L. A. Cordero, E. García-Río (*Editors*)

Proceedings of the

WORKSHOP ON RECENT TOPICS IN DIFFERENTIAL GEOMETRY

Santiago de Compostela (Spain)

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Preface

This volume records the Proceedings of the *Workshop on Recent Topics in Differential Geometry* held at the Institute of Mathematics and the Department of Geometry and Topology of the University of Santiago de Compostela (Spain), 16–19 July 1997.

The aim of this Workshop has been to recuperate the spirit that inspired the two previous Workshops held in Lancaster University (U.K., 1989) and in the University of La Laguna (Spain, 1990). That is, to celebrate an informal working meeting to bring together a number of experts and their research students, and to give the participants the opportunity of having stimulating discussions in a friendly atmosphere. We are most thankful to all participants for their contribution to the success of the meeting; special thanks are due to the invited speakers, Profs. A. Ferrández (University of Murcia, Spain) and O. Kowalski (Charles University, Czech Republic), and to those who chaired sessions. All authors are to be congratulated on the preparation of excellent papers by the agreed time, so allowing a speedy publication. In addition, our thanks go also to the Department of Geometry and Topology for publishing this proceedings volume in its series.

The Workshop would not have been possible without the support of the University of Santiago de Compostela, through the Institute of Mathematics and the Department of Geometry and Topology, and of the D.G.I.C.Y.T. (Project PB-94-0633-C02-01), to which we heartly thank.

Luis A. Cordero
Eduardo García-Río
Santiago de Compostela, January 1998

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GEOMETRY OF THE UNIT TANGENT SPHERE BUNDLE

E. Boeckx* and L. Vanhecke

1 Introduction

When studying the geometric properties of a Riemannian manifold (M, g), it is often worthwhile and inspiring to consider geometric objects naturally associated to (M, g). These can be special hypersurfaces such as small geodesic spheres and tubes around geodesics (see, e.g., [14], [15] and [20] in these proceedings), or bundles with (M, g) as base manifold. The tangent bundle TM and the unit tangent sphere bundle T_1M are particularly interesting. As an example, one can study the geodesics on (M, g) via the geodesic flow on TM or on T_1M ([2], [3]). In the same vein, A. L. Besse uses the unit tangent sphere bundle with this flow as his basic tool in [3].

It is therefore natural to make an in-depth study of these two bundles, equipped with "natural" metrics and other "natural" structures (such as an almost complex structure on TM and a contact metric structure on T_1M). By "natural" we mean that these structures are canonically determined by the geometric structure of M (i.e., the metric g and possibly other structural tensors).

One of the best known Riemannian metrics on the tangent bundle TM is the Sasaki metric g_S. Unfortunately, as a metric space, (TM, g_S) is not very interesting for our purposes. For instance, the fairly weak hypothesis to have constant scalar curvature already implies that (M, g) must be flat ([26]). Other natural metrics on TM have been introduced and studied. We refer to [36] for some examples. As concerns the Cheeger-Gromoll metric, see [25], [26].

More interesting is the geometric structure of the unit tangent sphere bundle T_1M. It is well-known that T_1M admits a contact metric structure $(\xi, \eta, \varphi, \bar{g})$, where the metric $\bar{g}$ is homothetic to the metric induced by the Sasaki metric g_S ([5]). Still, several aspects of the Riemannian geometry of $(T_1M, \bar{g})$, in particular about the curvature, have received only little attention until recently. In a series

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of papers ([10], [11], [12]), the present authors focus precisely on this study. This article aims to give a survey of some of these recent results.

After an introductory section, recalling briefly the basic facts about the tangent bundle and the unit tangent sphere bundle, we investigate which Riemannian spaces (M, g) have a unit tangent sphere bundle $(T_1M, \bar{g})$ with constant scalar curvature. We give several classes of examples and classify all two- and three-dimensional and all conformally flat manifolds with this property. Constant scalar curvature is only the first step in the search for curvature homogeneous manifolds within the class of unit tangent sphere bundles, which continues in Section 4. It was already known that the unit tangent sphere bundle of a two-point homogeneous space is homogeneous ([26], [34]), hence curvature homogeneous. Our results in this section seem to suggest that, up to local isometries, these might very well be the only ones. In Section 5, we determine all unit tangent sphere bundles which are Einstein, Ricci-parallel or (locally) symmetric, reproving in this way a result of D. Blair ([6]). Finally, in Section 6, we return to the contact metric structure on T_1M and study the reflections with respect to the integral curves of the characteristic vector field ξ.

2 The tangent bundle and the unit tangent sphere bundle

First, we collect the basic facts about the tangent bundle and the unit tangent sphere bundle of a Riemannian manifold. For more details and further information, we refer to [3], [10], [19], [23], [26], [27], [32], [35] and [36].

Let (M, g) be an n-dimensional $(n \geq 2)$ connected Riemannian manifold and ∇ its Levi Civita connection. The Riemann curvature tensor R is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for all vector fields X, Y and Z on M. The tangent bundle of (M, g), denoted by TM, consists of pairs (x, u) where x is a point in M and u a tangent vector to M at x. The mapping $\pi: TM \to M: (x, u) \mapsto x$ is the natural projection from TM onto M.

It is well-known that the tangent space to TM at (x, u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker \pi_{*|(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ with respect to ∇:

$$T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}.$$

The horizontal subspace $HTM_{(x,u)}$ consists of those vectors which are tangent at (x, u) to a curve $\gamma(t) = (\alpha(t), V(t))$ in TM satisfying $\nabla_{\dot{\alpha}(t)}V(t) = 0$.

For $X \in T_xM$, there exists a unique vector X^h at the point $(x, u) \in TM$ such that $X^h \in HTM_{(x,u)}$ and $\pi_*(X^h) = X$. X^h is called the *horizontal lift* of X to (x, u). There is also a unique vector X^v at the point (x, u) such that $X^v \in VTM_{(x,u)}$ and $X^v(df) = Xf$ for all functions f on M. X^v is called the *vertical lift* of X to (x, u). The map $X \mapsto X^h$, respectively $X \mapsto X^v$, is an isomorphism between T_xM and $HTM_{(x,u)}$, respectively T_xM and $VTM_{(x,u)}$. Similarly, one lifts vector fields on M to horizontal or vertical vector fields on TM. The expressions in local coordinates for these lifts are given in [10], for example.

The tangent bundle TM of a Riemannian manifold (M, g) can be endowed in a natural way with a Riemannian metric g_S, the so-called *Sasaki metric,* depending only on the Riemannian structure g of the base manifold M. It is uniquely determined by

$$g_S(X^h, Y^h) = g_S(X^v, Y^v) = g(X, Y) \circ \pi, \qquad g_S(X^h, Y^v) = 0$$

for all vector fields X and Y on M. There is also an almost complex structure J on TM given by

$$JX^h = X^v, \qquad JX^v = -X^h$$

for all vector fields X on M.

For these structures, we have the following result:

Theorem 2.1 ([19]) *The tangent bundle (TM, g_S, J) is almost Kählerian. It is a Kähler manifold only when (M, g) is flat.*

As for the metric structure of (TM, g_S), Kowalski proved

Theorem 2.2 ([23]) *The tangent bundle (TM, g_S) is locally symmetric if and only if (M, g) is flat.*

A stronger result was derived by E. Musso and F. Tricerri:

Theorem 2.3 ([26]) *The tangent bundle (TM, g_S) has constant scalar curvature if and only if (M, g) is flat.*

This last result shows that, although the Sasaki metric is a very natural Riemannian metric on TM, it is "extremely rigid" ([26]). There are two ways out: either we study other interesting metrics on TM (see, e.g., [25], [26], [28], [36]) or we consider the unit tangent sphere bundle T_1M with the metric induced from g_S. Here, we choose the second option.

The hypersurface T_1M of TM consists of the unit tangent vectors to (M, g) and is given implicitly by the equation $g_x(u, u) = 1$. A unit normal vector N to T_1M at $(x, u) \in T_1M$ is given by the vertical lift of u to (x, u): $N_{|(x,u)} = u^v$.

As the vertical lift of a vector (field) is not tangent to T_1M in general, we define the *tangential lift* of $X \in T_xM$ to $(x, u) \in T_1M$ by

$$X^t_{(x,u)} = (X - g(X, u)\, u)^v.$$

Clearly, the tangent space to T_1M at (x, u) is spanned by vectors of the form X^h and X^t where $X \in T_xM$.

We now describe the natural contact metric structure on T_1M. We refer to [5] for the basic concepts of contact geometry. We endow T_1M with the metric induced from the Sasaki metric g_S on TM, denoted also by g_S. Using the almost complex structure J on TM, we define a unit vector field ξ', a one-form η' and a $(1,1)$-tensor field φ' on T_1M by

$$\xi' = -JN, \qquad JX = \varphi' X + \eta'(X)\, N.$$

It is easily checked that $(T_1M, \xi', \eta', \varphi', g_S)$ is an almost contact metric manifold. However, $\bar{g}(X, \varphi' Y) = 2\, d\eta'(X,Y)$, so $(\xi', \eta', \varphi', g_S)$ is not a contact metric structure. This defect can be rectified by taking

$$\xi = 2\xi', \qquad 2\eta = \eta', \qquad \varphi = \varphi', \qquad 4\bar{g} = g_S.$$

Note that the metric $\bar{g}$ is obtained from the one induced from the Sasaki metric on TM by a homothetic change. So it is given explicitly by

$$\begin{aligned}
\bar{g}_{|(x,u)}(X^t, Y^t) &= \frac{1}{4}\Big(g(X,Y) - g(X,u)g(Y,u)\Big),\\
\bar{g}_{|(x,u)}(X^t, Y^h) &= 0,\\
\bar{g}_{|(x,u)}(X^h, Y^h) &= \frac{1}{4} g(X,Y).
\end{aligned}$$

The integral curves of the characteristic vector field ξ will be called *characteristic curves* in what follows. Note that ξ, or more precisely, ξ', also describes the *geodesic flow* (see [3]).

Making T_1M into a contact metric manifold is the best we can do in the context of contact geometry, as follows from the following result by Y. Tashiro:

Theorem 2.4 ([33]) *The natural contact metric structure on T_1M is K-contact if and only if (M,g) has constant curvature 1, in which case the structure on T_1M is Sasakian.*

With the metric $\bar{g}$ on T_1M in place, it is a fairly routine exercise to calculate the associated Levi Civita connection $\bar{\nabla}$, the Riemann curvature tensor $\bar{R}$, the Ricci tensor $\bar{\rho}$ and the scalar curvature $\bar{\tau}$. The formulas can be found, e.g., in [10], [11] and [35]. Here, we give only the expressions for $\bar{\rho}$ and $\bar{\tau}$. The Ricci tensor is given explicitly by

$$\begin{aligned}
\bar{\rho}_{|(x,u)}(X^t, Y^t) &= (n-2)\Big(g(X,Y) - g(X,u)g(Y,u)\Big)\\
&\quad + \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)E_i, R(u,Y)E_i),\\
\bar{\rho}_{|(x,u)}(X^t, Y^h) &= \frac{1}{2}\Big((\nabla_u \rho)(X,Y) - (\nabla_X \rho)(u,Y)\Big), \qquad (1)\\
\bar{\rho}_{|(x,u)}(X^h, Y^h) &= \rho_x(X,Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,E_i)X, R(u,E_i)Y),
\end{aligned}$$

and the scalar curvature $\bar{\tau}$ by

$$\bar{\tau}_{(x,u)} = 4\tau_x + 4(n-1)(n-2) - \xi_x(u,u) \tag{2}$$

where, as in [3] and [17], $\xi(u,v) = \sum_{i,j=1}^{n} g(R(u,E_i)E_j, R(v,E_i)E_j)$. Obviously, this $(0,2)$ tensor field ξ has nothing to do with the characteristic vector field ξ on an almost contact metric manifold. Note that the natural mapping $\pi\colon (T_1M, g_S) \to (M,g)$ is a Riemannian submersion, hence these curvature formulas may also be derived using O'Neill's formalism. (See, e.g., [4].)

3 Unit tangent sphere bundles with constant scalar curvature

From the formula (2) for the *scalar curvature* on T_1M it follows easily

Theorem 3.1 *The unit tangent sphere bundle* $(T_1M, \bar{g})$ *has constant scalar curvature* $\bar{\tau}$ *if and only if on* (M,g) *it holds*

$$\xi = \frac{|R|^2}{n} g, \tag{3}$$

$$4n\tau - |R|^2 = \textit{constant}. \tag{4}$$

Remark 1 The *algebraic* condition (3) has appeared in the literature before (see, e.g., [3], [17], [22]), but without a clear *geometric* meaning. An analytic interpretation is given in [4, p. 134]: an Einstein metric on a compact manifold is critical for the functional $SR(g) = \int_M |R_g|^2\, d\text{vol}$ restricted to those metrics g such that $\text{vol}(M) = 1$ if and only if $\xi = (|R|^2/n)\, g$. We can now give a nice geometric interpretation for Riemannian manifolds satisfying (4): on such manifolds, (3) holds if and only if their unit tangent sphere bundle has constant scalar curvature.

The case of (locally) reducible manifolds is easy to deal with:

Corollary 3.2 *The unit tangent sphere bundle* $(T_1M, \bar{g})$ *of a (local) product manifold* $(M,g) = (M_1^{n_1}, g_1) \times (M_2^{n_2}, g_2)$ *has constant scalar curvature if and only if the unit tangent sphere bundles of both* (M_1, g_1) *and* (M_2, g_2) *have constant scalar curvature and, additionally,*

$$\frac{|R_1|^2}{n_1} = \frac{|R_2|^2}{n_2}. \tag{5}$$

As immediate examples of Riemannian spaces whose unit tangent sphere bundles have constant scalar curvature, we have

0. Spaces of constant curvature.

1. Irreducible symmetric spaces and, more generally, isotropy irreducible homogeneous spaces.

2. Reducible symmetric spaces $(M,g) = (M_1, g_1) \times \cdots \times (M_k, g_k)$ with irreducible components (M_i, g_i) such that $|R_1|^2/n_1 = \ldots = |R_k|^2/n_k$.

3. Super-Einstein spaces ([22]): these are Einstein manifolds satisfying condition (3) with $|R|^2$ constant.

4. Harmonic spaces: as every harmonic space is super-Einstein (see, e.g., [3], [17]).

5. Four-dimensional orientable Einstein manifolds which are self-dual or anti-self-dual. For this result and more in the same direction, we refer to [11].

In low dimensions, we can give a complete classification.

Proposition 3.3 *The unit tangent sphere bundle $(T_1M, \bar{g})$ of a two-dimensional manifold (M,g) has constant scalar curvature $\bar{\tau}$ if and only if (M,g) has constant curvature.*

Proposition 3.4 *The unit tangent sphere bundle $(T_1M, \bar{g})$ of a three-dimensional manifold (M,g) has constant scalar curvature $\bar{\tau}$ if and only if (M,g) has constant curvature or (M,g) is a curvature homogeneous space with constant Ricci roots $\rho_1 = \rho_2 = 0 \neq \rho_3$.*

The proofs of these propositions use the explicit expressions for the curvature tensor R in terms of the scalar curvature τ and the Ricci tensor ρ, namely

$$R = \frac{\tau}{4} g \owedge g$$

in dimension two and

$$R = \rho \owedge g - \frac{\tau}{4} g \owedge g$$

in dimension three. Here $\owedge$ is the Kulkarni-Nomizu product of symmetric two-tensors defined as follows:

$$\begin{aligned}(h \owedge k)(X,Y,Z,V) &= h(X,Z)k(Y,V) + h(Y,V)k(X,Z) \\ &\quad - h(X,V)k(Y,Z) - h(Y,Z)k(X,V).\end{aligned}$$

There is another class of Riemannian manifolds where a similar curvature expression exists: for conformally flat manifolds, it holds

$$R = \frac{1}{n-2} \rho \owedge g - \frac{\tau}{2(n-1)(n-2)} g \owedge g.$$

As before, we use this formula for R to express the conditions (3) and (4). Further, we also use H. Takagi's classification of conformally flat locally homogeneous spaces ([30]) which is also valid for curvature homogeneous manifolds ([24]) to obtain

Theorem 3.5 *Let (M^n, g) be conformally flat and $n \geq 4$. The unit tangent sphere bundle $(T_1M, \bar{g})$ has constant scalar curvature if and only if (M, g) has constant curvature or n is even, say $n = 2k$, and (M, g) is locally isometric to the product $M^k(\kappa) \times M^k(-\kappa)$, $\kappa \neq 0$, or $n = 4$, $|\rho|^2$ is constant and $\tau = 0$.*

4 Curvature homogeneous unit tangent sphere bundles

Based on the results in the previous section, we now look for *curvature homogeneous* unit tangent sphere bundles. For the definition of curvature homogeneous spaces, see [29]. An extensive survey about this class of manifolds can be found in [9, Chapter 12]. We start with a positive result:

Theorem 4.1 ([26], [34]) *If (M, g) is a two-point homogeneous space, then its unit tangen sphere bundle $(T_1M, \bar{g})$ is a homogeneous Riemannian manifold.*

Proof. If h is an isometry of a Riemannian manifold (M, g), then it can be lifted to an isometry $\bar{h}$ of $(T_1M, \bar{g})$ defined by $\bar{h}(x, u) = (h(x), h_*(u))$. Clearly, if (M, g) is two-point homogeneous, then the lifted isometries act transitively on T_1M.

∎

Remark 2 To our knowledge, it is not known whether the converse of this theorem holds, i.e., whether a manifold (M, g) with locally homogeneous unit tangent sphere bundle $(T_1M, \bar{g})$ is necessarily locally isometric to a two-point homogeneous space.

Next, we start from the classification results in the previous section. In order to prove that some classes do *not* have a curvature homogeneous unit tangent sphere bundle, we will show that they are not even Ricci-curvature homogeneous.

A Riemannian manifold (M, g) is said to be *Ricci-curvature homogeneous* if for every pair of points $x, y \in M$, there exists a linear isometry $F: T_yM \to T_xM$ such that $F^*\rho_x = \rho_y$. As ρ is a symmetric $(0,2)$-tensor field and as such diagonalizable at each point, one can say equivalently that the matrices for ρ_x, respectively ρ_y, with respect to an orthonormal basis of T_xM, respectively of T_yM, must have the same eigenvalues (with the same multiplicities), or that their characteristic polynomials are identical. Obviously, every curvature homogeneous space is Ricci-curvature homogeneous, but the converse does not hold. E.g., every Einstein space is Ricci-curvature homogeneous, but not necessarily curvature homogeneous.

The proofs of the following propositions consist typically in comparing the matrices for $\bar{\rho}$ at different points $(x, u) \in T_1M$, using the formulas (1), and requiring that they have the same eigenvalues or the same characteristic polynomial. In this way, we prove

Proposition 4.2 *Let (M, g) be a two- or three-dimensional Riemannian manifold. Its unit tangent sphere bundle $(T_1M, \bar{g})$ is (Ricci-)curvature homogeneous if and only if (M, g) has constant curvature. In that case, $(T_1M, \bar{g})$ is even locally homogeneous.*

Proposition 4.3 *Let (M, g) be conformally flat. Then $(T_1M, \bar{g})$ is (Ricci-)curvature homogeneous if and only if (M, g) has constant curvature. In that case, $(T_1M, \bar{g})$ is even locally homogeneous.*

A final result in this framework deals with harmonic spaces. Up to local isometries, the only known examples so far, apart from the two-point homogeneous spaces, are the so-called *Damek-Ricci spaces*, that have only been discovered fairly recently. These are solvable Lie groups whose Lie algebras are solvable extensions of generalized Heisenberg algebras, equipped with a special left-invariant metric. We refer to [1] for the precise definitions, some geometric properties of these remarkable spaces and further references. As harmonic spaces, every Damek-Ricci space has a unit tangent sphere bundle with constant scalar curvature. Moreover, we have

Proposition 4.4 *The unit tangent sphere bundle $(T_1S, \bar{g})$ of a Damek-Ricci space S is (Ricci-)curvature homogeneous if and only if S is a symmetric space. In that case, S is two-point homogeneous and $(T_1S, \bar{g})$ is homogeneous.*

A final result deals with the case of product manifolds where at least one of the factors has a Codazzi Ricci tensor (i.e., $(\nabla_X\rho)(Y, Z) = (\nabla_Y\rho)(X, Z)$ for all vectors X, Y and Z). In particular, it settles the case of reducible symmetric manifolds.

Proposition 4.5 *Let (M, g) be locally isometric to the Riemannian product of (M_1, g_1) and (M_2, g_2) and suppose that the Ricci tensor ρ_1 of (M_1, g_1) is a Codazzi tensor. If $(T_1M, \bar{g})$ is (Ricci-)curvature homogeneous, then (M, g) is flat.*

The case of irreducible symmetric spaces is as yet undecided. Clearly, the symmetric spaces of rank one have a (Ricci-)curvature homogeneous unit tangent sphere bundle as they are two-point homogeneous. The authors strongly believe that these are the only ones. A proof or a refutation of their belief would be very welcome.

5 The Ricci tensor of the unit tangent sphere bundle

In Section 3, we considered the case of constant scalar curvature for the unit tangent sphere bundle. Here, we look at some stronger conditions, involving the Ricci tensor $\bar{\rho}$.

First, we have

Theorem 5.1 *The unit tangent sphere bundle $(T_1M, \bar{g})$ of a Riemannian manifold (M, g) is an Einstein manifold if and only if (M, g) is a surface of constant curvature* 0 *or* 1.

More generally, it holds

Theorem 5.2 *The unit tangent sphere bundle $(T_1M, \bar{g})$ of a Riemannian manifold (M, g) has parallel Ricci tensor if and only if (M, g) is flat or is a surface of constant curvature* 1.

The proofs of these two theorems use curvature invariants. As a consequence of Theorem 5.2, we have

Corollary 5.3 *The unit tangent sphere bundle $(T_1M, \bar{g})$ of a Riemannian manifold (M, g) is locally symmetric if and only if (M, g) is flat or is a surface of constant curvature* 1.

Proof. If (M, g) is the n-dimensional Euclidean space, then $(T_1M, \bar{g})$ is the Riemannian product of $\mathbb{R}^n$ and S^{n-1} with their standard metrics, hence symmetric. If (M, g) is locally isometric to the two-sphere of radius 1, then $(T_1M, \bar{g})$ has constant curvature 1. The converse follows from Theorem 5.2.

∎

This theorem has already been proved by D. Blair in [6], but he uses the contact metric structure of T_1M in an essential way. Our method of proof only involves metric information.

If we restrict our attention to the case where the manifold (M, g) has constant curvature c, more can be said about the Ricci curvature $\bar{\rho}$ (see [10]). As an example, it is easy to prove that $\bar{\rho}$ is cyclic parallel (i.e., $(\bar{\nabla}_X\bar{\rho})(X, X) = 0$ for all vectors X) if and only if $n = 2$ or $c \in \{0, 1\}$. As a consequence, one proves

Proposition 5.4 *The universal covering of $(T_1M^n(c), \bar{g})$ is a naturally reductive homogeneous space if and only if $n = 2$ or $c \in \{0, 1\}$.*

We will come back to this in the next section.

6 Characteristic reflections on unit tangent sphere bundles

Corollary 5.3 shows that requiring a unit tangent sphere bundle $(T_1M, \bar{g})$ to be locally symmetric, i.e., the geodesic reflections with respect to all points (x, u) are isometries, or, analytically, $\bar{\nabla}\bar{R} = 0$, is a very strong condition indeed. Still, on a unit tangent sphere bundle, as on every (almost) contact metric manifold, there exists a family of distinguished curves, namely the characteristic curves. It is natural, therefore, to determine when the reflections with respect to these curves, the so-called *characteristic reflections,* are isometries. We give the following

Definition 6.1 An (almost) contact metric manifold $(M, \xi, \eta, \varphi, g)$ will be called a *locally φ-symmetric space* if and only if all characteristic reflections are (local) isometries.

We note that the notion of a locally φ-symmetric space has already been introduced in the context of Sasakian geometry by T. Takahashi ([31]). He defines a Sasakian locally φ-symmetric space as a Sasakian manifold satisfying the curvature condition $g((\nabla_X R)(Y,Z)V,W) = 0$ for all vector fields X, Y, Z, V and W orthogonal to ξ, and he proves that this is equivalent to having characteristic reflections which are local automorphisms of the Sasakian structure. In [8], it is proved that this is also equivalent to the isometry property of the characteristic reflections. (For a slightly more general result, see [13].)

In [7], the authors generalize the notion of a locally φ-symmetric space to the class of contact metric manifolds in a way different from ours above: they simply take the curvature condition which holds in the Sasakian case as defining condition. Our (probably stronger) Definition 6.1 in the general almost contact metric case gives rise to an infinite list of curvature conditions (cf. [18]).

Theorem 6.2 *Let $(M, \xi, \eta, \varphi, g)$ be an almost contact metric manifold. If it is a locally φ-symmetric space, then the following hold:*

1) *the characteristic curves are geodesics,*
2) $g((\nabla^{2k}_{X \cdots X} R)(X,Y)X, \xi) = 0,$
3) $g((\nabla^{2k+1}_{X \cdots X} R)(X,Y)X, Z) = 0,$
4) $g((\nabla^{2k+1}_{X \cdots X} R)(X,\xi)X, \xi) = 0$

for all vectors X, Y and Z orthogonal to ξ and $k = 0, 1, 2, \ldots$. Moreover, if (M, g) is analytic, these conditions are also sufficient for the almost contact metric manifold to be a locally φ-symmetric space.

As every unit tangent sphere bundle is a contact metric manifold, the characteristic curves are geodesics and we are left with an infinite list of conditions on the Riemann curvature tensor R and its covariant derivatives.

Theorem 6.3 *The unit tangent sphere bundle $(T_1M, \xi, \eta, \varphi, \bar{g})$ is locally φ-symmetric if and only if (M, g) has constant curvature.*

Proof. A complete proof can be found in [10]. Here we only outline the major steps. We use the explicit expressions for the curvature tensor $\bar{R}$ of $(T_1M, \bar{g})$ in terms of the curvature tensor R of (M, g) and its covariant derivatives.

If we suppose that $(T_1M, \xi, \eta, \varphi, \bar{g})$ is locally φ-symmetric, it follows already from the condition $\bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{X}, \xi) = 0$ for $\bar{X}$ and $\bar{Y}$ orthogonal to ξ that (M, g) has constant curvature (via Cartan's criterion, see [16]).

Conversely, for a space of constant curvature, the tangent unit sphere bundle is analytic. By an induction argument, we show that the infinite list of curvature conditions holds.

∎

Remark 3 In the previous section, we mentioned that the only spaces of constant curvature for which the universal covering of their unit tangent sphere bundle is naturally reductive, are the two-dimensional ones and those with constant curvature 0 or 1. Yet, for every space of constant curvature, its unit tangent sphere bundle is locally φ-symmetric. To our knowledge, this gives the first examples of spaces which are not naturally reductive, but which do admit a one-dimensional foliation of geodesics such that the reflections with respect to these geodesics are local isometries. Many other naturally reductive examples are given in the study of flow geometry. See [21] for more details and references.

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VOLUME AND LOCAL HOMOGENEITY

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1 Introduction

In their search for an alternative proof of Gauss' theorema egregium, J. Bertrand, M. Diguet and V. Puiseux [3] discovered that the Gaussian curvature $K(m)$ of an arbitrary surface M at a point m can be defined in a very geometric way as

$$K(m) = \lim_{r\to 0} \frac{3(2\pi r - S_m(r))}{\pi r^3},$$

where $S_m(r)$ denotes the length of a geodesic circle of radius r centered at the point m, implying that there exists a close relationship between the Gaussian curvature of a surface and the length of the geodesic circles contained in it. This link was later studied in the framework of arbitrary Riemannian manifolds (where geodesic circles are now replaced by geodesic spheres or balls) by Vermeil [36], and for other classes of geometric objects, in particular tubes around curves and arbitrary submanifolds by Hotelling [20] and Weyl [37].

The existence of a relation between the curvature of a Riemannian manifold and the volume of geodesic spheres and tubes contained in it led some authors to state the following question: *To what extent is the curvature (or geometry) of a given pseudo-Riemannian manifold (M, g) influenced, or even determined, by the volume properties of certain naturally defined families of geometric objects in M?* In its full generality, this problem seems to be very difficult to handle. It is to be expected, however, that families of geometric objects (e.g., geodesic balls and spheres, cones, tubes and disks) in certain so-called "model spaces", i.e., manifolds with a high degree of symmetry (e.g., locally homogeneous manifolds, two-point homogeneous manifolds, space forms and g.o. spaces) will have nice properties and, conversely, that manifolds whose families of geometric objects satisfy such volume properties will have a highly symmetric geometric structure.

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As a particular approach to the general question stated above, one can therefore study the following problem: *Can we determine, using the volume properties of certain families of geometric objects, whether a pseudo-Riemannian manifold is (locally isometric to) a given model space?* To study this problem, one generally starts by investigating the direct problem of determining the volume properties of families of geometric objects in the model spaces. The next step is then to study the converse problem of determining if these volume properties are characteristic for the given model spaces, that is, to investigate whether these model spaces are (up to local isometry) the only pseudo-Riemannian manifolds having these properties.

Problems of this type have been investigated by many authors, and these investigations have led to nice characterizations of special families of model spaces, as well as to a number of interesting open problems which turn out to be very difficult to solve. In the rest of this paper we review some of the questions studied in this framework, we present the results and open problems obtained in this investigation, and collect a list of references to papers where these problems have been studied in detail. We omit the proofs but remark, as variations of geodesics are closely related to the study of the spheres, balls, tubes and disks, that normal and Fermi coordinates and Jacobi or Fermi vector fields are used intensively in the treatment.

It should be remarked that, in this paper, we restrict our attention to the *volume* properties of families of geometric objects. As it turns out, there also exists a strong relation between the curvature of a pseudo-Riemannian manifold and certain other properties of these families of geometric objects, such as their curvature (intrinsic geometry) and shape operator (extrinsic geometry). The reader interested in examples of such properties is referred to [17],[5] for more details and further references.

In the rest of this paper, we denote by (M,g) an n-dimensional, smooth, connected pseudo-Riemannian manifold, by ∇ its Levi Civita connection and by R and ρ the Riemann and Ricci curvature tensors associated to ∇, where we define

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all pairs of tangent vector fields X, Y. τ denotes the scalar curvature of (M,g). Also, to avoid problems with the domains of exponential maps we will, whenever necessary, assume that the geometric objects (geodesic spheres and balls, tubes, disks) considered here are sufficiently small, i.e., their radius is always smaller than the injectivity radius $i(m)$ of the manifold at the center m of the object or than the distance to the nearest focal point of the central axis of a tube.

2 Euclidean space and the volume conjecture

As a first example, we start from an n-dimensional Euclidean model space. Then it is a well-known classical result that the volume $V_m(r)$ of a ball $B_m(r)$ of radius r and centered at m is given by

$$(1)\qquad V_m(r) = \omega_n r^n, \qquad \omega_n = \left(\left(\frac{1}{2}n\right)!\right)^{-1} \pi^{(n/2)}.$$

(For the $(n-1)$-dimensional volume $S_m(r)$ of the sphere $G_m(r)$ of radius r, we have $S_m(r) = \frac{dV_m(r)}{dr} = n\omega_n r^{n-1}$.) In [18], A. Gray and the second author investigate whether this volume property for balls (or spheres) in (locally) Euclidean spaces is characteristic for these model spaces. In particular, they study Riemannian manifolds such that the volume $V_m(r)$ of every (sufficiently small) *geodesic ball*

$$B_m(r) = \{\exp_m(su) | u \in T_m M, \|u\| = 1, 0 \leq s \leq r\}$$

of radius r centered at a point $m \in M$ is given by (1). As no non-flat examples of such manifolds are known, they state the following so-called *volume conjecture* (an equivalent version can be stated using the volumes of the geodesic spheres $G_m(r) = \{\exp_m(ru) | u \in T_m M, \|u\| = 1\}$):

Conjecture 2.1 *Let (M, g) be an n-dimensional Riemannian manifold and suppose that the volume of every (sufficiently small) geodesic ball $B_m(r)$ is given by (1). Then (M, g) is locally flat.*

One possible approach to the study of this problem is to start from the power series expansion for the volume of a (small) geodesic ball, which is given for a general Riemannian manifold by

$$V_m(r) = \omega_n r^n \{1 + A(m)r^2 + B(m)r^4 + O(r^6)\},$$

where

$$(2)\qquad A = -\frac{\tau}{6(n+2)},$$

$$(3)\qquad B = \frac{1}{360(n+2)(n+4)}(-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 - 18\Delta\tau).$$

Expressing the volume property (1) then yields an infinite set of necessary conditions (in terms of curvature invariants) for the manifold to possess the volume property under consideration. The first two of these conditions take the form

$$\tau = 0, \quad 3\|R\|^2 - 8\|\rho\|^2 = 0.$$

Using these necessary conditions, one should then prove that the manifold is locally flat, i.e., $R = 0$.

In [18] the next term in the power series has been derived and furthermore it was shown that the volume conjecture is true in a number of special cases, which are collected in the following

Theorem 2.2 *The volume conjecture is true if any of the following additional assumptions are made:*

- $\dim M \leq 3$;
- *M has non-positive or non-negative Ricci curvature; in particular, this condition holds when M is an Einstein manifold;*
- *M is conformally flat;*
- *M is a Bochner flat Kähler manifold;*
- *M is a product of surfaces;*
- *M is four- or five-dimensional and its Ricci tensor is parallel;*
- *M is compact and the Laplacian of M has the same spectrum on functions as that of a compact flat manifold.*

In the special case where the manifold is locally symmetric, the following result was obtained in [13] (see [18] for some special cases):

Theorem 2.3 *Let (M, g) be a locally symmetric space such that the volume of every sufficiently small geodesic ball of radius r is the same as in Euclidean space. Then (M, g) is locally flat.*

Finally, G. Calvaruso and the second author [7] (see also [9]) were able to generalize this result to the framework of *semi-symmetric spaces*, i.e., Riemannian manifolds satisfying the condition $R_{XY} \cdot R = 0$ for all vector fields X, Y:

Theorem 2.4 *Let (M, g) be a semi-symmetric Riemannian manifold such that the volume of every sufficiently small geodesic ball of radius r is given by* (1). *Then (M, g) is locally flat.*

Apart from these special cases, however, the volume conjecture stated above has, to our knowledge, not been solved and it remains an intriguing open problem.

The volume conjecture is an attempt to characterize locally Euclidean spaces by means of the volume of their geodesic balls or spheres. A more general class of Riemannian manifolds with a high degree of symmetry is that of two-point homogeneous spaces which, apart from the Euclidean spaces, consists of the rank one symmetric spaces, i.e., spaces of constant curvature, Kähler spaces of constant holomorphic sectional curvature, quaternionic space forms, the Cayley plane and its non-compact dual. In [18], A. Gray and the second author also investigate whether the geodesic spheres and balls in these two-point homogeneous spaces have volume properties similar to those in Euclidean space, that is, they compute explicit expressions for the volumes of these geodesic balls and formulate volume conjectures for the rank one symmetric spaces which are similar to that for Euclidean space. To our knowledge, however, some of these volume conjectures have again only been solved in a few special cases, the main problems also remaining open.

Another well-known classical result from Euclidean geometry states that the volume of a (solid) cylinder in an n-dimensional Euclidean space is given by

$$\omega_{n-1} r^{n-1} h, \tag{4}$$

where r denotes the radius and h the height of the cylinder. More generally, it was shown by H. Hotelling [20] that the volume of a (solid) tube around an arbitrary curve σ in an n-dimensional Euclidean space is given by

$$\omega_{n-1} r^{n-1} L(\sigma),$$

where $L(\sigma)$ now denotes the length of the curve σ, implying that the volume of such a tube is independent of the embedding of the axial curve. A similar result was obtained in [20] for tubes around curves in n-dimensional spheres, i.e., spaces of positive constant curvature, and for tubes around arbitrary submanifolds in spaces of non-negative constant sectional curvature by H. Weyl [37]. In [19], the authors define a *(solid) tube* of radius r around a smooth curve $\sigma : [a, b] \to M$ in an arbitrary Riemannian manifold (M, g) as the set

$$P_\sigma(r) = \{\exp_{\sigma(t)}(su) | a \le t \le b, 0 \le s \le r, u \in \sigma'(t)^\perp, \|u\| = 1\}.$$

They compute explicit expressions for the volume of a (solid) tube around a curve σ in a two-point homogeneous space and obtain the following theorem, which generalizes the results of Hotelling and Weyl mentioned above.

Theorem 2.5 *The volume of a tube of radius r around a curve σ in a two-point-homogeneous space is given by*

$$V_\sigma(r) = kL(\sigma)S(r),$$

where $S(r)$ denotes the volume of an (arbitrary) geodesic sphere of radius r in M, $L(\sigma)$ is the length of the curve σ and k is a constant (depending on the dimension of M).

It should be remarked here that the volume of a geodesic sphere or a geodesic ball in any two-point homogeneous space is independent of its center m. This can easily be seen from the explicit expressions for these volumes, or from the fact that a two-point homogeneous space is locally homogeneous, and hence ball-homogeneous. (We refer to Section 5 for further details concerning ball-homogeneous manifolds.)

Starting from the explicit expressions for the volumes of tubes in two-point homogeneous spaces, A. Gray and the second author [19] then study Riemannian manifolds such that, for every small radius r and every sufficiently short geodesic γ, the (solid) tube of radius r around γ has the same volume as in a given two-point homogeneous space. It turns out that these properties are, indeed, characteristic for the model spaces under consideration. In particular, they prove the following

Theorem 2.6 *Let (M, g) be a Riemannian manifold and suppose that, for all small r and all sufficiently short geodesics γ, the tube of radius r around γ has the same volume as in a space of constant curvature. Then (M, g) is locally isometric to that space of constant curvature.*

Theorem 2.7 *Let (M, g, J) be a Kähler manifold (that is, its holonomy group is contained in $U(n)$) and suppose that, for all small r and all sufficiently short geodesics γ, the tube of radius r around γ has the same volume as in a Kähler manifold of constant holomorphic sectional curvature. Then (M, g, J) has constant holomorphic sectional curvature and is locally isometric to that space.*

A similar characterization was obtained for quaternionic space forms, under the additional assumption that the holonomy group of the manifold under consideration is a subgroup of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$. It should be remarked here that similar restrictions on the holonomy group were taken into account in the formulation of the volume conjectures for geodesic balls and spheres in two-point homogeneous spaces mentioned above, and that these restrictions cannot be dropped. Indeed, it was noted in [31] (see also [30],[35]) that there exist examples of so-called Damek-Ricci spaces (see [2] for further details concerning this class of manifolds) whose geodesic balls have the same volume as those in a quaternionic space form, but which are non-symmetric (and hence not locally isometric to the quaternionic space form). It also follows from a well-known result of Alekseevsky that a similar characterization (that is, taking into account holonomy restrictions) for the Cayley plane would be trivial. Indeed, it was shown in [1] that the manifolds whose holonomy group is contained in Spin(9) are either locally flat or locally isometric to the Cayley plane or its non-compact dual. We refer to [19] for the proofs of the results mentioned above.

