restrictions on it have consequences on the geometry of solutions in Riemannian signature (see [105] for details). Moreover, the weighted Einstein field equations give D a useful alternative characterization.

Lemma 6.1. *For any solution of the vacuum weighted Einstein field equations, the augmented Cotton tensor D satisfies*

$$(n-2)D = (n-1)\rho \wedge dh + g \wedge \iota_{\nabla h}\rho - \tau g \wedge dh \tag{6.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Proof. Substituting the definition of D in (6.2), and using the curvature decomposition $R = P \owedge g + W$, we get

$$\iota_{\nabla h}(P \owedge g) = \rho \wedge dh - \frac{\tau}{n-1} g \wedge dh - D.$$

Moreover, by the definition of P and the Kulkarni-Nomizu product,

$$\begin{aligned} \iota_{\nabla h}(P \owedge g) &= -\left(P \wedge dh + g \wedge \iota_{\nabla h} P\right) \\ &= -\tfrac{1}{n-2}\left(g \wedge \iota_{\nabla h}\rho + \rho \wedge dh - \tfrac{\tau}{n-1} g \wedge dh\right). \end{aligned}$$

Equating both expressions for $\iota_{\nabla h}(P \owedge g)$, the result follows. □

6.1.1 Solutions with vanishing augmented Cotton tensor

As a first step in understanding solutions to the vacuum weighted Einstein field equations, we consider those with vanishing augmented Cotton tensor. Recall that we are assuming that the character of ∇h does not change in M. Throughout this section and in the rest of the chapter, when considering the isotropic case, we will make use of the results in Chapter 5, in particular of Theorem 5.4, allowing us to only work with Kundt spacetimes (or one of their subfamilies). Hence, as one might expect, the bulk of the work will consist in arguments regarding non-isotropic solutions.

The following result shows that, if the augmented Cotton tensor vanishes, the underlying manifold is a warped product or a Brinkmann wave, depending on whether h is non-isotropic or isotropic, respectively.

Theorem 6.2. *Let* (M, g, h) *be a solution of the weighted Einstein field equations* (4.6) *with vanishing* D *tensor.*

1. *If* $g(\nabla h, \nabla h) \neq 0$*, then* (M, g) *is locally isometric to a warped product* $I \times_\varphi N$*, where* $I \subset \mathbb{R}$ *is an open interval,* N *is an* $(n-1)$*-dimensional Einstein manifold, and* ∇h *is tangent to* I*.*
2. *If* $g(\nabla h, \nabla h) = 0$*, then* (M, g) *is a Brinkmann wave with either vanishing or* 2*-step nilpotent Ricci operator.*

Proof. We analyze both cases separately. Assume first that $g(\nabla h, \nabla h) \neq 0$. Note that, for any non-isotropic solution, since ∇h is not lightlike, we can consider a local pseudo-orthonormal frame $\mathcal{B} = \{E_1, E_2, \ldots, E_n\}$, where $E_1 = \frac{\nabla h}{\|\nabla h\|}$ and $\|\nabla h\| = \sqrt{\varepsilon g(\nabla h, \nabla h)}$ ($\varepsilon = \pm 1$ depending on whether ∇h is spacelike or timelike, respectively). Furthermore, without loss of generality, we can take $g(E_2, E_2) = \varepsilon_2 = -\varepsilon$ and $g(E_i, E_i) = 1$ for $i > 2$.

On the one hand, in the expression (6.4) for the augmented Cotton tensor D, take $Y = \frac{E_1}{\|\nabla h\|}$ so that $g(\nabla h, Y) = \varepsilon$, and take $X = E_i$, $Z = E_j$, $i, j > 1$. Since D vanishes, we have $\rho(E_i, E_j) = \frac{\tau - \varepsilon\rho(E_1, E_1)}{n-1} g(E_i, E_j)$. Then, by equation (6.1),

$$\mathrm{Hes}_h(E_i, E_j) = -h\varepsilon \frac{\rho(E_1, E_1)}{n-1} g(E_i, E_j).$$

On the other hand, we can take $X = Y = E_1$ and $Z = E_i$, $i > 1$ to find $\rho(E_1, E_i) = \mathrm{Hes}_h(E_1, E_i) = 0$. It follows that the level hypersurfaces of h in M are totally umbilical and, furthermore, that the distribution generated by ∇h is totally geodesic. Consequently, by Theorem 1.3, (M, g) splits locally as a twisted product $I \times_{\tilde\varphi} N$, where $I \subset \mathbb{R}$ is an open interval, for some function $\tilde\varphi$ on $I \times N$. Moreover, since the mixed Ricci terms $\rho(E_1, E_i)$ vanish, from Theorem 1.4 it follows that the twisted product reduces to a warped product $I \times_\varphi N$ for some function φ on I.

Let t be a local coordinate parameterizing I by arc length with $E_1 = \nabla t = \varepsilon \partial_t$ and let $\varepsilon\alpha = \rho(E_1, E_1)$ and $\lambda = \frac{\tau - \alpha}{n-1}$. Then, we can write $\mathrm{Hes}_h(E_1, E_1) = h''$ and, by the weighted Einstein field equations (6.1), $\alpha = \varepsilon h^{-1} h'' + 2J$, so α depends only on t. Moreover, since τ is constant, $\rho(E_i, E_j) = \lambda g(E_i, E_j)$ depends only on t as well. We shall show that N is Einstein as follows. Consider the basis $\{\bar{E}_i = \varphi E_i\}_{i=2,\dots,n}$ which is orthonormal on N. From the expression of the Ricci tensor of a warped product (see (1.6)), we have

$$\begin{aligned}\rho^N(\bar{E}_i, \bar{E}_j) &= \rho(\bar{E}_i, \bar{E}_j) + \varepsilon g(\bar{E}_i, \bar{E}_j)\left(\tfrac{\varphi''}{\varphi} + (n-2)\tfrac{(\varphi')^2}{\varphi^2}\right)\\ &= \varphi^2\left(\varepsilon\lambda + \tfrac{\varphi''}{\varphi} + (n-2)\tfrac{(\varphi')^2}{\varphi^2}\right)\varepsilon g(E_i, E_j).\end{aligned}$$

Since $\rho^N(\bar{E}_i, \bar{E}_j)$ is a function defined on the fiber, it does not depend on t, which is a coordinate of the base. Hence, $\rho^N = \beta g^N$ for some $\beta \in \mathbb{R}$ and N is Einstein. Thus, assertion (1) holds.

Now, assume $g(\nabla h, \nabla h) = 0$. Then, it follows from Theorem 5.4 that the Ricci operator Ric is nilpotent. Moreover, there exists a pseudo-orthonormal basis $\{\nabla h, U, X_1, \dots, X_{n-2}\}$ such that the non-zero terms of the metric tensor are $g(\nabla h, U) = g(X_i, X_i) = 1$, $i = 1, \dots, n-2$, and the non-zero terms of the Ricci operator are given by

$$\mathrm{Ric}(U) = \nu\nabla h + \mu X_1 \quad\text{and}\quad \mathrm{Ric}(X_1) = \mu\nabla h$$

(see equation (5.2) and Lemma 5.2). Since $D = 0$, equation (6.4) evaluated on (U, U, X_1) yields

$$\begin{aligned}0 &= (n-1)\rho \wedge dh(U, U, X_1) + g \wedge \iota_{\nabla h}\rho(U, U, X_1) - \tau g \wedge dh(U, U, X_1)\\ &= -(n-1)\mu.\end{aligned}$$

Hence, $\mu = 0$, so the Ricci operator either vanishes or is two-step nilpotent. It follows from Theorem 5.4 that (M, g) is a Brinkmann wave. □

Note that the warped product structure of non-isotropic solutions with vanishing D tensor described in Theorem 6.2 (1) is analogous to the case of Riemannian signature discussed in [105]. This analogy also works when considering locally conformally flat non-isotropic solutions and comparing them to those studied in Riemannian signature in [79], as we will see in the following section.

6.1.2 Locally conformally flat solutions

We will start this subsection keeping the dimension of the manifold arbitrary in order to prove a general rigidity result, and then we will obtain a stronger variant of it in dimension four. Unsurprisingly, the vanishing of the Weyl tensor turns out to be more restrictive than the vanishing of the augmented Cotton tensor.

Theorem 6.3. *Let (M, g, h) be an n-dimensional locally conformally flat smooth metric measure spacetime. Then, (M, g, h) is a solution of the vacuum weighted Einstein field equations* (4.6) *if and only if one of the following is satisfied:*

1. *$g(\nabla h, \nabla h) \neq 0$ and (M, g) is locally isometric to a warped product $(I \times N, dt^2 \oplus \varphi^2 g^N)$, where $I \subset \mathbb{R}$ is an open interval, N is an $(n-1)$-dimensional manifold of constant sectional curvature, and $h(t)$ and $\varphi(t)$ satisfy the following system of ODEs:*

$$\begin{aligned}0 &= h'\varphi' - h\varphi'',\\ 0 &= h'' + (n-1)h\tfrac{\varphi''}{\varphi} + \varepsilon\tfrac{\tau}{n-1}h.\end{aligned} \tag{6.5}$$

2. *$g(\nabla h, \nabla h) = 0$ and (M, g, h) is a plane wave. Moreover, there exist local coordinates $\{u, v, x_1, \ldots, x_n\}$ such that the metric is given by*

$$g(u, v, x_1, \ldots, x_{n-2}) = 2dvdu + F(v, x_1, \ldots, x_{n-2})dv^2 + \sum_{i=1}^{n-2} dx_i^2,$$

where $F(v, x_1, \ldots, x_{n-2}) = -\frac{h''(v)}{(n-2)h(v)} \sum_{i=1}^{n-2} x_i^2 + \sum_{i=1}^{n-2} b_i(v)x_i + c(v)$.

Proof. We start with the case $g(\nabla h, \nabla h) \neq 0$. In dimension three, local conformal flatness is equivalent to the condition $dP = 0$, while for locally conformally flat manifolds of dimension $n \geq 4$, $W = 0$ implies $dP = 0$. In both cases, the augmented Cotton tensor D given by (6.3) vanishes identically. Thus, we apply Theorem 6.2 to obtain a local splitting into a warped product $I \times_\varphi N$ and adopt the notation of its proof. A warped product of the form $I \times_\varphi N$ is locally conformally flat if and only if the fiber N has constant sectional curvature (see Theorem 1.6).

Once the local splitting into a warped product has been established, let $\varepsilon = g(\partial_t, \partial_t)$ and $\varepsilon_i = g(E_i, E_i)$. We use the gradient $\nabla h = \varepsilon h' \partial_t$, the expressions (1.4) and (1.6) for the connection and the Ricci tensor of a warped product and the weighted Einstein field equations (6.1) to compute the Laplacian of h, the two eigenvalues of the Ricci operator ($\mathrm{Ric}(\nabla h) = \alpha \nabla h$ and $\mathrm{Ric}(E_i) = \lambda E_i$ for $i > 1$) and the scalar curvature in terms of h and φ:

$$\begin{aligned}
\Delta h &= \varepsilon g(\nabla_{\partial_t} \nabla h, \partial_t) + \textstyle\sum_{i=2}^n \varepsilon_i g(\nabla_{E_i} \nabla h, E_i) \overset{(1.4)}{=} \varepsilon \left(h'' + (n-1)\frac{h'\varphi'}{\varphi} \right), \\
\alpha &= \varepsilon \rho(\partial_t, \partial_t) \overset{(1.6)}{=} -\varepsilon(n-1)\frac{\varphi''}{\varphi} \overset{(6.1)}{=} \varepsilon \frac{h''}{h} + \frac{\tau}{n-1}, \\
\lambda &= \varepsilon_i \rho(E_i, E_i) = \frac{\tau - \alpha}{n-1} = \varepsilon \frac{\varphi''}{\varphi} + \frac{\tau}{n-1}.
\end{aligned}$$

Moreover, since $\tau = \alpha + (n-1)\lambda$, we have

$$-h\frac{\alpha}{n-1}\varepsilon_i = h\left(\lambda - \frac{\tau}{n-1}\right)\varepsilon_i \overset{(6.1)}{=} \mathrm{Hes}_h(E_i, E_i) \overset{(1.4)}{=} \varepsilon \frac{\varphi' h'}{\varphi}\varepsilon_i,$$

so $\alpha = -(n-1)\varepsilon\frac{\varphi' h'}{h\varphi}$. Equating this expression to the second expression for α above, we can solve for τ. In summary, we obtain the following four quantities:

$$\begin{aligned}
\Delta h &= \varepsilon \left(h'' + (n-1)\frac{h'\varphi'}{\varphi} \right), & \alpha &= -\varepsilon(n-1)\frac{\varphi''}{\varphi} = \varepsilon\frac{h''}{h} + \frac{\tau}{n-1}, \\
\lambda &= \frac{\tau-\alpha}{n-1} = \varepsilon\frac{\varphi''}{\varphi} + \frac{\tau}{n-1}, & \tau &= -(n-1)\varepsilon\left(\frac{h''}{h} + (n-1)\frac{h'\varphi'}{h\varphi}\right).
\end{aligned}$$

The non-diagonal terms of $G^h = h\rho - \mathrm{Hes}_h + \Delta h g$ vanish identically. Since $\varepsilon^2 = 1$, using the expressions above allows us to compute the diagonal terms as follows:

$$\begin{aligned} 0 = G^h(\partial_t,\partial_t) &= h\varepsilon\alpha - h'' + \varepsilon\Delta h \\ &= -(n-1)\tfrac{h\varphi''}{\varphi} - h'' + h'' + (n-1)\tfrac{h'\varphi'}{\varphi} \\ &= (n-1)\left(\tfrac{h'\varphi' - h\varphi''}{\varphi}\right), \end{aligned}$$

$$\begin{aligned} 0 = G^h(E_i,E_i) &= \left(h\lambda - \varepsilon\tfrac{\varphi' h'}{\varphi} + \Delta h\right)\varepsilon_i \\ &= \varepsilon\left(h\left(\tfrac{\varphi''}{\varphi} + \varepsilon\tfrac{\tau}{n-1}\right) + \left(h'' + (n-1)\tfrac{h'\varphi'}{\varphi}\right) - \tfrac{h'\varphi'}{\varphi}\right)\varepsilon_i \\ &= \varepsilon\left(h'' + h(n-1)\tfrac{\varphi''}{\varphi} + h\varepsilon\tfrac{\tau}{n-1}\right)\varepsilon_i, \end{aligned}$$

where we have used the relation $h'\varphi' - h\varphi'' = 0$ that we got from the first expression to simplify the second one. Hence, we obtain that the system of ODEs given in (6.5) are necessary and sufficient conditions for a warped product as above to be a solution of the field equations (4.6).

Assume now that $g(\nabla h, \nabla h) = 0$. Since $D = 0$, we use Theorem 6.2 to see that (M, g) is a Brinkmann wave with either vanishing or 2-step nilpotent Ricci tensor. Moreover, $\tau = J = 0$ and the only possibly non-zero term of the Ricci tensor is $\rho(U, U) = \nu$. In particular, if $\nu = 0$, then the manifold is flat. In any case, $\operatorname{Ric}\nabla h = 0$, $\operatorname{Ric} U = \nu\nabla h$ and $\operatorname{Ric} X_i = 0$ for all i, so (M, g) is Ricci-isotropic, meaning that $g(\operatorname{Ric}(Y), \operatorname{Ric}(Y)) = 0$ for all $Y \in \mathfrak{X}(M)$. Therefore, (M, g) is a pp-wave if and only if $R(\mathcal{D}^\perp, \mathcal{D}^\perp, \cdot, \cdot) = 0$, where $\mathcal{D}^\perp = \operatorname{span}\{\nabla h, X_1, \dots, X_{n-2}\}$ (see Section 1.3 and [86]).

From $W = 0$ and $\tau = 0$, we have that $R = P \owedge g = \frac{1}{n-2}\rho \owedge g$. Since $\operatorname{Ric} X_i = \operatorname{Ric}\nabla h = 0$, by directly substituting in the curvature expression we get that $R(\mathcal{D}^\perp, \mathcal{D}^\perp, \cdot, \cdot) = 0$ and that (M, g) is indeed a pp-wave.

Now, locally conformally flat pp-waves are plane waves that admit local Kundt-type coordinates $(u, v, x_1 \dots, x_{n-2})$ such that the metric takes the form

$$g(u, v, x_1, \dots, x_{n-2}) = 2dvdu + F(v, x_1, \dots, x_{n-2})dv^2 + \sum_{i=1}^{n-2} dx_i^2,$$

where $F(v, x_1, \dots, x_{n-2}) = \frac{a(v)}{n-2}\sum_{i=1}^{n-2} x_i^2 + \sum_{i=1}^{n-2} b_i(v)x_i + c(v)$ (see, for example, [14]). With respect to these coordinates, the only non-vanishing component of the Ricci tensor is $\rho(\partial_u, \partial_u) = -a(u)$. Moreover, because the distinguished parallel lightlike distribution of the pp-wave corresponds with ∇h by construction, it follows that ∇h is a multiple of ∂_u, so $h(v, u, x_1, \dots, x_{n-2}) = h(v)$. Now, a direct computation of the terms in (4.6) yields the only condition:

$$-a(v)h(v) - h''(v) = 0,$$

from where case (2) follows. □

It is important to note that, in Theorem 6.3 (2), although (M, g) is a pp-wave, and hence admits a parallel lightlike vector field, ∇h is not parallel in general. In fact, if it is parallel, then

$\mathrm{Hes}_h = h\rho = 0$ by the field equations, and given that $W = 0$ and $\tau = 0$, the resulting solution is flat.

Remark 6.4. The system of ODEs (6.5) was obtained in [79] for $\varepsilon = 1$ in the Riemannian setting. An analogous reasoning to that in [79] shows that from (6.5), it follows that

$$\begin{aligned} \gamma &= \varphi^{n-1}\varphi'' + \tfrac{\varepsilon\tau}{n(n-1)}\varphi^n, \\ \tfrac{\varepsilon\kappa}{n-2} &= (\varphi')^2 + \tfrac{2\gamma}{n-2}\varphi^{2-n} + \tfrac{\varepsilon\tau}{n(n-1)}\varphi^2, \end{aligned}$$

where γ, κ are real constants and $\rho^N = \kappa g^N$. The discussion of the solutions to these ODEs in [79] in terms of the constants τ, κ and γ, also applies to the Lorenztian case by substituting τ and κ with $\varepsilon\tau$ and $\varepsilon\kappa$ respectively. Note that these ODEs are also satisfied in the case $D = 0$ (not necessarily locally conformally flat), if we allow a generic Einstein fiber instead of a fiber of constant sectional curvature.

Isotropic locally conformally flat solutions are completely characterized, for arbitrary dimension, in Theorem 6.3 (2). For the non-isotropic case, we give a detailed description in dimension four as follows, where the nature of solutions depends on the sign of the scalar curvature, which is constant by Lemma 4.5.

Corollary 6.5. *Let (M, g, h) be a non-isotropic, non-flat solution of the weighted Einstein field equations* (4.6) *with* $\dim M = 4$ *and vanishing augmented Cotton tensor. Then M decomposes locally as a product $I \times N$, where $I \subset \mathbb{R}$ is an open interval with ∇h tangent to I; and N is a 3-dimensional manifold with constant sectional curvature κ. Moreover, the metric and the density functions satisfy one of the following:*

1. *g is a direct product metric $\varepsilon dt^2 + g^N$ with t a coordinate parameterizing I by arc length such that*
$$\begin{aligned} h(t) &= c_1 \sin\left(\tfrac{\sqrt{2\varepsilon\kappa}}{\varphi}t\right) + c_2\cos\left(\tfrac{\sqrt{2\varepsilon\kappa}}{\varphi}t\right), && \text{if } \varepsilon\kappa > 0, \\ h(t) &= c_1 e^{\frac{\sqrt{-2\varepsilon\kappa}}{\varphi}t} + c_2 e^{-\frac{\sqrt{-2\varepsilon\kappa}}{\varphi}t}, && \text{if } \varepsilon\kappa < 0. \end{aligned}$$
2. *g is a warped product metric $\varepsilon dt^2 + \varphi(t)^2 g^N$ with t a coordinate parameterizing I by arc length such that the density function h satisfies $h(t) = A\varphi'(t)$, $A \in \mathbb{R}^*$, and φ takes the following forms, depending on the sign of the scalar curvature τ of the product:*
$$\begin{aligned} \varphi(t)^2 &= \tfrac{6\kappa}{\tau} + c_1\sin\left(\sqrt{\tfrac{\varepsilon\tau}{3}}t\right) + c_2\cos\left(\sqrt{\tfrac{\varepsilon\tau}{3}}t\right), && \text{if } \varepsilon\tau > 0, \\ \varphi(t)^2 &= \tfrac{6\kappa}{\tau} + c_1 e^{\sqrt{-\frac{\varepsilon\tau}{3}}t} + c_1 e^{-\sqrt{-\frac{\varepsilon\tau}{3}}t}, && \text{if } \varepsilon\tau < 0, \\ \varphi(t)^2 &= \varepsilon\kappa t^2 + c_1 t + c_2, && \text{if } \tau = 0,\ c_1^2 \neq 4\varepsilon c_2\kappa, \end{aligned}$$

where A, c_1, c_2 are suitable integration constants so that $\varphi(t)^2, h(t) > 0$ for all $t \in I$.

Proof. From the first ODE in (6.5), it follows that either $\varphi' = 0$, so we have a Riemannian product, or $h(t) = A\varphi'(t)$ with $A \in \mathbb{R}^*$ such that $h > 0$ for all $t \in I$. In the first case, the remaining non-vanishing components of the weighted Einstein field equations (6.1) take the form

$$0 = G^h(E_i, E_j) = \varepsilon \left(h'' + \varepsilon \frac{\tau}{3} h \right) g(E_i, E_j),$$

where $\tau = \frac{6\kappa}{\varphi^2}$. We can solve the resulting ODE $0 = h'' + \frac{2\varepsilon\kappa}{\varphi^2} h$ to determine the density function in Item 1 of the corollary. Note that, if $\kappa = 0$, then the manifold is flat.

Now, assume that $\varphi' \neq 0$, so $h(t) = A\varphi'(t)$, and take $F(t) = \varphi(t)^2$. Then, we compute the scalar curvature of the warped product, in terms of κ and F, using (1.7), resulting in the equation $0 = \tau F - 3(2\kappa - \varepsilon F'')$. We solve this ODE to get the different forms of φ^2 in Item 2 of the corollary. Then, a direct computation shows that all components of the weighted Einstein field equations vanish. Note that, if $\tau = 0$ and $c_1^2 = 4\varepsilon c_2 \kappa$, the manifold is flat. □

For manifolds of dimensions other than 4, it is not possible to express all possible solutions in such a simple way as in Corollary 6.5. However, we refer to Section 6.2 for some generalizations of Kobayashi's locally conformally flat static spaces to Lorentzian signature that are of special interest, since they appear as submanifolds of higher-dimensional solutions. The following example will be used to illustrate this fact.

Example 6.6. Let $(I \times_\varphi N, g, h)$ be a 3-dimensional (Riemannian or Lorentzian) SMMS, with N a surface of constant Gauss curvature κ. By Theorem 1.6, this manifold is locally conformally flat. Therefore, if this triple is a non-isotropic, non-flat solution of the weighted Einstein field equations (4.6) with vanishing scalar curvature τ, then the system of ODEs (6.5) is satisfied. Moreover, particularizing the equations in Remark 6.4 to $n = 3$ and $\tau = 0$ yields $\gamma = \varphi^2 \varphi''$ and $\varepsilon\kappa = (\varphi')^2 + 2\gamma\varphi^{-1}$ for some constant $\gamma \in \mathbb{R}\backslash\{0\}$, since the manifold becomes flat if $\gamma = 0$. If $\gamma > 0$, this corresponds to case IV.I in [79], and if $\gamma < 0$, to case III.1 (substituting κ by $\varepsilon\kappa$). Notice that all solutions of this kind have two distinct Ricci eigenvalues.

6.2 Solutions with harmonic curvature. The diagonalizable case

We have already seen how local conformal flatness only allows for very specific warped product structures (see Theorem 6.3 and Corollary 6.5) and how they relate to the Riemannian static spaces discussed in [79]. In order to get a broader family of solutions with a more flexible geometry, we are going to focus on dimension four and impose a weaker restriction than local conformal flatness: harmonic Weyl tensor. As we have already pointed out, due to Lemma 4.5, this is equivalent to the harmonicity of the curvature tensor.

