JUAN MANUEL LORENZO NAVEIRO

SUBMANIFOLDS AND ACTIONS ON HOMOGENEOUS MANIFOLDS

161
2025

Publicaciones del Departamento de Geometría y Topología

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

JUAN MANUEL LORENZO NAVEIRO

SUBMANIFOLDS AND ACTIONS ON HOMOGENEOUS MANIFOLDS

161

2025

Publicaciones
del
Departamento
de Geometría y Topología

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

ESCOLA DE DOUTORAMENTO
INTERNACIONAL DA USC

TESE DE DOUTORAMENTO

Submanifolds and actions on homogeneous manifolds

Juan Manuel Lorenzo Naveiro

PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

SANTIAGO DE COMPOSTELA

ANO 2025

A presente tese foi dirixida por José Carlos Díaz Ramos. Defendida na Universidade de Santiago de Compostela o 7 de xullo de 2025.

Os resultados presentados nesta memoria foron obtidos no marco da axuda para contratos predoutorais para a formación de doutores PRE2020-094450 do Ministerio de Ciencia, Innovación e Universidades.

O autor tamén contou co financiamento do Ministerio de Ciencia e Innovación e a Axencia Estatal de Investigación no marco dos proxectos PID2019-105138GB-C21 e PID2022-138988NB-I00 (MICIU/AEI/10.13039/501100011033, ERDF, EU), e da Xunta de Galicia, nas modalidades de Grupo de Referencia Competitiva polos proxectos ED431C 2019/10 e ED431C 2023/31, e do Proxecto de Excelencia ED431F 2020/04.

Out of nothing, a new, another world I staged.
JÁNOS BOLYAI

Contents

Abstract

The objective of this Ph.D. thesis is to treat several classification problems concerning submanifolds and geometric structures on homogeneous manifolds with different degrees of symmetry.

There are three main research lines that we pursue in this thesis. The first one is that of polar actions on symmetric spaces, where we classify polar homogeneous foliations of codimension two on irreducible symmetric spaces of noncompact type, as well as polar homogeneous foliations on the Cayley hyperbolic plane and standard polar foliations on quaternionic hyperbolic spaces. The second line is the study of totally geodesic submanifolds; we classify totally geodesic submanifolds of the homogeneous nearly Kähler 6-manifolds and their cones with special holonomy. The last topic is that of kinematical algebras and homogeneous spacetimes, in which we classify $(3, 2)$-kinematical Lie algebras with spatial isotropy of dimension greater than two.

Resumo en galego

A simetría é probablemente un dos conceptos máis importantes na ciencia. De xeito intuitivo, entendemos que un obxecto é simétrico cando ten a mesma aparencia independentemente da perspectiva na que o vexamos. É por isto que un dos maiores logros das matemáticas modernas é a formalización desta idea, así como a súa aplicación a unha ampla clase de desafíos. En efecto, se un problema está descrito mediante un modelo matemático que ten algunha simetría inherente, entón esta pode ser aproveitada para reducir a complexidade do problema. Por exemplo, determinar as solucións dunha ecuación en derivadas parciais é en xeral unha tarefa complicada. Non obstante, se nos centramos unicamente no estudo de solucións que posúan certa simetría (o cal é suficiente en moitos casos), entón dita ecuación en derivadas parciais pode ser transformada nunha ecuación diferencial ordinaria ou incluso unha ecuación puramente alxébrica.

Felix Klein afirma no célebre *programa de Erlangen* [98] que a xeometría é precisamente o estudo da simetría. O seguinte extracto deste programa marca o nacemento da nosa concepción actual da xeometría:

> *Esquezamos a concepción concreta do espazo, que para o matemático non é esencial, e pensemos neste como unha variedade de n dimensións, é dicir, de tres dimensións, se nos atemos á idea usual dun punto como elemento do espazo. En analoxía coas transformación do espazo, falamos de transformacións da variedade; estas tamén forman grupos. Pero xa non hai, como no caso do espazo, un grupo distinguido por enriba dos demais polo seu significado; cada grupo é tan importante como os demais. Xorde así como unha xeneralización da xeometría o seguinte problema exhaustivo:*
>
> *Dada unha variedade e un grupo de transformacións da mesma; investigar as configuracións da variedade con respecto das propiedades que non son alteradas polas transformacións do grupo.*

Polo tanto, as simetrías dun espazo veñen descritas por un grupo de transformacións, mentres que a súa xeometría está gobernada polas propiedades que permanecen invariantes baixo a acción deste grupo. Un primeiro exemplo é a xeometría euclidiana, cuxo grupo epónimo está formado por transformacións ortogonais e translacións.

O tema xeral desta tese é o estudo de subvariedades e estruturas xeométricas en variedades homoxéneas con distintos graos de simetría. Dado un grupo de Lie G e unha variedade diferenciable M, dicimos que M é un G-*espazo homoxéneo* se hai unha acción transitiva de G en M. Os espazos homoxéneos, tamén coñecidos como *xeometrías de Klein*, proporcionan a linguaxe matemática na que se formula o programa de Erlangen. Isto dános a oportunidade perfecta para incorporar ferramentas da teoría de grupos e álxebras de Lie para atacar problemas xeométricos.

A meirande parte desta tese sitúase no eido da xeometría riemanniana, e máis precisamente, na teoría de subvariedades. Unha *variedade riemanniana* é un espazo que localmente se asemella ao espazo euclídeo. Polo tanto, o estudo de variedades riemannianas dá unha xeneralización directa do estudo de curvas e superficies en tres dimensións, mentres que a xeometría de subvariedades estuda as propiedades extrínsecas dunha variedade riemanniana en relación ao espazo ambiente onde esta estea situada.

A aplicación natural do formalismo de Klein en xeometría riemanniana aparece no estudo dos espazos homoxéneos riemannianos. Dicimos que unha variedade riemanniana é *homoxénea* se o seu grupo de isometrías actúa transitivamente nela. Ademais, durante a primeira parte desta disertación consideramos como variedades ambiente os espazos simétricos. Esencialmente, un *espazo simétrico* é unha variedade riemanniana na que cada punto posúe unha reflexión xeodésica (isto é, unha isometría que invirte as xeodésicas que pasan por dito punto). Os espazos simétricos foron introducidos orixinalmente por Cartan, e subministran unha familia abundante de exemplos, tales como os espazos forma, os espazos proxectivos, os espazos hiperbólicos, as grassmannianas ou os grupos de Lie compactos. A clasificación dos espazos simétricos riemannianos—tamén debida a Cartan—exhibe unha profunda conexión entre a álxebra e a xeometría. En particular, a teoría de espazos simétricos que non posúen factores euclídeos é paralela á de álxebras de Lie reais semisimples.

Esta tese está composta de tres partes principais, xunto cun capítulo inicial cos preliminares necesarios para poder desenvolver os contidos da tese. No que segue, describimos brevemente os contidos de cada parte desta tese, así como os resultados principais que obtivemos ao longo da mesma.

Accións polares en espazos simétricos

Unha acción propia e isométrica dun grupo de Lie G nunha variedade riemanniana M dise *polar* se existe unha subvariedade Σ de M que corta a todas as órbitas ortogonalmente. Dicimos neste caso que Σ é unha *sección*, e se Σ é chá coa métrica inducida de M, entón dicimos que a acción $\mathsf{G} \curvearrowright M$ é *hiperpolar*. Estas accións foron introducidas por Conlon [44] e xeneralizan moitos resultados clásicos en álxebra e xeometría, como son a existencia de coordenadas polares esféricas, o teorema espectral para operadores autoadxuntos, ou o teorema do toro maximal para grupos de Lie compactos. O noso obxectivo na primeira parte desta tese é avanzar na clasificación de accións polares sen órbitas singulares (tamén coñecidas como *foliacións polares homoxéneas*) en espazos simétricos de tipo non compacto.

No capítulo 2 tratamos a teoría xeral de accións isométricas e polares en variedades riemannianas.

Comezamos describindo a noción de acción propia e as súas propiedades elementais na sección 2.1. Dicimos que unha acción dun grupo de Lie G nunha variedade diferenciable M é *propia* se a aplicación $(g,p) \in \mathsf{G}\times M \mapsto (g\cdot p, p) \in M\times M$ é propia. No caso de que M sexa unha variedade riemanniana e G sexa un subgrupo de Lie do seu grupo de isometrías $I(M)$, pódese probar que a acción de G é propia se e só se G é un subgrupo pechado de $I(M)$. As accións propias xeneralizan as accións dos grupos de Lie compactos, e polo tanto moitas das boas propiedades que presentan as accións destes grupos consérvanse incluso ao pasar ao caso non compacto.

Na sección 2.2 presentamos o concepto de acción polar e recordamos as características principais das súas órbitas e seccións. En particular, probamos o seguinte resultado:

Teorema 2.10. *Sexa $\Sigma \subseteq M$ unha sección dunha acción polar $\mathsf{G} \curvearrowright M$. Entón Σ é unha subvariedade totalmente xeodésica de M.*

Se ben o Teorema 2.10 é *folclórico* na teoría de accións isométricas, sorprendentemente non fomos capaces de atopar unha demostración completa na literatura.

Finalmente, a sección 2.3 trata sobre o progreso actual no problema de clasificar accións polares en espazos simétricos (salvo equivalencia de órbitas).

O primeiro resultado nesta dirección débese a Dadok [49], quen probou que toda representación polar é equivalente á representación de isotropía dun espazo simétrico semisimple. Disto dedúcese inmediatamente a clasificación de accións polares en esferas redondas. Máis adiante, Podestà e Thorbergsson [146] obtiveron a clasificación de accións polares nos espazos proxectivos complexos e cuaterniónicos, así como no plano proxectivo de Cayley, completando a clasificación de accións polares en espazos simétricos compactos de rango un. A maioría do traballo detrás da clasificación de accións polares en espazos simétricos irreducibles de tipo compacto e rango superior a un débese a Kollross. En efecto, Kollross e Lytchak [109] probaron que se M é un espazo simétrico irreducible de tipo compacto e rango superior, entón toda acción polar e non trivial en M é automáticamente hiperpolar. Este resultado cambia por completo o paradigma respecto do caso de rango un, onde todo espazo (salvo a esfera S^2) admite unha acción polar non hiperpolar. A maiores, Kollross [106] demostrou que toda acción hiperpolar nestes espazos é de cohomoxeneidade un (isto é, as súas órbitas principais teñen codimensión un) ou equivalente a unha acción de Hermann. Isto remata a clasificación en espazos simétricos compactos e irreducibles, aínda que o caso reducible permanece aberto.

Nun gran contraste co caso compacto, a día de hoxe temos moi poucos resultados relativos a accións polares en espazos simétricos de tipo non compacto. De feito, os únicos espazos nesta familia onde as accións polares foron clasificadas por completo son os espazos hiperbólicos reais [169] e os espazos hiperbólicos complexos [52]. Por outro lado, as accións de cohomoxeneidade un están clasificadas. Isto é froito dun esforzo colectivo que comezou con [16] e rematou recentemente en [150]. A única clasificación xeral de accións polares en cohomoxeneidade superior foi obtida por Berndt, Díaz-Ramos e Tamaru [19], que determinaron todas as foliacións hiperpolares homoxéneas en espazos simétricos de tipo non compacto.

Foliacións polares homoxéneas de cohomoxeneidade dous en espazos simétricos de tipo non compacto

A primeira contribución orixinal desta tese atópase no capítulo 3, onde clasificamos as foliacións polares homoxéneas de cohomoxeneidade dous en espazos simétricos irreducibles de tipo non compacto. Dada unha acción polar de cohomoxeneidade dous nun espazo deste tipo, é doado comprobar que as súas seccións teñen curvatura constante non positiva. Tendo en conta isto e os resultados de [19], o problema redúcese a determinar todas as foliacións polares con sección homotética ao plano hiperbólico real $\mathbb{R}\mathsf{H}^2$.

Para poder describir os exemplos que aparecen na nosa clasificación precisamos establecer a seguinte notación (presentada con detalle na subsección 1.1.1).

Sexa M un espazo simétrico (conexo) irreducible de tipo non compacto. Entón podemos escribir $M = \mathsf{G}/\mathsf{K}$, onde $\mathsf{G} = I^0(M)$ é a compoñente identidade do grupo de isometrías e K é o subgrupo de isotropía dun punto $o \in M$. A existencia dunha reflexión xeodésica arredor de o implica que a álxebra de Lie $\mathfrak{g}$ de G admite unha involución $\theta\colon \mathfrak{g} \to \mathfrak{g}$, chamada *involución de Cartan*. Esta induce á súa vez unha $\mathbb{Z}_2$-graduación $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, chamada a *descomposición de Cartan* de $\mathfrak{g}$. Se $\mathcal{B}$ é a forma de Killing de $\mathfrak{g}$, entón a expresión $\langle X, Y \rangle = -\mathcal{B}(X, \theta Y)$ define un produto escalar en $\mathfrak{g}$. Ao ser M irreducible, temos que a súa métrica vén dada (módulo unha homotecia) pola restrición deste produto escalar a $\mathfrak{p}$.

Agora, consideramos un subespazo abeliano maximal $\mathfrak{a} \subseteq \mathfrak{p}$. Para cada covector $\lambda \in \mathfrak{a}^*$, defínese

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}\colon [H, X] = \lambda(H)X \text{ para todo } H \in \mathfrak{a}\}.$$

Se tanto λ como $\mathfrak{g}_\lambda$ son non nulos, dicimos que λ é unha *raíz (restrinxida)* de $\mathfrak{g}$ e $\mathfrak{g}_\lambda$ é un *espazo de raíz (restrinxido)*. O conxunto de todas as raíces denótase por Σ. Ademais, podemos escoller un subconxunto de raíces *positivas* $\Sigma^+ \subseteq \Sigma$, e dentro del o conxunto das raíces simples $\Lambda \subseteq \Sigma^+$, onde dicimos que unha raíz é *simple* se é positiva e non se pode escribir como suma de dúas raíces positivas. O subespazo $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ resulta ser unha álxebra de Lie nilpotente, e podemos descompoñer $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ como suma directa de espazos vectoriais. Esta descomposición dá lugar a un difeomorfismo $\mathsf{G} = \mathsf{KAN}$, onde A e N son os subgrupos conexos de G con álxebras de Lie $\mathfrak{a}$ e $\mathfrak{n}$. Tanto a descomposición a nivel de álxebras como a nivel de grupos coñécense como a *descomposición de Iwasawa*.

Tomemos agora unha raíz simple $\alpha \in \Lambda$, e consideremos $\mathfrak{b}_{\{\alpha\}} = \mathbb{R}H_\alpha \oplus (1 - \theta)(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha})$ (sendo $H_\alpha \in \mathfrak{a}$ o vector dual a $\alpha \in \mathfrak{a}^*$). O subconxunto $B_{\{\alpha\}} = \exp_o(\mathfrak{b}_{\{\alpha\}})$ é unha subvariedade totalmente xeodésica de M homotética a un espazo hiperbólico, chamada a *compoñente borde* de M asociada ao conxunto $\{\alpha\}$. É posible probar que toda acción isométrica en $B_{\{\alpha\}}$ admite unha extensión a unha acción isométrica en todo M mediante un procedemento coñecido como a *extensión canónica*.

Con esta notación en mente, estamos en condicións de enunciar o teorema principal deste capítulo:

Teorema A. *Sexa M un espazo simétrico conexo, irreducible e de tipo non compacto. Toda foliación polar non hiperpolar homoxénea de codimensión dous en M é equivalente á extensión canónica dunha foliación polar non hiperpolar homoxénea nunha compoñente borde de rango un en M.*

Máis explicitamente, temos o seguinte:

Teorema B. *Sexa $M = \mathsf{G}/\mathsf{K}$ un espazo simétrico riemanniano conexo, irreducible e de tipo non compacto. Entón, unha foliación polar non hiperpolar homoxénea de codimensión dous en M é equivalente á foliación inducida polo subgrupo conexo de G cuxa álxebra de Lie está dada por unha das seguintes posibilidades:*

(i) $(\ker \alpha) \oplus (\mathfrak{n} \ominus \ell_\alpha)$, *onde* $\alpha \in \Lambda$ *é unha raíz simple, e* ℓ_α *é unha recta en* $\mathfrak{g}_\alpha$, *ou*

(ii) $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha)$, *onde* $\alpha \in \Lambda$ *é unha raíz simple, e* $\mathfrak{v}_\alpha$ *é un subespazo abeliano bidimensional de* $\mathfrak{g}_\alpha$.

As accións mencionadas no Teorema B son construídas na sección 3.1. Ademais, como resultado de estudar a xeometría extrínseca das súas órbitas, deducimos o seguinte:

Corolario C. *Se* $\mathcal{F}$ *é unha foliación polar non hiperpolar homoxénea e de codimensión dous en* M*, entón* $\mathcal{F}$ *é harmónica se e só se* $\mathcal{F}$ *é equivalente á extensión canónica da foliación trivial nunha compoñente borde homotética ao plano hiperbólico* $\mathbb{R}\mathsf{H}^2$.

Podemos combinar entón o Teorema B cos resultados de [19], o cal nos dá a clasificación completa de foliacións polares homoxéneas en M:

Corolario D. *Unha foliación polar homoxénea en* M *é equivalente á foliación inducida polo subgrupo conexo de* G *cuxa álxebra de Lie é unha das seguintes:*

(a) $(\mathfrak{a} \ominus \mathfrak{v}) \oplus \mathfrak{n}$, *onde* $\mathfrak{v}$ *é un subespazo bidimensional de* $\mathfrak{a}$, *ou*

(b) $(\mathfrak{a} \ominus \ell) \oplus (\mathfrak{n} \ominus \ell_\alpha)$, *onde* $\alpha \in \Lambda$ *é unha raíz simple,* ℓ_α *é unha recta en* $\mathfrak{g}_\alpha$*, é* ℓ *é unha recta en* $\ker \alpha$, *ou*

(c) $\mathfrak{a} \oplus \big(\mathfrak{n} \ominus (\ell_\alpha \oplus \ell_\beta)\big)$, *onde* α, $\beta \in \Lambda$ *son raíces simples ortogonais, e* ℓ_λ *é unha recta en* $\mathfrak{g}_\lambda$, $\lambda \in \{\alpha, \beta\}$, *ou*

(d) $(\ker \alpha) \oplus (\mathfrak{n} \ominus \ell_\alpha)$, *onde* $\alpha \in \Lambda$ *é unha raíz simple, e* ℓ_α *é unha recta en* $\mathfrak{g}_\alpha$, *ou*

(e) $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha)$, *onde* $\alpha \in \Lambda$ *é unha raíz simple, e* $\mathfrak{v}_\alpha$ *é un subespazo abeliano bidimensional de* $\mathfrak{g}_\alpha$.

Observemos ademais que sempre é posible producir unha foliación polar non hiperpolar de cohomoxeneidade dous salvo que $\Sigma^+ = \{\alpha\}$ e $\mathfrak{g}_\alpha$ sexa unidimensional. Disto obtemos inmediatamente o seguinte corolario, que contrasta fortemente cos resultados do caso compacto:

Corolario E. *Se* M *é un espazo simétrico irreducible de tipo non compacto onde toda acción polar é hiperpolar, entón* M *é o plano hiperbólico real* $\mathbb{R}\mathsf{H}^2$.

Para probar o Teorema B, primeiro necesitamos estudar foliacións homoxéneas en variedades de Hadamard, o cal facemos na sección 3.2. Vemos aquí que toda foliación homoxénea está inducida pola acción libre dun grupo de Lie resoluble. Motivados por este feito, estudamos a estrutura xeral dunha subálxebra resoluble maximal dunha álxebra de Lie real semisimple, véxase a subsección 3.2.1. Finalmente, a sección 3.3 contén a proba do Teorema B, do cal se deducen o resto dos resultados principais neste capítulo.

Foliacións polares homoxéneas en espazos hiperbólicos

A segunda aportación desta tese atópase no capítulo 4, e versa sobre a clasificación de foliacións polares homoxéneas en espazos simétricos de tipo non compacto e rango un. Posto que xa temos clasificacións de accións polares nos espazos hiperbólicos reais e complexos, resta estudar o caso dos espazos hiperbólicos cuaterniónicos $\mathbb{H}\mathsf{H}^n$ e o do plano hiperbólico de Cayley $\mathbb{O}\mathsf{H}^2$.

O primeiro resultado que obtemos neste capítulo é a clasificación das foliacións polares homoxéneas nos planos hiperbólicos cuaterniónico e de Cayley. Esencialmente, probamos que calquera foliación deste estilo ten cohomoxeneidade inferior ou igual a dous, co cal todos os exemplos que aparecen neste caso son xa coñecidos. Mantendo a notación do capítulo anterior (e tendo en conta que neste caso $\mathfrak{a}^* = \mathbb{R}\alpha$ e $\Sigma^+ = \{\alpha, 2\alpha\}$), podemos enunciar a nosa clasificación do seguinte xeito:

Teorema A. *Sexa $M \in \{\mathbb{H}\mathsf{H}^2, \mathbb{O}\mathsf{H}^2\}$ o plano hiperbólico cuaterniónico ou o plano hiperbólico de Cayley. Entón as seguintes afirmacións son certas:*

(i) *Dado un subespazo vectorial $\mathfrak{b} \subseteq \mathfrak{a}$ e un subespazo $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ tal que $\dim \mathfrak{v} \leq 1$, o subgrupo conexo $\mathsf{S}_{\mathfrak{b},\mathfrak{v}} \subseteq \mathsf{G}$ con álxebra de Lie $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$ actúa polarmente en M inducindo unha foliación.*

(ii) *Toda foliación polar homoxénea non trivial en M é equivalente á foliación inducida por un subgrupo $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ dos descritos no ítem* (i).

(iii) *Dados dous subespazos $\mathfrak{b}$, $\mathfrak{b}' \subseteq \mathfrak{a}$ e subespazos $\mathfrak{v}$, $\mathfrak{v}' \subseteq \mathfrak{g}_\alpha$ con $\dim \mathfrak{v}$, $\dim \mathfrak{v}' \leq 1$, as accións dos grupos $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ e $\mathsf{S}_{\mathfrak{b}',\mathfrak{v}'}$ son equivalentes se e só se $\mathfrak{b} = \mathfrak{b}'$ e $\dim \mathfrak{v} = \dim \mathfrak{v}'$.*

En particular, estas foliacións teñen por sección un plano hiperbólico $\mathbb{R}\mathsf{H}^2(2)$ de curvatura de Gauss constante igual a $-1/4$.

Neste capítulo tamén tratamos o seguinte problema: nos espazos hiperbólicos $\mathbb{H}\mathsf{H}^n$ e $\mathbb{O}\mathsf{H}^2$, que subvariedades totalmente xeodésicas poden aparecer como seccións dunha acción polar? Kollross [108] demostrou que se $\Sigma \subseteq \mathbb{O}\mathsf{H}^2$ é sección dunha acción polar con órbitas singulares (e cohomoxeneidade superior a un), entón Σ ten curvatura seccional constante κ. Combinando isto coa clasificación de subvariedades totalmente xeodésicas en espazos simétricos de rango un [168], dedúcese que $\kappa \in \{-1/4, 1\}$. O resultado de Kollross, así como a súa proba, segue sendo válido no caso cuaterniónico, así que combinando isto co Teorema A, concluímos que só precisamos estudar o caso de foliacións polares homoxéneas en $\mathbb{H}\mathsf{H}^n$. Veremos que nesta última situación, a sección tamén ten que ser de curvatura constante, o que nos leva a:

Teorema B. *Sexa M o espazo hiperbólico cuaterniónico $\mathbb{H}\mathsf{H}^n$ ou o plano hiperbólico de Cayley $\mathbb{O}\mathsf{H}^2$. Se S é un grupo de Lie conexo actuando polarmente en M (con cohomoxeneidade maior que un) e $\Sigma \subseteq M$ é unha sección da acción $\mathsf{S} \curvearrowright M$, entón ou ben a acción de S é trivial ou ben Σ é un espazo hiperbólico real $\mathbb{R}\mathsf{H}^k$ con curvatura constante $\kappa \in \{-1, -1/4\}$.*

Cómpre ter en conta que no caso complexo todas as accións polares teñen por sección un espazo hiperbólico real de curvatura $\kappa = -1/4$, e o mesmo é certo para todas as accións construídas neste capítulo. Polo tanto, a pregunta natural que xorde destas observacións é se o valor $\kappa = -1$ pode ser eliminado do Teorema B.

O último resultado principal deste capítulo é a clasificación de foliacións polares estándar no espazo hiperbólico cuaterniónico. Dicimos que unha foliación homoxénea $\mathcal{F}$ nun espazo simétrico de tipo non compacto é *estándar* se vén dada pola acción dun subgrupo conexo de AN (a parte resoluble da descomposición de Iwasawa). Un corolario da clasificación de foliacións hiperpolares en espazos simétricos de tipo non compacto é que estas sempre son estándar, e o mesmo sucede coas foliacións polares de codimensión dous. Polo tanto, un primeiro paso natural para tratar a clasificación de foliacións polares en $\mathbb{H}\mathsf{H}^n$ é comezar co caso estándar.

Teorema C. *Sexa $M = \mathbb{H}\mathsf{H}^n$ o espazo hiperbólico cuaterniónico. Entón as seguintes afirmacións son certas:*

(i) *Dado un subespazo vectorial $\mathfrak{b} \subseteq \mathfrak{a}$ e un subespazo abeliano $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$, o subgrupo conexo $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ de $\mathsf{Sp}(1,n)$ con álxebra de Lie $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$ actúa polarmente en M inducindo unha foliación estándar.*

(ii) *Se $\mathcal{F}$ é unha foliación polar homoxénea non trivial e estándar, entón existe un subespazo vectorial $\mathfrak{b} \subseteq \mathfrak{a}$ e un subespazo abeliano $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ tal que $\mathcal{F}$ é isometricamente congruente á foliación inducida polo subgrupo $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$.*

(iii) *Dados dous subespazos $\mathfrak{b}$, $\mathfrak{b}' \subseteq \mathfrak{a}$ e dous subespazos abelianos $\mathfrak{v}$, $\mathfrak{v}' \subseteq \mathfrak{g}_\alpha$, as accións de $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ e $\mathsf{S}_{\mathfrak{b}',\mathfrak{v}'}$ son equivalentes se e só se $\mathfrak{b} = \mathfrak{b}'$ e $\dim \mathfrak{v} = \dim \mathfrak{v}'$.*

A organización do capítulo 4 é a seguinte. Na sección 4.1 describimos como é a estrutura alxébrica da álxebra de isometrías dun espazo hiperbólico $\mathbb{F}\mathsf{H}^n$. En particular, a parte nilpotente de calquera destas álxebras é unha *álxebra de Heisenberg xeneralizada*, co cal podemos facer uso das propiedades alxébricas destes obxectos para poder realizar cálculos de xeito eficiente nestes espazos. Na sección 4.2 repasamos a clasificación de subvariedades totalmente xeodésicas dos espazos hiperbólicos, xa que estas son as nosas candidatas a seccións de accións polares. Finalmente, a sección 4.3 contén as demostracións dos teoremas principais deste capítulo.

Subvariedades totalmente xeodésicas

Lembremos que unha subvariedade Σ dunha variedade riemanniana M dise *totalmente xeodésica* se as xeodésicas de Σ tamén son xeodésicas de M. Falando de xeito informal, se Σ é totalmente xeodésica entón a súa xeometría pode ser pensada como a restrición da xeometría ambiente a ela. En consecuencia, se coñecemos as subvariedades totalmente xeodésicas dun ambiente dado, entón podémonos facer unha idea de como é a xeometría global deste espazo. Isto motiva o problema de clasificar as subvariedades totalmente xeodésicas de variedades riemannianas (salvo por congruencia).

O capítulo 5, que é de carácter técnico, ten por obxectivo desenvolver a teoría de inmersións totalmente xeodésicas en variedades riemannianas. Estamos especialmente interesados no caso de variedades analíticas, pois os espazos homoxéneos están nesta categoría. Na sección 5.1 recordamos que toda subvariedade totalmente xeodésica está completamente determinada polo seu espazo tanxente nun punto (do mesmo modo que unha xeodésica está determinada pola súa velocidade nun punto). Así, se M é unha variedade riemanniana e $p \in M$, dicimos que un subespazo $V \subseteq T_pM$ é *totalmente xeodésico* se existe unha subvariedade totalmente xeodésica Σ de M que pasa por p con espazo tanxente V. Nesta sección tamén recordamos a caracterización de subespazos totalmente xeodésicos debida a Cartan. Máis adiante, en §5.2, presentamos o concepto de inmersión totalmente xeodésica compatible e demostramos que toda inmersión totalmente xeodésica factoriza por unha compatible mediante unha isometría local. Na sección 5.3 probamos que toda inmersión compatible pode ser estendida de modo único a unha inmersión compatible e inextendible (en analoxía co caso das xeodésicas). Na sección 5.4 xustificamos que toda subvariedade totalmente xeodésica e inextendible dunha variedade riemanniana analítica (respectivamente, homoxénea) é tamén analítica (respectivamente, homoxénea). O propósito da sección 5.5 é definir unha noción de maximalidade para inmersións totalmente xeodésicas que estende a idea usual de maximalidade para subvariedades embebidas.

Finalmente, na sección 5.6 comentamos as clasificacións coñecidas de subvariedades totalmente xeodésicas en espazos homoxéneos. A investigación destas subvariedades foi iniciada por Wolf [168], quen as clasificou nos espazos simétricos de rango un. Por outra banda, a clasificación de subvariedades totalmente xeodésicas en espazos simétricos de rango dous é o resultado do traballo de Chen–Nagano e Klein [42, 43, 99–101]. Aínda que o problema de clasificación permanece aberto para espazos simétricos de rango superior a dous, coñécense solucións parciais a este problema baixo hipóteses adicionais, véxase [41, 116]. Algúns autores tamén trataron problemas de clasificación en espazos homoxéneos non simétricos [58, 135], pero neste contexto os resultados son máis escasos.

Subvariedades totalmente xeodésicas das 6-variedades nearly Kähler homoxéneas e os seus conos G_2

A terceira contribución orixinal desta tese, recollida no capítulo 6, é a clasificación das subvariedades totalmente xeodésicas nas variedades estritamente nearly Kähler homoxéneas de dimensión 6, así como nos seus conos de cohomoxeneidade un e holonomía G_2.

Dicimos que unha variedade case hermitiana (M, J) é *nearly Kähler* se o tensor ∇J é totalmente antisimétrico, e *estritamente nearly Kähler* se $\nabla_X J \neq 0$ para todo $X \in TM$. A xeometría nearly Kähler en dimensión 6 é especialmente interesante por varios motivos. En primeiro lugar, esta é a dimensión máis baixa na que podemos atopar exemplos de variedades nearly Kähler que non son Kähler, e estes sempre son estritamente nearly Kähler. Toda 6-variedade M estritamente nearly Kähler resulta ser Einstein con curvatura de Ricci positiva. Ademais, se reescalamos a métrica de M para que a súa constante de Einstein sexa $\lambda = 5$, entón próbase que o cono riemanniano $\widehat{M}$ de M ten holonomía especial G_2. Esta foi a maneira na que se obtiveron os primeiros exemplos de variedades con holonomía G_2, véxase [35].

As variedades estritamente nearly Kähler, simplemente conexas e homoxéneas de dimensión 6 foron clasificadas por Butruille [36], quen probou que son exactamente os seguintes espazos dotados da métrica normal homoxénea:

$$\mathsf{S}^6 = \frac{\mathsf{G}_2}{\mathsf{SU}(3)}, \qquad \mathbb{C}\mathsf{P}^3 = \frac{\mathsf{Sp}(2)}{\mathsf{U}(1)\times\mathsf{Sp}(1)},$$
$$\mathsf{F}(\mathbb{C}^3) = \frac{\mathsf{SU}(3)}{\mathsf{T}^2}, \qquad \mathsf{S}^3\times\mathsf{S}^3 = \frac{\mathsf{SU}(2)^3}{\Delta\mathsf{SU}(2)}.$$

Estes espazos están descritos con detalle na sección 6.1. En particular, S^6 é a esfera redonda, co cal as súas subvariedades totalmente xeodésicas xa son coñecidas, así que só precisamos traballar coas tres variedades restantes. Estas últimas xorden como espazos totais de fibracións homoxéneas. Concretamente, temos as fibracións

$$\mathbb{C}\mathsf{P}^1 \to \mathbb{C}\mathsf{P}^3 \to \mathbb{H}\mathsf{P}^1, \qquad \mathbb{C}\mathsf{P}^1 \to \mathsf{F}(\mathbb{C}^3) \to \mathbb{C}\mathsf{P}^2, \qquad \mathsf{S}^3 \to \mathsf{S}^3\times\mathsf{S}^3 \to \mathsf{S}^3,$$

onde as dúas primeiras son as fibracións twistor das 4-variedades $\mathbb{H}\mathsf{P}^1 = \mathsf{S}^4$ e $\mathbb{C}\mathsf{P}^2$, mentres que a terceira é simplemente a proxección no primeiro factor.

Neste capítulo, aparte de clasificar as subvariedades totalmente xeodésicas destes tres espazos, estudamos a súa interacción coa estrutura case complexa do ambiente. Dicimos que unha subvariedade $\Sigma \subseteq M$ dunha variedade case hermitiana é *J-holomorfa* se J preserva $T\Sigma$, e *totalmente real* se para cada $p \in \Sigma$ os subespazos $T_p\Sigma$ e $J(T_p\Sigma)$ son ortogonais. Ademais, Σ é *lagrangiana* se é totalmente real e a súa dimensión é a metade da do ambiente. A maiores, tamén miramos como se comportan estas subvariedades respecto das fibracións homoxéneas descritas anteriormente. Se $F \to M \to B$ é unha submersión riemanniana, dicimos que unha subvariedade totalmente xeodésica $\Sigma \subseteq M$ está *ben posicionada* se para cada $p \in \Sigma$ temos

$$T_p\Sigma = (T_p\Sigma \cap \mathcal{V}_p) \oplus (T_p\Sigma \cap \mathcal{H}_p),$$

sendo $\mathcal{V}_p$ e $\mathcal{H}_p$ os subespazos vertical e horizontal de T_pM inducidos pola submersión riemanniana $M \to B$.

Moitos autores estudaron anteriormente as subvariedades totalmente xeodésicas dos nosos espazos de interese baixo hipóteses adicionais sobre o seu comportamento respecto da estrutura case complexa J [9, 29, 48, 55, 118]. Os métodos empregados nestes artigos empregan referencias especiais para poder clasificar subvariedades totalmente xeodésicas lagrangianas ou J-holomorfas. Non obstante, non existen motivos *a priori* para afirmar que unha subvariedade totalmente xeodésica nestes espazos deba ser J-holomorfa ou totalmente real. Utilizando as técnicas da teoría de espazos homoxéneos riemannianos, seremos capaces de resolver este problema con total xeneralidade.

Co motivo de poder atacar o problema de clasificación, é preciso desenvolver novas ferramentas para estudar subvariedades totalmente xeodésicas en espazos homoxéneos naturalmente redutivos, véxase a sección 6.2. En particular, introducimos a clase de subvariedades totalmente xeodésicas D-invariantes (sendo $D = \nabla - \nabla^c$ a diferenza entre a conexión de Levi-Civita e a conexión canónica) e probamos que admiten unha caracterización alxébrica similar á coñecida para subvariedades totalmente xeodésicas en espazos simétricos. Esta familia de subvariedades está estudada en detalle na subsección 6.2.1.

Enunciamos agora os resultados principais en relación a estas 6-variedades, cuxas demostracións poden ser consultadas na sección 6.5. No caso de $\mathbb{C}\mathsf{P}^3$, vemos que as súas subvariedades totalmente xeodésicas están descritas mediante o seguinte teorema.

Teorema A. *Sexa* Σ *unha subvariedade completa da variedade nearly Kähler homoxénea* $\mathbb{C}\mathsf{P}^3 = \mathsf{Sp}(2)/\mathsf{U}(1) \times \mathsf{Sp}(1)$ *de dimensión* $d \geq 2$. *Entón,* Σ *é totalmente xeodésica se e só se é congruente a unha das subvariedades descritas na Táboa 6.1.*

Táboa 6.1: Subvariedades totalmente xeodésicas de $\mathbb{C}\mathsf{P}^3$ de dimensión $d \geq 2$.

Subvariedade	**Relación con J**	**Comentarios**	**Ben posicionada?**
$\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)$	Lagrangiana	Órbita de $\mathsf{U}(2)$	Si
$\mathsf{S}^2\left(1/\sqrt{2}\right)$	J-holomorfa	Fibra de $\mathbb{C}\mathsf{P}^3 \to \mathsf{S}^4$	Si
$\mathsf{S}^2(1)$	J-holomorfa	Órbita de $\mathsf{SU}(2)$	Si
$\mathsf{S}^2\left(\sqrt{5}\right)$	J-holomorfa	Órbita de $\mathsf{SU}(2)_{\Lambda_3}$	Si

Para a variedade $\mathsf{F}(\mathbb{C}^3)$, a clasificación toma a seguinte forma:

Teorema B. *Sexa* Σ *unha subvariedade completa da variedade nearly Kähler homoxénea* $\mathsf{F}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathsf{T}^2$. *Entón,* Σ *é totalmente xeodésica se e só se é congruente a unha das subvariedades descritas na Táboa 6.2.*

Táboa 6.2: Subvariedades totalmente xeodésicas de $\mathsf{F}(\mathbb{C}^3)$ de dimensión $d \geq 2$.

Subvariedade	**Relación con J**	**Comentarios**	**Ben posicionada?**
$\mathsf{F}(\mathbb{R}^3)$	Lagrangiana	Órbita de $\mathsf{SO}(3)$	Si
$\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})$	Lagrangiana	Órbita de $\mathsf{SU}(2)$	Non
T^2_Λ	J-holomorfa	Órbita de T^2	Non
$\mathsf{S}^2\left(1/\sqrt{2}\right)$	J-holomorfa	Fibra de $\mathsf{F}(\mathbb{C}^3) \to \mathbb{C}\mathsf{P}^2$	Si
$\mathsf{S}^2\left(\sqrt{2}\right)$	J-holomorfa	Órbita de $\mathsf{SO}(3)$	Non
$\mathbb{R}\mathsf{P}^2\left(2\sqrt{2}\right)$	Totalmente real	Non inxectiva	Non

Por último, para o case produto $\mathsf{S}^3 \times \mathsf{S}^3$ obtemos o seguinte teorema de clasificación:

Teorema C. *Sexa* Σ *unha subvariedade completa da variedade nearly Kähler homoxénea* $\mathsf{S}^3 \times \mathsf{S}^3 = \mathsf{SU}(2)^3/\Delta\mathsf{SU}(2)$. *Entón,* Σ *é totalmente xeodésica se e só se é congruente a unha das subvariedades descritas na Táboa 6.3.*

Táboa 6.3: Subvariedades totalmente xeodésicas de $\mathsf{S}^3 \times \mathsf{S}^3$ de dimensión $d \geq 2$.

Subvariedade	Relación con J	Comentarios	Ben posicionada?
$\mathsf{S}^3\left(2/\sqrt{3}\right)$	Lagrangiana	Fibra de $\mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3$	Si
$\mathsf{S}^3_{\mathbb{C},1/3}(2)$	Lagrangiana	Órbita de $\Delta_{1,3}\mathsf{SU}(2) \times \mathsf{SU}(2)_2$	Si
T^2_Γ	J-holomorfa	Órbita dun toro bidimensional	Si
$\mathsf{S}^2(\sqrt{3/2})$	J-holomorfa	Órbita de $\Delta\mathsf{SU}(2)$	Non
$\mathsf{S}^2\left(2/\sqrt{3}\right)$	Totalmente real	Órbita de $\Delta\mathsf{SU}(2)$	Si

Referimos ao lector á sección 6.3 para máis detalles sobre cada un dos exemplos aparecendo nas táboas anteriores. Unha consecuencia directa dos Teoremas A, B e C é a seguinte:

Corolario D. *Sexa Σ unha subvariedade totalmente xeodésica maximal dunha variedade nearly Kähler homoxénea de dimensión 6 de curvatura non constante. Entón as seguintes afirmacións son certas:*

(i) *se Σ ten dimensión dous, entón Σ é unha curva J-holomorfa.*

(ii) *se Σ ten dimensión tres, entón Σ é unha subvariedade lagrangiana.*

Curiosamente, esta situación non se dá no caso de variedades Kähler, como se pode ver en [100]. Polo tanto, a pregunta natural que un se pode facer é se o Corolario D segue sendo certo se o noso ambiente é unha variedade nearly Kähler irreducible de curvatura non constante (e non necesariamente homoxénea).

Neste traballo tamén estudamos subvariedades totalmente xeodésicas en conos riemannianos, véxase a sección 6.4. Concretamente, vemos que as subvariedades totalmente xeodésicas maximais do cono $\widehat{M}$ sobre unha variedade analítica e completa M son de dous tipos:

(i) Conos sobre subvariedades totalmente xeodésicas maximais de M.

(ii) Hipersuperficies completas.

É claro que para coñecer as subvariedades do primeiro tipo é suficiente con estudar as subvariedades totalmente xeodésicas de M. Ademais, imos comprobar que toda hipersuperficie da segunda familia pode expresarse localmente como a gráfica dunha función $f\colon \Omega \subseteq M \to \mathbb{R}^+$ tal que a función $h = 1/f$ satisfai a *ecuación de Obata* $\operatorname{Hess} h = -hg$.

Volvendo ao caso dos conos de cohomoxeneidade un e holonomía G_2, comprobamos que estes non admiten hipersuperficies totalmente xeodésicas, co cal a clasificación das súas subvariedades totalmente xeodésicas dedúcese inmediatamente dos teoremas A, B e C. Concretamente, obtemos:

Teorema E. *Sexa M unha variedade nearly Kähler homoxénea de curvatura non constante e sexa Σ unha subvariedade totalmente xeodésica maximal do cono $\widehat{M}$ de dimensión maior que un. Entón Σ é o cono riemanniano dunha subvariedade totalmente xeodésica maximal S de M.*

Obsérvese que o cono dunha curva J-holomorfa nunha 6-variedade estritamente nearly Kähler M é sempre unha subvariedade *asociativa* de $\widehat{M}$ (é dicir, está calibrada pola 3-forma ϕ que determina a estrutura G_2), mentres que o dunha subvariedade lagrangiana de M resulta ser unha subvariedade *coasociativa* (isto é, está calibrada polo dual de Hodge de ϕ). Tendo en conta o Corolario D, concluímos o seguinte:

Corolario F. *Sexa Σ unha subvariedade totalmente xeodésica maximal do cono* G_2 *sobre unha* 6-*variedade estritamente nearly Kähler de curvatura non constante. Entón verifícanse as seguintes propiedades:*

(i) *se Σ é de dimensión tres, entón é unha subvariedade asociativa.*

(ii) *se Σ é de dimensión catro, entón é unha subvariedade coasociativa.*

Álxebras de Lie cinemáticas e espazotempos homoxéneos

A última parte desta tese trata sobre o *programa de Erlangen cinemático*, cuxo obxectivo principal é a clasificación de grupos de Lie cinemáticos e espazotempos homoxéneos.

No capítulo 7 presentamos as xeneralidades sobre álxebras de Lie cinemáticas e espazotempos homoxéneos. É coñecido que as leis da física son invariantes baixo certas transformacións, tales como as rotacións espaciais, as translacións no espazo e no tempo, e os *boosts*, o cal conduce á existencia de grupos de simetrías que preservan ditas leis. Bacry e Lévy-Leblond [12] introducen a noción de grupo cinemático como unha maneira de xeneralizar estes grupos de simetrías, e as súas álxebras de Lie coñécense como álxebras cinemáticas. Dicimos que unha *álxebra de Lie* (s, v)*-cinemática* (con isotropía espacial de dimensión d) é unha álxebra de Lie real $\mathfrak{g}$ que contén unha subálxebra $\mathfrak{r} \cong \mathfrak{so}(d)$ (coñecida como a *subálxebra rotacional*) de tal maneira que baixo a acción adxunta de $\mathfrak{r}$ temos a descomposición

$$\mathfrak{g} = \mathfrak{so}(d) \oplus \bigoplus^{v} \mathbb{R}^d \oplus \bigoplus^{s} \mathbb{R}.$$

Na ecuación anterior, estamos considerando $\mathbb{R}^d$ como a representación estándar de $\mathfrak{so}(d)$, mentres que $\mathbb{R}$ denota a representación trivial. Un grupo de Lie G dise (s, v)*-cinemático* se a súa álxebra de Lie é (s, v)-cinemática. Por exemplo, os grupos de Galilei, Poincaré e Carroll son $(1, 2)$-cinemáticos, como se pode ver na sección 7.1. Así, un tema de investigación activo na área de física matemática é a clasificación de grupos de Lie (s, v)-cinemáticos e as súas variedades homoxéneas.

Na sección 7.2 discutimos a clasificación das álxebras de Lie $(1, 2)$-cinemáticas. Esta clase de álxebras é a que máis atención recibiu historicamente, ao ser as álxebras de simetrías dos espazotempos homoxéneos con isotropía espacial completa. A clasificación dos grupos de Lie $(1, 2)$-

cinemáticos salvo revestimentos séguese dunha serie de traballos dos seguintes autores: Bacry–Lévy-Leblond [12], Bacry–Nuyts [13], Figueroa-O'Farrill [61, 62] e Andrzejewski–Figueroa-O'Farrill [8]. Posteriormente, Figueroa-O'Farrill e Prohazka clasificaron os espazotempos homoxéneos con isotropía espacial completa salvo por revestimentos [66]. Tamén comentamos nas seccións 7.3 e 7.4 outras familias de espazotempos homoxéneos que xa foron clasificadas, tales como os espazotempos aristotelianos [66] e os espazotempos de Lifshitz [65].

Clasificación de álxebras de Lie $(3, 2)$-cinemáticas

A última contribución orixinal desta tese é a clasificación das álxebras de Lie $(3, 2)$-cinemáticas con isotropía espacial de dimensión $d > 2$, e está recollida no capítulo 8. Este traballo forma parte dun proxecto a longo prazo no que procuramos clasificar os espazotempos homoxéneos de coisotropía espacial un (salvo revestimentos). Estes espazotempos están caracterizados pola existencia dunha dirección espacial que queda invariante baixo a acción da subálxebra rotacional.

Na sección 8.1 atopamos expresións xerais que determinan o corchete de Lie dunha álxebra de Lie $(3, 2)$-cinemática (con $d > 2$). Para isto, é conveniente escribir unha álxebra de Lie $(3, 2)$-cinemática como $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$, sendo $V = \mathbb{R}^d$ a representación estándar de $\mathfrak{so}(V) = \mathfrak{so}(d)$, $W = \mathbb{R}^2$ a representación bidimensional trivial de $\mathfrak{so}(V)$ e $\mathfrak{b} = \mathbb{R}^3$ a representación trivial de dimensión tres. A definición de álxebra de Lie $(3, 2)$-cinemática impón as seguintes relacións para o corchete que involucran es elementos $X, Y \in \mathfrak{so}(V)$, $v \in V$, $w \in W$ e $B \in \mathfrak{b}$:

$$[X, Y] = XY - YX, \qquad [X, v \otimes w] = Xv \otimes w, \qquad [X, B] = 0.$$

Así, para identificar o corchete de Lie por completo, precisamos saber como é a súa restrición a $\mathfrak{b} \times \mathfrak{b}$, a $\mathfrak{b} \times (V \otimes W)$ e a $(V \otimes W) \times (V \otimes W)$. Debido á existencia de isomorfismos excepcionais entre representacións de $\mathfrak{so}(d)$ en dimensión baixa, é necesario tratar por separado os casos $d > 3$ e $d = 3$.

Para o caso $d > 3$ obtemos a seguinte caracterización do corchete dunha álxebra de Lie $(3, 2)$-cinemática:

Teorema A. *Sexa $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ unha álxebra de Lie $(3, 2)$-cinemática con $\dim V = d > 3$. Entón o corchete de Lie de $\mathfrak{g}$ está determinado polos seguintes obxectos alxébricos:*

(i) *unha estrutura de álxebra de Bianchi*[1] *en $\mathfrak{b}$.*

(ii) *unha representación bidimensional de álxebras de Lie $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$,*

(iii) *e unha 2-forma $\mathfrak{b}$-equivariante $\varphi\colon \Lambda^2 W \to \mathfrak{b}$.*

Para todo $X \in \mathfrak{so}(V)$, $v, v_i \in V$, $w, w_i \in W$ e $B \in \mathfrak{b}$, a estrutura de álxebra de Lie está dada polos corchetes

$$\begin{aligned}
[X, v \otimes w] &= Xv \otimes w, \\
[X, B] &= 0, \\
[B, v \otimes w] &= v \otimes \rho(B)w, \\
[v_1 \otimes w_1, v_2 \otimes w_2] &= \alpha(w_1, w_2) v_1 \curlywedge v_2 + \langle v_1, v_2 \rangle \, \varphi(w_1 \wedge w_2),
\end{aligned}$$

[1]Dicimos que unha *álxebra de Bianchi* é unha álxebra de Lie real de dimensión tres.

onde $\alpha\colon W \times W \to \mathbb{R}$ *é a única forma bilinear simétrica e* $\mathfrak{b}$*-equivariante caracterizada pola condición*

$$\rho(\varphi(w_1 \wedge w_2))w_3 = \alpha(w_2, w_3)w_1 - \alpha(w_1, w_3)w_2, \quad w_i \in W.$$

Por outra banda, no caso $d = 3$ obtemos:

Teorema B. *Sexa* $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ *unha álxebra de Lie* $(3,2)$*-cinemática con* $\dim V = d = 3$*. Entón o corchete de* $\mathfrak{g}$ *está determinado, despois de empregar un isomorfismo linear que deixa fixos os elementos de* $\mathfrak{so}(V)$ *se fose necesario, polos seguintes obxectos alxébricos:*

(i) *unha estrutura de álxebra de Bianchi en* $\mathfrak{b}$*,*

(ii) *unha representación bidimensional de álxebras de Lie* $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$*,*

(iii) *unha* 2*-forma* $\mathfrak{b}$*-equivariante* $\varphi\colon \Lambda^2 W \to \mathfrak{b}$*,*

(iv) *e unha aplicación bilinear, simétrica e* $\mathfrak{b}$*-equivariante* $\sigma\colon W \times W \to W$*.*

Se consideramos a forma bilinear $\alpha \in (\mathsf{S}^2 W^*)^{\mathfrak{b}}$ *definida mediante*

$$\begin{aligned}\alpha(w_2, w_3)w_1 - \alpha(w_1, w_3)w_2 = {}& \rho(\varphi(w_1 \wedge w_2))w_3 \\ &+ \sigma(\sigma(w_2, w_3), w_1) - \sigma(\sigma(w_1, w_3), w_2),\end{aligned}$$

temos a maiores que α*,* σ *e* φ *deben satisfacer as seguintes condicións:*

$$\begin{aligned}0 &= \alpha(\sigma(w_1, w_3), w_2) - \alpha(\sigma(w_2, w_3), w_1),\\ 0 &= \varphi(\sigma(w_1, w_2) \wedge w_3 + \sigma(w_2, w_3) \wedge w_1 + \sigma(w_3, w_1) \wedge w_2).\end{aligned}$$

para todo $w_i \in W$*. A estrutura de álxebra de Lie está dada polos corchetes*

$$\begin{aligned}[X, v \otimes w] &= Xv \otimes w,\\ [X, B] &= 0,\\ [B, v \otimes w] &= v \otimes \rho(B)w,\\ [v_1 \otimes w_1, v_2 \otimes w_2] &= \alpha(w_1, w_2)v_1 \curlywedge v_2 + \langle v_1, v_2\rangle\, \varphi(w_1 \wedge w_2)\\ &\quad + (v_1 \times v_2) \otimes \sigma(w_1, w_2),\end{aligned}$$

para todo $X \in \mathfrak{so}(V)$*,* $v, v_i \in V$*,* $w, w_i \in W$ *e* $B \in \mathfrak{b}$*.*

Na sección 8.2 determinamos cando, dadas dúas álxebras $(3,2)$-cinemáticas $\mathfrak{g}$ e $\mathfrak{g}'$, existe un *isomorfismo relativo* $f\colon \mathfrak{g} \to \mathfrak{g}'$ (é dicir, un isomorfismo de álxebras que actúa trivialmente en $\mathfrak{so}(d)$). A partir dos resultados obtidos nesta sección, deducimos que a clasificación de álxebras de Lie $(3,2)$-cinemáticas salvo isomorfismo relativo se reduce á determinación de todas as representacións reais bidimensionais das álxebras de Bianchi módulo unha condición de equivalencia débil, así como algúns dos seus tensores invariantes de rango baixo. Este proceso é levado a cabo ao longo das seccións 8.3 e 8.4.

Por último, a sección 8.5 (axudada dos resultados da sección 8.A) contén a clasificación das álxebras $(3,2)$-cinemáticas con isotropía espacial de dimensión $d > 3$. Nótese que cada representación bidimensional $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ dunha álxebra de Bianchi $\mathfrak{b}$ induce unha estrutura de álxebra $(3,2)$-cinemática en $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$. Efectivamente, basta declarar que $\mathfrak{b}$ sexa subálxebra de $\mathfrak{g}$, que $V \otimes W$ sexa abeliano, e que $[B, v \otimes w] = v \otimes \rho(B)w$ para cada $B \in \mathfrak{b}$, $v \in V$ e $w \in W$. Esta solución da ecuación de Jacobi coñécese como a *extensión escindida abeliana* da suma directa $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ por $V \otimes W$ inducida por ρ.

Para poder enunciar os nosos resultados, fixamos unha base $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ de W, xunto cunha base $\{B_1, B_2, B_3\}$ de $\mathfrak{b}$. Tamén definimos o símbolo de Levi-Civita ϵ_{ij} en dimensión dous mediante as condicións $\epsilon_{11} = \epsilon_{22} = 0$ e $\epsilon_{12} = -\epsilon_{21} = 1$.

No caso $d > 3$, vemos que a clasificación toma a seguinte forma:

Teorema C. *Sexa $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ unha álxebra de Lie $(3,2)$-cinemática tal que $\dim V = d > 3$ e consideremos os datos alxébricos $(\mathfrak{b}, \rho, \varphi, \alpha)$ asociados con $\mathfrak{g}$. Entón, ocorre exactamente unha das seguintes situacións:*

(I) *O subespazo $V \otimes W$ é abeliano (é dicir, $\varphi = 0$). Neste caso, $\mathfrak{g}$ é relativamente isomorfa á extensión escindida abeliana de $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ por $V \otimes W$ inducida por exactamente unha das representacións construídas na sección 8.4.*

(II) *O subespazo $[V \otimes W, V \otimes W]$ é non nulo e está contido en $\mathfrak{b}$ (é dicir, $\varphi \neq 0$ pero $\alpha = 0$). Neste caso, $\mathfrak{g}$ é relativamente isomorfa a exactamente unha das álxebras descritas na Táboa 8.13.*

(III) *O subespazo $[V \otimes W, V \otimes W]$ proxecta de xeito non trivial aos subespazos $\mathfrak{so}(V)$ e $\mathfrak{b}$ (é dicir, $\alpha \neq 0$). Neste caso, $\mathfrak{g}$ é relativamente isomorfa a exactamente unha das álxebras descritas na Táboa 8.14.*

Por outra parte, no caso $d = 3$ aparece unha nova familia de álxebras exclusivas para esta dimensión:

Teorema D. *Sexa $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ unha álxebra de Lie $(3,2)$-cinemática con $\dim V = d = 3$ e consideremos os datos alxébricos $(\mathfrak{b}, \rho, \varphi, \alpha, \sigma)$ asociados con $\mathfrak{g}$. Entón, salvo isomorfismo relativo, ocorre exactamente unha das seguintes situacións:*

(I) *O subespazo $V \otimes W$ é abeliano (é dicir, tanto φ como σ anúlanse). Neste caso, $\mathfrak{g}$ é relativamente isomorfa á extensión escindida abeliana de $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ por $V \otimes W$ inducida por exactamente unha das representacións construídas na sección 8.4.*

(II) *O subespazo $[V \otimes W, V \otimes W]$ é non nulo e está contido en $\mathfrak{b}$ (é dicir, σ e α son cero mentres que $\varphi \neq 0$). Neste caso, $\mathfrak{g}$ é relativamente isomorfa a exactamente unha das álxebras descritas na Táboa 8.13.*

(III) *O subespazo $[V \otimes W, V \otimes W]$ está contido en $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ e proxecta de modo non trivial tanto en $\mathfrak{so}(V)$ como en $\mathfrak{b}$ (é dicir, $\sigma = 0$ pero $\alpha \neq 0$). Neste caso, $\mathfrak{g}$ é relativamente isomorfa a exactamente unha das álxebras descritas na Táboa 8.14.*

(IV) *O subespazo $V \otimes W$ é un ideal non abeliano de $\mathfrak{g}$ (é dicir, $\varphi = 0$ pero $\sigma \neq 0$) e $\mathfrak{g}$ non é relativamente isomorfa a unha extensión escindida abeliana. Neste caso, $\mathfrak{g}$ é relativamente isomorfa a exactamente unha das álxebras descritas na Táboa 8.15.*

O lector pode atopar as táboas mencionadas nos Teoremas C e D ao final da sección 8.5.

Introduction

Symmetry is perhaps the most important concept in science. In a very broad sense, we understand that an object is symmetric when it has the same appearance independently of how we look at it. As such, one of the key successes of modern mathematics is the formalization of this idea and its subsequent application to many challenges. Indeed, if a certain problem is described by a mathematical model that has an inherent symmetry, then this can be exploited in order to reduce the complexity of said problem. For instance, in general it is a hard task to determine the solutions of a partial differential equation. However, if we restrict our attention to the study of solutions possessing certain symmetry (which is sufficient in many cases), then the partial differential equation can be transformed into an ordinary differential equation or even a purely algebraic equation.

Felix Klein's celebrated *Erlangen programme* [98] states that geometry and symmetry are two sides of the same coin. The following excerpt of this programme marks the birth of geometry as we understand it today:

> *Let us now dispense with the concrete conception of space, which for the mathematician is not essential, and regard it only as a manifoldness of n dimensions, that is to say, of three dimensions, if we hold to the usual idea of the point as space element. By analogy with the transformations of space we speak of transformations of the manifoldness; they also form groups. But there is no longer, as there is in space, one group distinguished above the rest by its signification; each group is of equal importance with every other. As a generalization of geometry arises then the following comprehensive problem:*
>
> *Given a manifoldness and a group of transformations of the same; to investigate the configurations of the manifoldness with regard to such properties as are not altered by the transformations of the group.*

Thus, the symmetries of a space are described by means of a transformation group, whereas its geometry is governed by the properties that are invariant under this group. The first example of this is Euclidean geometry, whose eponymous group is comprised of orthogonal transformations and space translations.

The overarching theme of this dissertation is the study of submanifolds and geometric structures on homogeneous manifolds with varying degrees of symmetry. Given a Lie group G and a smooth manifold M, we say that M is a G-*homogeneous space* if it is endowed with a transitive action of G. Homogeneous manifolds are also known as *Klein geometries*, and they provide the mathematical framework in which the Erlangen programme is formulated. This language supplies the perfect opportunity to incorporate tools from the theory of Lie groups and their algebras in order to tackle geometric problems.

A large part of this thesis lies within the context of Riemannian geometry, and more precisely, in the field of submanifold geometry. A *Riemannian manifold* is a space locally modeled on the usual Euclidean space. Consequently, the study of Riemannian manifolds and their intrinsic properties provides a direct generalization of the study of curves and surfaces in three-dimensions, whereas the area of submanifold geometry studies the extrinsic properties of a Riemannian manifold lying inside a certain ambient space.

The natural application of Klein's formalism in Riemannian geometry appears in the study of homogeneous Riemannian manifolds. A Riemannian manifold is called *homogeneous* if its isometry group acts transitively on it. Moreover, during the first part of this dissertation we consider symmetric spaces as our ambient manifolds. Essentially, a *symmetric space* is a Riemannian manifold for which every point admits a geodesic reflection (that is, an isometry that reverses the geodesics passing through said point). Symmetric spaces were originally introduced by Cartan, and they provide an abundant family of examples, such as space forms, projective spaces, hyperbolic spaces, Grassmannians or compact Lie groups. The classification of symmetric spaces—also due to Cartan—exhibits a profound connection between algebra and geometry. In particular, the theory of symmetric spaces that do not have Euclidean factors is parallel to that of real semisimple Lie algebras.

In what follows we present the main objectives and contributions of this thesis.

Codimension two polar homogeneous foliations on symmetric spaces of noncompact type

An isometric action of a Lie group on a Riemannian manifold is *polar* if there exists a submanifold (known as a *section*) that meets every orbit orthogonally. We also say that the action is *hyperpolar* if it admits a flat section. It turns out that many classical results in algebra and geometry can be characterized in terms of these actions, such as the existence of spherical coordinates in Euclidean spaces, the spectral theorem for symmetric matrices or the maximal torus theorem. Polar actions were first introduced by Conlon [44], whose original interest was to find examples of variationally complete actions. In particular, *cohomogeneity one* actions (that is, actions that have a principal orbit of codimension one) are polar. The use of cohomogeneity one techniques has proved to be instrumental in producing new examples of Riemannian manifolds with desirable properties (such as special holonomy or Einstein metrics), see for example [28, 35, 67]. More generally, several authors have found applications of polar actions to invariant theory and submanifold geometry [86, 87, 142].

A natural problem to consider is the classification of polar actions on Riemannian manifolds up to orbit equivalence. In order for a Riemannian manifold M to admit nontrivial examples of polar actions, we need its isometry group to be large enough so as to possess nontrivial Lie subgroups. Moreover, one can show that sections of polar actions are totally geodesic submanifolds, so we also need our ambient spaces to possess examples of these submanifolds. The class of symmetric spaces meets both criteria, so they provide the optimal testing ground for producing and classifying polar actions.

The first result in this direction is due to Dadok [49], who showed that every polar representation is orbit equivalent to the isotropy representation of a semisimple symmetric space. While the classification problem in symmetric spaces of compact type is nearing its full conclusion [106, 109, 146], only partial classifications have been achieved in the noncompact setting [19, 22, 52, 169].

The first original contribution of this thesis (see Chapter 3) is the classification of polar actions without singular orbits (also known as *polar homogeneous foliations*) whose section is homothetic to a real hyperbolic plane. This, combined with the results in [19], yields the classification of all codimension two polar homogeneous foliations on these spaces. It turns out that there are two families of polar nonhyperpolar foliations of codimension two, which are obtained as lower-dimensional analogues of the so-called *foliations of horospherical type* and *foliations of solvable type* introduced by Berndt and Tamaru in [22].

As part of the proof of this classification result, we investigate homogeneous foliations on general Hadamard manifolds. More precisely, we show that if M is a Hadamard manifold and $\mathcal{F}$ is a polar homogeneous foliation on M, then there exists a solvable Lie group S that acts freely on M in such a way that the leaves of $\mathcal{F}$ are the orbits of S. As a consequence, M becomes the total space of an S-principal bundle (whose base is any section of the action). We also show that if M is a symmetric space of noncompact type, then S can be assumed to lie inside a minimal parabolic subgroup of the isometry group of M.

Polar homogeneous foliations on hyperbolic spaces

The second contribution of this thesis deals with the classification of polar homogeneous foliations on symmetric spaces of noncompact type and rank one. As of today, we only have full classifications of polar actions on real [169] and complex hyperbolic spaces [52], as well as some partial results concerning polar actions that preserve a totally geodesic submanifold on the remaining hyperbolic spaces [107, 108].

In Chapter 4, we classify polar homogeneous foliations on the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$ and derive some partial results concerning polar foliations on quaternionic hyperbolic spaces $\mathbb{H}\mathsf{H}^n$. More precisely, we show that $\mathbb{O}\mathsf{H}^2$ admits exactly three (nontrivial, nontransitive) polar homogeneous foliations up to orbit equivalence, two of which are of cohomogeneity one while the remaining one is of codimension two. Afterwards, we show that every polar homogeneous foliation—and consequently, every polar action—on $\mathbb{H}\mathsf{H}^n$ has a section of constant sectional curvature. Finally, we treat the case of standard polar foliations on $\mathbb{H}\mathsf{H}^n$. A homogeneous foliation on a symmetric space of noncompact type is *standard* if it arises from the action of a Lie subgroup of its solvable model. Our last result in this topic is the classification of standard polar homogeneous foliations on quaternionic hyperbolic spaces.

Totally geodesic submanifolds of the homogeneous nearly Kähler manifolds of dimension 6 and their G_2-cones

Recall that a submanifold Σ of a Riemannian manifold M is *totally geodesic* if every geodesic of Σ is also a geodesic of M. Loosely speaking, the geometry of a totally geodesic submanifold can be obtained as the restriction of that of its ambient space. Therefore, by knowing the totally geodesic submanifolds of a given ambient manifold we can get a grasp on its global geometry. This motivates the problem of classifying totally geodesic submanifolds of Riemannian manifolds up to congruence.

Many authors have dealt with the classification of totally geodesic submanifolds in the case of symmetric spaces. Firstly, Wolf [168] determined all totally geodesic submanifolds of rank one symmetric spaces, and the classification in symmetric spaces of rank two is the result of the work of Chen, Nagano and Klein [42, 43, 99–101]. For symmetric spaces of rank greater than two, this problem remains essentially open, although there have been some classification results for these submanifolds under additional hypotheses, see for example [41, 116]. While some authors have also dealt with the classification of totally geodesic submanifolds in non-symmetric homogeneous spaces [58, 135], the results in this more general setting are quite scarce.

The third original work of this thesis (see Chapter 6) is the classification of totally geodesic submanifolds of the (simply connected) homogeneous strictly nearly Kähler manifolds of dimension six having nonconstant curvature, as well as their cohomogeneity one G_2-cones.

Many authors have obtained partial results concerning totally geodesic submanifolds of these 6-manifolds under additional assumptions related to their interaction with the ambient almost complex structure [9, 29, 48, 55, 118]. In our work, we drop these hypotheses and classify these submanifolds in full generality. A corollary of our classification is that all of these submanifolds are automatically totally real or complex. In particular, the *maximal* totally geodesic submanifolds—that is, those which are not properly contained in another totally geodesic submanifold—are J-holomorphic curves and Lagrangian submanifolds.

An interesting feature of 6-dimensional strictly nearly Kähler manifolds is that their Riemannian cones have holonomy equal to the exceptional Lie group G_2. In this chapter we also develop a structural result that characterizes totally geodesic submanifolds of Riemannian cones in terms of submanifolds of the base space and (local) solutions to the so-called *Obata equation*. With these result, we show that the maximal totally geodesic submanifolds of these G_2-cones are precisely the cones of the maximal totally geodesic submanifolds of our 6-manifolds under investigation. Consequently, every such submanifold is either *associative* (that is, calibrated by the ambient G_2-structure) or *coassociative* (that is, calibrated by the Hodge dual of this G_2-structure).

$(3, 2)$-kinematical algebras

The final part of this thesis deals with what may be coined the *kinematical Erlangen programme*, whose main objective is to classify kinematical Lie groups and their homogeneous spacetimes. It is well-known that the laws of physics are invariant under certain transformations such as spatial rotations, space and time translations, and intertial boosts, leading to the existence of symmetry groups that preserve said laws. Bacry and Lévy-Leblond [12] introduced the notion of kinematical groups as a way to generalize these symmetry groups, and their corresponding Lie algebras are known as kinematical algebras. By an (s, v)-*kinematical Lie algebra* with d-dimensional spatial isotropy we mean a real Lie algebra $\mathfrak{g}$ containing a rotational subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$ and such that, under the adjoint action of $\mathfrak{r}$, $\mathfrak{g}$ decomposes as a direct sum of the rotational subalgebra, v copies of the standard representation $\mathbb{R}^d$, and s copies of the trivial representation

$\mathbb{R}$. Furthermore, a Lie group G is called an (s, v)-*kinematical group* if its Lie algebra is (s, v)-kinematical. Therefore, an active topic of research in the area of mathematical physics has been the classification of (s, v)-kinematical Lie groups and their homogeneous manifolds.

The case that has enjoyed the most attention is that of $(1, 2)$-kinematical groups, whose corresponding manifolds are spatially isotropic homogeneous spacetimes. Indeed, the classification of simply connected $(1, 2)$-kinematical groups (equivalently, of $(1, 2)$-kinematical algebras) is the result of combined work of the following authors: Bacry–Lévy-Leblond [12], Bacry–Nuyts [13], Figueroa-O'Farrill [61, 62] and Andrzejewski–Figueroa-O'Farrill [8]. In addition, spatially isotropic homogeneous spacetimes have been classified up to coverings by Figueroa-O'Farrill and Prohazka, see [66]. Some other families of spacetimes that have been completely determined are the Aristotelian ones (which are quotients of $(1, 1)$-kinematical groups [66]) and the Lifshitz ones (arising as quotients of $(2, 1)$-kinematical groups [65]).

The fourth and final original contribution of this doctoral thesis is the classification of $(3, 2)$-kinematical algebras with d-dimensional spatial isotropy, where $d > 2$. This work stems from our interest in classifying coisotropy one homogeneous spacetimes, which are characterized by the existence of a spatial direction that is preserved by the rotational subalgebra. In order to determine all of these algebras, we first exploit the representation theory of $\mathfrak{so}(d)$ to derive general expressions of the Lie bracket of a $(3, 2)$-kinematical algebra. As we will see, this bracket is determined by a three-dimensional Lie algebra (also known as a *Bianchi algebra*) and a two-dimensional representation of this algebra, together with some invariant tensors of low order. This allows us to transform the problem of classifying $(3, 2)$-kinematical algebras into a simple, albeit long, problem in low-dimensional representation theory.

Structure of the thesis

This thesis is comprised of three parts, together with an initial chapter containing preliminary material. We now briefly describe the general structure of this thesis.

Chapter 1 is devoted to presenting the basic concepts and notation that will be used throughout this thesis. More specifically, in Section 1.1 we review the theory of Lie groups and algebras, representation theory and smooth actions of Lie groups on manifolds. Afterwards, in Section 1.2 we recall the elementary facts on Riemannian manifolds, with a view toward studying their submanifolds. Section 1.3 deals with the main objects that we work with in this dissertation, which are homogeneous spaces. Lastly, in Section 1.4 we discuss the main properties and structure of Riemannian symmetric spaces, with special emphasis on those of noncompact type.

The first part of the thesis is concerned with polar actions on symmetric spaces and their classification up to orbit equivalence.

The objective of Chapter 2 is to discuss isometric and polar actions on Riemannian manifolds. Firstly, in Section 2.1 we introduce the notion of proper action and describe the basic properties of these actions. In Section 2.2 we present the definition of polar action. Moreover, we review the basic features of the orbits and sections of polar actions. Finally, in Section 2.3 we collect the known classification results pertaining polar actions on symmetric spaces.

In Chapter 3 we classify polar homogeneous foliations of codimension two on irreducible symmetric spaces of noncompact type. Due to previous work on hyperpolar foliations, this reduces to determining all polar foliations with a section homotethic to the real hyperbolic plane, and it turns out that there are two different families of foliations satisfying this property up to equivalence. These examples are presented in Section 3.1, and we also study the extrinsic geometry of their orbits. In Section 3.2 we study homogeneous foliations on Hadamard manifolds and prove that they are always given by a free action of a solvable Lie group. Motivated by this and the fact that the isometry group of a symmetric space of noncompact type is semisimple, we also restate a result of Mostow [126] that characterizes maximal solvable subalgebras of real semisimple Lie algebras. We end this chapter by proving the aforementioned classification result, see Section 3.3.

The purpose of Chapter 4 is to advance in the classification of polar homogeneous foliations on quaternionic hyperbolic spaces and the Cayley hyperbolic plane. In Section 4.1 we present the main features of rank one symmetric spaces of noncompact type that set them apart from those of higher rank. One sees that the nilpotent part of their Iwasawa decomposition is a so-called *generalized Heisenberg group*, so we can exploit the algebraic properties of these groups and their algebras in order to perform calculations with these spaces. We also review the classification of totally geodesic submanifolds of hyperbolic spaces—see Section 4.2. Lastly, in Section 4.3 we prove the three main theorems of this section. The first result is the classification of polar homogeneous foliations on the hyperbolic planes $\mathbb{H}\mathsf{H}^2$ and $\mathbb{O}\mathsf{H}^2$. Afterwards, we prove that a section of any polar action on either $\mathbb{H}\mathsf{H}^n$ or $\mathbb{O}\mathsf{H}^2$ is automatically a space of constant curvature. The third result is the classification of standard polar homogeneous foliations on the quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^n$.

In the second part of this dissertation we deal with the classification problem for totally geodesic submanifolds in homogeneous spaces.

Chapter 5 is concerned with presenting the general theory of totally geodesic immersions in Riemannian manifolds. In Section 5.1 we introduce the notion of totally geodesic subspace and review Cartan's local characterization of totally geodesic submanifolds via these subspaces. Section 5.2 is dedicated to discussing compatible totally geodesic immersions, which are those that are injective when viewed as maps to an adequate Grassmannian bundle. Afterwards, in Section 5.3 we show that every totally geodesic immersion can be extended into an *inextendable* one, and that this extension is unique up to reparametrization. In Section 5.4 we show that every inextendable totally geodesic submanifold a real analytic (respectively, homogeneous) Riemannian manifold is also real analytic (respectively, homogeneous). Later, in Section 5.5 we define a notion of maximality for totally geodesic submanifolds that extends the usual one for embedded submanifolds. We end this chapter by discussing the state of the art for the classification problem of totally geodesic submanifolds in Riemannian symmetric and homogeneous spaces.

The goal of Chapter 6 is to classify totally geodesic submanifolds in the simply connected homogeneous strictly nearly Kähler 6-manifolds, as well as their G_2-cones. Section 6.1 is dedicated to presenting the nearly Kähler ambient manifolds that we work with throughout this chapter and describing them as 3-symmetric spaces. In Section 6.2 we develop new techniques to study totally geodesic submanifolds of naturally reductive homogeneous spaces. In particular, we introduce the class of D-invariant totally geodesic submanifolds (see Subsection 6.2.1) and characterize them by an algebraic condition. In Section 6.3 we describe the totally geodesic submanifolds that arise in our classification result and study their interaction with both the ambient almost complex structure and the homogeneous fibration of which each ambient manifold is the total space. We then turn our attention to Riemannian cones in Section 6.4, where we recall their basic properties and derive a characterization of their totally geodesic submanifolds. Lastly, we provide the proofs of our classification theorems in Section 6.5.

The last part of this thesis deals with the classification of kinematical groups and their homogeneous spacetimes.

The purpose of Chapter 7 is to discuss some generalities regarding kinematical Lie algebras and homogeneous spacetimes. In Section 7.1 we present the notion of (s, v)-Lie algebra and provide some well-known examples. The rest of this chapter discusses the classification of some classes of homogeneous spacetimes up to coverings. These include the family of spatially isotropic spacetimes (see Section 7.2), the family of Aristotelian spacetimes (see Section 7.3) and the family of Lifshitz spacetimes (see Section 7.4).

Our journey ends in Chapter 8, where we obtain the classification of $(3, 2)$-kinematical algebras with spatial isotropy of dimension greater than two. The proof of this result is done in several steps, the first of them being the derivation of general formulae for the bracket of $(3, 2)$-kinematical algebras, see Section 8.1. We will see from the results in this section that the classification of these algebras requires us to determine all real two-dimensional representations of three-dimensional real Lie algebras. This is done in Section 8.4. Finally, Section 8.5 contains the proof of the classification theorem for $(3, 2)$-kinematical algebras. This chapter is also equipped with an appendix in which we study the orbit space of a certain action appearing in the proof of one of the classification theorems, see Section 8.A.

Chapter 1
Preliminaries

In this chapter we present the basic notation and terminology that are used throughout this thesis. Both the results presented in this work and the techniques for their proofs lie in the fields of differential geometry and Lie theory, so we aim to present a swift introduction to these two topics.

This chapter is organized as follows. In Section 1.1 we briefly review the theory of Lie groups and Lie algebras, with a view toward studying their actions and representations. Section 1.2 deals with the elementary definitions concerning Riemannian manifolds, geometry of submanifolds and Riemannian submersions. Afterwards, in Section 1.3 we present the basic concepts regarding homogeneous spaces, the core topic of interest in this thesis. Since a large part of this work is devoted to studying Riemannian homogeneous spaces, in Subsection 1.3.1 we develop the necessary tools for performing calculations with these objects. Finally, in Section 1.4 we give an overview of the structure theory of Riemannian symmetric spaces, focusing especially on those of noncompact type, see Subsection 1.4.1.

Before proceeding any further, let us make the following basic assumptions for the remainder of this thesis. All smooth manifolds are assumed to be second countable unless otherwise specified. If M is a smooth manifold and $p \in M$, the tangent space of M at p is denoted by T_pM. The disjoint union of all tangent spaces of M is known as the tangent bundle TM of M. Moreover, if $f\colon M \to N$ is a smooth map from M to another smooth manifold N the differential of f at p is written $f_{*p}\colon T_pM \to T_{f(p)}N$. The ring of real-valued smooth functions on M is denoted by $\mathcal{C}^\infty(M)$, and the $\mathcal{C}^\infty(M)$-module of vector fields on M is denoted by $\mathfrak{X}(M)$. More generally, if E is a vector bundle over M, we write $\Gamma(E)$ for the $\mathcal{C}^\infty(M)$-module of smooth sections of E.

1.1 Lie groups and Lie algebras

We recommend the references [92,102] for a detailed introduction to Lie groups and Lie algebras.

Let $\mathbb{K}$ be a field (here we only use $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$). A *Lie algebra* over $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ endowed with a skew-symmetric bilinear map $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \quad X, Y, Z \in \mathfrak{g}.$$

Given two vector subspaces $\mathfrak{a}$, $\mathfrak{b} \subseteq \mathfrak{g}$, we define $[\mathfrak{a},\mathfrak{b}]$ to be the linear span of $\{[X,Y]\colon X \in \mathfrak{a}, Y \in \mathfrak{b}\}$. A *Lie subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ satisfying $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$. If in addition we have $[\mathfrak{g},\mathfrak{h}] \subseteq \mathfrak{h}$, then $\mathfrak{h}$ is called an *ideal* of $\mathfrak{g}$. More generally, for a subset $\mathfrak{a} \subseteq \mathfrak{g}$, we define the *normalizer* of $\mathfrak{a}$ in $\mathfrak{g}$, denoted by $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$, as the set of all $X \in \mathfrak{g}$ satisfying $[X,\mathfrak{a}] \subseteq \mathfrak{a}$. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra if and only if $\mathfrak{h} \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$, and it is an ideal if and only if $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}$.

A linear map $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is called a *Lie algebra homomorphism* if it satisfies $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. In the case that $\mathfrak{h} = \mathfrak{gl}(V)$ is the Lie algebra of linear endomorphisms of V for some vector space V, we say that ϕ (and V) is a *representation* of $\mathfrak{g}$. Every Lie algebra $\mathfrak{g}$ possesses a natural representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, known as the *adjoint representation* of $\mathfrak{g}$, given by letting $\mathrm{ad}(X)Y = [X,Y]$ for all $X, Y \in \mathfrak{g}$. The kernel $\mathfrak{z}(\mathfrak{g}) = \ker \mathrm{ad}$ is called the *center* of $\mathfrak{g}$. Furthermore, we define the *Killing form* of $\mathfrak{g}$ as the symmetric bilinear form $\mathcal{B} \equiv \mathcal{B}_\mathfrak{g}$ defined via the equation $\mathcal{B}(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$ for all $X, Y \in \mathfrak{g}$.

For a Lie algebra $\mathfrak{g}$, we define its *derived series* as the sequence $(\mathfrak{g}^{(i)})_{i=1}^{\infty}$ obtained by letting $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$ for all i. We say that $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(i)} = 0$ for some $i \geq 0$. The (unique) largest solvable ideal $\mathfrak{rad}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is known as the *radical* of $\mathfrak{g}$. Similarly, the *lower central series* of $\mathfrak{g}$ is the sequence $(\mathfrak{g}_i)_{i=1}^{\infty}$ in which $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$ for all i. The algebra $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}_i = 0$ for some $i \geq 0$.

A Lie algebra is *simple* if it possesses no nonzero proper ideals, and *semisimple* if $\mathfrak{rad}(\mathfrak{g}) = 0$. Every semisimple Lie algebra can be decomposed as a direct sum of simple ideals. Moreover, a Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form $\mathcal{B}_\mathfrak{g}$ is nondegenerate. The algebra $\mathfrak{g}$ is called *reductive* if for each ideal $\mathfrak{a} \subseteq \mathfrak{g}$ there exists a complementary ideal $\mathfrak{b} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Every reductive Lie algebra has $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ and can be decomposed as $\mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, with $\mathfrak{z}(\mathfrak{g})$ abelian and $[\mathfrak{g}, \mathfrak{g}]$ semisimple.

Let $\mathfrak{g}$ be a real Lie algebra and consider a Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$. We say that $\mathfrak{k}$ is *compactly embedded* in $\mathfrak{g}$ if there exists an inner product on $\mathfrak{g}$ with respect to which the transformations $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ are skew-symmetric for all $X \in \mathfrak{k}$. If $\mathfrak{g}$ is compactly embedded in itself, then $\mathfrak{g}$ is called a *compact Lie algebra*. A real semisimple Lie algebra is *of noncompact type* if it has no nontrivial compact ideals.

If $\mathfrak{g}$ is a real Lie algebra, then we define its *complexification* as the complex Lie algebra $\mathfrak{g}(\mathbb{C}) = \mathfrak{g} \otimes \mathbb{C}$ whose Lie bracket is the $\mathbb{C}$-bilinear extension of the bracket of $\mathfrak{g}$. Moreover, if $\mathfrak{h}$ is a complex Lie algebra, then its *realification* is simply $\mathfrak{h}(\mathbb{R}) = \mathfrak{h}$ regarded as a real Lie algebra.

A (real) *Lie group* is a smooth manifold G equipped with a group structure such that the multiplication map $\mathsf{G} \times \mathsf{G} \to \mathsf{G}$ is smooth. We denote by e the identity element of G. Moreover, G^0 denotes the *identity component* of G (that is, the connected component of G containing e), which is an open normal subgroup of G. For each $g \in \mathsf{G}$ one has two commuting diffeomorphisms L_g and R_g of G, known as the *left multiplication* and *right multiplication* maps, defined by the equations $L_g(x) = gx$ and $R_g(x) = xg$. In particular, the composition $I_g = L_g \circ R_{g^{-1}}$ is known as *conjugation by* g.

A *Lie group homomorphism* is a smooth map $f\colon \mathsf{G} \to \mathsf{H}$ between Lie groups that is also an abstract group homomorphism. In particular, if $\mathsf{H} = \mathsf{GL}(V)$ is the group of linear automorphisms of some vector space V, we say that f (and V) is a *representation* of G. If $\mathsf{H} \subseteq \mathsf{G}$ is an abstract subgroup and the inclusion $\mathsf{H} \hookrightarrow \mathsf{G}$ is a Lie group homomorphism, we say that H is a *Lie subgroup of* G. The concepts of Lie group isomorphism and automorphism follow naturally, and we use the notation $\mathrm{Aut}(\mathsf{G})$ to refer to the group of automorphisms of G. It can be shown that both closed abstract subgroups and arcwise connected abstract subgroups of Lie groups are automatically Lie subgroups. In addition, if G is a Lie group and H is a closed subgroup, then the quotient G/H admits a unique smooth manifold structure such that the canonical projection $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ is a smooth submersion (in fact, a principal H-bundle).

A vector field $X \in \mathfrak{X}(\mathsf{G})$ is called *left-invariant* if it is preserved under the maps L_g for all $g \in \mathsf{G}$. The set $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ of all left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(\mathsf{G})$ of dimension $\dim \mathfrak{g} = \dim \mathsf{G}$, and we refer to it as the *Lie algebra of* G. In general, we denote Lie groups with uppercase sans serif letters and their corresponding Lie algebras with lowercase gothic letters. The Lie algebra $\mathfrak{g}$ can be canonically identified with the tangent space $T_e\mathsf{G}$. Moreover, one sees that a Lie group homomorphism $f\colon \mathsf{G} \to \mathsf{H}$ induces a Lie algebra homomorphism $f_*\colon \mathfrak{g} \to \mathfrak{h}$ by differentiation. In particular, for each $g \in \mathsf{G}$ the conjugation map I_g induces a Lie algebra automorphism $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$. The map $\mathrm{Ad}\colon \mathsf{G} \to \mathrm{Aut}(\mathfrak{g}) \subseteq \mathsf{GL}(\mathfrak{g})$ is known as the *adjoint representation* of G, and its differential is $\mathrm{Ad}_* = \mathrm{ad}$. It turns out that every left-invariant vector field is complete, a fact that allows us to define the so-called *(Lie) exponential map* $\mathrm{Exp}\colon \mathfrak{g} \to \mathsf{G}$ of G. Explicitly, for $X \in \mathfrak{g}$ the element $\mathrm{Exp}(X)$ is equal to $\alpha(1)$, where $\alpha\colon \mathbb{R} \to \mathsf{G}$ is the integral curve of X satisfying $\alpha(0) = e$. The Lie exponential map is smooth and its differential at e coincides with the canonical isomorphism $\mathfrak{g} \cong T_e\mathsf{G}$.

For an abstract Lie algebra $\mathfrak{g}$, the automorphism group $\mathrm{Aut}(\mathfrak{g})$ is a Lie subgroup of $\mathsf{GL}(\mathfrak{g})$ whose Lie algebra is the set $\mathfrak{Der}(\mathfrak{g})$ of all *derivations* of $\mathfrak{g}$, that is, the linear maps $D\colon \mathfrak{g} \to \mathfrak{g}$ that satisfy $D[X,Y] = [DX,Y] + [X,DY]$ for all X, $Y \in \mathfrak{g}$. Note that $\mathrm{ad}(\mathfrak{g}) \subseteq \mathfrak{Der}(\mathfrak{g})$ is a Lie subalgebra, known as the algebra of *inner derivations* of $\mathfrak{g}$. The connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$, denoted $\mathrm{Int}(\mathfrak{g})$, is the group of *inner automorphisms* of $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple, we have $\mathfrak{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$, and thus $\mathrm{Int}(\mathfrak{g})$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$.

We say that a Lie group G is *solvable* (respectively, *nilpotent*, *simple*, *semisimple*) if it is connected and $\mathfrak{g}$ is solvable (respectively, nilpotent, simple, semisimple).

1.1.1 Structure theory of real semisimple Lie groups

This subsection is devoted to presenting the main decomposition theorems for real semisimple Lie algebras and their corresponding groups, see [102, Chapter 6] for details and proofs.

Let us establish the following notation: if V is a vector space and it admits the direct sum decomposition $V = U \oplus W$, then for each $v \in V$ we write $v_U \in U$ and $v_W \in W$ for the unique elements that satisfy $v = v_U + v_W$. If V is a Euclidean vector space (that is, a vector space with an inner product) and no direct sum decomposition is specified, then for every subspace $U \subseteq V$ and $v \in V$ we denote by v_U the orthogonal projection of v onto U. Furthermore, the orthogonal complement of U in V is written as $V \ominus U$.

Throughout this section, G denotes a semisimple Lie group and $\mathfrak{g}$ its corresponding Lie algebra.

Consider a Lie algebra involution $\theta \in \mathrm{Aut}(\mathfrak{g})$ (that is, a Lie algebra automorphism whose square is the identity). Then θ determines a $\mathbb{Z}_2$-gradation $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \ker(1-\theta)$ is the even part of $\mathfrak{g}$ and $\mathfrak{p} = \ker(1+\theta)$ is the odd part of $\mathfrak{g}$. By definition, we have the relations $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$. The map θ is called a *Cartan involution* if the symmetric bilinear map

$$\mathcal{B}_\theta\colon (X,Y) \in \mathfrak{g} \times \mathfrak{g} \mapsto \mathcal{B}_\theta(X,Y) = \langle X,Y \rangle = -\mathcal{B}(X,\theta Y) \in \mathbb{R}$$

is positive definite on $\mathfrak{g}$. In this case, we say that the gradation $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a *Cartan decomposition* of $\mathfrak{g}$. Every real semisimple Lie algebra admits a Cartan involution, and it is unique up to inner automorphisms. Note that the restriction of $\mathcal{B}$ to $\mathfrak{k} \times \mathfrak{k}$ is negative definite, whereas its restriction to $\mathfrak{p} \times \mathfrak{p}$ is positive definite. Moreover, for every $X \in \mathfrak{g}$ we see that the adjoint transformation of $\operatorname{ad}(X)$ with respect to the inner product $\mathcal{B}_\theta$ is $\operatorname{ad}(X)^* = -\operatorname{ad}(\theta X)$. This implies that the elements of $\operatorname{ad}(\mathfrak{k})$ are all skew-symmetric operators, whereas the elements of $\operatorname{ad}(\mathfrak{p})$ are symmetric. In particular, $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}$. In fact, one sees that $\mathfrak{k}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$.

Let $\mathsf{K} \subseteq \mathsf{G}$ be the connected subgroup with Lie algebra $\mathfrak{k}$. Then K is closed in G and it contains the center $\mathsf{Z}(\mathsf{G})$ of G. Moreover, K is compact if and only if $\mathsf{Z}(\mathsf{G})$ is a finite group. This is the case, for example, if G is a Lie subgroup of $\mathsf{GL}(n, \mathbb{R})$ for some n (we say in this case that G is a *linear Lie group*). The *Cartan decomposition theorem* for the group G states that the map

$$(k, X) \in \mathsf{K} \times \mathfrak{p} \mapsto k \operatorname{Exp}(X) \in \mathsf{G}$$

is a global diffeomorphism. In particular, G has the same homotopy type as K and the quotient G/K is diffeomorphic to $\mathbb{R}^{\dim \mathfrak{p}}$.

Because $\mathfrak{k}$ normalizes $\mathfrak{p}$ and K is connected, the adjoint action of K induces a representation $\mathsf{K} \to \mathsf{GL}(\mathfrak{p})$. This representation has the property that any two maximal abelian subspaces of $\mathfrak{g}$ are conjugate by an element of K. We fix one such subspace $\mathfrak{a} \subseteq \mathfrak{p}$. Then the set $\operatorname{ad}(\mathfrak{a}) \subseteq \mathfrak{gl}(\mathfrak{g})$ is a commuting family of symmetric endomorphisms, which means that we can diagonalize all operators $\operatorname{ad} H$ (with $H \in \mathfrak{a}$) simultaneously. More precisely, we define for each covector $\lambda \in \mathfrak{a}^*$ the subspace

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \colon [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$, we say that λ is a *(restricted) root* of $\mathfrak{g}$ and $\mathfrak{g}_\lambda$ is a *(restricted) root space* of $\mathfrak{g}$. The set of restricted roots of $\mathfrak{g}$ is denoted by Σ. Note that:

- If $\lambda, \mu \in \{0\} \cup \Sigma$ are different covectors, then $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are orthogonal with respect to $\mathcal{B}_\theta$.
- For each $\lambda \in \mathfrak{a}^*$ we have $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, so in particular $\lambda \in \Sigma$ if and only if $-\lambda \in \Sigma$.
- If $\lambda, \mu \in \mathfrak{a}^*$, then $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.
- For every $\lambda \in \Sigma$, if we write $\mathfrak{k}_\lambda = (1+\theta)\mathfrak{g}_\lambda$ and $\mathfrak{p}_\lambda = (1-\theta)\mathfrak{g}_\lambda$, we have $\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} = \mathfrak{k}_\lambda \oplus \mathfrak{p}_\lambda$.
- We have $\mathfrak{p}_0 = \mathfrak{a}$, so that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$ and $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$.

The orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$$

is called the *root space decomposition* of $\mathfrak{g}_0$.

Observe that the inner product of $\mathfrak{g}$ allows us to identify $\mathfrak{a}$ with its dual space $\mathfrak{a}^*$. Explicitly, to each $\lambda \in \mathfrak{a}^*$ we assign the unique vector $H_\lambda \in \mathfrak{a}$ that satisfies $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{a}$. This in turn transforms $\mathfrak{a}^*$ into a Euclidean vector space by declaring $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$ for all $\lambda, \mu \in \mathfrak{a}^*$. It can be shown that the set Σ is an *abstract root system* in $\mathfrak{a}^*$. This means that the following assertions are true:

- The vector space $\mathfrak{a}$ is spanned by Σ.
- For each $\lambda \in \Sigma$, the reflection $s_\lambda \in \mathsf{O}(\mathfrak{a}^*)$ with respect to the hyperplane $\mathfrak{a}^* \ominus \mathbb{R}\lambda$ preserves Σ.
- The numbers $A_{\lambda,\mu} = \frac{2\langle\lambda,\mu\rangle}{|\lambda|^2}$ are integers for all λ, $\mu \in \Sigma$.

We remark that Σ may be *nonreduced*, in the sense that there may be roots whose double is also a root.

We say that a vector $H_0 \in \mathfrak{a}$ is a *regular element* if it lies in $\mathfrak{a} \setminus \bigcup_{\lambda\in\Sigma} \ker\lambda$. The set of regular elements in $\mathfrak{a}$ is open and dense in $\mathfrak{a}$. A fixed regular element $H_0 \in \mathfrak{a}$ determines a notion of positivity on Σ as follows: a root $\lambda \in \Sigma$ is said to be *positive* if $\lambda(H_0) > 0$ and *negative* if $\lambda(H_0) < 0$. Moreover, we say that λ is a *simple root* if it is positive and cannot be written as the sum of two positive roots. We write Σ^+ (respectively, Λ) to refer to the set of positive (respectively, simple) roots of $\mathfrak{g}$. Any two choices of Σ^+ (and thus of Λ) are conjugate under an element of $\mathsf{N}_\mathsf{K}(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in K. One sees that Λ is a basis for the vector space $\mathfrak{a}^*$.

We can associate a graph to the root system Σ. To do this, we assign a node to each simple root $\alpha \in \Lambda$, which we draw with the symbol $\circ$ if $2\alpha \notin \Sigma$, and with $\circledcirc$ if $2\alpha \in \Sigma$. For each α, $\beta \in \Lambda$ that are not orthogonal, we connect the nodes corresponding to the roots α and β with an edge of multiplicity $A_{\alpha,\beta}A_{\beta,\alpha} \in \{1, 2, 3\}$, together with an arrow pointing toward the root of least length (or none if both roots have the same norm). The resulting graph is known as the *Dynkin diagram* of Σ.

Let us define the vector subspace $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$. It turns out that $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$. The *Iwasawa decomposition theorem* (at the Lie algebra level) states that we can decompose $\mathfrak{g}$ as a direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

A similar situation arises at the Lie group level. Indeed, if we let A, N and AN be the connected Lie subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$, then the *Iwasawa decomposition theorem* (for Lie groups) states that the multiplication map

$$(k, a, n) \in \mathsf{K} \times \mathsf{A} \times \mathsf{N} \mapsto kan \in \mathsf{G}$$

is a global diffeomorphism. In addition, the multiplication map $\mathsf{A} \times \mathsf{N} \to \mathsf{AN}$ is also a global diffeomorphism. The exponential map of AN is a global diffeomorphism, so AN is diffeomorphic to a Euclidean space and every connected Lie subgroup of AN is closed (and also diffeomorphic to a lower-dimensional Euclidean space).

We now label the simple roots as $\Lambda = \{\alpha_1, \dots, \alpha_r\}$, where $r = \dim\mathfrak{a}$. Every root $\lambda \in \Sigma$ can be written as $\lambda = \sum_{i=1}^r c_i\lambda_i$, where the coefficients c_i are all integers having the same sign (here we interpret 0 as both a positive and negative number). The integer $l(\lambda) = \sum_{i=1}^r c_i$ is known as the *level* of λ. In particular, the sign of λ as an element of Σ coincides with the sign of $l(\lambda)$ and λ is simple if and only if $l(\lambda) = 1$. By defining

$$\mathfrak{g}^k = \bigoplus_{\substack{\lambda\in\Sigma \\ l(\lambda)=k}} \mathfrak{g}_\lambda,$$

we obtain a $\mathbb{Z}$-gradation of $\mathfrak{g}$ as $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^k$. This gradation satisfies $[\mathfrak{g}^k, \mathfrak{g}^1] = \mathfrak{g}^{k+1}$ and $[\mathfrak{g}^{-k}, \mathfrak{g}^{-1}] = \mathfrak{g}^{-k-1}$ for all $k \geq 1$. We set $\mathfrak{n}^k = \mathfrak{g}^k$ for each $k \geq 1$. Then $\mathfrak{n}^1$ generates $\mathfrak{n}$ as a Lie algebra. Moreover, we define $\mathfrak{p}^k = (1-\theta)\mathfrak{g}^k = (\mathfrak{g}^k \oplus \mathfrak{g}^{-k}) \cap \mathfrak{p}$. It can be shown that the root system Σ has a highest root. If m is the level of this root, then $\mathfrak{g}^k = 0$ whenever $|k| > m$, meaning that $\mathfrak{g} = \bigoplus_{k=-m}^{m} \mathfrak{g}^k$ and $\mathfrak{n} = \bigoplus_{k=1}^{m} \mathfrak{n}^k$.

We finish this subsection by mentioning some useful formulae that will be used in the next chapters:

- For every $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$, then $[X, (1 \pm \theta)Y] = (1 \pm \theta)[X, Y]$,
- For all $X \in \mathfrak{p}$ and $Y \in \mathfrak{g}$, we have $[X, (1 \pm \theta)Y] = (1 \mp \theta)[X, Y]$.
- If $\lambda \in \Sigma$ and $X, Y \in \mathfrak{g}_\lambda$, then $(1-\theta)[\theta X, Y] = 2\langle X, Y\rangle H_\lambda$.

Moreover, let us define the element

$$\delta = \frac{1}{2} \sum_{\lambda\in\Sigma^+} (\dim \mathfrak{g}_\lambda)\lambda \in \mathfrak{a}^*.$$

Then, for each simple root $\lambda \in \Lambda$, we have

$$2\langle \delta, \alpha\rangle = |\alpha|^2(\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}). \tag{1.1}$$

Remark 1.1. We are not aware of a reference that shows (1.1), so for the sake of completeness we include the proof of this equality (see also [54]).

If $s_\alpha\colon \mathfrak{a}^* \to \mathfrak{a}^*$ is the root reflection with respect to a simple root $\alpha \in \Lambda$, it follows from [102, Theorem 6.57] that s_α is induced by an element of $\mathsf{N}_\mathsf{K}(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in K. In particular, $\dim \mathfrak{g}_{s_\alpha(\lambda)} = \dim \mathfrak{g}_\lambda$ for every $\lambda \in \Sigma$. On the other hand, by [102, Lemma 2.61] s_α permutes all positive roots linearly independent from α, while sending α to $-\alpha$. As a consequence, we obtain $s_\alpha(\delta) = \delta - (\dim \mathfrak{g}_\alpha)\alpha - 2(\dim \mathfrak{g}_{2\alpha})\alpha$. Taking the inner product with α automatically yields (1.1).

1.1.2 Representations of Lie groups and algebras

We now describe the basic concepts related to representations of Lie groups and algebras. Here we only discuss this topic from the perspective of algebras, since the notions and results for groups are *mutatis mudandis* the same. For a detailed treatment of the representation theory of (mostly semisimple) Lie groups and Lie algebras we refer the reader to [68, 138]. All vector spaces in this subsection are assumed of finite dimension.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathbb{K} = \mathbb{C}$, then a (complex) *representation* of $\mathfrak{g}$ is a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ for some complex vector space V. If $\mathbb{K} = \mathbb{R}$, then a *real representation* of $\mathfrak{g}$ is a (real) Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ for some real vector space V, whereas a *complex representation* of $\mathfrak{g}$ is a (real) Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ for some complex vector space V. In all cases, we say that V is a $\mathfrak{g}$-*module*. If $X \in \mathfrak{g}$ and $v \in V$, we use the notation $X \cdot v = \rho(X)v$ whenever there is no ambiguity. In this thesis we mainly work with real representations of real Lie algebras.

We note that any functorial construction with vector spaces yields an analogue for representations. In particular, for any representations V, W of the Lie algebra $\mathfrak{g}$, the following vector spaces are also representations of $\mathfrak{g}$ in a natural way: the direct sum $V \oplus W$, the tensor product $V \otimes W$, the dual space V^*, the symmetric powers $\mathsf{S}^k V$ $(k \geq 0)$, the exterior powers $\Lambda^k V$ $(k \geq 0)$ and the space of linear maps $\mathrm{Hom}(V, W)$.

If $\mathfrak{g}$ is a real Lie algebra and V is a real representation of $\mathfrak{g}$, then its *complexification* $V(\mathbb{C}) = V \otimes \mathbb{C}$ becomes a complex representation of $\mathfrak{g}$. Similarly, if W is a complex representation of $\mathfrak{g}$, then the *realification* $W(\mathbb{R}) = W$ (viewed as a real vector space) becomes a real representation of $\mathfrak{g}$. We also define the *complex conjugate* representation $\overline{W}$ of W as the vector space W together with the $\mathbb{C}$-vector space structure given by $z \cdot w = \bar{z}w$ for $z \in \mathbb{C}$ and $w \in W$.

The notion of homomorphism between representations is defined in the natural way: given a Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-modules V and W, an *equivariant* map $f\colon V \to W$ is a linear map satisfying $f(X \cdot v) = X \cdot f(v)$ for all $X \in \mathfrak{g}$ and $v \in V$. We write $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ to denote the space of equivariant maps from V to W, and $\mathrm{End}_{\mathfrak{g}}(V) = \mathrm{Hom}_{\mathfrak{g}}(V, V)$ to refer to the (equivariant) *endomorphism ring* of V.

If V is a $\mathfrak{g}$-module, then an *invariant subspace* (or $\mathfrak{g}$-*submodule*) of V is a vector subspace $W \subseteq V$ satisfying $X \cdot v \in W$ for all $X \in \mathfrak{g}$ and $v \in W$. We say that V is *irreducible* if its only invariant subspaces are 0 and V, and *reducible* otherwise. The representation V is *completely reducible* if for every invariant subspace $W \subseteq V$ there exists another invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$. For instance, if $\mathfrak{g}$ is semisimple, then every representation of $\mathfrak{g}$ is completely reducible. Every completely reducible representation V can be decomposed as a sum of irreducible modules. Moreover, if $W \subseteq V$ is an irreducible submodule of V, then the sum of all submodules of V isomorphic to W is known as the *isotypical component* of V associated with W.

Assume V is a real representation of the real Lie algebra $\mathfrak{g}$. A *complex structure* on V is an $\mathbb{R}$-linear map $J\colon V \to V$ such that $J^2 = -\mathrm{id}_V$. A $\mathfrak{g}$-invariant complex structure on V turns it into a complex vector space for which the elements of $\mathfrak{g}$ act by $\mathbb{C}$-linear transformations.

Now, suppose that V is a complex representation of a real Lie algebra $\mathfrak{g}$. On the one hand, a real subspace $W \subseteq V$ is called a *real form* of V if $V = W(\mathbb{C})$. If W is also an invariant subspace, then $V = W(\mathbb{C})$ as a representation of $\mathfrak{g}$. On the other hand, a *real structure* on V is a $\mathbb{C}$-antilinear map $\tau\colon V \to V$ such that $\tau^2 = \mathrm{id}_V$. The set $\mathrm{Fix}(\tau)$ of fixed points of τ is a real form of V, and the correspondence $\tau \mapsto \mathrm{Fix}(\tau)$ establishes a bijection between the family of real structures on V and that of real forms of V. The $\mathfrak{g}$-invariant real forms correspond precisely to the $\mathfrak{g}$-invariant real structures. A $\mathbb{C}$-antilinear map $J\colon V \to V$ satisfying $J^2 = -\mathrm{id}_V$ is known as a *quaternionic structure* on V. The division algebra of quaternions is denoted by $\mathbb{H}$. Any $\mathfrak{g}$-invariant quaternionic structure on V turns it into an $\mathbb{H}$-vector space[1] for which the elements of $\mathfrak{g}$ act by quaternionic linear transformations.

Schur's lemma states that if V is an irreducible representation of the $\mathbb{K}$-Lie algebra $\mathfrak{g}$, then the endomorphism ring $\mathrm{End}_{\mathfrak{g}}(V)$ is a division algebra over $\mathbb{K}$. On the one hand, if $\mathbb{K} = \mathbb{C}$, then this forces $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}$, as $\mathbb{C}$ is algebraically closed. On the other hand, if $\mathbb{K} = \mathbb{R}$ and V is a real representation, Frobenius' theorem states that the (finite-dimensional) associative division

[1] In this thesis, by a quaternionic vector space we mean a right $\mathbb{H}$-module.

algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, meaning that $\mathrm{End}_{\mathfrak{g}}(V)$ is one of these algebras. Motivated by this, we say that the irreducible real representation V of $\mathfrak{g}$ is of *real* (respectively, *complex* or *quaternionic*) *type* if its endomorphism ring is $\mathbb{R}$ (respectively, $\mathbb{C}$ or $\mathbb{H}$). Moreover, one sees that $\mathrm{End}_{\mathfrak{g}}(V(\mathbb{C})) = \mathrm{End}_{\mathfrak{g}}(V) \otimes \mathbb{C}$. Combining this fact with the isomorphisms

$$\mathbb{R} \otimes \mathbb{C} = \mathbb{C}, \qquad \mathbb{C} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}, \qquad \mathbb{H} \otimes \mathbb{C} = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2),$$

we obtain that:

(a) The following conditions are equivalent to V being of real type:

- V admits no invariant complex structures.
- The complexification $V(\mathbb{C})$ is irreducible.

(b) The following conditions are equivalent to V being of complex type:

- $V = E(\mathbb{R})$ is the realification of an irreducible complex representation E of $\mathfrak{g}$ that does not admit an invariant real or quaternionic structure.
- The complexification $V(\mathbb{C})$ is a direct sum $W \oplus \overline{W}$ with W complex irreducible and $W \ncong \overline{W}$.

(c) The following conditions are equivalent to V being of quaternionic type:

- $V = E(\mathbb{R})$ is the realification of an irreducible complex representation E of $\mathfrak{g}$ admitting an invariant quaternionic structure (but not an invariant real structure).
- The complexification $V(\mathbb{C})$ is a direct sum $W \oplus \overline{W}$ with W complex irreducible and $W \cong \overline{W}$.

1.1.3 Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra. The Lie bracket can be viewed as a map $\Lambda^2\mathfrak{g} \to \mathfrak{g}$, and thus we can define a dual map $d\colon \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^* \subseteq \Lambda\mathfrak{g}^*$ by means of the equation $d\alpha(X,Y) = -\alpha([X,Y])$. It can be shown that d extends uniquely to a linear map $d\colon \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ of degree one satisfying $d^2 = 0$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for all $\alpha \in \Lambda^k\mathfrak{g}^*$ and $\beta \in \Lambda^l\mathfrak{g}^*$. The map d is known as the *Chevalley–Eilenberg* differential of $\mathfrak{g}$ and $\Lambda\mathfrak{g}^*$ is known as the *Chevalley–Eilenberg complex*.

Furthermore, let V be a representation of $\mathfrak{g}$. We may construct a differential d on $\Lambda\mathfrak{g}^* \otimes V = \mathrm{Hom}(\Lambda\mathfrak{g}, V)$ as follows: for $v \in V$ we define $dv \in \mathrm{Hom}(\mathfrak{g}, V)$ by the equation $dv(X) = X \cdot v$; in general, for $\omega \in \Lambda^k\mathfrak{g}^*$ and $v \in V$ we let $d(\omega \otimes v) = d\omega \otimes v + (-1)^k \omega \wedge dv$. The space of n-*cochains* of this complex is $C^n(\mathfrak{g}; V) = \Lambda^n\mathfrak{g}^* \otimes V = \mathrm{Hom}(\Lambda^n\mathfrak{g}, V)$. The space of n-*cocycles* is

$$Z^n(\mathfrak{g}; V) = \ker d\colon C^n(\mathfrak{g}; V) \to C^{n+1}(\mathfrak{g}; V),$$

whereas the space of n-*coboundaries* is

$$B^n(\mathfrak{g}; V) = \mathrm{im}\, d\colon C^{n-1}(\mathfrak{g}; V) \to C^n(\mathfrak{g}; V).$$

We have $B^n(\mathfrak{g};V) \subseteq Z^n(\mathfrak{g};V)$ and the quotient $H^n(\mathfrak{g};V) = Z^n(\mathfrak{g};V)/B^n(\mathfrak{g};V)$ is known as the n-th *Chevalley–Eilenberg cohomology group* of $\mathfrak{g}$ with values in V.

Observe that by definition $H^0(\mathfrak{g};V) = V^{\mathfrak{g}}$ is the space of *invariants* of V (that is, the elements of V that are annihilated by $\mathfrak{g}$). The map $d\colon C^1(\mathfrak{g};V) \to C^2(\mathfrak{g};V)$ is explicitly given by the formula

$$d\omega(X,Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y]).$$

1.1.4 Lie group actions

A good source of information on Lie group actions can be found in [113, Chapter 21] and in [92, Chapter 10].

Let G be a Lie group and M a smooth manifold. A (smooth, left) *action* of G on M (denoted $\mathsf{G} \curvearrowright M$) is a smooth map $(g,p) \in \mathsf{G}\times M \mapsto g{\cdot}p \in M$ that satisfies $e{\cdot}p = p$ and $g{\cdot}(h{\cdot}p) = (gh){\cdot}p$ for all g, $h \in \mathsf{G}$ and $p \in M$. One can define a *right action* $(p,g) \in M \times \mathsf{G} \mapsto p \triangleleft g \in M$ of G on M in a similar fashion. We will seldom make use of right actions in this thesis.

Given a smooth action $\mathsf{G} \curvearrowright M$, we see that each $g \in \mathsf{G}$ induces a diffeomorphism of M defined via $p \mapsto g \cdot p$. Whenever this does not cause confusion, we identify each element $g \in \mathsf{G}$ with this diffeomorphism, so we may view $g\colon M \to M$.

Suppose G is a Lie group and M, N are two manifolds equipped with an action of G. We say that a smooth map $f\colon M \to N$ is G-*equivariant* if $f(g \cdot p) = g \cdot f(p)$ for all $g \in \mathsf{G}$ and $p \in M$.

For each $p \in M$, the set $\mathsf{G} \cdot p = \{g \cdot p\colon g \in \mathsf{G}\} \subseteq M$ is called the *orbit* of p, whereas the subgroup $\mathsf{G}_p = \{g \in \mathsf{G}\colon g \cdot p = p\}$ is known as the *isotropy* or *stabilizer* subgroup of p. The *orbit space* of the action is by definition the set M/G of all orbits endowed with the topology that makes the canonical map $M \to M/\mathsf{G}$ a topological quotient map. One easily checks that G_p is a closed subgroup of G and the map $g\mathsf{G}_p \in \mathsf{G}/\mathsf{G}_p \mapsto g \cdot p \in \mathsf{G} \cdot p$ is a set bijection. As a consequence, $\mathsf{G} \cdot p$ admits a unique topology and smooth structure that makes the previous bijection a diffeomorphism, and with respect to this structure the inclusion $i\colon \mathsf{G} \cdot p \hookrightarrow M$ is an injective immersion. Note however that i is not an embedding in general. In fact, i is an embedding if and only if $\mathsf{G} \cdot p$ is locally compact with respect to the topology inherited from M (or equivalently, if $\mathsf{G} \cdot p$ is the intersection of an open and a closed subset of M).

The intersection $\mathsf{N} = \bigcap_{p \in M} \mathsf{G}_p$ of all isotropy subgroups is called the *ineffective kernel* of the action $\mathsf{G} \curvearrowright M$. The action of G is called *effective* if $\mathsf{N} = \{e\}$, and *almost effective* if N is discrete. Note that if the action of G is not effective, then it descends to an action of the Lie group G/N on M that is effective and has the same orbits as those of G. If all the isotropy subgroups of G are trivial, then we say that the action is *free*. The action is *transitive* if for some (hence all) $p \in M$ we have $M = \mathsf{G} \cdot p$. In this case M is globally diffeomorphic to the quotient G/G_p. We also say that M is a *homogeneous space* of G. Finally, if the action of G is free and transitive, we say that it is *simply transitive*.

Now, consider the Lie algebra $\mathfrak{g}$ of G. Using the action of G we may define a linear map $*\colon \mathfrak{g} \to \mathfrak{X}(M)$, where for each $X \in \mathfrak{g}$ and $p \in M$ we have

$$X^*_p = \frac{d}{dt}\Big|_{t=0} \operatorname{Exp}(tX) \cdot p.$$

We call X^* the *fundamental vector field* associated with X. One sees that the correspondence $*$ is a Lie algebra anti-homomorphism (that is, $[X,Y]^* = -[X^*,Y^*]$ for all X, $Y \in \mathfrak{g}$) with kernel $\mathfrak{n}$. Moreover, one has $T_p(\mathsf{G}\cdot p) = \{X_p^* \colon X \in \mathfrak{g}\}$ for each point $p \in M$. By definition, the isotropy subgroup G_p fixes p, so for each $g \in \mathsf{G}_p$ the differential g_{*p} is a linear automorphism of T_pM preserving $T_p(\mathsf{G}\cdot p)$. The map $g \in \mathsf{G}_p \mapsto g_{*p} \in \mathsf{GL}(T_pM)$ is known as the *isotropy representation* of G at p.

1.2 Riemannian geometry

In this section we discuss the main concepts and results from Riemannian geometry. We refer the reader to [114] and [145] for a detailed introduction.

Let M be a smooth manifold. A *Riemannian metric* on M is a smooth $(0,2)$-tensor field g on M that is symmetric and positive definite at each point. A *Riemannian manifold* is a smooth manifold M together with a Riemannian metric g. We also use the notation $g = \langle\cdot,\cdot\rangle$ to refer to the metric on M. The fundamental theorem of Riemannian geometry states that, given a Riemannian manifold M, there exists a unique linear connection ∇ on M with zero torsion and satisfying $\nabla g = 0$. We call ∇ the *Levi-Civita* connection of M.

If (M,g) and (N,h) are two Riemannian manifolds, then we say that M and N are *isometric* if there exists a smooth diffeomorphism $f\colon M \to N$ satisfying $g = f^*h$ (we say in this case that f is an *isometry*). A local diffeomorphism $f\colon M \to N$ that satisfies $g = f^*h$ is called a *local isometry*. We denote by $I(M)$ the isometry group of M, which turns out to be a Lie group, and by $I^0(M)$ its identity component. Moreover, a vector field $X \in \mathfrak{X}(M)$ is called a *Killing vector field* if for each $p \in M$ the operator $(\nabla X)_p$ is skew-symmetric. We denote by $\mathcal{K}(M)$ the Lie algebra of Killing fields on M. If $p \in M$ is any point, every Killing vector field X is completely determined by the vector $X_p \in T_pM$ and the endomorphism $(\nabla X)_p \in \mathfrak{so}(T_pM)$. In addition, the natural action of $I(M)$ on M induces an injective Lie algebra anti-homomorphism $X \in \mathfrak{i}(M) \mapsto X^* \in \mathcal{K}(M)$ whose image is the subalgebra of all complete Killing vector fields on M. If G is a Lie group acting on M, we say that the action $\mathsf{G} \curvearrowright M$ is *isometric* if each $g \in \mathsf{G}$ acts as an isometry of M. An isometric action $\mathsf{G} \curvearrowright M$ induces a Lie group homomorphism $\mathsf{G} \to I(M)$, and the image of this map is a Lie subgroup of M acting with the same orbits as G.

A vector field $X \in \mathfrak{X}(M)$ is called *parallel* if $\nabla X = 0$. If $\gamma\colon I \subseteq \mathbb{R} \to M$ is a (piecewise) smooth curve and $X \in \mathfrak{X}(\gamma)$ is a vector field defined along γ, we say that X is *parallel along* γ if $\nabla_{\gamma'}X = 0$. Given real numbers $a, b \in I$ and a tangent vector $v \in T_{\gamma(a)}M$, there exists a unique parallel vector field $X \in \mathfrak{X}(\gamma)$ satisfying $X(a) = v$. The vector $\mathcal{P}^{\gamma}_{a,b}(v) = X(b)$ is called the *parallel transport* of v to $\gamma(b)$ along γ. The parallel translation map $\mathcal{P}^{\gamma}_{a,b}\colon T_{\gamma(a)}M \to T_{\gamma(b)}M$ is readily seen to be a vector space isometry. For a point $p \in M$, the subgroup $\mathrm{Hol}_p(M) \subseteq \mathsf{O}(T_pM)$ consisting of all parallel translation maps along piecewise smooth loops based at p is called the *holonomy group* of M at p. One can define analogous concepts of parallelism and holonomy on arbitrary vector bundles equipped with a connection.

A curve γ in a Riemannian manifold is called a *geodesic* if γ' is parallel along γ. Every geodesic is completely determined by is position and velocity at a single point, and this allows us to define the so-called *(Riemannian) exponential map* of M. More precisely, we consider the set $\mathcal{E} \subseteq TM$ of all vectors $v \in TM$ such that the maximal geodesic $\gamma_v(t)$ with $\gamma_v'(0) = v$ is defined at $t = 1$ and define $\exp(v) = \gamma_v(1)$. The restriction of exp to $\mathcal{E}_p = \mathcal{E} \cap T_pM$ is denoted by $\exp_p$. We say that M is *complete* if every maximal geodesic of M is defined on all $\mathbb{R}$. The Hopf–Rinow theorem states that this condition is equivalent to $\mathcal{E}_p = T_pM$ for some (and thus all) $p \in M$, as well as M being a complete metric space with the Riemannian distance function.

It is worth noting that Killing fields on complete Riemannian manifolds are complete, and thus for a complete manifold M the Lie algebra of $I(M)$ is anti-isomorphic to the algebra of all Killing fields on M. In addition, assume X is a Killing vector field on the complete manifold M with flow maps $\phi_t^X\colon M \to M$. Choose any point $p \in M$ and consider the integral curve $\gamma\colon t \in \mathbb{R} \mapsto \phi_t^X(p) \in M$. Then the differential $(\phi_t^X)_{*p}$ and the parallel translation map $\mathcal{P}_{0,t}^{\gamma}$ are related by

$$(\mathcal{P}_{0,t}^{\gamma})^{-1} \circ (\phi_t^X)_p = e^{t(\nabla X)_p}, \quad t \in \mathbb{R}. \tag{1.2}$$

We say that a Riemannian manifold M is *irreducible* if it cannot be written as the Riemannian product of two manifolds of lower dimension. The de Rham decomposition theorem states that every complete and simply connected Riemannian manifold M can be decomposed as a product $M = M_0 \times M_1 \times \cdots \times M_k$ of irreducible Riemannian manifolds.

Suppose M is a Riemannian manifold with Levi-Civita connection ∇. The *Riemann curvature tensor* of M is the $(1,3)$-tensor field R on M characterized by the equation

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X,Y,Z \in \mathfrak{X}(M).$$

The *Ricci curvature* of M is the symmetric $(0,2)$-tensor field given by the equation $\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y)$. Furthermore, if $p \in M$ and $\sigma = \mathrm{span}\{X,Y\}$ is a two-dimensional plane in T_pM, we define the *sectional curvature* of σ as the following quantity (independent of the choice of X and Y):

$$\sec(\sigma) = \frac{\langle R(X,Y)Y, X\rangle}{|X|^2|Y|^2 - \langle X,Y\rangle^2}.$$

We say that M is *flat* if $R = 0$ and *Ricci-flat* if $\mathrm{Ric} = 0$. If $\mathrm{Ric} = \lambda\langle\cdot,\cdot\rangle$ for some real number λ, we say that M is *Einstein* and λ is the *Einstein constant* of M.

For each $p \in M$ and $X \in T_pM$ we define the associated *Jacobi operator* $R_X\colon T_pM \to T_pM$ given by $R_XY = R(Y,X)X$. Furthermore, we also consider the so-called *Cartan operator* $C_X\colon T_pM \to T_pM$ defined by $C_XY = (\nabla R)(X,X,Y,X)$. The operators R_X and C_X are both symmetric, meaning that we can decompose T_pM as the orthogonal direct sum of the eigenspaces of R_X, as well as the orthogonal direct sum of the eigenspaces of C_X.

1.2.1 Submanifold geometry

We describe here the fundamental properties of Riemannian submanifolds, see [17] for a thorough treatment of the topic.

Let $(\widetilde{M}, g)$ and (M,h) be Riemannian manifolds. A map $f\colon M \to \widetilde{M}$ satisfying $h = f^*g$ is called an *isometric immersion*. We say in this case that M is a *(Riemannian) submanifold* of $\widetilde{M}$. Two isometric immersions $f_i\colon M_i \to \widetilde{M}$ (where $i = 1,2$) are *congruent* if there exist global isometries $g\colon \widetilde{M} \to \widetilde{M}$ and $\varphi\colon M_1 \to M_2$ satisfying $g \circ f_1 = f_2 \circ \varphi$. If the isometric immersion $f\colon M \to \widetilde{M}$ is also an injective map, then we say that M is *injectively immersed*. In this case, we can endow $f(M)$ with a smooth structure and a Riemannian metric that turns $f\colon M \to f(M)$ into an isometry, and the inclusion $f(M) \hookrightarrow \widetilde{M}$ is an isometric immersion. As a consequence, one can always regard injectively immersed submanifolds as subsets of $\widetilde{M}$. For example, if H is a Lie subgroup of $I(\widetilde{M})$ and $p \in \widetilde{M}$, then the orbit $\mathsf{H} \cdot p$ is injectively immersed because it arises as the image of the injective immersion $\mathsf{H}/\mathsf{H}_p \to \mathsf{H} \cdot p \hookrightarrow \widetilde{M}$. A submanifold $M \subseteq \widetilde{M}$ is *extrinsically homogeneous* if it is the orbit of some Lie subgroup $\mathsf{H} \subseteq I(\widetilde{M})$.

We say that an injectively immersed submanifold $M \subseteq \widetilde{M}$ is *embedded* if the inclusion of M in $\widetilde{M}$ is a smooth embedding (that is, a smooth immersion that is a topological embedding as well). Because every isometric immersion is locally an embedding, in order to understand the local geometry of Riemannian submanifolds it suffices to work with the embedded ones.

From now on we fix a Riemannian manifold $\widetilde{M}$ and an embedded submanifold $M \subseteq \widetilde{M}$. For each $p \in M$, we have the orthogonal decomposition

$$T_p\widetilde{M} = T_pM \oplus \nu_pM,$$

where $\nu_pM = T_p\widetilde{M} \ominus T_pM$ is the *normal space* of M at p. The disjoint union νM of all normal spaces of M is known as the *normal bundle* of M.

Now, let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections of M and $\widetilde{M}$ respectively. The extrinsic geometry of M is governed by the so-called *second fundamental form*, which is the νM-valued symmetric bilinear tensor field $\mathrm{I\!I}$ on M given by

$$\mathrm{I\!I}(X,Y) = (\widetilde{\nabla}_XY)_{\nu M}, \quad X, Y \in \mathfrak{X}(M).$$

The trace $\mathcal{H} \in \Gamma(\nu M)$ of the second fundamental form of M is called the *mean curvature vector field*. Moreover, if $p \in M$ and $\xi \in \nu_pM$ is a normal vector, we define the *shape operator* $\mathcal{S}_\xi\colon T_pM \to T_pM$ by the condition $\langle \mathcal{S}_\xi X, Y\rangle = \langle \mathrm{I\!I}(X,Y), \xi\rangle$ for all X, $Y \in T_pM$. The symmetry of the second fundamental form implies that $\mathcal{S}_\xi$ is a self-adjoint operator.

The relationship between the connections ∇ and $\widetilde{\nabla}$ is given by means of the Gauss formula

$$\widetilde{\nabla}_XY = \nabla_XY + \mathrm{I\!I}(X,Y), \quad X, Y \in \mathfrak{X}(M).$$

Additionally, if one defines the *normal connection* $\nabla^\perp$ on νM by the condition $\nabla^\perp_X\xi = (\nabla_X\xi)_{\nu M}$ for $X \in \mathfrak{X}(M)$ and $\xi \in \Gamma(\nu M)$, then the Weingarten equation states that

$$\widetilde{\nabla}_X\xi = -\mathcal{S}_\xi X + \nabla^\perp_X\xi, \quad \text{for all } X \in \mathfrak{X}(M) \text{ and } \xi \in \Gamma(\nu M).$$

We now relate the curvature tensors R and $\widetilde{R}$ of M and $\widetilde{M}$. If X, Y, Z, T are vector fields on M, then we have the following equalities (known as the Gauss and Codazzi equations):

$$\begin{aligned}
\langle \widetilde{R}(X,Y)Z,T\rangle &= \langle R(X,Y)Z,T\rangle - \langle \mathrm{I\!I}(Y,Z), \mathrm{I\!I}(X,T)\rangle + \langle \mathrm{I\!I}(X,Z), \mathrm{I\!I}(Y,T)\rangle,\\
(\widetilde{R}(X,Y)Z)_{\nu M} &= (\nabla^\perp_X\mathrm{I\!I})(Y,Z) - (\nabla^\perp_Y\mathrm{I\!I})(X,Z).
\end{aligned}$$

We say that M is *totally geodesic* if its second fundamental form vanishes identically, and *minimal* if it has zero mean curvature. It is easy to show that M is totally geodesic if and only if every M-geodesic is also an $\widetilde{M}$-geodesic. Note that for a totally geodesic submanifold $M \subseteq \widetilde{M}$ and a point $p \in M$, the Gauss and Codazzi equations imply that T_pM is invariant under the ambient curvature tensor $\widetilde{R}$.

1.2.2 Riemannian submersions

We also need some elementary facts about Riemannian submersions, see [143, Section 5.5] for more details.

Consider a smooth submersion $\pi\colon \widetilde{M} \to M$ between Riemannian manifolds. For each point $p \in \widetilde{M}$ we call $\mathcal{V}_p = \ker \pi_{*p}$ the *vertical subspace* at p, whereas its orthogonal complement $\mathcal{H}_p = T_p\widetilde{M} \ominus \mathcal{V}_p$ is known as the *horizontal subspace* at p. This construction induces two smooth distributions $\mathcal{V}$ and $\mathcal{H}$ on $\widetilde{M}$, which we call the *vertical* and *horizontal* distributions. We say that π is a *Riemannian submersion* if for each $p \in \widetilde{M}$ the restriction $\pi_{*p}\colon \mathcal{H}_p \to T_{\pi(p)}M$ is a vector space isometry. The *fibers* of π are the embedded submanifolds $\pi^{-1}(q)$ (with $q \in M$).

Let $\pi\colon \widetilde{M} \to M$ be a Riemannian submersion. We define two tensors T and A on $\widetilde{M}$ by the following equations involving $X, Y \in \mathfrak{X}(\widetilde{M})$:

$$T_X Y = (\nabla_{X_\mathcal{V}} Y_\mathcal{V})_\mathcal{H} + (\nabla_{X_\mathcal{V}} Y_\mathcal{H})_\mathcal{V}, \qquad A_X Y = (\nabla_{X_\mathcal{H}} Y_\mathcal{H})_\mathcal{V} + (\nabla_{X_\mathcal{H}} Y_\mathcal{V})_\mathcal{H}.$$

These are known as the *O'Neill tensors*. It can be shown that the tensor T vanishes identically if and only if the fibers of π are totally geodesic. Additionally, we have $A = 0$ if and only if the horizontal distribution $\mathcal{H}$ is integrable. In the case that $\mathcal{H}$ is an integrable distribution, one sees that every integral manifold S of $\mathcal{H}$ is totally geodesic and the restricted projection $\pi\colon S \to M$ is a local isometry.

1.3 Homogeneous spaces

Homogeneous spaces provide the language in which Felix Klein formulated the celebrated *Erlangen programme*. Loosely speaking, Klein's key observation is that each *geometry* (for instance, Euclidean geometry, projective geometry, hyperbolic geometry, affine geometry...) has a transitive group of transformations (*haugtgruppe*, or *principal group* in Haskell's translation of the Erlangen programme [98]) that preserves the properties of the figures studied in each geometry. Because of this, Klein generalizes the notion of a geometry to that of a manifold endowed with a transitive action of a Lie group.

In this section we present the basic theory of homogeneous spaces. Further details may be found in [153]. We also recommend [24] and [105] for a perspective more tailored to Riemannian geometers, as these books have chapters focusing primarily on Riemannian homogeneous spaces.

By a *homogeneous space* we mean a smooth manifold M together with a transitive action of a Lie group G. If M is a G-homogeneous space and N is an H-homogeneous space, then we say that M and N are *isomorphic* if there exists a Lie group isomorphism $\phi\colon \mathsf{G} \to \mathsf{H}$ and a smooth diffeomorphism $f\colon M \to N$ satisfying $f(g \cdot p) = \phi(g) \cdot f(p)$ for all $g \in \mathsf{G}$ and $p \in M$. In particular, if M is a homogeneous space of G and we fix a point $o \in M$ with isotropy subgroup $\mathsf{K} = \mathsf{G}_o$, then M is isomorphic to the quotient G/K (endowed with the natural action of G). Because the action of G is transitive, different choices of o yield conjugate isotropy subgroups.

If G is a Lie group and K is a closed Lie subgroup of G, we say that (G, K) is a *Klein pair*. Two Klein pairs (G, K) and $(\mathsf{G}', \mathsf{K}')$ are *isomorphic* if there exists a Lie group isomorphism $\phi\colon \mathsf{G} \to \mathsf{G}'$ such that $\phi(\mathsf{K}) = \mathsf{K}'$. To each Klein pair (G, K) we can assign the G-homogeneous space $M = \mathsf{G}/\mathsf{K}$. Moreover, if M is a G-homogeneous space and $o \in M$ is any point, then we say that $(\mathsf{G}, \mathsf{G}_o)$ is a *Klein pair associated with* M. The correspondence $(\mathsf{G}, \mathsf{K}) \mapsto \mathsf{G}/\mathsf{K}$ descends to a bijection between isomorphism classes of Klein pairs and isomorphism classes of homogeneous spaces. We say that the Klein pair (G, K) is (almost) effective if the action $\mathsf{G} \curvearrowright \mathsf{G}/\mathsf{K}$ is (almost) effective. The ineffective kernel of the action $\mathsf{G} \curvearrowright \mathsf{G}/\mathsf{K}$ is $\mathsf{N} = \bigcap_{g \in \mathsf{G}} g\mathsf{K}g^{-1}$, and is precisely the largest normal subgroup of G inside K.

Suppose $M = \mathsf{G}/\mathsf{K}$ is a homogeneous space of a connected Lie group and consider the universal cover $\widetilde{\mathsf{G}}$ of G. Then we obtain a transitive action $\widetilde{\mathsf{G}} \curvearrowright M$ via the universal covering map $\pi\colon \widetilde{\mathsf{G}} \to \mathsf{G}$, and the isotropy subgroup of $\widetilde{\mathsf{G}}$ at $o = e\mathsf{K}$ is $\pi^{-1}(\mathsf{K})$. If $\widetilde{\mathsf{K}}$ is the identity component of $\pi^{-1}(\mathsf{K})$, then $\widetilde{M} = \widetilde{\mathsf{G}}/\widetilde{\mathsf{K}}$ is a simply connected space and the canonical projection $\widetilde{M} = \widetilde{\mathsf{G}}/\widetilde{\mathsf{K}} \to \widetilde{\mathsf{G}}/\pi^{-1}(\mathsf{K}) = M$ is the universal covering map of M. This shows that the universal cover of a homogeneous space $M = \mathsf{G}/\mathsf{K}$ is a homogeneous space with corresponding Klein pair $(\widetilde{\mathsf{G}}, \widetilde{\mathsf{K}})$.

We now establish a notion of Klein pair at the Lie algebra level. An *(infinitesimal) Klein pair* is a pair $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is a real Lie algebra and $\mathfrak{k} \subseteq \mathfrak{g}$ is a subalgebra. We say that the Klein pair $(\mathfrak{g}, \mathfrak{k})$ is *effective* if the only ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$ is the zero ideal. For instance, if (G, K) is a Klein pair of groups, then passing to the corresponding algebras gives an infinitesimal Klein pair $(\mathfrak{g}, \mathfrak{k})$, and the latter pair is effective precisely when the former is almost effective. Two Klein pairs $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}', \mathfrak{k}')$ are *isomorphic* if there exists a Lie algebra isomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ carrying $\mathfrak{k}$ to $\mathfrak{k}'$.

Observe that if $M = \mathsf{G}/\mathsf{K}$ is a homogeneous space (equivalently, if (G, K) is a Klein pair), then we have an associated infinitesimal Klein pair $(\mathfrak{g}, \mathfrak{k})$. Moreover, $(\mathfrak{g}, \mathfrak{k})$ is also a Klein pair associated with the universal cover $\widetilde{M} = \widetilde{\mathsf{G}}/\widetilde{\mathsf{K}}$ (with the same notation as before). However, not every infinitesimal Klein pair comes from a homogeneous space. Motivated by this, we say that a Klein pair $(\mathfrak{g}, \mathfrak{k})$ is *geometrically realizable* if there exists a (group) Klein pair (G, K) for which $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ and $\mathfrak{k} = \mathrm{Lie}(\mathsf{K})$. The homogeneous space $M = \mathsf{G}/\mathsf{K}$ is a *geometric realization* of $(\mathfrak{g}, \mathfrak{k})$. One checks that the isomorphism classes of geometrically realizable Klein pairs are in a bijective correspondence with the isomorphism classes of simply connected homogeneous spaces.

Fix a homogeneous space $M = \mathsf{G}/\mathsf{K}$ and set $o = e\mathsf{K}$. The action of G induces a vector space isomorphism $X + \mathfrak{k} \in \mathfrak{g}/\mathfrak{k} \mapsto X^*_o \in T_oM$. This map also establishes a K-module isomorphism between the adjoint representation of K on $\mathfrak{g}/\mathfrak{k}$ and the isotropy representation of $M = \mathsf{G}/\mathsf{K}$ at o. Moreover, there exists a vector space isomorphism between the space of all G-invariant tensor fields of type (p, q) on M and the space of all K-invariant tensors of type (p, q) on $T_oM \cong \mathfrak{g}/\mathfrak{k}$, which is given by sending the G-invariant tensor field T to its restriction to $T_oM \cong \mathfrak{g}/\mathfrak{k}$.

The homogeneous space $M = \mathsf{G}/\mathsf{K}$ is *reductive* if there exists a K-submodule $\mathfrak{p} \subseteq \mathfrak{g}$, known as a *reductive complement*, such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This allows us to identify $\mathfrak{p}$ with $\mathfrak{g}/\mathfrak{k} \cong T_oM$ in the natural manner as a representation of K. The direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is known as a *reductive decomposition*. If M is reductive, then we can define a G-invariant connection ∇^c on M by declaring

$$(\nabla^c_{X^*} Y^*)_o = -[X, Y]_{\mathfrak{p}}, \quad X, Y \in \mathfrak{p}.$$

This is known as the *canonical connection* of M. In general, the connection ∇^c may have nontrivial torsion. We remark that a tensor field T on M is G-invariant if and only if $\nabla^c T = 0$. The curvature tensor R^c of the canonical connection is characterized by the condition

$$R^c(X, Y)Z = -[[X, Y]_{\mathfrak{k}}, Z], \quad X, Y, Z \in \mathfrak{p}.$$

1.3.1 Riemannian homogeneous spaces

Let M be a Riemannian manifold. We say that M is *(Riemannian) homogeneous* if the isometry group $I(M)$ acts transitively on M. In this case, we can write $M = \mathsf{G}/\mathsf{K}$ as a quotient of a Lie subgroup[2] $\mathsf{G} \subseteq I(M)$. It turns out that M is automatically reductive, so we may choose a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The Riemannian metric $\langle \cdot, \cdot \rangle$ on M induces, by restriction, an inner product on $\mathfrak{p}$, which we also denote by $\langle \cdot, \cdot \rangle$. In particular, we have $\mathrm{Ad}(k) \in \mathsf{O}(\mathfrak{p})$ for all $k \in \mathsf{K}$ and $\mathrm{ad}(X) \in \mathfrak{so}(\mathfrak{p})$ for all $X \in \mathfrak{k}$.

The Levi-Civita connection ∇ of M is G-invariant and thus $D = \nabla - \nabla^c$ is a G-invariant $(2, 1)$-tensor field on M, called the *difference tensor*. The difference tensor at o can be recovered entirely from algebraic data. Indeed, let us define a symmetric bilinear map $U \colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ by

$$2\langle U(X, Y), Z \rangle = \langle [Z, X]_{\mathfrak{p}}, Y \rangle + \langle [X, Z]_{\mathfrak{p}}, Y \rangle, \quad X, Y, Z \in \mathfrak{p}. \tag{1.3}$$

Then the restriction of D to $T_oM \cong \mathfrak{p}$ satisfies

$$D_X Y = \frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y), \quad X, Y \in \mathfrak{p}.$$

As a consequence, we also have

$$(\nabla_{X^*} Y^*)_o = -\frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y), \quad X, Y \in \mathfrak{p}.$$

Moreover, the curvature tensor of M at o is given by

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - [[X, Y]_{\mathfrak{k}}, Z] - D_{[X,Y]_{\mathfrak{p}}} Z \tag{1.4}$$

for all X, Y, $Z \in \mathfrak{p}$. One can also compute the covariant derivative ∇R as follows: observe that R is invariant under G, and therefore $\nabla^c R = 0$. Using the fact that $D = \nabla - \nabla^c$ we readily obtain

$$\begin{aligned}(\nabla_V R)(X, Y, Z) = D_V(R(X, Y)Z) - R(D_V X, Y)Z - R(X, D_V Y)Z \\ - R(X, Y)D_V Z\end{aligned} \tag{1.5}$$

[2]From now on, whenever we write $M = \mathsf{G}/\mathsf{K}$ for a Riemannian homogeneous space we implicitly assume that G is, up to some finite covering, a transitive Lie subgroup of $I(M)$.

for every $X, Y, Z, V \in \mathfrak{p}$.

We say that the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is *naturally reductive* if $U = 0$, or equivalently, if the difference tensor is skew-symmetric. If this is the case, then one sees that the exponential map of M at $T_oM = \mathfrak{p}$ is given by $\exp_o(X) = \operatorname{Exp}(X) \cdot o$ for all $X \in \mathfrak{p}$. We also say that the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is *normal homogeneous* if there exists an $\operatorname{Ad}(\mathsf{G})$-invariant inner product q on $\mathfrak{g}$ for which $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ (with respect to q) and the restriction of q to $\mathfrak{p} \times \mathfrak{p}$ coincides with the Riemannian metric $\langle \cdot, \cdot \rangle$.