Finally, in [23], O. Kowalski and the second author introduce the notion of a geodesic disk of radius r, which generalizes the notion of a two-dimensional disk in three-dimensional Euclidean space. Given a point $m \in M$ and a (unit) vector $\xi \in T_mM$, the *geodesic disk* of radius r, centered at m and orthogonal to ξ is defined as the set

$$D_m^\xi(r) = \{\exp_m(su) | u \in T_mM, \|u\| = 1, g(u, \xi) = 0, 0 \le s \le r\}.$$

In [25], explicit expressions are computed for the volumes of geodesic disks in two-point homogeneous spaces. Starting from these expressions, one can then provide characterizations of two-point homogeneous Riemannian manifolds by means of the volumes of their small geodesic disks similar to those for tubes. We refer to [25] for the statements and proofs of these results.

3 Archimedes' theorem and its generalizations

A classical and well-known result of Archimedes states that, in a three-dimensional Euclidean space, the ratio of the volumes of a ball and its (solid) circumscribing tube is equal to $\frac{2}{3}$. It follows immediately from (1) and (4) that in a Euclidean space of arbitrary dimension this ratio is also a constant but it depends on the dimension. In [12],[34], M. Djorić and the second author investigate whether this volume property is characteristic for locally Euclidean spaces. To do this, they generalize the notion of a circumscribing tube to the framework of Riemannian geometry as follows. Let $B_m(r)$ be the geodesic ball of radius r centered at m and denote by γ a unit speed geodesic such that $\gamma(0) = m$. Then the *circumscribing tube* of $B_m(r)$ with axial geodesic γ is defined as the set

$$P_\gamma^c(r) = \{\exp_{\gamma(t)}(su) | t \in [-r, r], u \in \{\gamma'(t)\}^\perp, \|u\| = 1, 0 \leq s \leq r\}.$$

Using the power series expansions for the volumes of geodesic balls and tubes, they are then able to prove the following "converse" of Archimedes' result:

Theorem 3.1 *Let (M, g) be a Riemannian manifold and suppose that, for every $m \in M$, every axial geodesic γ through m and every sufficiently small radius r, the ratio of the volumes of $B_m(r)$ and $P_\gamma^c(r)$ is constant. Then (M, g) is locally flat.*

It follows from Theorem 2.5 that, in any two-point homogeneous space, the ratio of the volumes of a geodesic ball and its circumscribing tube is independent of the chosen axial geodesic (and of the center of the ball). Using the explicit expressions for the volumes of balls and tubes in two-point homogeneous spaces, these ratios can be computed explicitly. In [12], the authors investigate whether these ratios are characteristic for the given two-point homogeneous model spaces, and they obtain the following generalizations of Archimedes' theorem:

Theorem 3.2 *Let (M, g) be an n-dimensional Riemannian manifold such that, for all $m \in M$, all geodesics γ through m and all sufficiently small r, the ratio of the volumes of a geodesic ball $B_m(r)$ of radius r centered at m and its circumscribing tube (with axial geodesic γ) is the same as in a space of constant curvature (M_0, g_0). Then (M, g) is locally isometric to (M_0, g_0).*

Theorem 3.3 *Let (M, g, J) be an n-dimensional Kähler manifold such that, for all $m \in M$, all geodesics γ through m and all sufficiently small r, the ratio of the volumes of a geodesic ball $B_m(r)$ and its circumscribing tube (with axial geodesic γ) is the same as in a Kähler space of constant holomorphic sectional curvature (M_0, g_0, J_0). Then (M, g, J) is locally isometric to (M_0, g_0, J_0).*

A similar characterization can be obtained for quaternionic space forms, while it follows again from the result of Alekseevsky [1] that such an Archimedes-like characterization for the Cayley plane and its non-compact dual is trivial.

We refer to [12],[34] for similar considerations about the ratios of the $(n-1)$-dimensional volumes of the spheres and circumscribed tubes as well as for the *total* $(n-1)$-dimensional volumes of these spheres and tubes.

Another consequence of Theorem 2.5 is that, in a two-point homogeneous space, the volume of a circumscribing tube of a geodesic ball $B_m(r)$ of radius r and centered at $m \in M$ is given by

$$V_\sigma^c(r) = \alpha r S_m(r), \tag{5}$$

where α is a constant depending on the dimension. In [34], the second author considers the converse problem, and he states the following

Conjecture 3.4 *Let (M, g) be an n-dimensional Riemannian manifold such that for all $m \in M$, all axial geodesics σ and all sufficiently small r, the volume of the circumscribing tube of a geodesic ball $B_m(r)$ is given by $V_\sigma^c(r) = \alpha r S_m(r)$. Then (M, g) is locally isometric to a two-point homogeneous space.*

Up to now, the general solution of this problem has, to our knowledge, not been found, although we have the following partial result (see [34]).

Theorem 3.5 *Conjecture 3.4 holds for two- and three-dimensional manifolds. It also holds when M is reducible.*

4 A volume conjecture in Lorentzian geometry

As we have seen above, the results of Bertrand-Diguet-Puiseux and Vermeil give a nice geometric interpretation of the Gaussian (or scalar) curvature of a surface (or n-dimensional Riemannian manifold) in terms of volume defects of geodesic circles (or geodesic spheres and balls). It is well-known that, in Lorentzian geometry, there exists no notion of a geodesic ball of finite volume, which makes it impossible to generalize the Riemannian notion of a volume defect to this class of manifolds in a straightforward way. For this reason, F. and B. Gackstatter introduced, in [16], the notion of a truncated light cone, which can be defined as follows. Let

$\xi \in T_m M$ be a timelike vector of length -1. Then the *truncated light cone* of (sufficiently small) height T and with ξ as axis vector is the set

$$L_\xi(T) = \{\exp_m(u) | g(u,u) < 0, 0 \le -g(u,\xi) \le T\}.$$

In [16],[14],[15], the authors make a study of volume defects of these truncated light cones, and they show the existence of a strong link between the curvature of a Lorentzian manifold and these volume defects. It is easily seen that the volume $V_\xi(T)$ of a truncated light cone in four-dimensional Minkowski space is given by [16]

$$V_\xi(T) = \frac{1}{3}\pi T^4.$$

In [14], F. Gackstatter then investigates whether this volume property is characteristic for four-dimensional Minkowski space. Using the power series expansion for the volume of a truncated light cone and a technique similar to the one used in [18] and [19], he proves the following

Theorem 4.1 *Let (M,g) be a four-dimensional Lorentzian manifold such that every truncated light cone has the same volume as in four-dimensional Minkowski space. Then (M,g) is locally flat.*

This result was later generalized to higher-dimensional Lorentzian manifolds and completely solved by R. Schimming [28], while a similar problem was studied (and partially solved) in the more general framework of pseudo-Riemannian geometry in [29].

5 Locally homogeneous spaces and the volume of geodesic balls

We have already remarked that the volume of a geodesic ball or sphere in a two-point homogeneous space is independent of its center m. More generally, let (M,g) be a locally homogeneous Riemannian manifold, i.e., a Riemannian manifold such that for any pair of points p and q in M there exists a local isometry of (M,g) mapping p into q. It follows immediately from this definition that the volume of a (small) geodesic ball $B_m(r)$ or sphere $G_m(r)$ of radius r is independent of the center m. A Riemannian manifold is said to be *ball-homogeneous* if it satisfies this volume property for all geodesic balls or spheres, i.e., if the volume of every geodesic ball (or sphere) of sufficiently small radius r is independent of the center m. The problem of determining if this volume property is *characteristic* for locally homogeneous spaces, i.e., determining if every ball-homogeneous space is locally homogeneous, has been investigated extensively in a number of recent papers ([6],[7],[8]) but, although a number of partial results have again been obtained, the problem remains, in general, still unsolved. For a summary of the results concerning ball-homogeneous spaces and local homogeneity and for a discussion of several remarkable examples, we refer to [9].

6 Two-point homogeneous spaces and the volume of tubes and disks

One other immediate consequence of Theorem 2.5 is that, in any two-point homogeneous space, the volume of the circumscribing tube of a geodesic ball $B_m(r)$ is independent of the chosen axial geodesic and of the center m of the ball.

The definition of two-point homogeneity also implies that, for every pair of points m, m' in M and every pair of unit vectors $\xi \in T_mM$ and $\xi' \in T_{m'}M$, there exists an isometry ϕ of M such that

$$\phi(m) = m', \qquad \phi_{*m}(\xi) = \xi'.$$

As a consequence we find that, in any two-point homogeneous space, the volume of a geodesic disk $D^{\xi}_m(r)$ of radius r is independent of the chosen normal ξ and of the center m.

To study whether these volume properties of geodesic disks and circumscribing tubes are characteristic for two-point homogeneous Riemannian manifolds, O. Kowalski and the second author [23],[34] introduced the notions of (strong) disk-homogeneity and (strong) tube-homogeneity. A Riemannian manifold (M, g) is said to be *disk-homogeneous* [23] if the volume of a geodesic disk $D^{\xi}_m(r)$ is independent of the chosen normal $\xi \in T_mM$ and *strongly disk-homogeneous* if it is disk-homogeneous and, in addition, the volume of $D^{\xi}_m(r)$ is independent of the center m. Similarly, the manifold is said to be *tube-homogeneous* [34] if the volume of the circumscribing tube of a geodesic ball $B_m(r)$ is independent of the chosen axial geodesic and *strongly tube-homogeneous* if, in addition, this volume is independent of the center of the ball.

It is easily seen that any two-dimensional Riemannian manifold is (strongly) disk-homogeneous. However, apart from these manifolds there are no examples known of disk- or tube-homogeneous manifolds which are not locally isometric to a two-point-homogeneous space, and we can therefore state the following problem (see [23],[34]).

Question 6.1 *Let (M, g) be a (strongly) tube-homogeneous manifold or a (strongly) disk-homogeneous manifold of dimension $n \geq 3$. Is (M, g) locally isometric to a two-point homogeneous Riemannian manifold ?*

To our knowledge, none of these problems has been solved completely, although a number of partial answers were obtained, which we summarize in the following theorems. For the proofs of these results, and for more detailed information, we refer to [23],[24],[33],[34].

Theorem 6.2 *Let (M, g) be a strongly disk-homogeneous three- or four-dimensional Riemannian manifold. Then (M, g) is locally isometric to a two-point homogeneous manifold.*

Theorem 6.3 *Every three-dimensional disk-homogeneous Riemannian manifold has constant sectional curvature.*

Theorem 6.4 *Let (M, g) be a (strongly) tube-homogeneous three-dimensional Riemannian manifold. Then (M, g) has constant sectional curvature. Furthermore, every two-dimensional strongly tube-homogeneous Riemannian manifold has constant sectional curvature. Finally, a reducible tube-homogeneous space is locally Euclidean.*

In [24], O. Kowalski and the second author make a study of locally symmetric disk-homogeneous Riemannian manifolds, and they show that such a manifold is a so-called 3-stein manifold. (We recall that a Riemannian manifold is said to be *k-stein* if, for all $m \in M$, all $X \in T_mM$ and all $l \in \{1, 2, \ldots, k\}$,

$$\mathrm{tr} R_X^l = A_l(m) g(X, X)^l,$$

i.e., $\mathrm{tr} R_X^l$ is independent of the (unit) vector X, where $R_X = R_{X.}X$ denotes the Jacobi operator of M.) One can then use the classification of Gray-Carpenter-Willmore [10] to conclude that, apart from a few undecided cases (which can be written down explicitly), every locally symmetric disk-homogeneous space is locally isometric to a two-point homogeneous space. Using similar computations, M. Djorić and the authors were able to prove a similar result for locally symmetric tube-homogeneous spaces.

7 G.o. spaces and the volume of tubes and disks

As a final example, we consider the class of so-called *g.o. spaces* as model spaces. A Riemannian manifold is said to be a g.o. space if every geodesic γ in M is the orbit of a one-parameter group of isometries of (M, g). This class of Riemannian manifolds was studied extensively in [26], where it was shown that every naturally reductive homogeneous manifold is a g.o. space and that, conversely, every simply connected g.o. space of dimension ≤ 5 is also naturally reductive. It should be remarked, however, that there exist examples of six-dimensional g.o. spaces which are not naturally reductive (see for example [26],[27]).

Now, consider a geodesic γ in a g.o. space (M, g). Then it follows immediately from the definition that the geodesic disks $D_{\gamma(t)}^{\gamma'(t)}(r)$ of radius r, orthogonal to the geodesic γ and with center on the geodesic, have the same volume. Using a similar argument, it can easily be seen that the volume of a circumscribing tube $P_\gamma^c(r)$ (with axial geodesic γ) of a geodesic ball $B_{\gamma(t)}(r)$ of radius r does not change if the center $\gamma(t)$ of the ball moves along the axial geodesic γ. In [4], the authors investigated whether these volume properties are characteristic for g.o. spaces. To this purpose, a Riemannian manifold is said to be *weakly disk-homogeneous* if it has the property that, for every geodesic, the volume of a (sufficiently small) geodesic disk normal to the geodesic, with center on the geodesic and fixed small radius, does not change if the center moves along the geodesic, and it is said to be *weakly tube-homogeneous* if the same property holds for the circumscribing tubes. It is easily seen that every two-dimensional Riemannian manifold is weakly disk-homogeneous. Apart from these trivial examples, however, no other examples are known of weakly disk- or tube-homogeneous spaces which are not locally isometric to a g.o. space, and in [4] the authors studied the following

Question 7.1 *Let (M, g) be a weakly tube-homogeneous manifold or a weakly disk-homogeneous manifold of dimension $n \geq 3$. Is (M, g) locally isometric to a g.o. space ?*

To study this problem, one uses the power series expansion for the volume of a geodesic disk (see for example [23],[4]) or circumscribing tube ([12],[34]) to construct a set of necessary conditions for a Riemannian manifold to be weakly disk- or tube-homogeneous. Using the first of these conditions, one can then prove the following

Theorem 7.2 *Let (M, g) be a weakly tube-homogeneous manifold or a weakly disk-homogeneous manifold of dimension $n \geq 3$. Then (M, g) has constant scalar curvature τ and, moreover, its Ricci tensor is cyclic parallel, i.e.,*

$$\nabla_X \rho_{XX} = 0,$$

for all X tangent to M.

This theorem already has the following interesting consequences (see [4] for the proofs of these results).

Corollary 7.3 *A two-dimensional weakly tube-homogeneous Riemannian manifold has constant sectional curvature.*

Corollary 7.4 *Every semi-symmetric weakly tube-homogeneous manifold and every semi-symmetric weakly disk-homogeneous manifold of dimension $n \geq 3$ is locally symmetric. The same result holds when the additional condition of semi-symmetry is replaced by conformal flatness or, more generally, by the condition that the Ricci tensor of (M, g) is a Codazzi tensor, i.e.,*

$$\nabla_X \rho_{YZ} = \nabla_Y \rho_{XZ},$$

for all X, Y, Z tangent to M.

Taking into account the first and second necessary conditions and using the techniques from [22], one can also prove the following

Theorem 7.5 *Every three-dimensional weakly disk-homogeneous or tube-homogeneous Riemannian manifold is locally isometric to a naturally reductive homogeneous manifold.*

Finally, the authors considered in [4] the special case where (M, g) is a four-dimensional Einstein space. In this case, a well-known result by Jensen [21] states that every four-dimensional (locally) homogeneous Einstein manifold is (locally) symmetric, and Question 7.1 can therefore be reformulated (for this special case) as follows:

Question 7.6 *Let (M, g) be a four-dimensional weakly disk- or tube-homogeneous Einstein manifold. Is (M, g) locally symmetric ?*

In [4] the authors started by studying this question in the case where (M, g) is a 2-stein space, and obtained the following partial answer to the question.

Theorem 7.7 *Let (M, g) be a four-dimensional weakly disk- or tube-homogeneous 2-stein space and suppose that $\|\nabla R\|^2$ is constant on the manifold. Then (M, g) is locally symmetric.*

Using this result, one can then prove the following

Theorem 7.8 *Let (M, g, J) be a four-dimensional weakly disk- or tube-homogeneous Kähler manifold satisfying the condition that $\|\nabla R\|^2$ is a constant. Then (M, g) is locally symmetric.*

Finally, using the method developed in [11], the following partial answer was obtained.

Theorem 7.9 *Let (M, g) be a four-dimensional weakly disk- or tube-homogeneous Hermitian Einstein space for which $\|\nabla R\|^2$ is constant on M. Then (M, g) is locally symmetric.*

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BALL–HOMOGENEOUS SPACES

G. Calvaruso and L. Vanhecke

1 Introduction

Ball-homogeneous spaces have been introduced in 1982 by O. Kowalski and the second author as a natural generalization of locally homogeneous spaces [KV1], [KV2]. A Riemannian manifold (M, g) is said to be *ball-homogeneous* if the volume of every sufficiently small geodesic sphere (or ball) does not depend on the center of the ball but is only a function of the radius. Locally homogeneous spaces are trivial examples. P. Günther and F. Prüfer proved that also D'Atri spaces are ball-homogeneous [GP]. A *D'Atri space* is a Riemannian manifold all of whose local geodesic symmetries are volume-preserving (up to sign). Many examples of D'Atri spaces have been discovered and their geometry has been treated in several papers. See [KPV] for a survey. Up to now, there are no examples known which are not locally homogeneous and so, it is still an open problem whether there are D'Atri spaces which are not locally homogeneous. Even in the broader class of ball-homogeneous spaces, no examples which are not locally homogeneous spaces are known. Therefore, it is natural to consider the following

PROBLEM: Is a ball-homogeneous space necessarily locally homogeneous?

The above problem, in its full generality, seems difficult to solve, even in dimension three. For this reason one looks for partial answers by considering special classes of ball-homogeneous spaces. Moreover, it is also worthwhile to see what properties which hold for D' Atri spaces remain valid for the broader class of ball-homogeneous spaces.

The aim of this note is to give a short survey about some of the results obtained so far. We refer to [CTV], [CV1], [CV2] for more details.

Before exposing these results, we first note that ball-homogeneity implies conditions on an infinite number of scalar curvature invariants. Let (M, g) be a smooth, connected Riemannian manifold and $V_m(r)$ the volume of a geodesic ball $B_m(r)$ of sufficiently small radius r and center m. R denotes the Riemannian curvature tensor of (M, g) given by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all smooth vector fields X, Y on M. ϱ is the Ricci tensor of type $(0,2)$, τ the scalar curvature and Δ the Laplacian on (M,g).

If (M,g) is ball-homogeneous, then $V_m(r)$ does not depend on m. So, all the derivatives $\frac{d^k V_m(r)}{dr}$, $k \in \mathbb{N}$, evaluated at m, have to be globally constant and these derivatives are functions of the scalar curvature invariants. For arbitrary k, the explicit expressions are not known in the general case. Nevertheless, we have the following (see [GV] for more details):

Proposition 1.1 *Let (M,g) be an n-dimensional Riemannian manifold, m a point of M and $r > 0$. Then*

$$V_m(r) = \omega r^n (1 + A(m) r^2 + B(m) r^4 + C(m) r^6 + O(r^8))$$

where

$$\begin{aligned}
A &= -\frac{\tau}{6(n+2)}, \\
B &= \frac{-3\|R\|^2 + 8\|\varrho\|^2 + 5\tau^2 - 18\tau\Delta\tau}{360(n+2)(n+4)}, \\
C &= \frac{1}{720(n+2)(n+4)(n+6)} \Big(-\frac{5}{9}\tau^3 - \frac{8}{3}\tau\|\varrho\|^2 + \\
&\quad + \tau\|R\|^2 + \frac{64}{63}\check{\varrho} - \frac{64}{21} g(\varrho\otimes\varrho, \bar{R}) + \frac{32}{7} g(\varrho, \dot{R}) - \\
&\quad - \frac{110}{63}\check{R} - \frac{200}{63}\check{\check{R}} + \frac{45}{7}\|\nabla\tau\|^2 + \frac{45}{14}\|\nabla\varrho\|^2 + \\
&\quad + \frac{45}{7}\alpha(\varrho) - \frac{45}{14}\|\nabla R\|^2 + 6\tau\,\Delta\tau + \frac{48}{7} g(\Delta\varrho,\, \varrho) + \\
&\quad + \frac{54}{7} g(\nabla^2\tau,\, \varrho) - \frac{30}{7} g(\Delta R,\, R) - \frac{45}{7}\Delta^2\tau\Big)\,(m).
\end{aligned}$$

Here, we have with respect to an orthonormal basis,

$$\|\varrho\|^2 = \sum_{i,j} \varrho_{ij}^2, \qquad \|R\|^2 = \sum_{i,j,k,l} R_{ijkl}^2,$$

$$\check{\varrho} = \sum_{i,j,k} \varrho_{ij}\varrho_{jk}\varrho_{ki}, \qquad g(\varrho\otimes\varrho, \bar{R}) = \sum_{i,j,k,l} \varrho_{ij}\varrho_{kl}R_{ikjl},$$

$$\check{R} = \sum_{i,j,k,l,p,q} R_{ijkl}R_{klpq}R_{pqij}, \qquad \check{\check{R}} = \sum_{i,j,k,l,p,q} R_{ikjl}R_{kplq}R_{piqj},$$

$$g(\varrho, \dot{R}) = \sum_{i,j,p,q,r} \varrho_{ij}R_{ipqr}R_{jpqr}, \qquad \alpha(\varrho) = \sum_{i,j,k} \nabla_i\varrho_{jk}\nabla_k\varrho_{ij}.$$

2 Ball-homogeneous Riemannian products

Let (M,g) be an n-dimensional Riemannian manifold, $m \in M$ and denote by $\exp_m$ the exponential map centered at m. For $r < i(m)$, the injectivity radius at m, we denote by $G_m(r) = \exp_m S_m(r)$ the geodesic sphere with center m and radius r. $S_m(r)$ is the sphere with center 0 and radius r in T_mM.

The following extension of an analogous result for D'Atri spaces holds [CV2].

Theorem 2.1 *Let $(M,g) = (M_1,g_1) \times .. \times (M_r,g_r)$ be an analytic Riemannian product. Then (M,g) is ball-homogeneous if and only if each factor (M_i,g_i), $i = 1,..,r$, is ball-homogeneous.*

Proof. Obviously, it is enough to restrict to the case where $r = 2$. Let $\tilde{\Delta}^{(k)}$ be the *Euclidean Laplacian of order* k defined for a function φ by

$$\tilde{\Delta}^{(k)}[\varphi](m) = \frac{1}{(2k)!}\sum_{i_1...i_{2k}=1}^{n}\sum_{\sigma}\{\delta_{i_{\sigma(1)}i_{\sigma(2)}}...\delta_{i_{\sigma(2k-1)}i_{\sigma(2k)}}\nabla^{2k}_{i_1...i_{2k}}\varphi\}(m).$$

Here $\{e_1,...,e_n\}$ is an arbitrary local orthonormal basis, $\nabla^k_{i_1...i_k} = \nabla^k_{e_{i_1}...e_{i_k}}$, δ_{ij} denotes the Kronecker symbol and the summation is made over all permutations σ of the set $\{1,..,2k\}$. These Euclidean Laplacians are globally defined differential operators of order $2k$. By using the generalized Pizetti formula for mean-value operators, we may express the volume $m_r = \dfrac{dV_m(r)}{dr}$ of $G_m(r)$ by means of these Euclidean Laplacians. Explicitly, we have

$$m_r = 2\pi^{\frac{n}{2}}r^{n-1}\sum_{k=0}^{+\infty}(\frac{r}{2})^{2k}\frac{1}{k!\Gamma(\frac{n}{2}+k)}D^{2k}_M(m)$$

where

$$D^{2k}_M(m) = \tilde{\Delta}^{(k)}[\theta_m](m).$$

Here, $D^0_M(m) = 1$. Hence, (M,g) is ball-homogeneous if and only if D^{2k}_M is globally constant for each $k \in \mathbb{N}$.

Next, let $(M,g) = (M_1,g_1) \times (M_2,g_2)$. Then we have

$$\tilde{\Delta}^{(k)}_M = \sum_{\lambda=0}^{k}\binom{k}{\lambda}\tilde{\Delta}^{(\lambda)}_{M_1}\tilde{\Delta}^{(k-\lambda)}_{M_2}$$

and

$$\theta_M = \theta_{M_1}\cdot\theta_{M_2}.$$

So, for $m = (m_1,m_2)$, we have

$$D_M^{2k}(m) = \sum_{\lambda=0}^{k} \binom{k}{\lambda} D_{M_1}^{2\lambda}(m_1) D_{M_2}^{2k-2\lambda}(m_2).$$

The theorem follows now easily from this formula. Indeed, let (M_1, g_1) and (M_2, g_2) be ball-homogeneous. Then $D_{M_1}^{2p}$ and $D_{M_2}^{2q}$ are globally constant for all $p, q, \in I\!N$. Hence, D_M^{2p} is also globally constant.

Conversely, let (M, g) be a ball-homogeneous space. For $s \geq 1$ we have

$$D_M^{2s}(m) = D_{M_1}^{2s}(m_1) + D_{M_2}^{2s}(m_2) + \sum_{\lambda=1}^{s-1} \binom{s}{\lambda} D_{M_1}^{2\lambda}(m_1) D_{M_2}^{2s-2\lambda}(m_2).$$

An induction procedure yields

$$D_{M_1}^{2s}(m_1) + D_{M_2}^{2s}(m_2) = \text{const.}$$

and hence, $D_{M_1}^{2p}$ and $D_{M_2}^{2p}$ are constant for all $p \in I\!N$, which proves the required result.

∎

Clearly, because of Theorem 2.1, to decide whether a locally reducible analytic Riemannian manifold is ball-homogeneous or not, it suffices to investigate if the spaces in the local decomposition of such a manifold are ball-homogeneous or not.

3 Ball-homogeneous semi-symmetric spaces and a volume conjecture

A *semi-symmetric space* is a Riemannian manifold (M, g) whose curvature tensor R satisfies $R_{XY} \cdot R = 0$ for all vector fields X, Y. This is equivalent to saying that R_p, for each $p \in M$, is the same as the curvature tensor of a symmetric space. This last space may vary with p. This class of manifolds extends that of the locally symmetric spaces. We refer to [BKV2] for an extensive treatment and further references. In [B], it is proved that semi-symmetric locally homogeneous spaces and semi-symmetric D'Atri spaces are locally symmetric. Therefore, it is worthwhile to investigate whether this also holds for semi-symmetric ball-homogeneous spaces. In [CV2], we provided a positive answer, obtaining in this way that any semi-symmetric ball-homogeneous space is indeed locally homogeneous. Here, we give a brief sketch of the proof.

First, the following proposition of Z. I. Szabó describes the local structure of a semi-symmetric space (M, g).

Proposition 3.1 *There exists an open dense subset U of M such that around every point of U the manifold is locally isometric to the direct product of symmetric spaces, two-dimensional manifolds, spaces foliated by Euclidean leaves of codimension two, elliptic cones, hyperbolic cones, Euclidean cones and Kählerian cones.*

Further, we have (since the spaces of "cone type" do not have constant scalar curvature)

Proposition 3.2 *If τ is constant, then on U the semi-symmetric space (M,g) is locally isometric to a Riemannian product*

$$(M_s, g_s) \times (M_1, g_1) \times ... \times (M_r, g_r)$$

where (M_s, g_s) is a symmetric space and (M_i, g_i), $i = 1,..,r$, are locally irreducible Riemannian spaces foliated by totally geodesic Euclidean leaves of codimension two having constant scalar curvature.

Note that a semi-symmetric space (M,g) has constant scalar curvature on U if and only if (M,g) is curvature homogeneous on U, that is, for each pair of points $p,q \in U$, there exists a linear isometry $\varphi : T_pU \to T_qU$ such that $\varphi^* R_q = R_p$.

Further, the following result concerning the explicit description of the metric of an (M_i, g_i) has been given in [BKV1] (see also [BKV2]).

Proposition 3.3 *Let (M,g) be an $(n+2)$-dimensional locally irreducible semi-symmetric space foliated by n-dimensional Euclidean leaves and such that its scalar curvature is constant along each leaf. Then there exists a dense open subset U of M such that in a neighborhood of every point $p \in U$ there exist local coordinates $(w, x^1, .., x^{n+1})$ and an orthonormal coframe of the form*

$$\begin{cases} \omega^0 = f(w, x^1) dw \,, \\ \omega^i = dx^i + \sum_{j=1}^{n+1} D^i_j(w) x^j dw \,, \qquad i = 1,..,n+1 \,, \end{cases}$$

where $D^i_j(w) + D^j_i(w) = 0$. The scalar curvature of this metric is given by

$$\tau = -2f^{-1} f''_{x^1 x^1} \neq 0 \ .$$

Conversely, any local metric of this form is semi-symmetric and foliated by Euclidean leaves of codimension two with constant scalar curvature along the leaves.

Now, let (M,g) be a semi-symmetric ball-homogeneous space. From Proposition 1.1 we have that A, and hence τ, must be constant on M. So, (M,g) is curvature homogeneous on U. This implies that all scalar curvature invariants which do not involve the components of ∇R are globally constant on M. This fact, together with the constancy of B and C, leads to the following

Lemma 3.4 *Let (M,g) be a semi-symmetric ball-homogeneous space. Then the invariants τ and $5||\nabla R||^2 - 17||\nabla \varrho||^2 + 30\alpha(\varrho)$ are globally constant. Moreover, (M,g) is curvature homogeneous on U.*

We are now in a position to prove the main result of this section.

Theorem 3.5 *A semi-symmetric ball-homogeneous space is locally symmetric.*

Proof. Since τ is constant on M, the manifold may be described locally as in Proposition 3.2. The curvature invariants τ and $5||\nabla R||^2 - 17||\nabla\varrho||^2 + 30\alpha(\varrho)$ are additive. So, since they are globally constant on M, they are also constant on each factor of the local decomposition of M. In the following lemma we shall prove that this implies that each of the factors is locally symmetric. Since M is connected, it then follows from Proposition 3.2 that (M,g) is locally symmetric. ∎

So, we are left with

Lemma 3.6 *Let (M,g) be a locally irreducible curvature homogeneous semi-symmetric space foliated by totally geodesic Euclidean leaves of codimension two and such that the invariant $5||\nabla R||^2 - 17||\nabla\varrho||^2 + 30\alpha(\varrho)$ is constant on M. Then (M,g) is locally symmetric.*

Proof. We use the local coordinates and the orthonormal coframe given in Proposition 3.3. For the connection forms ω_i^j we then have

$$\begin{cases} \omega_1^0 = f^{-1} f'_{x_1} \omega^0 , & \\ \omega_j^0 = 0 , & j \geq 2 , \\ \omega_j^i = f^{-1} D_j^i(w) \omega^0 , & i,j \geq 1 , \end{cases}$$

where $\omega_j^i + \omega_i^j = 0$. So, the Riemann curvature tensor and the Ricci tensor of (M,g) are given by

$$R = 2\tau(\omega^0 \wedge \omega^1 \otimes \omega^0 \wedge \omega^1) ,$$

$$\varrho = \frac{\tau}{2}(\omega^0 \otimes \omega^0 + \omega^1 \otimes \omega^1) .$$

Next, let $\{e_0, .., e_{n+1}\}$ be the local orthonormal frame dual to $\{\omega^0, .., \omega^{n+1}\}$ and let X be a smooth vector field on M. Since

$$\nabla_X \omega^i = -\sum_j \omega_j^i(X)\omega^j , \qquad i = 1, .., n+1 ,$$

we obtain that the only possible non-vanishing components of ∇R are

$$\nabla_0 R_{10j0} = -\frac{1}{2}\tau f^{-1} D_j^1 \qquad \text{for } j \geq 2 ,$$

and those obtained from these ones by using the symmetry properties of ∇R.

We then derive for $\nabla\varrho$:

$$\begin{cases} \nabla_0\varrho_{1j} = \nabla_0\varrho_{j1} = -\frac{\tau}{2}f^{-1}D^1_j & \text{for } j \geq 2\,, \\ \nabla_i\varrho_{jk} = 0 & \text{in all the other cases}\,. \end{cases}$$

This yields

$$\begin{aligned} ||\nabla R||^2 &= 2\tau^2(f^{-1})^2\sum_{j=2}^{n+1}(D^1_j)^2\,, \\ ||\nabla\varrho||^2 &= \frac{\tau^2}{2}(f^{-1})^2\sum_{j=2}^{n+1}(D^1_j)^2\,, \\ \alpha(\varrho) &= \sum_{i,j,k=1}^{n+1}\nabla_i\varrho_{jk}\nabla_j\varrho_{ik} = 0\,. \end{aligned}$$

So, we finally get

$$5||\nabla R||^2 - 17||\nabla\varrho||^2 + 30\alpha(\varrho) = \frac{3}{2}\tau^2(f^{-1})^2\sum_{j=2}^{n+1}(D^1_j)^2\,.$$

Since D^1_j depends only on w and f depends on x^1, the constancy of this invariant implies that $D^1_j = 0$ for all $j = 2, .., n+1$. So, $||\nabla R||^2 = 0$, which implies the required result.

∎

Using this result, we are now able to provide for semi-symmetric spaces a positive answer to the following problem stated in [GV]: *Let (M, g) be a Riemannian manifold such that the volume of each geodesic ball of sufficiently small radius is the same as in a Euclidean space. Is (M, g) locally flat ?* Only partial answers are known and the general case is still an intriguing open problem. It should be noted that an (M, g) satisfying the hypothesis in the above problem is necessarily ball-homogeneous. M. Ferrarotti and the second author proved that the conjecture holds for locally symmetric spaces [FV]. It now follows at once from this result and Theorem 3.5 that the conjecture is also true for the broader class of semi-symmetric spaces. Hence, we have

Theorem 3.7 *Let (M, g) be a semi-symmetric space such that the volume of all sufficiently small geodesic spheres or balls is the same as in a Euclidean space. Then (M, g) is locally flat.*

4 Three-dimensional ball-homogeneous spaces

Now we turn to consideration of ball-homogeneous spaces of low dimensions. From Proposition 1.1 it is clear that a ball-homogeneous space (M, g) has constant curvature if $\dim M = 2$. Further, for $\dim M = 3$, we have the following relations for the scalar curvature invariants [GV]:

$$
\begin{aligned}
&||R||^2 = 4||\varrho||^2 - \tau^2\,,\\
&g(\varrho \otimes \varrho, \bar{R}) = \tfrac{5}{2}\tau||\varrho||^2 - \tfrac{1}{2}\tau^3 - 2\breve{\varrho}\,,\\
&g(\varrho, \dot{R}) = 4\tau||\varrho||^2 - \tau^3 - 2\breve{\varrho}\,.\\
(4.1) \qquad &\check{R} = 12\tau||\varrho||^2 - 3\tau^3 - 8\breve{\varrho}\,,\\
&\check{\bar{R}} = \tfrac{3}{2}\tau||\varrho||^2 - \tfrac{1}{4}\tau^3 - 2\breve{\varrho}\,,\\
&||\nabla R||^2 = 4||\nabla\varrho||^2 - ||\nabla\tau||^2\,,\\
&g(\Delta R,\, R) = 4g(\Delta\varrho,\, \varrho) - \tau\Delta\tau\,.
\end{aligned}
$$

We also note that we always have

$$
(4.2) \qquad \begin{cases} \tfrac{1}{2}\Delta||R||^2 = g(\Delta R,\, R) + ||\nabla R||^2\,,\\ \tfrac{1}{2}\Delta||\varrho||^2 = g(\Delta\varrho,\, \varrho) + ||\nabla\varrho||^2\,. \end{cases}
$$

Using these formulas and Proposition 1.1, we then get at once

Proposition 4.1 *Let (M, g) be a three-dimensional ball-homogeneous space. Then, τ, $||\varrho||^2$ and $256\breve{\varrho} + 9||\nabla\varrho||^2 + 90\alpha(\varrho)$ are constant on M.*

Now, we prove

Theorem 4.2 *A three-dimensional ball-homogeneous space (M, g) with at most two distinct Ricci eigenvalues is locally homogeneous.*

Proof. Let W be the dense open subset of M on which the multiplicities of the eigenvalues ϱ_1, ϱ_2 and ϱ_3 of the Ricci operator are locally constant. Let $p \in W$ and consider a neighborhood U of p where these multiplicities are constant. In what follows we shall show that all the scalar curvature invariants are constant on U. A continuity argument then shows that these invariants are constant on M. Then it follows from a criterion proved in [PTV] that (M, g) is locally homogeneous.

So, in what follows we concentrate on U and consider the following two cases: $\varrho_1 = \varrho_2 = \varrho_3$ and $\varrho_1 = \varrho_2 \neq \varrho_3$. For the first case we have at once that U is Einsteinian and hence, locally symmetric. For the second case and since

$2\varrho_1+\varrho_3=\tau$, $2\varrho_1^2+\varrho_3^2=||\varrho||^2$, it follows from Proposition 4.1 that ϱ_1 and ϱ_3 are constant on U. We consider the cases $\varrho_3\neq 0$ and $\varrho_3=0$.

a) *Case* $\varrho_3\neq 0$
This case has been treated in [K2]. There it is shown that there exists a local orthonormal frame $\{E_1,E_2,E_3\}$ with dual coframe $\{\omega^1,\omega^2,\omega^3\}$ such that

$$\begin{aligned}\nabla\varrho=(\varrho_3-\varrho_1)\ \{&(a\,\omega^1+b\,\omega^2)\otimes(\omega^1\otimes\omega^3+\omega^3\otimes\omega^1)+\\&+(c\,\omega^1+e\,\omega^2)\otimes(\omega^2\otimes\omega^3+\omega^3\otimes\omega^2)\}\,.\end{aligned}$$

This yields

$$||\nabla\varrho||^2=2(\varrho_1-\varrho_3)^2(a^2+b^2+c^2+e^2)\,,$$

$$\alpha(\varrho)=(\varrho_1-\varrho_3)^2(a^2+e^2+2bc)\,.$$

Further, since $\nabla_{E_3}\tau=0$, we have $a+e=0$. Moreover, it follows from a result in [K2] that

$$||\nabla\varrho||^2=2(\varrho_1-\varrho_3)^2(h^2-\varrho_3)$$

where $h=b-c$. So $h^2-\varrho_3=2a^2+b^2+c^2$, that is, $-\varrho_3=2a^2+2bc$. Hence, we have

$$(4.3)\qquad\left\{\begin{aligned}&||\nabla\varrho||^2=2(\varrho_1-\varrho_3)^2(h^2-\varrho_3)\,,\\&\alpha(\varrho)=-(\varrho_1-\varrho_3)^2\varrho_3\,.\end{aligned}\right.$$

Using Proposition 4.1 and the last formulas, we obtain that h is constant and this implies that U is locally homogeneous [K2] from which it follows that all scalar curvature invariants are constant on U.

b) *Case* $\varrho_3=0$
The condition $\varrho_3=0$ implies that (U,g) is semi-symmetric. So, using Theorem 3.5, we may conclude that (U,g) is locally symmetric. In particular, it is locally homogeneous.