Our analysis is divided into several sections depending on the structure of the Ricci operator Ric. We will prove shortly that, for solutions with harmonic curvature, ∇h is an eigenvalue of Ric (see Lemma 6.10). With this information, we can apply the discussion on self-adjoint operators in Lorentzian vector spaces (see Section 1.1.3) to Ric. Thus, if $g(\nabla h, \nabla h) \neq 0$, at each point of the manifold, Ric takes one of the following four forms: On the one hand, relative to an orthonormal frame $\mathcal{B}_1 = \{E_1 = \nabla h/\|\nabla h\|, E_2, E_3, E_4\}$,

$$\mathrm{Ric} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad \text{or} \quad \mathrm{Ric} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & -b & a & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \tag{6.6}$$

with $b \neq 0$. Recall that we refer to these structures as Type I.a and Type I.b, respectively. On the other hand, relative to a pseudo-orthonormal frame $\mathcal{B}_2 = \{E_1 = \nabla h/\|\nabla h\|, U, V, E_2\}$, where the only non-vanishing terms of the metric are $g(E_i, E_i) = 1$, $i = 1, 2$, $g(U, V) = 1$, there are two more possible forms:

$$\mathrm{Ric} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & \varepsilon & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} \quad \text{or} \quad \mathrm{Ric} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha \end{pmatrix}, \tag{6.7}$$

which we call Type II and Type III respectively.

For solutions with $g(\nabla h, \nabla h) = 0$, particularizing (5.2) in dimension four, and given that the Ricci operator is nilpotent by Theorem 5.4, there exists an adapted pseudo-orthonormal frame $\mathcal{B}_0 = \{\nabla h, U, X_1, X_2\}$ such that the non-zero terms of the metric tensor are $g(\nabla h, U) = g(X_i, X_i) = 1$, $i = 1, 2$, and the Ricci operator takes the form

$$\mathrm{Ric} = \begin{pmatrix} 0 & \nu & \mu & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.8}$$

Hence, the isotropic solution is Ricci-flat (corresponding to Type I.a), 2-step nilpotent ($\mu = 0$ and $\nu \neq 0$, Type II) or 3-step nilpotent ($\mu \neq 0$, Type III).

Henceforth, we will assume that the Ricci operator of any solution is of constant type in the manifold. Otherwise, one would restrict to an open subset where this happens. In this section, we treat the diagonalizable case (see Section 6.3 for a study of Type I.b and Section 6.4 for details on Type II and Type III). Solutions with harmonic curvature were previously considered in Riemannian signature in [78], where the Ricci operator is necessarily diagonalizable. Motivated by this work, we follow some of the arguments applied to static spaces to obtain all possible solutions in this setting (see also [54] for the study of eigendistributions of the Ricci operator on manifolds with harmonic curvature, whose arguments we will mimic at some instances, and [32] for related arguments for Ricci solitons). Unsurprisingly, some of the results in this section are reminiscent of those in [78]. However, the fact that we are working in Lorentzian signature allows for greater flexibility and gives rise to new geometric structures when the solution is isotropic. Moreover, if ∇h is timelike, we will see that the Ricci operator is necessarily diagonalizable, so all solutions where ∇h has this character are described below in Theorem 6.7.

Much like in the Riemannian case, the geometric structure of a non-isotropic solution strongly depends on the number of distinct eigenvalues of Ric. Arguments in [54] and [78] show that this number does not change in an open dense subset of M. Indeed, for $x \in M$, let $E_{\mathrm{Ric}}(x)$ be the number of distinct eigenvalues of Ric_x and set

$$M_{\mathrm{Ric}} = \{x \in M \colon E_{\mathrm{Ric}} \text{ is constant in a neighborhood of } x\}.$$

It is clear that M_{Ric} is open. To show that this subset is dense, take $x \in M$ and consider any open ball B centered at x. Since the rank of E_{Ric} is finite, there is a point $q \in B$ where $E_{\mathrm{Ric}}(q)$ is the maximum of E_{Ric} on B. Since a small variation of the eigenvalues cannot decrease the value $E_{\mathrm{Ric}}(q)$ and because it is maximum by definition, there is a neighborhood of q where $E_{\mathrm{Ric}} = E_{\mathrm{Ric}}(q)$, so $q \in M_{\mathrm{Ric}}$. Therefore, since the number of distinct eigenvalues heavily influences the geometry of solutions, we work locally around points that belong to M_{Ric} in the non-isotropic case.

Theorem 6.7. *Let (M, g, h) be a 4-dimensional smooth metric measure space with diagonalizable Ricci operator and harmonic curvature (not locally conformally flat).*

1. *If $g(\nabla h, \nabla h) = 0$, then (M, g) is a solution of equation* (4.6) *if and only if (M, g) is a Ricci-flat pp-wave and, in suitable local coordinates $\{u, v, x_1, x_2\}$, it can be written as*
$$g(u, v, x_1, x_2) = 2du dv + F(v, x_1, x_2)dv^2 + dx_1^2 + dx_2^2, \tag{6.9}$$
with $\Delta_x F = \partial^2_{x_1} F + \partial^2_{x_2} F = 0$, and $h(u, v, x_1, x_2) = v$.
2. *If $g(\nabla h, \nabla h) \neq 0$ and (M, g, h) is a solution to* (4.6)*, then $\left(M_{\mathrm{Ric}}, g\big|_{M_{\mathrm{Ric}}}\right)$ is locally isometric to:*
   - *(a) A direct product $I_2 \times \tilde{M}$, where $\tilde{M} = I_1 \times_\xi N$ is a warped product 3-dimensional solution with $\tilde{\tau} = 0$ and N a surface of constant Gauss curvature. Moreover, $h = c\xi'$ is defined on I_1.*
   - *(b) A direct product $N_1 \times N_2$ of two surfaces of constant Gauss curvature $\frac{\kappa}{2}$ and κ, respectively. The density function is defined on N_1 and is a solution to the Obata equation $\mathrm{Hes}_h^{N_1} = -\frac{\kappa h}{2} g^{N_1}$.*

Remark 6.8. Notice that the condition on the defining function F of the pp-wave metric in (6.9) resembles the Laplace equation. This is indeed a necessary and sufficient condition for a pp-wave to be Ricci-flat (see Remark 5.15). Thus, consider, for example, a solution of the form $F(v, x_1, x_2) = f(v)(x_1^2 - x_2^2)$ to build solutions with harmonic Weyl tensor but which are not locally conformally flat. Indeed, the non-vanishing components of the Weyl tensor are, up to symmetries,

$$W(\partial_v, \partial_{x_1}, \partial_v, \partial_{x_1}) = W(\partial_v, \partial_{x_2}, \partial_v, \partial_{x_2}) = -f(v).$$

Remark 6.9. Solutions in Theorem 6.7 (2.a) are built from 3-dimensional locally conformally flat solutions with vanishing scalar curvature (see Example 6.6), just by adding a 1-dimensional factor, and result in a multiply warped product of the form $I_1 \times I_2 \times_\xi N$. If $I_1 \times_\xi N$ is Riemannian with N of constant positive Gauss curvature ($\kappa > 0$), it was pointed out by Kobayashi that there is a solution in $\mathbb{R} \times S^2$ which contains a spatial slice of the well-known Schwarzschild spacetime. Thus, we construct a solution on the 4-dimensional spacetime $\mathbb{R} \times (\mathbb{R} \times_\varphi S^2)$, with the metric given by $g = -ds^2 \oplus g^{\mathrm{Sch}}$, where g^{Sch} stands for the spatial part of the Schwarzschild metric (see [114] for details on this solution). In contrast, solutions for $\kappa < 0$ are incomplete (cf. [78, Example 3]).

From Example 6.6, taking $\kappa = 0$ allows for an explicit expression for $\varphi(t)$. Indeed, if $\kappa = 0$, then $(\varphi')^2 + 2a\varphi^{-1} = 0$ and $a = \varphi^2\varphi''$. Thus, we can write $(\varphi')^2 + 2\varphi\varphi'' = 0$. Solving this ODE yields (after a translation of t, if needed) $\varphi(t) = K_1 t^{2/3}$ and $h(t) = K_2 t^{-1/3}$ for some suitable $K_1, K_2 \in \mathbb{R}^*$. This gives solutions with Ricci eigenvalues $\left\{\frac{4\varepsilon}{9t^2}, 0\right\}$ (simple) and $\left\{-\frac{2\varepsilon}{9t^2}\right\}$ (double).

Proof of Theorem 6.7 (1). If $g(\nabla h, \nabla h) = 0$, then by Theorem 5.4, Ric is nilpotent. Since it diagonalizes by assumption, the manifold is necessarily Ricci-flat (hence all solutions of this type have harmonic curvature tensor). Moreover, since we are working with 4-dimensional manifolds, by Theorem 5.14, the underlying manifold (M, g) is thus a *pp*-wave, and there exist local coordinates $\{u, v, x_1, x_2\}$ so that the metric is given by (6.9). A direct computation shows that the *pp*-wave is Ricci-flat if and only if the spacelike Laplacian vanishes: $\partial^2_{x_1} F + \partial^2_{x_2} F = 0$. In these coordinates, h is only a function of v and, since $\mathrm{Hes}_h = 0$, we have $h''(v) = 0$. Now, the coordinate v can be normalized so that $h(u, v, x_1, x_2) = v$. □

Non-isotropic solutions require a deeper analysis in order to provide the classification in Theorem 6.7 (2). Thus, throughout the rest of this section, all solutions are assumed to be non-isotropic. Following ideas developed in [78] for the Riemannian counterpart, we establish some preliminary results. Although we are focusing on 4-dimensional manifolds, they apply to solutions of arbitrary dimension.

Lemma 6.10. *For any n-dimensional solution (M, g, h) of the weighted Einstein field equations* (4.6) *with harmonic curvature,* $\mathrm{Ric}(\nabla h) = \lambda \nabla h$ *for some smooth function λ on M.*

Proof. Assume $\operatorname{div} R = dP = 0$. In (6.2), we can choose $X = \nabla h$, $Y \perp \nabla h$ and Z such that $g(Z, \nabla h) = 1$ to see that

$$\begin{aligned} 0 &= R(\nabla h, \nabla h, Y, Z) \\ &= dh(Z)(\rho - 2Jg)(Y, \nabla h) - dh(Y)(\rho - 2Jg)(Z, \nabla h) = \rho(Y, \nabla h) \end{aligned}$$

for every $Y \perp \nabla h$. Consequently, ∇h is an eigenvector of the Ricci operator. □

The fact that ∇h is a real eigenvector of Ric has important geometric consequences for vacuum solutions. For example, if ∇h is timelike, we consider the restriction of Ric to $\nabla h^\perp$. Since the metric is positive definite in $\nabla h^\perp$, and $\mathrm{Ric}\big|_{\nabla h^\perp}$ is self-adjoint, it follows that it diagonalizes in some orthonormal frame. Thus, the full Ricci operator Ric diagonalizes in a suitable pseudo-orthonormal frame $\mathcal{B}_1 = \{E_1, E_2, E_3, E_4\}$, where $E_1 = \nabla h / \|\nabla h\|$ and $\|\nabla h\| = \sqrt{\varepsilon g(\nabla h, \nabla h)}$ (with $\varepsilon = g(E_1, E_1) = -1$), $g(E_2, E_2) = -\varepsilon$ and $g(E_i, E_i) = 1$ for $i > 2$. We will refer to this as an *adapted frame*. Consequently, all solutions with ∇h timelike are described by Theorem 6.7.

In contrast, if ∇h is spacelike, this structure is also possible, with $\varepsilon = 1$, but other Jordan forms for Ric can also arise, as portrayed in (6.6) and (6.7). Nevertheless, since in this section we are assuming that Ric diagonalizes, let $\mathrm{Ric}\, E_i = \lambda_i E_i$, with

$$\varepsilon = g(E_1, E_1) = \pm 1, \;\; g(E_2, E_2) = -\varepsilon \;\; \text{and} \;\; g(E_i, E_i) = 1, \; i = 3, \dots, n.$$

From the vacuum equation (6.1), it follows that the Hessian operator hes_h diagonalizes in the frame $\mathcal{B}_1$, with $\mathrm{hes}_h\, E_i = h(\lambda_i - 2J)E_i$. In particular, this implies that $X(g(\nabla h, \nabla h)) = 2\,\mathrm{Hes}_h(\nabla h, X) = 0$ for all $X \perp \nabla h$, so the distribution generated by ∇h is totally geodesic. This also means that $\|\nabla h\|$ is constant on each connected component of the level sets of h, so we can write

$$\nabla_{E_i} E_1 = \frac{1}{\|\nabla h\|} \nabla_{E_i} \nabla h = \beta_i E_i \quad \text{with} \quad \beta_i = \frac{h(\lambda_i - 2J)}{\|\nabla h\|}. \tag{6.10}$$

Furthermore, we have

$$0 = E_i(g(\nabla h, E_j)) = g(\nabla_{E_i} \nabla h, E_j) + g(\nabla h, \nabla_{E_i} E_j) = g(\nabla h, \nabla_{E_i} E_j)$$

for $i, j > 1$, $i \neq j$. It follows that $\mathrm{span}\{E_2, \dots, E_4\}$ is closed under Lie bracket, and the distribution generated by $\mathrm{span}\{E_2, \dots, E_n\}$ is integrable, so M splits locally as a product $I \times N$, where I is an open interval to which E_1 is tangent, and N projects onto the leaves of the foliation generated by $\mathrm{span}\{E_2, \dots, E_n\}$. Moreover, we have

$$d\left(\frac{dh}{\|\nabla h\|}\right) = -\frac{1}{2\|\nabla h\|^3} d\|\nabla h\|^2 \wedge dh = 0,$$

since $\nabla_X(\|\nabla h\|^2) = 0$ for $X \perp \nabla h$. Thus, $dh/\|\nabla h\|$ is a closed form, and there is a local coordinate t such that $dt = dh/\|\nabla h\|$. Note that $\nabla t = \nabla h/\|\nabla h\| = E_1$, so that $\nabla_{E_1} E_1 = 0$, and $h = h(t)$. With this, the metric takes the form

$$g^M = \varepsilon dt^2 \oplus g^N, \tag{6.11}$$

with N Lorentzian if $\varepsilon = 1$ and Riemannian if $\varepsilon = -1$, where g^N possibly depends on t. We compute

$$\mathrm{Hes}_h(E_1, E_1) = g(\nabla_{\varepsilon \partial_t}(\varepsilon h' \partial_t), \varepsilon \partial_t) = h''$$

and, from (6.1), we have $\lambda_1 = \varepsilon h^{-1} h'' + 2J$, so λ_1 depends only on t. This will be important in what follows, but first, we prove the following lemma.

Lemma 6.11. *Let $f_1, \dots, f_k$ be smooth functions defined on the product manifold $(I \times N, \varepsilon dt^2 \oplus g^N)$. If the power sums $p_j = \sum_{i=1}^k f_i^j$ for $j = 1, \dots, k$ depend only on the coordinate t, then each f_i depends only on t as well.*

Proof. Let s_j be the j-th elementary symmetric polynomial in the variables $f_1, \dots, f_k$ (by convention, $s_0 = 1$). It is well-known that the first k elementary symmetric polynomials are generated by the first k power sums p_j through Newton's identities. Since $p_0 = 1, \dots, p_k$ depend only on t by assumption, then $s_0, \dots, s_k$ also depend only on t. Now, consider the following polynomial in $\mathbb{R}[f_1, \dots, f_k][x]$:

$$(x - f_1) \cdots (x - f_k) = \sum_{i=0}^{k} (-1)^i s_i(f_1, \dots, f_k) x^{k-i}.$$

Note that, for a fixed t, changing the point in N does not change the polynomial, since all the symmetric polynomials s_i depend only on t. Therefore, its roots, which are $f_1, \dots, f_k$, do not change from point to point either, so they only depend on t. □

We now apply this result to the eigenvalues of the Ricci operator.

Lemma 6.12. *Let (M, g, h) be a non-isotropic solution of dimension n of the weighted Einstein field equations* (4.6) *with harmonic curvature such that the Ricci operator* Ric *diagonalizes. Then, all eigenvalues of* Ric *depend only on the local coordinate t of the metric* (6.11).

Proof. From the harmonicity condition $\operatorname{div} R = 0$, it follows that the Ricci tensor satisfies $(\nabla_{E_i}\rho)(E_i, \nabla h) = (\nabla_{\nabla h}\rho)(E_i, E_i)$. Then, using (6.1), we see that, for $i \neq 1$,

$$\begin{aligned} (\nabla_{E_i}\rho)(E_i, \nabla h) &= E_i(\rho(E_i, \nabla h)) - \rho(\nabla_{E_i} E_i, \nabla h) - \rho(E_i, \nabla_{E_i} \nabla h) \\ &= (\lambda_1 - \lambda_i) g(E_i, \nabla_{E_i} \nabla h) \\ &= \varepsilon_i h(\lambda_1 - \lambda_i)(\lambda_i - 2J), \end{aligned}$$

where $\varepsilon_i = g(E_i, E_i)$. On the other hand, $(\nabla_{\nabla h}\rho)(E_i, E_i) = \varepsilon_i \nabla h(\lambda_i)$, so $\nabla h(\lambda_i) = h(\lambda_1 - \lambda_i)(\lambda_i - 2J)$. Now, since τ is constant and λ_1 depends only on t, we have that $\tau - \lambda_1 = \sum_{i=2}^n \lambda_i$ depends only on t. Moreover,

$$\begin{aligned} 0 = \nabla h(\tau) &= \nabla h(\lambda_1) + \textstyle\sum_{j=2}^n h(\lambda_1 - \lambda_j)(\lambda_j - 2J) \\ &= \nabla h(\lambda_1) + h(\lambda_1 + 2J) \textstyle\sum_{j=2}^n \lambda_j \\ &\quad - h\left(2J(n-1)\lambda_1 + \textstyle\sum_{j=2}^n \lambda_j^2\right). \end{aligned}$$

Since $\nabla t = \nabla h / \|\nabla h\|$ and $\|\nabla h\|$ depends only on t, every term in the equation above, except for $\sum_{j=2}^n \lambda_j^2$, depends only on t. Thus, $\sum_{j=1}^n \lambda_j^2$ depends only on t as well. We can perform this same process for any $k \in \{1, \dots, n-1\}$ by induction:

$$\begin{aligned} k^{-1} \nabla h\left(\textstyle\sum_{i=1}^n \lambda_i^k\right) &= \lambda_1^{k-1} \nabla h(\lambda_1) + \textstyle\sum_{j=2}^n \lambda_j^{k-1} \nabla h(\lambda_j) \\ &= \lambda_1^{k-1} \nabla h(\lambda_1) + h \textstyle\sum_{j=2}^n \lambda_j^{k-1}(\lambda_1 - \lambda_j)(\lambda_j - 2J) \\ &= \lambda_1^{k-1} \nabla h(\lambda_1) + h(\lambda_1 + 2J) \textstyle\sum_{j=2}^n \lambda_j^k \\ &\quad - h\left(2J\lambda_1 \textstyle\sum_{j=2}^n \lambda_j^{k-1} + \textstyle\sum_{j=2}^n \lambda_j^{k+1}\right). \end{aligned}$$

By assumption, every term in the equation above, except for $\sum_{j=2}^n \lambda_j^{k+1}$, depends only on t. Thus, $\sum_{j=1}^n \lambda_j^{k+1}$ depends only on t as well. As a result, applying Lemma 6.11, we have that each λ_i, $i = 1, \dots, n$ depends only on t. In particular, we have $E_i(\beta_j) = E_i(\lambda_j) = 0$ for all $i, j = 2, \dots, n$. □

It is well-known that the curvature tensor is harmonic if and only if the Ricci tensor is Codazzi, i.e., if the skew-symmetrization of $\nabla\rho$ vanishes. Previous results from [54] show that this Codazzi character imposes by itself some important restrictions on the geometry of the leaves of the eigendistributions of Ric. Indeed, a version of the following result was proved in [54] and extends from Riemannian to Lorentzian signature when Ric is diagonalizable. We include the proof here in the interest of completeness and because we will use the weighted Einstein field equations to provide additional information on the connection relations for a solution (see also [78]).

Lemma 6.13 (cf. [54])**.** *Let (M, g, h) be an n-dimensional non-isotropic solution of the weighted Einstein field equations* (4.6) *with Codazzi and diagonalizable Ricci tensor. Then, the distribution associated to each eigenvalue of* Ric *is integrable and their corresponding leaves are totally umbilical submanifolds of M.*

Proof. We work in an adapted local orthonormal frame $\mathcal{B}_1 = \{E_1, \ldots, E_n\}$ that diagonalizes the Ricci operator and such that $g(E_i, E_i) = \varepsilon_i$. Denote by $\Gamma_{ijk} = g(\nabla_{E_i} E_j, E_k)$ the corresponding Christoffel symbols. First, note that, due to the fact that $\mathcal{B}$ is pseudo-orthonormal, $\Gamma_{ijk} = -\Gamma_{ikj}$ for all i, j, k. We calculate the covariant derivative of the Ricci tensor,

$$\begin{aligned}(\nabla_{E_i}\rho)(E_j, E_k) &= E_i(\rho(E_j, E_k)) - \rho(\nabla_{E_i} E_j, E_k) - \rho(E_j, \nabla_{E_i} E_k)\\ &= \varepsilon_j \delta_{jk} E_i(\lambda_j) + (\lambda_j - \lambda_k)\Gamma_{ijk}.\end{aligned}$$

Now, since ρ is Codazzi, we have

$$\varepsilon_j \delta_{jk} E_i(\lambda_k) + (\lambda_j - \lambda_k)\Gamma_{ijk} = \varepsilon_i \delta_{ik} E_j(\lambda_k) + (\lambda_i - \lambda_k)\Gamma_{jik}. \tag{6.12}$$

From here, choose $i = 1$, rearrange the labels so that $j \to i$, $k \to j$, and let $i \neq j$, $i, j > 1$. We obtain

$$(\lambda_i - \lambda_j)\Gamma_{1ij} = (\lambda_1 - \lambda_j) g(\nabla_{E_i} E_1, E_j) = (\lambda_1 - \lambda_j)\beta_i g(E_i, E_j) = 0.$$

Hence, for every i, j such that $\lambda_i \neq \lambda_j$, $\Gamma_{1ij} = 0$. Furthermore, since $\nabla_{E_1} E_1 = 0$, we have $\Gamma_{1i1} = -\Gamma_{11i} = 0$ and it is clear that $\Gamma_{1ii} = 0$ since the adapted frame is pseudo-orthonormal. It follows that $\nabla_{E_1} E_i$ stays in the eigenspace associated to the eigenvalue λ_i, while being orthogonal to E_i.

Similarly, $\Gamma_{ii1} = -\varepsilon_i \beta_i$ (see (6.10)) and, by (6.12) and Lemma 6.12, we also have $(\lambda_j - \lambda_i)\Gamma_{iji} = 0$ if $i, j > 1$. Thus, $\Gamma_{iji} = -\Gamma_{iij} = 0$ if $\lambda_i \neq \lambda_j$ and $\Gamma_{iii} = 0$, so the component of $\nabla_{E_i} E_i$ that is perpendicular to E_1 also stays in the eigenspace associated with λ_i, while being orthogonal to E_i.

For the rest of the connection coefficients Γ_{ijk}, with $i, j, k > 1$, we use (6.12) and Lemma 6.12 to write $(\lambda_j - \lambda_k)\Gamma_{ijk} = (\lambda_i - \lambda_k)\Gamma_{jik}$. It follows that, if $\lambda_i = \lambda_k \neq \lambda_j$, $-\Gamma_{ikj} = \Gamma_{ijk} = 0$. In other words, if $E_i \neq E_k$ belong to the same eigenspace, then $\nabla_{E_i} E_k$ stays in it.

In summary, let E_i, E_j be vectors in the same eigenspace (we denote the set of indices corresponding to eigenvectors in the eigenspace associated to λ_i by $[i]$), and E_μ so that $\lambda_\mu \neq \lambda_i$. Then, in general, the connection relations read

$$\begin{aligned}\nabla_{E_1} E_1 &= 0, \quad \nabla_{E_i} E_1 = \beta_i E_i, \quad \nabla_{E_1} E_i = \textstyle\sum_{k\in[i]} \varepsilon_k \Gamma_{1ik} E_k,\\ \nabla_{E_i} E_j &= -\varepsilon_1 \varepsilon_i \beta_i \delta_{ij} E_1 + \textstyle\sum_{k\in[i]} \varepsilon_k \Gamma_{ijk} E_k,\\ \nabla_{E_i} E_\mu &= \textstyle\sum_{k\notin[i], k\neq 1,\mu} \varepsilon_k \Gamma_{i\mu k} E_k.\end{aligned} \tag{6.13}$$

In particular, for two vectors in the same eigenspace,

$$[E_i, E_j] = \sum_{k\in[i]} \varepsilon_k(\Gamma_{ijk} - \Gamma_{jik})E_k,$$

and so the distribution generated by all eigenvectors associated to λ_i is integrable. Moreover, the second fundamental form satisfies $II(E_i, E_j) = -\varepsilon_1\varepsilon_i\beta_i\delta_{ij}E_1$, so the tangent submanifold to this distribution is totally umbilical with mean curvature vector field $H = -\varepsilon_1\beta_i E_1$. □

From this point on, we focus on 4-dimensional solutions to attain the classification in Theorem 6.7 (2). Once we are working around a point in M_{Ric}, we perform specific analyses depending on whether the eigenvalues λ_2, λ_3 and λ_4 are all different; or at least two of them coincide. As it turns out, the first case is not admissible, independently of the causal (timelike or spacelike) character of ∇h.