If $M = \mathsf{G}/\mathsf{K}$ is a Riemannian homogeneous space, we say that a submanifold $N \subseteq M$ is *extrinsically homogeneous with respect to the presentation* $M = \mathsf{G}/\mathsf{K}$ if it is an orbit of some Lie subgroup $\mathsf{S} \subseteq \mathsf{G}$. Note that if S is a Lie subgroup of G, then the tangent space of the orbit $\mathsf{S} \cdot o$ at o is $T_o(\mathsf{S} \cdot o) = \mathfrak{s}_{\mathfrak{p}}$. The following lemma (taken from [119]) gives a formula for the second fundamental form of an extrinsically homogeneous submanifold of a homogeneous space. A first version of this formula was derived by Solonenko [155, Proposition 2.2.43] for the case of Riemannian symmetric spaces. An alternative expression can also be found in [3, Proposition 2.2].

Lemma 1.2. *Let $M = \mathsf{G}/\mathsf{K}$ be a Riemannian homogeneous space equipped with an arbitrary G-invariant metric and reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Assume that S is a Lie subgroup of G, and let $\mathfrak{s}_{\mathfrak{p}}$ and $\mathfrak{s}_{\mathfrak{p}}^{\perp}$ be the tangent and normal spaces to $\mathsf{S} \cdot o$ at o (regarded as subspaces of $\mathfrak{p}$). Let $V_{\mathfrak{s}_{\mathfrak{p}}^{\perp}}$ denote the orthogonal projection of $V \in \mathfrak{p}$ onto $\mathfrak{s}_{\mathfrak{p}}^{\perp}$. Then the second fundamental form of $\mathsf{S} \cdot o$ at o is given by*

$$\mathrm{I\!I}(X_{\mathfrak{p}}, Y) = ([X_{\mathfrak{k}}, Y] + D_{X_{\mathfrak{p}}} Y)_{\mathfrak{s}_{\mathfrak{p}}^{\perp}} \tag{1.6}$$

for all $X \in \mathfrak{s}$ and $Y \in \mathfrak{s}_{\mathfrak{p}}$. In particular, $\mathsf{S} \cdot o$ is totally geodesic if and only if $[X_{\mathfrak{k}}, Y] + D_{X_{\mathfrak{p}}} Y \in \mathfrak{s}_{\mathfrak{p}}$ for all $X \in \mathfrak{s}$ and $Y \in \mathfrak{s}_{\mathfrak{p}}$.

Proof. Choose arbitrary elements $X, Y \in \mathfrak{s}$, so that the vector fields X^* and Y^* are tangent to $\mathsf{S} \cdot o$ and their values at o are $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ respectively. We evaluate the covariant derivative $\nabla_{Y^*} X^*$ at o. We see that

$$\begin{aligned}
\nabla_{Y^*} X^* = \nabla_{Y_{\mathfrak{p}}^*} X^* = \nabla_{X^*} Y_{\mathfrak{p}}^* + [Y_{\mathfrak{p}}^*, X^*] = \nabla_{X_{\mathfrak{p}}^*} Y_{\mathfrak{p}}^* + [X, Y_{\mathfrak{p}}]^* \\
= &\nabla_{X_{\mathfrak{p}}^*} Y_{\mathfrak{p}}^* + [X_{\mathfrak{k}}, Y_{\mathfrak{p}}]^* + [X_{\mathfrak{p}}, Y_{\mathfrak{p}}]^* \\
= &-\frac{1}{2}[X_{\mathfrak{p}}, Y_{\mathfrak{p}}]_{\mathfrak{p}} + U(X_{\mathfrak{p}}, Y_{\mathfrak{p}}) + [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{p}}]_{\mathfrak{p}} = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + D_{X_{\mathfrak{p}}} Y_{\mathfrak{p}}.
\end{aligned}$$

Thus, projecting to the normal space and using the fact that the second fundamental form is symmetric, we obtain that

$$\mathrm{I\!I}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}) = (\nabla_{Y^*} X^*)_{\mathfrak{s}_{\mathfrak{p}}^{\perp}} = ([X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + D_{X_{\mathfrak{p}}} Y_{\mathfrak{p}})_{\mathfrak{s}_{\mathfrak{p}}^{\perp}},$$

as desired. □

An important class of Riemannian submersions involving homogeneous spaces is that of homogeneous fibrations, which we briefly describe. The main reference is [82]. Let $\mathsf{H} \subseteq \mathsf{K} \subseteq \mathsf{G}$ be a chain of inclusions of compact connected subgroups of the connected Lie group G, and endow G/K with a G-invariant Riemannian metric. Then there exists a left-invariant metric on G that is also right K-invariant, and a G-invariant metric on G/H that makes the canonical projection $\pi\colon \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K}$ a Riemannian submersion with totally geodesic fibers isometric to K/H. We denote by $\mathcal{V}$ and $\mathcal{H}$ the vertical and horizontal distributions associated with the Riemannian submersion π. In Chapter 6 we will be concerned with the case where G is a compact group with a bi-invariant metric and the homogeneous spaces $F = \mathsf{K}/\mathsf{H}$, $M = \mathsf{G}/\mathsf{H}$ and $B = \mathsf{G}/\mathsf{K}$ are endowed with the corresponding normal homogeneous metrics. In this case, the tangent space T_oM at $o = e\mathsf{H}$ is identified with $\mathfrak{p} = \mathfrak{g} \ominus \mathfrak{h}$, so the vertical and horizontal subspaces at o are $\mathcal{V}_o = \mathfrak{k} \ominus \mathfrak{h}$ and $\mathcal{H}_o = \mathfrak{p} \ominus \mathfrak{k}$. It turns out that the distributions $\mathcal{V}$ and $\mathcal{H}$ are G-invariant in the sense that for every $p \in M$ and $g \in \mathsf{G}$, we have $g_{*p}\mathcal{V}_p = \mathcal{V}_{g\cdot p}$ and $g_{*p}\mathcal{H}_p = \mathcal{H}_{g\cdot p}$.

1.4 Riemannian symmetric spaces

The study of symmetric spaces started with the (nearly) centennial work of Cartan [39], who was interested in understanding which Riemannian manifolds have parallel curvature tensor (known today as *locally symmetric spaces*). In fact, a manifold M satisfies $\nabla R = 0$ if and only if every point p of M has a locally defined geodesic reflection—that is, an isometry that reverses the geodesics around p. The classification of symmetric spaces was already achieved by Cartan in the aforementioned work. In our view, the quintessential reference for Riemannian symmetric spaces is [88]. We also recommend [105, Chapter XI] and [145, Chapter 8].

Let M be a Riemannian manifold. We say that M is a *(Riemannian) symmetric space* if for every point $p \in M$ there exists a global isometry $s_p\colon M \to M$ satisfying $s_p(p) = p$ and $(s_p)_{*p} = -\operatorname{id}_{T_pM}$. The map s_p is unique and satisfies $s_p^2 = \operatorname{id}_M$. We call s_p the *geodesic reflection* at p. Throughout this thesis, we work exclusively with connected Riemannian symmetric spaces.

Fix a (connected) symmetric space M. Then one can show that M is homogeneous, and in particular that the Lie group $\mathsf{G} = I^0(M)$ acts transitively on M. We fix a point $o \in M$ and consider the isotropy subgroup $\mathsf{K} = \mathsf{G}_o$, which is compact. Then M can be written as the homogeneous space G/K, and we may define an involutive automorphism $\Theta\colon \mathsf{G} \to \mathsf{G}$ by declaring $\Theta(g) = s_o g s_o$ for each $g \in \mathsf{G}$. If G^Θ denotes the fixed point subgroup of Θ, then we have $(\mathsf{G}^\Theta)^0 \subseteq \mathsf{K} \subseteq \mathsf{G}^\Theta$. The differential $\theta = \Theta_*$ is thus an involutive automorphism of $\mathfrak{g}$ for which $\ker(1-\theta) = \mathfrak{k}$. We say that θ is a *Cartan involution*[3] and the corresponding $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is its associated *Cartan decomposition*. In particular, the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is reductive and allows us to identify $\mathfrak{p}$ with T_oM. We can also describe $\mathfrak{k}$ and $\mathfrak{p}$ in purely geometric terms. Indeed, we have

$$\mathfrak{k} = \{X \in \mathfrak{g}\colon X^*_o = 0\}, \quad \mathfrak{p} = \{X \in \mathfrak{g}\colon (\nabla X^*)_o = 0\}.$$

Conversely, one can produce examples of symmetric spaces from purely algebraic data. We say that a Klein pair (G, K) is a *Riemannian symmetric pair* if the following conditions are satisfied:

[3] Despite this nomenclature, θ need not be a Cartan involution in the sense of Subsection 1.1.1.

- The group G is connected.
- There exists an involution $\Theta \in \mathrm{Aut}(\mathsf{G})$ such that $(\mathsf{G}^{\Theta})^0 \subseteq \mathsf{K} \subseteq \mathsf{G}^{\Theta}$.
- The subgroup $\mathrm{Ad}_{\mathfrak{g}}(\mathsf{K}) \subseteq \mathsf{GL}(\mathfrak{g})$ is compact.

Given a Riemannian symmetric pair (G, K), the compactness of $\mathrm{Ad}_{\mathfrak{g}}(\mathsf{K})$ implies that there exists a G-invariant metric on the homogeneous space $M = \mathsf{G}/\mathsf{K}$. Moreover, for $o = e\mathsf{K}$, the map $s_o\colon g\mathsf{K} \in M \to \Theta(g)\mathsf{K} \in M$ turns out to be the geodesic reflection at o, and if follows from homogeneity that M is a Riemannian symmetric space. Note that, even if the pair (G, K) is effective, the group G need not coincide with the identity component of $I(M)$. For the purposes of this thesis, whenever we consider a symmetric space $M = \mathsf{G}/\mathsf{K}$ we make the tacit assumption that G is a finite cover of $I^0(M)$. In particular, we only work with almost effective symmetric pairs having compact isotropy subgroups.

Let us fix a Riemannian symmetric space $M = \mathsf{G}/\mathsf{K}$ and set $o = e\mathsf{K}$. Because the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, we automatically deduce that M is naturally reductive and the difference tensor vanishes identically. Thus, the formula (1.4) for the curvature tensor reduces to

$$R(X,Y)Z = -[[X,Y],Z], \quad X,Y,Z \in \mathfrak{p}. \tag{1.7}$$

Moreover, it is also clear from (1.5) that $\nabla R = 0$.

We say that a vector subspace $\mathfrak{v} \subseteq \mathfrak{p}$ is a *Lie triple system* if $[[\mathfrak{v}, \mathfrak{v}], \mathfrak{v}] \subseteq \mathfrak{v}$. It turns out that (complete) totally geodesic submanifolds of M containing o are in a one to one correspondence with Lie triple systems in $\mathfrak{p}$. On the one hand, if $S \subseteq M$ is a totally geodesic submanifold of M and $o \in S$ (this is not a restrictive condition, as M is homogeneous), then the tangent space $\mathfrak{v} = T_oS \subseteq \mathfrak{p}$ is invariant under the curvature tensor, hence a Lie triple system by (1.7). On the other hand, if $\mathfrak{v} \subseteq \mathfrak{p}$ is a Lie triple system, then one checks that $\mathfrak{h} = [\mathfrak{v}, \mathfrak{v}] \oplus \mathfrak{v}$ is the smallest subalgebra of $\mathfrak{g}$ containing $\mathfrak{v}$. If $\mathsf{H} \subseteq \mathsf{G}$ is the connected subgroup of G with Lie algebra $\mathfrak{h}$, then the orbit $S = \mathsf{H} \cdot o$ is a complete totally geodesic submanifold of M with $o \in S$ and $T_oS = \mathfrak{v}$. A direct consequence of this argument is that every complete totally geodesic submanifold of M is extrinsically homogeneous. Furthermore, if S is a complete totally geodesic submanifold of M and $o \in S$, the geodesic reflection s_o preserves S and restricts to a geodesic reflection of S, meaning that S is also a Riemannian symmetric space. One can see that a totally geodesic submanifold S of M containing o is flat if and only if its corresponding Lie triple system is abelian. In this case, S is called a *flat* of M. It turns out that all maximal abelian subspaces of $\mathfrak{p}$ are conjugate under the action of K, which means that any two maximal flats of M are isometrically congruent. We define the *rank* of M as the dimension of any maximal flat of M (equivalently, of any maximal abelian subspace of $\mathfrak{p}$).

Note that our choice of presentation $M = \mathsf{G}/\mathsf{K}$ guarantees that $\mathfrak{g}$ is the isometry algebra of M and $\mathfrak{k}$ is the isotropy algebra at the point $o = e\mathsf{K}$. In particular, the restriction of the Killing form $\mathcal{B}_{\mathfrak{g}}$ to $\mathfrak{k} \times \mathfrak{k}$ is negative definite. We say that M is:

- of *Euclidean type* if $(\mathcal{B}_{\mathfrak{g}})|_{\mathfrak{p}\times\mathfrak{p}}$ is identically zero,
- of *compact type* if $(\mathcal{B}_{\mathfrak{g}})|_{\mathfrak{p}\times\mathfrak{p}}$ is negative definite,
- and of *noncompact type* if $(\mathcal{B}_{\mathfrak{g}})|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite.

One sees that symmetric spaces of Euclidean type are flat, whereas symmetric spaces of compact (respectively, noncompact) type have nonnegative (respectively, nonpositive) sectional curvature. Moreover, if M is of compact type then G is a compact semisimple Lie group, whereas if M is of noncompact type the group G is semisimple of the noncompact type. If M is a symmetric space, then its universal cover $\widetilde{M}$ is also seen to be symmetric, and the de Rham theorem allows us to decompose $\widetilde{M} = M_0 \times M_1 \times \cdots \times M_k$, where M_0 is a Euclidean space (called the *flat factor*) and each M_i with $i > 0$ is irreducible. It can be shown that each of the M_i is necessarily of compact or noncompact type, and thus $\widetilde{M}$ splits as the Riemannian product of some Euclidean space, a symmetric space of compact type and a symmetric space of noncompact type.

There exists a notion of duality between symmetric spaces of compact and noncompact type that generalizes the existing one between spherical and hyperbolic geometry. This can be described in Lie algebraic terms as follows: given a symmetric space $M = \mathsf{G}/\mathsf{K}$ of noncompact type, then $\mathfrak{g}$ is a real semisimple Lie algebra, and thus $\mathfrak{g}(\mathbb{C})$ is a complex semisimple Lie algebra. Denoting by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$, we define $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g}(\mathbb{C})$. It is readily checked that $\mathfrak{g}^*$ is a compact Lie subalgebra of $\mathfrak{g}(\mathbb{C})$, known as the *dual algebra* of $\mathfrak{g}$. Let G^* be the simply connected Lie group with Lie algebra $\mathfrak{g}^*$ and $\mathsf{K}^* \subseteq \mathsf{G}^*$ the connected subgroup corresponding to $\mathfrak{k}$. Then $M^* = \mathsf{G}^*/\mathsf{K}^*$ can be endowed with the metric coming from the opposite of the Killing form of $\mathfrak{g}^*$, so that it becomes a simply connected symmetric space of compact type, known as the *dual* of M. For instance, the symmetric spaces of noncompact type and rank one are the real hyperbolic spaces $\mathbb{R}\mathsf{H}^n$, the complex hyperbolic spaces $\mathbb{C}\mathsf{H}^n$, the quaternionic hyperbolic spaces $\mathbb{H}\mathsf{H}^n$ and the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$. Their compact duals are, respectively, the spheres S^n, the complex projective spaces $\mathbb{C}\mathsf{P}^n$, the quaternionic projective spaces $\mathbb{H}\mathsf{P}^n$ and the Cayley projective plane $\mathbb{O}\mathsf{P}^2$.

1.4.1 Symmetric spaces of noncompact type

In this thesis we mainly devote our attention to symmetric spaces of noncompact type. Because of this, it is pertinent to discuss the properties that are exclusive to this family. A nice reference that treats symmetric spaces of noncompact type extensively is [59].

Consider a symmetric space $M = \mathsf{G}/\mathsf{K}$ of noncompact type with $o = e\mathsf{K}$, and let $\Theta\colon \mathsf{G} \to \mathsf{G}$ be the involution induced from the geodesic reflection at o. The Lie algebra $\mathfrak{g}$ is semisimple of noncompact type, and in this case the differential $\theta = \Theta^*\colon \mathfrak{g} \to \mathfrak{g}$ is also a Cartan involution in the sense of Lie algebras. One can show that M is a *Hadamard* manifold (that is, a complete simply connected manifold of nonpositive curvature), and the Cartan–Hadamard theorem implies that the Riemannian exponential map of M at any point is a diffeomorphism. Consequently, K is a compact connected subgroup of G, so the center of G is finite. In fact, the center of $I^0(M)$ is trivial.

Let us choose an Iwasawa decomposition of $\mathfrak{g}$. More precisely, we take a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ with associated root system Σ, a set of positive roots $\Sigma^+ \subseteq \Sigma$ with corresponding simple roots Λ and define $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$. Let A, N and AN be the connected subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$. Note that $\mathsf{A} \cdot o$ is a maximal flat of M, and thus the rank of M coincides with $\dim \mathfrak{a}$.

A straightforward calculation shows that AN acts simply transitively on M. From this we deduce that the map $g \in \mathsf{AN} \mapsto g \cdot o \in M$ is an AN-equivariant global diffeomorphism, and we can equip AN with the Riemannian metric $\langle \cdot, \cdot \rangle_{\mathsf{AN}}$ that makes this map an isometry. This shows that every symmetric space of noncompact type is isometric to a solvable Lie group with a left-invariant Riemannian metric. We say that $M = \mathsf{AN}$ is the *solvable model* of M. In the case that the metric on M is induced from the Killing form of $\mathfrak{g}$, one can see that the Levi-Civita connection of AN is characterized by the condition

$$4\langle \nabla_X Y, Z\rangle_{\mathsf{AN}} = \langle [X,Y] + (1-\theta)[\theta X, Y], Z\rangle, \quad X, Y, Z \in \mathfrak{a} \oplus \mathfrak{n}.$$

Parabolic subalgebras

We finish this chapter by introducing the notion of parabolic subalgebras of a real semisimple Lie algebra and describing them by means of its root space decomposition. We also comment on their application to symmetric spaces of noncompact type. For the sake of convenience, we opt to take a purely algebraic approach to parabolic subalgebras based on [102] and [32], but one can find a more geometric treatment of these objects in [59].

From now on we fix a symmetric space $M = \mathsf{G}/\mathsf{K}$ of noncompact type and we take an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of its isometry algebra. We say that a Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ is *parabolic* if its complexification $\mathfrak{q}(\mathbb{C})$ contains a maximal solvable subalgebra of $\mathfrak{g}(\mathbb{C})$. The first example is $\mathfrak{q}_0 = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. It turns out that $\mathfrak{q}_0$ is a minimal parabolic subalgebra of $\mathfrak{g}$ and every minimal parabolic subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{q}_0$ under $\mathrm{Ad}(\mathsf{G})$. Consequently, an arbitrary subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ is parabolic if and only if $\mathrm{Ad}(g)\mathfrak{q}$ contains $\mathfrak{q}_0$ for some $g \in \mathsf{G}$.

To each subset of simple roots we can associate a parabolic subalgebra by the following procedure. Let $\Phi \subseteq \Lambda$ be any subset. We define $\Sigma_\Phi = \Sigma \cap (\operatorname{span}\Phi)$ to be the root subsystem of Σ generated by Φ and set $\Sigma_\Phi^+ = \Sigma^+ \cap \Sigma_\Phi$. We first consider the following subalgebras of $\mathfrak{a} \oplus \mathfrak{n}$:

$$\mathfrak{a}^\Phi = \bigoplus_{\alpha\in\Phi} \mathbb{R}H_\alpha, \qquad \mathfrak{a}_\Phi = \mathfrak{a} \ominus \mathfrak{a}^\Phi = \bigcap_{\alpha\in\Phi} \ker\alpha,$$

$$\mathfrak{n}^\Phi = \bigoplus_{\lambda\in\Sigma_\Phi^+} \mathfrak{g}_\lambda, \qquad \mathfrak{n}_\Phi = \mathfrak{n} \ominus \mathfrak{n}^\Phi = \bigoplus_{\lambda\in\Sigma^+\setminus\Sigma_\Phi^+} \mathfrak{g}_\lambda.$$

If we set

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma_\Phi} \mathfrak{g}_\lambda, \quad \mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi,$$

then it is clear that $\mathfrak{q}_\Phi$ contains $\mathfrak{q}_0$, so it is automatically a parabolic subalgebra. We call $\mathfrak{q}_\Phi$ the *parabolic subalgebra determined by* Φ. Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{q}_\Phi$ for some $\Phi \subseteq \Lambda$. Moreover, $\mathfrak{q}_\varnothing = \mathfrak{q}_0$ and $\mathfrak{q}_\Lambda = \mathfrak{g}$. We set $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$, which is a subalgebra of $\mathfrak{g}$ that normalizes $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$. Then we have the decomposition

$$\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi,$$

which is known as the *Langlands decomposition* of $\mathfrak{q}_\Phi$. The Lie algebra $\mathfrak{m}_\Phi$ is reductive, which implies that $\mathfrak{g}_\Phi = [\mathfrak{m}_\Phi, \mathfrak{m}_\Phi]$ is a semisimple Lie algebra.

Let us consider the connected subgroups Q_Φ, M_Φ, G_Φ, A_Φ and N_Φ with Lie algebras $\mathfrak{q}_\Phi$, $\mathfrak{m}_\Phi$, $\mathfrak{g}_\Phi$, $\mathfrak{a}_\Phi$ and $\mathfrak{n}_\Phi$ respectively. Then Q_Φ is the *parabolic subgroup* of G associated with Φ and the map

$$(m, a, n) \in \mathsf{M}_\Phi \times \mathsf{A}_\Phi \times \mathsf{N}_\Phi \mapsto man \in \mathsf{Q}_\Phi$$

is a global diffeomorphism, known as the *Langlands decomposition* of Q_Φ.

We now define $B_\Phi = \mathsf{M}_\Phi \cdot o = \mathsf{G}_\Phi \cdot o$. It is not hard to show that B_Φ is a complete totally geodesic submanifold of M whose tangent space at o is

$$\mathfrak{b}_\Phi = \mathfrak{m}_\Phi \cap \mathfrak{p} = \mathfrak{a}^\Phi \oplus \bigoplus_{\lambda \in \Sigma_\Phi^+} \mathfrak{p}_\lambda .$$

The submanifold B_Φ is usually known as a *boundary component* of M [32]. One sees that the isometry algebra of B_Φ is contained in $\mathfrak{g}_\Phi$. The Langlands decomposition of Q_Φ induces a global diffeomorphism

$$(m \cdot o, a, n) \in B_\Phi \times \mathsf{A}_\Phi \times \mathsf{N}_\Phi \mapsto (man) \cdot o \in M,$$

known as a *horospherical decomposition* of M.

Part I

Polar actions on symmetric spaces

Chapter 2
Isometric and polar actions on Riemannian manifolds

In this chapter we introduce the basic notions pertaining isometric actions on Riemannian manifolds, with a special focus on polar actions. We primarily follow the references [17, 83, 121, 142, 143].

Firstly, we introduce the concept of proper action. Broadly speaking, a Lie group G acts properly on a manifold M if for every compact set $K \subseteq M$ the elements of G that are sufficiently far from the identity carry K to a subset of $M \setminus K$. These actions generalize compact group actions, and thus have many of the desirable properties that are satisfied automatically in the compact case (such as the orbit space being Hausdorff and all orbits being properly embedded submanifolds of the ambient space).

Afterwards, we present one of the central topics of this thesis, which is that of polar actions. An isometric action of a connected Lie group G on a complete Riemannian manifold is said to be polar if it admits a section (that is, a submanifold meeting all orbits orthogonally). Polar actions were first introduced by Conlon [44], with the further restrictions that G is compact and that the sections are flat. Later on, Dadok [49] studied the case of polar representations in depth, showing that these representations arise from isotropy representations of symmetric spaces, see Theorem 2.7. Perhaps the first detailed treatment of polar actions in general was carried out by Palais and Terng in [142].

Many well known results in algebra and geometry can be stated in terms of polar actions. The most basic example of this phenomenon is the spectral theorem for self-adjoint operators, stating that every symmetric $n \times n$ matrix admits an orthogonal basis of eigenvectors. This result follows from the conjugation action of $\mathsf{SO}(n)$ on the vector space M of symmetric $n \times n$ matrices (of trace zero) being polar.

A more sophisticated application of polar actions appears in invariant theory. For example, suppose G is a compact connected Lie group. A continuous map $\chi\colon \mathsf{G} \to \mathbb{R}$ is a *class function* if it is constant along the adjoint orbits of G. The typical example of a class function is the character of a (say, real) representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, which is defined as the map $\chi\colon g \in \mathsf{G} \mapsto \operatorname{tr}\rho(g) \in \mathbb{R}$. The maximal torus theorem asserts that G admits a unique maximal torus T up to conjugacy. It turns out that a class function χ is completely determined by its restriction to T (because a maximal torus intersects all adjoint orbits). Moreover, the so-called *Weyl group* $\mathsf{W}(\mathsf{G},\mathsf{T}) = \mathsf{N}_{\mathsf{G}}(\mathsf{T})/\mathsf{Z}_{\mathsf{G}}(\mathsf{T})$ is a finite group acting on T in such a way that the restriction map induces an isomorphism of real algebras

$$\left\{ \begin{array}{c} \text{Class functions} \\ \chi\colon \mathsf{G} \to \mathbb{R} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathsf{W}(\mathsf{G},\mathsf{T})\text{-invariant continuous} \\ \chi\colon \mathsf{T} \to \mathbb{R} \end{array} \right\}$$

At the Lie algebra level, one has an analogous version of the above correspondence, known as the *Chevalley restriction theorem*. To state the theorem, let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T respectively and consider the real algebras $\mathbb{R}[\mathfrak{g}]$, $\mathbb{R}[\mathfrak{t}]$ consisting of all polynomial functions defined on $\mathfrak{g}$ and $\mathfrak{t}$ respectively. Denote by $\mathbb{R}[\mathfrak{g}]^{\mathsf{G}}$ (respectively, $\mathbb{R}[\mathfrak{t}]^{\mathsf{W}(\mathsf{G},\mathsf{T})}$) the real subalgebra of $\mathbb{R}[\mathfrak{g}]$ of invariant polynomials under the adjoint representation (respectively, the subalgebra of $\mathbb{R}[\mathfrak{t}]$ consisting of invariant polynomials under the action of the Weyl group). The Chevalley restriction theorem states that the map $f \in \mathbb{R}[\mathfrak{g}]^{\mathsf{G}} \mapsto f|_{\mathfrak{t}} \in \mathbb{R}[\mathfrak{t}]^{\mathsf{W}(\mathsf{G},\mathsf{T})}$ is an isomorphism, which reduces the theory of Ad-invariant polynomials to that of polynomials invariant under a finite group. These results become clear from the fact that if a Lie group G acts polarly on a manifold M with section Σ, then M/G is naturally homeomorphic to the orbit space $\Sigma/\Pi(\Sigma)$ under the action of the polar group (a generalization of the Weyl group for polar actions, see Subsection 2.2.4).

Another application of polar actions (with a similar philosophy) lies in the analysis of partial differential equations on Riemannian manifolds and the reduction of their complexity under the assumption of symmetry. In a very loose sense, given a smooth action $\mathsf{G} \curvearrowright M$, one can think of G-invariant objects in M as objects in the orbit space M/G. The issue in general is that M/G may not be a manifold in general, meaning that working in the orbit space can become quite difficult. In the polar case, the homeomorphism $M/\mathsf{G} \cong \Sigma/\Pi(\Sigma)$ allows us to view the quotient M/G as an *orbifold* (that is, a second countable and Hausdorff topological space that is locally homeomorphic to a finite quotient of the Euclidean space), which gives a significant advantage with respect to the case of general actions.

A particular case where this situation is extremely favorable is that of cohomogeneity one actions (which are always polar, see Example 2.3). The orbit space M/G of a cohomogeneity one action $\mathsf{G} \curvearrowright M$ is either $\mathbb{R}$, S^1, $[0,\infty)$ or $[0,1]$. Therefore, a G-invariant solution to a partial differential equation on M corresponds to a solution to a differential equation on one of the aforementioned spaces. A very simple example of this is problem of determining the space of $\mathsf{O}(n)$-invariant solutions to the Laplace equation $\Delta u = 0$ on $\mathbb{R}^n$. An $\mathsf{O}(n)$-invariant function $u\colon \mathbb{R}^n \to \mathbb{R}$ can be written as $u(x) = f(|x|)$ for some $f\colon [0,\infty) \to \mathbb{R}$. The Laplace equation becomes the ordinary differential equation $\frac{d}{dt}(t^{n-1}f'(t)) = 0$, and it can be solved explicitly without issues. With these ideas, several authors have been able to produce new examples of manifolds with exceptional holonomy [35] and inhomogeneous Einstein metrics [28], to name a few examples.

We are interested in treating polar actions and their classification from the point of view of submanifold geometry. To this end, we say that two isometric actions $\mathsf{G} \curvearrowright M$ and $\mathsf{G}' \curvearrowright M$ are said to be *orbit equivalent* if there exists an isometry $f\colon M \to M$ such that $f(\mathsf{G}\cdot p) = \mathsf{G}'\cdot f(p)$ for all $p \in M$. This notion of equivalence places the orbits of isometric actions in the spotlight, as it does not really take into account the groups G and G' more so than the orbit foliations on M induced by them. In particular, we may well have non-isomorphic groups giving rise to orbit equivalent actions. Our main focus during this part of the thesis is the classification problem for polar actions on symmetric spaces (of noncompact type) up to orbit equivalence. While this problem is nearing its full conclusion in the compact setting, little is known about polar actions on symmetric spaces of noncompact type. See Subsection 2.3 for a more detailed account of the current progress in this area.

We briefly describe the organization of this chapter. In Section 2.1, we give the formal definition of proper action and introduce all the relevant concepts surrounding it. Section 2.2 will be devoted to introducing polar actions, as well as describing the main features of their orbits and sections. Finally, in Section 2.3 we discuss the known results concerning the classification of polar actions on Riemannian symmetric spaces.

2.1 Proper isometric actions

Let M be a smooth manifold and G be a Lie group acting smoothly on M. We assume from now on that M is connected (but G need not be connected). We define an associated a *shear map* $\theta\colon \mathsf{G} \times M \to M \times M$ to the action $\mathsf{G} \curvearrowright M$ defined by the equation $\theta(g, p) = (g \cdot p, p)$. The action of G is called *proper* if θ is a proper map. It can be shown that the following conditions are equivalent to properness:

- For any pair of sequences $(g_n) \in \mathsf{G}$ and $(x_n) \in M$ such that (x_n) and $(g_n \cdot x_n)$ converge, the sequence (g_n) admits a convergent subsequence.
- Given a compact set $K \subseteq M$, the set $\{g \in \mathsf{G}\colon g \cdot K \cap K \neq \varnothing\}$ is compact.
- Given two points x, $y \in M$ there exist open neighborhoods U, $V \subseteq M$ of x and y respectively such that the set $\{g \in \mathsf{G}\colon g \cdot U \cap V \neq \varnothing\}$ has compact closure in G.

If G acts properly on M, then the orbit space M/G is Hausdorff, all isotropy subgroups are compact and the orbits of the action are closed in M. As a consequence, if $p \in M$ is arbitrary, the map $g\mathsf{G}_p \in \mathsf{G}/\mathsf{G}_p \to g \cdot p \in M$ is an embedding with image $\mathsf{G} \cdot p$. The orbit space can be endowed with a distance function $d\colon M/\mathsf{G} \times M/\mathsf{G} \to \mathbb{R}$, where $d(\mathsf{G} \cdot p, \mathsf{G} \cdot q)$ is merely the distance between the orbits $\mathsf{G} \cdot p$ and $\mathsf{G} \cdot q$ as closed subsets of M.

Let $\mathsf{G} \curvearrowright M$ be a proper action. We say that two orbits $\mathsf{G} \cdot p$ and $\mathsf{G} \cdot q$ *have the same orbit type* if the isotropy subgroups G_p and G_q are conjugate (equivalently, if there exists a G-equivariant bijection $f\colon \mathsf{G} \cdot p \to \mathsf{G} \cdot q$). This gives an equivalence relation on the orbit space M/G, whose equivalence classes are known as *orbit types*. Moreover, we define a partial ordering $\leq$ on the set of orbit types by letting $[\mathsf{G} \cdot p] \leq [\mathsf{G} \cdot q]$ whenever there exists a G-equivariant map $f\colon \mathsf{G} \cdot q \to \mathsf{G} \cdot p$ (equivalently, if G_q is conjugate to a subgroup of G_p). The fact that $\leq$ is indeed a partial ordering follows from the compactness of all isotropy subgroups.

Let $p \in M$ be any point. We say that the orbit $\mathsf{G} \cdot p$ is:

- *principal* if there exists a G-invariant neighborhood U of p such that for all $q \in U$ we have $[\mathsf{G} \cdot p] \geq [\mathsf{G} \cdot q]$ (in other words, if G_p does not properly contain the isotropy group of a point in U);
- *singular* if its dimension is smaller than that of any principal orbit;
- *exceptional* if it is not principal but has the same dimension as a principal orbits; and
- *regular* if it is either principal or exceptional.

Because M is assumed to be connected, one sees that all principal orbits have the same type, and the union M_{prin} of all principal orbits constitutes a dense open subset of M. The *cohomogeneity* of the action $\mathsf{G} \curvearrowright M$ is defined as the codimension of any of the G-principal orbits. We denote this number by $\operatorname{cohom}(\mathsf{G} \curvearrowright M)$. If the action $\mathsf{G} \curvearrowright M$ has no singular orbits (that is, if all orbits have the same dimension) we say that G *induces a (homogeneous) foliation* on M. The family $\mathcal{F} = \{\mathsf{G} \cdot p \colon p \in M\}$ is the *foliation* induced by the action.

For the purposes of this thesis, we will be interested in studying proper isometric actions on complete Riemannian manifolds. If M is a connected and complete Riemannian manifold and G is a Lie group acting isometrically on M, then there exists an induced Lie group homomorphism $\phi \colon \mathsf{G} \to I(M)$ defined by the equation $\phi(g)(p) = g \cdot p$. We may replace G with the quotient $\mathsf{G}/\ker\phi$ and assume directly that G is a Lie subgroup of $I(M)$ that acts effectively on M. Therefore, we are interested in understanding which subgroups of the isometry group $I(M)$ act properly on M. The proposition below, which is a combination of [51, Theorem 4] and [57, Theorem A], provides a characterization of these subgroups.

Proposition 2.1. *Let M be a complete Riemannian manifold and $\mathsf{G} \subseteq I(M)$ a Lie subgroup of its full isometry group. Then, the following conditions are equivalent:*

(i) *The natural action of G on M is proper.*

(ii) *G is a closed subgroup of $I(M)$.*

(iii) *For all $p \in M$, the orbit $\mathsf{G} \cdot p$ is closed in M and the isotropy subgroup G_p is closed in $I(M)$.*

(iv) *There exists a $p \in M$ such that the orbit $\mathsf{G} \cdot p$ is closed in M and the isotropy subgroup G_p is closed in $I(M)$.*

Observe that in items (iii) and (iv) we are requiring that the isotropy subgroups are closed in the full isometry group. It is clear that these subgroups are always closed in G.

One of the main features of proper actions is the existence of so-called slices. If G is a Lie group acting properly on M and $p \in M$, we say that a subset $S \subseteq M$ is a *slice* at p if there exists a G-invariant neighborhood $U \subseteq M$ containing p and a G-equivariant retraction $r \colon U \to \mathsf{G} \cdot p$ satisfying $S = r^{-1}(p)$. The celebrated *slice theorem* [141] states that every point $p \in M$ admits a slice.

Suppose S is a slice at $p \in M$. Then G_p leaves S invariant, so we can construct an associated bundle from the G_p-principal bundle $\mathsf{G} \to \mathsf{G} \cdot p$ and the action $\mathsf{G}_p \curvearrowright S$. By definition, the associated bundle is $\mathsf{G} \times_{\mathsf{G}_p} S = (\mathsf{G} \times S)/\mathsf{G}_p$, where the action of G_p is given by $k \cdot (g, s) = (gk^{-1}, k \cdot s)$. A consequence of the definitions is that $U = \mathsf{G} \cdot S$ is diffeomorphic to $\mathsf{G} \times_{\mathsf{G}_p} S$ via the natural map $[g, s] \mapsto g \cdot s$. Moreover, given any $x \in S$, the isotropy subgroup of x is $\mathsf{G}_x = (\mathsf{G}_p)_x \subseteq \mathsf{G}_p$, which shows that the orbit type $[\mathsf{G} \cdot p]$ is always minimum in a G-invariant neighborhood of p. In particular, $\mathsf{G} \cdot p$ is a principal orbit if and only if the induced action of G_p on S is trivial.

Now, suppose that M is a complete Riemannian manifold and G acts on M properly and isometrically. For every $p \in M$, it is easy to show that there exists an $\varepsilon > 0$ such that the exponential map $\exp_p\colon B_{T_pM}(0,\varepsilon) \to M$ is a diffeomorphism onto its image and the set $S = \exp_p(B_{\nu_p(\mathsf{G}\cdot p)}(0,\varepsilon))$ is a slice at p. We say in this case that S is a *geodesic slice*, and it is clear that in this case $\mathsf{G}\cdot S$ is the exponential image of the set $\{\xi \in \nu(\mathsf{G}\cdot p)\colon |\xi| < \varepsilon\}$. Because the exponential map $\exp_p\colon B_{\nu_p(\mathsf{G}\cdot p)}(0,\varepsilon) \to S$ is a G_p-equivariant diffeomorphism, the action of G_p on S is isomorphic to its action on an open ball in the normal space $\nu_p(\mathsf{G}\cdot p)$.

Motivated by the above fact, we define the *slice representation* of G_p at p as the representation $\mathsf{G}_p \to \mathsf{O}(\nu_p(\mathsf{G}\cdot p))$ given by $g\cdot\xi = g_{*p}\xi$. Observe that an orbit $\mathsf{G}\cdot p$ is principal if and only if the slice representation at p is trivial. Furthermore, the cohomogeneity of the slice representation at any point coincides with the cohomogeneity of the action $\mathsf{G} \curvearrowright M$. Note that a normal vector $\xi_p \in \nu_p(\mathsf{G}\cdot p)$ is fixed under the slice representation if and only if can be extended to an *equivariant normal vector field*, that is, a vector field $\xi \in \Gamma(\nu(\mathsf{G}\cdot p))$ satisfying $g_{*q}\xi_q = \xi_{g\cdot q}$ for all $g \in \mathsf{G}$ and $q \in \mathsf{G}\cdot p$. Equivariant normal vector fields may be used to compute the orbits of an isometric action from a given one. Indeed, if $\mathsf{G}\cdot p$ is any orbit and ξ is an equivariant normal vector field along $\mathsf{G}\cdot p$, an elementary calculation shows that

$$\mathsf{G}\cdot\exp_p(\xi_p) = \{\exp_x(\xi_x)\colon x \in \mathsf{G}\cdot p\}. \tag{2.1}$$

In general, the full set $Z = \exp_p(\nu_p(\mathsf{G}\cdot p))$ need not be a slice at p. However, we remark that Z meets every orbit of the G-action. Indeed, given another orbit $\mathcal{O} \subseteq M$, because M is complete and the orbits of G are closed we can find a geodesic segment $\gamma\colon [0,1] \to M$ that realizes the distance between $\mathsf{G}\cdot p$ and $\mathcal{O}$. After using the action of G if necessary, we can assume that $\gamma(0) = p$. The first variation formula for the length of γ readily implies that $\xi = \gamma'(0)$ is perpendicular to $\mathsf{G}\cdot p$, so $q = \gamma(1) = \exp_p(\xi) \in Z\cap\mathcal{O}$. One also sees that the isotropy subgroups $\mathsf{G}_{\gamma(t)}$ are the same for all $t \in (0,1)$, and they are contained in $\mathsf{G}_p \cap \mathsf{G}_q$. This result is known as *Kleiner's lemma* [4, Lemma 3.70]. At any rate, if p is contained in a principal orbit, because the set Z meets all orbits and the slice representation at p is trivial, we can reconstruct every orbit of the action from $\mathsf{G}\cdot p$ and the equivariant normal vector fields along this orbit.

Let M be a Riemannian manifold and suppose that a Lie group G acts properly and isometrically on M in such a way that all orbits have the same type (so they are all principal). Then the slice representation at any point is trivial, meaning that if S is a slice at $p \in M$ we have $\mathsf{G}\cdot S \cong \mathsf{G}\cdot p \times S$. Thus, the restriction of the canonical projection $\pi\colon M \to M/\mathsf{G}$ to S is a homeomorphism onto its image. These homeomorphisms may be used to define charts on M/G so that this quotient becomes a smooth manifold whose dimension equals the cohomogeneity of the action $\mathsf{G} \curvearrowright M$. According to [121, Section 29.21], the equation

$$\langle \pi_{*p}\xi, \pi_{*p}\eta\rangle = \langle \xi, \eta\rangle \quad p \in M,\ \ \xi,\eta \in \nu_p(\mathsf{G}\cdot p),$$

gives a well-defined Riemannian metric on M/G that makes the canonical projection π a Riemannian submersion. For a general proper isometric action $\mathsf{G} \curvearrowright M$, the same reasoning can be applied for the restriction of the action to the union M_{prin} of all principal orbits, so that $M_{\text{prin}}/\mathsf{G}$ becomes a Riemannian manifold in such a way that the projection $\pi\colon M_{\text{prin}} \to M_{\text{prin}}/\mathsf{G}$ becomes a Riemannian submersion. The distribution $\mathcal{H}$ on M_{prin} that assigns to each $p \in M_{\text{prin}}$ the normal space $\nu_p(\mathsf{G}\cdot p)$ is known as the *principal horizontal distribution*.

2.2 Polar actions

Let $\mathsf{G} \curvearrowright M$ be a proper isometric action of a Lie group G on a complete Riemannian manifold M. We say that the action of G is *polar* if there exists a connected, complete and injectively immersed submanifold $\Sigma \subseteq M$ that meets every orbit of M orthogonally. More precisely:

(i) the intersection $\mathsf{G} \cdot p \cap \Sigma$ is nonempty for all $p \in M$, and

(ii) the tangent spaces $T_p(\mathsf{G} \cdot p)$ and $T_p\Sigma$ are orthogonal for all $p \in \Sigma$.

The submanifold Σ is known as a *section*. If the action of G admits a flat section (with respect to the induced metric), then we say that the action is *hyperpolar*. Note that the condition (ii) is equivalent to the vector fields X^* (where $X \in \mathfrak{g}$) being orthogonal to Σ at every point.

Suppose G acts polarly on M and Σ is a section of the action. Then the map $h\colon (g,p) \in \mathsf{G} \times \Sigma \mapsto g \cdot p \in M$ is a surjective smooth map, so by Sard's theorem there exists some $(g,p) \in M$ such that $h_{*(g,p)}\colon T_g\mathsf{G} \oplus T_p\Sigma \to T_{g\cdot p}M$ is surjective. It is not hard to show from the definition of Σ that $\mathsf{G}\cdot p$ is a regular orbit and the dimension of Σ coincides with the codimension of $\mathsf{G}\cdot p$ in M. Therefore, we have $\dim \Sigma = \operatorname{cohom}(\mathsf{G} \curvearrowright M)$ and $T_x\Sigma = \nu_x(\mathsf{G}\cdot x)$ for all $x \in M$ belonging to a regular orbit.

Let us give some examples of polar actions to further illustrate the concept.

Example 2.2. Let $M = \mathbb{R}^n$ be the Euclidean plane and $\mathsf{G} = \mathsf{SO}(n)$. The standard representation $\mathsf{SO}(n) \curvearrowright \mathbb{R}^n$ is polar, since every one-dimensional subspace $\Sigma \subseteq \mathbb{R}^n$ is a section of the action. Note that this action is actually hyperpolar. Using the coordinates of the orbits of this action (that is, the spheres centered at $0 \in \mathbb{R}^n$) and the coordinates of any section, we can construct the usual (spherical) polar coordinates on $\mathbb{R}^n \setminus \{0\}$. This is the reason behind the nomenclature for polar actions.

Example 2.3. Let M be a complete Riemannian manifold and $\mathsf{G} \curvearrowright M$ a Lie group acting properly and isometrically on M with cohomogeneity one. If $\gamma\colon \mathbb{R} \to M$ is any geodesic and $X \in \mathfrak{g}$, the inner product $\langle X^*, \gamma' \rangle$ is constant along γ because X^* is a Killing vector field and γ' is parallel. Therefore, any geodesic that intersects a principal orbit orthogonally will meet all orbits orthogonally. The image $\Sigma = \gamma(\mathbb{R})$ is seen to be an injectively immersed submanifold of M [1, Theorem 6.1], meaning that every cohomogeneity one action on M is hyperpolar.

For instance, let $M = \mathbb{R}\mathsf{H}^2 = \mathsf{SL}(2,\mathbb{R})/\mathsf{SO}(2)$. Up to conjugacy, there are exactly three closed subgroups of $\mathsf{SL}(2,\mathbb{R})$ acting nontrivially and nontransitively, which are

$$\mathsf{K} = \mathsf{SO}(2), \qquad \mathsf{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}, \qquad \mathsf{N} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Because the actions of these groups are of cohomogeneity one, they are automatically hyperpolar.

Example 2.4. Suppose G is a compact connected Lie group. We say that a subgroup $\mathsf{T} \subseteq \mathsf{G}$ is a *maximal torus* if it is maximal among the subgroups of G isomorphic to a torus. Equivalently, a maximal torus of G is a maximal connected abelian subgroup of G. Because G is compact, we can

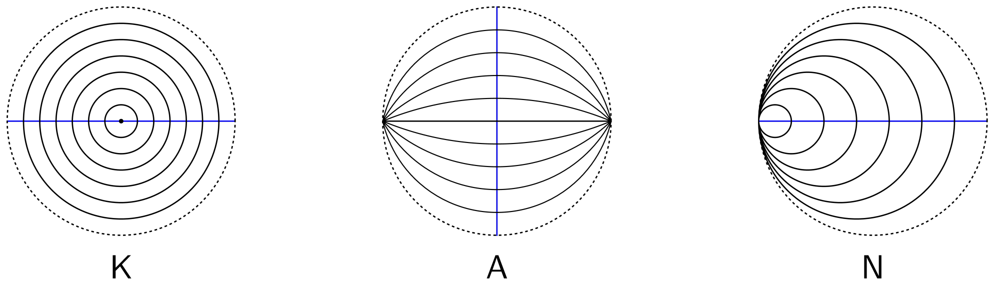


Figure 2.2.1: Cohomogeneity one actions on $\mathbb{R}\mathsf{H}^2$ and their sections.

endow it with a bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$. The adjoint action $\mathsf{G} \curvearrowright \mathsf{G}$ defined by $g{\cdot}x = gxg^{-1}$ for all g, $x \in \mathsf{G}$ is isometric with respect to the bi-invariant metric $\langle\cdot,\cdot\rangle$. Furthermore, this action is hyperpolar, as any maximal torus T of G is a section (see for example [4, Theorem 4.1]). As we will see later on, this implies that any two maximal tori in G are conjugate, so in particular they have the same dimension. This common dimension is known as the *rank* of G.

Example 2.5. Let $\mathsf{Sym}(n,\mathbb{R})$ be the vector space of all symmetric $n \times n$ matrices with real coefficients. We may decompose $\mathsf{Sym}(n,\mathbb{R}) = \mathbb{R}I \oplus \mathsf{Sym}_0(n,\mathbb{R})$, where $\mathsf{Sym}_0(n,\mathbb{R})$ is the space of symmetric matrices with zero trace. We can view $\mathsf{Sym}(n,\mathbb{R})$ as a representation of the special orthogonal group $\mathsf{SO}(n)$, where the action is given by $g \cdot X = gXg^{-1}$, so that $\mathbb{R}I \cong \mathbb{R}$ and $\mathsf{Sym}_0(n,\mathbb{R}) \cong \mathsf{S}^2_0\mathbb{R}^n$ are the irreducible submodules of this representation. The restricted representation $\mathsf{SO}(n) \curvearrowright \mathsf{Sym}_0(n,\mathbb{R})$ is hyperpolar, and the subspace Σ of all diagonal $n \times n$ matrices with trace zero is a section of this action. The fact that this action is polar is a restatement of the well-known *spectral theorem* for self-adjoint matrices: every symmetric $n \times n$ matrix can be diagonalized by an orthogonal matrix (because $\mathbb{R}I$ is the trivial module and it already consists of diagonal matrices, it is indeed enough to show this for traceless matrices). Motivated by this example, Palais and Terng [142] suggest that for a polar action $\mathsf{G} \curvearrowright M$ the elements of Σ should be regarded as *canonical forms* of the elements of M.

Examples 2.2 and 2.5 are cases of *polar representations*. At this point, it is also important to observe that both examples are actually the isotropy representations of a symmetric space (Example 2.2 corresponds to $M = \mathbb{R}\mathsf{H}^n = \mathsf{SO}^0(1,n)/\mathsf{SO}(n)$ while Example 2.5 corresponds to $\mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n)$). In general, we say that a representation $\rho\colon \mathsf{H} \to \mathsf{O}(V)$ of a compact Lie group H is an *s-representation* if there exists a simply connected semisimple symmetric space $M = \mathsf{G}/\mathsf{K}$ with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and a Lie group isomorphism $\phi\colon \mathsf{K} \to \mathsf{H}$ such that $\rho \circ \phi$ is isomorphic to the isotropy representation $\mathrm{Ad}\colon \mathsf{K} \to \mathsf{O}(\mathfrak{p})$. Because the isotropy representation of a symmetric space of noncompact type is the same as that of its dual of compact type (and vice-versa), we see that an s-representation is precisely the isotropy representation of a symmetric space of noncompact type.

A vast generalization of the above examples comes in the form of the following theorem, which states that s-representations are polar.

Theorem 2.6 [17, Theorem 2.3.15]**.** *Let $M = \mathsf{G}/\mathsf{K}$ be a simply connected semisimple Riemannian symmetric space and $o = e\mathsf{K}$. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition coming from the geodesic reflection at o, then the isotropy representation $\mathsf{K} \curvearrowright T_oM \cong \mathfrak{p}$ is polar. The sections of the isotropy representation are precisely the maximal abelian subspaces of $\mathfrak{p}$.*

Most remarkably, Dadok [49] shows that the converse of Theorem 2.6 is essentially true:

Theorem 2.7. *Let $\rho\colon \mathsf{H} \to \mathsf{O}(V)$ be a polar representation of a compact connected Lie group on an n-dimensional Euclidean vector space V. Then there exists an n-dimensional Riemannian symmetric space of noncompact type $M = \mathsf{G}/\mathsf{K}$ and a linear isometry $f\colon V \to T_{e\mathsf{K}}M$ that maps the orbits of H to the orbits of K under the isotropy representation. In other words, every polar representation is orbit equivalent to an s-representation.*

2.2.1 Sections of polar actions

In this subsection, we show that if $\mathsf{G} \curvearrowright M$ is a polar action and $\Sigma \subseteq M$ is a section of it, then Σ is a totally geodesic submanifold of M. Although this is a standard fact in the theory of isometric actions, surprisingly we have not been able to find a complete proof of this result. Some references that are usually cited for a proof are [142] (as well as the subsequent book [143]), by Palais and Terng, and [157] by Szenthe. In these texts the authors prove that the second fundamental form of Σ vanishes at all regular points contained in Σ. Thus, it is natural to suppose that the second fundamental form vanishes identically due to the density of M_{prin} in M. The issue here is that the set $\Sigma_{\text{prin}} = \Sigma \cap M_{\text{prin}}$ may not *a priori* be dense in Σ. While some authors comment on why this result should hold (see for example [4, Exercise 4.9 (iii)], [83, Proposition 1.3 (a)], [121, Theorem 30.9 (4)]), it appears that no detailed proofs are available.

In the note [120] (written in collaboration with Ivan Solonenko) we provide a complete proof that sections are totally geodesic, based on the route proposed in [17]. In particular, we deal with the density of Σ_{prin} in Σ. The results presented in this section are taken from that note.

We will need two lemmas to achieve our objective.

Lemma 2.8 [17, Exercise 2.11.5]**.** *Let $p \in M$ be such that $\mathsf{G} \cdot p$ is not a principal orbit, and let $V \subseteq \nu_p(\mathsf{G} \cdot p)$ be the subspace of fixed points of the slice representation at p. Then $\dim V$ is strictly smaller than the cohomogeneity of the action $\mathsf{G} \curvearrowright M$.*

Proof. Consider the orthogonal complement $V^\perp$ of V in $\nu_p(\mathsf{G} \cdot p)$, which is nonzero because the slice representation is not trivial. If $\xi \in \nu_p(\mathsf{G} \cdot p)$ is a regular point of the slice representation, then the codimension of $\mathsf{G}_p \cdot \xi$ in the normal space is the same as the cohomogeneity of the action of G. Write $\xi = \xi_V + \xi_{V^\perp}$ as the sum of its orthogonal projections onto V and $V^\perp$. Then the definition of V gives $\mathsf{G}_p \cdot \xi = \xi_V + \mathsf{G}_p \cdot \xi_{V^\perp}$, so that $\mathsf{G}_p \cdot \xi$ and $\mathsf{G}_p \cdot \xi_{V^\perp}$ have the same dimension. As $\mathsf{G}_p \cdot \xi_{V^\perp}$ is a compact submanifold of $V^\perp$ (which is diffeomorphic to a Euclidean space), its codimension in $V^\perp$ is positive. Therefore, the cohomogeneity of the action of G is equal to

$$\begin{aligned}\operatorname{cohom}(\mathsf{G} \curvearrowright M) &= \operatorname{codim}_{\nu_p(\mathsf{G}\cdot p)}(\mathsf{G}_p \cdot \xi) = \dim V + \operatorname{codim}_{V^\perp}(\mathsf{G}_p \cdot \xi) \\ &= \dim V + \operatorname{codim}_{V^\perp}(\mathsf{G}_p \cdot \xi_{V^\perp}) > \dim V,\end{aligned}$$

as claimed. □

Lemma 2.9. *If the set Σ_{prin} is not dense in Σ, then there exists an open subset $\Omega \subseteq \Sigma$ such that all of its points have the same orbit type but none of them is principal.*

Proof. By our assumption, there must be a nonempty open subset $\Xi \subseteq \Sigma$ containing no principal points. Let $\Xi' \subseteq \Xi$ stand for the subset of points whose isotropy subgroups have the smallest possible dimension among all points of Ξ. Pick $p \in \Xi'$ such that $\mathsf{G} \cdot p$ has the smallest possible number of connected components among all points of Ξ' (this is possible because all the isotropy subgroups of G are compact). Due to the slice theorem, there is a neighborhood U of p in M such that the orbit type of every $q \in U$ is greater than or equal to that of p. By construction, $\Omega = U \cap \Xi \subseteq \Sigma$ consists of nonprincipal points of the same orbit type. □

With these lemmas, we are now ready to attack the main result of interest.

Theorem 2.10. *Let Σ be a section of a polar action $\mathsf{G} \curvearrowright M$. Then Σ is a totally geodesic submanifold of M.*

Proof. We first show that the second fundamental form $\mathbb{II}$ of Σ vanishes at the points of Σ_{prin}. Let $p \in \Sigma_{\text{prin}}$ belong to a regular orbit, so that the normal space of Σ at p is $\nu_p\Sigma = T_p(\mathsf{G} \cdot p)$. Every normal vector to Σ at p is of the form X_p^* for some $X \in \mathfrak{g}$. Therefore, for any tangent vector $v \in T_p\Sigma$ we have $\langle \mathbb{II}(v, v), X^* \rangle = -\langle v, \nabla_v X^* \rangle = 0$, because ∇X^* is a skew-symmetric map. Because $\mathbb{II}(v, v)$ is a normal vector, we conclude that $\mathbb{II}(v, v) = 0$, and polarizing we obtain that $\mathbb{II}$ is the zero map at p.

We are now left with proving that Σ_{prin} is a dense subset of Σ. If this is not the case, then by Lemma 2.9 we can find an open subset $\Omega \subseteq \Sigma$ such that all orbits that meet Ω have the same nonprincipal type. Take any point $p \in \Omega$. We choose a geodesic slice S at p and let $U = \mathsf{G} \cdot p$, so that the map $F \colon (g, s) \in \mathsf{G} \times S \mapsto g \cdot s \in U$ induces a diffeomorphism between the associated bundle $\mathsf{G} \times_{\mathsf{G}_p} S$ and U. By shrinking Ω if necessary, we can assume that Ω is an embedded submanifold of M contained in U.

Consider the restriction $\widetilde{F} \colon F^{-1}(\Omega) \to \Omega$, which is a smooth submersion because F is a diffeomorphism. We claim that the map $\beta \colon (g, s) \in \mathsf{G} \times S \mapsto s \in S$ is constant along the fibers of $\widetilde{F}$. Indeed, consider elements $(g, s), (g', s') \in F^{-1}(\Omega)$ such that $g \cdot s = g' \cdot s'$. As $g \cdot s \in \Omega$, we have that s and p have the same orbit type, and because $s \in S$ we have $\mathsf{G}_s \subseteq \mathsf{G}_p$. Therefore, $\mathsf{G}_s = \mathsf{G}_p$, and the same argument applies for s'. Now, because $g \cdot s = g' \cdot s'$, we can choose an $h \in \mathsf{G}_p$ such that $(g', s') = (gh^{-1}, h \cdot s) = (gh^{-1}, s)$, yielding $s = s'$ as desired.

We consider the smooth map $f \colon \Omega \to S$ defined by the equality $f(g \cdot s) = s$ for all $(g, s) \in F^{-1}(\Omega)$. We compute the differential f_{*p}. In order to do this, we consider the commutative diagram

$$\begin{array}{ccc} T_{(e,p)}(F^{-1}(\Omega)) & & \\ {\scriptstyle F_{*(e,p)}}\big\downarrow & \searrow^{\beta_{*(e,p)}} & \\ T_p\Omega & \xrightarrow[f_{*p}]{} & T_pS \end{array}$$

The tangent space $T_{(e,p)}(\mathsf{G} \times S)$ is naturally identified with $\mathfrak{g} \oplus T_pS$, and for any pair $(X, Y) = \mathfrak{g} \oplus T_pS$ we see that $\beta_{*p}(X, Y) = Y$. Furthermore, the differential of F at (e, p) is given by

$F_{*(e,p)}(X,Y) = X_p^* + Y$. As the diagram above is commutative, we conclude that the differential of f at p is given by $f_{*p}(Y) = Y$ for all $Y \in T_p\Omega \subseteq T_pS$. From this we conclude that f is an immersion at p, so we may suppose after shrinking Ω once more that f is a smooth embedding with image $\widetilde{\Omega} = f(\Omega) \subseteq S$.

To finish, note that the points of $\widetilde{\Omega}$ are of the form $s = f(g \cdot s)$, where $(g,s) \in F^{-1}(\Omega)$. As we saw earlier, $\mathsf{G}_s = \mathsf{G}_p$, so s is fixed under the action of G_p. Thus, $\widetilde{\Omega}$ is pointwise fixed by G_p, so its tangent space $T_p\widetilde{\Omega}$ is contained in the space of fixed vectors of the slice representations. Since $\dim \widetilde{\Omega} = \dim \Omega = \dim \Sigma$ coincides with the cohomogeneity of the action, we arrive at a contradiction with Lemma 2.8. As a consequence, Σ_{prin} is dense in Σ, so $\mathrm{I\!I}$ vanishes identically on Σ, thus proving that Σ is a totally geodesic submanifold. □

Theorem 2.10 implies that sections are quite rigid. Indeed, if $p \in M$ is a regular point of the polar action $\mathsf{G} \curvearrowright M$ and Σ is a section of the action containing p, the fact that Σ is complete, totally geodesic and of the same dimension as $\operatorname{cohom}(\mathsf{G} \curvearrowright M)$ forces $\Sigma = \exp_p(\nu_p(\mathsf{G} \cdot p))$. Thus, a regular point of a polar action is contained in exactly one section.

Moreover, any two sections of a polar action $\mathsf{G} \curvearrowright M$ are congruent under the action of G. This is because for any two sections Σ_1, $\Sigma_2 \subseteq M$ of the action we can take a regular point $p \in \Sigma_1 \cap M_{\text{reg}}$ and find an element $g \in \mathsf{G}$ such that $g \cdot p \in \Sigma_2$. The submanifold $g \cdot \Sigma_1$ is a section of the action containing the regular point $g \cdot p$, so the uniqueness of a section through $g \cdot p$ implies that $\Sigma_2 = g \cdot \Sigma_1$. In particular, all sections of a given polar action are congruent under elements of G.

2.2.2 The principal horizontal distribution and normal holonomy

Recall that a proper isometric action $\mathsf{G} \curvearrowright M$ induces a submersion $\pi\colon M_{\text{prin}} \to M_{\text{prin}}/\mathsf{G}$. Clearly, if the action of G is polar with section Σ, the connected components of Σ_{prin} are integral manifolds of the principal horizontal distribution $\mathcal{H}$, so $\mathcal{H}$ is integrable. Applying the general theory of Riemannian submersions to π, we deduce that the associated O'Neill tensor A vanishes identically, and that for every point $p \in M_{\text{prin}}$, the equivariant normal vector fields along $\mathsf{G}\cdot p$ are parallel with respect to the normal connection $\nabla^\perp$ (this is because a normal vector field ξ along $\mathsf{G}\cdot p$ is equivariant if and only if it is *π-parallel*, in the sense that the map $x \in \mathsf{G} \cdot p \mapsto \pi_{*x}(\xi_x) \in T_{\pi(p)}(M_{\text{prin}}/\mathsf{G})$ is constant, see [143, Theorem 5.5.12]). Since the slice representation at a principal point is trivial, it follows that the normal bundle $\nu(\mathsf{G} \cdot p)$ of a principal orbit $\mathsf{G} \cdot p$ is *globally flat* (that is, it admits a globally defined parallel frame).

The above observation leads one to conjecture whether the converse is true: given a proper isometric action $\mathsf{G} \curvearrowright M$ whose principal horizontal distribution is integrable, can we guarantee that there exists a section for the action? The answer to this question is negative, at least with the definition that we have taken for polar actions. However, the integrability of $\mathcal{H}$ characterizes a condition slightly weaker than polarity. A proper isometric action $\mathsf{G} \curvearrowright M$ is *locally polar*[1] if there exists a complete Riemannian manifold Σ and an isometric immersion $\sigma\colon \Sigma \to M$ that meets every orbit orthogonally. The key difference between this definition and ours is that Σ is

[1] Some authors take this as the definition of polarity, see for example [83].

no longer assumed to be injectively immersed. Palais and Terng [143] conjectured that if $\mathcal{H}$ is an integrable distribution, then the action of G is locally polar, but only managed to show this in the case that the ambient space is real analytic. The general case was solved by Heintze, Liu and Olmos [86, Theorem A]. Summarizing:

Theorem 2.11. *Let $\mathsf{G} \curvearrowright M$ be a proper isometric action of a Lie group on a complete Riemannian manifold. The following conditions are equivalent:*

(i) *The action of G is locally polar.*

(ii) *The principal horizontal distribution $\mathcal{H}$ on M_{prin} is integrable.*

(iii) *The O'Neill tensor A associated with the submersion $\pi\colon M_{prin} \to M_{prin}/\mathsf{G}$ is identically zero.*

(iv) *Every G-equivariant normal vector field on a principal orbit is $\nabla^\perp$-parallel.*

A remarkable feature about the regular orbits of a polar action is the structure of their normal bundle. If G is a connected Lie group acting polarly on the complete Riemannian manifold M, we have already seen that any principal orbit $\mathcal{O} = \mathsf{G} \cdot p$ of the action $\mathsf{G} \curvearrowright M$ has globally flat normal bundle. Let $\mathrm{Hol}_p(\nu\mathcal{O}, \nabla^\perp)$ be the *holonomy group* of the normal bundle $\nu\mathcal{O}$ at p. More precisely, $\mathrm{Hol}_p(\nu\mathcal{O}, \nabla^\perp)$ is the subgroup of $\mathsf{O}(\nu_p\mathcal{O})$ consisting of all $\nabla^\perp$-parallel translation maps along piecewise smooth loops in $\mathcal{O}$ with base point p. The fact that $\nu\mathcal{O}$ is globally flat immediately yields that $\mathrm{Hol}_p(\nu\mathcal{O}, \nabla^\perp)$ is trivial.

The case of exceptional orbits is more delicate, as the slice representation at p is not trivial, so we cannot apply the previous argument directly. To treat this case, we consider the *connected slice representation* $\mathsf{G}_p^0 \curvearrowright \nu_p(\mathsf{G} \cdot p)$, which is the restriction of the slice representation to the identity component G_p^0 of G_p. This representation is trivial if $\mathsf{G} \cdot p$ is regular, as its cohomogeneity coincides with $\mathrm{cohom}(\mathsf{G} \curvearrowright M) = \dim \nu_p(\mathsf{G} \cdot p)$ and its orbits are connected (moreover, a point p belongs to an exceptional orbit if and only if its connected slice representation is trivial while its slice representation is not). This allows us to establish the following fact:

Proposition 2.12 [124, Lemma 3.1]**.** *If $\mathcal{O} \subseteq M$ is a regular orbit, then for every $p \in \mathcal{O}$ there exists a neighborhood $U \subseteq \mathsf{G}$ such that every normal vector $v \in \nu_p\mathcal{O}$ can be extended to a normal vector field ξ on $U \cdot p$ satisfying $\mathcal{L}_{X^*}\xi = 0$ for all $X \in \mathfrak{g}$. Moreover, ξ is parallel with respect to the normal connection $\nabla^\perp$.*

Remark 2.13. In the proposition above, the operator $\mathcal{L}$ denotes the Lie derivative acting on normal vector fields. More precisely, for a vector field $V \in \mathfrak{X}(\mathcal{O})$ with flow ϕ_t^V and $\xi \in \Gamma(\nu\mathcal{O})$, the Lie derivative $\mathcal{L}_V\xi$ is defined by

$$(\mathcal{L}_V\xi)_p = \frac{d}{dt}\Big|_{t=0} \left(\phi_t^V\right)_{*p}^{-1} \xi_{\phi_t^V(p)}, \quad p \in \mathcal{O}.$$

Proof. Let $p \in \mathcal{O}$ be any point, so that $\mathcal{O} = \mathsf{G} \cdot p$. Because G_p^0 is an open set in G_p, we may find an open subset $V \subseteq \mathsf{G}$ such that $V \cap \mathsf{G}_p = \mathsf{G}_p^0$. Moreover, because G is a Lie group, we can find an open neighborhood $U \subseteq V$ of e in G such that for every $g, h \in U$ we have $g^{-1}h \in V$. Note in particular that $U \subseteq V$.

Let $v \in \nu_p(\mathsf{G} \cdot p)$ be arbitrary, and define a vector field ξ along $U \cdot p$ by letting $\xi_{g \cdot p} = g_{*p} v$. We claim that ξ is well defined: if $g, h \in U$ are such that $g \cdot p = h \cdot p$, then $g^{-1}h \in V \cap \mathsf{G}_p = \mathsf{G}_p^0$, so $(g^{-1}h)_{*p} v = v$. Thus, $g_{*p} v = h_{*p} v$, which yields that ξ is well defined. This vector field satisfies the following local equivariance property: if $g, h \in U$ are elements such that $gh \in U$, then

$$g_{*h \cdot q} \xi_{h \cdot q} = g_{*h \cdot q} h_{*q} v = (gh)_{*q} v = \xi_{gh \cdot q}.$$

Fix any element $X \in \mathfrak{g}$ and $q \in U \cdot p$. The Lie derivative of ξ with respect to X^* is

$$(\mathcal{L}_{X^*} \xi)_q = \frac{d}{dt}\Big|_{t=0} (\mathrm{Exp}(-tX))_{*\,\mathrm{Exp}(tX) \cdot q} \xi_{\mathrm{Exp}(tX) \cdot q},$$

and for small values of t the element $\mathrm{Exp}(tX)$ and its inverse are contained in U, so the local equivariance of ξ gives

$$(\mathcal{L}_{X^*} \xi)_q = \frac{d}{dt}\Big|_{t=0} \xi_q = 0.$$

To see that ξ is $\nabla^\perp$-parallel, we fix a point $q \in U$ together with an $X \in \mathfrak{g}$, and extend ξ locally to a vector field η defined in an open set of M containing q. The calculation above then yields $[X^*, \eta] = \mathcal{L}_{X^*} \xi = 0$ on $U \cap \mathcal{O}$. Moreover, if $\Sigma \subseteq M$ is the section of the action containing q, we have $T_q \Sigma = \nu_q \mathcal{O}$ and ∇X^* sends $T_q \Sigma$ to $\nu_q \Sigma = T_q \mathcal{O}$, so $\nabla_\xi X^* \in T_q \mathcal{O}$. As a consequence,

$$\nabla^\perp_{X^*} \xi = (\nabla_{X^*} \xi)^\perp = (\nabla_\xi X^*)^\perp + [X^*, \eta]^\perp = 0 \quad \text{at } q,$$

so we conclude that $\nabla^\perp \xi = 0$, as desired. □

With this result, we can relate the normal holonomy of a regular orbit $\mathcal{O} = \mathsf{G} \cdot p$ with the slice representation of G at p.

Theorem 2.14. *Let $\mathsf{G} \curvearrowright M$ be a polar action of a connected Lie group G on a complete Riemannian manifold M. Suppose $p \in M$ is a point such that $\mathcal{O} = \mathsf{G} \cdot p$ is a regular orbit of the action. Then the following assertions are true:*

(i) *If $\gamma\colon [0,1] \to \mathcal{O}$ is a piecewise smooth curve with $\gamma(0) = p$ and $\gamma(1) = q$, then the parallel translation map $\mathcal{P}^\perp\colon \nu_p \mathcal{O} \to \nu_q \mathcal{O}$ along γ with respect to the normal connection $\nabla^\perp$ is of the form g_{*p}, where $g \in \mathsf{G}$ is an element sending p to q. Conversely, if $g \in \mathsf{G}$ and $q = g \cdot p$, there exists a piecewise smooth curve $\gamma\colon [0,1] \to \mathcal{O}$ such that the map $g_{*p}\colon \nu_p \mathcal{O} \to \nu_q \mathcal{O}$ is equal to the parallel transport map from p to q along γ.*

(ii) *The normal holonomy group $\mathrm{Hol}_p(\nu \mathcal{O}, \nabla^\perp)$ is the image of the slice representation $\mathsf{G}_p \to \mathsf{O}(\nu_p \mathcal{O})$. In particular, $\mathrm{Hol}_p(\nu \mathcal{O}, \nabla^\perp)$ is trivial if $\mathcal{O}$ is principal, and it is nontrivial and finite if $\mathcal{O}$ is exceptional.*

Proof. The key is to cover the orbit $\mathcal{O}$ by adequate open subsets on which parallel translation coincides with the action of an element of G. Let $x \in \mathcal{O}$ be arbitrary. By Proposition 2.12 we can find an open subset $U_x \subseteq \mathsf{G}$ containing e such that the normal bundle $\nu\mathcal{O}$ restricted to $U_x \cdot x$ admits a $\nabla^\perp$-parallel global frame $\xi_1, \dots, \xi_k$ that is annihilated by the operators $\mathcal{L}_{X^*}$ (for all $X \in \mathfrak{g}$). Choose an open neighborhood $\Omega_x \subseteq U_x$ of e in G such that if $g, h \in \Omega_x$, then $g^{-1} \in \Omega_x$ and $gh \in U_x$. If $y, z \in \Omega_x \cdot x$, we may write $y = g \cdot x$ and $z = h \cdot x$ for some $g, h \in \Omega_x$, so $z = hg^{-1} \cdot y$. Set $k = hg^{-1} \in U_x$. Then from the proof of Proposition 2.12 we have that every $\nabla^\perp$-parallel vector field $\xi \in \Gamma(\nu(\Omega_x \cdot x))$ satisfies $\xi_z = \xi_{k \cdot y} = k_{*y}\xi_y$. Therefore, given any curve α in $\Omega_x \cdot x$ joining y and z, we deduce that the $\nabla^\perp$-parallel translation from y to z along α is independent of α and coincides with the linear isometry $k_{*y} \colon \nu_y\mathcal{O} \to \nu_z\mathcal{O}$.

We now proceed to prove (i). Consider the open cover $\mathcal{U} = \{\Omega_x \cdot x \colon x \in \mathcal{O}\}$ of $\mathcal{O}$. If $\gamma \colon [0,1] \to \mathcal{O}$ is a piecewise smooth curve connecting p to q, then by applying the Lebesgue number lemma to the compact set $\gamma([0,1])$ we can find a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_k = 1$ such that for all $i = 1, \dots, k$ the image of $\gamma|_{[t_{i-1}, t_i]}$ lies on an open set of the form $\Omega_{x_i} \cdot x_i$ for some $x_i \in \mathcal{O}$. Thus, the parallel transport map from $\gamma(t_{i-1})$ to $\gamma(t_i)$ along γ coincides with the differential $(g_i)_{*\gamma(t_{i-1})}$ of the left multiplication by an element $g_i \in \mathsf{G}$. Therefore, the element $g = g_k g_{k-1} \cdots g_1 \in \mathsf{G}$ is such that the parallel translation map $\mathcal{P}^\perp$ from p to q along γ coincides with the map $g_{*p} \colon \nu_p\mathcal{O} \to \nu_q\mathcal{O}$.

The proof of the converse will be done in two steps.

Firstly, assume that there exists an $X \in \mathfrak{g}$ such that $g = \operatorname{Exp}(X)$ takes p to q. Define the curve $\gamma \colon t \in [0,1] \to \operatorname{Exp}(tX) \cdot p \in \mathcal{O}$. If $v \in \nu_p\mathcal{O}$ is any vector, we may extend it to a normal vector field $\xi \colon t \in [0,1] \to \operatorname{Exp}(tX)_{*p} v \in \nu\mathcal{O}$. This vector field is parallel along γ. Indeed, choose any $t_0 \in [0,1]$, define $x = \gamma(t_0)$ and let $\Omega_x \subseteq \mathsf{G}$ be as above. There is a unique $\nabla^\perp$-parallel normal vector field η along $\Omega_x \cdot x$ satisfying $\eta_x = \xi(t_0)$. As $\operatorname{Exp}(tX)$ is a one parameter subgroup of G and η is locally equivariant, for any $h \in \mathbb{R}$ sufficiently close to 0 we have

$$\eta_{\gamma(t_0+h)} = \eta_{\operatorname{Exp}(hX)\cdot x} = \operatorname{Exp}(hX)_{*x}\eta_x = \operatorname{Exp}(hX)_{*x}\xi(t_0) = \xi(t_0 + h).$$

This shows that ξ is the restriction of a $\nabla^\perp$-parallel vector field in a neighborhood of x, so ξ is parallel in a neighborhood of t_0. We conclude that ξ is globally $\nabla^\perp$-parallel along γ, so we have $\mathcal{P}^\perp v = \xi(1) = \operatorname{Exp}(X)_{*p} v$. Thus, the parallel translation map along γ with respect to the normal connection is simply (the differential of) left multiplication by $\operatorname{Exp}(tX)$.

Secondly, we suppose that $q \in \mathcal{O}$ is arbitrary, so there exists a $g \in \mathsf{G}$ carrying p to q. Since G is connected, we may find vectors $X_1, \dots, X_k \in \mathfrak{g}$ such that $g = \operatorname{Exp}(X_k) \cdots \operatorname{Exp}(X_1)$. We define recursively the curves $\gamma_1(t) = \operatorname{Exp}(tX_1) \cdot p$ and $\gamma_i(t) = \operatorname{Exp}(tX_i) \cdot \gamma_{i-1}(1)$ for all $t \in [0,1]$ and $i = 2, \dots, k$. The concatenation $\gamma = \gamma_1 \dots \gamma_k$ is a piecewise smooth curve joining p and q and the argument from the previous paragraph applied to each γ_i readily implies that the parallel translation map $\mathcal{P}^\perp \colon \nu_p\mathcal{O} \to \nu_q\mathcal{O}$ along γ coincides with g_{*p}. This finishes the proof of (i).

The first part of (ii) is immediate from (i) and the definition of $\operatorname{Hol}_p(\nu\mathcal{O}, \nabla^\perp)$. In particular, if $\mathcal{O}$ is a principal orbit, the slice representation is trivial, so $\operatorname{Hol}_p(\nu\mathcal{O}, \nabla^\perp)$ is the trivial group. In addition, if $\mathcal{O}$ is an exceptional orbit, the connected slice representation is trivial, so the full slice representation $\mathsf{G}_p \to \mathsf{O}(\nu_p\mathcal{O})$ factors through a group homomorphism $\mathsf{G}_p/\mathsf{G}_p^0 \to \mathsf{O}(\nu_p\mathcal{O})$ with image $\operatorname{Hol}_p(\nu\mathcal{O}, \nabla^\perp)$. The quotient $\mathsf{G}_p/\mathsf{G}_p^0$ is compact and discrete, hence finite, so we deduce that the normal holonomy group at p is nontrivial and finite. □

Suppose for a moment that M is simply connected. A direct consequence of [5, Theorem 1.4] is that each regular orbit of the G-action has trivial normal holonomy. Combining this with Theorem 2.14, we can conclude the following.

Corollary 2.15. *A polar action of a connected Lie group on a simply connected complete Riemannian manifold does not possess exceptional orbits.*

2.2.3 The slice representation of a polar action

We now show that all the slice representations corresponding to a polar action are also polar.

Proposition 2.16. *Let G be a connected Lie group acting polarly on the complete Riemannian manifold M. Given a section Σ of the action of G and a point $p \in \Sigma$, the slice representation $\mathsf{G}_p \curvearrowright \nu_p(\mathsf{G} \cdot p)$ is a polar representation. Furthermore, the tangent space $T_p\Sigma$ is a section of this representation.*

Proof. Let $\xi \in T_p\Sigma$ be any vector, and let us show that $T_p\Sigma$ is orthogonal to $\mathsf{G}_p \cdot \xi$ at ξ. This is equivalent to saying that for every $X \in \mathfrak{g}_p$ the vector

$$\bar{X}_\xi = \frac{d}{dt}\Big|_{t=0} (\mathrm{Exp}(tX))_{*p}\xi \in T_\xi(\nu_p(\mathsf{G} \cdot p)) \equiv \nu_p(\mathsf{G} \cdot p)$$

is perpendicular to $T_p\Sigma$. Because the flow of the induced Killing field X^* lies in G_p, the flow line $\mathrm{Exp}(tX) \cdot p$ is constant, and we may apply (1.2) to see that

$$\bar{X}_\xi = \frac{d}{dt}\Big|_{t=0} e^{t\nabla X^*}\xi = \nabla_\xi X^*.$$

Since the restriction of X^* to Σ is a normal vector field and Σ is totally geodesic, the vector $\nabla_\xi X^*$ is perpendicular to Σ. This demonstrates that the vector fields $\bar{X}$ are orthogonal to $T_p\Sigma$ for all $X \in \mathfrak{g}_p$.

We now claim that $T_p\Sigma$ meets a principal orbit of the action of G_p. To see this, choose an $\varepsilon > 0$ small enough so that the exponential map $\exp_p$ defines a diffeomorphism of $B_{T_pM}(0, \varepsilon)$ onto its image. Because Σ_{prin} is dense in Σ, we can find a vector $\xi \in T_p\Sigma$ with $|\xi| < \varepsilon$ and such that $q = \exp_p(\xi)$ belongs to a principal orbit of G. Then ξ belongs to a principal orbit of G_p, as desired. In particular, $T_p\Sigma$ is the normal space of $\mathsf{G}_p \cdot \xi$ at ξ (because $\dim \Sigma$ coincides with the cohomogeneity of the slice representation), so it meets every orbit. As a consequence, $T_p\Sigma$ is a section of the slice representation. □

Corollary 2.17. *Given a polar action $\mathsf{G} \curvearrowright M$ and a point $p \in M$, the isotropy subgroup G_p acts transitively on the set of all sections of the action containing p.*

Proof. Let Σ_1 and Σ_2 be two sections such that $p \in \Sigma_1 \cap \Sigma_2$. Then the tangent spaces $T_p\Sigma_1$ and $T_p\Sigma_2$ are sections of the slice representation $\mathsf{G}_p \curvearrowright \nu_p(\mathsf{G} \cdot p)$. As a consequence, there exists an element $g \in \mathsf{G}_p$ such that $g_{*p}(T_p\Sigma_1) = T_p\Sigma_2$, and because sections are totally geodesic we deduce that $g \cdot \Sigma_1 = \Sigma_2$. □

2.2.4 The polar group

Let $\mathsf{G} \curvearrowright M$ be a polar action and assume that it admits a closed section $\Sigma \subseteq M$ (so that all sections are closed). We define the *normalizer* and *centralizer* of Σ in G as

$$\mathsf{N}_{\mathsf{G}}(\Sigma) = \{g \in \mathsf{G} \colon g \cdot \Sigma = \Sigma\} \qquad \mathsf{Z}_{\mathsf{G}}(\Sigma) = \{g \in \mathsf{G} \colon g \cdot p = p \text{ for all } p \in \Sigma\}.$$

It is easy to check that both $\mathsf{N}_{\mathsf{G}}(\Sigma)$ and $\mathsf{Z}_{\mathsf{G}}(\Sigma)$ are closed Lie subgroups of G, and their Lie algebras are

$$\mathfrak{n}_{\mathfrak{g}}(\Sigma) = \{X \in \mathfrak{g} \colon X^*|_{\Sigma} \in \mathfrak{X}(\Sigma)\} = \{X \in \mathfrak{g} \colon X^*|_{\Sigma} = 0\} = \mathfrak{z}_{\mathfrak{g}}(\Sigma),$$

where the second equality comes from the fact that $X^*|_{\Sigma}$ is orthogonal to Σ whenever $X \in \mathfrak{g}$. Therefore, the quotient $\Pi(\Sigma) = \mathsf{N}_{\mathsf{G}}(\Sigma)/\mathsf{Z}_{\mathsf{G}}(\Sigma)$ is a discrete group, known as the *polar group* or *generalized Weyl group* of Σ. Because any two sections of a polar action differ by an element of G, the polar group is uniquely defined up to an inner automorphism of G.

Let $p \in \Sigma$ belong to a principal orbit and $\mathsf{H} = \mathsf{G}_p$ be the corresponding principal isotropy subgroup. As a consequence of the slice theorem, we have $\mathsf{H} = \mathsf{Z}_{\mathsf{G}}(\Sigma)$ and $\mathsf{N}_{\mathsf{G}}(\Sigma) \subseteq \mathsf{N}_{\mathsf{G}}(\mathsf{H})$, so the polar group $\Pi(\Sigma)$ is a discrete subgroup of $\mathsf{N}_{\mathsf{G}}(\mathsf{H})/\mathsf{H}$. It turns out that the action of $\Pi(\Sigma)$ on Σ is properly discontinuous and the inclusion $\Sigma \hookrightarrow M$ induces a homeomorphism (in fact, an isometry) between $\Sigma/\Pi(\Sigma)$ and M/G. A consequence of this is that for any $p \in \Sigma$ we have $\Pi(\Sigma) \cdot p = \mathsf{G} \cdot p \cap \Sigma$.

We can relate the polar group of the action $\mathsf{G} \curvearrowright M$ with the polar group of its slice representation at any point. Let $p \in M$ be an arbitrary point and consider the slice representation $\mathsf{G}_p \curvearrowright \nu_p M$. If $\Sigma \subseteq M$ is a section of the action of G, then by Proposition 2.16 the subspace $T_p\Sigma \subseteq \nu_p(\mathsf{G} \cdot p)$ is a section of the slice representation $\mathsf{G}_p \curvearrowright M$. Moreover, one sees that the polar group $\Pi(T_x\Sigma) = \Pi(\Sigma)_x \subseteq \mathsf{O}(T_x\Sigma)$ is a finite group.

Recall from Dadok's classification [49] that every polar representation is orbit equivalent to an s-representation. In the context of s-representations, Weyl groups have been extensively studied (see [102, Chapter 2, Section 6]). If $\mathsf{K} \curvearrowright \mathfrak{p}$ is the isotropy representation of a symmetric space of noncompact type $M = \mathsf{G}/\mathsf{K}$ and $\mathfrak{a} \subseteq \mathfrak{p}$ is a maximal abelian subspace (thus, a section), then there is an induced set of roots $\Sigma_{\mathfrak{a}} \subseteq \mathfrak{a}^*$ and $\Pi(\mathfrak{a})$ is generated by the reflections along all hyperplanes $\ker \lambda$ with $\lambda \in \Sigma_{\mathfrak{a}}$. As a consequence, given any polar representation $\mathsf{G} \curvearrowright V$ with section $\Sigma \subseteq V$ and polar group $\Pi(\Sigma)$, the action $\Pi(\Sigma) \curvearrowright \Sigma$ has the same orbits as a finite reflection group, implying that $\Pi(\Sigma) \subseteq \mathsf{O}(V)$ is a finite reflection group itself [2].

2.2.5 Criteria for polarity

In this subsection we describe some methods that allow us to determine when a proper isometric action is polar, with special focus on symmetric spaces. The first criterion of polarity is attributed to Gorodski [72] in the context of symmetric spaces of compact type. An analogous criterion in

[2]In general, two finite subgroups $\mathsf{G}, \mathsf{H} \subseteq \mathsf{O}(V)$ have the same orbits if and only if they are equal. It is perhaps curious to note that a finite subgroup of $\mathsf{O}(V)$ is completely determined by two of its principal orbits [122, Theorem 2.12].

the noncompact setting was developed by Berndt, Díaz-Ramos and Tamaru in [19, Theorem 4.1]. We give a unified approach that allows us to derive both criteria at once by using the algebraic similarities between symmetric spaces of compact and noncompact type.

To start, we give the following general criterion valid for arbitrary Riemannian manifolds.

Proposition 2.18 [53, Corollary 6]**.** *Let M be a complete connected Riemannian manifold and G a connected Lie group acting properly and isometrically on M. Fix a point $p \in M$ and suppose that there exists a connected, complete and injectively immersed totally geodesic submanifold $\Sigma \subseteq M$ containing p and satisfying the following three conditions:*

(i) *the tangent space $T_p\Sigma$ is contained in $\nu_p(\mathsf{G} \cdot p)$,*

(ii) *the slice representation $\mathsf{G}_p \curvearrowright \nu_p(\mathsf{G} \cdot p)$ is polar with section $T_p\Sigma$, and*

(iii) *for every $X \in \mathfrak{g}$, the covariant derivative $(\nabla X^*)_p$ sends $T_p\Sigma$ to $\nu_p\Sigma$.*

Then the action of G is polar and Σ is a section of the action.

Proof. We have to show that Σ meets every orbit perpendicularly. Firstly, let $X \in \mathfrak{g}$ be an arbitrary vector and consider its associated Killing vector field $X^* \in \mathcal{K}(M)$. Decompose the restriction of X^* to Σ as $X^*|_\Sigma = Y + Z$, where $Y \in \mathfrak{X}(\Sigma)$ and $Z \in \Gamma(\nu\Sigma)$. Because Σ is a totally geodesic submanifold and X^* is a Killing field, the vector field Y is a Killing field on Σ [105, Theorem 8.9]. By our assumptions, we have $Y_p = 0$. Furthermore, given v, $w \in T_p\Sigma$, we see that

$$0 = \langle \nabla_v X^*, w \rangle = \langle \nabla_v Y, w \rangle + \langle \nabla_v Z, w \rangle = \langle \nabla_v Y, w \rangle,$$

where the last equality follows from $\nabla_v Z$ being normal to Σ. Since the Levi-Civita connection of Σ is merely the restriction of ∇ to $\mathfrak{X}(\Sigma)$, we conclude from the previous equality that the operator $(\nabla Y)_p \in \mathfrak{so}(T_p\Sigma)$ vanishes identically. Therefore, $Y = 0$, which means that X^* is everywhere orthogonal to Σ. We deduce that the intersection of Σ with every orbit it meets is orthogonal.

We now show that Σ meets all orbits. Let $\mathcal{O} \subseteq M$ be any orbit of the G-action. Given any $q \in \mathcal{O}$, we can find a minimizing geodesic segment $\gamma \colon [0,1] \to M$ from $\mathsf{G} \cdot p$ to q. In particular, $\gamma'(0)$ is orthogonal to $\mathsf{G} \cdot p$. After translating γ by an appropriate element of G, we may suppose directly that $\gamma(0) = p$ (and $\gamma(1)$ is still in $\mathcal{O}$). Because $T_p\Sigma$ is a section for the slice representation $\mathsf{G}_p \curvearrowright M$, we can choose a $g \in \mathsf{G}_p$ satisfying $g_{*p}(\gamma'(0)) \in T_p\Sigma$. As a consequence, the curve $\beta(t) = g \cdot \gamma(t)$ is a geodesic contained in Σ such that $\beta(1) = g \cdot \gamma(1) \in \mathcal{O}$, so $\mathcal{O} \cap \Sigma \neq \varnothing$, and we conclude that Σ is a section of the action of G. □

There is a weak point in the hypotheses of Proposition 2.18, namely, that we need to have *a priori* a candidate for a section of the action $\mathsf{G} \curvearrowright M$ at p. Because totally geodesic submanifolds are determined by their tangent space at a point, this problem is reduced to choosing an adequate subspace $V \subseteq T_pM$. A way to find such a subspace is to consider the slice representation $\mathsf{G}_p \curvearrowright \nu_p(\mathsf{G} \cdot p)$ and choose a regular vector $\xi \in \nu_p(\mathsf{G} \cdot p)$. If the slice representation is polar, then the normal space $V = \nu_\xi(\mathsf{G}_p \cdot \xi)$ is the only possible section of this representation passing through ξ. One then has to check if V is the tangent space to a totally geodesic submanifold Σ of M (this problem will be treated more thoroughly in Chapter 5), and it follows that the action $\mathsf{G} \curvearrowright M$ is polar if and only if Σ satisfies the conditions in Proposition 2.18.

Because we are interested in studying polar actions on symmetric spaces, our next objective is to give a computationally efficient restatement of Proposition 2.18 in that setting. The main observations that allow us to simplify the conditions in the above proposition are the fact that totally geodesic submanifolds are characterized by Lie triple systems, and the explicit formula for the Levi-Civita connection at the origin.

Let (G, K) be an effective Riemannian symmetric pair and $M = \mathsf{G}/\mathsf{K}$ the associated symmetric space. Consider the corresponding Cartan involution $\theta\colon \mathfrak{g} \to \mathfrak{g}$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We denote both the Riemannian metric on M and the induced inner product on $\mathfrak{p}$ by $\langle \cdot, \cdot \rangle$. Let H be a Lie subgroup of G, and recall that for $o = e\mathsf{K}$ the tangent space $T_o(\mathsf{H} \cdot o)$ is naturally identified with the projection $\mathfrak{h}_{\mathfrak{p}}$. We let $\mathfrak{h}_{\mathfrak{p}}^{\perp} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}$ be the normal space of $\mathsf{H} \cdot o$ at o. If the action of H is polar, we may consider a section Σ containing o. The tangent space $\mathfrak{v} = T_o\Sigma \subseteq \mathfrak{h}_{\mathfrak{p}}^{\perp}$ is thus a Lie triple system and a section of the slice representation $\mathsf{H} \cap \mathsf{K} \curvearrowright \mathfrak{h}_{\mathfrak{p}}^{\perp}$. Moreover, for each $X \in \mathfrak{h}$ and $\xi, \eta \in \mathfrak{v}$ we have

$$0 = \langle \nabla_{\xi_o^*} X^*, \eta_o^* \rangle = \langle [\xi^*, X^*]_o, \eta_o^* \rangle = \langle [X, \xi]_{\mathfrak{p}}, \eta \rangle = \langle [X_{\mathfrak{k}}, \xi], \eta \rangle.$$

Proposition 2.18 shows that the above conditions characterize the polarity of the H-action. Therefore, we have arrived at the following characterization:

Proposition 2.19. *Let $M = \mathsf{G}/\mathsf{K}$ be a connected Riemannian symmetric space, $o = e\mathsf{K}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$. Given a closed connected Lie subgroup H of G, the action $\mathsf{H} \curvearrowright M$ is polar if and only if there exists a vector subspace $\mathfrak{v} \subseteq \mathfrak{h}_{\mathfrak{p}}^{\perp}$ satisfying the following conditions:*

(i) *$\mathfrak{v}$ is a Lie triple system in $\mathfrak{p}$,*

(ii) *the slice representation $\mathsf{H} \cap \mathsf{K} \curvearrowright \mathfrak{h}_{\mathfrak{p}}^{\perp}$ is polar with section $\mathfrak{v}$, and*

(iii) *the subspaces $\mathfrak{v}$ and $[\mathfrak{h}_{\mathfrak{k}}, \mathfrak{v}]$ are orthogonal.*

If this is the case, then the subset $\Sigma = \exp_o(\mathfrak{v})$ is a section of the H-action.

While the criterion above is valid for any symmetric space and regardless of the orbit type of $\mathsf{H} \cdot o$, we can further simplify the conditions when working with irreducible symmetric spaces of compact or noncompact type and assuming that $\mathsf{H} \cdot o$ is a principal orbit. On the one hand, if $M = \mathsf{G}/\mathsf{K}$ is an irreducible symmetric space of compact type, then we can rescale the metric on $\mathfrak{p}$ so that it is the opposite Killing form $-\mathcal{B}$ of $\mathfrak{g}$. If we consider the inner product $-\mathcal{B}$ defined globally on $\mathfrak{g}$, we have that the operators in $\operatorname{ad}(\mathfrak{p})$ are skew-symmetric. On the other hand, if $M = \mathsf{G}/\mathsf{K}$ is of noncompact type, we can rescale the metric on $\mathfrak{p}$ so that it coincides with the restriction of the Killing form $\mathcal{B}$ of $\mathfrak{g}$ to $\mathfrak{p} \times \mathfrak{p}$, and it can be extended to the inner product $\mathcal{B}_\theta \in \mathsf{S}^2\mathfrak{g}^*$. With respect to this inner product, the operators in $\operatorname{ad}(\mathfrak{p})$ are all symmetric. In both cases, we see that item (iii) in Proposition 2.18 is equivalent to

$$0 = \langle [X, \xi], \eta \rangle = \pm \langle X, [\xi, \eta] \rangle, \quad X \in \mathfrak{h}, \ \xi, \eta \in \mathfrak{v}.$$

The sign in the equation above is positive when M is of compact type, and negative when M is of noncompact type. Therefore, in this situation, item (iii) can be replaced with the condition $[\mathfrak{v}, \mathfrak{v}] \perp \mathfrak{h}$ (note that in order to consider the orthogonality of these subspaces we need to extend the inner product on $\mathfrak{p}$, as $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$).

Now, suppose that $\mathsf{H} \cdot o$ is a principal orbit. Then, because the dimension of any potential section is equal to the cohomogeneity of the action, its tangent space at o is necessarily $\mathfrak{h}_{\mathfrak{p}}^{\perp}$. The slice representation $\mathsf{H} \cap \mathsf{K} \to \mathsf{O}(\mathfrak{h}_{\mathfrak{p}}^{\perp})$ is trivial because $\mathsf{H} \cdot o$ is principal, and therefore it is polar with section $\mathfrak{h}_{\mathfrak{p}}^{\perp}$. Thus, items (i) and (ii) can be replaced by the condition that $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie triple system.

From this discussion we arrive at the polarity criterion that we use throughout this thesis:

Proposition 2.20. *Let $M = \mathsf{G}/\mathsf{K}$ be either a symmetric space of compact or noncompact type endowed with the metric induced by the Killing form of $\mathfrak{g}$ and denote $o = e\mathsf{K}$. Suppose $\mathsf{H} \subseteq \mathsf{G}$ is a closed connected subgroup and the orbit $\mathsf{H} \cdot o$ is principal. Then the action of H is polar if and only if the following conditions are met:*

(i) *The normal space $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie triple system in $\mathfrak{p}$.*

(ii) *The subspace $[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}]$ is orthogonal to $\mathfrak{h}$ with respect to the inner product $-\mathcal{B}$ (in the compact case) or $\mathcal{B}_\theta$ (in the noncompact case).*

Furthermore, if the above conditions are satisfied, then the set $\Sigma = \exp_o(\mathfrak{h}_{\mathfrak{p}}^{\perp})$ is the unique section of the H-action containing o.

Using the formula for the curvature tensor of a symmetric space, we also obtain:

Corollary 2.21. *Under the hypotheses of Proposition 2.20, the action of H is hyperpolar if and only if $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is an abelian subspace of $\mathfrak{p}$.*

2.3 Classifying polar actions on symmetric spaces: the state of the art

This section is devoted to presenting the known results concerning polar actions on (irreducible) symmetric spaces. The first result in this direction was Dadok's classification of polar representations, which can also be regarded as the classification of polar actions on Euclidean spaces that leave a point fixed. Later, motivated by their relation to polar actions on Hilbert spaces and Kac–Moody algebras, Heintze, Palais, Terng and Thorbergsson [87] posed the problem of classifying all hyperpolar actions on compact symmetric spaces. We separate this discussion into the compact and noncompact case, as there is a vast difference between the progress in each setting as well as the nature of the actions that appear.

2.3.1 Polar actions on symmetric spaces of compact type

As of today, we already have classifications of polar actions on every irreducible symmetric space of compact type. It is necessary to treat the cases of rank one and higher rank separately, since there is a vast difference between the results obtained in each situation. The classification of polar actions on compact rank one symmetric spaces is due to Podestà and Thorbergsson, whereas on spaces of higher rank this work was done by Kollross and collaborators.

Polar actions on spheres and projective spaces

Recall that the simply connected symmetric spaces of compact type and rank one are the spheres S^n and the projective spaces $\mathbb{C}\mathsf{P}^n$, $\mathbb{H}\mathsf{P}^n$ and $\mathbb{O}\mathsf{P}^2$. We present the classification of polar actions on these spaces, which is strongly related to Cartan's classification of symmetric spaces due to Dadok's theorem. See [146] by Podestà and Thorbergsson for details and proofs of the results.

The case of spheres can be dealt with quite quickly by means of the following observation: the full isometry group of S^n is the orthogonal group $\mathsf{O}(n+1)$, so every isometry of the round sphere extends uniquely to a linear isometry of $\mathbb{R}^{n+1}$. Therefore, we can attach to an isometric action $\mathsf{H} \curvearrowright \mathsf{S}^n$ a corresponding orthogonal representation $\rho\colon \mathsf{H} \to \mathsf{O}(n+1)$. Furthermore, H acts polarly on S^n if and only if ρ is a polar representation, so from Dadok's result we deduce:

Theorem 2.22. *Let $\mathsf{S}^n = \mathsf{SO}(n+1)/\mathsf{SO}(n)$ be the round sphere. Every polar action on S^n is orbit equivalent to the restriction of an s-representation on $\mathbb{R}^{n+1}$ to S^n.*

Remark 2.23. If a Lie group G acts properly and isometrically on a complete manifold M fixing a point $o \in M$, then G also acts properly and isometrically on the geodesic spheres sufficiently close to o. If $M = \mathbb{R}^{n+1}$, we see from the discussion above that an action on M fixing 0 is polar if and only if its restriction to the geodesic spheres centered at 0 is also polar. This result is not true in general, as there are actions on general symmetric spaces that fix a point and act polarly on its geodesic spheres while they do not act polarly on the total space, see [151].

We proceed to the case of $\mathbb{C}\mathsf{P}^n$, which is related to the theory of Hermitian symmetric spaces.

Let $M = \mathsf{G}/\mathsf{K}$ be a compact Hermitian symmetric space of complex dimension $n+1$ and rank r. This means that M possesses a complex structure J such that the geodesic reflections s_p are holomorphic isometries for all $p \in M$. The complex structure J on M lets us identify T_oM with $\mathbb{C}^{n+1}$ in such a way that the isotropy representation $\mathsf{K} \curvearrowright T_oM$ consists of unitary transformations. As a consequence, the action of K descends to an action $\mathsf{K} \curvearrowright \mathsf{P}_{\mathbb{C}}(T_oM) = \mathbb{C}\mathsf{P}^n$ on the complex projective space. This action turns out to be polar; given a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$, its projectivization $\Sigma = \mathsf{P}(\mathfrak{a}) \cong \mathbb{R}\mathsf{P}^{r-1}$ is a section of the K-action.

It turns out that this procedure exhausts all possible polar actions on $\mathbb{C}\mathsf{P}^n$ up to orbit equivalence:

Theorem 2.24. *Let $\mathbb{C}\mathsf{P}^n = \mathsf{SU}(n)/\mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(n-1))$ be the complex projective space. Every polar action on $\mathbb{C}\mathsf{P}^n$ is orbit equivalent to the action induced by an s-representation corresponding to a compact Hermitian symmetric space.*

The list of all irreducible Hermitian symmetric spaces can be found in [17, Table A.5]. All semisimple Hermitian symmetric spaces are simply connected and can be decomposed into a Riemannian product of irreducible Hermitian symmetric spaces.

The quaternionic case is treated similarly by considering s-representations corresponding to compact quaternionic Kähler symmetric spaces, but the procedure is slightly more involved. This is due to the issue that the Riemannian product of quaternionic Kähler manifolds is not quaternionic Kähler.

Let $M = \mathsf{G}/\mathsf{K}$ be a compact quaternionic Kähler symmetric space with quaternionic dimension $n+1$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This means by [24, Page 408] that M is irreducible and K can be written as a subgroup of the form $\mathsf{HSp}(1)$, where H and $\mathsf{Sp}(1)$ are normal in K. The $\mathsf{Sp}(1)$ factor induces a quaternionic structure on $\mathfrak{p}$ so that we may regard $\mathfrak{p} = \mathbb{H}^{n+1}$ as a right $\mathbb{H}$-module, and the action of H consists of quaternionic linear transformations, so K sends quaternionic lines to quaternionic lines. As a consequence, the action of K descends to an isometric action on $\mathbb{H}\mathsf{P}^n$ which turns out to be polar, where the section is constructed as in the complex case.

Now, consider a product $M = M_1 \times \cdots \times M_m$ of compact quaternionic Kähler symmetric spaces $M_i = \mathsf{G}_i/\mathsf{K}_i$ with Cartan decompositions $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ and define subgroups $\mathsf{H}_i \subseteq \mathsf{K}_i$ as above. The subgroup $\mathsf{H} = \mathsf{H}_1 \times \cdots \times \mathsf{H}_m \times \mathsf{Sp}(1)$ acts on $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m$ sending quaternionic lines to quaternionic lines. Moreover, the representation $\mathsf{H} \curvearrowright \mathfrak{p}$ is polar if and only if all but at most one of the M_i are of rank one. If this is the case, then the induced action of H on $\mathbb{H}\mathsf{P}^n$ is polar of cohomogeneity equal to $\operatorname{rank}(M) - 1$.

With this procedure, we have constructed all polar actions on $\mathbb{H}\mathsf{P}^n$ up to orbit equivalence:

Theorem 2.25. *Let $\mathbb{H}\mathsf{P}^n = \mathsf{Sp}(n+1)/(\mathsf{Sp}(1) \times \mathsf{Sp}(n))$ be the quaternionic projective space. Every polar action on $\mathbb{H}\mathsf{P}^n$ is orbit equivalent to the action induced by the isotropy representation of a product of k compact quaternionic Kähler symmetric spaces, where all but at most one of them are of rank one.*

We refer the reader to [24, Table 14.52] for the classification of irreducible quaternionic Kähler symmetric spaces.

The Cayley projective plane is not realizable as a set of octonionic lines in $\mathbb{O}^3$, so we cannot expect polar actions on $\mathbb{O}\mathsf{P}^2$ to come from isotropy representations of adequate symmetric spaces. Nevertheless, we have an explicit list of polar actions on $\mathbb{O}\mathsf{P}^2$ up to orbit equivalence. It should be noted that [146] missed one example in their classification, and this error was fixed in [73].

Theorem 2.26. *Let $\mathbb{O}\mathsf{P}^2 = \mathsf{F}_4/\mathsf{Spin}(9)$ be the Cayley projective plane. Then every polar action on $\mathbb{O}\mathsf{P}^2$ is of cohomogeneity at most two and is orbit equivalent to the action of exactly one of the subgroups of F_4 given in Table 2.1.*

A consequence of the classifications by Podestà and Thorbergsson is that every symmetric space of compact type and rank one (apart from the sphere S^2) admits polar actions that are not hyperpolar.

Table 2.1: Subgroups of F_4 acting polarly on $\mathbb{O}\mathsf{P}^2$

Cohomogeneity one	$\mathsf{Spin}(9)$	$\mathsf{Sp}(3)\mathsf{Sp}(1)$	$\mathsf{Sp}(3)\mathsf{U}(1)$	$\mathsf{Sp}(3)$	
Cohomogeneity two	$\mathsf{Spin}(8)$	$\mathsf{Spin}(7)\mathsf{SO}(2)$	$\mathsf{SU}(4)\mathsf{SU}(2)$	$\mathsf{SU}(3)\mathsf{SU}(3)$	$\mathsf{SO}(3)\mathsf{G}_2$

Polar actions on symmetric spaces of compact type and higher rank

For irreducible compact symmetric spaces of higher rank, the paradigm changes completely. Indeed, Biliotti and Gori classified in [26,27] all polar actions on irreducible Hermitian symmetric spaces of compact type up to orbit equivalence, and all examples are actually hyperpolar. Due to this phenomenon, Biliotti conjectured that every nontrivial polar action on a symmetric space of compact type and higher rank is hyperpolar. Although many authors worked in the resolution of this conjecture, Kollross and Lytchak were the ones to finally give a positive answer (see [109] and the references therein):

Theorem 2.27. *If M is an irreducible Riemannian symmetric space of compact type and rank greater than one, then every nontrivial polar action on M is hyperpolar.*

The task of classifying hyperpolar actions in this setting was carried out by Kollross [106]. The key observation that makes their classification possible is the following: consider an simply connected symmetric space of compact type $M = \mathsf{G}/\mathsf{K}$, and endow G with a corresponding bi-invariant metric, then a subgroup $\mathsf{H} \subseteq \mathsf{G}$ acts hyperpolarly on M if and only if $\mathsf{H} \times \mathsf{K}$ acts hyperpolarly on M. Recall that a simply connected irreducible symmetric space of compact type is either a compact simple Lie group endowed with a bi-invariant metric (Type I) or a quotient G/K with G simple (Type II). The above observation allows us to work exclusively with symmetric spaces of Type I.

Suppose G is a compact semisimple Lie group and H and K are Lie subgroups of G such that (G, H) and (G, K) are Riemannian symmetric pairs. It was shown by Hermann [91] that the product $\mathsf{H} \times \mathsf{K}$ acts hyperpolarly on G, so the group H acts hyperpolarly on G/K. The action $\mathsf{H} \curvearrowright \mathsf{G}/\mathsf{K}$ is known as a *Hermann action*. This procedure allows us to construct all polar actions of cohomogeneity greater than one:

Theorem 2.28 [106]**.** *If M is an irreducible symmetric space of compact type, then a hyperpolar action on M is either of cohomogeneity one or orbit equivalent to a Hermann action.*

The classification of cohomogeneity one actions on irreducible symmetric spaces of compact type can also be seen in [106]. As of today, the problem of determining all polar actions on reducible symmetric spaces of compact type remains open, even for actions of cohomogeneity one.

2.3.2 Polar actions on symmetric spaces of noncompact type

In comparison with the compact case, results concerning polar actions on symmetric spaces of noncompact type are quite scarce, and hardly any symmetric space of noncompact type enjoys a full classification of its polar actions. In this section, we discuss the most notable known results to date concerning polar actions on symmetric spaces of noncompact type.

By far, the case that has been studied most extensively is that of cohomogeneity one actions. If M is a symmetric space of noncompact type and G is a connected Lie group acting on M with cohomogeneity one, then a simple argument using the polar group shows that the orbit space M/G is either the real line $\mathbb{R}$ or the half open interval $[0,\infty)$. On the one hand, the case $M/\mathsf{G}=\mathbb{R}$ corresponds to G acting without singular orbits on M, so G induces a homogeneous foliation on M. The classification of homogeneous codimension one foliations on symmetric spaces of noncompact type will be described in Section 2.3.2, as it will be necessary for our work. On the other hand, the case $M/\mathsf{G}=[0,\infty)$ corresponds to M having a unique singular orbit (which is automatically minimal). This case becomes much more complicated, and the resolution of the classification problem for these actions is the result of a collective effort during over twenty years, starting in [16] and finally ending in [150], see the introduction of the latter paper and the references therein for an excellent survey on this topic.

In order to present the results, we make use of the Iwasawa decomposition associated with a symmetric space of noncompact type. We fix the following notation throughout the remainder of this section. Let $M=\mathsf{G}/\mathsf{K}$ be a symmetric space of noncompact type, where $\mathsf{G}=I^0(M)$ and K is the isotropy subgroup at some point $o\in M$. Consider the associated Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, a maximal abelian subspace $\mathfrak{a}\subseteq\mathfrak{p}$ together with its set of roots $\Sigma\subseteq\mathfrak{a}^*$, and choose a notion of positivity on Σ, which induces a subset $\Sigma^+\subseteq\Sigma$ of positive roots, as well as a subset $\Lambda\subseteq\Sigma^+$ of simple roots. These choices induce an Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ of the isometry algebra, together with an Iwasawa decomposition $\mathsf{G}=\mathsf{KAN}$ at the Lie group level.

Polar actions on hyperbolic spaces

Let us first discuss the known results for noncompact symmetric spaces of rank one. All polar actions on the real hyperbolic space $\mathbb{R}\mathsf{H}^n=\mathsf{SO}^0(1,n)/\mathsf{SO}(n)$ have been obtained up to orbit equivalence by Wu, and their classification can be derived from Dadok's theorem. We make use of the Iwasawa decomposition of $\mathfrak{so}(1,n)$, see Chapter 4 for more details. We choose an Iwasawa decomposition $\mathfrak{g}=\mathfrak{so}(1,n)=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ with $\mathfrak{k}=\mathfrak{so}(n)$, $\mathfrak{a}=\mathbb{R}$ and $\mathfrak{n}=\mathbb{R}^{n-1}$. We note that $\mathfrak{n}$ is an abelian subalgebra in this case. The corresponding group decomposition is denoted $\mathsf{SO}^0(1,n)=\mathsf{SO}(n)\mathsf{AN}$, and we have $\mathsf{K}_0=\mathsf{SO}(n-1)$. Wu's theorem can be stated as follows:

Theorem 2.29 [169]. *Let $M=\mathsf{SO}^0(1,n)/\mathsf{SO}(n)$ be the real hyperbolic space and $o=e\mathsf{K}$. Then, if $\mathsf{H}\subseteq\mathsf{SO}^0(1,n)$ is a closed connected subgroup of $\mathsf{SO}^0(1,n)$ acting polarly on M, the action is orbit equivalent to that of one of the following subgroups:*

(i) *A subgroup $\mathsf{H}\subseteq\mathsf{SO}(n)$ (that is, a subgroup fixing o) whose corresponding action on $\mathbb{R}^n\cong T_o\mathbb{R}\mathsf{H}^n$ is polar.*

(ii) *A subgroup of the parabolic subgroup $\mathsf{SO}(n-1)\mathsf{AN}$ whose Lie algebra is of the form $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{a}\oplus\mathfrak{v}$, where $\mathfrak{h}_0\subseteq\mathfrak{so}(n-1)$ and $\mathfrak{v}\subseteq\mathfrak{n}$, and the action of the isotropy subgroup H_o on the normal space $\mathfrak{h}_{\mathfrak{p}}^{\perp}=(1-\theta)(\mathfrak{n}\ominus\mathfrak{v})$ is polar. In this case, the orbit $\mathsf{H}\cdot o$ is a totally geodesic $\mathbb{R}\mathsf{H}^k$ inside M.*

(iii) *A subgroup of the parabolic subgroup* $\mathsf{SO}(n-1)\mathsf{AN}$ *whose Lie algebra is of the form* $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{v}$*, where* $\mathfrak{h}_0 \subseteq \mathfrak{so}(n-1)$ *and* $\mathfrak{v} \subseteq \mathfrak{n}$*, and the action of the isotropy subgroup* H_o *on* $(1-\theta)(\mathfrak{n} \ominus \mathfrak{v})$ *is polar. In this case, the orbit* $\mathsf{H} \cdot o$ *is a horosphere inside a totally geodesic* $\mathbb{R}\mathsf{H}^k$ *of* M*.*

More recently, this problem was also settled in the case of complex hyperbolic spaces, by the hand of Díaz-Ramos, Domínguez-Vázquez and Kollross. Consider the complex hyperbolic space $\mathbb{C}\mathsf{H}^n = \mathsf{SU}(1,n)/\mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(n))$. The Iwasawa decomposition of $\mathfrak{su}(1,n)$ takes the form $\mathfrak{su}(1,n) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where we have $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$, $\mathfrak{a} = \mathbb{R}$ and $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is a two-step nilpotent subalgebra consisting of two positive root spaces. The group K_0 is equal to $\mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(n-1))$ and we have $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, $\mathfrak{g}_{2\alpha} \cong \mathbb{R}$ as representations of K_0.

Theorem 2.30 [52, Theorem A]. *Let* $\mathbb{C}\mathsf{H}^n = \mathsf{SU}(1,n)/\mathsf{S}(\mathsf{U}(1)\times\mathsf{U}(n))$ *be the complex hyperbolic space. Consider a connected Lie subgroup* $\mathsf{H} \subseteq \mathsf{SU}(1,n)$ *whose Lie algebra* $\mathfrak{h}$ *is one of the following:*

(i) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{so}(1,k)$*, where* $0 \leq k \leq n$ *and* $\mathfrak{q} \subseteq \mathfrak{u}(n-k)$ *is a subalgebra such that its corresponding connected subgroup* $\mathsf{Q} \subseteq \mathsf{U}(n-k)$ *acts polarly on* $\mathbb{C}^{n-k}$ *with a totally real section.*

(ii) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{v} \oplus \mathfrak{g}_{2\alpha}$*, where* $\mathfrak{b}$ *is a subspace of* $\mathfrak{a}$*,* $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ *is a real subspace and* $\mathfrak{q}$ *is a subalgebra of* $\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n-1))$ *that normalizes* $\mathfrak{v}$ *and such that the action of its corresponding connected subgroup* Q *acts polarly on* $\mathfrak{g}_\alpha \ominus \mathfrak{v}$ *with a totally real section.*

Then the action of H *on* $\mathbb{C}\mathsf{H}^n$ *is polar. Conversely, every nontrivial polar action on* $\mathbb{C}\mathsf{H}^n$ *is orbit equivalent to one of the actions described above.*

In all the examples above, the section is a totally real hyperbolic space $\mathbb{R}\mathsf{H}^k$. We also note that the congruence problem for these actions is solved in [52, Theorem B].

The above examples are the only symmetric spaces of noncompact type whose polar actions have been fully classified. There has been some work on the cases of $\mathbb{H}\mathsf{H}^n$ and $\mathbb{O}\mathsf{H}^2$, essentially by Kollross.

Using the notion of duality (which we will describe later on), Kollross classified all polar actions on $\mathbb{H}\mathsf{H}^n$ induced by reductive algebraic subgroups. Suppose $\mathfrak{g}$ is a complex semisimple Lie algebra. Because $\mathfrak{z}(\mathfrak{g}) = 0$, we can identify $\mathfrak{g}$ with the linear Lie algebra $\mathrm{ad}(\mathfrak{g})$. We say that $\mathfrak{h} \subseteq \mathfrak{g}$ is an *algebraic subalgebra* of $\mathfrak{g}$ if it is the Lie algebra of an algebraic subgroup of $\mathsf{GL}(\mathfrak{g})$. We also say that $\mathfrak{h}$ is a *reductive subalgebra* if it is a reductive Lie algebra and the elements of $\mathfrak{z}(\mathfrak{h})$ are semisimple in $\mathfrak{g}$. A *reductive algebraic subalgebra* of $\mathfrak{g}$ is a reductive subalgebra of $\mathfrak{g}$ which is also algebraic. Finally, if $\mathfrak{g}$ is a real semisimple Lie algebra, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a *reductive algebraic subalgebra* if $\mathfrak{h}(\mathbb{C})$ is a reductive algebraic subalgebra of $\mathfrak{g}(\mathbb{C})$. A connected subgroup H of a real semisimple Lie group G is said to be *reductive algebraic* if its Lie algebra $\mathfrak{h}$ is a reductive algebraic subalgebra of $\mathfrak{g}$. It can be shown that if $M = \mathsf{G}/\mathsf{K}$ is a symmetric space of noncompact type and $\mathsf{H} \subseteq \mathsf{G}$ is a closed reductive algebraic subgroup, then there is a point $p \in M$ such that $\mathsf{H} \cdot p$ is a totally geodesic submanifold of M.

Theorem 2.31 [107, Theorem 10.1]. *Let* $\mathbb{H}\mathsf{H}^n = \mathsf{Sp}(1,n)/(\mathsf{Sp}(1)\times\mathsf{Sp}(n))$ *be the quaternionic hyperbolic space and* $\mathsf{H}\subseteq\mathsf{Sp}(1,n)$ *a reductive algebraic subgroup. The action* $\mathsf{H}\curvearrowright\mathbb{H}\mathsf{H}^n$ *is polar if and only if it is orbit equivalent to one of the following actions:*

(i) *The action of* $\mathsf{Sp}(1,k)\times\mathsf{Sp}(n_1)\times\cdots\times\mathsf{Sp}(n_\nu)\times\mathsf{L}$*, where* $\mathsf{L}\subseteq\mathsf{Sp}(m)$ *is a subgroup whose action is induced from a quaternionic Kähler symmetric space,* $1\le k\le n$ *and* $k+n_1+\cdots+n_\nu+m=n$*.*

(ii) *The action of* $\mathsf{U}(1,k)\times\mathsf{Sp}(n_1)\times\cdots\times\mathsf{Sp}(n_\nu)$*, where* $1\le k\le n$ *and* $k+n_1+\cdots+n_\nu=n$*.*

(iii) *The action of* $(\mathsf{Sp}(1)\mathsf{SO}^0(1,k))\times\mathsf{Sp}(n_1)\times\cdots\times\mathsf{Sp}(n_\nu)$*, where* $1\le k\le n$ *and* $k+n_1+\cdots+n_\nu=n$*.*

(iv) *The action of* $\mathsf{Sp}(1)\times\mathsf{L}$*, where* L *is a subgroup whose action on* $\mathbb{H}^n$ *is induced by a product of quaternionic Kähler symmetric spaces where at most one of the factors is of rank greater than one.*

On the other hand, Kollross [108] classified all polar actions on the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2=\mathsf{F}_4^{-20}/\mathsf{Spin}(9)$ that leave a totally geodesic submanifold invariant. More precisely:

Theorem 2.32. *Let* $\mathbb{O}\mathsf{H}^2=\mathsf{F}_4^{-20}/\mathsf{Spin}(9)$ *be the Cayley hyperbolic plane. If* $\mathsf{H}\subseteq\mathsf{F}_4^{-20}$ *is a closed connected subgroup acting polarly and nontrivially on* $\mathbb{O}\mathsf{H}^2$*. If* H *leaves a totally geodesic submanifold* $P\subsetneq\mathbb{O}\mathsf{H}^2$ *invariant, then there are two possibilities for* H*:*

(i) *The group* H *fixes a point in* $\mathbb{O}\mathsf{H}^2$ *and its action is orbit equivalent to the action of one of the following subgroups:*

$$\mathsf{Spin}(9),\quad \mathsf{Spin}(8),\quad \mathsf{Spin}(7)\mathsf{SO}(2),\quad \mathsf{Spin}(6)\mathsf{Spin}(3).$$

(ii) *The orbits of* H *coincide with those of the identity component* $\mathsf{N}(P)^0$ *of the normalizer* $\mathsf{N}(P)$*, where* $P\notin\{\mathbb{R}\mathsf{H}^3,\mathbb{R}\mathsf{H}^4\}$*. These groups are*

$$\begin{array}{lll} \mathsf{G}_2\mathsf{SO}^0(1,2), & \mathsf{SU}(3)\mathsf{SU}(1,2), & \mathsf{Sp}(1)\mathsf{Sp}(1,2),\\ \mathsf{Spin}(7)\mathsf{SO}^0(1,1), & \mathsf{Spin}(6)\mathsf{Spin}(1,2), & \mathsf{Spin}(3)\mathsf{Spin}(1,5),\\ \mathsf{SO}(2)\mathsf{Spin}(1,6), & \mathsf{Spin}(1,7), & \mathsf{Spin}(1,8). \end{array}$$

The results above still leave the problem of classifying polar actions on $\mathbb{H}\mathsf{H}^n$ and $\mathbb{O}\mathsf{H}^2$ that do not preserve a totally geodesic submanifold open. Furthermore, in the quaternionic case it is still unknown whether the classification by Kollross holds if the assumption that H is reductive algebraic can be replaced by H having a totally geodesic orbit.

Homogeneous codimension one foliations

As mentioned at the beginning of the section, homogeneous codimension one foliations on symmetric spaces of noncompact type have been fully classified. Berndt and Tamaru [22] gave the solution to this problem in the irreducible setting, while Solonenko [154] extended their result to the reducible case.

Suppose $M = \mathsf{G}/\mathsf{K}$ is a symmetric space of noncompact type. We construct two families of foliations on M:

(FS) Foliations of *solvable type* [21, Section 4]: Let $\alpha \in \Lambda$ be a simple root, and choose a line $\ell \subseteq \mathfrak{g}_\alpha$. The vector subspace $\mathfrak{s}_\alpha = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$ is a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ whose corresponding Lie subgroup $\mathsf{S}_\alpha \subseteq \mathsf{AN}$ acts on M freely and properly inducing a codimension one foliation $\mathcal{F}_\alpha$ on M. Since the adjoint action of K_0 on the unit sphere of $\mathfrak{g}_\alpha$ is transitive, the subgroup S_α is independent of ℓ. The orbit $\mathsf{S}_\alpha \cdot o$ is the unique minimal leaf of this foliation, and the rest of orbits can be obtained as the equidistant hypersurfaces of $\mathsf{S}_\alpha \cdot o$. If $\gamma\colon \mathbb{R} \to M$ is a unit speed geodesic meeting the S_α-orbits perpendicularly and $\gamma(0) = o$, then one sees that the orbit $\mathsf{S}_\alpha \cdot \gamma(t)$ is only congruent to itself and to $\mathsf{S}_\alpha \cdot \gamma(-t)$.

(FH) Foliations of *horospherical type* [21, Section 3]: Let $\ell \subseteq \mathfrak{a}$ be any line and construct the Lie subalgebra $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n} \subseteq \mathfrak{a} \oplus \mathfrak{n}$. This is an ideal of $\mathfrak{a} \oplus \mathfrak{n}$ whose corresponding connected subgroup $\mathsf{S}_\ell \subseteq \mathsf{AN}$ acts freely and properly on M with cohomogeneity one inducing a foliation $\mathcal{F}_\ell$. As S_ℓ is an ideal of AN and the action of the latter group is transitive, we see that for any $p, q \in M$ the orbits $\mathsf{S}_\ell \cdot p$ and $\mathsf{S}_\ell \cdot q$ are isometrically congruent. The orbits of this action are minimal submanifolds if and only if the covector $\delta = (1/2)\sum_{\lambda \in \Sigma^+} (\dim \mathfrak{g}_\lambda)\lambda$ annihilates ℓ. We say in this case that $\mathcal{F}$ is a *harmonic foliation*, since the minimality of all orbits is equivalent to the natural projection $\pi\colon M \to M/\mathsf{S}_\ell$ being a harmonic function.

With these two constructions, we have obtained all cohomogeneity one actions without singular orbits:

Theorem 2.33 [21, Main theorem], [154, Main Theorem]. *Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of noncompact type. A homogeneous codimension one foliation $\mathcal{F}$ on M is isometrically congruent to either a foliation of solvable type $\mathcal{F}_\alpha$ or a foliation of horospherical type $\mathcal{F}_\ell$. Moreover:*

(i) *Given two simple roots α, $\alpha' \in \Lambda$, the foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_{\alpha'}$ are isometrically congruent if and only if there exists an automorphism of the Dynkin diagram of M that sends α to α'.*

(ii) *If ℓ, $\ell' \subseteq \mathfrak{a}$ are one-dimensional subspaces, the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ are isometrically congruent if and only if there exists an automorphism of the Dynkin diagram of M that sends ℓ to ℓ'.*

Hyperpolar homogeneous foliations

The other general classification known prior to the work in this thesis was that of hyperpolar homogeneous foliations on irreducible symmetric spaces of noncompact type, due to Berndt, Díaz-Ramos and Tamaru [19].

Suppose $M = \mathsf{G}/\mathsf{K}$ is a symmetric space of noncompact type. We assume that the metric on M is induced from the Killing form $\mathcal{B}$ restricted to $\mathfrak{p}$ (recall that if M is irreducible then every G-invariant metric comes from a multiple of the Killing form on $\mathfrak{p}$). Choose a subset $\Phi \subseteq \Lambda$ such that any two roots $\alpha, \beta \in \Phi$ satisfy $\langle \alpha, \beta \rangle = 0$ (because α and β are simple, this condition is equivalent to saying that neither $\alpha + \beta$ nor $\alpha - \beta$ are roots). The subset Φ is said to be *orthogonal*. On each root space $\mathfrak{g}_\alpha$ with $\alpha \in \Phi$ we choose a one-dimensional subspace $\ell_\alpha \subseteq \mathfrak{g}_\alpha$, and we define $\ell_\Phi = \bigoplus_{\alpha \in \Phi} \ell_\alpha$. Finally, let $V \subseteq \mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \ker \alpha$ be any vector subspace. We define $\mathfrak{s}_{\Phi,V} = (\mathfrak{a}^\Phi \oplus V \oplus \mathfrak{n}) \ominus \ell_\Phi \subseteq \mathfrak{a} \oplus \mathfrak{n}$. Then $\mathfrak{s}_{\Phi,V}$ is a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ such that its corresponding connected subgroup $\mathsf{S}_{\Phi,V} \subseteq \mathsf{AN}$ acts hyperpolarly on M inducing a foliation $\mathcal{F}_{\Phi,V}$. The geometry of the $\mathsf{S}_{\Phi,V}$-orbits is studied in depth in [19, Section 6]. These foliations exhaust all possible hyperpolar homogeneous foliations up to orbit equivalence:

Theorem 2.34 [19, Main Theorem]**.** *If $M = \mathsf{G}/\mathsf{K}$ is a symmetric space of noncompact type whose metric is induced from the Killing form of $\mathfrak{g}$, then every hyperpolar homogeneous foliation $\mathcal{F}$ on M is isometrically congruent to a foliation of the form $\mathcal{F}_{\Phi,V}$ for some orthogonal subset $\Phi \subseteq \Lambda$ and some vector subspace $V \subseteq \mathfrak{a}_\Phi$.*

We remark that the congruence problem for the foliations $\mathcal{F}_{\Phi,V}$ remains unsolved as of today. Solonenko showed that the hypothesis on the metric can be removed [155, Proposition 4.1.4], so that the above result holds for any symmetric space of noncompact type.

The canonical extension method

We now give examples of polar (nonhyperpolar) homogeneous foliations on symmetric spaces of noncompact type constructed from the theory of parabolic subalgebras. The main interest of these actions, apart from being examples themselves, is that we may use their corresponding groups in order to construct isometric actions on any symmetric space of noncompact type from actions on a lower rank symmetric space by a procedure known as canonical extension. This method was originally developed by Berndt and Tamaru in [21], where their main interest was the construction of cohomogeneity one actions. We refer to [56] for a detailed study of the canonical extension method in a more general context.

Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of noncompact type and choose an Iwasawa decomposition of G (with the same conventions as in Section 2.3.2). Choose an arbitrary subset $\Phi \subseteq \Lambda$ of simple roots (we do not assume that Φ is orthogonal) and let $\mathfrak{q}_\Phi$ be the associated parabolic subalgebra. The subalgebra $\mathfrak{q}_\Phi$ is equipped with its Langlands decomposition $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, where we recall that

$$\mathfrak{m}_\Phi = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathbb{R} H_\alpha \oplus \bigoplus_{\lambda \in \Sigma_\Phi} \mathfrak{g}_\lambda, \qquad \mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \ker \alpha, \qquad \mathfrak{n}_\Phi = \bigoplus_{\lambda \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\lambda.$$

Moreover, if $\mathfrak{g}_\Phi = [\mathfrak{m}_\Phi, \mathfrak{m}_\Phi]$, the connected subgroup $\mathsf{G}_\Phi \subseteq \mathsf{G}$ with Lie algebra $\mathfrak{g}_\Phi$ acts on M in such a way that the orbit $B_\Phi = \mathsf{G}_\Phi \cdot o$ is a totally geodesic submanifold of M (the boundary component associated with Φ). The Lie subalgebra $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ exponentiates to a connected subgroup $\mathsf{A}_\Phi \mathsf{N}_\Phi \subseteq \mathsf{AN}$ that acts freely and properly with cohomogeneity equal to $|\Phi| + \sum_{\lambda \in \Sigma_\Phi^+} \dim \mathfrak{g}_\lambda$.

It can be shown that the action $\mathsf{A}_\Phi\mathsf{N}_\Phi \curvearrowright M$ is polar and B_Φ is the unique section of the action containing o. Moreover, the orbits of $\mathsf{A}_\Phi\mathsf{N}_\Phi$ are minimal submanifolds, as well as Einstein manifolds [158]. These orbits intersect the section B_Φ exactly once.

First, take an injectively immersed submanifold $S \subseteq B_\Phi$. The subset $\mathsf{A}_\Phi\mathsf{N}_\phi \cdot S$ is an injectively immersed submanifold of M, which we call the *canonical extension* of S. The codimension of $\mathsf{A}_\Phi\mathsf{N}_\Phi \cdot S$ in M coincides with the codimension of S in B_Φ. In addition, if S satisfies one of the following properties, then so does $\mathsf{A}_\Phi\mathsf{N}_\Phi \cdot S$: embedded, minimal, of parallel mean curvature, of globally flat normal bundle [56, Theorem 2.1]. We warn that if S is totally geodesic, the canonical extension $\mathsf{A}_\Phi\mathsf{N}_\Phi \cdot S$ need not be totally geodesic. For instance, any singleton $\{p\} \subseteq B_\Phi$ is a totally geodesic submanifold, whereas its canonical extension $\mathsf{A}_\Phi\mathsf{N}_\Phi \cdot p$ is minimal, but not totally geodesic.

Now, suppose that an isometric action $\mathsf{H}_\Phi \curvearrowright B_\Phi$ of a connected Lie group H_Φ is given. Because $\mathfrak{g}_\Phi$ contains the isometry algebra of B_Φ, we may assume without loss of generality that $\mathsf{H}_\Phi \subseteq \mathsf{G}_\Phi \subseteq \mathsf{M}_\Phi$ (this last group being the connected subgroup of G with Lie algebra $\mathfrak{m}_\Phi$). Because M_Φ normalizes $\mathsf{A}_\Phi\mathsf{N}_\Phi$, we see that $\mathsf{H} = \mathsf{H}_\Phi\mathsf{A}_\Phi\mathsf{N}_\Phi$ is a subgroup of Q_Φ whose Lie algebra is $\mathfrak{h} = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$. The action of H on M is known as the *canonical extension* of the action of H_Φ on B_Φ. By definition, if $p \in B_\Phi$ is arbitrary, we have $\mathsf{H}\cdot p = (\mathsf{A}_\Phi\mathsf{N}_\Phi)\cdot(\mathsf{H}_\Phi \cdot p)$, meaning that the orbits of H are precisely the canonical extensions of the orbits of H_Φ. Thus, the cohomogeneity of the action $\mathsf{H} \curvearrowright M$ coincides with the cohomogeneity of the action $\mathsf{H}_\Phi \curvearrowright B_\Phi$. It is not hard to show that if H_Φ acts polarly on M, then H acts polarly on M. In fact, if $\Sigma \subseteq B_\Phi$ is a section of $\mathsf{H}_\Phi \curvearrowright M$, then it is also a section of $\mathsf{H} \curvearrowright M$.

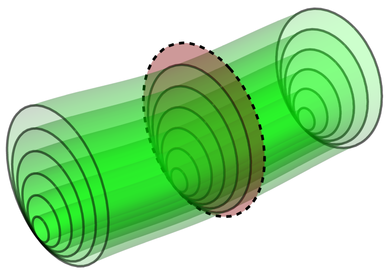


Figure 2.3.1: Canonical extension of the horosphere foliation $\mathsf{N} \curvearrowright \mathbb{R}\mathsf{H}^2$.

A word on duality

Because the problem of determining all polar actions on symmetric spaces has benefitted from greater advances in the compact setting, a natural idea is to attempt to use the duality of symmetric spaces to make progress in the noncompact setting. Using a technique proposed by Kollross [107], we exhibit how some polar actions on a symmetric space of noncompact type can be identified with polar actions on its compact dual.

Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of noncompact type, define $o = e\mathsf{K}$ and consider the induced Cartan involution $\theta\colon \mathfrak{g} \to \mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Recall that $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g}(\mathbb{C})$ is the dual algebra of $\mathfrak{g}$ and there is a unique simply connected symmetric space of compact type $M^* = \mathsf{G}^*/\mathsf{K}^*$ associated with the Klein pair $(\mathfrak{g}^*, \mathfrak{k})$.

A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *canonically embedded* if $\theta\mathfrak{h} = \mathfrak{h}$. This condition is equivalent to $\mathfrak{h}$ admitting the decomposition $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. If $\mathfrak{h}$ is canonically embedded in $\mathfrak{g}$, then we can define a dual subalgebra $\mathfrak{h}^* = (\mathfrak{h} \cap \mathfrak{k}) \oplus i(\mathfrak{h} \cap \mathfrak{p}) \subseteq \mathfrak{g}^*$. Suppose $\mathsf{H} \subseteq \mathsf{G}$ (respectively, $\mathsf{H}^* \subseteq \mathsf{G}^*$) is the connected subgroup of G with Lie algebra $\mathfrak{h}$ (respectively, the connected subgroup of G^* with Lie algebra H). We say that the action $\mathsf{H}^* \curvearrowright M^*$ is *dual* to the action $\mathsf{H} \curvearrowright M$. From [107, Theorem 5.1] we see that the action of H is (hyper)polar if and only if the action of H^* is (hyper)polar. This allows us to produce many examples of polar actions from the known results in the compact case. However, there are some limitations to this technique that do not allow us to classify all polar actions on symmetric spaces of noncompact type from those on symmetric spaces of compact type. For instance, not all polar actions admit a dual action. Indeed, if $\mathsf{H} \subseteq \mathsf{G}$ is such that its Lie algebra $\mathfrak{h}$ is canonically embedded, then the orbit $\mathsf{H} \cdot o$ is a totally geodesic submanifold (see Theorem 6.3 for the proof of a more general version of this assertion). It may well be the case that a given polar action $\mathsf{H} \curvearrowright M$ does not have a totally geodesic orbit at all, meaning that we cannot conjugate H so that its Lie algebra is canonically embedded in $\mathfrak{g}$. Moreover, a polar action on a symmetric space of compact type can be dual to two non-equivalent polar actions on its dual symmetric space (for example, if $M = \mathbb{R}\mathsf{H}^2$, the actions of K and A are both dual to the standard action $\mathsf{SO}(2) \curvearrowright \mathsf{S}^2$), so we do not have an injective correspondence between these actions. This phenomenon justifies the need to develop new techniques to tackle the noncompact case.

Chapter 3
Codimension two polar homogeneous foliations on symmetric spaces of noncompact type

The objective of this chapter is to present the classification of polar homogeneous foliations of codimension two on irreducible symmetric spaces of noncompact type. The content of this chapter corresponds to a joint work with José Carlos Díaz-Ramos that has been published in [54].

As we have seen in the previous chapter, the classification problem for polar actions on irreducible symmetric spaces of noncompact type has proven to be quite elusive. For actions of cohomogeneity greater than one, the only general classification that has been obtained thus far is that of hyperpolar homogeneous foliations [19]. While this result is formidable on its own, dropping any of the two key assumptions (that is, hyperpolarity and the absence of singular orbits) supposes a great increase in difficulty. On the one hand, there are no classifications of totally geodesic submanifolds on symmetric spaces of rank greater than two, and we do not possess a clear cut way to determine which totally geodesic submanifolds can arise as sections of polar actions (unlike in the compact case, where all polar actions are hyperpolar if the ambient space is of higher rank). On the other hand, while the orbit space of a cohomogeneity one action on a symmetric space of noncompact type is either $\mathbb{R}$ or $[0, \infty)$, there is little information about the orbit space of a polar action of higher cohomogeneity, meaning that we do not have a general picture of the singular set.

In this work we start the study of polar nonhyperpolar foliations. As such, we preserve the hypothesis that our actions of interest induce a foliation on the ambient space while replacing the condition of having a flat section by having a two-dimensional one. This is a very natural assumption to make given the current progress in the area, as it allows us to keep a certain degree of control on the possible sections. Indeed, if $\mathsf{H} \curvearrowright M$ is a polar action on a symmetric space of noncompact type with cohomogeneity two and section Σ, then Σ is a homogeneous Riemannian manifold diffeomorphic to $\mathbb{R}^2$. This means that Σ has constant Gaussian curvature $C \leq 0$, so either the action is hyperpolar or Σ is homothetic to a real hyperbolic plane. Consequently, our goal for this chapter is to classify polar homogeneous foliations whose section is a surface of constant negative curvature.