∎

It seems to be difficult to decide in full generality whether a three-dimensional ball-homogeneous space is necessarily locally homogeneous. Theorem 4.2 provides a positive answer when there are at most two distinct eigenvalues for the Ricci operator. This contrasts to the case of D'Atri spaces where it has been shown that every three-dimensional D'Atri space is locally isometric to a naturally reductive space and hence, is locally homogeneous [K1]. This leads to the consideration of special three-dimensional ball-homogeneous spaces, namely those equipped with Einstein-like metrics. So, we now consider (M^3, g) such that the Ricci tensor ϱ is a *Codazzi tensor*, that is, $(\nabla_X \varrho)(Y, Z) = (\nabla_Y \varrho)(X, Z)$, or is *cyclic-parallel*, that is, $(\nabla_X \varrho)(Y, Z) + (\nabla_Y \varrho)(Z, X) + (\nabla_Z \varrho)(X, Y) = 0$. This last condition means that ϱ is a Killing tensor. Note that in both cases τ is necessarily constant.

As it is well-known that when τ is constant, (M^3, g) has a Codazzi Ricci tensor if and only if it is conformally flat, we first state a useful theorem about conformally flat spaces. H. Takagi [T] has proved that any conformally flat locally homogeneous space is locally symmetric. Using the same proof, this result can be extended to the broader class of curvature homogeneous spaces. So, we have

Proposition 4.3 *A curvature homogeneous conformally flat Riemannian manifold is locally symmetric.*

Clearly, a three-dimensional Riemannian manifold is curvature homogeneous if and only if the eigenvalues of the Ricci operator are constant.

Theorem 4.4 *(M^3, g) is locally symmetric if and only if it is ball-homogeneous and its Ricci tensor is a Codazzi tensor.*

Proof. The "only if" part is trivial. So, we consider the "if" part. Since

$$\sum_{i,j,k} (\nabla_i \varrho_{jk} - \nabla_j \varrho_{ik})^2 = 2(||\nabla \varrho||^2 - \alpha(\varrho)),$$

we get that ϱ is a Codazzi tensor if and only if

$$||\nabla \varrho||^2 = \alpha(\varrho). \tag{4.4}$$

Moreover, we have

$$\sum_{i,j,k} \varrho_{jk} \nabla^2_{ij} \varrho_{ik} = \frac{1}{2} g(\nabla^2 \tau, \varrho) + \check{\varrho} - g(\varrho \otimes \varrho, \bar{R}). \tag{4.5}$$

Next, since ϱ is of Codazzi type, we get at once

$$\sum_{i,j,k} \varrho_{jk} \nabla^2_{ij} \varrho_{ik} = \frac{1}{2} \Delta ||\varrho||^2 - ||\nabla \varrho||^2. \tag{4.6}$$

Using now that τ and $||\varrho||^2$ are constant, we get

$$||\nabla \varrho||^2 = \frac{5}{2} \tau ||\varrho||^2 - \frac{1}{6} \tau^3 - 3\check{\varrho}. \tag{4.7}$$

Hence, (4.4), (4.7) and Proposition 4.1 yield that $\check{\varrho}$ is constant. From this, together with $\tau = \text{const.}$ and $||\varrho||^2 = \text{const.}$, we obtain that the eigenvalues of the

Ricci operator are constant. So, (M^3, g) is curvature homogeneous. Then the result follows from Proposition 4.3.

∎

Using an argument similar to the one used in the proof of Theorem 4.4, together with the well-known formulas about the scalar curvature invariants of a conformally flat manifold, it is possible to extend the result in Theorem 4.4. We have

Theorem 4.5 *An n-dimensional conformally flat ball-homogeneous space with at most three distinct Ricci roots is locally symmetric.*

Next, we turn to the case of a cyclic-parallel Ricci tensor and prove

Theorem 4.6 *(M^3, g) is locally isometric to a naturally reductive homogeneous space if and only if it is ball-homogeneous and has cyclic-parallel Ricci tensor.*

Proof. First, let (M^3, g) be locally isometric to a naturally reductive homogeneous space. Then it is a D'Atri space and so, it is clearly ball-homogeneous and ϱ is cyclic-parallel.

To prove the converse, we shall first show that (M^3, g) is a curvature homogeneous space. Since ϱ is cyclic-parallel, we have

$$2\alpha(\varrho) = -||\nabla\varrho||^2 \tag{4.8}$$

and moreover,

$$\sum_{i,j,k} \varrho_{jk}\nabla^2_{ij}\varrho_{ik} = -\frac{1}{4}\Delta||\varrho||^2 + \frac{1}{2}||\nabla\varrho||^2 . \tag{4.9}$$

Since $||\nabla\varrho||^2$ is constant, we get

$$||\nabla\varrho||^2 = 6\check{\varrho} - 5\tau||\varrho||^2 + \tau^3 . \tag{4.10}$$

Hence, the constancy of τ and $||\varrho||^2$, together with (4.8), (4.10) and Proposition 4.1, imply that $\check{\varrho}$ is constant. So, the eigenvalues of ϱ are constant and (M^3, g) is curvature homogeneous.

If ϱ has less than three distinct eigenvalues, the required result then follows from Theorem 4.2. If ϱ has three distinct eigenvalues, we have, since it is cyclic-parallel,

$$2\sum_{a,b} \varrho_{ab}\nabla_b\varrho_{ia} = -\sum_{a,b} \varrho_{ab}\nabla_i\varrho_{ab} = -\frac{1}{2}\nabla_i||\varrho||^2 = 0 .$$

Then, a theorem of K. Yamato [Y] implies that (M, g) is locally homogeneous and the conclusion follows from the fact that a three-dimensional locally homogeneous space with cyclic-parallel Ricci tensor is locally isometric to a naturally reductive homogeneous space [AGV].

∎

5 Four-dimensional ball-homogeneous Einstein spaces

Jensen's result about locally homogeneous Einstein spaces of dimension four shows that those spaces are locally symmetric. The same result holds for the broader class of four-dimensional curvature homogeneous Einstein spaces (see for example [SV]). Recently, K.P. Tod proved that the same property holds for four-dimensional Einstein D'Atri spaces [To]. We investigated whether it is possible to extend these results to the broader class of ball-homogeneous spaces. We now describe our results [CTV].

Let (M, g) be an Einstein manifold of dimension four which we suppose to be connected. As it is well-known, at each point $m \in M$ there exists a *Singer-Thorpe basis* $\{e_1, e_2, e_3, e_4\}$ for the tangent space T_mM, that is, an orthonormal basis such that the components of R are given by

$$(5.1)\qquad \begin{cases} R_{1212} = R_{3434} = a, \ R_{1313} = R_{2424} = b, \ R_{1414} = R_{2323} = c\,, \\ R_{1234} = \alpha, \ R_{1342} = \beta, \ R_{1423} = \gamma\,, \\ R_{ijkl} = 0 \text{ whenever three of the indices } i, j, k, l \text{ are distinct.} \end{cases}$$

With respect to a Singer-Thorpe basis, we have the following formulas for the scalar curvature invariants τ, $||R||^2$, $\check{R}$ and $\check{\check{R}}$:

$$(5.2)\qquad \begin{aligned} &\tau = 4(a+b+c)\,, \\ &||R||^2 = 8(a^2+b^2+c^2+\alpha^2+\beta^2+\gamma^2)\,, \\ &\check{R} = 16(a^3+b^3+c^3+3a\alpha^2+3b\beta^2+3c\gamma^2)\,, \\ &\check{\check{R}} = 24(abc+a\beta\gamma+b\alpha\gamma+c\alpha\beta)\,. \end{aligned}$$

An Einstein manifold (M, g) is called a *2-stein space* if $F(x) = \Sigma_{a,b} R^2_{xaxb}$ is independent of the unit vector $x \in T_mM$, for all $m \in M$. For a four-dimensional Einstein manifold we have

Lemma 5.1 *Let (M, g) be a four-dimensional Einstein manifold. Then (M, g) is a 2-stein space if and only if*

$$(5.3)\qquad \pm\,\alpha = a - \tau/12\ , \ \ \pm\beta = b - \tau/12\ , \ \ \pm\gamma = c - \tau/12$$

for each $m \in M$.

Moreover, a four-dimensional manifold is 2-stein if and only if it is a pointwise Osserman space. We recall that a Riemannian manifold is called a *globally*

Osserman space if the eigenvalues of the Jacobi operator $R_x = R(x, \cdot)x$ are independent of $m \in M$ and of the unit vector $x \in T_m M$. It is called a *pointwise Osserman space* if the eigenvalues of R_x only depend on m and not on x. A four-dimensional globally Osserman space is flat or locally isometric to a rank-one symmetric space. See [GSV], [BTV] for more details and references.

Using Proposition 1.1 and some well-known formulas about the scalar curvature invariants of an Einstein manifold, we have

Lemma 5.2 *Let (M, g) be a four-dimensional ball-homogeneous Einstein space. Then $||R||^2$ and $||\nabla R||^2$ are constant on M.*

We are now ready to prove

Theorem 5.3 *Let (M, g) be a four-dimensional 2-stein space. If (M, g) is ball-homogeneous, then (M, g) is flat or locally isometric to a rank-one symmetric space.*

Proof. With respect to a Singer-Thorpe basis, we have

$$\text{(5.4)} \qquad \begin{cases} a+b+c = \tau/4\,, \\ ab+bc+ca = -(1/32)||R||^2 + (5/192)\tau^2\,, \\ 96abc = \tau^3/12 - (1/6)\tau||R||^2 + (1/6)||\nabla R||^2\,. \end{cases}$$

Since τ, $||R||^2$ and $||\nabla R||^2$ are constant (Lemma 5.2), it follows that a, b and c are constant on M. So, since (M, g) is pointwise Osserman, it is a globally Osserman space and this completes the proof.

∎

We note that the hypothesis of ball-homogeneity can be replaced by the condition "*$||\nabla R||^2$ is constant*".

We consider now four-dimensional Kähler-Einstein spaces. If (M, g, J) is such a space and $m \in M$, then there exists an *adapted* Singer-Thorpe basis at m, that is, a Singer-Thorpe basis $\{e_1, e_2, e_3, e_4\}$ of $T_m M$ such that $e_2 = Je_1$ and $e_4 = Je_3$. We have

Theorem 5.4 *Let (M, g) be a four-dimensional Kähler-Einstein space. If (M, g) is ball-homogeneous, then (M, g) is locally symmetric.*

Proof. Using an adapted Singer-Thorpe basis, we obtain

$$\text{(5.5)} \qquad \begin{cases} \tau = 4(a+b+c)\,, \\ ||R||^2 = 8(a^2+3b^2+3c^2+2bc)\,, \\ \check{\check{R}} = 48bc(a-b-c)\,. \end{cases}$$

Further, expressing $\check{\check{R}}$ by means of τ^3, $\tau||R||^2$ and $||\nabla R||^2$, we have

$$(5.6)\qquad \begin{cases} \tau = 4(a+b+c)\,, \\ ||R||^2 = 8(a^2+3b^2+3c^2+2bc)\,, \\ \tau^3/48 - (\tau/24)||R||^2 + (1/6)||\nabla R||^2 = 48bc(a-b-c)\,. \end{cases}$$

Now, we substitute a from the first equation into the other equations to get

$$(5.7)\qquad \begin{cases} (b+c)^2 - (\tau/8)(b+c) - bc = \text{const.}, \\ bc(\tau/8 - b - c) = \text{const.} \end{cases}$$

since τ, $||R||^2$ and $||\nabla R||^2$ are constant on M. It follows that bc and $b+c$ are constant, that is, b and c, and hence also a, are constant on M. Then, (M,g) is curvature homogeneous and so, it is locally symmetric.

∎

Finally, using Theorem 5.3 and Theorem 5.4, we obtain

Theorem 5.5 *Let (M,g,J) be a connected four-dimensional Hermitian Einstein space. If (M,g) is ball-homogeneous, then (M,g) is locally symmetric.*

Proof. The proof follows essentially the same method as in [CSV] where a similar result is proved for D'Atri spaces. Let τ^* denote the *$*$-scalar curvature* of (M,g,J), that is, $\tau^* = \text{tr}Q^*$ where $g(Q^*x,y) = \varrho^*(x,y) = -(1/2)\text{tr}(z \mapsto R(x,Jy)Jz$, and let $H(x)$ be the *holomorphic sectional curvature* of the holomorphic plane determined by x. The form $\omega = \delta\Omega \circ J$, Ω being the *Kähler form* of (M,g,J), is related to τ and τ^* by the formula

$$(5.8)\qquad \tau - \tau^* = 2\delta\omega + ||\omega||^2\,.$$

Note that $\tau = \tau^*$ for the Kählerian case. In [CTV], we proved that on any four-dimensional ball-homogeneous Hermitian-Einstein space we have

$$(5.9)\qquad (\tau - 3\tau^*)(\tau + 3\tau^*)\omega(x) = 0\,.$$

Now, put

$$\begin{aligned} M_0 &= \left\{ m \in M \;/\; (\tau - 3\tau^*)(\tau + 3\tau^*) \neq 0 \text{ at } m \right\}, \\ M_1 &= \left\{ m \in M \;/\; \tau + 3\tau^* = 0 \text{ at } m \right\}, \\ M_2 &= \left\{ m \in M \;/\; \tau - 3\tau^* = 0 \text{ at } m \right\}. \end{aligned}$$

If $M_0 \neq \varnothing$, then, since $\omega(x) = 0$ on M_0, (M_0,g,J) is Kählerian and hence locally symmetric (Theorem 5.4). Then $\tau = \tau^* = \text{const.}$. So, $M_0 = \bar{M}_0$ and hence $M_0 = M$.

Next, if $M_0 = \varnothing$, we have $M = M_1 \cup M_2$. First, we consider the case $\tau \neq 0$. Then, $M_1 \cap M_2 = \varnothing$ and hence $M = M_1$ or $M = M_2$, since M is connected. Now, assume $M = M_2$. Then from the formula [CSV]

$$\begin{aligned} (5.10) \qquad ||R||^2 g(x,x)g(x,x) &= 16\,F(x) + 2\,(\tau - 3\tau^*)\,H(x) + \\ &\quad + (1/4)(3\tau^{*^2} - \tau^2)g(x,x)g(x,x)\,, \end{aligned}$$

for all $m \in M$ and $x \in T_m M$, we may conclude that (M, g) is a 2-stein space and hence, it is locally symmetric (Theorem 5.3). Next, let $M = M_1$. In this case we can construct an adapted Singer-Thorpe basis $\{e_1, e_2, e_3, e_4\}$ of $T_m M$ and with respect to such a basis we may prove, since τ, $||R||^2$ and $||\nabla R||^2$ are constant, that a, b and c are constant. So, (M, g) is curvature homogeneous and hence locally symmetric.

Finally, if $\tau = 0$, from (5.9) we get $\tau^* = 0$. Then (5.10) implies that (M, g) is 2-stein and thus locally symmetric.

∎

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CONSTANT MEAN CURVATURE SURFACES IN THE EUCLIDEAN SPACE

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Abstract.– We give an introduction to the subject of constant mean curvature surfaces; we first introduce them from the variational problem which they solve and prove Hopf theorem, then via the sinh-Gordon equation, we study periodic constant mean curvature surfaces showing the ideas which brought to the disproof of Hopf conjecture.

1 Introduction

This paper is intended as a brief review of the main recent results in the vast subject of constant mean curvature surfaces. Such surfaces can be introduced as solutions of a variational problem: if we look for closed surfaces, whose area is critical under deformations that keep the enclosed volume constant, we find that the differential equation which characterizes locally those surfaces is H =constant. Such surfaces are known as *soap bubbles*, since a soap film in equilibrium between two regions of different pressure, and subject only to the forces induced by the pressure and the surface tension, has critical area for deformations that mantain the enclosed volume fixed.

The problem of the existence of constant mean curvature immersions has been a crucial one in differential geometry for a long time. A constant mean curvature immersion in fact is the most appealing among all the immersions of a closed surface in $\mathbb{E}^3$, since such a surface cannot have a minimal immersion. Let us follow the generally adopted notation, and indicate by CMC surface a non-zero constant mean curvature, immersed, closed, smooth surface in $\mathbb{E}^3$, and by CMC immersion the corresponding immersion. We want to mention in this context the old result by Jellet ([Jel]) who in 1853 showed that star-shaped CMC surfaces are round spheres; Hopf demonstrated the same for topological CMC spheres, and Alexandrov for embedded CMC surfaces. More recently Barbosa and do Carmo ([B-dC]) proved that local minimizers of the variational problem are round spheres. Because of those results, it was concievable to believe that round spheres are the only CMC surfaces: this is what has been known as *Hopf*

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conjecture. This conjecture has been disproved, in higher dimension, by Hsiang ([Hs]) who constructed a counterexample in $\mathbb{E}^4$, and then by Wente ([We1]) who, surprisingly, produced a CMC immersion of a compact oriented surface of genus 1 in $\mathbb{E}^3$. Finally Kapouleas ([K1, K2]) achieved the construction of CMC surfaces of every genus $g \geq 2$. The genus 1 case has been deeply analized, and we now have a classification of CMC tori ([P-S, Bo, Ab1, Ab2, We2]). In another direction, the author recently proved the existence of CMC surfaces embedded in $\mathbb{R}^3$ endowed with a conformally flat metric, which are not spheres ([Co]).

2 First variation formula

Let us consider the general case of an immersed manifold:

$$\psi : \mathcal{M}^n \to \mathcal{N}^{n+k}.$$

Assuming $\mathcal{N}$ endowed with a riemannian metric h, we can introduce two main geometric invariants of the immersion:

(1) the first fundamental form g which is the restriction of the metric h to $\mathcal{M}$: $g = \psi_* h$

(2) the second fundamental form A which involves first order differentiations.

To define A we need to introduce on $\mathcal{M}$ the connection ∇ induced by the Levi-Civita connection ∇' of the ambient space $\mathcal{N}$. A priori there are two different connections on $\mathcal{M}$:

(1) ∇=Levi-Civita connection induced by the first fundamental form g of $\mathcal{M}$

(2) $(\nabla')^T$=projection of ∇' on $T\mathcal{M}$.

Those two connections agree, because both are symmetric and compatible with the metric of $\mathcal{M}$, so both are the Levi-Civita connection associated to g on $\mathcal{M}$, and by uniqueness they need to be the same. Hence

$$\forall X, Y \in T\mathcal{M} \qquad \nabla_X Y = (\nabla'_X Y)^T.$$

Using the orthogonal decomposition of the tangent space to $\mathcal{N}$ as the direct sum of the spaces tangent and normal to $\mathcal{M}$

$$T_p\mathcal{N} = T_p\mathcal{M} \oplus N_p\mathcal{M},$$

∇' gets decomposed as

$$\nabla'_X Y = (\nabla'_X Y)^T + (\nabla'_X Y)^\perp, \qquad X, Y \in T\mathcal{M}.$$

We define the second fundamental form of the immersion as the symmetric bilinear map $A : T\mathcal{M} \times T\mathcal{M} \to N\mathcal{M}$ given by

$$A(X,Y) = (\nabla'_X Y)^{\perp}, \qquad \text{for all} \quad X, Y \in T\mathcal{M}.$$

so that

$$\nabla'_X Y = \nabla_X Y + A(X,Y).$$

Using $x^1, \dots, x^n$ an orthonormal coordinate system around p, and $\partial_1, \dots, \partial_n$ the corresponding orthonormal frame for $T_p\mathcal{M}$, we write:

$$A_{ij} = A(\partial_i, \partial_j)$$

and define the mean curvature vector field

$$H = \frac{1}{n}\text{trace}_g(A) = \frac{1}{n} g^{ij} A_{ij}$$

(the convention to sum over repeated indeces is always assumed). H is a smooth vector field of vectors normal to $\mathcal{M}$.

By a smooth variation of ψ we mean a C^∞ mapping: $\Psi : \mathbb{R} \times \mathcal{M} \to N$ such that each $\psi_t = \Psi(t, \cdot) : \mathcal{M} \to N$ is an immersion, and $\Psi(0,x) = \psi(x)$. We will consider each ψ_t defined only locally in a neighbourhood of a point p of $\mathcal{M}$ (for x outside this neighbourhood, we will set $\psi_t(x) = \psi(x)$).

If $(\mathcal{M}, g)$ is a Riemannian manifold, we have a notion of volume on it:

$$\text{vol}(\psi(\mathcal{M})) = \int \sqrt{\det g_{ij}} dx^1 \dots dx^n$$

where $x^1, \dots, x^n$ are local coordinates around p, $\partial_i = \partial / \partial x^i, i = 1, \dots, n$ is a (not necessarily orthonormal) basis for the tanget space $T_p\mathcal{M}$ and $dx^i, i = 1 \dots, n$ is the dual basis for the cotangent space $T_p\mathcal{M}^*$. Calling dV_t the volume element of the immersion ψ_t, we have:

$$dV_t = \sqrt{\det g_{ij}(t)} dx^1 \dots dx^n$$

hence

$$\text{vol}(\psi_t(\mathcal{M})) = \int \sqrt{\det g_{ij}(t)} dx^1 \dots dx^n$$

where $g(t)$ is the first fundamental form of the immersion ψ_t.

First Variation Formula *The volume of $\psi_t(\mathcal{M})$ and the mean curvature H of $\psi(\mathcal{M}) = \psi_0(\mathcal{M})$ are related by the following formula:*

$$\frac{\partial}{\partial t}(vol(\psi_t(\mathcal{M})))|_{t=0} = -n \int < H, V > dV_0$$

where $V = \psi_*(\frac{\partial}{\partial t})|_{t=0}$, dV_0 *is the volume element of* $\psi_0(\mathcal{M})$ *and* $< \cdot, \cdot >$ *denotes the inner product of the riemannian structure of* N.

Proof. To simplify the notation let us abbreviate $\mathrm{vol}(\psi_t(\mathcal{M})) = \mathrm{vol}(\psi_t)$.

$$\begin{aligned}\frac{\partial}{\partial t}(\mathrm{vol}(\psi_t))|_{t=0} &= \int \partial_t \sqrt{\det g_{ij}(t)}|_{t=0} dx^1 \dots dx^n \\ &= \int \frac{1}{2\sqrt{\det(g_{ij}(0))}} \partial_t (\det g_{ij}(t))|_{t=o} dx^1 \dots dx^n.\end{aligned}$$

Now

$$\begin{aligned}\partial_t(\det g_{ij}(t))|_{t=0} &= \mathrm{trace} \partial_t g_{ij}(t)|_{t=o} \\ &= g^{ij}(0)(\partial_t g_{ij}(t))|_{t=0} \det g_{ij}(0)\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial}{\partial t}(\mathrm{vol}(\psi_t))|_{t=0} &= \frac{1}{2}\int g^{ij}(0)(\partial_t g_{ij}(t))|_{t=0}\sqrt{\det g_{ij}(0)} dx^1 \dots dx^n \\ &= \frac{1}{2}\int g^{ij}(\partial_t g_{ij}(t))|_{t=0} dV_0.\end{aligned}$$

So we need to compute $\partial_t g_{ij}(t)|_{t=0}$:

$$\begin{aligned}\partial_t g_{ij}(t)|_0 &= \partial_t < (\psi_t)_* \partial_i, (\psi_t)_* \partial_j >_{|t=0} \\ &= (\psi_t)_* \partial_t|_{t=0} < \partial_i, \partial_j > \\ &= V < \partial_i, \partial_j > \\ &= < \nabla'_V \partial_i, \partial_j > + < \partial_i, \nabla'_V \partial_j >\end{aligned}$$

where ∇' is the Levi-Civita connection on the manifold N, and the last equality holds because of the compatibility of ∇' with the metric of N.

We can interchange the order of covariant differentiation:

$$\nabla'_V \partial_i = \nabla'_{\partial_i} V$$

because

$$\nabla'_V \partial_i - \nabla'_{\partial_i} V = [V, \partial_i] = \psi_*[\partial_t, \partial_i] = 0.$$

This yields

$$\partial_t g_{ij}(t)|_{t=0} = \partial_i < V, \partial_j > - < V, \nabla'_{\partial_i}, \partial_j > + \partial_j < \partial_i, V > - < \nabla'_{\partial_j} \partial_i, V > .$$

To simplify the computation, we will assume that V is a normal variation, that is $< V, \partial_i >= 0$, but the formula does hold for any variation field. This gives:

$$\begin{aligned}\partial_t g_{ij}(t)|_{t=0} &= - < V, \nabla'_{\partial_i} \partial_j > - < \nabla'_{\partial_j} \partial_i, V > \\ &= -2 < \nabla'_{\partial_i} \partial_j, V >= -2 < (\nabla'_{\partial_i} \partial_j)^{\perp}, V > \\ &= -2 < A(\partial_i, \partial_j), V >= -2 < A_{ij}, V > .\end{aligned}$$

We are able to eliminate the tangent component of $\nabla'_{\partial_i}\partial_j$ since the vector field V is assumed to be orthogonal to $T_p\mathcal{M}$.

All of this finally gives:

$$\begin{aligned}\partial_t \mathrm{vol}(\psi_t)|_{t=0} &= \frac{1}{2}\int \frac{g^{ij}}{\sqrt{\det g_{ij}(0)}}(-2 < A_{ij}, V >) \det g_{ij}(0) dx^1 \dots dx^n \\ &= -\int < g^{ij}A_{ij}, V > dV_0 \\ &= -n\int < H, V > dV_0.\end{aligned}$$

□

An immersion is *critical for the volume* if $\partial_t \mathrm{vol}(\psi_t)|_{t=0} = 0$ for all variations ψ_t. This is equivalent to the system of non-linear elliptic partial differential equations $H = 0$. Any immersion satisfying $H = 0$ is called *minimal.*

Proposition 2.1 *If $X : \mathcal{M}^n \to \mathbb{E}^m$ is an immersion in the Euclidean m-space, the mean curvature vector H of this immersion satisfies*

$$nH = \Delta_g X$$

where Δ_g is the Laplace-Beltrami operator for the metric g on $\mathcal{M}$.

Proof. We consider the immersion $X = (x^1, \dots, x^m)$ as a vector $X \in T_p\mathbb{E}^m \doteq \mathbb{E}^m$. Then the Laplace-Beltrami operator on X is defined by

$$\Delta_g X = g^{ij}(\partial_i\partial_j X - (\nabla_{\partial_i}\partial_j)X).$$

Now

$$(\nabla_{\partial_i}\partial_j)X = X_*(\nabla_{\partial_i}\partial_j)$$
$$\partial_i\partial_j X = \bar{\nabla}_{\partial_i}\partial_j$$

with ∇ the Levi-Civita connection of $\mathcal{M}$ and $\bar{\nabla}$ the Euclidean connection. So we find:

$$\begin{aligned}\Delta_g X &= g^{ij}(\bar{\nabla}_{\partial_i}\partial_j - X_*(\nabla_{\partial_i}\partial_j)) \\ &= g^{ij}(\bar{\nabla}_{\partial_i}\partial_j)^\perp = g^{ij}A_{ij} = nH\end{aligned}$$

since $X_*(\nabla_{\partial_i}\partial_j) = (\bar{\nabla}_{\partial_i}\partial_j)^T$. □

Corollary 2.1 *An immersion $X : \mathcal{M} \to \mathbb{E}^m$ is minimal if and only if $\Delta_g X = 0$, that is to say the coordinate functions are harmonic.*

Corollary 2.2 *There is no minimal immersion of $\mathcal{M}^n$ $(n \geq 1)$ into $\mathbb{E}^m$ if $\mathcal{M}^n$ is closed (compact with empty boundary).*

Proof. By Hopf maximun principle a harmonic map on a closed set has to be constant. But if the coordinate functions of $\mathcal{M}$ are constant, $\mathcal{M}$ is mapped into a single point, hence it is not immersed. □

3 Hopf Theorem

In codimension 1, we want to study a more general problem: find the condition under which $\partial\Omega = \mathcal{M}$ has critical area, subject to variations that keep the volume of $\Omega \subset \mathcal{N}$ constant. Consider the unit vector ν normal to $\partial\Omega$, oriented by having it pointing ouside Ω. Since the codimension is 1, the dimension of the normal vector space $N_p\mathcal{M}$ is 1, hence ν is a basis, and the curvature vector field is a multiple of ν:

$$H = h\nu, \qquad \text{with} \quad h \quad \text{a smooth function.}$$

Consider a normal variation V of Ω: then also V can be written as a C^∞ multiple of ν

$$V = f\nu$$

so that by the first variation formula

$$\frac{d}{dt}\text{Area}(\partial\Omega_t)|_{t=0} = -n\int_{\mathcal{M}} fhdV_g.$$

We look for a variation V that is a critical point for the area and keeps the volume constant, i.e. that satisfies:

$$\frac{d}{dt}\text{Area}(\partial\Omega_t)|_{t=0} = -n\int_{\mathcal{M}} fhdV_g = 0$$

$$\frac{d}{dt}\text{vol}(\Omega_t)|_{t=0} = \int_{\mathcal{M}} fdV_g = 0.$$

Assuming $\partial\Omega$ closed, or taking compactly supported variations, the condition to have critical area with fixed volume is then H =constant, meaning that the function h is constant.

We will now study the case of surfaces immersed in the Euclidean space with constant mean curvature H, i.e. we are going to restrict our attention to immersions:

$$X : \mathcal{M}^2 \to \mathbb{E}^3$$

$$H = 1$$

(the choice of the constant as value for H is non restrictive, as a change of orientation changes the sign of H, and any non-zero value is obtainable via a homothetic expansion).

Let us fix a Riemann surface structure on $\mathcal{M}^2$, with $z = u + iv$ the local coordinate. We may define the *Hopf differential*

$$\Phi =< \frac{d^2X}{dz^2}, \nu > dz^2$$

with

$$\frac{d}{dz} = \frac{1}{2}(\frac{d}{du} - i\frac{d}{dv})$$

and ν the unit normal vector to $\mathcal{M}$, which is fixed by the orientation ($\mathcal{M}$ has to be oriented because $H \neq 0$). It turns out that:

$$\begin{aligned}
\Phi &= \frac{1}{4} < \frac{\partial^2 X}{\partial u^2} - \frac{\partial^2 X}{\partial v^2} - 2i\frac{\partial^2 X}{\partial u \partial v}, \nu > dz^2 \\
&= \frac{1}{4} < \bar{\nabla}_{\partial_u}\partial_u - \bar{\nabla}_{\partial_v}\partial_v - 2i\bar{\nabla}_{\partial_u}\partial_v, \nu > dz^2 \\
&= \frac{1}{4}\left(A(\partial_u, \partial_u) - A(\partial_v, \partial_v) - 2iA(\partial_u, \partial_v)\right) dz^2 \\
&= \frac{1}{4}(A_{11} - A_{22} - 2iA_{12})dz^2
\end{aligned}$$

using $\bar{\nabla}$ the connection on $\mathbb{E}^3$. Notice that, being the unit normal vector ν fixed, the coefficients A_{ij} of the second fundamental form of the immersion are functions on $\mathcal{M}$. They are real-valued functions, while Φ is a complex differential:

$$\operatorname{Re}\Phi = \frac{1}{4}(A_{11} - A_{22}), \qquad \operatorname{Im}\Phi = -\frac{1}{2}A_{12}.$$

The crucial observation of Hopf is that when H is constant, holomorphicity of Φ is equivalent to the Codazzi equations for the immersion. Φ is holomorphic if it satisfies Cauchy-Riemann equations:

$$\begin{aligned}
\partial_1(A_{11} - A_{22}) &= -2\partial_2 A_{12} \\
\partial_2(A_{11} - A_{22}) &= 2\partial_1 A_{12}.
\end{aligned}$$

We may replace the usual derivatives with covariant derivatives, since the Christoffel symbols cancel. We are going to use the notation $X_{;i}$ to mean covariant derivative of X with respect to ∂_i, so Cauchy- Riemann equations for Φ can be written

$$(A_{11} - A_{22})_{;1} = -2A_{12;2} \qquad (A_{11} - A_{22})_{;2} = 2A_{12;1}.$$

On the other hand, recall Codazzi equations:

$$< R(X,Y)Z, \xi > = < \nabla_X A(Y,Z) - \nabla_Y A(X,Z), \xi >, \qquad X, Y, Z \in T\mathcal{M}, \xi \in N\mathcal{M}$$

which in the Euclidean (flat) space read as the full symmetry of A:

$$A_{ij;k} = A_{ik;j}.$$

In the case of a 2-dimensional immersion, there are only 2 indeces, so Codazzi equations are:

$$A_{11;2} = A_{12;1} \qquad A_{22;1} = A_{21;2}.$$

Now from the constancy of H one has $(A_{11}+A_{22})_{;i}=0$ which gives

$$(A_{11}-A_{22})_{;1}=(A_{11}-A_{22})_{;1}-(A_{11}+A_{22})_{;1}=-2A_{22;1}.$$

This, together with the first Cauchy-Riemann equation, finally shows

$$A_{22;1}=A_{12;2}=A_{21;2}$$

that is the second Codazzi equation, and the first one is obtained in an analogous way.

Hopf Theorem *An immersed C^∞ sphere in $\mathbb{E}^3$ with H=constant has to be the round sphere* ([Ho]).

Proof. The steps in this proof are the following:

(1) An immersed surface has to satisfy Codazzi equations, and, as already noticed, since H is constant this equals to the holomorphicity of Hopf differential Φ. Moreover, a holomorphic quadratic differential on $\mathcal{M}^2$ is zero.

(2) From $\Phi=0$ we will establish the relation $A_{ij}=hg_{ij}$ (with h the mean curvature function): this is the strong part, since this property is enjoyed only by spheres (we will invoke the fundamental theorem of surface theory of Bonnet to show it).

Proof of 1. The Riemann surface structure of $\mathcal{M}^2$ is like that of the unit sphere $S^2(1)$, having two charts with two coordinate functions z, w satisfying, in the intersection of the charts, $w=1/z$. Then Hopf differential can be written, in the intersection of the two charts

$$\Phi=\phi(z)dz^2=\psi(w)dw^2$$

with ϕ,ψ holomorphic functions.

Since $dw=-\frac{1}{z^2}dz$, is

$$\phi(z)dz^2=\psi(1/z)(1/z^4)dz^2$$

and $\phi(z)=\psi(1/z)(\frac{1}{z^4})$ is an entire function on $\mathbb{C}$, bounded at ∞ since

$$\lim_{z\to\infty}\phi(z)=\lim_{z\to\infty}\psi(1/z)(1/z^4)=\psi(0)\cdot 0=0$$

being ψ well defined at 0. But then $\phi:\mathbb{C}\to\mathbb{C}$ an entire function bounded at infinity is, by Liouville theorem, a constant. Moreover this constant has to be 0 since this is the value ϕ assumes at ∞.

Proof of 2.

We actually proved that every holomorphic quadratic differential on $\mathcal{M}$ is zero. Now let us restrict to Hopf differential

$$\Phi=(A_{11}-A_{22}-2iA_{12})dz^2.$$

$\Phi \equiv 0$ means

$$A_{11} - A_{22} = 0, \qquad A_{12} = 0$$

that is

$$A_{ij} = \tilde{C}\delta_{ij} = Cg_{ij}.$$

C is just the mean curvature function h (so it is a constant):

$$h = \frac{1}{2}\text{trace}_g A = \frac{1}{2}g^{ij}A_{ij} = \frac{1}{2}C\text{trace}_g g = C \quad .$$

since $\text{trace}_g g = 2$.

Hence

$$A_{ij} = hg_{ij}.$$

This property specifies the surface $\mathcal{M}$ to be the standard unit sphere $S^2(1)$, as it can be seen by applying the following

Fundamental Theorem of Surface Theory (Bonnet) *Given g_{ij} and A_{ij} two forms on an open simply connected set $V \subset \mathbb{R}^2$ which satisfy Gauss and Codazzi equations, then there exists a (unique up to Euclidean motion) immersion $\phi : V \to \mathbb{E}^3$ such that g_{ij} and A_{ij} are the coefficients of the first and second fundamental forms, respectively* ([DoC1, pag.236]).

In fact, $A_{ij} = hg_{ij}$ gives $\det A = h^2 \det g$, so Gauss equation is

$$K = \frac{\det A}{\det g} = h^2 \qquad \text{a positive constant.}$$

This establishes the intrinsic geometry of $\mathcal{M}^2$: it has to be isometric to the sphere $S^2(r)$ of radius r. But by uniqueness in Bonnet theorem, the immersion of $\mathcal{M}^2$ and that of $S^2(r)$ are the same up to Euclidean motion, so $\mathcal{M}^2 = S^2(r)$. It is now enough to rescale to obtain $H = 1 = K$ and so $r = 1$. □

Notice that Hopf Theorem is essentially 2-dimensional, since to define Hopf differential is not possible in higher dimension. An analogous result valid in any dimension is

Alexandrov Theorem *The only compact embedded hypersurfaces $\mathcal{M}^n$ of constant mean curvature in $\mathbb{E}^{n+1}$ are the standard round spheres S^n* ([Al]).

Let us observe that the generalization to higher dimension requires the extra hypothesis of *embeddedness* which replaces that of *immersed.*

4 CMC Tori

Let us consider a torus $T^2 = \mathbb{C}/\text{lattice}$, where $\mathbb{C}$ is equipped with the standard complex structure for which $z = u + iv$ is a coordinate. Hopf differential can be written $\Phi = \phi dz^2$, with ϕ a doubly periodic function (since it is defined on the torus). Now, if the mean curvature is constant, ϕ is holomorphic and hence it is a constant by Liouville theorem. Moreover we may assume $\phi \neq 0$, otherwise we would have a sphere instead of a torus, since we saw that from the condition $\Phi \equiv 0$ specifies the intrinsic geometry of spheres. So without loss of generality

$$\Phi = -\frac{1}{2}dz^2.$$

Consider the immersion $T^2 \to \mathbb{E}^3$ with first and second fundamental forms

$$g = \frac{1}{4}e^{2w}(du^2 + dv^2),$$

$$A = \frac{1}{4}(e^{2w} - 1)du^2 + \frac{1}{4}(e^{2w} + 1)dv^2,$$

where (u, v) are the standard coordinates of $\mathbb{R}^2$, $w : \mathbb{R}^2 \to \mathbb{R}$ a smooth function. The mean curvature for this immersion is constant:

$$\begin{aligned} H &= \frac{1}{2}\text{trace}A = \frac{1}{2}\left(g^{11}A_{11} + g^{22}A_{22}\right) \\ &= \frac{1}{2}\left(\frac{e^{2w}-1}{4}\frac{4}{e^{2w}} + \frac{e^{2w}+1}{4}\frac{4}{e^{2w}}\right) = 1, \end{aligned}$$

so the hypothesis $\Phi = -\frac{1}{2}dz^2$ is consistent, and gives

$$\Phi = (A_{11} - A_{22} - 2iA_{12})dz^2 = -\frac{1}{2}dz^2$$

hence

$$\begin{aligned} A_{11} - A_{22} &= -\tfrac{1}{2} \\ A_{12} &= 0 \end{aligned}$$

so that the matrix A is diagonal, with $A_{11} = A_{22} - 1/2$. This is satisfied by the given A:

$$\begin{aligned} A_{22} - \frac{1}{2} &= \frac{e^{2w}+1}{4} - \frac{1}{2} \\ &= \frac{e^{2w}-1}{4} = A_{11}. \end{aligned}$$

So we found A and g so that we can have the expression $\Phi = -1/2dz^2$ for the Hopf differential. We now want to find conditions on the unknown function w in such a way to be able to integrate to get the immersion. Since $\mathbb{C}$ is simply connected, given g and A on $\mathbb{C}$ it exists a (unique up to rigid motions) immersion $X : \mathbb{C} \to \mathbb{E}^3$ that has them as first and second fundamental forms if and only if they satisfy Gauss and Codazzi equations. Codazzi equations holds since, being

H constant, its validity is equivalent to the holomorphicity of Φ. Let us the examine Gauss equation:

$$K = \frac{\det A}{\det g} \qquad \text{(ambient space is flat)}.$$

Being:

$$\det A = A_{11}A_{22} = \frac{1}{16}(e^{2w} - 1)(e^{2w} + 1)$$

$$\det g = g_{11}g_{22} = \frac{1}{16}e^{2w}e^{2w}.$$

Gauss equation reads

$$K = \frac{e^{4w} - 1}{e^{4w}} = 1 - e^{-4w}.$$

On the other hand, in terms of isothermal parameters $g_{ij} = \lambda^2\delta_{ij}$ it is valid the formula

$$K = -\frac{\Delta \log \lambda}{\lambda^2}$$

which in our case becomes

$$K = -\frac{4\Delta w}{e^{2w}}.$$

Confronting the two expressions for the Gauss curvature K we get

$$\frac{e^{2w} - e^{-2w}}{2} = -\Delta(2w).$$

Hence Gauss equation becomes the sinh-Gordon equation

$$\Delta(2w) + \sinh(2w) = 0$$

which is equivalent to

$$\Delta w + \sinh w \cosh w = 0.$$

What we obtain is the following

Proposition 4.1 *Given a smooth function $w : \mathbb{R}^2 \to \mathbb{R}$ which solves*

$$\Delta w + \sinh w \cosh w = 0$$

there is a unique up to Euclidean motion immersion $X : \mathbb{R}^2 \to \mathbb{E}^3$ with first and second fundamental forms given by

(1)
$$\begin{aligned} g &= \tfrac{1}{4}e^{2w}(du^2 + dv^2) \\ A &= \tfrac{1}{4}(e^{2w} - 1)du^2 + \tfrac{1}{4}(e^{2w} + 1)dv^2. \end{aligned}$$

Moreover, if $\nu : \mathbb{R}^2 \to S^2(1)$ *denotes the Gauss map of the immersion, we have*

$$\begin{aligned} \nu_u &= -(1-e^{-2w})X_u \\ \nu_v &= -(1+e_{-2w})X_v \end{aligned}$$

(see [K2, Prop.1.1]).