6.2.1 The three eigenvalues coincide: $\lambda_2 = \lambda_3 = \lambda_4$

If $\lambda_2 = \lambda_3 = \lambda_4$ then, from (6.13), the connection behaves as follows:

$$\begin{aligned} \nabla_{E_1}E_1 &= 0, \quad \nabla_{E_i}E_1 = \beta_i E_i, \quad \nabla_{E_1}E_i = \textstyle\sum_{k\neq i}\varepsilon_k\Gamma_{1ik}E_k, \\ \nabla_{E_i}E_j &= -\varepsilon_1\varepsilon_i\beta_i\delta_{ij}E_1 + \textstyle\sum_{k\neq i}\varepsilon_k\Gamma_{ijk}E_k, \end{aligned}$$

where $i, j, k \in \{2, 3, 4\}$, $i \neq k$. We consider the distribution $\mathrm{span}\{E_2, E_3, E_4\}$, which is integrable and whose tangent leaves are umbilical (see Lemma 6.13), with unit normal E_1. Also, notice that $\beta_2 = \beta_3 = \beta_4$ and the mean curvature vector of these leaves satisfies $\nabla_{E_i}(-\varepsilon_1\beta_2E_1) = -\varepsilon_1(\beta_2)^2E_i \perp E_1$. In other words, the mean curvature vector $-\varepsilon_1\beta_3E_1$ is parallel in the normal bundle $\mathrm{span}\{E_1\}$, so the leaves are indeed spherical. This formula comes from the fact that $\beta_2 = \frac{h(\lambda_2-2J)}{\|\nabla h\|}$, and since J is constant by Lemma 4.5 and λ_2 depends only on t by Lemma 6.12, so does β_2. Hence, by Theorem 1.3, the metric decomposes locally as a warped product $I \times_\varphi N$. Moreover, the Ricci eigenvalues λ_i are equal for $i = 2, 3, 4$, so applying the formula (1.6), it follows that N is Einstein. Since N is 3-dimensional, it is of constant sectional curvature. This implies that $I \times_\varphi N$ is locally conformally flat (see Theorem 1.6), so these solutions were already described in Theorem 6.3 (1), but do not fall into the scope of Theorem 6.7.

6.2.2 Two eigenvalues coincide: $\lambda_2 \neq \lambda_3 = \lambda_4$ or $\lambda_2 = \lambda_3 \neq \lambda_4$.

In order to fix a unique notation for all cases, we arrange the pseudo-orthonormal basis $\{E_1 = \nabla h/\|\nabla h\|, E_2, E_3, E_4\}$ so that $\lambda_2 \neq \lambda_3 = \lambda_4$ and we set $g(E_i, E_i) = \varepsilon_i$ for $i = 1, 2, 3$, so the unit timelike vector field could be E_1, E_2 or E_3. In this context, the geometry of the manifold is so restricted that it decomposes as a multiply warped product.

Lemma 6.14. *Let (M, g, h) be a 4-dimensional non-isotropic solution of the weighted Einstein field equations* (4.6) *with harmonic curvature tensor, such that the Ricci operator diagonalizes in* *the adapted local frame $\mathcal{B}_1 = \{E_1, \dots, E_4\}$. If there are two distinct eigenvalues $\lambda_2 \neq \lambda_3 = \lambda_4$, then (M, g) splits locally as a multiply warped product $I_1 \times_\varphi I_2 \times_\xi N$ with metric*

$$g = \varepsilon_1 dt^2 + \varepsilon_2 \varphi(t)^2 ds^2 + \xi(t)^2 \tilde{g}, \tag{6.14}$$

where $\tilde{g}$ is the metric of a Riemannian or Lorentzian surface of constant Gauss curvature κ and h is a function on t.

Proof. We adapt the relations in (6.13) to this context to see that

$$\begin{aligned}
&\nabla_{E_1} E_1 = 0,\ \nabla_{E_i} E_1 = \beta_i E_i,\ \nabla_{E_1} E_2 = 0,\\
&\nabla_{E_1} E_3 = \Gamma_{134} E_4,\ \nabla_{E_1} E_4 = \varepsilon_3 \Gamma_{143} E_3,\\
&\nabla_{E_2} E_2 = -\varepsilon_1 \varepsilon_2 \beta_2 E_1,\ \nabla_{E_2} E_3 = \Gamma_{234} E_4,\ \nabla_{E_2} E_4 = \varepsilon_3 \Gamma_{243} E_3\\
&\nabla_{E_3} E_3 = -\varepsilon_1 \varepsilon_3 \beta_3 E_1 + \Gamma_{334} E_4,\ \nabla_{E_4} E_4 = -\varepsilon_1 \beta_4 E_1 + \varepsilon_3 \Gamma_{443} E_3,\\
&\nabla_{E_3} E_4 = \varepsilon_3 \Gamma_{343} E_3,\ \nabla_{E_4} E_3 = \Gamma_{434} E_4,\ \nabla_{E_3} E_2 = \nabla_{E_4} E_2 = 0.
\end{aligned}$$

From the behavior of the connection, it follows that the tangent submanifolds to the distributions $\mathcal{D}_1 = \operatorname{span}\{E_1, E_2\}$ and $\mathcal{D}_2 = \operatorname{span}\{E_1, E_3, E_4\}$ are totally geodesic, since $\nabla \mathcal{D}_1 \subset \mathcal{D}_1$ and $\nabla \mathcal{D}_2 \subset \mathcal{D}_2$. We already know, by Lemma 6.13, that leaves tangent to $\mathcal{D}_3 = \operatorname{span}\{E_3, E_4\}$ are umbilical but, moreover, the mean curvature vector associated to these leaves is $-\varepsilon_1 \beta_3 E_1$, where $\beta_3 = \beta_4 = \frac{h(\lambda_3 - 2J)}{\|\nabla h\|}$. Since J is constant by Lemma 4.5 and λ_3 depends only on t by Lemma 6.12, we have that

$$\nabla_{E_i}(-\varepsilon_1 \beta_3 E_1) = -\varepsilon_1 \beta_3^2 E_i \perp E_1, E_2 \quad \text{for} \quad i \in \{3, 4\}.$$

Therefore, the mean curvature vector $-\varepsilon_1 \beta_3 E_1$ is parallel in $\operatorname{span}\{E_1, E_2\}$, so the leaves are indeed spherical. Since leaves tangent to $\operatorname{span}\{E_2\}$ are also spherical by the same argument, by Theorem 1.3, locally we can decompose the tangent submanifolds to the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ as warped products with 1-dimensional base. Furthermore, both warping functions depend on the variable t in the decomposition (6.11). Thus, the manifold decomposes locally as a multiply warped product as in (6.14), with h depending only on t.

Finally, since $\lambda_3 = \lambda_4$ depends only on t by Lemma 6.12, the Ricci tensor on the fiber $\tilde{g}$ has constant eigenvalues (indeed, otherwise λ_3 would depend at least on one coordinate of the fiber) and hence it is of constant Gauss curvature. □

Lemma 6.15. *Let (M, g, h) be a multiply warped product solution as in* (6.14) *with $h = h(t)$ and harmonic curvature. Then one of the warping functions, either φ or ξ, is constant.*

Proof. Let g be a multiply warped product metric as in (6.14). Let κ be the Gauss curvature of $\tilde{g}$. We choose local coordinates (x_2, x_3) on N so that

$$\tilde{g}(x_2, x_3) = \frac{1}{\left(1 + \frac{\kappa}{4}\left(\varepsilon_3 x_2^2 + x_3^2\right)\right)^2} (\varepsilon_3 dx_2^2 + dx_3^2).$$

Using these coordinates and introducing the exponentials $\varphi = e^{f_1}$ and $\xi = e^{f_2}$ to simplify expressions, we set local coordinates (t, x_1, x_2, x_3) on M such that

$$g = \varepsilon_1 dt^2 + \varepsilon_2 e^{2f_1(t)} dx_1^2 + e^{2f_2(t)} \tilde{g}.$$

Now, from (6.2) we see that

$$\begin{aligned} 0 &= R(\nabla h, \partial_{x_1}, \partial_t, \partial_{x_1}) - (\rho - 2Jg) \wedge dh(\partial_{x_1}, \partial_t, \partial_{x_1}) \\ &= \tfrac{2\varepsilon_1\varepsilon_2 e^{2f_1} h'}{3} \left(-\kappa\varepsilon_1 e^{-2f_2} - 2f_1'^2 - f_1' f_2' + 3f_2'^2 - 2f_1'' + 2f_2''\right), \end{aligned}$$

from where

$$e^{2f_2}\left(2\,(f_1')^2 + f_1' f_2' - 3\,(f_2')^2 + 2f_1'' - 2f_2''\right) + \kappa\varepsilon_1 = 0.$$

A direct computation also shows that

$$\tau = -2\varepsilon_1 \left(-\kappa\varepsilon_1 e^{-2f_2} + (f_1')^2 + 2f_1' f_2' + 3\,(f_2')^2 + f_1'' + 2f_2''\right).$$

Using these two expressions we obtain that

$$\begin{aligned} f_1'' &= -\tfrac{1}{6}\left(6\,(f_1')^2 + 6f_1' f_2' + \tau\varepsilon_1\right), \\ f_2'' &= \tfrac{1}{2} e^{-2f_2}\kappa\varepsilon_1 - \tfrac{3}{2}(f_2')^2 - \tfrac{1}{2} f_1' f_2' - \tfrac{\tau\varepsilon_1}{6}. \end{aligned}$$

Moreover, we compute

$$G^h(\partial_{x_1}, \partial_{x_1}) = \frac{1}{6} e^{2f_1}\varepsilon_1\varepsilon_2 \left(6\left(2f_2' h' + h''\right) + h\left(\tau\varepsilon_1 - 6f_1' f_2'\right)\right)$$

to get that $h'' = \frac{1}{6}\left(-12f_2' h' + 6hf_1' f_2' - h\tau\varepsilon_1\right)$. Thus, we have expressed the second derivatives of h, f_1 and f_2 in terms of lower order terms. Now, we use these relations to compute

$$\begin{aligned} G^h(\partial_t, \partial_t) &= \left(f_1' + 2f_2'\right) h' + \left(-\kappa\varepsilon_1 e^{-2f_2} + (f_2')^2 + 2f_1' f_2' + \tfrac{\tau\varepsilon_1}{2}\right) h, \\ G^h(\partial_{x_3}, \partial_{x_3}) &= \tfrac{8\varepsilon_1 e^{2f_2}}{\left(\kappa x_2^2\varepsilon_3 + \kappa x_3^2 + 4\right)^2} \left(\left(f_1' f_2' - (f_2')^2 + \kappa\varepsilon_1 e^{-2f_2}\right) h + 2\left(f_1' - f_2'\right) h'\right). \end{aligned}$$

This leads to a homogeneous linear system of two equations in the unknowns h' and h. Hence, because $h \neq 0$, the determinant of the associated matrix must vanish:

$$C_1 = e^{2f_2}\left(f_1' - f_2'\right)\left(3f_1' f_2' + \tau\varepsilon_1\right) - 3\kappa\varepsilon_1 f_1' = 0. \tag{6.15}$$

Differentiating with respect to t and simplifying second order terms using the expressions above, we get

$$f_1'\left(3\kappa\varepsilon_1 f_1' - e^{2f_2}\left(f_1' - f_2'\right)\left(4\,(f_2')^2 + 5f_1' f_2' + \tau\varepsilon_1\right)\right) = 0.$$

Thus, either $f_1' = 0$, in which case the lemma holds, or

$$C_2 = 3\kappa\varepsilon_1 f_1' - e^{2f_2}\left(f_1' - f_2'\right)\left(4\,(f_2')^2 + 5f_1' f_2' + \tau\varepsilon_1\right) = 0.$$

If $C_2 = 0$, then we compute

$$0 = C_1 + C_2 = -2e^{2f_2} f_2' \left(f_1' - f_2'\right)\left(f_1' + 2f_2'\right).$$

Hence, there are three possible cases. If $f_2' = 0$, the result follows. If $f_1' = f_2'$, then by (6.15), we have either $f_1' = 0$, so the result follows, or $\kappa = 0$, in which case the manifold is locally conformally flat. Finally, if $f_1' = -2f_2'$, then $f_1(t) = C - 2f_2(t)$ for a constant C, and $G^h(\partial_t, \partial_t) = -6h(f_2')^2$, so $f_2' = 0$. □

Lemma 6.15 above shows that at least one of the warping functions is constant. Notice that if both are constant, then we have a direct product. In this case, a direct computation of the equation $G^h = 0$ shows that, necessarily, $\kappa = 0$, so the manifold is flat. As a result, we can restrict our analysis of the multiply warped product solutions to the case where one of the warping functions is constant and the other one is strictly non-constant. We first analyze the case $\xi' \neq 0$.

Case $\varphi =$ constant

Lemma 6.16. *Let (M, g, h) be a multiply warped product solution as in* (6.14) *with φ constant, with harmonic curvature and $h = h(t)$. Then, (M, g, h) is a solution of* (4.6) *if and only if $\tau = 0$, $h = c\xi'$ and $I_1 \times_\xi N$ is one of the 3-dimensional locally conformally flat solutions portrayed in Example 6.6 for the density function h.*

Proof. Consider a multiply warped product structure as in (6.14) with φ constant. Normalize the coordinate s if necessary so that $\varphi = 1$. Because of the metric structure we have $R(\partial_t, \partial_s, \partial_t, \partial_s) = 0$ and $\rho(\partial_s, \partial_s) = 0$, so we obtain $0 = \tau = \frac{2\kappa - \varepsilon_1 4\xi\xi'' - 2\varepsilon_1(\xi')^2}{\xi^2}$ from (6.2). This implies that the scalar curvature of $I_1 \times_\xi N$ also vanishes.

From equation (4.6), we have that

$$G^h(\partial_t, \partial_t) = 2\frac{h'\xi' - h\xi''}{\xi}, \quad \text{and} \quad G^h(\partial_s, \partial_s) = \frac{\varepsilon_1\varepsilon_2\left(\xi h'' + 2h'\xi'\right)}{\xi}.$$

Hence, on the one hand, solving $h'\xi' - h\xi'' = 0$ we get $h = c\xi'$ for a constant $c \neq 0$ (recall that we are assuming that ξ is non-constant). On the other hand, since $h'\xi' - h\xi'' = 0$, we write

$$0 = \varepsilon_1\varepsilon_2 G^h(\partial_s, \partial_s) = h'' + 2\frac{\xi' h'}{\xi} = h'' + 2h\frac{\xi''}{\xi}.$$

Thus, we see that, for the SMMS $(I_1 \times_\xi N, h)$, the system of ODEs (6.5) for a locally conformally flat 3-dimensional solution with vanishing scalar curvature is satisfied. These spaces are further described by Example 6.6.

Conversely, take any 3-dimensional solution $(I_1 \times_\xi N, h)$, with (N, g^N) of constant Gauss curvature κ, $h = h(t)$, and vanishing scalar curvature. Consider the 4-dimensional manifold $(M, g) = I_1 \times I_2 \times_\xi N$. Then, $\tau = \frac{2\kappa - 4\varepsilon_1\xi\xi'' - 2\varepsilon_1(\xi')^2}{\xi^2} = 0$. Moreover, because the system of

ODEs (6.5) is satisfied, we have $h'\xi' - h\xi'' = 0$ and $\xi h'' + 2h'\xi' = 0$, which imply $G^h(\partial_t, \partial_t) = G^h(\partial_s, \partial_s) = 0$. Using that $h = c\xi'$ we compute:

$$\begin{aligned}\varepsilon_1 \xi^2 G^h(X,X) &= \left(\xi\left(h'\xi' + \xi h''\right) + h\left(\kappa\varepsilon_1 - \xi\xi'' - (\xi')^2\right)\right) g(X,X) \\ &= c(\kappa\varepsilon_1\xi' - (\xi')^3 + \xi^2\xi^{(3)}) g(X,X)\end{aligned}$$

for any vector X tangent to N. Since $\tau' = -\frac{4(\kappa\xi' - \varepsilon_1(\xi')^3 + \varepsilon_1\xi^2\xi^{(3)})}{\xi^3} = 0$, this term vanishes, so (M,g,h) is a 4-dimensional solution. Moreover, $\kappa\xi' - \varepsilon_1(\xi')^3 + \xi^2\xi^{(3)}\varepsilon_1 = 0$ is also the necessary and sufficient condition for the manifold to have harmonic curvature. □

Remark 6.17. Note that solutions given in Lemma 6.16 generically present three distinct eigenvalues for Ric. Indeed, in the frame $\{\partial_t, \partial_s, \partial_{x_1}, \partial_{x_2}\}$ for the multiply warped product metric (6.14), the Ricci operator takes the form

$$\mathrm{Ric} = \begin{pmatrix} -\varepsilon_1\frac{2\xi''}{\xi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \text{where} \quad \lambda = \frac{\kappa - \varepsilon_1((\xi')^2 + \xi\xi'')}{\xi^2},$$

so there is a zero eigenvalue corresponding to the I_2 factor and the number of eigenvalues reduces to two only if $I_1 \times_\xi N$ is Einstein (note that $\xi'' \neq 0$ because $h = c\xi'$ is non-constant by assumption). In this case, since the scalar curvature of $I_1 \times_\xi N$ vanishes, the underlying manifold is flat and $\xi'' = 0$, so this case does not appear among our solutions.

Case ξ = constant

Lemma 6.18. *Let (M,g,h) be a multiply warped product solution as in* (6.14)*, with ξ constant, with harmonic curvature and $h = h(t)$. Then (M,g) is a direct product of two surfaces of constant Gauss curvature and $h = c\varphi'$.*

Proof. Since the metric $\xi^2\tilde{g}$ on N has constant Gauss curvature $\frac{\kappa}{\xi^2}$, we can assume $\xi = 1$ in (6.14) by a change of coordinates and a redefinition of κ. Then, the Ricci operator is given by

$$\mathrm{Ric}(\partial_t) = -\frac{\varepsilon_1\varphi''}{\varphi}\partial_t, \quad Ric(\partial_s) = -\frac{\varepsilon_1\varphi''}{\varphi}\partial_s, \quad \mathrm{Ric}(X) = \kappa X \text{ for } X \text{ tangent to } N.$$

From (6.2), we have

$$0 = R(\nabla h, \partial_s, \partial_t, \partial_s) - (\rho - 2Jg) \wedge dh(\partial_s, \partial_t, \partial_s) = -\frac{2}{3}\varepsilon_2\varphi h'\left(\kappa\varphi + 2\varepsilon_1\varphi''\right).$$

This implies that $-\frac{\varepsilon_1\varphi''}{\varphi} = \frac{\kappa}{2}$, so the manifold is a direct product of two surfaces N_1 and N_2 with constant Gauss curvatures $\frac{\kappa}{2}$ and κ respectively. Moreover, we compute $0 = G^h(\partial_t, \partial_t) = \frac{h'\varphi' - h\varphi''}{\varphi}$, to see that $h = c\varphi'$ for a suitable integration constant $c \in \mathbb{R}^*$ (since φ is non-constant in this context). □

Remark 6.19. Assume we are in the conditions of Lemma 6.18. Then, we can use the product structure of (6.14) with $\xi = 1$ to write simple coordinate expressions for φ and h. Firstly, note that κ cannot vanish, since this results in a direct product of two flat surfaces (hence a flat solution), with constant density function h.

Now, for $\kappa \neq 0$, since $-\frac{\varepsilon_1 \varphi''}{\varphi} = \frac{\kappa}{2}$, the warping function φ takes the following forms, depending on the sign of the product $\varepsilon_1 \kappa$:

$$\begin{aligned} \varphi(t) &= c_1 \sin\left(\sqrt{\tfrac{\varepsilon_1\kappa}{2}}t\right) + c_2 \cos\left(\sqrt{\tfrac{\varepsilon_1\kappa}{2}}t\right), \quad &&\text{if } \varepsilon_1\kappa > 0, \\ \varphi(t) &= c_1 e^{\sqrt{-\frac{\varepsilon_1\kappa}{2}}t} + +c_1 e^{-\sqrt{-\frac{\varepsilon_1\kappa}{2}}t}, &&\text{if } \varepsilon_1\kappa < 0, \end{aligned}$$

where c_1 and c_2 are suitable integration constants so that $\varphi(t), h(t) > 0$ for all $t \in I$.

6.2.3 The three eigenvalues are different ($\lambda_2 \neq \lambda_3 \neq \lambda_4$)

Finally, assume that the three eigenvalues λ_2, λ_3 and λ_4 of the Ricci operator are distinct. We show in Lemma 6.21 below that this case is not admissible, with the proof being essentially a Lorentzian analogue to that given in [78] in Riemannian signature. Let $\varepsilon_1 = -\varepsilon_2 = \varepsilon$ and $\varepsilon_3 = \varepsilon_4 = 1$. From equation (6.13) we know that

$$\begin{aligned} \nabla_{E_1} E_1 &= 0, \quad \nabla_{E_i} E_1 = \beta_i E_i, \quad \nabla_{E_i} E_i = -\varepsilon\varepsilon_i\beta_i E_1, \\ \nabla_{E_1} E_i &= 0, \quad \nabla_{E_i} E_j = \varepsilon_k \Gamma_{ijk} E_k, \end{aligned} \tag{6.16}$$

where $\{i, j, k\} = \{2, 3, 4\}$ and $\Gamma_{ijk} = g(\nabla_{E_i} E_j, E_k)$. From these relations, we obtain the expression for the Ricci eigenvalues as follows.