The main result of this chapter is the following:

Theorem A. *Let M be a connected irreducible Riemannian symmetric space of noncompact type. Every codimension two polar nonhyperpolar homogeneous foliation on M is orbit equivalent to the canonical extension of a codimension two polar nonhyperpolar homogeneous foliation on a boundary component of rank one in M.*

More explicitly, we prove:

Theorem B. *Let $M = \mathsf{G}/\mathsf{K}$ be a connected irreducible Riemannian symmetric space of noncompact type. Then, a codimension two homogeneous polar nonhyperpolar foliation on M is orbit equivalent to the orbit foliation of the closed connected Lie group whose Lie algebra is given by one of the following possibilities:*

(i) $(\ker\alpha) \oplus (\mathfrak{n} \ominus \ell_\alpha)$, *where $\alpha \in \Lambda$ is a simple root, and ℓ_α is a line in $\mathfrak{g}_\alpha$, or*

(ii) $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha)$, *where $\alpha \in \Lambda$ is a simple root, and $\mathfrak{v}_\alpha$ is a 2-dimensional abelian subspace of $\mathfrak{g}_\alpha$.*

We will show in Section 3.1 that different choices of ℓ_α or $\mathfrak{v}_\alpha$ above give rise to congruent foliations. We also determine the mean curvature of their leaves. A direct consequence of this computation is that

Corollary C. *If $\mathcal{F}$ is a codimension two polar nonhyperpolar homogeneous foliation on M, then $\mathcal{F}$ is harmonic if and only if $\mathcal{F}$ is orbit equivalent to the canonical extension of the trivial foliation on a boundary component homothetic to the hyperbolic plane $\mathbb{R}\mathsf{H}^2$.*

Recall that a foliation $\mathcal{F}$ is harmonic if all of its leaves are minimal submanifolds of M. Equivalently, $\mathcal{F}$ is harmonic when the canonical projection from M to the space of leaves of $\mathcal{F}$ is a harmonic map.

As a result of combining Theorem B with [19], Corollary D states the complete classification of homogeneous polar foliations of codimension two. Choose an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of the isometry algebra $\mathfrak{g} = \mathfrak{i}(M)$ and let Σ be its corresponding root system. We remind that two roots λ, $\mu \in \Sigma$ are said to be strongly orthogonal if $\lambda \pm \mu$ is not a root. Two simple roots α, $\beta \in \Lambda$ are strongly orthogonal if and only if they are orthogonal, or equivalently, if they are not connected in the Dynkin diagram of Σ.

Corollary D. *A codimension two homogeneous polar foliation on M is orbit equivalent to the orbit foliation of a closed connected Lie group whose Lie algebra is:*

(a) $(\mathfrak{a} \ominus \mathfrak{v}) \oplus \mathfrak{n}$, *where $\mathfrak{v}$ is a 2-dimensional subspace of $\mathfrak{a}$, or*

(b) $(\mathfrak{a} \ominus \ell) \oplus (\mathfrak{n} \ominus \ell_\alpha)$, *where $\alpha \in \Lambda$ is a simple root, ℓ_α is a line in $\mathfrak{g}_\alpha$, and ℓ is a line in $\ker\alpha$, or*

(c) $\mathfrak{a} \oplus \big(\mathfrak{n} \ominus (\ell_\alpha \oplus \ell_\beta)\big)$, *where α, $\beta \in \Lambda$ are orthogonal simple roots, and ℓ_λ is a line in $\mathfrak{g}_\lambda$, $\lambda \in \{\alpha, \beta\}$, or*

(d) $(\ker\alpha) \oplus (\mathfrak{n} \ominus \ell_\alpha)$, *where $\alpha \in \Lambda$ is a simple root, and ℓ_α is a line in $\mathfrak{g}_\alpha$, or*

(e) $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha)$, *where $\alpha \in \Lambda$ is a simple root, and $\mathfrak{v}_\alpha$ is a 2-dimensional abelian subspace of $\mathfrak{g}_\alpha$.*

Examples (a) to (c) of Corollary D are hyperpolar.

Remark 3.1. The assumption that M is irreducible implies that all the G-invariant metrics on M are rescalings of the metric induced by the Killing form. If the space M is reducible, then the results presented above still hold if we impose the condition that the metric on M is (homothetic to) the one induced by the Killing form.

The trivial action is always polar and the whole space is a section of this action. On the other hand, cohomogeneity one polar actions are automatically hyperpolar. Note that one can always construct a foliation of the form (i) unless $\Sigma^+ = \{\alpha\}$ consists of only one simple root and $\dim \mathfrak{g}_\alpha = 1$. These conditions are satisfied if and only if M is a real hyperbolic plane. Thus,

Corollary E. *If M is an irreducible symmetric space of noncompact type where all polar actions are hyperpolar, then M is the real hyperbolic space $\mathbb{R}\mathsf{H}^2$.*

This contrasts sharply with the situation in the compact setting: polar actions on irreducible symmetric spaces of compact type and higher rank are always hyperpolar, as we saw in Subsection 2.3.1.

This chapter is organized as follows. In Section 3.1, we present the list of codimension two polar nonhyperpolar homogeneous foliations that appear in Theorem B. We also determine the curvature of their sections, the extrinsic geometry of their orbits, and prove Theorem A. In Section 3.2 we study homogeneous foliations on Hadamard manifolds and show that they are induced by free actions of solvable Lie groups. Moreover, we describe the structure of maximal solvable subalgebras of real semisimple Lie algebras. Finally, Section 3.3 will be devoted to the proof of Theorem B.

3.1 Examples of homogeneous polar foliations

We now introduce the two families of polar homogeneous foliations on $M = \mathsf{G}/\mathsf{K}$ whose section is homothetic to the hyperbolic plane, and describe their extrinsic geometry. We assume the notation used in Section 1.1.1.

Theorem 3.2. *Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of noncompact type and choose an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$. Let $\alpha \in \Lambda$ be a simple root, and consider the following subspaces of $\mathfrak{a} \oplus \mathfrak{n}$:*

(i) $\mathfrak{s}_\xi = (\mathfrak{a} \ominus \mathbb{R}H_\alpha) \oplus (\mathfrak{n} \ominus \mathbb{R}\xi)$, *where $\xi \in \mathfrak{g}_\alpha$ is a nonzero vector.*

(ii) $\mathfrak{s}_\mathfrak{v} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v})$, *where $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ is an abelian plane inside $\mathfrak{g}_\alpha$.*

The subspaces $\mathfrak{s}_\xi$ and $\mathfrak{s}_\mathfrak{v}$ are Lie subalgebras of $\mathfrak{a} \oplus \mathfrak{n}$. The corresponding connected subgroups S_ξ, $\mathsf{S}_\mathfrak{v}$ act polarly on M inducing a codimension two foliation whose section is a totally geodesic $\mathbb{R}\mathsf{H}^2$ with constant curvature $-|\alpha|^2$.

Proof. It is clear that $\mathfrak{s}_\xi$ and $\mathfrak{s}_\mathfrak{v}$ are subalgebras of $\mathfrak{a} \oplus \mathfrak{n}$, so we can consider the connected Lie subgroups S_ξ, $\mathsf{S}_\mathfrak{v}$ associated with these subalgebras. Since AN acts simply transitively on M and

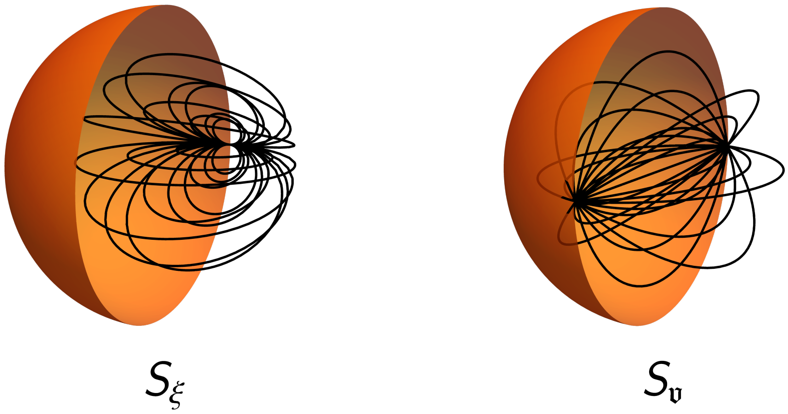


Figure 3.1.1: Orbits of the actions of S_ξ and $\mathsf{S}_\mathfrak{v}$ on the hyperbolic space $\mathbb{R}\mathsf{H}^3$.

$\mathrm{Exp}\colon \mathfrak{a}\oplus\mathfrak{n}\to \mathsf{AN}$ is a diffeomorphism, it follows that S_ξ and $\mathsf{S}_\mathfrak{v}$ are closed subgroups inducing a homogeneous foliation on M of codimension 2. It remains to show that both subgroups act polarly.

Firstly, if $\mathsf{S}=\mathsf{S}_\xi$, we see that $\mathfrak{s}_\mathfrak{p}^\perp=\operatorname{span}\{H_\alpha,(1-\theta)\xi\}$. Now, a direct computation shows that $[\mathfrak{s}_\mathfrak{p}^\perp,\mathfrak{s}_\mathfrak{p}^\perp]=\mathbb{R}(1+\theta)\xi$ is orthogonal to $\mathfrak{s}$, and $[\mathfrak{s}_\mathfrak{p}^\perp,[\mathfrak{s}_\mathfrak{p}^\perp,\mathfrak{s}_\mathfrak{p}^\perp]]$ is spanned by $[H_\alpha,(1+\theta)\xi]=|\alpha|^2(1-\theta)\xi$ and $[(1-\theta)\xi,(1+\theta)\xi]=-(1-\theta)[\theta\xi,\xi]=-2|\xi|^2H_\alpha$. We therefore obtain $[\mathfrak{s}_\mathfrak{p}^\perp,[\mathfrak{s}_\mathfrak{p}^\perp,\mathfrak{s}_\mathfrak{p}^\perp]]=\mathfrak{s}_\mathfrak{p}^\perp$, which means that $\mathfrak{s}_\mathfrak{p}^\perp$ is a Lie triple system. By applying Proposition 2.20, we deduce that S_ξ acts polarly, as desired. Note that if $\mathsf{S}_\mathfrak{p}^\perp\cdot o$ is the section through o, then $\mathsf{S}_\mathfrak{p}^\perp\cdot o$ is a closed, simply connected, totally geodesic surface whose tangent space is $\mathfrak{s}_\mathfrak{p}^\perp$. Its sectional curvature is seen to be

$$\begin{aligned}\sec(\mathsf{S}_\mathfrak{p}^\perp\cdot o)&=\frac{-\langle[[H_\alpha,(1-\theta)\xi],(1-\theta)\xi],H_\alpha\rangle}{|(1-\theta)\xi|^2|H_\alpha|^2}\\&=\frac{-|[H_\alpha,(1-\theta)\xi]|^2}{2|\xi|^2|\alpha|^2}=\frac{-|(1+\theta)|\alpha|^2\xi|^2}{2|\xi|^2|\alpha|^2}=-|\alpha|^2,\end{aligned}$$

so $\mathsf{S}_\mathfrak{p}^\perp\cdot o$ is a real space form of constant curvature $-|\alpha|^2$.

In the case $\mathsf{S}=\mathsf{S}_\mathfrak{v}$, the normal space is $\mathfrak{s}_\mathfrak{p}^\perp=(1-\theta)\mathfrak{v}$. Choose two orthogonal vectors $\xi,\eta\in\mathfrak{v}$ with norm $1/\sqrt{2}$. Since $[(1-\theta)\xi,(1-\theta)\eta]=-2[\xi,\theta\eta]\in\mathfrak{k}_0$, it follows that $[\mathfrak{s}_\mathfrak{p}^\perp,\mathfrak{s}_\mathfrak{p}^\perp]=\mathbb{R}[\xi,\theta\eta]$ is perpendicular to $\mathfrak{s}$. Furthermore, $\theta[\xi,\theta\eta]=[\xi,\theta\eta]$ yields

$$\begin{aligned}[(1-\theta)\xi,[\xi,\theta\eta]]&=(1-\theta)[\xi,[\theta\xi,\eta]]=-(1-\theta)[\eta,[\xi,\theta\xi]]\\&=(1-\theta)[\eta,|\xi|^2H_\alpha]=-|\xi|^2|\alpha|^2(1-\theta)\eta\in\mathfrak{s}_\mathfrak{p}^\perp,\end{aligned}$$

and a similar calculation gives $[(1-\theta)\eta,[\xi,\theta\eta]]=|\alpha|^2|\eta|^2(1-\theta)\xi\in\mathfrak{s}_\mathfrak{p}^\perp$, and thus $[\mathfrak{s}_\mathfrak{p}^\perp,[\mathfrak{s}_\mathfrak{p}^\perp,\mathfrak{s}_\mathfrak{p}^\perp]]=\mathfrak{s}_\mathfrak{p}^\perp$. Proposition 2.20 readily implies that the action of $\mathsf{S}_\mathfrak{v}$ is polar with section $\mathsf{S}_\mathfrak{p}^\perp\cdot o$. The same argument given in the previous paragraph allows us to determine the section by computing its curvature. In this case, taking into account our previous calculations,

$$\sec(\mathsf{S}_\mathfrak{p}^\perp\cdot o)=-\frac{\langle[[(1-\theta)\xi,(1-\theta)\eta],(1-\theta)\eta],(1-\theta)\xi\rangle}{|(1-\theta)\xi|^2|(1-\theta)\eta|^2}=-|\alpha|^2,$$

which finishes the proof. □

The previous theorem shows that the examples that appear in Theorem B give rise to homogeneous polar foliations. Furthermore, it follows from the next lemma that different choices of ξ in case (i) or of $\mathfrak{v}$ in (ii) give orbit equivalent actions.

Lemma 3.3. *Let $\alpha \in \Lambda$ and $k \geq 1$. Then, the group K_0 acts transitively on the set of abelian subspaces of dimension k of $\mathfrak{g}_\alpha$.*

Proof. Following [88, Chapter IX, §2], we consider the Lie subalgebra $\mathfrak{g}^\alpha$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. This Lie algebra is simple and its Cartan decomposition is $\mathfrak{g}^\alpha = \mathfrak{k}^\alpha \oplus \mathfrak{p}^\alpha$, with $\mathfrak{k}^\alpha = \mathfrak{k} \cap \mathfrak{g}^\alpha$, $\mathfrak{p}^\alpha = \mathfrak{p} \cap \mathfrak{g}^\alpha$. It turns out that $\mathbb{R}H_\alpha$ is a maximal abelian subspace of $\mathfrak{p}^\alpha$, and the root space decomposition of $\mathfrak{g}^\alpha$ is

$$\mathfrak{g}^\alpha = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus (\mathfrak{k}_0^\alpha \oplus \mathbb{R}H_\alpha) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},$$

where $\mathfrak{k}_0^\alpha$ is the centralizer of $\mathbb{R}H_\alpha$ in $\mathfrak{k}^\alpha$, and $\mathfrak{k}_0^\alpha = \mathfrak{k}_0 \cap \mathfrak{g}^\alpha$. Let G^α, K^α, K_0^α be the connected subgroups of G whose Lie algebras are $\mathfrak{g}^\alpha$, $\mathfrak{k}^\alpha$, and $\mathfrak{k}_0^\alpha$. Then, by [88, Chapter IX, Lemma 2.3], we have $\mathsf{K}^\alpha = \mathsf{K} \cap \mathsf{G}^\alpha$ and $\mathsf{K}_0^\alpha = \mathsf{K}_0 \cap \mathsf{G}^\alpha$. Therefore, in order to prove this lemma, it suffices to show that K_0^α acts transitively on the set of abelian subspaces of $\mathfrak{g}_\alpha$.

Obviously, $\mathsf{G}^\alpha/\mathsf{K}^\alpha$ is a Riemannian symmetric space of noncompact type and rank one, that is, a hyperbolic space $\mathbb{F}\mathsf{H}^n$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $n \geq 2$ ($n = 2$ if $\mathbb{F} = \mathbb{O}$). Note that $\mathfrak{g}_\alpha \cong \mathbb{F}^{n-1}$ and $\dim \mathfrak{g}_{2\alpha} = \dim_{\mathbb{R}} \mathbb{F} - 1$. If $\mathbb{F} = \mathbb{R}$, then $\mathfrak{g}_\alpha$ is abelian and $\mathsf{K}_0^\alpha \cong \mathsf{SO}(n-1)$ acts in the standard way on $\mathfrak{g}_\alpha$; this action is transitive on the Grassmannian of k-planes of $\mathbb{R}^{n-1}$. If $\mathbb{F} = \mathbb{O}$, then the only nonzero abelian subspaces of $\mathfrak{g}_\alpha \cong \mathbb{O}$ are 1-dimensional, and $\mathsf{K}_0^\alpha \cong \mathsf{Spin}(7)$ acts on $\mathbb{O}$ by its irreducible 8-dimensional spin representation, which is transitive in S^7 [31]. Finally, if $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, recall that abelian subspaces of $\mathfrak{g}_\alpha$ are precisely totally real subspaces of $\mathfrak{g}_\alpha \cong \mathbb{F}^{n-1}$. In these cases we have the standard action of $\mathsf{S}(\mathsf{U}(n-1)\mathsf{U}(1))$ on $\mathbb{C}^{n-1}$ if $\mathbb{F} = \mathbb{C}$, and the standard action of $\mathsf{Sp}(n-1)\mathsf{Sp}(1)$ on $\mathbb{H}^{n-1}$ if $\mathbb{F} = \mathbb{H}$. Thus, if $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are two totally real subspaces of $\mathfrak{g}_\alpha$ of dimension k, choose an orthonormal basis of $\mathfrak{v}_1$ and an orthonormal basis of $\mathfrak{v}_2$. Since $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are totally real, these two bases are not only orthonormal, but $\mathbb{F}$-orthonormal. By definition of $\mathsf{U}(n-1)$ or $\mathsf{Sp}(n-1)$ it is then clear that there is an element of K_0^α that maps one basis to the other. This finishes the proof. □

We now exhibit examples (i) and (ii) of Theorem B as canonical extensions of actions on a rank one boundary component. Let $\alpha \in \Lambda$ be a simple root, $\xi \in \mathfrak{g}_\alpha$ a unit vector and $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ an abelian plane. We consider the set $\Phi = \{\alpha\} \subseteq \Lambda$. Then, the subalgebras constructed in Section 1.4.1 take the form

$$\begin{aligned}
\mathfrak{l}_\Phi &= \mathfrak{g}_0 \oplus \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},\\
\mathfrak{a}_\Phi &= \ker \alpha,\\
\mathfrak{n}_\Phi &= \mathfrak{n} \ominus (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}),\\
\mathfrak{m}_\Phi &= \mathfrak{k}_0 \oplus \mathbb{R}H_\alpha \oplus \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},
\end{aligned}$$

and $B_\Phi = \mathsf{M}_\Phi \cdot o$ is a rank one noncompact symmetric space whose tangent space at o is $T_oB_\Phi = \mathbb{R}H_\alpha \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$. Consider the subalgebras $\hat{\mathfrak{s}}_\xi = (\mathfrak{g}_\alpha \ominus \mathbb{R}\xi) \oplus \mathfrak{g}_{2\alpha}$ and $\hat{\mathfrak{s}}_\mathfrak{v} = \mathbb{R}H_\alpha \oplus (\mathfrak{g}_\alpha \ominus \mathfrak{v}) \oplus \mathfrak{g}_{2\alpha}$ of

$\mathfrak{g}_\Phi = [\mathfrak{m}_\Phi, \mathfrak{m}_\Phi]$. The corresponding connected subgroups $\hat{\mathsf{S}}_\xi$ and $\hat{\mathsf{S}}_\mathfrak{v}$ act polarly on B_Φ inducing a foliation, due to Theorem 3.2. Recall that the canonical extensions of the actions of $\hat{\mathsf{S}}_\xi$ and $\hat{\mathsf{S}}_\mathfrak{v}$ are the actions of the subgroups $\hat{\mathsf{S}}_\xi \mathsf{A}_\Phi \mathsf{N}_\Phi$ and $\hat{\mathsf{S}}_\mathfrak{v} \mathsf{A}_\Phi \mathsf{N}_\Phi$, respectively. Observe that $\hat{\mathfrak{s}}_\xi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi = \mathfrak{s}_\xi$, while $\hat{\mathfrak{s}}_\mathfrak{v} \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi = \mathfrak{s}_\mathfrak{v}$, and this readily implies that these canonical extensions are precisely the actions of S_ξ and $\mathsf{S}_\mathfrak{v}$. We deduce that Theorem B implies Theorem A.

The remaining part of this section will be devoted to computing the mean curvature of the orbits in each example. To this end, we consider the solvable model $M = \mathsf{AN}$ discussed in Section 1.4.1. If $\mathfrak{s} \subseteq \mathfrak{a} \oplus \mathfrak{n}$, we refer to its orthogonal complement in $\mathfrak{a} \oplus \mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle_{\mathsf{AN}}$ as $\mathfrak{s}^\perp$. Note that if $\mathsf{S} \subseteq \mathsf{AN}$ is a Lie subgroup, the isometry $g \in \mathsf{AN} \mapsto g \cdot o \in M$ induces an orbit equivalence between the action of S on M and the action of S on AN by left multiplication.

Recall that if $M \subseteq \widetilde{M}$ is a submanifold of a Riemannian manifold with second fundamental form $\mathbb{II}$, we define the mean curvature vector of M at p as $\mathcal{H}_p = \sum_i \mathbb{II}(e_i, e_i)$, where $\{e_i\}_i$ is an orthonormal basis of T_pM. In other words, $\mathcal{H}$ is the trace of the second fundamental form. If $\mathsf{S} \subseteq \mathsf{AN}$ is a connected subgroup of AN, it is easy to see from the formula for the Levi-Civita connection that the second fundamental form of $\mathsf{S} \subseteq \mathsf{AN}$ at e satisfies the identity

$$\langle \mathbb{II}(X, X), \eta \rangle_{AN} = \frac{1}{4} \langle (1 - \theta)[\theta X, X], \eta \rangle \tag{3.1}$$

for each $X \in \mathfrak{s}$ and $\eta \in \mathfrak{s}^\perp$. We also remind that $\mathfrak{a}^*$ is endowed with a distinguished element $\delta = (1/2) \sum_{\lambda \in \Sigma^+} (\dim \mathfrak{g}_\lambda) \lambda$.

Let us start by discussing foliations of type (i).

Proposition 3.4. *Let $\alpha \in \Lambda$ and $\xi \in \mathfrak{g}_\alpha$ a vector such that $\langle \xi, \xi \rangle = 1$. All the orbits of S_ξ are isometrically congruent. Furthermore, the mean curvature vector of S_ξ at e is given by the following expression:*

$$\mathcal{H}_e = (\dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha} - 1)\, H_\alpha.$$

Proof. Observe that $\mathfrak{s}_\xi = \ker \alpha \oplus (\mathfrak{n} \ominus \mathbb{R}\xi)$ is an ideal of $\mathfrak{a} \oplus \mathfrak{n}$. As a consequence, if $g \in \mathsf{AN}$ is an arbitrary point, we have $\mathsf{S}_\xi \cdot g = g g^{-1} \mathsf{S}_\xi g = g \mathsf{S}_\xi$, because S_ξ is a normal subgroup of AN. Therefore, $\mathsf{S}_\xi \cdot g$ is isometric to S_ξ via the left multiplication by g.

We proceed to compute $\mathcal{H}_o$. For this, it suffices to determine the vectors $\mathbb{II}(H, H)$ and $\mathbb{II}(X, X)$ for each $H \in \mathfrak{a}$ and $X \in \mathfrak{g}_\lambda \ominus \mathbb{R}\xi$, where λ is any positive root. Given any $H \in \mathfrak{a}$, it is clear from (3.1) that $\mathbb{II}(H, H) = 0$. On the other hand, if $\lambda \in \Sigma^+$ and $X \in \mathfrak{g}_\lambda$ is a unit vector orthogonal to ξ, we obtain that $1 = \langle X, X \rangle_{\mathsf{AN}} = \frac{1}{2}|X|^2$ and $\langle \mathbb{II}(X, X), \eta \rangle_{\mathsf{AN}} = \frac{1}{2}\langle |X|^2 H_\lambda, \eta \rangle = \langle H_\lambda, \eta \rangle = \frac{\langle \lambda, \alpha \rangle}{|\alpha|^2} \langle H_\alpha, \eta \rangle_{\mathsf{AN}}$ for each vector $\eta \in \mathfrak{s}^\perp = \operatorname{span}\{H_\alpha, \xi\}$, which means that $\mathbb{II}(X, X) = \frac{\langle \lambda, \alpha \rangle}{|\alpha|^2} H_\alpha$. In conclusion,

$$\begin{aligned}
\mathcal{H}_e &= \frac{1}{|\alpha|^2} \left(\sum_{\lambda \in \Sigma^+ \setminus \{\alpha\}} (\dim \mathfrak{g}_\lambda) \langle \lambda, \alpha \rangle + (\dim \mathfrak{g}_\alpha - 1)|\alpha|^2 \right) H_\alpha \\
&= \frac{1}{|\alpha|^2} (\langle 2\delta, \alpha \rangle - |\alpha|^2) H_\alpha = (\dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha} - 1) H_\alpha,
\end{aligned}$$

where we have used (1.1) for the last equality. □

A direct consequence of the previous proposition is that the foliation induced by S_ξ consists of congruent minimal submanifolds if and only if $2\alpha \notin \Sigma$ and $\dim \mathfrak{g}_\alpha = 1$. If this is the case, then $\mathfrak{s}_\xi = \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ for $\Phi = \{\alpha\}$, and B_Φ is homothetic to the hyperbolic plane. Furthermore, $\hat{\mathfrak{s}}_\xi = 0$, which means that $\hat{\mathsf{S}}_\xi$ acts trivially on B_Φ.

We now consider the foliations from case (ii). In this setting, the orbits are not isometrically congruent, as their mean curvature does not have constant length. More precisely:

Proposition 3.5. *Let $\alpha \in \Lambda$ be a simple root and $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ an abelian subspace of dimension 2. Fix a vector $\xi \in \mathfrak{v}$ with $|\xi| = 1$, and denote by $\mathcal{H}_t$ the mean curvature vector of $\mathsf{S}_\mathfrak{v} \cdot \mathrm{Exp}(t\xi)$ at $\mathrm{Exp}(t\xi)$ and by $L_{\mathrm{Exp}(t\xi)} \colon \mathsf{AN} \to \mathsf{AN}$ the left translation by $\mathrm{Exp}(t\xi)$. Then,*

$$(L_{\mathrm{Exp}(-t\xi)})_{*\,\mathrm{Exp}(t\xi)}\mathcal{H}_t = \frac{t|\alpha|^2}{2+t^2|\alpha|^2}(\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha} - 1)(tH_\alpha - 2\xi).$$

In particular, the orbit through $\mathrm{Exp}(t\xi)$ is minimal if and only if $t = 0$.

Proof. Firstly, if $g = \mathrm{Exp}(t\xi) \in \mathsf{AN}$, we deduce that $\mathsf{S}_\mathfrak{v} \cdot g = gg^{-1}\mathsf{S}_\mathfrak{v} g = g(g^{-1}\mathsf{S}_\mathfrak{v} g)$ is isometric to $g^{-1}\mathsf{S}_\mathfrak{v} g$ by left translation. Thus, it suffices to compute the mean curvature $\tilde{\mathcal{H}}_t$ of $\tilde{\mathsf{S}} = g^{-1}\mathsf{S}_\mathfrak{v} g$ at e. To this end, we compute the Lie algebra $\tilde{\mathfrak{s}} = \mathrm{Ad}(g^{-1})\mathfrak{s}_\mathfrak{v}$ of $g^{-1}\mathsf{S}_\mathfrak{v} g$. Observe that the subspace $\mathrm{Ad}(g^{-1})\mathfrak{s}_\mathfrak{v} \subseteq \mathfrak{a} \oplus \mathfrak{n}$ is orthogonal to $\mathrm{Ad}(g)^*\mathfrak{v} = e^{-t\,\mathrm{ad}(\theta\xi)}\mathfrak{v}$ with respect to the inner product $\langle \cdot, \cdot \rangle$. Given any $\eta \in \mathfrak{v}$, we have

$$\begin{aligned} e^{-t\,\mathrm{ad}(\theta\xi)}\eta &\equiv \eta - t[\theta\xi, \eta] \pmod{\theta\mathfrak{n}} \\ &= \eta - t\langle \xi, \eta \rangle H_\alpha - \frac{t}{2}(1+\theta)[\theta\xi, \eta] \pmod{\theta\mathfrak{n}} \\ &\equiv \eta - t\langle \xi, \eta \rangle H_\alpha \pmod{\mathfrak{k}_0 \oplus \theta\mathfrak{n}}. \end{aligned}$$

Therefore, if we consider an orthonormal basis $\{\xi, \eta\}$ of $\mathfrak{v}$, it is immediate that the orthogonal complement of $\mathrm{Ad}(g^{-1})\mathfrak{s}_\mathfrak{v}$ in $\mathfrak{a} \oplus \mathfrak{n}$ is $\mathrm{span}\{tH_\alpha - \xi, \eta\}$. As a consequence, $\tilde{\mathfrak{s}} = \mathrm{Ad}(g^{-1})\mathfrak{s}_\mathfrak{v} = \ker \alpha \oplus (\mathfrak{n} \ominus \mathfrak{v}) \oplus \mathbb{R}(H_\alpha + t|\alpha|^2\xi)$. The normal space $\mathfrak{s}^\perp$ is given by $\tilde{\mathfrak{s}}^\perp = \mathbb{R}\eta \oplus \mathbb{R}(tH_\alpha - 2\xi)$.

Assume $H \in \ker \alpha$. In this case, we directly have from (3.1) that $\mathrm{I\!I\!I}(H, H) = 0$.

Now, suppose that $\lambda \in \Sigma^+$ and $X \in \mathfrak{g}_\lambda \ominus \mathfrak{v}$ is such that $1 = \langle X, X \rangle_{\mathsf{AN}} = \frac{1}{2}|X|^2$. Then, $\mathrm{I\!I\!I}(X, X)$ satisfies $\langle \mathrm{I\!I\!I}(X, X), \nu \rangle_{\mathsf{AN}} = \frac{1}{2}\langle |X|^2 H_\lambda, \nu \rangle = \langle H_\lambda, \nu \rangle = \langle \frac{t\langle \lambda, \alpha \rangle}{2+t^2|\alpha|^2}(tH_\alpha - 2\xi), \nu \rangle_{\mathsf{AN}}$ for every $\nu \in \mathfrak{s}^\perp$, and thus

$$\mathrm{I\!I\!I}(X, X) = \frac{t\langle \lambda, \alpha \rangle}{2+t^2|\alpha|^2}(tH_\alpha - 2\xi).$$

Finally, consider the vector $Y = H_\alpha + t|\alpha|^2\xi$, whose norm squared is given by $\langle Y, Y \rangle_{\mathsf{AN}} = |\alpha|^2 + \frac{1}{2}t^2|\alpha|^4$. Note that $(1-\theta)[\theta Y, Y] = 2t|\alpha|^4(tH_\alpha - (1-\theta)\xi)$, so we deduce from (3.1) that $\mathrm{I\!I\!I}(Y, Y) = \frac{t|\alpha|^4}{2}(tH_\alpha - 2\xi)$. As a consequence, the normalized vector $Z = Y/|Y|_{\mathsf{AN}}$ satisfies $\mathrm{I\!I\!I}(Z, Z) = \frac{t|\alpha|^2}{2+t^2|\alpha|^2}(tH_\alpha - 2\xi)$. From these calculations, we obtain that the mean curvature of $\tilde{S}$

at o is given by

$$\begin{aligned}\tilde{\mathcal{H}}_t &= \frac{t}{2+t^2|\alpha|^2}\Bigg(\sum_{\lambda\in\Sigma^+\setminus\{\alpha\}}(\dim\mathfrak{g}_\lambda)\langle\lambda,\alpha\rangle + (\dim\mathfrak{g}_\alpha - 1)|\alpha|^2\Bigg)(tH_\alpha - 2\xi)\\ &= \frac{t}{2+t^2|\alpha|^2}(\langle 2\delta,\alpha\rangle - |\alpha|^2)(tH_\alpha - 2\xi)\\ &= \frac{t|\alpha|^2}{2+t^2|\alpha|^2}(\dim\mathfrak{g}_\alpha + 2\dim\mathfrak{g}_{2\alpha} - 1)(tH_\alpha - 2\xi).\end{aligned}$$

Finally, note that the existence of an abelian plane inside $\mathfrak{g}_\alpha$ implies that $\dim\mathfrak{g}_\alpha + 2\dim\mathfrak{g}_{2\alpha} - 1$ is positive, so the orbit through $\mathrm{Exp}(t\xi)$ is minimal if and only if $t = 0$, as desired. □

In particular, the homogeneous foliation induced by $\mathsf{S}_\mathfrak{v}$ is never harmonic independently of the choice of $\mathfrak{v}$. From here, Corollary C follows immediately.

Corollary 3.6. *No polar homogeneous foliation constructed as in case* (i) *of Theorem B is orbit equivalent to a homogeneous foliation given in case* (ii).

3.2 Homogeneous foliations and solvable subgroups

The aim of this section is to give a general structure result for homogeneous foliations on Hadamard manifolds. We basically show that every homogeneous foliation arises from a free proper action of a solvable Lie group, meaning that its leaves are Riemannian solvmanifolds. Moreover, the leaves are isomorphic as Lie groups (but not necessarily isometric as Riemannian manifolds, see Section 3.1).

We start by mentioning the following result, which is a slight refinement of [52, Lemma 2.5].

Lemma 3.7. *Let M be a complete Riemannian manifold and H and $\tilde{\mathsf{H}}$ be connected, not necessarily closed, subgroups of the isometry group of M such that $\mathsf{H} \subseteq \tilde{\mathsf{H}}$. Suppose that there exists $o \in M$ such that $\mathsf{H}\cdot o = \tilde{\mathsf{H}}\cdot o$ is a closed subset of M, and the slice representation of $\tilde{\mathsf{H}}$ at o is trivial. Then H and $\tilde{\mathsf{H}}$ act with the same orbits.*

Proof. Let $p \in M$ be arbitrary. Since $\mathsf{H}\cdot o$ is closed in M, we may find a point $q \in \mathsf{H}\cdot o$ such that the distance from q to p is minimum among all points of $\mathsf{H}\cdot o$. The first variation formula implies that the minimizing geodesic joining q and p must leave $\mathsf{H}\cdot o$ perpendicularly. Thus, by homogeneity we may assume that $q = o$ and $p = \exp_o(\xi_o)$, with $\xi_o \in \nu_o(H\cdot o)$. Let $\xi \in \Gamma(\nu(\mathsf{H}\cdot o))$ be the unique $\tilde{\mathsf{H}}$-equivariant vector field whose value at o is ξ_o (which exists because ξ_o is fixed by the slice representation of $\tilde{\mathsf{H}}$). Since $\mathsf{H} \subseteq \tilde{\mathsf{H}}$, ξ is also the unique H-equivariant normal vector field along $\mathsf{H}\cdot o$ generated by ξ_o. Now it is clear from (2.1) that $\mathsf{H}\cdot p = \tilde{\mathsf{H}}\cdot p$, so H and $\tilde{\mathsf{H}}$ have the same orbits. □

We also need a result from [19] stating that homogeneous foliations on Hadamard manifolds are always induced by solvable groups (up to orbit equivalence). The original proof by Berndt, Díaz-Ramos and Tamaru contains a nontrivial step that needs some clarification. Namely, the authors work with a certain semisimple Lie group L and choose an Iwasawa decomposition $\mathsf{L} = \mathsf{KAN}$. The key to the proof is to apply Cartan's fixed point theorem [59, Theorem 1.4.6] to an isometric action of K, and this needs us to guarantee that K is a compact group (equivalently, that $\mathsf{Z}(\mathsf{L})$ is finite, see Subsection 1.1.1), something that is not done explicitly in [19]. For the sake of completeness, we repeat the proof of this result and fill in the missing detail from the original paper.

Proposition 3.8 [19, Proposition 2.2]**.** *Let M be a Hadamard manifold, and let H be a connected closed subgroup of the isometry group of M acting on M in such a way that the orbits of H form a foliation. Then, all the orbits of H are principal, and there is a connected closed solvable group S acting isometrically on M whose orbits coincide with the orbits of H.*

Proof. Firstly, it was shown in [11, Section 2] that the center of $I^0(M)$ is trivial, which means that $I^0(M)$ is a linear Lie group, and so is H.

We first prove that all the H-orbits are principal. This can be done directly by appealing to Corollary 2.15, but we can show this directly in the case that the ambient manifold has nonpositive curvature. Suppose that, on the contrary, there exists a point $p \in M$ such that its orbit $\mathsf{H} \cdot p$ is exceptional. By [92, Theorem 14.1.3], there exists a maximal compact subgroup $\mathsf{C} \subseteq \mathsf{H}$ containing H_p, and using Cartan's fixed point theorem together with the fact that isotropy subgroups are compact, we obtain that C is the isotropy subgroup of a point $q \in M$. As $\mathsf{H}_p \subseteq \mathsf{C} = \mathsf{H}_q$, we see that $\mathsf{H} \cdot q$ is also an exceptional orbit. However, by [92, Theorem 14.3.11], $\mathsf{H} \cdot q = \mathsf{H}/\mathsf{C}$ is diffeomorphic to $\mathbb{R}^k$ for some k, which implies that C is connected. In particular, the slice representation of H at q is trivial, contradicting the fact that $\mathsf{H} \cdot q$ is an exceptional orbit. Therefore, all H-orbits are principal.

Consider a Levi decomposition $\mathfrak{h} = \mathfrak{rad}(\mathfrak{h}) \rtimes \mathfrak{l}$, where $\mathfrak{l}$ is a maximal semisimple subalgebra of $\mathfrak{h}$, and let L be the connected subgroup of H with Lie algebra $\mathfrak{l}$. Then L is a linear Lie group, so $\mathsf{Z}(\mathsf{L})$ is finite. Take an Iwasawa decomposition $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, with corresponding Lie group decomposition $\mathsf{L} = \mathsf{KAN}$. Then K is compact because L has finite center, and using Cartan's fixed point theorem we may find a point $p \in M$ such that $\mathsf{K} \cdot p = \{p\}$. Let $\mathfrak{s} = \mathfrak{rad}(\mathfrak{h}) \rtimes (\mathfrak{a} \oplus \mathfrak{n})$, which is a solvable Lie algebra, and let S be the connected Lie subgroup of H with Lie algebra $\mathfrak{s}$. The condition $\mathsf{K} \cdot p = \{p\}$ implies that $T_p(\mathsf{S} \cdot p) = T_p(\mathsf{H} \cdot p)$, so $\mathsf{S} \cdot p$ is an open subset of $\mathsf{H} \cdot p$. In particular, $\mathsf{S} \cdot p$ is a complete totally geodesic submanifold of $\mathsf{H} \cdot p$, so it coincides with $\mathsf{H} \cdot p$. We apply Lemma 3.7 to conclude that H and S have the same orbits, and Proposition 2.1 yields that S is closed in $I(M)$, which finishes the proof. □

As a by-product of the proof of Proposition 3.8, we have seen that the leaves of a homogeneous foliation on a Hadamard manifold are diffeomorphic to a Euclidean space.

We now show that every homogeneous foliation on M arises from a free proper action (cf. [71, Lemma 1.2]).

Theorem 3.9. *Let M be a Hadamard manifold and $\mathcal{F}$ a homogeneous foliation on M. Then there exists a closed solvable subgroup S of $I^0(M)$ acting freely on M in such a way that its orbits are precisely the leaves of $\mathcal{F}$.*

Proof. We already know from Proposition 3.8 that there exists a connected closed solvable subgroup $\mathsf{H} \subseteq I^0(M)$ whose orbits on M are the leaves of $\mathcal{F}$. Let $\mathfrak{n}$ be the nilradical of the Lie algebra $\mathfrak{h}$ (that is, the largest nilpotent ideal of $\mathfrak{h}$) and take its corresponding connected subgroup $\mathsf{N} \subseteq \mathsf{H}$. Consider the adjoint representation $\mathrm{Ad}_{\mathfrak{h}} \colon \mathsf{H} \to \mathsf{GL}(\mathfrak{h})$, whose differential at e is $\mathrm{ad}_{\mathfrak{h}} \colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$. By [102, Proposition 1.40] we have

$$\mathfrak{n} = \{X \in \mathfrak{h} \colon \ \mathrm{ad}_{\mathfrak{h}}(X) \text{ is nilpotent}\}.$$

On the other hand, let K be the isotropy subgroup of H at a point $o \in M$. This group is compact and connected because $\mathsf{H} \cdot o$ is simply connected.

We prove that $\mathsf{K} \cap \mathsf{N} = \{e\}$. Since $\mathrm{ad}_{\mathfrak{h}}(\mathfrak{n})$ consists of nilpotent endomorphisms of $\mathfrak{h}$, we can apply Engel's theorem to find a basis of $\mathfrak{h}$ such that the matrices of $\mathrm{ad}_{\mathfrak{h}}(\mathfrak{n})$ are all strictly upper triangular relative to this basis. Therefore, the matrices of $\mathrm{Ad}_{\mathfrak{h}}(\mathsf{N})$ are upper triangular with ones in the diagonal. Moreover, because K is a compact Lie group, we may find an inner product on $\mathfrak{h}$ that is preserved under the elements of $\mathrm{Ad}_{\mathfrak{h}}(\mathsf{K})$. Let $g \in \mathsf{N} \cap \mathsf{K}$. The map $\mathrm{Ad}_{\mathfrak{h}}(g)$ is a Euclidean isometry, so it is complex diagonalizable, and its only eigenvalue is 1, so necessarily we have $\mathrm{Ad}_{\mathfrak{h}}(g) = \mathrm{id}_{\mathfrak{h}}$. Consequently, $g \in \mathsf{Z}(\mathsf{H}) \cap \mathsf{K}$, implying that g fixes $\mathsf{H} \cdot o$ pointwise. In particular, the restriction of g_{*o} to $T_o(\mathsf{H} \cdot o)$ is the trivial map, and because the slice representation at o is trivial we also have that g_{*o} is trivial on $\nu_o(\mathsf{H} \cdot o)$, which yields $g_{*o} = \mathrm{id}_{T_oM}$ and $g = e$.

Now, choose a vector subspace $\mathfrak{s} \subseteq \mathfrak{h}$ satisfying $\mathfrak{n} \subseteq \mathfrak{s}$ and $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{k}$ (this is possible because $\mathfrak{n} \cap \mathfrak{k} = 0$). Note that since $\mathfrak{h}$ is solvable, $[\mathfrak{h}, \mathfrak{h}]$ is a nilpotent ideal of $\mathfrak{h}$ [102, Proposition 1.39], so $[\mathfrak{h}, \mathfrak{s}] \subseteq [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{n} \subseteq \mathfrak{s}$, showing that $\mathfrak{s}$ is an ideal in $\mathfrak{h}$. We denote by S the connected subgroup of H with Lie algebra $\mathfrak{s}$ (which is clearly solvable). Then $\mathsf{S} \cdot o \subseteq \mathsf{H} \cdot o$ and $\dim \mathsf{S} \cdot o = \dim \mathsf{H} \cdot o$ by design. As a consequence, $\mathsf{S} \cdot o$ is an open and extrinsically homogeneous submanifold of $\mathsf{H} \cdot o$, thus forcing $\mathsf{S} \cdot o = \mathsf{H} \cdot o$. Therefore, $\mathsf{S} \cdot o$ is simply connected, so $\mathsf{S}_o = \mathsf{S} \cap \mathsf{K}$ is connected and discrete, hence trivial. Applying Proposition 2.1, we deduce that S is a closed subgroup of $I^0(M)$. The slice representations of S and H are both trivial, so by applying Lemma 3.7 we see that the orbits of H and S coincide. Lastly, recall that S does not possess exceptional orbits, as M is a Hadamard manifold, and thus all isotropy subgroups are trivial, which shows that S acts freely on M. This finishes the proof. □

Corollary 3.10. *Let M be a Hadamard manifold and $\mathcal{F}$ a homogeneous foliation on M. Then the following assertions are true:*

(i) *There exists a connected closed solvable subgroup $\mathsf{S} \subseteq I^0(M)$ acting freely on M and such that the orbits of S coincide with the leaves of $\mathcal{F}$ and the right action of S on M given by $p \triangleleft g = g^{-1} \cdot p$ turns the canonical projection $M \to M/\mathsf{S}$ into an S-principal bundle.*

(ii) *The principal horizontal distribution $\mathcal{H}$ on M defines a principal connection on the bundle $M \to M/\mathsf{S}$.*

(iii) *The foliation $\mathcal{F}$ is polar if and only if M is a flat principal bundle.*

(iv) *If S acts polarly on M with section Σ, then Σ meets every leaf of $\mathcal{F}$ exactly once and $M/\mathcal{F}$ is naturally isometric to Σ.*

Proof. The statements (i) and (ii) are direct consequences of Theorem 3.9 and the fact that $\mathcal{H}$ is an S-invariant distribution on M. As for (iii), note that $\mathcal{H}$ is a flat connection on M if and only if it is an integrable distribution, and since $M = M_{\text{prin}}$, it is clear that this condition is equivalent to the polarity of the action $\mathsf{S} \curvearrowright M$, so the claim follows.

We now show (iv). To see this, let $\Pi(\Sigma)$ be the polar group of Σ. It suffices to show that $\Pi(\Sigma)$ is trivial, as in that case we have $(\mathsf{S} \cdot p) \cap \Sigma = \Pi(\Sigma) \cdot p = \{p\}$ for all $p \in \Sigma$ and the restricted projection map $\Sigma \to M/\mathsf{S}$ becomes a global isometry. Let $g \in \mathsf{N}_{\mathsf{S}}(\Sigma)$ be any element and take any $p \in \Sigma$, so that $g \cdot p$ is also in Σ. Since Σ is complete and totally geodesic, there exists a vector $v \in T_p\Sigma = \nu_p(\mathsf{S} \cdot p)$ such that $g \cdot p = \exp_p(v)$. Moreover, using [160, §2] and the fact that all isotropy subgroups are trivial, we see that the normal exponential map $\exp^{\perp} \colon \nu(\mathsf{S} \cdot p) \to M$ is an S-equivariant diffeomorphism. We can write $g \cdot p = \exp^{\perp}_p(g_{*p}(v)) = \exp^{\perp}_{g \cdot p}(0)$, and the injectivity of $\exp^{\perp}$ gives $p = g \cdot p$. As p is arbitrary, we deduce that g fixes Σ pointwise, so $g \in \mathsf{Z}_{\mathsf{S}}(\Sigma)$. We thus obtain $\Pi(\Sigma) = \{e\}$, from which (iv) follows. □

3.2.1 Maximal solvable subalgebras of real semisimple Lie algebras

Theorem 3.9 implies that every homogeneous foliation on a Hadamard manifold M is induced by the action of a solvable Lie group $\mathsf{S} \subseteq I^0(M)$. As a consequence, its Lie algebra $\mathfrak{s}$ is contained in some maximal solvable subalgebra of $\mathfrak{i}(M)$. Moreover, recall that if M is a symmetric space of noncompact type, then $\mathfrak{i}(M)$ is semisimple. Motivated by these facts, we now describe the structure of maximal solvable subalgebras of real semisimple Lie algebras.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. We work with the notation in Subsection 1.1.1. We say that a subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ is a *Borel subalgebra* if it is a maximal solvable subalgebra of $\mathfrak{g}$. On the other hand, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Cartan subalgebra* if its complexification $\mathfrak{h}(\mathbb{C})$ is a Cartan subalgebra of the complex semisimple Lie algebra $\mathfrak{g}(\mathbb{C})$[1]. In particular, $\mathfrak{h}$ is abelian [102, Proposition 2.10]. Note, however, that Cartan subalgebras of real semisimple Lie algebras are not necessarily conjugate.

Any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is conjugate to a θ-stable subalgebra [102, Proposition 6.59]. Thus, we can assume that $\theta\mathfrak{h} = \mathfrak{h}$, which means that $\mathfrak{h}$ splits as a direct sum $\tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{a}}$, where $\tilde{\mathfrak{t}} \subseteq \mathfrak{k}$ and $\tilde{\mathfrak{a}} \subseteq \mathfrak{p}$. Both $\tilde{\mathfrak{t}}$ and $\tilde{\mathfrak{a}}$ are abelian subspaces of $\mathfrak{g}$. In this case, $\dim \tilde{\mathfrak{t}}$ is called the *compact dimension* of $\mathfrak{h}$, and $\dim \tilde{\mathfrak{a}}$ is called the *noncompact dimension* of $\mathfrak{h}$. The subalgebra $\tilde{\mathfrak{t}}$ is called the *torus part* of $\mathfrak{h}$, and $\tilde{\mathfrak{a}}$ is called the *vector part* of $\mathfrak{h}$. We have that $\tilde{\mathfrak{a}}$ induces a root space decomposition on $\mathfrak{g}$. Note that, in principle, $\tilde{\mathfrak{a}}$ does not have to be a maximal abelian subspace of $\mathfrak{p}$ in this case. Root spaces are defined analogously: for each $\tilde{\lambda} \in \tilde{\mathfrak{a}}^*$, let

$$\tilde{\mathfrak{g}}_{\tilde{\lambda}} = \{X \in \mathfrak{g} \colon \operatorname{ad}(H)X = \tilde{\lambda}(H)X \text{ for all } H \in \tilde{\mathfrak{a}}\},$$

and define $\tilde{\Sigma}$ to be the set of all $\tilde{\lambda} \in \tilde{\mathfrak{a}}^*$ such that $\tilde{\lambda} \neq 0$ and $\tilde{\mathfrak{g}}_{\tilde{\lambda}} \neq 0$. Since the family $\operatorname{ad}(\tilde{\mathfrak{a}})$ consists again of commuting self-adjoint endomorphisms, it follows that $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \big(\bigoplus_{\tilde{\lambda} \in \tilde{\Sigma}} \tilde{\mathfrak{g}}_{\tilde{\lambda}}\big)$. Observe that $\tilde{\mathfrak{t}} \subseteq \tilde{\mathfrak{g}}_0 \cap \mathfrak{k}$ since $\mathfrak{h}$ is abelian.

[1]If $\mathfrak{g}$ is a complex semisimple Lie algebra, then a *Cartan subalgebra* of $\mathfrak{g}$ is a nilpotent Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ that is equal to its own normalizer.

We now relate the previous decomposition to the root space decomposition induced by a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ containing $\tilde{\mathfrak{a}}$. Let $\Sigma' \subseteq \Sigma$ be the subset of roots that annihilate $\tilde{\mathfrak{a}}$. We then have the following equalities:

$$\tilde{\mathfrak{a}} = \bigcap_{\lambda \in \Sigma'} \ker \lambda, \qquad \tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \Big(\bigoplus_{\lambda \in \Sigma'} \mathfrak{g}_\lambda\Big), \qquad \tilde{\mathfrak{g}}_{\tilde{\lambda}} = \bigoplus_{\substack{\lambda \in \Sigma \\ \lambda|_{\tilde{\mathfrak{a}}} = \tilde{\lambda}}} \mathfrak{g}_\lambda.$$

Since Σ' is an abstract root system in $(\mathfrak{a} \ominus \tilde{\mathfrak{a}})^*$, we may give a notion of positivity on Σ that is compatible with that of Σ' and $\tilde{\Sigma}$, that is, $\lambda \in \Sigma$ is positive if and only if $\lambda \in \Sigma'$ is positive or $\lambda \in \Sigma \setminus \Sigma'$ and $\lambda|_{\tilde{\mathfrak{a}}} \in \tilde{\Sigma}^+$. In order to to this, we proceed as follows. One can define a notion of positivity on $\tilde{\Sigma}$ by fixing a regular element $\tilde{H}_0 \in \tilde{\mathfrak{a}}$ (that is, $\tilde{\lambda}(\tilde{H}_0) \neq 0$ for all $\tilde{\lambda} \in \tilde{\Sigma}$) and declaring $\tilde{\lambda} \in \tilde{\Sigma}$ to be positive if $\tilde{\lambda}(\tilde{H}_0) > 0$. We also take a regular element $H' \in \mathfrak{a} \ominus \tilde{\mathfrak{a}}$ and define $\lambda \in \Sigma'$ to be positive whenever $\lambda(H') > 0$. We now define $H_0 = \tilde{H}_0 + \varepsilon H'$, where ε is a positive constant. Note that for every $\lambda \in \Sigma'$, $\lambda(H_0) = \varepsilon\lambda(H')$, so $\lambda(H_0)$ and $\lambda(H')$ have the same sign. Furthermore, if $\lambda \in \Sigma \setminus \Sigma'$, we have $\lambda(H_0) = \lambda|_{\tilde{\mathfrak{a}}}(\tilde{H}_0) + \varepsilon\lambda(H')$. Since the set of roots is finite, we can choose $\varepsilon > 0$ sufficiently small so that $\lambda(H_0)$ and $\lambda|_{\tilde{\mathfrak{a}}}(\tilde{H}_0)$ have the same sign for all $\lambda \in \Sigma \setminus \Sigma'$.

By [126, Theorem 4.1], any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is of the form $\mathfrak{b} = \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{a}} \oplus \tilde{\mathfrak{n}}$ for an adequate choice of a Cartan subalgebra $\mathfrak{h} = \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{a}}$, a set of positive elements $\tilde{\Sigma}^+ \subseteq \tilde{\Sigma}$, and where $\tilde{\mathfrak{n}} = \bigoplus_{\tilde{\lambda} \in \tilde{\Sigma}^+} \tilde{\mathfrak{g}}_{\tilde{\lambda}}$. We aim to restate this description of $\mathfrak{b}$ directly in terms of the root system induced by $\mathfrak{a}$.

We consider the subset $\Phi \subseteq \Sigma'$ of simple roots associated with the positivity criterion in Σ'. Note that $\Phi \subseteq \Lambda$. Indeed, by the construction of our set of positive roots in Σ, we have $\Phi \subseteq \Sigma^+$. Suppose $\alpha \in \Phi$ is not simple, so that $\alpha = \beta + \gamma$ for two positive roots $\beta, \gamma \in \Sigma^+$. Since α is simple in Σ', we have that β and γ cannot be simultaneously in Σ', and combining this with the equation $0 = \alpha(\tilde{H}_0) = \beta(\tilde{H}_0) + \gamma(\tilde{H}_0)$, we deduce that neither β nor γ are in Σ', and either β or γ is negative, a contradiction.

To summarize, we have found a subset $\Phi \subseteq \Lambda$ of simple roots for which $\Sigma' = \Sigma_\Phi$ is the root system generated by Φ and the following identities hold: $\tilde{\mathfrak{a}} = \mathfrak{a}_\Phi$, $\tilde{\mathfrak{g}}_0 = \mathfrak{l}_\Phi$ and $\tilde{\mathfrak{n}} = \mathfrak{n}_\Phi$.

We have thus arrived at the following result.

Theorem 3.11. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. Then $\mathfrak{b}$ contains a Cartan subalgebra $\mathfrak{h}$. Furthermore, there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$, a choice of simple roots $\Lambda \subseteq \Sigma$, and a set $\Phi \subseteq \Lambda$ such that $\mathfrak{b} = \tilde{\mathfrak{t}} \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, where $\tilde{\mathfrak{t}}$ is an abelian subspace of $\mathfrak{k}_\Phi = \mathfrak{k} \cap \mathfrak{l}_\Phi = \mathfrak{k}_0 \oplus \big(\bigoplus_{\lambda \in \Sigma_\Phi^+} \mathfrak{k}_\lambda\big)$.*

We say that a Cartan subalgebra $\mathfrak{h}$ (resp. Borel subalgebra $\mathfrak{b}$) is *maximally compact* if its compact dimension is maximal, and *maximally noncompact* if its noncompact dimension is maximal. Since $\tilde{\mathfrak{a}}$ is abelian in $\mathfrak{p}$, we have that $\mathfrak{h}$ (resp. $\mathfrak{b}$) is maximally noncompact if and only if $\tilde{\mathfrak{a}}$ is a maximal abelian subspace of $\mathfrak{p}$, see for example [102, Proposition 6.47].

If a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ corresponds to a maximally noncompact Cartan subalgebra, then $\Phi = \varnothing$ is the empty set, $\tilde{\mathfrak{a}} = \mathfrak{a}_\varnothing = \mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, and $\mathfrak{t} = \tilde{\mathfrak{t}}$ is a maximal abelian subspace of $\mathfrak{k}_0$ [102, Proposition 6.47 and Lemma 6.62]. This implies $\mathfrak{n}_\varnothing = \mathfrak{n}$, and thus, $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

3.3 Proof of Theorem B

Now we prove that the examples appearing in Theorem B are the only examples of codimension two homogeneous polar foliations on symmetric spaces of noncompact type.

Let $M = \mathsf{G}/\mathsf{K}$ be a symmetric space of noncompact type endowed with the metric induced by the Killing form. We use the notation introduced in Subsection 1.1.1. Thus, K is the isotropy group at $o \in M$, we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a choice of maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ that determines a root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \big(\bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda\big)$, and a positivity criterion that selects a set of positive roots Σ^+. We denote by Λ the set of simple roots. We define $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$, and recall that $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$.

Assume that H is a connected closed subgroup of the isometry group G that acts polarly on M, and that the orbits of H on M induce a foliation. Theorem 3.9 says that there exists a solvable subgroup S of G acting freely and whose orbits coincide with the orbits of H. However, we will not make use of the assumption that S acts freely on M. Let $\mathfrak{s}$ be the Lie algebra of S. Then, $\mathfrak{s}$ is contained in a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. See Subsection 3.2.1 for further details. The next result states that we may assume that $\mathfrak{s}$ is contained in a maximally noncompact Borel subalgebra.

Proposition 3.12. *The leaves of a homogeneous polar foliation on $M = \mathsf{G}/\mathsf{K}$ coincide, up to isometric congruence, with the orbits of a connected closed solvable subgroup S of G whose Lie algebra $\mathfrak{s}$ is contained in a maximally noncompact Borel subalgebra of the form $\mathfrak{t}\oplus\mathfrak{a}\oplus\mathfrak{n}$, where $\mathfrak{t} \subseteq \mathfrak{k}_0$ is abelian.*

Proof. By Theorem 3.9 there is a solvable subgroup S of G on M and whose orbits form the homogeneous polar foliation under investigation. According to Theorem 3.11, we may assume that $\mathfrak{s}$ is contained in a maximal solvable subalgebra of the form $\tilde{\mathfrak{t}} \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, with $\tilde{\mathfrak{t}} \subseteq \mathfrak{k}_\Phi$ an abelian subspace, and $\Phi \subseteq \Lambda$ a subset of simple roots. In particular, the tangent space of $\mathsf{S}\cdot o$ at o, as a subspace of $\mathfrak{p}$, is contained in $(1-\theta)(\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi)$. As a consequence,

$$(\mathfrak{a} \ominus \mathfrak{a}_\Phi) \oplus (1-\theta)(\mathfrak{n} \ominus \mathfrak{n}_\Phi) = \Big(\bigoplus_{\alpha\in\Phi} \mathbb{R}H_\alpha\Big) \oplus \Big(\bigoplus_{\lambda\in\Sigma^+_\Phi} \mathfrak{p}_\lambda\Big) \subseteq \mathfrak{s}_{\mathfrak{p}}^\perp,$$

where $\mathfrak{s}_{\mathfrak{p}}^\perp = \{\xi \in \mathfrak{p} : \langle \xi, \mathfrak{s}\rangle = 0\}$. Let $\lambda \in \Sigma^+_\Phi$ be arbitrary, and $X \in \mathfrak{g}_\lambda$. Then, since $H_\lambda, (1-\theta)X \in \mathfrak{s}_{\mathfrak{p}}^\perp$, and the action of S is polar, it follows from Proposition 2.20 that the vector $[H_\lambda, (1-\theta)X] = (1+\theta)|\lambda|^2 X$ is orthogonal to $\mathfrak{s}$. Thus, $\mathfrak{s}$ is orthogonal to $\bigoplus_{\lambda\in\Sigma^+_\Phi} \mathfrak{k}_\lambda$, and is therefore contained in $(\tilde{\mathfrak{t}} \cap \mathfrak{k}_0) \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi \subseteq (\tilde{\mathfrak{t}} \cap \mathfrak{k}_0) \oplus \mathfrak{a} \oplus \mathfrak{n}$, with $\mathfrak{t} = \tilde{\mathfrak{t}} \cap \mathfrak{k}_0$ abelian. □

In view of Proposition 3.12, if S is a closed solvable subgroup of G acting polarly on M and such that its orbits induce a foliation on M, we may assume from now on that the Lie algebra $\mathfrak{s}$ of S is contained in a Borel subalgebra of the form $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{t} \subseteq \mathfrak{k}_0$ is abelian. We also suppose from this point onward that the action of S on M is polar, but not hyperpolar. The following observation will prove useful for our calculations:

Lemma 3.13. *If $\mathfrak{s} \subseteq \mathfrak{t}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is a subalgebra, then we have*

$$\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}} = \big\{X \in \mathfrak{a}\oplus\mathfrak{n} \colon \big\langle X, \mathfrak{s}_{\mathfrak{p}}^\perp\big\rangle = 0\big\},$$
$$\mathfrak{s}_{\mathfrak{p}}^\perp = \big\{H + (1-\theta)\xi \colon H \in \mathfrak{a}, \xi \in \mathfrak{n}, H + \xi \perp \mathfrak{s}\big\}.$$

Proof. Firstly, let $X \in \mathfrak{a} \oplus \mathfrak{n}$ be arbitrary. If $X \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, then there exists a $T \in \mathfrak{t}$ such that $T + X \in \mathfrak{s}$. Projecting onto $\mathfrak{p}$ and taking into account that $\mathfrak{t} \subseteq \mathfrak{k}$, we obtain $X_{\mathfrak{p}} \in \mathfrak{s}_{\mathfrak{p}}$, so $X_{\mathfrak{p}} \perp \mathfrak{s}_{\mathfrak{p}}^{\perp}$. Since $\mathfrak{k}$ is already orthogonal to $\mathfrak{s}_{\mathfrak{p}}^{\perp}$, we deduce $\langle X, \mathfrak{s}_{\mathfrak{p}}^{\perp} \rangle = 0$. Conversely, if $\langle X, \mathfrak{s}_{\mathfrak{p}}^{\perp} \rangle = 0$, then $2X_{\mathfrak{p}} = (1-\theta)X \in \mathfrak{s}_{\mathfrak{p}}$, so we can find a $T \in \mathfrak{k}$ such that $T + (1-\theta)X \in \mathfrak{s}$. Let us decompose $X = H + Y$ for some $H \in \mathfrak{a}$ and $Y \in \mathfrak{n}$. We obtain

$$T + (1-\theta)X = T + 2H + (1-\theta)Y = T - (1+\theta)Y + 2H + 2Y,$$

so the projection of $T + (1-\theta)X$ onto $\mathfrak{a} \oplus \mathfrak{n}$ is $2H + 2Y = 2X$, yielding $X \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$.

Secondly, let $V \in \mathfrak{p}$ be arbitrary. Because the orthogonal projection $\mathfrak{g} \to \mathfrak{p}$ restricted to $\mathfrak{a} \oplus \mathfrak{n}$ is an isomorphism, we can write $V = H + (1-\theta)\xi$ for some $H \in \mathfrak{a}$ and $\xi \in \mathfrak{n}$. As $\theta\mathfrak{n}$ is perpendicular to $\mathfrak{s}$, it is clear that V is orthogonal to $\mathfrak{s}$ if and only if $H + \xi$ is orthogonal to $\mathfrak{s}$, so the second equality follows. □

Recall from Proposition 2.20 that $\mathfrak{s}_{\mathfrak{p}}^{\perp}$ is a Lie triple system (but not an abelian subspace) and $[\mathfrak{s}_{\mathfrak{p}}^{\perp}, \mathfrak{s}_{\mathfrak{p}}^{\perp}]$ is orthogonal to $\mathfrak{s}$. Moreover, $[\mathfrak{s}_{\mathfrak{p}}^{\perp}, \mathfrak{s}_{\mathfrak{p}}^{\perp}] \oplus \mathfrak{s}_{\mathfrak{p}}^{\perp}$ is a reductive Lie algebra and the orbit through the origin of the subgroup $\mathsf{S}_{\mathfrak{p}}^{\perp}$ whose Lie algebra is this one is also a symmetric space. Since it is two-dimensional and not flat, it must be homothetic to a real hyperbolic plane $\mathbb{R}\mathsf{H}^2$. Because $\mathbb{R}\mathsf{H}^2$ has constant curvature, it follows from (1.7) that there exists a constant $C > 0$ such that $\operatorname{ad}(\xi)^2(\eta) = C\eta$ for any pair of orthonormal vectors $\xi, \eta \in \mathfrak{s}_{\mathfrak{p}}^{\perp}$.

Our next step is to prove that $\mathfrak{s}_{\mathfrak{p}}^{\perp}$ is contained in $\mathfrak{a} \oplus \mathfrak{p}^1$. We recall that $\mathfrak{n}^1 = \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_{\alpha}$ and $\mathfrak{p}^1 = (1-\theta)\mathfrak{n}^1 = \bigoplus_{\alpha\in\Lambda} \mathfrak{p}_{\alpha}$. We consider the vector subspace

$$\tilde{\mathfrak{s}} = \mathfrak{s} + (\mathfrak{n} \ominus \mathfrak{n}^1) = \mathfrak{s} + \bigoplus_{\lambda\in\Sigma^+\setminus\Lambda} \mathfrak{g}_{\lambda}.$$

Since $\mathfrak{t} \oplus \mathfrak{a}$ normalizes all root spaces and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{n} \ominus \mathfrak{n}^1$, it follows that $\tilde{\mathfrak{s}}$ is a subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ containing $\mathfrak{s}$. In particular, $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}} \subseteq \tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$, so the codimension of $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$ is less than or equal to two.

Lemma 3.14. *Let $\mathfrak{q}$ be a Lie subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\lambda \in \Sigma^+$. If $\mathfrak{g}_{\lambda} \subseteq \mathfrak{q}_{\mathfrak{a}\oplus\mathfrak{n}}$ and there exists $H \in \mathfrak{a} \cap \mathfrak{q}_{\mathfrak{a}\oplus\mathfrak{n}}$ such that $\lambda(H) \neq 0$, then $\mathfrak{g}_{\lambda} \subseteq \mathfrak{q}$.*

Proof. Take $X \in \mathfrak{g}_{\lambda} \subseteq \mathfrak{q}_{\mathfrak{a}\oplus\mathfrak{n}}$. Then there exist vectors $T, T' \in \mathfrak{t}$ such that $T + H, T' + X \in \mathfrak{q}$. In particular, $\operatorname{ad}(T)X + \lambda(H)X = [T + H, T' + X] \in \mathfrak{q}$. This means that the linear map $\operatorname{ad}(T) + \lambda(H)\operatorname{id}_{\mathfrak{g}_{\lambda}}$ preserves $\mathfrak{g}_{\lambda}$ and carries $\mathfrak{g}_{\lambda}$ to $\mathfrak{q}$. Since $T \in \mathfrak{t}$, the linear transformation $\operatorname{ad}(T)$ is skew-adjoint, so $\operatorname{ad}(T) + \lambda(H)\operatorname{id}_{\mathfrak{g}_{\lambda}}$ is a linear isomorphism and it follows that $\mathfrak{g}_{\lambda} \subseteq \mathfrak{q}$. □

Now we can rule out the possibility that $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$ has codimension zero.

Lemma 3.15. $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}} \neq \mathfrak{a} \oplus \mathfrak{n}$.

Proof. Assume that $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{n}$. Both $\mathfrak{a}$ and all root spaces corresponding to positive roots are contained in $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$. By Lemma 3.14, it follows that $\mathfrak{n} \subseteq \tilde{\mathfrak{s}}$.

Let m denote the maximum possible level of a root. We define $k \in \{0, \dots, m\}$ to be the smallest integer for which $\mathfrak{n}^{k+1} \oplus \cdots \oplus \mathfrak{n}^m \subseteq \mathfrak{s}$. We want to show that $k = 0$. On the contrary, assume that $k \geq 1$. Let $\lambda \in \Sigma^+$ be a root of level k. As $\mathfrak{n}^1$ generates $\mathfrak{n}$, the root space $\mathfrak{g}_\lambda$ is generated by elements of the form

$$\operatorname{ad}(X_1) \cdots \operatorname{ad}(X_{k-1}) X_k, \quad X_i \in \mathfrak{n}^1.$$

Since $\mathfrak{n} \subseteq \tilde{\mathfrak{s}} = \mathfrak{s} + (\mathfrak{n} \ominus \mathfrak{n}^1)$, we can choose $Y_1, \dots, Y_k \in \mathfrak{n} \ominus \mathfrak{n}^1$ such that $X_i + Y_i \in \mathfrak{s}$ for each $i \in \{1, \dots, k\}$. Hence, $\operatorname{ad}(X_1 + Y_1) \cdots \operatorname{ad}(X_{k-1} + Y_{k-1})(X_k + Y_k) \in \mathfrak{s}$. By using the fact that $[\mathfrak{n}^r, \mathfrak{n}^s] \subseteq \mathfrak{n}^{r+s}$, we have

$$\begin{aligned} &\operatorname{ad}(X_1 + Y_1) \cdots \operatorname{ad}(X_{k-1} + Y_{k-1})(X_k + Y_k) \equiv \\ &\qquad\qquad \operatorname{ad}(X_1) \cdots \operatorname{ad}(X_{k-1}) X_k \pmod{\mathfrak{n}^{k+1} \oplus \cdots \oplus \mathfrak{n}^m}, \end{aligned}$$

so we obtain $\operatorname{ad}(X_1) \cdots \operatorname{ad}(X_{k-1}) X_k \in \mathfrak{s}$. This means that $\mathfrak{g}_\lambda \subseteq \mathfrak{s}$, and as a result, $\mathfrak{n}^k \subseteq \mathfrak{s}$, contradicting the definition of k.

Therefore, $k = 0$ and $\mathfrak{n} \subseteq \mathfrak{s}$. In particular, $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{a}$ must be an abelian subspace, contradicting the fact that our action is not hyperpolar. Thus, the case $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{n}$ is not possible. ☐

Before analyzing the remaining possibilities for the codimension of $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$ we need the following result.

Lemma 3.16. *Assume that $V \in \mathfrak{a} \oplus \mathfrak{n}$ is nonzero and orthogonal to $\mathfrak{s}$. Then:*

(i) *If V is in $\mathfrak{a}$, then $\mathfrak{s}_\mathfrak{p}^\perp = \mathbb{R}V \oplus (1-\theta)\mathbb{R}\eta_\alpha$, where $\eta_\alpha \in \mathfrak{g}_\alpha$ is nonzero and $\alpha \in \Lambda$. Furthermore, V is proportional to H_α.*

(ii) *If $V \in \mathfrak{g}_\alpha$ for some $\alpha \in \Lambda$, then $\mathfrak{s}_\mathfrak{p}^\perp = (1-\theta)\big(\mathbb{R}V \oplus \mathbb{R}(aH_\alpha + \eta_\alpha)\big)$, where $a \in \mathbb{R}$, $\eta_\alpha \in \mathfrak{g}_\alpha \ominus \mathbb{R}V$, and $[V, \eta_\alpha] = 0$.*

(iii) *If V is of the form $H_\alpha + \xi_\alpha$, where $\alpha \in \Lambda$ and $\xi_\alpha \in \mathfrak{g}_\alpha$ is a nonzero vector, and $g = \operatorname{Exp}(-\xi_\alpha/|\xi_\alpha|^2) \in \mathsf{N}$, then $\operatorname{Ad}(g)\mathfrak{s} \subseteq \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is orthogonal to ξ_α. In particular, we have the equality $(\operatorname{Ad}(g)\mathfrak{s})_\mathfrak{p}^\perp = (1-\theta)\big(\mathbb{R}\xi_\alpha \oplus \mathbb{R}(aH_\alpha + \eta_\alpha)\big)$, for $a \in \mathbb{R}$ and $\eta_\alpha \in \mathfrak{g}_\alpha \ominus \mathbb{R}\xi_\alpha$. Furthermore, $[\xi_\alpha, \eta_\alpha] = 0$.*

Proof. We prove (i). Assume $V \in \mathfrak{a}$. Since $\mathfrak{a} \subseteq \mathfrak{p}$, this means $V \in \mathfrak{s}_\mathfrak{p}^\perp$. Choose any unit vector $\eta = \eta_0 + \sum_{\lambda \in \Sigma^+} (1-\theta)\eta_\lambda \in \mathfrak{s}_\mathfrak{p}^\perp$ orthogonal to V, where $\eta_0 \in \mathfrak{a}$ and $\eta_\lambda \in \mathfrak{g}_\lambda$ for each $\lambda \in \Sigma^+$. Since the action of S is polar nonhyperpolar, $[V, \eta]$ is a nonzero vector orthogonal to $\mathfrak{s}$. Note that

$$[V, \eta] = (1+\theta) \sum_{\lambda \in \Sigma^+} \lambda(V)\eta_\lambda,$$

and recalling that $\theta\mathfrak{n}$ is orthogonal to $\mathfrak{s}$, we obtain

$$(1-\theta) \sum_{\lambda \in \Sigma^+} \lambda(V)\eta_\lambda \in \mathfrak{s}_\mathfrak{p}^\perp \ominus \mathbb{R}V = \mathbb{R}\eta,$$

so $\eta_0 = 0$ and $\lambda(V) = \mu(V)$ for every pair of roots λ, $\mu \in \Sigma^+$ such that $\eta_\lambda, \eta_\mu \neq 0$.

Suppose μ, $\nu \in \Sigma^+$ are two different roots such that $\eta_\mu, \eta_\nu \neq 0$. Hence, $\langle H_{\mu-\nu}, V\rangle = \mu(V) - \nu(V) = 0$ and $\langle H_{\mu-\nu}, \eta\rangle = 0$, so $H_{\mu-\nu} \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. Therefore, we may choose a $T \in \mathfrak{t}$ such that $T + H_{\mu-\nu} \in \mathfrak{s}$. An analogous argument shows that $\eta_\mu + a\eta_\nu \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ for $a = -|\eta_\mu|^2/|\eta_\nu|^2 < 0$, so there is a vector $T' \in \mathfrak{t}$ such that $T' + \eta_\mu + a\eta_\nu \in \mathfrak{s}$. Hence,

$$[T, \eta_\mu] + a[T, \eta_\nu] + \langle \mu - \nu, \mu\rangle\eta_\mu + a\langle \mu - \nu, \nu\rangle\eta_\nu = [T + H_{\mu-\nu}, T' + \eta_\mu + a\eta_\nu] \in \mathfrak{s} \cap \mathfrak{n}.$$

Since $\operatorname{ad}(T)$ is skew-adjoint (because $T \in \mathfrak{k}$) we deduce $[T, \eta_\mu]$, $[T, \eta_\nu] \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. Because of this, the element $\langle \mu - \nu, \mu\rangle\eta_\mu + a\langle \mu - \nu, \nu\rangle\eta_\nu$ is also in $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. Observe that

$$\begin{vmatrix} 1 & a \\ \langle \mu - \nu, \mu\rangle & a\langle \mu - \nu, \nu\rangle \end{vmatrix} = -a|\mu - \nu|^2 > 0,$$

which implies that $\eta_\mu + a\eta_\nu$ and $\langle \mu - \nu, \mu\rangle\eta_\mu + a\langle \mu - \nu, \nu\rangle\eta_\nu$ are linearly independent vectors in $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. Therefore, η_μ, $\eta_\nu \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ must be orthogonal to η, contradicting the fact that they are nonzero. We thus obtain that only one of the η_λ can be nonzero, that is, $\eta = (1-\theta)\eta_\mu \in \mathfrak{p}_\mu$ for some $\mu \in \Sigma^+$.

We now prove that μ is simple. If $\mu = \beta + \gamma$ were a sum of positive roots β, $\gamma \in \Sigma^+$, then $\eta_\mu \in \mathfrak{g}_\mu = [\mathfrak{g}_\beta, \mathfrak{g}_\gamma]$, so we may write $\eta_\mu = \sum_{i=1}^k [X_i, Y_i]$, where $X_i \in \mathfrak{g}_\beta$ and $Y_i \in \mathfrak{g}_\gamma$ for each i. Clearly, $\mathfrak{g}_\beta + \mathfrak{g}_\gamma \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, which means that for each i there are vectors T_i, $T_i' \in \mathfrak{t}$ such that $T_i + X_i$, $T_i' + Y_i \in \mathfrak{s}$. As a consequence,

$$\sum_{i=1}^k [T_i, Y_i] + [X_i, T_i'] + [X_i, Y_i] = \sum_{i=1}^k [T_i + X_i, T_i' + Y_i] \in \mathfrak{s}.$$

Note that each $[T_i, Y_i]$ is in $\mathfrak{g}_\gamma \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ and each $[X_i, T_i']$ is in $\mathfrak{g}_\beta \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, which implies that $\eta_\mu = \sum_i [X_i, Y_i] \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, contradicting that η_μ is nonzero. We deduce that $\mu \in \Lambda$, so we may write $\mathfrak{s}_{\mathfrak{p}}^\perp = \mathbb{R}V \oplus (1-\theta)\eta_\alpha$, where $\alpha = \mu \in \Lambda$.

As V and $(1-\theta)\eta_\alpha$ are orthogonal, we know that the vector $\operatorname{ad}((1-\theta)\eta_\alpha)^2 V$ is nonzero and proportional to V. Taking the inner product with an arbitrary vector $H \in \mathfrak{a}$, we see that

$$\begin{aligned}\langle \operatorname{ad}((1-\theta)\eta_\alpha)^2 V, H\rangle &= \langle [(1-\theta)\eta_\alpha, V], [(1-\theta)\eta_\alpha, H]\rangle \\ &= \langle (1+\theta)\alpha(V)\eta_\alpha, (1+\theta)\alpha(H)\eta_\alpha\rangle = 2|\eta_\alpha|^2\alpha(V)\alpha(H) \\ &= \langle 2|\eta_\alpha|^2\alpha(V)H_\alpha, H\rangle,\end{aligned}$$

which means that V is proportional to $\operatorname{ad}((1-\theta)\eta_\alpha)^2 V = 2|\eta_\alpha|^2\alpha(V)H_\alpha$ (this vector is nonzero due to the action not being hyperpolar), and thus to H_α. This proves the first assertion.

Now we prove (ii). Assume $V \in \mathfrak{g}_\alpha$ for a simple root $\alpha \in \Lambda$. Choose a nonzero vector $\eta = \eta_0 + \sum_{\lambda\in\Sigma^+}(1-\theta)\eta_\lambda \in \mathfrak{s}_{\mathfrak{p}}^\perp$, where $\eta_0 \in \mathfrak{a}$ and $\eta_\lambda \in \mathfrak{g}_\lambda$ for each $\lambda \in \Sigma^+$, such that $\langle V, \eta\rangle = \langle V, \eta_\alpha\rangle = 0$. We may assume that η is not in $\mathfrak{a}$. Indeed, if $\eta \in \mathfrak{a}$, then the we are in the conditions of item (i), and we directly obtain that η is proportional to H_α, proving our claim. Let $\mu \in \Sigma^+$ be a positive root with $\eta_\mu \neq 0$.

For now, let us suppose that we can choose μ so that $\mu \neq \alpha$.

We first prove that $\eta_0 \in \mathbb{R}H_\mu$. Assume otherwise, so there exists a vector $H \in \mathfrak{a}$ such that $\langle H, \eta_0\rangle = 0$ and $\langle H, H_\mu\rangle = \mu(H) \neq 0$. Then $H \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, so there exists $T \in \mathfrak{t}$ such that $T + H$ is in $\mathfrak{s}$. On the other hand, $\eta_0 + a\eta_\mu$ is orthogonal to both V and η for $a = -|\eta_0|^2/|\eta_\mu|^2 < 0$, which implies that $\eta_0 + a\eta_\mu \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, and there exists $T' \in \mathfrak{t}$ such that $T' + \eta_0 + a\eta_\mu \in \mathfrak{s}$. In particular, $a[T, \eta_\mu] + a\mu(H)\eta_\mu = [T + H, T' + \eta_0 + a\eta_\mu] \in \mathfrak{s}$, so $0 = \langle a[T, \eta_\mu] + a\mu(H)\eta_\mu, \eta\rangle = a\mu(H)|\eta_\mu|^2$, a contradiction.

We now prove that $\eta_\lambda = 0$ for every $\lambda \in \Sigma^+ \setminus \{\mu\}$. If λ is a positive root linearly independent with μ, and $\eta_\lambda \neq 0$, we may find $H \in \mathfrak{a}$ such that $\mu(H) = 0 \neq \lambda(H)$. Since $\eta_0 \in \mathbb{R}H_\mu$, this implies that $H \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, and hence, there exists $T \in \mathfrak{t}$ such that $T + H \in \mathfrak{s}$. Furthermore, $\eta_\mu + b\eta_\lambda \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ with $b = -|\eta_\mu|^2/|\eta_\lambda|^2 < 0$, and there exists $T' \in \mathfrak{t}$ satisfying $T' + \eta_\mu + b\eta_\lambda \in \mathfrak{s}$. As a consequence, $[T, \eta_\mu] + b[T, \eta_\lambda] + b\lambda(H)\eta_\lambda = [T + H, T' + \eta_\mu + b\eta_\lambda] \in \mathfrak{s}$. In particular,

$$0 = \langle [T, \eta_\mu] + b[T, \eta_\lambda] + b\lambda(H)\eta_\lambda, \eta\rangle = b\lambda(H)|\eta_\lambda|^2,$$

a contradiction. On the other hand, if $2\mu \in \Sigma^+$ and $\eta_{2\mu} \neq 0$, a similar argument yields that there is $T \in \mathfrak{t}$ such that $T + H_\mu + a\eta_\mu \in \mathfrak{s}$, for $a = -\mu(\eta_0)/|\eta_\mu|^2$, and $T' \in \mathfrak{t}$ such that $T' + \eta_\mu + b\eta_{2\mu} \in \mathfrak{s}$, where $b = -|\eta_\mu|^2/|\eta_{2\mu}|^2 < 0$. Thus,

$$\begin{aligned} 0 &= \langle [T + H_\mu + a\eta_\mu, T' + \eta_\mu + b\eta_{2\mu}], \eta\rangle \\ &= \langle [T, \eta_\mu] + b[T, \eta_{2\mu}] + |\mu|^2\eta_\mu + 2b|\mu|^2\eta_{2\mu} + a[\eta_\mu, T'], \eta\rangle = -|\mu|^2|\eta_\mu|^2, \end{aligned}$$

contradicting our choice of μ. This implies $\eta_{2\mu} = 0$ (and an analogous argument shows that $\eta_{\mu/2} = 0$ in the case that $\mu/2 \in \Sigma^+$).

To summarize, we have obtained $\eta = aH_\mu + (1 - \theta)\eta_\mu$ for some constant $a \in \mathbb{R}$, and μ is a root different from α with $\eta_\mu \neq 0$.

Assume $a \neq 0$. Since $\mathfrak{s}_{\mathfrak{p}}^\perp$ is a Lie triple system, $\operatorname{ad}((1 - \theta)V)^2\eta$ is proportional to η. Since for any $H \in \mathfrak{a}$ we have

$$\begin{aligned} &\langle \operatorname{ad}((1 - \theta)V)^2\eta, H\rangle \\ &\quad = -\langle [(1 - \theta)V, \eta], [H, (1 - \theta)V]\rangle \\ &\quad = -\langle (1 + \theta)\big(-a\langle\alpha, \mu\rangle V + [V, \eta_\mu] - [\theta V, \eta_\mu]\big), \alpha(H)(1 + \theta)V\rangle \\ &\quad = 2a\langle\alpha, \mu\rangle|V|^2\alpha(H) = \langle 2a\langle\alpha, \mu\rangle|V|^2H_\alpha, H\rangle, \end{aligned}$$

it follows that $\mu = \alpha$ or $\mu = 2\alpha$. This last case is not possible, because by Proposition 2.20, $(1 + \theta)(-2a|\alpha|^2V - [\theta V, \eta_{2\alpha}]) = [(1 - \theta)V, aH_{2\alpha} + (1 - \theta)\eta_{2\alpha}]$ would be orthogonal to $\mathfrak{s}$. This would imply that $[\theta V, \eta_{2\alpha}]$ is proportional to V, and thus,

$$\begin{aligned} 0 &= [V, [\theta V, \eta_{2\alpha}]] = -[\theta V, [\eta_{2\alpha}, V]] - [\eta_{2\alpha}, [V, \theta V]] \\ &= -[|V|^2H_\alpha, \eta_{2\alpha}] = -2|\alpha|^2|V|^2\eta_{2\alpha}, \end{aligned}$$

contradicting the fact that $\eta_{2\alpha} \neq 0$. We conclude that $\mu = \alpha$.

Suppose now that $a = 0$, so $\eta \in \mathfrak{p}_\mu$. Since $\mathfrak{a}, \mathfrak{g}_{\alpha+\mu} \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, we obtain $\mathfrak{g}_{\alpha+\mu} \subseteq \mathfrak{s}$ by Lemma 3.14. The vector $[(1 - \theta)V, (1 - \theta)\eta_\mu] = (1 + \theta)([V, \eta_\mu] - [\theta V, \eta_\mu])$ is nonzero and orthogonal to $\mathfrak{s}$. Combining this with the fact that $\mathfrak{g}_{\alpha+\mu} \subseteq \mathfrak{s}$, we get $[V, \eta_\mu] = 0$, so $(1+\theta)[\theta V, \eta_\mu]$ is nonzero and orthogonal to $\mathfrak{s}$. This now implies $\mu - \alpha \in \Sigma^+$ and $[\theta V, \eta_\mu] \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{s}$. Furthermore, we must have $\mu = 2\alpha$, and thus, $[\theta V, \eta_\mu] \in ((\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{s}) \cap \mathfrak{g}_\alpha = \mathbb{R}V$. Therefore,

$$0 = [V, [\theta V, \eta_{2\alpha}]] = -[\eta_{2\alpha}, [V, \theta V]] = -2|\alpha|^2|V|^2\eta_{2\alpha},$$

which yields a contradiction.

We now assume $\eta_\mu = 0$ for every $\mu \in \Sigma^+ \setminus \{\alpha\}$. As a consequence, $\eta = \eta_0 + (1-\theta)\eta_\alpha$, with $\langle V, \eta_\alpha \rangle = 0$. We only need to prove that $\eta_0 \in \mathbb{R}H_\alpha$. Indeed, if η_0 is not proportional to H_α, there exists $H \in \mathfrak{a}$ such that $\langle H, \eta_0 \rangle = 0$ and $\alpha(H) \neq 0$. Therefore, $H \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, so we may find $T \in \mathfrak{t}$ satisfying $T + H \in \mathfrak{s}$. Similarly, by taking $\eta_0 + x\eta_\alpha$ with $x = -|\eta_0|^2/|\eta_\alpha|^2 < 0$, we obtain $\eta_0 + x\eta_\alpha \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, so $T' + \eta_0 + x\eta_\alpha \in \mathfrak{s}$ for an adequate $T' \in \mathfrak{t}$. Thus,

$$0 = \langle [T + H, T' + \eta_0 + x\eta_\alpha], \eta \rangle = x\alpha(H)|\eta_\alpha|^2,$$

contradiction. Hence, we may write $\eta = aH_\alpha + (1-\theta)\eta_\alpha$ with $a \in \mathbb{R}$, $\eta_\alpha \in \mathfrak{g}_\alpha \ominus \mathbb{R}V$, and $\mathfrak{s}_\mathfrak{p}^\perp = \mathbb{R}(1-\theta)V \oplus \mathbb{R}\eta$.

If $a = 0$, then note that $\mathfrak{a}$, $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, and Lemma 3.14 implies $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{s}$. Together with the fact that $[(1-\theta)V, (1-\theta)\eta_\alpha] = (1+\theta)([V, \eta_\alpha] - [\theta V, \eta_\alpha])$ is orthogonal to $\mathfrak{s}$ by Proposition 2.20, we get $[V, \eta_\alpha] = 0$.

If $a \neq 0$, we can take the triple bracket $[\eta, [\eta, (1-\theta)V]]$, which is in $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{a} \oplus \mathfrak{p}^1$. Then, for any $X \in \mathfrak{g}_{2\alpha}$:

$$\begin{aligned} 0 &= \langle [\eta, [\eta, (1-\theta)V]], X \rangle = \langle [\eta, (1-\theta)V], [\eta, X]] \rangle \\ &= \langle (1+\theta)(a|\alpha|^2V + [\eta_\alpha, V] - [\theta\eta_\alpha, V]), 2a|\alpha|^2X - [\theta\eta_\alpha, X] \rangle \\ &= -a|\alpha|^2\langle V, [\theta\eta_\alpha, X] \rangle + 2a|\alpha|^2\langle [\eta_\alpha, V], X \rangle = -3a|\alpha|^2\langle [V, \eta_\alpha], X \rangle, \end{aligned}$$

and this yields $[V, \eta_\alpha] = 0$, as stated. This finishes the proof of (ii).

To prove (iii), assume $V = H_\alpha + \xi_\alpha$ for a nonzero $\xi_\alpha \in \mathfrak{g}_\alpha$, where $\alpha \in \Lambda$, and consider $g = \operatorname{Exp}(-\xi_\alpha/|\xi_\alpha|^2) \in \mathsf{N}$. Then, the isomorphism $\operatorname{Ad}(g)$ preserves the subalgebra $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and therefore, $\operatorname{Ad}(g)\mathfrak{s} \subseteq \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Lie algebra of $g\mathsf{S}g^{-1}$, which induces a homogeneous polar foliation on M with a non flat section. Observe that

$$\operatorname{Ad}(g^{-1})^*(H_\alpha + \xi_\alpha) = e^{-\frac{1}{|\xi_\alpha|^2}\operatorname{ad}(\theta\xi_\alpha)}(H_\alpha + \xi_\alpha) = \xi_\alpha - \frac{|\alpha|^2}{2|\xi_\alpha|^2}\theta\xi_\alpha,$$

which means that $\operatorname{Ad}(g)\mathfrak{s} \subseteq \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is orthogonal to ξ_α, as desired. The rest of the assertion follows from (ii). □

Now we continue with the proof of Theorem B. Recall that $\tilde{\mathfrak{s}} = \mathfrak{s} + (\mathfrak{n} \ominus \mathfrak{n}^1)$. According to Lemma 3.15, $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}} \neq \mathfrak{a} \oplus \mathfrak{n}$, so either $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$ has codimension one or two in $\mathfrak{a} \oplus \mathfrak{n}$.

If $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$ has codimension two, we have $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, which is equivalent to $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{a} \oplus \mathfrak{p}^1$.

On the other hand, if $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}}$ is has codimension one in $\mathfrak{a} \oplus \mathfrak{n}$, then by [22, Proposition 5.4] we have $\tilde{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}} = (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi$, where ξ satisfies one of the following possibilities:[2]

[2]Note that Berndt and Tamaru's proof does not rely on the additional condition that they impose on $\mathfrak{s}$, namely, that $\mathfrak{s} \cap \mathfrak{t} = 0$.

(i) $\xi \in \mathfrak{a}$.

(ii) $\xi \in \mathfrak{g}_\alpha$ for a simple root $\alpha \in \Lambda$.

(iii) $\xi = H_\alpha + \xi_\alpha$, where $\xi_\alpha \in \mathfrak{g}_\alpha$ is a nonzero vector and $\alpha \in \Lambda$.

Note that ξ is also orthogonal to $\mathfrak{s}$. Hence, by Lemma 3.16, we obtain that $\mathfrak{s}_\mathfrak{p}^\perp$ may be assumed to be in $\mathfrak{a} \oplus \mathfrak{p}^1$ after conjugation by an element of N. The next step is to determine the orthogonal projection $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$.

Lemma 3.17. *The action of* S *is orbit equivalent to the action of a connected closed subgroup* $\bar{\mathsf{S}}$ *whose Lie algebra* $\bar{\mathfrak{s}}$ *is contained in* $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$*, the normal space* $\bar{\mathfrak{s}}_\mathfrak{p}^\perp$ *is contained in* $\mathfrak{a} \oplus \mathfrak{p}^1$*, and* $\bar{\mathfrak{s}}_\mathfrak{p}^\perp \cap \mathfrak{p}^1 \neq 0$*. Equivalently, the orthogonal projection of* $\bar{\mathfrak{s}}_\mathfrak{p}^\perp$ *on* $\mathfrak{a}$ *is not two-dimensional.*

Proof. Assume $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}^1$ is trivial. Let $\Psi = \{\alpha \in \Lambda \colon \pi_{\mathfrak{g}_\alpha}(\mathfrak{s}_\mathfrak{p}^\perp) \neq 0\}$, where $\pi_{\mathfrak{g}_\alpha} \colon \mathfrak{g} \to \mathfrak{g}_\alpha$ denotes the orthogonal projection. Since the action is not hyperpolar, Ψ is a nonempty subset of Λ. We prove $\mathfrak{a}^\Psi = \bigoplus_{\alpha\in\Psi} \mathbb{R} H_\alpha \subseteq \pi_\mathfrak{a}(\mathfrak{s}_\mathfrak{p}^\perp)$. Here $\pi_\mathfrak{a}$ denotes the orthogonal projection onto $\mathfrak{a}$. Indeed, assume $H \in \mathfrak{a} \ominus \pi_\mathfrak{a}(\mathfrak{s}_\mathfrak{p}^\perp)$. Then there exists a vector $T \in \mathfrak{t}$ such that $T + H \in \mathfrak{s}$. On the other hand, let $\alpha \in \Psi$. We may find two vectors $\xi = \xi_0 + \sum_{\beta\in\Lambda}(1-\theta)\xi_\beta$ and $\eta = \eta_0 + \sum_{\beta\in\Lambda}(1-\theta)\eta_\beta$ in $\mathfrak{s}_\mathfrak{p}^\perp$ such that $\xi_\alpha \neq 0$ and $\langle \xi_\alpha, \eta_\alpha \rangle = 0$. By our assumption, ξ_0 and η_0 are linearly independent, which implies that there exist unique constants x, $y \in \mathbb{R}$ such that $\xi_\alpha + x\xi_0 + y\eta_0 \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. In particular, we may find $T' \in \mathfrak{t}$ satisfying $T' + \xi_\alpha + x\xi_0 + y\eta_0 \in \mathfrak{s}$. Thus, $[T, \xi_\alpha] + \alpha(H)\xi_\alpha = [T + H, T' + \xi_\alpha + x\xi_0 + y\eta_0] \in \mathfrak{s}$. Taking the inner product with ξ, we deduce $\alpha(H) = 0$. All in all, we obtain $\mathfrak{a} \ominus \pi_\mathfrak{a}(\mathfrak{s}_\mathfrak{p}^\perp) \subseteq \mathfrak{a}_\Psi = \cap_{\alpha\in\Psi} \ker\alpha$, so $\mathfrak{a}^\Psi \subseteq \pi_\mathfrak{a}(\mathfrak{s}^\perp)$ (in particular, Ψ has either one or two elements).

In order to prove the result, we assume first that for all $\alpha \in \Psi$, the orthogonal projection $\pi_{\mathfrak{g}_\alpha}(\mathfrak{s}_\mathfrak{p}^\perp)$ is one-dimensional. Thus, we can take two orthogonal vectors $\xi = \xi_0 + \sum_{\alpha\in\Psi}(1-\theta)\xi_\alpha$ and $\eta = \eta_0 + \sum_{\alpha\in\Psi}(1-\theta)\eta_\alpha$ that span $\mathfrak{s}_\mathfrak{p}^\perp$. Since the action is polar nonhyperpolar, the vector $[\xi, \eta]$ is nonzero and orthogonal to $\mathfrak{s}$. Observe that, because $\Psi \subseteq \Lambda$, $[\theta\xi_\alpha, \eta_\alpha] \in \mathfrak{a}$ for all $\alpha \in \Psi$, and $\alpha - \beta \notin \Sigma$, we have

$$[\xi, \eta] = (1+\theta)\left(\sum_{\alpha\in\Psi}\Big(\alpha(\xi_0)\eta_\alpha - \alpha(\eta_0)\xi_\alpha\Big) + \sum_{\alpha,\beta\in\Psi}[\xi_\alpha, \eta_\beta]\right).$$

Since $\theta\mathfrak{n}$ and $\mathfrak{s}$ are orthogonal, we obtain

$$(1-\theta)\left(\sum_{\alpha\in\Psi}\Big(\alpha(\xi_0)\eta_\alpha - \alpha(\eta_0)\xi_\alpha\Big) + \sum_{\alpha,\beta\in\Psi}[\xi_\alpha, \eta_\beta]\right) \in \mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{a} \oplus \mathfrak{p}^1,$$

which means that all terms in $\mathfrak{p}^2$ cancel out and $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}^1 \neq 0$, a contradiction.

Now, assume that there exists $\alpha \in \Psi$ such that $\pi_{\mathfrak{g}_\alpha}(\mathfrak{s}_\mathfrak{p}^\perp)$ is two-dimensional. Since H_α is in $\pi_\mathfrak{a}(\mathfrak{s}_\mathfrak{p}^\perp)$, we may find $H_\alpha + \sum_{\beta\in\Psi}(1-\theta)\xi_\beta \in \mathfrak{s}_\mathfrak{p}^\perp$, with each $\xi_\beta \in \mathfrak{g}_\beta$, and $\xi_\alpha \neq 0$, for dimension reasons. Consider the element $g = \operatorname{Exp}(-\xi_\alpha/|\xi_\alpha|^2) \in \mathsf{N}$. Then the action of S is orbit equivalent to the action of $g\mathsf{S}g^{-1}$, whose Lie algebra is $\operatorname{Ad}(g)\mathfrak{s} \subseteq \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Note that the equality

$$\operatorname{Ad}(g^{-1})^*\Big(H_\alpha + \sum_{\beta\in\Psi}\xi_\beta\Big) = \sum_{\beta\in\Psi}\xi_\beta - \frac{|\alpha|^2}{2|\xi_\alpha|^2}\theta\xi_\alpha$$

and $\langle\theta\mathfrak{n},\mathfrak{s}\rangle = 0$ imply $\sum_{\beta\in\Psi}(1-\theta)\xi_\beta \in (\operatorname{Ad}(g)\mathfrak{s})^\perp_\mathfrak{p} \cap \mathfrak{p}^1$.

To conclude, it suffices to prove that $(\operatorname{Ad}(g)\mathfrak{s})^\perp_\mathfrak{p} \subseteq \mathfrak{a}\oplus\mathfrak{p}^1$. This is the case if the projection of $\operatorname{Ad}(g)\mathfrak{s} + (\mathfrak{n}\ominus\mathfrak{n}^1)$ onto $\mathfrak{a}\oplus\mathfrak{n}$ has codimension 2. As the projection of $\operatorname{Ad}(g)\mathfrak{s} + (\mathfrak{n}\ominus\mathfrak{n}^1)$ onto $\mathfrak{a}\oplus\mathfrak{n}$ cannot be $\mathfrak{a}\oplus\mathfrak{n}$ by Lemma 3.15, our assertion is false whenever this projection is of codimension one, that is, when the orthogonal complement of $\operatorname{Ad}(g)\mathfrak{s}$ in $\mathfrak{a}\oplus\mathfrak{n}$ is spanned by $\sum_{\beta\in\Psi}\xi_\beta$. By [22, Proposition 5.4], $\xi_\beta = 0$ for all simple roots $\beta\neq\alpha$, and by Lemma 3.16(ii) we have $(\operatorname{Ad}(g)\mathfrak{s})^\perp_\mathfrak{p} = (1-\theta)\big(\mathbb{R}\xi_\alpha \oplus (aH_\alpha+\eta_\alpha)\big)$ for $a\in\mathbb{R}$ and $\eta_\alpha\in\mathfrak{g}_\alpha$. Thus, we have $(\operatorname{Ad}(g)\mathfrak{s})^\perp_\mathfrak{p} \subseteq \mathfrak{a}\oplus\mathfrak{p}^1$, contradicting the fact that the projection of $\operatorname{Ad}(g)\mathfrak{s} + (\mathfrak{n}\ominus\mathfrak{n}^1)$ onto $\mathfrak{a}\oplus\mathfrak{n}$ has codimension one. □

Due to the previous lemma, we may assume that $\mathfrak{s}^\perp_\mathfrak{p}\cap\mathfrak{p}^1$ is a nonzero subspace of $\mathfrak{g}$.
First we need:

Lemma 3.18. *Let* S *be a closed subgroup of* G *whose Lie algebra* $\mathfrak{s}$ *satisfies* $\mathfrak{s}\subseteq\mathfrak{t}\oplus\mathfrak{a}\oplus\mathfrak{n}$*, and the orbits of* S *form a homogeneous foliation on* M*. Let* $V\in\mathfrak{n}$ *be a vector such that* $(1-\theta)V\in\mathfrak{s}^\perp_\mathfrak{p}$ *and* $g=\operatorname{Exp}(V)\in\mathsf{N}$*. Then* $\mathfrak{s}\cap\mathfrak{t} = \operatorname{Ad}(g)(\mathfrak{s}\cap\mathfrak{t}) = \operatorname{Ad}(g)(\mathfrak{s})\cap\mathfrak{k}$*.*

Proof. Since the orbit $\mathsf{S}\cdot o$ is principal, we have $[\mathfrak{s}\cap\mathfrak{t},\mathfrak{s}^\perp_\mathfrak{p}]=0$. Hence, we have $[\mathfrak{s}\cap\mathfrak{t},(1-\theta)V] = (1-\theta)[\mathfrak{s}\cap\mathfrak{t},V]=0$, which means $\operatorname{ad}(V)(\mathfrak{s}\cap\mathfrak{t})=0$ and $\operatorname{Ad}(g)(\mathfrak{s}\cap\mathfrak{t}) = e^{\operatorname{ad}(V)}(\mathfrak{s}\cap\mathfrak{t}) = \mathfrak{s}\cap\mathfrak{t}$. On the other hand, $\mathfrak{s}\cap\operatorname{Ad}(g^{-1})\mathfrak{k}$ is the isotropy algebra of S at $g^{-1}\cdot o$, and since all orbits have the same type, it follows that

$$\dim\operatorname{Ad}(g)\mathfrak{s}\cap\mathfrak{k} = \dim\mathfrak{s}\cap\operatorname{Ad}(g^{-1})\mathfrak{k} = \dim\mathfrak{s}\cap\mathfrak{k} = \dim\mathfrak{s}\cap\mathfrak{t}.$$

Since $\mathfrak{s}\cap\mathfrak{t} = \operatorname{Ad}(g)(\mathfrak{s}\cap\mathfrak{t}) \subseteq \operatorname{Ad}(g)(\mathfrak{s})\cap\mathfrak{k}$, the equality follows. □

The next result is needed later to handle the two examples in Theorem B simultaneously.

Proposition 3.19. *Let* S *be a connected closed subgroup of* G *inducing a homogeneous polar foliation on* M*. Assume that its Lie algebra is contained in a maximally noncompact Borel subalgebra* $\mathfrak{t}\oplus\mathfrak{a}\oplus\mathfrak{n}$*, and* $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{z}\oplus(\mathfrak{n}\ominus\mathfrak{v}_\alpha)$*, where* $\mathfrak{v}_\alpha$ *is an abelian subspace of* $\mathfrak{g}_\alpha$*,* $\alpha\in\Lambda$*, and* $\mathfrak{z}$ *is a subspace of* $\mathfrak{a}$*. Let* $\tilde{\mathsf{S}}$ *be the connected Lie subgroup of* G *whose Lie algebra is* $\tilde{\mathfrak{s}} = \mathfrak{z}\oplus(\mathfrak{n}\ominus\mathfrak{v}_\alpha)$*. Then,* S *and* $\tilde{\mathsf{S}}$ *have the same orbits.*

Proof. Denote by $\mathfrak{s}_\mathfrak{t}$ the orthogonal projection of $\mathfrak{s}$ onto $\mathfrak{t}$. We start by proving that $\hat{\mathfrak{s}} = \mathfrak{s}_\mathfrak{t}\oplus\tilde{\mathfrak{s}}$ is a Lie subalgebra of $\mathfrak{g}$. Since $[\mathfrak{g}_\lambda,\mathfrak{g}_\mu]\subseteq\mathfrak{g}_{\lambda+\mu}$, for λ, $\mu\in\Sigma^+$, and $\mathfrak{t}$ is abelian, centralizes $\mathfrak{a}$, and normalizes each root space, this amounts to proving the inclusion $[\mathfrak{s}_\mathfrak{t},\mathfrak{g}_\alpha\ominus\mathfrak{v}_\alpha]\subseteq\mathfrak{g}_\alpha\ominus\mathfrak{v}_\alpha$.

Let $U, V \in \mathfrak{v}_\alpha$ and $T \in \mathfrak{s}_\mathfrak{t}$. We choose $X \in \mathfrak{a} \oplus \mathfrak{n}$, such that $T + X \in \mathfrak{s}$. Since the action of S is polar, we know that $[\mathfrak{s}_\mathfrak{p}^\perp, \mathfrak{s}_\mathfrak{p}^\perp]$ is perpendicular to $\mathfrak{s}$. Thus,

$$\begin{aligned} 0 &= \langle [(1-\theta)U, (1-\theta)V],\, T + X\rangle = \langle -(1+\theta)[\theta U, V],\, T\rangle \\ &= -2\langle [U, \theta V],\, T\rangle = -2\langle U, [V, T]\rangle. \end{aligned}$$

This proves $[\mathfrak{s}_\mathfrak{t}, \mathfrak{v}_\alpha] \subseteq \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$.

Let $T \in \mathfrak{s} \cap \mathfrak{t}$ and $V \in \mathfrak{v}_\alpha$. Since $\mathfrak{s} \cap \mathfrak{t} \subseteq \mathfrak{s}_\mathfrak{t}$, we have $[T, V] \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$. Let $X \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$. Then there exists $T_X \in \mathfrak{t}$ such that $T_X + X \in \mathfrak{s}$. Thus, $[T, X] = [T,\, T_X + X] \in \mathfrak{s}$, and hence, $\langle [T, X], V\rangle = 0$. We have proved $[\mathfrak{s} \cap \mathfrak{t}, \mathfrak{v}_\alpha] = 0$.

Let $T\colon \tilde{\mathfrak{s}} \to \mathfrak{s}_\mathfrak{t} \ominus (\mathfrak{s} \cap \mathfrak{t})$, $X \mapsto T_X$, be defined by $T_X + X \in \mathfrak{s}$. This map is well-defined: if $T_X, T'_X \in \mathfrak{t}$ are such that $T_X + X,\, T'_X + X \in \mathfrak{s}$, subtracting, $T_X - T'_X \in \mathfrak{s} \cap \mathfrak{t}$. Note that T is surjective.

Given a nonzero $V \in \mathfrak{v}_\alpha$, we define $\Phi_V\colon \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha \to \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$ by $\Phi_V(X) = [T_X, V]$. We prove that Φ_V is self-adjoint. Indeed, given $X, Y \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha \subseteq \tilde{\mathfrak{s}}$, we obtain $[T_X, Y] + [X, T_Y] + [X, Y] = [T_X + X, T_Y + Y] \in \mathfrak{s}$, which means

$$\begin{aligned} 0 &= \langle V, [T_X + X, T_Y + Y]\rangle = -\langle [T_X, V], Y\rangle + \langle [T_Y, V], X\rangle \\ &= -\langle \Phi_V(X), Y\rangle + \langle X, \Phi_V(Y)\rangle. \end{aligned}$$

We now prove that $\Phi_V = 0$. Assume this is not the case, so by the spectral theorem, there exists a nonzero vector $X \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$ and a nonzero constant $\lambda \in \mathbb{R}$ such that $\Phi_V(X) = \lambda X$.

Observe that $[V, T_{[V,X]}] = 0$. Indeed, $[V, T_{[V,X]}] \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$, and given any $Y \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$ we obtain

$$\begin{aligned} 0 &= \langle [T_Y + Y, T_{[V,X]} + [V, X]],\, V\rangle = \langle [T_Y, [V, X]] + [Y, T_{[V,X]}],\, V\rangle \\ &= \langle [Y, T_{[V,X]}],\, V\rangle = -\langle Y,\, [V, T_{[V,X]}]\rangle, \end{aligned}$$

which implies $[V, T_{[V,X]}] = 0$.

Now, consider $g = \operatorname{Exp}(\frac{1}{\lambda}V)$ and $Z = T_X + X - \frac{1}{2\lambda}\bigl(T_{[V,X]} + [V, X]\bigr) \in \mathfrak{s}$. Then

$$\begin{aligned} \operatorname{Ad}(g)Z &= e^{\frac{1}{\lambda}\operatorname{ad}(V)}\Bigl(T_X + X - \frac{1}{2\lambda}T_{[V,X]} - \frac{1}{2\lambda}[V, X]\Bigr) \\ &= T_X + X - \frac{1}{2\lambda}T_{[V,X]} - \frac{1}{2\lambda}[V, X] + \frac{1}{\lambda}\bigl(-\lambda X + [V, X]\bigr) - \frac{1}{2\lambda^2}\lambda[V, X] \\ &= T_X - \frac{1}{2\lambda}T_{[V,X]} \in \operatorname{Ad}(g)(\mathfrak{s}) \cap \mathfrak{k}. \end{aligned}$$

By Lemma 3.18, we obtain $\operatorname{Ad}(g)Z \in \mathfrak{s} \cap \mathfrak{t}$, and thus, $Z \in \operatorname{Ad}(g^{-1})(\mathfrak{s} \cap \mathfrak{t}) = \mathfrak{s} \cap \mathfrak{t}$, a contradiction. We conclude that Φ_V is the zero map for every $V \in \mathfrak{v}_\alpha$. Since $\Phi_{\mathfrak{v}_\alpha} \equiv 0$ we have $[T_X, V] = 0$ for every $V \in \mathfrak{v}_\alpha$ and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$.

Now, let $H \in \mathfrak{z}$ and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$. Since the vectors $T_H + H$ and $T_X + X$ are in $\mathfrak{s}$, their Lie bracket $[T_H + H, T_X + X] = [T_H, X] + \alpha(H)X$ is also in $\mathfrak{s}$, and because $X \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$ we deduce that $[T_H, X] \in \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$. As a consequence, $\operatorname{ad}(T_H)$ preserves $\mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$, so it also preserves $\mathfrak{v}_\alpha$

as it is skew-symmetric. Similarly, suppose that $Y \in \mathfrak{g}_\lambda$ is any vector, with $\lambda \in \Sigma^+ \setminus \{\alpha\}$. We see that $[T_Y, X] + [Y, T_X] + [Y, X] = [T_Y + Y, T_X + X] \in \mathfrak{s}$, so taking the inner product with any $V \in \mathfrak{g}_\alpha$ we obtain that $0 = \langle [T_Y, X], V\rangle$. This means that $\operatorname{ad}(T_Y)(\mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha)$ is contained in $\mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$, and the skew-symmetry of $\operatorname{ad}(T_Y)$ yields $\operatorname{ad}(T_Y)(\mathfrak{v}_\alpha) \subseteq \mathfrak{v}_\alpha$.

All in all, we have seen that $[\mathfrak{s}_\mathfrak{t} \ominus (\mathfrak{s} \cap \mathfrak{t}), \mathfrak{v}_\alpha] \subseteq \mathfrak{v}_\alpha$. Let $T \in \mathfrak{s}_\mathfrak{t} \ominus (\mathfrak{s} \cap \mathfrak{t})$ and $V \in \mathfrak{v}_\alpha$. Then, the skew-symmetry of $\operatorname{ad}(T)$ implies that $[T, V] \in \mathfrak{v}_\alpha \ominus \mathbb{R}V$. Choose $X \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ such that $T + X \in \mathfrak{s}$, and let $W \in \mathfrak{v}_\alpha \ominus \mathbb{R}V$ be arbitrary. As the action of S is polar and $\mathfrak{v}_\alpha$ is abelian, we see that

$$-2[\theta V, W] = [(1-\theta)V, (1-\theta)W] \in \mathfrak{k}_0$$

is orthogonal to $\mathfrak{s}$, and as a consequence we deduce that

$$0 = \langle T + X, [\theta V, W]\rangle = \langle [T, V], W\rangle,$$

which means that $[T, V] = 0$, so $[\mathfrak{s}_\mathfrak{t} \ominus (\mathfrak{s} \cap \mathfrak{t}), \mathfrak{v}_\alpha] = 0$.

Combining what we have seen in the previous paragraph with $[\mathfrak{s} \cap \mathfrak{t}, \mathfrak{v}_\alpha] = 0$, we arrive at $[\mathfrak{s}_\mathfrak{t}, \mathfrak{v}_\alpha] = 0$. Therefore, by skew-symmetry of the elements of $\mathfrak{t}$, we get $\langle [\mathfrak{s}_\mathfrak{t}, \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha], \mathfrak{v}_\alpha\rangle = \langle \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha, [\mathfrak{s}_\mathfrak{t}, \mathfrak{v}_\alpha]\rangle = 0$. Since $\mathfrak{t}$ normalizes $\mathfrak{g}_\alpha$, we finally get $[\mathfrak{s}_\mathfrak{t}, \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha] \subseteq \mathfrak{g}_\alpha \ominus \mathfrak{v}_\alpha$, which in turn implies that $\hat{\mathfrak{s}} = \mathfrak{s}_\mathfrak{t} \oplus \mathfrak{z} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha) = \mathfrak{s}_\mathfrak{t} \oplus \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ is a Lie subalgebra of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

We can therefore consider the connected subgroup $\hat{\mathsf{S}}$ of G whose Lie algebra is $\hat{\mathfrak{s}}$. We prove that $\hat{\mathsf{S}}$, S and $\tilde{\mathsf{S}}$ have the same orbits. Note that *a priori* we do not know if $\hat{\mathsf{S}}$ is closed, and thus the action of $\hat{\mathsf{S}}$ may not be proper. Since $\mathsf{S} \subseteq \hat{\mathsf{S}}$ and $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}} = \hat{\mathfrak{s}}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{z} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha)$, we deduce that $\mathsf{S} \cdot o = \hat{\mathsf{S}} \cdot o$. The same argument may be applied to see that $\hat{\mathsf{S}} \cdot o = \tilde{\mathsf{S}} \cdot o$. In particular, $\mathsf{S} \cdot o = \hat{\mathsf{S}} \cdot o = \tilde{\mathsf{S}} \cdot o$ is simply connected (because AN is an exponential Lie group acting simply transitively on M), which means that the isotropy subgroups $\mathsf{S} \cap \mathsf{K}$ and $\hat{\mathsf{S}} \cap \mathsf{K}$ are connected. As a consequence, the slice representation of $\hat{\mathsf{S}}$ at o is trivial because $[\hat{\mathfrak{s}} \cap \mathfrak{k}, \mathfrak{s}_\mathfrak{p}^\perp] = [\mathfrak{s}_\mathfrak{t}, (\mathfrak{a} \ominus \mathfrak{z}) \oplus (1-\theta)\mathfrak{v}_\alpha] = (1-\theta)[\mathfrak{s}_\mathfrak{t}, \mathfrak{v}_\alpha] = 0$. Hence, using Lemma 3.7 twice, the groups S, $\hat{\mathsf{S}}$, and $\tilde{\mathsf{S}}$ act with the same orbits . □

Since we can assume that $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}^1$ has dimension one or two, we tackle these two possibilities separately.

3.3.1 The case $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{p}^1$

Assume $\mathfrak{s}_\mathfrak{p}^\perp$ is contained in $\mathfrak{p}^1$. We have that $\mathfrak{a}$ and $\mathfrak{n} \ominus \mathfrak{n}^1$ are subspaces of $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. A direct application of Lemma 3.14 gives $\mathfrak{n} \ominus \mathfrak{n}^1 \subseteq \mathfrak{s}$. Let $\xi = \sum_{\alpha\in\Lambda}(1-\theta)\xi_\alpha$ and $\eta = \sum_{\alpha\in\Lambda}(1-\theta)\eta_\alpha$ be orthonormal vectors in $\mathfrak{s}_\mathfrak{p}^\perp$, where ξ_α, $\eta_\alpha \in \mathfrak{g}_\alpha$. Since the action is polar nonhyperpolar, $[\xi, \eta] = (1+\theta)\big(\sum_{\alpha,\beta\in\Lambda}[\xi_\alpha, \eta_\beta] - [\theta\xi_\alpha, \eta_\beta]\big)$ is a nonzero vector orthogonal to $\mathfrak{s}$. By using the fact that $\mathfrak{n}\ominus\mathfrak{n}^1 \subseteq \mathfrak{s}$ and $[\theta\xi_\alpha, \eta_\beta] = 0$ when $\beta \neq \alpha$, we deduce $[\xi_\alpha, \eta_\alpha] = 0$, and $[\xi_\alpha, \eta_\beta] + [\xi_\beta, \eta_\alpha] = 0$. Thus, $[\xi, \eta] = -(1+\theta)\sum_{\alpha\in\Lambda}[\theta\xi_\alpha, \eta_\alpha]$.

Since $\mathfrak{s}_\mathfrak{p}^\perp$ is a two-dimensional Lie triple system, it determines a totally geodesic submanifold that is isometric to a real hyperbolic space. Hence, there exists $C > 0$ such that $\operatorname{ad}(\xi)^2\eta = C\eta$.

Thus, we have for every $\alpha \in \Lambda$,

$$\begin{aligned} C\langle \xi_\alpha, \eta_\alpha \rangle &= C\langle \xi_\alpha, \eta \rangle = \langle \xi_\alpha, \operatorname{ad}(\xi)^2 \eta \rangle = \langle [\xi, \xi_\alpha], [\xi, \eta] \rangle \\ &= -\Big\langle \sum_{\beta \in \Lambda} [\xi_\beta, \xi_\alpha] - [\theta \xi_\alpha, \xi_\alpha], (1+\theta) \sum_{\gamma \in \Lambda} [\theta \xi_\gamma, \eta_\gamma] \Big\rangle \\ &= \Big\langle |\xi_\alpha|^2 H_\alpha, (1+\theta) \sum_{\beta \in \Lambda} [\theta \xi_\beta, \eta_\beta] \Big\rangle = 0. \end{aligned}$$

Proposition 3.20. $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}} = (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{v}_\alpha$, *where* $\mathfrak{v}_\alpha$ *is an abelian subspace of* $\mathfrak{g}_\alpha$.

Proof. Let λ, $\mu \in \Lambda$ with ξ_λ, $\xi_\mu \neq 0$. Since $\langle \xi_\alpha, \eta_\alpha \rangle = 0$ for all simple roots $\alpha \in \Lambda$, taking $x = -|\xi_\lambda|^2/|\xi_\mu|^2 < 0$ we have $\xi_\lambda + x\xi_\mu \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. Hence, we may find a $T \in \mathfrak{t}$ such that $T + \xi_\lambda + x\xi_\mu \in \mathfrak{s}$. On the other hand, choose a vector $H \in \mathfrak{a}$ with $\lambda(H) = 0$ and $\mu(H) \neq 0$. Since $\mathfrak{a} \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, there exists $T' \in \mathfrak{t}$ such that $T' + H \in \mathfrak{s}$. As a consequence, $[T', \xi_\lambda] + x[T', \xi_\mu] + x\mu(H)\xi_\mu = [T' + H, T + \xi_\lambda + x\xi_\mu] \in \mathfrak{s}$. This means that $0 = \langle [T', \xi_\lambda] + x[T', \xi_\mu] + x\mu(H)\xi_\mu, \sum_\alpha (1-\theta)\xi_\alpha \rangle = x\mu(H)|\xi_\mu|^2$, which is a contradiction. Thus, $\xi = (1-\theta)\xi_\alpha$ for a fixed simple root $\alpha \in \Lambda$. The same argument can be applied to conclude that $\eta = (1-\theta)\xi_\beta$ for a simple root $\beta \in \Lambda$. We now prove that $\alpha = \beta$. Indeed, if $\alpha \neq \beta$, we have $[\xi, \eta] = (1-\theta)([\xi_\alpha, \eta_\beta] - [\theta\xi_\alpha, \eta_\beta]) = 0$, contradicting the fact that S has a non flat section.

Finally, $\mathfrak{v}_\alpha = \mathbb{R}\xi_\alpha \oplus \mathbb{R}\eta_\alpha$ is abelian due to the discussion at the beginning of this case. □

The expression obtained in Proposition 3.23 together with Proposition 3.19 with $\mathfrak{z} = \mathfrak{a}$ imply now that the action of S is orbit equivalent to item (ii) of Theorem B.

3.3.2 The case $\dim(\mathfrak{s}_{\mathfrak{p}}^{\perp} \cap \mathfrak{p}^1) = 1$

In this setting, we can choose two orthonormal vectors $\xi = \xi_0 + \sum_{\alpha \in \Lambda} (1-\theta)\xi_\alpha$ and $\eta = \sum_{\alpha \in \Lambda} (1-\theta)\eta_\alpha$ that span $\mathfrak{s}_{\mathfrak{p}}^{\perp}$, where $\xi_0 \in \mathfrak{a}$ is nonzero and $\xi_\alpha, \eta_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Lambda$.

Since $\alpha - \beta \notin \Sigma$ for α, $\beta \in \Lambda$, $\alpha \neq \beta$, we have

$$[\xi, \eta] = (1+\theta)\Big(\sum_{\alpha \in \Lambda} \big(\alpha(\xi_0)\eta_\alpha - [\theta\xi_\alpha, \eta_\alpha] \big) + \sum_{\alpha, \beta \in \Lambda} [\xi_\alpha, \eta_\beta] \Big).$$

Recall that $\operatorname{ad}(\xi)^2 \eta = C\eta$ for a positive constant $C \in \mathbb{R}$. In particular, for any $H \in \mathfrak{a}$,

$$\begin{aligned} 0 = \langle C\eta,\, H \rangle &= \langle \operatorname{ad}(\xi)^2 \eta,\, H \rangle = \langle [\xi, \eta],\, [\xi, H] \rangle = -\Big\langle [\xi, \eta],\, (1+\theta) \sum_{\gamma \in \Lambda} \gamma(H) \xi_\gamma \Big\rangle \\ &= -2 \sum_{\alpha \in \Lambda} \alpha(\xi_0)\alpha(H) \langle \xi_\alpha, \eta_\alpha \rangle = \Big\langle -2 \sum_{\alpha \in \Lambda} \alpha(\xi_0) \langle \xi_\alpha, \eta_\alpha \rangle H_\alpha,\, H \Big\rangle, \end{aligned}$$

which implies $-2\sum_{\alpha \in \Lambda} \alpha(\xi_0)\langle \xi_\alpha, \eta_\alpha \rangle H_\alpha = 0$. Therefore,

$$\alpha(\xi_0)\langle \xi_\alpha, \eta_\alpha \rangle = 0, \text{ for every } \alpha \in \Lambda. \tag{3.2}$$

On the other hand, since $\mathrm{ad}(\eta)^2\xi = C\xi$, for any $H \in \mathfrak{a}$,

$$\begin{aligned}C\langle \xi_0, H\rangle &= \langle C\xi, H\rangle = \langle \mathrm{ad}(\eta)^2\xi,\ H\rangle = \langle [\eta,\xi],\ [\eta,H]]\rangle\\ &= \Big\langle [\xi,\eta],\ (1+\theta)\sum_{\gamma\in\Lambda}\gamma(H)\eta_\gamma\Big\rangle = 2\sum_{\alpha\in\Lambda}\alpha(\xi_0)\alpha(H)|\eta_\alpha|^2\\ &= \Big\langle 2\sum_{\alpha\in\Lambda}\alpha(\xi_0)|\eta_\alpha|^2 H_\alpha,\ H\Big\rangle,\end{aligned}$$

we obtain

$$C\xi_0 = 2\sum_{\alpha\in\Lambda}\alpha(\xi_0)|\eta_\alpha|^2 H_\alpha. \tag{3.3}$$

Similarly, let $\alpha,\beta\in\Lambda$ be arbitrary, and $X\in\mathfrak{g}_{\alpha+\beta}$. Then,

$$\begin{aligned}0 &= \langle C\xi, X\rangle = \langle \mathrm{ad}(\eta)^2\xi,\ X\rangle = \langle [\eta,\xi],\ [\eta,X]\rangle\\ &= \Big\langle [\xi,\eta],\ \sum_{\mu\in\Lambda}\big([X,\eta_\mu]-[X,\theta\eta_\mu]\big)\Big\rangle = -\Big\langle \sum_{\gamma\in\Lambda}\gamma(\xi_0)\eta_\gamma,\ \sum_{\delta\in\Lambda}[X,\theta\eta_\delta]\Big\rangle\\ &= \sum_{\gamma,\delta\in\Lambda}\gamma(\xi_0)\langle[\eta_\gamma,\eta_\delta],\ X\rangle = \langle \alpha(\xi_0)[\eta_\alpha,\eta_\beta]+\beta(\xi_0)[\eta_\beta,\eta_\alpha], X\rangle.\end{aligned}$$

Consequently, for any two simple roots $\alpha,\beta\in\Lambda$, $[(\alpha-\beta)(\xi_0)\,\eta_\alpha,\eta_\beta]=0$.

Lemma 3.21. *We have $\langle\xi_\alpha,\eta_\alpha\rangle = 0$ for all $\alpha\in\Lambda$.*

Proof. We define $\Psi = \{\alpha\in\Lambda\colon \eta_\alpha\neq 0\}$. We show that $\langle\xi_\alpha,\eta_\alpha\rangle = 0$ for each $\alpha\in\Psi$.

From (3.3) the map $\alpha\in\Psi\mapsto\alpha(\xi_0)$ cannot be identically zero. Thus, fix $\alpha\in\Psi$ such that $\alpha(\xi_0)\neq 0$. Hence (3.2) already implies $\langle\xi_\alpha,\eta_\alpha\rangle = 0$.

Assume $\beta\in\Psi$ satisfies $\langle\xi_\beta,\eta_\beta\rangle\neq 0$. In particular, from (3.2) we have $\beta(\xi_0)=0$. If $\langle\alpha,\beta\rangle\neq 0$, the linear map $\mathrm{ad}(\eta_\beta)\colon\mathfrak{g}_\alpha\to\mathfrak{g}_{\alpha+\beta}$ is injective. From the equation $[(\alpha-\beta)(\xi_0)\,\eta_\alpha,\eta_\beta]=0$ we deduce that $(\alpha-\beta)(\xi_0)\eta_\alpha = 0$, so $\alpha(\xi_0)=\beta(\xi_0)=0$, contradiction. Thus, $\langle\alpha,\beta\rangle = 0$. Furthermore, note that $H_\beta\in\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ because $\beta(\xi_0)=0$, so there exists $T\in\mathfrak{t}$ such that $T+H_\beta\in\mathfrak{s}$. On the other hand, $\xi_0+x\eta_\alpha+y\eta_\beta\in\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ for $y=-|\xi_0|^2/\langle\xi_\beta,\eta_\beta\rangle\neq 0$ and $x=-y|\eta_\beta|^2/|\eta_\alpha|^2\neq 0$, so $T'+\xi_0+x\eta_\alpha+y\eta_\beta\in\mathfrak{s}$ for an adequate $T'\in\mathfrak{t}$. Thus,

$$\begin{aligned}0 &= \langle [T+H_\beta, T'+\xi_0+x\eta_\alpha+y\eta_\beta],\ \eta\rangle\\ &= \langle x[T,\eta_\alpha]+y[T,\eta_\beta]+y|\beta|^2\eta_\beta,\ \eta\rangle = y|\beta|^2|\eta_\beta|^2,\end{aligned}$$

which gives us a contradiction. Therefore, $\langle\xi_\beta,\eta_\beta\rangle = 0$ for all $\beta\in\Lambda$. □

Proposition 3.22. *There exists a simple root $\alpha\in\Lambda$ and a constant $a\in\mathbb{R}$ such that $\xi = aH_\alpha+(1-\theta)\xi_\alpha$. If $\xi_\alpha = 0$ (that is, if $\xi\in\mathfrak{a}$), then $\eta=(1-\theta)\eta_\alpha$.*

Proof. Firstly, suppose $\xi\in\mathfrak{a}$. Then, a direct application of Lemma 3.16 implies that $\eta = (1-\theta)\eta_\alpha$ for a simple root $\alpha\in\Lambda$. In particular, (3.3) is reduced to $C\xi_0 = 2\alpha(\xi_0)|\eta_\alpha|^2H_\alpha$, so $\xi_0\in\mathbb{R}H_\alpha$, and the proposition follows.

Now, assume $\xi_0 \notin \mathfrak{a}$, and let $\alpha \in \Lambda$ be a simple root such that $\xi_\alpha \neq 0$. We prove that $\xi = aH_\alpha + (1-\theta)\xi_\alpha$.

Suppose that ξ_0 is not proportional to H_α. Then there exists $H \in \mathfrak{a}$ such that $\langle H, \xi_0 \rangle = 0$ and $\alpha(H) \neq 0$. As a consequence, $H \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, and there exists $T \in \mathfrak{t}$ for which $T + H \in \mathfrak{s}$. On the other hand, $\xi_0 + x\xi_\alpha \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ for the number $x = -|\xi_0|^2/|\xi_\alpha|^2 < 0$ (because $\langle \xi_\alpha, \eta_\alpha \rangle = 0$ by Lemma 3.21), and we may choose $T' \in \mathfrak{t}$ such that $T' + \xi_0 + x\xi_\alpha \in \mathfrak{s}$. Thus, we obtain that $[T + H, T' + \xi_0 + x\xi_\alpha] = x[T, \xi_\alpha] + x\alpha(H)\xi_\alpha \in \mathfrak{s}$. Taking the inner product with ξ yields $x\alpha(H)|\xi_\alpha|^2 = 0$, a contradiction.

Hence $\xi_0 \in \mathbb{R}H_\alpha$ for any $\alpha \in \Lambda$. The fact that simple roots are linearly independent together with (3.3) implies $\xi_\beta = 0$ for every $\beta \in \Lambda \setminus \{\alpha\}$. □

So far we have proved that ξ must take the form $aH_\alpha + (1-\theta)\xi_\alpha$ for a nonzero $a \in \mathbb{R}$ and $\xi_\alpha \in \mathfrak{g}_\alpha$ (which may be zero). If $\xi_\alpha = 0$, then we also know that $\eta = (1-\theta)\eta_\alpha$. If $\xi_\alpha \neq 0$, then the third statement of Lemma 3.16 implies that the action of S is orbit equivalent to an action of another closed connected subgroup $\tilde{\mathsf{S}}$ for which the normal space of $\tilde{\mathsf{S}} \cdot o$ at o takes the form $\tilde{\mathfrak{s}}_{\mathfrak{p}}^{\perp} = \{(1-\theta)\xi_\alpha, bH_\alpha + (1-\theta)\nu_\alpha\}$ for a constant $b \in \mathbb{R}$ and a $\nu_\alpha \in \mathfrak{g}_\alpha$. Because of this, we may assume without loss of generality that $\mathfrak{s}_{\mathfrak{p}}^{\perp}$ is spanned by two orthogonal vectors $\xi = aH_\alpha + (1-\theta)\xi_\alpha$, $\eta = (1-\theta)\eta_\alpha$, with $a \neq 0$, ξ_α, $\eta_\alpha \in \mathfrak{g}_\alpha$. Recall from Lemma 3.16 that $[\xi_\alpha, \eta_\alpha] = 0$.

The key to finishing the proof lies in the following result:

Proposition 3.23. *Assume $\mathfrak{s}_{\mathfrak{p}}^{\perp} = \operatorname{span}\{\xi, \eta\}$ is generated by $\xi = aH_\alpha + (1-\theta)\xi_\alpha$ and $\eta = (1-\theta)\eta_\alpha$, where $a \neq 0$, ξ_α, $\eta_\alpha \in \mathfrak{g}_\alpha$ are orthogonal commuting vectors, and $\alpha \in \Lambda$. Then:*

(i) *If $\xi_\alpha = 0$, then the action of S has the same orbits as the action of the connected subgroup of G whose Lie algebra is $(\mathfrak{a} \ominus \mathbb{R}H_\alpha) \oplus (\mathfrak{n} \ominus \mathbb{R}\eta_\alpha)$.*

(ii) *If $\xi_\alpha \neq 0$, then there exists an abelian subspace $\mathfrak{v}_\alpha \subseteq \mathfrak{g}_\alpha$ such that the action of S is orbit equivalent to the action of the connected subgroup of G whose Lie algebra is $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha)$.*

Proof. If $\xi_\alpha = 0$, then $\mathfrak{s}_{\mathfrak{p}}^{\perp} = \mathbb{R}H_\alpha \oplus \mathbb{R}(1-\theta)\eta_\alpha$ and $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}} = (\mathfrak{a} \ominus \mathbb{R}H_\alpha) \oplus (\mathfrak{n} \ominus \mathbb{R}\eta_\alpha)$. Then, statement (i) follows directly from Proposition 3.19.

We prove (ii). We consider the element $g = \operatorname{Exp}(-\frac{a}{|\xi_\alpha|^2}\xi_\alpha) \in \mathsf{N}$. Since $\mathfrak{s}$ is orthogonal to $aH_\alpha + \xi_\alpha$ and η_α, it follows that $\operatorname{Ad}(g)\mathfrak{s}$ is orthogonal to the vectors $\operatorname{Ad}(g^{-1})^*(aH_\alpha + \xi_\alpha)$ and $\operatorname{Ad}(g^{-1})^*\eta_\alpha$. By direct computation,

$$\begin{aligned} \operatorname{Ad}(g^{-1})^*(aH_\alpha + \xi_\alpha) &= e^{-\frac{a}{|\xi_\alpha|^2}\operatorname{ad}(\theta\xi_\alpha)}(aH_\alpha + \xi_\alpha) \equiv \xi_\alpha \pmod{\theta\mathfrak{n}}, \\ \operatorname{Ad}(g^{-1})^*\eta_\alpha &= e^{-\frac{a}{|\xi_\alpha|^2}\operatorname{ad}(\theta\xi_\alpha)}\eta_\alpha \equiv \eta_\alpha - \frac{a}{|\xi_\alpha|^2}[\theta\xi_\alpha, \eta_\alpha] \pmod{\theta\mathfrak{n}}, \end{aligned}$$

and since $\operatorname{Ad}(g)\mathfrak{s} \subseteq \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, it follows that the vectors ξ_α and $\eta_\alpha - \frac{a}{|\xi_\alpha|^2}[\theta\xi_\alpha, \eta_\alpha]$ are orthogonal to $\operatorname{Ad}(g)\mathfrak{s}$. On the other hand, the action of S is polar, so $[\xi, \eta] = a|\alpha|^2(1+\theta)\eta_\alpha - 2[\theta\xi_\alpha, \eta_\alpha]$ is also orthogonal to $\mathfrak{s}$. As a consequence, we see that $\langle a|\alpha|^2\eta_\alpha - 2[\theta\xi_\alpha, \eta_\alpha], \mathfrak{s} \rangle = 0$. We deduce that

$$\operatorname{Ad}(g^{-1})^*(a|\alpha|^2\eta_\alpha - 2[\theta\xi_\alpha, \eta_\alpha]) \equiv a|\alpha|^2\eta_\alpha - \Big(2 + \frac{a^2|\alpha|^2}{|\xi_\alpha|^2}\Big)[\theta\xi_\alpha, \eta_\alpha] \pmod{\theta\mathfrak{n}}$$

is also orthogonal to $\mathrm{Ad}(g)\mathfrak{s}$. Because $\theta\mathfrak{n}$ is already orthogonal to $\mathrm{Ad}(g)\mathfrak{s}$, it follows that $a|\alpha|^2\eta_\alpha - \left(2 + \frac{a^2|\alpha|^2}{|\xi_\alpha|^2}\right)[\theta\xi_\alpha, \eta_\alpha]$ is perpendicular to $\mathrm{Ad}(g)\mathfrak{s}$. Since

$$\begin{vmatrix} 1 & -\frac{a}{|\xi_\alpha|^2} \\ a|\alpha|^2 & -2 - \frac{a^2|\alpha|^2}{|\xi_\alpha|^2} \end{vmatrix} = -2 < 0,$$

we deduce that η_α and $[\theta\xi_\alpha, \eta_\alpha]$ are both orthogonal to $\mathrm{Ad}(g)\mathfrak{s}$. We conclude that $(\mathrm{Ad}(g)\mathfrak{s})^\perp_\mathfrak{p} = (1-\theta)\mathfrak{v}_\alpha$, where $\mathfrak{v}_\alpha = \mathrm{span}\{\xi_\alpha, \eta_\alpha\}$, and thus, $g\mathsf{S}g^{-1}$ and the connected subgroup of G whose Lie algebra is $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}_\alpha)$ act with the same orbits due to Proposition 3.19. □

All things considered, the proof of Theorem B follows from the observation that Proposition 3.23(i) corresponds to case (i), and Proposition 3.23(ii) corresponds to case (ii).

Chapter 4

Polar homogeneous foliations on quaternionic hyperbolic spaces and the Cayley hyperbolic plane

This chapter deals with the classification of polar homogeneous foliations on symmetric spaces of noncompact type and rank one. We derive the classification of these foliations on the quaternionic and Cayley hyperbolic planes. Furthermore, a partial classification of polar homogeneous foliations on quaternionic hyperbolic spaces of higher dimension is given. The results of this chapter are part of an ongoing project in collaboration with José Carlos Díaz-Ramos that aims to finish the classification of polar homogeneous foliations on rank one symmetric spaces.

As we saw in Section 2.3.2, the only symmetric spaces of noncompact type on which polar actions have been classified are the real and complex hyperbolic spaces. In the quaternionic and Cayley cases, the works of Kollross [107, 108] provide partial classifications of polar actions that preserve a proper totally geodesic submanifold, and it turns out that all of the actions found in his results possess singular orbits. However, there are no results concerning polar actions without singular orbits on $\mathbb{H}\mathsf{H}^n$ or $\mathbb{O}\mathsf{H}^2$ (or more generally, actions that do not preserve a totally geodesic orbit). In this chapter we treat this type of actions employing the techniques developed in Chapter 3.

Recall that in Chapter 3 we constructed two families of polar homogeneous foliations $\mathcal{F}_\xi$ and $\mathcal{F}_\mathfrak{v}$ having codimension two on symmetric spaces of noncompact type from the Iwasawa decomposition of their isometry algebras. Moreover, these two families exhausted all possible cohomogeneity two polar homogeneous foliations up to orbit equivalence (see Theorem B in that chapter). If $M = \mathbb{F}\mathsf{H}^n = \mathsf{G}/\mathsf{K}$ is a symmetric space of noncompact type and rank one, then after choosing an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ of $\mathfrak{g}$ we see that $\mathfrak{a}$ is one-dimensional and there is a unique simple root $\alpha \in \Lambda$. As a consequence, M admits exactly two polar homogeneous foliations of codimension two up to orbit equivalence, induced by the connected subgroups S_ξ and $\mathsf{S}_\mathfrak{v}$ of G with Lie algebras $\mathfrak{s}_\xi = \mathfrak{n} \ominus \mathbb{R}\xi$ and $\mathfrak{s}_\mathfrak{v} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v})$, where $\xi \in \mathfrak{g}_\alpha$ is nonzero and $\mathfrak{v} \subseteq \mathfrak{g}_{2\alpha}$ is a two-dimensional abelian subspace. The choices of ξ and $\mathfrak{v}$ are irrelevant, as they always induce orbit equivalent foliations due to Lemma 3.3.