Notice that this proposition implies

$$H = 1$$

$$\nu^* g = \sinh^2 w du^2 + \cosh^2 w dv^2.$$

The simplest case is when as solution to

$$\Delta w + \sinh w \cosh w = 0$$

we take the function $w \equiv 0$. This gives

$$g = \frac{du^2 + dv^2}{4}, \qquad A = \frac{dv^2}{2}$$

and hence corresponds to a cylinder.

5 The Delaunay surfaces

These are a continuous family of rotationally invariant surfaces, parametrized by a parameter τ. They correspond to solutions of the sinh-Gordon equation which are translationally invariant, so the two possible cases arise considering either $w = w(u)$ or $w = w(v)$. In those cases, the sinh-Gordon equation reduces to the ordinary differential equation

$$w'' + \frac{1}{2} \sinh 2w = 0$$

which has first integral $w'^2 + \frac{1}{2}\cosh 2w$. The solutions are parametrized by $E \in [\frac{1}{2}, \infty)$, where w_E is the solution corresponding to E, and it satisfies

$$w_E'^2 + \frac{1}{2}\cosh 2w_E = E, \qquad w_E(0) = w_E'(0) > 0.$$

From standard ODE theory we know that w_E is smooth, depends smoothly on E, and is periodic with period $4A_\tau$, for $\tau = (4E+2)^{-1}$. Let us consider $w(u,v) = w_E(u)$. Then by the proposition in the previous section there exists an immersion $X_\tau : \mathbb{R}^2 \to \mathbb{E}^3$ with fundamental forms given by (1).

Let us now try to visualize this immersion, looking at the Gauss map $\nu : \mathbb{R}^2 \to S^2(1)$. Along vertical lines in the uv-plane one has w=constant; because of periodicity, vertical lines go to circles under the Gauss map ν. Moreover, ν respects orthogonality

$$< \nu_* \partial/\partial u, \nu_* \partial/\partial v >= 0$$

so the image under ν of lines in $\mathbb{R}^2$ which are vertical and horizontal, are lines in S^2 that are still orthogonal. The equality

$$\nu_* g = \sinh^2 w du^2 + \cosh^2 w dv^2$$

implies

$$\begin{aligned} |\nu_* \partial/\partial u|^2 &= < \nu_* \partial/\partial u, \nu_* \partial/\partial u >= \sinh^2 w \\ |\nu_* \partial/\partial v|^2 &= < \nu_a st \partial/\partial v, \nu_* \partial/\partial v >= \cosh^2 w. \end{aligned}$$

The vector $\nu_* \partial/\partial u$ measures horizontal velocity; when $\sinh^2 w = 0$ its lenght is zero, and this happens when $w = 0$. The sign of w is the same as the sign of the Gauss curvature K of the immersion, since $K = 1 - e^{-4w}$, so when $w > 0$ the horizontal velocity and the Gauss curvature are positive, at $w = 0$ both velocity and curvature are zero, and then for $w < 0$ both velocity and curvature are negative. The lenght of a circle image under ν of a vertical line is $L = \cosh^2 w \cdot l$ where l is the period along the v direction. This lenght never becomes zero, so in S^2 the circles images of vertical lines of $\mathbb{R}^2$ never reach the (east and west) poles. All of this gives an *embedded* Delaunay surface $D_\tau, \tau > 0$.

Define now $\tau = -(4E-2)^{-1}$, and consider $w(u, v) = w_E(v)$. Then w=constant are horizontal lines, which again go to circle applying ν, but this time their lenght does go to zero, since it is $L = \sinh^2 w \cdot l$ which is zero for $w = 0$. Hence in S^2 the circles images of horizontal lines of $\mathbb{R}^2$ reach the (north and south) poles. The relation between the signs of w and the Gauss curvature K is the same as before, so the difference is that in this case we obtain an *immersed* Delaunay surface $D_\tau, \tau \leq 0$ (see[K1] or the original work by Delaunay [De]).

In the embedded case, if we let $\tau \to 0$, the positively curved regions tend to spheres, while the negatively curved regions tend to points connecting them. If we enlarge those negatively curved regions by a factor of the order of τ^{-1}, we can see that those regions will tend to minimal surfaces which do keep the rotational invariance of the Delaunay, that is to say to catenoids (the only rotationally invariant minimal surfaces).

6 Wente Tori

In this section we will give an idea of the construction of *Wente tori*, the first example of non-spherical CMC surfaces in $\mathbb{E}^3$, which provides a counterexample to the conjecture of Hopf. Wente tori are a 1-parameter family of rotationally invariant, periodic CMC cylinders, which, for a countable dense set of values of the parameter, close up to CMC tori. We have seen that a CMC immersion arises starting from a solution w of the sinh-Gordon equation $\Delta w + \sinh w \cosh w = 0$. The idea is to construct a *doubly periodic* solution, which we may obtain by solving the sinh-Gordon equation on a rectangle with identically zero boundary data. In [We1] Wente examined the case of w defined on a torus $T^2 = \mathbb{C}/\Lambda$ with Λ a lattice generated by orthogonal vectors. In the subsequent paper [We2] examine more general lattices are studied.

Let us define the function $w : \mathbb{R}^2 \to \mathbb{R}$ by

$$w(u,v) = \log \frac{1 + U(u)V(v)}{1 - U(u)V(v)}$$

where $U(u)$ and $V(v)$ are functions of one variable, defined as multiples of the Jacobian elliptic function $cn_k(t)$ (see [Ch, Chp.VII]). U and V are periodic, with periods that we denote by $4u_0$ and $4v_0$ respectively. U is positive on $(-u_0, u_0)$ with a maximum at $u = 0$ and negative on $(u_0, 3u_0)$ with a minimum at $u = 2u_0$ (a similar statement holds for V). $w = w(u,v)$ satisfies the sinh-Gordon equation, and is doubly periodic, with periods $(4u_0, 0)$ and $(0, 4v_0)$.

The fundamental domain of w is the rectangle $(-u_0, 3u_0) \times (-v_0, 3v_0)$; the vertical lines $u = \pm u_0, \pm 3u_0, \pm 5u_0, \cdots$ and the horizontal ones $v = \pm v_o, \pm 3v_0, \pm 5v_0, \cdots$ are lines along which is $w = 0$, and are called *nodal lines.* The function w is odd symmetric under reflection with respect to those lines and even symmetric with respect to the lines $\{u = 2ku_0\}$ and $\{v = 2kv_0\}$, for any integer k. The solution w depends on two parameters, which represent the lenghts of the two sides of the rectangle on which w itself is defined. It can be shown that one of them is not a free parameter, so the fundamental rectangle is $\Omega(\tau)$, with $\tau > 0$. The immersion $X : \mathbb{R}^2 \to \mathbb{E}^3$ induced by w is periodic with period $(4u_0, 0)$ (see [K2, Prop.1.7]). Periodicity implies that X factors through a cylinder, which will be the Wente cylinder of parameter τ, defined on the fundamental rectangle $\Omega(\tau)$.

Let us try to visualize the Wente cylinder, by looking at the Gauss map $\nu : \mathbb{C} \to S^2(1)$ restricted to the fundamental domain of w. The three horizontal nodāl lines $v = -v_0, v = v_0, v = 3v_0$ in the fundamental domains are mapped to points, in fact their lenght in the image is

$$\mathrm{vol}_* g(\partial/\partial u, \partial/\partial u) = \sinh^2 w$$

which is zero when $w = 0$. Call A and B the points which are the image under ν of the two nodal lines that bound the fundamental domain $v = -v_0$ and $v = 3v_0$. The vertical lines go to curves on S^2 joining A and B. Among them, the nodal lines $u = -u_0$ and $u = 3u_0$ go to "short" curves, while the other nodal line $u = u_0$ goes to one of the two parts of the great circle that joins A and B. Since this vertical nodal line is a line of symmetry for w, the plane which cuts S^2 along this great circle is a plane of symmetry. We can see that not all of S^2 is covered by the image of ν: we are then interested in surfaces where A and B are very close, so at least most of S^2 will be covered by ν(fundamental domain). This is obtained by sending to zero the parameter τ, so that the fundamental domain will shrink to a point. If we call $\Omega^{\pm}(\tau)$ a rectangular domain where $w(u,v)$ is positive (respectively negative), we can see that the Gauss curvature of the immersion is positive on Ω^+ and negative on Ω^-. Moreover as $\tau \to 0$

(1) the area of the image surface $X(\Omega^+(\tau))$ approaches 4π

(2) the area of the image surface $X(\Omega^-(\tau))$ approaches zero

(see [Wel, Theorem5.1]). This suggests that the surface $X(\Omega^+(\tau))$ converges towards a sphere of radius 1, while the surface $X(\Omega^-(\tau))$ has negative Gauss curvature and connects the positively curved regions. It has small area and, if appropriately enlarged by a factor of order τ^{-1} it will closely follow an Enneper minimal surface.

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COMPLEX HOMOTOPY THEORY FOR COMPACT NILMANIFOLDS

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Abstract.– Following the Dolbeault homotopy theory introduced by Neisendorfer and Taylor in [22], we consider a minimal model for the Dolbeault complex of a compact nilmanifold $\Gamma\backslash G$ with a *nilpotent complex structure* and show that this model is formal if and only if $\Gamma\backslash G$ is a complex torus; thus a non-toral $\Gamma\backslash G$ has no (positive definite) Kähler metrics. Furthermore, we prove that (complex) tori are the only compact complex parallelizable nilmanifolds with indefinite Kähler metric.

1 Introduction

Let G be a simply-connected connected nilpotent Lie group. A well known theorem of Mal'čev [20] states that if there exists a basis of left invariant 1-forms for which the coefficients in the structure equations of G are rational numbers then there exists a lattice $\Gamma \subset G$ of maximal rank, and hence the quotient manifold $\Gamma\backslash G$ is compact. We call such a manifold *compact nilmanifold.*

Let us suppose the nilpotent Lie group G endowed with a left invariant almost complex structure J; then, the compact nilmanifold $\Gamma\backslash G$ inherits an almost complex structure by passing to the quotient. If the almost complex structure J on G is integrable, then the induced almost complex structure on $\Gamma\backslash G$ is also integrable; in this case we call $\Gamma\backslash G$ *compact complex nilmanifold.*

Compact complex nilmanifolds have deserved special attention from many authors during the last years, because they provide examples of compact complex manifolds possessing interesting unusual properties (see the papers listed in the bibliography).

In a series of recent papers we have characterized and studied a special class of compact complex nilmanifolds, namely *compact nilmanifolds with a nilpotent*

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complex structure (see [10, 9, 8]). The simplest nontrivial examples of such manifolds are the compact complex parallelizable nilmanifolds in the sense of Wang [29].

For a compact nilmanifold $\Gamma\backslash G$ with a nilpotent complex structure it has been proved in [10] that the computation of the Dolbeault cohomology $H^{*,*}(\Gamma\backslash G)$ can be reduced to the level of the Lie algebra $\mathfrak{g}$ of G; in fact, we have proved that there is a canonical isomorphism

$$H^{p,q}(\Gamma\backslash G) \cong H^{p,q}(\mathfrak{g}^{\mathbb{C}}),$$

where we denote by $H^{*,*}(\mathfrak{g}^{\mathbb{C}})$ the cohomology of the differential bigraded algebra $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar\partial)$, and $\bar\partial$ being the operator arising in the canonical decomposition $d = \partial + \bar\partial$ of the Chevalley–Eilenberg differential d in $\Lambda^*(\mathfrak{g}^{\mathbb{C}})^*$. Moreover, $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar\partial)$ is in fact a *minimal model* in the sense of Neisendorfer-Taylor [22] for the Dolbeault complex $(\Lambda^{*,*}(\Gamma\backslash G), \bar\partial)$ associated to the complex structure on $\Gamma\backslash G$.

Our main goal in this paper is to give a survey of some results on the formality of the above Dolbeault minimal model $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar\partial)$ of a compact nilmanifold $\Gamma\backslash G$ with a nilpotent complex structure.

The paper is structured as follows. In Section 2 we recall the definition and some basic results about nilpotent complex structures. In Section 3 we show that a compact nilmanifold $\Gamma\backslash G$ with a nilpotent complex structure is Dolbeault formal if and only if $\Gamma\backslash G$ is a complex torus (Theorem 3.2); then, as a consequence of a result of Neisendorfer and Taylor [22], it follows that a compact nilmanifold with a nilpotent complex structure does not admit (positive definite) Kähler metric unless it is a torus (Corollary 3.3). Let us emphasize that our proof of this fact for this particular class of compact nilmanifolds with nilpotent complex structures makes use exclusively of their Dolbeault cohomology. It is well known so far that the same result holds for an arbitrary even-dimensional compact nilmanifold (see Benson-Gordon [3] and Hasegawa [18]); but Benson-Gordon's proof holds from the failure of the Hard Lefschetz Theorem for any symplectic structure on a non-toral nilmanifold, and Hasegawa's is based on the study of the formality of a minimal model for the de Rham complex of a compact nilmanifold.

It is known that the Iwasawa manifold I_3 with its natural complex structure I is a compact complex parallelizable nilmanifold with no indefinite Kähler metric, because there does not exist a symplectic form on I_3 compatible with I [14]. This property of (I_3, I) is extended in Proposition 3.8 to any non-toral compact complex parallelizable nilmanifold. As a consequence, complex tori are the only compact complex parallelizable nilmanifolds with compatible indefinite Kähler metric (Corollary 3.9).

Nevertheless, other nilpotent complex structures J_θ on I_3 that possess indefinite Kähler metrics are constructed in Section 4. A natural question comes up: is it possible to define a nilpotent complex structure on any compact complex parallelizable nilmanifold that admits an indefinite Kähler metric? In Section 4 we remark that the answer to this question is negative.

2 Compact nilmanifolds with nilpotent complex structure

We are interested in a particular class of compact complex nilmanifolds which has been introduced in [10], namely compact nilmanifolds with a nilpotent complex structure.

Let G be a real nilpotent Lie group endowed with a left invariant complex structure J. Then, there exists an ascending series $\{\mathfrak{a}_l(J)\}_{l\geq 0}$ of the Lie algebra $\mathfrak{g}$ of G, associated to J in a natural way; this series is defined inductively by

$$(1)\quad \begin{aligned} &\mathfrak{a}_0(J)=\{0\}\,, \\ &\mathfrak{a}_l(J)=\{X\in\mathfrak{g}\mid [X,\mathfrak{g}]\subseteq\mathfrak{a}_{l-1}(J)\ \ \textit{and}\ \ [JX,\mathfrak{g}]\subseteq\mathfrak{a}_{l-1}(J)\}\,,\quad l\geq 1. \end{aligned}$$

Each $\mathfrak{a}_l(J)$ is a J-invariant ideal in $\mathfrak{g}$, and $\mathfrak{a}_l(J)\subseteq\mathfrak{g}_l$ for $l\geq 0$, where $\mathfrak{g}_l$ denotes the term in the usual ascending central series $\{\mathfrak{g}_l\}_{l\geq 0}$ of $\mathfrak{g}$.

Remark that $\{\mathfrak{a}_l(J)\}_{l\geq 0}$ depends on the complex structure J. In fact, in [10] it is exhibited a nilpotent Lie group G admitting two complex structures J and J' for which $\mathfrak{a}_l(J)\neq\mathfrak{a}_l(J')$ for all $l>0$. Moreover, this ascending series can degenerate at a step t with $\mathfrak{a}_t(J)\neq\mathfrak{g}$, that is, it may occur that $\mathfrak{a}_l(J)=\mathfrak{a}_t(J)\neq\mathfrak{g}$ for some $t\geq 0$ and for all $l\geq t$.

Definition 2.1 A left invariant complex structure J on G is called *nilpotent* if the series $\{\mathfrak{a}_l(J)\}_{l\geq 0}$ given by (1) satisfies $\mathfrak{a}_t(J)=\mathfrak{g}$ for some integer $t>0$.

In particular, if G is a complex nilpotent Lie group, J being the left invariant integrable almost complex structure on it, then each $\mathfrak{g}_l$ in its ascending central series is a complex Lie subalgebra of $\mathfrak{g}$ and hence $J(\mathfrak{g}_l)=\mathfrak{g}_l$ for $l\geq 0$. Therefore, $\mathfrak{a}_l(J)=\mathfrak{g}_l$ for all $l\geq 0$, which implies $\mathfrak{a}_s(J)=\mathfrak{g}_s=\mathfrak{g}$, and the complex structure J on G is nilpotent.

Nilpotent complex structures can be characterized in terms of the structure equations of the Lie group as follows. Let J be a left invariant complex structure on a nilpotent Lie group G of real dimension $2n$. Then there exists a basis $\{\omega_i,\overline{\omega}_i; 1\leq i\leq n\}$ satisfying

$$d\omega_i=\sum_{j<k}A_{ijk}\,\omega_j\wedge\omega_k+\sum_{j,k}B_{ijk}\,\omega_j\wedge\overline{\omega}_k\,,\quad 1\leq i\leq n\,,$$

where ω_i is of type (1,0) and $\overline{\omega}_i$ of type (0,1) with respect to J, for $1\leq i\leq n$.

Theorem 2.2 [10] *Let G be a nilpotent Lie group (of real dimension $2n$) with a left invariant complex structure J. Then J is nilpotent if and only if there exists a basis $\{\omega_i, \overline{\omega}_i; 1 \leq i \leq n\}$ for which the structure equations of G have the form*

$$d\omega_i = \sum_{j<k<i} A_{ijk}\, \omega_j \wedge \omega_k + \sum_{j,k<i} B_{ijk}\, \omega_j \wedge \overline{\omega}_k\,, \tag{2}$$

for $1 \leq i \leq n$.

A necessary condition for a s-step nilpotent Lie group G to have a nilpotent complex structure is the following: $\dim \mathfrak{g}_l \geq 2l$, for all $0 \leq l \leq s$. Other necessary conditions are proved in [10].

Corollary 2.3 *Let G be a nilpotent Lie group endowed with a nilpotent left invariant complex structure. Then the structure equations of G have the form (2). Conversely, the structure equations (2) define a nilpotent Lie group G with a nilpotent left invariant complex structure. Hence quotients of G have complex structures.*

Definition 2.4 A *compact nilmanifold with a nilpotent complex structure* is a complex manifold of the form $\Gamma\backslash G$ whose complex structure is inherited from a nilpotent left invariant complex structure on G by passing to the quotient.

If G is indeed a complex Lie group then $\Gamma\backslash G$ is a compact complex parallelizable nilmanifold in the sense of Wang [29]. The compact complex parallelizable nilmanifolds are precisely those for which the coefficients B_{ijk} in (2) vanish. But there are many compact nilmanifolds with a nilpotent complex structure which are not complex parallelizable but real parallelizable only (see [6, 7, 8, 15, 19, 27] and the examples in Section 4).

3 Dolbeault minimal models and formality

First, let us recall some definitions of the Dolbeault homotopy theory as developed by Neisendorfer and Taylor in [22].

A *differential bigraded algebra* (*dba*) $\mathcal{M} = (\mathcal{M}^{*,*}, \bar{\partial})$ is a bigraded commutative algebra $\mathcal{M}^{*,*}$ over $\mathbb{C}$ with a differential $\bar{\partial}$ of type (0,1) which is a derivation. It is further required that it be augmented over $\mathbb{C}$. Morphisms between *dba*'s are required to be bidegree preserving algebra maps which commute with the differentials.

One immediate example of a *dba* is the Dolbeault complex of a complex manifold M. If $\Lambda^{p,q}(M)$ denotes the space of complex valued differential forms of type (p,q) on M then $(\Lambda^{*,*}(M), \bar{\partial})$ is a *dba*, where $\Lambda^{*,*}(M) = \bigoplus_{p,q\geq 0} \Lambda^{p,q}(M)$ and $\bar{\partial}$ is the differential in the usual decomposition $d = \partial + \bar{\partial}$ of the exterior differential d of M. The cohomology of $(\Lambda^{*,*}(M), \bar\partial)$, denoted by $H^{*,*}(M)$, is the *Dolbeault cohomology* of M. Notice that $H^{*,*}(M)$ with the differential $\bar\partial \equiv 0$, that is $(H^{*,*}(M), 0)$, is again a *dba*.

A *dba* $\mathcal{M} = (\mathcal{M}^{*,*}, \bar\partial)$ is a *model* for the Dolbeault complex of a complex manifold M if there is a morphism of *dba*'s $\varphi : (\mathcal{M}^{*,*}, \bar\partial) \longrightarrow (\Lambda^{*,*}(M), \bar\partial)$ inducing an isomorphism on cohomology.

A model $\mathcal{M} = (\mathcal{M}^{*,*}, \bar\partial)$ is said to be *minimal* if the algebra $\mathcal{M}^{*,*}$ is free and the differential $\bar\partial$ is decomposable; it is said to be *formal* if there is a morphism of *dba*'s $\psi : (\mathcal{M}^{*,*}, \bar\partial) \longrightarrow (H^{*,*}(\mathcal{M}), 0)$ inducing the identity on cohomology. One can define Massey products for $H^{*,*}(\mathcal{M})$ in a standard way. Then, nonzero Massey products are obstructions to the formality of a model $\mathcal{M}$.

We shall say that a complex manifold M is *Dolbeault formal* if there exists a minimal model $\mathcal{M} = (\mathcal{M}^{*,*}, \bar\partial)$ for $(\Lambda^{*,*}(M), \bar\partial)$ which is formal.

Another immediate example of a *dba* is the following. Let $\mathfrak{g}$ be the Lie algebra of a nilpotent Lie group G with a left invariant complex structure J. Since J is left invariant on G, the complexifications of $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ admit the following decompositions:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{0,1}, \qquad (\mathfrak{g}^*)^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}.$$

Since $(\mathfrak{g}^{\mathbb{C}})^* \cong (\mathfrak{g}^*)^{\mathbb{C}}$, there is a natural bigraduation on the exterior algebra $\Lambda^*(\mathfrak{g}^{\mathbb{C}})^*$:

$$\Lambda^*(\mathfrak{g}^{\mathbb{C}})^* = \bigoplus_{p,q\geq 0} \Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^*,$$

where $\Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^* = \Lambda^p(\mathfrak{g}^{1,0}) \otimes \Lambda^q(\mathfrak{g}^{0,1})$. Since J is complex, the natural extension to $\Lambda^*(\mathfrak{g}^{\mathbb{C}})^*$ of the Chevalley–Eilenberg differential $d : \Lambda^*(\mathfrak{g}^{\mathbb{C}})^* \longrightarrow \Lambda^{*+1}(\mathfrak{g}^{\mathbb{C}})^*$ decomposes as $d = \partial + \bar\partial$ where

$$\partial : \Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^* \longrightarrow \Lambda^{p+1,q}(\mathfrak{g}^{\mathbb{C}})^*, \qquad \bar\partial : \Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^* \longrightarrow \Lambda^{p,q+1}(\mathfrak{g}^{\mathbb{C}})^*,$$

and $\partial^2 = \partial\bar\partial + \bar\partial\partial = \bar\partial^2 = 0$. Therefore, $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar\partial)$ is a *dba* which is canonically identified to the *dba* of complex valued left invariant differential forms on G.

Suppose that the Lie group G has compact quotients of the form $\Gamma\backslash G$, Γ being a lattice in G of maximal rank. Then each left invariant form on G descends to the quotient manifold $\Gamma\backslash G$, and its differential satisfies on $\Gamma\backslash G$ the same relations as it does on the Lie group G; therefore, there is a canonical morphism of *dba*'s

$$i : (\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar\partial) \longrightarrow (\Lambda^{*,*}(\Gamma\backslash G), \bar\partial). \tag{3}$$

Theorem 3.1 [10] *Let $\Gamma\backslash G$ be a compact nilmanifold with a nilpotent complex structure, and let $\mathfrak{g}$ be the Lie algebra of G. Then, the morphism* (3) *induces an isomorphism on cohomology. Therefore, $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*, \bar\partial)$ is a minimal model for the Dolbeault complex of $\Gamma\backslash G$.*

Therefore, if the structure equations of G are given by (2) then a minimal model for the Dolbeault complex of the compact nilmanifold $\Gamma\backslash G$ (of real dimension $2n$) with a nilpotent complex structure is the *dba*

$$\mathcal{M} = (\Lambda^{*,*}(x^1_{1,0},\ldots,x^n_{1,0},x^1_{0,1},\ldots,x^n_{0,1});\bar\partial), \tag{4}$$

where the generators have total degree 1 and bidegree as indexed, and $\bar\partial$ is given by

$$\bar\partial x^i_{1,0} = \sum_{j,k<i} B_{ijk}\, x^j_{1,0}\cdot x^k_{0,1}\,, \qquad \bar\partial x^i_{0,1} = \sum_{j<k<i} \bar A_{ijk}\, x^j_{0,1}\cdot x^k_{0,1}\,, \qquad 1\le i\le n. \tag{5}$$

Let $\mathfrak{g}$ be again the Lie algebra of a Lie group G endowed with a left invariant complex structure J, and let $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*,\bar\partial)$ be the associated *dba*. Let k_1 and k_2 be the integer numbers

$$\begin{aligned} k_1 &= \dim H^{1,0}(\mathfrak{g}^{\mathbb{C}}) = \dim\left(\mathfrak{g}_{1,0}/\pi_{1,0}([\mathfrak{g}_{1,0},\mathfrak{g}_{0,1}])\right), \\ k_2 &= \dim H^{0,1}(\mathfrak{g}^{\mathbb{C}}) = \dim(\mathfrak{g}_{0,1}/[\mathfrak{g}_{0,1},\mathfrak{g}_{0,1}]), \end{aligned} \tag{6}$$

where $\pi_{1,0}:\mathfrak{g}^{\mathbb{C}}\longrightarrow\mathfrak{g}_{1,0}$ denotes the canonical projection.

If the complex structure J is nilpotent then, from Theorem 2.2 and the equations in (2), it follows easily that $n\ge k_1\ge 1$ and $n\ge k_2\ge 2$. If, moreover, $\mathfrak{g}$ is a *complex* Lie algebra then $[\mathfrak{g}_{1,0},\mathfrak{g}_{0,1}]=0$, and therefore $k_1=n$. Hence, a compact nilmanifold $\Gamma\backslash G$ with nilpotent complex structure is complex parallelizable if and only if the Lie algebra $\mathfrak{g}$ of G satisfies $k_1=n$. Moreover, $\mathfrak{g}$ is Abelian if and only if $k_1=k_2=n$.

Theorem 3.2 *Let $\Gamma\backslash G$ be a compact nilmanifold with a nilpotent complex structure, and let $\mathcal{M}$ denote the minimal model for the Dolbeault complex of $\Gamma\backslash G$ given by* (4) *and* (5). *Then, $\mathcal{M}$ is formal if and only if $\Gamma\backslash G$ is a complex torus.*

Proof : A compact nilmanifold $\Gamma\backslash G$ is a torus if and only if the Lie algebra $\mathfrak{g}$ of G is Abelian. In [11] the authors prove that the model $\mathcal{M}$ is formal if and only if $k_1=k_2=n$, where $k_1=\dim H^{1,0}(\mathcal{M})=\dim H^{1,0}(\Gamma\backslash G)$ and $k_2=\dim H^{0,1}(\mathcal{M})=\dim H^{0,1}(\Gamma\backslash G)$ are given by (6). QED

According to Neisendorfer and Taylor [22], a minimal model for the Dolbeault complex of a compact Kähler manifold is formal. Therefore:

Corollary 3.3 *A compact nilmanifold with a nilpotent complex structure does not admit Kähler structure unless it is a complex torus.*

Neisendorfer and Taylor have also introduced the notion of *strict formality.* Compact Kähler manifolds are strictly formal, and strictly formal complex manifolds are, in particular, de Rham formal and Dolbeault formal. (In [22], examples of Dolbeault formal manifolds which are not de Rham formal and of Dolbeault formal manifolds which are not strictly formal are given.) In this context, a new consequence of Theorem 3.2 is the following:

Corollary 3.4 *A compact nilmanifold with a nilpotent complex structure is strictly formal if and only if it is a complex torus.*

Therefore, recalling that a compact nilmanifold is de Rham formal if and only if it is a torus [18], we obtain:

Corollary 3.5 *Let $\Gamma\backslash G$ be a compact nilmanifold with a nilpotent complex structure. The following conditions are equivalent:*

(i) *$\Gamma\backslash G$ is de Rham formal,*

(ii) *$\Gamma\backslash G$ is Dolbeault formal,*

(iii) *$\Gamma\backslash G$ is strictly formal,*

(iv) *$\Gamma\backslash G$ is a complex torus.*

Remark 3.6 In a recent survey about rational homotopy theory and geometry, A. Tralle (see the last Remark in page 458, or Problem 8, page 469, in [28]) asked the following question: are there complex symplectic manifolds whose non-Kählerness follows from the Dolbeault homotopy theory?

Reading Tralle's question as such, Theorem 3.2 and the examples that are described in Section 4, namely the Iwasawa manifold and the Kodaira-Thurston manifold, show that the answer to this question is affirmative. But in a private conversation A. Tralle told us that his question should be read as follows: are there complex symplectic (de Rham) formal manifolds whose non-Kählerness follows from the Dolbeault homotopy theory? With this rewording, Theorem 3.2 and Corollary 3.5 show that the family of compact nilmanifolds with a nilpotent complex structure will not provide an answer to Tralle's question.

Let M be a complex manifold. Frölicher's spectral sequence $\{E_r(M)\}_{r\geq 1}$, constructed in [16], relates the Dolbeault cohomology $H^{*,*}(M)$ to the de Rham cohomology $H^*(M)$ of M. An interesting problem in complex manifold theory is to understand which compact complex manifolds M have $E_2(M) \ncong E_\infty(M)$.

It is known that the Frölicher spectral sequence associated to the canonical complex structure of Iwasawa's manifold (see Example 1 in Section 4) satisfies $E_1 \ncong E_2 \cong E_\infty$ (see [17]). More in general, in [6] it is proved that $E_2(M) \cong E_\infty(M)$ for any compact complex parallelizable nilmanifold M. Moreover, examples of compact complex manifolds such that their associated Frölicher spectral sequences degenerate at higher levels have been constructed using compact nilmanifolds with nilpotent complex structure [6, 7, 9].

On the other hand, Tanré [26] proved that if G is a compact connected semisimple Lie group such that $T \hookrightarrow G \to G/T$ is a holomorphic principal bundle, T being a maximal torus of G, then $E_2(G) \ncong E_\infty(G)$ if and only if G is not Dolbeault formal. These conditions are satisfied by Pittie's example [24], where $G = SO(9)$.

In contrast to Tanré's result, for compact nilmanifolds we have:

Corollary 3.7 *Let $\Gamma\backslash G$ be a compact nilmanifold with a nilpotent complex structure. If $E_1(\Gamma\backslash G) \not\cong E_\infty(\Gamma\backslash G)$ then $\Gamma\backslash G$ is not Dolbeault formal.*

Proof : It follows immediately from Theorem 3.2, taking into account that a compact Kähler manifold M satisfies $E_1(M) \cong E_\infty(M)$ by Hodge theory (see [16]). QED

However, the converse of this result does not hold in general. Kodaira–Thurston's manifold KT is an example whose Dolbeault minimal model is not formal but with degenerate Frölicher spectral sequence (see Example 1 in Section 4).

Recall that Benson and Gordon [3] and Hasegawa [18] have proved that an even–dimensional compact nilmanifold has no Kähler structure unless it is a torus. The following is significant in this context.

Proposition 3.8 *Let $M = \Gamma\backslash G$ be a compact complex parallelizable nilmanifold of (complex) dimension n. If M is not a complex torus then M does not admit symplectic structure compatible with its complex parallelizable structure.*

Proof : Since $M = \Gamma\backslash G$ is complex parallelizable the coefficients B_{ijk} in (2) vanish or, equivalently, $[\mathfrak{g}_{1,0}, \mathfrak{g}_{0,1}] = 0$. Since the Frölicher spectral sequence degenerates at $E_2(M)$, in particular the second de Rham cohomology group decomposes as $H^2(M) \cong E_2^{2,0}(M) \oplus E_2^{1,1}(M) \oplus E_2^{0,2}(M)$, where $E_2^{1,1}(M) \cong H^1(\mathfrak{g}_{1,0}) \otimes H^1(\mathfrak{g}_{0,1})$.

Put $k = \dim H^1(\mathfrak{g}_{1,0})$, and let $\{[\alpha^1_{1,0}], \ldots, [\alpha^k_{1,0}]\}$ be a basis of $H^1(\mathfrak{g}_{1,0})$. Then $\{[\alpha^1_{0,1}], \ldots, [\alpha^k_{0,1}]\}$ is a basis of $H^1(\mathfrak{g}_{0,1})$, $\alpha^i_{0,1}$ being the conjugate of $\alpha^i_{1,0}$, $1 \leq i \leq k$. Therefore, any cohomology class $[\Omega]$ of type $(1,1)$ must be a linear combination of $[\alpha^i_{1,0} \wedge \alpha^j_{0,1}]$, for $1 \leq i, j \leq k$. If M is not a complex torus then $k < n$ and we conclude that $[\Omega]^n = 0$, that is, no closed form of bidegree $(1,1)$ on M can have maximal rank. QED

Corollary 3.9 *Complex tori are the only compact complex parallelizable nilmanifolds which admit indefinite Kähler metric compatible with its natural complex structure.*

However, there may exist other nilpotent complex structures on a compact complex parallelizable nilmanifold that do possess (compatible) indefinite Kähler metric as the discussion on Iwasawa's manifold in the next section will illustrate. (See [1] for a description of all invariant complex and symplectic structures on I_3 compatible with a standard positive definite metric and orientation on it.) Moreover, Kodaira-Thurston's manifold shows that there may also exist compact nilmanifolds with nilpotent complex structure admitting indefinite Kähler structure.

4 Examples

In this section two classical examples of compact nilmanifolds with nilpotent complex structure and indefinite Kähler metric are described. Therefore, Dolbeault formality provides a difference between indefinite Kähler metrics and (positive definite) Kähler metrics.

Example 1: The Kodaira–Thurston manifold
This manifold is the simplest example of a compact nilmanifold with nilpotent complex structure which is real parallelizable but not complex parallelizable.

Let G be the nilpotent Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & \bar{x} & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark that G is not a complex Lie group because right translations are not holomorphic.

Kodaira–Thurston's manifold KT is the compact nilmanifold with nilpotent complex structure obtained as $KT = \Gamma\backslash G$, where Γ is the subgroup of G consisting of those matrices whose entries $\{x, y\}$ are Gaussian integers. The functions x, y are natural complex coordinates on G. Since the complex 1-forms dx, $dy - \bar{x}dx$ are left invariant on G, they descend to 1-forms ω_1, ω_2 on KT such that

$$d\omega_1 = 0, \quad d\omega_2 = \omega_1 \wedge \bar{\omega}_1.$$

This manifold has been the first known example of a compact symplectic manifold which is also a complex manifold with no (positive definite) Kähler metric [19, 27, 5]. In fact, $\langle \omega_1, \omega_1, \bar{\omega}_1 \rangle$ defines a nonzero triple Massey product on KT.

However, KT possesses indefinite Kähler metrics [15]. Moreover, if ds^2 is an indefinite Kähler metric on KT which stems from a left invariant indefinite Kähler metric on the Lie group G with respect to its natural complex structure, then the Kähler form corresponding to ds^2 is given by

$$F = \sqrt{-1}\, ds^2(Z_1, \bar{Z}_2)\,(\omega_1 \wedge \bar{\omega}_2 + \bar{\omega}_1 \wedge \omega_2) + \sqrt{-1}\, ds^2(Z_1, \bar{Z}_1)\, \omega_1 \wedge \bar{\omega}_1,$$

where $\{Z_1, \bar{Z}_1, Z_2, \bar{Z}_2\}$ is the basis dual to $\{\omega_1, \bar{\omega}_1, \omega_2, \bar{\omega}_2\}$ (see [2]).

Also, it must be remarked that, since KT has complex dimension 2, its associated Frölicher spectral sequence is degenerate (see [19, 4]). In fact, $\dim E_1(KT) = \dim E_\infty(KT) = 12$. From Theorem 3.2 we conclude that the converse of Corollary 3.7 is not true.

Example 2: The Iwasawa manifold
The Iwasawa manifold I_3 can be realized as the compact quotient manifold $I_3 = \Gamma\backslash G$ of the complex nilpotent Lie group G of all complex matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

by the subgroup Γ of G consisting of those matrices whose entries $\{x, y, z\}$ are Gaussian integers. The functions x, y, z are natural complex coordinates on G, and the complex differential forms dx, dy and $dz - xdy$ on G are left invariant. Hence, they descend to holomorphic 1–forms ω_1, ω_2 and ω_3 on I_3 such that

$$\begin{cases} d\omega_1 = d\omega_2 = 0, \\ d\omega_3 = -\omega_1 \wedge \omega_2. \end{cases}$$

We shall denote this complex structure on I_3 by I and refer to it as the *natural* complex structure on I_3.