Lemma 6.20. *Let (M, g, h) be a 4-dimensional non-isotropic solution of the weighted Einstein field equations* (4.6) *with harmonic curvature tensor, such that the Ricci operator diagonalizes in the adapted local frame $\mathcal{B}_1 = \{E_1, \dots, E_4\}$ and the eigenvalues λ_2, λ_3 and λ_4 are pairwise distinct. Then, they take the following forms:*

$$\begin{aligned} -\varepsilon\lambda_2 &= \beta_2^2 + \varepsilon\beta_2' + \beta_2\beta_3 + \beta_2\beta_4 - 2\Gamma_{342}\Gamma_{432}, \\ -\varepsilon\lambda_3 &= \beta_3^2 + \varepsilon\beta_3' + \beta_3\beta_2 + \beta_3\beta_4 + 2\tfrac{\beta_2-\beta_4}{\beta_3-\beta_4}\Gamma_{342}\Gamma_{432}, \\ -\varepsilon\lambda_4 &= \beta_4^2 + \varepsilon\beta_4' + \beta_4\beta_2 + \beta_4\beta_3 + 2\tfrac{\beta_2-\beta_3}{\beta_4-\beta_3}\Gamma_{342}\Gamma_{432}. \end{aligned}$$

Proof. On the one hand, from (6.16) and for $i, j \neq 1$, we compute

$$\begin{aligned} \nabla_{[E_j,E_i]} E_i &= \nabla_{\nabla_{E_j} E_i} E_i - \nabla_{\nabla_{E_i} E_j} E_i = \varepsilon_k\varepsilon_j(\Gamma_{jik} - \Gamma_{ijk})\Gamma_{kij} E_j, \\ \nabla_{E_j}\nabla_{E_i} E_i &= -\varepsilon\varepsilon_i\beta_i \nabla_{E_j} E_1 = -\varepsilon\varepsilon_i\beta_i\beta_j E_j, \\ \nabla_{E_i}\nabla_{E_j} E_i &= \varepsilon_k \nabla_{E_i}(\Gamma_{jik} E_k) = \varepsilon_k E_i(\Gamma_{jik}) E_k + \varepsilon_k\varepsilon_j \Gamma_{jik}\Gamma_{ikj} E_j, \end{aligned}$$

while, on the other hand,

$$\begin{aligned} \nabla_{[E_1,E_i]} E_1 &= \nabla_{\nabla_{E_1} E_i} E_1 - \nabla_{\nabla_{E_i} E_1} E_1 = -\beta_i^2 E_i, \\ \nabla_{E_1}\nabla_{E_i} E_1 &= \nabla_{E_1}(\beta_i E_i) = \varepsilon\beta_i' E_i, \\ \nabla_{E_i}\nabla_{E_1} E_1 &= 0. \end{aligned}$$

Hence, we have the curvature components given by

$$\begin{aligned} R_{jiji} &= -\varepsilon\varepsilon_i\varepsilon_j\beta_i\beta_j + \varepsilon_k\{(\Gamma_{ijk} - \Gamma_{jik})\Gamma_{kij} - \Gamma_{jik}\Gamma_{ikj}\}, \\ R_{1i1j} &= -\varepsilon_i(\varepsilon\beta_i' + \beta_i^2)\delta_{ij}, \end{aligned}$$

so the components of the Ricci tensor take the form:

$$\begin{aligned} -\varepsilon\lambda_2 = \rho_{22} &= \varepsilon R_{1212} + R_{3232} + R_{4242} \\ &= \varepsilon\beta_2' + \beta_2^2 + \beta_2\beta_3 + \beta_2\beta_4 \\ &\quad +(\Gamma_{234} - \Gamma_{324})\Gamma_{423} - \Gamma_{324}\Gamma_{243} \\ &\quad +(\Gamma_{243} - \Gamma_{423})\Gamma_{324} - \Gamma_{423}\Gamma_{234} \\ &= \beta_2^2 + \varepsilon\beta_2' + \beta_2\beta_3 + \beta_2\beta_4 - 2\Gamma_{342}\Gamma_{432}, \end{aligned}$$

and

$$\begin{aligned} \lambda_3 = \rho_{33} &= \varepsilon R_{1313} - \varepsilon R_{2323} + R_{4343} \\ &= -\varepsilon\beta_3^2 - \beta_3' - \varepsilon(\beta_2\beta_3 + \beta_4\beta_3) + 2\varepsilon\Gamma_{243}\Gamma_{423} \\ &= -\varepsilon(\beta_3^2 + \varepsilon\beta_3' + \beta_3\beta_2 + \beta_3\beta_4 - 2\Gamma_{243}\Gamma_{423}), \end{aligned}$$

where we have used that $\Gamma_{jki} = -\Gamma_{jik}$. The computation of λ_4 is analogous to the previous one. Now, since the Ricci tensor is Codazzi due to the harmonicity of the curvature, so is the Hessian Hes_h due to equation (6.1) and the constancy of J given by Lemma 4.5. Hence, if we compute the covariant derivative of the Hessian tensor,

$$\begin{aligned} (\nabla_{E_i}\mathrm{Hes}_h)(E_j, E_k) &= E_i(\mathrm{Hes}_h(E_j, E_k)) - \mathrm{Hes}_h(\nabla_{E_i}E_j, E_k) \\ &\quad - \mathrm{Hes}_h(E_j, \nabla_{E_i}E_k) \\ &= \varepsilon_j\delta_{jk}E_i(\beta_j) + (\beta_j - \beta_k)\Gamma_{ijk}, \end{aligned}$$

and apply the Codazzi condition and the fact that $E_i(\beta_j) = 0$ for $i = 2, 3, 4$ by Lemma 6.12, it follows that $(\beta_j - \beta_k)\Gamma_{ijk} = (\beta_i - \beta_k)\Gamma_{jik}$. Notice that this is the same process that we used in Lemma 6.13 for the Ricci tensor. From these relations, knowing that $\Gamma_{jki} = -\Gamma_{jik}$, we write

$$\Gamma_{243}\Gamma_{423} = -\frac{\beta_2 - \beta_4}{\beta_3 - \beta_4}\Gamma_{342}\Gamma_{432}, \qquad \Gamma_{234}\Gamma_{324} = -\frac{\beta_2 - \beta_3}{\beta_4 - \beta_3}\Gamma_{342}\Gamma_{432},$$

from where the result follows. □

Lemma 6.21. *Let (M, g, h) be a 4-dimensional non-isotropic solution of the weighted Einstein field equations* (4.6) *with harmonic curvature tensor, such that the Ricci operator diagonalizes in the adapted local frame $\mathcal{B}_1 = \{E_1, \dots, E_4\}$. Then, the eigenvalues λ_2, λ_3 and λ_4 cannot be pairwise distinct.*

Proof. In addition to Lemma 6.20, we have two more possibilities in order to express these components of the Ricci tensor. The first one is using the expression $R_{1i1j} = -\varepsilon\varepsilon_i(\lambda_i - 2J)\delta_{ij}$ given by (6.2), which yields

$$-\varepsilon\lambda_i = \varepsilon_i R_{i1i1} - 2\varepsilon J = -\varepsilon\beta_i' - \beta_i^2 - 2\varepsilon J. \tag{6.17}$$

The other option is using the weighted Einstein field equations (6.1) itself, so we get

$$-\varepsilon\lambda_i = -\varepsilon\frac{h'}{h}\beta_i - 2\varepsilon J. \tag{6.18}$$

For the sake of clarity, let $\beta_2 = a$, $\beta_3 = b$, $\beta_4 = c$, and let $\Gamma = \Gamma_{342}\Gamma_{432}$. Then, we take the expressions given both by Lemma 6.20 and by (6.17) for the difference $R_{22} - R_{33}$ and add them to find

$$-2\varepsilon(\lambda_2 - \lambda_3) = (a-b)c - 2\frac{a+b-2c}{b-c}\Gamma,$$

while taking the expression given by (6.18) gives

$$-2\varepsilon(\lambda_2 - \lambda_3) = -2\varepsilon(a-b)\frac{h'}{h}.$$

Equating both expressions yields a first value for $h'h^{-1}$:

$$-\varepsilon\frac{h'}{h} = \frac{c}{2} - \frac{a+b-2c}{(a-b)(b-c)}\Gamma.$$

By the same process, using the components λ_2 and λ_3, we have another expression for $h'h^{-1}$:

$$-\varepsilon\frac{h'}{h} = \frac{b}{2} - \frac{a+c-2b}{(b-c)(c-a)}\Gamma.$$

We can now use both values to solve for Γ in terms of a, b and c. Indeed, take

$$\begin{aligned} P &= a^2 + b^2 + c^2 - ac - ab - bc \\ &= \tfrac{1}{2}(a-b)^2 + \tfrac{1}{2}(a-c)^2 + \tfrac{1}{2}(b-c)^2 \geq 0, \end{aligned}$$

with equality only being achievable when $a = b = c$. Then, we have

$$\Gamma = -\frac{(a-b)(a-c)(b-c)^2}{4P}.$$

Now, consider that $(\beta_4 - \beta_2)\Gamma_{34}^2 = (\beta_3 - \beta_2)\Gamma_{43}^2$. Thus, we can write

$$(\Gamma_{432})^2 = \Gamma_{432}\Gamma_{432} = \frac{c-a}{b-a}\Gamma_{342}\Gamma_{432} = -\frac{(a-c)^2(b-c)^2}{4P}.$$

It follows that either $a = c$ or $b = c$, which is a contradiction. Thus, β_2, β_3 and β_4 cannot be pairwise distinct, and the same holds for λ_2, λ_3 and λ_4. □

6.2.4 Proof of Theorem 6.7 (2).

Let (M, g, h) be a 4-dimensional solution of the field equations (4.6) with diagonalizable Ricci operator and harmonic curvature (not locally conformally flat), and such that $g(\nabla h, \nabla h) \neq 0$. Then, for any point in M_{Ric}, which is open and dense in M, we apply Lemmas 6.10-6.18 to find the admissible structures at the local level. Note that, in the hypotheses of Lemma 6.18, the fact that h satisfies the Obata equation on N_1 is immediate from $\mathrm{Hes}_h^{N_1} = \mathrm{Hes}_h\big|_{N_1} = h(\rho - \frac{\tau}{3}g)\big|_{N_1} = -\frac{\kappa h}{2}g^{N_1}$ (since $\tau = 3\kappa$). □

6.3 The case with complex eigenvalues

Throughout the previous section, we have discussed the admissible solutions with diagonalizable Ricci operator (the only possible case if ∇h is timelike), and we have seen that the geometric characteristics of such solutions are not dissimilar from those of Riemannian static spaces discussed in [78, 105] if the solution is non-isotropic. However, the fact that we are working in Lorentzian signature means that Ric does not diagonalize, in general, when ∇h is spacelike or lightlike. For isotropic solutions, it was shown in Theorem 5.4 that the eigenvalues of the Ricci operator are necessarily real (indeed, vanishing). In this section, we show that this is also the case for 4-dimensional non-isotropic solutions with harmonic curvature.

6.3.1 A note on Gröbner bases

In what follows, we will come across a system of polynomial equations $\{\mathfrak{P}_i = 0\}$ in several geometric variables of our SMMSs. Given the complexity of this system, in order to extract information from it, we will use the algebraic tool of Gröbner bases, which we briefly introduce here. We refer to [47] for details on the properties of Gröbner bases and some algorithms used to compute them.

Given a generic monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of total degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$, we can establish a one-to-one correspondence between monomials in the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$ and $\mathbb{Z}^n_{\geq 0}$ by considering the vector of exponents $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. A *monomial order* on $\mathbb{R}[x_1, \dots, x_n]$ is a relation $>$ on $\mathbb{Z}^n_{\geq 0}$ (equivalently, on the set of monomials x^α) which satisfies

1. $>$ is a total order on $\mathbb{Z}^n_{\geq 0}$.
2. If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.
3. $(\mathbb{Z}^n_{\geq 0}, >)$ is well-ordered, i.e., every non-empty subset of $\mathbb{Z}^n_{\geq 0}$ has a least element.

A few notable monomial orders are the following:

- *Lexicographic order*: $\alpha >_{\text{lex}} \beta$ if the leftmost non-zero entry in $\alpha - \beta$ is positive.
- *Graded lexicographic order*: $\alpha >_{\text{grlex}} \beta$ if $|\alpha| > \beta$ or $|\alpha| = |\beta|$ with $\alpha >_{\text{lex}} \beta$.
- *Graded reverse lexicographic order*: $\alpha >_{\text{grevlex}} \beta$ if $|\alpha| > \beta$ or $|\alpha| = |\beta|$ and the rightmost non-zero entry in $\alpha - \beta$ is negative.

Let $\mathfrak{P} = \sum_\alpha a_\alpha x^\alpha \in \mathbb{R}[x_1, \dots, x_n]$. Given a monomial order, we define the *leading term* of $\mathfrak{P}$, $LT(\mathfrak{P})$, as the monomial corresponding to the greatest $\alpha \in \mathbb{Z}^n_{\geq 0}$ such that $a_\alpha \neq 0$. For an ideal $\mathcal{I} \subset \mathbb{R}[x_1, \dots, x_n]$, let $LT(\mathcal{I})$ be the set of leading terms of all polynomials in $\mathcal{I}$, and $\langle LT(\mathcal{I}) \rangle$ the ideal generated by this set. In particular, if $\mathcal{I} = \langle \mathfrak{P}_i \rangle$, then $\langle LT(\mathfrak{P}_i) \rangle \subset \langle LT(\mathcal{I}) \rangle$, but equality is not always attained.

A *Gröbner basis* for $\mathcal{I}$, with respect to a certain monomial order, is a finite subset $\mathcal{G} = \{g_1, \dots, g_\nu\} \subset \mathcal{I}$ such that $\langle LT(g_1), \dots, LT(g_\nu) \rangle = \langle LT(\mathcal{I}) \rangle$. The following well-known result guarantees that every non-zero ideal $\mathcal{I} \in \mathbb{R}[x_1, \dots, x_n]$ admits a Gröbner basis:

Hilbert Basis Theorem: Every ideal $\mathcal{I} \in \mathbb{R}[x_1, \ldots, x_n]$ has a finite generating set.

Moreover, every Gröbner basis $\mathcal{G}$ for an ideal $\mathcal{I}$ is a generating set for $\mathcal{I}$. We can take advantage of this fact in the following way: Consider the set of solutions of the polynomial system given by the polynomials $\mathfrak{P}_i$, i.e., the set of vectors $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $\mathfrak{P}_i(\overrightarrow{a}) = 0$ for all i. These solutions also cancel out any polynomial in the ideal $\mathcal{I} = \langle \mathfrak{P}_i \rangle$. Since $\mathcal{I}$ is generated by $\mathcal{G}$, the solutions of the original system $\{\mathfrak{P}_i = 0\}$ are the same as those for the elements of $\mathcal{G}$. This means that we can try to solve large polynomial systems by obtaining simpler polynomials, sharing solutions with the original ones, as elements of a Gröbner basis.

There exist several algorithms for the computation of Gröbner bases, such as Buchberger's algorithm. Nevertheless, in any case, the construction of Gröbner basis is extremely dependent on the choice of monomial order, as well as the way one sorts the variables in the polynomials themselves. Indeed, a Gröbner basis might be simple or contain a small number of polynomials for a certain order, while the basis for a different order might become completely unmanageable. Thus, when working with this tool, it is essential to specify the order being used in order to make the computations reproducible.

6.3.2 The nonexistence result

The following result tells us that, in the case of harmonic curvature, solutions of Type I.b (see (6.6)) are not admissible.

Theorem 6.22. *Let (M, g, h) be a 4-dimensional solution of the weighted Einstein field equations* (4.6) *with harmonic curvature. Then, the Ricci operator of (M, g) has real eigenvalues.*

We emphasize that the harmonicity of the curvature is an essential assumption in Theorem 6.22, since there are solutions with non-real eigenvalues and non-harmonic curvature, as illustrated by the following example.

Example 6.23. In order to build a solution with complex eigenvalues for the Ricci operator, we consider a left-invariant metric on the Lie group $\mathbb{R}^3 \rtimes \mathbb{R}$, this is, a semi-direct extension of the Abelian group. Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the corresponding Lie algebra, where e_4 generates the $\mathbb{R}$ factor, and consider the Lie bracket given by

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = e_3, \quad [e_3, e_4] = -e_2.$$

The Lorentzian metric is given by $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = 1$.

Now, we look for an expression of the metric in local coordinates $(x, y, z, t) \in \mathbb{R}^4$. Using the relation $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$, for a 1-form ω, on the dual basis $\{e^1, e^2, e^3, e^4\}$ we obtain

$$de^1 = e^1 \wedge e^4, \quad de^2 = e^3 \wedge e^4, \quad de^3 = -e^2 \wedge e^4, \quad de^4 = 0.$$

By relating the basis $\{e^1, e^2, e^3, e^4\}$ with $\{dx, dy, dz, dt\}$ and integrating the corresponding equations we get a particular solution of the form $e^1 = e^{-t}dx$, $e^2 = \cos(t)dy + \sin(t)dz$, $e^3 = \sin(t)dy - \cos(t)dz$, $e^4 = dt$. Hence,

$$g = e^{2t}dx^2 + \cos(2t)(dy^2 - dz^2) + 2\sin(2t)dydz + dt^2.$$

Consider the positive density function $h(t) = e^{-t}$, whose gradient is $\nabla h = -e^{-t}\partial_t$, so we have $g(\nabla h, \nabla h) = e^{-2t} > 0$. Moreover, the Ricci and the Hessian operators are given by

$$\begin{aligned}
&\operatorname{Ric}(\partial_x) = \partial_t, \quad \operatorname{Ric}(\partial_y) = \partial_z, \quad \operatorname{Ric}(\partial_z) = -\partial_y, \quad \operatorname{Ric}(\partial_t) = \partial_t, \\
&\operatorname{hes}_h(\partial_x) = e^{-t}\partial_t, \quad \operatorname{hes}_h(\partial_y) = e^{-t}\partial_z, \\
&\operatorname{hes}_h(\partial_z) = -e^{-t}\partial_y, \quad \operatorname{hes}_h(\partial_t) = e^{-t}\partial_t,
\end{aligned}$$

so $(\mathbb{R}^4, g, h)$ is a solution of Type I.b with $\lambda = 1$, $\alpha = b = -1$ and $a = 0$. However, the curvature tensor is not harmonic, since

$$(\nabla_{\partial_x}\rho)(\partial_t, \partial_t) - (\nabla_{\partial_t}\rho)(\partial_x, \partial_t) = 2e^{2t} \neq 0.$$

In order to prove Theorem 6.22, assume, on the contrary, that (M, g, h) is a 4-dimensional solution of the weighted Einstein field equations (6.1) with ∇h spacelike and Ricci operator of Type I.b in M, as shown in (6.6). We work on an adapted pseudo-orthonormal basis $\mathcal{B}_1 = \{E_1 = \nabla h/\|\nabla h\|, E_2, E_3, E_4\}$ and see that, by the field equations, the Hessian operator is given by

$$\begin{aligned}
&\operatorname{hes}_h \nabla h = \tilde{\lambda}\nabla h, \quad \operatorname{hes}_h E_2 = \tilde{a}E_2 - \tilde{b}E_3, \\
&\operatorname{hes}_h E_3 = \tilde{b}E_2 + \tilde{a}E_3, \quad \operatorname{hes}_h E_4 = \tilde{\alpha}E_4,
\end{aligned}$$

where $\tilde{\lambda} = h(\lambda - 2J)$, $\tilde{a} = h(a - 2J)$, $\tilde{b} = hb$ and $\tilde{\alpha} = h(\alpha - 2J)$. We start by using the harmonicity of the curvature to obtain information on a, b λ and α, and on the components of the Levi-Civita connection, with the following two lemmas.

Lemma 6.24. *Let (M, g, h) be a solution of Type I.b such that (M, g) has harmonic curvature. Then, a, b, λ and α have vanishing derivatives in the direction of E_2, E_3 and E_4. Moreover,*

$$\begin{aligned}
\nabla h(a) &= h(b^2 + (\lambda - a)(a - 2J)), \\
\nabla h(b) &= hb(\lambda + 2J - 2a), \\
\nabla h(\alpha) &= h(\lambda - \alpha)(\alpha - 2J), \\
\nabla h(\lambda) &= -h(2b^2 - 2a^2 - \alpha^2 + (\lambda + 2J)(2a + \alpha) - 6J\lambda).
\end{aligned} \tag{6.19}$$

Proof. In the frame $\mathcal{B}_1$, and using the weighted Einstein field equations (6.1), we compute $(\nabla_{\nabla h}\rho)(E_2, \nabla h) = 0$ and $(\nabla_{E_2}\rho)(\nabla h, \nabla h) = \|\nabla h\|^2 E_2(\lambda)$. Since ∇h is spacelike, from the Codazzi condition $(\nabla_{\nabla h}\rho)(E_2, \nabla h) = (\nabla_{E_2}\rho)(\nabla h, \nabla h)$, we find $E_2(\lambda) = 0$. Similarly, we prove $E_3(\lambda) = E_4(\lambda) = 0$.

We continue to extract information from the Codazzi condition. On the one hand, we have

$$(\nabla_{\nabla h}\rho)(E_2, E_3) = -\nabla h(b) \quad \text{and} \quad (\nabla_{E_2}\rho)(\nabla h, E_3) = -(\lambda - a)\tilde{b} + b\tilde{a},$$

which gives the expression for $\nabla h(b)$. In the same way, we compute $(\nabla_{\nabla h}\rho)(E_i, E_i)$ and $(\nabla_{E_i}\rho)(\nabla h, E_i)$ for $i = 2, 3$ to find

$$\begin{aligned}
-\nabla h(a) + 2bg(\nabla_{\nabla h}E_2, E_3) &= (a - \lambda)\tilde{a} - b\tilde{b}, \\
\nabla h(a) + 2bg(\nabla_{\nabla h}E_2, E_3) &= (\lambda - a)\tilde{a} + b\tilde{b},
\end{aligned}$$

which yields $g(\nabla_{\nabla h}E_2, E_3) = 0$ and the expression for $\nabla h(a)$. On the other hand,

$$(\nabla_{\nabla h}\rho)(E_4, E_4) = \nabla h(\alpha) \quad \text{and} \quad (\nabla_{E_4}\rho)(\nabla h, E_4) = (\lambda - \alpha)\tilde{\alpha},$$

which provides an equation for $\nabla h(\alpha)$. Finally, since τ is constant by Lemma 4.5, and $\tau = \lambda + 2a + \alpha$, we have $0 = \nabla h(\lambda) + 2\nabla h(a) + \nabla h(\alpha)$, which yields the last equation in (6.19). Moreover, since we know $E_i(\lambda) = 0$ for $i = 2, 3, 4$ and τ is constant, it follows that $2E_i(a) = -E_i(\alpha)$. Thus, taking the derivative of this equation in the direction of E_i, for $i = 2, 3, 4$, it follows that $E_i(2a^2 + \alpha^2 - 2b^2) = 0$. Now, take

$$\begin{aligned} h^{-1}\nabla h(a^2 + \tfrac{1}{2}\alpha^2 - b^2) &= (\lambda + 2J)(2a^2 + \alpha^2 - 2b^2) - 2J\lambda(2a + \alpha) \\ &\quad -2a^3 - \alpha^3 + 6ab^2. \end{aligned}$$

Differentiating this expression in the direction of E_i yields $E_i(2a^3+\alpha^3-6ab^2) = 0$. In summary, we have three distinct expressions:

$$E_i(\alpha) = -2E_i(a), \quad E_i(2a^2 + \alpha^2 - 2b^2) = 0, \quad E_i(2a^3 + \alpha^3 - 6ab^2) = 0,$$

for $i = 2, 3, 4$. Using the first and second ones, we can write $E_i(b) = \frac{a-\alpha}{b}E_i(a)$, so now the third equation becomes

$$\begin{aligned} 0 &= 6a^2E_i(a) + 3\alpha^2E_i(\alpha) - 6b^2E_i(a) - 12abE_i(b) \\ &= -6((a - \alpha)^2 + b^2)E_i(a). \end{aligned}$$

Since $b \neq 0$, it follows that $E_i(a) = E_i(\alpha) = E_i(b) = 0$. □

Lemma 6.25. *Let (M, g, h) be a solution of Type I.b such that (M, g) has harmonic curvature. Let C be the matrix such that $C_{1i} = \nabla_{\nabla h}E_i$, $C_{i1} = \nabla_{E_i}\nabla h$ and $C_{ij} = \nabla_{E_i}E_j$, for $i, j \in \{2, 3, 4\}$. Then,*

$$C = \begin{pmatrix} \tilde{\lambda}\nabla h & 0 & 0 & 0 \\ \tilde{a}E_2 - \tilde{b}E_3 & \frac{\tilde{a}}{\|\nabla h\|^2}\nabla h + \frac{\alpha-a}{b}\Gamma E_4 & \frac{\tilde{b}}{\|\nabla h\|^2}\nabla h + \Gamma E_4 & \Gamma\left(\frac{\alpha-a}{b}E_2 - E_3\right) \\ \tilde{b}E_2 + \tilde{a}E_3 & \frac{\tilde{b}}{\|\nabla h\|^2}\nabla h - \Gamma E_4 & -\frac{\tilde{a}}{\|\nabla h\|^2}\nabla h + \frac{\alpha-a}{b}\Gamma E_4 & -\Gamma\left(E_2 + \frac{\alpha-a}{b}E_3\right) \\ \tilde{\alpha}E_4 & -\frac{(\alpha-a)^2+b^2}{2b^2}\Gamma E_3 & -\frac{(\alpha-a)^2+b^2}{2b^2}\Gamma E_2 & -\frac{\tilde{\alpha}}{\|\nabla h\|^2}\nabla h \end{pmatrix}$$

where $\Gamma = g(\nabla_{E_2}E_3, E_4)$.

Proof. The column C_{i1} is given by the weighted Einstein field equations (6.1) and the fact that $\nabla_{E_i}\nabla h = \mathrm{hes}_h E_i$. We also use $g(\nabla_{E_i}E_j, \nabla h) = -\mathrm{Hes}_h(E_i, E_j)$ to find the component in the direction of ∇h of $\nabla_{E_i}E_j$. Now, from the proof of Lemma 6.24, we know that $g(\nabla_{\nabla h}E_2, E_3) = -g(\nabla_{\nabla h}E_3, E_2) = 0$. Next, we compute $(\nabla_{\nabla h}\rho)(E_i, E_4) = (\nabla_{E_i}\rho)(\nabla h, E_4)$, with $i = 2, 3$, to find

$$\begin{aligned} (\alpha - a)g(E_2, \nabla_{\nabla h}E_4) + bg(E_3, \nabla_{\nabla h}E_4) &= 0, \\ (\alpha - a)g(E_3, \nabla_{\nabla h}E_4) - bg(E_2, \nabla_{\nabla h}E_4) &= 0. \end{aligned}$$

Since $b \neq 0$, we have $g(\nabla_{\nabla h} E_i, E_4) = -g(\nabla_{\nabla h} E_4, E_i) = 0$, for $i = 2,3,4$. Moreover, $g(\nabla_{\nabla h} E_i, \nabla h) = \mathrm{Hes}_h(E_i, \nabla h) = 0$ for $i = 2,3,4$. This completes the row C_{1i}.