In this chapter we construct lower-dimensional analogues of S_ξ and $\mathsf{S}_\mathfrak{v}$ that induce polar homogeneous foliations on $\mathbb{F}\mathsf{H}^n$. As the rank of $\mathbb{F}\mathsf{H}^n$ is one, the algebras $\mathfrak{s}_\xi$ and $\mathfrak{s}_\mathfrak{v}$ can be written jointly as $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$, where $\mathfrak{b}$ is a vector subspace of $\mathfrak{a}$ (that is, $\mathfrak{b} = 0$ or $\mathfrak{b} = \mathfrak{a}$) and $\mathfrak{v}$ is an abelian subspace of $\mathfrak{g}_\alpha$, both subject to the relation $\dim \mathfrak{b} + \dim \mathfrak{v} = 2$. By dropping this last assumption, we obtain new families of subalgebras of $\mathfrak{g}$ whose corresponding connected subgroups act polarly inducing a foliation. Explicitly, given a subspace $\mathfrak{b} \subseteq \mathfrak{a}$ and an abelian subspace $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$, we construct the subalgebra $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$. We will show that its corresponding subgroup $\mathsf{S}_{\mathfrak{b},\mathfrak{v}} \subseteq \mathsf{AN}$ acts polarly on $\mathbb{F}\mathsf{H}^n$ inducing a foliation. Moreover, any section of the action of $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ is a real hyperbolic space whose curvature is equal to the maximum sectional curvature attained by a 2-plane in $\mathbb{F}\mathsf{H}^n$. Throughout this chapter, we work with the symmetric metric on $\mathbb{F}\mathsf{H}^n$ that pinches its sectional curvature between -1 and $-1/4$.

Our first main result states that for the quaternionic and Cayley hyperbolic planes, the foliations induced by the subalgebras above give all polar homogeneous foliations. Observe that if $M \in \{\mathbb{H}\mathsf{H}^2, \mathbb{O}\mathsf{H}^2\}$, a subspace of $\mathfrak{g}_\alpha$ is abelian if and only if its dimension is not greater than one, see Section 4.1.

Theorem A. *Let $M \in \{\mathbb{H}\mathsf{H}^2, \mathbb{O}\mathsf{H}^2\}$ be the quaternionic or Cayley hyperbolic plane. Then the following assertions are true:*

(i) *Given a vector subspace $\mathfrak{b} \subseteq \mathfrak{a}$ and a subspace $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ such that $\dim \mathfrak{v} \leq 1$, the connected subgroup $\mathsf{S}_{\mathfrak{b},\mathfrak{v}} \subseteq \mathsf{G}$ with Lie algebra $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$ acts polarly on M inducing a foliation.*

(ii) *Any nontrivial polar homogeneous foliation on M is orbit equivalent to the orbit foliation of a subgroup $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ as in item* (i).

(iii) *Given subspaces $\mathfrak{b}$, $\mathfrak{b}' \subseteq \mathfrak{a}$ and subspaces $\mathfrak{v}$, $\mathfrak{v}' \subseteq \mathfrak{g}_\alpha$ with $\dim \mathfrak{v}$, $\dim \mathfrak{v}' \leq 1$, the actions of $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ and $\mathsf{S}_{\mathfrak{b}',\mathfrak{v}'}$ are orbit equivalent if and only if $\mathfrak{b} = \mathfrak{b}'$ and $\dim \mathfrak{v} = \dim \mathfrak{v}'$.*

A direct consequence of Theorem A is that up to orbit equivalence there are exactly four subgroups of G acting nontrivially and polarly inducing a foliation. These are:

$$\mathsf{S}_{0,0} = \mathsf{AN}, \quad \mathsf{S}_{\mathfrak{a},0} = \mathsf{N}, \quad \mathsf{S}_{0,\ell}, \quad \mathsf{S}_{\mathfrak{a},\ell}.$$

Here, ℓ denotes a one-dimensional subspace of $\mathfrak{g}_\alpha$. As a consequence, $\mathbb{F}\mathsf{H}^2$ admits exactly two cohomogeneity one homogeneous foliations (one of solvable type (FS) and another of horospherical type (FH)), and a unique cohomogeneity two foliation corresponding to item (i) in Theorem B.

We also address the problem of determining which (complete) totally geodesic submanifolds of $\mathbb{H}\mathsf{H}^n$ and $\mathbb{O}\mathsf{H}^2$ can arise as sections of polar actions. The classification of totally geodesic submanifolds of rank one symmetric spaces is a classical result dating back to Wolf [168]. Broadly speaking, every totally geodesic submanifold of the hyperbolic space $\mathbb{F}\mathsf{H}^n$ is a hyperbolic space $\mathbb{K}\mathsf{H}^k$, where $\mathbb{K}$ is a real division algebra contained in $\mathbb{F}$ and $k \leq n$. Our result builds on prior work by Kollross [108] and shows that if a totally geodesic submanifold $\mathbb{K}\mathsf{H}^k$ of $\mathbb{F}\mathsf{H}^n$ is the section of a polar action, then $\mathbb{K} = \mathbb{R}$.

Theorem B. *Let M be either the quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^n$ or the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$. If S is a connected Lie group acting polarly on M (with cohomogeneity greater than one) and $\Sigma \subseteq M$ is a section of the action $\mathsf{S} \curvearrowright M$, then either the action of S is trivial or Σ is a totally geodesic real hyperbolic space $\mathbb{R}\mathsf{H}^k$ of constant curvature $\kappa \in \{-1, -1/4\}$.*

The actions constructed in this chapter all have sections of constant curvature equal to $-1/4$. Moreover, it is known that in the complex case [52] all sections have curvature $-1/4$. At the moment, we do not know if the value $\kappa = -1$ can be removed from the statement of Theorem B.

Finally, we deal with polar homogeneous foliations on quaternionic hyperbolic spaces of higher dimension. Note that in all classifications of polar homogeneous foliations up to date (see [19, 21] and Chapter 3) one sees that every hyperpolar (or codimension two polar) homogeneous foliation is isometrically congruent to the orbit foliation induced by a connected subgroup of AN (with respect to an appropriate Iwasawa decomposition of the isometry algebra). We say that a homogeneous foliation arising from such a subgroup is standard. Thus, a natural first step to solve the quaternionic case is to classify all standard polar homogeneous foliations on $\mathbb{H}\mathsf{H}^n$. It turns out that we obtain the following:

Theorem C. *Let $M = \mathbb{H}\mathsf{H}^n$ be the quaternionic hyperbolic space. Then the following statements hold:*

(i) *Given a vector subspace $\mathfrak{b} \subseteq \mathfrak{a}$ and an abelian subspace $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$, the connected subgroup $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ of $\mathsf{Sp}(1,n)$ with Lie algebra $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$ acts polarly on M inducing a standard foliation.*

(ii) *If $\mathcal{F}$ is a nontrivial, standard and polar homogeneous foliation on M, then there exists a subspace $\mathfrak{b} \subseteq \mathfrak{a}$ and an abelian subspace $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ such that $\mathcal{F}$ is isometrically congruent to the orbit foliation induced by the subgroup $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$.*

(iii) *Given two subspaces $\mathfrak{b}$, $\mathfrak{b}' \subseteq \mathfrak{a}$ and two abelian subspaces $\mathfrak{v}$, $\mathfrak{v}' \subseteq \mathfrak{g}_\alpha$, the actions of $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ and $\mathsf{S}_{\mathfrak{b}',\mathfrak{v}'}$ are orbit equivalent if and only if we have $\mathfrak{b} = \mathfrak{b}'$ and $\dim \mathfrak{v} = \dim \mathfrak{v}'$.*

Observe that the statement of Theorem C is essentially identical to that of Theorem A, save for the assumption of the foliations under consideration being standard. From [169] and [52] we also know that every polar homogeneous foliation on $\mathbb{R}\mathsf{H}^n$ and $\mathbb{C}\mathsf{H}^n$ is standard (see also [18] for a direct proof in the complex case). All in all, previous experience suggests that Theorem C might also be true if we consider arbitrary polar homogeneous foliations on $\mathbb{H}\mathsf{H}^n$. Unfortunately, we have not been able to confirm or deny this conjecture thus far.

We now describe the structure of this chapter. In Section 4.1 we present the basic algebro-geometric properties of hyperbolic spaces and their relationship with generalized Heisenberg algebras. Section 4.2 is devoted to recalling the classification of totally geodesic submanifolds of the hyperbolic spaces $\mathbb{F}\mathsf{H}^n$. Lastly, in Section 4.3 we prove the main theorems of this chapter. In particular, we study the extrinsic geometry of the $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$-orbits. Note that this is only necessary for the case of $\mathbb{H}\mathsf{H}^n$ with $n \geq 3$, as in $\mathbb{F}\mathsf{H}^2$ the group $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ acts with cohomogeneity at most two.

4.1 Structure of hyperbolic spaces

This section is devoted to describing the main features of hyperbolic spaces that set them apart from symmetric spaces of noncompact type and higher rank. The solvable model $M = \mathsf{AN}$ allows us to view a rank one symmetric space as an extension of a one-dimensional Lie group by a two-step nilpotent (or abelian) Lie group. The subgroup N is an example of a generalized Heisenberg group, while M belongs to the class of Damek–Ricci spaces. By exploiting the algebraic structure of the Lie algebra $\mathfrak{n}$ as a generalized Heisenberg algebra, we will be able to perform general calculations with relative ease.

A detailed account of the classification and geometry of both generalized Heisenberg groups and Damek–Ricci spaces can be found in [23]. We also refer the reader to [46] for a treatment focused on rank one symmetric spaces. Section 4.2 is devoted to stating the classification of (complete) totally geodesic submanifolds of each hyperbolic space. Finally, Section 4.3 contains the proofs of Theorems A, B and C.

4.1.1 Generalized Heisenberg algebras

Let $\mathfrak{v}$ and $\mathfrak{z}$ be real vector spaces and $\beta\colon \Lambda^2\mathfrak{v} \to \mathfrak{z}$ a skew-symmetric bilinear map. We construct the vector space $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and endow it with the following algebraic data:

- An inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal, and
- A Lie algebra structure $[\cdot, \cdot]$ given by $[\mathfrak{n}, \mathfrak{z}] = 0$ and $[U, V] = \beta(U \wedge V)$ for all $U, V \in \mathfrak{v}$.

Each vector $Z \in \mathfrak{z}$ gives rise to a skew-symmetric endomorphism $J_Z \in \mathfrak{so}(\mathfrak{v})$ by letting $\langle J_Z U, V \rangle = \langle [U, V], Z \rangle$ for all U, $V \in \mathfrak{z}$. We say that $\mathfrak{n}$ is a *generalized Heisenberg algebra* if $J_Z^2 = -|Z|^2 \operatorname{id}_{\mathfrak{v}}$ for every $Z \in \mathfrak{v}$. In other words, we require $J\colon \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ to extend to an algebra homomorphism $J\colon \operatorname{Cl}(\mathfrak{z}) \to \operatorname{End}(\mathfrak{v})$ defined on the Clifford algebra of $\mathfrak{z}$ (with the aforementioned inner product). In particular, generalized Heisenberg algebras are closely related to Clifford modules. Thus, the classification of Clifford modules allows us to deduce the classification of generalized Heisenberg algebras, see [23, Section 3.1.2] for details.

A consequence of the definition is that the maps J_Z with $|Z| = 1$ become complex structures on the real vector space $\mathfrak{v}$. Because a complex structure is also a linear isomorphism, the Lie algebra $\mathfrak{n}$ is two-step nilpotent (except in the case $\mathfrak{z} = 0$, where $\mathfrak{n}$ is abelian) and its center is $\mathfrak{z}$. A subspace $\mathfrak{w} \subseteq \mathfrak{v}$ is said to be *totally real* if it is totally real with respect to all J_Z, $|Z| = 1$. Equivalently, $\mathfrak{w}$ is totally real if and only if it is an abelian subspace of $\mathfrak{v}$.

We will make use of the following formulae involving U, $V \in \mathfrak{v}$ and X, $Y \in \mathfrak{z}$:

$$J_X J_Y + J_Y J_X = -2\langle X, Y \rangle \operatorname{id}_{\mathfrak{v}}, \qquad \langle J_X U, J_X V \rangle = |X|^2 \langle U, V \rangle,$$
$$\langle J_X U, J_Y U \rangle = |U|^2 \langle X, Y \rangle, \qquad [V, J_X V] = |V|^2 X.$$

Let $V \in \mathfrak{v}$ be a nonzero vector and define $\mathfrak{J}V = \{J_Z V \colon Z \in \mathfrak{z}\}$. It is clear from the skew-symmetry of the operators J_Z that $\mathbb{R}V$ and $\mathfrak{J}V$ are orthogonal subspaces of $\mathfrak{v}$. Furthermore, it is easily checked that an element $W \in \mathfrak{v}$ belongs to $\mathfrak{v} \ominus (\mathbb{R}V \oplus \mathfrak{J}V)$ if and only if $\langle V, W \rangle = 0$ and $[V, W] = 0$.

4.1.2 Noncompact rank one symmetric spaces

Let $M = \mathbb{F}\mathsf{H}^n = \mathsf{G}/\mathsf{K}$ be a symmetric space of noncompact type and rank one. We choose G and K as in Table 4.1, so that G is a finite covering group of $I^0(M)$ and $\mathsf{K} = \mathsf{G}_o$ is the isotropy subgroup at a point $o \in M$. The corresponding Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and we denote by $\theta \in \mathrm{Aut}(\mathfrak{g})$ the Cartan involution. We recall that $\mathfrak{g}$ admits an inner product $\mathcal{B}_\theta$ given by $\mathcal{B}_\theta(X, Y) = -\mathcal{B}(X, \theta Y)$, and its restriction to $\mathfrak{p}$ gives a symmetric metric on M. This metric is the unique G-invariant metric on M up to scaling. Because M is a symmetric space of rank one, we have that any maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ is one-dimensional.

Table 4.1: Data associated with each hyperbolic space.

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{g}_{2\alpha}$
$\mathbb{R}\mathsf{H}^n$	$\mathsf{SO}^0(1,n)$	$\mathsf{SO}(n)$	$\mathsf{SO}(n-1)$	$\mathbb{R}^{n-1}$	0
$\mathbb{C}\mathsf{H}^n$	$\mathsf{SU}(1,n)$	$\mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(n))$	$\mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(n-1))$	$\mathbb{C}^{n-1}$	$\mathbb{R}$
$\mathbb{H}\mathsf{H}^n$	$\mathsf{Sp}(1,n)$	$\mathsf{Sp}(1) \times \mathsf{Sp}(n)$	$\mathsf{Sp}(1) \times \mathsf{Sp}(n-1)$	$\mathbb{H}^{n-1}$	$\mathbb{R}^3$
$\mathbb{O}\mathsf{H}^2$	F_4^{-20}	$\mathsf{Spin}(9)$	$\mathsf{Spin}(7)$	$\mathbb{O}$	$\mathbb{R}^7$

The set of roots is of the form $\Sigma = \{\pm\alpha, \pm 2\alpha\}$ for some $\alpha \in \mathfrak{a}^*$ (except in the case that $\mathbb{F} = \mathbb{R}$, where $\Sigma = \{\pm\alpha\}$), and thus the root space decomposition of $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha},$$

with $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$. We choose a notion of positivity by letting α be positive (in fact, simple), and the corresponding Iwasawa decomposition becomes $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$.

Let us briefly discuss the group K_0 and the positive root spaces as its representations:

- If $\mathbb{F} = \mathbb{R}$, then $\mathsf{K}_0 = \mathsf{SO}(n-1)$ and $\mathfrak{g}_\alpha = \mathbb{R}^{n-1}$ is its standard representation.
- If $\mathbb{F} = \mathbb{C}$, then $\mathsf{K}_0 = \mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(n-1))$, $\mathfrak{g}_\alpha = \mathbb{C}^{n-1}$ is its standard representation, whereas $\mathfrak{g}_{2\alpha} = \mathbb{R} = \mathrm{Im}\,\mathbb{C}$ is the trivial representation.
- If $\mathbb{F} = \mathbb{H}$, then $\mathsf{K}_0 = \mathsf{Sp}(1) \times \mathsf{Sp}(n-1)$, $\mathfrak{g}_\alpha = \mathbb{H}^{n-1}$ is its standard representation and $\mathfrak{g}_{2\alpha} = \mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$, where the action of K_0 is given by $(q, B) \cdot \lambda = q\lambda\bar{q}$.
- If $\mathbb{F} = \mathbb{O}$, then $\mathsf{K}_0 = \mathsf{Spin}(7)$ acts on $\mathfrak{g}_\alpha = \mathbb{O} \cong \mathbb{R}^8$ via the (unique) spin representation $\mathsf{Spin}(7) \to \mathsf{SO}(8)$, whereas $\mathfrak{g}_{2\alpha} = \mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$, where the action of K_0 is the standard representation $\mathsf{Spin}(7) \to \mathsf{SO}(7)$.

We consider on $\mathfrak{g}$ the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle X, Y \rangle = \frac{2}{\dim \mathfrak{g}_\alpha + 4 \dim \mathfrak{g}_{2\alpha}} \mathcal{B}_\theta(X, Y) = \frac{1}{2(n+2)} \mathcal{B}_\theta(X, Y), \quad X, Y \in \mathfrak{g}.$$

This inner product is taken so that the vector $H_\alpha \in \mathfrak{a}$ characterized by the equation $\alpha(H) = \langle H_\alpha, H\rangle$ has norm $|H_\alpha| = |\alpha| = \frac{1}{2}$. We normalize the metric on M so that its restriction to $T_oM \cong \mathfrak{p}$ is precisely $\langle\cdot,\cdot\rangle$.

Now, consider the nilpotent algebra $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathsf{AN}} = \frac{1}{2}\langle\cdot,\cdot\rangle$. For each $Z \in \mathfrak{g}_{2\alpha}$, define $J_Z \in \operatorname{End}(\mathfrak{g}_\alpha)$ as in Subsection 4.1.1. It is not hard to check that $J_Z V = -[\theta V, Z]$ for each $V \in \mathfrak{g}_\alpha$ and $J_Z^2 = -|Z|^2_{\mathsf{AN}} \operatorname{id}_{\mathfrak{g}_\alpha}$, so $\mathfrak{n}$ becomes a generalized Heisenberg algebra. Moreover, $\mathfrak{n}$ satisfies the so-called J^2 *condition*: given an element $V \in \mathfrak{g}_\alpha$ and orthogonal vectors $X, Y \in \mathfrak{g}_{2\alpha}$, there exists a $Z \in \mathfrak{g}_{2\alpha}$ (depending on X, Y and V) satisfying $J_X J_Y V = J_Z V$. By [47, Theorem 1.1], any generalized Heisenberg algebra that satisfies the J^2 condition is isometrically isomorphic to the nilpotent part of the Iwasawa decomposition associated with a rank one symmetric space.

Remark 4.1. In the case of the complex (respectively, quaternionic) hyperbolic space, we may relate the operators J_Z with the complex (respectively, quaternionic) structure of the ambient space. This fact becomes clear when we consider an appropriate realization of the hyperbolic spaces $\mathbb{C}\mathsf{H}^n$ and $\mathbb{H}\mathsf{H}^n$.

Firstly, suppose that $M = \mathbb{C}\mathsf{H}^n$. We consider on $\mathbb{C}^{1,n} = \mathbb{C}^{n+1}$ the sesquilinear product $\langle\cdot,\cdot\rangle$ of signature $(1, n)$ defined by

$$\langle x, y\rangle = -\overline{x_0}y_0 + \sum_{r=1}^{n} \overline{x_r}y_r.$$

Then the subset $\mathsf{S}^n(2) = \{x \in \mathbb{C}^{1,n} \colon \langle x, x\rangle = -4\}$ is a semi-Riemannian submanifold of $\mathbb{C}^{1,n}$ such that the signature of the induced metric is $(2n, 1)$. Furthermore, the canonical complex structure J on $\mathbb{C}^{1,n}$ restricts to a complex structure on $\mathsf{S}^n(2)$, which we still denote by J. Let $\pi\colon \mathsf{S}^n(2) \to \mathbb{C}\mathsf{P}^n$ be the canonical projection to the complex projective space. Then π is a smooth submersion onto its image $\pi(\mathsf{S}^n(2))$, and the fibers of π turn out to be the orbits of the standard action $\mathsf{U}(1) \curvearrowright \mathsf{S}^n(2)$. If we endow $\pi(\mathsf{S}^n(2))$ with the metric that turns π into a semi-Riemannian submersion, one shows that $\pi(\mathsf{S}^n(2))$ is isometric to $\mathbb{C}\mathsf{H}^n$. Under this realization, the action of $\mathsf{SU}(1,n)$ on $\mathbb{C}\mathsf{H}^n$ is the projectivization of the standard representation $\mathsf{SU}(1,n) \curvearrowright \mathbb{C}^{1,n}$. In addition, the complex structure J on $\mathsf{S}^n(2)$ is $\mathsf{U}(1)$-invariant, meaning that it descends to an almost complex structure J on $\mathbb{C}\mathsf{H}^n$, which is actually the canonical Kähler structure on $\mathbb{C}\mathsf{H}^n$. If $o = \mathbb{C}\boldsymbol{e}_0 = \pi(2\boldsymbol{e}_0)$, then the identification $T_o\mathbb{C}\mathsf{H}^n \equiv \mathfrak{p} \equiv \mathfrak{a} \oplus \mathfrak{n}$ allows us to define a complex structure J on $\mathfrak{a} \oplus \mathfrak{n}$, and one sees that $J\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$, $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$ and there exists a vector $Z \in \mathfrak{g}_{2\alpha}$ such that $|Z|^2 = 2$ and $J_Z = J|_{\mathfrak{g}_\alpha}$.

The quaternionic case is more involved. We choose the quaternionic sesquilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^{1,n} = \mathbb{H}^{n+1}$ defined by

$$\langle x, y\rangle = -\overline{x_0}y_0 + \sum_{r=1}^{n} \overline{x_r}y_r,$$

and consider the subset $\mathsf{S}^n(2) = \{x \in \mathbb{H}^{1,n} \colon \langle x, x\rangle = -4\}$. This is now a semi-Riemannian submanifold of signature $(4n, 3)$, and the complex structures I, J, K on $\mathbb{H}^{1,n}$ given by $Iv = vi$, $Jv = vj$ and $Kv = vk$ restrict to $\mathsf{S}^n(2)$. The canonical projection $\pi\colon \mathsf{S}^n(2) \to \mathbb{H}\mathsf{P}^n$ is a smooth submersion whose image is isometric to $\mathbb{H}\mathsf{H}^n$ when we give it the metric that makes π a semi-Riemannian submersion. In this case, the fibers of π are the orbits of the standard action of $\mathsf{Sp}(1)$. While the maps I, J, K are not $\mathsf{Sp}(1)$-invariant, the quaternionic structure $\mathcal{Q}$ generated by them is invariant under this action, so it descends to a quaternionic structure on $\mathbb{H}\mathsf{H}^n$, which is the usual quaternonic Kähler structure on $\mathbb{H}\mathsf{H}^n$. Let $o = \boldsymbol{e}_0\mathbb{H} = \pi(\boldsymbol{e}_0 2)$ (recall that $\mathbb{H}^n$ is a right $\mathbb{H}$-module). Then the action of $\mathsf{Sp}(1,n)$ on $\mathbb{H}\mathsf{H}^n$ is the projectivization of the standard representation of $\mathsf{Sp}(1,n)$. We can translate the quaternionic structure $\mathcal{Q}$ at o to a quaternionic structure $\mathcal{Q}$ on $\mathfrak{a} \oplus \mathfrak{n}$. It can be shown that $\mathfrak{g}_\alpha$ is $\mathcal{Q}$-invariant, the transformations of $\mathcal{Q}$ send $\mathfrak{a}$ to $\mathfrak{g}_{2\alpha}$ and for any $J \in \mathcal{Q}$ the restriction of J to $\mathfrak{g}_\alpha$ is of the form J_Z, where $Z \in \mathfrak{g}_{2\alpha}$.

Since the complex structures J_Z are induced by the Kähler (respectively, quaternionic Kähler) structure of $\mathbb{C}\mathsf{H}^n$ (respectively, $\mathbb{H}\mathsf{H}^n$), it follows that a vector subspace $\mathfrak{w} \subseteq \mathfrak{g}_\alpha$ is totally real in the sense of generalized Heisenberg algebras if and only if it is totally real with respect to the complex (respectively, quaternionic) structure of the tangent space $\mathfrak{a} \oplus \mathfrak{n}$.

Motivated by the aforementioned phenomena in the complex and quaternionic case, we define for each $V \in \mathfrak{g}_\alpha$ the subspace $\mathbb{F}V = \mathbb{R}V \oplus \mathfrak{J}V$. Observe that if $V \neq 0$ then $\dim \mathbb{F}V = \dim \mathbb{F}$ and a vector $W \in \mathfrak{g}_\alpha$ is orthogonal to $\mathbb{F}V$ if and only if $[V, W] = 0$ and $\langle V, W \rangle = 0$. For the proof of Theorem A we will heavily rely on the following fact: if $M = \mathbb{F}\mathsf{H}^2$ and $V \in \mathfrak{g}_\alpha \setminus \{0\}$, then $\mathbb{F}V = \mathfrak{g}_\alpha$.

Because the cohomogeneity of the isotropy representation $\mathsf{K} \curvearrowright \mathfrak{p}$ is one, we deduce that K acts transitively on each sphere of $\mathfrak{p}$ centered at the origin. In particular, two vectors $X, Y \in \mathfrak{p}$ are conjugate under K if and only if $|X| = |Y|$, and in that case their corresponding Jacobi operators are conjugate under $\mathrm{Ad}(\mathsf{K}) \subseteq \mathsf{O}(\mathfrak{p})$. This means that the Jacobi operators R_X, R_Y associated with two nonzero vectors X, $Y \in \mathfrak{p}$ have the same eigenvalues (counting multiplicities) up to some positive scalar dependent on the lengths of X and Y.

We will particularly make use of the Jacobi operators coming from vectors of $\mathfrak{a}$ and $\mathfrak{p}_\alpha$. On the one hand, the Jacobi operator $R_{H_\alpha} = -\operatorname{ad}(H_\alpha)^2 \in \operatorname{End}(\mathfrak{p})$ satisfies

$$R_{H_\alpha}X = \begin{cases} 0, & X \in \mathfrak{a}, \\ -\frac{1}{16}X & X \in \mathfrak{p}_\alpha, \\ -\frac{1}{4}X & X \in \mathfrak{p}_{2\alpha}. \end{cases} \tag{4.1}$$

On the other hand, if $V \in \mathfrak{g}_\alpha$ is a vector such that $|(1-\theta)V| = 1$, the Jacobi operator $R_{(1-\theta)V}$ is given by

$$R_{(1-\theta)V}X = \begin{cases} 0, & X \in \mathbb{R}(1-\theta)V, \\ -\frac{1}{4}X, & X \in \mathfrak{a} \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)\mathbb{F}V) \oplus \mathfrak{p}_{2\alpha}, \\ -X, & X \in (1-\theta)\mathfrak{J}V. \end{cases} \tag{4.2}$$

It is clear from the above equations that a rank one symmetric space has negative curvature $-1 \leq \sec \leq -\frac{1}{4}$ and is quarter pinched.

4.2 Totally geodesic submanifolds of $\mathbb{F}\mathsf{H}^n$

In this section we recall the classification of complete totally geodesic submanifolds of the hyperbolic spaces $\mathbb{F}\mathsf{H}^n$ under investigation. Wolf [168] classified totally geodesic submanifolds of the compact rank one symmetric spaces, and we may apply the duality of symmetric spaces to derive our classification on hyperbolic spaces. See also [108, Section 5] for a detailed treatment in the case of $\mathbb{O}\mathsf{H}^2$. From this point until the end of the chapter, we denote by $\mathbb{R}\mathsf{H}^k(c)$ the real hyperbolic space of constant curvature $-c^{-2}$. In particular, we have $\mathbb{R}\mathsf{H}^k(1) = \mathbb{R}\mathsf{H}^k$.

Let M be the n-dimensional hyperbolic space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Every complete totally geodesic submanifold of M is congruent to exactly one of the examples given in Table 4.2.

Table 4.2: Complete totally geodesic submanifolds of $\mathbb{F}\mathsf{H}^n$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Ambient space	**Submanifold**	**Comments**
$\mathbb{R}\mathsf{H}^n$	$\mathbb{R}\mathsf{H}^k$	$2 \leq k \leq n$
$\mathbb{C}\mathsf{H}^n$	$\mathbb{R}\mathsf{H}^k(2)$	$2 \leq k \leq n$
	$\mathbb{C}\mathsf{H}^k$	$2 \leq k \leq n-1$
	$\mathbb{R}\mathsf{H}^2$	

The case of $M = \mathbb{R}\mathsf{H}^n$ is by far the simplest. For each $2 \leq k \leq n$ there exists a unique complete k-dimensional totally geodesic submanifold of M, and it is isometric to $\mathbb{R}\mathsf{H}^k$. This submanifold can be obtained as the orbit through $o = e\mathsf{K}$ of the subgroup $\mathsf{SO}^0(1,k) \subseteq \mathsf{SO}^0(1,n)$ embedded in the standard manner.

Let us consider the complex case. The submanifolds $\mathbb{R}\mathsf{H}^k(2)$ and $\mathbb{C}\mathsf{H}^k$ appear as the orbits of the subgroups $\mathsf{SO}^0(1,k) \subseteq \mathsf{SU}(1,k)$ through the origin. One sees that $\mathbb{R}\mathsf{H}^k(2)$ is a totally real submanifold of $\mathbb{C}\mathsf{H}^n$, whereas $\mathbb{C}\mathsf{H}^k$ is a complex submanifold. Apart from these, we also have a totally geodesic $\mathbb{C}\mathsf{H}^1 \cong \mathbb{R}\mathsf{H}^2$ appearing as an orbit of $\mathsf{SU}(1,1)$. Unlike the submanifolds $\mathbb{R}\mathsf{H}^k(2)$, we see that $\mathbb{R}\mathsf{H}^2$ is a complex submanifold of $\mathbb{C}\mathsf{H}^n$.

Let M be either the quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^n$ or the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$. Then every complete totally geodesic submanifold of M is congruent to exactly one of the examples given in Table 4.3.

Let us discuss the quaternionic case first. On the one hand, the first three examples correspond to the orbits through $o = e\mathsf{K}$ of the subgroups $\mathsf{SO}^0(1,k)$, $\mathsf{SU}(1,k)$ and $\mathsf{Sp}(1,k)$, all embedded in the standard manner. In particular, the submanifold $\mathbb{R}\mathsf{H}^k(2)$ is totally real in $\mathbb{H}\mathsf{H}^n$ and has constant curvature equal to $-1/4$, whereas $\mathbb{H}\mathsf{H}^k$ is quaternionic. On the other hand, the orbit $\mathsf{Sp}(1,1) \cdot o$ is a totally geodesic $\mathbb{H}\mathsf{H}^1 \cong \mathbb{R}\mathsf{H}^4$ of constant curvature equal to -1. Because $\mathbb{R}\mathsf{H}^4$ has constant curvature, any k-dimensional real subspace of $T_o\mathbb{R}\mathsf{H}^4$ exponentiates to a totally geodesic $\mathbb{R}\mathsf{H}^k$ inside $\mathbb{R}\mathsf{H}^4$.

We now comment on the Cayley case. The submanifolds $\mathbb{R}\mathsf{H}^2(2) \subseteq \mathbb{C}\mathsf{H}^2 \subseteq \mathbb{H}\mathsf{H}^2$ appear as the orbits of the subgroups $\mathsf{SO}^0(1,2) \subseteq \mathsf{SU}(1,2) \subseteq \mathsf{Sp}(1,2)$ at o. The Lie algebra embedding

Table 4.3: Complete totally geodesic submanifolds of $\mathbb{F}\mathsf{H}^n$ with $\mathbb{F} \in \{\mathbb{H}, \mathbb{O}\}$.

Ambient space	**Submanifold**	**Comments**
$\mathbb{H}\mathsf{H}^n$	$\mathbb{R}\mathsf{H}^k(2)$	$2 \leq k \leq n$
	$\mathbb{C}\mathsf{H}^k(2)$	$2 \leq k \leq n$
	$\mathbb{H}\mathsf{H}^k$	$2 \leq k \leq n-1$
	$\mathbb{R}\mathsf{H}^k$	$2 \leq k \leq 4$
$\mathbb{O}\mathsf{H}^2$	$\mathbb{R}\mathsf{H}^2(2)$	
	$\mathbb{C}\mathsf{H}^2$	
	$\mathbb{H}\mathsf{H}^2$	
	$\mathbb{R}\mathsf{H}^k$	$2 \leq k \leq 8$

$\mathfrak{sp}(1,2) \hookrightarrow \mathfrak{f}_4^{-20}$ is given explicitly in [108, Proposition 5.3]. In addition, we have a totally geodesic $\mathbb{O}\mathsf{H}^1 \cong \mathbb{R}\mathsf{H}^8$ arising as the orbit of the action of an $\mathsf{SO}^0(1,8) \subseteq \mathsf{F}_4^{-20}$. Similarly to the quaternionic case, the fact that $\mathbb{O}\mathsf{H}^1$ has constant curvature means that every subspace of $T_o\mathbb{O}\mathsf{H}^1$ exponentiates to a totally geodesic submanifold isometric to $\mathbb{R}\mathsf{H}^k$.

4.3 Proofs of the main theorems

The rest of this chapter is focused on the proofs of Theorems A, B and C. This section is divided into three subsections, where each one is dedicated to one of the main theorems.

4.3.1 Proof of Theorem A

We now classify polar homogeneous foliations on $M = \mathbb{F}\mathsf{H}^2$. Our strategy and setup is similar to the one devised on Chapter 3. Note that items (i) and (iii) follow directly from Theorem B and [22]. Therefore, we are tasked with proving item (ii).

From now until the end of this subsection, we let $M = \mathbb{F}\mathsf{H}^2 = \mathsf{G}/\mathsf{K}$ be either the quaternionic or Cayley hyperbolic plane (where G and K are chosen as in Table 4.1) and we consider a closed connected subgroup $\mathsf{S} \subseteq \mathsf{G}$ that acts polarly on M inducing a homogeneous foliation. The action of S is assumed to be nontrivial and nontransitive. We let $\Sigma \subseteq M$ be the section through o of the action, and let $\mathfrak{s}_\mathfrak{p}^\perp$ be its tangent space. By virtue of Proposition 3.12, we can assume up to orbit equivalence that the Lie algebra $\mathfrak{s}$ of S is contained in a maximally noncompact subalgebra of the form $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, with $\mathfrak{t} \subseteq \mathfrak{k}_0$ an abelian subspace. We now define the vector subspace $\tilde{\mathfrak{s}} = \mathfrak{s} + \mathfrak{g}_{2\alpha}$, which is also a Lie subalgebra of $\mathfrak{g}$. Lemma 3.15 then shows that the projection of $\tilde{\mathfrak{s}}$ to $\mathfrak{a} \oplus \mathfrak{n}$ is not the whole $\mathfrak{a} \oplus \mathfrak{n}$. As a consequence, there exists a nonzero vector in $\mathfrak{a} \oplus \mathfrak{n}$ that is orthogonal to both $\mathfrak{s}$ and $\mathfrak{g}_{2\alpha}$. We may write this vector as $H+\xi$, where $H \in \mathfrak{a}$ and $\xi \in \mathfrak{g}_\alpha$, and because $H+\xi$ is orthogonal to $\mathfrak{s}$, we obtain that $H+(1-\theta)\xi$ is orthogonal to $\mathfrak{s}_\mathfrak{p}$. All things considered, we have shown the following:

Lemma 4.2. *The subspace* $\mathfrak{s}_\mathfrak{p}^\perp \cap (\mathfrak{a} \oplus \mathfrak{p}_\alpha)$ *is nonzero.*

Let us consider a nonzero vector $\xi \in \mathfrak{s}_\mathfrak{p}^\perp \cap (\mathfrak{a} \oplus \mathfrak{p}_\alpha)$. Again, we may decompose $\xi = aH_\alpha + (1-\theta)\xi_\alpha$ for some $a \in \mathbb{R}$ and $\xi_\alpha \in \mathfrak{g}_\alpha$.

If both a and ξ_α are nonzero, we may rescale ξ so that $a=1$. Define $g = \operatorname{Exp}(-\xi_\alpha/|\xi_\alpha|^2)$ and consider the conjugate subgroup $g\mathsf{S}g^{-1} \subseteq \mathsf{G}$. The actions of S and $g\mathsf{S}g^{-1}$ are obviously orbit equivalent, and the Lie algebra $\operatorname{Ad}(g)\mathfrak{s}$ of $g\mathsf{S}g^{-1}$ is orthogonal to

$$\begin{aligned}\operatorname{Ad}(g^{-1})^*(H_\alpha+\xi_\alpha) &= e^{\frac{1}{|\xi_\alpha|^2}\operatorname{ad}(\xi_\alpha)^*}(H_\alpha+\xi_\alpha) = e^{-\frac{1}{|\xi_\alpha|^2}\operatorname{ad}(\theta\xi_\alpha)}(H_\alpha+\xi_\alpha)\\ &\equiv H_\alpha+\xi_\alpha-\frac{|\xi_\alpha|^2}{|\xi_\alpha|^2}H_\alpha = \xi_\alpha \pmod{\theta\mathfrak{n}}.\end{aligned}$$

Since $\theta\mathfrak{n}$ is already orthogonal to $\operatorname{Ad}(g)\mathfrak{s}$, we obtain that ξ_α is perpendicular to $\operatorname{Ad}(g)\mathfrak{s}$. In other words, we may assume (up to orbit equivalence) that either $\mathfrak{a} \subseteq \mathfrak{s}_\mathfrak{p}^\perp$ or that $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}_\alpha \neq 0$.

The case $\mathfrak{a} \subseteq \mathfrak{s}_\mathfrak{p}^\perp$

Let us suppose first that $H_\alpha \in \mathfrak{s}_\mathfrak{p}^\perp$. In particular, we see that $\mathfrak{s}_\mathfrak{p}^\perp$ is invariant under the Jacobi operator $R_{H_\alpha} = -\operatorname{ad}(H_\alpha)^2 \in \operatorname{End}(\mathfrak{p})$. As a consequence, $\mathfrak{s}_\mathfrak{p}^\perp$ can be decomposed into the eigenspaces of the restriction $R_{H_\alpha}|_{\mathfrak{s}_\mathfrak{p}^\perp}$, which are precisely the intersections of $\mathfrak{s}_\mathfrak{p}^\perp$ with the eigenspaces of R_{H_α}. Thus, from (4.1) we have the splitting

$$\mathfrak{s}_\mathfrak{p}^\perp = \mathfrak{a} \oplus (1-\theta)\mathfrak{v} \oplus (1-\theta)\mathfrak{w},$$

where $\mathfrak{v}$ is a real subspace of $\mathfrak{g}_\alpha$ and $\mathfrak{w}$ is a subspace of $\mathfrak{g}_{2\alpha}$.

We first show that $\mathfrak{v}$ (and therefore $\mathfrak{g}_\alpha \ominus \mathfrak{v}$) is invariant under J_X for every $X \in \mathfrak{w}$. Let $U \in \mathfrak{v}$ and $X \in \mathfrak{w}$. Because the action of S is polar, we see that $(1+\theta)J_X U = [(1-\theta)U, (1-\theta)X] \perp \mathfrak{s}$, and since $\theta\mathfrak{n}$ is perpendicular to $\mathfrak{s}$, we conclude that $J_X U$ is perpendicular to $\mathfrak{s}$. This gives $J_X U \in \mathfrak{v}$, as desired.

Now, note that the maximum dimension of a totally geodesic submanifold of M is equal to $\frac{1}{2}\dim M = \dim \mathfrak{g}_\alpha$ (corresponding to $\mathbb{R}\mathsf{H}^4 \subseteq \mathbb{H}\mathsf{H}^2$, $\mathbb{C}\mathsf{H}^2 \subseteq \mathbb{H}\mathsf{H}^2$ and $\mathbb{R}\mathsf{H}^8 \subseteq \mathbb{O}\mathsf{H}^2$). Since the dimension of Σ is $1+\dim\mathfrak{v}+\dim\mathfrak{w}$, we see that $\mathfrak{v} \neq \mathfrak{g}_\alpha$, so we can choose a nonzero vector $V \in \mathfrak{g}_\alpha \ominus \mathfrak{v}$. Let $X \in \mathfrak{w}$ be any vector. The elements V and $J_X V$ belong to $\mathfrak{g}_\alpha \ominus \mathfrak{v} \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, so there exist T, $T' \in \mathfrak{t}$ satisfying $T+V$, $T'+J_X V \in \mathfrak{s}$. In particular, we see that $[T, J_X V] + [V, T'] + 2|V|^2|\alpha|^2 X = [T+V, T'+J_X V] \in \mathfrak{s}$. Taking the inner product with $(1-\theta)X \in \mathfrak{s}_\mathfrak{p}^\perp$ we deduce that $2|V|^2|\alpha|^2|X|^2 = 0$. As a consequence, $X=0$, which means that $\mathfrak{w}$ is trivial.

Finally, we claim that $\mathfrak{v}$ is at most one-dimensional. Indeed, suppose that $\mathfrak{v} \neq 0$. Then the restriction of R_{H_α} to $\mathfrak{s}_\mathfrak{p}^\perp \ominus \mathfrak{a} = (1-\theta)\mathfrak{v}$ is a multiple of the identity map. As Σ is a symmetric space of rank one, we deduce from this that Σ is a space of constant curvature. Now, the sectional curvature of Σ is $-\frac{1}{4}$, and a look at Table 4.3 reveals that Σ is congruent to $\mathbb{R}\mathsf{H}^2(2)$. In particular, $\mathfrak{v}$ is one-dimensional.

In conclusion, the normal space $\mathfrak{s}_\mathfrak{p}^\perp$ is either $\mathfrak{a}$ or $\mathfrak{a} \oplus (1-\theta)\ell$ for a line $\ell \subseteq \mathfrak{g}_\alpha$.

We can summarize our discussion in the following result:

Proposition 4.3. *If $\mathfrak{a} \subseteq \mathfrak{s}_\mathfrak{p}^\perp$, then the tangent space $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ is of the form $\mathfrak{n} \ominus \mathfrak{v}$, where $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ is a subspace such that* $\dim \mathfrak{v} \leq 1$.

The case $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}_\alpha \neq 0$

Now we suppose that there exists a nonzero vector $\xi \in \mathfrak{g}_\alpha$ with $(1-\theta)\xi \in \mathfrak{s}_\mathfrak{p}^\perp$. Without loss of generality, we may choose ξ so that $|\xi| = 1$. Once again, we know that $\mathfrak{s}_\mathfrak{p}^\perp$ is invariant under the Jacobi operator $R_{(1-\theta)\xi}$, so using (4.2) we may write

$$\mathfrak{s}_\mathfrak{p}^\perp = (1-\theta)(\mathbb{R}\xi \oplus \mathfrak{v}) \oplus \mathfrak{w},$$

where $\mathfrak{v} \subseteq \mathfrak{g}_\alpha \ominus \mathbb{R}\xi = \mathfrak{J}\xi$ is a real subspace and $\mathfrak{w} \subseteq \mathfrak{a} \oplus \mathfrak{p}_{2\alpha}$.

We start by claiming that the space $\mathbb{R}\xi \oplus \mathfrak{v}$ is invariant under J_X for all $X \in \mathfrak{w}_{\mathfrak{g}_{2\alpha}}$. To see this, let $\eta \in \mathbb{R}\xi \oplus \mathfrak{v}$ and $tH_\alpha + (1-\theta)X \in \mathfrak{w}$, where $t \in \mathbb{R}$ and $X \in \mathfrak{g}_{2\alpha}$. Since the action of S is polar, we see that $(1+\theta)(\frac{t}{4}\eta - J_X\eta) = [tH_\alpha + (1-\theta)X, (1-\theta)\eta]$ is orthogonal to $\mathfrak{s}$. This combined with the fact that $\theta\mathfrak{n}$ and η are already orthogonal to $\mathfrak{s}$ gives $J_X\eta \perp \mathfrak{s}$, so $J_X\eta \in \mathbb{R}\xi \oplus \mathfrak{v}$, as desired. In particular, the skew-symmetry of J_X also implies that $\mathfrak{g}_\alpha \ominus (\mathbb{R}\xi \oplus \mathfrak{v})$ is J_X-invariant. Because $\dim \Sigma \leq \frac{1}{2}\dim M = \dim \mathfrak{g}_\alpha$, either $\mathfrak{s}_\mathfrak{p}^\perp = \mathfrak{p}_\alpha$ or the orthogonal complement $\mathfrak{g}_\alpha \ominus (\mathbb{R}\xi \oplus \mathfrak{v})$ is nonzero.

Lemma 4.4. *If the normal space $\mathfrak{s}_\mathfrak{p}^\perp$ is contained in $\mathfrak{p}_\alpha$, then* $\dim \mathfrak{s}_\mathfrak{p}^\perp = 1$. *In particular, $\mathfrak{s}_\mathfrak{p}^\perp$ is not equal to $\mathfrak{p}_\alpha$.*

Proof. Suppose $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{p}_\alpha$ and $\dim \mathfrak{s}_\mathfrak{p}^\perp \geq 2$. Observe that both $\mathfrak{a}$ and $\mathfrak{g}_{2\alpha}$ are contained in $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. Applying Lemma 3.14 to the root 2α, we deduce that $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{s}$. Now, because $\dim \mathfrak{s}_\mathfrak{p}^\perp \geq 2$ and $\mathfrak{g}_\alpha \ominus \mathbb{R}\xi = \mathfrak{J}\xi$, we may find a nonzero vector $X \in \mathfrak{g}_{2\alpha}$ such that $(1-\theta)J_X\xi \in \mathfrak{s}_\mathfrak{p}^\perp$. As a consequence, the vector $(1+\theta)(\frac{1}{2}X - [\theta\xi, J_X\xi]) = [(1-\theta)\xi, (1-\theta)J_X\xi]$ is perpendicular to $\mathfrak{s}$ due to the polarity of the action. This contradicts the fact that $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{s}$, so necessarily $\dim \mathfrak{s}_\mathfrak{p}^\perp = 1$. ☐

Lemma 4.5. *The subspace $\mathfrak{w}$ is contained in $\mathfrak{a}$.*

Proof. It suffices to prove that the projection of $\mathfrak{w}$ onto $\mathfrak{g}_{2\alpha}$ is trivial. Because of Lemma 4.4 and its preceding discussion, we know that there exists a nonzero vector $V \in \mathfrak{g}_\alpha \ominus (\mathbb{R}\xi \oplus \mathfrak{v})$. Now, let $tH_\alpha + (1-\theta)X \in \mathfrak{w}$, where $t \in \mathbb{R}$ and $X \in \mathfrak{g}_{2\alpha}$. Recall that $\mathfrak{g}_\alpha \ominus (\mathbb{R}\xi \oplus \mathfrak{v}) \subseteq \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ is invariant under J_X, so we deduce that V, $J_XV \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. As a consequence, we may select vectors $T, T' \in \mathfrak{t}$ such that $T + V, T' + J_XV \in \mathfrak{s}$. Therefore, we have $[T, J_XV] + [V, T'] + \frac{1}{2}|V|^2X = [T + V, T' + J_XV] \in \mathfrak{s}$, and we obtain

$$0 = \big\langle [T, J_XV] + [V, T'] + \frac{1}{2}|V|^2X, tH_\alpha + (1-\theta)X \big\rangle = \frac{1}{2}|V|^2|X|^2,$$

thus forcing $X = 0$. This proves the desired assertion. ☐

Because $\mathfrak{w}$ is contained in the one-dimensional space $\mathfrak{a}$, we see that either $\mathfrak{w} = \mathfrak{a}$ or $\mathfrak{w} = 0$. The case $\mathfrak{w} = \mathfrak{a}$ was already dealt with in Section 4.3.1, so we may assume directly that $\mathfrak{w} = 0$, giving $\mathfrak{s}_{\mathfrak{p}}^{\perp} = (1-\theta)(\mathbb{R}\xi \oplus \mathfrak{v}) \subseteq \mathfrak{p}_{\alpha}$. Lemma 4.4 then implies that $\dim \mathfrak{s}_{\mathfrak{p}}^{\perp} = 1$, so $\mathfrak{v} = 0$ and $\mathfrak{s}_{\mathfrak{p}}^{\perp} = \mathbb{R}(1-\theta)\xi$. We therefore conclude with:

Proposition 4.6. *If $\mathfrak{s}_{\mathfrak{p}}^{\perp} \cap \mathfrak{p}_{\alpha} \neq 0$, then the tangent space $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ is of the form $(\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \ell)$, where $\mathfrak{b} \subseteq \mathfrak{a}$ is a vector subspace and $\ell \subseteq \mathfrak{p}_{\alpha}$ is a line.*

We are now ready to conclude the proof of Theorem A(ii). Indeed, by combining Propositions 4.3 and 4.6 we deduce that, up to orbit equivalence, the tangent space $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$ is of the form $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}}$ for a subspace $\mathfrak{b} \subseteq \mathfrak{a}$ and a subspace $\mathfrak{v} \subseteq \mathfrak{p}_{\alpha}$ of dimension at most one. Therefore, S falls under the hypotheses of Proposition 3.19 (with $\mathfrak{z} = \mathfrak{a} \ominus \mathfrak{b}$ and $\mathfrak{v}_{\alpha} = \mathfrak{v}$ in the notation therein). This guarantees that the orbits of S are equal to the orbits of the connected subgroup $\mathsf{S}_{\mathfrak{b},\mathfrak{v}} \subseteq \mathsf{G}$ whose Lie algebra is $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}}$.

4.3.2 Proof of Theorem B

Our next goal is to prove that any section of a polar action on a (quaternionic or Cayley) hyperbolic space has necessarily constant curvature.

Let $\mathsf{S} \curvearrowright M$ be a polar action on $M \in \{\mathbb{H}\mathsf{H}^n, \mathbb{O}\mathsf{H}^2\}$. We split the proof into two cases: the case where S induces a foliation and the case where S has a singular orbit.

In the setting of polar foliations, we only need to focus on the quaternionic case, as we already know that every polar homogeneous foliation on $\mathbb{O}\mathsf{H}^2$ has a section of constant curvature. We first need the following "reverse" version of the J^2 condition.

Lemma 4.7. *Let $M = \mathbb{F}\mathsf{H}^n$ be a hyperbolic space with $\mathbb{F} \in \{\mathbb{H}, \mathbb{O}\}$. Given nonzero vectors $V \in \mathfrak{g}_{\alpha}$ and $Z \in \mathfrak{g}_{2\alpha}$, there exist elements X, $Y \in \mathfrak{g}_{2\alpha}$ such that $\{X, Y, Z\}$ is an orthogonal set and $J_Z V = J_X J_Y V$.*

Proof. Choose a nonzero vector $X \in \mathfrak{g}_{2\alpha}$ orthogonal to Z, which is possible since $\dim \mathfrak{g}_{2\alpha} > 1$. Then the usual J^2 condition implies that there exists a $Y' \in \mathfrak{g}_{2\alpha}$ such that $J_X J_Z V = J_{Y'} V$. Thus, $J_Z V = -\frac{2}{|X|^2} J_X J_{Y'} V = J_X J_Y V$, where $Y = -2Y'/|X|^2$. Note that $J_{Y'} V$ is orthogonal to both $J_X V$ and $J_Z V$, so the following equations hold:

$$0 = \langle J_Z V, J_{Y'} V \rangle = \frac{1}{2}|V|^2 \langle Z, Y' \rangle, \qquad 0 = \langle J_X V, J_{Y'} V \rangle = \frac{1}{2}|V|^2 \langle X, Y' \rangle.$$

We deduce that $\{X, Y, Z\}$ is an orthogonal set, as required. □

Proposition 4.8. *Suppose $\mathsf{S} \subseteq \mathsf{Sp}(1,n)$ is a closed connected subgroup of $\mathsf{Sp}(1,n)$ acting polarly on $M = \mathbb{H}\mathsf{H}^n$ in such a way that its orbits form a homogeneous foliation. Let Σ be a section of the action. Then either Σ is a space of constant curvature or the action of S is trivial.*

Proof. From the classification of totally geodesic submanifolds in $\mathbb{H}\mathsf{H}^n$ (see Table 4.3), it suffices to show that Σ is not isometric to a complex or quaternionic hyperbolic space.

Firstly, let us suppose that $\Sigma = \mathbb{H}\mathsf{H}^k$ for some $k \in \{2, \dots, n\}$. This means that Σ is a quaternionic submanifold of M. In addition, if $p \in \Sigma$ is any point, the subspace $T_p(\mathsf{S} \cdot p) = \nu_p \Sigma$ is also quaternionic, so we see that the orbits of the action $\mathsf{S} \curvearrowright M$ are quaternionic. Because a quaternionic submanifold of a quaternionic Kähler manifold is totally geodesic [76, Theorem 5], we deduce that $\mathsf{S} \cdot p$ is a totally geodesic submanifold of M for all $p \in M$. If the action of S is not transitive, then any two distinct orbits $\mathsf{S} \cdot p$ and $\mathsf{S} \cdot q$ are totally geodesic (hence minimal), and [3, Corollary 5.2] guarantees that both $\mathsf{S} \cdot p$ and $\mathsf{S} \cdot q$ consist of one point. Since p and q are arbitrary, we conclude that the action of S is trivial.

Now, assume that $\Sigma = \mathbb{C}\mathsf{H}^k$ for some $2 \leq k \leq n$. Arguing as in the beginning of Section 4.3.1, we can assume (up to orbit equivalence) that S is solvable and its Lie algebra $\mathfrak{s}$ is contained in a maximally noncompact subalgebra of the form $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{t} \subseteq \mathfrak{k}_0$ is abelian. Additionally, if the action of S is not transitive, then we can also assume that either $\mathfrak{a} \subseteq \mathfrak{s}_\mathfrak{p}^\perp$ of that $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}_\alpha \neq 0$.

Suppose that $\mathfrak{a} \subseteq \mathfrak{s}_\mathfrak{p}^\perp$. Then the restriction of the Jacobi operator R_{H_α} to $\mathfrak{s}_\mathfrak{p}^\perp$ has the same spectral decomposition as that of the Jacobi operator of a vector in $T\mathbb{C}\mathsf{H}^k$. In other words, there exists a vector $X \in \mathfrak{g}_{2\alpha}$ and a subspace $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ such that $\mathfrak{s}_\mathfrak{p}^\perp = \mathfrak{a} \oplus (1-\theta)\mathfrak{v} \oplus \mathbb{R}(1-\theta)X$. As the action of S is polar, we have for every $\xi \in \mathfrak{v}$ that $(1+\theta)J_X\xi = -(1+\theta)[\theta\xi, X] = [(1-\theta)\xi, (1-\theta)X]$ is orthogonal to $\mathfrak{s}$. This implies that $\mathfrak{v}$ is invariant under J_X, and so is the orthogonal complement $\mathfrak{g}_\alpha \ominus \mathfrak{v}$. As the dimension of Σ is at most $2n < \dim \mathfrak{g}_\alpha$, we have $\mathfrak{v} \neq \mathfrak{g}_\alpha$. Choose a nonzero vector $V \in \mathfrak{g}_\alpha$ that is orthogonal to $\mathfrak{v}$. We have V, $J_X V \in \mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$, so we may choose elements T, $T' \in \mathfrak{t}$ such that $T + V$, $T' + J_X V \in \mathfrak{s}$. Its bracket

$$[T, J_X V] + [V, T'] + \tfrac{1}{2}|V|^2 X = [T + V, T' + J_X V]$$

is also in $\mathfrak{s}$. This readily gives a contradiction with the fact that $(1-\theta)X$ is orthogonal to $\mathfrak{s}$.

Finally, assume $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}_\alpha \neq 0$. We consider a vector $\xi \in \mathfrak{g}_\alpha$ such that $|\xi| = 1$ and $(1-\theta)\xi \in \mathfrak{s}_\mathfrak{p}^\perp$. The restriction of the Jacobi operator $R_{(1-\theta)\xi}$ to $\mathfrak{s}_\mathfrak{p}^\perp$ has the same eigenvalues (with multiplicities) as the Jacobi operator of a unit vector in $T\mathbb{C}\mathsf{H}^k$, and this yields $\mathfrak{s}_\mathfrak{p}^\perp = \mathbb{R}(1-\theta)\xi \oplus \mathbb{R}(1-\theta)J_Z\xi \oplus \mathfrak{w}$, where $Z \in \mathfrak{g}_{2\alpha}$ is a nonzero vector and $\mathfrak{w}$ is a real subspace of $\mathfrak{a} \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)\mathbb{H}\xi) \oplus \mathfrak{p}_{2\alpha}$. Owing to Lemma 4.7, we may choose vectors X, $Y \in \mathfrak{g}_{2\alpha}$ such that $\{X, Y, Z\}$ is an orthogonal set and $J_Z\xi = J_X J_Y \xi$. In particular, the vectors $J_X\xi$ and $J_Y\xi$ are perpendicular to $\mathfrak{s}_\mathfrak{p}^\perp$, which means that $J_X\xi$, $J_Y\xi$ are in $\mathfrak{s}_{\mathfrak{a}\oplus\mathfrak{n}}$. Choose elements T_X, $T_Y \in \mathfrak{t}$ satisfying $T_X + J_X\xi$, $T_Y + J_Y\xi \in \mathfrak{s}$. Then we have $[T_X, J_Y\xi] + [J_X\xi, T_Y] + [J_X\xi, J_Y\xi] = [T_X + J_X\xi, T_Y + J_Y\xi] \in \mathfrak{s}$. Observe that

$$\begin{aligned} [J_X\xi, J_Y\xi] &= -\frac{2}{|X|^2}[J_X\xi, J_X^2 J_Y\xi] = -\frac{2}{|X|^2}[J_X\xi, J_X J_Z\xi] \\ &= \frac{2}{|X|^2}[J_X\xi, J_Z J_X\xi] = \frac{|J_X\xi|^2}{|X|^2} Z = \frac{1}{2}Z, \end{aligned} \tag{4.3}$$

so we deduce that the vector $[T_X, J_Y\xi] + [J_X\xi, T_Y] + \frac{1}{2}Z$ is in $\mathfrak{s}$. Moreover, as the action of S is polar, the vector $(1+\theta)\left(\frac{1}{2}Z - [\theta\xi, J_Z\xi]\right) = [(1-\theta)\xi, (1-\theta)J_Z\xi]$ is perpendicular to $\mathfrak{s}$. As a consequence, we obtain

$$0 = \left\langle [T_X, J_Y\xi] + [J_X\xi, T_Y] + \frac{1}{2}Z, (1+\theta)\left(\frac{1}{2}Z - [\theta\xi, J_Z\xi]\right)\right\rangle = \frac{1}{4}|Z|^2,$$

contradicting the fact that Z is nonzero.

We have seen that the two possible situations give rise to a contradiction, so we conclude that no polar homogeneous foliation on M can have a complex hyperbolic space as its section. This finishes the proof. □

We now consider the case in which S acts with singular orbits. It was shown by Kollross [108, Lemma 6.6] that no polar action on $\mathbb{O}\mathsf{H}^2$ admitting a singular orbit can have $\mathbb{C}\mathsf{H}^2$ or $\mathbb{H}\mathsf{H}^2$ as its section. His proof is actually valid for the quaternionic case as well:

Proposition 4.9. *Let M be either the quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^n$ or the Cayley hyperbolic plane $\mathbb{O}\mathsf{H}^2$. If S is a Lie group acting polarly on M admitting a singular orbit and $\Sigma \subseteq M$ is a section of the action $\mathsf{S} \curvearrowright M$, then Σ has constant curvature.*

Proof. Let $p \in \Sigma$ be a point such that $\mathsf{S} \cdot p$ is singular. Such a p exists because Σ meets all orbits. The slice representation $\mathsf{S}_p \curvearrowright \nu_p(\mathsf{S} \cdot p)$ is a nontrivial polar representation with section $T_p\Sigma$. In particular, the polar group $\Pi(T_p\Sigma) = \Pi(\Sigma)_p$ is a nontrivial finite group generated by reflections along hyperplanes of $T_p\Sigma$.

Let $g \in \mathsf{N}_{\mathsf{S}_p}(T_p\Sigma)$ be an element such that $g_{*p} \in \mathsf{O}(T_p\Sigma)$ is the reflection along a hyperplane $V \subseteq T_p\Sigma$. Then $g \cdot \Sigma = \Sigma$ and the restriction $g \colon \Sigma \to \Sigma$ is an involutive isometry. Denote by $\mathcal{H} \subseteq \Sigma$ the connected component of $\mathrm{Fix}(g)$ containing p. We have that $\mathcal{H}$ is a totally geodesic hypersurface of Σ whose tangent space at p is V. From Tables 4.2 and 4.3 we see that the only totally geodesic submanifolds of M admitting a totally geodesic hypersurfaces are those of constant curvature, so our claim follows. □

Theorem B now follows from combining the information in Table 4.3 and Propositions 4.8 and 4.9.

4.3.3 Proof of Theorem C

We now work on classifying polar homogeneous foliations on the quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^n = \mathsf{Sp}(1,n)/(\mathsf{Sp}(1) \times \mathsf{Sp}(n))$ coming from a connected subgroup of AN. Following Solonenko [155], we say that a homogeneous foliation $\mathcal{F}$ on a symmetric space of noncompact type $M = \mathsf{G}/\mathsf{K}$ is *standard* if there exists an Iwasawa decomposition $\mathsf{G} = \mathsf{KAN}$ for which $\mathcal{F}$ is the orbit foliation induced by a connected subgroup $\mathsf{H} \subseteq \mathsf{AN}$.

We begin by studying the actions of the subgroups $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ and the mean curvature of their orbits. Because these groups are analogues of the ones constructed in Chapter 3, it should come as no surprise that the calculations presented here are very similar to those in Section 3.1.

Let us show that item (i) is satisfied.

Proposition 4.10. *Let $M = \mathbb{H}\mathsf{H}^n$ be the quaternionic hyperbolic space. Consider a vector subspace $\mathfrak{b}$ of $\mathfrak{a}$ and a totally real subspace $\mathfrak{v}$ of $\mathfrak{g}_\alpha$. Then, the subspace $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$ is a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Moreover, the connected subgroup $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ of $\mathsf{Sp}(1,n)$ with Lie algebra $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}}$ acts freely and properly on M in such a way that its orbits form a polar homogeneous foliation on M. Any section of the action $\mathsf{S}_{\mathfrak{b},\mathfrak{v}} \curvearrowright M$ is congruent to $\mathbb{R}\mathsf{H}^k(2)$.*

Proof. The fact that $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}}$ is a Lie subalgebra of $\mathfrak{g}$ contained in $\mathfrak{a} \oplus \mathfrak{n}$ is immediate. To show that its corresponding group $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ acts polarly, we make use of Proposition 2.20. The normal space of $\mathsf{S}_{\mathfrak{b},\mathfrak{v}} \cdot o$ at o is $(\mathfrak{s}_{\mathfrak{b},\mathfrak{v}})^{\perp}_{\mathfrak{p}} = \mathfrak{b} \oplus (1-\theta)\mathfrak{v}$, which is totally real in $\mathfrak{p}$ because $\mathfrak{v}$ is totally real and the quaternionic structure of $\mathfrak{p}$ sends $\mathfrak{a}$ to $\mathfrak{p}_{2\alpha}$. In particular, $(\mathfrak{s}_{\mathfrak{b},\mathfrak{v}})^{\perp}_{\mathfrak{p}}$ is a Lie triple system in $\mathfrak{p}$ whose corresponding totally geodesic submanifold Σ is a totally real $\mathbb{R}\mathsf{H}^k(2)$.

It remains to check that $\left[(\mathfrak{s}_{\mathfrak{b},\mathfrak{v}})^{\perp}_{\mathfrak{p}}, (\mathfrak{s}_{\mathfrak{b},\mathfrak{v}})^{\perp}_{\mathfrak{p}}\right]$ is orthogonal to $\mathfrak{s}$. Let $\xi, \eta \in \mathfrak{g}_\alpha$ be two commuting orthogonal vectors. Then we have

$$[(1-\theta)\xi, (1-\theta)\eta] = -(1+\theta)[\theta\xi, \eta] \in \mathfrak{k}_0.$$

Furthermore, $[H_\alpha, (1-\theta)\xi] = (1+\theta)\frac{1}{4}\xi$. As a consequence, we have $\left[(\mathfrak{s}_{\mathfrak{b},\mathfrak{v}})^{\perp}_{\mathfrak{p}}, (\mathfrak{s}_{\mathfrak{b},\mathfrak{v}})^{\perp}_{\mathfrak{p}}\right] \subseteq \mathfrak{k}_0 + (1+\theta)\mathfrak{v}$, which clearly implies the desired condition.

We thus conclude that the action of $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ induces a polar homogeneous foliation with section $\Sigma = \mathbb{R}\mathsf{H}^2(2)$, as desired. □

Geometry of the orbits of $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$

We now study the extrinsic geometry of the orbits of our examples, with a view towards proving item (iii) in Theorem C. This time, instead of using the solvable model $M = \mathsf{AN}$ to study the second fundamental form of the orbits, we make use of the formula for the second fundamental form developed in Lemma 1.2. A direct consequence of (1.6) is that if $M = \mathsf{G}/\mathsf{K}$ is a Riemannian symmetric space and S is a Lie subgroup of G, then the second fundamental form $\mathbb{II}$ of $\mathsf{S} \cdot o$ at o satisfies

$$\mathbb{II}((1-\theta)X, (1-\theta)X) = [(1+\theta)X, (1-\theta)X]_{\mathfrak{s}^{\perp}_{\mathfrak{p}}}, \quad \text{for all } X \in \mathfrak{s}.$$

We start by considering the subgroups of the form $\mathsf{S} = \mathsf{S}_{\mathfrak{a},\mathfrak{v}}$. The Lie algebra $\mathfrak{s} = \mathfrak{s}_{\mathfrak{a},\mathfrak{v}} = \mathfrak{n} \ominus \mathfrak{v}$ is an ideal of $\mathfrak{a} \oplus \mathfrak{n}$, meaning that S is a normal subgroup of AN. Thus, the orbits of S are mutually congruent under isometries in AN, so in order to understand their extrinsic geometry we only need to focus on the orbit $\mathsf{S} \cdot o$.

Let us compute the mean curvature vector $\mathcal{H}_o$ of $\mathsf{S} \cdot o$ at o. Take a vector $U \in \mathfrak{g}_\alpha \ominus \mathfrak{v}$ such that $2|U|^2 = |(1-\theta)U|^2 = 1$. Then we see that the second fundamental form of $\mathsf{S} \cdot o$ at o satisfies

$$\begin{aligned}\mathbb{II}((1-\theta)U, (1-\theta)U) &= [(1+\theta)U, (1-\theta)U]_{\mathfrak{s}^{\perp}_{\mathfrak{p}}} = ((1-\theta)[U+\theta U, U])_{\mathfrak{s}^{\perp}_{\mathfrak{p}}} \\ &= 2|U|^2 H_\alpha = H_\alpha.\end{aligned}$$

Similarly, for a vector $X \in \mathfrak{g}_{2\alpha}$ such that $(1-\theta)X$ unit length, we obtain

$$\begin{aligned}\mathbb{II}((1-\theta)X, (1-\theta)X) &= [(1+\theta)X, (1-\theta)X]_{\mathfrak{s}^{\perp}_{\mathfrak{p}}} = ((1-\theta)[\theta X, X])_{\mathfrak{s}^{\perp}_{\mathfrak{p}}} \\ &= 4|X|^2 H_\alpha = 2H_\alpha.\end{aligned}$$

Therefore, the mean curvature vector of $\mathsf{S} \cdot o$ at o is $\mathcal{H}_o = (\dim(\mathfrak{g}_\alpha \ominus \mathfrak{v}) + 6)H_\alpha$.

We summarize this discussion as follows.

Proposition 4.11. *Let $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ be a totally real subspace. The orbits of the action $\mathsf{S}_{\mathfrak{a},\mathfrak{v}} \curvearrowright \mathbb{H}\mathsf{H}^n$ are all mutually congruent. Moreover, the mean curvature vector $\mathcal{H}_o$ of the orbit $\mathsf{S}_{\mathfrak{a},\mathfrak{v}} \cdot o$ at o is given by*

$$\mathcal{H}_o = (\dim(\mathfrak{g}_\alpha \ominus \mathfrak{v}) + 6)H_\alpha = (4n - k + 3)H_\alpha,$$

where $k = \dim \mathfrak{v} + 1$ is the cohomogeneity of the action $\mathsf{S}_{\mathfrak{a},\mathfrak{v}} \curvearrowright \mathbb{H}\mathsf{H}^n$.

We proceed to study the actions of the groups $\mathsf{S}_{0,\mathfrak{v}}$. Fix a totally real subspace $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ and let $\mathsf{S} = \mathsf{S}_{0,\mathfrak{v}}$. The section $\Sigma = \exp_o(\mathfrak{s}_\mathfrak{p}^\perp) = \exp_o((1-\theta)\mathfrak{v})$ meets every orbit exactly once, meaning that each S-orbit has a unique representative in Σ. Furthermore, it is not hard to show that, for each $\xi \in \mathfrak{v}$, the subspace $\mathfrak{c} = \operatorname{span}\{H_\alpha, \xi\}$ of $\mathfrak{a} \oplus \mathfrak{n}$ is a Lie subalgebra and the orbit $\mathsf{C} \cdot o$ of its corresponding connected subgroup $\mathsf{C} \subseteq \mathsf{AN}$ is a totally geodesic $\mathbb{R}\mathsf{H}^2(2)$ inside $\mathbb{H}\mathsf{H}^n$. This means that every point of Σ can be written as $\operatorname{Exp}(uH_\alpha + \xi) \cdot o$ for some $u \in \mathbb{R}$ and $\xi \in \mathfrak{v}$.

Let us fix a vector $\xi \in \mathfrak{g}_\alpha$ with $|\xi| = 1$ and focus on the points $\operatorname{Exp}(t\xi) \cdot o \in \Sigma$. We aim to compute the mean curvature of $\mathsf{S} \cdot (\operatorname{Exp}(t\xi) \cdot o)$ at $\operatorname{Exp}(t\xi) \cdot o$. Writing $g_t = \operatorname{Exp}(t\xi)$ and $\mathsf{S}_t = g_t^{-1}\mathsf{S} g_t$, we have $\mathsf{S} \cdot (g_t \cdot o) = g_t \cdot (\mathsf{S}_t \cdot o)$, so the orbits $\mathsf{S} \cdot g_t$ and $\mathsf{S}_t \cdot o$ are congruent under g_t. The Lie algebra $\mathfrak{s}_t = \operatorname{Ad}(g_t^{-1})\mathfrak{s}$ is contained in $\mathfrak{a} \oplus \mathfrak{n}$ and orthogonal to $\operatorname{Ad}(g_t)^*\mathfrak{v} = e^{-t \operatorname{ad}(\theta\xi)}\mathfrak{v}$. Observe that if $\eta \in \mathfrak{v} \ominus \mathbb{R}\xi$, we have

$$e^{-t\operatorname{ad}(\theta\xi)}\xi \equiv \xi - tH_\alpha \pmod{\theta\mathfrak{n}}, \qquad e^{-t\operatorname{ad}(\theta\xi)}\eta \equiv \eta \pmod{\mathfrak{k}_0 \oplus \theta\mathfrak{n}},$$

meaning that $\mathfrak{s}_t$ is orthogonal to $\mathfrak{v} \ominus \mathbb{R}\xi$ and $\xi - tH_\alpha$. For dimensional reasons, we deduce that $\mathfrak{s}_t$ is the orthogonal complement of $\mathbb{R}(\xi - tH_\alpha) \oplus (\mathfrak{v} \ominus \mathbb{R}\xi)$ in $\mathfrak{a} \oplus \mathfrak{n}$. A straightforward calculation gives

$$\mathfrak{s}_t = \mathbb{R}(4H_\alpha + t\xi) \oplus (\mathfrak{g}_\alpha \ominus \mathfrak{v}) \oplus \mathfrak{g}_{2\alpha}.$$

Let us compute the mean curvature vector $\mathcal{H}'_t$ of $\mathsf{S}_t \cdot o$ at o. Firstly, if $U \in \mathfrak{g}_\alpha \ominus \mathfrak{v}$ is such that $|(1-\theta)U| = 1$, then

$$\begin{aligned}\mathrm{I\!I}((1-\theta)U, (1-\theta)U) &= [(1+\theta)U, (1-\theta)U]_{(\mathfrak{s}_t)_\mathfrak{p}^\perp} = (H_\alpha)_{(\mathfrak{s}_t)_\mathfrak{p}^\perp} \\ &= \frac{t}{8+t^2}(tH_\alpha - (1-\theta)\xi)\end{aligned}$$

Secondly, for any $X \in \mathfrak{g}_{2\alpha}$ we see that

$$\begin{aligned}\mathrm{I\!I}((1-\theta)X, (1-\theta)X) &= [(1+\theta)X, (1-\theta)X]_{(\mathfrak{s}_t)_\mathfrak{p}^\perp} = 2(H_\alpha)_{(\mathfrak{s}_t)_\mathfrak{p}^\perp} \\ &= \frac{2t}{8+t^2}(tH_\alpha - (1-\theta)\xi)\end{aligned}$$

Finally, for the tangent vector $8H_\alpha + t(1-\theta)\xi \in (\mathfrak{s}_t)_\mathfrak{p}$ we see that

$$\begin{aligned}\mathrm{I\!I}(8H_\alpha + t(1-\theta)\xi, 8H_\alpha + t(1-\theta)\xi) &= [t(1+\theta)\xi, 8H_\alpha + t(1-\theta)\xi]_{(\mathfrak{s}_t)_\mathfrak{p}^\perp} \\ &= t((1-\theta)[\xi, 8H_\alpha + t(1-\theta)\xi])_{(\mathfrak{s}_t)_\mathfrak{p}^\perp} \\ &= 2t(tH_\alpha - (1-\theta)\xi)_{(\mathfrak{s}_t)_\mathfrak{p}^\perp} \\ &= 2t(tH_\alpha - (1-\theta)\xi),\end{aligned}$$

meaning that for the unit vector $Z = \frac{1}{\sqrt{16+2t^2}}(8H_\alpha + t(1-\theta)\xi)$ we obtain

$$\mathrm{I\!I}(Z,Z) = \frac{t}{8+t^2}(tH_\alpha - (1-\theta)\xi).$$

Consequently, the mean curvature vector of $\mathsf{S}_t \cdot o$ at o is given by

$$\mathcal{H}'_t = \frac{t(7+\dim(\mathfrak{g}_\alpha \ominus \mathfrak{v}))}{8+t^2}(tH_\alpha - (1-\theta)\xi).$$

Proposition 4.12. *Let $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ be a totally real subspace and $\xi \in \mathfrak{v}$ a vector such that $\langle \xi, \xi \rangle = 1$. If $g_t = \operatorname{Exp}(t\xi) \in \mathsf{N}$, then the mean curvature vector $\mathcal{H}_t$ of the orbit $\mathsf{S}_{0,\mathfrak{v}} \cdot (g_t \cdot o)$ at $g_t \cdot o$ is given by the expression*

$$\begin{aligned}\mathcal{H}_t &= (g_t)_{*o}\left(\frac{t(7+\dim(\mathfrak{g}_\alpha \ominus \mathfrak{v}))}{8+t^2}(tH_\alpha - (1-\theta)\xi)\right)\\ &= (g_t)_{*o}\left(\frac{t(4n-k+3)}{8+t^2}(tH_\alpha - (1-\theta)\xi)\right),\end{aligned}$$

where $k = \dim \mathfrak{v}$ is the cohomogeneity of the action $\mathsf{S}_{0,\mathfrak{v}} \curvearrowright \mathbb{H}\mathsf{H}^n$.

A direct consequence of Proposition 4.12 is that the action of $\mathsf{S}_{0,\mathfrak{v}}$ possesses non-congruent orbits, as we have found a one-parameter family of orbits whose mean curvature vector has variable length. This combined with Proposition 4.11 gives that two subgroups of the form $\mathsf{S}_{0,\mathfrak{v}}$ and $\mathsf{S}_{\mathfrak{a},\mathfrak{v}}$ never induce orbit equivalent actions on the quaternionic hyperbolic space $\mathbb{H}\mathsf{H}^n$.

Now, the proof of Theorem C(iii) is straightforward. Given two subgroups $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ and $\mathsf{S}_{\mathfrak{b}',\mathfrak{v}'}$ whose actions on $\mathbb{H}\mathsf{H}^n$ are orbit equivalent, we see that necessarily $\mathfrak{b} = \mathfrak{b}'$. In addition, we must have $\dim \mathfrak{v} = \dim \mathfrak{v}'$ in order for $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ and $\mathsf{S}_{\mathfrak{b}',\mathfrak{v}'}$ to act with the same cohomogeneity. Conversely, if $\mathfrak{b} = \mathfrak{b}'$ and $\mathfrak{v} = \mathfrak{v}'$, then the groups $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ and $\mathsf{S}_{\mathfrak{b}',\mathfrak{v}'}$ are conjugate under an element of K_0 owing to Lemma 3.3, so their actions are automatically orbit equivalent. This concludes the proof.

Classification of standard polar homogeneous foliations

Let us finish the proof of Theorem C by showing that item (ii) is satisfied. Recall that we are only interested in classifying standard polar homogeneous foliations on $\mathbb{H}\mathsf{H}^n$. By definition, any foliation of this kind is induced by a connected subgroup of AN, which means that this problem is equivalent to determining the subgroups of AN that act polarly on $\mathbb{H}\mathsf{H}^n$. Therefore, to prove item (ii) it is necessary and sufficient to show:

Proposition 4.13. *Let $M = \mathbb{H}\mathsf{H}^n$ be the quaternionic hyperbolic space and $\mathsf{S} \subseteq \mathsf{AN}$ a connected subgroup whose orbits form a polar homogeneous foliation on M. Then there exists a subspace $\mathfrak{b} \subseteq \mathfrak{a}$ and a totally real subspace $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ such that the actions of S and $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$ are orbit equivalent.*

Proof. Recall from Lemma 4.2 that $\mathfrak{s}_\mathfrak{p}^\perp$ contains at least one vector in $\mathfrak{a} \oplus \mathfrak{p}_\alpha$, and we can suppose that either $\mathfrak{a} \subseteq \mathfrak{s}_\mathfrak{p}^\perp$ or $\mathfrak{s}_\mathfrak{p}^\perp \cap \mathfrak{p}_\alpha \neq 0$.

Firstly, assume $\mathfrak{a} \subseteq \mathfrak{s}_\mathfrak{p}^\perp$. In this case, using the fact that $\mathfrak{s}_\mathfrak{p}^\perp$ is invariant under R_{H_α}, we can write $\mathfrak{s}_\mathfrak{p}^\perp = \mathfrak{a} \oplus (1-\theta)(\mathfrak{v} \oplus \mathfrak{w})$, where $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$ and $\mathfrak{w} \subseteq \mathfrak{g}_{2\alpha}$. As $\mathfrak{s} \subseteq \mathfrak{a} \oplus \mathfrak{n}$, this forces $\mathfrak{s} = (\mathfrak{g}_\alpha \ominus \mathfrak{v}) \oplus (\mathfrak{g}_{2\alpha} \ominus \mathfrak{w})$. From Theorem B, we know that $\mathfrak{s}_\mathfrak{p}^\perp$ is tangent to a totally geodesic real hyperbolic space, which means that either $\mathfrak{v} = 0$ or $\mathfrak{w} = 0$. If $\mathfrak{v} = 0$, then from $\mathfrak{g}_{2\alpha} = [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \subseteq \mathfrak{s}$ we obtain $\mathfrak{w} = 0$, and this yields $\mathfrak{s} = \mathfrak{n}$. Now, if $\mathfrak{w} = 0$, we have $\mathfrak{s} = (\mathfrak{g}_\alpha \ominus \mathfrak{v}) \oplus \mathfrak{g}_{2\alpha}$. Let ξ, $\eta \in \mathfrak{v}$ be arbitrary. As the action of S is polar, we see that $(1+\theta)([\xi,\eta] - [\theta\xi,\eta]) = [(1-\theta)\xi, (1-\theta)\eta]$ is orthogonal to $\mathfrak{s}$. Note that $(1+\theta)[\theta\xi,\eta] \in \mathfrak{k}_0$ is automatically orthogonal to $\mathfrak{s}$, so from this we deduce that $[\xi,\eta] \perp \mathfrak{s}$. Because $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{s}$, this means that $[\xi,\eta] = 0$, so $\mathfrak{v}$ is an abelian subspace of $\mathfrak{s}$.

Secondly, suppose that $\mathfrak{s}_\mathfrak{p}^\perp$ contains a nonzero vector in $\mathfrak{p}_\alpha$. Choose an element $\xi \in \mathfrak{g}_\alpha$ such that $|\xi| = 1$ and $(1-\theta)\xi \in \mathfrak{s}_\mathfrak{p}^\perp$. As $\exp_o(\mathfrak{s}_\mathfrak{p}^\perp)$ is a space of constant curvature, it follows that $\mathfrak{s}_\mathfrak{p}^\perp$ is contained either in the quaternionic line spanned by $(1-\theta)\xi$ or in a totally real subspace of $\mathfrak{p}$.

We first deal with the case $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathbb{R}(1-\theta)\xi \oplus (1-\theta)\mathfrak{J}\xi \subseteq \mathfrak{p}_\alpha$. In particular, $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{s}$. If there exists an $X \in \mathfrak{g}_{2\alpha}$ such that $(1-\theta)J_X\xi \in \mathfrak{s}_\mathfrak{p}^\perp$, then the polarity of the action gives that $(1+\theta)\big(\frac{1}{2}X - [\theta\xi, J_X\xi]\big) = [(1-\theta)\xi, (1-\theta)J_X\xi]$ is orthogonal to $\mathfrak{s}$. Because $\mathfrak{g}_{2\alpha}$ is contained in $\mathfrak{s}$, this forces $X = 0$. As a consequence, $\mathfrak{s}_\mathfrak{p}^\perp = \mathbb{R}(1-\theta)\xi$ is one-dimensional and $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{R}\xi)$.

We now assume that $\mathfrak{s}_\mathfrak{p}^\perp$ is totally real. Choose any $X \in \mathfrak{g}_{2\alpha}$, and note that because $\mathfrak{s}_\mathfrak{p}^\perp$ is totally real and $\mathfrak{s} \subseteq \mathfrak{a} \oplus \mathfrak{n}$, we get $J_X\xi \in \mathfrak{s}$. This means that $[\mathfrak{J}\xi, \mathfrak{J}\xi]$ is contained in $\mathfrak{s}$, and (4.3) gives $[\mathfrak{J}\xi, \mathfrak{J}\xi] = \mathfrak{g}_{2\alpha}$. As a consequence, $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{a} \oplus \mathfrak{p}_\alpha$. If $\mathfrak{s}_\mathfrak{p}^\perp \subseteq \mathfrak{p}_\alpha$, then we are done. Otherwise, we can find an orthogonal basis of $\mathfrak{s}_\mathfrak{p}^\perp$ of the form $H_\alpha + (1-\theta)\nu$, $(1-\theta)\eta_1, \ldots, (1-\theta)\eta_k$, where ν and each η_i are in $\mathfrak{p}_\alpha$. Observe that the η_i commute, and the polarity of the S-action implies that for each $i = 1, \ldots, k$,

$$(1+\theta)\frac{1}{4}\eta_i + (1+\theta)([\nu,\eta_i] - [\theta\nu,\eta_i]) = [H_\alpha + (1-\theta)\nu, (1-\theta)\eta_i] \perp \mathfrak{s}.$$

Since η_i and $\mathfrak{k}_0 \oplus \theta\mathfrak{n}$ are orthogonal to $\mathfrak{s}$, the above condition reduces to $[\nu,\eta_i] \perp \mathfrak{s}$, and because $\mathfrak{g}_{2\alpha} \subseteq \mathfrak{s}$ we conclude $[\nu,\eta_i] = 0$. Define $g = \operatorname{Exp}\big(-\frac{1}{|\nu|^2}\nu\big) \in \mathsf{N}$. The action of S is orbit equivalent to the action of $g\mathsf{S}g^{-1} \subseteq \mathsf{AN}$ and the Lie algebra $\operatorname{Ad}(g)\mathfrak{s}$ of $g\mathsf{S}g^{-1}$ is orthogonal to the vectors

$$\begin{aligned}\operatorname{Ad}(g^{-1})^*(H_\alpha + \nu) &= e^{-\frac{1}{|\nu|^2}\operatorname{ad}(\theta\nu)}(H_\alpha + \nu) \equiv H_\alpha + \nu - H_\alpha = \nu \pmod{\theta\mathfrak{n}},\\ \operatorname{Ad}(g^{-1})^*\eta_i &= e^{-\frac{1}{|\nu|^2}\operatorname{ad}(\theta\nu)}\eta_i \equiv \eta_i - [\theta\nu,\eta_i] \pmod{\theta\mathfrak{n}}.\end{aligned}$$

Because $[\theta\nu,\eta_i] \in \mathfrak{k}_0$ is automatically orthogonal to $\operatorname{Ad}(g)\mathfrak{s}$, we deduce that $\operatorname{Ad}(g)\mathfrak{s}$ is perpendicular to the abelian subspace $\mathfrak{v} = \{\nu, \eta_1, \ldots, \eta_k\}$. For dimension reasons, this gives $\operatorname{Ad}(g)\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v})$, so we are done. □

The above proposition implies that item (ii) holds, thus putting an end to the proof of Theorem C.

Part II

Totally geodesic submanifolds

Chapter 5
Totally geodesic submanifolds of Riemannian manifolds

Recall that if $f\colon \Sigma \to M$ is an isometric immersion between Riemannian manifolds, then f is *totally geodesic* if for every geodesic γ of Σ the composition $f \circ \gamma$ is a geodesic of M. The aim of this chapter is to develop a general theory of totally geodesic immersions in Riemannian manifolds, paying special attention to the real analytic case. We follow the ideas in [17, Section 10.3], [86, Appendix A], [89] and [165]. Furthermore, we introduce a novel result characterizing which totally geodesic immersions to a real analytic Riemannian manifold are inextendable, as well as a concept of maximality for totally geodesic immersions (also in real analytic spaces) that extends the usual idea of maximality between embedded submanifold with respect to the inclusion relation. These new results have been presented in the article [119, Section 3], written in collaboration with Alberto Rodríguez-Vázquez.

As we have seen in Section 1.4, when our ambient space under consideration is a symmetric space, its totally geodesic submanifolds are automatically injectively immersed. However, we quickly lose this behavior when considering more general ambient spaces. For instance, it is well known that a geodesic on a complete Riemannian manifold need not be injective or periodic, so it may not correspond to an immersion of a one-dimensional Riemannian manifold. Therefore, in order to conduct a global study of totally geodesic submanifolds, it is crucial to allow our immersions to have self-intersections.

When we change our focus to immersions, there are certain redundancies that we need to treat. A first issue appears when considering reparametrizations of a given immersion. Indeed, if $f\colon \Sigma \to M$ is a totally geodesic immersion and $h\colon \Sigma' \to \Sigma$ is an isometry, then the composition $f \circ h\colon \Sigma' \to M$ is also a totally geodesic immersion. While f and $f \circ h$ are different immersions *a priori*, we should treat them as equal, as they convey the same information. This suggests the need to introduce a suitable notion of equivalence between totally geodesic immersions. A second redundancy is given by (surjective) local isometries. For example, consider the standard embedding $f\colon \mathbb{R}\mathsf{P}^2 \to \mathbb{R}\mathsf{P}^3$, which is totally geodesic. Its composition with the universal cover $h\colon \mathsf{S}^2 \to \mathbb{R}\mathsf{P}^2$ is also a totally geodesic immersion, but it is not an embedding. Because of this, the map $f \circ h$ carries undesirable repetitions that we need to avoid.

The key to circumvent the aforementioned problems is to regard totally geodesic immersions as immersions to the Grassmannian bundle. In order to motivate this idea, let us discuss the case of one-dimensional totally geodesic immersions, or equivalently, unit speed geodesics. Consider a geodesic $\gamma\colon I \subseteq \mathbb{R} \to M$ in a Riemannian manifold. Then this geodesic induces a curve $\overline{\gamma}\colon I \to TM$ given by $\overline{\gamma}(t) = (\gamma(t), \gamma'(t))$, which is an integral curve of the so-called geodesic vector field $G\colon TM \to TTM$. Conversely, the projection of an integral curve of the geodesic vector field to M is a geodesic of M. From the general theory of ordinary differential equations, we know that either $\overline{\gamma}$ is an injective immersion of I in TM, or it is a closed curve, so it descends to an injective immersion of S^1 to TM. In other words, a unit speed geodesic of M is uniquely characterized by an injectively immersed integral manifold of the distribution $\mathcal{D} = \operatorname{span} G$. We are also interested in identifying each geodesic $\gamma(t)$ with any possible reparametrizations of the form $\beta(t) = \gamma(t_0 \pm t)$. For this, we introduce the projectivized tangent bundle $\mathsf{P}(TM)$ whose fiber at each $p \in M$ is the real projective space $\mathsf{P}(T_pM)$. The geodesic γ induces an immersion $\widetilde{\gamma}\colon I \to \mathsf{P}(TM)$ by letting $\widetilde{\gamma}(t) = (\gamma(t), \mathbb{R}\gamma'(t))$. Similarly to the case of $\overline{\gamma}$, we see that either $\widetilde{\gamma}$ is injective or it descends to an injective immersion of the circle. Not only that, but any other geodesic β of M arises as a reparametrization of γ if and only if $\operatorname{im}\widetilde{\beta} = \operatorname{im}\widetilde{\gamma}$, meaning that γ is determined up to reparametrizations by the injectively immersed submanifold $\widetilde{\gamma}(I) \subseteq \mathsf{P}(TM)$.

The natural extension of the arguments above to k-dimensional totally geodesic submanifolds is to replace $\mathsf{P}(TM)$ with the Grassmannian bundle $\mathsf{G}_k(TM)$, whose fiber at $p \in M$ is the Grassmannian $\mathsf{G}_k(T_pM)$ of k-planes in T_pM. A totally geodesic immersion $f\colon \Sigma^k \to M$ induces a map $\widetilde{f}\colon \Sigma \to \mathsf{G}_k(TM)$, and we will show that (after perhaps taking a quotient of Σ) this new map is injective. We say in this case that f is compatible, so we are interested in understanding compatible totally geodesic immersions up to reparametrization. Just as in the one-dimensional case, the compatible totally geodesic immersion f is determined up to reparametrizations by the set $\widetilde{f}(\Sigma)$.

While we will not make use of this fact, it is worth noting that Tsukada [165] gives a characterization of the subsets $A \subseteq \mathsf{G}_k(TM)$ that take the form $A = \widetilde{f}(\Sigma)$ for a totally geodesic immersion $f\colon \Sigma \to M$. The main idea is to define a distribution $\mathcal{D}$ on $\mathsf{G}_k(TM)$ such that a connected subset $A \subseteq \mathsf{G}_k(TM)$ satisfies the above property if and only if A is an integral manifold of $\mathcal{D}$. In general, the distribution $\mathcal{D}$ is not involutive, which is expected due to the fact that not every element of $\mathsf{G}_k(TM)$ is the tangent space of a totally geodesic submanifold. However, it is shown in the same paper that given a (non-involutive) distribution Δ on a smooth manifold N, every connected integral manifold of N can be extended to a maximal one. In addition, two different maximal integral manifolds have empty intersection, which means in our particular case that every totally geodesic immersion to M can be extended to an *inextendable* one, and two inextendable compatible totally geodesic immersions $f_i\colon \Sigma_i \to M$ satisfying $\widetilde{f}_1(\Sigma_1) \cap \widetilde{f}_2(\Sigma_2) \neq \varnothing$ only differ by a reparametrization $h\colon \Sigma_1 \to \Sigma_2$. We will prove these statements without working explicitly with the distribution $\mathcal{D}$.

This chapter is structured as follows. In Section 5.1 we describe the local characterization of totally geodesic submanifolds via so-called totally geodesic subspaces. We define a notion of compatibility for totally geodesic immersions in Section 5.2 and prove that we can restrict our attention to these kinds of immersions. Section 5.3 is dedicated to showing how every totally geodesic submanifold can be extended into an extendable one. In Section 5.4 we prove that any totally geodesic submanifold of a homogeneous (respectively, real analytic) Riemannian manifold is also homogeneous (respectively, real analytic), and in Section 5.5 we use this result as leverage to define a notion of maximality for totally geodesic immersions. Finally, Section 5.6 is devoted to presenting the known classifications of totally geodesic submanifolds in homogeneous and symmetric spaces.

Before diving deeper, we need to make the following key observation. Suppose that a totally geodesic immersion $f\colon \Sigma \to M$ is given. For each $p \in \Sigma$, let $\mathcal{E}_p^\Sigma \subseteq T_p\Sigma$ be the domain of the exponential map $\exp_p$ (and make the analogous definition for points of M). As f carries geodesics to geodesics, we have

$$f(\exp_p(v)) = \exp_{f(p)}(f_{*p}(v)), \quad p \in \Sigma, v \in \mathcal{E}_p^\Sigma. \tag{5.1}$$

In particular, suppose that for a certain $\varepsilon > 0$, the exponential maps are defined on the open balls $B_{T_p\Sigma}(0,\varepsilon)$ and $B_{T_{f(p)}M}(0,\varepsilon)$ and $\exp_{f(p)}\colon B_{T_pM}(0,\varepsilon) \to M$ is a diffeomorphism onto its image. Then, since f_{*p} is injective, we deduce from (5.1) that $f \circ \exp_p$ is injective on $B_{T_p\Sigma}(0,\varepsilon)$, and as a consequence $\exp_p$ is injective on said ball. In particular, $\exp_p$ is a smooth diffeomorphism onto its image, and the restriction of f to $\exp_p(B_{T_p\Sigma}(0,\varepsilon))$ is also a smooth embedding.

5.1 Local existence of totally geodesic submanifolds

Let $f\colon \Sigma \to M$ be a totally geodesic immersion and $p \in \Sigma$. For now, let us argue locally, so that we may assume $\Sigma \subseteq M$ is an embedded submanifold and $f = \iota$ is the inclusion of Σ in M. The formula (5.1) implies that (after shrinking Σ if necessary) Σ is completely determined by the vector subspace $V = T_p\Sigma \subseteq T_pM$. Because of this, a totally geodesic submanifold of M is locally determined by its tangent space at a point. It is therefore natural to establish the following definition:

Let M be a Riemannian manifold, $p \in M$ and $V \subseteq T_pM$ a vector subspace. We say that V is a *totally geodesic subspace* if there exists a totally geodesic immersion $f\colon \Sigma \to M$ and a point $x \in \Sigma$ such that $p = f(x)$ and $V = f_{*x}(T_x\Sigma)$. In particular, V is a totally geodesic subspace if and only if there exists a $\delta > 0$ such that the subset $S = \exp_p(B_V(0,\delta))$ is an embedded totally geodesic submanifold of M. A first characterization of totally geodesic subspaces was given by Cartan.

Theorem 5.1 [17, Theorem 10.3.3]**.** *Let M be a Riemannian manifold, $p \in M$ and $V \subseteq T_pM$. Then V is a totally geodesic subspace if and only if there exists a $\delta > 0$ such that for every geodesic $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma'(0) \in B_V(0,\delta)$ the parallel translate $\mathcal{P}^\gamma_{0,1}V$ is curvature-invariant, that is,*

$$R(\mathcal{P}^\gamma_{0,1}V, \mathcal{P}^\gamma_{0,1}V)\mathcal{P}^\gamma_{0,1}V \subseteq \mathcal{P}^\gamma_{0,1}V. \tag{5.2}$$

For the purposes of this thesis we are interested in the case that M is a real analytic Riemannian manifold. In this context, one has a convenient restatement of Cartan's theorem.

Choose a point $p \in M$, a vector subspace $V \subseteq T_pM$ and $v \in V \cap \mathcal{E}_p^M$. We consider the maximal geodesic $\gamma(t) = \exp_p(tv)$. Given vectors X, Y, $Z \in V$ and $\xi \in V^\perp = T_pM \ominus V$, we may extend them to parallel vector fields $X(t)$, $Y(t)$, $Z(t)$ and $\xi(t)$ along γ. Since the geodesics of a real analytic manifold are real analytic and the differential equation defining parallel vector fields has real analytic coefficients when viewed in real analytic coordinates, it follows that the four vector fields are analytic, and thus the function

$$u(t) = \langle R(X(t), Y(t))Z(t), \xi(t)\rangle$$

is also analytic. Observe that (5.2) holds if and only if $u(1) = 0$ for all choices of X, Y, Z and ξ, so V is totally geodesic if and only if u vanishes identically for all choices of X, Y, Z and ξ. Because the metric and the input fields are parallel, the derivatives of u at 0 are

$$\begin{aligned} u'(0) &= \langle (\nabla_v R)(X,Y,Z), \xi \rangle = \langle (\nabla R)(v,X,Y,Z), \xi \rangle, \\ u''(0) &= \langle (\nabla_v(\nabla R))(v,X,Y,Z), \xi \rangle = \langle (\nabla^2 R)(v,v,X,Y,Z), \xi \rangle, \\ &\vdots \\ u^{(k)}(0) &= \langle (\nabla^k R)(v,\dots,v,X,Y,Z), \xi \rangle. \end{aligned}$$

Therefore, u vanishes in a neighborhood of zero if and only if the inner products

$$\langle (\nabla^k R)(v,\dots,v,X,Y,Z), \xi \rangle$$

are all zero, and in that case u is identically zero. We have thus obtained the following characterization:

Proposition 5.2 [165, Corollary 2.2]**.** *Let M be a real analytic Riemannian manifold, $p \in M$ and $V \subseteq T_pM$. Then the following conditions are equivalent:*

(i) *V is a totally geodesic subspace.*

(ii) *V is invariant under the tensors $\nabla^k R$ for all $k \geq 0$.*

(iii) *For all X, Y, Z, $v \in V$ and $k \geq 0$ the vector $(\nabla^k R)(v,\dots,v,X,Y,Z)$ is also in V.*

If M is not real analytic, we still have (i) ⇒ (ii) ⇒ (iii) but (iii) may not imply (i). From the definition, it is clear that for a totally geodesic subspace $V \subseteq T_pM$ and a geodesic γ of M satisfying $\gamma'(0) \in V$, the parallel translates $\mathcal{P}^\gamma_{0,t}V$ are totally geodesic subspaces for small values of t. In the real analytic case, it turns out that this property holds for arbitrary values of t:

Corollary 5.3. *Let M be a real analytic Riemannian manifold, $p \in M$ and $V \subseteq T_pM$. If V is a totally geodesic subspace and $\gamma \colon I \to M$ is a maximal geodesic satisfying $\gamma(0) = p$ and $\gamma'(0) \in V$, then $\mathcal{P}^\gamma_{0,t}V$ is a totally geodesic subspace of $T_{\gamma(t)}M$ for all $t \in I$.*

Proof. Consider the nonempty subset

$$J = \{t \in I \colon \mathcal{P}^\gamma_{0,t}V \text{ is totally geodesic}\}.$$

The definition of totally geodesic subspace yields that J is an open subset of I. We prove that J is also closed in I. For this, fix a basis $X_1,\dots,X_k$ of V and a basis $\xi_1,\dots,\xi_l$ of $V^\perp$. Then the corresponding parallel vector fields $X_1(t),\dots,X_k(t)$ and $\xi_1(t),\dots,\xi_l(t)$ give bases of $\mathcal{P}^\gamma_{0,t}V$ and $(\mathcal{P}^\gamma_{0,t}V)^\perp = \mathcal{P}^\gamma_{0,t}V^\perp$ respectively. Proposition 5.2 shows that the element $t \in I$ belongs to J if and only if we have

$$\begin{aligned} 0 &= \langle R(X_{i_1}(t), X_{i_2}(t))X_{i_3}(t), \xi_j(t) \rangle, \\ 0 &= \langle (\nabla^s R)(X_{i_1}(t),\dots,X_{i_1}(t), X_{i_2}(t), X_{i_3}(t), X_{i_4}(t)), \xi_j(t) \rangle, \end{aligned}$$

for all $s \geq 0$, i_1, i_2, i_3, $i_4 \in \{1,\dots,k\}$ and $1 \leq j \leq l$. This means that J is the zero locus of a system of equations determined by continuous functions, so J is closed in I. As I is connected, we deduce $J = I$, which proves the result. □

5.2 Compatible totally geodesic immersions

We define $\mathsf{G}_k(TM)$ to be the Grassmannian bundle of k-planes in TM. As a set, it is given by

$$\mathsf{G}_k(TM) = \{(p, V) \colon p \in M, V \subseteq T_pM \text{ is a } k\text{-dimensional subspace}\}.$$

We refer to the pair (p, V) simply as V if there is no ambiguity. We also consider the map $\rho \colon (p, V) \in \mathsf{G}_k(TM) \mapsto p \in M$. Then $\mathsf{G}_k(TM)$ admits a natural topology and smooth structure turning $\rho \colon \mathsf{G}_k(TM) \to M$ into a fiber bundle with fiber the Grassmannian $\mathsf{G}_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$.

Now let $f \colon \Sigma^k \to M$ be a totally geodesic immersion. Then f induces a smooth map $\widetilde{f} \colon \Sigma \to \mathsf{G}_k(TM)$ defined by

$$\widetilde{f}(x) = f_{*x}(T_x\Sigma), \quad x \in \Sigma.$$

This map satisfies $f = \rho \circ \widetilde{f}$. Furthermore, if $\alpha \colon [0, 1] \to \Sigma$ is a smooth curve with $\alpha(0) = x$ and $\alpha(1) = y$, then because parallel transport commutes with totally geodesic immersions we have

$$\widetilde{f}(y) = \mathcal{P}_{0,1}^{f\circ\alpha} \widetilde{f}(x).$$

We say that f is *compatible* if:

(i) Σ is connected.

(ii) $\widetilde{f}$ is injective.

Remark 5.4. Assume $f \colon E \to F$ and $g \colon F \to M$ are totally geodesic immersions. It may be the case that $g \circ f$ is not compatible, see Remark 6.14. However, if f is compatible and g is injective, then $g \circ f$ is compatible. We prove this as follows.

Suppose $x, y \in E$ satisfy $\widetilde{g \circ f}(x) = \widetilde{g \circ f}(y)$. In particular, we have $g(f(x)) = g(f(y))$, so $f(x) = f(y)$ because g is injective. Write $z = f(x) = f(y)$, so we have $\widetilde{g \circ f}(x) = g_{*z}(\widetilde{f}(x)) = g_{*z}(\widetilde{f}(y)) = \widetilde{g \circ f}(y)$. As g_{*z} is injective, we deduce $\widetilde{f}(x) = \widetilde{f}(y)$, and compatibility of f yields $x = y$.

It is stated in [86, Appendix A] that any totally geodesic immersion factors through a compatible one. For the sake of completeness we include a proof of this result and state it in a more general manner.

Lemma 5.5. *Let $f \colon \Sigma^k \to M^m$ be a totally geodesic immersion, where Σ need not be second countable or connected, and let $\mathcal{R}$ be the equivalence relation on Σ defined by*

$$x\mathcal{R}y \Leftrightarrow \widetilde{f}(x) = \widetilde{f}(y). \tag{5.3}$$

Consider the quotient space $\Upsilon = \Sigma/\mathcal{R}$ with corresponding quotient map $\pi \colon \Sigma \to \Upsilon$. Then there exists a unique smooth structure and Riemannian metric on Υ such that π is a surjective local isometry. Furthermore, the map $g \colon \Upsilon \to M$ defined by $g(\pi(x)) = f(x)$ for every $x \in \Sigma$ is a totally geodesic immersion such that $\widetilde{g}$ is injective.

Proof. Note that g is well defined and continuous because π is a quotient map. We may also define a map $h\colon \Upsilon \to \mathsf{G}_k(TM)$ via $h(\pi(x)) = \widetilde{f}(x)$ for each $x \in \Sigma$, and h is also well defined and continuous. Note that h is injective by the definition of $\mathcal{R}$. Our first objective is to prove that Υ is a (perhaps not second countable) smooth manifold.

Firstly, the space Υ is Hausdorff due to the fact that $\mathsf{G}_k(TM)$ is a Hausdorff space and h is an injective continuous map.

We now aim to endow Υ with a smooth structure. Let $p \in \Upsilon$ be arbitrary, $z = g(p)$ and $V = h(p)$. Choose an $\varepsilon > 0$ such that $\exp_z\colon B_{T_zM}(0, 2\varepsilon) \to M$ is a well-defined diffeomorphism onto its image and there exists $x \in \pi^{-1}(p)$ such that $\exp_x$ is defined on $B_{T_x\Sigma}(0, 2\varepsilon)$. The map $f\colon B_\Sigma(x, 2\varepsilon) \to M$ is therefore a totally geodesic diffeomorphism onto its image. In particular, the induced map $\widetilde{f}\colon B_\Sigma(x, 2\varepsilon) \to \mathsf{G}_k(TM)$ is an injective immersion, and combining that with the compactness of the closed ball $B_\Sigma[x, \varepsilon]$ we deduce that $\widetilde{f}\colon B_\Sigma(x, \varepsilon) \to \mathsf{G}_k(TM)$ is also a smooth embedding.

We note that $\pi(B_\Sigma(x, \varepsilon))$ is open in Υ, or equivalently, that $\pi^{-1}(\pi(B_\Sigma(x, \varepsilon)))$ is open in Σ. Indeed, let $y \in \pi^{-1}(\pi(B_\Sigma(x, \varepsilon)))$ be arbitrary. Then there exists some $y' \in B_\Sigma(x, \varepsilon)$ such that $\widetilde{f}(y') = \widetilde{f}(y)$. Choose $\delta > 0$ such that the open balls $B_{T_y\Sigma}(0, 2\delta)$, $B_{T_{y'}\Sigma}(0, 2\delta)$ and $B_{T_{f(y')}M}(0, 2\delta)$ are contained in the domain of the exponential map, $\exp$ is a diffeomorphism at each of these, and $B_\Sigma(y', \delta) \subseteq B_\Sigma(x, \varepsilon)$. We claim that $B_\Sigma(y, \delta) \subseteq \pi^{-1}(\pi(B_\Sigma(x, \varepsilon)))$. Indeed, if $a \in B_\Sigma(y, \delta)$, we can choose $v \in T_y\Sigma$ with $|v| < \delta$ and $a = \exp_y(v)$. We then have $f(a) = \exp_{f(y)}(f_{*y}(v))$ and $\widetilde{f}(a)$ is the parallel translate of $f_{*y}(T_y\Sigma) = \widetilde{f}(y)$ along the M-geodesic $\gamma(t) = f(\exp_y(tv)) = \exp_{f(y)}(tf_{*y}(v))$. Let $w \in T_{y'}\Sigma$ be the unique tangent vector satisfying $f_{*y'}(w) = f_{*y}(v)$ (this vector exists because of the equality $\widetilde{f}(y) = \widetilde{f}(y')$) and define $b = \exp_{y'}(w) \in B_\Sigma(y', \delta) \subseteq B_\Sigma(x, \varepsilon)$. By construction we have $\gamma(t) = f(\exp_{y'}(tw))$, so $f(a) = \gamma(1) = f(b)$. Furthermore, we see that

$$\widetilde{f}(b) = \mathcal{P}^\gamma_{0,1}\widetilde{f}(y') = \mathcal{P}^\gamma_{0,1}\widetilde{f}(y) = \widetilde{f}(a).$$

This shows that $\pi(a) = \pi(b)$ is in $\pi(B_\Sigma(x, \varepsilon))$, and thus $\pi(B_\Sigma(x, \varepsilon))$ is open in Υ. It follows that π is an open map.

In order to introduce local coordinates on $\pi(B_\Sigma(x, \varepsilon))$, we note that the restriction $\pi\colon B_\Sigma(x, \varepsilon) \to \pi(B_\Sigma(x, \varepsilon))$ is a bijective open map (because the restriction of f to this subset is injective), so it is a homeomorphism. Then the map $\exp_x^{-1} \circ \pi^{-1}$ yields a homeomorphism from $\pi(B_\Sigma(x, \varepsilon))$ to $B_{T_x\Sigma}(0, \varepsilon) \cong B_{\mathbb{R}^k}(0, \varepsilon)$ which provides a local coordinate system near p. We now have to show that the transition functions associated with these local coordinate systems are smooth maps.

Suppose $x_1, x_2 \in \Sigma$ are two points, and $\varepsilon_1, \varepsilon_2 > 0$ are such that the exponential map is defined on $B_{T_{x_i}\Sigma}(0, 2\varepsilon_i)$ and a well-defined diffeomorphism on $B_{T_{f(x_i)}M}(0, 2\varepsilon_i)$ for $i \in \{1, 2\}$, and let $B_i = B_\Sigma(x_i, \varepsilon_i)$. We also define the local coordinate systems $\varphi_i\colon \pi(B_i) \to B_{T_{x_i}\Sigma}(0, \varepsilon_i)$ via

$$\varphi_i(q) = \exp_{x_i}^{-1}\left(\left(\pi|_{B_i}\right)^{-1}(q)\right).$$

Suppose $\pi(B_1) \cap \pi(B_2) \neq \varnothing$. We have

$$\varphi_2 \circ \varphi_1^{-1}(u) = \exp_{x_2}^{-1}\left(\left(\pi|_{B_2}\right)^{-1}\left(\pi(\exp_{x_1}(u))\right)\right), \quad u \in \pi(B_1) \cap \pi(B_2),$$

so the smoothness of the transition functions is equivalent to the smoothness of the map $(\pi|_{B_2})^{-1} \circ (\pi|_{B_1})$. We actually have that this composition is the same as the composition $(f|_{B_2})^{-1} \circ (f|_{B_1})$. Indeed, for any $c \in B_1 \cap \pi^{-1}(B_2)$ the element $(\pi|_{B_2})^{-1} \circ (\pi|_{B_1})(c)$ is the only element c' in B_2 with $\pi(c) = \pi(c')$. By definition, this is equal to the condition $\widetilde{f}(c) = \widetilde{f}(c')$, and since the restriction of f to each B_i is an embedding, this condition is also equivalent to $f(c) = f(c')$, so the claimed equality holds. In particular, $(\pi|_{B_2})^{-1} \circ (\pi|_{B_1})$ is smooth, and thus the transition functions are smooth. This shows that Υ is indeed a smooth manifold. Note that the definition of our local charts also implies that π is a surjective local diffeomorphism.

Observe that h is an injective immersion because $\widetilde{f} = h \circ \pi$ is an immersion and π is a local diffeomorphism. Composing with ρ we also obtain that $f = g \circ \pi$ and g is a smooth immersion. As a consequence, we may endow Υ with the pullback metric that turns g into an isometric immersion. As f and g are isometric immersions, we deduce that π is a surjective local isometry.

We check that g is a totally geodesic immersion. Indeed, let $p \in \Upsilon$ and $v \in T_p\Upsilon$. Choose any $x \in \Sigma$ such that $\pi(x) = p$ and $w = (\pi_{*x})^{-1}(v) \in T_x\Sigma$. If $\beta(t) = \exp_x(tw)$ is the maximal Σ-geodesic with initial conditions $\beta(0) = x$ and $\beta'(0) = w$, then the projection $\gamma(t) = \pi(\beta(t))$ is a Υ-geodesic with initial conditions $\gamma(0) = p$, $\gamma'(0) = v$. As f is totally geodesic, we see that $g(\gamma(t)) = f(\beta(t))$ is a geodesic in M. As a consequence, g carries geodesics of Σ to geodesics of M, so it is a totally geodesic map.

To finish, we show that $\widetilde{g}$ is injective. Indeed, if $p \in \Upsilon$ and $x \in \Sigma$ is such that $p = \pi(x)$, we see that

$$\widetilde{g}(p) = g_{*p}(T_p\Upsilon) = g_{*p}(\pi_{*x}(T_x\Sigma)) = f_{*x}(T_x\Sigma) = \widetilde{f}(x) = h(p),$$

so $\widetilde{g} = h$ is an injective map by construction. This finishes the proof. □

Corollary 5.6. *Let $f\colon \Sigma \to M$ be a totally geodesic immersion where Σ is connected, and define $\pi\colon \Sigma \to \Upsilon$ and $g\colon \Upsilon \to M$ as in Lemma 5.5, so that g is a totally geodesic immersion satisfying $f = g \circ \pi$. Then g is compatible.*

Proof. This follows from noting that in this case Υ is also connected. □

Observe that in general totally geodesic immersions are not assumed to be injective, so they may not be embeddings. Moreover, for a compatible totally geodesic immersion $f\colon \Sigma \to M$, the induced map $\widetilde{f}$ may not be an embedding even though it is an injective immersion.

Example 5.7. Let $M = \mathsf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the standard flat torus. If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is any irrational number, we may consider the geodesic

$$\gamma\colon t \in \mathbb{R} \to [t, \theta t] \in \mathsf{T}^2.$$

Since θ is irrational, $\gamma\colon \mathbb{R} \to \mathsf{T}^2$ is an injective totally geodesic immersion, so it is compatible. Note that $\mathsf{G}_1(T\mathsf{T}^2) = \mathsf{T}^2 \times \mathbb{R}\mathsf{P}^1$ is a trivial bundle and $\widetilde{\gamma}$ is given by the formula

$$\widetilde{\gamma}(t) = ([t, \theta t], \mathbb{R}(1, \theta)).$$

Since the slice $\mathsf{T}^2 \times \{\mathbb{R}(1, \theta)\}$ is embedded in $\mathsf{G}_1(T\mathsf{T}^2)$ diffeomorphic to T^2, we see that if $\widetilde{\gamma}$ were an embedding then γ would also be an embedding. This is not possible because $\gamma(\mathbb{R})$ is one-dimensional and dense in T^2, so it is not open in the closure $\overline{\gamma(\mathbb{R})} = \mathsf{T}^2$.

However, we have the following result that allows us to give factorizations of compatible totally geodesic immersions.

Proposition 5.8. *Let* $f\colon E \to M$ *and* $g\colon F \to M$ *be compatible totally geodesic immersions where* $\dim E = \dim F$. *The following conditions are equivalent:*

(i) *There exists a local isometry* $\phi\colon E \to F$ *such that* $f = g \circ \phi$.

(ii) $\widetilde{f}(E) \subseteq \widetilde{g}(F)$.

Furthermore, the map ϕ *is injective and unique.*

Proof. Firstly, suppose (i) holds. Then for every $x \in E$ we see that

$$\widetilde{f}(x) = f_{*x}(T_x E) = g_{*\phi(x)}(\phi_{*x}(T_x E)) = g_{*\phi(x)}(T_{\phi(x)} F) = \widetilde{g}(\phi(x)) \in \widetilde{g}(F),$$

so the inclusion (ii) is true, and the uniqueness of ϕ follows. Note that the equality $\widetilde{f} = \widetilde{g} \circ \phi$ also implies that ϕ is injective.

Conversely, suppose that $\widetilde{f}(E) \subseteq \widetilde{g}(F)$ and let $\phi\colon E \to F$ be the map given by $\phi(x) = \widetilde{g}^{-1}(\widetilde{f}(x))$. By construction, $\widetilde{g} \circ \phi = \widetilde{f}$, and composing with ρ we get $g \circ \phi = f$. It suffices to show that ϕ is smooth. To prove this, let $x \in E$ and choose an $\varepsilon > 0$ such that $B_E(x, 2\varepsilon)$, $B_F(\phi(x), 2\varepsilon)$ and $B_M(f(x), 2\varepsilon)$ are normal coordinate balls. Then the maps $f\colon B_E(x,\varepsilon) \to M$ and $g\colon B_F(\phi(x), \varepsilon) \to M$ are embeddings with the same image

$$f(B_E(x,\varepsilon)) = g(B_F(\phi(x),\varepsilon)) = \exp_{f(x)}(B_{\widetilde{f}(x)}(0,\varepsilon)).$$

Thus, we have a well defined diffeomorphism $\psi\colon B_E(x,\varepsilon) \to B_F(\phi(x),\varepsilon)$ satisfying $f = g \circ \psi$ on $B_E(x,\varepsilon)$. As a consequence, $\widetilde{f} = \widetilde{g} \circ \psi$ on that set, so the restriction of ϕ to $B_E(x,\varepsilon)$ is smooth because it coincides with ψ. Since x is arbitrary, ϕ is globally smooth. The fact that ϕ is a local isometry is a direct consequence of the chain rule. □

5.3 Equivalence and extendability of totally geodesic immersions

Let M be a Riemannian manifold and $f\colon E \to M$, $g\colon F \to M$ two compatible totally geodesic immersions. We say that f and g are *equivalent* if there exists a global isometry $\phi\colon E \to F$ such that $f = g \circ \phi$. This condition clearly defines an equivalence relation on any set of compatible totally geodesic immersions to M. We also say that the pair (F, g) (or simply g) *extends* (E, f) (or simply f) if there exists an injective local isometry $\phi\colon E \to F$ such that $f = g \circ \phi$. In particular, we have $\dim E = \dim F$. By Proposition 5.8, this is equivalent to $\widetilde{f}(E) \subseteq \widetilde{g}(F)$.

Consider the collection of all equivalence classes of compatible totally geodesic immersions to M. If $[E, f]$ and $[F, g]$ are two equivalence classes, we write $[E, f] \leq [F, g]$ if (F, g) extends (E, f).

Lemma 5.9. *The relation $\leq$ is well defined and gives a partial ordering on any set of equivalence classes of totally geodesic immersions. In particular, two compatible totally geodesic immersions $f\colon E \to M$ and $g\colon F \to M$ are equivalent if and only if $\widetilde{f}(E) = \widetilde{g}(F)$.*

Proof. The fact that $\leq$ is well defined is easy to check, and the reflexive and transitive characters of $\leq$ are immediate. To check skew-symmetry, suppose that $[E, f] \leq [F, g]$ and $[F, g] \leq [E, f]$. Then there exist injective local isometries $\phi\colon E \to F$ and $\psi\colon F \to E$ satisfying $f = g \circ \phi$ and $g = f \circ \psi$. In particular, $f = f \circ \psi \circ \phi$, and thus $\widetilde{f} = \widetilde{f} \circ \psi \circ \phi$. Since $\widetilde{f}$ is injective, it follows that $\psi \circ \phi = \mathrm{Id}_E$. Similarly, $\phi \circ \psi = \mathrm{Id}_F$, so ϕ and ψ are mutually inverse. This means that both maps are isometries, so $[E, f] = [F, g]$. □

Fix a totally geodesic subspace $V \subseteq T_pM$. We aim to show that there exists a compatible totally geodesic immersion $f\colon \Sigma \to M$ with $V \in \widetilde{f}(\Sigma)$ such that it is maximal with respect to the extension relation. We say in this case that $f\colon \Sigma \to M$ is *inextendable*.

We start with a topological lemma that will help with issues of second countability.

Lemma 5.10. *Let M and Σ be smooth manifolds such that M is second countable and Σ is connected (but not necessarily second countable). If there exists an immersion $f\colon \Sigma \to M$, then Σ is also second countable.*

Proof. Endow M with a Riemannian metric, which is possible because M is second countable. The pullback f^*g defines a Riemannian metric on Σ, and thus Σ becomes a metric space with the Riemannian distance function. By [129, Theorem 41.4], Σ is paracompact, and thus second countable by [129, §41, Exercise 10]. □

Theorem 5.11 [165, Theorem 3.5]**.** *Let M be a Riemannian manifold, $p \in M$ and $V \subseteq T_pM$ a totally geodesic subspace. Then the following statements are true:*

(i) *There exists a totally geodesic immersion $f\colon \Sigma \to M$ which is compatible, inextendable, and such that $V \in \widetilde{f}(\Sigma)$.*

(ii) *If $g\colon E \to M$ is a compatible totally geodesic immersion with $V \in \widetilde{g}(E)$, then f is an extension of g.*

(iii) *The immersion $f\colon \Sigma \to M$ is unique up to equivalence.*

Proof. Let $\mathcal{G}_V \subseteq \mathsf{G}_k(TM)$ be the set of all subspaces $W \in \mathsf{G}_k(TM)$ for which there exists a compatible totally geodesic immersion $h\colon F \to M$ satisfying $V, W \in \widetilde{h}(F)$. We now consider a set of compatible totally geodesic immersions $\{(F_i, h_i)\}_{i \in I}$ such that $V \in \widetilde{h}_i(F_i)$ for all $i \in I$ and $\mathcal{G}_V = \bigcup_{i \in I} \widetilde{h}_i(F_i)$. The disjoint union $F = \bigsqcup_{i \in I} F_i$ is a Riemannian manifold (which may not be second countable) and the map $h\colon F \to M$ defined by the condition $h|_{F_i} = h_i$ for each $i \in I$ is a totally geodesic immersion. We may now apply Lemma 5.5 to obtain that the set $\Sigma = F/\mathcal{R}$, where $\mathcal{R}$ is the equivalence relation given by identifying the elements $x \in F_i$ and $y \in F_j$ if $\widetilde{h}_i(x) = \widetilde{h}_j(y)$, is a Riemannian manifold and the map $f\colon \Sigma \to M$ induced by h is a totally geodesic immersion satisfying that $\widetilde{f}$ is injective and $\widetilde{f}(\Sigma) = \mathcal{G}_V$. Observe that Σ is connected because the manifolds F_i are all connected and have a common point of intersection in the quotient (since $V \in \bigcap_{i\in I} \widetilde{h}_i(F_i)$). Therefore, $f\colon \Sigma \to M$ is a compatible immersion. By Lemma 5.10 we have that Σ is also second countable.

We show that $f\colon \Sigma \to M$ satisfies the three assertions given above.

Suppose that $g\colon E \to M$ is another compatible totally geodesic immersion with $V \in \widetilde{g}(E)$. Then, as $\widetilde{g}(E) \subseteq \mathcal{G}_V = \widetilde{f}(\Sigma)$, we have from Proposition 5.8 that f extends g, which proves (ii). In particular, if g is an extension of f, then f and g are equivalent, so f is inextendable, yielding (i). This also proves the uniqueness of f, so (iii) also holds. □

Using a similar argument, we can show that totally geodesic immersions may be "glued together".

Proposition 5.12. *Let $f\colon E \to M$ and $g\colon F \to M$ be two compatible totally geodesic immersions such that $\widetilde{f}(E) \cap \widetilde{g}(F) \neq \varnothing$. Then there exists a compatible totally geodesic immersion $h\colon \Sigma \to M$ extending both f and g.*

Proof. One can deduce this statement directly from item (ii) in Theorem 5.11 by simply choosing any $V \in \widetilde{f}(E) \cap \widetilde{g}(F)$ and letting $h\colon \Sigma \to M$ be the inextendable totally geodesic immersion satisfying $V \in \widetilde{h}(\Sigma)$.

An alternative proof that allows us to obtain a "minimal" extension is to consider the (non-compatible) totally geodesic immersion $f \sqcup g\colon E \sqcup F \to M$, taking the equivalence relation $\mathcal{R}$ in $E \sqcup F$ given by (5.3) and defining $\Sigma = (E \sqcup F)/\mathcal{R}$ with $h\colon \Sigma \to M$ the induced map. The condition $\widetilde{f}(E) \cap \widetilde{g}(F) \neq \varnothing$ implies that Σ is connected, and Lemma 5.5 implies that h is compatible. It is clear that h extends both f and g. □

5.3.1 A characterization of inextendability

In the case that the ambient space M is a real analytic Riemannian manifold, we show that for a compatible totally geodesic immersion, the condition of inextendability is equivalent to that of mapping maximal geodesics to maximal geodesics.

Let $f\colon \Sigma \to M$ be a compatible totally geodesic immersion. We say that Σ and M *share maximal geodesics* if for every maximal geodesic $\gamma\colon I \to \Sigma$ the composition $f \circ \gamma\colon I \to M$ is a maximal geodesic of M.

Remark 5.13. Let $f\colon \Sigma \to M$ be a compatible totally geodesic immersion such that Σ and M share maximal geodesics. Suppose $\gamma\colon I \subseteq \mathbb{R} \to M$ is a geodesic and there exist $t_0 \in I$ and $x \in \Sigma$ such that $\gamma(t_0) = f(x)$ and $\gamma'(t_0) \in \widetilde{f}(x) = f_{*x}(T_x\Sigma)$. Then, from the definition it is clear that there exists a unique geodesic $\bar{\gamma}\colon I \to \Sigma$ such that $\bar{\gamma}(t_0) = x$ and $f(\bar{\gamma}(t)) = \gamma(t)$ for all $t \in I$.

We start by proving that in general the condition of sharing maximal geodesics is stronger than inextendability.

Lemma 5.14. *Let $f\colon \Sigma^k \to M^n$ be a compatible totally geodesic immersion, $x \in \Sigma$ a point in Σ, and $V = \widetilde{f}(x)$. Assume $\gamma\colon [0,1) \to \Sigma$ is a geodesic that satisfies:*

(i) *The curve γ cannot be extended to the right, but $f \circ \gamma\colon [0,1) \to M$ admits an extension to a geodesic segment $\sigma\colon [0,1] \to M$.*

(ii) *The vector subspace $W = \mathcal{P}^{\sigma}_{0,1} \subseteq T_{\sigma(1)}M$ is totally geodesic.*

Then $W \notin \widetilde{f}(\Sigma)$ and f admits a proper extension $g\colon \widetilde{\Sigma} \to M$ such that $W \in \widetilde{g}(\widetilde{\Sigma})$.

Proof. Let $q = \sigma(1) \in M$. We first prove that $W \notin \widetilde{f}(\Sigma)$. Indeed, suppose that for a certain $y \in \Sigma$ we have $W = \widetilde{f}(y)$. Since $-\sigma'(1) = -\mathcal{P}^{\sigma}_{0,1}\sigma'(0) \in W = f_{*y}(T_y\Sigma)$, there exists a geodesic $\alpha\colon [0,\delta) \to \Sigma$ satisfying $\alpha(0) = y$ and $\alpha'(0) = -(f_{*y})^{-1}\sigma'(1)$. Suppose without loss of generality that $\delta < 1$. The composition $f \circ \alpha$ satisfies $f(\alpha(0)) = q$ and $(f \circ \alpha)'(0) = -\sigma'(1)$, so $f(\alpha(t)) = \sigma(1-t)$. Furthermore, we have

$$\begin{aligned}\widetilde{f}(\alpha(t)) &= f_{*\alpha(t)}(T_{\alpha(t)}\Sigma) = f_{*\alpha(t)}(\mathcal{P}^{\alpha}_{0,t}\Sigma) = \mathcal{P}^{f\circ\alpha}_{0,t} f_{*y}(T_y\Sigma) = \mathcal{P}^{\sigma}_{1,1-t}W \\ &= \mathcal{P}^{\sigma}_{0,1-t}V = \mathcal{P}^{f\circ\gamma}_{0,1-t}V = \mathcal{P}^{\gamma}_{0,1-t} f_{*x}(T_x\Sigma) = f_{*\gamma(1-t)}(T_{\gamma(1-t)}\Sigma) \\ &= \widetilde{f}(\gamma(1-t)),\end{aligned}$$

so using that $\widetilde{f}$ is injective we see that $\alpha(t) = \gamma(1-t)$ for all $t \in [0,\delta)$. Because α is continuous at 0, we see that the limit $\lim_{t\to 1^-}\gamma(t)$ exists and coincides with y, but this contradicts the fact that γ is not extendable to the right. We deduce that W is not in the image of $\widetilde{f}$.

We now consider an $\varepsilon > 0$ sufficiently small so that $\exp_q\colon B_{T_qM}(0,2\varepsilon) \to M$ is a diffeomorphism and $S = \exp_q(B_W(0,\varepsilon))$ is a totally geodesic submanifold of M. As $-\sigma'(1) \in W$, there exists a $\delta > 0$ such that $\sigma(t) \in S$ for all $t \in (1-\delta, 1]$. In particular, we have for all $t \in (1-\delta, 1)$ that

$$\widetilde{f}(\gamma(t)) = f_{*\gamma(t)}(T_x\Sigma) = \mathcal{P}^{f\circ\gamma}_{0,t}V = \mathcal{P}^{\sigma}_{0,t}\mathcal{P}^{\sigma}_{1,0}V = \mathcal{P}^{\sigma}_{1,t}V = \widetilde{i}(\sigma(t)),$$

where $i\colon S \hookrightarrow M$ is the inclusion map. Applying Lemma 5.12 to f and i, we obtain that f admits a proper extension g, constructed from f and i, such that W is in the image of $\widetilde{g}$. □

Proposition 5.15. *Let $f\colon \Sigma^k \to M^n$ be a compatible totally geodesic immersion. If Σ and M share maximal geodesics, then f is inextendable.*

Proof. Assume $g\colon E \to M$ extends f, and consider an injective local isometry $\phi\colon \Sigma \to E$ satisfying $f = g \circ \phi$. Replacing Σ with $\phi(\Sigma)$, we may suppose directly that $\Sigma \subseteq E$ is an open set and $f = g|_{\Sigma}$. If we show that Σ is also closed, then we may conclude that $\Sigma = E$ and $f = g$.

If Σ is not closed in E, then we can find a geodesic $\gamma\colon [0,1] \to E$ such that $\gamma(t) \in \Sigma$ for all $t \in [0,1)$ and $\gamma(1) \in E \setminus \Sigma$. Write $x = \gamma(0) \in \Sigma$. The composition $f \circ \gamma\colon [0,1] \to M$ is also a geodesic with $f(\gamma(0)) = f(x)$ and $(f \circ \gamma)'(0) \in \widetilde{f}(x)$, and since Σ and M share maximal geodesics we may find a geodesic $\beta\colon [0,1] \to \Sigma$ satisfying $\beta(0) = x$ and $f(\beta(t)) = f(\gamma(t))$ for all $t \in [0,1]$. By uniqueness of E-geodesics, we have $\beta = \gamma$, so $\gamma(1) = \beta(1) \in \Sigma$, which gives a contradiction. □

If M is a real analytic Riemannian manifold, it turns out that the converse of Proposition 5.15 is also true.

Proposition 5.16. *Let M^n be a real analytic Riemannian manifold and consider an inextendable compatible totally geodesic immersion $f\colon \Sigma^k \to M^n$. Then Σ and M share maximal geodesics.*

Proof. We argue by contradiction. Suppose that Σ and M do not share maximal geodesics. Then we may find a geodesic $\gamma\colon [0,1) \to \Sigma$ that cannot be extended further to the right while the composition $f \circ \gamma$ admits a proper extension $\sigma\colon [0,1] \to M$. Write $x = \gamma(0)$, $p = f(x)$ and $q = \sigma(1)$. We also consider the vector subspace $V = \widetilde{f}(x) \subseteq T_pM$, which is obviously totally geodesic. Define $W = \mathcal{P}^{\sigma}_{0,1}V \subseteq T_qM$. Because of Corollary 5.3, we know that W is a totally geodesic subspace of T_qM. Applying Lemma 5.14 we conclude that f is not inextendable, contradicting our initial assumption. □

We have thus arrived at the following result:

Proposition 5.17. *Let M be a connected real analytic Riemannian manifold and $f\colon \Sigma \to M$ a compatible totally geodesic immersion. Then f is inextendable if and only if Σ and M share maximal geodesics.*

As a direct application of the Hopf–Rinow theorem and Proposition 5.17, one obtains the following corollary which generalizes a result by Hermann [89] to the case of non-complete ambient manifolds.

Corollary 5.18. *Let M be a connected real analytic Riemannian manifold, $p \in M$ and V a totally geodesic subspace of T_pM. If $f\colon \Sigma \to M$ is the inextendable compatible totally geodesic immersion associated with V, then Σ is complete if and only if the exponential map $\exp_p$ is defined on all V.*

Proof. Let $x \in \Sigma$ be the unique point satisfying $\widetilde{f}(x) = V$, so that $f(x) = p$ and $f_{*x}(T_x\Sigma) = V$.

Firstly, assume that Σ is complete. If $v \in T_x\Sigma$ is an arbitrary vector, the maximal Σ-geodesic $\gamma(t) = \exp_x(tv)$ is defined on all $\mathbb{R}$, and so is the composition $f(\gamma(t)) = \exp_p(tf_{*x}(v))$. Therefore, the subspace V is contained in the domain of the exponential map $\exp_p$.

Secondly, assume that $\exp_p$ is defined on all V. Take an arbitrary vector $v \in T_x\Sigma$ and consider the maximal Σ-geodesic $\gamma\colon t \in I_v \subseteq \mathbb{R} \mapsto \exp_x(tv) \in \Sigma$. Because f is an inextendable totally geodesic immersion, the composition $f \circ \gamma\colon I_v \to M$ is also a maximal geodesic of M, whose initial conditions are $f(\gamma(0)) = p$, $(f \circ \gamma)'(0) = f_{*x}(v) \in V$. As a consequence, $I_v = \mathbb{R}$. This yields that $\exp_x$ is defined on all $x \in T_x\Sigma$, so by the Hopf–Rinow theorem we conclude that Σ is complete. □

Corollary 5.19 [89]**.** *Let M be a connected complete real analytic Riemannian manifold and $f\colon \Sigma \to M$ an inextendable compatible totally geodesic immersion. Then Σ is a complete Riemannian manifold. In other words, every totally geodesic submanifold of M can be uniquely extended to a complete one.*

5.4 Preservation of global properties

As shown in Section 1.4, every complete totally geodesic submanifold of a symmetric space is also a symmetric space. Therefore, it is natural to expect that other global properties of a given ambient space are inherited by its totally geodesic submanifolds.

Let us consider the case that the ambient manifold M is real analytic. Because its totally geodesic submanifolds are locally parametrized by the Riemannian exponential map of M (which is a real analytic map), it is natural to expect that these are analytic as well. It turns out that our intuition is correct.

Proposition 5.20. *Let M^n be a real analytic Riemannian manifold and $f\colon \Sigma^k \to M$ a totally geodesic immersion. Then Σ admits a unique real analytic structure (refining its smooth structure) such that the metric of Σ is real analytic and f is a real analytic map.*

Proof. Let $\mathcal{A}$ be the maximal atlas defining the smooth structure of Σ. We aim to extract a real analytic atlas $\mathcal{A}_\omega \subseteq \mathcal{A}$. In order to do this, let $x \in \Sigma$ and consider an $\varepsilon \equiv \varepsilon_x > 0$ such that $B_{T_x\Sigma}(0, 2\varepsilon)$ and $B_{T_{f(x)}M}(0, 2\varepsilon)$ are contained in the domains of their respective exponential maps, and $\exp_{f(x)}\colon B_{T_{f(x)}M}(0, 2\varepsilon) \to M$ is a smooth embedding. Then $\exp_x\colon B_{T_x\Sigma}(0, \varepsilon) \to \Sigma$ and $f\colon B_\Sigma(x, \varepsilon) \to M$ are also embeddings. As a consequence, the corresponding inverse map $\exp_x^{-1}\colon B_\Sigma(x, \varepsilon) \to B_{T_x\Sigma}(0, \varepsilon)$ is a local coordinate chart (formally, one would have to compose this map with a linear isomorphism $T_x\Sigma \to \mathbb{R}^k$ in order to obtain a chart, but we may omit this step as linear isomorphisms of $\mathbb{R}^k$ are diffeomorphisms). We claim that the family

$$\mathcal{F} = \left\{ \left(B_\Sigma(x, \varepsilon_x), \exp_x^{-1} \right) : x \in \Sigma \right\}$$

is a real analytic atlas of M. It is clear that the coordinate charts under consideration cover Σ, so we need to prove that the transition maps are analytic.

Take two points x_1, $x_2 \in \Sigma$ and for each x_i let $\varepsilon_i = \varepsilon_{x_i} > 0$. If $B_i = B_\Sigma(x_i, \varepsilon)$ and $\varphi_i = \exp_{x_i}^{-1}\colon B_i \to B_{T_{x_i}\Sigma}(0, \varepsilon_i)$ is its corresponding coordinate system, we see that

$$\begin{aligned}(\varphi_2 \circ \varphi_1^{-1})(u) &= \left(\exp_{x_2}^{-1} \circ \exp_{x_1} \right)(u) \\ &= \Big((f_{*x_2})^{-1} \circ (\exp_{f(x_2)}^{-1} \circ \exp_{f(x_1)}) \circ f_{*x_1} \Big)(u)\end{aligned}$$

for all $u \in \varphi_1(B_1 \cap B_2)$, where by $\exp_{f(x_2)}^{-1}$ we are referring to the inverse of the diffeomorphism $\exp_{f(x_2)}\colon B_{T_{f(x_2)}M}(0, \varepsilon_2) \to B_M(f(x_2), \varepsilon_2)$. As the exponential map of a real analytic manifold is real analytic, the composition $\exp_{f(x_2)}^{-1} \circ \exp_{f(x_1)}$ is real analytic, so the transition map $\varphi_2 \circ \varphi_1^{-1}$ is also analytic. We conclude that the local coordinate systems defined as above induce a real analytic atlas on M. We denote by $\mathcal{A}_\omega$ the maximal real analytic atlas containing $\mathcal{F}$, which by construction is contained in $\mathcal{A}$.

To show that f is real analytic, choose $x \in \Sigma$ and $\varepsilon > 0$ so that $\exp_x$ and $\exp_{f(x)}$ give well defined embeddings on the open balls of radius 2ε centered at the origin. Then we have $f = \exp_{f(x)} \circ f_{*x} \circ \exp_x^{-1}$ on $B_\Sigma(x, \varepsilon)$, so f is real analytic due to it being the composition of real analytic maps. As x is arbitrary, we obtain that f is globally analytic. In particular, the metric of Σ is the pullback of the metric of M by an analytic map, so it is analytic as well.