The compact complex nilmanifold (I_3, I) is, apart from a torus, the simplest example of a complex parallelizable nilmanifold. It is easy to check that $\langle\bar{\omega}_1, \bar{\omega}_1, \bar{\omega}_2\rangle$ defines a nonzero triple Massey product on (I_3, I). Therefore, according with Proposition 3.8, no symplectic structure compatible with I can exist on I_3.

However, there exist other nilpotent structures on I_3 that do posses (compatible) indefinite Kähler metric.

Let $\{X_1, Y_1, X_2, Y_2, X_3, Y_3\}$ be the basis dual to $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$, where $\alpha_i = \mathfrak{Re}\,(\omega_i)$, $\beta_i = \mathfrak{Im}\,(\omega_i)$, for $i = 1, 2, 3$. In [14] it is proved that the almost complex structure J_θ given by

$$J_\theta X_1 = \cos\theta\, X_2 + \sin\theta\, Y_2, \qquad J_\theta Y_1 = -\sin\theta\, X_2 + \cos\theta\, Y_2, \qquad J_\theta X_3 = Y_3,$$

is integrable for each $\theta \in \mathbb{R}$. The 1–forms τ_1, τ_2, τ_3 given by

$$\tau_1 = \alpha_1 + \sqrt{-1}\,(\cos\theta\, \alpha_2 + \sin\theta\, \beta_2),$$

$$\tau_2 = \beta_1 + \sqrt{-1}\,(-\sin\theta\, \alpha_2 + \cos\theta\, \beta_2),$$

$$\tau_3 = 2(\alpha_3 + \sqrt{-1}\,\beta_3),$$

is a basis of complex forms of type $(1,0)$ with respect to J_θ. In terms of τ_1, τ_2, τ_3 and their conjugates the structure equations become

$$\begin{cases} d\tau_1 = d\tau_2 = 0, \\ d\tau_3 = \lambda\,(\tau_1 \wedge \bar{\tau}_2 + \tau_2 \wedge \bar{\tau}_1) + \mu\,(\tau_1 \wedge \bar{\tau}_1 - \tau_2 \wedge \bar{\tau}_2), \end{cases} \tag{7}$$

where $\lambda = \cos\theta + \sqrt{-1}\sin\theta$ and $\mu = \sin\theta - \sqrt{-1}\cos\theta$. These equations prove that J_θ is a nilpotent complex structure on I_3 for each $\theta \in \mathbb{R}$. Since

$\dim H^{1,0}(I_3, J_\theta) = 2 < 3$, we conclude that (I_3, J_θ) is not a complex parallelizable manifold.

Let

$$ds^2_\theta = \frac{1}{8}(\bar\lambda\,\tau_1\#\bar\tau_3 + \lambda\,\bar\tau_1\#\tau_3 - \bar\mu\,\tau_2\#\bar\tau_3 - \mu\,\bar\tau_2\#\tau_3) - \frac{1}{4}\mathfrak{Re}\,(\lambda+\mu)\sqrt{-1}\,(\tau_1\#\bar\tau_2 - \bar\tau_1\#\tau_2),$$

where $\#$ denotes the symmetric product. Then ds^2_θ is an indefinite Kähler metric compatible with J_θ, and its corresponding Kähler form F_θ is the symplectic form given by

$$F_\theta = \frac{1}{4}(\bar\lambda\,\tau_2\wedge\bar\tau_3 + \lambda\,\bar\tau_2\wedge\tau_3 + \bar\mu\,\tau_1\wedge\bar\tau_3 + \mu\,\bar\tau_1\wedge\tau_3) + \frac{1}{4}\mathfrak{Re}\,(\lambda+\mu)(\tau_1\wedge\bar\tau_2 + \bar\tau_1\wedge\tau_2).$$

It is easy to check that $\langle \tau_1, \tau_1\wedge\tau_2, \mu\,\bar\tau_1 + \lambda\,\bar\tau_2\rangle$ defines a nonzero triple Massey product on (I_3, J_θ), for each $\theta \in \mathbb{R}$.

Integrating the structure equations (7) it is easy to see that (I_3, J_θ) can be viewed as the quotient of the nilpotent Lie group

$$G_\theta = \left\{ \begin{pmatrix} 1 & \lambda\bar z - \mu\bar v & \mu\bar z + \lambda\bar v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z, v, w \in \mathbb{C} \right\}$$

by the subgroup Γ_θ of G_θ consisting of those matrices whose entries $\{z, v, w\}$ are Gaussian integers.

Finally, a direct computation shows that the Frölicher spectral sequence associated to J_θ satisfies $E_1 \not\cong E_2 \cong E_\infty$. More precisely,

$$\dim E_1(I_3, J_\theta) = 48 > \dim E_2(I_3, J_\theta) = \dim E_\infty(I_3, J_\theta) = 36,$$

for each $\theta \in \mathbb{R}$.

Remark 4.1 In [12] a family of compact complex parallelizable nilmanifolds $N_{\mathbb{C}}(r,1)$, $r \geq 1$ have been constructed as quotients of the generalized complex Heisenberg group. Although $N_{\mathbb{C}}(1,1) \equiv I_3$, the compact nilmanifolds $N_{\mathbb{C}}(r,1)$ for $r \geq 2$ do not admit symplectic forms. Therefore, on the compact nilmanifolds $N_{\mathbb{C}}(r,1)$, $r \geq 2$, there do not exist complex structures, neither nilpotent nor non-nilpotent, which admit compatible indefinite Kähler metrics.

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GAMMA MANIFOLDS AND STOCHASTIC GEOMETRY

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Abstract.– Families of gamma and gamma-related distributions yield parametric statistical models in the form of Riemannian statistical 2-manifolds which are topologically $\mathbb{R}^+ \times \mathbb{R}^+$. These constitute a representation of certain stochastic line, rectangle and cylinder processes having the random or chaotic case as a 1-dimensional submanifold and a 2-parameter family of of departures from the random state. A source of statistically natural metrics on statistical manifolds is the Fisher information matrix, based on the expectation of products of gradients of the log-likelihood function, that is the statistical covariance matrix of the derivatives of log-likelihood. In this preliminary report we collect some of the geometrical features of such manifolds and juxtapose them with physical features of stochastic processes for which they provide models.

Statistical Manifolds

The geometric representation of statistical models by Riemannian manifolds seems to have its origin in the work of C.R. Rao [16] who introduced the metric tensor in terms of the Fisher information matrix [10]. The thesis of Rao's student M. Deng [5] contains a summmary of the developments and a number of results that we shall use below. For a collection of formulae for commonly occurring statistical distributions see Kokoska and Nevison [13].

Following Amari [1] (cf. also [7]), let M be the parameter space of a statistical model S, that is an n-dimensional smooth family of probability density functions defined on some fixed event space Ω of unit measure,

$$\int_\Omega p_\theta = 1 \quad \text{for all } \theta \in M.$$

For each sequence $X = \{X_1, X_2, \ldots, X_n\}$, of independent identically distributed observed values, the *likelihood function* lik_X on M which measures the likelihood of the sequence arising from different $p_\theta \in S$ is defined by

$$lik_X : M \to [0,1] : \theta \mapsto \prod_{i=1}^{n} p_\theta(X_i).$$

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Statisticians use the likelihood function, or *log-likelihood* its logarithm $l = \log lik$, in the evaluation of goodness of fit of statistical models. The so-called 'method of maximum likelihood' is used to obtain optimal fitting of the parameters in a distribution to observed data. We illustrate this for the particular case of a gamma distribution in the next section.

We denote by $\mathcal{E}$ the expectation operator (measure) for functions defined on Ω; in particular, the mean $\mathcal{E}(x) = \bar{x}$, and variance $\mathcal{E}(x^2) - \bar{x}^2 = Var(x)$, which will be functions of θ. Then, at each point $\theta \in M$, the covariance of partial derivatives of the log-likelihood function is a matrix with entries the expections

$$g_{ij} = \mathcal{E}\left(\frac{\partial l}{\partial \theta^i}\frac{\partial l}{\partial \theta^j}\right) = -\mathcal{E}\left(\frac{\partial^2 l}{\partial \theta^i \partial \theta^j}\right) \quad \text{(for coordinates } (\theta^i) \text{ about } \theta \in M)$$

which gives rise to a positive definite matrix. It induces a Riemannian metric g on M, called the expected information metric for the parametric statistical model S, which has statistical significance [10, 16, 1]. Families of connections which contain the Levi-Civita connection of g turn out to have importance also [1], and are actually part of larger systems of connections for which stability results follow [7]. Deng in her thesis [5] used the same coordinates but took the metric $h_{ij} = g_{ij} + \delta_{ij}$.

Gamma Manifolds

The family of gamma distributions has event space $\Omega = \mathbb{R}^+$ and probability density functions given by

$$S = \{p(x; \mu, \beta) | \mu, \beta \in \mathbb{R}^+\}$$

so here $M = \mathbb{R}^+ \times \mathbb{R}^+$ and the random variable is $x \in \Omega = \mathbb{R}^+$ with

$$p(x; \mu, \beta) = \left(\frac{\beta}{\mu}\right)^\beta \frac{x^{\beta-1}}{\Gamma(\beta)} e^{-x\beta/\mu} \qquad (\star)$$

Then $\bar{x} = \mu$ and $Var(x) = \mu^2/\beta$ and we see that μ controls the mean of the distribution while β controls its variance and hence the shape.

To see the role of the log-likelihood function here, consider the case that we have a set $X = \{X_1, X_2, \ldots, X_n\}$ of measurements, which we consider to be a sample drawn from independent identically distributed random variables, to which we wish to fit the best (ie maximum likelihood) gamma distribution. So we need a procedure to optimize the choice of μ, β. For independent events X_i, with identical distribution $p(x; \mu, \beta)$, their joint probability density is the product of the marginal densities so a measure of the 'likelihood' of finding such a set of events is

$$lik_X(\mu, \beta) = \prod_{i=1}^{n} p(X_i; \mu, \beta).$$

We seek a choice of μ, β to maximize this product and since the log function is monotonic increasing it is simpler to maximize the logarithm

$$l_X(\mu,\beta) = \log lik_X(\mu,\beta) = \log[\prod_{i=1}^{n} p(X_i;\mu,\beta)].$$

Substitution gives us

$$\begin{aligned} l_X(\mu,\beta) &= \sum_{i=1}^{n}[\beta(\log\beta - \log\mu) + (\beta-1)\log X_i - \frac{\beta}{\mu}X_i - \log\Gamma(\beta)] \\ &= n\beta(\log\beta - \log\mu) + (\beta-1)\sum_{i=1}^{n}\log X_i - \frac{\beta}{\mu}\sum_{i=1}^{n}X_i - n\log\Gamma(\beta). \end{aligned}$$

Then, solving for $\partial_\mu l_X(\mu,\beta) = \partial_\beta l_X(\mu,\beta) = 0$ in terms of properties of the X_i, we obtain the maximum likelihood estimates $\hat{\mu}, \hat{\beta}$ of μ, β in terms of the mean and mean logarithm of the X_i

$$\begin{aligned} \hat{\mu} &= \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \\ \log\hat{\beta} - \frac{\Gamma'(\hat{\beta})}{\Gamma(\hat{\beta})} &= \overline{\log X} - \log\bar{X} \end{aligned}$$

where $\overline{\log X} = \frac{1}{n}\sum_{i=1}^{n}\log X_i$.

The usual information metric g, used by Lauritzen [14], is given by

$$ds_g^2 = \frac{\beta}{\mu^2}\,d\mu^2 + \left(\psi'(\beta) - \frac{1}{\beta}\right)\,d\beta^2 \quad \text{for } \mu,\beta \in \mathbb{R}^+,$$

where $\psi(\beta) = \frac{\Gamma'(\beta)}{\Gamma(\beta)}$ is the digamma function, the logarithmic derivative of the gamma function. Deng's metric [5] h yields

$$ds_h^2 = (1 + \frac{\beta}{\mu^2})\,d\mu^2 + \left(1 + \psi'(\beta) - \frac{1}{\beta}\right)\,d\beta^2 \quad \text{for } \mu,\beta \in \mathbb{R}^+.$$

Lauritzen [14] and Deng [5] have computed some geometrical properties of statistical 2-manifolds based on the family of gamma distributions, among other commonly occurring distributions, using their respective metrics. In particular, Lauritzen computed the 1-parameter family of α-connections of Amari [1] and obtained the following expression for their corresponding sectional curvatures $k(\alpha)$:

$$k(\alpha) = -R^{(\alpha)}_{1212}\,g^{11}g^{22} = \frac{1-\alpha^2}{4}\,\frac{\psi'(\beta) + \beta\psi''(\beta)}{\beta^2\psi'(\beta)}.$$

Using the Brioschi intrinsic formula for surfaces (cf. Gray [9] page 393, wherein also can be found the Mathematica functions for various geometric objects) we have for a metric $ds^2 = E\,d\mu^2 + G\,d\beta^2$ the Gaussian curvature

$$K = \frac{-1}{\sqrt{EG}} \left\{ \frac{\partial}{\partial_\mu} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial_\mu} \right) + \frac{\partial}{\partial_\beta} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial_\beta} \right) \right\}$$

Then the Gaussian curvature of our gamma manifold (M, g) can be computed using Gray's Mathematica code to yield:

$$K_g(\mu, \beta) = \frac{3 - 4\,\beta\,\psi'(\beta) - \beta^2\,\psi''(\beta)}{2\,\beta\,(-1 + \beta\,\psi'(\beta))^2} \quad \text{for } \mu, \beta \in \mathbb{R}^+.$$

We note that $K_g(\mu, \beta) \to -1$ as $\beta \to 0$. Further geometric features of gamma manifolds will be presented elsewhere.

Using Deng's metric and the Brioschi formula, the manifold (M, h) has Gaussian curvature

$$K_h(\mu, \beta) = \frac{\mu^2 + \beta^2 + \beta^2\,\psi'(\beta) + \beta^2\,(\mu^2 + \beta)\,\psi''(\beta)}{4\,(\mu^2 + \beta)^2\,(-1 + \beta + \beta\,\psi'(\beta))^2} \quad \text{for } \mu, \beta \in \mathbb{R}^+,$$

and $K_h(\mu, \beta) \to 0$ as $\beta \to 0$ for $\mu > 0$. This is certainly more interesting geometrically than the previous case K_g, but differs from the following expression for Gaussian curvature of (M, h) reported by Deng on page 63 of [5]:

$$K_h^*(\mu, \beta) = \frac{\mu^2\beta^2\psi(\beta)}{(\mu^2\beta + \beta^2 + \mu^2\psi(\beta))(\mu^2 + \beta)(\beta + \psi(\beta))}$$

Further families of gamma distributions arise in the multivariate case, for example see Johnson and Kotz [12], which can accommodate correlation among the event space variables and introduce higher dimensional parameter spaces.

Stochastic Geometry

For some background to stochastic geometry see Baddeley [2] and Stoyan et al. [17] and for stochastic processes in general see Cox and Hall [4] and Papoulis [15], for example. The simplest stochastic geometric structure is perhaps a 'random' distribution of points along a line—which serves as a model for a queuing process. However, this begs the question of what is meant by 'random' and the usual model is that of a Poisson point process [4, 11], then the successive gaps between points are drawn independently from an exponential distribution, $p(x; \mu, 1)$ in $(\star)$. In higher dimensions, a Poisson point process in a bounded region arises when the locations of points are independently assigned with the expected number in a zone proportional to the measure of the zone. Then the probability that a zone of measure A contains n points is

$$P(n) = e^{-\rho A}\frac{(\rho A)^n}{n!}$$

where ρ is the density of the underlying point process. In the case of dimension 1, the Poisson process on a line yields the exponential distribution for gaps by substitution of $n = 0$.

By a planar network of random line segments ('fibres') we mean the realisation of a Poisson point process [4, 11] in a (large) bounded region $U \subset \mathbb{R}^2$, together with a uniform distribution of angles to a fixed direction for lines drawn through the points. Such a structure partitions U into polygons in the case of segments of infinite length and it turns out that the mean number of sides per polygon is four and on each line we have a 1-dimensional Poisson process. Corte and Lloyd [3] approximated the distribution of areas of the polygons so formed as the product of two exponential distributions, which are known to give a good approximation to the inter-crossing lengths in a random network [6]. In fact, practical applications to flow of fluids through stochastic fibrous networks require rather a high density of fibres and so the fact that they are not of infinite length is rather unimportant. We repeated the analysis using the gamma distribution as a generalisation of inter-crossing length distributions for more general stochastic fibre networks [8]; then the exponential distribution is a special case. We note also that the gamma distribution has the Chi-square distribution as another special case, it corresponds to an Erlang distribution and a Beta distribution, and has as limiting case a Normal distribution (cf. [13]).

The gamma distribution has a probability density function given by ($\star$) above. The exponential distribution is a gamma distribution with $\beta = 1$ and hence also $Var(x) = \mu^2$. Thus, μ and β are the parameters in the family and represent possible departures in S from the random case $\beta = 1$; such departures arise from clumping of fibres (larger variance than random) or dispersion of fibres (smaller variance than random). For clumped or 'flocculated' stochastic fibre networks, we expect $\beta < 1$ and so $\frac{\mu^2}{\beta} > \mu^2$ and the variance to increase with increasing fibre clumping. For dispersed stochastic fibre networks, we expect $\beta > 1$ and $\frac{\mu^2}{\beta} < \mu^2$ and the variance to decrease with increasing uniformity.

We consider the product of two independent identical gamma distributions $p(x)$ and $p(y)$ such that $xy = a$ where a is the area of a rectangular pore. The probability density of a will be given by:

$$h(a) = \int_0^\infty \frac{1}{x}p(x)p(\frac{a}{x})\,dx.$$

Evaluation of this integral gives us,

$$h(a) = \frac{2\,a^{\beta-1}\,\beta^{2\,\beta}\,K_0(2\beta\sqrt{a}/\mu)}{\mu^{2\,\beta}\Gamma(\beta)^2} \qquad (\star\star)$$

and K_0 is the zeroth order modified Bessel function of the second kind. The distribution given by equation $(\star\star)$ has mean, $\bar{a} = \mu^2$ and variance $Var(a) = \frac{(1+2\,\beta)\,\mu^4}{\beta^4}$.

Following Corte and Lloyd, we define an equivalent pore radius r which is given by $a = \pi r^2$. The probability of finding an equivalent pore radius $r_1 \leq r \leq r_2$, is given by:

$$\int_{\pi r_1{}^2}^{\pi r_2{}^2} p(a)\,da = \int_{r_1}^{r_2} p(\pi r^2)\,2\pi r\,dr\,.$$

So the probability density function for equivalent pore radii is:

$$q(r) = 2\,\pi r\,p(\pi r^2)\,,$$

which gives us:

$$q(r) = 4\pi^\beta \left(\frac{\beta}{\mu}\right)^{2\beta} r^{2\beta-1}\frac{K_0(z)}{\Gamma(\beta)^2}$$

where $z = 2\beta r\sqrt{\pi}/\mu$ and $\int_0^\infty q(r)\,dr = 1$. The mean and variance of $q(r)$ are given by:

$$\bar{r} = \frac{\mu\Gamma(\beta+\frac{1}{2})^2}{\beta\sqrt{\pi}\,\Gamma(\beta)^2} \quad \text{and} \quad Var(r) = \bar{r}^2\left(\frac{\beta^2\,\Gamma(\beta)^4}{\Gamma(\beta+\frac{1}{2})^4} - 1\right).$$

For a *random* network, $\beta = 1$ and the distribution of pore radii has mean, $\bar{r} = \frac{\mu\sqrt{\pi}}{4\beta}$, and variance, $Var(r) = (\frac{\mu}{\beta})^2\left(\frac{1}{\pi} - \frac{\pi}{16}\right)$ in agreement with Corte and Lloyd [3].

Departures from randomness, in the form of dispersion or clumping, will be representable as curves in the gamma manifold (M, g) and can be quantified by their length. This will be taken up elsewhere.

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RECENT RESULTS ON THE COHOMOLOGY AND HOMOLOGY OF POISSON MANIFOLDS

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Abstract.– We give a survey concerning recent developments on the homology and cohomology of Poisson manifolds.

1 Introduction

Since its introduction by Lichnerowicz, Poisson manifolds have gained an increasing interest in geometry and physics. Indeed, they are the natural setting for hamiltonian dynamics, and at the same time they present interesting geometrical properties.

The existence of a Poisson structure on a manifold M provides a natural cohomology and homology. The Lichnerowicz-Poisson cohomology is obtained by considering the complex of p-vectors endowed with the natural operator $\sigma(P) = -[\Lambda, P]$, where Λ is the Poisson tensor and $[\,,\,]$ is the Schouten-Nijenhuis bracket. This cohomology provides a good framework for deformation and quantization. The LP-cohomology is similar to the de Rham cohomology on forms, but it depends on the existence of the Poisson structure. In fact, for symplectic manifolds, they are isomorphic, but the result does not hold for arbitrary non-symplectic Poisson manifolds. Its computation is far to be a trivial matter, and in Section 3 we collect a huge number of results in this direction.

The Poisson or canonical homology is obtained by defining the Koszul operator $\delta = [i(\Lambda), d]$ on forms. For symplectic manifolds, it is isomorphic to the de Rham cohomology of M, but the result fails for arbitrary Poisson manifolds. The canonical homology was extensively studied by Brylinski who introduced the so-called double canonical complex. Also, Brylinski proposed the study of a symplectic harmonic theory which led to discuss very interesting problems on symplectic manifolds. The spectral sequences associated to the double canonical complex was also extensively studied in recent years. Canonical homology and

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LP-cohomology are related by a natural pairing as was remarked by Bhaskara and Viswanath. All these problems are discussed in Section 4.

Finally, we remark that these concepts were recently extended in two directions: for Jacobi manifolds [10, 11, 12, 31, 32, 33, 34], and for multibrackets [24, 25, 26, 27, 28].

2 Poisson structures

In this section we shall describe several ways of defining a Poisson structure on a differentiable manifold. Also, we shall show several outstanding examples of this structure, as well as properties and related topics.

Let M be an m-dimensional manifold and denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields and by $C^\infty(M)$ the algebra of smooth functions on M. A *Poisson bracket* on M is a bilinear mapping $\{\,,\}$ on $C^\infty(M)$ satisfying the following properties:

i) skew-symmetry: $\{f,g\} = -\{g,f\}$,

ii) Leibniz rule: $\{f,gh\} = \{f,g\}h + g\{f,h\}$,

iii) Jacobi identity: $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$,

for $f,g,h \in C^\infty(M)$. A *Poisson manifold* is a smooth manifold equipped with a Poisson bracket.

- From *i*) and *iii*) we observe that the algebra $C^\infty(M)$ of smooth functions on M is a Lie algebra with respect to the Poisson bracket. Indeed, it is a *Poisson algebra* (i.e. a Lie algebra with an associative multiplication satisfying the Leibniz rule).

- Property *i*) is equivalent to the vanishing of the Poisson bracket $\{f,f\}$ for any $f \in C^\infty(M)$ (i.e. the conservation law of energy).

- Taking into account the Leibniz rule, the mapping $f \mapsto X_f(g) = \{f,g\}$ defines a vector field X_f on M, called the *Hamiltonian vector field* associated to f.

- The Jacobi identity is equivalent to the property $X_{\{f,g\}} = [X_f, X_g]$, which means that the mapping $f \in C^\infty(M) \mapsto X_f \in \mathfrak{X}(M)$ is a Lie algebra homomorphism. The kernel of this mapping consists of the *Casimir* functions. Also, the Jacobi identity is equivalent to the fact that X_f is a derivation of $(C^\infty(M), \{,\})$, i.e. $X_f\{g,h\} = \{X_f g, h\} + \{g, X_f h\}$.

Another way of defining a Poisson structure is by considering a skew-symmetric tensor field Λ of type $(2,0)$ satisfying $[\Lambda,\Lambda] = 0$, where $[\,,]$ is the Schouten-Nijenhuis bracket [3, 43]. We say that Λ is the *Poisson tensor.*

- The relation between the Poisson tensor with the Poisson bracket is given by the following formula:

$$\{f, g\} = \Lambda(df, dg), \quad f, g \in C^\infty(M).$$

The property $[\Lambda, \Lambda] = 0$ is equivalent to the Jacobi identity.

- We can consider the associated mapping $\# : \Lambda^1(M) \longrightarrow \mathfrak{X}(M)$ defined by $\#(\alpha) = i(\alpha)\Lambda$ (i.e. $\#(\alpha)(\beta) = \Lambda(\alpha, \beta)$, for $\beta \in \Lambda^1(M)$). For $x_0 \in M$, we define the *rank* of Λ at x_0 as $rank\,\Lambda_{x_0} = rank\,\#_{x_0}$.

Next, we shall describe several examples.

Symplectic manifolds. Let (M, ω) be a symplectic manifold of dimension $2n$, that is, $\omega \in \Lambda^2(M)$ is closed ($d\omega = 0$) and non-degenerate everywhere ($\omega^n \neq 0$). For a smooth function $f \in C^\infty(M)$, the associated Hamiltonian vector field is defined by

$$i(X_f)\omega = df.$$

Therefore, the Poisson bracket is

$$\{f, g\} = -\omega(X_f, X_g).$$

For symplectic manifolds, the associated mapping $\#$ is an isomorphism and the Casimir functions are the constants (that is, the Poisson bracket is non-degenerate) (see [1, 36]).

Structures on $\mathbb{R}^m$. Let M be a Poisson manifold and take local coordinates $(x_1, \cdots, x_m)$, then the Poisson tensor Λ is locally represented as

$$\Lambda = \frac{1}{2}\Lambda_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where the coefficients $\Lambda_{ij} = \{x_i, x_j\}$ satisfy certain conditions: $\Lambda_{ij} = -\Lambda_{ji}$ and

$$\sum_{l=1}^{m} \left(\Lambda_{lj}\frac{\partial \Lambda_{ik}}{\partial x_l} + \Lambda_{li}\frac{\partial \Lambda_{kj}}{\partial x_l} + \Lambda_{lk}\frac{\partial \Lambda_{ji}}{\partial x_l} \right) = 0,$$

for $i, j, k = 1, \cdots, m$. Assume now that $M = \mathbb{R}^m$ and that $x_1, \ldots, x_m$ are the standard coordinates. There exists an special interest (with mathematical and physical motivations) in the study and classification of polynomial Poisson structures (i.e. Λ_{ij} are polynomial). Special attention is given to homogeneous structures (see [42]):

i) If each Λ_{ij} is a constant, then the Poisson structure Λ is said to be *constant*.

ii) If each Λ_{ij} is a homogeneous linear polynomial, then the Poisson structure Λ is said to be *linear* [52] (see Lie-Poisson structures).

iii) If each Λ_{ij} is a homogeneous quadratic polynomial, then the Poisson structure Λ is said to be *quadratic* [2, 14, 39]. Each quadratic Poisson structure corresponds to a solution of the classical Yang-Baxter equation on $End(\mathbb{R}^n)$ and viceversa.

Lie-Poisson structures. Let $\mathfrak{g}$ be a Lie algebra of finite dimension, then the dual space $\mathfrak{g}^*$ has a natural Poisson structure defined by

$$\{\varphi, \psi\}(\gamma) = \gamma([d_\gamma\varphi, d_\gamma\psi]),$$

where $\gamma \in \mathfrak{g}^*$ and $\varphi, \psi \in C^\infty(\mathfrak{g}^*)$, therefore $d_\gamma\varphi, d_\gamma\psi \in \mathfrak{g}$ since one can look them as linear mappings from $T_\gamma\mathfrak{g}^* \cong \mathfrak{g}^*$ to $\mathbb{R}$. This bracket is called the *Lie-Poisson structure* on $\mathfrak{g}^*$.

Let $\nu_1, \cdots, \nu_m$ be a basis for $\mathfrak{g}$ with structure constants c_{ij}^k, that is, $[\nu_i, \nu_j] = c_{ij}^k \nu_k$, and let $\mu_1, \cdots, \mu_m$ be the corresponding coordinate functions on $\mathfrak{g}^*$. The above bracket is then defined by

$$\{\mu_i, \mu_j\} = c_{ij}^k \mu_k,$$

that is, it is a linear Poisson structure. This is the well-known example due to Lie [38] and studied by Kirillov, Kostant, Soriau and others.

Some constructions of Poisson structures.

- Let (M_i, Λ_i) $(i = 1, 2)$ be two Poisson manifolds, then the product manifold $M_1 \times M_2$ is a Poisson manifold with the structure $\Lambda_1 + \Lambda_2$.
- Let (M, Λ) be a Poisson manifold, then the tangent space TM is a Poisson manifold with the complete lift Λ^c of Λ to TM.
- Given two Poisson tensors Λ_1 and Λ_2 on a manifold M, then the tensor field $\Lambda_1 + \Lambda_2$ is a Poisson tensor if and only if $[\Lambda_1, \Lambda_2] = 0$, and in this case we say that the two Poisson structures are *compatible*.

Poisson-Lie groups. Let G be a Lie group, endowed with a Poisson structure Λ. Then, (G, Λ) is a *Poisson-Lie group* if the multiplication $G \times G \longrightarrow G$ is a Poisson mapping. A necesary and sufficient condition for a pair (G, Λ) to be a Poisson-Lie group is that

$$\Lambda_{g_1 g_2} = L_{g_1 *}\Lambda_{g_2} + R_{g_2 *}\Lambda_{g_1}\ ,\ \forall g_1, g_2 \in G.$$

In particular, we must have $\Lambda_e = 0$, for e the unit of G (see [41]).

A mapping $\phi : (M_1, \Lambda_1) \longrightarrow (M_2, \Lambda_2)$ between two Poisson manifolds is called a *Poisson morphism* if

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi,\ \forall f, g \in C^\infty(M_2),$$

or, equivalently, $\phi^* : C^\infty(M_1) \longrightarrow C^\infty(M_2)$ is a Poisson-algebra homomorphism.

It can be seen that the Hamiltonian vector fields are infinitesimal automorphisms of the Poisson tensor, that is, $L_{X_f}\Lambda = 0$. As a consequence, the flow Φ_t associated to X_f consists of Poisson morphisms.

The local structure of Poisson manifolds is described by the splitting theorem due to Weinstein [52]: let (M, Λ) be a Poisson manifold of dimension m and $x_0 \in M$ such that $rank\,\Lambda_{x_0} = 2r$, then there exists an open neighborhood U of x_0 such that $(U, \Lambda_{|U})$ is "Poisson-isomorphic" to a product manifold $S \times N$, where S is a $2r$-dimensional symplectic manifold and N a Poisson manifold of rank 0 in the image of x_0.

We can interpret this result as follows: there exists a coordinate neighborhood $(U; q^i, p_i, z^a)$ (for $i = 1, \cdots, r$ and $a = 1, \cdots, m-2r$) of x_0 such that

$$\{q^i, q^j\} = \{p_i, p_j\} = 0, \quad \{q^i, p_j\} = \delta_i^j, \quad \{q^i, z^a\} = \{p_i, z^a\} = 0,$$

and $\{z^a, z^b\}$ is a function of $z^1, \cdots, z^{m-2r}$ that vanishes at x_0. Notice that for constant rank Poisson structures $\Lambda_N = 0$ or equivalently, $\{z^a, z^b\} = 0$.

Poisson manifolds are foliated by symplectic leaves [50, 52]. Given a Poisson manifold (M, Λ), it can be defined a distribution $S(M)$ on M (called the *characteristic distribution*) by means of the Hamiltonian vector fields:

$$S_{x_0}(M) = \{v \in T_{x_0}(M) \mid \exists f \in C^\infty(M) : X_f(x_0) = v\}.$$

It is known that the characteristic distribution is completely integrable, so it defines a foliation on M, and the Poisson structure induces symplectic structures on the leaves of the foliation (each leaf has dimension equal to the rank of Λ at the points of the leaf).

In particular, the symplectic leaves of the Lie-Poisson structure of the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ are the orbits of the coadjoint representation of any connected Lie group G whose Lie algebra is $\mathfrak{g}$ [52].

3 Lichnerowicz-Poisson cohomology

Denote by $\mathcal{V}^p(M)$ the space of tensor fields of type $(p, 0)$ (in particular, $\mathcal{V}^1(M) = \mathfrak{X}(M)$). We define the contravariant exterior derivative $\partial : \mathcal{V}^p(M) \longrightarrow \mathcal{V}^{p+1}(M)$ by

$$\partial P = -[\Lambda, P], \quad for \quad P \in \mathcal{V}^p(M)$$

Koszul [30] has proved that $\partial^2 = 0$. Thus, it defines a differential complex $(\mathcal{V}^*(M), \partial)$ and its cohomology is called the *LP (Lichnerowicz-Poisson)-cohomology* [37]. We denote by $H^p_{LP}(M)$ (or, in case of confusion, $H^p_{LP}(M; \Lambda)$) the cohomology group of order p. The LP-cohomology can also be defined for Poisson algebras (see [21]).

We have the following results:

- $H^0_{LP}(M) = \{f \in C^\infty(M) \,|\, X_f(g) = 0, \forall g \in C^\infty(M)\}$ = the space of Casimir functions. Then, it is clear that for a (connected) symplectic manifold $H^0_{LP}(M) = \mathbb{R}$.

- $H^1_{LP}(M) = \{X \in \mathfrak{X}(M) \,|\, L_X\Lambda = 0\}/\{X_f \,|\, f \in C^\infty(M)\}$, that is, it is the quotient space of the infinitesimal automorphisms by the Hamiltonian vector fields.

- $\partial\Lambda = 0$, so the Poisson tensor defines a fundamental class $[\Lambda] \in H^2_{LP}(M)$. If $[\Lambda] = 0$ (that is, there exists a vector field A such that $L_A\Lambda = -\Lambda$), then it is said that (M, Λ) is a *homogeneous Poisson manifold* or also an *exact Poisson manifold* [13]. For instance, the Lie-Poisson structure of a coadjoint Lie algebra $\mathfrak{g}^*$ is exact. This kind of Poisson manifolds play an important role in the study of Jacobi manifolds [13].

- $\partial(P \wedge Q) = \partial(P) \wedge Q + (-1)^p P \wedge \partial(Q)$, so we can define a LP-cup product in the LP-cohomology algebra.

- The LP-cohomology satisfies the Mayer-Vietoris exact sequence property [49, 50], i.e. if U, V are open subsets of M, there is an exact sequence of the form

$$\cdots \to H^k_{LP}(U \cup V) \to H^k_{LP}(U) \oplus H^k_{LP}(V) \to H^k_{LP}(U \cap V) \to H^{k+1}_{LP}(U \cup V) \to \cdots$$

Unlike for the de Rham cohomology, the LP-cohomology has no functorial character. The best assertion we can make is that for (M_i, Λ_i) $(i = 1, 2)$ Poisson manifolds and $\phi : M_1 \longrightarrow M_2$ a Poisson morphism which is a local diffeomorphism, then one has an induced homomorphism

$$\phi^* : H^k_{LP}(M_2) \longrightarrow H^k_{LP}(M_1).$$

Moreover, if ϕ is a diffeomorphism, then ϕ^* is an isomorphism.

If we extend the mapping $\#$ associated to a Poisson tensor Λ to a mapping $\# : \Lambda^k(M) \longrightarrow \mathcal{V}^k(M)$ by putting

$$\#(\lambda)(\alpha_1, \cdots, \alpha_k) = (-1)^k \lambda(\#(\alpha_1), \cdots, \#(\alpha_k)),$$

for $\lambda \in \Lambda^k(M)$ and $\alpha_1, \cdots, \alpha_k \in \Lambda^1(M)$, then it can be proved [37, 3] that $\partial \circ \# = -\# \circ d$. Therefore, we have induced homomorphisms in cohomology

$$\# : H^{\cdot}_{DR}(M) \longrightarrow H^{\cdot}_{LP}(M),$$

which is an isomorphism if the Poisson structure comes from a symplectic structure of M [37]. However, the above result does not hold for non-symplectic Poisson manifolds as the following examples show.

- The compact manifold $M(k,n)$. In [18] we have considered the completely solvable four-dimensonal manifold $M(k,n)$ defined by the 1-forms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ such that

$$d\alpha_1 = -k\alpha_1 \wedge \alpha_3, \quad d\alpha_2 = k\alpha_2 \wedge \alpha_3, \quad d\alpha_3 = 0, \quad d\alpha_4 = n\alpha_1 \wedge \alpha_2,$$

where k is a real number such that $e^k + e^{-k}$ is different from 2 and n a non-zero real number. $M(k,n)$ is a compact quotient of a completely solvable Lie group $G(k,n)$ by a discrete and uniform subgroup. Let $\{X_1, X_2, X_3, X_4\}$ be the dual basis of vector fields, then

$$[X_1, X_2] = -nX_4, \quad [X_1, X_3] = kX_1, \quad [X_2, X_3] = -kX_2,$$

all the other brackets being zero. Hence, $G = X_3 \wedge X_4$ is a Poisson tensor coming from a left invariant Poisson structure on $G(k,n)$. It can be seen that $H^2_{DR}(M(k,n)) = 0$, however $dim\, H^2_{LP}(M(k,n)) \geq 2$.

- Vaisman [50] considered the quadratic Poisson structure

$$\Lambda = (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

on the noncompact $\mathbb{R}^2$; he proved that $H^1_{LP}(\mathbb{R}^2) \neq 0$, however $H^1_{DR}(\mathbb{R}^2) = 0$.

Nevertheless, it was not clear what $H^2_{LP}(M(k,n))$ and $H^1_{LP}(\mathbb{R}^2)$ are in the above examples. In general, the computation of LP-cohomology is a difficult problem, because of the lack of a powerful method. Now, we shall comment some results in this direction.

For a *regular* Poisson manifold (M, Λ) (that is, the rank of Λ is constant everywhere), the computation of the LP-cohomology was started by Lichnerowicz [37]. Later Vaisman and Xu have obtained new results.

- Vaisman studied in [49] the LP-cohomology for regular Poisson manifolds by observing that there is a Serre-Hochschild spectral sequence that converges to the LP-cohomology. By using a distribution S' transverse to the symplectic foliation S of (M, Λ), he obtains new expressions for this cohomology. In particular, some simple expressions are obtained when the symplectic foliation is either transversally Riemannian or transversally symplectic. For example, in the simple case where M is a symplectic manifold of finite type and N is a smooth manifold with the zero Poisson structure, we have for the Poisson product manifold $M \times N$ the following result:

$$H^k_{LP}(M \times N) = \bigoplus_{q=0}^{k} \left[H^q_{DR}(M) \otimes \Lambda^{k-q}(N) \right].$$

- P. Xu computes in [54] the LP-cohomology by means of symplectic groupoids. For an integrable Poisson manifold (i.e. Poisson manifolds admitting a global symplectic groupoid) the LP-cohomology is naturally isomorphic to the de Rham cohomology of the left invariant forms on the symplectic groupoid. Such idea was suggested by Vorob'ev and Karasev in [51], where they computed the LP-cohomology groups of degrees 1,2 or 3 for certain classes of Poisson manifolds.

If (M, Λ) is not regular the situation becomes more involved. Some results in the linear case and for Lie-Poisson structures have been obtained in [20, 40, 46]. And for quadratic structures in [47].