Let $\Gamma_{ijk} = g(\nabla_{E_i} E_j, E_k)$ (notice that $\Gamma_{ijk} = -\Gamma_{ikj}$) and set also $\nabla_i \rho_{jk} = (\nabla_{E_i}\rho)(E_j, E_k)$. Then, compute $\nabla_2 \rho_{33} = 2b\Gamma_{223}$ and $\nabla_3 \rho_{23} = 0$ to find $\Gamma_{223} = 0$. Analogously, from $\nabla_3 \rho_{22} = \nabla_2 \rho_{32}$, we have $\Gamma_{332} = 0$. Moreover, from $\nabla_i \rho_{44} = \nabla_4 \rho_{i4}$ for $i = 2,3$ it follows that

$$0 = b\Gamma_{443} + (\alpha - a)\Gamma_{442}, \qquad 0 = (\alpha - a)\Gamma_{443} - b\Gamma_{442},$$

from where $\Gamma_{443} = \Gamma_{442} = 0$. Hence, the only non-vanishing Γ_{ijk} (up to symmetries) are Γ_{4ij} and Γ_{ij4}, where $i,j = 2,3$. Finally, we use $\nabla_4 \rho_{ii} = \nabla_i \rho_{4i}$ to find

$$2b\Gamma_{423} = -(\alpha - a)\Gamma_{224} - b\Gamma_{234}, \qquad 2b\Gamma_{423} = -(\alpha - a)\Gamma_{334} + b\Gamma_{324},$$

while $\nabla_3 \rho_{24} = \nabla_2 \rho_{34} = \nabla_4 \rho_{23}$ gives two more relations:

$$0 = (\alpha - a)\Gamma_{234} - b\Gamma_{224}, \qquad 0 = (\alpha - a)\Gamma_{324} + b\Gamma_{334}.$$

Setting $\Gamma_{234} = \Gamma$, the indeterminate system given by these four equations yields

$$\Gamma_{224} = \Gamma_{334} = \frac{\alpha - a}{b}\Gamma, \qquad \Gamma_{324} = -\Gamma, \qquad \Gamma_{423} = -\frac{(\alpha - a)^2 + b^2}{2b^2}\Gamma,$$

which completes the remaining terms of the matrix C. □

The two lemmas above exhaust the amount of information we can extract from the harmonicity of the curvature tensor. However, more compatibility conditions can be obtained through the Jacobi identity of vector fields and the restrictions that the weighted Einstein field equations impose on the curvature tensor.

Lemma 6.26. *Let (M, g, h) be a solution of Type I.b of the weighted Einstein field equations such that (M, g) has harmonic curvature, with the connection coefficients given by Lemma 6.25. Then, $E_i(\Gamma) = 0$ for $i = 2,3,4$.*

Proof. We use the Jacobi identity of vector fields to write

$$[[E_4, E_2], E_3] + [[E_2, E_3], E_4] + [[E_3, E_4], E_2] = 0.$$

With the notation of Lemma 6.25, the Lie brackets take the form

$$\begin{aligned} [E_4, E_2] &= -\tfrac{\alpha - a}{b}\Gamma E_2 - \tfrac{(\alpha-a)^2 - b^2}{2b^2}\Gamma E_3, \qquad [E_2, E_3] = 2\Gamma E_4, \\ [E_3, E_4] &= -\tfrac{\alpha - a}{b}\Gamma E_3 + \tfrac{(\alpha-a)^2 - b^2}{2b^2}\Gamma E_2. \end{aligned}$$

Moreover, by Lemma 6.24, we have that $E_i(\alpha) = E_i(a) = E_i(b) = 0$ for $i = 2,3,4$. Hence,

$$\begin{aligned} [[E_4, E_2], E_3] &= \tfrac{\alpha - a}{b} E_3(\Gamma) E_2 + \tfrac{(\alpha-a)^2 - b^2}{2b^2} E_3(\Gamma) E_3 - 2\tfrac{\alpha - a}{b}\Gamma^2 E_4, \\ [[E_2, E_3], E_4] &= -2E_4(\Gamma) E_4, \\ [[E_3, E_4], E_2] &= 2\tfrac{\alpha - a}{b}\Gamma^2 E_4 + \tfrac{\alpha - a}{b} E_2(\Gamma) E_3 - \tfrac{(\alpha-a)^2 - b^2}{2b^2} E_2(\Gamma) E_2. \end{aligned}$$

Taking the sum of the three brackets, it follows that $E_4(\Gamma)=0$. Moreover, from the components in the direction of E_2 and E_3 respectively, we have

$$\begin{aligned} 0 &= \tfrac{\alpha-a}{b}E_3(\Gamma)-\tfrac{(\alpha-a)^2-b^2}{2b^2}E_2(\Gamma),\\ 0 &= \tfrac{(\alpha-a)^2-b^2}{2b^2}E_3(\Gamma)+\tfrac{\alpha-a}{b}E_2(\Gamma). \end{aligned}$$

The determinant associated to this homogeneous system is $\left(\frac{(\alpha-a)^2+b^2}{2b^2}\right)^2\neq 0$, so the only solution is $E_2(\Gamma)=E_3(\Gamma)=0$. □

Lemma 6.27. *Let (M,g,h) be a solution of Type I.b of the weighted Einstein field equations such that (M,g) has harmonic curvature, with the connection coefficients given by Lemma 6.25. Then, the following equations are satisfied:*

$$0=\tfrac{(\alpha-a)^2+b^2}{b^2}\Gamma^2-2(a-J)-\tfrac{h^2}{\|\nabla h\|^2}((a-2J)^2+b^2+(a-2J)(\alpha-2J)),\tag{6.20}$$

$$0=\tfrac{(\alpha-a)((\alpha-a)^2+b^2)}{b^2}\Gamma^2+2b^2+\tfrac{h^2}{\|\nabla h\|^2}b^2(\alpha-2J),\tag{6.21}$$

$$0=\tfrac{(\alpha-a)^2+b^2}{b^2}\Gamma^2+\alpha-J+\tfrac{h^2}{\|\nabla h\|^2}(a-2J)(\alpha-2J).\tag{6.22}$$

Proof. Recall that $E_1=\frac{\nabla h}{\|\nabla h\|}$ and denote

$$R_{ijkl}=R(E_i,E_j,E_k,E_l)=g((\nabla_{[E_i,E_j]}-[\nabla_{E_i},\nabla_{E_j}])E_k,E_l).$$

We will use Lemmas 6.24, 6.25 and 6.26 to compute the different components of the curvature tensor. For example,

$$\begin{aligned} R(E_2,E_4)E_3 &= \tfrac{\alpha-a}{b}\Gamma\nabla_{E_2}E_3+\tfrac{(\alpha-a)^2-b^2}{2b^2}\Gamma\nabla_{E_3}E_3\\ &\quad+\tfrac{(\alpha-a)^2+b^2}{2b^2}\Gamma\nabla_{E_2}E_2+\tfrac{\tilde b}{\|\nabla h\|^2}\nabla_{E_4}\nabla h+\Gamma\nabla_{E_4}E_4\\ &= \left(\tfrac{(\alpha-a)((\alpha-a)^2+b^2)}{b^3}\Gamma^2+\tfrac{\tilde b\tilde\alpha}{\|\nabla h\|^2}\right)E_4. \end{aligned}$$

Other components follow analogously, and the following are of interest:

$$\begin{aligned} -R_{3434}=R_{2424} &= \tfrac{(\alpha-a)^2+b^2}{b^2}\Gamma^2+\tfrac{h^2}{\|\nabla h\|^2}(a-2J)(\alpha-2J),\\ R_{2434} &= \tfrac{(\alpha-a)((\alpha-a)^2+b^2)}{b^3}\Gamma^2+\tfrac{h^2}{\|\nabla h\|^2}b(\alpha-2J),\\ R_{2323} &= -2\tfrac{(\alpha-a)^2+b^2}{b^2}\Gamma^2+\tfrac{h^2}{\|\nabla h\|^2}\left((a-2J)^2+b^2\right). \end{aligned}\tag{6.23}$$

On the other hand, we use equation (6.2) to compute the following components involving E_1:

$$R_{2121}=a-2J,\qquad R_{2131}=b,\qquad R_{4141}=-\alpha+2J.\tag{6.24}$$

Now, using the definition of the Ricci tensor, we have

$$\begin{aligned} -a &= \rho_{22}=R_{2121}+R_{2323}+R_{2424},\\ -b &= \rho_{23}=R_{2131}+R_{2434},\\ \alpha &= \rho_{44}=R_{4141}-R_{2424}+R_{3434}. \end{aligned}$$

Substituting in the expressions given by (6.23) and (6.24), the result follows. □

6.3.3 Proof of Theorem 6.22

Let (M, g, h) be a 4-dimensional Type I.b solution of the weighted Einstein field equations (4.6) such that (M, g) has harmonic curvature. For this solution, Lemmas 6.24-6.27 stated throughout this section apply. Let $H = \frac{h^2}{\|\nabla h\|^2}$. We analyze two cases separately: $\alpha = a$ and $\alpha \neq a$.

Case $\alpha = a$: Equations (6.20), (6.21) and (6.22) in Lemma 6.27 reduce to

$$\begin{aligned} 0 &= 2\,(a-J) + H(2\,(a-2J)^2 + b^2) - \Gamma^2, \\ 0 &= 2b^2 + Hb^2\,(a-2J)\,, \\ 0 &= a - J + H\,(a-2J)^2 + \Gamma^2. \end{aligned}$$

Since $b \neq 0$, we solve for a in the second expression to get $a = 2\frac{JH-1}{H}$. The remaining two equations become

$$0 = b^2 H + \frac{4}{H} + 2J - \Gamma^2, \qquad 0 = \frac{2}{H} + J + \Gamma^2,$$

so adding both yields

$$0 = b^2 H + \frac{6}{H} + 3J. \tag{6.25}$$

Now, notice that $\nabla h(H) = 2h(1 - H(\lambda - 2J))$. Using this, the fact that $\lambda = 6J - 3a = \frac{6}{H}$, and the expression for $\nabla h(b)$ given by Lemma 6.24, we differentiate (6.25) in the direction of ∇h. This gives $0 = 2\frac{h}{H^2}(5b^2H^2 - 12JH + 30)$. Hence,

$$0 = 5b^2H^2 - 12JH + 30. \tag{6.26}$$

Combining (6.25) and (6.26), it follows that $0 = 6 + b^2H^2$, which is not possible.

Case $\alpha \neq a$: This case requires some fairly long, although straightforward, computations, which we present schematically. Firstly, we compute the (6.21) $-$ $(\alpha - a)$(6.20) and (6.22) $-$ (6.20) to remove the terms involving Γ and obtain two polynomials in $\mathbb{R}[J, a, b, \alpha, H]$, that must vanish at every point of the manifold for our solution:

$$\begin{aligned} \mathfrak{P}_1 &= -ab^2H - 8J^2aH - 4JaH\alpha + 6Ja^2H + 2Ja + aH\alpha^2 - a^3H \\ &\quad +2a\alpha - 2a^2 - 2Jb^2H + 2b^2H\alpha + 2b^2 + 8J^2H\alpha - 2JH\alpha^2 - 2J\alpha, \\ \mathfrak{P}_2 &= -8JaH + 2aH\alpha + a^2H + 2a + b^2H + 12J^2H - 4JH\alpha - 3J + \alpha. \end{aligned}$$

Now, use $\nabla h(H) = 2h(1 - H(\lambda - 2J))$, $\lambda = 6J - 2a - \alpha$ and the derivatives given by

Lemma 6.24 to compute two new polynomials $\mathfrak{P}_3 = \frac{\nabla h(\mathfrak{P}_1)}{h}$ and $\mathfrak{P}_4 = \frac{\nabla h(\mathfrak{P}_2)}{2h}$:

$$\begin{aligned}
\mathfrak{P}_3 &= 8Jab^2H - 11ab^2H\alpha + 4a^2b^2H - 22ab^2 - 96J^3aH - 64J^2aH\alpha \\
&+108J^2a^2H + 24J^2a + 6Ja^2H\alpha + 28JaH\alpha^2 - 40Ja^3H + 30Ja\alpha \\
&-34Ja^2 + a^3H\alpha - 3a^2H\alpha^2 - 3aH\alpha^3 + 5a^4H - 6a^2\alpha + 4a\alpha^2 \\
&+10a^3 - 36J^2b^2H + 30Jb^2H\alpha + 30Jb^2 - 3b^2H\alpha^2 - b^4H + 2b^2\alpha \\
&+96J^3H\alpha - 44J^2H\alpha^2 - 24J^2\alpha + 6JH\alpha^3 + 4J\alpha^2, \\
\mathfrak{P}_4 &= -ab^2H - 24J^2aH + 8JaH\alpha + 8Ja^2H + 6Ja - aH\alpha^2 \\
&-a^2H\alpha - a^3H - 2a^2 + b^2H\alpha + 2b^2 + 24J^3H \\
&-12J^2H\alpha - 6J^2 + 2JH\alpha^2 + 3J\alpha - \alpha^2.
\end{aligned}$$

Finally, we compute $\mathfrak{P}_5 = \frac{\nabla h(\mathfrak{P}_3)}{h}$ and $\mathfrak{P}_6 = \frac{\nabla h(\mathfrak{P}_4)}{h}$, which provides two additional polynomials:

$$\begin{aligned}
\mathfrak{P}_5 &= 676J^2ab^2H - 514Jab^2H\alpha - 92Ja^2b^2H - 900Jab^2 + 94a^2b^2H\alpha \\
&+51ab^2H\alpha^2 - 20a^3b^2H + 20ab^4H + 12ab^2\alpha + 280a^2b^2 \\
&-1824J^4aH - 1536J^3aH\alpha + 2744J^3a^2H + 456J^3a + 240J^2a^2H\alpha \\
&+1012J^2aH\alpha^2 - 1564J^2a^3H + 692J^2a\alpha - 840J^2a^2 + 70Ja^3H\alpha \\
&-234Ja^2H\alpha^2 - 210JaH\alpha^3 + 404Ja^4H - 246Ja^2\alpha - 202Ja\alpha^2 \\
&+460Ja^3 - 17a^4H\alpha + 15a^3H\alpha^2 + 27a^2H\alpha^3 + 15aH\alpha^4 - 40a^5H \\
&+20a^3\alpha + 46a^2\alpha^2 + 14a\alpha^3 - 80a^4 - 840J^3b^2H + 696J^2b^2H\alpha \\
&+672J^2b^2 - 138Jb^2H\alpha^2 - 16Jb^4H + 38Jb^2\alpha + 9b^2H\alpha^3 - 9b^4H\alpha \\
&-18b^2\alpha^2 - 24b^4 + 1824J^4H\alpha - 1208J^3H\alpha^2 - 456J^3\alpha \\
&+312J^2H\alpha^3 + 148J^2\alpha^2 - 30JH\alpha^4 - 12J\alpha^3, \\
\mathfrak{P}_6 &= -7ab^2H\alpha + 4a^2b^2H - 22ab^2 - 336J^3aH + 160J^2aH \\
&+184J^2a^2H\alpha + 84J^2a - 36JaH\alpha^2 - 48Ja^2H\alpha - 48Ja^3H \\
&-12Ja\alpha - 50Ja^2 + 3aH\alpha^3 + 5a^2H\alpha^2 + 5a^3H\alpha + 5a^4H + 2a\alpha^2 \\
&+2a^2\alpha + 10a^3 - 24J^2b^2H + 24Jb^2H\alpha + 38Jb^2 - 3b^2H\alpha^2 \\
&-b^4H - 2b^2\alpha + 240J^4H - 168J^3H\alpha - 60J^3 + 52J^2H\alpha^2 \\
&+42J^2\alpha - 6JH\alpha^3 - 22J\alpha^2 + 4\alpha^3.
\end{aligned}$$

Thus, at each point of the manifold, we have the system of polynomial equations $\{\mathfrak{P}_i = 0 | i = 1, \dots, 6\}$, where $\mathfrak{P}_i \in \mathbb{R}[J, a, b, \alpha, H]$. Following the discussion in Section 6.3.1, we look for a simpler polynomial by computing a Gröbner basis $\mathcal{G}$ for $\mathcal{I}$ using graded lexicographic order. As a result, we obtain a basis with 13 polynomials, which include the following one:

$$\mathfrak{G} = 16b^8 + 8b^6\alpha^2 + b^4\alpha^4 \in \mathcal{G}.$$

Since $\mathfrak{G} \in \langle \mathfrak{P}_i \rangle$ and $\mathfrak{P}_i$ vanishes for all i, $\mathfrak{G}$ must vanish as well. We conclude that $b = 0$ necessarily, contradicting the assumption that the solution is of Type I.b. □

6.4 Non-diagonalizable cases with real eigenvalues

In this section, we focus on the two remaining cases, that is, those whose Ricci operator is non-diagonalizable with real eigenvalues. Hence, we will tackle 4-dimensional solutions (M, g, h) of the weighted Einstein field equations with ∇h spacelike, such that the Ricci operator is of Type II or Type III, as given by (6.7). We also include in this section any isotropic solutions that are either 2-step nilpotent or 3-step nilpotent. In this case, we will use an adapted frame which gives a Ricci operator in the form of (6.8). As in the previous two sections, all solutions are assumed to have harmonic curvature and the results will complete the proof of one of our main results (Theorem 6.36). The aforementioned proof is included at the end of the chapter, in Section 6.5.

6.4.1 Type II solutions

We begin by analyzing solutions with Ricci operator of Type II. We already know from Theorem 5.4 that isotropic solutions have nilpotent Ricci operator. Hence, we consider the case with ∇h spacelike first.

Assume Ric is of Type II with ∇h spacelike. Then, there exists a pseudo-orthonormal frame $\mathcal{B}_2 = \{E_1 = \nabla h / \|\nabla h\|, U, V, E_2\}$ as in (6.7), where E_1, E_2 are spacelike and U, V lighlike, and so that the Ricci operator is given by

$$\operatorname{Ric} \nabla h = \lambda \nabla h, \quad \operatorname{Ric} U = \alpha U + \varepsilon V, \quad \operatorname{Ric} V = \alpha V, \quad \operatorname{Ric} E_2 = \beta E_2.$$

Now, from the weighted Einstein field equations (6.1), it follows that the Hessian operator $\operatorname{hes}_h = \nabla\nabla h$ has the following form in this frame:

$$\operatorname{hes}_h \nabla h = \tilde{\lambda} \nabla h, \quad \operatorname{hes}_h U = \tilde{\alpha} U + \varepsilon h V, \quad \operatorname{hes}_h V = \tilde{\alpha} V, \quad \operatorname{hes}_h E_2 = \tilde{\beta} E_2,$$

where $\tilde{\lambda} = h(\lambda - 2J)$, $\tilde{\alpha} = h(\alpha - 2J)$ and $\tilde{\beta} = h(\beta - 2J)$.

Lemma 6.28. *Let (M, g, h) be a Type II solution with ∇h spacelike. Then M splits as a direct product $I \times N$, with metric $g^M = dt^2 \oplus g^N$ (with g^N possibly dependent on t), where $\partial_t = E_1 = \nabla h / \|\nabla h\|$.*

Proof. Since $\operatorname{hes}_h E_1 = \tilde{\lambda} E_1$, the distribution generated by ∇h is totally geodesic. Furthermore, we see that

$$\begin{aligned} 0 &= U(g(\nabla h, V)) = g(\nabla_U \nabla h, V) + g(\nabla h, \nabla_U V) = \tilde{\alpha} + g(\nabla h, \nabla_U V), \\ 0 &= V(g(\nabla h, U)) = g(\nabla_V \nabla h, U) + g(\nabla h, \nabla_V U) = \tilde{\alpha} + g(\nabla h, \nabla_V U), \end{aligned}$$

so subtracting both expressions we get $g([U,V],\nabla h) = 0$. We proceed analogously in order to verify that $g([U,E_2],\nabla h) = g([V,E_2],\nabla h) = 0$. Thus, $\mathrm{span}\{U,V,E_2\}$ is closed under Lie bracket, and the distribution generated by $\mathrm{span}\{U,V,E_2\}$ is integrable. By the same argument that was used in Section 6.2 to obtain (6.11), it follows that, locally, M splits as a product $I\times N$, with the metric $g^M = dt^2 \oplus g^N$, where t is such that $dt = dh/\|\nabla h\|$ and $h = h(t)$. Note that, since ∇h is spacelike, $\partial_t = \nabla t = E_1$. □

Notice that, from Lemma 6.28 we can compute $\mathrm{Hes}_h(\partial_t,\partial_t) = h''$, and $\lambda = h^{-1}h'' + 2J$, so λ depends only on t. This is indeed true for the three eigenvalues.

Lemma 6.29. *Let (M,g,h) be a Type II solution with ∇h spacelike. Then, all eigenvalues of the Ricci operator depend only on the local coordinate t. Thus, for the adapted frame $\mathcal{B}_2 = \{E_1,U,V,E_2\}$ one has*

$$U(\alpha) = V(\alpha) = E_2(\alpha) = 0 \quad \textit{and} \quad U(\beta) = V(\beta) = E_2(\beta) = 0.$$

Proof. Since the curvature tensor is harmonic, the Ricci tensor is Codazzi. Hence, on the one hand, we have

$$\begin{aligned}(\nabla_{\nabla h}\rho)(U,V) &= \nabla h(\rho(U,V)) - \rho(\nabla_{\nabla h}U,V) - \rho(U,\nabla_{\nabla h}V)\\ &= \nabla h(\alpha) - \alpha g(\nabla_{\nabla h}U,V) - \alpha g(U,\nabla_{\nabla h}V) - \varepsilon g(V,\nabla_{\nabla h}V)\\ &= \nabla h(\alpha),\end{aligned}$$

where we have used $\nabla h(g(U,V)) = 0$ and $\nabla h(g(V,V)) = 0$. On the other hand,

$$\begin{aligned}(\nabla_U\rho)(\nabla h,V) &= -\rho(\nabla_U\nabla h,V) - \rho(\nabla h,\nabla_U V)\\ &= -\alpha g(\nabla_U\nabla h,V) - \lambda g(\nabla h,\nabla_U V) = (\lambda-\alpha)\tilde{\alpha},\end{aligned}$$

so we end up with $\nabla h(\alpha) = h(\lambda-\alpha)(\alpha-2J)$. We can also write

$$(\nabla_{\nabla h}\rho)(E_2,E_2) = \nabla h(\beta) \quad \text{and} \quad (\nabla_{E_2}\rho)(\nabla h,E_2) = (\lambda-\beta)\tilde{\beta}$$

so that $\nabla h(\beta) = h(\lambda-\beta)(\beta-2J)$. Now, since $\tau = \lambda + 2\alpha + \beta$ is constant by Lemma 4.5 and λ depends only on t, we have that

$$\begin{aligned}0 = \nabla h(\tau) &= \nabla h(\lambda) + 2\nabla h(\alpha) + \nabla h(\beta)\\ &= \nabla h(\lambda) + 2h(\lambda-\alpha)(\alpha-2J) + h(\lambda-\beta)(\beta-2J)\\ &= \nabla h(\lambda) + h(\lambda+2J)(2\alpha+\beta) - h\left(6J\lambda + (2\alpha^2+\beta^2)\right).\end{aligned}$$

Since every term in this expression except for $2\alpha^2+\beta^2$ depends only on t, $2\alpha^2+\beta^2$ depends on t as well. Thus, for any vector field $X \in \partial_t^\perp$, we have

$$2X(\alpha) + X(\beta) = 0, \quad 2\alpha X(\alpha) + \beta X(\beta) = 0.$$

The determinant associated to this homogeneous system is $2(\beta-\alpha)$, so either $X(\alpha) = X(\beta) = 0$ or $\alpha = \beta$. In both cases, it follows that α,β depend only on t. Alternatively, applying Lemma 6.11 to the power sums $2\alpha+\beta$ and $2\alpha^2+\beta^2$ also yields the result. □

The fact that the Ricci tensor is Codazzi, together with the information already obtained, allow us to compute some Christoffel symbols as follows.

Lemma 6.30. *Let (M, g, h) be a Type II solution with ∇h spacelike. Then, for the adapted frame $\mathcal{B}_2 = \{\nabla h, U, V, E_2\}$, the following equations are satisfied:*

$$
\begin{aligned}
0 &= (\alpha - \beta) g(\nabla_{E_2} V, E_2), \\
0 &= (\alpha - \beta) g(\nabla_{\nabla h} U, E_2) + \varepsilon g(\nabla_{\nabla h} V, E_2), \\
0 &= (\alpha - \beta) g(\nabla_{\nabla h} V, E_2), \\
0 &= g(\nabla_V V, U), \\
0 &= (\alpha - \beta) g(\nabla_V V, E_2), \\
0 &= (\alpha - \beta) g(E_2, \nabla_U U) + \varepsilon g(E_2, \nabla_U V) - 2\varepsilon g(U, \nabla_{E_2} V), \\
0 &= (\alpha - \beta) g(\nabla_V U, E_2) + \varepsilon g(\nabla_V V, E_2), \\
0 &= (\alpha - \beta) g(\nabla_U V, E_2), \\
0 &= (\alpha - \beta) g(\nabla_{E_2} U, E_2) + \varepsilon g(\nabla_{E_2} V, E_2).
\end{aligned}
$$

Proof. Using Lemma 6.29 and the Codazzi character of the Ricci tensor, we further analyze the connection coefficients for the different vectors. For example,

$$(\nabla_V \rho)(E_2, E_2) = V(\beta) - 2\rho(\nabla_V E_2, E_2) = -2\beta g(\nabla_V E_2, E_2) = 0,$$

and similarly, $(\nabla_{E_2} \rho)(V, E_2) = (\alpha - \beta) g(\nabla_{E_2} V, E_2)$. Equating both expressions yields $(\alpha - \beta) g(\nabla_{E_2} V, E_2) = 0$. The remaining components of the covariant derivative of the Ricci tensor are computed in a similar manner, and we omit details. □

Once we have obtained enough information on the Levi-Civita connection with respect to the adapted frame, we are ready to give the following classification result for Type II solutions, where we distinguish the isotropic case from that in which ∇h is spacelike.