Finally, given another maximal real analytic atlas $\mathcal{A}'_\omega \subseteq \mathcal{A}$ such that $f\colon \Sigma \to M$ is real analytic with respect to $\mathcal{A}'_\omega$, we show that $\mathcal{F} \subseteq \mathcal{A}'_\omega$. Indeed, given a local smooth chart $\exp_x^{-1}\colon B_\Sigma(x,\varepsilon_x) \to B_{T_x\Sigma}(0,\varepsilon_x)$ in $\mathcal{F}$ we have the commutative diagram

$$\begin{array}{ccc} B_{T_x\Sigma}(0,\varepsilon_x) & \xrightarrow{f_{*x}} & B_{T_{f(x)}M}(0,\varepsilon_x) \\ {\scriptstyle \exp_x}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{f(x)}} \\ B_\Sigma(x,\varepsilon_x) & \xrightarrow{f} & B_M(f(x),\varepsilon_x) \end{array}$$

where the horizontal and rightmost maps are real analytic diffeomorphisms. Therefore, $\exp_x$ is an analytic diffeomorphism, which means that $(B_\Sigma(x,\varepsilon_x), \exp_x^{-1})$ belongs to $\mathcal{A}'_\omega$, proving our assertion. Maximality of $\mathcal{A}_\omega$ gives $\mathcal{A}_\omega = \mathcal{A}'_\omega$, so the real analytic structure of M is unique. □

We now show that a complete totally geodesic submanifold Σ of a homogeneous space M is also homogeneous. This is proved in [105, Corollary 8.10] for the case that $\Sigma \subseteq M$ is injectively immersed and complete. However, the proof by Kobayashi and Nomizu can be adapted to the general case with ease.

Proposition 5.21. *Let M be a Riemannian homogeneous space and $f\colon \Sigma \to M$ an inextendable compatible totally geodesic immersion. Then Σ is also a Riemannian homogeneous space.*

Proof. We can assume without loss of generality that M is connected, and thus complete. First of all, observe that Σ is also a complete Riemannian manifold due to Corollary 5.19, since homogeneous spaces are real analytic. In particular, every Killing vector field in Σ is complete. In order to show that Σ is homogeneous, it suffices to show that every tangent vector $v \in T\Sigma$ can be extended to a Killing vector field on Σ.

Let $x \in \Sigma$ and $v \in T_x\Sigma$ be any tangent vector. We define $p = f(x)$ and $w = f_{*x}(v) \in T_pM$. Since M is homogeneous, we may choose a Killing vector field $X \in \mathcal{K}(M)$ satisfying $X_p = w$. For each $y \in \Sigma$, let us decompose $X_{f(y)} = X^\top_{f(y)} + X^\perp_{f(y)}$, where $X^\top_{f(y)}$ and $X^\perp_{f(y)}$ are the orthogonal projections of $X_{f(y)}$ to $\widetilde{f}(y)$ and $T_{f(y)}M \ominus \widetilde{f}(y)$ respectively. We define a smooth vector field $Z \in \mathfrak{X}(\Sigma)$ by the equation

$$Z_y = (f_{*y})^{-1}\left(X^\top_{f(y)}\right), \quad y \in \Sigma.$$

By construction, it is clear that $Z_x = v$. Thus, it suffices to show that Z is a Killing vector field on Σ. Given $y \in \Sigma$ and $u \in T_y\Sigma$, we aim to prove that $\langle \nabla_u Z, u\rangle = 0$. To do this, we extend u to a vector field $U \in \mathfrak{X}(\Sigma)$. Because f is an immersion, we may choose:

- an open subset $\widetilde{\Omega} \subseteq M$,
- an open subset $\Omega \subseteq \Sigma$ such that $u \in \Omega$, the restriction of f to Ω is an embedding, and $f(\Omega) \subseteq \widetilde{\Omega}$,
- and a vector field $\overline{U} \in \mathfrak{X}(\widetilde{\Omega})$ that is f-related to $U|_\Omega$.

Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of Σ and M respectively. Using the fact that f is totally geodesic, we obtain that

$$\langle \nabla_U Z, U \rangle = \langle \overline{\nabla}_{\overline{U}} X^{\top}, \overline{U} \rangle \circ f \quad \text{on } \Omega.$$

Furthermore, because $f(\Omega)$ is a totally geodesic submanifold of M and X is a Killing vector field, we deduce that

$$\begin{aligned} 0 = \langle \overline{\nabla}_{\overline{U}} X, \overline{U} \rangle &= \langle \overline{\nabla}_{\overline{U}} X^{\top}, \overline{U} \rangle + \langle \overline{\nabla}_{\overline{U}} X^{\perp}, \overline{U} \rangle \\ &= \langle \overline{\nabla}_{\overline{U}} X^{\top}, \overline{U} \rangle + \overline{U} \langle X^{\perp}, \overline{U} \rangle - \langle X^{\perp}, \overline{\nabla}_{\overline{U}} \overline{U} \rangle = \langle \overline{\nabla}_{\overline{U}} X^{\top}, \overline{U} \rangle \end{aligned}$$

along $f(\Omega)$, where the last equality follows from $X^{\perp}$ being orthogonal to both $\overline{U}$ and $\overline{\nabla}_{\overline{U}} \overline{U}$. Because of this, we obtain that $\langle \nabla_u Z, u \rangle = 0$. As a consequence, Z is a (complete) Killing vector field on Σ extending v. This shows that Σ is a Riemannian homogeneous space. □

We emphasize that Proposition 5.21 shows only that the group $I(\Sigma)$ acts transitively on Σ. This does not imply that Σ is an extrinsically homogeneous submanifold of M. For instance, this is not possible if f is not injective, because extrinsically homogeneous submanifolds are injectively immersed. An explicit example of a totally geodesic submanifold which is not injectively immersed will be given in Section 6.3.2. Moreover, there are examples of totally geodesic submanifolds of homogeneous spaces that are embedded but not extrinsically homogeneous, see [97, 135].

5.5 Maximal totally geodesic submanifolds in analytic Riemannian manifolds

We now concern ourselves with defining a notion of maximality for totally geodesic submanifolds. Indeed, if M is a real analytic Riemannian manifold and $\Sigma_1, \Sigma_2 \subseteq M$ are two inextendable and embedded totally geodesic submanifolds, one can wonder if $\Sigma_1 \subseteq \Sigma_2$. In this case, it is easy to see that $\Sigma_1 \subseteq \Sigma_2$ if and only if there exists a point $p \in \Sigma_1 \cap \Sigma_2$ such that $T_p\Sigma_1 \subseteq T_p\Sigma_2$. Therefore, the study of inclusions between embedded totally geodesic submanifolds of M containing the point p is equivalent to that of inclusions between totally geodesic subspaces of T_pM. For general totally geodesic immersions, the situation is more involved, and one needs to introduce the following "pullback-type" construction to make sense of the inclusion relationship.

Proposition 5.22. *Let M be a connected real analytic Riemannian manifold, $p \in M$ and V_1, $V_2 \subseteq T_pM$ two totally geodesic subspaces. For each $i \in \{1, 2\}$, consider the inextendable compatible totally geodesic immersion $f_i \colon \Sigma_i \to M$ satisfying $V_i \in \widetilde{f}_i(\Sigma_i)$ and let $x_i \in \Sigma_i$ be the unique point such that $\widetilde{f}(x_i) = V_i$. Then the following assertions are equivalent:*

(i) $V_1 \subseteq V_2$.

(ii) *There exists a connected Riemannian manifold E, a surjective local isometry $\pi\colon E \to \Sigma_1$, a compatible and inextendable totally geodesic immersion $h\colon E \to \Sigma_2$ and a point $z \in E$ such that $f_1 \circ \pi = f_2 \circ h$, $\pi(z) = x_1$ and $h(z) = x_2$. In other words, the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{h} & \Sigma_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f_2} \\ \Sigma_1 & \xrightarrow{f_1} & M \end{array}$$

Furthermore, if $\Sigma_2 \subseteq M$ is injectively immersed and $f_2 = \iota\colon \Sigma_2 \hookrightarrow M$, one can take $E = \Sigma_1$, $\pi = \mathrm{Id}_{\Sigma_1}$, $h\colon \Sigma_1 \to \Sigma_2$ given by $h(x) = f_1(x)$ and $z = x_1$, so $V_1 \subseteq V_2$ if and only $f_1(\Sigma_1) \subseteq \Sigma_2$.

Proof. Start by assuming (ii). Then we have

$$\begin{aligned} V_1 = \widetilde{f}_1(x_1) = \widetilde{f}_1(\pi(z)) &= (f_1)_{*\pi(z)}(T_{\pi(z)}\Sigma_1) = (f_1 \circ \pi)_{*z}(T_z E) \\ &= (f_2 \circ h)_{*z}(T_z E) = (f_2)_{*x_2}(h_{*z}(T_z E)) \subseteq (f_2)_{*x_2}(T_{x_2}\Sigma_2) = \widetilde{f}_2(x_2) = V_2, \end{aligned}$$

which proves (i).

Now, suppose (i) is true, and let $W = (f_2)^{-1}_{*x_2}(V_1)$, which is a totally geodesic subspace of $T_{x_2}\Sigma_2$. We can construct an inextendable compatible totally geodesic immersion $h\colon E \to \Sigma_2$ such that $W \in \widetilde{h}(E)$, and there exists a unique $z \in E$ for which $h(z) = x_2$ and $\widetilde{h}(z) = W$. As M and Σ_2 are real analytic, we may apply Proposition 5.17 twice to see that the composition $f_2 \circ h\colon E \to M$ sends maximal geodesics of E to maximal geodesics of M. However, $f_2 \circ h$ need not be compatible. Let $\mathcal{R}$ be the equivalence relation on E defined by

$$x\mathcal{R}y\colon \Leftrightarrow \widetilde{f_2 \circ h}(x) = \widetilde{f_2 \circ h}(y).$$

By Lemma 5.5, the quotient space $E/\mathcal{R}$ admits a unique smooth structure and Riemannian metric such that the natural projection $\rho\colon E \to E/\mathcal{R}$ is a surjective local isometry and the map $g\colon E/\mathcal{R} \to M$ given by $g([x]) = f_2(h(x))$ is a compatible totally geodesic immersion. Because $f_2 \circ h$ sends maximal geodesics to maximal geodesics and ρ is a surjective local isometry, the immersion g sends the maximal geodesics of $E/\mathcal{R}$ to maximal geodesics of M, so g is inextendable by Proposition 5.17. Observe that

$$\widetilde{g}([z]) = g_{*[z]}(T_{[z]}E/\mathcal{R}) = (g \circ \rho)_{*z}(T_z E) = (f_2 \circ h)_{*z}(T_z E) = (f_2)_{*x_2}(W) = V_1,$$

so by uniqueness of f_1 there exists a global isometry $\phi\colon E/\mathcal{R} \to \Sigma_1$ such that $g = f_1 \circ \phi$. By considering $\pi = \phi \circ \rho\colon E \to \Sigma_1$, we obtain the equalities $f_1 \circ \pi = g \circ \rho = f_2 \circ h$ and $\pi(z) = x_1$ because $\widetilde{f}_1$ is injective and

$$\begin{aligned} \widetilde{f}_1(\pi(z)) = (f_1)_{*\pi(z)}(T_{\pi(z)}\Sigma_1) &= (f_1 \circ \pi)_{*z}(T_z E) = (g \circ \rho)_{*z}(T_z E) \\ &= V_1 = \widetilde{f}_1(x_1). \end{aligned}$$

Therefore, (ii) holds.

Finally, note that if $\Sigma_2 \subseteq M$ and $f_2 = \iota$ is the inclusion map, the composition $\iota \circ h$ in the previous paragraph is also a compatible totally geodesic immersion, so $E/\mathcal{R} = E$ and we obtain in this case a global isometry $\phi\colon E \to \Sigma_1$ satisfying $\iota \circ h = f_1 \circ \phi$. By replacing E with Σ_1 and h with $h \circ \phi^{-1}$, we obtain $\iota \circ h = f_1$, so $h\colon \Sigma_1 \to \Sigma_2$ is simply the restriction in codomain of f_1, and $f_1(\Sigma_1) \subseteq \Sigma_2$. □

Motivated by the previous proposition, we say that an inextendable compatible totally geodesic immersion $f\colon \Sigma \to M$ (or simply, Σ) is *maximal* if it is not a global isometry and whenever we have another inextendable compatible totally geodesic immersion $f'\colon \Sigma' \to M$, a Riemannian manifold E, a surjective local isometry $\pi\colon E \to \Sigma$ and a compatible totally geodesic immersion $h\colon E \to \Sigma'$ satisfying $f' \circ h = f \circ \pi$, we have that f' is either a global isometry or equivalent to f. From Proposition 5.22, the following conditions are equivalent:

- $f\colon \Sigma \to M$ is maximal.
- For all $x \in \Sigma$, $\widetilde{f}(x) = f_{*x}(T_x\Sigma)$ is a maximal totally geodesic subspace of $T_{f(x)}M$.
- There exists an $x \in \Sigma$ such that $\widetilde{f}(x)$ is a maximal totally geodesic subspace of $T_{f(x)}M$.

5.6 Totally geodesic submanifolds of homogeneous spaces

In this section we discuss the current progress in the classification problem for totally geodesic submanifolds in Riemannian homogeneous spaces. Since we are working with possibly non-injective immersions of these submanifolds, we first have to give a reasonable notion of equivalence between two totally geodesic immersions that extends the usual notion of congruence for embedded submanifolds. Let M be a Riemannian manifold and consider two (inextendable, compatible) totally geodesic immersions $f_i\colon \Sigma_i \to M$ (with $i = 1, 2$). We say that the immersions f_1 and f_2 (or the submanifolds Σ_1 and Σ_2) are *congruent* if there exists a global isometry $g \in I(M)$ such that $g \circ f_1$ and f_2 are equivalent. Observe that when $\Sigma_1, \Sigma_2 \subseteq M$ are injectively immersed, this definition of congruence is equivalent to the usual definition of congruence (that is, the existence of $g \in I(M)$ such that $g(\Sigma_1) = \Sigma_2$). The main problem that we deal with in this part of the thesis is the classification of inextendable compatible totally geodesic immersions $f\colon \Sigma \to M$ (with $\dim \Sigma \geq 2$) in Riemannian manifolds up to congruence.

Generically, a Riemannian manifold does not have any totally geodesic submanifolds of dimension greater than one [130], so their classification becomes trivial. This means that in order to find nontrivial examples of totally geodesic submanifolds we need to work with ambient spaces that possess a rich structure. There are two main reasons that make Riemannian homogeneous spaces a natural choice of manifolds on which to carry out this task. The first one is that if M is a Riemannian homogeneous space, one can fix a base point $o \in M$ and study all totally geodesic submanifolds passing through o, as any totally geodesic submanifold is automatically congruent to one containing o. In practice, we choose a presentation $M = \mathsf{G}/\mathsf{K}$ admitting a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, so that the problem reduces to that of classifying totally geodesic subspaces in $\mathfrak{p}$. The second one is that homogeneous spaces are real analytic, meaning that totally geodesic subspaces have an algebraic characterization by Proposition 5.2. Indeed, a subspace $\mathfrak{v} \subseteq \mathfrak{p}$ is totally geodesic if and only if is invariant under all tensors $\nabla^k R$ with $k \geq 0$. Nevertheless, this equivalent condition remains extremely hard to check, as it requires computing the covariant derivatives of the curvature tensor and solving the (countably infinite) system of polynomial equations corresponding the curvature-invariance condition. Therefore, in order to attack this problem in homogeneous spaces, it will prove convenient to apply this characterization only as a necessary condition while finding other sufficient conditions that are easier to check in order to determine when a tangent subspace is totally geodesic. In fact, we remark that at no point in this thesis we calculate covariant derivatives of R of order greater than one.

5.6.1 The symmetric case

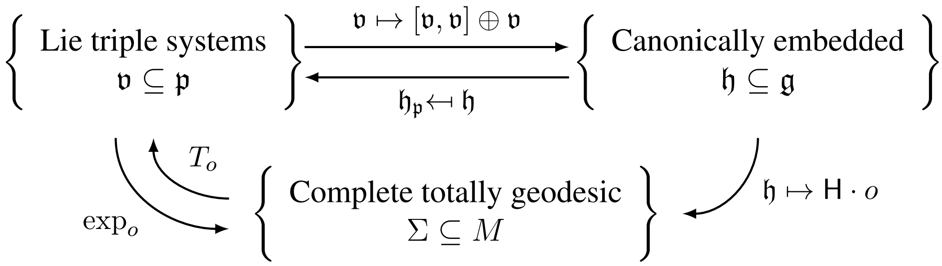
Most of the classification results concerning totally geodesic submanifolds to this day are focused on symmetric spaces. The clear advantage of working in this setting is that for a symmetric space M the covariant derivative ∇R vanishes identically, so understanding its totally geodesic submanifolds amounts to knowing the curvature-invariant subspaces of the tangent space at any point. Moreover, every complete totally geodesic submanifold of a symmetric space is injectively immersed—in fact, extrinsically homogeneous—meaning that it is not necessary to work with general immersions in this case.

Let (G, K) be a Riemannian symmetric pair with involution $\Theta\colon \mathsf{G} \to \mathsf{G}$ and consider the associated symmetric space $M = \mathsf{G}/\mathsf{K}$. We write $o = e\mathsf{K}$. The group involution $\Theta \in \mathrm{Aut}(\mathsf{G})$ induces a Lie algebra involution $\theta\colon \mathfrak{g} \to \mathfrak{g}$ and a corresponding reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We recall from (1.7) that a subspace $\mathfrak{v}$ of $\mathfrak{p} \equiv T_o M$ is curvature-invariant (equivalently, totally geodesic) if and only if it is a Lie triple system in $\mathfrak{p}$. Moreover, we have a series of correspondences between totally geodesic submanifolds of M and certain algebraic objects related to the symmetric pair (G, K), which we briefly describe in the following.

1. Given a Lie triple system $\mathfrak{v} \subseteq \mathfrak{p}$, its image $\Sigma = \exp_o(\mathfrak{v})$ under the Riemannian exponential map is a complete totally geodesic submanifold of M containing o. Conversely, if Σ is a complete totally geodesic submanifold of M with $o \in \Sigma$, then $\mathfrak{v} = T_o\Sigma$ is a Lie triple system. The two correspondences described above are mutually inverse.

2. If $\mathfrak{v}$ is a Lie triple system in $\mathfrak{p}$, then $\mathfrak{h} = [\mathfrak{v}, \mathfrak{v}] \oplus \mathfrak{v}$ is the smallest subalgebra of $\mathfrak{g}$ containing $\mathfrak{p}$ and it is canonically embedded in $\mathfrak{g}$ (that is, $\theta\mathfrak{h} = \mathfrak{h}$). Similarly, if $\mathfrak{h}$ is a canonically embedded subalgebra of $\mathfrak{g}$, then $\mathfrak{h}_\mathfrak{p}$ is a Lie triple system in $\mathfrak{p}$. Note that for a Lie triple system in $\mathfrak{p}$ we have $([\mathfrak{v}, \mathfrak{v}] \oplus \mathfrak{v})_\mathfrak{p} = \mathfrak{v}$, whereas for a canonically embedded subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we only have $[\mathfrak{h}_\mathfrak{p}, \mathfrak{h}_\mathfrak{p}] \oplus \mathfrak{h}_\mathfrak{p} \subseteq \mathfrak{h}$, meaning that the map $\mathfrak{v} \mapsto [\mathfrak{v}, \mathfrak{v}] \oplus \mathfrak{v}$ is only a right inverse of the map $\mathfrak{h} \mapsto \mathfrak{h}_\mathfrak{p}$. Observe that it is never a left inverse, because for instance the canonically embedded subalgebra $\mathfrak{h} = \mathfrak{k}$ has $[\mathfrak{k}_\mathfrak{p}, \mathfrak{k}_\mathfrak{p}] \oplus \mathfrak{k}_\mathfrak{p} = 0 \neq \mathfrak{k}$.

We summarize this discussion by means of the following diagram:



Recall that a (non-flat) irreducible symmetric space is necessarily of compact type or of noncompact type. Moreover, if $M = \mathsf{G}/\mathsf{K}$ is a simply connected irreducible symmetric space of compact type, then one can associate a dual symmetric space of noncompact type to it by taking the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ and considering the dual algebra $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g}(\mathbb{C})$ and taking $M^* = \mathsf{G}^*/\mathsf{K}^*$ to be the simply connected symmetric space coming from the Klein pair $(\mathfrak{g}^*, \mathfrak{k})$. A straightforward calculation shows that a vector subspace $\mathfrak{v} \subseteq \mathfrak{p}$ is a Lie triple system if and only if $i\mathfrak{v} \subseteq i\mathfrak{p}$ is a Lie triple system in $\mathfrak{p}$. Consequently, we have a bijective correspondence between the set of (complete) totally geodesic submanifolds of M passing through $o = e\mathsf{K}$ and those of M^* containing $o = e\mathsf{K}^*$. In particular, to classify totally geodesic submanifolds in irreducible Riemannian symmetric spaces, it is enough to restrict oneself to either the compact or the noncompact setting.

For symmetric spaces of compact type, the issue when working with Lie triple systems is that they do not convey the topological information needed to reconstruct the corresponding totally geodesic submanifold (the simplest example is that of a one-dimensional subspace of $\mathfrak{p}$, which may correspond to an injective immersion of $\mathbb{R}$ or of S^1). In the noncompact case, this problem is avoided entirely, as the Riemannian exponential map is a global diffeomorphism and every totally geodesic submanifold is diffeomorphic to some Euclidean space.

Furthermore, for a symmetric space of noncompact type $M = \mathsf{G}/\mathsf{K}$ there are some known results concerning canonically embedded subalgebras of the isometry algebra $\mathfrak{g}$. For instance, the celebrated Karpelevich–Mostow theorem [96, 127] says that any semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is canonically embedded with respect to some Cartan decomposition. In other words, every connected semisimple subgroup H of G has a totally geodesic orbit. A partial generalization of this result can be found in [70, Chapter 6, Theorem 3.6]: an algebraic subalgebra of $\mathfrak{g}$ is canonically embedded with respect to a Cartan decomposition of $\mathfrak{g}$ if and only if it is a reductive subalgebra. Of course, it is sufficient (albeit not necessary) to know all subalgebras that are canonically embedded with respect to some Cartan decomposition in order to find all totally geodesic submanifolds of M.

We also remark a result of Sanmartín-López and Solonenko [150] which states that totally geodesic submanifolds of symmetric spaces of noncompact type can be realized as subgroups of their solvable model:

Proposition 5.23 [150, Proposition 5.1]**.** *Let* $M = \mathsf{G}/\mathsf{K}$ *be a symmetric space of noncompact type and* Σ *a complete totally geodesic submanifold of* M *containing* o*. Then there exist an Iwasawa decomposition* $\mathsf{G} = \mathsf{KAN}$ *and a connected Lie subgroup* $\mathsf{H} \subseteq \mathsf{AN}$ *of the form* $\mathsf{H} = (\mathsf{H} \cap \mathsf{A})(\mathsf{H} \cap \mathsf{N})$ *such that* $\Sigma = \mathsf{H} \cdot o$*.*

The classification problem for totally geodesic submanifolds in symmetric spaces dates back to the seminal paper by Wolf [168], who determined all totally geodesic submanifolds of (compact) rank one symmetric spaces; see Section 4.2 to see the explicit classification in their noncompact duals. Afterwards, Chen and Nagano [42, 43] gave a classification on symmetric spaces of rank two. This was later revised by Klein [99–101], who found some examples missed in the original work of Chen and Nagano. As of today, we do not have complete classifications of totally geodesic submanifolds on symmetric spaces with rank greater than two. However, full classifications have been achieved in products of rank one symmetric spaces [148] by Rodríguez-Vázquez. Moreover, Kollross and Rodríguez-Vázquez determined all maximal totally geodesic submanifolds in exceptional symmetric spaces up to isometry [110].

Several authors have dealt with totally geodesic submanifolds that present some additional properties. For instance, there is a nice interaction between isometries of a Riemannian manifold and its totally geodesic submanifolds. Let M be a Riemannian manifold and $\Omega \subseteq I(M)$ a set of isometries of M. Then, every connected component of the fixed point set

$$\mathrm{Fix}(\Omega) = \{x \in M \colon f(x) = x \text{ for all } f \in \Omega\}$$

is seen to be a totally geodesic submanifold of M [103, Theorem 5.1]. A particular case of this phenomenon is given by taking $\Omega = \{f\}$ with f an involutive isometry. If M is a Riemannian manifold, we say that a connected injectively immersed submanifold $\Sigma \subseteq M$ is *reflective* if there exists an involutive isometry $g \in I(M)$ such that Σ is a connected component of $\mathrm{Fix}(g)$. In the case of symmetric spaces, one can apply the Cartan–Ambrose–Hicks theorem to characterize reflective submanifolds as follows:

Theorem 5.24. *Let $M = \mathsf{G}/\mathsf{K}$ be a simply connected Riemannian symmetric space and consider a vector subspace $\mathfrak{v}$ of $\mathfrak{p} \equiv T_oM$. Then, there exists a reflective submanifold $\Sigma \subseteq M$ with $o \in \Sigma$ and $T_oM = \mathfrak{v}$ if and only if both $\mathfrak{v}$ and $\mathfrak{v}^\perp = \mathfrak{p} \ominus \mathfrak{v}$ are Lie triple systems in $\mathfrak{p}$. Moreover, the totally geodesic submanifold $\Sigma^\perp = \exp_o(\mathfrak{v}^\perp)$ is also a reflective submanifold of M.*

The classification of reflective submanifolds on irreducible symmetric spaces was carried out in a series of papers by Leung, see [116] and the references therein. We remark that the study of reflective submanifolds coming from geodesic reflections on irreducible compact symmetric spaces is due to Chen and Nagano, and is also known as (M_+, M_-)-theory. A fixed point component of a geodesic reflection is known as a *polar*, whereas its corresponding orthogonal totally geodesic submanifold is called a *meridian*. See [41] for a summary of the main results concerning polars and meridians.

Another invariant of symmetric spaces associated with their totally geodesic submanifolds is their so-called index. Given a Riemannian manifold M, we define the *index* of M (denoted by $i(M)$) as the smallest codimension of its totally geodesic submanifolds. This notion was originally introduced by Onishchik in [140]. As the culmination of the work by Berndt, Olmos and Rodríguez [20], the index of all irreducible symmetric spaces has been computed. A by-product of their calculations is the (positive) resolution of the so-called index conjecture: for every irreducible Riemannian symmetric space M different from $\mathsf{G}_2/\mathsf{SO}(4)$ and $\mathsf{G}_2^2/\mathsf{SO}(4)$, the index $i(M)$ coincides with the smallest codimension of a reflective submanifold of M (known as the *reflective index* of M).

5.6.2 The non-symmetric case

While some authors have studied totally geodesic submanifolds in more general (non-symmetric) homogeneous spaces, this field has remained comparatively unexplored due to the sheer increase in difficulty when considering these kinds of manifolds. Suppose $M = \mathsf{G}/\mathsf{K}$ is an n-dimensional naturally reductive homogeneous space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a naturally reductive decomposition. Tsukada [165] showed that the problem of detecting totally geodesic subspaces is still given by a finite system of polynomial equations. Indeed, let $\mathfrak{R}(\mathfrak{p})$ be the space of algebraic curvature tensors on $\mathfrak{p}$ (in particular, $R \in \mathfrak{R}(\mathfrak{p})$). For each $X \in \mathfrak{p}$, let $\mathfrak{R}_X$ be the smallest D_X-invariant subspace of $\mathfrak{R}(\mathfrak{p})$ containing R (recall that D is the difference tensor) and define $d(X) = \dim \mathfrak{R}(X)$.

Theorem 5.25 [165, Theorem 2.3]**.** *Suppose $M = \mathsf{G}/\mathsf{K}$ is a naturally reductive homogeneous space and $\mathfrak{v} \subseteq \mathfrak{p}$ is a vector subspace. If for each $X \in \mathfrak{v}$ we have*

$$(\nabla^k R)(U, \dots, U, X, Y) \in \mathfrak{v}, \quad \textit{for all } 0 \leq k \leq d(U) \textit{ and } X, Y \in \mathfrak{v},$$

then $\mathfrak{v}$ is a totally geodesic subspace.

A direct corollary is that if $\mathfrak{v}$ is totally geodesic if and only if it is invariant by $\nabla^k R$ for all $k \leq \max_{X \in \mathfrak{v}} d(X)$. Note that the latter integer is bounded above by $\dim \mathfrak{R}(\mathfrak{p}) = \frac{n^2(n^2-1)}{12}$. However, this criterion is still not feasible to apply directly in practice due to the complications in computing covariant derivatives of R and the rapid growth of $\dim \mathfrak{R}(\mathfrak{p})$ as a function of n.

The second difficulty that arises in this context is that not every totally geodesic submanifold of a Riemannian homogeneous space is injectively immersed. For instance, it is known[1] that every geodesic on a homogeneous space is either injective or periodic, meaning that all one-dimensional totally geodesic submanifolds in it are injectively immersed.

Remark 5.26. For the sake of completeness, let us include the proof of the statement above. Let M be a Riemannian homogeneous space and consider a unit speed geodesic $\gamma\colon \mathbb{R} \to M$ that is not injective. Without loss of generality, we may assume that there exists a $T > 0$ satisfying $\gamma(0) = \gamma(T) = p$. Because M is homogeneous, we may choose a Killing vector field $X \in \mathcal{K}(M)$ satisfying $X_p = \gamma'(0)$. The restriction $J(t) = X_{\gamma(t)}$ is a *Jacobi vector field* along γ (that is, it satisfies the differential equation $\nabla^2_{\gamma'} J + R(J, \gamma')\gamma' = 0$), and from this it is easy to show that there exist constants a, $b \in \mathbb{R}$ such that $\langle J(t), \gamma'(t)\rangle = at + b$ for all $t \in \mathbb{R}$. By our construction, we have $b = |\gamma'(0)|^2 = 1$ and $a = \langle \nabla_{\gamma'(0)} X, \gamma'(0)\rangle = 0$, so $\langle J, \gamma'\rangle$ is constantly equal to one. In other words, $\gamma'(t)$ is the orthogonal projection of $J(t) = X_{\gamma(t)}$ to $\mathbb{R}\gamma'(t)$ for all t, so there exists a normal vector field $\xi(t)$ along γ such that $X_{\gamma(t)} = \gamma'(t) + \xi(t)$. Letting $t = T$ we obtain $X_p = \gamma'(T) + \xi(T)$, and thus $1 = |X_p|^2 = 1 + |\xi(T)|^2$, giving $\xi(T) = 0$ and $\gamma'(T) = X_p = \gamma'(0)$. Consequently, $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$, so γ is periodic and descends to an injective immersion of a circle to M.

In contrast, one can already find examples of (compatible) totally geodesic immersions from a complete surface to a homogeneous space that are not injective (an example of this phenomenon

[1] While we do not know of a reference with an explicit proof of this result, it seems to be *vox populi* within the field. The statement and proof presented here were communicated to us by Carlos E. Olmos.

will be provided in Subsection 6.3.2). This shows that much more care is needed when handling totally geodesic submanifolds on (naturally reductive) homogeneous spaces.

Perhaps the class of non-symmetric homogeneous spaces that has enjoyed the most attention in this context is that of Lie groups endowed with a left-invariant Riemannian metric. Observe that compact Lie groups with a bi-invariant metric are automatically symmetric spaces, whereas this fact does not remain true for arbitrary left-invariant metrics. Eberlein [58] classified totally geodesic submanifolds of (simply connected, nonsingular) 2-step nilpotent Lie groups. Moreover, Kim, Nikolayevsky and Park [97] gave a partial classification of totally geodesic submanifolds in Damek-Ricci spaces.

In this area, the study of totally geodesic subgroups is quite prominent, as the determination of these subgroups can be done entirely at the Lie algebra level. Indeed, if G is a Lie group endowed with a left-invariant metric, we obtain by restriction a Euclidean inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$, and the Levi-Civita connection restricts to an $\mathbb{R}$-bilinear operator $\nabla \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. A *totally geodesic subalgebra* of $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ satisfying $\nabla_X Y \in \mathfrak{h}$ for all X, $Y \in \mathfrak{h}$. This condition is clearly equivalent to $\mathfrak{h}$ being a Lie subalgebra whose corresponding connected subgroup $\mathsf{H} \subseteq \mathsf{G}$ is a totally geodesic submanifold. Totally geodesic subalgebras of nilpotent Lie algebras have been treated in [37, 38].

At this point, it is important to note that none of the results mentioned above provide fully explicit classifications of totally geodesic submanifolds, which should serve as a testament to the complexity of the problem in general. Some of the non-symmetric homogeneous spaces for which we have a precise description of their totally geodesic submanifolds are Hopf–Berger spheres [135] and, more recently, Stiefel manifolds of orthogonal 2-frames [74].

Let us conclude this chapter with a note on totally geodesic hypersurfaces. The existence of a totally geodesic hypersurface on a Riemannian manifold is quite restrictive, and thus, many authors have worked on understanding the structure of a space admitting such a hypersurface (under varying degrees of ambient symmetry). A first result in this direction was provided by Iwahori [93], who showed that the only irreducible Riemannian symmetric spaces admitting totally geodesic hypersurfaces are those of constant curvature. Later on, it was shown that this statement remains true when the ambient manifold is normal homogeneous [161], and more generally, when it is naturally reductive [164]. Finally, Nikolayevsky [132] described the general structure of a (simply connected) homogeneous space admitting a totally geodesic hypersurface.

Chapter 6
Totally geodesic submanifolds of the homogeneous nearly Kähler 6-manifolds and their G_2-cones

The purpose of this chapter is to derive the classification of totally geodesic submanifolds of the homogeneous strictly nearly Kähler manifolds of dimension 6, as well as those of their cohomogeneity one G_2-cones. Most of the content in this chapter corresponds to a joint work with Alberto Rodríguez-Vázquez (Université Libre de Bruxelles, Belgium) that has been collected in [119]. Furthermore, the results in Subsection 6.4.2 have been obtained during a research visit to Thomas Leistner (University of Adelaide, Australia).

Recall that an almost Hermitian structure on a Riemannian manifold M is a $(1,1)$ tensor field J preserving the Riemannian metric g and satisfying the identity $J^2 = -\operatorname{id}$. Gray and Hervella [80] showed that there are 16 natural classes of almost Hermitian structures. A nice example of those are Kähler structures, which are characterized by the equation $\nabla J = 0$. Among the non-integrable Hermitian structures, the nearly Kähler ones are particularly noteworthy. An almost Hermitian structure J on a Riemannian manifold M is *nearly Kähler* if it satisfies

$$(\nabla_X J)(Y) = -(\nabla_Y J)(X) \quad \text{for all vector fields } X \text{ and } Y \text{ of } M.$$

Nearly Kähler manifolds provide examples of Riemannian manifolds with special weak holonomy. The concept of weak holonomy was introduced by Gray, see [79] for a formal definition. The motivation behind his paper is the celebrated Ambrose-Singer theorem [6], which exhibits a deep connection between the holonomy group of a Riemannian manifold and its curvature tensor. The natural question to ask therefore is whether for a Riemannian manifold M one can find groups other than its holonomy group that provide information about its curvature. It turns out that a Riemannian manifold is nearly Kähler if and only if it has weak holonomy $\mathsf{U}(n)$.

The study of nearly Kähler geometry is particularly interesting in dimension 6, as strictly nearly Kähler 6-manifolds (that is, those which are not Kähler) are automatically Einstein, and their Riemannian cones are 7-manifolds with holonomy groups contained in G_2, see [14]. Indeed, in [35], Bryant constructed the first examples of manifolds with holonomy exactly equal to G_2, one of which is a Riemannian cone over the flag manifold $\mathsf{F}(\mathbb{C}^3)$ equipped with its homogeneous nearly Kähler metric.

Although investigations about nearly Kähler manifolds began in the 1950s, significant progress has been made in recent decades. In 2005, Butruille [36] classified simply connected, homogeneous strictly nearly Kähler manifolds of dimension six. These are:

$$\mathsf{S}^6 = \mathsf{G}_2/\mathsf{SU}(3), \qquad \mathbb{C}\mathsf{P}^3 = \mathsf{Sp}(2)/(\mathsf{U}(1)\times\mathsf{Sp}(1)),$$
$$\mathsf{F}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathsf{T}^2, \qquad \mathsf{S}^3\times\mathsf{S}^3 = \mathsf{SU}(2)^3/\Delta\mathsf{SU}(2).$$

All of the above spaces are examples of 3-symmetric spaces. A consequence of Butruille's list is the positive resolution of Gray and Wolf's conjecture: every homogeneous nearly Kähler manifold (M, J) is a 3-symmetric space for which J is the canonical almost complex structure, see Section 6.1 for further details. Furthermore, Cortés and Vásquez [45] classified all locally homogeneous strictly nearly Kähler manifolds in six dimensions, showing in particular that all of them are quotients of $\mathsf{S}^3 \times \mathsf{S}^3$. More recently, in 2017, Foscolo and Haskins [67] produced the first inhomogeneous nearly Kähler structures on S^6 and $\mathsf{S}^3 \times \mathsf{S}^3$. These structures are of cohomogeneity one and are $\mathsf{SU}(2) \times \mathsf{SU}(2)$-invariant. The aforementioned examples constitute all the known strictly nearly Kähler 6-manifolds to this day.

The main goal of this chapter is to classify totally geodesic submanifolds in homogeneous strictly nearly Kähler 6-manifolds. For this purpose, we develop some general tools for the study of totally geodesic submanifolds of naturally reductive homogeneous spaces. Moreover, we classify maximal totally geodesic submanifolds of the G_2-cones over homogeneous strictly nearly Kähler 6-manifolds.

In this chapter, we introduce the class of D-invariant totally geodesic submanifolds of a reductive homogeneous space, where D denotes the difference tensor, see Subsection 1.3.1. In the setting of naturally reductive homogeneous spaces, which includes symmetric spaces, D-invariant totally geodesic submanifolds are orbits of Lie groups acting isometrically, and they admit a nice algebraic description similar to that of Lie triple systems in symmetric spaces. Furthermore, totally geodesic submanifolds of symmetric spaces are trivially D-invariant. Thus, the class of D-invariant totally geodesic submanifolds seems to be a natural generalization of totally geodesic submanifolds of symmetric spaces. As we will see, not all totally geodesic submanifolds in a naturally reductive homogeneous space are D-invariant. However, it will follow from out classification that all maximal totally geodesic submanifolds in the homogeneous nearly Kähler 6-manifolds are D-invariant.

As we have seen, there are exactly four examples of simply connected homogeneous nearly Kähler 6-manifolds. The sphere $\mathsf{S}^6 = \mathsf{G}_2/\mathsf{SU}(3)$, whose nearly Kähler structure is induced by octonionic multiplication, was the first one that appeared. Since $\mathsf{S}^6 = \mathsf{G}_2/\mathsf{SU}(3)$ is isotropy irreducible, it carries the round metric. Thus, its totally geodesic submanifolds are open parts of intersections of vector subspaces of $\mathbb{R}^7$ passing through the center of the unit sphere $\mathsf{S}^6 \subseteq \mathbb{R}^7$. Each of the remaining three examples appears as the total space of a homogeneous fibration

$$F = \mathsf{K}/\mathsf{H} \to M = \mathsf{G}/\mathsf{H} \to B = \mathsf{G}/\mathsf{K}$$

induced by a triple of compact Lie groups $\mathsf{H} \subseteq \mathsf{K} \subseteq \mathsf{G}$. In the cases of $\mathbb{C}\mathsf{P}^3$ and $\mathsf{F}(\mathbb{C}^3)$, these fibrations are also examples of twistor fibrations. A twistor fibration of an oriented Riemannian 4-manifold N is a fiber bundle $\pi\colon M \to N$, where each fiber over $p \in N$ is equal to the set of complex structures of T_pN which preserve the orientation and the Riemannian metric of N. It turns out that π is an S^2-bundle over N, and the twistor space M admits two different natural almost Hermitian structures. One of them is called the Atiyah–Hitchin–Singer structure and it is integrable if and only if the 4-manifold N is self-dual, see [10]. The other one is the Eells–Salamon structure, which can be obtained from the Atiyah–Hitchin–Singer structure by changing the sign only in the fibers. The nearly Kähler structures of $\mathbb{C}\mathsf{P}^3$ and $\mathsf{F}(\mathbb{C}^3)$ that we are considering are precisely the Eells–Salamon structures that we get when N is the round sphere S^4 and the complex projective plane $\mathbb{C}\mathsf{P}^2$, respectively. Moreover, the celebrated Eells–Salamon correspondence, see [60], states a one-to-one correspondence between (branched) minimal surfaces in N, and non-vertical J-holomorphic curves in M, that is, J-invariant immersions of 2-manifolds in M.

When studying totally geodesic submanifolds of the total space of a Riemannian submersion $F \to M \to B$, it is also relevant to consider their behavior with respect to the underlying Riemannian submersion. Following [135], we say that a totally geodesic submanifold Σ of the total space of a Riemannian submersion M is *well-positioned* if

$$T_p\Sigma = (\mathcal{V}_p \cap T_p\Sigma) \oplus (\mathcal{H}_p \cap T_p\Sigma) \quad \text{for all } p \in \Sigma,$$

where $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal distributions associated with the Riemannian submersion $F \to M \to B$. It turns out that if a totally geodesic submanifold is well-positioned, the metric of the total space can be rescaled in the direction of the fibers while preserving the totally geodesic property for all these new metrics, see [50, Lemma 3.12]. In this work, we find several examples of not well-positioned totally geodesic submanifolds.

We are also interested in understanding the interaction between totally geodesic submanifolds of the nearly Kähler spaces under investigation and their ambient almost complex structure. A way to measure how a submanifold fails to be complex is by using the notion of Kähler angle, see for example [30]. We say that a submanifold Σ of an almost Hermitian manifold M has *constant Kähler angle* $\Phi(\Sigma) = \varphi \in [0, \pi/2]$ if

$$|(Jv)_{T_p\Sigma}|^2 = \cos^2(\phi)|v|^2 \quad \text{for all } v \in T_p\Sigma \text{ and every } p \in M.$$

The submanifolds satisfying $\Phi(\Sigma) = 0$ or $\Phi(\Sigma) = \pi/2$ are exactly those submanifolds which are almost complex or totally real, respectively. An interesting question is to determine the possible constant Kähler angles of the totally geodesic submanifolds of an almost Hermitian manifold. Of course, this question is only interesting for spaces with non-constant curvature, since in $\mathbb{C}^n$ every number in $[0, \pi/2]$ can be realized as the constant Kähler angle of a totally geodesic submanifold. In the setting of Hermitian symmetric spaces, not all totally geodesic submanifolds Σ satisfy $\Phi(\Sigma) \in \{0, \pi/2\}$. For instance, Klein realized in [100] that there is a totally geodesic 2-sphere in the Hermitian symmetric space $\mathsf{G}_2^+(\mathbb{R}^5) = \mathsf{SO}(5)/(\mathsf{SO}(3) \times \mathsf{SO}(2))$ with constant Kähler angle $\arccos(1/5)$. Even more, Rodríguez-Vázquez proved in [148] that every rational number in $[0, 1]$ can be realized as the arccosine of the Kähler angle of a totally geodesic submanifold embedded in a Hermitian symmetric space of large enough rank.

There is a relatively large number of articles focusing on the investigation of totally geodesic submanifolds of nearly Kähler homogeneous 6-manifolds under strong assumptions. In these works, the authors make use of special frames to carry out the classification for Lagrangian totally geodesic submanifolds, or totally geodesic J-holomorphic curves. Tojo [163] showed that every totally geodesic Lagrangian submanifold of a compact 3-symmetric space is extrinsically homogeneous (in fact, D-invariant). In $\mathbb{C}\mathsf{P}^3$, the Lagrangian totally geodesic submanifolds were classified independently by Aslan [9] and Liefsoens [118]. In the flag manifold $\mathsf{F}(\mathbb{C}^3)$, the Lagrangian totally geodesic submanifolds were classified in [156], and the totally geodesic

J-holomorphic curves were classified in [48]. In $\mathsf{S}^3 \times \mathsf{S}^3$, the totally geodesic Lagrangian submanifolds were classified in [55]. Although the authors originally listed six congruence classes of totally geodesic submanifolds in $\mathsf{S}^3 \times \mathsf{S}^3$, there are just two different ones: either a round sphere or a Berger sphere, where the latter was first constructed in [125]. Moreover, Bolton, Dillen, Dioos and Vrancken [29] determined all totally geodesic J-holomorphic curves in $\mathsf{S}^3 \times \mathsf{S}^3$. It is important to remark that there are no known results obstructing the existence of totally geodesic submanifolds Σ of nearly Kähler 6-manifolds when: Σ^3 is not Lagrangian, Σ^2 is not J-holomorphic, or Σ has dimension 4. In this chapter, we generalize the aforementioned partial classifications following an entirely different approach. By employing tools from the theory of Riemannian homogeneous spaces, we address the classification problem of totally geodesic submanifolds in its full generality.

In what follows we state the main results of this chapter. From now on, we denote by $\mathsf{S}^n(r)$ the n-dimensional sphere of radius r, and by $\mathbb{R}\mathsf{P}^n(r)$ its $\mathbb{Z}_2$-quotient under the antipodal map. Moreover, let us consider the sphere S^3 with the Berger metric g_τ given by taking the round metric (of radius one) and rescaling the vertical subspace of the Hopf fibration $\mathsf{S}^1 \to \mathsf{S}^3 \to \mathsf{S}^2$ by a factor of $\tau > 0$. Then $\mathsf{S}^3_{\mathbb{C},\tau}(r)$ denotes the sphere S^3 equipped with the Riemannian metric $r^2 g_\tau$, and we denote by $\mathbb{R}\mathsf{P}^3_{\mathbb{C},\tau}(r)$ its $\mathbb{Z}_2$-quotient. We also denote by $\mathsf{T}^2_\Lambda = \mathbb{R}^2/\Lambda$ the torus induced by a lattice $\Lambda \subseteq \mathbb{R}^2$.

Theorem A. *Let Σ be a complete submanifold of the homogeneous nearly Kähler manifold $\mathbb{C}\mathsf{P}^3 = \mathsf{Sp}(2)/\mathsf{U}(1) \times \mathsf{Sp}(1)$ of dimension $d \geq 2$. Then, Σ is totally geodesic if and only if it is congruent to one of the submanifolds listed in Table 6.1.*

Table 6.1: Totally geodesic submanifolds of $\mathbb{C}\mathsf{P}^3$ of dimension $d \geq 2$.

Submanifold	**Relationship with J**	**Comments**	**Well-positioned?**
$\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)$	Lagrangian	Orbit of $\mathsf{U}(2)$	Yes
$\mathsf{S}^2\left(1/\sqrt{2}\right)$	J-holomorphic	Fiber of $\mathbb{C}\mathsf{P}^3 \to \mathsf{S}^4$	Yes
$\mathsf{S}^2(1)$	J-holomorphic	Orbit of $\mathsf{SU}(2)$	Yes
$\mathsf{S}^2\left(\sqrt{5}\right)$	J-holomorphic	Orbit of $\mathsf{SU}(2)_{\Lambda_3}$	No

As far as we know, the totally geodesic $\mathsf{S}^2(\sqrt{5})$ has not appeared previously in the literature. This is an orbit of the group $\mathsf{SU}(2)_{\Lambda_3}$, which is the maximal connected subgroup of $\mathsf{Sp}(2)$ induced by the 4-dimensional complex irreducible representation of $\mathsf{SU}(2)$ (note that this representation is of symplectic type). All the examples in this theorem are maximal. The non-vertical totally geodesic J-holomorphic curves are $\mathsf{S}^2(1)$ and $\mathsf{S}^2(\sqrt{5})$. Their associated minimal surfaces in S^4 under the Eells–Salamon correspondence are a totally geodesic 2-sphere in S^4, and the Veronese embedding of the projective plane in S^4, respectively.

Theorem B. *Let Σ be a complete submanifold of the homogeneous nearly Kähler manifold $\mathsf{F}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathsf{T}^2$ of dimension $d \geq 2$. Then, Σ is totally geodesic if and only if it is congruent to one of the submanifolds listed in Table 6.2.*

Table 6.2: Totally geodesic submanifolds of $\mathsf{F}(\mathbb{C}^3)$ of dimension $d \geq 2$.

Submanifold	**Relationship with J**	**Comments**	**Well-positioned?**
$\mathsf{F}(\mathbb{R}^3)$	Lagrangian	Orbit of $\mathsf{SO}(3)$	Yes
$\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})$	Lagrangian	Orbit of $\mathsf{SU}(2)$	No
T^2_Λ	J-holomorphic	Orbit of T^2	No
$\mathsf{S}^2\left(1/\sqrt{2}\right)$	J-holomorphic	Fiber of $\mathsf{F}(\mathbb{C}^3) \to \mathbb{C}\mathsf{P}^2$	Yes
$\mathsf{S}^2\left(\sqrt{2}\right)$	J-holomorphic	Orbit of $\mathsf{SO}(3)$	No
$\mathbb{R}\mathsf{P}^2\left(2\sqrt{2}\right)$	Totally real	Not injectively immersed	No

To the best of our knowledge, $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ is the first example in the literature of a totally geodesic immersed submanifold of dimension $d \geq 2$ with self-intersections in a simply connected homogeneous space. Moreover, this is the only non-maximal example, it is not D-invariant, and not extrinsically homogeneous, i.e. an orbit of a subgroup of the isometry group of the ambient space. The non-vertical totally geodesic J-holomorphic curves are T^2_Λ and $\mathsf{S}^2(\sqrt{2})$. Their associated minimal surfaces in $\mathbb{C}\mathsf{P}^2$ under the Eells–Salamon correspondence are the Clifford torus in $\mathbb{C}\mathsf{P}^2$, that is, $\{[z_0 : z_1 : z_2] \in \mathbb{C}\mathsf{P}^2 : |z_0| = |z_1| = |z_2|\}$, and a totally geodesic $\mathbb{R}\mathsf{P}^2$ in $\mathbb{C}\mathsf{P}^2$, respectively.

Theorem C. *Let Σ be a complete submanifold of the homogeneous nearly Kähler manifold $\mathsf{S}^3 \times \mathsf{S}^3 = \mathsf{SU}(2)^3/\Delta\mathsf{SU}(2)$ of dimension $d \geq 2$. Then, Σ is totally geodesic if and only if it is congruent to one of the submanifolds listed in Table 6.3.*

Table 6.3: Totally geodesic submanifolds of $\mathsf{S}^3 \times \mathsf{S}^3$ of dimension $d \geq 2$.

Submanifold	**Relationship with J**	**Comments**	**Well-positioned?**
$\mathsf{S}^3\left(2/\sqrt{3}\right)$	Lagrangian	Fiber of $\mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3$	Yes
$\mathsf{S}^3_{\mathbb{C},1/3}(2)$	Lagrangian	Orbit of $\Delta_{1,3}\mathsf{SU}(2) \times \mathsf{SU}(2)_2$	Yes
T^2_Γ	J-holomorphic	Orbit of a two-dimensional torus	Yes
$\mathsf{S}^2(\sqrt{3/2})$	J-holomorphic	Orbit of $\Delta\mathsf{SU}(2)$	No
$\mathsf{S}^2\left(2/\sqrt{3}\right)$	Totally real	Orbit of $\Delta\mathsf{SU}(2)$	Yes

Interestingly, $\mathsf{S}^2(2/\sqrt{3})$ is not a D-invariant totally geodesic submanifold, but is extrinsically homogeneous. This together with the characterization of D-invariant totally geodesic submanifolds given in Theorem 6.3 gives a counterexample to Proposition 2 in [2], see Remark 6.16. Furthermore, $\mathsf{S}^2(2/\sqrt{3})$ is the only non-maximal example in the list above.

As a consequence of Theorem A, Theorem B, and Theorem C, we have:

Corollary D. *Let Σ be a maximal totally geodesic submanifold of a homogeneous nearly Kähler 6-manifold of non-constant curvature. Then the following statements hold:*

(i) *if Σ has dimension two, then Σ is a J-holomorphic curve.*

(ii) *if Σ has dimension three, then Σ is a Lagrangian submanifold.*

Thus, every totally geodesic submanifold Σ of a homogeneous nearly Kähler 6-manifold of non-constant curvature satisfies $\Phi(\Sigma) \in \{0, \pi/2\}$. This raises the question whether this also holds for (not necessarily homogeneous) irreducible strictly nearly Kähler manifolds.

In this chapter, we also study totally geodesic submanifolds of Riemannian cones. It can be checked that for every totally geodesic submanifold Σ of a Riemannian manifold M, the cone over Σ is a totally geodesic submanifold of the cone over M, see Lemma 6.22. However, there might be totally geodesic submanifolds of a Riemannian cone that do not arise as cones, see Section 6.4.2. In Section 6.4, we prove a structure result for totally geodesic submanifolds in cones, see Theorem 6.25. As a consequence of this, we deduce that maximal totally geodesic submanifolds of Riemannian cones are either cones over a totally geodesic submanifold or totally geodesic hypersurfaces, see Corollary 6.26. It was observed in [14] that Riemannian cones are intimately linked to special holonomy. For instance, a special class of Sasakian[1] manifolds is that of Sasakian–Einstein manifolds, whose investigation has led to the construction of many inhomogenous Einstein metrics on spheres, see [33]. It turns out that the holonomy of the cone over a Sasakian–Einstein manifold is contained in $\mathsf{SU}(n)$. Similarly, Riemannian cones over strictly nearly Kähler 6-manifolds have its holonomy contained in G_2. This holonomy reduction is equivalent to the existence of a parallel 3-form ϕ that is locally modeled on the associative 3-form on $\mathbb{R}^7$, or alternatively, the existence of a torsion-free G_2-structure. Another class of G_2-structures defined on 7-dimensional manifolds is that of nearly parallel G_2-structures. A G_2-structure ϕ is *nearly parallel* if it satisfies $\star d\phi = c\phi$, for $c \in \mathbb{R} \setminus \{0\}$, where $\star$ denotes the Hodge star operator. It can also be proved that cones over nearly parallel G_2-manifolds have holonomy contained in $\mathsf{Spin}(7)$. Indeed, the first examples of manifolds with exceptional holonomy G_2 and $\mathsf{Spin}(7)$ were constructed in [35], and they are cones over the homogeneous nearly Kähler 6-manifold $\mathsf{F}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathsf{T}^2$, and the homogeneous nearly parallel G_2-manifold $\mathsf{B}^7 = \mathsf{SO}(5)/\mathsf{SO}(3)$. We prove that Sasakian–Einstein, strictly nearly Kähler 6-manifolds, and nearly parallel G_2-manifolds do not admit totally geodesic hypersurfaces, see Theorem 6.23.

In view of the structure theorem for maximal totally geodesic submanifolds in Riemannian cones, it is clear that the case of hypersurfaces deserves special attention. It turns out that these hypersurfaces are related to the so-called *Obata equation* $\operatorname{Hess} h = -hg$, where g denotes the metric of the base space and $\operatorname{Hess} h$ denotes the Hessian of the smooth function $h \in \mathcal{C}^\infty(M)$. More precisely, given a totally geodesic hypersurface of a Riemannian cone, it is either the cone of a totally geodesic hypersurface of the base space or it is given locally by the graph of a smooth function whose reciprocal satisfies the Obata equation, see Theorem 6.31.

[1] We say that a Riemannian manifold M is *Sasakian* if its Riemannian cone is Kähler.

The Obata equation was first introduced in [133], and it sits as the cornerstone of many rigidity theorems in Riemannian geometry. Most notably, the Obata rigidity theorem [133, Theorem A] states that given any $r > 0$, the only n-dimensional complete Riemannian manifold admitting a globally defined solution h of the equation $\operatorname{Hess} h = -r^2 hg$ is the round sphere $\mathsf{S}^n(r)$. One can find generalizations of the Obata rigidity theorem in [85, 159, 170]. In the context of determining totally geodesic hypersurfaces of Riemannian cones, we need to understand the existence of local solutions for this equation. Brinkmann [34] proved that if a Riemannian manifold M admits a smooth function $h \in \mathcal{C}^\infty(M)$ whose Hessian is conformal to the metric (that is, $\operatorname{Hess} h = \lambda g$ for some $\lambda \in \mathcal{C}^\infty(M)$), then M is locally a warped product of an interval with another Riemannian manifold. As we will see, the existence of a local solution forces the base space to be locally isometric to a sine-cone, see Theorem 6.32.

Moreover, we consider the classification problem of totally geodesic submanifolds in cones with holonomy G_2 over homogeneous nearly Kähler manifolds. As a consequence of Corollary 6.20, these are the only Riemannian cones with holonomy equal to G_2 equipped with a metric of cohomogeneity one, and thus with the highest possible degree of symmetry, contributing to a rich presence of totally geodesic submanifolds.

Theorem E. *Let M be a homogeneous nearly Kähler 6-manifold of non-constant curvature and let Σ be a maximal totally geodesic submanifold of the G_2-cone $\widehat{M}$ over M of dimension greater than one. Then Σ is the Riemannian cone of a maximal totally geodesic submanifold S of M.*

Notice that combining Theorem E with the classification of totally geodesic submanifolds in cones over space forms (see Proposition 6.28) and three-dimensional Berger spheres (see Proposition 6.30), one can list all totally geodesic submanifolds in cohomogeneity one G_2-cones and thus obtain the full classification.

Moreover, Riemannian cones over J-holomorphic curves or Lagrangian submanifolds of a nearly Kähler 6-manifold give rise to associative or coassociative manifolds of the corresponding G_2-cone over M, respectively. By definition, *associative* and *coassociative* submanifolds are the submanifolds of a G_2-manifold calibrated by ϕ and the Hodge dual of ϕ, respectively; see [84] and [95, Chapters 4 and 12] for an introduction to calibrated geometry. As a consequence of Corollary D and Theorem E, one has the following:

Corollary F. *Let Σ be a maximal totally geodesic submanifold of the G_2-cone over a homogeneous nearly Kähler 6-manifold of non-constant curvature. Then the following statements hold:*

(i) *if Σ has dimension three, then Σ is an associative submanifold.*

(ii) *if Σ has dimension four, then Σ is a coassociative submanifold.*

Both Corollaries D and F seem to indicate that for totally geodesic submanifolds of nearly Kähler 6-manifolds and their G_2-cones, there is a strong link between this purely Riemannian property and the underlying nearly Kähler and G_2-structures, respectively. Consequently, we find that it would be very interesting to investigate whether both corollaries hold true without the homogeneity assumptions.

Let us briefly describe the structure of this chapter. Firstly, in Section 6.1 we introduce the homogeneous nearly Kähler 6-manifolds that we will be working with. These are presented both as 3-symmetric spaces and as total spaces of homogeneous fibrations. In Section 6.2 we develop novel techniques for the classification of totally geodesic submanifolds in naturally reductive homogeneous spaces. In particular, we introduce the class of D-invariant totally geodesic submanifolds on the one hand, while on the other hand we provide an algebraic criterion to determine when a totally geodesic submanifold is well-positioned with respect to a homogeneous fibration. Section 6.3 is dedicated to presenting the examples of totally geodesic submanifolds appearing in Theorems A, B and C, and describing their relationship with the ambient almost complex structure and homogeneous fibration. In Section 6.4 we recall the basic geometric properties of Riemannian cones and derive a structural result for their totally geodesic submanifolds. Finally, in Section 6.5 we provide the proofs of the main theorems.

6.1 Homogeneous nearly Kähler manifolds

In this section we present the ambient spaces that we work with throughout the rest of this chapter. The classification of homogeneous nearly Kähler manifolds in dimension six was done by Butruille [36]. Indeed, every simply connected Riemannian manifold satisfying the previous conditions is homothetic to either the sphere S^6, the complex projective space $\mathbb{C}\mathsf{P}^3$, the flag manifold $\mathsf{F}(\mathbb{C}^3)$, or $\mathsf{S}^3 \times \mathsf{S}^3$. Since S^6 carries its natural round metric, its totally geodesic submanifolds are well-known, so we only need to focus on the other three spaces. It turns out that these manifolds are examples of 3-symmetric spaces. The main reference for the description of 3-symmetric spaces is [81].

Let (M, J) be an almost Hermitian manifold. We say that M is *nearly Kähler* if for every vector field $X \in \mathfrak{X}(M)$ we have $(\nabla_X J)X = 0$. Moreover, M is *strictly nearly Kähler* if it is nearly Kähler and $\nabla_X J \neq 0$ for all nonzero $X \in TM$. One sees that a six-dimensional nearly Kähler manifold is strictly nearly Kähler if and only if it is not Kähler. On the other hand, we say that a 3-*symmetric space* M is a connected Riemannian manifold M together with a family of isometries $s_p \colon M \to M$ for each $p \in M$ that satisfy the following conditions: $s_p^3 = \mathrm{id}_M$ for all $p \in M$, p is an isolated fixed point of s_p, and each s_p is holomorphic with respect to the so-called *canonical almost complex structure* J defined via

$$(s_p)_{*p} = -\frac{1}{2}\,\mathrm{id}_{T_pM} + \frac{\sqrt{3}}{2} J_p, \quad p \in M. \tag{6.1}$$

Any 3-symmetric space is automatically homogeneous [77, Theorem 4.8]. Conversely, one can construct a 3-symmetric space in terms of algebraic data. Indeed, let G be a connected Lie group and K a closed subgroup of G. Assume that there exists an automorphism $\Theta \colon \mathsf{G} \to \mathsf{G}$ of order three such that $\mathsf{G}_0^\Theta \subseteq \mathsf{K} \subseteq \mathsf{G}^\Theta$, where G^Θ is the fixed point set of Θ and G_0^Θ is its identity component. It turns out that $M = \mathsf{G}/\mathsf{K}$ is a reductive homogeneous space in a way that $\theta = \Theta_*$ preserves the reductive complement. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a reductive decomposition of $\mathfrak{g}$ satisfying $\theta\mathfrak{p} = \mathfrak{p}$. Then, any inner product on $\mathfrak{p}$ that is invariant under $\mathrm{Ad}(\mathsf{K})$ and θ gives rise to a G-invariant metric on M that turns M into a 3-symmetric space, where the isometry of order three at $o = e\mathsf{K}$ is given by $s_o(x\mathsf{K}) = \Theta(x)\mathsf{K}$. We say that $(\mathsf{G}, \mathsf{K}, \Theta)$ is the *triple* associated with the 3-symmetric space M. The corresponding almost complex structure is the G-invariant tensor field J defined at o by (6.1). By [77, Proposition 5.6], the almost Hermitian manifold (M, J) is nearly Kähler if and only if the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is naturally reductive.

We now proceed to describe our six-dimensional examples, exhibiting them as 3-symmetric spaces.

The complex projective space $\mathbb{C}\mathsf{P}^3$

Consider $\mathbb{H}^2$ as a right $\mathbb{C}$-vector space, so that the projective space $\mathsf{P}(\mathbb{H}^2)$ is exactly $\mathbb{C}\mathsf{P}^3$. The natural action of $\mathsf{G} = \mathsf{Sp}(2)$ on $\mathbb{C}\mathsf{P}^3$ is transitive, and the isotropy subgroup of $o = [1 : 0]$ is $\mathsf{K} = \mathsf{U}(1) \times \mathsf{Sp}(1)$, so that $\mathbb{C}\mathsf{P}^3$ can be viewed as the quotient $\mathsf{Sp}(2)/\mathsf{U}(1) \times \mathsf{Sp}(1)$. The Killing form of $\mathfrak{g} = \mathfrak{sp}(2)$ is $\mathcal{B}(X,Y) = 12\,\mathrm{Re}\,\mathrm{tr}_{\mathbb{H}}(XY)$, so $-\mathcal{B}$ is an $\mathrm{Ad}(\mathsf{Sp}(2))$-invariant inner product in $\mathfrak{g}$, but we renormalize it so that the inner product on $\mathfrak{g}$ is $\langle X, Y\rangle = -2\,\mathrm{Re}\,\mathrm{tr}_{\mathbb{H}}(XY)$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{u}(1) \oplus \mathfrak{sp}(1)$ in $\mathfrak{sp}(2)$. Once again, we endow $\mathbb{C}\mathsf{P}^3$ with the homogeneous metric induced by the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{p}$. We also consider the element $\omega = \mathrm{diag}(e^{\frac{2\pi i}{3}}, 1) \in \mathsf{K}$. Then the conjugation $\Theta = I_\omega$ defines an inner automorphism of order three in G, whose fixed point set is $\mathsf{Sp}(2)^\Theta = \mathsf{U}(1) \times \mathsf{Sp}(1)$, and $(\mathsf{Sp}(2), \mathsf{U}(1) \times \mathsf{Sp}(1), \Theta)$ is the triple associated with the 3-symmetric space $\mathbb{C}\mathsf{P}^3$. The nearly Kähler complex structure J is defined as $J = \frac{2}{\sqrt{3}}\Theta_* + \frac{1}{\sqrt{3}}\,\mathrm{id}_{\mathfrak{p}}$.

We use the orthonormal basis $\{e_1, \dots, e_6\}$ of $\mathfrak{p}$ defined by

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$e_4 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad e_5 = \frac{1}{2}\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \qquad e_6 = \frac{1}{2}\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

The isotropy representation allows us to decompose $\mathfrak{p}$ as the direct sum of two irreducible submodules $\mathfrak{p}_1 = \mathrm{span}\{e_1, e_2\}$ and $\mathfrak{p}_2 = \mathrm{span}\{e_3, \dots, e_6\}$. Indeed, the subrepresentation $\mathfrak{p}_1$ of $\mathsf{U}(1) \times \mathsf{Sp}(1)$ is isomorphic to $\mathbb{C}$ with the action given by $(\lambda, \mu) \cdot z = \lambda^2 z$, whereas $\mathfrak{p}_2$ is isomorphic to the representation $\mathbb{R}^4$ of $\mathsf{U}(1) \times \mathsf{Sp}(1)$ under $(\lambda, \mu)x = \mu x \bar{\lambda}$. In particular, $\mathsf{U}(1) \times \mathsf{Sp}(1)$ acts transitively on the unit sphere of each $\mathfrak{p}_i$.

The isometry group of $\mathbb{C}\mathsf{P}^3$ is $I(\mathbb{C}\mathsf{P}^3) = (\mathsf{Sp}(2)/\mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where the outer $\mathbb{Z}_2$ is generated by conjugation by $\mathrm{diag}(j, 1) \in \mathsf{Sp}(2)$ (see for example [152]).

Now, consider the chain of subgroups $\mathsf{U}(1) \times \mathsf{Sp}(1) \subseteq \mathsf{Sp}(1) \times \mathsf{Sp}(1) \subseteq \mathsf{Sp}(2)$. This gives rise to the homogeneous fibration $\mathbb{C}\mathsf{P}^1 \to \mathbb{C}\mathsf{P}^3 \to \mathbb{H}\mathsf{P}^1 = \mathsf{S}^4$, which is precisely the twistor fibration, whose fiber is a totally geodesic $\mathbb{C}\mathsf{P}^1 = \mathsf{S}^2$. The decomposition of $\mathfrak{p}$ into the vertical and horizontal subspaces of this submersion is given by $\mathcal{V}_o = \mathfrak{p}_1$ and $\mathcal{H}_o = \mathfrak{p}_2$.

The flag manifold $\mathsf{F}(\mathbb{C}^3)$

Recall that a full flag in $\mathbb{C}^3$ is a chain $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = \mathbb{C}^3$ (also denoted by (V_1, V_2)) of subspaces such that $\dim_{\mathbb{C}} V_k = k$ for each k. We denote by $\mathsf{F}(\mathbb{C}^3)$ the space of all flags in $\mathbb{C}^3$, which is naturally identified with the quotient of the Stiefel manifold of orthonormal bases of $\mathbb{C}^3$ under the standard action of $\mathsf{U}(1)^3$. The group $\mathsf{G} = \mathsf{SU}(3)$ acts transitively on $\mathsf{F}(\mathbb{C}^3)$, and if o denotes the standard flag $0 \subseteq \mathbb{C}e_1 \subseteq \operatorname{span}\{e_1, e_2\} \subseteq \mathbb{C}^3$, its isotropy subgroup is the maximal torus T^2 of diagonal matrices in $\mathsf{SU}(3)$, so we have $\mathsf{F}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathsf{T}^2$.

Let us endow $\mathsf{F}(\mathbb{C}^3)$ with a reductive decomposition and a Riemannian metric. Note that the Killing form of $\mathfrak{g} = \mathfrak{su}(3)$ satisfies $\mathcal{B}(X,Y) = 6\operatorname{tr}(XY)$ for all $X, Y \in \mathfrak{g}$. As a consequence, the negative Killing form gives a bi-invariant metric on $\mathsf{SU}(3)$. However, for the sake of convenience, we rescale this metric so that the inner product in $\mathfrak{g}$ is $\langle X, Y\rangle = -\operatorname{tr}(XY)$ for all $X, Y \in \mathfrak{su}(3)$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{t} = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ in $\mathfrak{g}$. Then, the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{p}$ induces an $\operatorname{Ad}(\mathsf{T}^2)$-invariant inner product on $\mathfrak{p}$, that is, a G-invariant metric on $\mathsf{F}(\mathbb{C}^3)$. This metric is homothetic to the standard homogeneous metric on M. We also consider the automorphism $\Theta = I_\omega\colon \mathsf{SU}(3) \to \mathsf{SU}(3)$, where $\omega = \operatorname{diag}(e^{\frac{2\pi i}{3}}, 1, e^{-\frac{2\pi i}{3}}) \in \mathsf{T}^2$. Then Θ is an automorphism of order three, whose fixed point set is precisely $\mathsf{SU}(3)^\Theta = \mathsf{T}^2$, so $(\mathsf{SU}(3), \mathsf{T}^2, \Theta)$ is the triple associated with the 3-symmetric space $\mathsf{F}(\mathbb{C}^3)$. The corresponding almost complex structure J at $\mathfrak{p}$ is determined by the equation $J = \frac{2}{\sqrt{3}}\Theta_* + \frac{1}{\sqrt{3}}\operatorname{id}_{\mathfrak{p}}$. We consider the orthonormal basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{p}$, where

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad e_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},$$

$$e_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{pmatrix}, \qquad e_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}, \qquad e_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}.$$

It is easy to check that the tangent space $\mathfrak{p}$ splits as the direct sum of irreducible submodules $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$, where each $\mathfrak{p}_k = \operatorname{span}\{e_{2k-1}, e_{2k}\}$ is isomorphic to $\mathbb{C}$. To be more precise, if $g = \operatorname{diag}(e^{ix}, e^{iy}, e^{-i(x+y)})$ is an arbitrary element of T^2, then $\operatorname{Ad}(g)$ acts on $\mathfrak{p}_1$ as multiplication by $e^{i(x-y)}$, on $\mathfrak{p}_2$ as multiplication by $e^{i(x+2y)}$, and on $\mathfrak{p}_3$ as multiplication by $e^{i(2x+y)}$. Note that $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are pairwise non-isomorphic as representations of T^2. Furthermore, if $g \in \mathsf{U}(3)$ is a permutation matrix, then the map $a\mathsf{T}^2 \mapsto gag^{-1}\mathsf{T}^2$ is an isometry fixing o and whose differential at o permutes the irreducible submodules of $\mathfrak{p}$, and every permutation of these submodules can be achieved in this way. For example, the transposition $(1,2)$ interchanges $\mathfrak{p}_2$ and $\mathfrak{p}_3$ and the cycle $(1,2,3)$ sends each $\mathfrak{p}_i$ to $\mathfrak{p}_{i+1}$ (where we are taking indices modulo 3).

Consider the chain of inclusions $\mathsf{T}^2 \subseteq \mathsf{U}(2) \subseteq \mathsf{SU}(3)$. The corresponding homogeneous fibration is $\mathbb{C}\mathsf{P}^1 \to \mathsf{F}(\mathbb{C}^3) \to \mathbb{C}\mathsf{P}^2$ (explicitly, it takes the flag $(V_1, V_2) \in \mathsf{F}(\mathbb{C}^3)$ to $V_2^\perp \in \mathbb{C}\mathsf{P}^2$) and the fiber $\mathbb{C}\mathsf{P}^1 = \mathsf{U}(2)\cdot o$ is totally geodesic. The vertical and horizontal subspaces of this fibration at o are precisely $\mathcal{V}_o = \mathfrak{p}_1$ and $\mathcal{H}_o = \mathfrak{p}_2 \oplus \mathfrak{p}_3$.

We now determine the full isometry group of $\mathsf{F}(\mathbb{C}^3)$. This computation was done by Shankar in [152] when $\mathsf{F}(\mathbb{C}^3)$ carries a metric of positive sectional curvature. However, the homogeneous metric that we are considering in $\mathsf{F}(\mathbb{C}^3)$ does not have positive sectional curvature. In our case, we may calculate the isometry group of $\mathsf{F}(\mathbb{C}^3)$ via the following approach (based on the proof of [139, §4, Proposition 6 and §16, Theorem 3]). Firstly, the effectivized version of the presentation $\mathsf{SU}(3)/\mathsf{T}^2$ is $\mathsf{PSU}(3)/(\mathsf{T}^2/\mathbb{Z}_3)$, and we may apply [167, Theorem 5.1] to conclude that $I^0(\mathsf{F}(\mathbb{C}^3)) = \mathsf{PSU}(3)$. As for the group of components $I(\mathsf{F}(\mathbb{C}^3))/I^0(\mathsf{F}(\mathbb{C}^3))$, since the flag manifold is simply connected, this group is equal to H/H^0, where H is the isotropy subgroup of $I(\mathsf{F}(\mathbb{C}^3))$ at o and H^0 its identity component. This follows from the long exact sequence of homotopy groups associated with the fibration $\mathsf{H} \hookrightarrow I(\mathsf{F}(\mathbb{C}^3)) \to \mathsf{F}(\mathbb{C}^3)$. Now, the conjugation map

$$C\colon \mathsf{H} \to \mathsf{Aut}(\mathsf{PSU}(3), \mathsf{T}^2/\mathbb{Z}_3) = \{\varphi \in \mathsf{Aut}(\mathsf{PSU}(3)) : \varphi \text{ preserves } \mathsf{T}^2/\mathbb{Z}_3\}$$

is injective by [152, Proposition 1.7]. In addition, any $\varphi \in \mathsf{Aut}(\mathsf{PSU}(3), \mathsf{T}^2/\mathbb{Z}_3)$ descends to a diffeomorphism $\bar{\varphi}$ of $\mathsf{F}(\mathbb{C}^3)$, which is actually an isometry because φ preserves the Killing form and the metric on the flag manifold is induced by it. It is easy to show that $\bar{\varphi} \in \mathsf{H}$ and $C(\bar{\varphi}) = \varphi$, so C is an isomorphism and $\mathsf{H} = \mathsf{Aut}(\mathsf{PSU}(3), \mathsf{T}^2/\mathbb{Z}_3)$. Now, since $\mathsf{Aut}(\mathsf{PSU}(3)) = \mathrm{Ad}(\mathsf{PSU}(3)) \rtimes \mathbb{Z}_2$, where the outer $\mathbb{Z}_2$ is generated by complex conjugation, the computation of H/H^0 reduces to that of $\frac{\mathsf{N}_{\mathsf{PSU}(3)}(\mathsf{T}^2/\mathbb{Z}_3)}{\mathsf{Z}_{\mathsf{PSU}(3)}(\mathsf{T}^2/\mathbb{Z}_3)} \rtimes \mathbb{Z}_2$. The first factor is merely the Weyl group $\mathsf{W}(\mathsf{PSU}(3)) = \mathfrak{S}_3$, so we have obtained $\mathsf{H}/\mathsf{H}^0 = \mathfrak{S}_3 \rtimes \mathbb{Z}_2$, and we conclude that the full isometry group is $I(\mathsf{F}(\mathbb{C}^3)) = \mathsf{PSU}(3) \rtimes (\mathfrak{S}_3 \rtimes \mathbb{Z}_2)$. We remark that an alternative and more specific approach for computing the isometry group of $\mathsf{F}(\mathbb{C}^3)$ was described in the note [7].

The almost product $\mathsf{S}^3 \times \mathsf{S}^3$

This space is obtained via the Ledger-Obata construction from the group $\mathsf{SU}(2)$ (see for example [112]). We consider the product $\mathsf{G} = \mathsf{SU}(2)^3$ and the subgroup $\mathsf{K} = \Delta\mathsf{SU}(2)$ obtained by embedding $\mathsf{SU}(2)$ diagonally in G. Then the group G acts on $M = \mathsf{S}^3 \times \mathsf{S}^3 = \mathsf{SU}(2) \times \mathsf{SU}(2)$ via the equation $(g, h, k) \cdot (x, y) = (gxk^{-1}, hyk^{-1})$, and the isotropy subgroup at $o = (I, I)$ is K, so we obtain that $M = \mathsf{G}/\mathsf{K}$. The Killing form of $\mathfrak{su}(2)$ is $\mathcal{B}(X, Y) = 4\,\mathrm{tr}(XY)$, and the direct sum $\mathcal{B} \oplus \mathcal{B} \oplus \mathcal{B}$ is precisely the Killing form of $\mathfrak{g}$ whose inverse yields the standard homogeneous metric on M. Similarly to the previous cases, we consider the renormalized metric given by $\langle (X_1, X_2, X_3), (Y_1, Y_2, Y_3) \rangle = -\,\mathrm{tr}(X_1Y_1) - \mathrm{tr}(X_2Y_2) - \mathrm{tr}(X_3Y_3)$. We denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k} = \Delta\mathfrak{su}(2)$ in $\mathfrak{g}$ and we consider the order three automorphism $\Theta\colon \mathsf{G} \to \mathsf{G}$ defined by $\Theta(g, h, k) = (h, k, g)$. The nearly Kähler complex structure J is given as $J = \frac{2}{\sqrt{3}}\Theta_* + \frac{1}{\sqrt{3}}\,\mathrm{id}_{\mathfrak{p}}$. Furthermore, if $L_g\colon \mathsf{S}^3 \to \mathsf{S}^3$ denotes left multiplication by $g \in \mathsf{S}^3$, the almost product structure of $\mathsf{S}^3 \times \mathsf{S}^3$ is the G-invariant tensor P of type $(1, 1)$ defined by

$$P(v, w) = \left((L_{ab^{-1}})_{*b}\, w, (L_{ba^{-1}})_{*a}\, v\right), \quad v \in T_a\mathsf{S}^3, w \in T_b\mathsf{S}^3.$$

The restriction of P to $T_o(\mathsf{S}^3 \times \mathsf{S}^3)$ is identified with the $\mathrm{Ad}(\mathsf{K})$-invariant map $P\colon \mathfrak{p} \to \mathfrak{p}$ given by $P(X, Y, Z) = (Y, X, Z)$.

Consider the following basis of $\mathfrak{su}(2)$:

$$H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{6.2}$$

Then we can give a basis $\{e_1,\dots,e_6\}$ of $\mathfrak{p}$ as follows:

$$e_1=\frac{1}{\sqrt{12}}(H,-2H,H), \qquad e_2=\frac{1}{\sqrt{12}}(E,-2E,E), \qquad e_3=\frac{1}{\sqrt{12}}(F,-2F,F),$$
$$e_4=\frac{1}{2}(H,0,-H), \qquad e_5=\frac{1}{2}(E,0,-E), \qquad e_6=\frac{1}{2}(F,0,-F).$$

Consider the inclusions $\Delta\mathsf{SU}(2)\subseteq\Delta_{1,3}\mathsf{SU}(2)\times\mathsf{SU}(2)_2\subseteq\mathsf{SU}(2)\times\mathsf{SU}(2)\times\mathsf{SU}(2)$. These give rise to the homogeneous fibration $\mathsf{S}^3\hookrightarrow\mathsf{S}^3\times\mathsf{S}^3\to\mathsf{S}^3$ defined by the projection on the first factor. Once again the fibers are totally geodesic. The vertical and horizontal subspaces at o are

$$\mathcal{V}_o=\mathfrak{p}_1=\operatorname{span}\{e_1,e_2,e_3\}, \quad \mathcal{H}_o=\mathfrak{p}_2=\operatorname{span}\{e_4,e_5,e_6\}.$$

The isometry group of $\mathsf{S}^3\times\mathsf{S}^3$ is $I(\mathsf{S}^3\times\mathsf{S}^3)=(\mathsf{SU}(2)^3/\Delta\mathbb{Z}_2)\rtimes\mathfrak{S}_3$ where $\mathfrak{S}_3$ denotes the symmetric group on three elements acting in the natural manner on $\mathsf{S}^3\times\mathsf{S}^3$, see for example [166, Lemma 3.3] for a proof.

6.2 Totally geodesic immersions in naturally reductive homogeneous spaces

In this section, we introduce new techniques for studying totally geodesic submanifolds in naturally reductive homogeneous spaces. Because homogeneous spaces are complete and real analytic, by Corollary 5.19 their inextendable totally geodesic submanifolds are complete. These submanifolds are also homogeneous as Riemannian manifolds due to Proposition 5.21, but they need not be extrinsically homogeneous.

We denote by $M=\mathsf{G}/\mathsf{K}$ a naturally reductive homogeneous space endowed with a reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Since M is homogeneous, we may only consider totally geodesic submanifolds passing through $o=e\mathsf{K}$, which is equivalent to studying totally geodesic subspaces of $T_oM\equiv\mathfrak{p}$. In this setting, we have the following characterization of these subspaces due to Tojo:

Theorem 6.1 (Tojo's criterion [162])**.** *Let $M=\mathsf{G}/\mathsf{K}$ be a naturally reductive homogeneous space with reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Assume $\mathfrak{v}\subseteq\mathfrak{p}$ is a vector subspace and consider for each $X\in\mathfrak{v}$ the operator $D_X\colon\mathfrak{p}\to\mathfrak{p}$. Then, the following conditions are equivalent:*

(i) *There exists a totally geodesic submanifold Σ of M passing through o with tangent space $\mathfrak{v}$.*

(ii) *For each $X\in\mathfrak{v}$, we have $R(X,e^{-D_X}\mathfrak{v})e^{-D_X}\mathfrak{v}\subseteq e^{-D_X}\mathfrak{v}$.*

(iii) *For each $X\in\mathfrak{v}$, the subspace $e^{-D_X}\mathfrak{v}$ is R-invariant.*

We now give a geometric interpretation of the subspace $e^{-tD_X}\mathfrak{v}$. Consider the geodesic $\gamma(t)=\operatorname{Exp}(tX)\cdot o$ with initial condition $X\in\mathfrak{v}$. Then there are two vector space isomorphisms that we can establish between $\mathfrak{p}=T_oM$ and $T_{\gamma(t)}M$: parallel translation $\mathcal{P}^{\gamma}_{0,t}\colon T_oM\to T_{\gamma(t)}M$

and the pushforward of the flow of X^*, given by $\operatorname{Exp}(tX)_{*o}\colon T_oM \to T_{\gamma(t)}M$. Due to (1.2), both maps are related by

$$\operatorname{Exp}(tX)_{*o}^{-1} \circ \mathcal{P}_{0,t}^{\gamma} = e^{-tD_X}. \tag{6.3}$$

Suppose that $\mathfrak{v}$ is totally geodesic. Let $f\colon \Sigma \to M$ be the complete totally geodesic immersion such that $\mathfrak{v} = \widetilde{f}(p)$ and take $v = (f_{*p})^{-1}(X) \in T_p\Sigma$, $g = \operatorname{Exp}(-tX)$. From (6.3) and the fact that f commutes with parallel translations we see that $e^{-tD_X}\mathfrak{v} = \widetilde{g \circ f}(\exp_p(tv))$, yielding the following result:

Corollary 6.2 [162, Proposition 3.5]**.** *If $\mathfrak{v} \subseteq \mathfrak{p}$ is a totally geodesic subspace, then for every $X \in \mathfrak{v}$ the subspace $e^{-D_X}\mathfrak{v}$ is also totally geodesic, and the corresponding totally geodesic submanifolds are congruent.*

6.2.1 Totally geodesic submanifolds invariant under D

We now study a particular class of totally geodesic submanifolds of $M = \mathsf{G}/\mathsf{K}$. Consider the canonical connection ∇^c associated with the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the difference tensor $D = \nabla - \nabla^c$. We say that an immersion $f\colon \Sigma \to M$ is *D-invariant* (and Σ is a *D-invariant* submanifold) if for every $x \in \Sigma$ the subspace $\widetilde{f}(x) \subseteq T_{f(x)}M$ is invariant under D. It is immediate that a D-invariant submanifold is totally geodesic if and only if for every X, $Y \in \mathfrak{X}(\Sigma)$ the covariant derivative $\nabla^c_X Y$ remains tangent to Σ.

These submanifolds are related to certain subalgebras of $\mathfrak{g}$. We say that a Lie subalgebra $\mathfrak{s}$ is *canonically embedded* in $\mathfrak{g}$ if it splits with respect to the reductive decomposition, that is,

$$\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{k}) \oplus (\mathfrak{s} \cap \mathfrak{p}) = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{s}_{\mathfrak{p}}.$$

Note that this definition extends that of canonically embedded subalgebras given in the case of symmetric spaces. The following theorem gives an algebraic characterization of D-invariant totally geodesic submanifolds passing through the origin. Furthermore, it gives an explicit method to construct them from their tangent space at o. The proof can be obtained by combining the theorem in [149, p. 11] and the first result in [90, §2]. However, we include it for the sake of completeness.

Theorem 6.3. *Let $M = \mathsf{G}/\mathsf{K}$ be a naturally reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{v} \subseteq \mathfrak{p}$ a vector subspace. The following conditions are equivalent:*

(i) *The subspace $\mathfrak{v}$ is invariant under the tensors R and D.*

(ii) *The subspace $\mathfrak{v}$ is invariant under the tensors R^c and D.*

(iii) *There exists a connected Lie subgroup $\mathsf{S} \subseteq \mathsf{G}$ such that its Lie algebra $\mathfrak{s}$ is canonically embedded in $\mathfrak{g}$ and the tangent space to the orbit $\mathsf{S} \cdot o$ at o is $\mathfrak{v}$.*

(iv) *There exists a connected, injectively immersed, and complete D-invariant totally geodesic submanifold Σ such that $o \in \Sigma$ and $T_o\Sigma = \mathfrak{v}$.*

Furthermore, if any of the four previous conditions hold, we have:

(1) *a Lie subgroup satisfying the conditions of item* (iii) *is the connected subgroup* S *with Lie algebra*

$$\mathfrak{s} = [\mathfrak{v}, \mathfrak{v}] + \mathfrak{v} = [\mathfrak{v}, \mathfrak{v}]_{\mathfrak{k}} \oplus \mathfrak{v},$$

(2) *the totally geodesic submanifold* Σ *passing through* o *with tangent space* $\mathfrak{v}$ *is* $\Sigma = \mathsf{S} \cdot o$.

Proof. Firstly, note that the formula

$$R^c(X, Y)Z = -[[X, Y]_{\mathfrak{k}}, Z] = R(X, Y)Z - D_X D_Y Z + D_Y D_X Z + 2D_{D_X Y} Z$$

implies that any D-invariant subspace of $\mathfrak{p}$ is invariant under R if and only if it is invariant under R^c, so (i) and (ii) are equivalent.

Now, suppose that $\mathfrak{v}$ satisfies (ii). We prove that $\mathfrak{s} = [\mathfrak{v}, \mathfrak{v}] + \mathfrak{v}$ is a Lie subalgebra of $\mathfrak{v}$. This amounts to checking that $[\mathfrak{v}, [\mathfrak{v}, \mathfrak{v}]]$ and $[[\mathfrak{v}, \mathfrak{v}], [\mathfrak{v}, \mathfrak{v}]]$ are contained in $\mathfrak{s}$. Let X, Y, $Z \in \mathfrak{v}$. Then we have

$$[[X, Y], Z] = [[X, Y]_{\mathfrak{k}}, Z] + [[X, Y]_{\mathfrak{p}}, Z] = -R^c(X, Y)Z + 2[D_X Y, Z] \in \mathfrak{s}.$$

In particular, $[\mathfrak{v}, \mathfrak{s}] \subseteq \mathfrak{s}$. Similarly, by the Jacobi identity, we see that

$$[[\mathfrak{v}, \mathfrak{v}], [\mathfrak{v}, \mathfrak{v}]] \subseteq [\mathfrak{v}, [\mathfrak{v}, [\mathfrak{v}, \mathfrak{v}]]] \subseteq [\mathfrak{v}, \mathfrak{s}] \subseteq \mathfrak{s}.$$

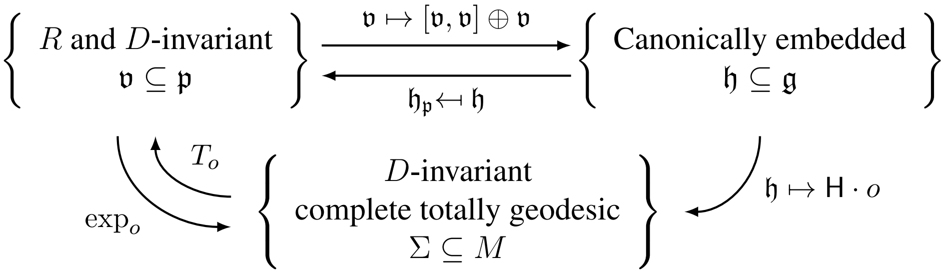
Therefore, $\mathfrak{s}$ is a Lie subalgebra. Because $\mathfrak{v}$ is D-invariant, we see that $\mathfrak{s}_{\mathfrak{p}} = \mathfrak{v} \subseteq \mathfrak{s}$, and from this inclusion it follows that $\mathfrak{s} = \mathfrak{s}_{\mathfrak{k}} \oplus \mathfrak{v}$, which proves that $\mathfrak{s}$ is canonically embedded. It is also immediate from the description of $\mathfrak{s}$ that $\mathfrak{s}_{\mathfrak{k}} = [\mathfrak{v}, \mathfrak{v}]_{\mathfrak{k}}$. As a consequence, if we consider the Lie subgroup S of G whose Lie algebra is $\mathfrak{s}$, then the tangent space $T_o(\mathsf{S} \cdot o)$ coincides with $\mathfrak{s}_{\mathfrak{p}} = \mathfrak{v}$, and therefore (iii) holds.

We now prove that (iii) implies (iv). Assume $\mathsf{S} \subseteq \mathsf{G}$ is a Lie subgroup whose Lie algebra is canonically embedded in $\mathfrak{g}$ and $T_o(\mathsf{S} \cdot o) = \mathfrak{s}_{\mathfrak{p}} = \mathfrak{v}$. It is clear from (1.6) that $\mathrm{I\!I}$ is zero at o. Because $\mathsf{S} \cdot o$ is extrinsically homogeneous, this implies that the second fundamental form vanishes everywhere, and thus $\mathsf{S} \cdot o$ is totally geodesic. We now prove that $\mathsf{S} \cdot o$ is D-invariant, which by G-invariance of D is equivalent to checking that $\mathfrak{v}$ is D-invariant. Given $X, Y \in \mathfrak{s}_{\mathfrak{p}} \subseteq \mathfrak{s}$, we have $D_X Y = (1/2)[X, Y]_{\mathfrak{p}} \in \mathfrak{s}_{\mathfrak{p}}$, so the claim follows.

Finally, it is immediate that (iv) implies (i) from the definition of D-invariant submanifolds and Theorem 5.1. □

Corollary 6.4. *Every complete D-invariant totally geodesic submanifold of a naturally reductive homogeneous space $M = \mathsf{G}/\mathsf{K}$ is extrinsically homogeneous with respect to the given presentation of M, and thus injectively immersed.*

Remark 6.5. Theorem 6.3 is also a refinement of [135, Lemma 3.1], which states that, for a general reductive homogeneous space, a subspace $\mathfrak{v} \subseteq \mathfrak{p}$ invariant under R and D is tangent to a complete totally geodesic submanifold. Also, notice that the class of D-invariant totally geodesic submanifolds includes all totally geodesic submanifolds of symmetric spaces, since in a symmetric space $D = 0$. Thus, in the symmetric setting the subspaces $\mathfrak{v}$ appearing in the previous theorem are the Lie triple systems. As a consequence, in naturally reductive homogeneous spaces we can express the relationships between D-invariant totally geodesic submanifolds, canonically embedded subalgebras, and subspaces of $\mathfrak{p}$ invariant under both R and D, by means of the following diagram:



Note that this diagram is essentially the same as the one in Subsection 5.6.1.

Remark 6.6. The case $\mathfrak{v} = \mathfrak{p}$ in Theorem 6.3 is part of a result by Kostant [111], which implies in particular that the connected (normal) subgroup with Lie algebra $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$ acts transitively on M.

Remark 6.7. It is worth noting that in the irreducible setting the conditions of Theorem 6.3 do not depend on the naturally reductive decomposition that we choose. Indeed, from [134, Theorem 2.1], we see that if $M = \mathsf{G}/\mathsf{K}$ is a simply connected irreducible naturally reductive space which is not symmetric, then the canonical connection ∇^c is unique. Therefore, given any naturally reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$, the subspaces $\mathfrak{v} \subseteq \mathfrak{p}$ that are invariant under R and D correspond under the identification $\mathfrak{p} \equiv T_oM$ to the subspaces $V \subseteq T_oM$ that are invariant under R and $\nabla - \nabla^c$, and the uniqueness of the canonical connection implies that these subspaces are always the same regardless of the decomposition. In a similar way, if M is a nearly Kähler 3-symmetric space and one restricts their attention to the reductive decompositions invariant under the automorphism of order three, then all of their associated canonical connections coincide by [163, Lemma 3.1], and the same argument applies.

6.2.2 Totally geodesic surfaces

As an application of Corollary 6.2, we derive a necessary condition for the existence of totally geodesic surfaces with a given tangent plane.

Let $\mathfrak{v} \subseteq \mathfrak{p}$ be a 2-dimensional subspace, and assume that $\mathfrak{v}$ is the tangent plane at o of the complete totally geodesic surface Σ of M. Fix a nonzero element $X \in \mathfrak{v}$ and choose any $Y \in \mathfrak{v} \setminus \{0\}$ that is orthogonal to X, so that $\{X, Y\}$ is an orthogonal basis of $\mathfrak{v}$. Since Σ is homogeneous and two-dimensional, it follows that Σ is a space of constant curvature $\kappa \in \mathbb{R}$, and the same can be said for the totally geodesic submanifold Σ_t associated with $e^{-tD_X}\mathfrak{v}$ for all $t \in \mathbb{R}$. This implies in particular that the restriction of ∇R to the tangent space of Σ and Σ_t at any of their points is the zero tensor. Furthermore, we have $e^{-tD_X}\mathfrak{v} = \operatorname{span}\{X, e^{-tD_X}Y\}$, since $D_X X = 0$ due to the skew-symmetry of D. Because Σ_t has curvature κ, it follows that $e^{-tD_X}Y$ is an eigenvector of the Jacobi operator R_X with eigenvalue $\kappa|X|^2$, as well as an element of $\ker C_X$ (recall that C_X is the Cartan operator associated with X given by $C_X Y = (\nabla_X R)(X,Y,X)$, see Section 1.2). One can argue similarly with the so-called *Cartan operators of order* j given by $C_X^j Y = \nabla^j R(X,\dots,X,Y,X)$, because they vanish identically on $\mathfrak{v}$. Since the subspace of $\mathfrak{p}$ generated by the curve $e^{-tD_X}Y$ is the span of all vectors of the form $D_X^k Y$ with $k \geq 0$, we have obtained the following:

Proposition 6.8. *Let $M = \mathsf{G}/\mathsf{K}$ be a naturally reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Choose orthogonal vectors X, $Y \in \mathfrak{p}$ and suppose that $\mathfrak{v} = \operatorname{span}\{X, Y\}$ is the tangent space of a totally geodesic surface Σ of M. Then we have the inclusion*

$$\operatorname{span}\{D_X^k Y \colon k \geq 0\} \subseteq \ker(R_X - \kappa|X|^2 \operatorname{id}_{\mathfrak{p}}) \cap \bigcap_{j=1}^{\infty} \ker C_X^j.$$

6.2.3 Well-positioned totally geodesic submanifolds and homogeneous fibrations

We now study the case that $M = \mathsf{G}/\mathsf{H}$ is also the total space of the homogeneous fibration induced by the inclusions $\mathsf{H} \subseteq \mathsf{K} \subseteq \mathsf{G}$ (observe the change of notation). Let $B = \mathsf{G}/\mathsf{K}$ be the base space and $F = \mathsf{K}/\mathsf{H}$ be the fiber of the given submersion. Consider a totally geodesic immersion $f \colon \Sigma \to M$. We say that Σ is *well-positioned* at $p \in \Sigma$ (with respect to the fibration $M \to B$) if

$$\widetilde{f}(p) = \left(\widetilde{f}(p) \cap \mathcal{V}_{f(p)}\right) \oplus \left(\widetilde{f}(p) \cap \mathcal{H}_{f(p)}\right).$$

Furthermore, Σ is said to be *well-positioned* if it is well-positioned at every point $p \in \Sigma$. The next result allows us to give an algebraic characterization for a totally geodesic submanifold to be well-positioned.

Lemma 6.9. *Let $F \to M \to B$ be the homogeneous fibration induced by the chain of inclusions $\mathsf{H} \subseteq \mathsf{K} \subseteq \mathsf{G}$, where H, K and G are compact and the Riemannian metrics on F, M and B are induced from a bi-invariant metric on G. Let $f \colon \Sigma \to M$ be a complete totally geodesic immersion passing through the point o with tangent space $\mathfrak{v}$. The following conditions are equivalent:*

(i) *Σ is well-positioned with respect to the submersion $M \to B$,*

(ii) *the subspace $e^{-D_X}\mathfrak{v}$ splits with respect to the decomposition $\mathfrak{p} = \mathcal{V}_o \oplus \mathcal{H}_o$ for all $X \in \mathfrak{v}$.*

Proof. Let $p \in \Sigma$ be such that $\widetilde{f}(p) = \mathfrak{v}$. Since we are assuming that Σ is connected and complete, every point of Σ is of the form $q = \exp_p(v)$ for a certain $v \in T_p\Sigma$. Consider the geodesic $\gamma(t) = f(\exp(tv)) = \operatorname{Exp}(tX) \cdot o$ (where $X = f_{*p}(v)$) connecting p and q. Then, we have $\widetilde{f}(q) = \mathcal{P}^{\gamma}_{0,1}\mathfrak{v}$. As $\mathcal{V}$ and $\mathcal{H}$ are invariant under G, we see that $\mathcal{V}_{f(q)} = \operatorname{Exp}(X)_{*o}\mathcal{V}_o$ and $\mathcal{H}_{f(q)} = \operatorname{Exp}(X)_{*o}\mathcal{H}_o$. Therefore, using (6.3) we see that Σ is well-positioned at q if and only if the subspace $\operatorname{Exp}(X)^{-1}_{*o}(\widetilde{f}(q)) = e^{-D_X}\mathfrak{v}$ splits with respect to the decomposition $\mathfrak{p} = \mathcal{V}_o \oplus \mathcal{H}_o$. As $\mathfrak{v} = f_{*p}(T_p\Sigma)$, the equivalence follows. □

Corollary 6.10. *Let $F \to M \to B$ be as in Lemma 6.9, and let $\Sigma \subseteq M$ be a D-invariant totally geodesic submanifold passing through o. Then Σ is well-positioned if and only if it is well-positioned at o.*

Proof. This follows from noting that the D-invariance of $\mathfrak{v}$ implies that $e^{-D_X}\mathfrak{v} = \mathfrak{v}$ for all $X \in \mathfrak{v}$. $\square$

6.3 The examples

In this section we describe the totally geodesic submanifolds of $\mathbb{C}\mathsf{P}^3$, $\mathsf{F}(\mathbb{C}^3)$ and $\mathsf{S}^3 \times \mathsf{S}^3$ that appear in the classification, and determine their isometry type. We indicate whether the examples are well-positioned with respect to the homogeneous fibrations given in Section 6.1.

Let us recall some definitions about special submanifolds of almost Hermitian manifolds. If (M^{2n}, J) is an almost Hermitian manifold and $f\colon \Sigma \to M$ is an immersion, we say that f (and Σ) is *totally real* if for all $p \in \Sigma$ the subspaces $f_{*p}(T_p\Sigma)$ and $Jf_{*p}(T_p\Sigma)$ of $T_{f(p)}M$ are orthogonal. If f is totally real and $T_pM = f_{*p}(T_p\Sigma) \oplus Jf_{*p}(T_p\Sigma)$ (that is, if $\dim\Sigma = n$), then Σ is a *Lagrangian* submanifold. Separately, f (and Σ) is *almost complex* (or *J-holomorphic*) if $f_{*p}(T_p\Sigma)$ is invariant under J for all $p \in \Sigma$. Furthermore, if Σ is a surface, we refer to it as an *almost complex surface* or a *J-holomorphic curve*.

Remark 6.11. Many of the totally geodesic submanifolds that appear in this section are isometric to a sphere with a round or complex Berger metric. We can compute the radius r of the sphere $\mathsf{S}^n(r)$, as well as the parameters of the Berger sphere $\mathsf{S}^3_{\mathbb{C},\tau}(r)$ from its sectional curvature. Indeed, it is well known that the sectional curvature of $\mathsf{S}^n(r)$ is equal to $1/r^2$. In the case of $\mathsf{S}^3_{\mathbb{C},\tau}(r)$, the parameters r and τ can be obtained from the equations $\tau = r^2\sec(U,X)$ and $4 - 3\tau = r^2\sec(X,Y)$, where U is a vertical vector and X, Y are horizontal vectors with respect to the Hopf fibration (see [74]).

6.3.1 The complex projective space

We describe the totally geodesic examples of the complex projective space $\mathbb{C}\mathsf{P}^3$ equipped with a homogeneous nearly Kähler metric.

The real projective space [9, Example 3.9]

Consider the subgroup $\mathsf{U}(2)^j \subseteq \mathsf{Sp}(2)$ whose Lie algebra is given by

$$\begin{aligned}\mathfrak{u}(2)^j &= \operatorname{span}\left\{\begin{pmatrix} j & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0\\ 0 & j\end{pmatrix}, \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & j\\ j & 0\end{pmatrix}\right\}\\ &= (\mathfrak{u}(2)^j \cap \mathfrak{k}) \oplus (\mathfrak{u}(2)^j \cap \mathfrak{p}).\end{aligned}$$

Then, $\mathfrak{u}(2)^j$ is canonically embedded in $\mathfrak{sp}(2)$, so the orbit $\mathsf{U}(2)^j \cdot o$ is a totally geodesic submanifold of $\mathbb{C}\mathsf{P}^3$ whose tangent space is $\operatorname{span}\{e_1, e_3, e_5\}$. The isotropy subgroup $\mathsf{U}(2)^j \cdot o$ is equal to $\mathbb{Z}_2 \times \mathsf{U}(1)$, so $\mathsf{U}(2)^j \cdot o$ is diffeomorphic to a real projective space $\mathbb{R}\mathsf{P}^3$. The induced metric is Berger-like. Indeed, this totally geodesic submanifold is isometric to $\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)$. Let us write $\mathfrak{p}_{\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)} = \operatorname{span}\{e_1, e_3, e_5\}$. A computation gives $J(\mathfrak{p}_{\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)}) = \mathfrak{p} \ominus \mathfrak{p}_{\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)}$, and since $\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)$ is extrinsically homogeneous we see that $\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)$ is a Lagrangian submanifold. Finally, note that $\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)$ is well-positioned at o, so by Corollary 6.10, it is well-positioned.

The fiber of the twistor fibration

Recall that the fibers of the twistor fibration are totally geodesic surfaces in $\mathbb{C}\mathsf{P}^3$. In particular, the orbit through o is $(\mathsf{Sp}(1) \times \mathsf{Sp}(1)) \cdot o = \mathsf{Sp}(1)_f \cdot o$, where $\mathsf{Sp}(1)_f$ denotes the image of the standard embedding of $\mathsf{Sp}(1)$ in $\mathsf{Sp}(2)$ in the first block. The isotropy subgroup $(\mathsf{Sp}(1)_f)_o$ coincides with $\mathsf{U}(1)$, so $\mathsf{Sp}(1)_f \cdot o$ is diffeomorphic to a sphere. Its tangent space at o is $\mathfrak{p}_{\mathsf{Sp}(1)_f \cdot o} = \mathfrak{p}_1$. The sectional curvature in this case is $\sec(\mathfrak{p}_1) = 2$, so $\mathsf{Sp}(1)_f \cdot o$ is a round sphere of radius $1/\sqrt{2}$. Furthermore, the fact that $J(\mathfrak{p}_1) = \mathfrak{p}_1$ implies that $\mathsf{Sp}(1)_f \cdot o$ is an almost complex surface in $\mathbb{C}\mathsf{P}^3$. By definition, $\mathsf{Sp}(1)_f \cdot o$ is well-positioned.

The horizontal sphere $\mathsf{SU}(2) \cdot o$

Consider the standard embedding of $\mathsf{SU}(2)$ in $\mathsf{Sp}(2)$. Since the Lie algebra $\mathfrak{su}(2)$ is given by

$$\mathfrak{su}(2) = \operatorname{span}\{H, E, F\} = (\mathfrak{su}(2) \cap \mathfrak{k}) \oplus (\mathfrak{su}(2) \cap \mathfrak{p}),$$

it follows that $\mathfrak{su}(2)$ is canonically embedded, and the orbit $\mathsf{SU}(2) \cdot o$ is a totally geodesic submanifold of $\mathbb{C}\mathsf{P}^3$ with tangent space $\mathfrak{p}_{\mathsf{SU}(2)\cdot o} = \operatorname{span}\{e_3, e_4\}$. The isotropy subgroup of $\mathsf{SU}(2)$ at o is the canonical $\mathsf{U}(1)$, so $\mathsf{SU}(2) \cdot o$ is diffeomorphic to a sphere. Furthermore, its sectional curvature is given by $\sec(\mathfrak{p}_{\mathsf{SU}(2)\cdot o}) = 1$, so this submanifold is a round sphere of radius 1. Finally, note that $\mathfrak{p}_{\mathsf{SU}(2)\cdot o}$ is J-invariant, so this sphere is also an almost complex surface in $\mathbb{C}\mathsf{P}^3$. Note that $\mathsf{SU}(2) \cdot o$ is well-positioned by Corollary 6.10. Indeed, its tangent space at every point is always contained in the horizontal subspace of the twistor fibration.

The sphere $\mathsf{SU}(2)_{\Lambda_3} \cdot o$

Consider the unique complex irreducible representation Λ_3 of $\mathsf{SU}(2)$ of dimension four. Since this representation is unitary and of symplectic type, it restricts to a Lie group homomorphism $\mathsf{SU}(2) \to \mathsf{Sp}(2)$. To get an explicit description of this map at the Lie algebra level (which is enough for our purposes), it suffices to see that the linear map $f \colon \mathfrak{su}(2) \to \mathfrak{sp}(2)$ defined via

$$H \mapsto \begin{pmatrix} i & 0 \\ 0 & 3i \end{pmatrix}, \qquad E \mapsto \begin{pmatrix} 2j & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}, \qquad F \mapsto \begin{pmatrix} -2k & -i\sqrt{3} \\ -i\sqrt{3} & 0 \end{pmatrix}$$

is a Lie algebra homomorphism which is also irreducible as a representation, so by uniqueness it must be equal to Λ_3.

We denote by $\mathsf{SU}(2)_{\Lambda_3}$ the image of the previous homomorphism. The Lie algebra of this group satisfies

$$\begin{aligned}\mathfrak{su}(2)_{\Lambda_3} &= \operatorname{span}\Big\{\operatorname{diag}(i,3i), \sqrt{2}e_1+\sqrt{3}e_3, \sqrt{2}e_2+\sqrt{3}e_4\Big\}\\ &= (\mathfrak{su}(2)_{\Lambda_3}\cap\mathfrak{k})\oplus(\mathfrak{su}(2)_{\Lambda_3}\cap\mathfrak{p}),\end{aligned}$$

so it is canonically embedded in $\mathfrak{sp}(2)$. As a consequence, the orbit $\mathsf{SU}(2)_{\Lambda_3}\cdot o$ is a totally geodesic submanifold of $\mathbb{C}\mathsf{P}^3$ with tangent space

$$\mathfrak{p}_{\mathsf{SU}(2)_{\Lambda_3}\cdot o} = \operatorname{span}\Big\{\sqrt{2}e_1+\sqrt{3}e_3, \sqrt{2}e_2+\sqrt{3}e_4\Big\}.$$

The isotropy subgroup at o is the $\mathsf{U}(1)$ subgroup with Lie algebra generated by $\operatorname{diag}(i,3i)$, so this orbit is actually a sphere. Since the sectional curvature of the plane $\mathfrak{p}_{\mathsf{SU}(2)_{\Lambda_3}\cdot o}$ is $1/5$, we see that $\mathsf{SU}(2)_{\Lambda_3}\cdot o$ is a sphere of radius $\sqrt{5}$. One sees that $\mathfrak{p}_{\mathsf{SU}(2)_{\Lambda_3}\cdot o}$ is J-invariant, and by homogeneity it follows that $\mathsf{SU}(2)_{\Lambda_3}\cdot o$ is an almost complex submanifold of $\mathbb{C}\mathsf{P}^3$. Clearly, $\mathsf{SU}(2)_{\Lambda_3}\cdot o$ is not well-positioned at o, so it is not well-positioned.

6.3.2 The flag manifold

We describe the totally geodesic examples of the flag manifold $\mathsf{F}(\mathbb{C}^3)$ equipped with a homogeneous nearly Kähler metric.

The real flag manifold $\mathsf{F}(\mathbb{R}^3)$ [156, Example 3.1]

There is a natural embedding of the real flag manifold $\mathsf{F}(\mathbb{R}^3)$ in $\mathsf{F}(\mathbb{C}^3)$ which is induced by the usual inclusion of $\mathbb{R}^3$ in $\mathbb{C}^3$. This submanifold can also be seen as the orbit $\mathsf{SO}(3)\cdot o$ of the standard $\mathsf{SO}(3)\subseteq\mathsf{SU}(3)$, and the corresponding isotropy subgroup is $\mathsf{SO}(3)_o=\mathbb{Z}_2\oplus\mathbb{Z}_2$, so we get $\mathsf{F}(\mathbb{R}^3)=\mathsf{SO}(3)/\mathbb{Z}_2\oplus\mathbb{Z}_2=\mathsf{Sp}(1)/\mathsf{Q}_8$, where $\mathsf{Q}_8=\{\pm1,\pm i,\pm j,\pm k\}$. Observe that $\mathfrak{so}(3)$ is canonically embedded in $\mathfrak{su}(3)$, since $\mathfrak{so}(3)\subseteq\mathfrak{p}$. Thus, Theorem 6.3 allows us to conclude that $\mathsf{F}(\mathbb{R}^3)$ is totally geodesic in M, and its tangent space is precisely $\mathfrak{p}_{\mathsf{F}(\mathbb{R}^3)}=\mathfrak{so}(3)=\operatorname{span}\{e_1,e_3,e_5\}$. A direct computation shows that $\mathsf{F}(\mathbb{R}^3)$ has constant curvature equal to $1/8$. Furthermore, we have the equality $J(\mathfrak{so}(3))=\operatorname{span}\{e_2,e_4,e_6\}$, implying that the inclusion $\mathsf{F}(\mathbb{R}^3)\subseteq\mathsf{F}(\mathbb{C}^3)$ is Lagrangian. Finally, note that $\mathsf{F}(\mathbb{R}^3)$ is well-positioned at o, so $\mathsf{F}(\mathbb{R}^3)$ is well-positioned by Corollary 6.10.

The Berger sphere [156, Example 3.2]

Let $\mathsf{SU}(2)_{(1,0,1)}$ denote the subgroup of $\mathsf{SU}(3)$ that fixes $(1,0,1)\in\mathbb{C}^3$. This subgroup is conjugate to the standard $\mathsf{SU}(2)$ inside $\mathsf{SU}(3)$. The Lie algebra $\mathfrak{su}(2)_{(1,0,1)}$ is the set of all $X\in\mathfrak{su}(3)$ such that $X(1,0,1)=0$, and its projection to $\mathfrak{p}$ is spanned by $\{e_1+e_3,e_2-e_4,e_6\}$. It is easy to check that the isotropy subgroup of $\mathsf{SU}(2)_{(1,0,1)}$ at o is trivial, so the corresponding orbit $\mathsf{SU}(2)_{(1,0,1)}\cdot o$ is diffeomorphic to a 3-sphere. A direct application of (1.6) yields that $\mathsf{SU}(2)_{(1,0,1)}\cdot o$ is a totally geodesic submanifold of $\mathsf{F}(\mathbb{C}^3)$ isometric to $\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})$ and whose tangent space is given by $\mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})} = \operatorname{span}\{e_1+e_3, e_2-e_4, e_6\}$. The subspace $\mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})}$ is also invariant under D. However, the Lie algebra $\mathfrak{su}(2)_{(1,0,1)}$ is not canonically embedded in $\mathfrak{su}(3)$. In this case, the connected subgroup given by Theorem 6.3 is actually the subgroup $\mathsf{U}(2)_{(1,0,1)}$ that fixes the complex line generated by $(1,0,1)$. A direct calculation shows that $J(\mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})}) = \mathfrak{p} \ominus \mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})}$, so the Berger sphere is Lagrangian. Note from the expression of $\mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})}$ that $\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})$ is not well-positioned.

The torus [48, Example 3.3]

Consider the torus $\mathsf{H} \subseteq \mathsf{SU}(3)$ with Lie algebra $\mathfrak{h} = \operatorname{span}\{e_1+e_3+e_5, e_2+e_4-e_6\}$. Observe that $\mathfrak{h} \subseteq \mathfrak{p}$, so it is a canonically embedded subalgebra of $\mathfrak{g}$, and the orbit $\mathsf{H}\cdot o$ is a totally geodesic surface. Since $\mathsf{H}\cdot o$ is a quotient of H by a finite group, it is a compact abelian Lie group itself and hence diffeomorphic to a torus. However, it turns out that $\mathsf{H}\cdot o$ is not isometric to the standard flat torus. Indeed, in order to determine the isometry type of $\mathsf{H}\cdot o$, we compute the preimage $\exp_o^{-1}(o)$. For this, we need a description of the Riemannian exponential map $\exp_o\colon \mathfrak{h} \to \mathsf{H}\cdot o \subseteq M$, which is merely the restriction of the Riemannian exponential map of M. We define the orthonormal vectors $X = \frac{1}{\sqrt{3}}(e_1+e_3+e_5)$, $Y = \frac{1}{\sqrt{3}}(e_2+e_4-e_6)$ of $\mathfrak{h}$. Then the exponential map of $\mathfrak{h}$ satisfies that $\exp_o(uX+vY) = e^{uX}e^{vY}\cdot o$, and this element is equal to o if and only if $e^{uX}e^{vY}$ is a diagonal matrix. A calculation shows that $e^{vX}e^{vY} = (a_{ij})$ has the following non-diagonal entries:

$$\begin{aligned}
a_{12} = a_{23} &= \frac{1}{3}\left(e^{\frac{iv}{\sqrt{6}}}\left(\sqrt{3}\sin\left(\frac{u}{\sqrt{2}}\right) + \cos\left(\frac{u}{\sqrt{2}}\right)\right) - e^{-i\sqrt{\frac{2}{3}}v}\right),\\
a_{21} = a_{32} &= \frac{1}{3}\left(e^{\frac{iv}{\sqrt{6}}}\left(\cos\left(\frac{u}{\sqrt{2}}\right) - \sqrt{3}\sin\left(\frac{u}{\sqrt{2}}\right)\right) - e^{-i\sqrt{\frac{2}{3}}v}\right),\\
a_{13} &= \frac{1}{3}\left(e^{\frac{iv}{\sqrt{6}}}\left(\sqrt{3}\sin\left(\frac{u}{\sqrt{2}}\right) - \cos\left(\frac{u}{\sqrt{2}}\right)\right) + e^{-i\sqrt{\frac{2}{3}}v}\right),\\
a_{31} &= \frac{1}{3}\left(e^{-i\sqrt{\frac{2}{3}}v} - e^{\frac{iv}{\sqrt{6}}}\left(\sqrt{3}\sin\left(\frac{u}{\sqrt{2}}\right) + \cos\left(\frac{u}{\sqrt{2}}\right)\right)\right).
\end{aligned}$$

Consequently, the solutions to the equation $\exp_o(uX+vY) = o$ are given by the lattice $\Lambda = \operatorname{span}_{\mathbb{Z}}\left\{\left(\sqrt{2}\pi, \sqrt{2}\pi/\sqrt{3}\right), \left(0, 2\sqrt{2}\pi/\sqrt{3}\right)\right\}$. Since $\exp_o$ is $\mathfrak{h}$-equivariant, in the sense that it satisfies the equation $\exp_o(T+S) = \operatorname{Exp}(T)\cdot\exp_o(S)$, it follows that $\exp_o$ is actually a Riemannian covering map, so $\mathsf{H}\cdot o$ is isometric to the quotient $\mathbb{R}^2/\Lambda$. We refer to this orbit as $\mathsf{T}^2_\Lambda = \mathsf{H}\cdot o = \mathbb{R}^2/\Lambda$. Note that $\mathsf{H}\cdot o$ is not a product $\mathsf{S}^1(r_1)\times\mathsf{S}^1(r_2)$, since the closest points in $\Lambda\setminus\{(0,0)\}$ to the origin are those in $\left\{\left(\pm\sqrt{2}\pi, \pm\sqrt{6}\pi/3\right), \left(\pm\sqrt{2}\pi, \mp\sqrt{6}\pi/3\right), \left(0, \pm 2\sqrt{6}\pi/3\right)\right\}$, and thus there exist three different closed geodesics of minimum length, as opposed to two in the case of $\mathsf{S}^1(r)\times\mathsf{S}^1(r)$ or one in the case of $\mathsf{S}^1(r_1)\times\mathsf{S}^1(r_2)$ with $r_1 \neq r_2$. If we let $\mathfrak{p}_{\mathsf{T}^2_\Lambda} = \mathfrak{h}$ be the tangent space of this surface, then $J(\mathfrak{p}_{\mathsf{T}^2_\Lambda}) = \mathfrak{p}_{\mathsf{T}^2_\Lambda}$, and by homogeneity it follows that T^2_Λ is an almost complex surface in $\mathsf{F}(\mathbb{C}^3)$. However, it is clear from the expression of $\mathfrak{p}_{\mathsf{T}^2_\Lambda}$ that T^2_Λ is not well-positioned.

The fiber of the submersion $\mathsf{F}(\mathbb{C}^3) \to \mathbb{C}\mathsf{P}^2$ [48, Example 3.1]

Recall that the fibers of the submersion $\mathsf{F}(\mathbb{C}^3) \to \mathbb{C}\mathsf{P}^2$ are totally geodesic. The fiber through o is $\mathbb{C}\mathsf{P}^1 = \mathsf{U}(2) \cdot o = \mathsf{SU}(2) \cdot o$, where the isotropy subgroup of $\mathsf{SU}(2)$ at o is $\mathsf{U}(1)$. The tangent space, as said before, is $\mathfrak{p}_{\mathbb{C}\mathsf{P}^1} = \mathfrak{p}_1$. Since the sectional curvature of $\mathfrak{p}_1$ is 2, it follows that $\mathsf{SU}(2) \cdot o$ is isometric to the round sphere of radius $1/\sqrt{2}$. Furthermore, $J(\mathfrak{p}_1) = \mathfrak{p}_1$, so $\mathsf{SU}(2) \cdot o$ is an almost complex surface in $\mathsf{F}(\mathbb{C}^3)$. Clearly, $\mathbb{C}\mathsf{P}^1$ is well-positioned as it is a fiber itself.

The sphere [48, Example 3.2]

Consider the real form $E = \operatorname{span}\{(0,1,0),(1,0,-1),(i,0,i)\}$ of $\mathbb{C}^3$, and let $\sigma\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the associated real structure. Then the normalizer

$$\mathsf{SO}(3)^\sigma = \{g \in \mathsf{SU}(3)\colon g(E) = E\} = \{g \in \mathsf{SU}(3)\colon g\sigma = \sigma g\}$$

is a subgroup of $\mathsf{SU}(3)$ conjugate to the standard $\mathsf{SO}(3)$. The corresponding Lie algebra is given by $\mathfrak{so}(3)^\sigma = \operatorname{span}\{\operatorname{diag}(i,0,-i), e_1+e_3, e_2+e_4\}$, and in particular it is canonically embedded in $\mathfrak{su}(3)$. One sees that the isotropy subgroup $\mathsf{SO}(3)^\sigma_o$ is the $\mathsf{U}(1)$ subgroup generated by $\mathfrak{so}(3)^\sigma \cap \mathfrak{t}$, so we obtain that $\mathsf{SO}(3)^\sigma \cdot o$ is a totally geodesic submanifold of $\mathsf{F}(\mathbb{C}^3)$ that is diffeomorphic to a sphere. Its tangent space at o is $\mathfrak{p}_{\mathfrak{so}(3)^\sigma} = \operatorname{span}\{e_1 + e_3, e_2 + e_4\}$, and this plane has sectional curvature $1/2$, so $\mathsf{SO}(3)^\sigma \cdot o$ is isometric to a two-dimensional sphere of radius $\sqrt{2}$. The equality $J(\mathfrak{p}_{\mathfrak{so}(3)^\sigma}) = \mathfrak{p}_{\mathfrak{so}(3)^\sigma}$ implies that $\mathsf{SO}(3)^\sigma \cdot o$ is an almost complex surface in $\mathsf{F}(\mathbb{C})^3$. Since $\mathsf{SO}(3)^\sigma \cdot o$ is not well-positioned at o, it is not well-positioned.

Real projective planes inside $\mathsf{F}(\mathbb{R}^3)$

Recall that $\mathsf{F}(\mathbb{R}^3)$ is a Lagrangian submanifold with constant sectional curvature. In particular, every 2-plane inside $\mathfrak{p}_{\mathsf{F}(\mathbb{R}^3)}$ gives rise to a totally geodesic surface inside $\mathsf{F}(\mathbb{R}^3)$ (hence inside $\mathsf{F}(\mathbb{C}^3)$). We describe these examples.

As we saw earlier, $\mathsf{F}(\mathbb{R}^3)$ can be regarded as the quotient $\mathsf{Sp}(1)/\mathsf{Q}_8$ with a metric of constant curvature equal to $1/8$. As a consequence, $\mathsf{F}(\mathbb{R}^3)$ is isometric to the quotient $\mathsf{S}^3(2\sqrt{2})/\mathsf{Q}_8$, and the projection map $\pi\colon \mathsf{S}^3(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3)$ is a Riemannian covering map. This projection is equivariant with respect to the double cover $\mathsf{Sp}(1) \to \mathsf{SO}(3)$.

We view $\mathbb{H} \equiv \mathbb{R}^4$. Consider the totally geodesic embedding $h\colon \mathsf{S}^2(2\sqrt{2}) \hookrightarrow \mathsf{S}^3(2\sqrt{2})$ defined by $h(x,y,z) = xi + yj + zk$. Then $\pi \circ h$ is also a totally geodesic immersion of the 3-sphere satisfying $\pi \circ h(-x,-y,-z) = \pi \circ h(x,y,z)$ for all $(x,y,z) \in \mathsf{S}^2(2\sqrt{2})$, so it factors through an isometric immersion $\phi\colon \mathbb{R}\mathsf{P}^2(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3)$ defined via

$$\phi([x:y:z]) = (xi + yj + zk)\mathsf{Q}_8, \quad (x,y,z) \in \mathsf{S}^2(2\sqrt{2}). \tag{6.4}$$

As the projection $\mathsf{S}^2(2\sqrt{2}) \to \mathbb{R}\mathsf{P}^2(2\sqrt{2})$ is also a covering map, we deduce that ϕ is a totally geodesic immersion. Note that ϕ is not injective, as the points $[2\sqrt{2} : 0 : 0]$, $[0 : 2\sqrt{2} : 0]$ and $[0 : 0 : 2\sqrt{2}]$ have the same image.

Proposition 6.12. *The map $\phi\colon \mathbb{R}\mathsf{P}^2(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3)$ defined by* (6.4) *is a non-injective inextendable compatible totally geodesic immersion.*

Proof. Since $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ is complete, we only need to show that ϕ is compatible. This is equivalent to proving the following assertion: for every pair of different points $p = [x : y : z]$ and $q = [x' : y' : z'] \in \mathbb{R}\mathsf{P}^2(2\sqrt{2})$ such that $\phi(p) = \phi(q)$, the vector spaces $\phi_{*p}(T_p\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$ and $\phi_{*q}(T_q\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$ are different subspaces of $T_{\phi(p)}\mathsf{F}(\mathbb{R}^3)$.

Let $p = [x : y : z]$ and $q = [x' : y' : z']$ be as above. Then $\phi(p) = \phi(q)$ implies that there is an element $\lambda \in \mathsf{Q}_8$ such that $x'i + y'j + z'k = (xi + yj + zk)\lambda$. Changing the sign of the homogeneous coordinates of q if necessary, we may assume that $\lambda \in \{i, j, k\}$. We deal with the case $\lambda = i$, as the other two cases can be treated in an analogous manner. In this setting, we obtain that $x'i + y'j + z'k = (xi + yj + zk)i = -x - yk + zj$, which yields $x = x' = 0$, $y' = z$ and $z' = -y$, so $p = [0 : y : z]$ and $q = [0 : z : -y]$.

Let us compute $\phi_{*p}(T_p\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$. On the one hand, we can identify the tangent space of $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ at $p = [0 : y : z]$ with the tangent space $T_{(0,y,z)}\mathsf{S}^2(2\sqrt{2}) \equiv \mathbb{R}(0, y, z)^\perp = \mathrm{span}\{(1, 0, 0), (0, -z, y)\}$. Moreover, we can also view the tangent space $T_{(yj+zk)\mathsf{Q}_8}\mathsf{F}(\mathbb{R}^3)$ as $T_{(yj+zk)}\mathsf{S}^3(2\sqrt{2}) \equiv \mathbb{R}(yj + zk)^\perp$. Under these identifications, $\phi_{*p}(T_p\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$ is spanned by $\phi_{*p}(1, 0, 0) = i$, and $\phi_{*p}(0, -z, y) = -zj + yk$. We now determine $\phi_{*q}(T_q\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$. For this, we have identifications $T_q\mathbb{R}\mathsf{P}^2(2\sqrt{2}) \equiv \mathbb{R}(0, z, -y)^\perp = \mathrm{span}\{(1, 0, 0), (0, y, z)\}$ and $T_{(-zj+yk)}\mathsf{F}(\mathbb{R}^3) \equiv \mathbb{R}(-zj + yk)^\perp$. We obtain that $\phi_{*q}(T_q\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$ is generated by the vectors $\phi_{*q}(1, 0, 0) = i$ and $\phi_{*q}(0, y, z) = yj + zk$. In order to finish, observe that the composition of the isomorphisms

$$\mathbb{R}(yj + zk)^\perp \to T_{(yj+zk)\mathsf{Q}_8}\mathsf{F}(\mathbb{R}^3) = T_{(zj-yk)\mathsf{Q}_8}\mathsf{F}(\mathbb{R}^3) \to \mathbb{R}(zj - yk)^\perp$$

is simply right multiplication by i, so $\phi_{*p}(T_p\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$, regarded as a subspace of $\mathbb{R}(zj-yk)^\perp$, is spanned by 1 and $yj + zk$. Thus, we obtain that the images of ϕ_{*p} and ϕ_{*q} are different, and therefore ϕ is a compatible immersion. □

The next lemma shows that, up to congruence, $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ is the unique totally geodesic surface of $\mathsf{F}(\mathbb{R}^3)$.

Lemma 6.13. *Let $\psi\colon \Sigma \to \mathsf{F}(\mathbb{R}^3)$ be a compatible totally geodesic immersion of a complete two-dimensional Riemannian manifold. Then ψ is congruent to ϕ under an element of* $\mathsf{SO}(3)$. *In particular, ψ and ϕ are congruent as immersions into $\mathsf{F}(\mathbb{C}^3)$ as well.*

Proof. Let $a \in \mathsf{Sp}(1)$ be arbitrary, and take the map $\phi_a\colon \mathbb{R}\mathsf{P}^2(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3) = \mathsf{S}^3(2\sqrt{2})/\mathsf{Q}_8$ given by $\phi_a([x : y : z]) = a(xi + yj + zk)\mathsf{Q}_8$. Since left multiplication by a is an isometry, ϕ_a is also a compatible totally geodesic immersion of $\mathbb{R}\mathsf{P}^2$ congruent to ϕ. We show that all totally geodesic surfaces arise in this manner.

Let $\pi\colon \mathsf{S}^3(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3)$ be the canonical projection map, and consider the totally geodesic sphere $\mathsf{S}^2_1(2\sqrt{2}) \subseteq \mathsf{S}^3(2\sqrt{2})$ obtained as the intersection of $\operatorname{Im}\mathbb{H}$ with our 3-sphere. Take any point $p = \pi(z) \in \mathsf{F}(\mathbb{R}^3)$ and a two-dimensional subspace $V \subseteq T_{\pi(z)}\mathsf{F}(\mathbb{R}^3)$. As π is a Riemannian covering map, we may regard V as a subspace of $T_z\mathsf{S}^3(2\sqrt{2}) \equiv \mathbb{R}z^\perp$, where we are considering the standard inner product on $\mathbb{H} \equiv \mathbb{R}^4$. Let $a \in \mathsf{Sp}(1)$ be orthogonal to V and z and consider the great sphere $\mathsf{S}^2_a = a \cdot \mathsf{S}^2_1(2\sqrt{2})$. Note that S^2_a coincides with the great sphere obtained by intersecting $\mathsf{S}^3(2\sqrt{2})$ with the subspace $V \oplus \mathbb{R}z$, and thus the map $h_a\colon \mathsf{S}^2(2\sqrt{2}) \to \mathsf{S}^3(2\sqrt{2})$

defined by $h_a(x, y, z) = a(xi + yj + zk)$ is the unique compatible totally geodesic immersion passing through z with tangent space V. Since $h_a(-x, -y, -z) = -h_a(x, y, z)$, we see that the map h_a descends to the map $\phi_a \colon \mathbb{R}\mathsf{P}^2(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3)$, so ϕ_a passes through $p = \pi(z)$ with tangent space V. As p and V are arbitrary, we conclude that every complete compatible totally geodesic immersion from a surface to $\mathsf{F}(\mathbb{R}^3)$ is equivalent to one of the form ϕ_a, and is thus congruent to $\phi \colon \mathbb{R}\mathsf{P}^2(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3)$. The element in $\mathsf{SO}(3)$ that achieves this congruence is the image of a under the double cover $\mathsf{Sp}(1) \to \mathsf{SO}(3)$. □

Clearly, the fact that $\phi(\mathbb{R}\mathsf{P}^2(2\sqrt{2}))$ is contained in a Lagrangian submanifold implies that ϕ is totally real. Note that none of these submanifolds are well-positioned. Indeed, the totally geodesic $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ corresponding to $\mathfrak{v} = \mathrm{span}\{e_1 + e_3, e_5\}$ is not well-positioned (since it is not well-positioned at o), and because all of these submanifolds are congruent to this $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ by an element of $\mathsf{SO}(3)$, it follows that no totally geodesic $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ is well-positioned. Also, as $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ is not injectively immersed, it can not arise as an extrinsically homogeneous submanifold of $\mathsf{F}(\mathbb{C}^3)$.

Remark 6.14. Let us consider the unit speed geodesic γ of $\mathbb{R}\mathsf{P}^2(2\sqrt{2})$ given by the expression $\gamma(t) = \left[\cos \frac{t}{2\sqrt{2}} : \sin \frac{t}{2\sqrt{2}} : 0\right]$. Then γ descends to an injective totally geodesic immersion $f \colon \mathsf{S}^1 = \mathbb{R}/(2\sqrt{2}\pi\mathbb{Z}) \to \mathbb{R}\mathsf{P}^2(2\sqrt{2})$ defined via $f([t]) = \left[\cos \frac{t}{2\sqrt{2}} : \sin \frac{t}{2\sqrt{2}} : 0\right]$. Thus, f is a compatible totally geodesic immersion. We now take the compatible totally geodesic immersion $\phi \colon \mathbb{R}\mathsf{P}^2(2\sqrt{2}) \to \mathsf{F}(\mathbb{R}^3)$ defined as in (6.4). The composition $\beta = \phi \circ f \colon \mathsf{S}^1 \to \mathsf{F}(\mathbb{R}^3)$ is not compatible. Indeed, a short calculation yields

$$\begin{aligned} \phi(\gamma(0)) &= \phi(\gamma(\sqrt{2}\pi)) = 2\sqrt{2}\mathsf{Q}_8, \\ (\phi \circ \gamma)'(0) &= (\phi \circ \gamma)'(\sqrt{2}\pi) = \frac{d}{dt}\bigg|_{t=0} 2\sqrt{2}\left(\cos \frac{t}{2\sqrt{2}} i + \sin \frac{t}{2\sqrt{2}} j\right)\mathsf{Q}_8, \end{aligned}$$

so $\widetilde{\beta}([0]) = \widetilde{\beta}([\sqrt{2}\pi])$, implying that $\widetilde{\beta}$ is not injective.

6.3.3 The almost product $\mathsf{S}^3 \times \mathsf{S}^3$

We describe the totally geodesic examples of the almost product $\mathsf{S}^3 \times \mathsf{S}^3$ equipped with a homogeneous nearly Kähler metric.

The fiber of $\mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3$ [55, Example 3.1]

Let $\Sigma = \mathsf{S}^3$ be the fiber of the projection $(x, y) \mapsto x$, which we know from Subsection 6.1 that is a totally geodesic submanifold of $\mathsf{S}^3 \times \mathsf{S}^3$, and it coincides with the orbit $(\Delta_{1,3}\mathsf{SU}(2) \times \mathsf{SU}(2)_2) \cdot o$. It is immediate to check that the normalizer of Σ in G is precisely $\mathsf{N}_\mathsf{G}(\Sigma) = \Delta_{1,3}\mathsf{SU}(2) \times \mathsf{SU}(2)_2$, and the restricted action $\mathsf{N}_\mathsf{G}(\Sigma) \curvearrowright \Sigma$ satisfies

$$(g, h, g) \cdot (I, x) = (I, hxg^{-1}), \quad g, h, x \in \mathsf{SU}(2), \tag{6.5}$$

so this action coincides with the double cover $\mathsf{Spin}(4) = \mathsf{SU}(2) \times \mathsf{SU}(2) \to \mathsf{SO}(4)$ acting on S^3. As a consequence, Σ is isometric to a round sphere. A direct calculation yields that its sectional curvature is $3/4$, so we actually have $\Sigma = \mathsf{S}^3\left(2/\sqrt{3}\right)$. The tangent space of Σ through o is $\mathfrak{p}_{\mathsf{S}^3\left(2/\sqrt{3}\right)} = \mathfrak{p}_1$. A direct calculation yields $J(\mathfrak{p}_1) = \mathfrak{p}_2$, so $\mathsf{S}^3\left(2/\sqrt{3}\right)$ is a Lagrangian submanifold. As $\mathsf{S}^3\left(2/\sqrt{3}\right)$ is the fiber, it is obviously well-positioned.

The Berger sphere [55, Example 3.4]

Consider the subgroup

$$\mathsf{B} = \left\{\left(g, k, HgH^{-1}\right) \in \mathsf{G} \colon g \in \mathsf{SU}(2), k \in \mathsf{U}(1)\right\},$$

where $H \in \mathsf{SU}(2)$ is the element defined in (6.2) and $\mathsf{U}(1)$ is embedded in $\mathsf{SU}(2)$ diagonally. The Lie algebra $\mathfrak{b} \subseteq \mathfrak{g}$ satisfies $\mathfrak{b} = \mathbb{R}(H, H, H) \oplus \operatorname{span}\{e_1, e_5, e_6\} = (\mathfrak{b} \cap \mathfrak{k}) \oplus (\mathfrak{b} \cap \mathfrak{p})$, so the orbit $\mathsf{B} \cdot o$ is a totally geodesic submanifold of $\mathsf{S}^3 \times \mathsf{S}^3$. As the isotropy subgroup B_o is merely the diagonally embedded $\mathsf{U}(1)$, it follows that $\mathsf{B} \cdot o$ is diffeomorphic to a 3-sphere. More precisely, $\mathsf{B} \cdot o$ is the Berger sphere $\mathsf{S}^3_{\mathbb{C},1/3}(2)$. Its tangent space at o is $\mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/3}(2)} = \mathfrak{b} \cap \mathfrak{p} = \operatorname{span}\{e_1, e_5, e_6\}$. One sees that $J(\mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/3}(2)}) = \mathfrak{p} \ominus \mathfrak{p}_{\mathsf{S}^3_{\mathbb{C},1/3}(2)}$, so $\mathsf{S}^3_{\mathbb{C},1/3}(2)$ is a Lagrangian submanifold. Finally, a direct application of Corollary 6.10 yields that $\mathsf{S}^3_{\mathbb{C},1/3}(2)$ is well-positioned.

Remark 6.15. We note that although the authors in [55] provide six examples of Lagrangian totally geodesic submanifolds of $\mathsf{S}^3 \times \mathsf{S}^3$, the first three are congruent to the round sphere given in Subsection 6.3.3 and the last three are congruent to the Berger sphere described in Subsection 6.3.3. This is a consequence of Theorem C.