- For a Poisson-Lie group G with Lie algebra $\mathfrak{g}$, its dual $\mathfrak{g}^*$ becomes a Lie algebra and its associated Lie group G^* is also a Poisson-Lie group. In [40] and [20], it has been studied the LP-cohomology for (dual) Poisson-Lie groups, and from these results follows that

  $$H^k_{LP}(\mathfrak{g}^*) = H^k(\mathfrak{g}) \otimes \{\, \textit{Casimir functions} \},$$

  where $H^{\cdot}(\mathfrak{g})$ denotes the cohomology of the Lie algebra $\mathfrak{g}$.

- Nakanishi considers in [46] some particular Lie-Poisson structures, as for instance $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, and using the Chevalley-Eilenberg complex, he determines the first cohomology space and relates that space with the space of Casimir functions which are flat at the origin.

- In a recent article of Nakanishi [47], the LP-cohomology of the quadratic Poisson structures on the plane $\mathbb{R}^2$ is computed (including the above example of Vaisman). He proves that if Λ is exact, then $H^0_{LP}(\mathbb{R}^2) = \mathbb{R}$ and for degrees 1,2 the LP-cohomology groups are infinite dimensional; but, if Λ is not exact, then $H^0_{LP}(\mathbb{R}^2) = \mathbb{R}$ and $H^k_{LP}(\mathbb{R}^2) = \mathbb{R} \oplus \mathbb{R}$, for $k = 1, 2$. For quadratic Poisson structures in R^m $(m > 2)$ the problem is still open.

- Recently, Ginzburg has studied the behaviour of the Poisson cohomology respect to momentum mappings and reduction [19].

4 Canonical homology

For a Poisson manifold (M, Λ), Koszul [30] introduced the differential operator $\delta : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M)$ defined by

$$\delta = [i(\Lambda), d] = i(\Lambda) \circ d - d \circ i(\Lambda),$$

where $i(\Lambda)$ denotes the contraction by Λ. Alternatively, Brylinski [5] gave the following expression for δ:

$$\delta(f_0\, df_1 \wedge \cdots \wedge df_k) = \sum_{1\le i\le k} (-1)^{i+1}\, \{f_0, f_i\}\, df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k$$
$$+ \sum_{1\le i<j\le k} (-1)^{i+j}\, f_0\, d\{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k\,.$$

Using the Jacobi identity it can be proved that $\delta^2 = 0$, so we obtain the canonical complex $(\Lambda^{\cdot}(M), \delta)$ whose homology $H^{can}_{\cdot}(M)$ is called the *canonical homology* (or also the *Poisson homology*). As for the LP-cohomology, the canonical homology can be considered for Poisson algebras [21].

In contrast with the LP-cohomology, canonical homology has a functorial character, that is, if $\phi : (M_1, \Lambda_1) \longrightarrow (M_2, \Lambda_2)$ is a Poisson morphism, we have induced homomorphism in homology

$$\phi^* : H^{can}_k(M_1) \longrightarrow H^{can}_k(M_2).$$

However, the canonical complex is not multiplicative.

For a symplectic manifold (M, ω) of dimension $2n$, *the symplectic star operator* can be defined [5] by

$$\beta \wedge *(\alpha) = \frac{1}{n!}\Lambda^k(G)(\beta, \alpha)\omega^n\,,$$

for $\alpha, \beta \in \Lambda^k(M)$ (see also [35]). Brylinski proved that

$$(1) \qquad *^2 = -Id, \quad \delta\alpha = (-1)^{k+1} * d * (\alpha), \quad for\, \alpha \in \Lambda^k(M).$$

An immediate consequence of these properties is that for symplectic manifolds the canonical homology is isomorphic to the de Rham cohomology:

$$(2) \qquad H^{can}_k(M) \cong H^{2n-k}_{DR}(M)\,.$$

Now, taking into account properties (1) and the Riemann-Hodge theory, Brylinski asked about the possibility of a symplectic Hogde theory. More precisely, he made the following conjecture [5]: if M is a compact symplectic manifold, any de Rham cohomology class has a *symplectically harmonic* representative ν (i.e. $d\nu = \delta\nu = 0$). More generally, he considered the problem for arbitrary Poisson manifolds:

Problem A: Give conditions on a compact Poisson manifold M which ensure that any de Rham cohomology class in $H^k_{DR}(M)$ has a *harmonic (with respect to the Poisson structure)* representative ν, that is, $d\nu = \delta\nu = 0$.

- Compact Kähler manifolds. The first evidence of the above conjecture was proved by Brylinski himself [5], that is, Problem A is satisfied for compact Kähler manifolds. First, it is proved a relation between the Hodge and symplectic star operators, then the result follows from the decomposition Hodge theorem for compact Kähler manifolds (i.e. each harmonic k-form can be decomposed as a sum of harmonic forms of pure type (p,q), with respect to the bigraduation induced by the complex structure) and the Riemann-Hodge theory (about the existence of a -Riemannian- harmonic representative for each de Rham cohomology class).

- Compact symplectic manifolds. The conjecture is not true in general as it was shown in [15, 45, 55]. In [15] we have shown a counterexample in dimension 4, the Kodaira-Thurston manifold. The Kodaira-Thurston manifold is a 4-dimensional compact nilmanifold $\Gamma\backslash G$ (that is, the quotient space of right cosets of a connected, simply-connected nilpotent Lie group G by a lattice Γ), determined by a basis of left invariant 1-forms $\{\alpha\}_{i=1}^{4}$ such that $d\alpha_1 = d\alpha_2 = d\alpha_4 = 0$ and $d\alpha_3 = -\alpha_1 \wedge \alpha_2$. By topological reasons this manifold has no Kähler structures [8], but it is symplectic. Consider the symplectic form $\omega = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4$. Using the symplectic relations [15] $[L,d] = 0$, $[L,\delta] = -d$, $i(\Lambda) = -*L*$ and $*(\beta) = -(n-1)!L^{n-1}(\beta)$, for a 1-form β, we have shown that there exists a de Rham cohomology class of degree 3 $[\alpha_2 \wedge \alpha_3 \wedge \alpha_4] \in H^3_{DR}(KT)$ without symplectically harmonic representative. Independently, Mathieu [45] (and also Yan [55]) obtained the following result: A compact symplectic manifold satisfies the Brylinski conjecture if and only if it satisfies the Hard Lefschetz theorem. Mathieu's proof involves the representation theory of quivers and Lie superalgebras, and Yan studies a special type of infinite dimensional $sl(2)$-representation (called an $sl(2,\mathbb{C})$-module of finite H-spectrum); then, both apply these results to study the space of differential forms on M and its subspace of symplectically harmonic forms.

- Compact almost cosymplectic manifolds. Let us recall that an almost cosymplectic manifold [4] is a $(2n+1)$-dimensional manifold with a closed 2-form Φ and a closed 1-form η such that $\Phi^n \wedge \eta \neq 0$. Roughly speaking, almost cosymplectic manifolds may be considered as the odd-dimensional counterpart of the symplectic manifolds, and as for them there exists a natural Poisson structure associted to it (see [6, 16]). Moreover, cosymplectic manifolds appear to be the closest odd-dimensional analogous of Kähler manifolds [9]. In [22] it has been proved that Problem A is satisfied for compact cosymplectic manifolds. After defining the almost cosymplectic star operator, it is considered an homological operator δ_2 related to the Koszul operator (but different to it, also the homology of both complexes is different). Moreover, a cosymplectic decomposition Hodge theorem is proved (by using some results of [9]). Then, comparing almost cosymplectic and Hodge star operators and using the Riemann-Hodge theory, the Problem A is solved for the operator δ_2. Finally, Problem A is obtained by comparing δ and δ_2. Also in [22] it is proved that Problem A is not satisfied for compact almost cosymplectic manifolds by showing a counterexample.

Imitating Connes [7], Brylinski introduced the double complex $C_{\cdot,\cdot}(M)$ defined by $C_{k,l}(M) = \Lambda^{l-k}(M)$, for $k, l \geq 0$. This double complex has d for horizontal diferential and δ for vertical differential (both of degree -1). Notice that $d\delta + \delta d = 0$. This double complex is concentrated in the first quadrant and we can consider the periodic double complex $C^{per}_{\cdot,\cdot}(M)$ such that $C^{per}_{k,l}(M) = \Lambda^{l-k}(M)$, for $k, l \in \mathbb{Z}$.

Therefore, there are two spectral sequences (of homological type) associated with this periodic double complex: $\{E^r_{\cdot,\cdot}, \delta_r\}$, $\{'E^r_{\cdot,\cdot}, '\delta_r\}$ (called the *first and second spectral sequences*). Both of these spectral sequences converge to the total homology $H^D_\cdot(M)$, that is, the homology of the complex $(C_\cdot(M), D = d + \delta)$, where $C_k(M) = \oplus_{p+q=k} C^{per}_{p,q}(M)$. The canonical homology is the first term of the first spectral sequence, $E^1_{p,q}(M) \cong H^{can}_{q-p}(M)$, and the de Rham cohomology is the first term of the second spectral sequence, $'E^1_{p,q}(M) \cong H^{q-p}_{DR}(M)$ (for more details see [5, 16, 17, 18]).

By using the relation [16]

$$ki(\Lambda)di(\Lambda)^{k-1} = i(\Lambda)^k d + (k-1)di(\Lambda)^k, \text{ for } k \in \mathbb{N},$$

we have shown [16] that the second spectral sequence degenerates at the first term $'E^1(M)$ for any Poisson manifold, therefore $'E^1(M) \cong {'E^2(M)} \cong \cdots \cong {'E^\infty(M)}$. As a consequence, the total homology is a topological invariant, and finite dimensional.

Moreover, Brylinski asked [5] about the degeneration of the first spectral sequence:

Problem B. Give conditions on a compact Poisson manifold which ensure the degeneracy at $E^1(M)$ of the first spectral sequence.

- Symplectic manifolds. It was proved in [5] that Problem B is satisfied for compact symplectic manifolds. More generally, we have shown [16] that, by means of the symplectic star operator, both spectral sequences are isomorphic for symplectic manifolds. Then, by the degeneration of the second spectral sequence at the first term for any Poisson manifold (in particular, for symplectic manifolds), we recover Brylinski's result.

- Almost cosymplectic manifolds. In [16] it is shown an example of almost cosymplectic manifold for which the first spectral sequence does not degenerate at the first term. Moreover, we study in such paper the canonical homology of almost cosymplectic manifolds.

Also in [5] it was suggested a relation between the answers of the two problems. In [17] we observed that both problems are completely independent.

- Problem A $\not\Rightarrow$ Problem B. In [17] we consider the Kodaira-Thurston manifold with a degenerate Poisson structure (of constant rank 2) satisfying that any de Rham cohomology class has a harmonic (respect to the Poisson structure) representative (i.e. Problem A), but the first spectral sequence does not degenerate at the first term.
- Problem B $\not\Rightarrow$ Problem A. As we have shown the Kodaira-Thurston manifold with a symplectic structure does not satisfy Brylinski conjecture, but (as for any symplectic manifold) its first spectral sequence degenerates at the first term.

As for the LP-cohomology, there is no an isomorphism between the canonical homology and the de Rham cohomology (2). Indeed, for an arbitrary Poisson manifold we have

$$H_k^{can}(M) \not\cong H_{DR}^{m-k}(M).$$

Some counterexamples have been shown in [16] (a compact almost cosymplectic manifold) and in [18] (a compact Poisson manifold). In [50] it can be seen that the above quadratic Poisson structure considered by Vaisman is again a counterexample for the isomorphism (2), in fact, we have $H_2^{can}(\mathbb{R}^2) = 0 \neq H_{DR}^0(\mathbb{R}^2) = \mathbb{R}$. Finally, although for compact symplectic and almost cosymplectic manifolds [16] the canonical homology and the LP-cohomology are isomorphic, again the Vaisman example gives a couterexample to this isomorphism for arbitrary Poisson manifolds (remember that $H_{LP}^0(\mathbb{R}^2) = \mathbb{R}$).

- Vaisman [50], using the same technique as for the LP-cohomology has obtained some results about the canonical homology of regular Poisson manifolds. In particular, for a $2n$-dimensional symplectic manifold M of finite type and N a smooth manifold with the zero Poisson structure, we have the following result:

$$H_k^{can}(M \times N) = \bigoplus_{q=0}^{k} \left[H_{DR}^{2n-k+q}(M) \otimes \Lambda^q(N) \right].$$

- In [3] it has been proved that for $\alpha \in \Lambda^k(M)$ and $Q \in \mathcal{V}^{k-1}(M)$ one has:

$$i(\partial Q)\alpha = i(Q)(\delta\alpha) + (-1)^k \delta(i(Q)\alpha).$$

Therefore the natural pairing $i(P)\alpha$, for $\alpha \in \Lambda^k(M)$ and $P \in \mathcal{V}^k(M)$, yields a pairing

$$H_{LP}^k(M) \times H_k^{can}(M) \longrightarrow H_0^{can}(M).$$

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RIEMANNIAN VERSUS LORENTZIAN SUBMANIFOLDS. SOME OPEN PROBLEMS

A. Ferrández

All results contained in this paper have been made in collaboration with my colleagues Manuel Barros, Pascual Lucas and Miguel A. Meroño.

Contents

This is a survey of the following three subjects: B-scrolls, r-elastic curves and Willmore-Chen submanifolds. B-scrolls arose as the first important example of indefinite submanifolds having no Riemannian counterpart. They have played an essential role in a series of classification results of indefinite submanifolds which point out substantial differences between indefinite and Riemannian submanifolds (see [1], [2], [3], [4], [5], [15] and [16]). Then, following Pinkall, [24], we define indefinite Hopf cylinders and find a nice characterization of B-scrolls with constant mean curvature in $\mathbb{H}^3_1(-1)$ in terms of them (see [12] and [9]). Now two remarkable facts should be noticed. On one hand, looking at parametrizations of indefinite Hopf cylinders, we bring to mind the Betchov-Da Rios soliton equation (see [18], [25], [26], [27], [28] and [29]). Then we find solutions of this equation lying on B-scrolls: they are helices. Furthermore, we give a rational one-parameter family of closed solutions and show that the only soliton solutions are the null geodesics of the corresponding B-scroll (see [9]). On the other hand,

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we see that Hopf surfaces in $\mathbb{H}_1^3(-1)$ shaped on closed curves in the hyperbolic plane $\mathbb{H}^2(-1/4)$ are Lorentzian Hopf tori. Then we first determine the isometry group of Lorentzian Hopf tori and, secondly, we try to get solutions of the Willmore problem in $\mathbb{H}_1^3(-1)$. The latter will be solved, following again Pinkall, by means of the Langer and Singer viewpoint on elastic curves ([19], [20] and [21]) and the symmetric criticality principle of Palais [23].

As far as helices is concerned, we start by recalling that a general helix in a Euclidean 3-space is defined by the property that its tangent indicatrix is a planar curve. The straight line perpendicular to this plane is called the axis of the general helix. First of all we need a good definition appropriate for the new ambient spaces. Moreover, we have to consider both degenerate and non-degenerate general helices in $\mathbb{L}^3$, according to the causal character of its axis. Therefore, to define general helices in 3-dimensional De Sitter $\mathbb{S}_1^3$ and anti De Sitter $\mathbb{H}_1^3$ spaces we follow the idea of Langer and Singer, [21], and use Killing vector fields along curves. Namely, let M be a non flat 3-dimensional Lorentzian space form. A curve γ in M is said to be a general helix if there exists a Killing vector field V along γ with constant length and orthogonal to the acceleration vector field of γ. V will be the axis of γ. The helix is said to be degenerate or non-degenerate according to V is, respectively. In [6] Barros has shown that general helices are geodesics either of right general cylinders or of Hopf cylinders, according to the curve lies in $\mathbb{R}^3$ or $\mathbb{S}^3$, respectively. Now, general helices in $\mathbb{L}^3$ are geodesics in right general cylinders or in flat B-scrolls, according to the helix is non-degenerate or degenerate, respectively. In non flat 3-dimensional Lorentzian space forms the Lancret thorem underlines deep differences between pseudo-spherical and pseudo-hyperbolic spaces. The former has no non trivial general helices, the latter being nicely similar to $\mathbb{L}^3$. Whence roles played by $\mathbb{H}_1^3$ and $\mathbb{S}_1^3$ correspond with those played by $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively (see [10]).

The Willmore-Chen variational problem is the natural extension of the Willmore one. The first non trivial examples of Willmore-Chen submanifolds were given by Barros and Garay in [13]. We aim to find Willmore-Chen submanifolds in a pseudo-hyperbolic space $\mathbb{H}_r^n$. That will be done in several steps. After writing $\mathbb{H}_r^n$ as a warped product, we characterize $SO(r+1)$-invariant submanifolds of $\mathbb{H}_r^n$. Then we extend the concept of elastic curves to r-elastic curves and apply the symmetric criticality principle. As a consequence Willmore-Chen submanifolds in $\mathbb{H}_r^n$ are characterized in terms of r-generalized free elaticae in the once punctured unit sphere Σ^{n-r} (see [11]). Furthermore, following the classification of closed free elasticae in the standard 2-sphere obtained by Langer and Singer, [21], we show that there exist infinitely many Lorentzian Willmore tori in the 3-dimensional anti De Sitter space. Examples of Willmore tori in non-standard 3-spheres have been recently given by Barros in [7]. The same author has also found wide families of Willmore tori in warped product manifolds (see [8]).

1 Indefinite Hopf cylinders ([9,12])

Following Pinkall [24], we look for pseudo-Riemannian submersions

$$\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4),\ s = 0, 1.$$

Idea:
identify $\mathbb{H}^3_1(-1)$ with an appropriate subset of maps $\mathbb{R}^4_2 \to \mathbb{R}^4_2$.
How to do that:
P be a 2-dimensional subspace in $\mathbb{R}^4_2$ and $\{x, y\}$ an orthonormal basis of P.
Define maps

$$\begin{aligned} &f: P \to P, && f(x) = y, && f(y) = -x,\\ &g: P \to P, && g(x) = y, && g(y) = x,\\ &h: P \to P, && h(x) = -y, && h(y) = -x, \end{aligned}$$

which will be called **rotation, first reflection** and **second reflection** on P, respectively.

Let $\{e_1, e_2, e_3, e_4\}$ be the usual basis of $\mathbb{R}^4_2$ equipped with $(g_{ij}) = \mathrm{diag}[-1, -1, 1, 1]$.
Set $P_i = \mathrm{span}\{e_1, e_i\}$, $i = 2, 3, 4$ such that $\mathbb{R}^4_2 = P_i \oplus P_i^\perp$.
Consider the following maps

$$\begin{aligned} \rho &= f \times f: P_2 \oplus P_2^\perp \to P_2 \oplus P_2^\perp,\\ \sigma &= g \times h: P_3 \oplus P_3^\perp \to P_3 \oplus P_3^\perp,\\ \iota &= g \times g: P_4 \oplus P_4^\perp \to P_4 \oplus P_4^\perp. \end{aligned}$$

Then $\mathcal{F} = \mathrm{span}\{1, \rho, \sigma, \iota\}$ is a 4-dimensional vector space over $\mathbb{R}$ and the following identities hold

$$\begin{aligned} &\rho^2 = -1, && \sigma\rho = -\iota, && \iota\rho = \sigma,\\ &\rho\sigma = \iota, && \sigma^2 = 1, && \iota\sigma = \rho,\\ &\rho\iota = -\sigma, && \sigma\iota = -\rho, && \iota^2 = 1. \end{aligned}$$

Let $\varphi : \mathcal{F} \to \mathbb{R}^4_2$ be the isomorphism given by

$$\varphi(1) = e_1,\ \varphi(\rho) = e_2,\ \varphi(\sigma) = e_3,\ \varphi(\iota) = e_4.$$

Then φ becomes an isometry when $\mathcal{F}$ is endowed with the metric $\varphi^*(g_0)$, g_0 being the standard scalar product on $\mathbb{R}^4_2$.

Both metrics will be denoted by $\langle\,,\rangle$.

Write $\omega = a + b\rho + c\sigma + d\iota \in \mathcal{F}$, a standing for $a \cdot 1$, a, b, c and d being real numbers.

Define

$$\overline{\omega} = -a + b\rho + c\sigma + d\iota.$$

Then

$$\langle \omega, \omega \rangle = \omega\overline{\omega} = \overline{\omega}\omega.$$

In general

$$\langle \omega_1, \omega_2 \rangle = p_1(\omega_1\overline{\omega}_2),$$

p_1 denoting the projection over the subspace spanned by the identity map.

Hence

$$\overline{\omega_1\omega_2} = -\overline{\omega_2}\,\overline{\omega_1}$$

and

$$\langle \omega_1\omega_2, \omega_1\omega_2 \rangle = -\langle \omega_1, \omega_1 \rangle \langle \omega_2, \omega_2 \rangle .$$

Now set

$$\begin{aligned} \mathbb{H}^3_1(-r^2) &\equiv \{\omega \in \mathcal{F}: \omega\overline{\omega} = -r^2\}, \\ \mathbb{H}^2(-r^2) &\equiv \operatorname{span}\{1, \sigma, \iota\} \subset \mathbb{H}^3_1(-r^2), \\ \mathbb{H}^2_1(-r^2) &\equiv \operatorname{span}\{1, \rho, \sigma\} \subset \mathbb{H}^3_1(-r^2). \end{aligned}$$

Define $\pi_s : \mathbb{H}^3_1(-1) \to \mathbb{H}^2_s(-1/4)$ by

$$\pi_s(\omega) = \frac{1}{2}\tilde{\omega}\omega,$$

where $\omega \to \tilde{\omega}$ denote the antiautomorphism of $\mathcal{F}$ given by

$$\tilde{\omega} = a - b\rho + c\sigma + d\iota, \qquad \text{or} \qquad \tilde{\omega} = a + b\rho + c\sigma - d\iota,$$

according to the base manifold is $\mathbb{H}^2(-1/4)$ or $\mathbb{H}^2_1(-1/4)$, respectively.

As usual, we define $e^{\theta x}$, $\theta \in \mathcal{F}$, by

$$\begin{aligned} &\cos(x) + \sin(x)\theta, && \text{if } \theta^2 = -1, \\ &\cosh(x) + \sinh(x)\theta, && \text{if } \theta^2 = 1. \end{aligned}$$

That means that the fibers are topologically $\mathbb{S}^1$ and $\mathbb{H}^1$, respectively.

Remark Writing $\sigma = f \times f$ and $\iota = f \times f$, then we obtain in the Euclidean space $\mathbb{R}^4$ the standard quaternionic structure, which was already used by U. Pinkall to describe the usual Hopf fibration $\mathbb{S}^3(1) \to \mathbb{S}^2(1)$.

Let $\overline{\nabla}$ and ∇ be the semi-Riemannian connections of $\mathbb{H}^3_1(-1)$ and $\mathbb{H}^2_s(-1/4)$, respectively, and denote by overbars the lifts of corresponding objects on the base $\mathbb{H}^2_s(-1/4)$.

Then

$$\begin{aligned} \overline{\nabla}_{\overline{X}}\overline{Y} &= \overline{\nabla_X Y} + (-1)^s(\langle JX, Y \rangle \circ \pi_s)V, \\ \overline{\nabla}_{\overline{X}}V &= \overline{\nabla}_V \overline{X} = \theta\overline{X}, \\ \overline{\nabla}_V V &= 0, \end{aligned}$$

where J denotes the standard complex structure of $\mathbb{H}^2_s(-1/4)$ and $\theta = \rho$ when $s = 0$ or $\theta = \iota$ when $s = 1$.

Let $\beta: I \to \mathbb{H}^2_s(-1/4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature κ.

Consider a horizontal lift $\overline{\beta} : I \to \mathbb{H}^3_1(-1)$ of β with Frenet frame $\left\{\overline{T}, \xi_2^*, \xi_3^*\right\}$ and curvatures κ^* and τ^*.

Now, from the Frenet equations, we can deduce that $\xi_2^* = \overline{\xi_2}$ and $\overline{\kappa} = \kappa \circ \pi_s$. In particular ξ_2^* lies in the horizontal distribution along $\overline{\beta}$ and it has the same causal character as ξ_2. Also it is not difficult to see that $\tau^* = \pm 1$ and $\xi_3^* = \pm V$, that is, the binormal ξ_3^* of $\overline{\beta}$ coincides with the unit tangent to the fibers through each point of $\overline{\beta}$.

Proposition

(1) *The horizontal lifts of unit speed curves in* $\mathbb{H}^2(-1/4)$ *are spacelike curves in* $\mathbb{H}^3_1(-1)$ *with torsion* ± 1.

(2) *The horizontal lifts of unit speed timelike curves in* $\mathbb{H}^2_1(-1/4)$ *are timelike curves in* $\mathbb{H}^3_1(-1)$ *with torsion* ± 1.

By pulling back via π_s a non-null curve β in $\mathbb{H}^2_s(-1/4)$ we get the total horizontal lift of β, which is a flat immersed surface M_β in $\mathbb{H}^3_1(-1)$, that will be called the **indefinite Hopf cylinder associated to** β

Notice that if $s = 0$, M_β is a Lorentzian surface, whereas if $s = 1$, M_β is Riemannian or Lorentzian, according to β be spacelike or timelike, respectively.

Theorem

Let M be a Lorentzian surface immersed into $\mathbb{H}^3_1(-1)$. Then M is the semi-Riemannian Hopf cylinder in $\mathbb{H}^3_1(-1)$ associated to a unit speed curve β in $\mathbb{H}^2_s(-1/4)$ if and only if M is the B-scroll over any horizontal lift $\overline{\beta}$ of β.

Let $\beta: I \to \mathbb{H}^2_s(-1/4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature function κ.

Let $\overline{\beta}$ be a horizontal lift of β to $\mathbb{H}^3_1(-1)$ with Frenet frame $\{\overline{T}, \overline{\xi}_2, \xi_3^*\}$ and curvature $\overline{\kappa} = \kappa \circ \pi_s$ and $\tau = 1$. Recall that ξ_3^* is nothing but the unit tangent vector field to the fibers along $\overline{\beta}$.

Then the Hopf cylinder M_β can be orthogonally parametrized as

$$X(t,z) = \begin{cases} \cos(z)\overline{\beta}(t) + \sin(z)\xi_3^*(t), & \text{if } s = 0 \\ \cosh(z)\overline{\beta}(t) + \sinh(z)\xi_3^*(t), & \text{if } s = 1 \end{cases}$$

Setting, as usual, $X_t = \frac{\partial X}{\partial t}$ and $X_z = \frac{\partial X}{\partial z}$, then $\{X_t, X_z\}$ is an orthonormal frame of $T_{X(t,z)}M_\beta$ along X and a direct computation shows that the shape operator S of M_β in this frame can be written as

$$\begin{aligned} S(X_t) &= \overline{\kappa} X_t + \varepsilon X_z, \\ S(X_z) &= X_t, \end{aligned}$$

where $\varepsilon = +1$ if M_β is Riemannian and $\varepsilon = -1$ if M_β is Lorentzian.

Notice that a unit normal vector field to M_β into $\mathbb{H}^3_1(-1)$ is obtained from the complete horizontal lift of ξ_2 and it is, of course, $\overline{\xi}_2$ along each horizontal lift of β.

As a consequence we have that M_β is a flat surface, as we said before, and its mean curvature function α is given by $\alpha = \overline{\kappa}/2$.

According to the description of curves with constant curvature in $\mathbb{H}^2_s(-1/4)$ we can give the following description of Hopf cylinders of constant mean curvature.

Proposition

Let β be a unit speed curve in $\mathbb{H}^2_s(-1/4)$ with constant curvature κ. Then one of the following statements holds:

(1) M_β is a minimal complex circle ($\kappa = 0$).

(2) M_β is a non-minimal complex circle ($0 < \kappa^2 < 4$).

(3) M_β is the Hopf cylinder over the horocycle ($s = 0$, $\kappa^2 = 4$) or over the pseudo-horocycle ($s = 1$, $\kappa^2 = 4$).

(4) M_β is one of the following semi-Riemannian products

$$\begin{aligned} &(4.1) \quad \mathbb{H}^1_1(-r^2) \times \mathbb{S}^1(r^2-1) \quad \text{if } s = 0 \text{ and } \kappa^2 > 4, \\ &(4.2) \quad \mathbb{H}^1(-r^2) \times \mathbb{S}^1_1(r^2-1) \quad \text{if } s = 1 \text{and } \kappa^2 > 4. \end{aligned}$$

(5) M_β is the Riemannian product $\mathbb{H}^1(-r^2) \times \mathbb{H}^1(-1+r^2)$ with r satisfying

$$\frac{1-2r^2}{r\sqrt{1-r^2}} = \kappa.$$

It should be noticed that the above cases (1) through (4) correspond to the Lorentzian character of M_β and so, according to the above theorem, it can be considered as the classification of B-scrolls with constant mean curvature in $\mathbb{H}^3_1(-1)$. The remainder case corresponds with the Riemannian character of M_β.

2 Lorentzian Hopf tori ([9])

Hopf surfaces in $\mathbb{H}^3_1(-1)$ shaped on closed curves in $\mathbb{H}^2(-1/4)$ are Lorentzian flat tori. Now we want to determine the isometry group of these surfaces.

We use standard computations involving the structure equations of the induced connection and [17] to get a similar result to that of Pinkall:

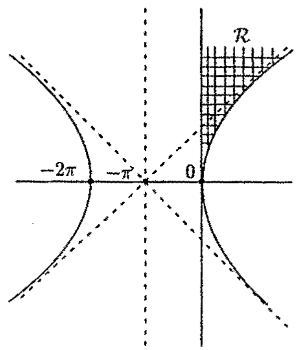
Theorem

Let β be a closed embedded curve in $\mathbb{H}^2(-1/4)$ of length L enclosing an area A. Then M_β is isometric to $\mathbb{L}^2/\Lambda$, Λ being the lattice in the Lorentzian plane $\mathbb{L}^2$ generated by the vectors $(2\pi, 0)$ and $(2A, L)$.

Remark It is worth noting that $(2A, L)$ is only constrained by the isoperimetric inequality in $\mathbb{H}^2(-1/4)$

$$L^2 \geq 4\pi A + 4A^2.$$

Hence the vector $(2A, L)$ must be spacelike. Therefore $(2A, L)$ lies in the shaded region $\mathcal{R}$



3 Willmore tori in $\mathbb{H}_1^3(-1)$ ([9])

Inspired again by Pinkall's paper, we look for Willmore tori in $\mathbb{H}_1^3(-1)$ associated to elastic curves in $\mathbb{H}^2(-1/4)$.

A unit-speed curve γ in M_ν^n is said to be an **elastica** (or **elastic curve**) if it is an extremal point of the functional

$$\mathfrak{G}_\lambda(\gamma) = \int_0^L (\langle \nabla_T T, \nabla_T T\rangle + \lambda) ds = \int_0^1 (\langle \nabla_T T, \nabla_T T\rangle + \lambda) v dt,$$

for some λ, where ds and L stand for the arclength on γ and the length of γ, respectively. It is called a **free elastica** if $\lambda = 0$ (see [21]).

The Euler-Lagrange equation associated to this variational problem is

$$2\nabla_T^3 T + \varepsilon_1 \nabla_T((3\varepsilon_2\kappa^2 - \lambda)T) - 2R(\nabla_T T, T)T = 0.$$

Frenet equations for γ:

$$\begin{aligned} \nabla_T T &= \varepsilon_2 \kappa \xi_2, \\ \nabla_T \xi_2 &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau \xi_3, \\ \nabla_T \xi_3 &= \varepsilon_2 \tau \xi_2 + \delta, \end{aligned}$$

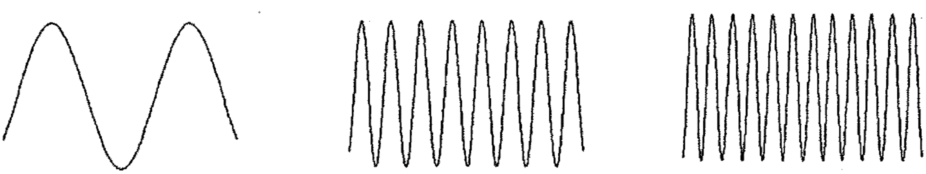
where $\delta \in \operatorname{span}\{T, \xi_2, \xi_3\}^\perp$, $\langle \xi_i, \xi_i \rangle = \varepsilon_i$ and τ is the torsion function (the second curvature if $n > 3$). Assume now that M_ν^n is of constant curvature c. Then the

Euler-Lagrange equation can be rewritten as follows

$$\begin{aligned} &2\varepsilon_2\kappa'' + \varepsilon_1\kappa^3 - 2\varepsilon_3\kappa\tau^2 + \varepsilon_1\varepsilon_2(2c-\lambda)\kappa = 0,\\ &2\kappa'\tau + \kappa\tau' = 0,\\ &\kappa\tau\delta = 0. \end{aligned}$$

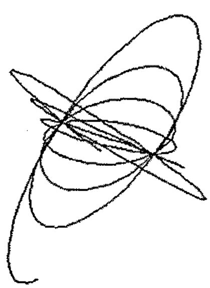
Taking $u = \kappa^2$ these equations can be solved by standard techniques in terms of elliptic functions.

For instance, a qualitative description of elasticae in the Lorentz-Minkowski plane $\mathbb{L}^2$ is given as follows. In general, the elasticae in $\mathbb{L}^2$ are curves which oscillates around a geodesic, so that the parameter λ, in some sense, could be viewed as the wavelength. That length increases or decreases according to $\varepsilon_1\lambda$ does. In the following we skecht some of these curves.

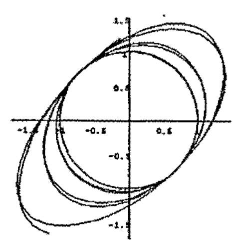


$\varepsilon_1\lambda > 0$ $\lambda = 0$ $\varepsilon_1\lambda < 0$

As for the pseudo-hyperbolic plane $\mathbb{H}_1^2(-1)$ the behaviour of the elastic curves is essentially the same as in $\mathbb{L}^2$, they also oscillate around geodesics. In particular, we can draw a free elastica oscillating around the central circle in $\mathbb{H}_1^2(-1)$.



Free elastica



Projection on xy-plane

Let M_s^2 be a surface in an indefinite 3-space $\widetilde{M}_\mu^3$ of constant curvature c.

We define the operator W over sections of the normal bundle of M_s^2 into $\widetilde{M}_\mu^3$ as follows

$$W : \mathfrak{N}M \to \mathfrak{N}M, \qquad W(\xi) = (\Delta^D + 2\langle H, H\rangle I - \widetilde{A})\xi,$$

$\tilde{A}$ standing for Simons operator.

A cross section ξ will be called a Willmore section if $W(\xi) = 0$. Suppose that M is compact and consider the Willmore functional

$$\mathcal{W}(M) = \int_M (\langle H, H\rangle + c)dv.$$

Then the operator W naturally appears provided that one computes the first variation formula for $\mathcal{W}$.

Now Willmore surfaces are nothing but the extremal points of the Willmore functional and they are characterized from the fact that their mean curvature vector fields are Willmore fields.

Proposition

Let $\pi_s : \mathbb{H}^3_1(-1) \rightarrow \mathbb{H}^2_s(-1/4)$ and $\beta : I \rightarrow \mathbb{H}^2_s(-1/4)$ be as before. Then the Hopf cylinder M_β satisfies $WH = \mu H$, $\mu \in \mathbb{R}$, if and only if β is an elastica in $\mathbb{H}^2_s(-1/4)$.

We know that the fibers of $\pi_0 : \mathbb{H}^3_1(-1) \rightarrow \mathbb{H}^2(-1/4)$ are circles, and so compact, whereas the fibers of $\pi_1 : \mathbb{H}^3_1(-1) \rightarrow \mathbb{H}^2_1(-1/4)$ are not compact. Therefore to find compact Hopf surfaces we have to consider Hopf torus shaped on closed curves in $\mathbb{H}^2(-1/4)$

In the anti-De Sitter world we have known that a Hopf torus M_β is a Willmore surface in $\mathbb{H}^3_1(-1)$ if and only if β is an elastica in $\mathbb{H}^2(-1/4)$ with $\lambda = -4$. However we have recently known from D. Singer (private communication) that cannot be hold. Thus we have to say that *there is no (Lorentzian) Willmore Hopf torus in* $\mathbb{H}^3_1(-1)$.

4 The Betchov-Da Rios equation ([9])

The Betchov-Da Rios (BDR) equation $u' \wedge u'' = \dot{u}$, also called localized induction equation in 3-dimensional hydrodynamics, is a soliton equation for space curves $u(t, s)$, where $u' = \partial u/\partial t$ and $\dot{u} = \partial u/\partial s$.

It is a straigthforward computation that, in general, the standard parametrization $X(t, z)$ of M_β is not a solution of BDR.

We ask for the classification of $h \in \mathrm{Diff}(\mathbb{R}^2)$ in order to $Y = X \circ h$ be a solution of BDR equation in $\mathbb{H}^3_1(-1)$.

We completely solve this problem.

Let η be a unit normal vector field to M_β in $\mathbb{H}^3_1(-1)$. Then η can be written as follows:

$$\eta = \begin{cases} -\sin(z)\overline{T}(t) + \varepsilon_1 \cos(z)\xi_2^*(t), & s = 0, \\ -\sinh(z)\overline{T}(t) + \varepsilon_1 \cosh(z)\xi_2^*(t), & s = 1. \end{cases}$$

A straightforward computation yields $Y(u,v)$ is a solution of BDR equation if and only if the following PDE system holds:

$$\begin{aligned} t_v &= (-1)^s\varepsilon_1 t_u z_u(t_u\overline{\kappa}+2z_u),\\ z_v &= t_u^2(t_u\overline{\kappa}+2z_u),\\ 0 &= t_u z_{uu}-z_u t_{uu}. \end{aligned}$$

Solving we get

Theorem

Let β be an arc length parametrized curve in $\mathbb{H}^2_s(-1/4)$ and M_β its Hopf cylinder in $\mathbb{H}^3_1(-1)$. For any $h \in Diff(\mathbb{R}^2)$, take $Y = X\circ h : \mathbb{R}^2 \to M_\beta$, X being the standard covering of $\mathbb{R}^2$ over M_β. Then Y is a solution of BDR soliton equation in $\mathbb{H}^3_1(-1)$ if and only if the following statements hold:

(i) *β has constant curvature, say κ, in $\mathbb{H}^2_s(-1/4)$;*

(ii) *$h(u,v) = (t(u,v), z(u,v))$ is given by*

$$\begin{aligned} t(u,v) &= au+(-1)^s ag\rho v + c_1,\\ z(u,v) &= agu+\varepsilon_1 a\rho v + c_2, \end{aligned}$$

where $(\varepsilon_1-(-1)^s g^2)a^2 = \varepsilon$, ε_1 being the causal character of β, ε the causal character of the u-curves, $g\in\mathbb{R}-\{-\kappa/2\}$, $\rho = \varepsilon_1(\kappa+2g)a^2$ is the curvature of the u-curves in $\mathbb{H}^3_1(-1)$ and a, c_1, c_2 are arbitrary constants.

Corollary 1

Let M_β be a Lorentzian Hopf cylinder in $\mathbb{H}^3_1(-1)$ of constant mean curvature. Then the only soliton solutions of BDR equation in $\mathbb{H}^3_1(-1)$ lying in M_β are the null geodesics of M_β.