Theorem 6.31. *Let (M, g, h) be a 4-dimensional solution of the weighted Einstein field equations* (4.6) *with harmonic curvature and Ricci operator of Type II.*

1. *If $g(\nabla h, \nabla h) > 0$, then (M, g) is a Kundt spacetime.*

2. *If $g(\nabla h, \nabla h) = 0$, then* Ric *is 2-step nilpotent and (M, g) is a pp-wave. Moreover, there exist local coordinates $\{u, v, x_1, x_2\}$ such that*

$$g_{ppw}(u, v, x_1, x_2) = 2\, du\, dv + F(v, x_1, x_2)\, dv^2 + dx_1^2 + dx_2^2$$

with $h = h(v)$ and $\Delta_x F = \partial^2_{x_1} F + \partial^2_{x_2} F = \frac{-2h''(v)}{h(v)}$.

Proof. Assume first that ∇h is spacelike. We work in the adapted pseudo-orthonormal frame $\mathcal{B}_2$ so that Ric is given by (6.7). From Lemma 6.30, either if $\alpha = \beta$ or $\alpha \neq \beta$, we have

$$g(\nabla_V V, U) = 0, \quad g(\nabla_V V, E_2) = 0.$$

Moreover, $g(\nabla_V V, V) = 0$ (since $g(V, V) = 0$), and

$$g(\nabla_V V, \nabla h) = -\operatorname{Hes}_h(V, V) = \tilde{\alpha} g(V, V) = 0.$$

Hence, $\nabla_V V = 0$ and V is geodesic.

In what follows, we prove that V satisfies $\nabla_X V = \omega(X)V$ for some 1-form ω and $X \perp V$ as in (1.8) to show that the spacetime is Kundt (recall this coordinate-free definition of Kundt spacetimes from Section 1.3). We check directly from Lemma 6.30 that $g(\nabla_{E_2} V, E_2) = 0$ and $g(\nabla_{\nabla h} V, E_2) = 0$. Furthermore, from the structure of Hes_h, it follows that

$$g(\nabla_{E_2} V, \nabla h) = -\operatorname{Hes}_h(V, E_2) = 0, \quad g(\nabla_{\nabla h} V, \nabla h) = -\operatorname{Hes}_h(V, \nabla h) = 0.$$

Finally, since V is lightlike, $g(\nabla_{\nabla h} V, V) = g(\nabla_{E_2} V, V) = 0$. Hence, for any $X \perp V$, we can write $\nabla_X V = \omega(X)V$ for some 1-form ω satisfying $\omega(V) = 0$, $\omega(\nabla h) = g(\nabla_{\nabla h} V, U)$ and $\omega(E_2) = g(\nabla_{E_2} V, U)$. Consequently, (M, g) is Kundt.

Now, we consider the isotropic case, i.e., the one where ∇h is lightlike. We work in the pseudo-orthonormal frame $\mathcal{B}_0 = \{\nabla h, U, X_1, X_2\}$ so that Ric takes the form of (6.8) with $\mu = 0$. Since Ric is 2-step nilpotent, the image of every vector field in this frame vanishes except for $\mathrm{Ric}(U) = \nu \nabla h$.

Moreover, the field equations (6.1) reduce to $h\rho = \operatorname{Hes}_h$, so we have $\nabla_{\nabla h} \nabla h = \nabla_{X_1} \nabla h = \nabla_{X_2} \nabla h = 0$, and $\nabla_U \nabla h = h\nu \nabla h$. Therefore, ∇h is recurrent and (M, g) is a Brinkmann wave. Notice that (M, g) is also Ricci-isotropic, that is, $\mathrm{Ric}(X) = 0$ for any X orthogonal to the recurrent lightlike vector field ∇h.

Next, we compute

$$\begin{aligned} 0 = \rho(X_1, U) &= R(X_1, U, U, \nabla h) + R(X_1, X_2, U, X_2), \text{ and} \\ 0 = \rho(X_1, X_1) &= 2R(X_1, U, X_1, \nabla h) + R(X_1, X_2, X_1, X_2). \end{aligned}$$

Additionally, since both the Cotton tensor dP and the Schouten scalar J vanish, we obtain from (6.2) that $R(\nabla h, X, Y, Z) = \rho \wedge dh(X, Y, Z)$, so

$$\begin{aligned} R(\nabla h, U, U, X_1) &= \rho \wedge dh(U, U, X_1) = 0, \text{ and} \\ R(\nabla h, X_1, U, X_1) &= \rho \wedge dh(X_1, U, X_1) = 0. \end{aligned}$$

Hence $R(X_1, X_2, X_2, U) = 0$ and $R(X_1, X_2, X_1, X_2) = 0$. Proceeding analogously, we prove that $R(X_1, X_2, X_1, U) = 0$. In summary, we have $R(\nabla h^{\perp}, \nabla h^{\perp}, -, -) = 0$, and given that (M, g) is a Ricci-isotropic Brinkmann wave, we conclude that (M, g) is a pp-wave (see Section 1.3).

Adopt canonical coordinates for a pp-wave so that the metric is given as in (1.13) with $\partial_u F = 0$. Then, the curvature tensor is harmonic if and only if $\Delta_x F = \partial^2_{x_1} F + \partial^2_{x_2} F = \lambda(v)$ is a function of the coordinate v. Moreover, a direct computation shows that, for such a metric, the only possibly non-vanishing component of G^h is

$$G^h(\partial_v, \partial_v) = \frac{1}{2}\left(-2h''(v) - h(v)\left(\partial^2_{x_1}F + \partial^2_{x_2}F\right)\right).$$

Hence, from $G^h(\partial_v, \partial_v) = 0$, we obtain $\partial^2_{x_1}F + \partial^2_{x_2}F = \frac{-2h''(v)}{h(v)}$. □

Remark 6.32. Note, from Theorem 6.31, that Type II solutions of the weighted Einstein field equations with ∇h spacelike are Kundt spacetimes where the distinguished lightlike vector field is V in the adapted frame $\mathcal{B}_2 = \{\nabla h, U, V, E_2\}$. Indeed, the covariant derivative of V satisfies

$$\nabla_V V = 0, \qquad \nabla_{\nabla h} V = g(\nabla_{\nabla h} V, U)V, \qquad \nabla_{E_2} V = g(\nabla_{E_2} V, U)V.$$

Associated to any Kundt spacetime, there exist canonical local coordinates as in (1.10). However, not every Kundt spacetime has Ricci operator of Type II.

If $\alpha \neq \beta$ in (6.7), using the conditions of the previous results for Type II solutions, we can obtain more specialized coordinates as follows. From the relations obtained in Lemmas 6.28, 6.29 and 6.30 we get that

$$\nabla_{\nabla h}\nabla h = \tilde{\lambda}\nabla h, \quad \nabla_{E_2}\nabla h = \tilde{\beta}E_2, \quad \nabla_{E_2}E_2 \parallel \nabla h, \text{ and } \nabla_{\nabla h}E_2 = 0.$$

Hence, the distribution $\operatorname{span}\{\nabla h, E_2\}$ is totally geodesic and $\operatorname{span}\{U, V\}$ is an integrable distribution, so the splitting in Lemma 6.28 can be further specialized. Thus, there exist local coordinates $\{t, e_2, u, v\}$ so that $h = h(t)$ and the metric takes the form

$$g(t, e_2, u, v) = dt^2 + r(t, e_2)de_2^2 + 2H(t, e_2, u, v)dvdu + F(t, e_2, u, v)dv^2.$$

Working with these local coordinates, a direct computation of the Hessian operator of h shows that the eigenvalues are: h'', $\frac{h'\partial_t r}{2r}$, and $\frac{h'\partial_t H}{2H}$. Recall that, by Lemma 6.29, the Ricci eigenvalues only depend on the coordinate t. Since Ric and hes_h are related by (6.1), the eigenvalues of the Hessian are also only dependent on t. Hence, r and H decompose as $r(t, e_2) = r_1(t)r_2(e_2)$ and $H(t, e_2, v, u) = H_0(t)H_1(e_2, v, u)$. Moreover, a direct computation of G^h yields

$$G^h(\partial_u, \partial_{e_2}) = \frac{h}{2H_1^2}\left(\partial_{e_2}H_1\partial_u H_1 - H_1\partial_{e_2}\partial_u H_1\right).$$

From where $\partial_{e_2}H_1\partial_u H_1 - H_1\partial_{e_2}\partial_u H_1 = 0$, which induces an extra decomposition on the function H_1 of the form $H_1(e_2, u, v) = H_2(e_2, v)H_3(u, v)$. Hence, the metric can be written as

$$\begin{aligned} g(t, e_2, u, v) &= dt^2 + r_1(t)r_2(e_2)de_2^2 + F(t, e_2, u, v)dv^2 \\ &\quad +2H_0(t)H_2(e_2, v)H_3(u, v)dudv. \end{aligned} \tag{6.27}$$

Generically, metrics given by (6.27) have Ricci operator of type II. However, they do not have harmonic Weyl tensor nor do they satisfy the weighted Einstein field equations in general.

Remark 6.33. In Chapter 7, we will provide a classification of solutions of dimension four realized on the family of pure radiation waves, with the metric given by (1.13), in both the general case (Theorem 7.2) and for solutions with harmonic curvature (Corollary 7.5). In the latter case, isotropic solutions are realized on *pp*-waves and non-isotropic ones, on plane waves. In both instances, they are of Type II. This contrasts with the broader family of Kundt spacetimes, where we can find Type III solutions with harmonic curvature tensor (see Example 6.35).

6.4.2 Type III solutions

The last family of solutions to consider is that of those with Ricci operator of Type III as portrayed in (6.7) or in (6.8), depending on whether ∇h is spacelike or lightlike. We already know that there are no solutions of this kind with ∇h timelike. In both possible cases, all solutions are realized on Kundt spacetimes, as shown in the following result.

Theorem 6.34. *Let (M, g, h) be a 4-dimensional solution of the weighted Einstein field equations* (4.6) *with harmonic curvature and Ricci operator of Type III. Then:*

1. *If $g(\nabla h, \nabla h) > 0$, (M, g) is a Kundt spacetime. Moreover, there exist local coordinates as in* (1.10) *where $h = h(v, x_1, x_2)$.*

2. *If $g(\nabla h, \nabla h) = 0$ then (M, g) is a Kundt spacetime. Moreover, the Ricci operator is 3-step nilpotent and there exist local coordinates as in* (1.10) *with $h = h(v)$.*

Proof. We assume (M, g, h) is a 4-dimensional solution with harmonic curvature and Ricci operator of Type III. We assume first that $g(\nabla h, \nabla h) > 0$. According to (6.7) and taking into account Lemma 6.10, there exists a suitable adapted frame $\mathcal{B}_2 = \{\nabla h, U, V, E_2\}$ on which the Ricci operator is given by

$$\operatorname{Ric} \nabla h = \lambda \nabla h, \quad \operatorname{Ric} U = \alpha U, \quad \operatorname{Ric} V = \alpha V + E_2 \quad \text{and} \quad \operatorname{Ric} E_2 = \alpha E_2 + U.$$

The treatment of solutions of this type is similar to that of the previous case. However, proving that the gradient of the Ricci eigenvalues has no component in $\nabla h^{\perp}$ is simpler. Indeed, we have

$$\begin{aligned}(\nabla_{\nabla h}\rho)(U, \nabla h) &= \nabla h(\rho(U, \nabla h)) - \rho(\nabla_{\nabla h} U, \nabla h) - \rho(U, \nabla_{\nabla h} \nabla h) \\ &= -\lambda g(\nabla_{\nabla h} U, \nabla h) - \alpha g(U, \nabla_{\nabla h} \nabla h) = 0,\end{aligned}$$

where we have used $\nabla_{\nabla h} \nabla h = h(\lambda - 2J)\nabla h$ and $g(U, \nabla h) = 0$. Now, we can write

$$\begin{aligned}(\nabla_U \rho)(\nabla h, \nabla h) &= U(\rho(\nabla h, \nabla h)) - 2\rho(\nabla_U \nabla h, \nabla h) \\ &= U(\lambda)\|\nabla h\|^2 + \lambda U(g(\nabla h, \nabla h)) - 2\lambda g(\nabla_U \nabla h, \nabla h) \\ &= U(\lambda)\|\nabla h\|^2.\end{aligned}$$

Since the Ricci tensor is Codazzi and $\|\nabla h\|^2 > 0$, it follows that $U(\lambda) = 0$. Moreover, $\tau = \lambda + 3\alpha$ is constant by Lemma 4.5, so $U(\alpha) = 0$. We can similarly compute the analogous covariant derivatives for V and E_2 instead of U, yielding $V(\lambda) = V(\alpha) = 0$ and $E_2(\lambda) = E_2(\alpha) = 0$.

In order to show that (M, g) is a Kundt spacetime, we are going to see that the lightlike vector U satisfies

$$\nabla_U U = 0, \qquad \nabla_{\nabla h} U = g(\nabla_{\nabla h} U, V)U, \qquad \nabla_{E_2} U = g(\nabla_{E_2} U, V)U.$$

The process of computing the covariant derivatives of the Ricci tensor and applying the Codazzi condition is the same as in previous instances, so we omit details. First, we compute $(\nabla_U \rho)(U, V) = -g(\nabla_U U, E_2)$ and $(\nabla_V \rho)(U, U) = 0$, so $g(\nabla_U U, E_2) = 0$. Similarly, we have

$$(\nabla_U \rho)(E_2, V) = g(\nabla_U U, V) \quad \text{and} \quad (\nabla_V \rho)(E_2, U) = 0,$$

so $g(\nabla_U U, V) = 0$. The component in the direction of ∇h is easier to compute: $g(\nabla_U U, \nabla h) = -\operatorname{Hes}_h(U, U) = 0$. Similarly, since U is lightlike, it is immediate that $g(\nabla_U U, U) = 0$. Thus, we have proved that $\nabla_U U = 0$.

Now, we need to compute $\nabla_{\nabla h} U$ and $\nabla_{E_2} U$. For the first derivative, we write $g(\nabla_{\nabla h} U, U) = 0$ and $g(\nabla_{\nabla h} U, \nabla h) = -\operatorname{Hes}_h(U, \nabla h) = 0$. Therefore, we only need to determine the component given by $g(\nabla_{\nabla h} U, E_2)$. To that end, consider the covariant derivatives $(\nabla_{\nabla h}\rho)(U, V) = \nabla h(\alpha) - g(\nabla_{\nabla h} U, E_2)$ and $(\nabla_V \rho)(\nabla h, U) = h(\alpha - 2J)(\lambda - \alpha)$. Moreover, $h(\alpha - 2J) = -\frac{1}{3} h\lambda$ so, by the Codazzi condition on the Ricci tensor, we have

$$\nabla h(\alpha) = g(\nabla_{\nabla h} U, E_2) + \frac{h}{3}(\alpha - \lambda)\lambda. \tag{6.28}$$

On the other hand, we can also compute $(\nabla_{\nabla h}\rho)(E_2, E_2) = \nabla h(\alpha) + 2g(E_2, \nabla_{\nabla h} U)$ and $(\nabla_{E_2}\rho)(\nabla h, E_2) = \frac{h}{3}(\lambda - \alpha)\lambda$, so

$$\nabla h(\alpha) = -2g(\nabla_{\nabla h} U, E_2) + \frac{h}{3}(\alpha - \lambda)\lambda. \tag{6.29}$$

Combining (6.28) and (6.29), it follows that $g(\nabla_{\nabla h} U, E_2) = 0$.

Finally, for the derivative $\nabla_{E_2} U$, since $g(\nabla_{E_2} U, \nabla h) = -\operatorname{Hes}_h(E_2, U) = 0$, we only need to compute $g(\nabla_{E_2} U, E_2)$. To that end, we use the fact that $(\nabla_{E_2}\rho)(V, U) = g(E_2, \nabla_{E_2} U) = (\nabla_V \rho)(E_2, U) = 0$.

Thus, we can write $\nabla_{\nabla h} U = g(\nabla_{\nabla h} U, V) U$ and $\nabla_{E_2} U = g(\nabla_{E_2} U, V) U$. In summary, we have $\nabla_U U = 0$ and that, for every $X \perp U$, $\nabla U = \omega \otimes U$, for the 1-form ω defined in $U^\perp$, and given by $\omega(U) = 0$, $\omega(\nabla h) = g(\nabla_{\nabla h} U, V)$ and $\omega(E_2) = g(\nabla_{E_2} U, V)$. By the characterization given by (1.8), the underlying manifold (M, g) is a Kundt spacetime where U is the distinguished lightlike vector field.

Additionally, in local coordinates (1.10), the distinguished lightlike geodesic vector field is ∂_u, which is orthogonal to ∇h. Hence $h = h(v, x_1, x_2)$ and Theorem 6.34 (1) follows.

Now, if $g(\nabla h, \nabla h) = 0$ on an open subset, then the Kundt character and the nilpotency of Ric follow from Theorem 5.4. In local coordinates as in (1.10), since the distinguished geodesic lightlike vector field in these coordinates is ∂_u, we have that $\nabla h \parallel \partial_u$, so a direct computation yields $h(u, v, x_1, x_2) = h(v)$. □

Example 6.35. The structure of isotropic solutions with 3-step nilpotent Ricci operator is very rigid. However, we can build examples for any arbitrary nowhere constant density function $h = h(v)$. Consider the Kundt metric given by

$$g = dv\,(2du + F(u, v, x)dv + 2\omega(u, v, x)dx_1) + g(v, x)(dx_1^2 + dx_2^2),$$

with

$$\begin{aligned} F(u, v, x_1, x_2) &= \tfrac{u^2 h(v)^4}{C x_1^2} - \tfrac{12 C x_1^2(\log(x_1) - 1) h'(v)^2}{h(v)^6} \\ \omega(u, v, x_1, x_2) &= -\tfrac{3 C x_1 h'(v)}{h(v)^5} - \tfrac{2u}{x_1}, \qquad g(v, x_1, x_2) = \tfrac{C}{h(v)^4}, \end{aligned}$$

where $C \neq 0$. This is an isotropic solution for the weighted Einstein field equations where Ric has 3-step nilpotent Ricci operator and harmonic Weyl tensor, but it is not locally conformally flat since the Weyl tensor does not vanish (for example, $W(\partial_u, \partial_v, \partial_v, \partial_{x_1}) = \frac{3h'(v)}{2x_1 h(v)} \neq 0$).

6.5 The complete classification result

After analyzing solutions with all four admissible structures (Types I.a and I.b, Type II and Type III), we summarize the results obtained in Sections 6.2–6.4 to give the complete statement of the main theorem of this chapter.

Theorem 6.36. *Let (M, g, h) be a 4-dimensional solution of the weighted Einstein field equations* (4.6) *such that (M, g) has harmonic curvature tensor (not locally conformally flat). Assume that the Jordan normal form of the Ricci operator* Ric *is constant in M. Then, the eigenvalues of* Ric *are real and one of the following is satisfied:*

1. Ric *diagonalizes on (M, g) and $g(\nabla h, \nabla h) \neq 0$. Furthermore, there exists an open dense subset M_{Ric} of M such that, for every $p \in M_{\mathrm{Ric}}$, (M, g) is isometric on a neighborhood of p to:*
   - *(a) A direct product $I_2 \times \tilde{M}$, where $\tilde{M} = I_1 \times_\xi N$ is a warped product 3-dimensional solution with $\tilde{\tau} = 0$ and N a surface of constant Gauss curvature.*
   - *(b) A direct product $N_1 \times N_2$ of two surfaces of constant Gauss curvature $\frac{\kappa}{2}$ and κ, respectively.*
2. *(M, g) is a Kundt spacetime and, depending on the causal character of ∇h, one of the following applies:*
   - *(a) If $g(\nabla h, \nabla h) = 0$, then* Ric *is nilpotent and ∇h determines the lightlike parallel line field. Moreover, if* Ric *vanishes or is 2-step nilpotent, the underlying manifold is a pp-wave.*
   - *(b) If $g(\nabla h, \nabla h) \neq 0$, then ∇h is spacelike and the distinguished lightlike vector field is orthogonal to ∇h.*

Proof. Let (M, g, h) be a 4-dimensional solution of the weighted Einstein field equations (4.6) such that (M, g) has harmonic curvature tensor (not locally conformally flat). Additionally, assume that Ric does not change type in M.

For Type I.a (diagonalizable) solutions, applying Theorem 6.7 in the non-isotropic case gives Theorem 6.36 (1), whereas Ricci-flat isotropic solutions fall into Theorem 6.36 (2.a).

In the non-diagonalizable case, we first use Theorem 6.22 to prove that there are no solutions of Type I.b., so all remaining admissible solutions are of Type II or Type III. Hence, we apply Theorems 6.31 and 6.34 to complete Theorem 6.7 (2.a) in the isotropic case, and Theorem 6.7 (2.b) in the non-isotropic case. □

Chapter 7
Some notable examples

As a last aspect of our study of the weighted Einstein field equations, in this chapter we go over some remarks and classification results that will help us to better understand the geometry of solutions under some of the conditions considered throughout this dissertation. The results in Sections 7.1 and 7.2 are included in the work [23].

Outline of the chapter

We begin with a brief remark on the role of the cosmological term Λ on non-isotropic solutions. Then, in Sections 7.1 and 7.2, we move on to the main part of the chapter, which is the classification of solutions of the weighted field equations with vanishing cosmological term given by (4.6),

$$h\rho - \operatorname{Hes}_h + \Delta h g = 0,$$

realized on manifolds in the family of pure radiation waves. Tying into the discussion in Chapter 6, and given their significance in General Relativity, we focus on 4-dimensional spacetimes.

Throughout Section 7.1, we carry out a systematic search of solutions of (4.6) realized on 4-dimensional pr-waves and provide a complete classification (Theorem 7.2). As it turns out, the Ricci operator is necessarily nilpotent for any solution, not only for isotropic ones (recall that this is guaranteed by Theorem 5.4); and there are no solutions with ∇h timelike. We also discuss the existence of solutions with different metric-measure structures given by the aforementioned classification. Finally, in Section 7.2 we provide some examples of solutions with special geometric features. Firstly, we prove a stronger rigidity result (Corollary 7.5) for 4-dimensional pr-wave solutions with harmonic curvature, which are linked to the results in Chapter 6. Secondly, in Section 7.2.2 we build some geodesically complete families of solutions realized on relevant types of plane waves, such as Cahen-Wallach symmetric spaces.

A remark on the role of the cosmological constant

In Chapter 6, we focused on the field equations (4.6) without cosmological term, i.e., $\Lambda = 0$. Also, due to Lemma 5.2, if a solution to the weighted Einstein field equation is isotropic, then it always has vanishing cosmological constant. Nevertheless, this implication does not hold if ∇h is not lightlike. Furthermore, any value of Λ can be realized by appropriate local solutions.

Indeed, consider an n-dimensional Einstein manifold (M, g), with $\rho = \frac{\tau}{n} g$, that satisfies equation (4.8), then

$$\operatorname{Hes}_h = \left(\frac{h\tau}{n} + \Delta h + \Lambda \right) g.$$

Notice that solutions of this equation are necessarily solutions of the local Möbius equation $\mathrm{Hes}_h = \frac{\Delta h}{n} g$ (see Section 1.4 for details on the significance of this geometric equation and its relation to the generalized Obata equation).