The torus [29, Example 1]

Let T^2 be the connected subgroup of G with Lie algebra $\mathfrak{t} = \operatorname{span}\{e_1, e_4\}$. As T^2 is contained in the torus $\mathsf{U}(1) \times \mathsf{U}(1) \times \mathsf{U}(1)$ (where $\mathsf{U}(1)$ is the diagonal subgroup of $\mathsf{SU}(2)$), it follows that T^2 is a two-dimensional torus. Furthermore, as $\mathfrak{t} \subseteq \mathfrak{p}$, we see that $\mathfrak{t}$ is canonically embedded, so $\mathsf{T}^2 \cdot o$ is a totally geodesic surface of $\mathsf{S}^3 \times \mathsf{S}^3$ diffeomorphic to a torus. Consider the exponential map $\exp_o \colon \mathfrak{t} \to \mathsf{T}^2 \cdot o$, which is $\mathfrak{t}$-equivariant in the sense that $\exp_o(T + S) = \operatorname{Exp}(T) \cdot \exp_o(S)$. This means that $\exp_o$ is a Riemannian covering map, and $\mathsf{T}^2 \cdot o$ is isometric to the quotient of $\mathfrak{t} = \mathbb{R}^2$ by the lattice $\Gamma = \exp_o^{-1}(o)$. Now, given $u, v \in \mathbb{R}$, we see that $\exp_o(ue_1 + ve_4)$ is equal to $(g_1, g_2, g_3) \cdot o$, where

$$g_1 = \operatorname{diag}\left(e^{\frac{1}{6}i\left(\sqrt{3}u+3v\right)}, e^{-\frac{1}{6}i\left(\sqrt{3}u+3v\right)}\right), \qquad g_2 = \operatorname{diag}\left(e^{-\frac{iu}{\sqrt{3}}}, e^{\frac{iu}{\sqrt{3}}}\right),$$
$$g_3 = \operatorname{diag}\left(e^{\frac{1}{6}i\left(\sqrt{3}u-3v\right)}, e^{-\frac{1}{6}i\left(\sqrt{3}u-3v\right)}\right),$$

so the corresponding lattice is $\Gamma = \operatorname{span}_{\mathbb{Z}}\left\{(2\pi/\sqrt{3}, 2\pi), (4\pi/\sqrt{3}, 0)\right\}$. Thus, $\mathsf{T}^2 \cdot o$ is isometric to the flat torus $\mathsf{T}^2_\Gamma = \mathbb{R}^2/\Gamma$. It turns out that the closest points in $\Gamma \setminus \{(0,0)\}$ to the origin are those in the set

$$\left\{\pm\left(\frac{2\pi}{\sqrt{3}}, 2\pi\right), \pm\left(\frac{4\pi}{\sqrt{3}}, 0\right), \pm\left(\frac{2\pi}{\sqrt{3}}, -2\pi\right)\right\},$$

so T^2_Γ admits three closed geodesics of minimum length $\frac{4\pi}{\sqrt{3}}$, unlike any Riemannian product of the form $\mathsf{S}^1(a) \times \mathsf{S}^1(b)$. By construction, we have that $\mathfrak{p}_{\mathsf{T}^2_\Gamma} = \operatorname{span}\{e_1, e_4\}$ is a J-invariant subspace, so T^2_Γ is an almost complex surface inside $\mathsf{S}^3 \times \mathsf{S}^3$. By Corollary 6.10, T^2_Γ is a well-positioned totally geodesic submanifold of $\mathsf{S}^3 \times \mathsf{S}^3$.

Not well-positioned totally geodesic spheres [29, Example 2]

Let $g = \left(e^{\frac{i\pi}{3}F}, I, e^{-\frac{i\pi}{3}F}\right)$ and take the subgroup $g\mathsf{K}g^{-1} \subseteq \mathsf{G}$. Its Lie algebra $\operatorname{Ad}(g)\mathfrak{k}$ satisfies

$$\operatorname{Ad}(g)\mathfrak{k} = \mathbb{R}(F, F, F) \oplus \operatorname{span}\{e_1 + e_5, e_2 - e_4\} = (\operatorname{Ad}(g)\mathfrak{k} \cap \mathfrak{k}) \oplus (\operatorname{Ad}(g)\mathfrak{k} \cap \mathfrak{p}),$$

so $\operatorname{Ad}(g)\mathfrak{k}$ is a canonically embedded subalgebra of $\mathfrak{g}$ and the orbit $(g\mathsf{K}g^{-1})\cdot o$ is a totally geodesic surface. The isotropy subgroup $(g\mathsf{K}g^{-1})\cdot o$ is isomorphic to $\mathsf{U}(1)$, and thus $(g\mathsf{K}g^{-1})\cdot o$ is isometric to a round sphere. A simple computation yields that its sectional curvature is $2/3$, so $(g\mathsf{K}g^{-1}) \cdot o$ is a round sphere with radius $\sqrt{3/2}$. Its tangent space at o is given by $\mathfrak{p}_{\mathsf{S}^2(\sqrt{3/2})} = \operatorname{Ad}(g)\mathfrak{k} \cap \mathfrak{p} = \operatorname{span}\{e_1 + e_5, e_2 - e_4\}$. One sees that J preserves this subspace, and by homogeneity we deduce that $\mathsf{S}^2(\sqrt{3/2})$ is an almost complex surface in $\mathsf{S}^3 \times \mathsf{S}^3$. Clearly, $\mathsf{S}^2(\sqrt{3/2})$ is not well-positioned at o.

Great spheres inside $\mathsf{S}^3\left(2/\sqrt{3}\right)$

Let Σ be a totally geodesic surface inside the Lagrangian round S^3, so it is merely a great sphere inside S^3. Then $\Sigma \subseteq \mathsf{S}^3 \times \mathsf{S}^3$ is automatically a totally real totally geodesic submanifold. Moreover, Σ is also homogeneous. Indeed, because $\mathsf{S}^3 = \mathsf{SO}(4)/\mathsf{SO}(3)$ is a symmetric space, its totally geodesic submanifolds are homogeneous, so Σ is an orbit of a subgroup $\mathsf{H} \subseteq \mathsf{SO}(4)$. Let $\phi\colon \mathsf{N}_\mathsf{G}(\mathsf{S}^3) \to \mathsf{SO}(4)$ be the double cover defined as the composition of the isometric action defined in (6.5) with the projection of $\mathsf{S}^3 \times \mathsf{S}^3$ onto its second factor. The preimage $\mathsf{L} = \phi^{-1}(\mathsf{H})$ is a subgroup of G whose orbit at any $p \in \Sigma$ coincides with Σ, so Σ is extrinsically homogeneous. For instance, one can take the diagonal subgroup $\mathsf{K} \subseteq \mathsf{N}_\mathsf{G}(\mathsf{S}^3)$, and the orbit $\mathsf{K} \cdot (I, H)$ is an example of these spheres. Similarly, if Σ' is another totally geodesic surface inside the round S^3, there exists an element $g \in \mathsf{SO}(4)$ such that $g \cdot \Sigma = \Sigma'$, which implies that any element $h \in \phi^{-1}(g)$ also satisfies $h \cdot \Sigma = \Sigma'$. As Σ is contained in the fiber of $\mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3$, its tangent space at every point is contained in the vertical subspace, so Σ is well-positioned.

It is worth noting that even though theses spheres are extrinsically homogeneous, they are not D-invariant. Indeed, observe that the difference tensor restricted to $\mathfrak{p}_1$ is given by

$$\frac{1}{2}[(X, -2X, X), (Y, -2Y, Y)]_\mathfrak{p} = \frac{1}{2}(-[X, Y], 2[X, Y], -[X, Y]), \quad X, Y \in \mathfrak{su}(2),$$

which means that the D-invariant subspaces of $\mathfrak{p}_1$ are in a one-to-one correspondence with the Lie subalgebras of $\mathfrak{su}(2)$. As $\mathfrak{su}(2)$ admits no codimension one subalgebras, it follows that no two-dimensional subspace of $\mathfrak{p}_1$ (and thus no totally geodesic sphere inside the fiber S^3) is D-invariant.

Remark 6.16. The round S^2 described in Section 6.3.3 serves as a counterexample to [2, Proposition 2]. In this result, the authors claim that for a compact geodesic orbit space $M = \mathsf{G}/\mathsf{K}$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a subspace $\mathfrak{v} \subseteq \mathfrak{p}$ is tangent to an extrinsically homogeneous totally geodesic submanifold if and only if it generates a canonically embedded subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ satisfying $\mathfrak{s}_\mathfrak{p} = \mathfrak{v}$ and $U(\mathfrak{v}, \mathfrak{v}) \subseteq \mathfrak{v}$ (recall that U is defined by (1.3)). In the naturally reductive setting, because $U = 0$, the proposition would imply that every extrinsically homogeneous totally geodesic submanifold of M is automatically D-invariant. However, this sphere is extrinsically homogeneous with respect to the presentation $\mathsf{S}^3 \times \mathsf{S}^3 = \mathsf{SU}(2)^3/\Delta\mathsf{SU}(2)$ and it is not D-invariant.

6.4 Riemannian cones and totally geodesic submanifolds

In this section we start by recalling the definition and the basic properties of Riemannian cones. After that we prove a structure result for totally geodesic submanifolds of Riemannian cones. We refer the reader to [115] for a detailed account on semi-Riemannian cones.

Let M be a Riemannian manifold, which for our purposes is assumed to be real analytic and complete. We define its *Riemannian cone* as the warped product $\widehat{M} = \mathbb{R}^+ \times_f M$, where $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is the identity map. More explicitly, if $g = \langle \cdot, \cdot \rangle$ denotes the metric on M and $r\colon \widehat{M} \to \mathbb{R}^+$ is the projection on the first factor, the inner product on $\widehat{M}$ is given by $\widehat{g} = \mathrm{d}r^2 + r^2 g$. The submanifolds $\{\tau\} \times M$ with $\tau > 0$ are called the *links* of $\widehat{M}$.

Every vector field $X \in \mathfrak{X}(\mathbb{R}^+)$ (respectively, $X \in \mathfrak{X}(M)$) admits a natural extension to $\widehat{M}$, which we also denote by X. In particular, if ∂_r is the unit radial vector field on $\mathbb{R}^+$, then its natural extension to $\widehat{M}$ is called the *cone vector field* or the *radial vector field*. Note that at every point (τ, p) of $\widehat{M}$ we have the orthogonal decomposition $T_{(\tau,p)}\widehat{M} = \mathbb{R}(\partial_r)_{(\tau,p)} \oplus T_pM$. The Levi-Civita connection $\widehat{\nabla}$ of $\widehat{M}$ is characterized by the following equations for $X, Y \in \mathfrak{X}(M)$:

$$\widehat{\nabla}_{\partial_r}\partial_r = 0, \quad \widehat{\nabla}_X \partial_r = \widehat{\nabla}_{\partial_r} X = \frac{1}{r}X, \quad \widehat{\nabla}_X Y = \nabla_X Y - r\langle X, Y\rangle \partial_r. \tag{6.6}$$

As a consequence, the curvature tensor $\widehat{R}$ is determined by the conditions

$$\begin{aligned} &\widehat{R}(\partial_r, \cdot)\cdot = \widehat{R}(\cdot, \partial_r)\cdot = \widehat{R}(\cdot, \cdot)\partial_r = 0, \\ &\widehat{R}(u, v)w = R(u, v)w - \langle v, w\rangle u + \langle u, w\rangle v, \quad u, v, w \in T_pM. \end{aligned} \tag{6.7}$$

Let $X = a\partial_r + v \in T_{(\tau,p)}\widehat{M}$ be arbitrary. From the equations above we see that the Jacobi operator associated with X satisfies

$$\begin{aligned} \widehat{R}_X(b\partial_r + w) &= \widehat{R}(b\partial_r + w, a\partial_r + v)(a\partial_r + v) = \widehat{R}(w, v)v \\ &= R_v w - |v|^2 w + \langle v, w\rangle v. \end{aligned}$$

Although the process is more tedious, it is possible to compute the covariant derivatives of the curvature tensor from (6.6) and (6.7). For instance, one can show that

$$(\widehat{\nabla}_x \widehat{R})(u, v, w) = (\nabla_x R)(u, v, w) - \langle x, \widehat{R}(u, v)w\rangle \tau \partial_r \tag{6.8}$$

for all $(\tau, p) \in \widehat{M}$ and $x, u, v, w \in T_pM$. We will make use of this formula later on.

Remark 6.17. Let $M = \mathsf{S}^n(1)$ be the standard round sphere of radius one. Then, its cone is the punctured Euclidean space $\mathbb{R}^{n+1} \setminus \{0\}$. However, if $r \neq 1$, the cone of $\mathsf{S}^n(r)$ is not flat due to (6.7). This illustrates that two homothetic manifolds may not have homothetic cones.

We can also describe the geodesics of $\widehat{M}$ in terms of those of M. Let $(\tau, p) \in \widehat{M}$ be any point and consider the tangent vector $w = a\partial_r + v$, where $a \in \mathbb{R}$ and $v \in T_pM$ are arbitrary. From [115, Equation 2.7], we see that the geodesic $\widehat{\gamma}(t) = \widehat{\exp}_{(\tau,p)}(tw) = (\rho(t), \alpha(t))$ is given in a neighborhood of $t = 0$ by

$$\rho(t) = \sqrt{(at+\tau)^2 + |v|^2\tau^2 t^2}, \quad \alpha(t) = \exp_p(f(t)v), \tag{6.9}$$

where

$$f(t) = \begin{cases} \frac{1}{|v|} \arctan\left(\frac{|v|\tau t}{at+\tau}\right), & v \neq 0, \\ 0, & v = 0. \end{cases}$$

As a consequence, the maximal interval of definition of $\widehat{\gamma}(t)$ contains the interval

$$I_a = \begin{cases} \mathbb{R}, & a = 0, \\ \left(-\frac{\tau}{a}, \infty\right), & a > 0, \\ \left(-\infty, -\frac{\tau}{a}\right) & a < 0. \end{cases} \tag{6.10}$$

Note that I_a only depends on a. A consequence of (6.9) is that if γ is a geodesic of $\widehat{M}$, its projection onto M is a pregeodesic of M. Observe that $\widehat{M}$ is never complete. However, it is clear that it is an analytic Riemannian manifold. The following lemma actually shows that the only incomplete geodesics are those tangent to the cone vector field.

Lemma 6.18. *Let M be a complete Riemannian manifold, $(\tau, p) \in \widehat{M}$ a point in its Riemannian cone and $w = a\partial_r + v \in T_{(\tau,p)}M$ a unit vector, where $a \in \mathbb{R}$ and $v \in T_pM$. Consider the maximal geodesic $\widehat{\gamma}(t)$ such that $\widehat{\gamma}(0) = (\tau, p)$ and $\widehat{\gamma}'(0) = w$. The following statements hold:*

(i) *If $v = 0$, then the maximal interval of definition of $\widehat{\gamma}(t)$ is precisely I_a.*

(ii) *If $v \neq 0$, then $\widehat{\gamma}(t)$ is defined on all $\mathbb{R}$.*

Proof. Firstly, assume that $v = 0$. Without loss of generality, we may also suppose that a is positive, so $a = 1$. From (6.9), we see that the curve $\beta\colon (-\tau, \infty) \to \widehat{M}$ defined by $\beta(t) = (t + \tau, p)$ is a geodesic of $\widehat{M}$ with initial conditions $\beta(0) = (\tau, p)$, $\beta'(0) = \partial_r$. Since $t + \tau$ converges to zero as t converges to $-\tau$, the curve β is a maximal geodesic, thus proving (i).

Now, assume that $v \neq 0$, so that the curve $\beta\colon I_a \to \widehat{M}$, $\beta(t) = (\rho(t), \alpha(t))$ defined by (6.9) is a geodesic of $\widehat{M}$ with $\beta(0) = p$, $\beta'(0) = w$. Note that the derivative

$$\begin{aligned} \rho'(t) &= \frac{2a(at+\tau) + 2|v|^2\tau^2 t}{2\rho(t)} = \frac{a\tau + (a^2 + |v|^2\tau^2)t}{\sqrt{(at+\tau)^2 + |v|^2\tau^2 t^2}} \\ &= \frac{a\tau + t}{\sqrt{(at+\tau)^2 + |v|^2\tau^2 t^2}} \end{aligned}$$

vanishes at $t_0 = -a\tau \in I_a$ (this last inclusion holds because $|a| \in (0,1)$ and $\tau > 0$). As a consequence, $\beta'(t_0) = \alpha'(t_0) \in T_{\alpha(t_0)}M$. Looking at (6.10), we obtain that the geodesic $\widehat{\exp}_{\beta(t_0)}(\beta'(t_0))$ is defined on all of $\mathbb{R}$, and by uniqueness it must coincide with the curve $\beta(t+t_0)$. Thus, β can be extended to all $\mathbb{R}$, so (ii) holds. □

Suppose that $f\colon M \to N$ is a smooth map. We define its associated *cone map* as the map $\widehat{f}\colon \widehat{M} \to \widehat{N}$ given by $\widehat{f}(\tau,p) = (\tau, f(p))$. It can easily be checked that $\widehat{f}$ is an isometric immersion (respectively, an isometry) if and only if f is an isometric immersion (respectively, an isometry). In the following, we provide some information about the isometry group of Riemannian cones.

Proposition 6.19. *Let M and N be two complete Riemannian manifolds. Then, every isometry $f\colon \widehat{M} \to \widehat{N}$ is the cone map of an isometry $g\colon M \to N$. In particular, if the cones $\widehat{M}$ and $\widehat{N}$ are isometric, then M and N are also isometric.*

Proof. We take the subset $\mathcal{C}_{(\tau,p)} = \{X \in T_{(\tau,p)}\widehat{M}\colon \widehat{\exp}_{(\tau,p)}(tX) \text{ is defined on } \mathbb{R}\}$ of $T_{(\tau,p)}\widehat{M}$ for each $(\tau,p) \in \widehat{M}$. It is clear from Lemma 6.18 that $\mathcal{C}_{(\tau,p)}$ coincides with $T_{(\tau,p)}\widehat{M} \setminus \mathbb{R}(\partial_r)_{(\tau,p)}$.

Now, let $f\colon \widehat{M} \to \widehat{N}$ be an isometry and fix $(\tau,p) \in \widehat{M}$ with image $(s,q) = f(\tau,p)$. Since f is an isometry, it sends $\mathcal{C}_{(\tau,p)}$ to $\mathcal{C}_{(s,q)}$, and thus $f_{*(\tau,p)}((\partial_r)_{(\tau,p)}) = \pm(\partial_r)_{(s,q)}$. The equality $f_{*(\tau,p)}((\partial_r)_{(\tau,p)}) = -(\partial_r)_{(s,q)}$ is not possible, because in that case the maximal geodesic $\gamma(t) = \widehat{\exp}_{(\tau,p)}(t\partial_r)$ would be mapped to the maximal geodesic $\beta(t) = \widehat{\exp}_{(s,q)}(-t\partial_r)$, which is not possible because the first geodesic is defined for all $t \geq 0$, while the second one is not. We deduce that $f_{*(\tau,p)}((\partial_r)_{(\tau,p)}) = (\partial_r)_{f(\tau,p)}$. As a consequence, the maximal geodesic $\gamma(t) = \widehat{\exp}_{(\tau,p)}(t\partial_r)$ is mapped to the maximal geodesic $\beta(t) = \widehat{\exp}_{(s,q)}(t\partial_r)$, and their corresponding intervals of definition are $(-\tau, 0)$ and $(-s, 0)$. Hence, $\tau = s$. All in all, we have seen that f sends the link $\{\tau\} \times M$ to $\{\tau\} \times N$ for each $\tau \in \mathbb{R}^+$, so f takes the form $f(\tau,p) = (\tau, h(\tau,p))$ for a map $h\colon \widehat{M} \to N$. Furthermore, at each $(\tau,p) \in \widehat{M}$ we have

$$\begin{aligned}(\partial_r)_{f(\tau,p)} &= f_{*(\tau,p)}((\partial_r)_{\tau,p}) = \left.\frac{d}{dt}\right|_{t=0} (\tau + t, h(\tau+t,p)) \\ &= (\partial_r)_{f(\tau,p)} + h_{*(\tau,p)}((\partial_r)_{(\tau,p)}),\end{aligned}$$

which means that $h_{*(\tau,p)}((\partial_r)_{(\tau,p)}) = 0$, so h does not depend on τ. In other words, there exists a map $g\colon M \to N$ such that $h(\tau,p) = g(p)$ for all $(\tau,p) \in \widehat{M}$. The fact that f is an isometry readily implies that g is also an isometry. □

Corollary 6.20. *For a complete Riemannian manifold M, the map $f \in I(M) \mapsto \widehat{f} \in I(\widehat{M})$ is a Lie group isomorphism.*

Many geometric properties of Riemannian manifolds can be translated into geometric properties of their cones. For instance, as a consequence of (6.7), M is an Einstein manifold with $\mathrm{Ric} = (n-1)\langle\cdot,\cdot\rangle$ if and only if $\widehat{M}$ is Ricci-flat. It turns out that nearly Kähler structures on six-dimensional manifolds are related to G_2-structures on their cones. We briefly describe this relationship, see [14] for details. For the general theory of G_2-manifolds, we refer the reader to [95, Chapter 10].

Let M be a six-dimensional strictly nearly Kähler manifold. Then M is Einstein with positive Ricci curvature [78, Theorem 5.2] and after rescaling the metric we may assume that the Einstein constant of M is $\lambda = 5$. In that case, one defines a three-form $\phi \in \Omega^3(\widehat{M})$ via the equations (for $X, Y, Z \in \mathfrak{X}(M)$)

$$\begin{aligned} \phi(X, Y, Z) &= r^3 \langle Y, (\nabla_X J) Z \rangle, \\ \phi(\partial_r, X, Y) &= -\phi(X, \partial_r, Y) = \phi(X, Y, \partial_r) = r^2 \langle X, JY \rangle, \end{aligned}$$

and checks that ϕ is a parallel three-form inducing a G_2 structure on $\widehat{M}$. In addition, the restricted holonomy group of $\widehat{M}$ is precisely G_2 whenever M is not locally isometric to S^6. Conversely, suppose that $\widehat{M}$ is a G_2-manifold whose structure is given by the parallel three-form ϕ. Then the almost complex structure J defined on M by $\langle X, JY \rangle = \phi(\partial_r, X, Y)$ is strictly nearly Kähler and M is an Einstein manifold with $\mathrm{Ric} = 5\langle \cdot, \cdot \rangle$.

Remark 6.21. Notice that the metrics of the nearly Kähler manifolds $\mathbb{C}\mathsf{P}^3$, $\mathsf{F}(\mathbb{C}^3)$ and $\mathsf{S}^3 \times \mathsf{S}^3$ that we are considering have Einstein constants $5/2$, $5/2$ and $5/3$, respectively. Therefore, one would have to rescale these metrics by $1/2$, $1/2$, and $1/3$ to obtain the G_2-cones over them. However, for our purposes this is not a problem, since the totally geodesic property is preserved under rescalings of the ambient manifold and, as we will see, the maximal totally geodesic submanifolds of these G_2-cones are cones over the totally geodesic submanifold of a homogeneous nearly Kähler 6-manifold.

There is also a relationship between submanifolds of M that have a nice interaction with J and calibrated cones inside $\widehat{M}$. The notions of calibrated geometry were introduced in the seminal paper [84] by Harvey and Lawson. We remind that a *calibration* on a Riemannian manifold N is a closed differential form $\omega \in \Omega^k(M)$ satisfying $\omega(v_1, \ldots, v_k) \leq 1$ whenever $v_1, \ldots, v_k$ are unit vectors in TN. A k-dimensional oriented submanifold S of N is *calibrated* if the restriction of ω to S is equal to the Riemannian volume form of S, and it follows that S is a minimal submanifold. It can be shown that for the case of a G_2-manifold (N, ϕ), both ϕ and its Hodge dual $\star\phi$ are calibrations [84, Theorem 1.4 and Theorem 1.16]. We say in this case that S is *associative* (respectively, *coassociative*) if it is calibrated with respect to ϕ (respectively, $\star\phi$). Coming back to the case that $N = \widehat{M}$ is the cone of a six-dimensional strict nearly Kähler manifold, it is known that the cone of a J-holomorphic curve is an associative submanifold, whereas the cone of a Lagrangian submanifold is a coassociative submanifold.

6.4.1 Totally geodesic submanifolds of Riemannian cones

We now discuss the relationship between the totally geodesic submanifolds of a Riemannian cone (over a complete real analytic manifold) and those of its base. We are interested in determining the maximal totally geodesic submanifolds of the cone $\widehat{M}$ over M.

This first result shows that every totally geodesic submanifold of the base induces a totally geodesic submanifold of the cone by means of the cone map.

Lemma 6.22. *Let M be a Riemannian manifold and $\phi\colon S \to M$ an isometric immersion of a k-dimensional submanifold S. Then the following statements hold:*

(i) *The immersion ϕ is totally geodesic if and only if the cone map $\widehat{\phi}\colon \widehat{S} \to \widehat{M}$ is totally geodesic.*

(ii) *The totally geodesic immersion ϕ is compatible if and only if the cone map $\widehat{\phi}\colon \widehat{S} \to \widehat{M}$ is compatible.*

Proof. First of all, as being totally geodesic is a local property, we may suppose that $S \subseteq M$ is embedded and ϕ is the inclusion map. As a consequence, $\widehat{S} = \mathbb{R}^+ \times S$ as a subset of $\widehat{M}$.

Firstly, assume that S is totally geodesic in M. Given $(\tau, p) \in \widehat{S}$ and $w = a\partial_r + v \in T_{(\tau,p)}\widehat{S}$ (so $v \in T_pS$), we know by (6.9) that the geodesic $\widehat{\gamma}(t) = \widehat{\exp}_{(\tau,p)}(tw)$ is of the form $\widehat{\gamma}(t) = (\rho(t), \beta(t))$, where $\beta(t)$ is a pregeodesic of M such that $\beta'(0) = v$. Since S is totally geodesic, there exists an $\varepsilon > 0$ such that $\beta(t) \in S$ for all $t \in (-\varepsilon, \varepsilon)$, so $\widehat{\gamma}(t) \in \widehat{S}$ for all $t \in (-\varepsilon, \varepsilon)$. Therefore, $\widehat{S}$ is totally geodesic in $\widehat{M}$.

Conversely, suppose that $\widehat{S}$ is totally geodesic, and let $p \in S$, $v \in T_pS$. The geodesic $\widehat{\gamma}(t) = \widehat{\exp}_{(1,p)}(tw)$ is locally of the form $(\rho(t), \beta(t))$, where $\beta(t) = \exp_p(f(t)v)$ for a diffeomorphism $f(t)$ such that $f(0) = 0$ and $f'(0) = 1$. Thus, as $\widehat{S}$ is totally geodesic, there exists an $\varepsilon > 0$ such that $\widehat{\gamma}(t) \in \widehat{S}$ for $|t| < \varepsilon$, which means that $\exp_p(f(t)v) \in S$ for the same values of t. As f^{-1} is continuous at $t = 0$, it follows that there exists $\delta > 0$ such that $\exp_p(sv) \in S$ for $|s| < \delta$. We conclude that S is totally geodesic in M, proving (i).

Finally, observe that since $\widehat{\phi}_{*(\tau,p)}(T_{(\tau,p)}\widehat{S}) = \mathbb{R}(\partial_r)_{\phi(\tau,p)} \oplus \phi_{*p}(T_pS)$ for all $(\tau, p) \in \widehat{S}$, it follows that the induced map of $\widehat{\phi}$ is an injection of $\widehat{S}$ to $\mathsf{G}_{k+1}(T\widehat{M})$ if and only if the induced map of ϕ is an injection of S to $\mathsf{G}_k(TM)$. This yields (ii). □

It was shown in [94] that certain types of Riemannian manifolds with special holonomy do not admit totally geodesic hypersurfaces. We deduce that the same result holds for Sasakian–Einstein, 6-dimensional nearly Kähler and nearly parallel G_2-manifolds.

Theorem 6.23. *Let M be a complete Riemannian manifold with non-constant sectional curvature. Assume that M satisfies one of the following conditions:*

(i) *M^{2n+1} is Sasakian–Einstein,*

(ii) *M^6 is a 6-dimensional strictly nearly Kähler manifold,*

(iii) *M^7 is a nearly parallel G_2-manifold.*

Then, M does not admit a totally geodesic hypersurface.

Proof. Observe that in all three cases M is an Einstein manifold with positive Ricci curvature. As the (non)existence of totally geodesic hypersurfaces is a purely local question, we may suppose that M is simply connected. Furthermore, as their existence is also independent of rescalings of the metric, we may also suppose that the Einstein constant of M is equal to $\dim M - 1$.

Let Σ be a totally geodesic hypersurface of M. By Lemma 6.22, $\widehat{\Sigma}$ is a totally geodesic hypersurface of $\widehat{M}$. By Gallot's Theorem (see [69]), $\widehat{M}$ is locally irreducible since M has non-constant sectional curvature. Moreover, by [14], we know that:

(i) If M^{2n+1} is a Sasakian–Einstein manifold, then the restricted holonomy of $\widehat{M}$ is contained in $\mathsf{SU}(n+1)$.

(ii) If M^6 is a strictly nearly Kähler manifold, then the restricted holonomy of $\widehat{M}$ is contained in G_2.

(iii) If M^7 is a nearly parallel G_2-manifold, then the restricted holonomy of $\widehat{M}$ is contained in $\mathsf{Spin}(7)$.

Now, since $\widehat{M}$ is Einstein (indeed Ricci-flat), by [94, Theorem 4.3] the restricted holonomy of the cone $\widehat{M}$ is $\mathsf{SO}(T_p\widehat{M})$. This contradicts the fact that the holonomy of $\widehat{M}$ is contained in one of three aforementioned groups, yielding the result. □

The following result is concerned with the extendability of cones over totally geodesic submanifolds of the base.

Lemma 6.24. *Let M be a complete real analytic Riemannian manifold and $\phi\colon S \to M$ a compatible totally geodesic immersion of a k-dimensional complete submanifold S. Then $\widehat{\phi}\colon \widehat{S} \to \widehat{M}$ is an inextendable compatible totally geodesic immersion.*

Proof. We have already seen in Lemma 6.22 that $\widehat{\phi}\colon \widehat{S} \to \widehat{M}$ is compatible, so we only need to show inextendability.

Let $(\tau, p) \in \widehat{S}$ be arbitrary and $w = a(\partial_r)_{(\tau,p)} + v$ be a nonzero tangent vector, where $v \in T_pM$. We consider the $\widehat{S}$-geodesic $\gamma(t) = \exp_{(\tau,p)}(tw)$. If $v \neq 0$, then γ is defined on all $\mathbb{R}$ due to Lemma 6.18, so $\widehat{\phi} \circ \gamma$ is also globally defined. Otherwise, we have $v = a(\partial_r)_{(\tau,p)}$, and Lemma 6.18 implies that γ is defined precisely on I_a. Because $\widehat{\phi}_{*(\tau,p)}((\partial_r)_{(\tau,p)}) = (\partial_r)_{(\tau,\phi(p))}$, the maximal $\widehat{M}$-geodesic $\exp_{(\tau,\phi(p))}(t\widehat{\phi}_{*(\tau,p)}(w))$ is also defined exactly on I_a, so it coincides with $\widehat{\phi} \circ \gamma$. We conclude that $\widehat{\phi}$ sends maximal geodesics of $\widehat{S}$ to maximal geodesics of $\widehat{M}$, so it is inextendable by Proposition 5.17. □

We can now prove the following characterization of totally geodesic submanifolds in cones:

Theorem 6.25. *Let M be a connected n-dimensional complete real analytic Riemannian manifold and consider its Riemannian cone $\widehat{M}$. Suppose Σ is a k-dimensional manifold (where $1 \leq k \leq n$), $f\colon \Sigma \to \widehat{M}$ is an inextendable compatible totally geodesic immersion, and let $x \in \Sigma$, $(\tau, p) = f(x)$ and $V = \widetilde{f}(x) = f_{*x}(T_x\Sigma)$. Then exactly one of the following two situations occur:*

(i) *The vector $(\partial_r)_{(\tau,p)}$ is in V. In this case, Σ is incomplete, the vector field ∂_r is everywhere tangent to the immersion f and there exists a complete compatible totally geodesic immersion $g\colon S \to M$ such that f and $\widehat{g}$ are equivalent.*

(ii) *The vector $(\partial_r)_{(\tau,p)}$ is not in V. In this case, Σ is complete, the vector field ∂_r is nowhere tangent to the immersion and there exists:*

- *a complete compatible totally geodesic immersion $g\colon S \to M$,*
- *a complete compatible totally geodesic immersion $h\colon E \to \widehat{S}$, where E is a hypersurface in $\widehat{S}$,*
- *and a surjective local isometry $\rho\colon E \to \Sigma$,*

such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{h} & \widehat{S} \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\widehat{g}} \\ \Sigma & \xrightarrow{f} & \widehat{M} \end{array}$$

Proof. We work with the projection $\pi\colon \widehat{M} \to M$. Let $W = \pi_{*(\tau,p)}(V) \subseteq T_pM$, which is precisely the orthogonal projection of V onto T_pM. We have that $V \subseteq \mathbb{R}(\partial_r)_{(\tau,p)} \oplus W$ and the dimension of W is either $k-1$ or k, depending on whether $(\partial_r)_{(\tau,p)}$ is in V or not.

Firstly, suppose that $(\partial_r)_{(\tau,p)} \in V$, so $V = \mathbb{R}(\partial_r)_{(\tau,p)} \oplus W$. Then Σ is not complete because the geodesic $\exp_{(\tau,p)}(t(\partial_r)_{(\tau,p)})$ is not defined on all $\mathbb{R}$, and for every $y \in \Sigma$ the tangent space $\widetilde{f}(y)$ contains $(\partial_r)_{f(y)}$ (otherwise, Σ would be complete by Corollary 5.18). Therefore, the radial vector field ∂_r is everywhere tangent to the immersion $f\colon \Sigma \to \widehat{M}$. Now, consider an $\varepsilon > 0$ such that $\widehat{\exp}_{(\tau,p)}$ is a diffeomorphism of the ball $B_{T_{(\tau,p)}\widehat{M}}(0,\varepsilon)$ onto its image and the set

$$F = \widehat{\exp}_{(\tau,p)}(V \cap B_{T_{(\tau,p)}\widehat{M}}(0,\varepsilon))$$

is an embedded totally geodesic submanifold of $\widehat{M}$. Then ∂_r is everywhere tangent to F and the restriction of π to F has constant rank equal to $k-1$, so the constant rank theorem implies that (perhaps after shrinking ε) the image $\pi(F)$ is a $(k-1)$-dimensional embedded submanifold of M and $\pi\colon F \to \pi(F)$ is a surjective submersion. Let $(s,q) \in F$ be any point and consider a nonzero $w \in T_q\pi(F) = \pi_{*(s,q)}(T_{(s,q)}F)$. Then, since $(\partial_r)_{(s,q)}$ is tangent to F, the vector $w \in T_{(s,q)}\widehat{M}$ is also tangent to F. As F is totally geodesic, we may choose $\delta > 0$ such that the $\widehat{M}$-geodesic $\widehat{\gamma}(t) = \widehat{\exp}_{(s,q)}(tw)$ is in F for all $t \in (-\delta,\delta)$, and as a consequence the curve $\pi(\widehat{\gamma}(t))$ is also contained in $\pi(F)$ for $t \in (-\delta,\delta)$. Recall from (6.9) that $\pi(\widehat{\gamma}(t)) = \exp_q(f(t)w)$, where the map $f\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism satisfying $f(0) = 0$. Because of this, the geodesic $\exp_q(tw)$ is contained in $\pi(F)$ for small values of t. This proves that $\pi(F)$ is a totally geodesic submanifold of M, and in particular the subspace W is totally geodesic in T_pM. Consider the complete compatible totally geodesic immersion $g\colon S \to M$ associated with W, and let $y \in S$ be the unique point with $g(y) = p$ and $\widetilde{g}(y) = W$. The cone map $\widehat{g}\colon \widehat{S} \to \widehat{M}$ is an inextendable compatible totally geodesic immersion by Lemma 6.24 and satisfies $\widehat{g}_{*(\tau,y)}(T_{(\tau,y)}\widehat{S}) = \mathbb{R}(\partial_r)_{(\tau,p)} \oplus W = V$, so f and $\widehat{g}$ are equivalent by uniqueness.

Secondly, assume that $(\partial_r) \notin V$, so that V is a hyperplane in $\mathbb{R}(\partial_r)_{(\tau,p)} \oplus W$. In this setting, Σ is complete by Corollary 5.18 and for all $y \in \Sigma$ the tangent space $\widetilde{f}(y)$ does not contain the vector $(\partial_r)_{f(y)}$ (otherwise, Σ would admit a non-complete geodesic). Thus, the radial vector field is nowhere tangent to $f\colon \Sigma \to M$. We now argue in a similar way as in the previous paragraph. Let $\varepsilon > 0$ be such that $\widehat{\exp}_{(\tau,p)}$ is a diffeomorphism of $B_{T_{(\tau,p)}\widehat{M}}(0, \varepsilon)$ to its image and

$$F = \widehat{\exp}_{(\tau,p)}(V \cap B_{T_{(\tau,p)}\widehat{M}}(0, \varepsilon))$$

is an embedded totally geodesic submanifold of $\widehat{M}$. Then, as ∂_r is nowhere tangent to F, the restriction of π to M has constant rank equal to k, so we may shrink ε so as to have that $\pi(F)$ is a k-dimensional embedded submanifold of M and the restricted projection $\pi\colon F \to \pi(F)$ is a diffeomorphism. The same argument as above shows that $\pi(F)$ is a totally geodesic submanifold, so in particular W is a totally geodesic subspace of T_pM. Let $g\colon S \to M$ be the associated complete compatible totally geodesic extension and $y \in S$ the unique point with $g(y) = (\tau, p)$ and $\widetilde{g}(y) = W$. Then the cone map $\widehat{g}\colon \widehat{S} \to \widehat{M}$ is the inextendable compatible totally geodesic immersion associated with $\mathbb{R}(\partial_r)_{(\tau,p)} \oplus W$. Because $V \subseteq \mathbb{R}(\partial_r)_{(\tau,p)} \oplus W$, we may use Proposition 5.22 to conclude. □

Corollary 6.26. *Let M be an analytic Riemannian manifold and $\widehat{M}$ its Riemannian cone. If Σ is a maximal totally geodesic submanifold of $\widehat{M}$, then Σ is either a hypersurface of $\widehat{M}$ or the cone over a maximal totally geodesic submanifold S of M.*

Theorem 6.25 reduces the classification of (maximal) totally geodesic submanifolds of cones to that of the totally geodesic submanifolds of the base manifold, and separately to that of totally geodesic hypersurfaces in the cone. We note that these hypersurfaces may not arise as cones over totally geodesic hypersurfaces in the base space, as we will see in Example 6.29 and more generally in Subsection 6.4.2.

Remark 6.27. Let M be a Riemannian manifold and suppose that Σ is a totally geodesic hypersurface of M that is not tangent to the cone vector field ∂_r. We may assume without loss of generality that $\Sigma \subseteq \widehat{M}$ is embedded, and we choose a point $(\tau, p) \in \Sigma$. The tangent space $V = T_{(\tau,p)}\Sigma \subseteq T_{(\tau,p)}\widehat{M}$ is a totally geodesic hyperplane satisfying $\partial_r \notin V$. This means that there exists a unique (possibly zero) vector $\eta \in T_pM$ such that $T_{(\tau,p)}\widehat{M} \ominus V = \mathbb{R}(\partial_r + \eta)$. Let $X = a\partial_r + v \in V$ be arbitrary, where $a \in \mathbb{R}$ and $v \in V$. Then, since V is $\widehat{R}_X$-invariant and $\widehat{R}_X$ is symmetric, it follows that $\partial_r + \eta$ is an eigenvector of $\widehat{R}_X$. However, by (6.7) the image of $\widehat{R}_X$ is contained in T_pM, and because $\partial_r + \eta$ is not tangent to the link we must have $\widehat{R}_X(\partial_r + \eta) = 0$, so $\widehat{R}(\eta, v)v = \widehat{R}(\partial_r + \eta, X)X = 0$ for all $X \in V$. As the orthogonal projection of V onto T_pM is a linear isomorphism, we deduce that if $V = \mathbb{R}(\partial_r + \eta)^\perp$ is a totally geodesic hyperplane, then $\widehat{R}(\eta, v)v = 0$ for all $v \in T_pM$.

Proposition 6.28. *Let M be a space of constant sectional curvature $\kappa \in \mathbb{R}$. Then, every totally geodesic submanifold of dimension $d \geq 2$ of the Riemannian cone $\widehat{M}$ is a cone over a totally geodesic submanifold S of M if and only if $\kappa \neq 1$.*

Proof. Due to Theorem 6.25, we may focus only on totally geodesic hypersurfaces. Let M be a connected complete Riemannian manifold of constant sectional curvature $\kappa \in \mathbb{R}$ and dimension $n \geq 2$, and suppose that $\widehat{M}$ admits a totally geodesic hypersurface Σ that is not tangent to the cone vector field. By shrinking Σ if necessary, we can assume that $\Sigma \subseteq \widehat{M}$ is embedded and every $(\tau, p) \in \Sigma$ is such that $V = T_{(\tau,p)}\Sigma$ does not contain ∂_r, so its orthogonal complement must be generated by a vector of the form $\partial_r + \eta$ for a certain $\eta \in T_pM$. Now, from Remark 6.27 and (6.7) we deduce that $0 = \widehat{R}(\eta, v)v = (\kappa - 1)(|v|^2\eta - \langle \eta, v\rangle v)$ for all $v \in T_pM$, which means that either $\kappa = 1$ or $\eta = 0$. If $\kappa \neq 1$, we deduce that $V = T_{(\tau,p)}M$ for all $(\tau, p) \in \Sigma$, so Σ is an integral manifold of the distribution $\mathcal{D} = \partial_r^\perp$ on $\widehat{M}$. The maximal integral manifolds of $\mathcal{D}$ are precisely the links $\{\tau\} \times M$ for each $\tau > 0$, so Σ is actually an open subset of a leaf $\{\tau_0\} \times M$ for a certain $\tau_0 > 0$. However, the last equation in (6.6) shows that the leaves are never totally geodesic, so we arrive at a contradiction. We conclude that if $\kappa \neq 1$, the inextendable totally geodesic hypersurfaces of $\widehat{M}$ are precisely the cones over the complete totally geodesic hypersurfaces of M.

If M has constant sectional curvature equal to 1, then $\widehat{M}$ is flat by (6.7), so for every point $(\tau, p) \in \widehat{M}$ and every hyperplane $V \subseteq T_{(\tau,p)}\widehat{M}$ there exists an inextendable compatible totally geodesic hypersurface $\Sigma \subseteq \widehat{M}$ such that $(\tau, p) \in \Sigma$ and $T_{(\tau,p)}\Sigma = V$. In particular, Σ is not (contained in) a cone over a totally geodesic hypersurface of M if and only if $(\partial_r)_{(\tau,p)} \notin V$. □

Example 6.29. Let us assume that $M = \mathsf{S}^n(1)$. Then $\widehat{M}$ is isometric to $\mathbb{R}^{n+1} \setminus \{0\}$ in such a way that the cones over the totally geodesic submanifolds of $\widehat{M}$ are of the form $V \setminus \{0\}$, where V is an arbitrary vector subspace of $\mathbb{R}^{n+1}$. In particular, any affine hyperplane $\Sigma \subseteq \mathbb{R}^{n+1}$ not containing the origin is a totally geodesic hypersurface that does not appear as a cone over a totally geodesic hypersurface of $\mathsf{S}^n(1)$.

Proposition 6.30. *Let M be equal to either $\mathsf{S}^3_{\mathbb{C},\tau}(r)$ or $\mathbb{R}\mathsf{P}^3_{\mathbb{C},\tau}(r)$ and let $\widehat{M}$ denote the Riemannian cone over M. Then, $\widehat{M}$ admits a totally geodesic hypersurface if and only if $\tau = r = 1$.*

Proof. Let $M = \mathsf{S}^3_{\mathbb{C},\tau}(r)$ be a three-dimensional Berger sphere of radius r and deformation parameter τ. We show that $\widehat{M}$ does not admit totally geodesic hypersurfaces unless $r = \tau = 1$ (that is, M is the unit round sphere).

We first establish some notation. Recall that $\mathsf{S}^3_{\mathbb{C},\tau}(r) = \mathsf{U}(2)/\mathsf{U}(1)$ as a homogeneous space, and we have a reductive decomposition $\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{p}$, where $\mathfrak{u}(1) = \mathbb{R}K$ and $\mathfrak{p} = \operatorname{span}\{E, X, Y\}$ for the matrices

$$K = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \qquad E = \frac{1}{r\sqrt{\tau}} \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \qquad X = \frac{1}{r} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \frac{1}{r} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Furthermore, if $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathfrak{p}$ induced by the Berger metric on M, then E, X and Y are orthonormal vectors with respect to this metric. Furthermore, the vertical and horizontal subspaces at $o = e\mathsf{U}(1)$ with respect to the Hopf fibration are $\mathcal{V}_o = \mathbb{R}E$ and $\mathcal{H}_o = \operatorname{span}\{X, Y\}$.

Let us suppose that $M \neq \mathsf{S}^3(1)$ and Σ is a totally geodesic hypersurface of $\widehat{M}$. We may assume that Σ is embedded in M. Because M is homogeneous, Corollary 6.20 allows us to suppose that Σ passes through a point of the form (t, o) with tangent space $V \subseteq T_{(t,o)}\widehat{M} \equiv \mathbb{R}\partial_r \oplus \mathfrak{p}$. As M does not admit totally geodesic hypersurfaces [135, Theorem A], we have that $\partial_r \notin V$, so $V^\perp$ must be spanned by a vector of the form $\partial_r + \eta$, where $\eta \in \mathfrak{p}$. We may write $\eta = a_1 E + a_2 X + a_3 Y$ for some constants a_1, a_2, $a_3 \in \mathbb{R}$. From Remark 6.27 we also know that $\widehat{R}(\eta, Z)Z = 0$ for all $Z \in \mathfrak{p}$. A polarization argument shows that the previous condition is equivalent to $\widehat{R}(\eta, Z)W + \widehat{R}(\eta, W)Z = 0$ for all $Z, W \in \mathfrak{p}$.

Firstly, suppose that $\tau \neq r^2$. Then the equations

$$\begin{aligned} 0 =& \widehat{R}(\eta, E)Y + \widehat{R}(\eta, Y)E = \left(1 - \frac{\tau}{r^2}\right)(a_3 E + a_1 Y), \\ 0 =& \widehat{R}(\eta, E)X + \widehat{R}(\eta, X)E = \left(1 - \frac{\tau}{r^2}\right)(a_2 E + a_1 X), \end{aligned}$$

imply that $a_1 = a_2 = a_3 = 0$, so $\eta = 0$ and $V = \mathfrak{p}$. However, from (6.8) and the fact that $\widehat{R}$ is not identically zero on $\mathfrak{p}$ we deduce that $\mathfrak{p}$ is not a totally geodesic subspace, giving us a contradiction.

Secondly, suppose that $\tau = r^2$ and $r \neq 1$. Since

$$0 = \widehat{R}(\eta, X)Y + \widehat{R}(\eta, Y)X = \frac{4(r^2-1)}{r^2}(a_3 X + a_2 Y),$$

we obtain $a_2 = a_3 = 0$. As a consequence, $\eta \in \mathcal{V}_o$, and V contains the horizontal subspace $\mathcal{H}_o$. Using (6.8), we obtain that $\left(4 - \frac{4}{r^2}\right) t\partial_r = (\widehat{\nabla}_X \widehat{R})(X, Y, Y) \in V$, so $\partial_r \in V$, which again yields a contradiction.

All in all, we have shown that $\widehat{M}$ does not admit totally geodesic hypersurfaces except in the case $M = \mathsf{S}^3(1)$. Since the natural projection $\mathsf{S}^3_{\mathbb{C},\tau}(r) \to \mathbb{R}\mathsf{P}^3_{\mathbb{C},\tau}(r)$ is a Riemannian covering map, the same result holds for the three-dimensional Berger projective space. □

6.4.2 Totally geodesic hypersurfaces and the Obata equation

Let M be a complete real analytic Riemannian manifold. In view of Propositions 6.28 and 6.30, one could think that the existence of a totally geodesic hypersurface in $\widehat{M}$ not tangent to the cone direction is possible only when M has constant sectional curvature equal to one. In this section we provide a plethora of examples of Riemannian cones admitting such a hypersurface. Moreover, we show that a Riemannian cone admits a totally geodesic hypersurface that is not tangent to the cone vector field if and only if its base is locally a sine cone.

Let M be a complete real analytic Riemannian manifold and let Σ be a totally geodesic hypersurface of $\widehat{M}$ that is not tangent to the cone vector field ∂_r. We assume without loss of generality that $\Sigma \subseteq M$ is embedded. Fix a point $(\tau, p) \in \Sigma$. Because ∂_r is not tangent to Σ, the restriction of the standard projection $\pi\colon \widehat{M} \to M$ to Σ is a local diffeomorphism, so by shrinking Σ if necessary we can assume that $\pi\colon \Sigma \to M$ is a diffeomorphism onto the open set $\Omega = \pi(\Sigma) \subseteq M$. Denote by $\sigma\colon \Omega \to \Sigma$ the inverse map of $\pi|_\Sigma$. Since $\pi(\sigma(x)) = x$ for all

$x \in \Sigma$, there exists a smooth function $f\colon \Omega \to \mathbb{R}^+$ such that $\sigma(x) = (f(x), x)$ for all $x \in \Sigma$. By definition, we have $f(p) = \tau$ and

$$\Sigma = \{(f(x), x)\colon x \in \Omega\}$$

is the graph of f. We conclude that a complete totally geodesic hypersurface of $\widehat{M}$ is locally the graph of a smooth positive function $f\colon \Omega \to \mathbb{R}^+$. Note that the tangent space of Σ at a point $(t, x) \in \Sigma$ is

$$T_{(t,x)}\Sigma = \{df_x(v)\partial_r + v\colon v \in T_xM\}.$$

The discussion above motivates the following question: given an open subset $\Omega \subseteq M$ and a smooth function $f\colon \Omega \to \mathbb{R}^+$, when is the graph Σ of f a totally geodesic submanifold of the cone $\widehat{M}$? To answer this, we observe that the vector field

$$\xi = \partial_r - \frac{1}{f^2}\operatorname{grad} f = \partial_r + \operatorname{grad}\frac{1}{f}$$

Is normal to Σ at all points. Note that the length $\sqrt{\widehat{g}(\xi,\xi)}$ is not constant. Choose any $x \in \Omega$ and consider a tangent vector $v \in T_xM$. Then $\mathcal{S}_\xi(df_x(v)\partial_r + v) = (\widehat{\nabla}_{df_x(v)\partial_r+v}\xi)^\top$ is the orthogonal projection of the covariant derivative $\widehat{\nabla}_{df_x(v)\partial_r+v}\xi$ to $T_{(t,x)}\Sigma$. A straightforward calculation using (6.6) gives

$$\widehat{\nabla}_{df_x(v)\partial_r+v}\xi = -\frac{df_x(v)}{f(x)^3}(\operatorname{grad} f)_x + \frac{1}{f(x)}v + \nabla_v \operatorname{grad}\frac{1}{f} + \frac{df_x(v)}{f(x)}(\partial_r)_x.$$

Taking the inner product with an arbitrary vector of the form $df_x(w)\partial_r + w$, where $w \in T_xM$, we obtain

$$\begin{aligned}\widehat{g}(\widehat{\nabla}_{df_x(v)\partial_r+v}\xi, df_x(w)\partial_r + w)\\ = f(x)\Big(\langle v, w\rangle + f(x)\Big\langle \nabla_v \operatorname{grad}\frac{1}{f}, w\Big\rangle\Big).\end{aligned} \tag{6.11}$$

Recall that if N is a Riemannian manifold and $h\colon N \to \mathbb{R}$ is a smooth function, the *Hessian* of h is the symmetric $(0, 2)$-tensor field $\operatorname{Hess} h = \nabla dh$. It is characterized by the equation $\operatorname{Hess} h(v, w) = \langle \nabla_v \operatorname{grad} h, w\rangle$. We conclude from (6.11) that the graph of f is totally geodesic if and only if the map $h = 1/f$ satisfies the differential equation

$$\operatorname{Hess} h = -h\langle\cdot,\cdot\rangle. \tag{6.12}$$

Equation (6.12) is known as the *Obata equation*. From the calculations above, we deduce the following result.

Theorem 6.31. *Let M be a complete real analytic Riemannian manifold and $(\tau, p) \in \widehat{M}$ a point of its Riemannian cone. Fix a vector $\eta \in T_pM$ and consider the hyperplane $V = T_{(\tau,p)}M \ominus \mathbb{R}(\partial_r + \eta)$. Then the following conditions for V are equivalent.*

(i) *There exists a complete totally geodesic hypersurface* Σ *of* $\widehat{M}$ *passing through* (τ, p) *with tangent space* V.

(ii) *There exists an open subset* $\Omega \subseteq M$ *containing* p *and a smooth function* $h\colon \Omega \to \mathbb{R}^+$ *satisfying the Obata equation* (6.12), *together with the initial conditions* $h(p) = 1/\tau$ *and* $(\operatorname{grad} h)_p = \eta$.

If any (hence both) of the above conditions hold, then the hypersurface Σ *is the inextendable extension of the graph of* $f = 1/h$.

Let (M, g) be a Riemannian manifold. We define the *sine-cone* of M as the warped product $SC(M) = (0, \pi) \times_{\sin} M$. Explicitly, the metric on $SC(M)$ is given by $g_{SC(M)} = dr^2 + (\sin r)^2 g$. The next result shows that if a Riemannian manifold admits a local solution h of the Obata equation, then the domain of h is locally isometric to a sine-cone.

Theorem 6.32. *Let* M *be a Riemannian manifold and* $p \in M$. *Suppose that there exists a positive local solution* h *to the Obata equation* (6.12) *defined near* p. *Then, there exists an open interval* $I \subseteq (0, \frac{\pi}{2})$, *a Riemannian manifold* N *and a neighborhood* Ω *of* p *in* M *that is isometric to the warped product* $I \times_{\sin} N$. *Furthermore, the isometry* $\Omega \cong I \times_{\sin} N$ *can be chosen in such a way that the restriction* $h\colon \Omega \to \mathbb{R}$ *is given by* $h(t, x) = A \cos t$ *for a certain* $A > 0$.

Conversely, given a warped product $M = I \times_{\sin} N$ *satisfying the above conditions, the functions* $h\colon (t, x) \in M \mapsto A \cos t \in \mathbb{R}$ *are solutions to* (6.12).

Proof. By [145, Theorem 4.3.3] we can suppose, after shrinking M if necessary, that $M = I \times_\rho N$ is the warped product of an open interval $I \subseteq \mathbb{R}$ with a Riemannian manifold N in such a way that h is constant along the fibers $\{t\} \times N$ with $t \in I$, so we can think of $h \equiv h(t)$ as a function on I. In particular, we have $\operatorname{grad} h = h'(t)\partial_t$. We can compute $\nabla \operatorname{grad} h$ using the formulas in [137, Chapter 7, Proposition 35]. Indeed, let $X \in \mathfrak{X}(N)$ be any vector field. Then we have

$$\nabla_{\partial_t} \operatorname{grad} h = h''(t)\partial_t, \quad \nabla_X \operatorname{grad} h = \frac{h'(t)\rho'(t)}{\rho(t)} X.$$

As a consequence, the Obata equation turns into the following system of ordinary differential equations:

$$\begin{cases} h''(t) + h(t) = 0, \\ h'(t)\rho'(t) + \rho(t)h(t) = 0. \end{cases} \tag{6.13}$$

The first condition in (6.13) yields that $h(t)$ is an $\mathbb{R}$-linear combination of $\sin t$ and $\cos t$. By shifting the t coordinate and shrinking M, we may assume without loss of generality that I is contained in $(0, \frac{\pi}{2})$ and $h(t) = A \cos t$ for some $A > 0$. Thus, the second equation in (6.13) becomes $\rho'(t) = (\tan t)^{-1}\rho(t)$. This implies $\rho(t) = B \sin t$ for some $B > 0$. By rescaling the metric on N, we may suppose that $B = 1$, which proves our assertion.

Finally, note that the converse is clear from the fact that the functions $\rho(t) = \sin t$ and $h(t) = A \cos t$ are solutions to (6.13). □

Because the fiber N in Theorem 6.32 can be chosen arbitrarily, the above statement yields a vast array of Riemannian manifolds whose cones admit totally geodesic hypersurfaces that are not tangent to the cone direction. The abundance of examples highlights the appropriateness of further exploring this class of totally geodesic hypersurfaces in cones.

Remark 6.33. Suppose M^2 is a two-dimensional manifold and $h\colon \Omega \subseteq M \to \mathbb{R}$ is a local solution to the Obata equation. Then Theorem 6.32 allows us to write Ω locally as a warped product $I \times_{\sin} J$ of two intervals. It is easy to see that the metric on $I \times_{\sin} J$ is precisely the round metric of radius one. Because of this, we conclude that the Gaussian curvature of M is $K = 1$ at all points of Ω.

6.5 Proofs of the main theorems

In this section we provide the proofs of the main theorems of this chapter. We go through each one of the homogeneous nearly Kähler 6-manifolds with non-constant sectional curvature, and classify their totally geodesic submanifolds.

6.5.1 The complex projective space

Lemma 6.34. *If $\mathfrak{v} \subseteq \mathfrak{p}$ is a totally geodesic subspace and $\mathfrak{v}$ contains a vertical or a horizontal vector, then $\mathfrak{v}$ is well-positioned.*

Proof. If $X \in \mathfrak{v}$ is a unit vertical vector, we may assume by means of the isotropy representation that $X = e_1$. Since the spectrum of $R_{e_1}\colon \mathfrak{p} \ominus \mathbb{R}e_1 \to \mathfrak{p} \ominus \mathbb{R}e_1$ consists of the eigenvalues 2, with eigenspace $\mathbb{R}e_2$, and $1/8$, with eigenspace $\mathfrak{p}_2$, the claim follows from the R_{e_1}-invariance of $\mathfrak{v}$. Similarly, if X is horizontal we may suppose that $X = e_3$, and in this case the eigenvalues of $R_{e_3}\colon \mathfrak{p} \ominus \mathbb{R}e_3 \to \mathfrak{p} \ominus \mathbb{R}e_3$ are $\frac{1}{8}$, with eigenspace $\mathfrak{p}_1$, 1, with eigenspace $\mathbb{R}e_4$, and $5/8$, with eigenspace $\operatorname{span}\{e_5, e_6\}$, so the result also holds in this case. ☐

Proposition 6.35. *There are no four-dimensional totally geodesic submanifolds in $\mathbb{C}\mathrm{P}^3$.*

Proof. Assume, on the contrary, that there exists a totally geodesic submanifold Σ of M of dimension four passing through o, and let $\mathfrak{v} \subseteq \mathfrak{p}$ be its corresponding totally geodesic subspace. From Lemma 6.34 and by dimension reasons, we know that $\mathfrak{v}$ is well-positioned. We distinguish three possibilities according to the dimension of $\mathfrak{v} \cap \mathfrak{p}_1$.

If $\mathfrak{v} \cap \mathfrak{p}_1 = \mathfrak{p}_1$, then $\mathfrak{v} \cap \mathfrak{p}_2$ is two-dimensional. By using the isotropy representation if necessary, we may suppose that $e_3 \in \mathfrak{v}$. We can therefore consider a basis of $\mathfrak{v}$ of the form $\{e_1, e_2, e_3, a_4e_4 + a_5e_5 + a_6e_6\}$. In particular, $a_6e_5 - a_5e_6$ is orthogonal to $\mathfrak{v}$, and the equality

$$0 = \langle R(e_1, e_2)(a_4e_4 + a_5e_5 + a_6e_6), a_6e_5 - a_5e_6\rangle = \frac{3(a_5^2 + a_6^2)}{4}$$

yields $a_5 = a_6 = 0$, so actually $\mathfrak{v} = \operatorname{span}\{e_1, e_2, e_3, e_4\}$. This is a contradiction due to the fact that $(\nabla_{e_1} R)(e_1, e_2, e_3) = -\frac{3}{4\sqrt{2}}e_6 \notin \mathfrak{v}$, so this case is not possible.

If $\mathfrak{v} \cap \mathfrak{p}_1$ is one-dimensional (which forces $\dim \mathfrak{v} \cap \mathfrak{p}_2 = 3$), we may use the isotropy representation to assume that $\mathfrak{v}$ contains e_1 and e_3. In particular, $e_5 = 4\sqrt{2}(\nabla_{e_3}R)(e_3, e_1, e_3)$ also belongs to $\mathfrak{v}$. As a consequence, $\mathfrak{v}$ admits a basis of the form $\{e_1, e_3, e_5, a_4e_4 + a_6e_6\}$, which means that $a_6e_4 - a_4e_6 \in \mathfrak{p} \ominus \mathfrak{v}$, and

$$0 = \langle R(e_3, e_5)(a_4e_4 + a_6e_6), a_6e_4 - a_4e_6\rangle = -\frac{1}{8}(a_4^2 + a_6^2),$$

so $a_4 = a_6 = 0$, another contradiction.

If $\mathfrak{v} \cap \mathfrak{p}_1 = 0$, then $\mathfrak{v} = \mathfrak{p}_2$, which is also not possible, since $(\nabla_{e_3}R)(e_3, e_4, e_6) = -\frac{1}{4\sqrt{2}}e_1$ is not in $\mathfrak{v}$. In conclusion, no such $\mathfrak{v}$ can exist, and the claim follows. □

Proposition 6.36. *Let Σ be a complete totally geodesic submanifold of $\mathbb{C}\mathsf{P}^3$ with $\dim\Sigma = 3$. Then Σ is congruent to the standard $\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(2)$.*

Proof. Let Σ be such a submanifold, and assume without loss of generality that Σ passes through o with tangent space $\mathfrak{v}$. Once again, by dimension reasons we see that $\mathfrak{v}\cap\mathfrak{p}_2 \neq 0$, and Lemma 6.34 implies that $\mathfrak{v}$ is well-positioned. We consider three cases according to the dimension of $\mathfrak{v} \cap \mathfrak{p}_1$.

If $\mathfrak{v} \cap \mathfrak{p}_1 = 0$, then $\mathfrak{v} \subseteq \mathfrak{p}_2$ is a hyperplane, and since the isotropy representation is transitive on the unit sphere of $\mathfrak{p}_2$, we may assume that $\mathfrak{v} = \operatorname{span}\{e_4, e_5, e_6\}$. However, since $R(e_4, e_5)e_6 = \frac{1}{8}e_3$, we obtain a contradiction.

If $\mathfrak{v} \cap \mathfrak{p}_1$ is one-dimensional, then by using the isotropy representation we may suppose that $\mathfrak{v}$ contains e_1 and e_3. Note that $4\sqrt{2}(\nabla_{e_3}R)(e_3, e_1, e_3) = e_5$ also belongs to $\mathfrak{v}$, which gives $\mathfrak{v} = \operatorname{span}\{e_1, e_3, e_5\} = \mathfrak{p}_{\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(\sqrt{2})}$. Therefore, in this case we obtain $\Sigma = \mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(\sqrt{2})$.

Finally, if $\mathfrak{v} \cap \mathfrak{p}_1 = \mathfrak{p}_1$, then by using the isotropy representation we can assume that $\mathfrak{v} = \operatorname{span}\{e_1, e_2, e_3\}$. This is not possible, since $R(e_1, e_2)e_3 = \frac{3}{4}e_4$ is not in $\mathfrak{v}$. This finishes the proof. □

Proposition 6.37. *Let Σ be a complete totally geodesic surface inside $\mathbb{C}\mathsf{P}^3$. Then Σ is congruent to one of the spheres described in Table 6.1.*

Proof. Suppose that $\mathfrak{v} \subseteq \mathfrak{p}$ is a totally geodesic plane, and consider the corresponding complete totally geodesic submanifold Σ of M. Notice that Σ must be intrinsically homogeneous, and thus a space of constant curvature since it is of dimension two. Also, note that either $\mathfrak{v}$ is completely contained in one of the irreducible K-submodules of $\mathfrak{p}$ or it contains a vector that projects nontrivially onto $\mathfrak{p}_1$ and $\mathfrak{p}_2$ at the same time. Since the case $\mathfrak{v} = \mathfrak{p}_1$ already corresponds to Σ being the fiber of the twistor fibration, we may skip this case.

Assume that $\mathfrak{v} \subseteq \mathfrak{p}_2$. Using the isotropy representation if necessary, we can assume $e_3 \in \mathfrak{v}$. One sees that the kernel of the Cartan operator C_X is spanned by e_3 and e_4, so we must have $\mathfrak{v} = \operatorname{span}\{e_3, e_4\} = \mathfrak{p}_{\mathsf{SU}(2)\cdot o}$, since Σ has constant curvature, which means that $\Sigma = \mathsf{SU}(2) \cdot o$.

Finally, suppose that there exists a vector $X \in \mathfrak{v}$ such that $X_{\mathfrak{p}_1}$ and $X_{\mathfrak{p}_2}$ are nonzero. By using the isotropy representation and rescaling, we can assume that $X = e_1 + \lambda e_3$ for a certain $\lambda > 0$. In this case, $\ker C_X$ is spanned by X and $Y = 3\lambda e_2 + (6-\lambda^2)e_4$, so necessarily $\mathfrak{v} = \operatorname{span}\{X, Y\}$, since Σ has constant curvature. In particular, we have $0 = 4\sqrt{2}\langle(\nabla_X R)(Y, X, Y), e_5\rangle = -\lambda\,(2\lambda^4 + 3\lambda^2 - 9)$, and this is only possible if $\lambda = \sqrt{3/2}$. Therefore,

$$\mathfrak{v} = \operatorname{span}\Big\{\sqrt{2}e_1 + \sqrt{3}e_3, \sqrt{2}e_2 + \sqrt{3}e_4\Big\} = \mathfrak{p}_{\mathsf{SU}(2)_{\Lambda_3}\cdot o}.$$

As a consequence, we see that in this case $\Sigma = \mathsf{SU}(2)_{\Lambda_3} \cdot o$. This finishes the proof. □

Proof of Theorem A. The theorem follows from combining Theorem 6.23, Proposition 6.35, Proposition 6.36, and Proposition 6.37. □

6.5.2 The flag manifold

Proposition 6.38. *The flag manifold* $\mathsf{F}(\mathbb{C}^3)$ *does not admit any codimension two totally geodesic submanifolds.*

Proof. Suppose that $\mathsf{F}(\mathbb{C}^3)$ admits a totally geodesic submanifold of dimension four. This means that there exists a totally geodesic subspace $\mathfrak{v} \subseteq \mathfrak{p}$ with $\dim \mathfrak{v} = 4$. By a dimension argument, one sees that the intersection $\mathfrak{v} \cap (\mathfrak{p}_1 \oplus \mathfrak{p}_2)$ is nontrivial, and using both the isotropy representation of T^2 and conjugating by a permutation matrix if necessary, we may suppose that $\mathfrak{v}$ admits a nonzero vector of the form $X = e_1 + \lambda e_3$, where $\lambda \in \mathbb{R}$ is a nonnegative number. The Cartan operator C_X is diagonalizable with eigenvalues 0, $\frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}$ and $-\frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}$, and corresponding eigenspaces

$$\begin{aligned}
\ker C_X &= \operatorname{span}\{e_1, e_3, e_5, \lambda e_2 + e_4\},\\
\ker\left(C_X - \frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}\,\mathrm{id}_{\mathfrak{p}}\right) &= \mathbb{R}\left(e_2 - \lambda e_4 + \sqrt{1+\lambda^2}e_6\right),\\
\ker\left(C_X + \frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}\,\mathrm{id}_{\mathfrak{p}}\right) &= \mathbb{R}\left(-e_2 + \lambda e_4 + \sqrt{1+\lambda^2}e_6\right).
\end{aligned}$$

First, assume that $\lambda > 0$, so the three eigenvalues given above are pairwise distinct. We prove that $\mathfrak{v}$ coincides with the kernel of the Cartan operator C_X.

If $C_X|_{\mathfrak{v}}$ is not identically zero, then $\mathfrak{v}$ contains a vector of the form $Y = \varepsilon e_2 - \lambda\varepsilon e_4 + \sqrt{1+\lambda^2}e_6$, where $\varepsilon \in \{\pm 1\}$.

If $\lambda \neq 1$, then we can construct a basis of $\mathfrak{v}$ with the vectors

$$\begin{aligned}
X &= e_1 + \lambda e_3,\\
Y &= \varepsilon e_2 - \lambda\varepsilon e_4 + \sqrt{1+\lambda^2}e_6,\\
U &= 8R(X,Y)X = -2\varepsilon\left(5\lambda^2+8\right)e_2 + 2\varepsilon\lambda\left(8\lambda^2+5\right)e_4 - \left(1+\lambda^2\right)^{3/2}e_6,\\
V &= 8\sqrt{2}(\nabla_X R)(X,Y,Y)\\
&= -3\lambda\sqrt{\lambda^2+1}\left(3\lambda^2+5\right)\varepsilon e_1 - 3\sqrt{\lambda^2+1}\left(5\lambda^2+3\right)\varepsilon e_3 - 6\lambda\left(\lambda^2-1\right)e_5.
\end{aligned}$$

Therefore, the vector

$$T = -\lambda\sqrt{\lambda^2+1}\left(5\lambda^2+3\right)e_2 - \sqrt{\lambda^2+1}\left(3\lambda^2+5\right)e_4 + 2\lambda\left(\lambda^2-1\right)\varepsilon e_6$$

is orthogonal to $\mathfrak{v}$. Thus, $0 = \langle R(X,U)X, T\rangle = 36\varepsilon\lambda^3\left(\lambda^2-1\right)\sqrt{\lambda^2+1}$, which is a contradiction.

If $\lambda = 1$, the equation $R(X,Y)X + \frac{13}{4}Y = 3\sqrt{2}e_6$ implies that the vectors $X = e_1 + e_3$, $Z = e_2 - e_4$, and $T = e_6$ are in $\mathfrak{v}$. We can therefore complete X, Z and T to a basis of $\mathfrak{v}$ by adding a vector of the form $U = c_1e_1 + c_2e_2 - c_1e_3 + c_2e_4 + c_3e_5$, where $c_i \in \mathbb{R}$ for each $i \in \{1,2,3\}$. In particular, we have that the vector $-c_2e_1 + c_1e_2 + c_2e_3 + c_1e_4$ is orthogonal to $\mathfrak{v}$, so

$$0 = 2\langle R(X,Z)U, -c_2e_1 + c_1e_2 + c_2e_3 + c_1e_4\rangle = -9(c_1^2 + c_2^2),$$

which forces $c_1 = c_2 = 0$. Therefore $\mathfrak{v} = \operatorname{span}\{X, Z, e_5, e_6\}$. However, we have $8R(X,e_5)e_6 = 3(e_2 + e_4) \notin \mathfrak{v}$, which yields a contradiction.

From all of the above, we see that $\mathfrak{v}$ must coincide with $\ker C_X = \operatorname{span}\{e_1, e_3, e_5, \lambda e_2 + e_4\}$. However, this is not possible either, since

$$\frac{8}{3}R(e_1,e_3)(\lambda e_2 + e_4) = -e_2 + \lambda e_4 \notin \mathfrak{v}.$$

Thus, the case $\lambda > 0$ is not possible.

Now, suppose that $\lambda = 0$, and we deduce that $e_1 \in \mathfrak{v}$. Since the intersection of $\mathfrak{v}$ with $\mathfrak{p}_2 \oplus \mathfrak{p}_3$ is at least two-dimensional, we can use the isotropy representation to assume that there is a tangent vector of the form $Y = e_3 + \mu e_5$, where $\mu \in \mathbb{R}$. However, $\mu = 0$, for if $\mu \neq 0$, by means of the full isotropy representation we can conjugate $\mathfrak{v}$ to a new totally geodesic subspace containing a tangent vector of the form $e_1 + \mu e_3$, and we may use the previous case to derive a contradiction. Now this yields that $e_1 + e_3$ is also in $\mathfrak{v}$, which is yet another contradiction. We conclude that this case is also not possible, and therefore there are no codimension two totally geodesic subspaces in $\mathfrak{p}$. ☐

Proposition 6.39. *Let $\Sigma \subseteq \mathsf{F}(\mathbb{C}^3)$ be a complete totally geodesic submanifold of dimension three. Then Σ is congruent to either $\mathsf{F}(\mathbb{R}^3)$ or to $\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})$.*

Proof. We need to classify three-dimensional totally geodesic subspaces of $\mathfrak{p}$. Let $\mathfrak{v} \subseteq \mathfrak{p}$ be such a subspace, and Σ the corresponding complete totally geodesic submanifold. Then we know that there exists a vector $X \in \mathfrak{v} \cap (\mathfrak{p}_1 \oplus \mathfrak{p}_2)$, and using the full isotropy representation, we may assume that it is of the form $X = e_1 + \lambda e_3$ for a certain $\lambda \geq 0$. The Cartan operator C_X is diagonalizable with eigenvalues 0, $\frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}$, and $-\frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}$, and corresponding eigenspaces

$$\begin{aligned}
\ker C_X &= \operatorname{span}\{e_1, e_3, e_5, \lambda e_2 + e_4\},\\
\ker\left(C_X - \frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}\right) &= \mathbb{R}\left(e_2 - \lambda e_4 + \sqrt{1+\lambda^2}e_6\right),\\
\ker\left(C_X + \frac{3\lambda\sqrt{1+\lambda^2}}{2\sqrt{2}}\right) &= \mathbb{R}\left(-e_2 + \lambda e_4 + \sqrt{1+\lambda^2}e_6\right).
\end{aligned}$$

First, assume that $\lambda > 0$, so the three eigenvalues given above are pairwise distinct. We prove in this case that either $\Sigma = \mathsf{SU}(2)_{(1,0,1)} \cdot o$ or $\mathfrak{v} \subseteq \ker C_X$. Indeed, if there is a vector of the form $Y = \varepsilon e_2 - \lambda\varepsilon e_4 + \sqrt{1+\lambda^2}e_6$ in $\mathfrak{v}$, where $\varepsilon \in \{\pm 1\}$, then we may construct a basis of $\mathfrak{v}$ by adding the vector

$$Z = 8R(X,Y)X = -2\left(5\lambda^2 + 8\right)\varepsilon e_2 + 2\lambda\left(8\lambda^2 + 5\right)\varepsilon e_4 - \left(\lambda^2 + 1\right)^{3/2} e_6 \in \mathfrak{v}.$$

In particular, $-\lambda e_1 + e_3$ is orthogonal to $\mathfrak{v}$, and we must have

$$0 = 4\langle R(X,Y,Y), -\lambda e_1 + e_3\rangle = 3\lambda(\lambda^2 - 1).$$

This forces $\lambda = 1$. Note that $4R(X,Y)X = -13\varepsilon e_2 + 13\varepsilon e_4 - \sqrt{2}e_6 \in \mathfrak{v}$. This means that $\mathfrak{v}$ is spanned by $e_1 + e_3$, $e_2 - e_4$ and e_6, so Σ coincides with $\mathsf{SU}(2)_{(1,0,1)} \cdot o$. Now, suppose that $\mathfrak{v} \subseteq \ker C_X$. Then either $\mathfrak{v} = \operatorname{span}\{e_1, e_3, e_5\}$ (which yields $\Sigma = \mathsf{F}(\mathbb{R}^3)$) or using the isotropy representation we can find a basis of $\mathfrak{v}$ given by vectors of the form

$$\begin{aligned} X &= e_1 + \lambda e_3, \\ Y &= a_1 e_1 + a_3 e_3 + a_5 e_5 + (\lambda e_2 + e_4), \\ Z &= c_1 e_1 + c_3 e_3 + c_5 e_5, \end{aligned}$$

for some constants a_i, $c_j \in \mathbb{R}$. In particular, the vectors e_6 and $e_2 - \lambda e_4$ are in $\mathfrak{p} \ominus \mathfrak{v}$. Now, we also see that $0 = \langle R(X,Y)X, e_2 - \lambda e_4\rangle = \frac{3\lambda(\lambda^2-1)}{4}$, which means that $\lambda = 1$. On the other hand, we have

$$\langle (\nabla_X R)(X,Y,Y), e_6\rangle = \frac{3(a_3 - a_1)}{4\sqrt{2}}, \qquad \langle (\nabla_X R)(X,Y,Y), e_2 - e_4\rangle = \frac{-3a_5}{2\sqrt{2}},$$
$$\langle (\nabla_X R)(X,Z,Y), e_6\rangle = \frac{3(c_3 - c_1)}{4\sqrt{2}}, \qquad \langle (\nabla_X R)(X,Z,Y), e_2 - e_4\rangle = \frac{-3c_5}{2\sqrt{2}}.$$

Since all of these inner products are zero, we deduce that $a_5 = c_5 = 0$, $a_1 = a_3$ and $c_1 = c_3$. In particular, Z and X are proportional, a contradiction.

We now assume $\lambda = 0$, so $e_1 \in \mathfrak{v}$. Since $\mathfrak{p} \cap (\mathfrak{p}_2 \oplus \mathfrak{p}_3)$ is nonzero, we may use the isotropy representation to suppose that a vector of the form $e_3 + \mu e_5$ belongs to $\mathfrak{v}$. Note that if $\mu \neq 0$, then using an element of the full isotropy group permuting the factors of the isotropy representation, we can carry this setting to the one in the previous paragraph, so we may assume that $\mu = 0$, and thus $e_3 \in \mathfrak{v}$. As a consequence, $e_1 + e_3 \in \mathfrak{v}$, and we may use the arguments in the case $\lambda > 0$ to derive the same conclusions, and the proposition is proved. □

Proposition 6.40. *Every complete totally geodesic surface of* $\mathsf{F}(\mathbb{C}^3)$ *is congruent to one of the following:*

(i) *the totally geodesic* T^2_Λ,

(ii) *the Berger sphere* $\mathsf{S}^3_{\mathbb{C},1/4}(\sqrt{2})$,

(iii) *the fiber* $\mathbb{C}\mathsf{P}^1$,

(iv) *or a totally geodesic* $\mathbb{R}\mathsf{P}^2\left(2\sqrt{2}\right) \subseteq \mathsf{F}(\mathbb{R}^3)$.

Proof. Let $\mathfrak{v}$ be a totally geodesic subspace of dimension 2 in $\mathfrak{p}$. We define $r \in \{1,2,3\}$ to be the largest number such that there exists a vector $X \in \mathfrak{v}$ that has nontrivial projection onto r of the irreducible submodules of $\mathfrak{p}$.

The case $r = 3$. Let $X \in \mathfrak{v}$ be a vector that projects nontrivially onto each of the submodules $\mathfrak{p}_i$. Then, by means of the isotropy representation, we know that X is (up to T^2-conjugacy and scaling) of the form $X = e_1 + a_3e_3 + a_5e_5 + a_6e_6$, where a_3, $a_5^2 + a_6^2 \neq 0$.

First, suppose that $a_6 \neq 0$. Then, one sees that the kernel of the Cartan operator C_X is spanned by X and $Y = a_3a_5e_1 + a_3a_6e_2 + a_5e_3 + a_6e_4 + a_3e_5$. As a consequence, if X is tangent to a totally geodesic surface Σ, then its tangent space $\mathfrak{v} = T_o\Sigma$ is precisely $\mathfrak{v} = \operatorname{span}\{X, Y\}$. In particular, since Σ is intrinsically homogeneous and therefore of constant sectional curvature, we have the equations

$$\begin{aligned} 0 =& \langle (\nabla_X R)(X, Y, Y), e_4\rangle = \frac{3a_3^2a_6(1 - a_5^2 - a_6^2)}{8\sqrt{2}}, \\ 0 =& \langle (\nabla_X R)(X, Y, Y), e_5\rangle = \frac{3a_3a_6^2(a_3^2 - 1)}{8\sqrt{2}}, \end{aligned}$$

which imply $a_3 \in \{\pm 1\}$ and $a_5^2 + a_6^2 = 1$. Therefore, we may rewrite $\mathfrak{v}$ as the span of

$$\begin{aligned} X &= e_1 + \varepsilon e_3 + \cos\phi e_5 + \sin\phi e_6, \\ Y &= \varepsilon\cos\phi e_1 + \varepsilon\sin\phi e_2 + \cos\phi e_3 + \sin\phi e_4 + \varepsilon e_5, \end{aligned}$$

where $\varepsilon \in \{\pm 1\}$ and $\sin\phi \neq 0$. It turns out that $\mathfrak{v} = \mathfrak{z}_{\mathfrak{p}}(X)$ is the centralizer of X in $\mathfrak{p}$, and in particular it is (maximal) abelian. If $\varepsilon = 1$, then the element $k = \operatorname{diag}(e^{i\phi/3}, 1, e^{-i\phi/3}) \in \mathsf{T}^2$ carries $\mathfrak{p}_{\mathsf{T}^2_\Lambda} = \operatorname{span}\{e_1 + e_3 + e_5, e_2 + e_4 - e_6\}$ to $\mathfrak{z}_{\mathfrak{p}}(X) = \mathfrak{v}$, which means that Σ is congruent to T^2_Λ. Similarly, if $\varepsilon = -1$, the element $k = \operatorname{diag}(e^{i\phi/3}, e^{i\pi/3}, e^{-i(\phi+\pi)/3}) \in \mathsf{T}^2$ carries $\mathfrak{p}_{\mathsf{T}^2_\Lambda}$ to $\mathfrak{z}_{\mathfrak{p}}(X) = \mathfrak{v}$, and thus Σ is congruent to T^2_Λ.

Now, assume that $a_6 = 0$. In this case, the kernel of the Cartan operator C_X is equal to $\mathfrak{so}(3) = \mathfrak{p}_{\mathsf{F}(\mathbb{C}^3)}$. Thus, the only totally geodesic surfaces containing X are projective planes contained in $\mathsf{F}(\mathbb{R}^3)$.

The case $r = 2$. We can find a vector $X \in \mathfrak{v}$ that has nontrivial projection onto two of the three irreducible submodules. By using the full isotropy representation and rescaling, we can assume that $X = e_1 + \lambda e_3$ for a certain $\lambda > 0$. Take any vector $Y \in \mathfrak{v} \ominus \mathbb{R}X$.

On the one hand if, $D_XY = 0$, then $Y \in \ker D_X \ominus \mathbb{R}X = \mathbb{R}(e_2 + \lambda e_4)$, so we may directly assume that $Y = e_2 + \lambda e_4$, and we have $\mathfrak{v} = \operatorname{span}\{X, Y\}$. In particular, observe that $-\lambda e_2 + e_4$ is orthogonal to $\mathfrak{v}$, and the condition

$$0 = \langle R(X, Y)X, -\lambda e_2 + e_4\rangle = 3\lambda(\lambda^2 - 1)$$

forces $\lambda = 1$. Thus, $\mathfrak{v} = \operatorname{span}\{e_1 + e_3, e_2 + e_4\}$, so $\Sigma = \mathsf{SO}(3)^\sigma \cdot o$.

On the other hand, if $D_XY \neq 0$, by Proposition 6.8, the vectors D_X^kY (for $k \geq 0$) must lie in a common eigenspace of R_X (of dimension greater than one because Y and D_XY are orthogonal) and in the kernel of C_X.

Moreover, the spectrum of the Jacobi operator R_X consists of the (pairwise distinct) eigenvalues

$$0, \qquad \frac{\lambda^2+1}{8},$$
$$\frac{17+17\lambda^2+3\sqrt{25\lambda^4-14\lambda^2+25}}{16}, \qquad \frac{17+17\lambda^2-3\sqrt{25\lambda^4-14\lambda^2+25}}{16},$$

and the only eigenspace of dimension greater than one is that of $\frac{\lambda^2+1}{8}$, which is actually the direct sum $\mathbb{R}(-\lambda e_1+e_3)\oplus\mathfrak{p}_3$. On the other hand, the kernel of the Cartan operator C_X is $\operatorname{span}\{e_1,e_3,e_5,\lambda e_2+e_4\}$. Thus, $Y\in\operatorname{span}\{-\lambda e_1+e_3,e_5\}$, and in particular $\mathfrak{v}$ is contained in $\mathfrak{so}(3)$, so Σ is contained in $\mathsf{F}(\mathbb{R}^3)$.

The case $r=1$. Here, we simply have $\mathfrak{p}=\mathfrak{p}_k$ for $k\in\{1,2,3\}$, so actually Σ is congruent to the fiber $\mathbb{C}\mathsf{P}^1$ of the submersion $\mathsf{F}(\mathbb{C}^3)\to\mathbb{C}\mathsf{P}^2$. □

Proof of Theorem B. The result now follows by combining Theorem 6.23, Proposition 6.38, Proposition 6.39, and Proposition 6.40. □

6.5.3 The almost product $\mathsf{S}^3\times\mathsf{S}^3$

Lemma 6.41. *Let $\mathfrak{v}\subseteq\mathfrak{p}$ be a totally geodesic subspace. If $\mathfrak{v}$ contains a nonzero vertical or horizontal vector, then $\mathfrak{v}$ is well-positioned.*

Proof. Suppose that $\mathfrak{v}$ contains a nonzero vector $X\in\mathfrak{p}_1$ (respectively, $X\in\mathfrak{p}_2$). Since the isotropy representation of $\Delta\mathsf{SU}(2)$ is transitive in the unit sphere of $\mathfrak{p}_1$ (respectively, $\mathfrak{p}_2$), we may suppose that $X=e_1$ (respectively, $X=e_4$). Note that the matrices of R_{e_1} and R_{e_4} are given by $R_{e_1}=\operatorname{diag}(0,3/4,3/4,0,1/12,1/12)$ and $R_{e_4}=\operatorname{diag}(0,1/12,1/12,0,3/4,3/4)$, which means that

$$\mathfrak{v}=(\mathfrak{v}\cap\mathbb{R}e_1)\oplus(\mathfrak{v}\cap\operatorname{span}\{e_2,e_3\})\oplus(\mathfrak{v}\cap\mathbb{R}e_4)\oplus(\mathfrak{v}\cap\operatorname{span}\{e_5,e_6\})$$

in both cases. This last equation implies directly that $\mathfrak{v}$ is well-positioned. □

Proposition 6.42. *The space $M=\mathsf{S}^3\times\mathsf{S}^3$ does not admit any codimension two totally geodesic submanifolds.*

Proof. Suppose on the contrary that there exists a four-dimensional totally geodesic submanifold Σ of M, and without loss of generality assume that Σ passes through o with tangent space $\mathfrak{v}$. A dimension argument yields that $\mathfrak{v}\cap\mathfrak{p}_1$ is nonzero. Since the isotropy representation is transitive on the unit sphere of $\mathfrak{p}_1$, we can further assume that $e_1\in\mathfrak{v}$. Because $\mathfrak{v}$ is R_{e_1}-invariant, using the eigenspace decomposition of R_{e_1} (obtained in the proof of the previous lemma) we get

$$\mathfrak{v}=(\mathfrak{v}\cap\operatorname{span}\{e_1,e_4\})\oplus(\mathfrak{v}\cap\operatorname{span}\{e_2,e_3\})\oplus(\mathfrak{v}\cap\operatorname{span}\{e_5,e_6\}).$$

In particular it follows that either $e_4\in\mathfrak{v}$ or $e_4\in\mathfrak{p}\ominus\mathfrak{v}$.

Firstly, suppose that $e_4\in\mathfrak{v}$. If e_2 and e_3 are also tangent to $\mathfrak{v}$, then $\mathfrak{v}=\operatorname{span}\{e_1,e_2,e_3,e_4\}$, which is a contradiction because $12R(e_1,e_2)e_4=-5e_5\notin\mathfrak{v}$. Similarly, if e_5 and e_6 are tangent to Σ we deduce that $\mathfrak{v}=\operatorname{span}\{e_1,e_4,e_5,e_6\}$, which is not possible either, as in that case we would have $6R(e_1, e_4)e_5 = e_2 \notin \mathfrak{v}$. Thus, we see that $\dim \mathfrak{v} \cap \operatorname{span}\{e_2, e_3\} = \dim \mathfrak{v} \cap \operatorname{span}\{e_5, e_6\} = 1$. Conjugating by an adequate element in K, we can assume that $\mathfrak{v} \cap \operatorname{span}\{e_2, e_3\} = \mathbb{R}e_2$. As a consequence, $e_5 = -6R(e_1, e_4)e_2 \in \mathfrak{v}$, and we obtain $\mathfrak{v} = \operatorname{span}\{e_1, e_2, e_4, e_5\}$. However, the equality $6\sqrt{3}(\nabla_{e_1} R)(e_1, e_4, e_2) = e_6$ implies that $\mathfrak{v}$ is not ∇R-invariant, which yields a contradiction once again. Therefore, e_4 is not tangent to Σ, so it must be normal to $\mathfrak{v}$.

We suppose that $e_1 \in \mathfrak{v}$ and $e_4 \in \mathfrak{p} \ominus \mathfrak{v}$. By dimensional reasons, and using the isotropy representation if necessary, we may find constants $\lambda, c_5, c_6 \in \mathbb{R}$ such that $e_2 + \lambda e_3, c_5 e_5 + c_6 e_6$ is in $\mathfrak{v}$. Moreover,

$$0 = \langle R(e_1, e_2 + \lambda e_3)(c_5 e_5 + c_6 e_6), e_4 \rangle = \frac{5}{12}\left(c_5 + \lambda c_6\right),$$
$$0 = \langle (\nabla_{e_1} R)(e_1, e_2 + \lambda e_3, c_5 e_5 + c_6 e_6), e_4 \rangle = \frac{5}{12\sqrt{3}}\left(\lambda c_5 - c_6\right),$$

which implies $c_5 = c_6 = 0$, a contradiction. We conclude that the existence of Σ is not possible. □

Proposition 6.43. *Let $\Sigma \to \mathsf{S}^3 \times \mathsf{S}^3$ be a three-dimensional complete totally geodesic submanifold. Then Σ is congruent to either the round sphere S^3 (viewed as the first factor) or the Berger sphere $\mathsf{S}^3_{\mathbb{C},1/3}(2)$.*

Proof. Suppose that Σ is such a submanifold, and assume without loss of generality that Σ passes through o with tangent space $\mathfrak{v}$.

We start by assuming that $\mathfrak{v}$ is well-positioned, so $\mathfrak{v} = (\mathfrak{v} \cap \mathfrak{p}_1) \oplus (\mathfrak{v} \cap \mathfrak{p}_2)$. We consider several cases according to the dimension of $\mathfrak{v} \cap \mathfrak{p}_1$.

If $\mathfrak{v} \cap \mathfrak{p}_1 = 0$, then $\mathfrak{v} = \mathfrak{p}_2$. However, the equation $3\sqrt{3}(\nabla_{e_4} R)(e_4, e_5, e_4) = e_3$ implies that $\mathfrak{v}$ is not ∇R-invariant, a contradiction.

If $\mathfrak{v} \cap \mathfrak{p}_1$ is a one-dimensional subspace, we may use the isotropy representation to assume that this intersection is spanned by e_1. Now, since $\mathfrak{v} \cap \mathfrak{p}_2$ is two-dimensional, it must intersect $\operatorname{span}\{e_5, e_6\}$, and we may use an element of K that fixes e_1 if necessary to assume that $e_5 \in \mathfrak{v}$. As a consequence, $e_6 = 3\sqrt{3}(\nabla_{e_5} R)(e_5, e_1, e_5)$ is also in $\mathfrak{v}$, so $\mathfrak{v} = \operatorname{span}\{e_1, e_5, e_6\}$, which implies that $\Sigma = \mathsf{S}^3_{\mathbb{C},1/3}(2)$ is the Berger sphere.

If $\mathfrak{v} \cap \mathfrak{p}_1$ is two-dimensional, then $\mathfrak{v} \cap \mathfrak{p}_2$ is one-dimensional, and we can assume that it is spanned by e_4. Arguing in the same fashion as above, we can also assume that $e_2 \in \mathfrak{v}$, and therefore we have $e_6 = 3\sqrt{3}(\nabla_{e_4} R)(e_2, e_4, e_4) \in \mathfrak{v}$, but this contradicts our hypothesis that $\mathfrak{v} \cap \mathfrak{p}_2$ is one-dimensional.

Lastly, if $\mathfrak{v} \cap \mathfrak{p}_1 = \mathfrak{p}_1$, then we actually have $\mathfrak{v} = \mathfrak{p}_1$, so Σ is simply the fiber of the fibration $\mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3$.

Now, suppose that $\mathfrak{v}$ is not well-positioned, so that $\mathfrak{v} \cap \mathfrak{p}_1 = \mathfrak{v} \cap \mathfrak{p}_2 = 0$ by Lemma 6.41. We start by proving (up to the isotropy representation) that $\mathfrak{v}$ admits a vector of the form $X = e_1 + \lambda e_4$ for a certain $\lambda \in \mathbb{R} \setminus \{0\}$.

Firstly, note that since $\dim \mathfrak{v} = 3$, the intersection $\mathfrak{v} \cap \operatorname{span}\{e_1, e_4, e_5, e_6\}$ is nontrivial. Conjugating by an adequate element of K and rescaling, we can assume that a vector of the form $X = e_1 + \rho \cos\theta e_4 + \rho \sin\theta e_5$ is in $\mathfrak{v}$, where $\rho \geq 0$ and $\theta \in [0, 2\pi)$. The condition

$\mathfrak{v} \cap \mathfrak{p}_1 = \mathfrak{v} \cap \mathfrak{p}_2 = 0$ forces $\rho > 0$, and the orthogonal projection maps $\mathfrak{v} \to \mathfrak{p}_i$ are vector space isomorphisms. Note that if $\theta \in \{0, \pi\}$, then $X = e_1 \pm \rho e_4$, and our assertion is proved. On the other hand, if $\theta \notin \{0, \pi\}$, we define polynomials

$$f(x) = \frac{1}{432}\left(-\rho^2 + 12x - 1\right)\left(-4\rho^2\cos(2\theta) + \rho^2(4 - 27x) + 9x(4x - 3)\right),$$
$$g(x) = \frac{1}{144}\left(-32\rho^2\cos(2\theta) + 9\rho^4 + 50\rho^2 + 144x^2 - 120\left(\rho^2 + 1\right)x + 9\right).$$

One sees that the product fg is precisely the characteristic polynomial of $R_X \in \mathrm{End}(\mathfrak{p} \ominus \mathbb{R}X)$, so in particular we obtain that $f(R_X)g(R_X) = 0$. Furthermore, f and g are relatively prime. As a consequence, because $\mathfrak{v}$ is an R_X-invariant subspace, it may be decomposed as

$$\mathfrak{v} = \mathbb{R}X \oplus (\mathfrak{v} \cap \ker f(R_X)) \oplus (\mathfrak{v} \cap \ker g(R_X)).$$

It turns out that

$$\ker f(R_X) = \mathrm{span}\{e_2, \rho\sin\theta e_1 - e_5, \rho\cos\theta e_1 - e_4\},$$
$$\ker g(R_X) = \mathrm{span}\{e_3, e_6\},$$

and as the projections of $\mathfrak{v}$ onto $\mathbb{R}e_3$ and $\mathbb{R}e_6$ are both nontrivial, we obtain that there is a vector in $\mathfrak{v}$ of the form $e_3 + \lambda e_6$ (where $\lambda \neq 0$), and by conjugating via the isotropy representation we can change $\mathfrak{v}$ so that $e_1 + \lambda e_4 \in \mathfrak{v}$.

We now let $Y = e_1 + \lambda e_4 \in \mathfrak{v}$, where $\lambda \in \mathbb{R} \setminus \{0\}$. The Jacobi operator $R_Y \in \mathrm{End}(\mathfrak{p} \ominus \mathbb{R}Y)$ has three different eigenvalues 0, $\frac{1+\lambda^2}{12}$ and $\frac{3(1+\lambda^2)}{4}$, with corresponding eigenspaces $\mathbb{R}(\lambda e_4 - e_1)$, $\mathrm{span}\{\lambda e_2 - e_5, \lambda e_3 - e_6\}$ and $\mathrm{span}\{e_2 + \lambda e_5, e_3 + \lambda e_6\}$. Note that the isotropy subgroup K_{e_1} of e_1 fixes Y and acts transitively on the set of lines in both $\mathrm{span}\{\lambda e_2 - e_5, \lambda e_3 - e_6\}$ and $\mathrm{span}\{e_2 + \lambda e_5, e_3 + \lambda e_6\}$. Now, the fact that $\dim \mathfrak{v} = 3$ implies that $\mathfrak{v}$ must contain a nonzero eigenvector from one of the last two eigenspaces. Our next step is to prove that $\lambda^2 = 3$ or $\lambda^2 = 1/3$.

If $\mathfrak{v} \cap \mathrm{span}\{\lambda e_2 - e_5, \lambda e_3 - e_6\} \neq 0$, we can conjugate by an element of K_{e_1} to assume that $Z = \lambda e_2 - e_5 \in \mathfrak{v}$ is tangent to Σ. If we assume that $\lambda^2 \notin \{3, 1/3\}$, then we can give a basis of $\mathfrak{v}$ with the vectors

$$Y = e_1 + \lambda e_4,$$
$$Z = \lambda e_2 - e_5,$$
$$T = 3\sqrt{3}(\nabla_Z R)(Z, Y, Z) = \lambda(1 - 3\lambda^2)e_3 + (3\lambda^2 - 1)e_6.$$

As a consequence, $e_3 + \lambda e_6$ is orthogonal to $\mathfrak{v}$, and we deduce that

$$0 = 3\sqrt{3}\langle(\nabla_Y R)(Y, Z, Y), e_3 + \lambda e_6\rangle = \lambda\left(3 - \lambda^2\right)\left(\lambda^2 + 1\right),$$

a contradiction, so we deduce that $\lambda^2 \in \{3, 1/3\}$.

Similarly, in the case that $\mathfrak{v} \cap \operatorname{span}\{e_2 + \lambda e_5, e_3 + \lambda e_6\} \neq 0$, we can use an element of K_{e_1} to assume that $Z = e_2 + \lambda e_5$ also belongs to $\mathfrak{v}$. Here, if $\lambda^2 \notin \{3, 1/3\}$, then we can construct a basis of $\mathfrak{v}$ with the vectors

$$\begin{aligned} Y &= e_1 + \lambda e_4, \\ Z &= e_2 + \lambda e_5, \\ T &= \frac{3\sqrt{3}}{\lambda}(\nabla_Y R)(Y, Z, Y) = \lambda(\lambda^2 - 3)e_3 + (3 - \lambda^2)e_6. \end{aligned}$$

In particular, $e_3 + \lambda e_6 \in \mathfrak{p} \ominus \mathfrak{v}$, and we have

$$0 = \langle (\nabla_T R)(Y, Z, T), e_3 + \lambda e_6 \rangle = \frac{(\lambda^2 - 3)^2 (\lambda^2 + 1)(3\lambda^2 - 1)}{6\sqrt{3}},$$

contradicting our assumption.

All in all, we have proved that the totally geodesic subspace $\mathfrak{v}$ contains $Y = e_1 + \lambda e_4$, and $\lambda^2 = 3$ or $\lambda^2 = 1/3$. Now, consider the isometry $s_o \colon \mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3 \times \mathsf{S}^3$ defined in Subsection 6.1. One sees that the differential $(s_o)_{*o}$ satisfies the identities

$$\begin{aligned} (s_o)_{*o}(e_1) &= -\frac{1}{2}(e_1 + \sqrt{3}e_4) & (s_o)_{*o}^{-1}(e_1) &= -\frac{1}{2}(e_1 - \sqrt{3}e_4), \\ (s_o)_{*o}(e_4) &= \frac{1}{2}(\sqrt{3}e_1 - e_4), & (s_o)_{*o}^{-1}(e_4) &= -\frac{1}{2}(\sqrt{3}e_1 + e_4). \end{aligned}$$

Using these equations, we see that either $(s_o)_{*o}(\mathfrak{v})$ or $(s_o)_{*o}^2(\mathfrak{v})$ is a totally geodesic subspace that contains either e_1 or e_4, so by Lemma 6.41 it is well-positioned, and thus Σ is congruent to $\mathsf{S}^2(2/\sqrt{3})$ or $\mathsf{S}^3_{\mathbb{C},1/3}(2)$. □

Proposition 6.44. *Let $\Sigma \to \mathsf{S}^3 \times \mathsf{S}^3$ be a complete totally geodesic surface. Then Σ is congruent to either T^2_Γ, the not well-positioned $\mathsf{S}^3(\sqrt{3/2})$, or a great sphere inside the round $\mathsf{S}^3\left(2/\sqrt{3}\right)$.*

Proof. Let Σ be a complete totally geodesic surface, and assume without loss of generality that it passes through o with tangent space $\mathfrak{v}$. We distinguish two situations for $\mathfrak{v}$ according to whether it is well-positioned or not.

First, suppose that $\mathfrak{v}$ is well-positioned. If $\mathfrak{v} \subseteq \mathfrak{p}_1$, then Σ is merely a round sphere inside the round S^3. If $\mathfrak{v} \subseteq \mathfrak{p}_2$, we may suppose that $\mathfrak{v} = \operatorname{span}\{e_4, e_5\}$, but the equation

$$3\sqrt{3}(\nabla_{e_4} R)(e_4, e_5, e_4) = e_3$$

implies that $\mathfrak{v}$ is not ∇R-invariant, a contradiction. Now, suppose that $\mathfrak{v} \cap \mathfrak{p}_1$ and $\mathfrak{v} \cap \mathfrak{p}_2$ are both one-dimensional. By using the isotropy representation, we may suppose that $\mathfrak{v} \cap \mathfrak{p}_1 = \mathbb{R}e_1$. The Jacobi operator R_{e_1} preserves $\mathfrak{p}_2$ and its restriction to $\mathfrak{p}_2$ has eigenvalues 0 (with eigenspace $\mathbb{R}e_4$) and $\frac{1}{12}$ (with eigenspace $\operatorname{span}\{e_5, e_6\}$). Thus, we can further assume (up to the action of K) that either $\mathfrak{v} = \operatorname{span}\{e_1, e_4\}$ or $\mathfrak{v} = \operatorname{span}\{e_1, e_5\}$. The first case simply yields $\Sigma = \mathsf{T}^2_\Gamma$, while the second case gives a contradiction because $3\sqrt{3}(\nabla_{e_5} R)(e_5, e_1, e_5) = e_6$. This completes the case when $\mathfrak{v}$ is well-positioned.

Suppose $\mathfrak{v}$ is not well-positioned. In particular $\mathfrak{v}_{\mathfrak{p}_1} \neq 0$, and this combined with the isotropy representation lets us assume that $\mathfrak{v}$ contains a vector of the form $X = e_1 + \rho\cos\theta e_4 + \rho\sin\theta e_5$, where $\rho > 0$ and $\theta \in [0, 2\pi)$.

Suppose $\mathfrak{v}$ is invariant under D, which yields $\mathfrak{v} \subseteq \ker D_X$ as it is of dimension two. If $\theta \in \{0, \pi\}$, then $\ker D_X = \operatorname{span}\{e_1, e_4\}$, which forces $\mathfrak{v} = \operatorname{span}\{e_1, e_4\}$, contradicting our hypothesis that $\mathfrak{v}$ is not well-positioned. If $\theta \notin \{0, \pi\}$, the kernel of D_X is spanned by X and $Y = \rho\cos\theta e_1 + \rho\sin\theta e_2 - e_4$, so $\mathfrak{v} = \operatorname{span}\{X, Y\}$. As a consequence, we see that the vector $Z = -\rho\sin\theta e_1 + \rho\cos\theta e_2 + e_5$ is orthogonal to $\mathfrak{v}$, and thus $0 = \langle R(X,Y)X, Z\rangle = \frac{4}{3}\rho^2\sin 2\theta$, which forces $\theta = \pi/2$ or $\theta = 3\pi/2$. We group these cases together by writing $X = e_1 + te_5$, $Y = te_2 - e_4$ for a nonzero $t \in \mathbb{R}$. As a consequence, $e_2 + te_4$ is orthogonal to $\mathfrak{v}$, and the inner product $\langle R(X,Y)X, e_2 + te_4\rangle = \frac{4}{3}t(t^2 - 1)$ vanishes, so $t = \pm 1$. The cases $t = 1$ and $t = -1$ give rise to congruent submanifolds. Indeed, the element $k = (\operatorname{diag}(i, -i), \operatorname{diag}(i, -i), \operatorname{diag}(i, -i)) \in \mathsf{K}$ satisfies the equations $\operatorname{Ad}(k)(e_1 + e_5) = e_1 - e_5$ and $\operatorname{Ad}(k)(e_2 - e_4) = -e_2 - e_4$. As a consequence, we see that $\mathfrak{v}$ is conjugate to $\operatorname{span}\{e_1 + e_5, e_2 - e_4\}$, so Σ is congruent to the not well-positioned $\mathsf{S}^2(\sqrt{3/2})$.

Now, assume that $\mathfrak{v}$ is not D-invariant. Arguing as before, we may suppose that $\mathfrak{v}$ contains a vector of the form $X = e_1 + \rho\cos\theta e_4 + \rho\sin\theta e_5$ for $\rho > 0$ and $0 \le \theta < 2\pi$. On the one hand, one sees that $C_X = 0$ if $\rho = \sqrt{3}$ and $\theta \in \{0, \pi\}$, but in this case $\mathfrak{v}$ is congruent to a well-positioned example. Indeed, we argue as in the end of Proposition 6.43. Consider the isometry $s_o \colon \mathsf{S}^3 \times \mathsf{S}^3 \to \mathsf{S}^3 \times \mathsf{S}^3$ defined as in Subsection 6.1. Then, if $\theta = 0$ we have $X = -2(s_o)_{*o}(e_1)$, so $\mathfrak{v}$ is congruent to $(s_o)_{*o}^{-1}(\mathfrak{v})$, and this subspace is well-positioned by Lemma 6.41, so we may apply the conclusions from the previous case. Similarly, if $\theta = \pi$, then $X = -2(s_o)_{*o}^{-1}(e_1)$, and $(s_o)_{*o}(\mathfrak{v})$ is well-positioned. On the other hand, if $\rho \neq \sqrt{3}$ or $\theta \neq \pi$, we have

$$
\begin{aligned}
C_X e_1 &= \frac{\rho^2\sin 2\theta}{6\sqrt{3}} e_3 + \frac{\rho^3\sin\theta}{3\sqrt{3}} e_6, \\
C_X e_2 &= \frac{\rho^2\sin^2\theta}{3\sqrt{3}} e_3 - \frac{\rho\,(\rho^2 - 3)\cos\theta}{3\sqrt{3}} e_6, \\
C_X e_4 &= -\frac{\rho\,(\rho^2 - 2)\sin\theta}{3\sqrt{3}} e_3 - \frac{\rho^2\sin 2\theta}{6\sqrt{3}} e_6, \\
C_X e_5 &= \frac{\rho\,(\rho^2 - 3)\cos\theta}{3\sqrt{3}} e_3 - \frac{\rho^2\sin^2\theta}{3\sqrt{3}} e_6.
\end{aligned}
$$

A straightforward computation gives that these four vectors generate $\operatorname{span}\{e_3, e_6\}$, so we have $\operatorname{span}\{e_3, e_6\} \subseteq \operatorname{im} C_X$. In particular, since C_X is a symmetric endomorphism, we have $\mathfrak{p} = \ker C_X \oplus \operatorname{im} C_X$ orthogonally, so $\ker C_X$ is orthogonal to e_3 and e_6. As a consequence, $\mathfrak{v}$ is spanned by X and a vector of the form $Y = a_1e_1 + a_2e_2 + a_4e_4 + a_5e_5$, where $a_1, a_2, a_4, a_5 \in \mathbb{R}$. However,

$$
D_X Y = \frac{a_4\rho\sin\theta - a_5\rho\cos\theta + a_2}{2\sqrt{3}} e_3 + \frac{a_1\rho\sin\theta - a_2\rho\cos\theta - a_5}{2\sqrt{3}} e_6
$$

is in $\ker C_X$ by Proposition 6.8 and in $\operatorname{span}\{e_3, e_6\} \subseteq \mathfrak{p} \ominus \ker C_X$, which forces $D_X Y = 0$, and thus $\mathfrak{v}$ is D-invariant, contradicting our assumption. This finishes the proof. $\square$

Proof of Theorem C. The result follows from combining Theorem 6.23, Proposition 6.42, Proposition 6.43, and Proposition 6.44. □

Finally, we can also prove Theorem E.

Proof of Theorem E. Let M be a homogeneous nearly Kähler 6-manifold with non-constant curvature. Since totally geodesic submanifolds are preserved under Riemannian coverings and the universal cover of $\widehat{M}$ is the cone of the universal cover of M, we may assume that M is simply connected, so M is either $\mathbb{C}\mathsf{P}^3$, $\mathsf{F}(\mathbb{C}^3)$ or $\mathsf{S}^3 \times \mathsf{S}^3$.

Let Σ be a complete totally geodesic submanifold of the G_2-cone $\widehat{M}$. By Corollary 6.26, Σ is either a hypersurface of $\widehat{M}$ or the cone of a maximal totally geodesic submanifold of M. The first case is not possible due to [94, Theorem 1.2], so we conclude that $\Sigma = \widehat{S}$ for a maximal totally geodesic submanifold $S \subseteq M$, which yields the desired result. □

Remark 6.45. Let $M \in \{\mathbb{C}\mathsf{P}^3, \mathsf{F}(\mathbb{C}^3), \mathsf{S}^3 \times \mathsf{S}^3\}$. A look at Tables 6.1, 6.2 and 6.3 shows that two (complete) totally geodesic submanifolds of M are congruent if and only if they are isometric. Combining this with Proposition 6.19, we deduce that the congruence classes of maximal totally geodesic submanifolds of $\widehat{M}$ are in a bijective correspondence with the congruence classes of maximal totally geodesic submanifolds of M.

We also note that in order to obtain the classification of all totally geodesic submanifolds of $\widehat{M}$, it suffices to iterate Corollary 6.26 and take into account Propositions 6.28 and 6.30.

Part III

Non-Lorentzian homogeneous spacetimes

“Well, in our country,” said Alice, still panting a little, “you’d generally get to somewhere else if you run very fast for a long time, as we’ve been doing.”

“A slow sort of country!” said the Queen. “Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!”

LEWIS CARROLL

Chapter 7
Kinematical Lie algebras and homogeneous spacetimes

The purpose of this chapter is to present the basic concepts concerning kinematical Lie groups and algebras, as well as their corresponding homogeneous spacetimes. We primarily follow [15, 123].

In a broad sense, the key idea behind the introduction of kinematical groups and their homogeneous manifolds is to develop an analogue of Klein's Erlangen programme for spacetimes. A first approach to this problem was given in the seminal work by Bacry and Lévy-Leblond [12], who worked in four dimensions and were motivated by the fact that the laws of physics are invariant under four kinds of transformations: spatial rotations, spatial translations, time translations and boosts (also known as inertial transformations). This is the case for Galilei spacetime (which is the model geometry appearing in Newtonian mechanics) and Minkowski spacetime (the basis of Einstein's theory of special relativity), which only disagree in their families of boosts. The change of paradigm is thus to determine *a priori* the general structure that a Lie group G must have in order to be considered the symmetry group of a certain spacetime, and understand the homogeneous spacetimes with symmetry group G as some adequate quotients of G.

The considerations by Bacry and Lévy-Leblond naturally lead them to the notion of kinematical Lie groups as the symmetry groups of spatially isotropic homogeneous spacetimes. Assume M^{d+1} is a spatially isotropic homogeneous spacetime with (connected) symmetry group G. Then G has to be comprised of (abstract) spatial rotations, spacetime translations and boosts. At the Lie algebra level, this means that $\mathfrak{g}$ contains a subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$ (known as the rotational subalgebra) such that, under the adjoint action of $\mathfrak{so}(d)$, $\mathfrak{g}$ admits the decomposition $\mathfrak{so}(d) \oplus \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$ as one copy of the adjoint representation (representing spatial rotations), two copies of the standard representation (representing space translations and boosts) and one copy of the trivial representation (representing time translations). A Lie algebra satisfying these conditions is called a $(1,2)$-kinematical Lie algebra[1]. Moreover, the isotropy algebra $\mathfrak{k}$ at any $o \in M$ is formed by rotations and boosts, meaning that $\mathfrak{r} \subseteq \mathfrak{k}$ and $\mathfrak{k} = \mathfrak{so}(d) \oplus \mathbb{R}^d$ as an $\mathfrak{so}(d)$-module. We say in this case that $\mathfrak{k}$ is an admissible subalgebra of $\mathfrak{g}$. With this in mind, the Erlangen programme for spatially isotropic homogeneous spacetimes can be formulated as follows:

What are the homogeneous manifolds $M = \mathsf{G}/\mathsf{K}$ for which the Lie algebra $\mathfrak{g}$ is $(1,2)$-kinematical and the isotropy algebra $\mathfrak{k}$ is admissible?[2]

[1] Originally named *kinematical algebra* in [12].

[2] Because the manifold M is not assumed to have a preferred geometric structure from the start, we need to keep track of the Klein pair (G, K) that represents it.

Because a connected homogeneous space $M = \mathsf{G}/\mathsf{K}$ and its universal cover have isomorphic infinitesimal Klein pairs (see Section 1.3), it suffices to work with simply connected homogeneous manifolds, or equivalently, with infinitesimal Klein pairs that are effective and geometrically realizable. This problem is therefore split into two: determining all effective Klein pairs formed by a $(1, 2)$-kinematical algebra and an admissible subalgebra (which is an algebraic problem centered around the representation theory of $\mathfrak{so}(d)$) and determining which of these admit geometric realizations (which is a topological problem), see Section 7.2 for details.

The definitions above can be generalized to accommodate for different families of symmetries (and thus different kinematical homogeneous spacetimes). We say that an (s, v)-kinematical algebra is a real Lie algebra $\mathfrak{g}$ containing a rotational subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$ and such that, as a representation of $\mathfrak{so}(d)$, $\mathfrak{g}$ admits the decomposition $\mathfrak{so}(d) \oplus \bigoplus^v \mathbb{R}^d \oplus \bigoplus^s \mathbb{R}$. Apart from the case $(s, v) = (1, 2)$, many other families of kinematical algebras have been studied due to their applications in physics, see [123] and the references therein. Some examples are $(s, v) = (1, 1)$ (known as Aristotelian algebras or boostless kinematical algebras [66]), $(s, v) = (2, 1)$ (known as Lifshitz algebras [64]) and $(s, v) = (3, 1)$ (known as ambient Aristotelian algebras [123]). We will discuss the classification of Aristotelian and Lifshitz algebras in Sections 7.3 and 7.4.

Another interesting aspect of treating homogeneous spacetimes with Klein's original philosophy is the apparition of new geometric structures modeled on these homogeneous manifolds. Formally, this is done by means of Cartan geometries [153, Chapter 5]. A *Cartan geometry* on a smooth manifold M modeled on the homogeneous space G/K is a principal K-bundle $P \to M$ together with a $\mathfrak{g}$-valued 1-form on P (known as the *Cartan connection*) satisfying the following three properties:

- the map $\omega_u \colon T_uP \to \mathfrak{g}$ is a vector space isomorphism for all $u \in P$,
- $(R_k)^*\omega = \mathrm{Ad}(k^{-1})\omega$ for all $k \in \mathsf{K}$, where R_k denotes right multiplication by k, and
- for every $X \in \mathfrak{k}$ we have $\omega(X^*) = X$.

For example, providing a Cartan geometry on M modeled on Minkowski spacetime is the same as endowing it with a Lorentzian metric g and a linear connection ∇ satisfying $\nabla g = 0$. Due to the fundamental theorem of pseudo-Riemannian geometry, a Lorentzian manifold admits a unique connection that is both torsion-free and adapted to the metric. However, for other families of spacetimes one may not have existence nor uniqueness results concerning torsion-free adapted connections. For instance, it turns out that a Cartan geometry on M modeled on Galilei spacetime is determined by a nowhere-vanishing one-form $\tau \in \Omega^1(M)$, a positive semidefinite bilinear form $\lambda \in \mathsf{S}^2(TM)$ whose radical is spanned by τ (the triple (M, τ, λ) is then known as a Galilean manifold), and a linear connection ∇ that annihilates both τ and λ. However, for general τ and λ, it may be the case that no torsion-free adapted connections exist (we say in this case that M has nontrivial intrinsic torsion). The intrinsic torsion of some of these spacetime structures is studied in detail in [63].

This chapter is organized as follows. Section 7.1 is dedicated to presenting the notion of kinematical Lie algebras, as well as exhibiting the Galilei, Poincaré and Carroll algebra as examples of $(1, 2)$-kinematical algebras. Moreover, we describe Galilean (respectively, Carrollian) manifolds and show they can be obtained as null reductions (respectively, hypersurfaces) of Lorentzian manifolds. In Section 7.2 we present the definition and classification of simply connected and spatially isotropic homogeneous spaces. Similarly, Section 7.3 contains the classification of Aristotelian Lie algebras and their homogeneous spacetimes up to coverings. Finally, we present in Section 7.4 the classification of Lifshitz algebras and simply connected spatially isotropic Lifshitz spacetimes.

7.1 Kinematical Lie algebras

Let s, v and d be positive integers. An (s,v)-*kinematical Lie algebra* (with d-dimensional spatial isotropy) is a real Lie algebra $\mathfrak{g}$ together with an embedding $\mathfrak{so}(d) \hookrightarrow \mathfrak{g}$ such that, under the adjoint action of $\mathfrak{so}(d)$, $\mathfrak{g}$ admits the decomposition

$$\mathfrak{g} = \mathfrak{so}(d) \oplus \bigoplus^{v} \mathbb{R}^d \oplus \bigoplus^{s} \mathbb{R},$$

where $\mathbb{R}^d$ is the standard representation of $\mathfrak{so}(d)$ and $\mathbb{R}$ denotes the trivial representation. In other words, we require that $\mathfrak{so}(d)$ commutes with each of the $\mathbb{R}$ summands, whereas the action of $\mathfrak{so}(d)$ in any of the $\mathbb{R}^d$ summands is given by $[X, v] = Xv$. We say that the subalgebra $\mathfrak{so}(d)$ is the *rotational subalgebra* of $\mathfrak{g}$. From now on, we use the abbreviation (s,v)-KLA to refer to an (s,v)-kinematical Lie algebra. A Lie group G whose Lie algebra is an (s,v)-KLA is said to be an (s,v)-*kinematical Lie group*.

Suppose $\mathfrak{g}$ is an (s,v)-KLA with d-dimensional spatial isotropy. We write $V = \mathbb{R}^d$ for the standard representation of $\mathfrak{so}(d) = \mathfrak{so}(V)$, $W = \mathbb{R}^v$ and $S = \mathbb{R}^s$ for the trivial representations of dimension v and s respectively. We may thus express

$$\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus S.$$

This alternate description will prove to be useful in future calculations, see Chapter 8.

To get a firmer grasp on this concept, let us describe the main examples of kinematical algebras by looking at three different spacetimes.

Example 7.1. Consider Minkowski spacetime $M = \mathbb{R}^{d,1}$, which is simply the vector space $\mathbb{R}^{d+1}$ together with the Lorentzian inner product γ defined by

$$\gamma((x,t),(y,s)) = \langle x, y\rangle - c^2 ts = \sum_{i=1}^{d} x_i y_i - c^2 ts.$$

The parameter $c > 0$ denotes the speed of light, and it is clear that different values of c give isometric Lorentzian vector spaces. The subgroup of $\mathsf{Aff}(\mathbb{R}^{d+1})$ that stabilizes this inner product is the *Poincaré group* $\mathsf{Poin}(d) = \mathsf{O}(d,1) \ltimes \mathbb{R}^{d+1}$. Its Lie algebra $\mathfrak{poin}(d) = \mathfrak{so}(d,1) \ltimes \mathbb{R}^{d+1}$ is easily seen to be

$$\mathfrak{poin}(d) = \left\{ \begin{pmatrix} X & c^2 v & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} : X \in \mathfrak{so}(d), v, a \in \mathbb{R}^d, s \in \mathbb{R} \right\}. \tag{7.1}$$

The symmetries in $\mathfrak{poin}(d)$ can be decomposed into four types of infinitesimal transformations:

- spatial rotations $(x,t) \mapsto (Xx, 0)$, with $X \in \mathfrak{so}(d)$;
- spatial translations $(x,t) \mapsto (a, 0)$, with $a \in \mathbb{R}^d$;
- *Lorentz boosts* $(x,t) \mapsto (tc^2 v, \langle v, x\rangle)$, with $v \in \mathbb{R}^d$;
- and time translations $(x,t) \mapsto (0, s)$, with $s \in \mathbb{R}$.

It is not hard to show that $\mathfrak{so}(d)$ acts on the set of spatial rotations by the adjoint representation, whereas the action of $\mathfrak{so}(d)$ on both spatial translations and boosts is equivalent to its standard representation. Furthermore, $\mathfrak{so}(d)$ acts trivially on time translations. Therefore, $\mathfrak{poin}(d)$ is a $(1,2)$-KLA with d-dimensional space isotropy. Clearly, the geometries associated with the homogeneous space $\mathbb{R}^{d,1} = \mathsf{Poin}(d)/\mathsf{O}(d,1)$ are Lorentzian manifolds.

Example 7.2. Let $M = \mathbb{R}^{d+1}$ be the $(d+1)$-dimensional *Galilei spacetime*. This is the vector space $\mathbb{R}^{d+1}$ endowed with the so-called *clock* one-form $\tau \colon \mathbb{R}^{d+1} \to \mathbb{R}$ defined by $\tau(x,t) = t$ and the *ruler* bilinear form $\lambda \colon \mathsf{S}^2\mathbb{R}^d \to \mathbb{R}$ given by

$$\lambda((x,0),(y,0)) = \langle x, y\rangle = \sum_{i=1}^{d} x_i y_i.$$

Note that λ is not defined on $\mathbb{R}^{d+1}$, but we can identify it with a globally defined bilinear form on $(\mathbb{R}^{d+1})^*$ as follows: the form λ gives an isomorphism $\mathbb{R}^d \cong (\mathbb{R}^d)^*$ and thus a bilinear form μ on $(\mathbb{R}^d)^*$. Dualizing the inclusion $\mathbb{R}^d \subseteq \mathbb{R}^{d+1}$ we obtain a surjective linear map $(\mathbb{R}^{d+1})^* \to (\mathbb{R}^d)^*$, which is merely the restriction operator. Therefore, we can identify λ with the symmetric bilinear form

$$\bar{\lambda} \colon (\mathbb{R}^{d+1})^* \otimes (\mathbb{R}^{d+1})^* \to (\mathbb{R}^d)^* \otimes (\mathbb{R}^d)^* \to (\mathbb{R}^d) \otimes (\mathbb{R}^d) \to \mathbb{R}$$

explicitly given by

$$\bar{\lambda}(\alpha,\beta) = \mu(\alpha|_{\mathbb{R}^d}, \beta|_{\mathbb{R}^d}), \quad \alpha, \beta \in (\mathbb{R}^{d+1})^*.$$

The form $\bar{\lambda}$ is symmetric, positive semidefinite and with radical $\mathbb{R}\tau$. We call the bilinear map $\bar{\lambda}$ the *spatial cometric* of Galilei spacetime.

The group of affine transformations of $\mathbb{R}^{d+1}$ preseving τ and λ (equivalently, $\bar{\lambda}$) is known as the *Galilei group*

$$\mathsf{Gal}(d) = \left\{ \begin{pmatrix} R & v & a \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} : R \in \mathsf{O}(d), v, a \in \mathbb{R}^d, s \in \mathbb{R} \right\} \subseteq \mathsf{GL}(d+2, \mathbb{R}).$$

Its Lie algebra is called the *Galilei algebra*, and it is easily seen to be

$$\mathfrak{gal}(d) = \left\{ \begin{pmatrix} X & v & a \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} : X \in \mathfrak{so}(d), v, a \in \mathbb{R}^d, s \in \mathbb{R} \right\}.$$

Both the Galilei and Poincaré algebras share the same spatial rotations, as well as spacetime translations. The key difference between these two algebras is given by their boosts. Indeed, the Galilei algebra is obtained by adding the (infinitesimal) *Galilei boosts* $(x,t) \mapsto (tv,0)$ (where $v \in \mathbb{R}^d$) to the aforementioned transformations. Arguing exactly as in the previous example, one sees that the Galilei algebra is a $(1,2)$-KLA with d-dimensional spatial isotropy.

Let $\mathsf{K} \subseteq \mathsf{Gal}(d)$ be the stabilizer of the origin, which is usually called the *homogeneous Galilei group*. One sees that $\mathsf{Gal}(d) = \mathsf{K} \ltimes \mathbb{R}^{d+1}$ and $\mathbb{R}^{d+1} = \mathsf{Gal}(d)/\mathsf{K}$. The geometries modeled on the Klein pair $(\mathsf{Gal}(d), \mathsf{K})$ are known as Galilean manifolds. A *Galilean manifold* is a $(d+1)$-dimensional smooth manifold M endowed with a nowhere-vanishing one-form $\tau \in \Gamma(T^*M)$ and a positive semidefinite $(2,0)$ tensor $\lambda \in \Gamma(\mathsf{S}^2TM)$ whose radical at each $p \in M$ is spanned by τ_p.

One can produce examples of Galilean manifolds as appropriate quotients of Lorentzian manifolds. Let (N,g) be a Lorentzian manifold with metric g and suppose that H is a one-dimensional connected Lie group acting properly and isometrically on N with a single orbit type. We also assume that for one (hence every) nonzero vector $X \in \mathfrak{h}$ the induced Killing field X^* satisfies $g(X^*,X^*) = 0$. The quotient manifold $M = N/\mathsf{H}$ can be naturally endowed with a Galilean structure as follows. Firstly, let $\widetilde{\tau} \in \Omega^1(N)$ denote the one-form dual to X^*. One can see that there exists a nowhere-vanishing $\tau \in \Omega^1(M)$ for which $\widetilde{\tau} = \pi^*\tau$, where $\pi\colon N \to M$ is the canonical projection. Secondly, if α and β are in $\Omega^1(M)$, let X_α and X_β be the unique vector fields in N that are dual to $\pi^*\alpha$ and $\pi^*\beta$ respectively. Then there exists a function $\lambda(\alpha,\beta) \in \mathcal{C}^\infty(M)$ such that $g(X_\alpha, X_\beta) = \lambda(\alpha,\beta) \circ \pi$. The equation above defines a section λ of S^2TM. The triple (M,τ,λ) turns out to be a Galilean manifold, known as a *null reduction* of (N,g).

Note how, by replacing v with $c^{-2}v$ and letting $c \to \infty$ in (7.1), the algebra $\mathfrak{poin}(d)$ transforms into $\mathfrak{gal}(d)$. This procedure will be formalized later on, and it is known as a Lie algebra contraction. We say in particular that $\mathfrak{gal}(d)$ is the *non-relativistic limit* of $\mathfrak{poin}(d)$.

Example 7.3. We define *Carroll spacetime* as the vector space $M = \mathbb{R}^{d+1}$ equipped with the vector $\kappa = (0,1) \in \mathbb{R}^{d+1}$ and the symmetric bilinear form $h\colon \mathsf{S}^2\mathbb{R}^{d+1} \to \mathbb{R}$ defined by

$$h((x,t),(y,s)) = \langle x,y \rangle = \sum_{i=1}^{d} x_i y_i.$$

The map h is positive semidefinite with radical $\mathbb{R}\kappa$. The subgroup of $\mathsf{Aff}(d+1)$ that preserves both κ and h is known as the *Carroll group*

$$\mathsf{Car}(d) = \left\{ \begin{pmatrix} R & 0 & a \\ v^t & 1 & s \\ 0 & 0 & 1 \end{pmatrix} : R \in \mathsf{O}(d), a, v \in \mathbb{R}^d, s \in \mathbb{R} \right\},$$

whose Lie algebra is called the *Carroll algebra*

$$\mathfrak{car}(d) = \left\{ \begin{pmatrix} X & 0 & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} : X \in \mathfrak{so}(d), a, v \in \mathbb{R}^d, s \in \mathbb{R} \right\}.$$

The Carroll algebra is obtained from space and time translations, spatial rotations and so-called *Carrollian boosts* $(x,t) \mapsto (0, \langle v, x\rangle)$ (where $v \in \mathbb{R}^d$). Once again, $\mathfrak{car}(d)$ is seen to be an $(1,2)$-KLA, as spatial rotations act on Carrollian boosts via the standard representation. The subgroup $\mathsf{K} \subseteq \mathsf{Car}(d)$ that stabilizes $0 \in \mathbb{R}^{d+1}$ is known as the *homogeneous Carroll group*, and it can be checked that $\mathsf{Car}(d) = \mathsf{K} \ltimes \mathbb{R}^{d+1}$. The geometries associated with the Carroll spacetime $\mathbb{R}^{d+1} = \mathsf{Car}(d)/\mathsf{K}$ are known as Carrollian manifolds. We say that a *Carrollian manifold* is a smooth manifold M endowed with a nowhere-vanishing vector field $\kappa \in \mathfrak{X}(M)$ and a positive semidefinite $h \in \Gamma(\mathsf{S}^2(T^*M))$ whose radical at every $p \in M$ is $\mathbb{R}\kappa_p$.

The Carroll group was originally introduced by Lévy-Leblond [117], and it arises as a contraction of the Poincaré group by letting the speed of light $c \to 0$. This becomes apparent from taking the expression (7.1) and taking $c = 0$. We say that $\mathfrak{car}(d)$ is the *ultra-relativistic limit* of $\mathfrak{poin}(d)$. Because a material body in Minkowski spacetime cannot travel faster than the speed of light, it cannot experience any motion in Carroll spacetime.[3] This *otherworldly* feature motivated Lévy-Leblond to name this spacetime after Lewis Carroll's *Alice in Wonderland*.

The archetypical examples of Carrollian manifolds are null hypersurfaces inside Lorentzian manifolds. Indeed, let (N,g) be a Lorentzian manifold and $M \subseteq N$ an embedded null hypersurface (that is, for every $p \in M$ we have $T_pM \cap \nu_pM \neq 0$). Suppose $\xi \in \mathfrak{X}(M)$ is a generator of the normal distribution $\nu(M)$ (so that ξ is also tangent to M at every point). Then, if h is the pullback of g to M, one can see that the triple (M, ξ, h) is a Carrollian manifold.

Remark 7.4. Let us make precise the notion that the Galilei and Carroll algebra are obtained as limits of the Poincaré algebra. In order to do this, we need to introduce the idea of contraction of a Lie algebra.

Take the vector space $E = \mathbb{R}^n$. The set of Lie algebra structures on E can be identified as the algebraic variety $\mathcal{J} \subseteq \Lambda^2 E^* \otimes E$ consisting of all skew-symmetric bilinear products on E that satisfy the Jacobi identity. Being an algebraic variety, it is clear that $\mathcal{J}$ is closed in $\Lambda^2 E^* \otimes E$ with respect to the usual topology. The variety $\mathcal{J}$ has a natural action of $\mathsf{GL}(E)$, where for each $\lambda \in \mathcal{J}$ and $g \in \mathsf{GL}(E)$ we let

$$(g \cdot \lambda)(v, w) = g\lambda(g^{-1}v, g^{-1}w).$$

It is clear that the isomorphism classes of n-dimensional real Lie algebras is in a bijective correspondence with the orbits of the action $\mathsf{GL}(E) \curvearrowright \mathcal{J}$. Now, consider two Lie algebra structures λ, $\mu \in \mathcal{J}$. We say that λ *degenerates* to μ if μ lies in the closure of $\mathsf{GL}(E) \cdot \lambda$ in $\mathcal{J}$ (with respect to the usual topology). Moreover, suppose that $\lambda \in \mathcal{J}$ and $g \colon (0,1] \to \mathsf{GL}(E)$ is a continuous map. If the curve $t \in (0,1] \mapsto g(t) \cdot \lambda \in \mathcal{J}$ has a limit when $t \to 0$, we say that $\mu = \lim_{t \to 0^+} g(t) \cdot \lambda$ is a *contraction* of λ. Obviously, a contraction of a Lie algebra is also a degeneration of it.

Now, let us obtain the Galilei algebra as a contraction of the Poincaré algebra. For ease of notation, we write $\mathfrak{poin}_c(d)$ to denote the Poincaré algebra with parameter c (recall that all of these are conjugate in $\mathsf{Aff}(d+1, \mathbb{R})$). We define for each $c > 0$ a linear isomorphism

[3] However, other particles such as Minkowski tachyons can move in this spacetime—see [65].

$\phi_c\colon \mathfrak{poin}(d) \to \mathfrak{poin}_c(d)$ by the formula

$$\phi_c \begin{pmatrix} X & v & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X & v & a \\ c^{-2}v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix}.$$

We also consider the Lie bracket $[\cdot,\cdot]_c$ on $\mathfrak{poin}(d)$ that makes ϕ_c an isomorphism. It is not hard to show that $[\cdot,\cdot]_c = g_c \cdot [\cdot,\cdot]$, where $g_c \in \mathsf{GL}(\mathfrak{poin}(d))$ is given by

$$g_c \begin{pmatrix} X & v & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X & cv & a \\ cv^t & 0 & c^{-1}s \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly, by letting $c \to \infty$, the brackets $[\cdot,\cdot]_c$ converge to a Lie bracket $[\cdot,\cdot]_\infty$ on $\mathfrak{poin}(d)$ that makes the map

$$\phi_\infty\colon \begin{pmatrix} X & v & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{poin}(d) \mapsto \begin{pmatrix} X & v & a \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gal}(d)$$

a Lie algebra isomorphism. Thus, the Galilei algebra is a contraction of the Poincaré algebra.

To obtain the Carroll algebra, the procedure is similar. For each $c > 0$, one takes the Lie bracket $[\cdot,\cdot]^c$ on $\mathfrak{poin}(d)$ that makes the vector space isomorphism

$$\psi_c \begin{pmatrix} X & v & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X & c^2 v & a \\ v & 0 & s \\ 0 & 0 & 0 \end{pmatrix}.$$

also a Lie algebra isomorphism. Then we have $[\cdot,\cdot]^c = h_c \cdot [\cdot,\cdot]$ for the linear isomorphism $h_c \in \mathsf{GL}(\mathfrak{poin}(d))$ defined by

$$h_c \begin{pmatrix} X & v & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X & c^{-1}v & a \\ c^{-1}v^t & 0 & c^{-1}s \\ 0 & 0 & 0 \end{pmatrix}.$$

The brackets $[\cdot,\cdot]^c$ converge when $c \to 0$ to a Lie bracket $[\cdot,\cdot]^0$ on $\mathfrak{poin}(d)$ that makes the linear isomorphism

$$\psi_0\colon \begin{pmatrix} X & v & a \\ v^t & 0 & s \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{poin}(d) \mapsto \begin{pmatrix} X & 0 & a \\ v & 0 & s \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{car}(d)$$

a Lie algebra isomorphism as well. Therefore, the Carroll algebra is also obtained as a contraction of the Poincaré algebra.

7.2 Spatially isotropic homogeneous spacetimes

In this section we present the definition and classification of spatially isotropic homogeneous spacetimes.

By a *spatially isotropic kinematical Klein pair* (or simply *kinematical Klein pair*), we mean an infinitesimal Klein pair $(\mathfrak{g}, \mathfrak{k})$ in which $\mathfrak{g}$ is a $(1,2)$-kinematical algebra, $\mathfrak{k} \subseteq \mathfrak{g}$ contains the rotational subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$ of $\mathfrak{g}$, and $\mathfrak{k} = \mathfrak{so}(d) \oplus \mathbb{R}^d$ as an $\mathfrak{so}(d)$-module. In addition, a *spatially isotropic homogeneous spacetime* is an almost effective homogeneous manifold $M = \mathsf{G}/\mathsf{K}$ whose associated Klein pair $(\mathfrak{g}, \mathfrak{k})$ satisfies the above condition.

Observe that for a kinematical Klein pair $(\mathfrak{g}, \mathfrak{k})$, we may find a basis

$$\mathfrak{B} = \{J_{\alpha\beta}, P_\alpha, B_\alpha, H \colon 1 \le \alpha, \beta \le n, \alpha \ne \beta\}, \tag{7.2}$$

such that the vectors $J_{\alpha\beta}$ span the rotational subalgebra of $\mathfrak{g}$, the vector subspaces

$$\operatorname{span}\{P_\alpha \colon 1 \le \alpha \le d\} \quad \text{and} \quad \operatorname{span}\{B_\alpha \colon 1 \le \alpha \le d\}$$

are standard representations of $\mathfrak{so}(d)$, $\mathbb{R}H$ commutes with $\mathfrak{so}(d)$, and

$$\mathfrak{k} = \operatorname{span}\{J_{\alpha\beta}, B_\alpha \colon 1 \le \alpha, \beta \le d\}.$$

Furthermore, the Lie bracket of $\mathfrak{g}$ satisfies the following equations involving elements of $\mathfrak{B}$:

$$\begin{aligned}
[J_{\alpha\beta}, J_{\mu\nu}] &= \delta_{\beta\mu} J_{\alpha\nu} - \delta_{\alpha\mu} J_{\beta\nu} - \delta_{\beta\nu} J_{\alpha\mu} + \delta_{\alpha\nu} J_{\beta\mu},\\
[J_{\alpha\beta}, P_\mu] &= \delta_{\beta\mu} P_\alpha - \delta_{\alpha\mu} P_\beta,\\
[J_{\alpha\beta}, B_\mu] &= \delta_{\beta\mu} B_\alpha - \delta_{\alpha\mu} B_\beta,\\
[J_{\alpha\beta}, H] &= 0.
\end{aligned} \tag{7.3}$$

Moreover, after fixing a choice of basis $\mathfrak{B}$, one can define two linear isomorphisms Π, Θ of $\mathfrak{g}$ as follows:

$$\begin{aligned}
&\Pi(J_{\alpha\beta}) = J_{\alpha\beta}, && \Pi(B_\alpha) = -B_\alpha, && \Pi(P_\alpha) = -P_\alpha, && \Pi(H) = H,\\
&\Theta(J_{\alpha\beta}) = J_{\alpha\beta}, && \Theta(B_\alpha) = -B_\alpha, && \Theta(P_\alpha) = P_\alpha, && \Theta(H) = -H.
\end{aligned}$$

The maps Π and Θ are known respectively as the *parity* and *time reversal* transformations associated with the Klein pair $(\mathfrak{g}, \mathfrak{k})$.

The classification of spatially isotropic homogeneous spacetimes is the result of a series of papers culminating in [66]. This is achieved in three steps:

(1) Firstly, one has to determine all $(1,2)$-KLAs up to isomorphism. By itself, this problem has been treated quite extensively. Indeed, the original work by Bacry and Lévy-Leblond [12] classifies $(1,2)$-KLAs with $d = 3$ and under the further assumption that the parity and time reversal maps are also Lie algebra automorphisms. However, this extra hypothesis is *by no means compelling* [12], and was later lifted in subsequent work by Bacry and Nuyts [13]. Afterwards, using techniques involving deformation theory of Lie algebras (see [131] for an introduction to the topic), the classification of $(1,2)$-KLAs in all dimensions was obtained (see [62] for the case $d = 3$, which recovers the work of Bacry and Nuyts, [61] for the case $d > 3$ and [8] for the case $d = 2$).