Corollary 2

Let β be a closed curve of constant curvature in $\mathbb{H}^2(-1/4)$ with length L enclosing an oriented area A. Then for any rational number q, the slope

$$g = \frac{2\pi}{L}\left(q+\frac{A}{\pi}\right)$$

defines a unique closed helix in $\mathbb{H}^3_1(-1)$ and therefore a closed solution of BDR equation in $\mathbb{H}^3_1(-1)$ living in the Hopf torus M_β. Furthermore, the closed solution is either spacelike, or timelike or null according to $q\in(q_1,q_2)$, $q\in\mathbb{R}-(q_1,q_2)$, $q\in\{q_1,q_2\}$, respectively, where $q_1 = -\frac{A}{\pi}-\frac{L}{2\pi}$ and $q_2 = -\frac{A}{\pi}+\frac{L}{2\pi}$.

5 General helices in 3-dimensional Lorentzian space forms ([10])

Helices got as solutions of BDR brought us to mind a Barros' idea: look out general helices.

A curve of constant slope or **general helix** in Euclidean space $\mathbb{R}^3$ is defined by the property that its tangent indicatrix is a planar curve. The straight line perpendicular to this plane is called the axis of the general helix.

A classical result stated by M.A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: "A necessary and sufficient condition in order to a curve be a general helix is that the ratio of curvature to torsion be constant".

For a given couple of one variable functions (eventually curvature and torsion parametrized by arclength) one might like to get an arclength parametrized curve for which the couple works as the curvature and torsion functions. This problem is usually referred as "the solving natural equations problem"

The natural equations for general helices can be integrated, not only in $\mathbb{R}^3$, but also in the 3-sphere $\mathbb{S}^3$ (the hyperbolic space is poor in this kind of curves and only helices are general helices). Indeed Barros, [6], has shown that general helices are geodesics either of right general cylinders or of Hopf cylinders, according to the curve lies in $\mathbb{R}^3$ or $\mathbb{S}^3$.

What about general helices in the 3-dimensional Lorentzian space forms?

A non-null curve γ immersed in $\mathbb{L}^3$ is called a general helix if its tangent indicatrix is contained in some plane, say π, of $\mathbb{L}^3$. Since π can be either degenerate or non-degenerate, then both cases are distinguished by calling degenerate and non-degenerate general helices, respectively.

We will point out a remarquable and deep difference between the behaviour of general helices in Euclidean and Lorentzian geometries:

While in $\mathbb{R}^3$ general helices are geodesics in right general cylinders, as classically is shown, we will prove that general helices in $\mathbb{L}^3$ are geodesics in either right general cylinders or flat B-scrolls, according to the general helix is non-degenerate or degenerate.

This nice difference between Euclidean and Lorentzian geometries (from the point of view of the behaviour of general helices) confirms once more the important role of the notion of B-scroll in Lorentzian geometries.

General helices in 3-dimensional De Sitter $\mathbb{S}^3_1$ and anti De Sitter $\mathbb{H}^3_1$ spaces are considered with the help of the idea of Langer and Singer (see [21]): use Killing vector field along a curve in a 3-dimensional real space form.

The Lancret theorem in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ underlines deep differences between the pseudospherical and pseudohyperbolic spaces. The pseudohyperbolic case is nicely analogous to the Lorentz-Minkowskian case, whereas in the pseudospherical case there are no nontrivial general helices. From this point of view, *the roles played by the non flat Lorentzian space forms $\mathbb{H}^3_1$ and $\mathbb{S}^3_1$ correspond with those played by the non flat Riemannian space forms $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively.*

Let $\gamma(t)$ be a non-null immersed curve in a 3-dimensional Lorentzian space form M with sectional curvature c and let $v(t) = |\gamma'(t)|$ be the speed of γ.

Let us consider a variation of γ, $\Gamma = \Gamma(t,z): I \times (-\varepsilon, \varepsilon) \to M$ with $\Gamma(t,0) = \gamma(t)$. In particular one can choose $\varepsilon > 0$ in such a way that all t-curves of the variation have the same causal character as that of γ. Associated with Γ there are two vector fields along Γ, $V(t,z) = \frac{\partial \Gamma}{\partial z}(t,z)$ and $W(t,z) = \frac{\partial \Gamma}{\partial t}(t,z)$. In particular $V(t) = V(t,0)$ is the variational vector field along γ and $W(t,z)$ is the tangent vector fields of the t-curves. We will use the notation $V = V(t,z)$, $v = v(t,z)$, $\kappa = \kappa(t,z)$, etc. with the obvious meanings. Also, if s denotes the arclength parameter of the t-curves, we will write $v(s,z)$, $V(s,z)$, $\kappa(s,z)$, etc. for the corresponding reparametrizations.

A straightforward but long computation allows us to obtain formulas for $\frac{\partial v}{\partial z}(t,0)$, $\frac{\partial \kappa^2}{\partial z}(t,0)$ and $\frac{\partial \tau^2}{\partial z}(t,0)$ which we collect, along with another standard identity, in the following lemma.

Lemma

(1) $[V,W] = 0$;

(2) $\dfrac{\partial v}{\partial z}(t,0) = -\varepsilon_1 g v$, *with* $g = \langle \overline{\nabla}_T V, T\rangle$;

(3) $\dfrac{\partial \kappa^2}{\partial z}(t,0) = 2\varepsilon_2 \langle \overline{\nabla}_T^2 V, \overline{\nabla}_T T\rangle + 4\varepsilon_1 g\kappa^2 + 2\varepsilon_2 \langle R(V,T)T, \overline{\nabla}_T T\rangle$;

(4) $\dfrac{\partial \tau^2}{\partial z}(t,0) = -2\varepsilon_2 \langle \dfrac{1}{\kappa}\overline{\nabla}_T^3 V - \dfrac{\kappa'}{\kappa^2}\overline{\nabla}_T^2 V + \varepsilon_1(\varepsilon_2 \kappa + \dfrac{c}{\kappa})\overline{\nabla}_T V - \varepsilon_1 \dfrac{c\kappa'}{\kappa^2} V, \tau B\rangle$,

where $\langle , \rangle$ *denotes the Lorentzian metric of* M *and* $\kappa' = \dfrac{\partial \kappa}{\partial t}(t,0)$.

Without loss of generality we can assume γ to be arclength parametrized.

A vector field $V(s)$ along γ, which infinitesimally preserves unit speed parametrization (that means $\frac{\partial v}{\partial z}(t,0) = 0$ for a V-variation of γ) is said to be a Killing vector field along γ if this evolves in the direction of V whithout changing shape, only position. In other words, the curvature and torsion functions of γ remain unchanged as the curve evolves.

Hence Killing vector fields along γ are characterized by the equations

$$\frac{\partial v}{\partial z}(t,0) = \frac{\partial \kappa^2}{\partial z}(t,0) = \frac{\partial \tau^2}{\partial z}(t,0) = 0.$$

Then V is a Killing vector field along γ if and only if it satisfies the following conditions:

a) $\langle \overline{\nabla}_T V, T\rangle = 0$,

b) $\langle \overline{\nabla}_T^2 V, N\rangle + \varepsilon_1 c\langle V, N\rangle = 0$,

c) $\langle \dfrac{1}{\kappa}\overline{\nabla}_T^3 V - \dfrac{\kappa'}{\kappa^2}\overline{\nabla}_T^2 V + \varepsilon_1(\varepsilon_2\kappa + \dfrac{c}{\kappa})\overline{\nabla}_T V - \varepsilon_1 c\dfrac{\kappa'}{\kappa^2} V, \tau B\rangle = 0.$

Now when M is simply connected, since the restriction to γ of any Killing field $\widetilde{V}$ of M is a Killing vector field along γ, one concludes from a well known dimension argument, the following lemma.

Lemma

Let M be a complete, simply connected, Lorentzian space form and γ a non-null immersed curve in M. A vector field V on γ is a Killing vector field along γ if and only if it extends to a Killing field $\widetilde{V}$ on M.

The Lancret theorem in $\mathbb{L}^3$

Let γ be a non-null immersed curve in $\mathbb{L}^3$ with curvature and torsion functions κ and τ, respectively. Then the following statements are equivalent:

(a) *γ is a general helix in $\mathbb{L}^3$;*

(b) *There exists a constant length Killing vector field V along γ which is orthogonal to the acceleration vector field of γ;*

(c) *There exists a constant r such that $\tau = r\kappa$.*

Moreover a general helix γ is degenerate if and only if $r = \pm 1$ and its normal vector field is spacelike. The Killing vector field V in (b) is not uniquely determined if γ is a helix (κ and τ both are constant); however, in this case, V can be uniquely determined, up to constants, once it is chosen parallel along γ (say otherwise, its extended Killing vector field in $\mathbb{L}^3$ is a translation vector field).

Solving natural equation for non-degenerate general helices

Let β be a non-null immersed curve in $\mathbb{L}^3$. Then β is a non-degenerate general helix if and only if it is a geodesic in some right cylinder whose directrix and generatrix are both non-null.

Solving natural equation for degenerate general helices

Let β be a non-null immersed curve in $\mathbb{L}^3$. Then β is a degenerate general helix if and only if it is a geodesic in some flat B-scroll in $\mathbb{L}^3$.

How to define general helices in non-flat 3-dimensional Lorentzian spaces forms?

Definition

A curve γ in M is said to be a general helix if there exists a Killing vector field V along γ with constant length and orthogonal to the acceleration vector field of γ.

We will say that V is **an axis** of the general helix γ.

Obvious examples of general helices in M are the following. Curves with torsion vanishing anywhere, where the unit binormal works as an axis. Helices are also general helices, where any vector field chosen in the rectifying plane having constant coordinates relative to T and B runs as an axis.

We can follow notation and terminology used in $\mathbb{L}^3$ to say that zero torsion curves are non-degenerate general helices, because the axis B is obviously non-null. As for curves with both constant curvature and torsion we know that for any pair of constants a and b the vector field along γ given by $V(s) = aT + bB$ is always a Killing vector field. Of course, when $\varepsilon_2 = -1$, i.e., the rectifying plane is positive definite at any point, all Killing vector fields $V(s)$ are non-null and we will say that the general helix is non-degenerate. However, if $\varepsilon_2 = 1$, i.e., the rectifying plane is Lorentzian, we have Killing vector fields along γ being either spacelike, or timelike, or null. It does not allow us to decide if such a general helix is degenerate or not. However, we can determine a unique Killing vector field along the helix by forcing it to be parallel along γ. The helix is said to be **degenerate** or **non-degenerate** according to V is null or non-null, respectively.

The Lancret theorem in the De Sitter space

A non-null immersed curve γ in $\mathbb{S}^3_1$ is a general helix if and only if either

(1) $\tau \equiv 0$ *and γ is a curve in some totally geodesic surface of $\mathbb{S}^3_1$; or*

(2) γ *is a helix in $\mathbb{S}^3_1$ (i.e. curvature κ and torsion τ constants).*

The Lancret theorem in the anti De Sitter space

A non-null immersed curve γ in $\mathbb{H}^3_1$ is a general helix if and only if either

(1) $\tau \equiv 0$ *and γ is a curve in some totally geodesic surface of $\mathbb{H}^3_1$. The curve admits only one axis which agrees with its binormal, being parallel along the curve and non-null. The general helix is non-degenerate; or*

(2) γ *is a helix in $\mathbb{H}^3_1$. It admits a plane (the rectifying plane) of axes but only one is parallel along γ. This parallel axis is null, and so γ is degenerate, if and only if $\varepsilon_2 = +1$ and $\tau = \pm\kappa$. Otherwise γ is non-degenerate; or*

(3) *there exists a certain constant b such that the curvature κ and the torsion τ functions of γ are related by $\tau = b\kappa \pm 1$. The curve admits a unique axis which can not be parallel along γ. It is null, and so γ is degenerate, if and only if $b = \pm 1$ and γ has spacelike normal vector ($\varepsilon_2 = +1$).*

Solving natural equation for non-degenerate general helices in $\mathbb{H}^3_1(-1)$

Let β a non-null immersed curve in $\mathbb{H}^3_1$. Then β is a non-degenerate general helix if and only if it is a geodesic in some Hopf cylinder M_γ.

Solving natural equation for degenerate general helices in $\mathbb{H}^3_1(-1)$

Let β a non-null immersed curve in $\mathbb{H}^3_1$. Then β is a degenerate general helix if and only if it is a geodesic in some flat B-scroll over a null curve.

6 Willmore tori and Willmore-Chen submanifolds into pseudo-Riemannian spaces ([11])

Two problems

(i) Find examples of Willmore surfaces in the anti De Sitter space.

(ii) Find examples of Willmore-Chen submanifolds in pseudo-Riemannian spaces (with non zero index).

6.1 Willmore tori in non standard anti de Sitter 3-space

Let $\pi:(M,g)\to(B,h)$ be a pseudo-Riemannian submersion.

A very interesting deformation of the metric g by changing the relative scales of B and the fibres (see [14]).

The canonical variation g_t, $t>0$, of g by setting

$$\begin{aligned} g_t|_{\mathcal{V}} &= t^2\, g|_{\mathcal{V}}, \\ g_t|_{\mathcal{H}} &= g|_{\mathcal{H}}, \\ g_t(\mathcal{V},\mathcal{H}) &= 0, \end{aligned}$$

where $\mathcal{V}$ and $\mathcal{H}$ stand for vertical and horizontal distributions, respectively, associated with the submersion. Thus we obtain a one-parameter family of pseudo-Riemannian submersions $\pi_t:(M,g_t)\to(B,h)$ with the same horizontal distribution $\mathcal{H}$, for all $t>0$.

Let us consider the canonical variation of the indefinite Hopf fibration

$$\pi=\pi_0\colon \mathbb{H}^3_1\to\mathbb{H}^2(-1/2)$$

to get a one-parameter family of pseudo-Riemannian submersions

$$\pi_t\colon(\mathbb{H}^3_1,g_t)\to(\mathbb{H}^2(-1/2),g_0).$$

Let γ be a unit speed curve immersed in $\mathbb{H}^2(-1/2)$. Set $\mathcal{T}_{\gamma,t}=\pi_t^{-1}(\gamma)$. Then $\mathcal{T}_{\gamma,t}$ is a Lorentzian flat surface immersed in $\mathbb{H}^3_1$, that will be called **the Lorentzian Hopf cylinder over** γ.

As the fibres of π_t are $\mathbb{H}^1_1$, which topologically are circles, then $\mathcal{T}_{\gamma,t}$ is a Hopf torus in $(\mathbb{H}^3_1,g_t)$, provided that γ is a closed curve.

Proposition

Let S be an immersed surface into $(\mathbb{H}^3_1,g_t)$. Then S is G-invariant if and only if S is a Lorentzian Hopf cylinder $\mathcal{T}_{\gamma,t}=\pi_t(\gamma)$ over a certain curve γ immersed in the hyperbolic 2-plane $(\mathbb{H}^2(-1/2),g_0)$.

Theorem

Let $\pi_t\colon(\mathbb{H}^3_1,g_t)\to(\mathbb{H}^2(-1/2),g_0)$, $t>0$, be the canonical variation of the pseudo-Riemannian Hopf fibration. Let γ be a closed immersed curve in $(\mathbb{H}^2(-1/2),g_0)$ and $\mathcal{T}_{\gamma,t}=\pi_t^{-1}(\gamma)$ its Lorentzian Hopf torus. Then $\mathcal{T}_{\gamma,t}$ is a Willmore surface in $(\mathbb{H}^3_1,g_t)$ if and only if γ is an elastica in $(\mathbb{H}^2(-1/2),g_0)$ with Lagrange multiplier $\lambda=-4t^2$.

Proof

The Willmore functional on $\mathcal{M}=\{\phi\colon T\to(\mathbb{H}^3_1,g_t):\ \phi \text{ is an immersion}\}$ is

$$\Omega(\phi)=\int_T(\langle H,H\rangle+R^t)dv,$$

H and R^t standing for the mean curvature vector field of T and the sectional curvature of $(\mathbb{H}_1^3, g_t)$, measured with respect to the tangent plane to (T, ϕ), respectively. It is clear that, for any $e^{i\theta} \in \mathbb{S}^1$, we have that $\Omega(\phi) = \Omega(e^{i\theta} \cdot \phi)$. Now let us denote by $\mathcal{C}$ the set of critical points of Ω in $\mathcal{M}$, i.e., $\mathcal{C}$ is the set of genus one Willmore surfaces. Let $\mathcal{M}_G$ be the submanifold of $\mathcal{M}$ made up by those immersions of T which are $(G = \mathbb{S}^1)$-invariant and let $\mathcal{C}_G$ be the set of critical points of Ω restricted to $\mathcal{M}_G$. The principle of symmetric criticality of Palais, [23], can be used here to find that $\mathcal{C} \cap \mathcal{M}_G = \mathcal{C}_G$. Now from the above Proposition we obtain that $\mathcal{C}_G = \{\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma) : \gamma \text{ is an immersed closed curve in } (\mathbb{H}^2(-1/2), g_0)\}$. To compute $\Omega(\mathcal{T}_{\gamma,t})$, i.e., the Willmore functional on $\mathcal{C}_G$, we first notice that $\alpha = \frac{1}{2}\kappa$, κ being the curvature function of γ in $(\mathbb{H}^2(-1/2), g_0)$.

On the other hand

$$R^t = -g_t(ti\overline{X}, ti\overline{X}) = -t^2.$$

Let L be the length of γ. As the fibres of g_t are circles of radii t, we have

$$\Omega(\mathcal{T}_{\gamma,t}) = \int_{\pi_t^{-1}(\gamma)} (\alpha^2 + R^t)dv = \int_0^L \int_0^{2\pi t} \left(\frac{1}{4}\kappa^2 - t^2\right) dsdr = \frac{\pi t}{4} \int_0^L (\kappa^2 - 4t^2)ds$$

and the proof finishes.

Then we give, for $t \in (0, 1)$, infinitely many Willmore tori in $(\mathbb{H}_1^3, g_t)$.

6.2 Willmore-Chen submanifolds in the hyperbolic space

We give a new method to construct critical points of the Willmore-Chen functional in the pseudo-hyperbolic space $\mathbb{H}_r^n = \mathbb{H}_r^n(-1)$.

First step: write $\mathbb{H}_r^n$ as a warped product with base space the standard hyperbolic space $\mathbb{H}^{n-r}$.

Second step: use the conformal invariance of the Willmore-Chen variational problem to make a conformal change of the canonical metric of $\mathbb{H}_r^n$.

Third step: use the principle of symmetric criticality of R. Palais to reduce the problem to a variational one for closed curves in the once punctured standard $(n-r)$-sphere.

Given $0 < r < n$, let

$$\mathbb{H}^{n-r} = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-r} : -x_0^2 + \langle x, x\rangle = -1 \text{ and } x_0 > 0\}$$

the hyperbolic $(n-r)$-space and

$$\mathbb{H}_r^n = \{(\xi, \eta) \in \mathbb{R}^{r+1} \times \mathbb{R}^{n-r} : -\langle \xi, \xi\rangle + \langle \eta, \eta\rangle = -1\}$$

the pseudo-hyperbolic n-space. They are hypersurfaces in $\mathbb{R}_1^{n-r+1}$ and $\mathbb{R}_{r+1}^{n+1}$, respectively. The induced metrics on these spaces, from those in the corresponding pseudo-Euclidean spaces, define standard metrics h_0 on $\mathbb{H}_r^n$ and g_0 on $\mathbb{H}^{n-r}$, both with constant curvature -1.

Let $\mathbb{S}^r$ be the standard unit r-sphere endowed with its canonical metric $d\sigma^2$. Consider the mapping

$$\Phi: \mathbb{H}^{n-r} \times \mathbb{S}^r \to \mathbb{H}^n_r$$

defined by

$$\Phi((x_0, x), u) = (x_0 u, x).$$

It is not difficult to see that Φ defines a diffeomorphism whose inverse is $\Phi^{-1}(\xi, \eta) = ((|\xi|, \eta), \xi/|\xi|)$. For any curve $\beta(t) = ((x_0(t), x(t)), u(t))$ in $\mathbb{H}^{n-r} \times \mathbb{S}^r$ we have

$$\left|\mathrm{d}\Phi_{\beta(t)}(\beta'(t))\right|^2 = -x_0'(t)^2 + |x'(t)|^2 - x_0(t)^2|u'(t)|^2.$$

Let $f : \mathbb{H}^{n-r} \to \mathbb{R}$ be the positive function given by $f(x_0, x) = x_0$ and consider the metric $g = g_0 - f^2 d\sigma^2$ on $\mathbb{H}^{n-r} \times \mathbb{S}^r$. The pseudo-Riemannian manifold $(\mathbb{H}^{n-r} \times \mathbb{S}^r, g)$ is called the **warped product** of base $(\mathbb{H}^{n-r}, g_0)$ and fibre $(\mathbb{S}^r, -d\sigma^2)$ with warping function f.

It is usually denoted by $(\mathbb{H}^{n-r}, g_0) \times_f (\mathbb{S}^r, -d\sigma^2)$ or $\mathbb{H}^{n-r} \times_f (-\mathbb{S}^r)$ when the metrics on the base and fibre are understood (see [14] and [22]).

Φ is an isometry between $\mathbb{H}^{n-r} \times_f (-\mathbb{S}^r)$ and $(\mathbb{H}^n_r, h_0)$.

A new metric h on $\mathbb{H}^n_r$ is defined by

$$h = \frac{1}{f^2} h_0 = \frac{1}{f^2} g_0 - d\sigma^2,$$

with the obvious meaning by removing the pulling back via Φ.

Thus $(\mathbb{H}^n_r, h)$ is the pseudo-Riemannian product of $(\mathbb{H}^{n-r}, \frac{1}{f^2} g_0)$ and $(\mathbb{S}^r, -d\sigma^2)$.

Finally it is not difficult to see that $(\mathbb{H}^{n-r}, \frac{1}{f^2} g_0)$ has constant sectional curvature 1, so that it can be identified, up to isometries, with the once punctured standard $(n-r)$-sphere $(\Sigma^{n-r}, d\sigma^2)$.

Consequently, $(\mathbb{H}^n_r, h)$ is nothing but $(\Sigma^{n-r}, d\sigma^2) \times (\mathbb{S}^r, -d\sigma^2)$, up to isometries.

$SO(r+1)$-invariant submanifolds in $\mathbb{H}^n_r$

For any immersed curve $\gamma : [0, L] \to \mathbb{H}^{n-r}$, we define the semi-Riemannian $(r+1)$-submanifold $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^r)$. It is clear that Υ_γ has index r and we will refer to Υ_γ as the cylinder over γ.

Now let $G = SO(r+1)$ be the group of isometries of $(\mathbb{S}^r, -d\sigma^2)$.

Then G acts transitively on $(\mathbb{S}^r, -d\sigma^2)$.

So we define an action of G on $\mathbb{H}^n_r$ as follows

$$a \cdot (\xi, \eta) = \Phi(a \cdot \Phi^{-1}(\xi, \eta)) = (a(\xi), \eta),$$

for any $a \in G$.

This action is realized through isometries of $(\mathbb{H}^n_r, h_0)$. The following statement characterizes the cylinders over curves in $\mathbb{H}^{n-r}$ as symmetric points of the above mentioned G-action.

Proposition

Let M be an $(r+1)$-dimensional submanifold in $\mathbb{H}^n_r$. Then M is G-invariant if and only if M is a cylinder Υ_γ over a certain curve γ in $\mathbb{H}^{n-r}$.

Critical points of $\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds$

Now we deal with the functional

$$\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds,$$

defined on the manifold of regular closed curves (or curves satisfying given first order boundary data) in a given pseudo-Riemannian manifold, where r stands for any natural number (even though all computations also hold if r is a real number). Notice that we write the integrand in that form to point out that it is an even function of the curvature κ. Also $\mathcal{F}^1$ agrees with $\mathcal{G}$, which is the elastic energy functional for free elasticae.

Let $\gamma: I \subset \mathbb{R} \to \mathbb{S}^m$ be a unit speed curve in the unit m-sphere with curvatures $\{\kappa, \tau, \ldots\}$ and Frenet frame $\{T = \gamma', \xi_2, \ldots, \xi_m\}$. Given a variation $\Gamma := \Gamma(s,t) : I \times (-\varepsilon, \varepsilon) \to \mathbb{S}^m$ of γ, with $\Gamma(s,0) = \gamma(s)$, we have the associated variation vector field $W(s) = \frac{\partial \Gamma}{\partial t}(s,0)$ along γ. We will use the notation and terminology of Langer-Singer. Set $V(s,t) = \frac{\partial \Gamma}{\partial s}$, $W(s,t) = \frac{\partial \Gamma}{\partial t}$, $v(s,t) = |V(s,t)|$, $T(s,t) = \frac{1}{v} V(s,t)$, $\kappa(s,t) = |\nabla_T T|^2$, ∇ being the Levi-Civita connection of $\mathbb{S}^m$. The following lemma collects some basic facts which we will use to find the Euler-Lagrange equations relative to $\mathcal{F}^r$.

Langer and Singer Lemma ([21])

With the above notation, the following assertions hold:

$$\begin{aligned}
[V,W] &= 0; \\
\frac{\partial v}{\partial t} &= \langle \nabla_T W, T\rangle v; \\
[W,T] &= -\langle \nabla_T W, T\rangle T; \\
[[W,T],T] &= T(\langle \nabla_T W, T\rangle)T; \\
\frac{\partial \kappa^2}{\partial t} &= 2\langle \nabla_T^2 W, \nabla_T T\rangle - 4\langle \nabla_T W, T\rangle \kappa^2 + 2\langle R(W,T)T, \nabla_T T\rangle,
\end{aligned}$$

R being the Riemann curvature tensor of $\mathbb{S}^m$.

Now $\frac{\partial}{\partial t}\big|_{t=0} \mathcal{F}^r(\Gamma(s,t)) = 0$ allows us to get the following Euler equation, which characterizes the critical points of $\mathcal{F}^r$ on the quoted manifolds of curves:

$$\begin{aligned}
&(\kappa^2)^{(r-1)/2} \nabla_T^3 T \\
&+2\frac{\mathrm{d}}{\mathrm{ds}}((\kappa^2)^{(r-1)/2}) \nabla_T^2 T
\end{aligned}$$

$$+\{(\kappa^2)^{(r-1)/2}+\frac{\mathrm{d}^2}{\mathrm{d}s^2}((\kappa^2)^{(r-1)/2})+\frac{2r+1}{r+1}(\kappa^2)^{(r+1)/2}\}\nabla_T T$$
$$+\frac{2r+1}{r+1}\frac{\mathrm{d}}{\mathrm{d}s}((\kappa^2)^{(r+1)/2})T=0.$$

From here and the Frenet equations for γ, we find the following characterization of the critical points of $\mathcal{F}^r$.

Proposition

Let γ be a regular curve in $\mathbb{S}^m$ with curvatures $\{\kappa,\tau,\delta,\ldots\}$. Then γ is a critical point of

$$\mathcal{F}^r(\gamma)=\int_\gamma(\kappa^2)^{(r+1)/2}ds$$

if and only if the following equations hold:

$$\begin{aligned} r\kappa''+\frac{r}{r+1}\kappa^3-\kappa\tau^2+\kappa+\frac{r(r-1)}{\kappa}(\kappa')^2 &= 0,\\ (\kappa^2)^r\tau &= 0,\\ \delta &= 0. \end{aligned}$$

In particular, γ lies in some $\mathbb{S}^2$ or $\mathbb{S}^3$ totally geodesic in $\mathbb{S}^m$.

From now on we will call r-**generalized elasticae** to the critical points of $\mathcal{F}^r$. In particular, free elasticae are nothing but 1-generalized elasticae.

A key result

Characterize the cylinders in $(\mathbb{H}^n_r,h_0)$ which are Willmore-Chen submanifolds.

Theorem

Let γ be a fully immersed closed curve in the hyperbolic space $\mathbb{H}^{n-r}$. The cylinder $\Upsilon_\gamma=\Phi(\gamma\times\mathbb{S}^r)$ in $(\mathbb{H}^n_r,h_0)$ is a Willmore-Chen submanifold if and only if γ is a generalized free elastica in the once punctured unit sphere $(\Sigma^{n-r},d\sigma^2)$. In particular, $n-r\leq 3$.

The proof is mainly based on the symmetric criticality principle of Palais.

Some examples

To find examples of non trivial Willmore-Chen submanifolds in the pseudo-hyperbolic space $(\mathbb{H}^n_r,h_0)$ we apply the latter Theorem.

Example 1.1

Let γ be an immersed closed curve in the hyperbolic 2-plane. The Lorentzian cylinder $\Upsilon_\gamma=\Phi(\gamma\times\mathbb{S}^1)$ is a Willmore torus in the 3-dimensional anti De Sitter space $(\mathbb{H}^3_1,h_0)$ if and only if γ is a free elastica in the once punctured unit 2-sphere $(\Sigma^2,d\sigma^2)$.

The complete classification of closed free elasticae in the standard 2-sphere was achieved by J.L. Langer and D.A. Singer, which can be briefly and geometrically described as follows ([21]): Up to rigid motions in the unit 2-sphere, the family of closed free elasticae consists of a geodesic γ_0, say the equator, and an integer two parameter family $\{\gamma_{m,n} : 0 < m < n,\ m, n \in \mathbb{Z}\}$, where $\gamma_{m,n}$ means that it closes up after n periods and m trips around the equator γ_0.

As a consequence we have

Example 1.2

There exist infinitely many Lorentzian Willmore tori in the 3*-dimensional anti De Sitter space. This family includes* $\{\Upsilon_{\gamma_{m,n}} : 0 < m < n,\ m, n \in \mathbb{Z}\}$ *and* Υ_{γ_0}.

A second case we will consider is $n - r = 3$. Then we look for critical points of $\mathcal{F}^r(\gamma)$ inside the family of helices in the standard once punctured 3-sphere $(\Sigma^3, d\sigma^2)$.

Let γ be a helix in $(\Sigma^3, d\sigma^2)$ with curvature κ and torsion τ. Assume that γ is a not a geodesic; otherwise, it is a trivial solution. Then γ is an r-generalized free elastica if and only if

$$\frac{r}{r+1}\kappa^2 - \tau^2 + 1 = 0.$$

A long and messy computation leads to

Theorem

For any natural number r*, there exists a one parameter family* $\{\gamma_q\}_{q \in \mathbb{Q}\setminus\{0\}}$ *of closed helices in* $(\Sigma^3, d\sigma^2)$ *which are* r*-generalized free elastica.*

As a consequence we obtain

Example 2

Let r *be any natural number. For any non zero rational number* q*, there exists an* $(r+1)$*-dimensional Willmore-Chen submanifold* $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^r)$ *in the pseudo-hyperbolic space* $(\mathbb{H}_r^{r+3}, h_0)$*,* γ *being an* r*-generalized free elastic closed helix in the once punctured unit* 3*-sphere* $(\Sigma^3, d\sigma^2)$ *whose slope* ℓ *is computed as above.*

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DIVERGENCE THEOREMS IN SEMI–RIEMANNIAN GEOMETRY

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Abstract.– A survey of divergence theorems in semi–Riemannian geometry is made by including their proofs.

1 Introduction

In Riemannian geometry, integral formulas have been in vogue since Gauss, and through their use many beautiful global results have been obtained. Perhaps the divergence theorem is the most well–known integral formula in Riemannian geometry as well as a very powerful tool for obtaining global results. However, a divergence theorem in semi–Riemannian geometry has not been available until recent years, perhaps because of not many people were working on integral formulas in semi–Riemannian geometry. Duggal was the first one who questioned the validity of a divergence theorem in semi–Riemannian geometry in one of his works on integral formulas in semi–Riemannian geometry [D]. The main difficulty in stating a divergence theorem for a semi–Riemannian manifold with boundary is that the boundary may become degenerate at some of its points and hence there exists no well–defined unit outward normal at such points. Ünal overcame this difficulty by assuming several conditions on the degenerate part of the boundary and stated two semi–Riemannian divergence theorems [Ü]. Also, recently the authors defined divergence of a vector field along a map between Riemannian manifolds and generalized the Riemannian divergence theorem to a divergence theorem for the vector field along a map between Riemannian manifolds [G-RK].

The purpose of this review is to collect the results of Ünal and the authors in semi–Riemannian geometry. Here we will provide a more general proof to a lemma of Ünal [Ü, Lemma 3.3] which shows the nondegenerate part of the boundary must be an open subset of the boundary and reprove his semi–Riemannian divergence theorems. Then we will adapt the proof of divergence theorem for

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a vector field along a map between Riemannian manifolds [G-RK, Theorem 10] to semi–Riemannian setting and obtain the generalized semi–Riemannian divergence theorem. Finally we will make an application of this theorem to harmonic maps.

2 Semi–Riemannian Divergence Theorem

The most important step in the proof of semi–Riemannian divergence theorem is to show that the nondegenerate part of the boundary of a semi–Riemannian manifold is an open subset of the boundary. In fact this makes the integration over the nondegenerate boundary of a semi–Riemannian manifold meaningful. For this, we will prove a lemma which also has certain applications in semi–Riemannian geometry.

A symmetric $(0,2)$–tensor field g on a manifold M is called *nondegenerate of index* ν at $p \in M$ if g_p is a nondegenerate bilinear form of index ν on T_pM, that is, g_p is an inner product of index ν on T_pM.

Lemma 2.1 *Let M be an n–dimensional manifold and g be a symmetric $(0,2)$–tensor field on M. Then the set of points where g is nondegenerate of index ν is an open subset of M, where $0 \leq \nu \leq n$.*

Proof. If the set of points where g is nondegenerate of index ν is empty, then the claim follows trivially. Suppose the set of points where g is nondegenerate of index ν is not empty. We will show that, any point p in this set has a neighborhood such that g is nondegenerate of index ν at every point of this neighborhood. For, let g be nondegenerate of index ν at $p \in M$. Then there exists an orthonormal basis $\{x_1, \ldots, x_\nu, x_{\nu+1}, \ldots, x_n\}$ for (T_pM, g_p) such that

$$g_p(x_i, x_i) = \begin{cases} -1 & \text{for} \quad 1 \leq i \leq \nu \\ 1 & \text{for} \quad \nu+1 \leq i \leq n. \end{cases}$$

Extend $\{x_1, \ldots, x_\nu, x_{\nu+1}, \ldots, x_n\}$ to a local basis frame $\{X_1, \ldots, X_\nu, X_{\nu+1}, \ldots, X_n\}$ for TM on a neighborhood U of p and let g_{ij} be functions on U defined by

$$g_{ij} = g(X_i, X_j), \quad 1 \leq i, j \leq n.$$

First note that, since $\det\,[g_{ij}(p)] = (-1)^\nu$, $\det\,[g_{ij}] \neq 0$ on a neigborhood of p in U. Hence by reducing U to this neighborhood if necessary, we may assume that g_q is nondegenerate of some index ν_q at each point $q \in U$. Also since

$$g_{ij}, \quad 1 \leq i, j \leq n,$$

are continuous functions on U, for $\varepsilon_+ = \frac{1}{n-\nu+2}$, there exists a neighborhood U_+ of p in U such that

$$|\, g_{ii}(q) - 1 \,| < \varepsilon_+, \quad \text{for} \quad \nu+1 \leq i \leq n$$

and

$$| g_{ij}(q) | < \varepsilon_+, \quad \text{for } i \neq j,\ \nu+1 \leq i \leq n$$

whenever $q \in U_+$. Now let

$$W_{+_q} = \text{ span } \{X_{\nu+1}(q), \ldots, X_n(q)\}$$

at each $q \in U_+$. We will show that $g(x,x) > 0$ for every $0 \neq x \in W_{+_q}$ at each $q \in U_+$. Indeed, for any $0 \neq x = \sum_{i=\nu+1}^{n} \lambda_i X_i(q) \in W_{+_q}$,

$$\begin{aligned} g(x,x) &= \sum_{i,j=\nu+1}^{n} \lambda_i \lambda_j g_{ij}(q) \\ &= \sum_{i=\nu+1}^{n} \lambda_i^2 g_{ii}(q) + 2 \sum_{\nu+1 \leq i < j \leq n} \lambda_i \lambda_j g_{ij}(q) \\ &> \sum_{i=\nu+1}^{n} \lambda_i^2 (1 - \varepsilon_+) - 2 \sum_{\nu+1 \leq i < j \leq n} | \lambda_i || \lambda_j | \varepsilon_+ \\ &= \sum_{i=\nu+1}^{n} \lambda_i^2 - \varepsilon_+ \sum_{i=\nu+1}^{n} \lambda_i^2 - 2\varepsilon_+ \sum_{\nu+1 \leq i < j \leq n} | \lambda_i || \lambda_j | \\ &\geq \sum_{i=\nu+1}^{n} \lambda_i^2 - \varepsilon_+ \sum_{i=\nu+1}^{n} \lambda_i^2 - \varepsilon_+ \sum_{\nu+1 \leq i < j \leq n} (\lambda_i^2 + \lambda_j^2) \\ &= \sum_{i=\nu+1}^{n} \lambda_i^2 - \varepsilon_+ (2\lambda_{\nu+1}^2 + 3\lambda_{\nu+2}^2 + \cdots + (n - \nu + 1)\lambda_n^2) > 0. \end{aligned}$$

Similarly it can be shown that there exists a neighborhood U_- of p in U such that $g(x,x) < 0$ for every $x \in W_{-_q}$ at each $q \in U_-$, where

$$W_{-_q} = \text{ span } \{X_1(q), \ldots, X_\nu(q)\}.$$

Hence on $U' = U_+ \cap U_-$, g_q is positive definite on W_{+_q} and negative definite on W_{-_q} at each $q \in U'$. Then $n - \nu_q \geq n - \nu$ and $\nu_q \geq \nu$ at each $q \in U'$. Thus $\nu_q = \nu$ for every $q \in U'$, that is, g is nondegenerate of index ν at every $q \in U'$. □

Corollary 2.1 *Let M be a manifold and g be a symmetric $(0,2)$–tensor field on M. Then the set of points where g is nondegenerate is an open subset of M.*

Proof. Immediate from Lemma 2.1. □

Recall that a symmetric nondegenerate $(0,2)$–tensor field g of constant index ν on a manifold M is called a *metric tensor on M*. In fact, if M is connected then the assumption of constant index is redundant.

Corollary 2.2 *Let M be a connected manifold. If g is a symmetric nondegenerate $(0,2)$–tensor field on M then g is a metric tensor on M.*

Proof. Let g be of index ν_p at $p \in M$. Then by Lemma 2.1, the set of points where g is of index ν_p is open in M. Again by Lemma 2.1, the complement of this set is also open. Thus by the connectedness of M, g is of constant index $\nu_p = \nu$. □

A semi–Riemannian manifold (M, g) with boundary ∂M (possibly $\partial M = \varnothing$) is a manifold with boundary ∂M (possibly $\partial M = \varnothing$) with a metric tensor g. Note that if (M, g) is a semi–Riemannian manifold with boundary ∂M, where g is of index ν, then the induced $(0,2)$–tensor field $g_{\partial M}$ on ∂M is also symmetric. But $g_{\partial M}$ may not be a metric tensor on the connected components of ∂M because it may fail to be nondegenerate at some points of the connected components of ∂M. The points of ∂M where $g_{\partial M}$ fails to be nondegenerate can also be characterized by the causal characters of the vectors orthogonal to ∂M. Note that, if $p \in \partial M$ then $T_p\partial M$ is a hyperspace in T_pM with orthogonal space $(T_p\partial M)^\perp$ of dimension 1. Thus, $T_p\partial M$ is a degenerate in T_pM if and only if $(T_p\partial M)^\perp$ is a degenerate in T_pM if and only if every nonzero vector orthogonal to $T_p\partial M$ is null. Hence, $T_p\partial M$ is nondegenerate if and only if the vectors orthogonal to $T_p\partial M$ are nonnull. In fact, if $T_p\partial M$ is nondegenerate then, nonzero vectors orthogonal to $T_p\partial M$ are spacelike if and only if $(g_{\partial M})_p$ is of index ν and, nonzero vectors orthogonal to $T_p\partial M$ are timelike if and only if $(g_{\partial M})_p$ is of index $\nu - 1$.