For illustrative purposes, since 3-dimensional Einstein manifolds have constant sectional curvature, one can solve the local Möbius equation on the de Sitter and the Anti-de-Sitter spacetimes of dimension three to provide simple examples of solutions of (4.6) with $\Lambda \neq 0$ as follows:

1. We consider the de Sitter space with coordinates (x, y, z) and metric

$$g_{dS} = \kappa^2 \left(-\cos^2 y dx^2 + dy^2 + \sin^2 y dz^2\right).$$

The scalar curvature is given by $\tau = \frac{6}{\kappa^2}$. A direct calculation shows that a function of the form $h(x, y, z) = -\frac{\kappa^2 \Lambda}{2} + \sin(y)(c_1 \cos(z) + c_2 \sin(z))$ gives solutions to the weighted Einstein field equations for constants c_1, c_2. Since

$$\begin{aligned}\|\nabla h\|^2 = \tfrac{1}{\kappa^2} (\cos^2(y)\,(c_2 \sin(z) + c_1 \cos(z))^{\,2} \\ + (c_2 \cos(z) - c_1 \sin(z))^{\,2}) > 0\end{aligned}$$

the gradient of h is spacelike. Thus, there exist local solutions of (4.8) for arbitrary Λ.

2. We consider the Anti-de Sitter space with coordinates (x, y, z) and metric

$$g_{AdS} = \kappa^2 \left(-\cosh^2 y dx^2 + dy^2 + \sinh^2 y dz^2\right).$$

The scalar curvature is given by $\tau = -\frac{6}{\kappa^2}$. Functions of the form $h(x, y, z) = \frac{\kappa^2 \Lambda}{2} + \sinh(y)(c_1 \cos(z) + c_2 \sin(z))$ provide solutions to (4.8) for constants c_1 and c_2. Note that the gradient of h is always spacelike, since

$$\begin{aligned}\|\nabla h\|^2 = \tfrac{1}{\kappa^2} \left(\cosh^2(y)\,(c_2 \sin(z) + c_1 \cos(z))^{\,2}\right. \\ \left. + (c_2 \cos(z) - c_1 \sin(z))^{\,2}\right) > 0.\end{aligned}$$

Therefore, there are solutions with spacelike ∇h for arbitrary Λ in this spacetime as well.

7.1 Solutions realized on pure radiation waves

In Chapters 5 and 6, we showed how several families of Lorentzian manifolds with a distinguished lightlike vector field, such as *pp*-waves or, more broadly, Kundt spacetimes, play an essential role in our classification results for solutions of the weighted Einstein field equations (4.6). Due to the complexity of the general Kundt metric (1.10), attempting to classify all solutions realized on manifolds of this type becomes unmanageable, even in the 4-dimensional case. Nevertheless, there are simpler subfamilies of Kundt spacetimes which are still physically and geometrically significant and therefore provide interesting examples. With this in mind, we consider the family of 4-dimensional *pr*-waves (see Section 1.3). Recall from (1.13) that the metric for this family of Brinkmann waves can be written in suitable local coordinates (u, v, x, y) as

$$g = 2dudv + F(u, v, x, y)dv^2 + dx^2 + dy^2. \tag{7.1}$$

Like in the rest of Part II of this thesis, our analysis is local in nature, and we work in open sets where $\nabla h \neq 0$ and its causal character remains invariant.

We begin our arguments to provide the classification of 4-dimensional solutions realized on pr-waves with the following lemma, which gives the general form of the metric and the density function of every possible solution.

Lemma 7.1. *Let (M, g, h) be a 4-dimensional, non-flat solution of the weighted Einstein field equations* (4.6)*, realized on a pr-wave with metric given by* (7.1) *in the coordinates (u, v, x, y). Then, the Ricci operator is nilpotent, and the density function h takes the form*

$$h(v, x, y) = h_x(v)x + h_y(v)y + h_0(v), \tag{7.2}$$

while the function F that defines the pr-wave is given by

$$F(u, v, x, y) = F_1(v, x, y)u + F_0(v, x, y), \tag{7.3}$$

for suitable smooth functions h_x, h_y, F_1 and F_0.

Proof. Consider coordinates (u, v, x, y) so that the metric of the pr-wave is written as in (7.1). Then, the scalar curvature of the manifold is given by $\tau = \partial_u^2 F$. Since, by Lemma 4.5, τ is constant for any solution, it follows that F takes the form $F(u, v, x, y) = \frac{\tau}{2}u^2 + F_1(v, x, y)u + F_0(v, x, y)$. We simplify notation and denote $G^h(\partial_u, \partial_u) = G^h_{uu}$, $G^h(\partial_u, \partial_x) = G^h_{ux}$, $G^h(\partial_x, \partial_y) = G^h_{xy}$ and so on to compute the following components of the weighted Einstein tensor:

$$G^h_{uu} = -\partial_u^2 h, \quad G^h_{ux} = -\partial_u\partial_x h, \quad G^h_{uy} = -\partial_u\partial_y h, \quad G^h_{xy} = -\partial_x\partial_y h.$$

Hence h can be written as $h(u, v, x, y) = h_1(v)u + h_x(v, x) + h_y(v, y)$ for certain h_1, h_x and h_y. Now, take the component $G^h_{yy} = -h_1(\tau u + F_1) + 2h_1' + \partial_x^2 h_x$. Differentiate with respect to u to find $0 = \partial_u G^h_{yy} = -\tau h_1$. On the other hand, we compute the component

$$2G^h_{vx} = -2\partial_v\partial_x h_x + (h_x + h_y + 2uh_1)\partial_x F_1 + h_1\partial_x F_0,$$

hence $0 = \partial_u G^h_{vx} = h_1\partial_x F_1$. Similarly, we find $0 = \partial_u G^h_{vy} = h_1\partial_y F_1$. Thus, there are two possibilities: either $h_1 = 0$ or $h_1 \neq 0$, $\tau = 0$ and $F_1 = F_1(v)$. We analyze them separately.

Case 1: $h_1 \neq 0$, $\tau = 0$ and $F_1 = F_1(v)$

We will see that this case results in a flat manifold. Let $h_1 \neq 0$, $\tau = 0$ and $F_1 = F_1(v)$. The only non-vanishing components (up to symmetries) of the curvature tensor for a pr-wave with $F(u, v, x, y) = F_1(v)u + F_0(v, x, y)$ are

$$\begin{aligned} 2R(\partial_x, \partial_v, \partial_v, \partial_x) &= \partial_x^2 F_0, \quad 2R(\partial_y, \partial_v, \partial_v, \partial_y) = \partial_y^2 F_0, \\ 2R(\partial_y, \partial_v, \partial_v, \partial_x) &= \partial_x\partial_y F_0. \end{aligned}$$

Moreover, the weighted Einstein field equations yield $0 = 2\partial_x G^h_{vy} = h_1 \partial_x \partial_y F_0$. Since $h_1 \neq 0$, we have $R(\partial_y, \partial_v, \partial_v, \partial_x) = 0$. Additionally, we compute

$$\begin{aligned} 0 &= G^h_{xx} = -h_1 F_1 + 2h_1' + \partial_y^2 h_y, \\ 0 &= G^h_{yy} = -h_1 F_1 + 2h_1' + \partial_x^2 h_x, \\ 0 &= 2G^h_{uv} = -h_1 F_1 + 2h_1' + 2\partial_x^2 h_x + 2\partial_y^2 h_y. \end{aligned}$$

On the one hand, $0 = G^h_{xx} - G^h_{yy} = \partial_y^2 h_y - \partial_x^2 h_x$ and, on the other hand, $G^h_{xx} + G^h_{yy} - 4G^h_{uv} = -3\left(\partial_x^2 h_x + \partial_y^2 h_y\right)$, so $\partial_x^2 h_x = \partial_y^2 h_y = 0$. Now, we compute $0 = 2\partial_x G^h_{vx} = h_1 \partial_x^2 F_0$ and $0 = 2\partial_y G^h_{vy} = h_1 \partial_y^2 F_0$. Since $h_1 \neq 0$, we obtain that $\partial_x^2 F_0 = \partial_y^2 F_0 = 0$ and, hence, $R(\partial_x, \partial_v, \partial_v, \partial_x) = R(\partial_y, \partial_v, \partial_v, \partial_y) = 0$. Thus, all components of the curvature tensor vanish and the underlying manifold (M, g) is flat, contrary to our assumption.

Case 2: $h_1 = 0$

Then, by the weighted Einstein tensor components $G^h_{xx} = \partial_y^2 h_y$ and $G^h_{yy} = \partial_x^2 h_x$, we find that h takes the form in (7.2). Moreover, from $G^h_{uv} = \frac{1}{2}\tau h$, and since $h > 0$, we have $\tau = 0$, so F is given by (7.3). For this form of F, the Ricci operator is nilpotent. □

With this lemma, we are in a position to prove the main classification result of this chapter.

Theorem 7.2. *Let (M, g, h) be a 4-dimensional, non-flat solution of the weighted Einstein field equations* (4.6) *realized on a pr-wave. Then ∇h is lightlike or spacelike, and the following holds:*

1. *If ∇h is lightlike, then (M, g) is a pp-wave with harmonic curvature. Moreover, there exist local coordinates as in* (7.1) *with $\partial_x^2 F + \partial_y^2 F = \gamma(v)$ and $\partial_u F = 0$ such that the density function $h = h(v)$ satisfies $2h'' + h\gamma = 0$.*

2. *If ∇h is spacelike, then* Ric *is nilpotent and, moreover:*

   (a) If Ric *is 2-step nilpotent, then (M, g) is a pp-wave. Moreover, there exist coordinates as in* (7.1) *with $\partial_u F = 0$ and density function of the form $h(v, x, y) = h_0(v) + (x + Ay)h_x$, with $A \in \mathbb{R}$ and $h_x \neq 0$, satisfying*

   $$0 = -2G^h_{vv} = 2h_0'' + h_x(\partial_x F + A\partial_y F) + h(\partial_x^2 F + \partial_y^2 F). \tag{7.4}$$

   (b) If Ric *is 3-step nilpotent, then there exist (u, v, x, y) such that the density function takes the form $h(v, x, y) = h_0(v) + (x + Ay)h_x(v)$, with $A \in \mathbb{R}$ and $h_x' \neq 0$, and the metric is as in* (7.1) *with*

   $$F(u, v, x, y) = F_0(v, x, y) + u\left(\frac{2h_x'(v)\log(h(v, x, y))}{h_x(v)} + \alpha(v)\right)$$

satisfying

$$\begin{aligned} 0 &= -2h_x G^h_{vv} \\ &= 2h_x'(h_0' + (x+Ay)h_x')\log(h_0 + (x+Ay)h_x) \\ &\quad + h_x^2(\partial_x F_0 + A\partial_y F_0 + (x+Ay)(\partial_x^2 F_0 + \partial_y^2 F_0)) \\ &\quad + h_x(\alpha(h_0' + (x+Ay)h_x') + 2h_0'' + 2(x+Ay)h_x'') \\ &\quad + h_x h_0(\partial_x^2 F_0 + \partial_y^2 F_0). \end{aligned} \tag{7.5}$$

Proof. From Lemma 7.1, h and F are given by (7.2) and (7.3), respectively. A direct computation shows that $\|\nabla h\|^2 = h_x^2 + h_y^2$, so ∇h cannot be timelike. Thus, assume first that (M, g, h) is an isotropic solution, so we get that $h_x = h_y = 0$ and $h(u, v, x, y) = h(v)$. Now, we see that

$$2G^h_{vx} = h\,\partial_x F_1, \text{ and } 2G^h_{vy} = h\,\partial_y F_1,$$

so $\partial_x F_1 = \partial_y F_1 = 0$ and $F_1(v, x, y) = F_1(v)$. But this implies that the only non-vanishing component of the Ricci operator is $\mathrm{Ric}(\partial_v) = -\frac{1}{2}\left(\partial_x^2 F_0 + \partial_y^2 F_0\right)\partial_u$, so Ric is 2-step nilpotent and the pr-wave is indeed a pp-wave (see [86]). Thus, there exist specific coordinates (7.1) with $\partial_u F = 0$, where the only non-zero component of G^h is $2G^h(\partial_v, \partial_v) = -2h'' - h\left(\partial_x^2 F + \partial_y^2 F\right)$, so $\partial_x^2 F + \partial_y^2 F = -\frac{2h''(v)}{h(v)} = \gamma(v)$. This is a sufficient condition for a pp-wave to have harmonic curvature (see [19]), and Theorem 7.2 (1) follows.

Assume now that ∇h is spacelike. Recall that, by Lemma 7.1, h and F are given by (7.2) and (7.3), respectively. Since $\|\nabla h\|^2 = h_x^2 + h_y^2$, we assume without loss of generality that $h_x \neq 0$ (otherwise, interchange the x and y coordinates). Under this condition, we compute the component $0 = G^h_{vx} = -h_x' + \frac{1}{2}(h_0 + xh_x + yh_y)\partial_x F_1$ of the field equations. We solve this PDE to find

$$F_1(v, x, y) = \alpha(v, y) + 2\frac{\log(h_0(v) + xh_x(v) + yh_y(v))h_x'(v)}{h_x(v)}.$$

For this form of F_1, we compute $0 = 2\partial_x G^h_{vy} = h_x\partial_y\alpha$ so $\alpha = \alpha(v)$. Moreover, in this case $0 = G^h_{vy} = \frac{h_y h_x'}{h_x} - h_y'$. Hence, we have $h_y(v) = Ah_x(v)$ for some $A \in \mathbb{R}$ (this includes the case $h_y = 0$). With this, all components of the weighted Einstein tensor vanish, except for G^h_{vv}, which is given by expression (7.5).

The Ricci operator is given by

$$\begin{aligned} \mathrm{Ric}(\partial_u) &= 0, & \mathrm{Ric}(\partial_v) &= \star\partial_u + \tfrac{h_x'}{h}\partial_x + \tfrac{Ah_x'}{h}\partial_y, \\ \mathrm{Ric}(\partial_x) &= \tfrac{h_x'}{h}\partial_u, & \mathrm{Ric}(\partial_y) &= \tfrac{Ah_x'}{h}\partial_u, \end{aligned}$$

where the expression of the coefficient $\star$ is irrelevant for the nilpotency of Ric. Notice that Ric is 3-step nilpotent if and only if $h_x' \neq 0$. This case corresponds to Theorem 7.2 (2.b).

Now, assume $h_x' = 0$. In this case, the Ricci operator becomes 2-step nilpotent, so the pr-wave is indeed a pp-wave and we can assume $\partial_u F = 0$. Finally, a straightforward calculation shows that the only remaining non-vanishing component of the weighted Einstein field equations is given by (7.4). □

In view of the classification in Theorem 7.2, the Ricci operator is necessarily nilpotent for any solution. Moreover, due to the structure of self-adjoint operators in Lorentzian vector spaces (see Section 1.1.3), the degree of nilpotency of a self-adjoint operator is three at most, irrespective of the dimension of the underlying space. For these solutions, on the one hand, Theorem 7.2 (1) and (2.a) result in examples with 2-step nilpotent Ricci operator and, on the other hand, Theorem 7.2 (2.b) results in examples with 3-step nilpotent Ricci operator. Therefore, all possible cases of nilpotency are exhausted for 4-dimensional non-isotropic solutions realized on pr-waves.

Remark 7.3. Note, from Theorem 7.2 (1), that any isotropic solution on a pr-wave is a pp-wave with harmonic curvature. Furthermore, any pp-wave with harmonic curvature gives rise to an isotropic solution, since there always exists a local solution of the ODE $2h'' + h\gamma = 0$ for a given $\gamma(v) = \partial_x^2 F + \partial_y^2 F$. In particular, if γ is constant, the ODE reduces to the harmonic oscillator equation. Thus, for $\gamma < 0$, the density function takes the form $h(v) = Ae^{\frac{\sqrt{-\gamma}v}{\sqrt{2}}} + Be^{-\frac{\sqrt{-\gamma}v}{\sqrt{2}}}$, so the solution can be extended to all $\mathbb{R}$ for appropriate values of A and B. We refer to the next section for examples of geodesically complete solutions.

Remark 7.4. Non-isotropic solutions on pr-waves are described in Theorem 7.2 (2). Those with 2-step nilpotent Ricci operator are realized on pp-waves, but not every pp-wave gives rise to a solution, since equation (7.4) imposes restrictions on the function F. However, given a (real) analytic density function of the form $h = h_0(v) + (x + Ay)h_x$, there always exist local analytic solutions of (7.4). Indeed, it is a second order quasi-linear PDE that we can write as

$$\partial_x^2 F = -\frac{1}{h}\left(2h_0'' + h_x(\partial_x F + A\partial_y F)\right) - \partial_y^2 F. \tag{7.6}$$

We consider the non-characteristic hypersurface $x = 0$ for this PDE and set analytic initial data $F\big|_{x=0} = \varphi_0$ and $\partial_x F\big|_{x=0} = \varphi_1$. Now, the Cauchy-Kovalevskaya Theorem guarantees that there exists an analytic solution F to equation (7.6) (see, for example, [58]), thus giving rise to a solution with the prescribed density.

The situation is similar for 3-step nilpotent solutions that are realized on pr-waves. Although not every pr-wave results in a solution of the weighted field equations, for a given analytic density function of the form $h(v, x, y) = h_0(v) + (x + Ay)h_x(v)$, there exist forms of F that give rise to pr-waves (M, g) so that (M, g, h) is a solution of (4.6). Indeed, since $h_x \neq 0$ and $h(v, x, y) = h_0(v) + h_x(v)(x + Ay) > 0$, the hypersurface $x = 0$ is non-characteristic for the PDE (7.5). Thus, for analytic functions h_0, h_x, α and analytic boundary conditions, due to the Cauchy-Kovalevskaya Theorem, there is a unique local analytic solution of the corresponding Cauchy problem.

7.2 Solutions with special geometric features

Our purpose in this section is to use the local description in Theorem 7.2 to provide explicit examples of solutions on pr-waves that show different properties and behavior. In the 4-dimensional case, non-isotropic solutions with harmonic curvature either present specific product structures or are realized on Kundt spacetimes (see Theorem 6.36), with the latter case being less rigid. Therefore, to improve our understanding of these Kundt solutions, we begin by focusing on classifying all pr-waves which give rise to a solution with this curvature property. Then, we turn our attention to examples of solutions realized on geodesically complete families of pr-waves (such as Cahen-Wallach spacetimes), and explain obstructions to geodesic completeness for non-isotropic solutions.

7.2.1 pr-waves with harmonic curvature

By Theorem 7.2 (1), isotropic solutions on pr-waves have harmonic curvature. This is not the case in general for non-isotropic solutions, so we give a stronger rigidity result, with an explicit description in local coordinates, for solutions on pr-waves which have harmonic curvature.

Corollary 7.5. *Let (M, g, h) be a 4-dimensional, non-flat solution of the weighted Einstein field equations, realized on a pr-wave with harmonic curvature. Then the following holds:*

1. *If ∇h is lightlike, then (M, g) is a pp-wave. In coordinates as in (7.1) with $\partial_u F = 0$, F satisfies $\partial_x^2 F + \partial_y^2 F = \gamma(v)$ and the density $h = h(v)$ is subject to the ODE $2h'' + h\gamma = 0$.*
2. *If ∇h is spacelike, then (M, g) is a plane wave. Moreover, the metric in (7.1) takes the form*
$$F(v, x, y) = F_x(v)x^2 + F_y(v)y^2 - 2A(F_x(v) + 2F_y(v))xy$$
with $A \in \mathbb{R}$ and F_x, F_y functions subject to the relation $(2 - A^2)F_x + (1 - 2A^2)F_y = 0$. The density function has the form $h(u, v, x, y) = h_0(v) + (x + Ay)h_x$ with h_0 obeying $h_0''(v) + (F_x(v) + F_y(v))h_0(v) = 0$.

Proof. It was shown in the proof of Theorem 7.2 (1) that isotropic solutions of the field equations are realized on manifolds with harmonic curvature, so Corollary 7.5 (1) holds and we focus on the case with ∇h spacelike. From Theorem 7.2, the Ricci operator is 2 or 3-step nilpotent. If Ric is 3-step nilpotent, then F takes the form given in Theorem 7.2 (2.b). In this case, we compute

$$0 = (\nabla_{\partial_u}\rho)(\partial_v, \partial_v) - (\nabla_{\partial_v}\rho)(\partial_u, \partial_v) = \frac{(A^2 + 1)\, h_x(v)h_x'(v)}{h^2}.$$

Hence $h_x' = 0$ and we conclude that the Ricci operator is 2-step nilpotent. Since the image of Ric is totally isotropic, the underlying manifold is a pp-wave (see Section 1.3).

Thus, adopting the notation in Theorem 7.2 (2.a), we take coordinates (u, v, x, y) as in (7.1) with $\partial_u F = 0$, such that the density function has the form $h(u, v, x, y) = h_0(v) + (x + Ay)h_x$ with $h_x \neq 0$. Recall that, in these coordinates, the curvature is harmonic if and only if $\partial_x^2 F + \partial_y^2 F = \beta(v)$, for an arbitrary function $\beta \neq 0$.

Moreover, the only non-vanishing term of the weighted Einstein tensor is given by (7.4). Using the fact that $\partial_x^2 F + \partial_y^2 F = \beta(v)$, the crucial equation becomes $2h_0'' + h_x(\partial_x F + A\partial_y F) + h\beta = 0$. Differentiating with respect to x and y yields

$$0 = h_x(A\partial_x\partial_y F + (2\partial_x^2 F + \partial_y^2 F)), \quad 0 = h_x(\partial_y\partial_x F + A(2\partial_y^2 F + \partial_x^2 F)). \tag{7.7}$$

Since $h_x \neq 0$, we get that $A\partial_x\partial_y F + (2\partial_x^2 F + \partial_y^2 F) = 0$ and that $\partial_y\partial_x F + A(2\partial_y^2 F + \partial_x^2 F) = 0$. If $A = 0$, then $\partial_y\partial_x F = 0$, while if $A \neq 0$, combining both equations yields $(1 + A^2)\partial_x\partial_y F + 3A\beta = 0$, and hence $\partial_x\partial_y F = -\frac{3A\beta(v)}{1+A^2}$. Thus, for any value of A, we have $\partial_x^2\partial_y F = \partial_x\partial_y^2 F = 0$. Moreover, differentiating $A\partial_x\partial_y F + (2\partial_x^2 F + \partial_y^2 F) = 0$ with respect to x and y, we get that $\partial_x^3 F = \partial_y^3 F = 0$ too.

It follows that F is a polynomial of order two in the variables x, y, whose coefficients are smooth functions of v. Hence, the underlying manifold is a plane wave, and can be further normalized so that

$$F(v, x, y) = F_y(v)y^2 + F_x(v)x^2 + F_{xy}(v)xy$$

for some smooth functions F_y, F_x and F_{xy}. With this, from (7.7) we get

$$0 = AF_{xy} + 2(2F_x + F_y), \quad 0 = F_{xy} + 2A(2F_y + F_x).$$

We can solve the second equation above to find $F_{xy} = -2A(2F_y + F_x)$, and substituting this into the first one yields $(2 - A^2)F_x + (1 - 2A^2)F_y = 0$.

Now, equation (7.4) reduces to

$$0 = h_0''(v) + (F_x + F_y)h_0(v) - h_x x\left(\left(A^2 - 2\right)F_x + \left(2A^2 - 1\right)F_y\right),$$

and, since $(2 - A^2)F_x + (1 - 2A^2)F_y = 0$, we have that h_0 satisfies $h_0''(v) + (F_x + F_y)h_0(v) = 0$. This completes the proof of Corollary 7.5 (2). □

7.2.2 Geodesically complete solutions

The family of plane waves as portrayed in Section 1.3 appears in Corollary 7.5 (2), since every pr-wave with harmonic curvature that results in a non-isotropic solution of the field equations is indeed a plane wave. Inspired by this fact and taking into account that plane waves on $\mathbb{R}^n$ are geodesically complete (see [30]), one may look for global solutions in this context. However, one of the difficulties in finding global solutions of the weighted Einstein field equations is that some geodesically complete spacetimes do not admit a globally defined density function. Indeed, notice that non-isotropic solutions described in Corollary 7.5 (2) cannot be extended to all $(x, y) \in \mathbb{R}^2$. In this section we illustrate this fact and provide some global examples.

Cahen-Wallach spaces

Cahen-Wallach spaces are the only indecomposable but not irreducible symmetric manifolds (see [27, 28]). They are plane waves, hence geodesically complete, and can be written in coordinates (u, v, x, y) on $\mathbb{R}^4$ as in (7.1) with $F(v, x, y) = ax^2 + by^2$. Since Cahen-Wallach spaces are symmetric, they have harmonic curvature and we can apply Corollary 7.5 directly to obtain the following families of solutions:

- *Global isotropic solutions.* The density function $h = h(v)$ satisfies the equation $0 = h'' + (a+b)h$. Thus, only spacetimes with $a+b < 0$ result in vacuum global solutions for an appropriate density function. Indeed, $a + b \geq 0$ yields densities which turn nonpositive for certain values of v. Moreover if $a+b<0$, h has the form $h(v)=c_1e^{v\sqrt{-a-b}}+c_2e^{-v\sqrt{-a-b}}$, $c_1,c_2\in\mathbb{R}$. To define a global solution we can take $c_1,c_2\geq 0$ (allowing for c_1 or c_2 to vanish) so as to keep $h>0$ for all $v\in\mathbb{R}$.