(2) The second step consists in determining all kinematical Klein pairs associated with each $(1,2)$-kinematical algebra. If we decompose a $(1,2)$-KLA $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathbb{R}$, where $V = \mathbb{R}^d$ and $W = \mathbb{R}^2$ is the trivial representation of $\mathfrak{so}(V)$, then this amounts to finding all one-dimensional subspaces ℓ of W for which $\mathfrak{k} = \mathfrak{so}(V) \oplus (V \otimes \ell)$ is a Lie subalgebra of $\mathfrak{g}$. This calculation is performed in [66, Section 4.1].

(3) Finally, one has to determine which of the Klein pairs $(\mathfrak{g}, \mathfrak{k})$ obtained in the previous step are both effective and geometrically realizable. On the one hand, checking effectiveness is quite straightforward, as one easily sees that the pair $(\mathfrak{g}, \mathfrak{k})$ is not effective if and only if $V \otimes \ell$ is an ideal of $\mathfrak{g}$ (in the above notation). On the other hand, there are no known systematic approaches to determining when a general Klein pair $(\mathfrak{g}, \mathfrak{k})$ is geometrically realizable, making this issue harder to settle. Of course, if G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and K is the connected subgroup of G with Lie algebra $\mathfrak{k}$, then $(\mathfrak{g}, \mathfrak{k})$ is geometrically realizable if and only if K is closed in G.[4] However, this requires us to have an explicit description of G, which is not always easy. At any rate, it turns out that all effective kinematical Klein pairs are geometrically realizable. This is proved in [66, Section 4.2] after performing a case by case study.

We are now in a position to state the classification of homogeneous spacetimes with full spatial isotropy up to coverings:

Theorem 7.5 [66]. *Let $M = \mathsf{G}/\mathsf{K}$ be a simply connected spatially isotropic homogeneous spacetime of dimension $d+1$ and consider its associated kinematical Klein pair $(\mathfrak{g}, \mathfrak{k})$. Then $(\mathfrak{g}, \mathfrak{k})$ is isomorphic to a Klein pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{k}})$ such that:*

- *as a vector space, $\bar{\mathfrak{g}} = \operatorname{span}\{J_{\alpha\beta}, P_\alpha, B_\alpha, H \colon 1 \leq \alpha, \beta \leq d\}$,*
- *the subalgebra $\bar{\mathfrak{k}} = \operatorname{span}\{J_{\alpha\beta}, B_\alpha \colon 1 \leq \alpha, \beta \leq d\}$, and*
- *the Lie bracket of $\bar{\mathfrak{g}}$ is determined by* (7.3) *and exactly one of the possibilities appearing in Table 7.1.*

In particular, M is isomorphic to the simply connected homogeneous spacetime $\overline{M}$ associated with $(\bar{\mathfrak{g}}, \bar{\mathfrak{k}})$.

A remarkable feature about these spacetimes is that (except for the last four two-dimensional examples in Table 7.1) they always carry an invariant Riemannian, Galilean, Lorentzian or Carrollian structure. We refer the reader to [66, Figures 3, 4 and 5] for the pictures describing all possible limits between the simply connected spatially isotropic homogeneous spacetimes.

[4] In general, checking whether a connected subgroup of a Lie group is closed is also a hard problem, see [92, Corollary 14.5.6] for some criteria.

Table 7.1: Simply connected spatially isotropic homogeneous spacetimes.

d	**Nonzero brackets in addition to** (7.3)					**Comments**
≥ 1	$[H, B_\alpha] = -P_\alpha$		$[B_\alpha, B_\beta] = J_{\alpha\beta}$	$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$		Minkowski
≥ 2	$[H, B_\alpha] = -P_\alpha$	$[H, P_\alpha] = -B_\alpha$	$[B_\alpha, B_\beta] = J_{\alpha\beta}$	$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$	$[P_\alpha, P_\beta] = -J_{\alpha\beta}$	de Sitter
≥ 1	$[H, B_\alpha] = -P_\alpha$	$[H, P_\alpha] = B_\alpha$	$[B_\alpha, B_\beta] = J_{\alpha\beta}$	$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$	$[P_\alpha, P_\beta] = J_{\alpha\beta}$	Anti de Sitter
≥ 1	$[H, B_\alpha] = P_\alpha$		$[B_\alpha, B_\beta] = -J_{\alpha\beta}$	$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$		Euclidean space $\mathbb{R}^{d+1}$
≥ 1	$[H, B_\alpha] = P_\alpha$	$[H, P_\alpha] = -B_\alpha$	$[B_\alpha, B_\beta] = -J_{\alpha\beta}$	$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$	$[P_\alpha, P_\beta] = -J_{\alpha\beta}$	Sphere S^{d+1}
≥ 1	$[H, B_\alpha] = P_\alpha$	$[H, P_\alpha] = B_\alpha$	$[B_\alpha, B_\beta] = -J_{\alpha\beta}$	$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$	$[P_\alpha, P_\beta] = J_{\alpha\beta}$	Real hyperbolic space $\mathbb{R}\mathsf{H}^{d+1}$
≥ 1	$[H, B_\alpha] = -P_\alpha$					Galilei
≥ 1	$[H, B_\alpha] = -P_\alpha$	$[H, P_\alpha] = -B_\alpha$				Galilean de Sitter
≥ 1	$[H, B_\alpha] = -P_\alpha$	$[H, P_\alpha] = \gamma B_\alpha + (1+\gamma) P_\alpha$				Torsional Galilean de Sitter, $\gamma \in (-1, 1]$
≥ 1	$[H, B_\alpha] = -P_\alpha$	$[H, P_\alpha] = B_\alpha$				Galilean anti de Sitter
≥ 1	$[H, B_\alpha] = -P_\alpha$	$[H, P_\alpha] = (1+\chi^2) B_\alpha + 2\chi P_\alpha$				Torsional Galilean anti de Sitter, $\chi > 0$
2	$[H, B_\alpha] = -P_\alpha$	$[H, P_\alpha] = (1+\gamma) P_\alpha - \chi \sum_\beta \epsilon_{\alpha\beta} P_\beta + \gamma B_\alpha - \chi \sum_\beta \epsilon_{\alpha\beta} B_\beta$				$\gamma \in [-1, 1), \chi > 0$
≥ 2				$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$		Carroll
≥ 2		$[H, P_\alpha] = -B_\alpha$		$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$	$[P_\alpha, P_\beta] = -J_{\alpha\beta}$	Carrollian de Sitter
≥ 2		$[H, P_\alpha] = B_\alpha$		$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H$	$[P_\alpha, P_\beta] = J_{\alpha\beta}$	Carrollian anti de Sitter
≥ 1	$[H, B_\alpha] = B_\alpha$	$[H, P_\alpha] = -P_\alpha$		$[B_\alpha, P_\beta] = \delta_{\alpha\beta} H + J_{\alpha\beta}$		Carrollian light cone
1	$[H, B] = -P$			$[B, P] = -H - 2P$		
1	$[H, B] = H$			$[B, P] = -P$		
1	$[H, B] = (1+\chi) H$			$[B, P] = (1-\chi) P$		$\chi > 0$
1	$[H, B] = -P$			$[B, P] = -(1+\chi^2) H - 2\chi P$		$\chi > 0$

The notation $\epsilon_{\alpha\beta}$ denotes the Levi-Civita symbol.

7.3 Aristotelian algebras

By definition, an *Aristotelian Lie algebra* is a $(1,1)$-kinematical Lie algebra. Essentially, one can regard Aristotelian algebras as analogues of $(1,2)$-KLAs with their boosts removed. An *Aristotelian Klein pair* is an infinitesimal Klein pair $(\mathfrak{g},\mathfrak{k})$ where $\mathfrak{g}$ is an Aristotelian Lie algebra and $\mathfrak{k}=\mathfrak{so}(d)$ is its rotational subalgebra. Finally, an almost effective homogeneous manifold $M=\mathsf{G}/\mathsf{K}$ is called an *Aristotelian homogeneous spacetime* if its associated Klein pair $(\mathfrak{g},\mathfrak{k})$ is Aristotelian.

Example 7.6. For every $d\geq 1$, the common intersection of the Galilei, Poincaré and Carroll groups

$$\begin{aligned}\mathsf{Ari}(d)&=\mathsf{Gal}(d)\cap\mathsf{Poin}(d)=\mathsf{Gal}(d)\cap\mathsf{Car}(d)=\mathsf{Poin}(d)\cap\mathsf{Car}(d)\\&=\mathsf{Gal}(d)\cap\mathsf{Poin}(d)\cap\mathsf{Car}(d)\\&=\left\{\begin{pmatrix}R&0&a\\0&1&s\\0&0&1\end{pmatrix}:R\in\mathsf{O}(d),a\in\mathbb{R}^d,s\in\mathbb{R}\right\}\end{aligned}$$

is known as the *Aristotle group*. Its Lie algebra

$$\mathfrak{ari}(d)=\left\{\begin{pmatrix}X&0&a\\0&0&s\\0&0&0\end{pmatrix}:X\in\mathfrak{so}(d),a\in\mathbb{R}^d,s\in\mathbb{R}\right\}$$

is readily seen to be an Aristotelian algebra in the above sense, and it is known as the *static Aristotelian algebra*.

The vector space $\mathbb{R}^{d+1}$ can be realized as the quotient $\mathsf{Ari}(d)/\mathsf{O}(d)$. Furthermore, the absence of boosts implies that, as a homogeneous space of $\mathsf{Ari}(d)$, $\mathbb{R}^{d+1}=\mathbb{R}^d\times\mathbb{R}$ decomposes as the product of the Euclidean space $\mathbb{R}^d$ and the real line $\mathbb{R}$ (that gives an absolute time coordinate). Its associated geometries are known as Aristotelian manifolds. Since the Aristotle group preserves a Riemannian, Galilean, Lorentzian and Carrollian structure at the same time, one can give several equivalent definitions of Aristotelian manifolds in terms of different sets of characteristic tensors, see for example [123, Proposition 2.3]. For instance, let us define them from the purely Riemannian perspective. We say that an *Aristotelian manifold* is a Riemannian manifold M equipped with a unit vector field $Z\in\mathfrak{X}(M)$.

Aristotelian Klein pairs arise naturally as reductions of noneffective kinematical Klein pairs. This is because for a noneffective kinematical Klein pair $(\mathfrak{g},\mathfrak{k})$, if $\mathfrak{B}$ is a basis of $\mathfrak{g}$ taking the form (7.2), the ineffective kernel of this pair is necessarily $\mathfrak{n}=\operatorname{span}\{B_a\colon a=1,\ldots,d\}$. From this, it is clear that the pair $(\mathfrak{g}/\mathfrak{n},\mathfrak{k}/\mathfrak{n})$ is effective and Aristotelian.

Aristotelian spacetimes were classified up to coverings in [66, Appendix A]. In reality, this problem is already equivalent to determining all Aristotelian Lie algebras up to isomorphism. Indeed, if $\mathfrak{g}$ is an Aristotelian Lie algebra with rotational subalgebra $\mathfrak{k}=\mathfrak{so}(d)$, then by definition the only possible Aristotelian Klein pair that can arise from $\mathfrak{g}$ is $(\mathfrak{g},\mathfrak{k})$. The pair $(\mathfrak{g},\mathfrak{k})$ is automatically effective because the definition of (s,v)-KLAs prevents $\mathfrak{k}=\mathfrak{so}(d)$ from containing a

$\mathfrak{g}$-ideal. Moreover, $(\mathfrak{g},\mathfrak{k})$ is geometrically realizable because $\mathfrak{k}=\mathfrak{so}(d)$ is a compact semisimple Lie algebra (for $d\geq 3$).

We can now state the classification theorem for simply connected Aristotelian spacetimes (equivalently, of Aristotelian algebras):

Theorem 7.7. *Let $M=\mathsf{G}/\mathsf{K}$ be a simply connected Aristotelian homogeneous spacetime of dimension $d+1$ with associated Klein pair $(\mathfrak{g},\mathfrak{k})$. Then $(\mathfrak{g},\mathfrak{k})$ is isomorphic to a Klein pair $(\bar{\mathfrak{g}},\bar{\mathfrak{k}})$ such that:*

- *the Lie algebra $\bar{\mathfrak{g}}$ possesses a basis of the form*

$$\mathfrak{B}=\{J_{\alpha\beta},P_\alpha,H\colon 1\leq\alpha,\beta\leq d,\alpha\neq\beta\},$$

- *the subalgebra $\bar{\mathfrak{k}}\cong\mathfrak{so}(d)$ is spanned by $\{J_{\alpha\beta}\colon\alpha\neq\beta\}$,*
- *and the Lie bracket of $\bar{\mathfrak{g}}$ is determined by the following equations:*

$$\begin{aligned}[][J_{\alpha\beta},J_{\mu\nu}]&=\delta_{\beta\mu}J_{\alpha\nu}-\delta_{\alpha\mu}J_{\beta\nu}-\delta_{\beta\nu}J_{\alpha\mu}+\delta_{\alpha\nu}J_{\beta\mu},\\ [J_{\alpha\beta},P_\mu]&=\delta_{\beta\mu}P_\alpha-\delta_{\alpha\mu}P_\beta,\\ [J_{\alpha\beta},H]&=0,\end{aligned}\tag{7.4}$$

  together with exactly one of the possibilities appearing in Table 7.2.

In particular, M is isomorphic to the simply connected Aristotelian homogeneous spacetime $\overline{M}$ associated with the Klein pair $(\bar{\mathfrak{g}},\bar{\mathfrak{h}})$.

Table 7.2: Simply connected Aristotelian spacetimes.

d	**Nonzero brackets in addition to** (7.4)		**Comments**
≥ 0			Static
≥ 1	$[H,P_\alpha]=P_\alpha$		Torsional static
≥ 2		$[P_\alpha,P_\beta]=J_{\alpha\beta}$	$\mathbb{R}\times\mathbb{R}\mathsf{H}^d$
≥ 2		$[P_\alpha,P_\beta]=-J_{\alpha\beta}$	$\mathbb{R}\times\mathsf{S}^d$
2		$[P_\alpha,P_\beta]=\delta_{\alpha\beta}H$	Heisenberg group

7.4 Lifshitz algebras

We say that a *Lifshitz Lie algebra* is a $(2,1)$-kinematical Lie algebra. Moreover, a $(2,1)$-kinematical Lie group is called a *Lifshitz Lie group*. Note that by definition a Lifshitz algebra $\mathfrak{g}$ with rotational subalgebra $\mathfrak{so}(d)$ admits the $\mathfrak{so}(d)$-module decomposition $\mathfrak{g}=\mathfrak{so}(d)\oplus\mathbb{R}^d\oplus\mathbb{R}\oplus\mathbb{R}$.

Example 7.8. For each $d \geq 1$, we define the *Lifshitz spacetime* as the smooth manifold $M = (0,\infty) \times \mathbb{R} \times \mathbb{R}^d$ (with global coordinates $(r, t, x_1, \ldots, x_d)$) endowed with the metric $g = -r^{-2z}dt^2 + r^{-2}(dr^2 + \sum_{i=1}^{d} dx_i^2)$. It can be shown that the group $I(M)$ of isometries of M acts transitively on M. Furthermore, for generic values of z its Lie algebra $\mathfrak{i}(M)$ admits the basis $\{J_{\alpha\beta}, P_\alpha, H, D \colon 1 \leq \alpha, \beta \leq d, \alpha \neq \beta\}$, where the Killing vector fields associated with the elements of this basis are

$$J^*_{\alpha\beta} = -x_\alpha \partial_{x_\beta} + x_\beta \partial_{x_\alpha}, \qquad P^*_\alpha = \partial_{x_\alpha},$$
$$H^* = \partial_t, \qquad D^* = r\partial_r + \sum_{i=1}^{d} x_i \partial_{x_i} + zt\partial_t.$$

A direct calculation shows that $\mathfrak{i}(M)$ is a Lifshitz Lie algebra, where the rotational subalgebra $\mathfrak{so}(d)$ is spanned by the vectors $J_{\alpha\beta}$, the vectors P_α become its standard representation under the adjoint action, and $\mathrm{span}\{H, D\}$ commutes with $\mathfrak{so}(d)$.

The Klein pairs of the form $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is a Lifshitz algebra and $\mathfrak{k} = \mathfrak{so}(d)$ is its rotational subalgebra, are known as *spatially isotropic Lifshitz* Klein pairs, and the same argument as in the Aristotelian case implies that they are automatically effective and geometrically realizable. An effective geometric realization $M^{d+2} = \mathsf{G}/\mathsf{K}$ of such a pair is known as a *spatially isotropic homogeneous Lifshitz spacetime*.

The classification of Lifshitz Lie algebras and their corresponding homogeneous spacetimes (up to coverings) has been carried out by Figueroa-O'Farrill, Grassie and Prohazka [64]. More precisely, we have:

Theorem 7.9. *Let $M = \mathsf{G}/\mathsf{K}$ be a simply connected spatially isotropic homogeneous Lifshitz spacetime and consider its associated Klein pair $(\mathfrak{g}, \mathfrak{k})$. Then $(\mathfrak{g}, \mathfrak{k})$ is isomorphic to a Klein pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{k}})$ for which:*

- *the Lie algebra $\bar{\mathfrak{g}}$ admits a basis of the form*

$$\mathfrak{B} = \{J_{\alpha\beta}, P_\alpha, H, D \colon 1 \leq \alpha, \beta \leq d, \alpha \neq \beta\},$$

- *the subalgebra $\bar{\mathfrak{k}} = \mathfrak{so}(d)$ is spanned by the elements $J_{\alpha\beta}$, and*
- *the Lie bracket of $\bar{\mathfrak{g}}$ is determined by the equations*

$$\begin{aligned} [J_{\alpha\beta}, J_{\mu\nu}] &= \delta_{\beta\mu} J_{\alpha\nu} - \delta_{\alpha\mu} J_{\beta\nu} - \delta_{\beta\nu} J_{\alpha\mu} + \delta_{\alpha\nu} J_{\beta\mu}, \\ [J_{\alpha\beta}, P_\mu] &= \delta_{\beta\mu} P_\alpha - \delta_{\alpha\mu} P_\beta, \\ [J_{\alpha\beta}, H] &= [J_{\alpha\beta}, D] = 0, \end{aligned} \tag{7.5}$$

together with exactly one of the possibilities described in Table 7.3.

In particular, M is isomorphic as a homogeneous space to the simply connected Lifshitz spacetime $\overline{M}$ associated with the Klein pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$.

Table 7.3: Simply connected Lifshitz spacetimes.

d	**Nonzero brackets in addition to** (7.5)			**Comments**
≥ 2				$\mathbb{R}^d \times \mathbb{R}^2$
≥ 2	$[D, H] = H$			$\mathsf{Aff}(1) \times \mathbb{R}$
≥ 2		$[D, P_\alpha] = zP_\alpha$		$\mathsf{TS} \times \mathbb{R}$
≥ 2	$[D, H] = zH$	$[D, P_\alpha] = zP_\alpha$		Lifshitz spacetime, $z \neq 0$
≥ 2			$[P_\alpha, P_\beta] = J_{\alpha\beta}$	$\mathbb{R}\mathsf{H}^d \times \mathbb{R}^2$
≥ 2			$[P_\alpha, P_\beta] = -J_{\alpha\beta}$	$\mathsf{S}^d \times \mathbb{R}^2$
≥ 2	$[D, H] = H$		$[P_\alpha, P_\beta] = J_{\alpha\beta}$	$\mathbb{R}\mathsf{H}^d \times \mathsf{Aff}(1)$
≥ 2	$[D, H] = H$		$[P_\alpha, P_\beta] = -J_{\alpha\beta}$	$\mathsf{S}^d \times \mathsf{Aff}(1)$
2			$[P_\alpha, P_\beta] = \epsilon_{\alpha\beta} H$	$\mathsf{N} \times \mathbb{R}$
2	$[D, H] = 2H$	$[D, P_\alpha] = P_\alpha$	$[P_\alpha, P_\beta] = \epsilon_{\alpha\beta} H$	N-bundle over $\mathbb{R}$

Here, TS denotes the torsional static Aristotelian spacetime and N is the three-dimensional Heisenberg group.

Observe that a direct consequence of the classification in Table 7.3 is that every Lifshitz Lie algebra $\mathfrak{g}$ can be decomposed as a semidirect product $\mathbb{R}D \ltimes \mathfrak{h}$ of an Aristotelian algebra with the one-dimensional subspace generated by D, and the action of $\mathrm{ad}(D)$ on $\mathfrak{h}$ is diagonalizable, meaning that D is a *grading element* for $\mathfrak{h}$. Moreover, one can see that all of these homogeneous spacetimes carry a G-invariant Lorentzian inner product.

Remark 7.10. There are two further classes of homogeneous spacetimes that we can associate to Lifshitz algebras, which are obtained by considering isotropy subalgebras of the form $\mathfrak{k} = \mathfrak{so}(d) \oplus \mathbb{R}$. The $(d+1)$-dimensional spacetimes for which the scalar part $\mathbb{R}$ acts effectively are known as *Lifshitz–Weyl* spacetimes, whereas those for which $\mathbb{R}$ is not effective reduce to Aristotelian spacetimes (and the generator D of $\mathbb{R}$ is known as a *scalar charge*). These manifolds have also been determined up to coverings in [64], see Tables 3 and 4 in the aforementioned paper for their classification.

Chapter 8

Classification of $(3,2)$-kinematical Lie algebras

The purpose of this chapter is to derive the classification of $(3,2)$-kinematical algebras with spatial isotropy of dimension larger than two up to relative isomorphism. The results presented here correspond to an ongoing work with José Miguel Figueroa-O'Farrill (University of Edinburgh, United Kingdom), in which we aim to classify all simply connected coisotropy one homogeneous spacetimes.

Recall from Chapter 7 that spatially isotropic homogeneous spacetimes have been fully classified up to coverings. These spacetimes have the key property that their symmetry algebras contain all spatial rotations. Therefore, a natural continuation of this programme is to lift the assumption of spatial isotropy, and the clear first step is to consider coisotropy one spacetimes, which are those admitting a spatial direction that breaks the rotational symmetry.

Our long term goal is to understand $(d+2)$-dimensional homogeneous spacetimes whose symmetry algebras are formed by spatial translations, time translations, inertial boosts and space rotations preserving a fixed direction. This means that if $\mathfrak{g}$ is the symmetry algebra of such a spacetime, then it possesses a rotational subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$. Moreover, under the adjoint representation of $\mathfrak{r}$, both spatial translations and boosts must decompose as the sum $\mathbb{R}^d \oplus \mathbb{R}$ of the standard representation (corresponding to all vectors orthogonal to our distinguished direction) with the trivial one (corresponding to the line generated by this direction), while time translations still form a one-dimensional trivial representation. Taking all of this into account, we conclude that $\mathfrak{g}$ takes the form

$$\mathfrak{g} = \mathfrak{so}(d) \oplus \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R},$$

meaning that $\mathfrak{g}$ is a $(3,2)$-kinematical Lie algebra. The isotropy algebra $\mathfrak{k}$, being comprised of rotations and boosts, must admit the $\mathfrak{so}(d)$-module decomposition $\mathfrak{k} = \mathfrak{so}(d) \oplus \mathbb{R}^d \oplus \mathbb{R}$.

From the above considerations, we can now give a precise definition of coisotropy one homogeneous spacetimes. We say that a (spatial) *coisotropy one* Klein pair is a Klein pair $(\mathfrak{g}, \mathfrak{k})$ in which $\mathfrak{g}$ is a $(3,2)$-kinematical Lie algebra, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{so}(d)$, and $\mathfrak{k} = \mathfrak{so}(d) \oplus \mathbb{R}^d \oplus \mathbb{R}$ as a representation of $\mathfrak{so}(d)$. Moreover, a *coisotropy one* homogeneous spacetime is a $(d+2)$-dimensional (almost effective, connected) homogeneous manifold $M = \mathsf{G}/\mathsf{K}$ whose associated Klein pair $(\mathfrak{g}, \mathfrak{k})$ is of coisotropy one. This is a much richer class of spacetime geometries and their classification becomes substantially harder.

In order to classify coisotropy one homogeneous spacetimes up to coverings, we follow a similar strategy as with spatially isotropic spacetimes (see Section 7.2). Therefore, the initial part of this process consists in classifying all $(3,2)$-kinematical Lie algebras.

Our first objective in this chapter is to obtain general expressions for the Lie bracket of a $(3,2)$-KLA with d-dimensional spatial isotropy. Observe that for $d = 1$, a $(3,2)$-KLA is merely a five-dimensional real Lie algebra. These algebras have been classified up to isomorphism in [128]. In this chapter we determine the Lie bracket of a $(3,2)$-KLA with $d \geq 3$. The case $d = 2$ turns out to be significantly more involved than $d \geq 3$, and we intend to tackle this case separately in the near future.

The existence of the exceptional isomorphism $\mathfrak{so}(3) \cong \mathbb{R}^3$ between the adjoint and standard representations of $\mathfrak{so}(3)$ requires us to consider two cases separately: the generic case $d > 3$ and the case $d = 3$. We write an arbitrary $(3,2)$-KLA as $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$, where $V = \mathbb{R}^d$ is the d-dimensional Euclidean space endowed with its usual inner product $\langle \cdot, \cdot \rangle$, $W = \mathbb{R}^2$ is the two-dimensional trivial representation of $\mathfrak{so}(V)$ and $\mathfrak{b} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ is the three-dimensional trivial representation. Then from the definition of kinematical algebra we know that the Lie bracket of $\mathfrak{g}$ satisfies

$$[X, Y] = XY - YX, \qquad [X, v \otimes w] = Xv \otimes w, \qquad [X, B] = 0,$$

for every X, $Y \in \mathfrak{so}(V)$, $v \in V$, $w \in W$ and $B \in \mathfrak{b}$. Thus, in order to fully determine the Lie bracket of $\mathfrak{g}$ we need to know its restriction to $\mathfrak{b} \times \mathfrak{b}$, $\mathfrak{b} \times (V \otimes W)$ and $(V \otimes W) \times (V \otimes W)$.

In the following, we state the main results of this chapter. We recall that for v_1, $v_2 \in V$, the operator $v_1 \curlywedge v_2 \in \mathfrak{so}(V)$ is defined by $(v_1 \curlywedge v_2)v = \langle v_2, v \rangle v_1 - \langle v_1, v \rangle v_2$.

For generic values of d, we may encode the Lie bracket of a $(3,2)$-KLA by means of Theorem A.

Theorem A. *Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ be a $(3,2)$-KLA with $\dim V = d > 3$. Then the Lie bracket of $\mathfrak{g}$ is determined by the following algebraic data:*

(i) *a Bianchi[1] Lie algebra structure on $\mathfrak{b}$,*

(ii) *a two-dimensional Lie algebra representation $\rho \colon \mathfrak{b} \to \mathfrak{gl}(W)$,*

(iii) *and a $\mathfrak{b}$-equivariant $\mathfrak{b}$-valued 2-form $\varphi \colon \Lambda^2 W \to \mathfrak{b}$.*

For all $X \in \mathfrak{so}(V)$, $v, v_i \in V$, $w, w_i \in W$ and $B \in \mathfrak{b}$, the Lie algebra structure is given by the brackets

$$\begin{aligned} [X, v \otimes w] &= Xv \otimes w, \\ [X, B] &= 0, \\ [B, v \otimes w] &= v \otimes \rho(B)w, \\ [v_1 \otimes w_1, v_2 \otimes w_2] &= \alpha(w_1, w_2) v_1 \curlywedge v_2 + \langle v_1, v_2 \rangle \varphi(w_1 \wedge w_2), \end{aligned} \tag{8.1}$$

where $\alpha \colon W \times W \to \mathbb{R}$ is the unique $\mathfrak{b}$-invariant symmetric bilinear form given by

$$\rho(\varphi(w_1 \wedge w_2))w_3 = \alpha(w_2, w_3)w_1 - \alpha(w_1, w_3)w_2, \quad w_i \in W. \tag{8.2}$$

[1] A three-dimensional real Lie algebra is known as a *Bianchi algebra*.

Observe that (8.1) defines a $(3, 2)$-KLA structure for $d \leq 3$ as well.

The case $d = 3$ becomes more involved, as one has to take into account two extra terms coming from the exceptional isomorphism $\mathfrak{so}(V) \cong V$ induced by the three-dimensional cross product $\times$ on V. However, we show that one of these can always be brought to zero. This leads to:

Theorem B. *Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ be a $(3, 2)$-KLA with $\dim V = d = 3$. Then the Lie bracket of $\mathfrak{g}$ is determined, after perhaps performing a suitable linear isomorphism that leaves $\mathfrak{so}(V)$ unchanged, by the following algebraic data:*

(i) *a Bianchi Lie algebra structure on $\mathfrak{b}$,*

(ii) *a two-dimensional Lie algebra representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$,*

(iii) *a $\mathfrak{b}$-valued 2-form $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ that is $\mathfrak{b}$-equivariant,*

(iv) *and a $\mathfrak{b}$-equivariant W-valued symmetric bilinear form $\sigma\colon W \times W \to W$.*

Letting $\alpha \in (\mathsf{S}^2 W^*)^{\mathfrak{b}}$ be defined by the relation

$$\begin{aligned} \alpha(w_2, w_3)w_1 - \alpha(w_1, w_3)w_2 = \rho(\varphi(w_1 \wedge w_2))w_3 \\ + \sigma(\sigma(w_2, w_3), w_1) - \sigma(\sigma(w_1, w_3), w_2), \end{aligned} \tag{8.3}$$

we have that α, σ and φ must satisfy the following equations:

$$\begin{aligned} 0 &= \alpha(\sigma(w_1, w_3), w_2) - \alpha(\sigma(w_2, w_3), w_1), \\ 0 &= \varphi(\sigma(w_1, w_2) \wedge w_3 + \sigma(w_2, w_3) \wedge w_1 + \sigma(w_3, w_1) \wedge w_2). \end{aligned} \tag{8.4}$$

for all $w_i \in W$. The Lie algebra structure is given by the brackets

$$\begin{aligned} [X, v \otimes w] &= Xv \otimes w, \\ [X, B] &= 0, \\ [B, v \otimes w] &= v \otimes \rho(B)w, \\ [v_1 \otimes w_1, v_2 \otimes w_2] &= \alpha(w_1, w_2)v_1 \curlywedge v_2 + \langle v_1, v_2 \rangle \varphi(w_1 \wedge w_2) \\ &\quad + (v_1 \times v_2) \otimes \sigma(w_1, w_2), \end{aligned} \tag{8.5}$$

for all $X \in \mathfrak{so}(V)$, $v, v_i \in V$, $w, w_i \in W$ and $B \in \mathfrak{b}$.

As a consequence of Theorems A and B, we see that in order to classify all $(3, 2)$-kinematical algebras with $d \geq 3$ we need to know all (real) two-dimensional representations of Bianchi Lie algebras as well as some of their low-order invariants. For the purpose of classifying $(3, 2)$-KLAs, it is actually enough to classify these representations up to a weaker notion of equivalence than the usual one. More precisely, we say that two representations $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$, $\rho'\colon \mathfrak{b} \to \mathfrak{gl}(W')$ of a Lie algebra $\mathfrak{b}$ are called weakly equivalent if there exist a linear isomorphism $f\colon W \to W'$ and a Lie algebra automorphism $\psi \in \mathrm{Aut}(\mathfrak{b})$ such that $f(\rho(X)w) = \rho'(\psi(X))f(w)$ for all $X \in \mathfrak{b}$ and $w \in W$. Observe that the usual notion of equivalence between representations is recovered by taking $\psi = \mathrm{id}_{\mathfrak{b}}$ in the previous definition.

While the representation theory of real semisimple Lie algebras has been treated extensively (see for example [138]), only the Bianchi algebras $\mathfrak{b}_{\text{VIII}} = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{b}_{\text{IX}} = \mathfrak{su}(2)$ are semisimple (in fact, simple), see Table 8.5. The remaining seven families in the Bianchi classification consist of solvable Lie algebras, whose representations are less known due to their sheer abundance and lack of structure results akin to the ones in the semisimple case. In Section 8.4 we find all two-dimensional representations of Bianchi algebras up to weak equivalence.

The last part of this chapter is devoted to classifying all $(3,2)$-KLAs with $d \geq 3$ up to relative isomorphism (that is, an isomorphism that acts trivially on the rotational subalgebra). Note that if $\mathfrak{b}$ is a Bianchi algebra and $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ is a representation of $\mathfrak{b}$, then $V \otimes W$ is a representation of the direct sum $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ and we can define a $(3,2)$-KLA structure on $\mathfrak{g} = \mathfrak{h} \oplus (V \otimes W)$ by imposing that $\mathfrak{h}$ is a subalgebra, $V \otimes W$ is an abelian subspace, and

$$[X, v \otimes w] = Xv \otimes w, \qquad\qquad [B, v \otimes w] = v \otimes \rho(B)w$$

for all $X \in \mathfrak{so}(V)$, $v \in V$, $w \in W$ and $B \in \mathfrak{b}$. This is known as the split abelian extension of $\mathfrak{h}$ by $V \otimes W$.

In order to give the classification results, we take a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ of W together with a basis $\{B_1, B_2, B_3\}$ of $\mathfrak{b}$. We also consider the Levi-Civita symbol ϵ_{ij} in two dimensions, which is defined by the conditions $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = 1$.

On the one hand, for $d > 3$, we obtain:

Theorem C. *Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ be a $(3,2)$-KLA for which $\dim V = d > 3$ and consider the algebraic data $(\mathfrak{b}, \rho, \varphi, \alpha)$ associated with $\mathfrak{g}$. Then, exactly one of the following situations occurs:*

- (I) *The subspace $V \otimes W$ is abelian (that is, $\varphi = 0$). In this case, $\mathfrak{g}$ is relatively isomorphic to the split abelian extension of $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ by $V \otimes W$ induced by exactly one of the representations given in Section 8.4.*
- (II) *The subspace $[V \otimes W, V \otimes W]$ is a nonzero subspace of $\mathfrak{b}$ (that is, $\varphi \neq 0$ but $\alpha = 0$). In this case, $\mathfrak{g}$ is relatively isomorphic to exactly one of the $(3,2)$-KLAs described in Table 8.13.*
- (III) *The subspace $[V \otimes W, V \otimes W]$ has nontrivial projection both onto $\mathfrak{so}(V)$ and $\mathfrak{b}$ (that is, $\alpha \neq 0$). In this case, $\mathfrak{g}$ is relatively isomorphic to exactly one of the $(3,2)$-KLAs described in Table 8.14.*

On the other hand, in the case that V is three-dimensional, the classification takes the following form:

Theorem D. *Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ be a $(3,2)$-KLA for which $\dim V = d = 3$ and consider the algebraic data $(\mathfrak{b}, \rho, \varphi, \alpha, \sigma)$ associated with $\mathfrak{g}$. Then, up to relative isomorphism, exactly one of the following situations occurs:*

- (I) *The subspace $V \otimes W$ is abelian (that is, φ and σ vanish). In this case, $\mathfrak{g}$ is relatively isomorphic to the split abelian extension of $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ by $V \otimes W$ obtained from exactly one of the representations described in Section 8.4.*

(II) *The subspace $[V\otimes W, V\otimes W]$ is a nonzero subspace of $\mathfrak{b}$ (that is, σ and α are zero whereas $\varphi \neq 0$). In this case, $\mathfrak{g}$ is relatively isomorphic to exactly one of the $(3,2)$-KLAs appearing in Table 8.13.*

(III) *The subspace $[V\otimes W, V\otimes W]$ is contained in $\mathfrak{h} = \mathfrak{so}(V)\oplus\mathfrak{b}$ and has nontrivial projection both onto $\mathfrak{so}(V)$ and $\mathfrak{b}$ (that is, $\sigma = 0$ whereas $\alpha \neq 0$). In this case, $\mathfrak{g}$ is relatively isomorphic to exactly one of the $(3,2)$-KLAs appearing in Table 8.14.*

(IV) *The subspace $V\otimes W$ is a nonabelian ideal of $\mathfrak{g}$ (that is, $\varphi = 0$ whereas $\sigma \neq 0$) and $\mathfrak{g}$ is not relatively isomorphic to a split abelian extension. In this case, $\mathfrak{g}$ is relatively isomorphic to exactly one of the $(3,2)$-KLAs appearing in Table 8.15.*

The algebras in class (IV) are exclusive to $d = 3$.

Having obtained all $(3,2)$-kinematical algebras with $d \geq 3$ up to relative isomorphism, the next step in our project is to determine all (effective and geometrically realizable) coisotropy one Klein pairs that one can obtain from these algebras up to relative automorphisms (this is actually sufficient because we have to keep track of the rotational subalgebra).

We now describe the organization of this chapter. In Section 8.1 we analyze the general structure of a $(3,2)$-KLA (with $d \geq 3$) so as to prove Theorems A and B. In Section 8.2 we define the notion of relative isomorphism between $(3,2)$-KLAs and derive criteria for discerning when two $(3,2)$-KLAs are relatively isomorphic in terms of the algebraic data provided in Theorems A and B. In Section 8.3 we determine all Lie subalgebras of the algebra $\mathfrak{gl}(2,\mathbb{R})$ by means of the Goursat Lemma, as knowing these subalgebras is necessary in order to classify two-dimensional representations. Afterwards, in Section 8.4 we recall the classification of Bianchi algebras and compute all of their real two-dimensional representations up to weak equivalence. Finally, Section 8.5 contains the proofs of Theorems C and D.

We have also included an appendix to this chapter (see Section 8.A) in which we study the orbit space of the action $\mathsf{GL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^* \curvearrowright \mathrm{Hom}(\mathsf{S}^2\mathbb{R}^2, \mathbb{R}^2)$ defined by

$$((h,\mu)\cdot\sigma)(x,y) = h\sigma(h^{-1}x, h^{-1}y) - \mu(h^{-1}x)y - \mu(h^{-1}y)x.$$

This action arises in the classification of $(3,2)$-KLAs with $d = 3$, and we will relate the determination of its orbit space to a classical problem in invariant theory.

8.1 Structure of $(3,2)$-kinematical Lie algebras

The aim of this section is to prove general structure theorems for $(3,2)$-KLAs with d-dimensional spatial isotropy (where $d \geq 3$) in order to simplify their classification problem. To this effect, we consider for a d-dimensional Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ the $\mathfrak{so}(V)$-representation

$$\mathfrak{g} = \mathfrak{so}(V)\oplus V\oplus V\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R},$$

where V denotes the standard representation and $\mathbb{R}$ denotes the trivial representation. We remark that for $d > 3$ (the *generic* case), the $\mathfrak{so}(V)$-modules $\mathfrak{so}(V)$, V and $\mathbb{R}$ are pairwise non-isomorphic, but if $d = 3$ then $\mathfrak{so}(V) \cong V$. We also define $W = \mathbb{R}^2$ and $\mathfrak{b} = \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, so that we may rewrite our $(3,2)$-KLA as

$$\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b},$$

where under $\mathrm{ad}_{\mathfrak{so}(V)}$ we have $V \otimes W$ as the tensor product of the standard representation and the two-dimensional trivial representation of $\mathfrak{so}(V)$, and $\mathfrak{b}$ is the three-dimensional trivial representation of $\mathfrak{so}(V)$. Note that for $d > 3$, the isotypical components of $\mathfrak{g}$ as an $\mathfrak{so}(V)$-representation are precisely $\mathfrak{so}(V)$, $V \otimes W$ and $\mathfrak{b}$.

We first treat the generic case and then separately the case $d = 3$, which becomes more complicated, just as with the classification of spatially isotropic kinematical Lie algebras.

8.1.1 Generic case: $d > 3$

We now proceed to the proof of Theorem A. Consider an arbitrary $(3,2)$-KLA $\mathfrak{g}$ for which $d > 3$. In order to understand the Lie bracket of $\mathfrak{g}$, we need to exploit the Jacobi identity, which tells us that the bracket $\Lambda^2\mathfrak{g} \to \mathfrak{g}$ is actually a $\mathfrak{g}$-module homomorphism. In particular, it is also an $\mathfrak{so}(V)$-equivariant map. Because the projection maps

$$\mathfrak{g} \to \mathfrak{so}(V), \qquad \mathfrak{g} \to V \otimes W, \qquad \mathfrak{g} \to \mathfrak{b},$$

are all $\mathfrak{so}(V)$-equivariant, so are their compositions

$$\Lambda^2\mathfrak{g} \to \mathfrak{so}(V), \qquad \Lambda^2\mathfrak{g} \to V \otimes W, \qquad \Lambda^2\mathfrak{g} \to \mathfrak{b},$$

with the bracket.

Let us consider the restriction of the bracket to $\Lambda^2\mathfrak{b}$, which is a three-dimensional trivial $\mathfrak{so}(V)$-module. Because $\mathfrak{so}(V)$ is a nontrivial irreducible representation and all the irreducible subrepresentations of $V \otimes W$ are standard representations, the induced maps $\Lambda^2\mathfrak{b} \to \mathfrak{so}(V)$ and $\Lambda^2\mathfrak{b} \to V \otimes W$ are identically zero. This means that the image of the Lie bracket restricted to $\mathfrak{b}$ is contained in $\mathfrak{b}$. In other words, the subspace $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$.

We now discuss the Lie bracket of elements of $\mathfrak{b}$ with elements of $V \otimes W$. The Jacobi identity, together with the equality $[\mathfrak{so}(V), \mathfrak{b}] = 0$, implies that the maps

$$\lambda = \mathrm{proj}_{\mathfrak{so}(V)} \circ \mathrm{ad}(B)\colon V \otimes W \to \mathfrak{so}(V) \quad \text{and} \quad \xi = \mathrm{proj}_{\mathfrak{b}} \circ \mathrm{ad}(B)\colon V \otimes W \to \mathfrak{b}$$

are $\mathfrak{so}(V)$-equivariant for every $B \in \mathfrak{b}$. If $\lambda \neq 0$, then the restriction of λ to any irreducible invariant subspace of $V \otimes W$ not contained in $\ker\lambda$ induces an isomorphism $V \cong \mathfrak{so}(V)$, which is not possible for $d > 3$, so $\lambda = 0$. Similarly, if $\xi \neq 0$, its restriction to an irreducible invariant subspace not contained in $\ker\xi$ would give an injective map $V \to \mathfrak{b}$, which is not possible for $d > 3$, so $\xi = 0$ as well. All in all, we deduce that $[\mathfrak{b}, V \otimes W] \subseteq V \otimes W$, and we now describe explicitly this bracket.

For any $B \in \mathfrak{b}$, the map $\mathrm{ad}(B)\colon V \otimes W \to V \otimes W$ is $(\mathfrak{so}(V) \oplus \mathfrak{b})$-equivariant. Since V is an irreducible $\mathfrak{so}(V)$-module, equivariance under $\mathfrak{so}(V)$ implies[2] that there exists a linear map

[2] We may use that $\mathrm{ad}(B) \in \mathrm{End}_{\mathfrak{so}(V)}(V \otimes W) = \mathrm{End}_{\mathfrak{so}(V)}(V) \otimes \mathrm{End}(W)$, but since V is an irreducible representation of real type, we have $\mathrm{End}_{\mathfrak{so}(V)}(V) = \mathbb{R}\,\mathrm{id}_V$, from where the result follows.

$\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ such that $[B, v\otimes w] = v\otimes\rho(B)w$ for all $B \in \mathfrak{b}$, $v \in V$ and $w \in W$. Furthermore, the $\mathfrak{b}$-equivariance of the Lie bracket readily implies that ρ is a Lie algebra homomorphism. In summary, we have shown that the Lie bracket restricted to $\mathfrak{b} \times (V \otimes W)$ is governed by a two-dimensional representation of $\mathfrak{b}$.

The only bracket left for us to understand is that of elements in $V \otimes W$ with themselves, which defines an $(\mathfrak{so}(V) \oplus \mathfrak{b})$-equivariant linear map $\Lambda^2(V \otimes W) \to \mathfrak{g}$. Note that $\Lambda^2(V \otimes W)$ admits the $\mathfrak{so}(V)$-module decomposition

$$\Lambda^2(V \otimes W) = \left(\Lambda^2 V \otimes \mathsf{S}^2 W\right) \oplus \left(\mathsf{S}^2_0 V \otimes \Lambda^2 W\right) \oplus \left(\mathbb{R}_{\mathrm{tr}} \otimes \Lambda^2 W\right),$$

where $\mathsf{S}^2 V = \mathsf{S}^2_0 V \oplus \mathbb{R}_{\mathrm{tr}}$ is the decomposition into traceless and trace symmetric 2-tensors; that is, the dual $\mathbb{R}^*_{\mathrm{tr}}$ is the line spanned by the Euclidean inner product in $\mathsf{S}^2 V^*$. Clearly, the decomposition above is also an $(\mathfrak{so}(V) \oplus \mathfrak{b})$-module decomposition. Using this decomposition, we see that the Lie bracket yields an $(\mathfrak{so}(V) \oplus \mathfrak{b})$-equivariant map

$$\left(\Lambda^2 V \otimes \mathsf{S}^2 W\right) \oplus \left(\mathsf{S}^2_0 V \otimes \Lambda^2 W\right) \oplus \left(\mathbb{R}_{\mathrm{tr}} \otimes \Lambda^2 W\right) \to \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}. \tag{8.6}$$

We use the $\mathfrak{so}(V)$-equivariant isomorphism $\Lambda^2 V \to \mathfrak{so}(V)$ which sends $v_1 \wedge v_2 \mapsto v_1 \curlywedge v_2$, where $v_1 \curlywedge v_2 \in \mathfrak{so}(V)$ is the skew-symmetric endomorphism of V defined by

$$(v_1 \curlywedge v_2)v = \langle v, v_2\rangle\, v_1 - \langle v, v_1\rangle\, v_2.$$

The $\mathfrak{so}(V)$-modules $\Lambda^2 V$, V, $\mathsf{S}^2_0 V$ and $\mathbb{R}_{\mathrm{tr}}$ are pairwise non-isomorphic and irreducible for $d \neq 4$. For $d = 4$, $\Lambda^2 V$ decomposes into two irreducible three-dimensional submodules, neither of which is isomorphic to either V, $\mathsf{S}^2_0 V$ or $\mathbb{R}_{\mathrm{tr}}$. This implies that the only nontrivial components of (8.6) are

$$\Lambda^2 V \otimes \mathsf{S}^2 W \to \mathfrak{so}(V) \qquad \text{and} \qquad \mathbb{R}_{\mathrm{tr}} \otimes \Lambda^2 W \to \mathfrak{b}. \tag{8.7}$$

In order to study the expression of this bracket, we need to understand the space of equivariant maps $\mathrm{Hom}_{\mathfrak{so}(V)}(\Lambda^2 V, \mathfrak{so}(V))$, which is different depending on whether $d > 4$ or $d = 4$. Because of this, it is pertinent to divide our investigation in two cases at this point. However, we will see that in reality both cases lead to the same conclusion.

The case $d > 4$

If $\dim V = d > 4$, then $\mathfrak{so}(V) \cong \Lambda^2 V$ is irreducible of real type, so $\mathrm{Hom}_{\mathfrak{so}(V)}(\mathfrak{so}(V), \Lambda^2 V)$ is one-dimensional and generated by the isomorphism that sends $u \wedge v \mapsto u \curlywedge v$. Taking this into account, we deduce that the first map in (8.7) is determined by some $\alpha \in \mathsf{S}^2 W^*$, whereas the second map is determined by some $\varphi\colon \Lambda^2 W \to \mathfrak{b}$. In terms of these maps, the Lie bracket becomes

$$[v_1 \otimes w_1, v_2 \otimes w_2] = \alpha(w_1, w_2) v_1 \curlywedge v_2 + \langle v_1, v_2\rangle\, \varphi(w_1 \wedge w_2).$$

The equivariance of the bracket under $\mathfrak{b}$, which is part of the Jacobi identity, yields that α is $\mathfrak{b}$-invariant and φ is $\mathfrak{b}$-equivariant.

A consequence of the $\mathfrak{b}$-equivariance of φ is that the image of $\rho \circ \varphi$ lies in $\mathfrak{sl}(W)$. To see this, note that for every $B \in \mathfrak{b}$ and $w_1, w_2 \in W$, we have

$$\operatorname{tr}(\rho(B))\varphi(w_1 \wedge w_2) = \varphi(B \cdot (w_1 \wedge w_2)) = [B, \varphi(w_1 \wedge w_2)],$$

so applying ρ to both sides we obtain

$$\operatorname{tr}(\rho(B))\rho(\varphi(w_1 \wedge w_2)) = [\rho(B), \rho(\varphi(w_1 \wedge w_2))],$$

and the right hand side has trace zero because of the identity $\operatorname{tr}(XY) = \operatorname{tr}(YX)$, so by letting $B = \varphi(w_1 \wedge w_2)$ we deduce that

$$\operatorname{tr}(\rho(\varphi(w_1 \wedge w_2)))^2 = 0,$$

which implies that $\operatorname{im}(\rho \circ \varphi) \subseteq \mathfrak{sl}(W)$.

Summarizing, by imposing the skew-symmetry and $(\mathfrak{so}(V) \oplus \mathfrak{b})$-equivariance conditions, we have obtained an explicit expression of the Lie bracket of $\mathfrak{g}$ in terms of the Lie algebra structure of $\mathfrak{b}$, a 2-dimensional representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$, a $\mathfrak{b}$-invariant symmetric bilinear form $\alpha \in \mathsf{S}^2 W^*$ and a $\mathfrak{b}$-equivariant $\mathfrak{b}$-valued 2-form $\varphi\colon \Lambda^2 W \to \mathfrak{b}$. We write $\alpha \in (\mathsf{S}^2 W^*)^{\mathfrak{b}}$ and $\varphi \in \operatorname{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$.

In addition, we can determine further relations between these objects by imposing the full Jacobi identity. To this end, we define the *Jacobiator* $\operatorname{Jac}\colon \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by the equation

$$\operatorname{Jac}(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]].$$

This is actually a $\mathfrak{g}$-valued 3-form, and for $\mathfrak{g}$ to be a Lie algebra it needs to be identically zero. Equivariance under $\mathfrak{so}(V) \oplus \mathfrak{b}$ says that $\operatorname{Jac}(x, y, z) = 0$ whenever one of the arguments lies in $\mathfrak{so}(V)$ or $\mathfrak{b}$. Hence we need to compute the values of Jac whenever all of its arguments lies in $V \otimes W$.

We calculate the nested bracket for all $v_i \in V$ and $w_i \in W$:

$$\begin{aligned}[][[v_1 \otimes w_1, v_2 \otimes w_2], v_3 \otimes w_3] = \alpha(w_1, w_2)\left(\langle v_2, v_3\rangle\, v_1 \otimes w_3 - \langle v_1, v_3\rangle\, v_2 \otimes w_3\right) \\ + \langle v_1, v_2\rangle\, v_3 \otimes \rho(\varphi(w_1 \wedge w_2))w_3.\end{aligned}$$

The Jacobi identity then gives

$$\operatorname{Jac}(v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3) = \underset{1,2,3}{\mathfrak{S}}[[v_1 \otimes w_1, v_2 \otimes w_2], v_3 \otimes w_3] = 0,$$

where the symbol $\mathfrak{S}_{1,2,3}$ denotes cyclic summation with respect to the indices 1, 2 and 3. Rearranging terms, the Jacobi identity becomes

$$\underset{1,2,3}{\mathfrak{S}}\big(\langle v_1, v_2\rangle\, v_3 \otimes (\rho(\varphi(w_1 \wedge w_2))w_3 + \alpha(w_1, w_3)w_2 - \alpha(w_2, w_3)w_1)\big) = 0.$$

In particular this has to hold when the vectors v_i are such that $\langle v_1, v_2\rangle \neq 0$ and $v_3 \neq 0$, but $\langle v_3, v_1\rangle = \langle v_3, v_2\rangle = 0$. In that case, only the first term survives, giving the following relation:

$$\rho(\varphi(w_1 \wedge w_2))w_3 = \alpha(w_2, w_3)w_1 - \alpha(w_1, w_3)w_2 \tag{8.8}$$

for all $w_i \in W$. Conversely, if the condition (8.8) is satisfied, then the Jacobi identity is satisfied.

In summary, we have proved that the Jacobiator is identically zero precisely when the maps $\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ and $\alpha \in (\mathsf{S}^2 W^*)^{\mathfrak{b}}$ satisfy (8.8). It turns out that this completely determines α from ρ and φ. Indeed, because W is two-dimensional we have an isomorphism of $\mathfrak{gl}(W)$-modules

$$\alpha \in \mathsf{S}^2 W^* \mapsto \phi_\alpha \in \mathrm{Hom}(\Lambda^2 W, \mathfrak{sl}(W)), \tag{8.9}$$

given by

$$\phi_\alpha(w_1 \wedge w_2) w_3 = \alpha(w_2, w_3) w_1 - \alpha(w_1, w_3) w_2.$$

In particular, it is also an isomorphism of $\mathfrak{b}$-modules, so it restricts to a bijective correspondence between $(\mathsf{S}^2 W^*)^{\mathfrak{b}}$ and $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{sl}(W))$. As a consequence, if $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ is $\mathfrak{b}$-equivariant, the composition $\rho \circ \varphi$ is in $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{sl}(W))$, so (8.9) yields a unique $\alpha \in (\mathsf{S}^2 W^*)^{\mathfrak{b}}$ such that (8.8) is satisfied.

The case $d = 4$

If $\dim V = d = 4$, then $\Lambda^2 V \cong \mathfrak{so}(V)$ is not an irreducible representation. Indeed, choose an orientation on V and for each $k \in \{0, \dots, 4\}$ let $\star\colon \Lambda^k V \to \Lambda^{4-k} V$ be the Hodge star operator. We see that in degree 2 the map $\star$ is an $\mathfrak{so}(V)$-equivariant involution of $\Lambda^2 V$, whose eigenspaces are both three-dimensional. This allows us to decompose $\Lambda^2 V = \Lambda^+ \oplus \Lambda^-$ as the direct sum of two three-dimensional irreducible submodules of $\mathfrak{so}(V)$. Moreover, Λ^+ and Λ^- are of real type (for dimension reasons) and not isomorphic as representations, so any element of $\mathrm{End}_{\mathfrak{so}(V)}(\Lambda^2 V)$ has to preserve both Λ^+ and Λ^-. As a consequence, we see that $\mathrm{End}_{\mathfrak{so}(V)}(\Lambda^2 V) \cong \mathbb{R} \oplus \mathbb{R}$ as a real algebra.

We deduce from the above discussion that $\mathrm{Hom}_{\mathfrak{so}(V)}(\Lambda^2 V, \mathfrak{so}(V)) \cong \mathbb{R}^2$ as a vector space. Let us define an $\mathfrak{so}(V)$-equivariant map $F\colon \Lambda^2 V \to \mathfrak{so}(V)$ by the condition $F(v_1 \wedge v_2) v_3 = \star(v_1 \wedge v_2 \wedge v_3)$. Because the expression $F(v_1 \wedge v_2) v_3$ is totally skew-symmetric on v_1, v_2 and v_3, it follows that F is not proportional to the generic isomorphism $v_1 \wedge v_2 \mapsto v_1 \curlywedge v_2$. Therefore, these two maps generate the whole space $\mathrm{Hom}_{\mathfrak{so}(V)}(\Lambda^2 V, \mathfrak{so}(V))$.

The argument is now similar to the one from the case $d > 4$. Indeed, $\mathfrak{so}(V)$-equivariance yields that the first map in (8.7) is given by two maps α, $\beta \in \mathsf{S}^2 W^*$, whereas the second map is still determined by some $\varphi \in \mathrm{Hom}(\Lambda^2 W, \mathfrak{b})$. Explicitly, this bracket becomes

$$[v_1 \otimes w_1, v_2 \otimes w_2] = \alpha(w_1, w_2) v_1 \curlywedge v_2 + \beta(w_1, w_2) F(v_1 \wedge v_2) + \langle v_1, v_2 \rangle \varphi(w_1 \wedge w_2)$$

for all v_1, $v_2 \in V$ and w_1, $w_2 \in W$. Moreover, since this bracket is $\mathfrak{b}$-equivariant, we readily obtain that α, $\beta \in \mathsf{S}^2 W^*$ and $\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$.

We now compute the Jacobiator on elements of $V \otimes W$. The nested bracket in this case is

$$\begin{aligned}[][[v_1 \otimes w_1, v_2 \otimes w_2], v_3 \otimes w_3] = {} & \alpha(w_1, w_2)(\langle v_2, v_3 \rangle v_1 - \langle v_1, v_3 \rangle v_2) \otimes w_3 \\ & + \beta(w_1, w_2) \star (v_1 \wedge v_2 \wedge v_3) \otimes w_3 \\ & + \langle v_1, v_2 \rangle v_3 \otimes \rho(\varphi(w_1 \wedge w_2)) w_3,\end{aligned}$$

and therefore the condition that $\mathrm{Jac} = 0$ on $\Lambda^3(V \otimes W)$ gives, after rearranging,

$$\begin{aligned} 0 = &\underset{1,2,3}{\mathfrak{S}} \langle v_1, v_2 \rangle v_3 \otimes \big(\alpha(w_3, w_1) w_2 - \alpha(w_2, w_3) w_1 + \rho(\varphi(w_1 \wedge w_2)) w_3\big) \\ &+ \underset{1,2,3}{\mathfrak{S}} \beta(w_1, w_2) \star (v_1 \wedge v_2 \wedge v_3) \otimes w_3. \end{aligned} \tag{8.10}$$

In particular, we may choose $v_1 = v_2$ to be orthogonal to v_3, so that (8.10) reduces to the familiar condition (8.8). Moreover, if we choose $w_1 = w_2 = w_3 = w$ and take three linearly independent vectors v_1, v_2, $v_3 \in V$, then (8.10) simply becomes $3\beta(w, w)w = 0$, which combined with the symmetry of β yields $\beta = 0$. Conversely, if $\beta = 0$ and (8.8) is satisfied, then (8.10) holds. We conclude from this that the Lie bracket on $V \otimes W$ takes exactly the same form as in the case $d > 4$.

Gathering our calculations, we have shown that the algebra structure on $\mathfrak{g}$ is determined by a Bianchi algebra $\mathfrak{b}$, a Lie algebra representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ and a $\mathfrak{b}$-equivariant map $\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$. Conversely, if we start with a triple $(\mathfrak{b}, \rho, \varphi)$ satisfying the aforementioned conditions, it is clear that the bilinear operation on $\mathfrak{g}$ defined by (8.1) induces a $(3, 2)$-KLA structure on $\mathfrak{g}$. This finishes the proof of Theorem A.

Remark 8.1. Let us compute the expression of the isomorphism (8.9) in coordinates. Suppose $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis of W, and let $\alpha \in \mathsf{S}^2 W^*$ have coordinates $\alpha_{ij} = \alpha(\boldsymbol{e}_i, \boldsymbol{e}_j)$. Then the matrix of the endomorphism $\phi_\alpha(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)\colon W \to W$ is readily given by

$$[\phi_\alpha(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)] = \begin{pmatrix} \alpha_{12} & \alpha_{22} \\ -\alpha_{11} & -\alpha_{12} \end{pmatrix}.$$

As a consequence, given a linear map $\phi\colon \Lambda^2 W \to \mathfrak{sl}(W)$, if (ϕ_{ij}) is the matrix of $\phi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$ with respect to our basis, then $\phi = \phi_\alpha$ for the bilinear form α with coefficients

$$(\alpha_{ij}) = \begin{pmatrix} -\phi_{21} & \phi_{11} \\ \phi_{11} & \phi_{12} \end{pmatrix}.$$

This means that given the maps $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ and $\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$, the composed map $\rho \circ \varphi\colon \Lambda^2 W \to \mathfrak{sl}(W)$, and the unique symmetric form $\alpha \in (\mathsf{S}^2 W^*)^{\mathfrak{b}}$ satisfying (8.8) is given in coordinates by

$$(\alpha_{ij}) = \begin{pmatrix} -\rho_{21} & \rho_{11} \\ \rho_{11} & \rho_{12} \end{pmatrix},$$

where (ρ_{ij}) is the matrix of $\rho(\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2))$ with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

8.1.2 Algebras with $d = 3$

In this section we prove Theorem B. The main difference when $d = 3$ is that now $\Lambda^2 V \cong V$ as $\mathfrak{so}(V)$-modules.

Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ be a $(3, 2)$-KLA with three-dimensional spatial isotropy. As a representation of $\mathfrak{so}(V)$, we see that $\mathfrak{g}$ contains two isotypical components: the component

$\mathfrak{so}(V) \oplus (V \otimes W)$ corresponding to the standard representation and the component $\mathfrak{b}$ corresponding to the trivial one.

We claim that $\mathfrak{b}$ is once again a Lie subalgebra of $\mathfrak{g}$. Indeed, since $\mathfrak{b}$ is a trivial $\mathfrak{so}(V)$-module, so is $\Lambda^2\mathfrak{b}$ and hence, by $\mathfrak{so}(V)$-equivariance, the only possible nonzero component of the bracket is $\Lambda^2\mathfrak{b} \to \mathfrak{b}$, making $\mathfrak{b}$ into a Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{so}(V) \cong V$ as $\mathfrak{so}(V)$-modules via the map $v_1 \wedge v_2 \mapsto v_1 \times v_2$, the bracket between any $B \in \mathfrak{b}$ and $v \otimes w \in V \otimes W$ can now have a component along $\mathfrak{so}(V)$, which gives

$$[B, v \otimes w] = v \otimes \rho(B)w + \lambda_B(w)\varepsilon(v),$$

for some linear maps $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ and $\lambda\colon \mathfrak{b} \to W^*$ and where $\varepsilon(v) \in \mathfrak{so}(V)$ is defined by $\varepsilon(v)u = v \times u$. It follows that $\varepsilon(u \times v) = -u \curlywedge v$.

The $(\mathfrak{b}, \mathfrak{b}, V \otimes W)$ Jacobi identity is tantamount to imposing $\mathfrak{b}$-equivariance of this bracket. Let $B, C \in \mathfrak{b}$ and $v \otimes w \in V \otimes W$. Then the Jacobi identity becomes

$$[B, [C, v \otimes w]] - [C, [B, v \otimes w]] = [[B, C], v \otimes w],$$

and this has two components:

- a component along $V \otimes W$:

$$v \otimes (\rho(B)\rho(C) - \rho(C)\rho(B) - \rho([B, C]))\, w = 0,$$

  which implies that $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ is a representation;

- and a component along $\mathfrak{so}(V)$:

$$\big(\lambda_B(\rho(C)w) - \lambda_C(\rho(B)w) - \lambda_{[B,C]}(w)\big)\, \epsilon(v) = 0. \tag{8.11}$$

Proposition 8.2. *The condition* (8.11) *implies that* $\lambda \in C^1(\mathfrak{b}; W^*)$ *is a Chevalley–Eilenberg cocycle and therefore determines a cohomology class* $[\lambda]$ *in* $H^1(\mathfrak{b}; W^*)$.

Proof. Let $C^p(\mathfrak{b}; W^*) = \mathrm{Hom}(\Lambda^p\mathfrak{b}, W^*)$ and let $\partial\colon C^p(\mathfrak{b}; W^*) \to C^{p+1}(\mathfrak{b}; W^*)$ denote the Chevalley–Eilenberg differential. We only need the expressions of this differential for $p = 0, 1$. If $\theta \in W^*$, then $\partial\theta \in C^1(\mathfrak{b}; W^*)$ is given by

$$(\partial\theta)(B) = \rho^*(B)\theta = -\theta \circ \rho(B),$$

and if $\lambda \in C^1(\mathfrak{b}, W^*)$ then $\partial\lambda \in C^2(\mathfrak{b}; W^*)$ is given by

$$\begin{aligned}\partial\lambda(B, C) &= \rho^*(B) \circ \lambda_C - \rho^*(C) \circ \lambda_B - \lambda_{[B,C]}\\ &= \lambda_B \circ \rho(C) - \lambda_C \circ \rho(B) - \lambda_{[B,C]}.\end{aligned}$$

From this expression, it is clear that (8.11) is equivalent to $\partial\lambda = 0$, so λ defines a class $[\lambda] \in H^1(\mathfrak{b}; W^*)$. □

It remains to consider the bracket of two elements in $V \otimes W$, which gives a linear map

$$\left(\Lambda^2 V \otimes \mathsf{S}^2 W\right) \oplus \left(\mathsf{S}^2_0 V \otimes \Lambda^2 W\right) \oplus \left(\mathbb{R}_{\mathrm{tr}} \otimes \Lambda^2 W\right) \to \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}.$$

Equivariance under $\mathfrak{so}(V)$ now says that the only possible nonzero components of this linear map are

$$\begin{aligned} \Lambda^2 V \otimes \mathsf{S}^2 W &\to \mathfrak{so}(V) \oplus (V \otimes W) \\ \mathbb{R}_{\mathrm{tr}} \otimes \Lambda^2 W &\to \mathfrak{b}. \end{aligned} \tag{8.12}$$

The first linear map itself breaks up into two components, as in the generic case. The first component is

$$\Lambda^2 V \otimes \mathsf{S}^2 W \to \mathfrak{so}(V),$$

which, given the isomorphism $\Lambda^2 V \cong \mathfrak{so}(V)$ from the previous section, is characterized by a symmetric bilinear form $\alpha\colon W \times W \to \mathbb{R}$. The second component is new:

$$\Lambda^2 V \otimes \mathsf{S}^2 W \to V \otimes W.$$

The isomorphism $\Lambda^2 V \to V$ is essentially the vector cross product $v_1 \wedge v_2 \mapsto v_1 \times v_2$ and hence this component is characterized by a W-valued symmetric bilinear form $\sigma\colon W \times W \to W$. The second linear map in equation (8.12) is as in the generic case and is characterized by a $\mathfrak{b}$-valued 2-form $\varphi\colon \Lambda^2 W \to \mathfrak{b}$.

In summary, the bracket is given by

$$[v_1 \otimes w_1, v_2 \otimes w_2] = \alpha(w_1, w_2) v_1 \curlywedge v_2 + \langle v_1, v_2\rangle\, \varphi(w_1 \wedge w_2) + (v_1 \times v_2) \otimes \sigma(w_1, w_2),$$

for all $v_i \in V$ and $w_i \in W$.

The Jacobi identity with $B \in \mathfrak{b}$ says that the brackets are $\mathfrak{b}$-equivariant. Let $B \in \mathfrak{b}$. Then for every $v_1, v_2 \in V$ and $w_1, w_2 \in W$ we have

$$\begin{aligned} [B, [v_1 \otimes w_1, v_2 \otimes w_2]] = &\ \langle v_1, v_2\rangle\, [B, \varphi(w_1 \wedge w_2)] + (v_1 \times v_2) \otimes \rho(B)\sigma(w_1, w_2) \\ &+ \lambda_B(\sigma(w_1, w_2))\varepsilon(v_1 \times v_2). \end{aligned}$$

The Jacobi identity reads as

$$[B, [v_1 \otimes w_1, v_2 \otimes w_2]] = [[B, v_1 \otimes w_1], v_2 \otimes w_2] + [v_1 \otimes w_1, [B, v_2 \otimes w_2]]$$

and it has three components according to $\mathfrak{g} = \mathfrak{so}(V) \oplus \mathfrak{b} \oplus (V \otimes W)$. The $\mathfrak{so}(V)$-component says that

$$\left(\alpha(\rho(B)w_1, w_2) + \alpha(w_1, \rho(B)w_2)\right) v_1 \curlywedge v_2 - \lambda_B(\sigma(w_1, w_2))\varepsilon(v_1 \times v_2) = 0.$$

Using that $\varepsilon(v_1 \times v_2) = -v_1 \curlywedge v_2$, we arrive at

$$\alpha(\rho(B)w_1, w_2) + \alpha(w_1, \rho(B)w_2) + \lambda_B(\sigma(w_1, w_2)) = 0,$$

which measures the lack of $\mathfrak{b}$-equivariance of α:

$$(B \cdot \alpha)(w_1, w_2) = \lambda_B(\sigma(w_1, w_2)).$$

The $\mathfrak{b}$-component simply says that $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ is $\mathfrak{b}$-equivariant:

$$\varphi(\rho(B)w_1 \wedge w_2) + \varphi(w_1 \wedge \rho(B)w_2) = [B, \varphi(w_1 \wedge w_2)].$$

Finally, taking the $V \otimes W$-component yields

$$\begin{aligned} 0 = \sigma(\rho(B)w_1, w_2) + \lambda_B(w_1)w_2 + \sigma(w_1, \rho(B)w_2) + \lambda_B(w_2)w_1 \\ - \rho(B)\sigma(w_1, w_2), \end{aligned} \tag{8.13}$$

which says that λ also measures the failure of $\sigma\colon \Lambda^2 W \to W$ to be $\mathfrak{b}$-equivariant:

$$(B \cdot \sigma)(w_1, w_2) = \lambda_B(w_1)w_2 + \lambda_B(w_2)w_1. \tag{8.14}$$

As before, it now remains to consider the Jacobi identity with three vectors in $V \otimes W$. The nested bracket has again three components:

$$\begin{aligned} [[v_1 \otimes w_1, v_2 \otimes w_2], v_3 \otimes w_3]_{\mathfrak{so}(V)} &= \alpha(\sigma(w_1, w_2), w_3)(v_1 \times v_2) \curlywedge v_3 \\ &\quad + \langle v_1, v_2 \rangle \, \lambda_{\varphi(w_1 \wedge w_2)}(w_3)\varepsilon(v_3), \\ [[v_1 \otimes w_1, v_2 \otimes w_2], v_3 \otimes w_3]_{\mathfrak{b}} &= \langle v_1 \times v_2, v_3 \rangle \, \varphi(\sigma(w_1, w_2) \wedge w_3), \\ [[v_1 \otimes w_1, v_2 \otimes w_2], v_3 \otimes w_3]_{V \otimes W} &= \alpha(w_1, w_2) \left(\langle v_2, v_3 \rangle \, v_1 - \langle v_1, v_3 \rangle \, v_2 \right) \otimes w_3 \\ &\quad + \langle v_1, v_2 \rangle \, v_3 \otimes \rho(\varphi(w_1 \wedge w_2))w_3 \\ &\quad + ((v_1 \times v_2) \times v_3) \otimes \sigma(\sigma(w_1, w_2), w_3). \end{aligned}$$

Each component must vanish separately in the Jacobi identity. The $\mathfrak{so}(V)$ terms give the equation

$$\underset{1,2,3}{\mathfrak{S}} \big(\alpha(\sigma(w_1, w_2), w_3)(v_1 \times v_2) \curlywedge v_3 + \langle v_1, v_2 \rangle \, \lambda_{\varphi(w_1 \wedge w_2)}(w_3)\varepsilon(v_3) \big) = 0, \tag{8.15}$$

whereas the $\mathfrak{b}$ terms give

$$\underset{1,2,3}{\mathfrak{S}} \langle v_1 \times v_2, v_3 \rangle \, \varphi(\sigma(w_1, w_2) \wedge w_3) = 0 \tag{8.16}$$

and the $V \otimes W$ terms give

$$\begin{aligned} \underset{1,2,3}{\mathfrak{S}} \big(\alpha(w_1, w_2) \left(\langle v_2, v_3 \rangle \, v_1 - \langle v_1, v_3 \rangle \, v_2 \right) \otimes w_3 + \langle v_1, v_2 \rangle \, v_3 \otimes \rho(\varphi(w_1 \wedge w_2))w_3 \\ + ((v_1 \times v_2) \times v_3) \otimes \sigma(\sigma(w_1, w_2), w_3) \big) = 0. \end{aligned} \tag{8.17}$$

Using the fact that $u \curlywedge v = -\varepsilon(u \times v)$ and the well-known cross product identity

$$(v_1 \times v_2) \times v_3 = \langle v_1, v_3 \rangle \, v_2 - \langle v_2, v_3 \rangle \, v_1, \tag{8.18}$$

we may rearrange the terms in (8.15) to deduce

$$\mathfrak{S}_{1,2,3} \left(\alpha(\sigma(w_3, w_1), w_2) - \alpha(\sigma(w_3, w_2), w_1) + \lambda_{\varphi(w_1 \wedge w_2)}(w_3)\right) \langle v_1, v_2 \rangle \, \varepsilon(v_3) = 0 \tag{8.19}$$

This has to hold for every choice of v_1, v_2 and v_3, and hence we may take them, in particular, to be linearly independent, so that this component of the Jacobi identity becomes simply

$$\lambda_{\varphi(w_1 \wedge w_2)}(w_3) = \alpha(\sigma(w_3, w_2), w_1) - \alpha(\sigma(w_3, w_1), w_2) \tag{8.20}$$

for all $w_i \in W$. It is clear that if this equation holds, then so does (8.19).

The expression $\langle v_1 \times v_2, v_3 \rangle$ in (8.16) is cyclically invariant, hence that equation simply becomes

$$\varphi(\sigma(w_1, w_2) \wedge w_3) + \varphi(\sigma(w_2, w_3) \wedge w_1) + \varphi(\sigma(w_3, w_1) \wedge w_2) = 0.$$

In (8.17), the expression $(v_1 \times v_2) \times v_3$ can be expanded using the cross product identity (8.18). Inserting this into (8.17) and rearranging, we may rewrite that equation as

$$\begin{aligned} \mathfrak{S}_{1,2,3} \langle v_1, v_2 \rangle \, v_3 \otimes (\rho(\varphi(w_1 \wedge w_2))w_3 - \alpha(w_2, w_3)w_1 + \alpha(w_1, w_3)w_2 \\ + \sigma(\sigma(w_2, w_3), w_1) - \sigma(\sigma(w_1, w_3), w_2)) = 0. \end{aligned} \tag{8.21}$$

Choosing $v_i \in V$ so that $\langle v_1, v_2 \rangle$ and v_3 are nonzero but $\langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0$, we obtain the equation

$$\begin{aligned} \alpha(w_2, w_3)w_1 - \alpha(w_1, w_3)w_2 = \rho(\varphi(w_1 \wedge w_2))w_3 + \sigma(\sigma(w_2, w_3), w_1) \\ - \sigma(\sigma(w_1, w_3), w_2) \end{aligned} \tag{8.22}$$

for all $w_i \in W$. Conversely, if this holds, then so does (8.17).

Similarly to the case $d > 3$, the form α can be recovered from ρ, φ and σ. Indeed, suppose that we are given maps $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ and $\sigma\colon \mathsf{S}^2 W \to W$ such that φ is $\mathfrak{b}$-equivariant, and define a map $\phi\colon \Lambda^2 W \to \mathfrak{gl}(W)$ via

$$\phi(w_1 \wedge w_2)w_3 = \rho(\varphi(w_1 \wedge w_2))w_3 + \sigma(\sigma(w_2, w_3), w_1) - \sigma(\sigma(w_1, w_3), w_2).$$

This map actually takes values in $\mathfrak{sl}(W)$. Indeed, the first summand yields a traceless endomorphism because of $\mathfrak{b}$-equivariance of φ. As for the other two summands, we may recognize them as the commutator $[\widehat{\sigma}_{w_1}, \widehat{\sigma}_{w_2}]$, where $\widehat{\sigma}_w$ is the endomorphism taking $w' \mapsto \sigma(w, w')$ for all $w \in W$. In particular, the trace $\operatorname{tr}([\widehat{\sigma}_{w_1}, \widehat{\sigma}_{w_2}]) = 0$, so we conclude that $\operatorname{im}\phi \subseteq \mathfrak{sl}(W)$. Owing to the isomorphism (8.9), we can guarantee the existence of a unique $\alpha \in \mathsf{S}^2 W^*$ that is related to ρ, φ and σ via (8.22). In particular, if σ is also $\mathfrak{b}$-equivariant, then ϕ is $\mathfrak{b}$-invariant (due to equivariance of σ and φ), and $\alpha \in (\mathsf{S}^2 W^*)^{\mathfrak{b}}$.

We now compute the coordinates of α from the rest of ingredients. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be a basis for W. We denote by (ρ_{ij}) and $([\widehat{\sigma}_1, \widehat{\sigma}_2]_{ij})$ the matrices of the linear endomorphisms $\rho(\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2))$

and $[\widehat{\sigma}_{e_1}, \widehat{\sigma}_{e_2}]$ relative to this basis. Then, arguing in the same fashion as in the case $d > 3$, we obtain

$$\begin{aligned}
\alpha_{11} &= -\rho_{21} - [\widehat{\sigma}_1, \widehat{\sigma}_2]_{21}, \\
\alpha_{12} &= \rho_{11} + [\widehat{\sigma}_1, \widehat{\sigma}_2]_{11} = -\rho_{22} - [\widehat{\sigma}_1, \widehat{\sigma}_2]_{22}, \\
\alpha_{22} &= \rho_{12} + [\widehat{\sigma}_1, \widehat{\sigma}_2]_{12}.
\end{aligned}$$

We may summarize this discussion as follows:

Theorem 8.3. *Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ be a $(3,2)$-KLA with $\dim V = d = 3$. Then the Lie bracket of $\mathfrak{g}$ is determined by the following algebraic data:*

(i) *a (Bianchi) Lie algebra structure on $\mathfrak{b}$,*
(ii) *a two-dimensional Lie algebra representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$,*
(iii) *a $\mathfrak{b}$-valued 2-form $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ that is $\mathfrak{b}$-equivariant,*
(iv) *a cocycle representative λ for a cohomology class $[\lambda] \in H^1(\mathfrak{b}; W^*)$,*
(v) *and a W-valued symmetric bilinear form $\sigma\colon W \times W \to W$.*

Furthermore, let $\alpha \in \mathsf{S}^2 W^*$ be the unique symmetric bilinear form satisfying

$$\begin{aligned}
\alpha(w_2, w_3)w_1 - \alpha(w_1, w_3)w_2 &= \sigma(\sigma(w_2, w_3), w_1) - \sigma(\sigma(w_1, w_3), w_2) \\
&\quad + \rho(\varphi(w_1 \wedge w_2))w_3
\end{aligned}$$

for all $w_i \in W$. Then the maps α, φ, λ, σ and ρ must satisfy the following equations:

$$\begin{aligned}
0 &= \sigma(\rho(B)w_1, w_2) + \lambda_B(w_1)w_2 + \sigma(w_1, \rho(B)w_2) + \lambda_B(w_2)w_1 \\
&\qquad - \rho(B)\sigma(w_1, w_2), \\
0 &= \alpha(\rho(B)w_1, w_2) + \alpha(w_1, \rho(B)w_2) + \lambda_B(\sigma(w_1, w_2)), \\
0 &= \alpha(\sigma(w_1, w_3), w_2) - \alpha(\sigma(w_2, w_3), w_1) + \lambda_{\varphi(w_1 \wedge w_2)}(w_3), \\
0 &= \varphi(\sigma(w_1, w_2) \wedge w_3) + \varphi(\sigma(w_2, w_3) \wedge w_1) + \varphi(\sigma(w_3, w_1) \wedge w_2),
\end{aligned} \tag{8.23}$$

for all $w_i \in W$. The Lie algebra structure is given by the brackets

$$\begin{aligned}
[X, v \otimes w] &= Xv \otimes w, \\
[X, B] &= 0, \\
[B, v \otimes w] &= v \otimes \rho(B)w + \lambda_B(w)\varepsilon(v), \\
[v_1 \otimes w_1, v_2 \otimes w_2] &= \alpha(w_1, w_2)v_1 \curlywedge v_2 + \langle v_1, v_2 \rangle\, \varphi(w_1 \wedge w_2) \\
&\quad + (v_1 \times v_2) \otimes \sigma(w_1, w_2),
\end{aligned}$$

for all $X \in \mathfrak{so}(V)$, $v, v_i \in V$, $w, w_i \in W$ and $B \in \mathfrak{b}$.

In order to deduce Theorem B from Theorem 8.3, we need to show that every $(3,2)$-KLA with $d = 3$ is isomorphic to one whose corresponding cocycle λ vanishes.

Remark 8.4. A natural question is whether the construction depends on the cocycle representative or only on its cohomology class. Let λ' be another cocycle representative for $[\lambda] \in H^1(\mathfrak{b}; W^*)$, so that $\lambda' = \lambda - \partial\mu$. Let $j_\mu \in \mathsf{GL}(\mathfrak{g})$ be the invertible endomorphism defined by

$$j_\mu(v \otimes w + X + B) = v \otimes w - \mu(w)\varepsilon(v) + X + B,$$

with inverse

$$j_\mu^{-1}(v \otimes w + X + B) = v \otimes w + \mu(w)\varepsilon(v) + X + B,$$

for all $v \in V$, $w \in W$, $X \in \mathfrak{so}(V)$ and $B \in \mathfrak{b}$. Then the transformed Lie bracket $[x,y]' := j_\mu^{-1}[j_\mu(x), j_\mu(y)]$ agrees with that in (8.5) using λ' instead of λ and where α and σ are changed to

$$\begin{aligned} \alpha &\mapsto \alpha' = \alpha - \mu \circ \sigma + \mu \otimes \mu \\ \sigma &\mapsto \sigma' = \sigma - \mu \otimes \mathrm{id}_W - \mathrm{id}_W \otimes \mu. \end{aligned}$$

In particular, if $[\lambda] = 0 \in H^1(\mathfrak{b}; W^*)$, we may eliminate λ from the Lie bracket by a general linear transformation on $\mathfrak{g}$.

This remark is pertinent in that indeed $[\lambda] = 0$, as the next result shows.

Proposition 8.5. *The condition* (8.14) *implies that* $[\lambda] = 0 \in H^1(\mathfrak{b}; W^*)$ *is zero; that is,* $\lambda = \partial\mu$ *for some* $\mu \in W^*$.

Proof. The form $\sigma : W \times W \to W$ is equivalent, by currying, to a linear map $\widehat{\sigma} : W \to \mathrm{End}(W)$ sending $w \mapsto \widehat{\sigma}_w$, where $\widehat{\sigma}_w(w') = \sigma(w, w')$. This allows us to abstract w_2 from (8.13) in order to obtain the following equation in $\mathrm{End}(W)$:

$$\rho(B) \circ \widehat{\sigma}_{w_1} - \widehat{\sigma}_{w_1} \circ \rho(B) - \widehat{\sigma}_{\rho(B)w_1} = \lambda_B(w_1)\,\mathrm{id}_W + w_1 \otimes \lambda_B.$$

We now take the trace $\mathrm{tr} : \mathrm{End}(W) \to \mathbb{R}$ and use that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ to arrive at

$$-\,\mathrm{tr}\,\widehat{\sigma}_{\rho(B)w_1} = 3\lambda_B(w_1).$$

Let us define $\mu \in W^*$ by $\mu := \frac{1}{3}\,\mathrm{tr} \circ \widehat{\sigma}$. In terms of μ, and abstracting w_1 from the previous equation, we deduce that

$$\lambda_B = -\mu \circ \rho(B) = \rho^*(B)\mu,$$

where ρ^* is the dual representation of ρ. In other words, we have $\lambda = \partial\mu$. ☐

Due to Remark 8.4, we may eliminate λ in Theorem 8.3, so we deduce that Theorem B holds.

8.2 Relative isomorphisms between $(3,2)$-kinematical Lie algebras

In this section we discuss the problem of determining all isomorphisms between $(3,2)$-KLAs with $d \geq 3$ that preserve rotations. More precisely, let $\mathfrak{g}$ and $\mathfrak{g}'$ be two $(3,2)$-KLAs and write them as

$$\mathfrak{g} = \mathfrak{so}(d) \oplus \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}, \quad \mathfrak{g}' = \mathfrak{so}(d') \oplus \mathbb{R}^{d'} \oplus \mathbb{R}^{d'} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R},$$

where we have decomposed $\mathfrak{g}$ (respectively, $\mathfrak{g}'$) as a direct sum of $\mathfrak{so}(d)$-modules (respectively, $\mathfrak{so}(d')$-modules). Since

$$\dim \mathfrak{g} = \frac{d(d-1)}{2} + 2d + 3$$

and similarly with $\mathfrak{g}'$, it is clear that for $\mathfrak{g}$ and $\mathfrak{g}'$ to be isomorphic we need $d = d'$. Fix embeddings $i\colon \mathfrak{so}(d) \hookrightarrow \mathfrak{g}$ and $j\colon \mathfrak{so}(d) \hookrightarrow \mathfrak{g}'$ whose images are precisely the rotational subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$. We say that a map $f\colon \mathfrak{g} \to \mathfrak{g}'$ is a *relative isomorphism* if it is a Lie algebra isomorphism and $j = f \circ i$. If $\mathfrak{g} = \mathfrak{g}'$, we say that f is a *relative automorphism* of $\mathfrak{g}$. We denote by $\mathrm{Aut}_0(\mathfrak{g})$ the set of relative automorphisms of $\mathfrak{g}$ (which is clearly a Lie group).

Given $(3,2)$-KLAs $\mathfrak{g}$, $\mathfrak{g}'$, we aim to determine when $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic in terms of their corresponding algebraic objects considered in Theorems A and B, as well as the set of relative isomorphisms between $\mathfrak{g}$ and $\mathfrak{g}'$ (equivalently, the relative automorphism group of a $(3,2)$-KLA).

8.2.1 Generic case: $d > 3$

Consider two $(3,2)$-KLA $\mathfrak{g}$ and $\mathfrak{g}'$, which we write as

$$\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}, \quad \mathfrak{g}' = \mathfrak{so}(V) \oplus (V \otimes W') \oplus \mathfrak{b}',$$

where $\dim V = d > 3$. Then, according to Theorem A, the Lie brackets of $\mathfrak{g}$ and $\mathfrak{g}'$ are determined by triples $(\mathfrak{b}, \rho, \varphi)$ and $(\mathfrak{b}', \rho', \varphi')$, where $\mathfrak{b}$ is a Bianchi algebra, $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ is a two-dimensional representation and $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ is a $\mathfrak{b}$-equivariant map (and likewise for $\mathfrak{b}'$, ρ' and φ').

Suppose a relative isomorphism $f\colon \mathfrak{g} \to \mathfrak{g}'$ is given. As $f|_{\mathfrak{so}(V)}$ is the identity map, f is also an isomorphism of $\mathfrak{so}(V)$-modules. In particular, f sends the isotypical components of $\mathfrak{g}$ to those of $\mathfrak{g}'$. For $d > 3$, the isotypical components of $\mathfrak{g}$ are $\mathfrak{so}(V)$, $V \otimes W$ and $\mathfrak{b}$, and identically for $\mathfrak{g}'$. Thus, we have $f(V \otimes W) = V \otimes W'$ and $f(\mathfrak{b}) = \mathfrak{b}'$, and the equivariance of $f|_{V \otimes W}$ yields that there exists a map $h\colon W \to W'$ such that $f(v \otimes w) = v \otimes h(w)$ for all $v \otimes w \in V \otimes W$. We have thus shown that f decomposes as

$$f = \mathrm{id}_{\mathfrak{so}(V)} \oplus (\mathrm{id}_V \otimes h) \oplus \psi$$

for some linear isomorphism $h\colon W \to W'$ and some Lie algebra isomorphism $\psi\colon \mathfrak{b} \to \mathfrak{b}'$.

Let $B \in \mathfrak{b}$, $v \otimes w \in V \otimes W$ be arbitrary. The equations

$$\begin{aligned} f([B, v \otimes w]) &= f(v \otimes \rho(B)w) = v \otimes h(\rho(B)w), \\ [f(B), f(v \otimes w)] &= [\psi(B), v \otimes h(w)] = v \otimes \rho'(\psi(B))h(w) \end{aligned}$$

imply that the maps $h \circ \rho(B)$ and $\rho'(\psi(B)) \circ h$ are equal. For a linear isomorphism $g\colon W \to W'$, we define a map $\mathrm{Ad}(g)\colon \mathfrak{gl}(W) \to \mathfrak{gl}(W')$ via $\mathrm{Ad}(g)T = g \circ T \circ g^{-1}$. Then we can rewrite the previous condition as $\rho' \circ \psi = \mathrm{Ad}(h) \circ \rho$. Furthermore, choose $v_1, v_2 \in V$ and $w_1, w_2 \in W$. We have

$$\begin{aligned} f([v_1 \otimes w_1, v_2 \otimes w_2]) &= f(\alpha(w_1, w_2) v_1 \curlywedge v_2 + \langle v_1, v_2 \rangle \varphi(w_1 \wedge w_2)) \\ &= \alpha(w_1, w_2) v_1 \curlywedge v_2 + \langle v_1, v_2 \rangle \psi(\varphi(w_1 \wedge w_2)), \end{aligned}$$

whereas

$$\begin{aligned}[f(v_1 \otimes w_1), f(v_2 \otimes w_2)] &= [v_1 \otimes h(w_1), v_2 \otimes h(w_2)] \\ &= \alpha'(h(w_1), h(w_2)) v_1 \curlywedge v_2 \\ &\quad + \langle v_1, v_2\rangle \varphi'(h(w_1) \wedge h(w_2)) \\ &= (h^*\alpha')(w_1, w_2) v_1 \curlywedge v_2 + \langle v_1, v_2\rangle (h^*\varphi')(w_1 \wedge w_2).\end{aligned}$$

Therefore, we must have $\alpha = h^*\alpha'$ and $\psi \circ \varphi = h^*\varphi'$. It is not hard to see that the latter equation, together with $\rho' \circ \psi = \operatorname{Ad}(h) \circ \rho$, implies the former.

Conversely, suppose $h\colon W \to W'$ is a linear isomorphism and $\psi\colon \mathfrak{b} \to \mathfrak{b}'$ is a Lie algebra isomorphism such that $\rho' \circ \psi = \operatorname{Ad}(h) \circ \rho$ and $\psi \circ \varphi = h^*\varphi'$. Then from the equalities above, it is clear that the map $f\colon \mathfrak{g} \to \mathfrak{g}'$ defined by $f = \mathrm{id}_{\mathfrak{so}(V)} \oplus (\mathrm{id}_V \otimes h) \oplus \psi$ is a relative isomorphism.

We summarize this discussion in the following result:

Proposition 8.6. *Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ and $\mathfrak{g}' = \mathfrak{so}(V) \oplus (V \otimes W') \oplus \mathfrak{b}'$ be two $(3,2)$-KLAs for which $\dim V = d > 3$, and let $(\mathfrak{b}, \rho, \varphi)$ and $(\mathfrak{b}', \rho', \varphi')$ be the algebraic data defining the Lie brackets of $\mathfrak{g}$ and $\mathfrak{g}'$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic relative to $\mathfrak{so}(V)$ if and only if there exist*

(i) *a vector space isomorphism $h\colon W \to W'$,*
(ii) *and a Lie algebra isomorphism $\psi\colon \mathfrak{b} \to \mathfrak{b}'$,*

satisfying the following equations:

$$\rho' \circ \psi = \operatorname{Ad}(h) \circ \rho \qquad \text{and} \qquad \psi \circ \varphi = h^*\varphi'.$$

The map

$$f = \mathrm{id}_{\mathfrak{so}(V)} \oplus (\mathrm{id}_V \otimes h) \oplus \psi$$

is a relative isomorphism from $\mathfrak{g}$ to $\mathfrak{g}'$. Furthermore, every relative isomorphism arises in this manner.

8.2.2 Case $d = 3$

We now study the possible isomorphisms between $(3,2)$-KLAs for which $d = 3$. We consider two $(3,2)$-KLAs of the form

$$\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}, \quad \mathfrak{g}' = \mathfrak{so}(V) \oplus (V \otimes W') \oplus \mathfrak{b}',$$

where $\dim V = d = 3$. Theorem 8.3 states that the Lie algebra structures of $\mathfrak{g}$ and $\mathfrak{g}'$ are determined by the 6-tuples $(\mathfrak{b}, \rho, \varphi, \lambda, \alpha, \sigma)$ and $(\mathfrak{b}', \rho', \varphi', \lambda', \alpha', \sigma')$, where $\mathfrak{b}$ is a Bianchi algebra, $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ is a representation, $\varphi \in \operatorname{Hom}(\Lambda^2 W, \mathfrak{b})$, $\alpha \in \mathsf{S}^2 W^*$, $\lambda \in B^1(\mathfrak{b}; W^*)$ is a coboundary and $\sigma\colon W \times W \to W$ is a symmetric bilinear form such that the relations (8.23) are satisfied (and similarly for $\mathfrak{b}', \rho', \varphi', \lambda', \alpha', \sigma'$). It is clear from Remark 8.4 that we may eliminate λ and λ'

via adequate relative automorphisms, so we only focus on determining the isomorphisms from $\mathfrak{g}$ to $\mathfrak{g}'$ in the case that both λ and λ' are zero.

Suppose that $f\colon \mathfrak{g} \to \mathfrak{g}'$ is a relative isomorphism. Then f is also an $\mathfrak{so}(V)$-module isomorphism, so it preserves the isotypical decompositions of $\mathfrak{g}$ and $\mathfrak{g}'$. For $d = 3$, the isotypical components of $\mathfrak{g}$ are $\mathfrak{so}(V) \oplus (V \otimes W)$ and $\mathfrak{b}$, as $\mathfrak{so}(V) \cong V$ (and the same holds for $\mathfrak{g}'$). This implies that $f(\mathfrak{b}) = \mathfrak{b}'$, while $f(V \otimes W)$ is a subspace of $\mathfrak{so}(V) \oplus (V \otimes W')$ complementary to $\mathfrak{so}(V)$. The restriction $\psi\colon \mathfrak{b} \to \mathfrak{b}'$ of f to $\mathfrak{b}$ is a Lie algebra isomorphism, while the maps $v \otimes w \mapsto f(v \otimes w)_{\mathfrak{so}(V)}$ and $v \otimes w \mapsto f(v \otimes w)_{V \otimes W}$ are both $\mathfrak{so}(V)$-equivariant. Since $\mathrm{Hom}_{\mathfrak{so}(V)}(V, V) \cong \mathbb{R}$ and $\mathrm{Hom}_{\mathfrak{so}(V)}(V, \mathfrak{so}(V)) \cong \mathbb{R}$ is generated by the map $\varepsilon\colon V \to \mathfrak{so}(V)$, $\varepsilon(v)w = v \times w$, we see that there exists a covector $\mu \in W^*$ and a linear isomorphism $h\colon W \to W'$ such that

$$f(v \otimes w) = \mu(w)\varepsilon(v) + v \otimes h(w), \quad v \in V, w \in W.$$

Consider $B \in \mathfrak{b}$, $v \in V$ and $w \in W$. Because f is an isomorphism, the elements

$$f([B, v \otimes w]) = f(v \otimes \rho(B)w) = \mu(\rho(B)w)\varepsilon(v) + v \otimes h(\rho(B)w)$$

and

$$[f(B), f(v \otimes w)] = [\psi(B), \mu(w)\varepsilon(v) + v \otimes h(w)] = v \otimes \rho'(\psi(B))h(w)$$

are equal, and because B, v, w are arbitrary, this is possible if and only if

$$h(\rho(B)w) = \rho'(\psi(B))h(w), \quad \mu(\rho(B)w) = 0,$$

for all choices of B, v, w. The first condition is equivalent to the equation

$$\rho' \circ \psi = \mathrm{Ad}(h) \circ \rho, \tag{8.24}$$

while the second condition can be read as

$$\partial\mu = 0, \tag{8.25}$$

where we recall that $\partial\colon C^\bullet(\mathfrak{b}; W^*) \to C^{\bullet+1}(\mathfrak{b}; W^*)$ is the Chevalley–Eilenberg differential. In other words, μ is invariant under the dual representation of ρ.

Now, let $v_1, v_2 \in V$ and $w_1, w_2 \in W$. We have

$$\begin{aligned}
&f([v_1 \otimes w_1, v_2 \otimes w_2]) \\
&\quad = f(\alpha(w_1, w_2)v_1 \curlywedge v_2 + (v_1 \times v_2) \otimes \sigma(w_1, w_2) + \langle v_1, v_2\rangle\varphi(w_1 \wedge w_2)) \\
&\quad = (\alpha(w_1, w_2) - \mu(\sigma(w_1, w_2)))v_1 \curlywedge v_2 + (v_1 \times v_2) \otimes h(\sigma(w_1, w_2)) \\
&\qquad + \langle v_1, v_2\rangle\psi(\varphi(w_1 \wedge w_2))
\end{aligned}$$

while using the equality $[\varepsilon(v_1), \varepsilon(v_2)] = \varepsilon(v_1 \times v_2) = -v_1 \curlywedge v_2$ we see that

$$\begin{aligned}
&[f(v_1 \otimes w_1), f(v_2 \otimes w_2)] \\
&\quad = [\mu(w_1)\varepsilon(v_1) + v_1 \otimes h(w_1), \mu(w_2)\varepsilon(v_2) + v_2 \otimes h(w_2)] \\
&\quad = ((h^*\alpha')(w_1, w_2) - \mu(w_1)\mu(w_2))v_1 \curlywedge v_2 \\
&\qquad + \langle v_1, v_2\rangle(h^*\varphi')(w_1 \wedge w_2) \\
&\qquad + (v_1 \times v_2) \otimes (\mu(w_1)h(w_2) + \mu(w_2)h(w_1) + (h^*\sigma')(w_1, w_2)).
\end{aligned}$$

These two elements are equal for all choices of v_i, w_i if and only if

$$h \circ \sigma = h^*\sigma' + \mu \otimes h + h \otimes \mu, \quad \psi \circ \varphi = h^*\varphi'. \tag{8.26}$$

and

$$\alpha = h^*\alpha' + \mu \circ \sigma - \mu \otimes \mu. \tag{8.27}$$

Conversely, suppose that we have a linear isomorphism $h\colon W \to W'$, a Lie algebra isomorphism $\psi\colon \mathfrak{b} \to \mathfrak{b}'$ and a one-form $\mu \in W^*$ that satisfy the relations (8.24), (8.25) and (8.26). Note that we do not require (8.27) to be satisfied. Consider the linear map $f\colon \mathfrak{g} \to \mathfrak{g}'$ defined by $f|_{\mathfrak{so}(V)\oplus\mathfrak{b}} = \mathrm{id}_{\mathfrak{so}(V)} \oplus \psi$ and

$$f(v \otimes w) = \mu(w)\varepsilon(v) + v \otimes h(w), \quad v \in V, w \in W,$$

We claim that f is a relative isomorphism (which in particular implies (8.27)). To see this, consider the Lie bracket $[\cdot,\cdot]_*$ on $\mathfrak{g}'$ that turns $f\colon \mathfrak{g} \to (\mathfrak{g}', [\cdot,\cdot]_*)$ into a Lie algebra isomorphism. This Lie bracket satisfies

$$\begin{aligned} [X, v \otimes w']_* &= f([X, v \otimes h^{-1}(w') - \mu(h^{-1}(w'))\varepsilon(v)]) = Xv \otimes w', \\ [X, B']_* &= f([X, \psi^{-1}(B)]) = 0, \end{aligned}$$

for all $X \in \mathfrak{so}(V)$, $v \in V$, $w' \in W'$ and $B' \in \mathfrak{b}'$, so $(\mathfrak{g}', [\cdot,\cdot]_*)$ is a $(3,2)$-KLA. From the definition of f and the equations (8.24), (8.25) and (8.26) we see that the algebraic data associated with $[\cdot,\cdot]_*$ is $\mathfrak{b}_* = \mathfrak{b}'$, $\rho_* = \rho'$, $\varphi_* = \varphi'$ and $\sigma_* = \sigma'$, and because α_* is obtained from the rest of ingredients, we have necessarily $\alpha_* = \alpha'$. This gives that $\mathfrak{g}$ and $\mathfrak{g}'$ (with the original bracket) are isomorphic, and (8.27) holds.

We conclude the following:

Proposition 8.7. *Let $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ and $\mathfrak{g}' = \mathfrak{so}(V) \oplus (V \otimes W') \oplus \mathfrak{b}'$ be two $(3,2)$-KLAs for which $\dim V = d = 3$ and let $(\mathfrak{b}, \rho, \varphi, \lambda, \sigma)$ and $(\mathfrak{b}', \rho', \varphi', \lambda', \sigma')$ be the algebraic data defining the Lie brackets of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. Assume furthermore that both λ and λ' vanish. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic relative to $\mathfrak{so}(V)$ if and only if there exist*

(i) *a linear isomorphism $h\colon W \to W'$,*

(ii) *a Lie algebra isomorphism $\psi\colon \mathfrak{b} \to \mathfrak{b}'$,*

(iii) *and a covector $\mu \in W^*$,*

satisfying the following equations:

$$\begin{aligned} \rho' \circ \psi &= \mathrm{Ad}(h) \circ \rho, & \psi \circ \varphi &= h^*\varphi', \\ h \circ \sigma &= h^*\sigma' + \mu \otimes h + h \otimes \mu, & \partial\mu &= 0. \end{aligned}$$

The map $f\colon \mathfrak{g} \to \mathfrak{g}'$ defined by

$$\begin{aligned} f(X) &= X, \\ f(v \otimes w) &= \mu(w)\varepsilon(v) + v \otimes h(w), \\ f(B) &= \psi(B) \end{aligned}$$

for all $X \in \mathfrak{so}(V)$, $v \otimes w \in V \otimes W$ and $B \in \mathfrak{b}$, is a relative isomorphism from $\mathfrak{g}$ to $\mathfrak{g}'$. Furthermore, every relative isomorphism arises in this manner.

8.3 Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R})$

As we have already seen in Section 8.1, part of the information that describes the general structure of a $(3,2)$-KLA is given by a two-dimensional representation of a Bianchi Lie algebra. Because of this, it is necessary to develop a systematic approach to determine all possible representations of this type by understanding their possible kernels and images (that is, the ideals of all Bianchi algebras and the subalgebras of $\mathfrak{gl}(2,\mathbb{R})$). Thus, the aim of this section is to provide a classification result of all subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ up to conjugacy. Although this classification has been done several times in the literature [40, 144, 147], there are some discrepancies in the results of these papers. Because of this, we shall repeat this calculation in this thesis as a safety measure.

The key observation is that $\mathfrak{gl}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}I$ is actually a direct sum of a simple Lie algebra and a one-dimensional abelian algebra. In general, one can compute the subalgebras of a direct sum of Lie algebras by means of the so-called Goursat lemma. This result was introduced originally in [75] in order to compute the subgroups of a product of two groups, but the result and its proof are essentially the same in the category of Lie algebras. In what follows, we briefly describe the lemma.

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of Lie algebras and consider the projections $\pi_i \colon \mathfrak{g} \to \mathfrak{g}_i$, which are surjective Lie algebra homomorphisms. To each Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we associate the following pieces of data:

- $\mathfrak{h}_i = \pi_i(\mathfrak{h})$, which is a Lie subalgebra of $\mathfrak{g}_i$ for each $i \in \{1,2\}$,
- $\mathfrak{q} = \mathfrak{h} \cap \mathfrak{g}_2$, which is an ideal of $\mathfrak{h}_2$,
- and the (well-defined) surjective Lie algebra homomorphism $f \colon \mathfrak{h}_1 \to \mathfrak{h}_2/\mathfrak{q}$ given by letting
$$f(X) = Y + \mathfrak{q}, \quad \text{where } (X,Y) \in \mathfrak{h}.$$

Lemma 8.8 (Goursat). *Assume the notation from above. The map $\mathfrak{h} \mapsto (\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{q}, f)$ establishes a bijective correspondence between the set of all Lie subalgebras of $\mathfrak{g}$ and the set of all 4-tuples $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{q}, f)$ such that each $\mathfrak{h}_i$ is a Lie subalgebra of $\mathfrak{g}_i$, $\mathfrak{q}$ is an ideal of $\mathfrak{h}_2$ and $f \colon \mathfrak{h}_1 \to \mathfrak{h}_2/\mathfrak{q}$ is a surjective homomorphism.*

The inverse map is given as follows: for a 4-tuple $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{q}, f)$ satisfying the aforementioned properties, the corresponding subalgebra is defined as
$$\mathfrak{h} = \{(X,Y) \in \mathfrak{g} \colon X \in \mathfrak{h}_1, f(X) = Y + \mathfrak{q}\}.$$

Since we are interested in determining subalgebras of $\mathfrak{g}$ up to conjugacy, we need to take into account how the action of an (inner) automorphism of $\mathfrak{g}$ affects the aforementioned algebraic data. In general, suppose that we are given a representation $\rho\colon \mathsf{G} \to \mathrm{Inn}(\mathfrak{g})$, where $\mathrm{Inn}(\mathfrak{g})$ is the group of inner automorphisms of $\mathfrak{g}$. Because $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$, they are invariant under inner automorphims, so ρ restricts to two subrepresentations $\rho_i \colon \mathsf{G} \to \mathrm{Aut}(\mathfrak{g}_i)$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $g \in \mathsf{G}$. We consider the conjugate subalgebra $\bar{\mathfrak{h}} = \rho(g)\mathfrak{h}$. Then, for their corresponding 4-tuples $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{q}, f)$ and $(\bar{\mathfrak{h}}_1, \bar{\mathfrak{h}}_2, \bar{\mathfrak{q}}, \bar{f})$ we have
$$\bar{\mathfrak{h}}_i = \rho_i(g)\mathfrak{h}_i, \quad \bar{\mathfrak{q}} = \rho_2(g)\mathfrak{q}, \quad \bar{f} = \kappa(g) \circ f \circ \rho_1(g)^{-1}, \tag{8.28}$$

where $\kappa(g)\colon \mathfrak{h}_2/\mathfrak{q} \to \bar{\mathfrak{h}}_2/\bar{\mathfrak{q}}$ sends $Y + \mathfrak{q} \mapsto \rho_2(g)Y + \bar{\mathfrak{q}}$. As a consequence, the Lie subalgebras of $\mathfrak{g}$ up to G-conjugacy are in a bijective correspondence with the orbits of the action given by (8.28).

We now focus on the case at hand. Decompose $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}I$. First we need to know the subalgebras of $\mathfrak{sl}(2,\mathbb{R})$ up to conjugacy. This can be done directly by hand or by appealing to the structure of maximal solvable subalgebras of real semisimple Lie algebras (see for example [126]). We obtain that apart from the obvious subalgebras 0 and $\mathfrak{sl}(2,\mathbb{R})$, these subalgebras are up to conjugacy

$$\begin{aligned} \mathfrak{t}_0(2,\mathbb{R}) &= \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}, & \mathfrak{so}(2) &= \mathbb{R}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \\ \mathfrak{a} &= \mathbb{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, & \mathfrak{n} &= \mathbb{R}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \end{aligned} \tag{8.29}$$

On the other hand, given any 4-tuple $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{q}, f)$, the pair $(\mathfrak{h}_2, \mathfrak{q})$ is either $(0,0)$, $(\mathbb{R}I, \mathbb{R}I)$ or $(\mathbb{R}I, 0)$. Note that in the first two cases we have $\mathfrak{h}_2/\mathfrak{q} = 0$, and thus f is necessarily the zero map. If $\mathfrak{q} = \mathfrak{h}_2 = 0$, then the corresponding subalgebra is $\mathfrak{h} = \mathfrak{h}_1$, whereas if $\mathfrak{q} = \mathfrak{h}_2 = \mathbb{R}I$, the corresponding subalgebra is $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathbb{R}I$. The subalgebras obtained by adding $\mathbb{R}I$ to a nonzero proper subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ are

$$\begin{aligned} \mathfrak{t}_0(2,\mathbb{R}) \oplus \mathbb{R}I &= \mathfrak{t}(2,\mathbb{R}) = \left\{\begin{pmatrix}x & y\\ 0 & z\end{pmatrix} : x,y,z \in \mathbb{R}\right\}, \\ \mathfrak{so}(2) \oplus \mathbb{R}I &= \mathbb{C}, \\ \mathfrak{a} \oplus \mathbb{R}I &= \mathfrak{diag}(2,\mathbb{R}) = \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}, \\ \mathfrak{n} \oplus \mathbb{R}I &= \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}. \end{aligned}$$

We now consider the case that $\mathfrak{h}_2 = \mathbb{R}I$ and $\mathfrak{q} = 0$. Because we need the existence of a surjective homomorphism $\mathfrak{h}_1 \to \mathfrak{h}_2/\mathfrak{q} = \mathbb{R}I$, $\mathfrak{h}_1$ is either $\mathfrak{t}_0(2,\mathbb{R})$ or any of the one-dimensional subalgebras.

For the case $\mathfrak{h}_1 = \mathfrak{t}_0(2,\mathbb{R})$, a surjective homomorphism $f\colon \mathfrak{t}_0(2,\mathbb{R}) \to \mathbb{R}I$ has to vanish in the ideal $\mathfrak{n} \subseteq \mathfrak{t}_0(2,\mathbb{R})$, and we have

$$f\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = yI, \quad y \in \mathbb{R}\setminus\{0\}.$$

The normalizer of $\mathfrak{t}_0(2,\mathbb{R})$ in $\mathsf{GL}(2,\mathbb{R})$ is the subgroup $\mathsf{T}(2,\mathbb{R})$ of all upper triangular matrices with nonzero determinant, and it acts trivially on the space of surjective homomorphisms $\mathfrak{t}_0(2,\mathbb{R}) \to \mathbb{R}I$, so different values of y yield nonconjugate subalgebras of the form

$$\operatorname{span}\left\{\begin{pmatrix}y+1 & 0\\ 0 & y-1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix}\cos\theta & 0\\ 0 & \sin\theta\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\} = \mathfrak{s}_\theta,$$

where
$$(\cos\theta,\sin\theta) = \frac{-1}{\sqrt{2(y^2+1)}}(y+1,y-1).$$
This establishes a bijective correspondence between $y \in \mathbb{R}\setminus\{0\}$ and $\theta \in (\frac{\pi}{4},\frac{5\pi}{4})\setminus\{\frac{3\pi}{4}\}$, and because θ and $\theta+\pi$ yield the same subalgebra we can directly suppose that $\theta \in [0,\pi)\setminus\{\frac{\pi}{4},\frac{3\pi}{4}\}$. Note that the case $\theta=\frac{\pi}{4}$ gives $\mathfrak{h}=\mathfrak{n}\oplus\mathbb{R}I$ and the case $\theta=\frac{3\pi}{4}$ gives $\mathfrak{h}=\mathfrak{t}_0(2,\mathbb{R})$.

Suppose that $\mathfrak{h}_1=\mathfrak{so}(2)$. The map f satisfies
$$f\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = yI, \quad y\in\mathbb{R}\setminus\{0\}.$$
The stabilizer of $\mathfrak{so}(2)$ in $\mathsf{GL}(2,\mathbb{R})$ is
$$\mathsf{CO}(2)\rtimes\mathbb{Z}_2 = \left\{\begin{pmatrix}a & -b\\ b & a\end{pmatrix} : a^2+b^2\neq 0\right\}\rtimes\left\langle\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\rangle$$
(the notation $\langle g\rangle$ denotes the subgroup generated by g), where $\mathsf{CO}(2)$ leaves f unchanged and the outer $\mathbb{Z}_2$ allows us to change the sign of f. Thus, a complete set of nonconjugate subalgebras is obtained by letting $y>0$, and these are of the form
$$\mathbb{R}\begin{pmatrix}y & -1\\ 1 & y\end{pmatrix}.$$
Note that the algebra above in the limit case $y=0$ is precisely $\mathfrak{so}(2)$.

Now, suppose $\mathfrak{h}_1=\mathfrak{a}$. A surjective map $f\colon\mathfrak{a}\to\mathbb{R}I$ is of the form
$$f\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = yI, \quad y\in\mathbb{R}\setminus\{0\}.$$
The stabilizer of $\mathfrak{a}$ is the semidirect product
$$\mathsf{Diag}(2,\mathbb{R})\rtimes\mathbb{Z}_2 = \left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} : ab\neq 0\right\}\rtimes\left\langle\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\rangle,$$
where only the outer $\mathbb{Z}_2$ acts nontrivially on f by changing its sign. We conclude that a complete family of nonconjugate subalgebras is obtained by letting $y<0$, giving us
$$\mathbb{R}\begin{pmatrix}y+1 & 0\\ 0 & y-1\end{pmatrix} = \mathbb{R}\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}, \quad x=\frac{y+1}{y-1}\in(-1,1).$$
Different values of x give rise to nonconjugate subalgebras. Observe that the case $x=-1$ corresponds to $\mathfrak{a}$ whereas the case $x=1$ gives $\mathbb{R}I$.

Finally, suppose that $\mathfrak{h}_1=\mathfrak{n}$. Then the map f is such that
$$f\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = yI, \quad y\neq 0.$$

The normalizer of $\mathfrak{n}$ is $\mathsf{T}(2,\mathbb{R})$ and its action allows us to rescale f freely, which means that we can simply set $y = 1$ and obtain the subalgebra

$$\mathbb{R}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We can now collect our findings in the following result.

Theorem 8.9. *Every nonzero proper subalgebra of* $\mathfrak{gl}(2,\mathbb{R})$ *is conjugate to exactly one of the subalgebras given in Table 8.1.*

Table 8.1: Representatives of the conjugacy classes of proper subalgebras of $\mathfrak{gl}(2,\mathbb{R})$.

Subalgebra	**Dimension**	**Comments**
$\mathfrak{sl}(2,\mathbb{R})$	3	Simple
$\mathfrak{t}(2,\mathbb{R})$	3	Solvable
$\mathfrak{diag}(2,\mathbb{R})$	2	Abelian
$\mathfrak{s}_\theta = \operatorname{span}\left\{\begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$	2	$\theta \in [0,\pi)$, abelian $\Leftrightarrow \theta = \frac{\pi}{4}$
$\mathbb{C} \cong \mathbb{R}I \oplus \mathfrak{so}(2)$	2	Abelian
$\mathbb{R}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$	1	$\lvert x\rvert \leq 1$
$\mathbb{R}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1	
$\mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	1	
$\mathbb{R}\begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$	1	$y \geq 0$

Among the one-dimensional subalgebras, $\mathfrak{so}(1,1)$ corresponds to the first one with $x = -1$ and $\mathfrak{so}(2)$ corresponds to the last with $y = 0$.

Due to Theorems A and B, it is important to determine which subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ admit nonzero invariant symmetric bilinear forms $\alpha \in \mathsf{S}^2(\mathbb{R}^2)^*$, as well as which admit a nonzero $\mathbb{R}^2$-valued symmetric bilinear form $\sigma \in \mathbb{R}^2 \otimes \mathsf{S}^2(\mathbb{R}^2)^*$. From the determination of the relative isomorphisms in Section 8.2, it is also important to determine which subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ leave a nonzero covector in $(\mathbb{R}^2)^*$ invariant.

This latter condition is easy to see by inspection of the conjugacy classes of subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ in Table 8.1. The result is tabulated in Table 8.2.

Table 8.2: Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ that leave a linear form invariant.

Subalgebra $\mathfrak{h}$	$((\mathbb{R}^2)^*)^{\mathfrak{h}}$
$\mathfrak{s}_0 = \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}$	$\mathbb{R}\boldsymbol{e}^2$
$\mathbb{R}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$	$\mathbb{R}\boldsymbol{e}^1$
$\mathbb{R}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$	$\mathbb{R}\boldsymbol{e}^2$

Up to a change of basis and scaling, there are only three nontrivial symmetric bilinear forms on $\mathbb{R}^2$, which are given by

$$\boldsymbol{e}^1 \otimes \boldsymbol{e}^1 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^2, \quad \boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1, \quad \boldsymbol{e}^2 \otimes \boldsymbol{e}^2,$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2$ is a basis for $\mathbb{R}^2$ and $\boldsymbol{e}^1, \boldsymbol{e}^2$ is the canonically dual basis. The corresponding stabilizer subalgebras are

$$\mathfrak{so}(2) = \mathbb{R}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \mathfrak{so}(1,1) = \mathbb{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \mathfrak{s}_0 = \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}.$$

Therefore, we can also list the Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ that admit an invariant quadratic form in Table 8.3.

Table 8.3: Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ that leave a quadratic form invariant.

Subalgebra $\mathfrak{h}$	$\alpha \in (\mathsf{S}^2(\mathbb{R}^2)^*)^{\mathfrak{h}}$
$\mathfrak{s}_0 = \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}$	$\boldsymbol{e}^2 \otimes \boldsymbol{e}^2$
$\mathfrak{so}(2)$	$\boldsymbol{e}^1 \otimes \boldsymbol{e}^1 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$
$\mathfrak{so}(1,1)$	$\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1$
$\mathbb{R}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$	$\boldsymbol{e}^2 \otimes \boldsymbol{e}^2$
$\mathbb{R}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$	$\boldsymbol{e}^1 \otimes \boldsymbol{e}^1$

In the last line, we have conjugated the invariant quadratic form under an element of $\mathsf{GL}(2,\mathbb{R})$ so that its stabilizer subalgebra agrees with one in Table 8.1.

Finally, the determination of the Lie subalgebras that admit a nonzero invariant bilinear map $\sigma\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is a routine linear algebra exercise and we only list the results in Table 8.4.

Table 8.4: Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ that leave invariant some $\sigma \in \mathsf{S}^2(\mathbb{R}^2)^* \otimes \mathbb{R}^2$.

Subalgebra $\mathfrak{h}$	$\sigma \in (\mathbb{R}^2 \otimes \mathsf{S}^2(\mathbb{R}^2)^*)^{\mathfrak{h}}$
$\mathfrak{s}_0 = \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}$	$\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$
$\mathfrak{s}_{\arctan(1/2)} = \operatorname{span}\left\{\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}$	$\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$
$\mathbb{R}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$	$\boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1$, $\boldsymbol{e}_2 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)$
$\mathbb{R}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\end{pmatrix}$	$\boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1$
$\mathbb{R}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$	$\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$, $\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$

8.4 Real two-dimensional representations of Bianchi Lie algebras

As discussed in Section 8.1, every $(3,2)$-KLA $\mathfrak{g}$ has an underlying Bianchi Lie algebra $\mathfrak{b}$ and a two-dimensional real representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$. Furthermore, if $\mathfrak{g}'$ is another $(3,2)$-KLA (with associated representation $\rho'\colon \mathfrak{b}' \to \mathfrak{gl}(W')$) which differs from $\mathfrak{g}$ via a relative isomorphism, due to the results in Propositions 8.6 and 8.7 there exists a Lie algebra isomorphism $\psi\colon \mathfrak{b} \to \mathfrak{b}'$ and a linear isomorphism $h\colon W \to W'$ such that $\rho'(\psi(B)) = h\rho(B)h^{-1}$ for all $B \in \mathfrak{b}$. This motivates the following definition.

Let $\mathfrak{b}$, $\mathfrak{b}'$ be two Lie algebras and $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$, $\rho'\colon \mathfrak{b}' \to \mathfrak{gl}(W')$ linear representations of $\mathfrak{b}$ and $\mathfrak{b}'$ respectively. We say that ρ and ρ' are *weakly equivalent* or *weakly isomorphic* if there exists a linear isomorphism $h\colon W \to W'$ and a Lie algebra isomorphism $\psi\colon \mathfrak{b} \to \mathfrak{b}'$ satisfying $\rho' \circ \psi = \mathrm{Ad}(h) \circ \rho$. The weak equivalence class of the representation ρ is also called its *(weak) isoclass*. Recall here that $\mathrm{Ad}(h)\colon \mathfrak{gl}(W) \to \mathfrak{gl}(W')$ is defined by $\mathrm{Ad}(h)T = h \circ T \circ h^{-1}$. Thus, for the classification of $(3,2)$-KLAs it suffices to know the isomorphism classes of all Bianchi algebras and, for each Bianchi algebra $\mathfrak{b}$, the weak isoclasses of two-dimensional representations of $\mathfrak{b}$.

The classification of three-dimensional Lie algebras was carried out by Bianchi (see for example [25]). It follows from Bianchi's work that every three-dimensional Lie algebra is isomorphic to an algebra $\mathfrak{b} = \mathrm{span}\{B_1, B_2, B_3\}$ whose Lie bracket is determined by exactly one of the possibilities listed in Table 8.5.

Table 8.5: Isomorphism classes of 3-dimensional real Lie algebras.

Bianchi	**Nonzero brackets**			**Structure**
I				Abelian
II			$[B_2, B_3] = B_1$	Nilpotent, unimodular
III $\cong$ VI$_1$			$[B_2, B_3] = 2B_2$	Solvable
IV		$[B_1, B_3] = B_1$	$[B_2, B_3] = B_1 + B_2$	Solvable
V		$[B_1, B_3] = B_1$	$[B_2, B_3] = B_2$	Solvable
VI$_{1\neq c\geq 0}$		$[B_1, B_3] = (c-1)B_1$	$[B_2, B_3] = (c+1)B_2$	Solvable
VII$_{c\geq 0}$		$[B_1, B_3] = cB_1 - B_2$	$[B_2, B_3] = B_1 + cB_2$	Solvable
VIII	$[B_1, B_2] = -B_3$	$[B_1, B_3] = -B_2$	$[B_2, B_3] = B_1$	Simple
IX	$[B_1, B_2] = B_3$	$[B_1, B_3] = -B_2$	$[B_2, B_3] = B_1$	Simple

Throughout the rest of this section we list all the possible representations $\rho\colon \mathfrak{b} \to \mathfrak{gl}(2,\mathbb{R})$ up to weak equivalence. For each such representation, we also determine the possible invariant $\alpha \in \mathsf{S}^2W^*$, $\sigma\colon \mathsf{S}^2W \to W$ and $\varphi\colon \Lambda^2W \to \mathfrak{b}$, as well as recording the calculation of the $\mathfrak{b}$-invariant vectors in W^*. We also compute the subgroup $\mathrm{Aut}(\rho)$ of $\mathsf{GL}(W) \times \mathrm{Aut}(\mathfrak{b})$ that preserves ρ. More precisely,

$$\operatorname{Aut}(\rho) = \{(h, \psi) \in \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b}) \colon \rho \circ \psi = \operatorname{Ad}(h) \circ \rho\} .$$

We call $\operatorname{Aut}(\rho)$ the *weak automorphism group* of ρ.

8.4.1 Bianchi I

This is the abelian three-dimensional Lie algebra and hence $\operatorname{Aut}(\mathfrak{b}) = \mathsf{GL}(\mathfrak{b})$, which acts transitively on one- and two-dimensional vector subspaces of $\mathfrak{b}$. The image $\rho(\mathfrak{b}) \subseteq \mathfrak{gl}(2,\mathbb{R})$ must be an abelian Lie algebra and they can be read off from Table 8.1: $\mathfrak{diag}(2,\mathbb{R})$, $\mathfrak{s}_{\pi/4}$, $\mathbb{C} \cong \mathbb{R}I \oplus \mathfrak{so}(2)$, any one-dimensional subalgebra and the zero subalgebra.

I.1: $\operatorname{im}\rho = \mathfrak{diag}(2,\mathbb{R})$

We may choose a basis $\{B_1, B_2, B_3\}$ for $\mathfrak{b}$ such that $\rho(B_3) = 0$, whereas

$$\rho(B_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(B_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For this representation, we have that α and σ are both zero, by inspection of Table 8.3 and Table 8.4. Similarly, because $\Lambda^2 W$ is a nontrivial representation and $\mathfrak{b}$ is the trivial representation, there cannot be any nonzero equivariant map $\Lambda^2 W \to \mathfrak{b}$, so $\varphi = 0$. Finally, there are no nonzero $\mathfrak{b}$-invariants in W^*. The group of weak automorphisms of ρ is

$$\begin{aligned} \operatorname{Aut}(\rho) = &\left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right) : h_1 h_2 a_{33} \neq 0 \right\} \\ &\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right) : h_1 h_2 a_{33} \neq 0 \right\}. \end{aligned}$$

I.2: $\operatorname{im}\rho = \mathfrak{s}_{\pi/4}$

We may choose a basis $\{B_1, B_2, B_3\}$ for $\mathfrak{b}$ such that $\rho(B_3) = 0$, whereas

$$\rho(B_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(B_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The situation here is just like the previous case: α, σ and $(W^*)^{\mathfrak{b}}$ are all zero. Since $\Lambda^2 W$ is a nontrivial representation, φ vanishes identically as well. We also have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} a_{22} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right) : h_1 a_{22} a_{33} \neq 0 \right\}.$$

I.3: $\operatorname{im}\rho = \mathbb{C}$

We may choose a basis $\{B_1, B_2, B_3\}$ for $\mathfrak{b}$ such that $\rho(B_3) = 0$, whereas

$$\rho(B_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(B_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have again that φ vanishes identically. From the tables, it follows that so do α, σ and $(W^*)^{\mathfrak{b}}$. The group $\operatorname{Aut}(\rho)$ is equal to

$$\left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -\varepsilon h_2 & \varepsilon h_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right) : a_{33}(h_1^2 + h_2^2) \neq 0, \varepsilon = \pm 1 \right\}.$$

I.4: $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \leq 1$

We may choose a basis $\{B_1, B_2, B_3\}$ for $\mathfrak{b}$ such that $\rho(B_2) = \rho(B_3) = 0$ and

$$\rho(B_1) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

with $|x| \leq 1$. Here all $\alpha, \sigma, \varphi, (W^*)^{\mathfrak{b}}$ are zero except in the following cases:

- $x = -1$, which corresponds to $\mathfrak{so}(1,1)$. Here we have $\sigma = 0$, but $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)$. Similarly, $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$ can be nonzero.
- $x = \frac{1}{2}$. Here $\sigma \in \mathbb{R}(\boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1)$, but $\alpha, \varphi, (W^*)^{\mathfrak{b}}$ are zero.
- $x = 0$. Here $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1) \oplus \mathbb{R}(\boldsymbol{e}_2 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1))$,$(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^1$, $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^1)$, but φ still vanishes.

The stabilizer $\operatorname{Aut}(\rho)$ of ρ depends on x. Indeed, we have:

- If $x = 1$, then

$$\operatorname{Aut}(\rho) = \left\{ \left(h, \begin{pmatrix} 1 & 0 \\ u & A \end{pmatrix} \right) : h, A \in \mathsf{GL}(2,\mathbb{R}), u \in \mathbb{R}^2 \right\}.$$

- If $x = -1$, then

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ u & A \end{pmatrix} \right) : u \in \mathbb{R}^2, h_1 h_2 (\det A) \neq 0 \right\}$$
$$\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ u & A \end{pmatrix} \right) : u \in \mathbb{R}^2, h_1 h_2 (\det A) \neq 0 \right\}.$$

- If $x \neq \pm 1$, then

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ u & A \end{pmatrix} \right) : u \in \mathbb{R}^2, h_1 h_2 (\det A) \neq 0 \right\}.$$

I.5: $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

We may choose a basis $\{B_1, B_2, B_3\}$ for $\mathfrak{b}$ such that $\rho(B_2) = \rho(B_3) = 0$ and

$$\rho(B_1) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

Here all α, σ and $(W^*)^{\mathfrak{b}}$ are zero from the tables and φ is readily seen to be zero. The group of weak automorphisms of ρ is

$$\operatorname{Aut}(\rho) = \left\{\left(\begin{pmatrix}h_1 & h_2\\ 0 & h_1\end{pmatrix}, \begin{pmatrix}1 & 0\\ u & A\end{pmatrix}\right) : h_1(\det A) \neq 0\right\}.$$

I.6: $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

We may choose a basis $\{B_1, B_2, B_3\}$ for $\mathfrak{b}$ such that $\rho(B_2) = \rho(B_3) = 0$ and

$$\rho(B_1) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Here $\alpha \in \mathbb{R}(\boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$ and

$$\sigma \in \mathbb{R}\left(\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2\right) \oplus \mathbb{R}\left(\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2\right).$$

Also, φ is unconstrained. From the tables, $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$. One sees that the stabilizer of ρ is

$$\operatorname{Aut}(\rho) = \left\{\left(\begin{pmatrix}ah_1 & h_2\\ 0 & h_1\end{pmatrix}, \begin{pmatrix}a & 0\\ u & B\end{pmatrix}\right) : ah_1(\det B) \neq 0\right\}.$$

I.7: $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix}y & -1\\ 1 & y\end{pmatrix}$ **with** $y \geq 0$

We may choose basis $\{B_1, B_2, B_3\}$ for $\mathfrak{b}$ such that $\rho(B_2) = \rho(B_3) = 0$ and

$$\rho(B_1) = \begin{pmatrix}y & -1\\ 1 & y\end{pmatrix}.$$

Different values of y give rise to nonconjugate subalgebras, and thus to representations which are not weakly equivalent. Here σ always vanishes and so do α and φ unless $y = 0$, which corresponds to $\mathfrak{so}(2)$. In that case, $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^1 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$. Similarly, $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$ can be nonzero. From the tables, we see that $(W^*)^{\mathfrak{b}} = 0$. If $y = 0$, the group of weak automorphisms of ρ is

$$\operatorname{Aut}(\rho) = \left\{\left(\begin{pmatrix}h_1 & h_2\\ -\varepsilon h_2 & \varepsilon h_1\end{pmatrix}, \begin{pmatrix}\varepsilon & 0\\ u & A\end{pmatrix}\right) : (h_1^2 + h_2^2)(\det A) \neq 0, \varepsilon = \pm 1\right\}.$$

I.8: $\operatorname{im}\rho = 0$

Here both the representation W and the adjoint representation $\mathfrak{b}$ are trivial and hence every linear map is trivially invariant. Clearly, we have $\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b})$.

Summary

We summarize the results of this section in Table 8.6.

Table 8.6: Two-dimensional real representations of Bianchi I.

Label	$\rho(B_1)$	$\rho(B_2)$	$\rho(B_3)$	**Remarks**	α	σ	φ	$(W^*)^{\mathfrak{b}}$
I.1	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	0					
I.2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	0					
I.3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0					
I.4_x	$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$	0	0	$\lvert x\rvert \leq 1$	$x = -1, 0$	$x = 0, \frac{1}{2}$	$x = -1$	$\mathbb{R}\boldsymbol{e}^1$ $(x = 0)$
I.5	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	0	0					
I.6	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	0	0		✓	✓	✓	$\mathbb{R}\boldsymbol{e}^2$
I.7_y	$\begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$	0	0	$y \geq 0$	$y = 0$		$y = 0$	
I.8	0	0	0		✓	✓	✓	W^*

Using the basis $\{B_1, B_2, B_3\}$ in Table 8.5 for the Bianchi I Lie algebra $\mathfrak{b}$, each row is a representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(2,\mathbb{R})$. The representation has some parameters which in the absence of any Remarks take any possible real values. We also indicate for each representation whether there are any invariant $\alpha \in \mathsf{S}^2(\mathbb{R}^2)^*$, $\sigma \in \mathbb{R}^2 \otimes \mathsf{S}^2(\mathbb{R}^2)^*$, $\varphi \in \mathfrak{b} \otimes \Lambda^2(\mathbb{R}^2)^*$. A ✓ indicates existence of a nonzero such invariant for any (allowed) value of the parameters, whereas a blank cell indicates its absence. Any other expression indicates the values of any parameter where a nonzero invariant exists. The last column is the $\mathfrak{b}$-invariant subspace of W^*: it is assumed to be zero unless otherwise noted.

8.4.2 Bianchi II

This is the Heisenberg Lie algebra with brackets $[B_2, B_3] = B_1$ relative to a choice of basis $\{B_1, B_2, B_3\}$. The automorphism group is given by

$$\mathrm{Aut}(\mathfrak{b}) = \left\{ \begin{pmatrix} \det A & a^t \\ 0 & A \end{pmatrix} : A \in \mathsf{GL}(2,\mathbb{R}),\ a \in \mathbb{R}^2 \right\}.$$

The proper ideals are the first derived ideal $[\mathfrak{b},\mathfrak{b}] = \mathbb{R}B_1$ and the span of B_1 and any nonzero vector in the (B_2, B_3)-plane. Since $\mathrm{Aut}(\mathfrak{b})$ acts transitively on such nonzero vectors, we are free to consider only the ideal spanned by $\{B_1, B_2\}$.

It follows from Table 8.1 that there are no nilpotent nonabelian Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R})$; hence $\ker\rho \neq 0$ for any two-dimensional representation ρ. We discuss in turn the cases $\ker\rho = \mathfrak{b}$, $\ker\rho = \mathbb{R}B_1$ and $\ker\rho = \mathrm{span}\{B_1, B_2\}$. For $\ker\rho = \mathbb{R}B_1$, because $\mathfrak{b}/\mathbb{R}B_1$ is abelian, $\mathrm{im}\,\rho$ can be any two-dimensional abelian subalgebra of $\mathfrak{gl}(2,\mathbb{R})$, so it is conjugate to either $\mathfrak{diag}(2,\mathbb{R})$, $\mathfrak{s}_{\pi/4}$ or $\mathbb{C}$.

II.1: $\ker\rho = \mathbb{R}B_1$ **and** $\mathrm{im}\,\rho = \mathfrak{diag}(2,\mathbb{R})$

We can choose bases for W and $\mathfrak{b}$ such that $\rho(B_1) = 0$, whereas

$$\rho(B_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(B_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

From the tables in Section 8.3, we obtain $\alpha = 0$, $\sigma = 0$ and $(W^*)^{\mathfrak{b}} = 0$. Since $\Lambda^2 W$ is a nontrivial $\mathfrak{b}$-module and the only one-dimensional ideal of $\mathfrak{b}$ is central, there are no nonzero equivariant maps $\Lambda^2 W \to \mathfrak{b}$, hence $\varphi = 0$. The weak automorphism group of ρ is

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) : h_1 h_2 \neq 0 \right\}$$
$$\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & a_1 & a_2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) : h_1 h_2 \neq 0 \right\}.$$

II.2: $\ker\rho = \mathbb{R}B_1$ **and** $\mathrm{im}\,\rho = \mathfrak{s}_{\pi/4}$

We can choose bases for W and $\mathfrak{b}$ such that $\rho(B_1) = 0$, whereas

$$\rho(B_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(B_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the tables in Section 8.3, it follows that $\alpha = 0$, $\sigma = 0$ and $(W^*)^{\mathfrak{b}} = 0$. Since B_2 is an isomorphism when acting on either $\Lambda^2 W$, but is nilpotent in the adjoint representation, there can be no nonzero equivariant maps $\Lambda^2 W \to \mathfrak{b}$, which yields $\varphi = 0$. We also have

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} dh_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} d & a_1 & a_2 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix} \right) : dh_1 \neq 0 \right\}.$$

II.3: $\ker\rho = \mathbb{R}B_1$ **and** $\operatorname{im}\rho = \mathbb{C}$

We can choose bases for W and $\mathfrak{b}$ such that $\rho(B_1) = 0$, whereas

$$\rho(B_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(B_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This is very similar to the previous case, since $\rho(B_2)$ takes the same form. All of α, σ, φ and $(W^*)^{\mathfrak{b}}$ are zero. We also see that

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -\varepsilon h_2 & \varepsilon h_1 \end{pmatrix}, \begin{pmatrix} \varepsilon & a_1 & a_2 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \right) : h_1^2 + h_2^2 \neq 0, \varepsilon = \pm 1 \right\}.$$

II.4: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \leq 1$

We can choose bases for W and $\mathfrak{b}$ such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

with $|x| \leq 1$. Different values of x give rise to diferent weak equivalence classes of representations.

All of $\alpha, \sigma, \varphi, (W^*)^{\mathfrak{b}}$ are zero except for the following values of x:

- $x = 0$. Here $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^1)$, $\sigma \in \operatorname{span}\{\boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1, \boldsymbol{e}_2 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)\}$ and $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^1$. The map φ is zero.
- $x = \frac{1}{2}$. Here $\sigma \in \mathbb{R}(\boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1)$ and all other are zero.
- $x = -1$, which is $\mathfrak{so}(1,1)$. Here one sees that $\sigma = 0$ and $(W^*)^{\mathfrak{b}} = 0$, whereas $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)$ and $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$.

As for the group of weak automorphisms, we have:

- If $x = 1$, one sees that

$$\operatorname{Aut}(\rho) = \left\{ \left(h, \begin{pmatrix} a & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & 1 \end{pmatrix} \right) : a(\det h) \neq 0 \right\}.$$

- If $x = -1$, we have

$$\begin{aligned} \operatorname{Aut}(\rho) = &\left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & 1 \end{pmatrix} \right) : a h_1 h_2 \neq 0 \right\} \\ &\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} -a & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & -1 \end{pmatrix} \right) : a h_1 h_2 \neq 0 \right\}. \end{aligned}$$

- If $x \neq \pm 1$, we obtain

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & 1 \end{pmatrix} \right) : ah_1h_2 \neq 0 \right\}.$$

II.5: $\ker\rho = \mathrm{span}\{B_1, B_2\}$ **and** $\mathrm{im}\,\rho = \mathbb{R}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

We can choose bases for W and $\mathfrak{b}$ such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From the tables, α, σ and $(W^*)^{\mathfrak{b}}$ are all zero. Since the action of B_3 on $\Lambda^2 W$ is an isomorphism but its adjoint action on $\mathfrak{b}$ is nilpotent, we have that φ is zero as well. The stabilizer subgroup of ρ is

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & 1 \end{pmatrix} \right) : ah_1 \neq 0 \right\}.$$

II.6: $\ker\rho = \mathrm{span}\{B_1, B_2\}$ **and** $\mathrm{im}\,\rho = \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

We can choose bases for W and $\mathfrak{b}$ such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the tables, we see that $\alpha \in \mathbb{R}(\boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$,

$$\sigma \in \mathrm{span}\{\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2, \boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2\}$$

and $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$. It follows that $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$. We also have

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} dh_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} ad & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & d \end{pmatrix} \right) : adh_1 \neq 0 \right\}.$$

II.7: $\ker\rho = \mathrm{span}\{B_1, B_2\}$ **and** $\mathrm{im}\,\rho = \mathbb{R}\begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$ **with** $y \geq 0$

Here we can choose bases for W and $\mathfrak{b}$ such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$$

for some $y \geq 0$. Furthermore, different values of y yield different weak isoclasses of represena-tions.

Here $\sigma = 0$ always and the same is true for α and φ unless $y = 0$, in which case we have $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^1 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$ and $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$. From the tables we read off that $(W^*)^{\mathfrak{b}} = 0$. As for the group of weak automorphisms, we see that if $y = 0$, it satisfies

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -\varepsilon h_2 & \varepsilon h_1 \end{pmatrix}, \begin{pmatrix} \varepsilon a & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & \varepsilon \end{pmatrix} \right) : a(h_1^2 + h_2^2) \neq 0, \varepsilon = \pm 1 \right\},$$

while for $y \neq 0$ we have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}, \begin{pmatrix} a & b_1 & b_2 \\ 0 & a & c \\ 0 & 0 & 1 \end{pmatrix} \right) : a(h_1^2 + h_2^2) \neq 0 \right\}.$$

II.8: $\ker \rho = \mathfrak{b}$

Here α and σ are not constrained at all, whereas $\operatorname{im} \varphi \in \mathbb{R}B_1$ and all of W^* is invariant. We trivially have $\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b})$.

Summary

We summarize the results of this section in Table 8.7, with the same notation as that explained in Table 8.6.

Table 8.7: Two-dimensional real representations of Bianchi II.

Label	$\rho(B_1)$	$\rho(B_2)$	$\rho(B_3)$	**Remarks**	α	σ	φ	$(W^*)^{\mathfrak{b}}$
$\mathrm{II}.1$	0	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$					
$\mathrm{II}.2$	0	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$					
$\mathrm{II}.3$	0	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$					
$\mathrm{II}.4_x$	0	0	$\begin{pmatrix}x&0\\0&1\end{pmatrix}$	$\|x\| \leq 1$	$x=-1,0$	$x=0,\frac{1}{2}$	$x=-1$	$\mathbb{R}\boldsymbol{e}^1$ $(x=0)$
$\mathrm{II}.5$	0	0	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$					
$\mathrm{II}.6$	0	0	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$		✓	✓	✓	$\mathbb{R}\boldsymbol{e}^2$
$\mathrm{II}.7_y$	0	0	$\begin{pmatrix}y&-1\\1&y\end{pmatrix}$	$y \geq 0$	$y=0$		$y=0$	
$\mathrm{II}.8$	0	0	0		✓	✓	✓	W^*

8.4.3 Bianchi III

This is the unique nonabelian decomposable Bianchi Lie algebra with B_1 central and $[B_2, B_3] = 2B_2$. It is the special case of Bianchi VI_c with $c = 1$, but it is convenient to treat it separately and then simply assume that $c \neq 1$ when we treat Bianchi VI_c in Section 8.4.6. The automorphism group is given by

$$\mathrm{Aut}(\mathfrak{b}) = \left\{ \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} : ab \neq 0 \right\}.$$

The proper ideals are $\mathbb{R}B_1$, $\mathbb{R}B_2$ and $\mathrm{span}\{B_2, \sin\theta B_1 + \cos\theta B_3\}$ for $\theta \in [0, \pi)$. Up to the action of the automorphisms, there are two cases for the two-dimensional ideal: if $\theta \neq \frac{\pi}{2}$, we may shift $B_3 \mapsto B_3 - \tan\theta B_1$ to obtain the ideal $\mathrm{span}\{B_2, B_3\}$; whereas if $\theta = \frac{\pi}{2}$, then we just have the ideal $\mathrm{span}\{B_1, B_2\}$.

The Lie algebra $\mathfrak{b}$ is actually isomorphic to $\mathfrak{t}(2, \mathbb{R})$. Indeed, an isomorphism is the map $\iota\colon \mathfrak{b} \to \mathfrak{t}(2, \mathbb{R})$ given by

$$\iota(B_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \iota(B_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \iota(B_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.30}$$

This means that the possible kernels for representations $\rho : \mathfrak{b} \to \mathfrak{gl}(W)$ are 0, $\mathbb{R}B_1$, $\mathbb{R}B_2$,

$\operatorname{span}\{B_1, B_2\}$, $\operatorname{span}\{B_2, B_3\}$ and $\mathfrak{b}$. Note that in the case $\ker\rho = \mathbb{R}B_2$, the quotient $\mathfrak{b}/\mathbb{R}B_1$ is nonabelian, so the image of ρ is conjugate to either $\mathfrak{s}_{\pi/4}$, $\mathfrak{diag}(2,\mathbb{R})$ or $\mathbb{C}$.

III.1: $\ker\rho = 0$

In this case, ρ is an isomorphism $\mathfrak{b} \to \mathfrak{t}(2,\mathbb{R})$. It follows from the tables that α, σ and $(W^*)^{\mathfrak{b}}$ are all zero. Up to an automorphism of $\mathfrak{b}$, we can take $\rho = \iota$, the isomorphism $\iota\colon \mathfrak{b} \xrightarrow{\cong} \mathfrak{t}(2,\mathbb{R})$ in (8.30). It follows that $\rho(B_1)\,(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = 2\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$, so that $[B_1, \varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)] = 2\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$, but B_1 is central, hence $\varphi = 0$. The weak automorphisms of ρ are given by

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} bt & \frac{tv}{2} \\ 0 & t \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bt \neq 0 \right\}.$$

III.2: $\ker\rho = \mathbb{R}B_1$

In this case we have (with respect to an appropriate basis of W) $\operatorname{im}\rho = \mathfrak{s}_\theta$ for $\theta \neq \frac{\pi}{4}$, since $\mathfrak{b}/\mathbb{R}B_1$ is nonabelian. We have that $\rho(B_1) = 0$ and

$$\rho(B_2) = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} x & s \\ 0 & y \end{pmatrix},$$

where, in particular, $r \neq 0$. Imposing $[\rho(B_2), \rho(B_3)] = 2\rho(B_2)$ says that $y = x + 2$. We can rescale $B_2 \mapsto \frac{1}{r}B_2$ to put $r = 1$ and we may shift $B_3 \mapsto B_3 - sB_2$ to set $s = 0$. In summary, we have

$$\rho(B_1) = 0, \qquad \rho(B_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} x & 0 \\ 0 & x+2 \end{pmatrix},$$

for any $x \in \mathbb{R}$ (and different values of x give different isoclasses).

From the tables, we have that α, σ and $(W^*)^{\mathfrak{b}}$ are zero except in the following cases:

- $x = -2$ ($\theta = 0$) when $\alpha \in \mathbb{R}(\boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$, $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$ and

$$\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2);$$

- $x = -4$ ($\cot\theta = 2$), when $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$.

Since $\rho(B_i)(\boldsymbol{e}^1 \wedge \boldsymbol{e}^2) = 0$ for $i = 1, 2$, it follows that $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \operatorname{span}\{B_1, B_2\}$, whereas $\rho(B_3)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = 2(x+1)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$. Therefore $\varphi = 0$ unless

- $x = -2$, in which case $[B_3, \varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)] = -2\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$, so that $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_2$;
- or $x = -1$, in which case $[B_3, \varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)] = 0$, so that $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$.

As for the weak automorphism group, we have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} bt & \frac{tv}{2} \\ 0 & t \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abt \neq 0 \right\}.$$

III.3/4: $\ker\rho = \mathbb{R}B_2$ **and** $\operatorname{im}\rho = \mathfrak{diag}(2,\mathbb{R})$

Here $\rho(B_2) = 0$ and

$$\rho(B_1) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}.$$

We have two cases to consider up to the action of $\operatorname{Aut}(\mathfrak{b})$ depending on whether or not $x = 0$.

If $x \neq 0$, we may rescale B_1 so that $x = 1$ and we may shift B_3 so that $s = 0$, resulting in

$$\rho(B_1) = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}, \qquad \rho(B_2) = 0 \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}, \tag{8.31}$$

where $t \neq 0$. One sees that two pairs of parameters $(t,y), (t',y') \in (\mathbb{R}\setminus\{0\}) \times \mathbb{R}$ yield weakly equivalent representations if and only if $(t,y) = (t',y')$ or $yy' \neq 0$ and $(t',y') = (-\frac{t}{y}, \frac{1}{y})$, so a complete set of isoclasses is obtained by restricting ourselves to the subset

$$\big\{(t,y) \in (\mathbb{R}\setminus\{0\}) \times [-1,1] \colon y \neq 1 \text{ or } t > 0\big\}.$$

If $x = 0$, then we may rescale B_1 so that $y = 1$ and then we may shift B_3 by a multiple of B_1 so that $t = 0$. For uniformity of notation, we relabel s as t and arrive at

$$\rho(B_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho(B_2) = 0 \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \tag{8.32}$$

where $t \neq 0$. It turns out that different values of t yield nonequivalent representations.

We treat these two cases separately. In both cases, from the tables it follows that $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ are zero.

In the first case, when ρ is given by (8.31), we see that $\varphi = 0$ unless $(t,y) = (-2,-1)$, where we have $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_2$.

As for the weak automorphism group, we have that if $y = -1$ then

$$\begin{aligned}\operatorname{Aut}(\rho) = &\left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1h_2 \neq 0 \right\} \\ &\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & t \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1h_2 \neq 0 \right\},\end{aligned}$$

whereas for $y \neq -1$ we get that

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1h_2 \neq 0 \right\}.$$

In the second case, when ρ is given by equation (8.32), one checks that $\varphi = 0$.

The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1h_2 \neq 0 \right\}.$$

III.5/6: $\ker\rho = \mathbb{R}B_2$ **and** $\operatorname{im}\rho = \mathfrak{s}_{\pi/4}$

Here we have

$$\rho(B_1) = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} r & s \\ 0 & r \end{pmatrix}.$$

We have to distinguish two cases depending on whether or not $y = 0$.

If $y \neq 0$, we may use $\operatorname{Aut}(\mathfrak{b})$ to rescale $y = 1$ and to set $r = 0$, resulting in

$$\rho(B_1) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}.$$

The normalizer of $\operatorname{im}\rho$ in $\mathsf{GL}(2,\mathbb{R})$ is the group $\mathsf{T}(2,\mathbb{R})$ of invertible upper triangular matrices and we may conjugate by $\mathsf{T}(2,\mathbb{R})$ in order to set $s = 1$, resulting in

$$\rho(B_1) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, different values of x give different weak isoclasses of representations. From the tables it follows that α, σ and $(W^*)^{\mathfrak{b}}$ are zero. Since $\rho(B_1)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = 2\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$, it follows that $[B_1, \varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)] = 2\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$, which says $\varphi = 0$, since B_1 is central. The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1 \neq 0 \right\}.$$

If $y = 0$, then $x \neq 0$ and we may use $\operatorname{Aut}(\mathfrak{b})$ to rescale $x = 1$ and set $s = 0$. Relabelling the parameters for uniformity of notation, we find

$$\rho(B_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix},$$

where $t \neq 0$. Different values of t correspond to non-isomorphic representations. From the tables, it follows that α, σ and $(W^*)^{\mathfrak{b}}$ are zero. Moreover, we see that $\varphi = 0$ unless $t = -1$, in which case $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_2$. The group of weak automorphisms of ρ is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} ah_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1 \neq 0 \right\}.$$

III.7/8: $\ker\rho = \mathbb{R}B_2$ **and** $\operatorname{im}\rho = \mathbb{C}$

Here we have

$$\rho(B_1) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} t & -s \\ s & t \end{pmatrix}.$$

We must again distinguish between two cases, depending on whether or not $x = 0$.

If $x = 0$, then $y \neq 0$. We may use $\operatorname{Aut}(\mathfrak{b})$ to set $y = 1$ and $s = 0$, resulting in

$$\rho(B_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix},$$

with $t \neq 0$. One sees that each value of t corresponds to a different isoclass of representations. From the tables, it follows that α, σ and $(W^*)^{\mathfrak{b}}$ are zero. The map φ vanishes unless $t = -1$, in which case $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_2$. The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -\varepsilon h_2 & \varepsilon h_1 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : b(h_1^2 + h_2^2) \neq 0, \varepsilon = \pm 1 \right\}.$$

If $x \neq 0$, we may use $\operatorname{Aut}(\mathfrak{b})$ to rescale $x = 1$ and eliminate the diagonal terms of $\rho(B_3)$, resulting in

$$\rho(B_1) = \begin{pmatrix} 1 & -y \\ y & 1 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix},$$

where $t \neq 0$ and where we have relabelled parameters for uniformity. It turns out that two pairs $(t, y), (t', y') \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ give rise to weakly equivalent representations if and only if $(t', y') = \pm(t, y)$, so we obtain a complete set of isoclasses by restricting ourselves to $t > 0$. From the tables, α, σ and $(W^*)^{\mathfrak{b}}$ are zero. From the fact that $\rho(B_1)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = 2\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$, it follows that $[B_1, \varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)] = 2\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$, which says that $\varphi = 0$, since B_1 is central.

The group of weak automorphisms is given by

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : b(h_1^2 + h_2^2) \neq 0 \right\}.$$

III.9: $\ker \rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im} \rho = \mathbb{R} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \leq 1$

In this case we have $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix} \qquad \text{with } t \neq 0.$$

We see that two pairs $(t, x), (t', x') \in (\mathbb{R} \setminus \{0\}) \times [-1, 1]$ give weakly equivalent representations if and only if $(t, x) = (t', x')$ or $x = -1$ and $t' = -t$, so we obtain a complete set of representatives by restricting ourselves to the subset

$$\{(t, x) \in (\mathbb{R} \setminus \{0\}) \times [-1, 1] \colon x \neq -1 \text{ or } t > 0\}.$$

From the tables, $(W^*)^{\mathfrak{b}} = 0$ unless $x = 0$, where we have $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^1$. Moreover, we have $\alpha = 0$ unless $x = -1$, in which case $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)$, or $x = 0$, in which case $\alpha \in \mathbb{R}\boldsymbol{e}^1 \otimes \boldsymbol{e}^1$. The map σ is zero except in the following scenarios:

- $x = 0$, in which case $\sigma \in \operatorname{span}\{\boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1, \boldsymbol{e}_2 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)\}$;
- or $x = \frac{1}{2}$, in which case $\sigma \in \mathbb{R}\boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1$.

Moreover, φ is zero unless

- $x = -1$, in which case $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$;
- or $t(x+1) = -2$, in which case $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_2$.

Finally, one sees that if $x = 1$, we have

$$\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b}),$$

whereas for $x < 1$ we see that

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1h_2 \neq 0 \right\}.$$

III.10: $\ker \rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im} \rho = \mathbb{R} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

In this case we take $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}, \qquad \text{where } t \neq 0.$$

It is clear that different values of t yield different (weak) isoclasses of representations. The 2-form φ vanishes identically unless $t = -1$, in which case $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_2$. From the tables, we have that $(W^*)^{\mathfrak{b}}$, α, and σ are all zero.

The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1 \neq 0 \right\}.$$

III.11: $\ker \rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im} \rho = \mathbb{R} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Here $\rho(B_1) = \rho(B_2) = 0$ and we may conjugate $\rho(B_3)$ to its Jordan normal form to obtain

$$\rho(B_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the tables, $\alpha \in \mathbb{R}\boldsymbol{e}^2 \otimes \boldsymbol{e}^2$, $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$ and

$$\sigma \in \operatorname{span}\{\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2, \boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2\}.$$

The map φ takes values in $\mathbb{R}B_1$. Finally, we have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1h_2 \neq 0 \right\}.$$

III.12: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$

Here $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} ty & -t \\ t & ty \end{pmatrix}, \qquad \text{with } t \neq 0 \text{ and } y \geq 0.$$

Different values of (t, y) yield different weak isoclasses. From the tables, σ and $(W^*)^{\mathfrak{b}}$ are always zero, whereas α vanishes unless $y = 0$, in which case we have $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^1 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$. Moreover, $\varphi = 0$ unless

- $y = 0$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_1$;
- or $ty = -1$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_2$.

The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : ab(h_1^2 + h_2^2) \neq 0 \right\}.$$

III.13: $\ker\rho = \operatorname{span}\{B_2, B_3\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \leq 1$

Here $\rho(B_2) = \rho(B_3) = 0$ and, since rescaling B_1 is an automorphism of $\mathfrak{b}$, we can just take

$$\rho(B_1) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

with $|x| \leq 1$. Different values of x yield different weak isoclasses. The conditions on $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ are as in the case treated in Section 8.4.3 and will not be repeated here. Now $\varphi = 0$ unless $x = -1$ and $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$. As for the weak automorphism group, we have that:

- if $x = 1$ then
$$\operatorname{Aut}(\rho) = \left\{ \left(h, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : b(\det h) \neq 0 \right\}.$$
- If $x = -1$, then
$$\begin{aligned} \operatorname{Aut}(\rho) = &\left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1h_2 \neq 0 \right\} \\ &\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1h_2 \neq 0 \right\}. \end{aligned}$$

- For $x \in (-1, 1)$, we have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1h_2 \neq 0 \right\}.$$

III.14: $\ker \rho = \operatorname{span}\{B_2, B_3\}$ **and** $\operatorname{im} \rho = \mathbb{R} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Here $\rho(B_2) = \rho(B_3) = 0$ and

$$\rho(B_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The conditions on $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ are as in the case treated in Section 8.4.3 and will not be repeated here. A calculation shows that φ is zero. The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : bh_1 \neq 0 \right\}.$$

III.15: $\ker \rho = \operatorname{span}\{B_2, B_3\}$ **and** $\operatorname{im} \rho = \mathbb{R} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Here $\rho(B_2) = \rho(B_3) = 0$ and

$$\rho(B_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The conditions on $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ are as in the case treated in Section 8.4.3 and will not be repeated here. Now $\rho(B)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = 0$ for all $B \in \mathfrak{b}$, hence $\operatorname{im} \varphi$ is central. A calculation shows that

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} ah_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1 \neq 0 \right\}.$$

III.16: $\ker \rho = \operatorname{span}\{B_2, B_3\}$ **and** $\operatorname{im} \rho = \mathbb{R} \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$ **with** $y \geq 0$

Here $\rho(B_2) = \rho(B_3) = 0$ and

$$\rho(B_1) = \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}, \quad y \geq 0.$$

One sees that different values of y correspond to different isoclasses for ρ. The conditions on $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ are as in the case treated in Section 8.4.3 and will not be repeated here. The 2-form φ is zero unless $y = 0$, in which case $\operatorname{im} \varphi$ is central. We also have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -\varepsilon h_2 & \varepsilon h_1 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : b(h_1^2 + h_2^2) \neq 0, \varepsilon = \pm 1 \right\}.$$

III.17: $\rho = 0$

In this case, α, σ are unconstrained and $(W^*)^{\mathfrak{b}} = W^*$, whereas φ takes values in the center. Trivially, $\mathrm{Aut}(\rho) = \mathsf{GL}(W) \times \mathrm{Aut}(\mathfrak{b})$.

Summary

We summarize the results of this section in Table 8.8, with the same notation as that explained in Table 8.6.

Table 8.8: Two-dimensional real representations of Bianchi III.

Label	$\rho(B_1)$	$\rho(B_2)$	$\rho(B_3)$	**Remarks**	α	σ	φ	$(W^*)^{\mathfrak{b}}$
$\mathrm{III}.1$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$					
$\mathrm{III}.2_x$	0	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}x&0\\0&x+2\end{pmatrix}$		$x=-2$	$x=-2,-4$	$x=-1,-2$	$\mathbb{R}\boldsymbol{e}^2$ $(x=-2)$
$\mathrm{III}.3_{t,y}$	$\begin{pmatrix}1&0\\0&y\end{pmatrix}$	0	$\begin{pmatrix}0&0\\0&t\end{pmatrix}$	$t\neq 0$ $\lvert y\rvert\leq 1$ $y\neq 1$ or $t>0$			$(t,y)=(-2,-1)$	
$\mathrm{III}.4_t$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	0	$\begin{pmatrix}t&0\\0&0\end{pmatrix}$	$t\neq 0$				
$\mathrm{III}.5_x$	$\begin{pmatrix}1&x\\0&1\end{pmatrix}$	0	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$					
$\mathrm{III}.6_t$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	0	$\begin{pmatrix}t&0\\0&t\end{pmatrix}$	$t\neq 0$			$t=-1$	
$\mathrm{III}.7_t$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	0	$\begin{pmatrix}t&0\\0&t\end{pmatrix}$	$t\neq 0$			$t=-1$	
$\mathrm{III}.8_{t,y}$	$\begin{pmatrix}1&-y\\y&1\end{pmatrix}$	0	$\begin{pmatrix}0&-t\\t&0\end{pmatrix}$	$t>0$				
$\mathrm{III}.9_{t,x}$	0	0	$\begin{pmatrix}tx&0\\0&t\end{pmatrix}$	$t\neq 0$ $\lvert x\rvert\leq 1$ $x\neq -1$ or $t>0$	$x=-1,0$	$x=0,\frac{1}{2}$	$\begin{cases}x=-1\\t(x+1)=-2\end{cases}$	$\mathbb{R}\boldsymbol{e}^1$ $(x=0)$
$\mathrm{III}.10_t$	0	0	$\begin{pmatrix}t&t\\0&t\end{pmatrix}$	$t\neq 0$			$t=-1$	
$\mathrm{III}.11$	0	0	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$		✓	✓	✓	$\mathbb{R}\boldsymbol{e}^2$
$\mathrm{III}.12_{t,y}$	0	0	$\begin{pmatrix}ty&-t\\t&ty\end{pmatrix}$	$t\neq 0$ $y\geq 0$	$y=0$		$\begin{cases}y=0\\ty=-1\end{cases}$	
$\mathrm{III}.13_x$	$\begin{pmatrix}x&0\\0&1\end{pmatrix}$	0	0	$\lvert x\rvert\leq 1$	$x=-1,0$	$x=0,\frac{1}{2}$	$x=-1$	$\mathbb{R}\boldsymbol{e}^1$ $(x=0)$
$\mathrm{III}.14$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	0	0					
$\mathrm{III}.15$	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	0	0		✓	✓	✓	$\mathbb{R}\boldsymbol{e}^2$
$\mathrm{III}.16_y$	$\begin{pmatrix}y&-1\\1&y\end{pmatrix}$	0	0	$y\geq 0$	$y=0$		$y=0$	
$\mathrm{III}.17$	0	0	0		✓	✓	✓	W^*

8.4.4 Bianchi IV

This is the solvable Lie algebra with brackets $[B_1, B_3] = B_1$ and $[B_2, B_3] = B_1 + B_2$. The automorphism group is given by

$$\mathrm{Aut}(\mathfrak{b}) = \left\{ \begin{pmatrix} a & b & r \\ 0 & a & s \\ 0 & 0 & 1 \end{pmatrix} : a \neq 0 \right\}.$$

The proper ideals are the first derived ideal $[\mathfrak{b}, \mathfrak{b}] = \mathrm{span}\{B_1, B_2\}$ and the one-dimensional ideal $\mathbb{R}B_1$.

IV.1: $\ker \rho = \mathbb{R}B_1$

The quotient Lie algebra $\mathfrak{b}/\mathbb{R}B_1$ is isomorphic to the affine Lie algebra $\mathfrak{aff}(1, \mathbb{R})$, so that $\mathrm{im}\,\rho$ is conjugate to $\mathfrak{s}_\theta$ for $\theta \neq \frac{\pi}{4}$ and $0 \leq \theta < \pi$.

We can choose bases such that $\rho(B_1) = 0$, whereas

$$\rho(B_2) = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(B_3) = \begin{pmatrix} x & v \\ 0 & x+1 \end{pmatrix}.$$

Using $\mathrm{Aut}(\mathfrak{b})$, we can set $u = 1$ and $v = 0$, so that in the end

$$\rho(B_1) = 0, \quad \rho(B_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(B_3) = \begin{pmatrix} x & 0 \\ 0 & x+1 \end{pmatrix},$$

where $x \in \mathbb{R}$, and x determines ρ up to weak equivalence.

From the tables and from calculations we see that $\alpha, \sigma, \varphi, (W^*)^{\mathfrak{b}}$ are all zero except for the following values of x:

- $x = -2$: $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$, but $\alpha, \varphi, (W^*)^{\mathfrak{b}}$ are zero;
- $x = -1$: $\alpha \in \mathbb{R}(\boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$, $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$, $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$ and $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$.
- and $x = -\frac{1}{2}$: $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)$, but $\sigma, \varphi, (W^*)^{\mathfrak{b}}$ are zero.

The group of weak automorphisms is

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} ah & sh \\ 0 & h \end{pmatrix}, \begin{pmatrix} a & b & r \\ 0 & a & s \\ 0 & 0 & 1 \end{pmatrix} \right) : ah \neq 0 \right\}.$$

IV.2: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \le 1$

We can find bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix},$$

where $t \neq 0$. It turns out that two pairs (t, x) and (t', x') yield equivalent representations if and only if $(t, x) = (t', x')$ or $x = x' = -1$ and $t' = -t$, so we obtain a complete set of isoclasses by restricting ourserlves to the subset

$$\big\{(t, x) \in (\mathbb{R} \setminus \{0\}) \times [-1, 1] \colon x \neq -1 \text{ or } t > 0\big\}.$$

The quadratic form $\alpha \in \mathsf{S}^2 W^*$ is zero unless $x = 0$, where $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^1)$ or $x = -1$, which corresponds to $\mathfrak{so}(1,1)$, in which case $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)$. There are no nonzero invariants in W^* unless $x = 0$, in which case $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^1$.

The symmetric bilinear form $\sigma \in W \otimes \mathsf{S}^2 W^*$ is zero except when $x = 0$, in which case $\sigma \in \operatorname{span}\{\boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1, \boldsymbol{e}_2 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1)\}$, or when $x = \frac{1}{2}$, in which case $\sigma \in \mathbb{R}(\boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1)$.

The 2-form $\varphi \in \mathfrak{b} \otimes \Lambda^2 W^*$ is zero, unless $t(x+1) = -1$, in which case we have $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$.

Finally, if $x = 1$ we have

$$\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b}),$$

whereas for $x \neq 1$ we see that

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & b & r \\ 0 & a & s \\ 0 & 0 & 1 \end{pmatrix} \right) : ah_1h_2 \neq 0 \right\}.$$

IV.3: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

We can find bases such that

$$\rho(B_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho(B_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(B_3) = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix},$$

where $t \neq 0$ determines ρ up to weak isomorphism.

From the tables, it follows that α, σ and $(W^*)^{\mathfrak{b}}$ are all zero. The map φ vanishes unless $t = -\frac{1}{2}$, in which case $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$.

We see that

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & b & r \\ 0 & a & s \\ 0 & 0 & 1 \end{pmatrix} \right) : ah_1 \neq 0 \right\}.$$

IV.4: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

We can find bases such that $\rho(B_1) = \rho(B_2) = 0$ and by putting $\rho(B_3)$ in Jordan normal form we can assume that

$$\rho(B_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

From the tables, it follows that $\alpha \in \mathbb{R}(\boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$, $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$ and

$$\sigma \in \operatorname{span}\left\{\boldsymbol{e}_1 \otimes \left(\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1\right) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2, \boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2\right\}.$$

Since $\mathfrak{b}$ has zero center, it follows that $\varphi = 0$. The group of weak automorphisms is once again

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & b & r \\ 0 & a & s \\ 0 & 0 & 1 \end{pmatrix} \right) : ah_1 \neq 0 \right\}.$$

IV.5: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$ **with** $y \geq 0$

We can find bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} ty & -t \\ t & ty \end{pmatrix},$$

where $t \neq 0$ and $y \geq 0$ determine the isoclass of ρ. From the tables, we see that $\alpha = 0$ unless $y = 0$, in which case $\alpha \in \mathbb{R}(\boldsymbol{e}^1 \otimes \boldsymbol{e}^1 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$. It follows that σ and $(W^*)^{\mathfrak{b}}$ are always zero. The map φ vanishes identically unless $ty = -\frac{1}{2}$ in which case $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \mathbb{R}B_1$.

Finally, we have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}, \begin{pmatrix} a & b & r \\ 0 & a & s \\ 0 & 0 & 1 \end{pmatrix} \right) : a(h_1^2 + h_2^2) \neq 0 \right\}.$$

IV.6: $\rho = 0$

Since the center of $\mathfrak{b}$ is zero, it follows that φ is zero. The other data α, σ are unconstrained, but we are free to use $\mathsf{GL}(W)$ to bring them to normal forms. Also, of course, $(W^*)^{\mathfrak{b}} = W^*$. We obviously have $\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b})$.

Summary

We summarize the results of this section in Table 8.9, with the same notation as that explained in Table 8.6.

Table 8.9: Two-dimensional real representations of Bianchi $\mathbb{IV}$.

Label	$\rho(B_1)$	$\rho(B_2)$	$\rho(B_3)$	**Remarks**	α	σ	φ	$(W^*)^{\mathfrak{b}}$
$\mathbb{IV}.1_x$	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x+1 \end{pmatrix}$		$x = -\frac{1}{2}, -1$	$x = -1, -2$	$x = -1$	$\mathbb{R}\boldsymbol{e}^2$ $(x = -1)$
$\mathbb{IV}.2_{t,x}$	0	0	$\begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix}$	$\lvert x\rvert \leq 1$ $t \neq 0$ $x \neq -1$ or $t > 0$	$x = -1, 0$	$x = 0, \frac{1}{2}$	$t(x+1) = -1$	$\mathbb{R}\boldsymbol{e}^1$ $(x = 0)$
$\mathbb{IV}.3_t$	0	0	$\begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$	$t \neq 0$			$t = -\frac{1}{2}$	
$\mathbb{IV}.4$	0	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		✓	✓		$\mathbb{R}\boldsymbol{e}^2$
$\mathbb{IV}.5_{t,y}$	0	0	$\begin{pmatrix} ty & -t \\ t & ty \end{pmatrix}$	$t \neq 0$ $y \geq 0$	$y = 0$		$ty = -\frac{1}{2}$	
$\mathbb{IV}.6$	0	0	0		✓	✓		W^*

8.4.5 Bianchi V

This is the solvable algebra with nonzero brackets $[B_1, B_3] = B_1$ and $[B_2, B_3] = B_2$. The automorphism group is

$$\operatorname{Aut}(\mathfrak{b}) = \left\{ \begin{pmatrix} A & u \\ 0 & 1 \end{pmatrix} : A \in \mathsf{GL}(2,\mathbb{R}),\ u \in \mathbb{R}^2 \right\} \cong \mathsf{Aff}(2,\mathbb{R}).$$

The proper ideals are $[\mathfrak{b},\mathfrak{b}] = \operatorname{span}\{B_1, B_2\}$ and any line therein. Since the affine group acts transitively on lines, we may choose $\mathbb{R}B_1$ without loss of generality. Hence up to weak automorphisms, the possible values of $\ker\rho$ can be taken to be $\mathbb{R}B_1$, $\operatorname{span}\{B_1, B_2\}$ and $\mathfrak{b}$.

V.1: $\ker\rho = \mathbb{R}B_1$

In this case $\operatorname{im}\rho \cong \mathfrak{b}/\mathbb{R}B_1$ is nonabelian and hence isomorphic to $\mathfrak{s}_\theta$ for $\theta \neq \frac{\pi}{4}$. We can therefore choose bases such that $\rho(B_1) = 0$, whereas

$$\rho(B_2) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} x & r \\ 0 & t \end{pmatrix}.$$

By demanding that $[\rho(B_2), \rho(B_3)] = \rho(B_2)$, we see that $t = x+1$. Using $\operatorname{Aut}(\mathfrak{b})$ we may rescale B_2 (so we can set $s = 1$) and shift B_3 by any multiple of B_2 (so we can set $r = 0$), resulting in

$$\rho(B_1) = 0, \qquad \rho(B_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} x & 0 \\ 0 & x+1 \end{pmatrix},$$

where x can be any real number (and it determines ρ up to weak equivalence).

From the tables, we have that

- $(W^*)^{\mathfrak{b}} = 0$ unless $x = -1$, in which case $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$;
- $\alpha = 0$ unless $x = -1$, in which case $\alpha \in \mathbb{R}(\boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$;
- $\sigma = 0$ unless
  - $x = -1$, in which case $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$;
  - or $x = -2$, in which case $\sigma \in \mathbb{R}\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$.

The 2-form φ vanishes unless $x = -1$, in which case $\operatorname{im}\varphi \subseteq \operatorname{span}\{B_1, B_2\}$. In this case, the group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} a_{22}h & u_2 h \\ 0 & h \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & u_1 \\ 0 & a_{22} & u_2 \\ 0 & 0 & 1 \end{pmatrix} \right) : a_{11}a_{22}h \neq 0 \right\}.$$

V.2: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \leq 1$

We can find bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix},$$

where $t \neq 0$. As with previous cases, we obtain a complete set of isoclasses by restricting ourselves to the subset

$$\big\{(t, x) \in (\mathbb{R} \setminus \{0\}) \times [-1, 1] \colon x \neq -1 \text{ or } t > 0\big\}.$$

The situation for $\alpha, \sigma, (W^*)^\flat$ is similar to any other case with such $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated here. It is easy to see that $\varphi = 0$ unless $t(x+1) = -1$, in which case $\operatorname{im}\varphi \subseteq \operatorname{span}\{B_1, B_2\}$. As for the weak automorphism group, we see that

- If $x = 1$, this group is
$$\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\rho).$$
- Otherwise, we have
$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} A & u \\ 0 & 1 \end{pmatrix} \right) : h_1 h_2 (\det A) \neq 0 \right\}.$$

V.3: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

We can find bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix},$$

where $t \neq 0$ determines ρ up to weak isomorphism. The situation for $\alpha, \sigma, (W^*)^\flat$ is similar to any other case with such $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated here. The 2-form φ vanishes identically unless $t = -\frac{1}{2}$, in which case $\operatorname{im}\varphi \subseteq \operatorname{span}\{B_1, B_2\}$. The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} A & u \\ 0 & 1 \end{pmatrix} \right) : h_1 (\det A) \neq 0 \right\}.$$

V.4: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

We can find bases such that $\rho(B_1) = \rho(B_2) = 0$ and putting $\rho(B_3)$ in Jordan normal form we get

$$\rho(B_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

The situation for $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ is similar to any other case with such $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated here. Here $\varphi = 0$ since $\mathfrak{b}$ has trivial center. We also obtain

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} A & u \\ 0 & 1 \end{pmatrix} \right) : h_1(\det A) \neq 0 \right\}.$$

V.5: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$ **with** $y \geq 0$

We can find bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} ty & -t \\ t & ty \end{pmatrix},$$

where $t \neq 0$ and $y \geq 0$ determine the isoclass of ρ. The situation for $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ is similar to any other case with such $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated here. It is easily checked that $\varphi = 0$ except when $2ty = -1$, in which case $\operatorname{im}\varphi \subseteq \operatorname{span}\{B_1, B_2\}$. The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}, \begin{pmatrix} A & u \\ 0 & 1 \end{pmatrix} \right) : (\det A)(h_1^2 + h_2^2) \neq 0 \right\}.$$

V.6: $\rho = 0$

Here α and σ are unconstrained and all of W^* is invariant, but since $\mathfrak{b}$ has trivial center, φ is zero. The stabilizer of ρ is trivially $\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b})$.

Summary

We summarize the results of this section in Table 8.10, with the same notation as that explained in Table 8.6.

Table 8.10: Two-dimensional real representations of Bianchi $\mathbb{V}$.

Label	$\rho(B_1)$	$\rho(B_2)$	$\rho(B_3)$	**Remarks**	α	σ	φ	$(W^*)^\mathfrak{b}$
$\mathbb{V}.1_x$	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x+1 \end{pmatrix}$		$x=-1$	$x=-1,-2$	$x=-1$	$\mathbb{R}\boldsymbol{e}^2$ $(x=-1)$
$\mathbb{V}.2_{t,x}$	0	0	$\begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix}$	$\lvert x\rvert \le 1$ $t \neq 0$ $x \neq -1$ or $t > 0$	$x=-1,0$	$x=0,\frac{1}{2}$	$t(x+1)=-1$	$\mathbb{R}\boldsymbol{e}^1$ $(x=0)$
$\mathbb{V}.3_t$	0	0	$\begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$	$t \neq 0$			$t=-\frac{1}{2}$	
$\mathbb{V}.4$	0	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		✓	✓		$\mathbb{R}\boldsymbol{e}^2$
$\mathbb{V}.5_{t,y}$	0	0	$\begin{pmatrix} ty & -t \\ t & ty \end{pmatrix}$	$t \neq 0$ $y \ge 0$	$y=0$		$2ty=-1$	
$\mathbb{V}.6$	0	0	0		✓	✓		W^*

8.4.6 Bianchi $\mathrm{VI}_{c\neq 1}$

This is the Lie algebra with brackets $[B_1, B_3] = (c-1)B_1$ and $[B_2, B_3] = (c+1)B_2$ for $c \geq 0$ and $c \neq 1$, since $c = 1$ is Bianchi III, which was already treated in Section 8.4.3.

The automorphism group depends on whether $c > 0$ or $c = 0$. If $c > 0$,

$$\operatorname{Aut}(\mathfrak{b}) = \left\{ \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} : ab \neq 0 \right\}.$$

If $c = 0$ there are additional automorphisms

$$\operatorname{Aut}(\mathfrak{b})_- = \left\{ \begin{pmatrix} 0 & a & u \\ b & 0 & v \\ 0 & 0 & -1 \end{pmatrix} : ab \neq 0 \right\}.$$

The proper ideals are $\mathbb{R}B_1$, $\mathbb{R}B_2$ and $\operatorname{span}\{B_1, B_2\}$, hence the possible $\ker\rho$ are $\mathbb{R}B_1$, $\mathbb{R}B_2$, $\operatorname{span}\{B_1, B_2\}$ and $\mathfrak{b}$. Note that for $c = 0$ the ideals $\mathbb{R}B_1$ and $\mathbb{R}B_2$ are conjugate.

VI.1: $\ker\rho = \mathbb{R}B_1$

If $\ker\rho = \mathbb{R}B_1$, then $\mathfrak{b}/\ker\rho \cong \mathfrak{aff}(1,\mathbb{R})$ and hence we can take $\operatorname{im}\rho = \mathfrak{s}_\theta$ for $\theta \neq \frac{\pi}{4}$. This means that we can choose bases where $\rho(B_1) = 0$, but

$$\rho(B_2) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} x & r \\ 0 & y \end{pmatrix},$$

where $s \neq 0$ and $x, y \in \mathbb{R}$, $(x, y) \neq (0, 0)$. Using $\operatorname{Aut}(\mathfrak{b})$ we can rescale B_2 to make $s = 1$ and translate B_3 so that $r = 0$. By demanding that $[\rho(B_2), \rho(B_3)] = (c+1)\rho(B_2)$, we find that $y = x + c + 1$. In summary, we have

$$\rho(B_1) = 0, \qquad \rho(B_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} x & 0 \\ 0 & x+c+1 \end{pmatrix},$$

where $\cot\theta = \frac{x}{x+c+1}$. The parameter $x \in \mathbb{R}$ determines ρ up to weak isomorphism. From the tables, $\alpha = 0$ and $(W^*)^\mathfrak{b} = 0$ unless $x+c+1 = 0$, in which case $\alpha \in \mathbb{R}\boldsymbol{e}^2 \otimes \boldsymbol{e}^2$ and $(W^*)^\mathfrak{b} = \mathbb{R}\boldsymbol{e}^2$; $\sigma = 0$ except in the following cases:

- $x + c + 1 = 0$, in which case $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$;
- or $x + 2c + 2 = 0$, in which case $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$.

The map φ is identically zero except in the following scenarios:

- $x + c + 1 = 0$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_2$;
- and $x = -c$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_1$.

As for the group of weak automorphisms, we obtain

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} bh_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & \frac{(c+1)h_2}{h_1} \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1 \neq 0 \right\}.$$

VI.2: $\ker\rho = \mathbb{R}B_2$

First of all, note that ρ is not weakly equivalent to a representation in the family $\overline{\mathrm{VI}}.1$ if and only if $c > 0$, so we assume that this is the case. Again, $\operatorname{im}\rho$ can be taken to be $\mathfrak{s}_\theta$ for $\theta \neq \frac{\pi}{4}$ in $[0,\pi)$. Using $\operatorname{Aut}(\mathfrak{b})$ and by demanding that we get a representation, we find

$$\rho(B_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \rho(B_2) = 0 \qquad \text{and} \qquad \rho(B_3) = \begin{pmatrix} x & 0 \\ 0 & x+c-1 \end{pmatrix},$$

for $x \in \mathbb{R}$ (which determines ρ up to weak equivalence) and where now $\cot\theta = \frac{x}{x+c-1}$. The situation with α and σ is just as in the previous case in terms of θ, but the relation between x and θ is different. From the tables, $\alpha = 0$ and $(W^*)^{\mathfrak{b}} = 0$ unless $x = 1-c$, in which case $\alpha \in \mathbb{R}\boldsymbol{e}^2 \otimes \boldsymbol{e}^2$ and $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$. Furthermore, $\sigma = 0$ except in the following cases:

- $x = 1-c$, in which case $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes (\boldsymbol{e}^1 \otimes \boldsymbol{e}^2 + \boldsymbol{e}^2 \otimes \boldsymbol{e}^1) + 2\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$;
- or $x = 2(1-c)$, in which case $\sigma \in \mathbb{R}(\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2)$.

In addition, the map φ is only nonzero in the following two cases:

- $x = 1-c$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_1$;
- or $x = -c$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_2$.

A calculation shows that the group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} ah_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & \frac{(c-1)h_2}{h_1} \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1 \neq 0 \right\}.$$

VI.3: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \leq 1$

We can choose bases where $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix}, \qquad \text{where } t \neq 0 \text{ and } |x| \leq 1.$$

We briefly discuss isoclasses. For $c = 0$, one sees that (t,x) and (t',x') yield weakly isomorphic representations if and only if $x = x'$ and $t' = \pm t$, so a complete set of isoclasses is taken by restricting ourselves to $(t,x) \in \mathbb{R}^+ \times [-1,1]$. Meanwhile, for $c \neq 0$ one gets a complete set of isoclasses by restricting ourselves to

$$\big\{(t,x) \in (\mathbb{R} \setminus \{0\}) \times [-1,1] \colon x \neq -1 \text{ or } t > 0\big\}.$$

The situation with $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ can be read off from other cases with the same $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated.

The map φ is seen to be zero except in the following two scenarios:

- $t(x+1) = 1-c$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_1$;
- or $t(x+1) = -1-c$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_2$.

As for the weak automorphisms, we have:

- If $x = 1$, then
$$\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times (\operatorname{Aut}(\mathfrak{b}) \setminus \operatorname{Aut}(\mathfrak{b})_-),$$
where the second factor is the full automorphism group of $\mathfrak{b}$ for $c > 0$.
- If $x = -1$ and $c = 0$, then
$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1h_2 \neq 0 \right\}$$
$$\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & u \\ b & 0 & v \\ 0 & 0 & -1 \end{pmatrix} \right) : abh_1h_2 \neq 0 \right\}.$$
- If $x \neq \pm 1$ or $x = -1$ and $c > 0$, then
$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1h_2 \neq 0 \right\}.$$

VI.4: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho \in \mathbb{R}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

We can choose bases where $\rho(B_1) = \rho(B_2) = 0$ and
$$\rho(B_3) = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}, \qquad \text{where } t \neq 0.$$

One sees that for $c > 0$ each t determines the isoclass of ρ, while for $c = 0$ we have to restrict ourselves to $t > 0$ to get a complete set of isoclasses. The situation with $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ can be read off from other cases with the same $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated. From the fact that $[B_3, \varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)] = 2t\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$, we see that $\varphi = 0$ except in two cases:

- $t = \frac{1-c}{2}$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_1$;
- or $t = -\frac{1+c}{2}$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_2$.

The weak automorphism group is
$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : abh_1 \neq 0 \right\}.$$

VI.5: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho \in \mathbb{R}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

We can choose bases where $\rho(B_1) = \rho(B_2) = 0$ and we can conjugate $\rho(B_3)$ to its Jordan normal form to get

$$\rho(B_3) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

The situation with $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ can be read off from other cases with the same $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated. Now from the fact that $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$ is $\mathfrak{b}$-invariant, it follows that $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$ must be central, but the center of $\mathfrak{b}$ is trivial and hence $\varphi = 0$.

Furthermore, the weak automorphism group is

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix}h_1 & h_2\\ 0 & h_1\end{pmatrix}, \begin{pmatrix}a & 0 & u\\ 0 & b & v\\ 0 & 0 & 1\end{pmatrix} \right) : abh_1 \neq 0 \right\},$$

and for $c = 0$ we get the additional automorphisms

$$\operatorname{Aut}(\rho)_- = \left\{ \left(\begin{pmatrix}h_1 & h_2\\ 0 & -h_1\end{pmatrix}, \begin{pmatrix}0 & a & u\\ b & 0 & v\\ 0 & 0 & -1\end{pmatrix} \right) : abh_1 \neq 0 \right\}.$$

VI.6: $\ker\rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im}\rho \in \mathbb{R}\begin{pmatrix}y & -1\\ 1 & y\end{pmatrix}$ **with** $y \geq 0$

We can choose bases where $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix}ty & -t\\ t & ty\end{pmatrix}, \qquad \text{where } t \neq 0 \text{ and } y \geq 0.$$

If $c > 0$, then t and y determine ρ up to isomorphism, whereas for $c = 0$ we have to restrict ourselves to $t > 0$ in order to get a complete set of isoclasses. The situation with $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ can be read off from other cases with the same $\operatorname{im}\rho$ (see, e.g., Section 8.4.4) and will not be repeated.

The map φ vanishes identically except in two cases:

- $ty = \frac{1-c}{2}$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_1$;
- or $ty = -\frac{1+c}{2}$, in which case $\operatorname{im}\varphi \subseteq \mathbb{R}B_2$.

Finally, for $c > 0$ or $c = 0$ and $y \neq 0$ we have

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix}h_1 & h_2\\ -h_2 & h_1\end{pmatrix}, \begin{pmatrix}a & 0 & u\\ 0 & b & v\\ 0 & 0 & 1\end{pmatrix} \right) : ab(h_1^2 + h_2^2) \neq 0 \right\},$$

while for $(c, y) = (0, 0)$ we obtain

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}, \begin{pmatrix} a & 0 & u \\ 0 & b & v \\ 0 & 0 & 1 \end{pmatrix} \right) : ab(h_1^2 + h_2^2) \neq 0 \right\} \bigcup \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ h_2 & -h_1 \end{pmatrix}, \begin{pmatrix} 0 & a & u \\ b & 0 & v \\ 0 & 0 & -1 \end{pmatrix} \right) : ab(h_1^2 + h_2^2) \neq 0 \right\}.$$

VI.7: $\rho = 0$

As usual α, σ are unconstrained and all of W^* is invariant, but $\operatorname{im}\varphi$ is central and since $c \neq 1$ it is zero. The group of weak automorphisms is trivially $\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b})$.

Summary

We summarize the results of this section in Table 8.11, with the same notation as that explained in Table 8.6.

Table 8.11: Two-dimensional real representations of Bianchi $\mathbb{VI}_{1\neq c\geq 0}$.

Label	$\rho(B_1)$	$\rho(B_2)$	$\rho(B_3)$	**Remarks**	α	σ	φ	$(W^*)^\flat$
$\mathbb{VI}.1_{c,x}$	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x+c+1 \end{pmatrix}$		$x=-(1+c)$	$\begin{cases} x=-(1+c) \\ x=-2(1+c) \end{cases}$	$\begin{cases} x=-(1+c) \\ x=-c \end{cases}$	$\mathbb{R}\boldsymbol{e}^2$ $(x=-1-c)$
$\mathbb{VI}.2_{c,x}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	0	$\begin{pmatrix} x & 0 \\ 0 & x+c-1 \end{pmatrix}$	$c>0$	$x=1-c$	$\begin{cases} x=1-c \\ x=2(1-c) \end{cases}$	$\begin{cases} x=1-c \\ x=-c \end{cases}$	$\mathbb{R}\boldsymbol{e}^2$ $(x=1-c)$
$\mathbb{VI}.3_{c,t,x}$	0	0	$\begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix}$	$c=0, \lvert x\rvert\leq 1, t>0$ or $\begin{cases} c>0, t\neq 0, \lvert x\rvert \leq 1 \\ x\neq -1 \text{ or } t>0 \end{cases}$	$x=-1,0$	$x=0,\frac{1}{2}$	$t(x+1)=\pm 1-c$	$\mathbb{R}\boldsymbol{e}^1$ $(x=0)$
$\mathbb{VI}.4_{c,t}$	0	0	$\begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$	$t\neq 0$ $c>0$ or $t>0$			$t=\frac{1}{2}(\pm 1-c)$	
$\mathbb{VI}.5_c$	0	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		✓	✓		$\mathbb{R}\boldsymbol{e}^2$
$\mathbb{VI}.6_{c,t,y}$	0	0	$\begin{pmatrix} ty & -t \\ t & ty \end{pmatrix}$	$y\geq 0$ $t\neq 0$ $c>0$ or $t>0$	$y=0$		$ty=\frac{1}{2}(\pm 1-c)$	
$\mathbb{VI}.7_c$	0	0	0		✓	✓		W^*

8.4.7 Bianchi VII$_{c\geq 0}$

This is the solvable Lie algebra with nonzero brackets $[B_1, B_3] = cB_1 - B_2$ and $[B_2, B_3] = B_1 + cB_2$, where $c \geq 0$. The automorphism group depends on whether or not $c = 0$. If $c > 0$, then

$$\mathrm{Aut}(\mathfrak{b}) = \left\{ \begin{pmatrix} a & b & u \\ -b & a & v \\ 0 & 0 & 1 \end{pmatrix} : a^2 + b^2 > 0 \right\}.$$

If $c = 0$ then there are additional automorphisms

$$\mathrm{Aut}(\mathfrak{b})_- = \left\{ \begin{pmatrix} a & b & u \\ b & -a & v \\ 0 & 0 & -1 \end{pmatrix} : a^2 + b^2 > 0 \right\}.$$

There is a unique proper ideal $[\mathfrak{b}, \mathfrak{b}] = \mathrm{span}\{B_1, B_2\}$, so for every two-dimensional representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$, the space $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is trivial because otherwise the image of a nonzero equivariant map $\Lambda^2 W \to \mathfrak{b}$ would be a one-dimensional ideal. This means that $\varphi = 0$ in every case.

The possible $\ker\rho$ are $[\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{b}$ itself. Since the quotient $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ is a one-dimensional Lie algebra, we have four cases to consider.

VII.1: $\ker\rho = \mathrm{span}\{B_1, B_2\}$ **and** $\mathrm{im}\,\rho = \mathbb{R}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ **with** $|x| \leq 1$

We may choose bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix} \qquad \text{with } t \neq 0.$$

If $c = 0$, a complete set of isoclasses is obtained by assuming $t > 0$, whereas for $c > 0$ we get a complete set of isoclasses by restricting ourselves to the set

$$\big\{(t, x) \in (\mathbb{R} \setminus \{0\}) \times [-1, 1]\colon x \neq -1 \text{ or } t > 0\big\}.$$

We may read off the results for $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ from any other case with the same $\mathrm{im}\,\rho$ (see, e.g., Section 8.4.4) and will not be repeated here. The map $\varphi = 0$ automatically. If $x = 1$, then the weak automorphism group is

$$\mathrm{Aut}(\rho) = \mathsf{GL}(W) \times (\mathrm{Aut}(\mathfrak{b}) \setminus \mathrm{Aut}(\mathfrak{b})_-).$$

For $x = -1$ and $c = 0$, we obtain

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & b & u \\ -b & a & v \\ 0 & 0 & 1 \end{pmatrix} \right) : h_1 h_2 (a^2 + b^2) \neq 0 \right\}$$
$$\bigcup \left\{ \left(\begin{pmatrix} 0 & h_2 \\ h_1 & 0 \end{pmatrix}, \begin{pmatrix} a & b & u \\ b & -a & v \\ 0 & 0 & -1 \end{pmatrix} \right) : h_1 h_2 (a^2 + b^2) \neq 0 \right\}.$$

Finally, for $x \neq \pm 1$ or $c > 0$ and $x \neq 1$, we get

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} a & b & u \\ -b & a & v \\ 0 & 0 & 1 \end{pmatrix} \right) : h_1 h_2 (a^2 + b^2) \neq 0 \right\}.$$

VII.2: $\ker \rho = \mathrm{span}\{B_1, B_2\}$ **and** $\mathrm{im}\, \rho = \mathbb{R} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

We may choose bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix} \qquad \text{with } t \neq 0.$$

If $c > 0$, then t determines ρ up to isomorphism, whereas for $c = 0$ we have to assume $t > 0$ to get a complete set of isoclasses. This is very similar to the previous case: we may read off $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ from, e.g., Section 8.4.4. As explained earlier, $\varphi = 0$. The weak automorphism group is

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & b & u \\ -b & a & v \\ 0 & 0 & 1 \end{pmatrix} \right) : h_1 (a^2 + b^2) \neq 0 \right\}.$$

VII.3: $\ker \rho = \mathrm{span}\{B_1, B_2\}$ **and** $\mathrm{im}\, \rho = \mathbb{R} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

We may choose bases such that $\rho(B_1) = \rho(B_2) = 0$ and by conjugating $\rho(B_3)$ to its Jordan form we obtain

$$\rho(B_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This is very similar to the previous case: we may read off $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ from, e.g., Section 8.4.4. As before, $\varphi = 0$. The group of weak automorphisms is generically

$$\mathrm{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & h_1 \end{pmatrix}, \begin{pmatrix} a & b & u \\ -b & a & v \\ 0 & 0 & 1 \end{pmatrix} \right) : h_1 (a^2 + b^2) \neq 0 \right\},$$

while for $c = 0$ we get extra automorphisms of the form

$$\mathrm{Aut}(\rho)_- = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ 0 & -h_1 \end{pmatrix}, \begin{pmatrix} a & b & u \\ b & -a & v \\ 0 & 0 & -1 \end{pmatrix} \right) : h_1 (a^2 + b^2) \neq 0 \right\}.$$

VII.4: $\ker \rho = \operatorname{span}\{B_1, B_2\}$ **and** $\operatorname{im} \rho = \mathbb{R} \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}$ **with** $y \geq 0$

We may choose bases such that $\rho(B_1) = \rho(B_2) = 0$ and

$$\rho(B_3) = \begin{pmatrix} ty & -t \\ t & ty \end{pmatrix} \qquad \text{with } t \neq 0.$$

If $c > 0$, then the pair (t, y) determines ρ up to weak isomorphism, whereas for $c = 0$ we have to assume $t > 0$ to obtain a complete set of isoclasses. We may read off $\alpha, \sigma, (W^*)^{\mathfrak{b}}$ from, e.g., Section 8.4.4. Again $\varphi = 0$ automatically. For $y = 0$ and $c = 0$, the weak automorphism group is seen to be

$$\left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -\varepsilon h_2 & \varepsilon h_1 \end{pmatrix}, \begin{pmatrix} a & b & u \\ -\varepsilon b & \varepsilon a & v \\ 0 & 0 & \varepsilon \end{pmatrix} \right) : (h_1^2 + h_2^2)(a^2 + b^2) \neq 0, \varepsilon = \pm 1 \right\}.$$

Otherwise, we obtain

$$\operatorname{Aut}(\rho) = \left\{ \left(\begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}, \begin{pmatrix} a & b & u \\ -b & a & v \\ 0 & 0 & 1 \end{pmatrix} \right) : (h_1^2 + h_2^2)(a^2 + b^2) \neq 0 \right\}.$$

VII.5: $\rho = 0$

Here α, σ are unconstrained and all of W^* is invariant, whereas φ is zero since $\mathfrak{b}$ has trivial center. We obviously have $\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b})$.

Summary

We summarize the results of this section in Table 8.12, with the same notation as that explained in Table 8.6.

Table 8.12: Two-dimensional real representations of Bianchi $\mathbb{VII}_{c\geq 0}$.

Label	$\rho(B_1)$	$\rho(B_2)$	$\rho(B_3)$	**Remarks**	α	σ	φ	$(W^*)^{\mathfrak{b}}$
$\mathbb{VII}.1_{c,t,x}$	0	0	$\begin{pmatrix} tx & 0 \\ 0 & t \end{pmatrix}$	$c=0, t>0, \|x\|\leq 1$ or $\begin{cases} c>0, t\neq 0, \|x\|\leq 1 \\ x\neq -1 \text{ or } t>0 \end{cases}$	$x=-1,0$	$x=0,\frac{1}{2}$		$\mathbb{R}\boldsymbol{e}^1$ $(x=0)$
$\mathbb{VII}.2_{c,t}$	0	0	$\begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$	$t\neq 0$ $c>0$ or $t>0$				
$\mathbb{VII}.3_{c}$	0	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		✓	✓		$\mathbb{R}\boldsymbol{e}^2$
$\mathbb{VII}.4_{c,t,y}$	0	0	$\begin{pmatrix} ty & -t \\ t & ty \end{pmatrix}$	$y\geq 0$ $t\neq 0$ $c>0$ or $t>0$	$y=0$			
$\mathbb{VII}.5_{c}$	0	0	0		✓	✓		W^*

8.4.8 Bianchi VIII

This is $\mathfrak{b} = \mathfrak{sl}(2,\mathbb{R})$, which is simple. The Lie brackets are $[B_1, B_2] = -B_3$, $[B_2, B_3] = B_1$ and $[B_1, B_3] = -B_2$. There are no proper ideals. So we have to consider two cases: either $\rho = 0$ or else ρ is injective.

VIII.1: $\ker \rho = 0$

If ρ is injective, then ρ is equivalent to the standard representation. We know from the tables that $\alpha = 0$, $(W^*)^{\mathfrak{b}} = 0$ and $\sigma\colon \mathsf{S}^2 W \to W$ has to be zero, since $\mathsf{S}^2 W$ is the adjoint representation, which is irreducible and inequivalent to the 2-dimensional defining representation. Because $\mathfrak{sl}(2,\mathbb{R})$ has no one-dimensional ideals, it follows directly that any equivariant map $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ is zero. The group of weak automorphisms is

$$\operatorname{Aut}(\rho) = \left\{(g, \operatorname{Ad}(g))\colon g \in \mathsf{GL}(2,\mathbb{R})\right\}.$$

VIII.2: $\rho = 0$

Here, α, σ are unconstrained and all of W^* is invariant, but because $\mathfrak{b}$ has zero center, φ is zero. The group of weak automorphisms is clearly

$$\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b}).$$

8.4.9 Bianchi IX

This is $\mathfrak{b} = \mathfrak{su}(2)$, which is simple and has no nontrivial two-dimensional representations. Hence the only possibility here is $\rho = 0$. Hence α, σ are unconstrained and all of W^* is invariant, but since $\mathfrak{b}$ has trivial center, we have $\varphi = 0$. Clearly, the group of weak automorphisms is $\operatorname{Aut}(\rho) = \mathsf{GL}(W) \times \operatorname{Aut}(\mathfrak{b})$.

8.5 Classification of $(3, 2)$-KLAs

In this section we classify $(3, 2)$-kinematical algebras with spatial isotropy of dimension larger than 2. In particular, we provide the proofs of Theorems C and D.

As we have seen in Section 8.1, for every $(3, 2)$-KLA $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ (with $\dim V = d \geq 3$) the subspace $\mathfrak{b}$ is a Lie subalgebra, and therefore the direct sum $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$ is also a Lie subalgebra. Moreover, the adjoint representation $\operatorname{ad}_{\mathfrak{h}}$ decomposes $\mathfrak{g}$ into a direct sum of modules $\mathfrak{g} = \mathfrak{h} \oplus (V \otimes W)$, where V is the vector representation of $\mathfrak{so}(V)$ and W is a (real) two-dimensional representation of the Bianchi algebra $\mathfrak{b}$.

Now, suppose that we are given a Bianchi algebra $\mathfrak{b}$ and a two-dimensional representation W of $\mathfrak{b}$, so that $V \otimes W$ becomes a representation of $\mathfrak{h} = \mathfrak{so}(V) \oplus \mathfrak{b}$. Then we can construct a $(3, 2)$-KLA structure on $\mathfrak{g} = \mathfrak{h} \oplus (V \otimes W)$ by viewing $V \otimes W$ as the $2d$-dimensional abelian algebra and setting $\mathfrak{g} = \mathfrak{h} \ltimes (V \otimes W)$. We say in this case that $\mathfrak{g}$ is the *split abelian extension* of $\mathfrak{h}$ by $V \otimes W$:

$$0 \longrightarrow V \otimes W \longrightarrow \mathfrak{g} \underset{\curvearrowleft}{\longrightarrow} \mathfrak{h} \longrightarrow 0. \tag{8.33}$$

We shall take the existence of these $(3,2)$-KLA structures as accounted for and not mention them explicitly for the remainder of this section. We concentrate instead on those $(3,2)$-KLAs $\mathfrak{g}$ with nontrivial Lie bracket $\Lambda^2(V \otimes W) \to \mathfrak{g}$.

8.5.1 $(3,2)$-KLAs with $d > 3$

Let $\mathfrak{g}$ be a $(3,2)$-KLA for which $d > 3$. Due to Theorem A the Lie bracket of $\mathfrak{g}$ is determined by a two-dimensional real representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ of a Bianchi Lie algebra $\mathfrak{b}$ and a $\mathfrak{b}$-equivariant map $\varphi\colon \Lambda^2 W \to \mathfrak{b}$. In order to reconstruct the Lie bracket from this data, we also need the $\mathfrak{b}$-invariant symmetric bilinear form $\alpha\colon W \times W \to \mathbb{R}$ defined by relation (8.8).

Moreover, let $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ be a representation. by Proposition 8.6 two equivariant maps $\varphi, \varphi'\colon \Lambda^2 W \to \mathfrak{b}$ induce relatively isomorphic algebras (together with $\mathfrak{b}$ and ρ) if and only if there exists an element $(h, \psi) \in \mathrm{Aut}(\rho)$ such that $\psi \circ \varphi = h^*\varphi' = (\det h)\varphi'$. As a consequence, we obtain the following restatement of Proposition 8.6:

Proposition 8.10. *Let $\mathfrak{b}$ be a Bianchi algebra and $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ a two-dimensional representation. Then, the relative isomorphism classes of $(3,2)$-KLAs with $d > 3$ associated with $\mathfrak{b}$ and ρ are in a one-to-one correspondence with the orbits of the action $\mathrm{Aut}(\rho) \curvearrowright \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ given by*

$$(h, \psi) \cdot \varphi = \frac{1}{\det h}(\psi \circ \varphi). \tag{8.34}$$

Therefore, we need to calculate the orbits of the action (8.34) for each of the representations constructed in Section 8.4 to determine the relative isomorphism classes of $(3,2)$-KLAs with $d > 3$. We note that the orbit of 0 corresponds to the split abelian extension.

We first establish some terminology. Let G be a group and $\mathsf{G} \curvearrowright X$ an action of G on a set X. We say that a subset $S \subseteq X$ is a *complete set of normal forms* for the action if the restriction of the canonical projection $X \to X/\mathsf{G}$ to S is a bijection (in other words, if every element of X is in the same orbit as exactly one element of S).

Moreover, it will prove useful to generalize the notion of orbit equivalent actions (see Chapter 2) to a purely algebraic context. Let $\mathsf{G} \curvearrowright X$ and $\mathsf{H} \curvearrowright Y$ be two group actions. We say that the actions are *orbit equivalent* if there exists a set bijection $f\colon X \to Y$ such that for each $x \in X$ we have $f(\mathsf{G} \cdot x) = \mathsf{H} \cdot f(y)$. The map f is said to be an *orbit equivalence map.*

It is clear that if $f\colon X \to Y$ is an orbit equivalence map and $S \subseteq X$ is a complete set of normal forms for the action $\mathsf{G} \curvearrowright X$, then $f(S)$ is a complete set of normal forms for the action $\mathsf{H} \curvearrowright Y$.

Before embarking on the classification, we fix the following notation: by $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ we denote a basis of W, and by $\{B_1, B_2, B_3\}$ a basis of $\mathfrak{b}$; for each $i, j \in \{1, 2\}$ we write

$$\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = \sum_{k=1}^{3} \varphi^k B_k, \quad \alpha(\boldsymbol{e}_i, \boldsymbol{e}_j) = \alpha_{ij}.$$

If (ρ_{ij}) is the matrix of $\rho(\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2))$ with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ of W, then the coefficients α_{ij} can be computed by

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} -\rho_{21} & \rho_{11} \\ \rho_{11} & \rho_{12} \end{pmatrix}.$$

Because the invariant α is computed directly from φ, we do not focus on it during the classification process, and we simply give its corresponding value in each isomorphism class.

We now determine the orbit spaces of the action (8.34) associated with each two-dimensional representation in Section 8.4.

Bianchi I

We see from Table 8.6 that the two-dimensional representations W of $\mathfrak{b}$ having nonzero invariant maps $\varphi\colon \Lambda^2 W \to \mathfrak{b}$ are $\mathrm{I}.4_{x=-1}$, $\mathrm{I}.6$, $\mathrm{I}.7_{y=0}$ and $\mathrm{I}.8$.

For the case of $\mathrm{I}.4_{x=-1}$, the action of $\mathrm{Aut}(\rho)$ on $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ has three orbits labeled the condition $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \{0, B_1, B_2\}$.

For the cases of $\mathrm{I}.6$ and $\mathrm{I}.7_{y=0}$, one sees that the orbits of the action $\mathrm{Aut}(\rho) \curvearrowright \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ are labeled by $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \{0, \pm B_1, B_2\}$.

Finally, for the trivial representation $\mathrm{I}.8$, because the projection of $\mathrm{Aut}(\rho)$ onto $\mathrm{Aut}(\mathfrak{b}) = \mathsf{GL}(\mathfrak{b})$ is surjective, there are two possible normal forms of φ, obtained by letting $\varphi = 0$ or $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = B_1$.

Bianchi II

From Table 8.7 we obtain that the representations W of $\mathfrak{b}$ for which $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is nontrivial are $\mathrm{II}.4_{x=-1}$, $\mathrm{II}.6$, $\mathrm{II}.7_{y=0}$ and $\mathrm{II}.8$. In all cases we have $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) = \Lambda^2 W^* \otimes \mathbb{R}B_1$. Because $\mathbb{R}B_1$ is the only one-dimensional ideal of $\mathfrak{b}$ and it is contained in the kernel of any two-dimensional representation ρ of $\mathfrak{b}$, we deduce immediately that $\alpha = 0$ for any choice of φ.

It is easy to check that in all cases the map $\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) \mapsto \varphi^1 \in \mathbb{R}$ is an orbit equivalence map between the action of $\mathrm{Aut}(\rho)$ on $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ and the standard action of $(\mathbb{R} \setminus \{0\})$ on $\mathbb{R}$. Consequently, every nonzero $\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ lies in the same orbit as $(\boldsymbol{e}^1 \wedge \boldsymbol{e}^2) \otimes B_1$.

Bianchi III

In this setting, we see from Table 8.8 that the representations W of $\mathfrak{b}$ for which the space $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is nontrivial are the following: $\mathrm{III}.2_{x=-1}$, $\mathrm{III}.2_{x=-2}$, $\mathrm{III}.3_{(t,y)=(-2,-1)}$, $\mathrm{III}.6_{t=-1}$, $\mathrm{III}.7_{t=-1}$, $\mathrm{III}.9_{x=-1}$, $\mathrm{III}.9_{t(x+1)=-2}$, $\mathrm{III}.10_{t=-1}$, $\mathrm{III}.11$, $\mathrm{III}.12_{y=0}$, $\mathrm{III}.12_{ty=-1}$, $\mathrm{III}.13_{x=-1}$, $\mathrm{III}.15$, $\mathrm{III}.16_{y=0}$ and $\mathrm{III}.17$.

In all of the above cases, the space $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is one-dimensional—in fact, spanned by B_i, where $i \in \{1, 2\}$ depending on the case. Furthermore, except for the representations $\mathrm{III}.2_{x=-2}$, $\mathrm{III}.13_{x=-1}$, $\mathrm{III}.15$ and $\mathrm{III}.16_{y=0}$, the action of $\mathrm{Aut}(\rho)$ on $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is orbit equivalent to the standard action of $\mathbb{R} \setminus \{0\}$ on $\mathbb{R}$, meaning that any nonzero φ can be renormalized to $(\boldsymbol{e}^1 \wedge \boldsymbol{e}^2) \otimes B_i$ by an adequate element of $\mathrm{Aut}(\rho)$. In the remaining cases, the action is orbit equivalent to the standard action of $\mathbb{R}^+$ on $\mathbb{R}$, so we obtain a complete set of normal forms for φ by letting $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \{0, \pm B_i\}$.

Bianchi IV

For this Bianchi algebra $\mathfrak{b}$, we see from Table 8.9 that the representations W admitting nonzero invariant elements in $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ are $\mathbb{IV}.1_{x=-1}$, $\mathbb{IV}.2_{t(x+1)=-1}$, $\mathbb{IV}.3_{t=-1/2}$ and $\mathbb{IV}.5_{ty=-1/2}$. We have $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) = \Lambda^2 W^* \otimes \mathbb{R}B_1$ in all cases and, except for the case of $\mathbb{IV}.1_{x=-1}$, the action of $\mathrm{Aut}(\rho)$ on this space is orbit equivalent to the action of $\mathbb{R} \setminus \{0\}$ on $\mathbb{R}$, so we have a unique nonzero normal form for φ given by letting $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = B_1$. In the remaining case, the action is orbit equivalent to the standard action $\mathbb{R}^+ \curvearrowright \mathbb{R}$, and thus the three possible normal forms for φ are given by $\varphi = 0$ and $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = \pm B_1$.

Bianchi V

From Table 8.10 we see that the two-dimensional representations W of $\mathfrak{b}$ that admit nonzero invariant 2-forms in $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ are $\mathbb{V}.1_{x=-1}$, $\mathbb{V}.2_{t(x+1)=-1}$, $\mathbb{V}.3_{t=-1/2}$ and $\mathbb{V}.5_{ty=-1/2}$. In all cases we have that the image of φ lies in $\mathrm{span}\{B_1, B_2\}$.

For the case of $\mathbb{V}.1_{x=-1}$, it can be shown that the map

$$\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) \mapsto (\varphi^1, \varphi^2) \in \mathbb{R}^2$$

establishes an orbit equivalence between the action of $\mathrm{Aut}(\rho)$ on $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ and the standard representation of the group

$$\mathsf{L} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \neq 0, d > 0 \right\}$$

on the plane. Now, a complete set of normal forms for the action $\mathsf{L} \curvearrowright \mathbb{R}^2$ is given by

$$\{(0,0), (1,0), (0, \pm 1)\},$$

and therefore we have four orbits of the $\mathrm{Aut}(\rho)$-action labeled by the possibilities $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \{0, B_1, \pm B_2\}$.

The remaining cases can be tackled at the same time. Indeed, in all of these the map $\varphi \in \mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) \mapsto (\varphi^1, \varphi^2) \in \mathbb{R}^2$ is an orbit equivalence map between the action of $\mathrm{Aut}(\rho)$ on $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ and the standard representation of $\mathsf{GL}(2, \mathbb{R})$ on $\mathbb{R}$. Because of this, every nonzero φ can be renormalized to the map $(\boldsymbol{e}^1 \wedge \boldsymbol{e}^2) \otimes B_1$.

Bianchi VI

From Table 8.11, we see that the two-dimensional representations W of $\mathfrak{b}$ for which the space $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is nonzero are the following ones: $\mathbb{VI}.1_{x+c+1=0}$, $\mathbb{VI}.1_{x=-c}$, $\mathbb{VI}.2_{x=1-c}$, $\mathbb{VI}.2_{x=-c}$, $\mathbb{VI}.3_{t(x+1)=1-c}$, $\mathbb{VI}.3_{t(x+1)=-1-c}$, $\mathbb{VI}.4_{2t=-1-c}$,$\mathbb{VI}.6_{2ty=1-c}$ and $\mathbb{VI}.6_{2ty=-1-c}$.

Note that in all of these cases we have $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) = \Lambda^2 W^* \otimes \mathbb{R}B_i$ for $i = 1$ or $i = 2$ depending on the representation.

In all cases but $\mathrm{VI}.1_{x+c+1=0}$ and $\mathrm{VI}.2_{x=1-c}$, the action of $\mathrm{Aut}(\rho)$ on the space $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is orbit equivalent to the standard action $(\mathbb{R} \setminus \{0\}) \curvearrowright \mathbb{R}$, and therefore every nonzero φ can be renormalized to $(\boldsymbol{e}^1 \wedge \boldsymbol{e}^2) \otimes B_1$.

For the two remaining representations, the action of $\mathrm{Aut}(\rho)$ on $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$ is orbit equivalent to the usual action of $\mathbb{R}^+$ on $\mathbb{R}$, and thus there are three possible normal forms of φ given by letting $\varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \in \{0, \pm B_i\}$.

Bianchi VII, VIII and IX

For these cases there is nothing to do, because if $\mathfrak{b}$ is any of these Bianchi algebras and W is a two-dimensional representation of $\mathfrak{b}$, we get $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) = 0$ automatically.

Summary

We now gather the results of our calculations. Tables 8.13 and 8.14 contain for each two-dimensional representation ρ the nonzero orbit representatives of the standard action of $\mathrm{Aut}(\rho)$ on $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b})$, which thus give $(3,2)$-KLA structures valid for all $d \neq 3$. On the one hand, the normal forms of φ in Table 8.13 are such that $\rho \circ \varphi = 0$, and thus the corresponding bilinear form α is zero. These normal forms correspond to relative isomorphism classes of algebras in class (II). On the other hand, the normal forms in Table 8.14 have a nonvanishing corresponding α, which we have also described. Any form in this table correspond to an algebra in class (III).

Since it is clear from Proposition 8.10 that isomorphic $(3,2)$-KLAs have to belong to the same class, we deduce that Theorem C holds.

Table 8.13: Isomorphism classes of generic $(3,2)$-KLAs in class (II).

Representation	$B=\varphi(\boldsymbol{e}_1\wedge\boldsymbol{e}_2)$	**Representation**	$B=\varphi(\boldsymbol{e}_1\wedge\boldsymbol{e}_2)$
I.$4_{x=-1}$	B_2	IV.$1_{x=-1}$	$\pm B_1$
I.6	B_2	IV.$2_{t(x+1)=-1}$	B_1
I.$7_{y=0}$	B_2	IV.$3_{t=-1/2}$	B_1
I.8	B_1	IV.$5_{ty=-1/2}$	B_1
II.$4_{x=-1}$	B_1	V.$1_{t=-1}$	B_1
II.6	B_1	V.$2_{t(x+1)=-1}$	B_1
II.$7_{y=0}$	B_1	V.$3_{t=-1/2}$	B_1
II.8	B_1	V.$5_{ty=-1/2}$	B_1
III.$2_{x=-1}$	B_1	VI.$1_{x=-c}$	B_1
III.$3_{(t,y)=(-2,-1)}$	B_2	VI.$2_{x=-c}$	B_2
III.$6_{t=-1}$	B_2	VI.$3_{t(x+1)=1-c}$	B_1
III.$7_{t=-1}$	B_2	VI.$3_{t(x+1)=-1-c}$	B_2
III.$9_{x=-1}$	B_1	VI.$4_{2t=1-c}$	B_1
III.$9_{t(x+1)=-2}$	B_2	VI.$4_{2t=-1-c}$	B_2
III.$10_{t=-1}$	B_2	VI.$6_{2ty=1-c}$	B_1
III.11	B_1	VI.$6_{2ty=-1-c}$	B_2
III.$12_{y=0}$	B_1		
III.$12_{ty=-1}$	B_2		
III.17	B_1		

The Lie bracket in $V\otimes W$ is given by
$[u\otimes\boldsymbol{e}_i, v\otimes\boldsymbol{e}_j]=\epsilon_{ij}\langle u,v\rangle B.$

Table 8.14: Isomorphism classes of generic $(3,2)$-KLAs in class (III).

Representation	$B = \varphi(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$	(α_{ij})
I.$4_{x=-1}$	B_1	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
I.6	$\pm B_1$	$\begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}$
I.$7_{y=0}$	$\pm B_1$	$\begin{pmatrix} \mp 1 & 0 \\ 0 & \mp 1 \end{pmatrix}$
III.$2_{x=-2}$	$\pm B_2$	$\begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}$
III.$13_{x=-1}$	B_1	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
III.15	$\pm B_1$	$\begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}$
III.$16_{y=0}$	$\pm B_1$	$\begin{pmatrix} \mp 1 & 0 \\ 0 & \mp 1 \end{pmatrix}$
V.$1_{t=-1}$	$\pm B_2$	$\begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}$
VI.$1_{x+c+1=0}$	$\pm B_2$	$\begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}$
VI.$2_{x=1-c}$	$\pm B_1$	$\begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}$

The Lie bracket in $V \otimes W$ is given by
$[u \otimes \boldsymbol{e}_i, v \otimes \boldsymbol{e}_j] = \alpha_{ij} u \curlywedge v + \epsilon_{ij} \langle u, v \rangle B.$

8.5.2 $(3,2)$-KLAs with $d=3$

This section is devoted to proving Theorem D.

Due to Theorem B, a $(3,2)$-KLA $\mathfrak{g}=\mathfrak{so}(V)\oplus(V\otimes W)\oplus\mathfrak{b}$ with $\dim V=3$ is specified (after performing an adequate relative isomorphism) by the 4-tuple $(\rho,\varphi,\alpha,\sigma)$, where $\rho\colon\mathfrak{b}\to\mathfrak{gl}(W)$ is a real two-dimensional representation of a Bianchi Lie algebra $\mathfrak{b}$, $\varphi\colon\Lambda^2W\to\mathfrak{b}$, $\alpha\colon W\times W\to\mathbb{R}$ and $\sigma\colon W\times W\to W$ are $\mathfrak{b}$-equivariant bilinear maps subject to the equations (8.4). In terms of this data, the Lie brackets of $\mathfrak{g}$ are such that $\mathfrak{so}(V)\oplus\mathfrak{b}$ is a Lie subalgebra acting on $V\otimes W$ in the standard way, while the restriction of the bracket to $\Lambda^2(W\otimes W)$ is given by (8.12). Furthermore, from Proposition 8.7 it follows that after fixing $\mathfrak{b}$ and ρ, the weak isomorphism classes of $(3,2)$-KLAs associated with these are in a bijective correspondence with the orbits of the group

$$\mathrm{Aut}(\rho)\ltimes(W^*)^{\mathfrak{b}}=\left\{(\psi,h,\mu)\in\mathrm{Aut}(\mathfrak{b})\times\mathsf{GL}(W)\times(W^*)^{\mathfrak{b}}\colon\rho\circ\psi=\mathrm{Ad}(h)\circ\rho\right\}$$

acting on solutions (φ,α,σ) of equations (8.4), where the action of (ψ,h,μ) on (φ,α,σ) is $(\varphi',\alpha',\sigma')$ with

$$h^*\varphi'=\psi\circ\varphi,\quad h^*\alpha'=\alpha-\mu\circ\sigma+\mu\otimes\mu,\quad h^*\sigma'=h\circ\sigma-\mu\otimes h-h\otimes\mu.$$

Our first observation is that if $\sigma=0$ then we have the same data (φ,α) and the same equations as in the generic case $\dim V>3$. The discussion in Subsection 8.5.1 therefore applies and we may restrict ourselves to representations $\rho\colon\mathfrak{b}\to\mathsf{GL}(W)$ with nonzero σ.

A second observation is that a closer look at equations (8.4) shows that the Bianchi Lie algebra $\mathfrak{b}$ is directly involved only in determining $\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2W,\mathfrak{b})$, while all the other conditions depend on the image $\mathrm{im}\,\rho\subseteq\mathfrak{gl}(W)$. This results in many solutions (φ,α,σ) being identical for different ρ.

We choose bases $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ for W and $\{B_1,B_2,B_3\}$ for $\mathfrak{b}$. Moreover, we define scalars α_{ij}, σ_{ij}^k and φ^a by

$$\alpha_{ij}=\alpha(\boldsymbol{e}_i,\boldsymbol{e}_j),\qquad\sigma(\boldsymbol{e}_i,\boldsymbol{e}_j)=\sum_{k=1}^{2}\sigma_{ij}^k\boldsymbol{e}_k,\qquad\varphi(\boldsymbol{e}_i\wedge\boldsymbol{e}_j)=\epsilon_{ij}\sum_{a=1}^{3}\varphi^aB_a,$$

where $\epsilon_{ij}=-\epsilon_{ji}$ is the Levi-Civita symbol normalized so that $\epsilon_{12}=1$. We also let $\boldsymbol{e}^i$, $i\in\{1,2\}$, be the canonical dual basis for W^*.

We now summarize the results via the subalgebras $\mathrm{im}\,\rho$ of $\mathfrak{gl}(2,\mathbb{R})$ listed in Table 8.4 and, of course, the trivial representation $\rho=0$.

Case $\mathrm{im}\,\rho=\mathfrak{s}_0$

This scenario corresponds to the representations Ⅲ.$2_{x=-2}$, Ⅳ.$1_{x=-1}$, Ⅴ.$1_{x=-1}$, Ⅵ.$1_{x=-(1+c)}$ and Ⅵ.$2_{x=1-c}$. For these representations, we have

$$\mathrm{Hom}_{\mathfrak{b}}(\mathsf{S}^2W,W)=\mathbb{R}(\boldsymbol{e}_1\otimes(\boldsymbol{e}^1\otimes\boldsymbol{e}^2+\boldsymbol{e}^2\otimes\boldsymbol{e}^1)+2\boldsymbol{e}_2\otimes\boldsymbol{e}^2\otimes\boldsymbol{e}^2).$$

Suppose that $\sigma\in\mathrm{Hom}_{\mathfrak{b}}(\mathsf{S}^2W,W)$ and $\varphi\in\mathrm{Hom}_{\mathfrak{b}}(\Lambda^2W,\mathfrak{b})$ satisfy (8.4). Because $(W^*)^{\mathfrak{b}}=\mathbb{R}\boldsymbol{e}^2$ and the covector $\mu_2\boldsymbol{e}^2$ acts on σ by sending σ_{12}^1 to $\sigma_{12}^1-\mu_2$, we see that the pair (σ,φ) admits a renormalization for which $\sigma=0$, which means that none of these representations admit a $(3,2)$-KLA structure exclusive to $d=3$.

Case $\operatorname{im}\rho = \mathfrak{s}_{\arctan(1/2)}$

This corresponds to $\mathbb{III}.2_{x=-4}$, $\mathbb{IV}.1_{x=-2}$, $\mathbb{V}.1_{x=-2}$, $\mathbb{VI}.1_{x=-2(1+c)}$ and $\mathbb{VI}.2_{x=2(1-c)}$. In all cases one sees that $\operatorname{Hom}_{\mathfrak{b}}(\Lambda^2 W, \mathfrak{b}) = 0$, and the only nonzero component of σ is σ^1_{22}. In particular, we get by direct calculation that $\alpha = 0$. Note that in this case $(W^*)^{\mathfrak{b}} = 0$, so that if σ is nonzero it does not belong to the orbit of the zero map. For all representations ρ in this class, the projection of the group $\operatorname{Aut}(\rho)$ to $\mathsf{GL}(W)$ contains matrices of the form

$$h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $a/d^2 \neq 0$ but otherwise arbitrary (perhaps after compensating with a suitable automorphism of $\mathfrak{b}$). The action of such h on $\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$ is to rescale it by a/d^2 and hence if σ^1_{22} is nonzero, we may assume that $\sigma^1_{22} = 1$. We conclude that these representations admit a unique $(3,2)$-KLA structure exclusive to $d = 3$.

Case $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

This corresponds to $\mathbb{I}.4_{x=0}$, $\mathbb{II}.4_{x=0}$, $\mathbb{III}.9_{x=0}$, $\mathbb{III}.13_{x=0}$, $\mathbb{IV}.2_{x=0}$, $\mathbb{V}.2_{x=0}$, $\mathbb{VI}.3_{x=0}$ and $\mathbb{VII}.1_{x=0}$. We have in these cases that $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^1$ and the possible nonzero components of σ are σ^1_{11} and σ^2_{12}, without any restrictions. Suppose that the triple $(\sigma, \varphi, \alpha)$ satisfies (8.4). Because the action of $\mu_1 \boldsymbol{e}^1$ sends σ^1_{11} to $\sigma^1_{11} - 2\mu_1$ and σ^2_{12} to $\sigma^2_{12} - \mu_1$, we can renormalize σ so that $\sigma^2_{12} = 0$. If $\sigma \neq 0$, then we see that

$$\sigma(\boldsymbol{e}_1, \boldsymbol{e}_1) \wedge \boldsymbol{e}_2 + \sigma(\boldsymbol{e}_1, \boldsymbol{e}_2) \wedge \boldsymbol{e}_1 + \sigma(\boldsymbol{e}_2, \boldsymbol{e}_1) \wedge \boldsymbol{e}_1 = \sigma^1_{11}(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$$

does not vanish, so the second equation in (8.4) forces $\varphi = 0$. By direct computation, one also sees that $\alpha = 0$. Because the projection of $\operatorname{Aut}(\rho)$ onto $\mathsf{GL}(W)$ contains the subgroup of all nonsingular diagonal matrices in all cases, we can rescale σ^1_{11} freely, so we conclude that these representations admit a unique $(3,2)$-KLA structure exclusive to $d = 3$ given by letting $\sigma^1_{11} = 1$ and $\varphi = 0$.

Case $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

This case corresponds to the representations $\mathbb{I}.4_{x=1/2}$, $\mathbb{II}.4_{x=1/2}$, $\mathbb{III}.9_{x=1/2}$, $\mathbb{III}.13_{x=1/2}$, $\mathbb{IV}.2_{x=1/2}$, $\mathbb{V}.2_{x=1/2}$, $\mathbb{VI}.3_{x=1/2}$ and $\mathbb{VII}.1_{x=1/2}$. In all of these cases we have $(W^*)^{\mathfrak{b}} = 0$ and the only nonzero component of σ is σ^2_{11}. Furthermore, the action of $\operatorname{Aut}(\rho)$ allows us to rescale σ^2_{11} freely, so if $\sigma \neq 0$ we can renormalize σ so that $\sigma^2_{11} = 1$. Observe that

$$\sigma(\boldsymbol{e}_1, \boldsymbol{e}_1) \wedge \boldsymbol{e}_1 = -\sigma^2_{11} \boldsymbol{e}_1 \wedge \boldsymbol{e}_2,$$

so the second equation in (8.4) implies that $\varphi = 0$. As a consequence, $\alpha = 0$, and we conclude that all of these representations admit a unique $(3,2)$-KLA structure exclusive to $d = 3$ obtained by letting $\sigma^2_{11} = 1$ and $\varphi = 0$.

Case $\operatorname{im}\rho = \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

This case corresponds to representations $\mathrm{I}.6$, $\mathrm{II}.6$, $\mathrm{III}.11$, $\mathrm{III}.15$, $\mathrm{IV}.4$, $\mathrm{V}.4$, $\mathrm{VI}.5$ and $\mathrm{VII}.3$. The possible nonzero components of σ are σ^1_{22} and $\sigma^2_{22} = 2\sigma^1_{12}$ (with σ^1_{12} and σ^1_{22} free). We also have $(W^*)^{\mathfrak{b}} = \mathbb{R}\boldsymbol{e}^2$ and the action of $\mu_2\boldsymbol{e}^2$ on σ sends

$$\sigma^1_{12} \mapsto \sigma^1_{12} - \mu_2 \quad \text{and} \quad \sigma^1_{22} \mapsto \sigma^1_{22}.$$

Because of this, we can renormalize any solution $(\varphi, \alpha, \sigma)$ so that $\sigma^1_{12} = 0$. For such a solution, assuming that $\sigma \neq 0$, we have $\sigma(\boldsymbol{e}_2, \boldsymbol{e}_2) \wedge \boldsymbol{e}_2 = \sigma^1_{22}\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$, so the second equation in (8.4) yields $\varphi = 0$. A calculation shows that $\alpha = 0$, and in all cases the action of $\operatorname{Aut}(\rho)$ on $\mathbb{R}\boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2$ is orbit equivalent to the standard action of $\mathbb{R} \setminus \{0\}$ on $\mathbb{R}$, so we can renormalize σ so that $\sigma^1_{22} = 1$. Thus, these representations admit exactly one $(3, 2)$-KLA structure exclusive to $d = 3$ given by $\sigma^1_{22} = 1$ and $\varphi = 0$.

Case $\rho = 0$

This corresponds to representations $\mathrm{I}.8$, $\mathrm{II}.8$, $\mathrm{III}.17$, $\mathrm{IV}.6$, $\mathrm{V}.6$, $\mathrm{VI}.7$, $\mathrm{VII}.5$, $\mathrm{VIII}.2$ and the only real two-dimensional representation of Bianchi IX.

We show that any solution $(\varphi, \alpha, \sigma)$ of (8.4) admits a renormalization with either $\sigma = 0$ or $\varphi = 0$. Indeed, from the results in Section 8.A we see that σ may be assumed to have trace zero, or in other words, that for every $w_1 \in W$ the operator $\widehat{\sigma}_{w_1} \in \operatorname{End}(W)$ given by $\widehat{\sigma}_{w_1}(w_2) = \sigma(w_1, w_2)$ is in $\mathfrak{sl}(W)$. In particular, this means that for every $w_2, w_3 \in W$ we have

$$0 = \widehat{\sigma}_{w_1}(w_2 \wedge w_3) = \sigma(w_1, w_2) \wedge w_3 + w_2 \wedge \sigma(w_1, w_3).$$

As a consequence, the expression $\sigma(w_1, w_2) \wedge w_3$ is totally symmetric in the three variables, so the second equation in (8.4) becomes

$$3\varphi(\sigma(w_1, w_2) \wedge w_3) = 0, \quad w_1, w_2, w_3 \in W.$$

Thus, because φ is either zero or an isomorphism, we deduce from the equation above that either $\sigma = 0$ (so the corresponding $(3, 2)$-KLA structure is generic) or $\varphi = 0$. Note that if $\varphi = 0$, then the coefficients of α are

$$\begin{aligned} \alpha_{11} &= (\sigma^2_{22} - \sigma^1_{12})\sigma^2_{11} + (\sigma^1_{11} - \sigma^2_{12})\sigma^2_{12}, \\ \alpha_{12} &= \sigma^1_{12}\sigma^2_{12} - \sigma^1_{22}\sigma^2_{11}, \\ \alpha_{22} &= (\sigma^1_{11} - \sigma^2_{12})\sigma^1_{22} + (\sigma^2_{22} - \sigma^1_{12})\sigma^1_{12}. \end{aligned}$$

We conclude that in order to determine the $(3, 2)$-KLA structures exclusive to $d = 3$, it is necessary and sufficient to understand the orbit space of the natural action of $\mathsf{GL}(W) \ltimes W^*$ on $\operatorname{Hom}(\mathsf{S}^2 W, W)$. This is done in Section 8.A, where we relate this classification to the classical problem of determining the normal forms for binary cubic forms of weight -1 under the action of $\mathsf{GL}(2, \mathbb{R})$. Summarizing the results in that section, we have the following normal forms σ_s (where $s = 1, \ldots, 4$) for nonzero σ, where we list only the nonzero coefficients σ^k_{ij} and α_{ij}.

- $(\sigma_1)^1_{11} = -(\sigma_1)^2_{12} = \frac{1}{3}$ and $(\sigma_1)^1_{22} = -1$, hence $(\alpha_1)_{11} = -\frac{2}{9}$, $(\alpha_1)_{22} = -\frac{2}{3}$;
- $(\sigma_2)^1_{11} = -(\sigma_2)^2_{12} = \frac{1}{3}$ and $(\sigma_2)^1_{22} = 1$, hence $(\alpha_2)_{11} = -\frac{2}{9}$, $(\alpha_2)_{22} = \frac{2}{3}$;
- $(\sigma_3)^1_{12} = -(\sigma_3)^2_{22} = \frac{1}{3}$, hence $(\alpha_3)_{22} = -\frac{2}{9}$;
- and $(\sigma_4)^1_{22} = 1$, hence $\alpha_4 = 0$.

Recall that it is possible that a general nonzero σ is in the same orbit as the zero tensor.

It will be simpler for us to consider different renormalizations of these σ_s so that their corresponding bilinear forms α_s vanish. Let us define modified bilinear maps τ_s as follows:

- $\tau_1 = (-\operatorname{id}_W, \frac{2}{3}\boldsymbol{e}^1) \cdot \sigma_1$, with nonzero components $\tau^1_{11} = \tau^1_{12} = \tau^1_{22} = 1$;
- $\tau_2 = (-\operatorname{id}_W, \frac{2}{3}\boldsymbol{e}^1) \cdot \sigma_2$, with nonzero components $\tau^1_{11} = \tau^1_{12} = -\tau^1_{22} = 1$;
- $\tau_3 = (h, \frac{1}{3}\boldsymbol{e}^2) \cdot \sigma_3 = \boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1$, where

$$h = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix};$$

- and $\tau_4 = (h, 0) \cdot \sigma_4 = \boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1$, where h is the same matrix as above.

Then by construction each τ_s is in the same orbit as σ_s, and a calculation shows that the bilinear form β_s constructed from τ_s via (8.3) vanishes identically. We therefore obtain that every solution of (8.4) taking the form $(\varphi = 0, \alpha, \sigma)$ is in the same orbit as exactly one of either $(0,0,0)$ or a $(0,0,\tau_s)$ with $s \in \{1,\dots,4\}$.

Summary

Let us now gather our results and prove Theorem D. From our calculations, we have seen that every $(3,2)$-KLA $\mathfrak{g} = \mathfrak{so}(V) \oplus (V \otimes W) \oplus \mathfrak{b}$ with $\dim V = d = 3$ is relatively isomorphic to one that has an associated two-dimensional representation $\rho\colon \mathfrak{b} \to \mathfrak{gl}(W)$ and such that $[\mathfrak{b}, V \otimes W] \subseteq V \otimes W$ (in other words, $\lambda = 0$ in the notation of Theorem 8.3). Moreover, the remaining objects φ and σ determining the Lie bracket of $\mathfrak{g}$ can be brought to a normal form where either $\varphi = 0$ or $\sigma = 0$. On the one hand, if $\sigma = 0$, then we are back at the problem of determining the normal forms of φ up to the action of $\operatorname{Aut}(\rho)$, so we conclude that $\mathfrak{g}$ is either a split abelian extension or one of the algebras appearing in Tables 8.13 and 8.14. On the other hand, if $\varphi = 0$, then either σ can be brought to zero by an element of $(W^*)^{\mathfrak{b}}$ (so $\mathfrak{g}$ is actually a split abelian extension in disguise) or it may be transformed into exactly one of the normal forms described in this section. This last case corresponds to $\mathfrak{g}$ being in class (IV). The canonical forms for σ (and thus all $(3,2)$-KLAs with $d = 3$ and in class (IV)) are collected in Table 8.15, and they all have its corresponding α equal to zero.

All in all, Theorem D is proved.

Table 8.15: Isomorphism classes of $(3,2)$-KLAs with $d=3$ and in class (IV).

Representation	σ_{11}	σ_{12}	σ_{22}
Ⅰ.$4_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅰ.$4_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅰ.6	0	0	$\boldsymbol{e}_1$
Ⅰ.8	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅰ.8	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅰ.8	$\boldsymbol{e}_1$	0	0
Ⅰ.8	$\boldsymbol{e}_2$	0	0
Ⅱ.$4_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅱ.$4_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅱ.6	0	0	$\boldsymbol{e}_1$
Ⅱ.8	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅱ.8	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅱ.8	$\boldsymbol{e}_1$	0	0
Ⅱ.8	$\boldsymbol{e}_2$	0	0
Ⅲ.$2_{x=-4}$	0	0	$\boldsymbol{e}_1$
Ⅲ.$9_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅲ.$9_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅲ.11	0	0	$\boldsymbol{e}_1$
Ⅲ.$13_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅲ.$13_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅲ.15	0	0	$\boldsymbol{e}_1$
Ⅲ.17	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅲ.17	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅲ.17	$\boldsymbol{e}_1$	0	0
Ⅲ.17	$\boldsymbol{e}_2$	0	0
Ⅳ.$1_{x=-2}$	0	0	$\boldsymbol{e}_1$
Ⅳ.$2_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅳ.$2_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅳ.4	0	0	$\boldsymbol{e}_1$
Ⅳ.6	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅳ.6	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅳ.6	$\boldsymbol{e}_1$	0	0
Ⅳ.6	$\boldsymbol{e}_2$	0	0

Representation	σ_{11}	σ_{12}	σ_{22}
Ⅴ.$1_{x=-2}$	0	0	$\boldsymbol{e}_1$
Ⅴ.$2_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅴ.$2_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅴ.4	0	0	$\boldsymbol{e}_1$
Ⅴ.6	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅴ.6	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅴ.6	$\boldsymbol{e}_1$	0	0
Ⅴ.6	$\boldsymbol{e}_2$	0	0
Ⅵ.$1_{x+2c+2=0}$	0	0	$\boldsymbol{e}_1$
Ⅵ.$1_{x=2(1-c)}$	0	0	$\boldsymbol{e}_1$
Ⅵ.$3_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅵ.$3_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅵ.5	0	0	$\boldsymbol{e}_1$
Ⅵ.7	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅵ.7	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅵ.7	$\boldsymbol{e}_1$	0	0
Ⅵ.7	$\boldsymbol{e}_2$	0	0
Ⅶ.$1_{x=0}$	$\boldsymbol{e}_1$	0	0
Ⅶ.$1_{x=1/2}$	$\boldsymbol{e}_2$	0	0
Ⅶ.3	0	0	$\boldsymbol{e}_1$
Ⅶ.5	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅶ.5	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅶ.5	$\boldsymbol{e}_1$	0	0
Ⅶ.5	$\boldsymbol{e}_2$	0	0
Ⅷ.2	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅷ.2	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅷ.2	$\boldsymbol{e}_1$	0	0
Ⅷ.2	$\boldsymbol{e}_2$	0	0
Ⅸ	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_1$
Ⅸ	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
Ⅸ	$\boldsymbol{e}_1$	0	0
Ⅸ	$\boldsymbol{e}_2$	0	0

The Lie bracket in $V \otimes W$ is given by
$[u \otimes \boldsymbol{e}_i, v \otimes \boldsymbol{e}_j] = (u \times v) \otimes \sigma_{ij}$.

8.A The action of $\mathsf{GL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^*$ on $\operatorname{Hom}(\mathsf{S}^2\mathbb{R}^2, \mathbb{R}^2)$

Let $\sigma \in \operatorname{Hom}(\mathsf{S}^2\mathbb{R}^2, \mathbb{R}^2)$ and let $(h, \mu) \in \mathsf{G} := \mathsf{GL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^*$. The action is defined, for all $x, y \in \mathbb{R}^2$, by

$$((h,\mu)\cdot\sigma)(x,y) = h\sigma(h^{-1}x, h^{-1}y) - \mu(h^{-1}x)y - \mu(h^{-1}y)x.$$

We are interested in bringing such σ to a normal form: a unique representative for every G-orbit.

Let us define a *trace map* $\operatorname{Tr}\colon \operatorname{Hom}(\mathsf{S}^2\mathbb{R}^2, \mathbb{R}^2) \to (\mathbb{R}^2)^*$ by

$$(\operatorname{Tr}\sigma)(x) = \operatorname{tr}\widehat{\sigma}_x,$$

for every $x \in \mathbb{R}^2$, where $\widehat{\sigma}_x \in \operatorname{End}(\mathbb{R}^2)$ is the endomorphism sending $y \mapsto \sigma(x,y)$ and tr is the usual trace of endomorphisms. Relative to canonical dual bases $\boldsymbol{e}_1, \boldsymbol{e}_2$ for $\mathbb{R}^2$ and $\boldsymbol{e}^1, \boldsymbol{e}^2$ for $(\mathbb{R}^2)^*$, we have $\sigma(\boldsymbol{e}_i, \boldsymbol{e}_j) = \sigma_{ij}^k \boldsymbol{e}_k$ and the trace is given by $\operatorname{Tr}\sigma = \sigma_{ij}^j \boldsymbol{e}^i$.

Lemma 8.11. *For all* $(h,\mu) \in \mathsf{G}$ *and symmetric bilinear* $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$*, we have*

$$\operatorname{Tr}((h,\mu)\cdot\sigma) = (h^{-1})^*(\operatorname{Tr}\sigma - 3\mu).$$

Proof. For all $x \in \mathbb{R}^2$, we have an identity between endomorphisms

$$((h,\mu)\cdot\widehat{\sigma})_x = h \circ \widehat{\sigma}_{h^{-1}x} \circ h^{-1} - \mu(h^{-1}x)\operatorname{id}_{\mathbb{R}^2} - x \otimes (\mu \circ h^{-1}),$$

so that taking the trace

$$\begin{aligned}
\operatorname{Tr}((h,\mu)\cdot\sigma)(x) &= \operatorname{tr}((h,\mu)\cdot\widehat{\sigma})_x \\
&= \operatorname{tr}\left(h \circ \widehat{\sigma}_{h^{-1}x} \circ h^{-1} - \mu(h^{-1}x)\operatorname{id}_{\mathbb{R}^2} - x \otimes (\mu \circ h^{-1})\right) \\
&= \operatorname{tr}\widehat{\sigma}_{h^{-1}x} - 2\mu(h^{-1}x) - \mu(h^{-1}x) \\
&= (\operatorname{Tr}\sigma)(h^{-1}x) - 3\mu(h^{-1}x) \\
&= (h^{-1})^*(\operatorname{Tr}\sigma - 3\mu)(x),
\end{aligned}$$

which gives the desired result upon abstracting x. □

It follows from this lemma, that given any σ, we may act with $(\operatorname{id}_{\mathbb{R}^2}, \frac{1}{3}\operatorname{tr}\sigma)$ to arrive at a σ' with $\operatorname{tr}\sigma' = 0$. Furthermore, a direct consequence of 8.11 is that the normalizer of $U = \ker\operatorname{tr}$ is the canonical subgroup $\mathsf{GL}(2,\mathbb{R})$ (embedded in G via the map $h \mapsto (h, 0)$). We conclude that the G-orbits in $\operatorname{Hom}(\mathsf{S}^2\mathbb{R}^2, \mathbb{R}^2)$ are in a bijective correspondence with the $\mathsf{GL}(2,\mathbb{R})$-orbits in U, a real four-dimensional space with basis

$$\begin{aligned}
\sigma_1 &= \boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^1, \\
\sigma_2 &= \boldsymbol{e}_2 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2 - \boldsymbol{e}_1 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^2 - \boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^1, \\
\sigma_3 &= \boldsymbol{e}_1 \otimes \boldsymbol{e}^2 \otimes \boldsymbol{e}^2, \\
\sigma_4 &= \boldsymbol{e}_2 \otimes \boldsymbol{e}^1 \otimes \boldsymbol{e}^1.
\end{aligned} \tag{8.35}$$

The subgroup $\mathsf{SL}(2,\mathbb{R}) \subseteq \mathsf{GL}(2,\mathbb{R})$ preserves a symplectic structure ϵ on $\mathbb{R}^2$, which we normalize by $\epsilon(\boldsymbol{e}_i \wedge \boldsymbol{e}_j) = \epsilon_{ij}$, the Levi-Civita symbol with $\epsilon_{12} = 1$. Given $\sigma \in \mathrm{Hom}(\mathsf{S}^2\mathbb{R}^2, \mathbb{R}^2)$ we may contract with ϵ to give a cubic $q \in \mathrm{Hom}(\mathsf{S}^2\mathbb{R}^2 \otimes \mathbb{R}^2, \mathbb{R})$ defined by the equation

$$q(x, y, z) = \epsilon(\sigma(x, y) \wedge z).$$

This cubic form is symmetric in the first two entries, but more is true.

Lemma 8.12. *If* $\mathrm{Tr}\,\sigma = 0$ *then* q *is totally symmetric.*

Proof. The condition $\mathrm{Tr}\,\sigma = 0$ is equivalent to the operators $\widehat{\sigma}_x$ being in $\mathfrak{sl}(2,\mathbb{R})$ for all $x \in \mathbb{R}^2$. Given any x, y, $z \in \mathbb{R}^2$, the fact that $\widehat{\sigma}_x$ has zero trace yields $\sigma(x, y) \wedge z + y \wedge \sigma(x, z) = 0$. Applying the symplectic structure ϵ we deduce $q(x, y, z) = q(x, z, y) = q(z, x, y)$. Therefore, q is invariant under a transposition and a 3-cycle, so it is totally symmetric. □

This means that to every "traceless" σ we can associate a binary cubic form Q via

$$Q(x) = \epsilon(\sigma(x, x), x), \qquad \text{for } x \in \mathbb{R}^2,$$

from where q can be reconstructed by polarization. As a corollary, the space U of traceless symmetric bilinear maps $\sigma\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is isomorphic to $\mathsf{S}^3(\mathbb{R}^2)^*$ as a representation of $\mathsf{SL}(2,\mathbb{R})$.

The action of $\mathsf{GL}(2,\mathbb{R})$ on U is not the natural one, but is instead twisted by the reciprocal of the determinant, reflecting the fact that ϵ is not invariant under $\mathsf{GL}(2,\mathbb{R})$. Indeed, as a representation of $\mathsf{GL}(2,\mathbb{R})$, $U \cong \mathsf{S}^3(\mathbb{R}^2)^* \otimes \Lambda^2(\mathbb{R}^2)^*$. In classical notation (see, e.g., [136]), Q is a binary cubic form of weight -1. The determination of the normal forms of such Q is a classical problem in invariant theory and its solution is described, for example, in [136, Section 2]. Let us summarize the necessary results.

Let Q be a cubic binary form of weight -1. This means that, using (x, y) for our coordinates in $\mathbb{R}^2$,

$$Q(x, y) = a_3x^3 + 3a_2x^2y + 3a_1xy^2 + a_0y^3$$

and under a $\mathsf{GL}(2,\mathbb{R})$ transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{8.36}$$

$Q \mapsto Q'$, where

$$Q(x, y) = (\alpha\delta - \beta\gamma)^{-1} Q'(x', y').$$

Recall that a homogeneous polynomial $J(\boldsymbol{a}, \boldsymbol{x}) := J(a_0, a_1, a_2, a_3, x, y)$ is said to be a *covariant of* Q *of weight* k (written $\mathrm{wt}\, J = k$) if $J \mapsto J'$ under the transformation (8.36), where

$$J(\boldsymbol{a}, \boldsymbol{x}) = (\alpha\delta - \beta\gamma)^k\, J'(\boldsymbol{a}', \boldsymbol{x}').$$

A covariant J of a binary form Q is subject to two natural gradations in addition to its weight: its *degree* $\deg J$ (the degree as a polynomial in $\boldsymbol{x}$) and its *order* $\mathrm{ord}\, J$ (the degree as a polynomial in $\boldsymbol{a}$). These are not independent, but satisfy the relation:

$$\deg J + 2\,\mathrm{wt}\, J = (\deg Q + 2\,\mathrm{wt}\, Q)\, \mathrm{ord}\, J.$$

A binary cubic form Q has four independent covariants: Q itself of order 1, degree 3 and weight -1; the *discriminant* Δ, defined by

$$\Delta := a_0^2 a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 - 3a_1^2a_2^2 + 4a_1^3a_3,$$

of degree 0, order 4 and weight 2; the *Hessian* H defined by

$$H := Q_{xx}Q_{yy} - Q_{xy}^2,$$

with subscripts denoting partial differentiation, of degree 2, order 2 and weight 0; and the Jacobian T of Q and H, defined by

$$T := Q_x H_y - Q_y H_x,$$

which we shall not need in our analysis. The discriminant Δ of a cubic Q agrees up to a nonzero multiple with the discriminant of the Hessian H as a binary quadratic form, so we do not get any further covariants in this way.

With every binary cubic form $Q(x, y)$ there is an associated polynomial $\widehat{Q}(z) := Q(z, 1)$, from where we may reconstruct Q via

$$Q(x, y) = y^3 \widehat{Q}\left(\frac{x}{y}\right).$$

The function $\widehat{Q}$ is really to be understood as a function on the projective space $\mathbb{R}\mathsf{P}^1$ on which $\mathsf{GL}(2, \mathbb{R})$ acts via fractional linear transformations.

The nature of roots of $\widehat{Q}$ (points in $\mathbb{R}\mathsf{P}^1$ where it vanishes) is intimately linked to the possible values of the covariants Δ and H. Two basic facts are the following: $\Delta = 0$ if and only if $\widehat{Q}$ has a repeated root; $H = 0$ if and only if $\widehat{Q}$ has a triple root.

If $\Delta \neq 0$, then $\widehat{Q}$ has three simple roots. There are two possibilities: either they are all real ($\Delta > 0$) or there are two complex roots appearing as a complex conjugate pair ($\Delta < 0$). If $\Delta = 0$, but $H \not\equiv 0$, then $\widehat{Q}$ has a double root. If $H \equiv 0$ then $\widehat{Q}$ has a triple root, which is at $z = 0$ if and only if $Q = 0$.

It is then a matter of studying the action of $\mathsf{GL}(2, \mathbb{R})$ via fractional linear transformations on the roots of $\widehat{Q}$, thought of as points in $\mathbb{R}\mathsf{P}^1$ or $\mathbb{C}\mathsf{P}^1$. This results in the following normal forms:

- if $\Delta > 0$, then $\widehat{Q}$ has three simple real roots which we can place at $-1, 1, \infty$, resulting in $\widehat{Q}(z) = z^2 - 1$ and hence
$$Q(x, y) = yx^2 - y^3;$$
- if $\Delta < 0$, then $\widehat{Q}$ has two complex conjugate roots and a real root, which we can place at $-i, i, \infty$, resulting in $\widehat{Q}(z) = z^2 + 1$ and hence
$$Q(x, y) = yx^2 + y^3;$$
- if $\Delta = 0$, but $H \not\equiv 0$, then $\widehat{Q}$ has a repeated root, which we can place at ∞ while placing the simple root at 0, resulting in $\widehat{Q}(z) = z$ and hence
$$Q(x, y) = y^2 x;$$

- if $H \equiv 0$, but $Q \not\equiv 0$, we have a triple root, which we can place at ∞, resulting in $\widehat{Q}(z) = 1$ and hence
$$Q(x,y) = y^3;$$
- and finally we have the trivial case where $Q \equiv 0$.

We now apply this to the case at hand. Consider a symmetric bilinear form $\sigma\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{Tr}\sigma = 0$. In terms of the basis $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ described in (8.35) we can write $\sigma = \sigma^1_{11}\sigma_1 - \sigma^1_{12}\sigma_2 + \sigma^1_{22}\sigma_3 + \sigma^2_{11}\sigma_4$. We find that the binary cubic form Q arising from σ is given by
$$Q(x,y) = -\sigma^2_{11}x^3 + 3\sigma^1_{11}x^2y + 3\sigma^1_{12}xy^2 + \sigma^1_{22}y^3. \tag{8.37}$$
The above normal forms for Q thus translate into the following:

- $Q(x,y) = x^2y - y^3$ corresponds to
$$\sigma = \frac{1}{3}\sigma_1 - \sigma_3;$$
- $Q(x,y) = x^2y + y^3$ corresponds to
$$\sigma = \frac{1}{3}\sigma_1 + \sigma_3;$$
- $Q(x,y) = xy^2$ corresponds to
$$\sigma = -\frac{1}{3}\sigma_2;$$
- $Q(x,y) = y^3$ corresponds to
$$\sigma = \sigma_3;$$
- and of course the trivial case $\sigma = 0$.

It is perhaps curious that the Hessian H of the binary cubic form Q in (8.37) is given by
$$\frac{1}{18}H(x,y) = \alpha_{11}x^2 + 2\alpha_{12}xy + \alpha_{22}y^2,$$
where α_{ij} are the coefficients of the unique symmetric bilinear form α related to σ by the condition
$$\alpha(w_2,w_3)w_1 - \alpha(w_1,w_3)w_2 = \sigma(\sigma(w_2,w_3),w_1) - \sigma(\sigma(w_1,w_3),w_2), \quad w_i \in \mathbb{R}^2.$$

Bibliography

[1] A. V. Alekseevski, D. V. Alekseevski, Riemannian G-manifold with one-dimensional orbit space, *Ann. Global Anal. Geom.* **11** (1993), no. 3, 197–211.

[2] D. V. Alekseevski, Y. G. Nikonorov, Compact Riemannian manifolds with homogeneous geodesics, *SIGMA Symmetry Integrability Geom. Methods Appl.* **5** (2009), Paper 093, 16 pp.

[3] D. V. Alekseevski, A. J. Di Scala, Minimal homogeneous submanifolds of symmetric spaces, Lie groups and symmetric spaces, *Amer. Math. Soc. Transl. Ser. 2* **210** (2003), 11–25.

[4] M. M. Alexandrino, R. G. Bettiol, *Lie groups and geometric aspects of isometric actions*, Springer, Cham, 2015.

[5] M. M. Alexandrino, D. Töben, Singular Riemannian foliations on simply connected spaces, *Differential Geom. Appl.* **24** (2006), no. 4, 383–397.

[6] W. Ambrose, I. M. Singer, A theorem on holonomy, *Trans. Amer. Math. Soc.* **75** (1953), 428–443.

[7] M. Anarella, M. Liefsoens, Isometry groups of nearly Kähler manifolds, `arXiv:2411.05675` [math.DG].

[8] T. Andrzejewski, J. M. Figueroa-O'Farrill, Kinematical Lie algebras in 2+1 dimensions, *J. Math. Phys.* **59** (2018), no. 6, 061703, 21 pp.

[9] B. Aslan, Special Lagrangians in nearly Kähler $\mathbb{CP}^3$, *J. Geom. Phys.* **184** (2023), Paper No. 104713, 18 pp.

[10] M. F. Atiyah, N. J. Hitchin, I. M. Singer, Self-duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A* **362** (1978), no. 1711, 425–461.

[11] R. Azencott, E. N. Wilson, Homogeneous manifolds with negative curvature. I, *Trans. Amer. Math. Soc.* **215** (1976), 323–362.

[12] H. Bacry, J. M. Lévy-Leblond, Possible kinematics, *J. Mathematical Phys.* **9** (1968), 1605–1614.

[13] H. Bacry, J. Nuyts, Classification of ten-dimensional kinematical groups with space isotropy, *J. Math. Phys.* **27** (1986), no. 10, 2455–2457.

[14] C. Bär, Real Killing spinors and holonomy, *Comm. Math. Phys.* **154** (1993) no. 3, 509–521.

[15] E. A. Bergshoeff, J. M. Figueroa-O'Farrill, J. Gomis, A non-lorentzian primer, SciPost Physics Lecture Notes, 2023, p. 069.

[16] J. Berndt, M. Brück, Cohomogeneity one actions on hyperbolic spaces, *J. Reine Angew. Math.* **541** (2001), 209–235.

[17] J. Berndt, S. Console, C. E. Olmos, *Submanifolds and Holonomy (2nd ed.)*, CRC Press, Boca Raton, FL, 2016.

[18] J. Berndt, J. C. Díaz-Ramos, Homogeneous polar foliations of complex hyperbolic spaces, *Comm. Anal. Geom.* **20** (2012), no. 3, 435–454.

[19] J. Berndt, J. C. Díaz-Ramos, H. Tamaru, Hyperpolar homogeneous foliations on symmetric spaces of noncompact type, *J. Differential Geom.* **86** (2010), 191–235.

[20] J. Berndt, C. E. Olmos, The index conjecture for symmetric spaces, *J. Reine Angew. Math.* **772** (2021), 187–222.

[21] J. Berndt, H. Tamaru, Cohomogeneity one actions on symmetric spaces of noncompact type, *J. Reine Angew. Math.* **683** (2013), 129–159.

[22] J. Berndt, H. Tamaru, Homogeneous codimension one foliations on noncompact symmetric spaces, *J. Differential Geom.* **63** (2003), no. 1, 1–40.

[23] J. Berndt, F. Tricerri, L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Math., 1598 Springer-Verlag, Berlin, 1995.

[24] A. L. Besse, *Einstein manifolds*, Classics Math., Springer-Verlag, Berlin, 2008.

[25] L. Bianchi, On the three-dimensional spaces which admit a continuous group of motions, *Gen. Relativity Gravitation* **33** (2001), no. 12, 2171–2253 (2002). Translated from Italian by R. Jantzen.

[26] L. Biliotti, Coisotropic and polar actions on compact irreducible Hermitian symmetric spaces, *Trans. Amer. Math. Soc.* **358** (2006), no. 7, 3003–3022.

[27] L Biliotti, A. Gori, Coisotropic and polar actions on complex Grassmannians, *Trans. Amer. Math. Soc.* **357** (2005), no. 5, 1731–1751.

[28] C. Böhm, Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces, *Invent. Math.* **134** (1998), no. 1, 145–176.

[29] J. Bolton, F. Dillen, B. Dioos, L. Vrancken, Almost complex surfaces in the nearly Kähler $S^3 \times S^3$, *Tohoku Math. J. (2)* **67** (2015), no. 1, 1–17.

[30] J. Bolton, G. R. Jensen, M. Rigoli, L. M. Woodward, On conformal minimal immersions of S^2 into $\mathbb{C}\mathrm{P}^n$, *Math. Ann.* **279** (1988), no. 4, 599–620.

[31] A. Borel, Le plan projectif des octaves et les sphères comme espaces homogènes (French), *C. R. Acad. Sci. Paris* **230** (1950), 1378–1380.

[32] A. Borel, L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2006.

[33] C. P. Boyer, K. Galicki, J. Kollár, Einstein metrics on spheres, *Ann. of Math.* (2) **162** (2005), no. 1, 557–580.

[34] H. W. Brinkmann, Einstein spaces which are mapped conformally on each other, *Math. Ann.* **94** (1925), no. 1, 119–145.

[35] R. L. Bryant, Metrics with exceptional holonomy, *Ann. of Math.* (2) **126** (1987), no. 3, 525–576.

[36] J.-B. Butruille, Classification des variétés approximativement kähleriennes homogènes (French), *Ann. Global Anal. Geom.* **27** (2005), no. 3, 201–225.

[37] G. Cairns, A. Hinić Galić, Y. Nikolayevsky, Totally geodesic subalgebras of filiform nilpotent Lie algebras, *J. Lie Theory* **23** (2013), no. 4, 1051–1074.

[38] G. Cairns, A. Hinić Galić, Y. Nikolayevsky, Totally geodesic subalgebras of nilpotent Lie algebras, *J. Lie Theory* **23** (2013), no. 4, 1023–1049.

[39] É. Cartan, Sur une classe remarquable d'espaces de Riemann (French), *Bull. Soc. Math. France* **54** (1926), 214–264.

[40] Y. Y. Chapovskyi, S. D. Koval, O. Zhur, Subalgebras of Lie algebras. Example of $\mathfrak{sl}_3(\mathbb{R})$ revisited, `arXiv:2403.02554` [math-ph].

[41] B. Chen, Geometry and topology of maximal antipodal sets and related topics, *Rom. J. Math. Comput. Sci.* **13** (2023), no. 2, 6–25.

[42] B. Chen, T. Nagano, Totally geodesic submanifolds of symmetric spaces, I, *Duke Math. J.* **44** (4) (1977), 745–755.

[43] B. Chen, T. Nagano, Totally geodesic submanifolds of symmetric spaces, II, *Duke Math. J.* **45** (2) (1978), 405–425.

[44] L. Conlon, Variational completeness and K-transversal domains, *J. Differential Geometry* **5** (1971), 135–147.

[45] V. Cortés, J. J. Vásquez, Locally homogeneous nearly Kähler manifolds, *Ann. Global Anal. Geom.* **48** (2015), no. 3, 269–294.

[46] M. Cowling, A. Dooley, A. Korányi, F. Ricci, An approach to symmetric spaces of rank one via groups of Heisenberg type, *J. Geom. Anal.* **8** (1998), no. 2, 199–237.

[47] M. Cowling, A. Dooley, A. Korányi, F. Ricci, H-type groups and Iwasawa decompositions, *Adv. Math.* **87** (1991), no. 1, 1–41.

[48] K. Cwiklinski, L. Vrancken, Almost complex surfaces in the nearly Kähler flag manifold, *Results Math.* **77** (2022), no. 3, Paper no. 134, 17 pp.

[49] J. Dadok, Polar coordinates induced by actions of compact Lie groups, *Trans. Amer. Math. Soc.* **288** (1985), no. 1, 125–137.

[50] O. Dearricott, J. S. Eschenburg, Totally geodesic embeddings of 7-manifolds in positively curved 13-manifolds, *Manuscripta Math.* **114** (2004), no. 4, 447–456.

[51] J. C. Díaz-Ramos, Proper isometric actions, `arXiv:0811.0547` [math.DG].

[52] J. C. Díaz-Ramos, M. Domínguez-Vázquez, A. Kollross, Polar actions on complex hyperbolic spaces, *Math. Z.* **287** (2017), 1183–1213.

[53] J. C. Díaz-Ramos, A. Kollross, Polar actions with a fixed point, *Differential Geom. Appl.* **29** (2011), no. 1, 20–25.

[54] J. C. Díaz-Ramos, J. M. Lorenzo-Naveiro, Codimension two polar homogeneous foliations on symmetric spaces of noncompact type, *Adv. Math.* **472** (2025), 110295.

[55] B. Dioos, Z. Hu, L. Vrancken, L. X. Wang, Y. Zhang, Lagrangian submanifolds in the 6-dimensional nearly Kähler manifolds with parallel second fundamental form, *J. Geom. Phys.* **108** (2016), 21–37.

[56] M. Domínguez-Vázquez, Canonical extension of submanifolds and foliations in noncompact symmetric spaces, *Int. Math. Res. Not. IMRN 2015*, no. 22, 12114–12125.

[57] W. E. Dydo, Proper G-spaces, *J. Differential Geom.* **9** (1974), 565–569.

[58] P. B. Eberlein, Geometry of 2-step nilpotent groups with a left invariant metric. II, *Trans. Amer. Math. Soc.* **343** (1994), no. 2, 805–828.

[59] P. B. Eberlein, *Geometry of nonpositively curved manifolds*, University of Chicago Press, Chicago, London, 1996.

[60] J. Eells, S. Salamon, Twistorial construction of harmonic maps of surfaces into four-manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **12** (1985), no. 4, 589–640.

[61] J. M. Figueroa-O'Farrill, Higher-dimensional kinematical Lie algebras via deformation theory, *J. Math. Phys.* **59** (2018), no. 6, 061702, 21 pp.

[62] J. M. Figueroa-O'Farrill, Kinematical Lie algebras via deformation theory, *J. Math. Phys.* **59** (2018), no. 6, 061701, 29 pp.

[63] J. M. Figueroa-O'Farrill, On the intrinsic torsion of spacetime structures, `arXiv:2009.01948` [hep-th].

[64] J. M. Figueroa-O'Farrill, R. Grassie, S. Prohazka, Lifshitz symmetry: Lie algebras, spacetimes and particles, *SciPost Phys.* **14** (2023), no. 3, Paper No. 035, 42 pp.

[65] J. M. Figueroa-O'Farrill, A. Pérez, S. Prohazka, Carroll/fracton particles and their correspondence, *J. High Energy Phys.* **2023**, no. 6, Paper No. 207, 49 pp.

[66] J. M. Figueroa-O'Farrill, S. Prohazka, Spatially isotropic homogeneous spacetimes, *J. High Energy Phys.* **2019**, no. 1, 229, front matter+77 pp.

[67] L. Foscolo, M. Haskins, New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$, *Ann. of Math.* (2) **185** (2017), no. 1, 59–130.

[68] W. Fulton, J. Harris, *Representation theory. A first course*, Grad. Texts in Math., 129. Read. Math. Springer-Verlag, New York, 1991.

[69] S. Gallot, Équations différentielles caractéristiques de la sphère (French), *Ann. Sci. École Norm. Sup.* (4) **12** (1979), no. 2, 235–267.

[70] V. V. Gorbatsevich, A. L. Onishchik, È. B. Vinberg, *Lie Groups and Lie Algebras, III*, Springer, Berlin, 1994.

[71] C. S. Gordon, E. N. Wilson, Isometry groups of Riemannian solvmanifolds, *Trans. Amer. Math. Soc.* **307** (1988), no. 1, 245–269.

[72] C. Gorodski, Polar actions on compact symmetric spaces which admit a totally geodesic principal orbit, *Geom. Dedicata* **103**, (2004), 193–204.

[73] C. Gorodski, A. Kollross, Some remarks on polar actions, *Ann. Global Anal. Geom.* **49** (2016), no. 1, 43–58.

[74] C. Gorodski, A. Kollross, A. Rodríguez-Vázquez, Totally geodesic submanifolds and polar actions on Stiefel manifolds, *J. Geom. Anal.* **35** (2025), no. 2, Paper No. 41, 21 pp.

[75] E. Goursat, Sur les substitutions orthogonales et les divisions régulières de l'espace (French), *Ann. Sci. École Norm. Sup.* (3) **6** (1889), 9–102.

[76] A. Gray, A note on manifolds whose holonomy group is a subgroup of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, *Michigan Math. J.* **16** (1969), 125–128.

[77] A. Gray, Riemannian manifolds with geodesic symmetries of order 3, *J. Differential Geometry* **7** (1972), 343–369.

[78] A. Gray, The structure of nearly Kähler manifolds, *Math. Ann.* **223** (1976), no. 3, 233–248.

[79] A. Gray, Weak holonomy groups, *Math. Z.* **123** (1971), 290–300.

[80] A. Gray, L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.* (4) **123** (1980), 35–58.

[81] A. Gray, J. A. Wolf, Homogeneous spaces defined by Lie group automorphisms. I, *J. Differential Geometry* **2** (1968), 77–114. *IRMA Lect. Math. Theor. Phys.* **16**, European Mathematical Society (EMS), Zürich, 2010.

[82] D. Gromoll, G. Walschap, *Metric foliations and curvature*, Birkhäuser Verlag, Basel, 2009.

[83] K. Grove, W. Ziller, Polar manifolds and actions, *J. Fixed Point Theory Appl.* **11** (2012), no. 2, 279–313.

[84] R. Harvey, H. B. Lawson, Calibrated geometries, *Acta Math.* **148** (1982), 47–157.

[85] C. He, P. Petersen, W. Wylie, Warped product rigidity, *Asian J. Math.* **19** (2015), no. 1, 135–170.

[86] E. Heintze, X. Liu, C. E. Olmos, Isoparametric submanifolds and a Chevalley-type restriction theorem, *Integrable systems, geometry, and topology,* 151–190. *AMS/IP Stud. Adv. Math.*, **36**, American Mathematical Society, Providence, RI, 2006.

[87] E. Heintze, R. Palais, C. L. Terng, G. Thorbergsson, *Hyperpolar actions on symmetric spaces*, Geometry, topology, & physics, 214–245. Conf. Proc. Lecture Notes Geom. Topology, IV International Press, Cambridge, MA, 1995

[88] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, **80**, Academic Press, Inc., New York-London, 1978.

[89] R. Hermann, Existence in the large of totally geodesic submanifolds of Riemannian spaces, *Bull. Amer. Math. Soc.* **66** (1960), 59–61.

[90] R. Hermann, Totally geodesic orbits of groups of isometries, *Indag. Math.* **24** (1962), 291–298;

[91] R. Hermann, Variational completeness for compact symmetric spaces, *Proc. Amer. Math. Soc.* **11** (1960), 544–546.

[92] J. Hilgert, K.-H. Neeb, *Structure and geometry of Lie groups*, Springer Monographs in Mathematics, Springer, New York, 2012;

[93] N. Iwahori, On discrete reflection groups on symmetric Riemannian manifolds, in *Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965)*, pp. 57–62, Nippon Hyoronsha Co., Ltd., Tokyo.

[94] T. Jentsch, A. Moroianu, U. Semmelmann, Extrinsic hyperspheres in manifolds with special holonomy, *Differential Geom. Appl.* **31** (2013), no.1, 104–111.

[95] D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics, 12. Oxford University Press, Oxford, 2007.

[96] I. Karpelevich, Surfaces of transitivity of semisimple group of motions of a symmetric space, *Dokl. Akad. Nauk* **93** (1953), 401–404.

[97] S. Kim, Y. Nikolakevsky, J. Park, Totally geodesic submanifolds of Damek-Ricci spaces, *Rev. Mat. Iberoam.* **37** (2021), no. 4, 1321–1332.

[98] F. Klein, A comparative review of recent researches in geometry, *Bull. Amer. Math. Soc.* **2** (1893), no. 10, 215–249.

[99] S. Klein, Totally geodesic submanifolds of the complex and the quaternionic 2-Grassmannians, *Trans. Amer. Math. Soc.* **361** (2009), no. 9, 4927–4967.

[100] S. Klein, Totally geodesic submanifolds of the complex quadric, *Differential Geom. Appl.* **26** (2008), no. 1, 79–96.

[101] S. Klein, Totally geodesic submanifolds of the exceptional Riemannian symmetric spaces of rank 2, *Osaka J. Math.* **47** (2010), no. 4, 1077–1157.

[102] A. W. Knapp, *Lie groups beyond an introduction*, second edition, Progress in Mathematics, 140, Birkhäuser Boston, Boston, MA, 2002.

[103] S. Kobayashi, *Transformation groups in differential geometry*, Springer-Verlag, Berlin, 1995.

[104] S. Kobayashi, K. Nomizu, *Foundations of differential geometry. Vol. I*, John Wiley & Sons, Inc., New York, 1996.

[105] S. Kobayashi, K. Nomizu, *Foundations of differential geometry. Vol. II*, John Wiley & Sons, Inc., New York, 1996.

[106] A. Kollross, A classification of hyperpolar and cohomogeneity one actions, *Trans. Amer. Math. Soc.* **354** (2002), no. 2, 571–612.

[107] A. Kollross, Duality of symmetric spaces and polar actions, *J. Lie Theory* **21** (2011), no. 4, 961–986.

[108] A. Kollross, Octonions, triality, the exceptional Lie algebra F4 and polar actions on the Cayley hyperbolic plane, *Internat. J. Math.* **31** (2020), no. 7, 2050051, 28 pp.

[109] A. Kollross, A. Lytchak, Polar actions on symmetric spaces of higher rank, *Bull. London Math. Soc.* **45** (2013), no. 2, 341–350.

[110] A. Kollross, A. Rodríguez-Vázquez, Totally geodesic submanifolds in exceptional symmetric spaces, *Adv. Math.* **418** (2023), Paper No. 108949, 37 pp.

[111] B. Kostant, On differential geometry and homogeneous spaces. I, II, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 258–261, 354–357.

[112] A. J. Ledger, M. Obata, Affine and Riemannian s-manifolds, *J. Differential Geometry* **2** (1968), 451–459.

[113] J. M. Lee, *Introduction to smooth manifolds*, second edition, Graduate Texts in Mathematics, 218, Springer, New York, 2013.

[114] J. M. Lee, *Riemannian geometry. An introduction to curvature*, Graduate Texts in Mathematics, 176, Springer-Verlag, New York, 1997.

[115] T. Leistner, Semi-Riemannian cones, *Geometry, Lie Theory and applications–the Abel Symposium 2019*, 193–222.

[116] D. S. P. Leung, Reflective submanifolds. IV. Classification of real forms of Hermitian symmetric spaces, *J. Differential Geom.* **14** (1979), 179–185.

[117] J. M. Lévy-Leblond, Une nouvelle limite non-relativiste du groupe de Poincaré (French), *Ann. Inst. H. Poincaré Sect. A (N.S.)* **3** (1965), 1–12.

[118] M. Liefsoens, Nearly Kähler $\mathbb{C}\mathrm{P}^3$: From the Hopf fibration to Lagrangian submanifolds, M.Sc. thesis, June 2022.

[119] J. M. Lorenzo-Naveiro, A. Rodríguez-Vázquez, Totally geodesic submanifolds of the homogeneous nearly Kähler 6-manifolds and their G_2-cones, `arXiv:2411.11261` [math.DG].

[120] J. M. Lorenzo-Naveiro, I. Solonenko, Sections of polar actions, `arXiv:2111.05280` [math.DG].

[121] P. W. Michor, *Topics in differential geometry*, Grad. Stud. Math., 93, American Mathematical Society, Providence, RI, 2008.

[122] D. G. Mixon, B. Vose, Recovering a group from few orbits, `arXiv:2411.17434` [math.RT].

[123] K. Morand, Possible ambient kinematics, *J. Math. Phys.* **64** (2023), no. 11, Paper No. 112502, 50 pp.

[124] M. Morimoto, Non-existence of exceptional orbits under polar actions on Hilbert spaces, `arXiv:2307.02782` [math.DG].

[125] A. Moroianu, U. Semmelmann, Generalized Killing spinors and Lagrangian graphs, *Differential Geom. Appl.* **37** (2014), 141–151.

[126] G. D. Mostow, On maximal subgroups of real Lie groups, *Ann. of Math. (2)* **74** (1961), 503–517.

[127] G. D. Mostow, Some new decomposition theorems for semi-simple groups, *Mem. Amer. Math. Soc.* **14** (1955), 31–54.

[128] G. M. Mubarakzjanov, Classification of real structures of Lie algebras of fifth order (Russian), *Izv. Vysš. Učebn. Zaved. Matematika* **1963** (1963), no. 3(34), 99–106.

[129] J. R. Munkres, *Topology (Second edition)*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000.

[130] T. Murphy, F. Wilhelm, Random manifolds have no totally geodesic submanifolds, *Michigan Math. J.* **68** (2019), no. 2, 323–335.

[131] A. Nijenhuis, R. W. Richardson, Deformations of Lie algebra structures, *J. Math. Mech.* **17** (1967), 89–105.

[132] Y. Nikolayevsky, Totally geodesic hypersurfaces of homogeneous spaces, *Israel J. Math.* **207** (2015), no. 1, 361–375.

[133] M. Obata, Certain conditions for a Riemannian manifold to be isometric to a sphere, *J. Math. Soc. Japan* **14** (1962), 333–340.

[134] C. E. Olmos, S. Reggiani, A note on the uniqueness of the canonical connection of a naturally reductive space, *Monatsh. Math.* **172** (2013), no. 3-4, 379-386.

[135] C. E. Olmos, A. Rodríguez-Vázquez, Hopf fibrations and totally geodesic submanifolds, `arXiv:2302.11711` [math.DG]. To appear in *J. Eur. Math. Soc.*

[136] P. J. Olver, *Classical invariant theory*, London Mathematical Society Student Texts, 44, Cambridge Univ. Press, Cambridge, 1999.

[137] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[138] A. L. Onishchik, *Lectures on real semisimple Lie algebras and their representations*, ESI Lect. Math. Phys., European Mathematical Society (EMS), Zürich, 2004.

[139] A. L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosius Barth Verlag, Leipzig, 1994.

[140] A. L. Onishchik, Totally geodesic submanifolds of symmetric spaces. *Geometric methods in problems of algebra and analysis*, No. 2 pp. 64–85, 161, Yaroslavl State University, Yaroslavl, 1980.

[141] R. S. Palais, On the existence of slices for actions of non-compact Lie groups, *Ann. of Math.* (2) **73** (1961), 295–323.

[142] R. S. Palais, C.-L. Terng, A general theory of canonical forms, *Trans. Amer. Math. Soc.* **300** (1987), no. 2, 771–789.

[143] R. S. Palais, C.-L. Terng, *Critical point theory and submanifold geometry*, Lecture Notes in Math., 1353 Springer-Verlag, Berlin, 1988.

[144] J. Patera, P. Winternitz, Subalgebras of real three- and four-dimensional Lie algebras, *J. Mathematical Phys.* **18** (1977), no. 7, 1449–1455.

[145] P. Petersen, *Riemannian geometry (3rd ed.)*, Grad. Texts in Math., 171, Springer, Cham, 2016.

[146] F. Podestà, G. Thorbergsson, Polar actions on rank-one symmetric spaces, *J. Differential Geom.* **53** (1999), no. 1, 131–175.

[147] R. O. Popovych, V. M. Boyko, M. O. Nesterenko, M. W. Lutfullin, Realizations of real low-dimensional Lie algebras, *J. Phys. A* **36** (2003), no. 26, 7337–7360.

[148] A. Rodríguez-Vázquez, Totally geodesic submanifolds in products of rank one symmetric spaces, `arXiv:2205.14720` [math.DG]. To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci.*

[149] A. Sagle, A note on triple systems and totally geodesic submanifolds in a homogeneous space, *Nagoya Math. J.* **32** (1968), 5–20.

[150] V. Sanmartín-López, I. Solonenko, Classification of cohomogeneity-one actions on symmetric spaces of noncompact type, `arXiv:2501.05553` [math.DG].

[151] A. J. Di Scala, Polar actions on Berger spheres, *Int. J. Geom. Methods Mod. Phys.* **3** (2006), no. 5-6, 1019–1023.

[152] K. Shankar, Isometry groups of homogeneous spaces with positive sectional curvature, *Differential Geom. Appl.* **14** (2001), no.1, 57–78.

[153] R. W. Sharpe, *Differential geometry. Cartan's generalization of Klein's Erlangen program*, Grad. Texts in Math., 166, Springer-Verlag, New York, 1997.

[154] I. Solonenko, Homogeneous codimension-one foliations on reducible symmetric spaces of noncompact type, `arXiv:2112.02189` [math.DG].

[155] I. Solonenko, Homogeneous hypersurfaces in Riemannian symmetric spaces, Doctoral thesis, 1 Feb 2024.

[156] R. Storm, Lagrangian submanifolds of the nearly Kähler full flag manifold $F_{1,2}(\mathbb{C}^3)$, *J. Geom. Phys.* **158** (2020), 103844, 16 pp.

[157] J. Szenthe, Orthogonally transversal submanifolds and the generalizations of the Weyl group, *Period. Math. Hungar.* **15** (1984), no. 4, 281–299.

[158] H. Tamaru, Parabolic subgroups of semisimple Lie groups and Einstein solvmanifolds, *Math. Ann.* **351** (2011), no. 1, 51–66.

[159] Y. Tashiro, Complete Riemannian manifolds and some vector fields, *Trans. Amer. Math. Soc.* **117** (1965), 251–275.

[160] D. Töben, Singular Riemannian foliations on nonpositively curved manifolds, *Math. Z.* **255** (2007), no. 2, 427–436.

[161] K. Tojo, Normal homogeneous spaces admitting totally geodesic hypersurfaces, *J. Math. Soc. Japan* **49** (1997), no. 4, 781–815.

[162] K. Tojo, Totally geodesic submanifolds of naturally reductive homogeneous spaces, *Tsukuba J. Math.* **20** (1996), no. 1, 181–190.

[163] K. Tojo, Totally real totally geodesic submanifolds of compact 3-symmetric spaces, *Tohoku Math. J.* (2) **53** (2001), no. 1, 131–143.

[164] K. Tsukada, Totally geodesic hypersurfaces of naturally reductive homogeneous spaces, *Osaka J. Math.* **33** (1996), no. 3, 697–707.

[165] K. Tsukada, Totally geodesic submanifolds of Riemannian manifolds and curvature-invariant subspaces, *Kodai Math. J.* **19** (1996), no. 3, 395–437.

[166] J. J. Vásquez, Isospectral nearly Kähler manifolds, *Abh. Math. Semin. Univ. Hambg.* **88** (2018), no.1, 23–50.

[167] M. Y. Wang, W. Ziller, On normal homogeneous Einstein manifolds, *Ann. Sci. École Norm. Sup.* (4) **18** (1985), no.4, 563–633.

[168] J. A. Wolf, Elliptic spaces in Grassmann manifolds, *Illinois J. Math.* **7** (1963), 447–462.

[169] B. Wu, Isoparametric submanifolds of hyperbolic spaces, *Trans. Amer. Math. Soc.* **331** (1992), no. 2, 609–626.

[170] G. Wu, R. Ye, A note on Obata's rigidity theorem, *Commun. Math. Stat.* **2** (2014), no. 3-4, 231–252.