- *Local non-isotropic solutions.* By virtue of Corollary 7.5 (2), the form of F is restricted to $F(v,x,y)=a(x^2-2y^2)$ and the density function takes the form $h(v,x)=h_0(v)+h_x x$ with $h_x\neq 0$ and $0=h_0''-ah_0$. Thus, for a fixed value of v, the density turns non-positive for large enough values of x. Hence, only local solutions are admissible.

A family of manifolds with recurrent curvature

Consider a plane wave on $\mathbb{R}^4$, given by the metric (7.1) with

$$F(v,x,y)=f(v)(ax^2+by^2), \tag{7.8}$$

in the usual coordinates (u,v,x,y), for a certain non-constant function $f(v)$. One characteristic property of these spacetimes is that they have recurrent curvature, this is, $\nabla R=\sigma\otimes R$ for a 1-form σ, but they are not locally symmetric (see [64, 115]). Whenever $a=b$, the resulting manifolds are locally conformally flat and they are referred to as *Egorov spaces*. Note that Egorov spaces such that $f(v)$ is a constant are Cahen-Wallach manifolds belonging to the family of *ε-spaces*. Egorov spaces and ε-spaces are notable examples of Lorentzian manifolds with large isometry groups (see [29]).

Spacetimes given by (7.8) are not homogeneous in general, but they have harmonic curvature tensor, so we can apply Corollary 7.5 to find the following families of solutions:

1. *Global isotropic solutions.* For a density function of the form $h=h(v)$ we have that ∇h is lightlike. Moreover, the only non-vanishing component of (4.6) becomes (see Corollary 7.5 (1))

$$0=G^h_{vv}=-h''-(a+b)fh. \tag{7.9}$$

Hence, we can choose appropriate values of f that provide global solutions. For example, we consider the following:

(a) Let $h=h(v)$ and set $f(v)=\frac{1}{h(v)}$. Then (7.9) reduces to

$$0=-G_{vv}=(a+b)+h''(v).$$

Thus, $h(v)=-\frac{a+b}{2}v^2+c_1v+c_2$ is the general solution of the ODE. Choosing $a+b<0$ and $c_2>-\frac{c_1^2}{2(a+b)}$, results in $h(v)>0$ for all $v\in\mathbb{R}$, giving rise to a global solution.

(b) Let $h=h(v)$ and $f(v)=-\frac{1+(a+b)e^{4v}}{a+b}$. Now, the ODE (7.9) is

$$h(v)\left(1+(a+b)e^{4v}\right)-h''(v)=0.$$

The general solution of this ODE for $a+b>0$ is

$$h(v) = e^{-v}\left(c_1 \cosh\left(\frac{1}{2}e^{2v}\sqrt{a+b}\right) + c_2 \sinh\left(\frac{1}{2}e^{2v}\sqrt{a+b}\right)\right).$$

Thus, for $a+b>0$, we obtain geodesically complete solutions of (4.6) by taking $c_1 > |c_2|$.

2. *Local non-isotropic solutions.* By Corollary 7.5 (2), we have $F(v,x,y) = af(v)(x^2-2y^2)$ and the density function takes the form $h(v,x) = h_0(v) + h_x x$, with $h_x \neq 0$ and $h_0''(v) - af(v)h_0(v) = 0$. As for Cahen-Wallach spaces, for a fixed v, the density turns non-positive for large enough values of x, so only local solutions are admissible. For example, based on the form of solutions given in the isotropic case, we have:

   (a) Let $f(v) = \frac{1}{h_0(v)}$ to see that $F(v,x,y) = a\frac{x^2-2y^2}{\frac{a}{2}v^2+c_1v+c_2}$ and $h(v,x) = \frac{a}{2}v^2 + c_1 v + c_2 + h_x x$ define a local non-isotropic solution for $a \neq 0$ and $h_x \neq 0$.

   (b) Let $f(v) = \frac{1}{a} + e^{4v}$ for $a > 0$ to obtain that $F(v,x,y) = \left(1 + ae^{4v}\right)\left(x^2 - 2y^2\right)$ and $h(v) = e^{-v}\left(c_1 \cosh\left(\frac{1}{2}\sqrt{a}e^{2v}\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{a}e^{2v}\right)\right) + xh_x$ define non-isotropic solutions for positive values of x if $h_x > 0$.

Bibliography

[1] D. Bakry and M. Émery; Diffusions Hypercontractives, in: Séminaire de probabilités XIX, 1983/84, 177–206, Lecture Notes in Math. 1123, Springer, Berlin, 1985.

[2] D. Bakry and Z. Qian; Some new results on eigenvectors via dimension, diameter, and Ricci curvature. *Adv. Math.* **155** (1) (2000), 98–153.

[3] A.L. Besse; Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer-Verlag, Berlin, 1987.

[4] R. L. Bishop and B. O'Neill; Manifolds of negative curvature. *Trans. Amer. Math. Soc.* **145** (1969), 1–49.

[5] E. Blancheton; Equations aux variations de la Relativité Générale. *C. R. Acad. Sci. Paris* **245**, 284–287 (1957).

[6] M. Blau and M. O'Loughlin; Homogeneous plane waves. *Nucl. Phys. B* **654** (2003), 135–176.

[7] M. Boucetta, A. Meliani and A. Zeghib; Kundt Three Dimensional Left Invariant Spacetimes. *J. Math. Phys.* **63** (2022), 112501.

[8] J. P. Bourguignon; Une stratification de l'espace des structures riemanniennes. *Compos. Math.* **30** (1975), 1–41.

[9] J. Brännlund, A. Coley and S. Hervik; Supersymmetry, holonomy and Kundt spacetimes. *Class. Quantum Grav.* **25**, No. 19 (2008), 195007.

[10] H. W. Brinkmann; Einstein spaces which are mapped conformally on each other. *Math. Ann.* **94** (1925), 119–145.

[11] H. W. Brinkmann; Riemann spaces conformal to Einstein spaces. *Math. Ann.* **91** (1924), 269–278.

[12] M. Brozos-Vázquez, G. Calvaruso, E. García-Río and S. Gavino-Fernández; Three-dimensional Lorentzian homogeneous Ricci solitons. *Israel. J. Math.* **188** (2012), 385–403; Corrigendum **255** (2) (2023), 975–984.

[13] M. Brozos-Vázquez and E. García-Río; Four-dimensional neutral signature self-dual gradient Ricci solitons. *Indiana Univ. Math. J.* **65** (6) (2016), 1921–1943.

[14] M. Brozos-Vázquez, E. García-Río and S. Gavino-Fernández; Locally conformally flat Lorentzian quasi-Einstein manifolds. *Monatsh. Math.* **173** (2014), 175–186.

[15] M. Brozos-Vázquez, E. García-Río and S. Gavino-Fernández; Locally conformally flat Lorentzian gradient Ricci solitons. *J. Geom. Anal.* **23** (2013), 1196–1212.

[16] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević and R. Vázquez-Lorenzo; *The Geometry of Walker Manifolds*. Synthesis Lectures on Mathematics and Statistics, Springer, Cham (2009).

[17] M. Brozos-Vázquez, E. García-Río, P. Gilkey and X. Valle-Regueiro; A natural linear equation in affine geometry: the affine quasi-Einstein equation. *Proc. Amer. Math. Soc.* **146** (8) (2018), 3485–3497.

[18] M. Brozos-Vázquez, E. García Río and D. Mojón-Álvarez; Conformally weighted Einstein manifolds: The uniqueness problem. *J. Differential Equations* **450** (2026), 113711.

[19] M. Brozos-Vázquez, E. García-Río and X. Valle-Regueiro; Isotropic quasi-Einstein manifolds. *Class. Quantum Grav.* **36** (24) (2019), 245005.

[20] M. Brozos-Vázquez, E. García-Río and R. Vázquez-Lorenzo; Complete locally conformally flat manifolds of negative curvature. *Pacific J. Math.* **226**, No. 2, (2006), 201–219.

[21] M. Brozos-Vázquez, E. García-Río and R. Vázquez-Lorenzo; Some remarks on locally conformally flat static space-times. *J. Math. Phys.* **46** (2005), 11 pp.

[22] M. Brozos-Vázquez and D. Mojón-Álvarez; The vacuum weighted Einstein field equations. *Math. Z.* **310** (2025), 44, 38 pp.

[23] M. Brozos-Vázquez and D. Mojón-Álvarez; The vacuum weighted Einstein field equations on pure radiation waves. To appear in *Progress in Lorentzian Geometry: GeLoMer 2024, Merida, Mexico, January 29–February 2*. Springer Proceedings in Mathematics & Statistics **512**, Springer, Cham (2025).

[24] M. Brozos-Vázquez and D. Mojón-Álvarez; Rigidity of weighted Einstein smooth metric measure spaces. *J. Math. Pures Appl.* **181** (2024), 91–112.

[25] M. Brozos-Vázquez and D. Mojón-Álvarez; Vacuum Einstein field equations in smooth metric measure spaces: the isotropic case. *Class. Quantum Grav.* **39** (13) (2022), 135013, 20 pp.

[26] H. A. Buchdahl; Non-Linear Lagrangians and Cosmological Theory. *MNRAS* **150** (1) (1970), 1–8.

[27] M. Cahen and N. Wallach; Lorentzian symmetric spaces. *Bull. Amer. Math. Soc.* **76** (1970), 585–591.

[28] M. Cahen, J. Leroy, M. Parker, F. Tricerri and L. Vanhecke; Lorentz manifolds modelled on a Lorentz symmetric space. *J. Geom. Phys.* **7** (1990), 571–581.

[29] G. Calvaruso and E. García-Río; Algebraic properties of curvature operators in Lorentzian manifolds with large isometry groups. SIGMA **6** (2010), 005.

[30] A. M. Candela, J. L. Flores and M. Sánchez; On general plane fronted waves. Geodesics. *Gen. Relativ. Gravit.* **35** (2003), 631–649.

[31] H.-D. Cao; Recent progress on Ricci solitons. *Recent Advances in Geometric Analysis* **138**. Advanced Lectures in Mathematics (11), International Press, Somerville (2010).

[32] H.-D. Cao and Q. Chen; On Bach-flat gradient shrinking Ricci solitons. *Duke Math. J.* **162** (6) (2013), 1149–1169.

[33] J. S. Case; The Frank-Lieb approach to sharp Sobolev inequalities. *Commun. Contemp. Math.* **23** (3) (2021), 2050015, 16 pp.

[34] J. S. Case; The weighted σ_k-curvature of a smooth metric measure space. *Pacific. J. Math.* **299** (2) (2019), 339–399.

[35] J. S. Case; A notion of the weighted σ_k-curvature for manifolds with density. *Adv. Math.* **295** (2016), 150–194.

[36] J. S. Case; A Yamabe-type problem on smooth metric measure spaces. *J. Differential Geom.*, **101** (3) (2015), 467–505.

[37] J. S. Case; The energy of a smooth metric measure space and applications. *J. Geom. Anal.* **25** (1) (2015), 616–667.

[38] J. S. Case; Conformal invariants measuring the best constants for Gagliardo-Nirenberg-Sobolev inequalities. *Calc. Var. Partial Differ. Equ.* **48** (3-4) (2013), 507–526.

[39] J. S. Case; Sharp metric obstructions for quasi-Einstein metrics. *J. Geom. Phys.* **64** (2013), 12–30.

[40] J. S. Case; Smooth metric measure spaces and quasi-Einstein metrics. *Int. J. Math.* **23** (10) (2012), 1250110.

[41] J. S. Case; Smooth metric measure spaces, quasi-Einstein metrics, and tractors. *Centr. Eur. J. Math.* **10** (2012), 1733–1762.

[42] J. S. Case; Singularity theorems and the Lorentzian splitting theorem for the Bakry-Émery-Ricci tensor. *J. Geom. Phys.* **60** (2010), 477–490.

[43] J. Case, Y.-J. Shu and G. Wei; Rigidity of quasi-Einstein metrics. *Differential Geom. Appl.* **29** (2011), 93–100.

[44] G. Catino; Generalized quasi-Einstein manifolds with harmonic Weyl tensor. *Math. Z.* **271** (2012), 751–756.

[45] G. Catino, C. Mantegazza, L. Mazzieri and M. Rimoldi; Locally conformally flat quasi-Einstein manifolds. *J. Reine Angew. Math.* **675** (2013), 181–189.

[46] D. D. Chow, C. Pope, and E. Sezgin; Kundt spacetimes as solutions of topologically massive gravity. *Class. Quantum Grav.* **27**, No. 10 (2010), p. 105002.

[47] D. Cox, D. Little and D. O'Shea: *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Undergraduate Texts in Mathematics, Springer, Cham (2015).

[48] A. Coley, S. Hervik, G. Papadopoulos and N. Pelavas; Kundt spacetimes. *Class. Quantum Grav.* **26**, No. 10 (2009) 105016.

[49] A. Coley, S. Hervik and N. Pelavas; Lorentzian spacetimes with constant curvature invariants in three dimensions. *Class. Quantum Grav.* **25**, No. 2, (2008), 025008.

[50] A. Coley, S. Hervik and N. Pelavas; On spacetimes with constant scalar invariants. *Class. Quantum Grav.* **23**, No. 9 (2006), 3053–3074.

[51] J. Corvino, M. Eichmair and P. Miao; Deformation of scalar curvature and volume. *Math. Ann.* **357**, No. 2 (2013), 551–584.

[52] G. de Rham; Sur la réductibilité d'un espace de Riemann. *Comment. Math. Helv.* **26** (1952), 328–344.

[53] M. Del Pino and J. Dolbeaut; Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.* **81** (9) (2002), 847–875.

[54] A. Derdziński; Classification of certain compact Riemannian manifolds with harmonic curvature and nonparallel Ricci tensor. *Math. Z.* **172** (1980), 273–280.

[55] F. Dobarro and B. Ünal; Curvature of multiply warped products. *J. Geom. Phys.* **55** (1) (2005), 75–106.

[56] T. Eguchi and A. J. Hanson; Self-dual solutions to Euclidean gravity. *Ann. Phys.* **120** (1) (1979), 82–106.

[57] J. Ehlers and W. Kundt; Exact solutions of the gravitational field equations. *Gravitation: an introduction to current research*, Wiley, New York (1962), 49–101.

[58] L. C. Evans; Partial differential equations, *Graduate Studies in Mathematics* **19**, American Mathematical Society, Providence, RI, 2010.

[59] M. Fernández-López and E. García-Río; Rigidity of shrinking Ricci solitons. *Math. Z.* **269** (2011), 461–466.

[60] M. Fernández-López, E. García-Río and D. N. Kupeli; A local analytic characterization of Schwarzschild metrics. *J. Geom. Phys.* **45**, No. 3-4 (2003), 309–322.

[61] M. Fernández-López, E. García-Río, D. N. Kupeli and B. Ünal; A curvature condition for a twisted product to be a warped product. *Manuscripta Math.* **106** (2001), 213–217.

[62] A. E. Fischer and J. E. Marsden; Deformations of the scalar curvature. *Duke Math. J.* **42** (3) (1975), 519–547.

[63] A. E. Fischer and J. E. Marsden; Linearization stability of nonlinear partial differential equations. *Differ. Geom., Proc. Symp. Pure Math.* **27**, Part 2, Stanford 1973, (1975), 219–263.

[64] A. Galaev; Some applications of the Lorentzian holonomy algebras. *J. Geom. Symmetry Phys.* **26** (2012), 13–31.

[65] E. García-Río, P. Gilkey and S. Nikčević; Homogeneity of Lorentzian three-manifolds with recurrent curvature. *Math. Nachr.* **287** (1) (2014), 32–47.

[66] A. Gebarowski; Doubly warped products with harmonic Weyl conformal curvature tensor. *Colloq. Math.* **67** (1994), 73–89.

[67] A. Guarnizo, L. Castañeda and J. M. Tejeiro; Boundary term in metric $f(R)$ gravity: field equations in the metric formalism. *Gen. Relativ. Gravit.* **42** (2010), 2713–2728.

[68] G. Hall; *Symmetries and curvature structure in General Relativity.* World Scientific Lecture Notes in Physics **46**, World Scientific Pub. Co., Singapore (2004).

[69] R. S. Hamilton; Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), 255–306.

[70] M. Hanounah, I. Kath, L. Mehidi and A. Zeghib; Topology and dynamics of compact plane waves. *J. Reine Angew. Math.* **820** (2025), 87–113.

[71] C. He, P. Petersen and W. Wylie; On the classification of warped product Einstein metrics. *Comm. Anal. Geom.* **20** (2) (2012), 271–311.

[72] S. Hiepko; Eine innere Kennzeichnung der verzerrten Produkte. *Math. Ann.* **241** (1979), 209–215.

[73] P. K. Ho and J. Shin; Deformation of the weighted scalar curvature. SIGMA **19** (2023), 087.

[74] B. Hua and J.-Y. Wu; Gap theorems for ends of smooth metric measure spaces. *Proc. Amer. Math. Soc.* **150** (2022), 4947–4957.

[75] A. Hurtado, V. Palmer and C. Rosales; Intrinsic and extrinsic comparison results for isoperimetric quotients and capacities in weighted manifolds. *J. Math. Anal. Appl.* **492** (2) (2020), 124488, 41 pp.

[76] S. Hwang; The critical point equation on a three dimensional compact manifold. *Proc. Amer. Math. Soc.* **131** (2003), 3221–3230.

[77] D.-S. Kim and Y.-H. Kim; Compact Einstein warped product spaces with nonpositive scalar curvature. *Proc. Amer. Math. Soc.* **131** (8) (2003), 2573–2576.

[78] J. Kim and J. Shin; Four-dimensional static and related critical spaces with harmonic curvature. *Pacific J. Math.* **295** (2) (2018), 429–462.

[79] O. Kobayashi; A differential equation arising from scalar curvature function. *J. Math. Soc. Japan* **34** (1982), 665–675.

[80] O. Kobayashi and M. Obata; Conformally-flatness and static space-time. J. Hano et al. (eds.), *Manifolds and Lie Groups*, Progress in Mathematics **14**. Birkhäuser, Boston, MA (1981), 197–206.

[81] W. Kühnel; *Differential geometry: Curves-Surfaces-Manifolds. Third Edition*. Student Mathematical Library **77**. American Mathematical Society (2015).

[82] W. Kühnel; Conformal transformations between Einstein spaces. *Conformal geometry*, Semin., MPI, Bonn/FRG 1985-86, Aspects Math.: E, 12 (1988), 105–146.

[83] W. Kühnel and H.-B. Rademacher; Conformal transformations of pseudo-Riemannian manifolds. D. V. Alekseevsky et al. (eds.). *Recent developments in pseudo-Riemannian geometry*, ESI Lect. Math. Phys., Eur. Math. Soc. (2008), 261–298.

[84] J. Lafontaine; Sur la géométrie d'une généralisation de l'équation différentielle d'Obata. *J. Math. Pures Appl.*, IX. Sér. 62 (1983), 63–72.

[85] J. M. Lee; *Introduction to Riemannian manifolds (Second Edition)*. Graduate Texts in Mathematics **176**. Springer, Cham, 2018.

[86] Th. Leistner; Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds. *Differential Geom. Appl.* **24**, no. 5, (2006), 458–478.

[87] A. Lichnerowicz, Propagateurs et commutateurs en relativité générale, *Inst. Hautes Etudes Sci. Publ. Mat.* **10** (1961), 293–344.

[88] J. Lott; Some geometric properties of the Bakry-Émery-Ricci tensor. *Comment. Math. Helv.* **78** (2003), 865–883.

[89] D. Lovelock; The Einstein tensor and its generalizations. *J. Math. Phys.* **12**, (1971), 498–501.

[90] P. Miao and L.-F. Tam; On the volume functional of compact manifolds with boundary with constant scalar curvature. *Calc. Var. Partial Differ. Eq.* **36** (2) (2009), 141–171.

[91] J. W. Moffat and V. T. Toth; Modified Jordan-Brans-Dicke theory with scalar current and the Eddington-Robertson γ-parameter. *Int. J. Mod. Phys. D* **21** (12) (2012), 1250084.

[92] F. Morgan; *Geometric measure theory: a beginner's guide. Fifth edition*. Academic Press (2016).

[93] O. Munteanu and N. Sesum; On gradient Ricci solitons. *J. Geom. Anal.* **23**, (2013), 539–561.

[94] B. L. Neto; A note on critical point metrics of the total scalar curvature functional. *J. Math. Anal. Appl.* **424** (2), (2015), 1544-1548.

[95] I. Robinson and A. Trautman; Some spherical gravitational waves in general relativity. *Proc. Roy. Soc. Lond. A* **265** (1962), 463–473.

[96] M. Obata; Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Japan* **14** (3) (1962), 333–340

[97] B. O'Neill; *Semi-Riemannian geometry. With applications to relativity.* Pure and Applied Mathematics **103**, Academic Press, Inc., New York, 1983.

[98] T. Ortín; *Gravity and strings*, 2nd Ed. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (2015).

[99] B. Osgood and D. Stowe; The Schwarzian derivative and conformal mapping of Riemannian manifolds. *Duke Math. J.* **67** (1) (1992), 57–99.

[100] G. Perelman; The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159 [math.DG] (2002).

[101] A. Peres; Some gravitational waves. *Phys. Rev. Lett.* **3** (1959), 571.

[102] S. Pigola, M. Rigoli, M. Rimoldi and A. Setti; Ricci almost solitons. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **10** (4) (2011), 757-799.

[103] J. Podolskỳ and M. Ẑofka; General Kundt spacetimes in higher dimensions. *Class. Quantum Grav.* **26** (10) (2009), 105008.

[104] R. Ponge and H. Reckziegel; Twisted products in pseudo-Riemannian geometry. *Geom. Dedicata* **48** (1993), 15–25.

[105] J. Qing and W. Yuan; A note on static spaces and related problems. *J. Geom. Phys.* **74** (2013), 18–27.

[106] I. Quiros *et al.*; Brans–Dicke Galileon and the variational principle. *Eur. J. Phys.* **37** (5) (2016), 055605.

[107] A. Romero and M. Sánchez; On completeness of certain families of semi-Riemannian manifolds. *Geom. Dedicata* **53** (1) (1994), 103–117.

[108] C. Rosales, A. Cañete, V. Bayle and F. Morgan; On the isoperimetric problem in Euclidean space with density. *Calc. Var. Partial Differ. Eq.* **31** (2008), 27–46.

[109] M. Rupert and E. Woolgar; Bakry-Émery black holes. *Class. Quantum Grav.* **31** (2014), 025008 (16pp).

[110] M. Sánchez; On the geometry of generalized Robertson-Walker spacetimes: geodesics. *Gen. Relativ. Gravitation* **30** (6) (1998), 915–932.

[111] R. Schoen; Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. M. Giaquinta, (ed.). *Topics in Calculus of Variations*, Lec. Notes Math. **1365**. Springer, Berlin, Heidelberg (1989), 120–154.

[112] Y. Shen; A note on Fischer-Marsden's conjecture. *Proc. Amer. Math. Soc.* **125** (3) (1997), 901–905.

[113] J. F. Silva Filho; Quasi-Einstein manifolds endowed with a parallel vector field. *Monatsh. Math.* **179** (2016), 305–320.

[114] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt; *Exact solutions of Einstein's field equations*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (2009).

[115] A. Walker; On Ruse's Spaces of Recurrent Curvature. *Proc. London Math. Soc.* **52** (1950), 36–64.

[116] G. Wei and W. Wylie; Comparison geometry for the Bakry-Émery Ricci tensor. *J. Differential Geom.* **83** (2009), 377–405.

[117] E. Woolgar; Scalar-tensor gravitation and the Bakry-Émery-Ricci tensor. *Class. Quantum Grav.* **30** (2013), 085007 (8pp).

[118] E. Woolgar and W. Wylie; Curvature-dimension bounds for Lorentzian splitting theorems. *J. Geom. Phys.* **132** (2018), 131–145.

[119] H. Wu, On the de Rham decomposition theorem. *Illinois J. Math.* **8** (1964), 291–311.

[120] G. Wu and R. Ye; A Note on Obata's Rigidity Theorem. *Commun. Math. Stat.* **2** (2014), 231–252.

[121] X. Xu; On the existence and uniqueness of solutions of Möbius equations. *Trans. Amer. Math. Soc.* **337**, No. 2 (1993), 927–945.

[122] K. Yano and T. Nagano; Einstein spaces admitting a one parameter group of conformal transformations. *Ann. of Math.* **69** (1959), 451–460.

[123] S. T. Yau; A survey of Calabi-Yau manifolds. *Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry*. Surv. Differ. Geom. **13** (1). International Press, Sommerville (2009), 277–318.

[124] S. T. Yau; Remarks on conformal transformations. *J. Differential Geom.* **8** (1973), 369–381.