Definition 2.1 Let (M, g) be a semi–Riemannian manifold with boundary ∂M. Define the subsets ∂M_+ and ∂M_- of ∂M to be the sets of points in ∂M where the nonzero vectors orthogonal to ∂M are spacelike and timelike, respectively. Also define the subset ∂M_0 of M to be the set of points in ∂M where the nonzero vectors orthogonal to ∂M are null.

Remark 2.1 Note that $\partial M = \partial M_+ \cup \partial M_- \cup \partial M_0$ and ∂M_+, ∂M_- and ∂M_0 are pairwise disjoint.

Lemma 2.2 *Let (M, g) be a semi–Riemannian manifold with boundary ∂M. Then ∂M_+ and ∂M_- are open subsets of ∂M.*

Proof. Since ∂M_+ and ∂M_- are the points of ∂M where $g_{\partial M}$ is nondegenerate of index ν and $\nu - 1$, respectively, it follows from Lemma 2.1 that ∂M_+ and ∂M_- are open subsets of ∂M. □

Remark 2.2 Let (M, g) be a semi–Riemannian manifold with boundary ∂M. Then since ∂M_+ and ∂M_- are open submanifolds of ∂M by Lemma 2.2, $\partial M' = \partial M_+ \cup \partial M_-$ is also an open submanifold of ∂M, which we call the *nondegenerate boundary of* (M, g).

Let (M, g) be an oriented semi–Riemannian manifold with boundary ∂M. Let μ be the semi–Riemannian volume form on (M, g) (see [AMR, p. 456]) and let N

be the unit outward normal vector field to ∂M defined on $\partial M'$: Then note that, the exterior form $\mu_{\partial M'}$ defined on $\partial M'$ by

$$\mu_{\partial M'} = i_N \mu,$$

where i is the interior product, is the induced semi–Riemannian volume form on the semi–Riemannian manifolds $(\partial M_+, g_{\partial M_+})$ and $(\partial M_-, g_{\partial M_-})$ when restricted to ∂M_+ and ∂M_-, respectively.

Theorem 2.1 [Ü] ***(Semi–Riemannian Divergence Theorem)*** *Let (M, g) be an oriented semi–Riemannian manifold with boundary ∂M (possibly $\partial M = \varnothing$) and semi–Riemannian volume form μ. Let Z be a vector field on M with compact support. If ∂M_0 has measure zero in ∂M (see [BJ, p. 57]) then,*

$$\int_M (div\ Z)\mu = \int_{\partial M_+} g(Z, N)\mu_{\partial M'} - \int_{\partial M_-} g(Z, N)\mu_{\partial M'}.$$

Proof. Recall from the Gauss' Theorem ([AMR, p. 483]) that,

$$\int_M (\text{div } Z)\mu = \int_{\partial M} i_Z \mu,$$

where i is the interior product. Then since ∂M_0 has measure zero in ∂M,

$$\int_M (\text{div } Z)\mu = \int_{\partial M'} i_Z \mu = \int_{\partial M_+} i_Z \mu + \int_{\partial M_-} i_Z \mu.$$

Now let $p \in \partial M'$ and let $\{N(p), e_1, \ldots, e_{n-1}\}$ be an orthonormal basis for T_pM, where $n = \dim M$. Then at p, since

$$Z = g(N, N)g(N, Z)N + \sum_{i=1}^{n-1} g(e_i, e_i)g(Z, e_i)e_i,$$

$$\begin{aligned}
(i_Z\mu)(e_1, \ldots, e_{n-1}) &= \mu(Z, e_1, \ldots, e_{n-1}) \\
&= \mu(g(N, N)g(N, Z)N + \sum_{i=1}^{n-1} g(e_i, e_i)g(Z, e_i)e_i, e_1, \ldots, e_{n-1}) \\
&= \mu(g(N, N)g(N, Z)N, e_1, \ldots, e_{n-1}) \\
&= g(N, N)g(N, Z)\mu(N, e_1, \ldots, e_{n-1}) \\
&= g(N, N)g(N, Z)(i_N\mu)(e_1, \ldots, e_{n-1}) \\
&= g(N, N)g(N, Z)\mu_{\partial M'}(e_1, \ldots, e_{n-1}).
\end{aligned}$$

Hence,

$$i_Z \mu = \begin{cases} g(Z, N)\mu_{\partial M'} & \text{on } \partial M_+ \\ -g(Z, N)\mu_{\partial M'} & \text{on } \partial M_-. \end{cases}$$

Thus it follows that

$$\int_M (\text{div } Z)\mu = \int_{\partial M_+} g(Z,N)\mu_{\partial M'} - \int_{\partial M_-} g(Z,N)\mu_{\partial M'}.$$

□

The assumption of vanishing measure of ∂M_0 in ∂M may be replaced by another condition in the semi-Riemannian divergence theorem.

Theorem 2.2 [Ü] *Let (M,g) be an oriented semi-Riemannian manifold with boundary ∂M (possibly $\partial M = \varnothing$) and semi-Riemannian volume form μ. Let Z be a vector field on M with compact support. If Z is tangent to ∂M at the points of ∂M_0 then*

$$\int_M (\mathit{div}\ Z)\mu = \int_{\partial M_+} g(Z,N)\mu_{\partial M'} - \int_{\partial M_-} g(Z,N)\mu_{\partial M'}.$$

Proof. Since Z is tangent to ∂M at the points of ∂M_0, $i_Z\mu = 0$ at the points of ∂M_0. Hence by the Gauss' Theorem,

$$\int_M (\text{div } Z)\mu = \int_{\partial M'} i_Z\mu = \int_{\partial M_+} i_Z\mu + \int_{\partial M_-} i_Z\mu.$$

Then as in Theorem 2.1, we obtain

$$\int_M (\text{div } Z)\mu = \int_{\partial M_+} g(Z,N)\mu_{\partial M'} - \int_{\partial M_-} g(Z,N)\mu_{\partial M'}.$$

□

For some applications of Theorems 2.1 and 2.2 in 2–dimensional Minkowski space, see [Ü].

3 Generalized semi–Riemannian Divergence Theorem

Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between semi–Riemannian manifolds (M_1, g_1) and (M_2, g_2) and let $p_2 = f(p_1)$ for each $p_1 \in M_1$. Then the *adjoint* of the tangent map $f_{*p_1} : T_{p_1}M_1 \to T_{p_2}M_2$ of f at p_1 is defined to be the unique linear map ${}^*f_{*p_1} : T_{p_2}M_2 \to T_{p_1}M_1$ by

$$g_{1p_1}(x, {}^*f_{*p_1}y) = g_{2p_2}(f_{*p_1}x, y)$$

for all $x \in T_{p_1}M_1$ and $y \in T_{p_2}M_2$.

Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between semi–Riemannian manifolds (M_1, g_1) and (M_2, g_2). Let $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ be the Levi Civita connections of (M_1, g_1) and (M_2, g_2), respectively, and let $\overset{2}{\nabla}$ also denote the pullback of $\overset{2}{\nabla}$ along f. Now, if Z is a vector field along f, that is, $Z : M_1 \to TM_2$ is a map with $Z(p_1) \in T_{p_2}(M_2)$ for each $p_1 \in M_1$, then define a bundle homomorphism

$$^*f_* \overset{2}{\nabla} Z : TM_1 \to TM_1$$

by

$$(^*f_* \overset{2}{\nabla} Z)(x) = {}^*f_{*p_1}(\overset{2}{\nabla}_x Z),$$

where $x \in T_{p_1}M_1$.

Definition 3.1 Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between semi–Riemannian manifolds (M_1, g_1) and (M_2, g_2), and let Z be a vector field along f. Then the *divergence of* Z is defined by

$$div\ Z = trace\ {}^*f_* \overset{2}{\nabla} Z.$$

Remark 3.1 Note that, if $n_1 = \dim M_1$ then,

$$\begin{aligned} \text{div}\ Z &= \text{trace}\ {}^*f_* \overset{2}{\nabla} Z \\ &= \sum_{i=1}^{n_1} g_1(X_i, X_i) g_1((^*f_* \overset{2}{\nabla} Z)X_i, X_i) \\ &= \sum_{i=1}^{n_1} g_1(X_i, X_i) g_1(\overset{2}{\nabla}_{X_i} Z, f_*X_i), \end{aligned}$$

where $\{X_1, \ldots, X_{n_1}\}$ is an orthonormal local basis frame for TM_1. Here also note that, if we set $(M_1, g_1) = (M_2, g_2) = (M, g)$ and $f = \text{id}$, then a vector field along f can be considered as a vector field on M and hence, divergence of a vector field along f reduces to the usual divergence of a vector field on M.

Definition 3.2 Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between semi–Riemannian manifolds (M_1, g_1) and (M_2, g_2). The *second fundamental form* ∇f_* of f is defined by

$$(\nabla f_*)(X, Y) = \overset{2}{\nabla}_X f_*Y - f_*(\overset{1}{\nabla}_X Y),$$

where X and Y are vector fields on M_1.

Remark 3.2 Note that $(\nabla f_*)(X, Y)$ is a vector field along f. Also it can be easily shown that ∇f_* is $C^\infty(M_1)$–linear in its arguments and symmetric.

Definition 3.3 Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between semi–Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then the *tension field* $\tau(f)$ of f is defined to be the trace of ∇f_* with respect to g_1, that is,

$$\tau(f) = \sum_{i=1}^{n_1} g_1(X_i, X_i)(\nabla f_*)(X_i, X_i),$$

where $n_1 = \dim M_1$ and $\{X_1, \ldots, X_{n_1}\}$ is an orthonormal local basis frame for TM_1. Also f is called *harmonic* if $\tau(f) = 0$.

Remark 3.3 Note that the tension field of a map $f : (M_1, g_1) \to (M_2, g_2)$ between semi–Riemannian manifolds (M_1, g_1) and (M_2, g_2) is also a vector field along f by Remark 3.2.

Now by using the notation and terminology of section 2, we will state the generalized semi–Riemannian divergence theorem as follows.

Theorem 3.1 *(Generalized semi–Riemannian Divergence Theorem)* *Let (M_1, g_1) be an oriented semi–Riemannian manifold with boundary ∂M_1 (possibly, $\partial M_1 = \varnothing$) and semi–Riemannian volume form μ_1, and let (M_2, g_2) be a semi–Riemannian manifold. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map and Z be a vector field along f with compact support. If $\partial(M_1)_0$ has measure zero in ∂M_1, then*

$$\begin{aligned}\int_{M_1} (div\ Z)\mu_1 &+ \int_{M_1} g_2(Z, \tau(f))\mu_1 \\ &= \int_{\partial(M_1)_+} g_2(Z, f_* N_1)\mu_{1\partial(M_1)'} - \int_{\partial(M_1)_-} g_2(Z, f_* N_1)\mu_{1\partial(M_1)'},\end{aligned}$$

where N_1 is the unit outward normal to ∂M_1 defined on $\partial(M_1)'$.

Proof. Let ${}^*f_*Z$ be a vector field on M_1 defined by

$$({}^*f_*Z)(p_1) = {}^*f_{*p_1}(Z(p_1))$$

at each $p_1 \in M_1$. Now let $\{X_1, \ldots, X_{n_1}\}$ be an adapted moving frame near p_1, that is, $\{X_1, \ldots, X_{n_1}\}$ is an orthonormal local basis frame for TM_1 with $(\overset{1}{\nabla} X_i)_{p_1} = 0$ for $i = 1, \ldots, n_1$, where $n_1 = \dim M_1$ (see [P, p. 152]). Then

$$\begin{aligned}(\ \mathrm{div}\ {}^*f_*Z)(p_1) &= \left[\sum_{i=1}^{n_1} g_1(X_i, X_i) g_1(\overset{1}{\nabla}_{X_i} ({}^*f_*Z), X_i)\right](p_1) \\ &= \left[\sum_{i=1}^{n_1} g_1(X_i, X_i) X_i g_1({}^*f_*Z, X_i)\right](p_1) \\ &= \left[\sum_{i=1}^{n_1} g_1(X_i, X_i) X_i g_2(Z, f_*X_i)\right](p_1) \\ &= \left[\sum_{i=1}^{n_1} g_1(X_i, X_i) g_2(\overset{2}{\nabla}_{X_i} Z, f_*X_i)\right](p_1) \\ &\quad + \left[\sum_{i=1}^{n_1} g_1(X_i, X_i) g_2(Z, \overset{2}{\nabla}_{X_i} f_*X_i)\right](p_1) \\ &= \left[\sum_{i=1}^{n_1} g_1(X_i, X_i) g_2(\overset{2}{\nabla}_{X_i} Z, f_*X_i)\right](p_1) \\ &\quad + \left[\sum_{i=1}^{n_1} g_1(X_i, X_i) g_2(Z, (\nabla f_*)(X_i, X_i))\right](p_1) \\ &= [\ \mathrm{div}\ Z + g_2(Z, \tau(f))]\,(p_1).\end{aligned}$$

Thus

$$\operatorname{div} {}^*f_*Z = \operatorname{div} Z + g_2(Z, \tau(f)).$$

Now by applying the semi–Riemannian divergence theorem (Theorem 2.1) to ${}^*f_*Z$ on (M_1, g_1), we obtain

$$\begin{aligned}\int_{M_1} (\operatorname{div} {}^*f_*Z)\mu_1 &= \int_{\partial(M_1)_+} g_1({}^*f_*Z, N_1)\mu_{1\partial(M_1)'} - \int_{\partial(M_1)_-} g_1({}^*f_*Z, N_1)\mu_{1\partial(M_1)'} \\ &= \int_{\partial(M_1)_+} g_2(Z, f_*N_1)\mu_{1\partial(M_1)'} - \int_{\partial(M_1)_-} g_2(Z, f_*N_1)\mu_{1\partial(M_1)'}.\end{aligned}$$

Thus, since $\operatorname{div} {}^*f_*Z = \operatorname{div} Z + g_2(Z, \tau(f))$, we have

$$\begin{aligned}&\int_{M_1} (\operatorname{div} Z)\mu_1 + \int_{M_1} g_2(Z, \tau(f))\mu_1 \\ &\qquad = \int_{\partial(M_1)_+} g_2(Z, f_*N_1)\mu_{1\partial(M_1)'} - \int_{\partial(M_1)_-} g_2(Z, f_*N_1)\mu_{1\partial(M_1)'}.\end{aligned}$$

□

Finally we will make an application of generalized semi–Riemannian divergence theorem to harmonic maps. Note that the following proposition generalizes [YI, Proposition 2.5] where $\overset{2}{\nabla} \tau(f) = 0$ is assumed.

Proposition 3.1 *Let (M_1, g_1) be an oriented compact semi–Riemannian manifold with semi–Riemannian volume form μ_1 and let (M_2, g_2) be a Riemannian manifold. If $f : (M_1, g_1) \to (M_2, g_2)$ is a map with $\int_{M_1} (div\ \tau(f))\mu_1 \geq 0$ then f is harmonic.*

Proof. Since $\partial M_1 = \varnothing$, by Theorem 3.1,

$$\int_{M_1} (\operatorname{div} \tau(f))\mu_1 + \int_{M_1} g_2(\tau(f), \tau(f))\mu_1 = 0.$$

Thus, since $\int_{M_1} (\operatorname{div} \tau(f))\mu_1 \geq 0$ and g_2 is a Riemannian metric tensor, it follows that $g_2(\tau(f), \tau(f)) = 0$, and hence $\tau(f) = 0$. □

Note here that Proposition 3.1 fails to hold if (M_1, g_1) is not compact. Indeed, the map $f : (\mathbb{R}, dt \otimes dt) \to (\mathbb{R}, dt \otimes dt)$ given by $f(t) = \frac{1}{2}t^2$ has $\operatorname{div} \tau(f) = 0$ but f is not harmonic.

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SOME RESULTS ON OSSERMAN SEMI-RIEMANNIAN SPACES

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1 Introduction

One of the central topics in Differential Geometry is the study of the curvature of the space. Following Osserman [22, pag. 731]:

> "The notion of curvature is one of the central concepts of differential geometry; one could argue that it is *the* central one, distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic, or topological. In the words of Marcel Berger, curvature is the *Nº 1 Riemannian invariant and the most natural. Gauss and then Riemann saw it instantly.*
> Curvature also plays a key role in physics. The magnitude of a force required to move an object at constant speed along a curved path is, according to Newton's laws, a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein, by the curvature of the space–time."

If (M, g) is a manifold equipped with a (definite or indefinite) metric tensor, the associated Levi Civita connection ∇ allows us to define the curvature of (M, g) by means of the tensor field R given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

An important component of the curvature tensor is the one given by the Jacobi operators. They arise in a natural way in the study of geodesic variations as a measure of the geodesic deviation. Given a geodesic γ, the second order differential equation along γ,

$$X'' + R_\gamma X = 0,$$

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is called the Jacobi differential equation and its solutions are the Jacobi vector fields along γ. $R_\gamma X = R(X, \gamma')\gamma'$ is called the *Jacobi operator* along the geodesic γ. The investigation of the properties of the Jacobi vector fields is basic in a number of geometrical problems, specially due to their relation with coordinated vector fields in system of normal and Fermi coordinates, which allow to describe the local geometry of the manifold from the knowledge of the Jacobi vector fields. However, it is rather difficult to determine explicitly the general solution of the Jacobi differential equation, except in those cases of manifolds with a simple curvature tensor. Even this way, many properties can be derived from the knowledge of the Jacobi operators.

Considering the Jacobi operator R_x, associated to a tangent vector $x \in T_pM$, as the endomorphism of the tangent space to M at p defined by $R_x y = R(y, x)x$, for $y \in T_pM$, the identities of the curvature tensor show that R_x is self–adjoint, that is, $g(R_x y, z) = g(y, R_x z)$, for all y, $z \in T_pM$. Therefore, we will pay special attention to the study of the spectrum of those operators. In general, the eigenvalues of the Jacobi operators depend on both, the basepoint and the direction: $\lambda_i(p, x)$. However, they take a simpler form for special classes of Riemannian or semi–Riemannian manifolds: it is well known that those eigenvalues are constant on the unit sphere bundle for two–point homogeneous Riemannian manifolds. Naturally arises the question of whether those are the only Riemannian spaces satisfying such condition on the eigenvalues of the Jacobi operators. Spaces satisfying such condition are called *Osserman spaces* and we will refer to the problem of classifying them as the *Osserman problem.*

Next we will review some of the most interesting known results on the study of the Osserman problem in Riemannian and Lorentzian geometry, before treating the more general case of semi–Riemannian manifolds.

1.1 Osserman Riemannian spaces

For positive definite metrics, Osserman [22] made the following conjecture:

Conjecture *If the eigenvalues of the Jacobi operators of a Riemannian manifold (M, g) are constant on the unit sphere bundle, then M is flat or locally isometric to a rank-one symmetric space.*

Chi has solved this conjecture in many cases. In particular, he has proved the following result.

Theorem 1.1 [9, Th.0] *Let (M^n, g) be a connected Osserman Riemannian space. If $n \neq 4k$, $k > 1$, then M has constant sectional curvature or it is a Kähler manifold of constant holomorphic sectional curvature.*

The essential lines of the proof are as follows. For each point $p \in M$ let S_pM denote the unit sphere in T_pM. Each eigenvalue λ of the Jacobi operators induces a smooth distribution on the sphere S_pM given by

$$\mathcal{D}_\lambda : x \in S_pM \mapsto \mathcal{D}_\lambda(x) = Ker(R_x - \lambda Id),$$

whose dimension is indicated by the multiplicity of the eigenvalue λ.

Now, if the dimension of M is odd, then S_pM is even–dimensional, and the nonexistence of continuous distributions on such spheres shows that, in this case, the Jacobi operators may have only one constant eigenvalue. Next, by using the Schur lemma, it follows that (M, g) is a space of constant curvature.

In an analogous way, if dim $M = 4k + 2$, then again topological restrictions show that the number of distinct eigenvalues of the Jacobi operators is one or two, in the later case one of them necessarily having multiplicity one. From this, one constructs a complex structure on M which gives to the manifold a structure of generalized complex space. Finally, the second Bianchi identity shows that (M, g) has constant sectional curvature or it is a Kähler manifold of constant holomorphic sectional curvature. (See also [17] for a more recent proof of the last case discussed above).

Chi has also studied the Osserman condition on manifolds equipped with some additional structures. In this sense we point out a result for Osserman Kähler spaces.

Theorem 1.2 [9, Th.0] *Let (M, g, J) be a Kähler manifold with nonnegative or nonpositive sectional curvature. If M is Osserman, then the holomorphic sectional curvature is constant.*

The key argument in the proof of this result is the following one. First of all we recall that, when dealing with Riemannian metrics, the eigenvalues of the Jacobi operator point out the extreme values of the sectional curvature, while the corresponding eigenspaces show the directions where those extreme values are reached. Now, in the particular case of Kähler metrics of nonpositive or nonnegative sectional curvature, an eigenspace corresponding to the maximal (resp., minimal) value of the sectional curvature must be holomorphic (cf. [8, p. 362]), and therefore the constancy of the holomorphic sectional curvature follows from the constancy of the eigenvalues of the Jacobi operators for Kähler manifolds of nonpositive or nonnegative sectional curvature.

The existence of distinguished eigenspaces of the Jacobi operators also influences on the geometry of the manifold. (See, for example [19], where it is shown that a Kähler manifold (M, g, J) is a complex space form if and only if for each unit vector x, Jx is an eigenvector of the Jacobi operator R_x, i.e., $R(Jx, x)x \sim Jx$). Therefore, in solving the Osserman problem is natural to impose some additional conditions on the behaviour of the eigenspaces of the Jacobi operators. This idea was developed by Chi who characterized the rank–one symmetric spaces by using the Osserman property with some natural additional conditions:

Theorem 1.3 [11, Th.1] *The locally rank–one symmetric spaces of nonconstant sectional curvature are determined by the following two axioms:*

Axiom 1. *The manifold is an Osserman space in which the Jacobi operators have two distinct eigenvalues λ and μ.*

Axiom 2. *If $y \in \langle x\rangle \oplus Ker(R_x - \lambda Id)$, then $\langle y\rangle \oplus Ker(R_y - \lambda Id) = \langle x\rangle \oplus Ker(R_x - \lambda Id)$.*

In view of all known results, the solution of the Osserman conjecture could arise in a two step proof. The first step consisting in the determination of all the possible Osserman curvature tensors which may be realized at a single point of a manifold. The second one, by making use of the second Bianchi identity, will allow to show that such spaces are locally symmetric. This approach has been initially considered by Gilkey, who constructed the first examples of curvature tensors which are Osserman but not corresponding to any rank–one symmetric spaces [16]. This construction makes use of the existence of certain Clifford–module structures and, the theory of normal coordinates shows the existence of Riemannian manifolds where such curvature tensors are realized at a point.

These examples motivated the study of those Riemannian manifolds which are Osserman at each point but where the eigenvalues of the Jacobi operators may change from point to point: *pointwise Osserman manifolds* [17]. They are trivially Osserman in dimension three but there is a large number of four–dimensional pointwise Osserman manifolds which are not (globally) Osserman, like the generalized complex space forms [20]. (We refer to [17] for more results about the relation between the pointwise and the global Osserman conditions).

A related problem to the Osserman one is the determination of those Riemannian manifolds all of whose *geodesic spheres are isoparametric* [27]. Since any isoparametric geodesic sphere has constant mean curvature, such spaces are necessarily harmonic and it has been shown in [17] that they are also globally Osserman. In spite of the known results about harmonic and Osserman spaces, the problem of determinig those Riemannian manifolds whose geodesic spheres are isoparametric is still open.

1.2 Osserman Lorentzian spaces

When considering indefinite metrics, a unit vector may be spacelike or timelike. This fact motivates the study of the Osserman problem in a separate way as corresponding to timelike or spacelike geodesics. Thus, we have:

Definition 1.1 *A semi–Riemannian manifold (M, g) is said to be an spacelike (resp., timelike) Osserman space if the (posibly complex) eigenvalues of the Jacobi operators R_x are constant for all unit spacelike (resp., timelike) vectors x. (M, g) will be called an Osserman space if it is spacelike and timelike Osserman at the same time.*

Note that when we consider an Osserman semi–Riemannian space, the eigenvalues of the Jacobi operators are not constant, since the sign of such eigenvalues changes depending on the causal character of the direction.

The analysis of the Osserman condition in Lorentzian geometry becomes simpler than in the Riemannian case. First of all, note that indefinite complex or quaternionic structures are allowed in Lorentzian signature. Secondly, note that the sectional curvature of indefinite metrics presents certain pathologies as concerns boundedness when comparing to the Riemannian case [21]. This facts led to a complete answer to the Osserman problem for Lorentzian manifolds. Indeed, it has been shown in [13] that a timelike Osserman Lorentzian manifold is of constant curvature and that the same holds for spacelike Osserman Lorentzian manifolds of dimension ≤ 4. Later on, Blažić, Bokan and Gilkey [2] completed the spacelike case and provided the following result:

Theorem 1.4 *Let (M, g) be a Lorentzian manifold. The following conditions are equivalent:*

(i) *The sectional curvature is constant at $p \in M$.*

(ii) *(M, g) is spacelike Osserman at p.*

(iii) *(M, g) is timelike Osserman at p.*

Note that, as a consequence of previous theorem, the pointwise and the global Osserman conditions become equivalent in the Lorentzian setting.

2 Indefinite Kähler Osserman spaces

The study of the Osserman condition for semi–Riemannian manifolds with metric of non–Lorentzian signature presents significant differences with respect to both the Riemannian and Lorentzian cases. It was shown in [15] the existence of Osserman semi–Riemannian spaces with metric of any signature (p, q), $p, q \geq 2$, which are not locally symmetric (even not locally homogeneous).

The purpose of this section is to study the Osserman condition under the additional assumption of the existence of an indefinite Kähler structure. We will show that, even in this case, the situation presents important differences again with respect to the Riemannian case (cf. Theorem 1.2). Indeed, a semi–Riemannian manifold is of nonpositive or nonnegative sectional curvature if and only if the sectional curvature is constant. Further, if the manifold is assumed to be Kähler, it must be necessarily flat. Therefore, in investigating the Osserman problem, other kind of boundedness conditions should be considered in the semi–Riemannian setting.

At this point, it is worthwhile to recall that the holomorphic sectional curvature of a positive definite almost Hermitian manifold is a real function defined on the unit sphere bundle. Therefore, it is bounded at each point $p \in M$. However, for indefinite almost Hermitian metrics, the holomorphic sectional curvature is a real function defined on the unit pseudosphere bundle. Since the pseudosphere is noncompact, the existence of bounds for the holomorphic sectional curvature is not guaranteed in the indefinite case. It has been shown in [1], [4] that the holomorphic sectional curvature is bounded from above and from below on spacelike or timelike holomorphic planes if and only if it is constant. Other kind of bounded conditions was investigated in [5], where it is shown that the holomorphic sectional curvature of an indefinite almost Hermitian manifold is bounded from above (resp., from below) on holomorphic planes of signature (++) and from below (resp., from above) on holomorphic planes of signature (−−) if and only if the manifold is null holomorphically flat. This last condition, when coupled with the Osserman one, immediately implies the constancy of the holomorphic sectional curvature for indefinite Kähler manifolds.

However, one side bounds do not suffice to have constant holomorphic sectional curvature, even if the manifold is assumed to be Osserman. Indeed, in what follows we will point out the existence of *locally symmetric, not locally symmetric and even not locally homogeneous examples of Osserman indefinite Kähler manifolds with nonpositive or nonnegative holomorphic sectional curvature but not of constant holomorphic sectional curvature.* We refer to [6] for more details.

Let $\mathbb{R}^4$ denote the 4–dimensional real Euclidean space, with usual coordinates (x^1, x^2, x^3, x^4). We define on $\mathbb{R}^4$ the metric g_ϕ given by

$$g_\phi = dx^1 \otimes dx^3 + dx^2 \otimes dx^4 + \sum_{i,j=1}^{2} \phi_{ij} dx^i \otimes dx^j, \tag{2.1}$$

where $\phi = (\phi_{ij})$ is any symmetric $(0,2)$–tensor field on $\mathbb{R}^2$.

The Levi Civita connection is then given by

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1} &= \left\{\frac{1}{2}\frac{\partial \phi_{11}}{\partial x^1}\right\} \frac{\partial}{\partial x^3} + \left\{-\frac{1}{2}\frac{\partial \phi_{11}}{\partial x^2} + \frac{\partial \phi_{12}}{\partial x^1}\right\} \frac{\partial}{\partial x^4}, \\ \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^2} &= \left\{\frac{1}{2}\frac{\partial \phi_{11}}{\partial x^2}\right\} \frac{\partial}{\partial x^3} + \left\{\frac{1}{2}\frac{\partial \phi_{22}}{\partial x^1}\right\} \frac{\partial}{\partial x^4}, \\ \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^2} &= \left\{-\frac{1}{2}\frac{\partial \phi_{22}}{\partial x^1} + \frac{\partial \phi_{12}}{\partial x^2}\right\} \frac{\partial}{\partial x^3} + \left\{\frac{1}{2}\frac{\partial \phi_{22}}{\partial x^2}\right\} \frac{\partial}{\partial x^4}, \end{aligned} \tag{2.2}$$

and, therefore, the only nonvanishing components of the curvature tensor are those given by

$$
(2.3)\quad
\begin{aligned}
R\left(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\right)\frac{\partial}{\partial x^1} &= \left\{\frac{1}{2}\frac{\partial^2\phi_{11}}{\partial x^2\partial x^2}+\frac{1}{2}\frac{\partial^2\phi_{22}}{\partial x^1\partial x^1}-\frac{\partial^2\phi_{12}}{\partial x^1\partial x^2}\right\}\frac{\partial}{\partial x^4},\\
R\left(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\right)\frac{\partial}{\partial x^2} &= -\left\{\frac{1}{2}\frac{\partial^2\phi_{11}}{\partial x^2\partial x^2}+\frac{1}{2}\frac{\partial^2\phi_{22}}{\partial x^1\partial x^1}-\frac{\partial^2\phi_{12}}{\partial x^1\partial x^2}\right\}\frac{\partial}{\partial x^3}.
\end{aligned}
$$

The above expressions allow us to obtain the following,

Theorem 2.1 [6] *$(\mathbb{R}^4, g_\phi)$ is an Osserman semi–Riemannian space with metric of signature $(+,+,-,-)$. The characteristic polynomial of the Jacobi operators is always $p_\lambda(R_x) = \lambda^4$, while the minimal polynomial $m_\lambda(R_x)$ is determined by the function*

$$
(2.4)\qquad F(x^1,x^2) = \frac{1}{2}\frac{\partial^2\phi_{11}}{\partial x^2\partial x^2}+\frac{1}{2}\frac{\partial^2\phi_{22}}{\partial x^1\partial x^1}-\frac{\partial^2\phi_{12}}{\partial x^1\partial x^2},
$$

in the following way:

(i) *$(\mathbb{R}^4, g_\phi)$ has zero sectional curvature (i.e., $m_\lambda(R_x) = \lambda$) at any point where F vanishes,*

(ii) *the minimal polynomial is $m_\lambda(R_x) = \lambda^2$ at those points where F is different from zero.*

Moreover, $(\mathbb{R}^4, g_\phi)$ is locally symmetric if and only if the function F is constant.

In what follows we will endow the space $(\mathbb{R}^4, g_\phi)$ with a Kähler structure. Note that the tensor field ϕ is said to be *Hermitian* with respect to a complex structure K on $\mathbb{R}^2$ if it satisfies the condition $\phi(KX, KY) = \phi(X, Y)$ for all vector fields $X, Y \in \mathbb{R}^2$. Next consider on $(\mathbb{R}^4, g_\phi)$ the complex structure, J, defined by

$$
J\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^2},\quad J\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^4}.
$$

Then we have

Theorem 2.2 [6] *$(\mathbb{R}^4, g_\phi, J)$ is an Osserman indefinite Kähler space if and only if ϕ is Hermitian with respect to the usual complex structure on $\mathbb{R}^2$ $(J\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^2})$. Moreover, the sign of the holomorphic sectional curvature of $(\mathbb{R}^4, g_\phi, J)$ is determined at each point by the sign of $\Delta\phi_{11}$, where Δ denotes the Laplacian on $\mathbb{R}^2$.*

Remark 2.1 1. Note that if $(\mathbb{R}^4, g_\phi, J)$ is a locally symmetric Osserman indefinite Kähler manifold, then its holomorphic sectional curvature is nonpositive or nonnegative but nonconstant unless be flat.

2. The product of the manifolds $(\mathbb{R}^4, g_\phi, J)$ and the Euclidean spaces $\mathbb{R}^n_\nu$ allows us to extend the previous examples to higher dimensions. In particular, we obtain examples of indefinite Kähler manifolds of any signature $(2p, 2q)$, for $p, q \geq 1$, with nonnegative or nonpositive holomorphic sectional curvature, which are Osserman but not locally symmetric (or even not locally homogeneous).

Remark 2.2 1. The semi–Riemannian metric g_ϕ on $\mathbb{R}^4$ can be interpreted as the *deformed complete lift* of the usual Riemannian metric on $\mathbb{R}^2$ to its tangent bundle $\mathbb{R}^4 = T\mathbb{R}^2$ with respect to the symmetric tensor field ϕ, [6].

2. Note that the deformed complete lift of the usual Minkowskian metric on $\mathbb{R}^2_1$ to its tangent bundle $\mathbb{R}^4 = T\mathbb{R}^2$ leads to similar results in the framework of para–Kähler geometry, [6].

3 Special Osserman manifolds

In this last section we will present some results in the study of semi–Riemannian manifolds with simple Jacobi operators. More precisely, we define the class of *special Osserman manifolds* as follows:

Definition 3.1 *A semi–Riemannian manifold (M, g) is called an special Osserman manifold if it satisfies the following two axioms:*

Axiom I. *For each unit vector x, the Jacobi operator R_x is diagonalizable with exactly two distinct eigenvalues: $\varepsilon_x\lambda$ and $\varepsilon_x\mu$, where $\varepsilon_x = g(x, x)$ and λ, $\mu \in \mathbb{R}$.*

Axiom II. *If z is a unit vector in $E_\lambda(x)$, then $E_\lambda(x) = E_\lambda(z)$ and, moreover, if $y \in Ker(R_x - \varepsilon_x\mu Id)$, then $x \in Ker(R_y - \varepsilon_y\mu Id)$, where for any unit vector x, $E_\lambda(x) = \langle x\rangle \oplus Ker(R_x - \varepsilon_x\lambda Id)$.*

First of all, we will briefly explain the motivation for the two conditions in the above definition. Since all known examples of nonsymmetric Osserman semi–Riemannian spaces have nondiagonalizable Jacobi operators (cf. [6], [14]), it seems natural to assume such condition in order to approach the general problem. Moreover, note that in [3], four–dimensional Osserman manifolds with diagonalizable Jacobi operators are characterized, by showing that they are indefinite real, complex or paracomplex space forms. In all this cases, only two distinct eigenvalues occur.

Now, it is clear that if the Jacobi operators of a semi–Riemannian manifold are diagonalizable with only one eigenvalue, it is a space of constant sectional curvature. (Note that, even in this case, the diagonalizability of the Jacobi operators is a necessary condition, because of the nonsymmetric examples in the previous section). Therefore, the first nontrivial case is the one corresponding to two distinct eigenvalues.

On the other hand, Axiom II allows us to decompose the tangent space at any point of an special Osserman manifold as direct sum of subspaces $E_\lambda(\cdot)$. For a better understanding of this axiom, note that indefinite Kähler manifolds of constant holomorphic sectional curvature, para–Kähler manifolds of constant paraholomorphic sectional curvature, indefinite quaternionic Kähler manifolds of constant quaternionic sectional curvature and paraquaternionic Kähler manifolds of constant paraquaternionic sectional curvature are examples of special Osserman manifolds. A further look at indefinite Kähler manifolds of constant holomorphic sectional curvature c, shows that the Jacobi operator R_x associated to a unit vector x is diagonalizable with two distinct eigenvalues, $\varepsilon_x c$ and $\varepsilon_x(c/4)$, $\varepsilon_x c$ of multiplicity one. Thus, Axiom I holds. With respect to Axiom II, if we denote by J the complex structure, then the c–eigenspace of R_x coincides with the span of Jx and, therefore, $E_c(x) = \langle x, Jx\rangle$. Thus, we see that if y is a unit vector in $E_c(x)$ then $\langle y, Jy\rangle = \langle x, Jx\rangle$, because $E_c(x)$ is a complex subspace. Similarly, we have that the $(c/4)$–eigenspace of R_x is J–invariant and is given by $\langle x, Jx\rangle^\perp$. This shows that if $y \in \langle x, Jx\rangle^\perp$ then $x \in \langle y, Jy\rangle^\perp$. Therefore, Axiom II can be interpreted as the fact that $E_c(x)$ is a J–invariant subspace of the tangent space. A similar interpretation can be given in the other three classes of special Osserman manifolds.

In what follows we will show a classification of the special Osserman manifolds. This problem is approached in two steps, as suggested in [17]. In the first one, we determine the form of the curvature tensors which may occur at an arbitrary point of an special Osserman manifold. In the second step, by making use of the second Bianchi identity, we classify the special Osserman manifolds of dimension different from 16 and 32.

3.1 Pointwise description of the curvature tensor

As we have pointed out before, the significance of Axiom II is the possibility of decomposing the tangent space T_pM at a point p of an special Osserman manifold in the form

$$T_pM = E_\lambda(x) \oplus E_\lambda(y) \oplus E_\lambda(z) \oplus \cdots$$

Now, let $E_\lambda(\xi)$ be one of the subspaces in the decomposition above and $\{\xi, \xi_1, \dots, \xi_\tau\}$ an orthonormal basis of $E_\lambda(\xi)$, where τ denotes the multiplicity of λ. Define on $E_\lambda(\xi)^\perp$ the structures $\phi_1, \dots, \phi_\tau$ given by

$$\phi_i\eta = \frac{3}{2(\lambda - \mu)} R(\xi, \xi_i)\eta, \quad \eta \in E_\lambda(x)^\perp. \tag{3.5}$$

Note that ϕ_i leaves invariant each one of the subspaces $E_\lambda(\cdot)$ contained in $E_\lambda(\xi)^\perp$. Thus, on any subspace $E_\lambda(\cdot)$ we have the structures $\phi_1, \dots, \phi_\tau$ defined before. A long calculation shows that these structures anticommute with each other and moreover that they are complex or paracomplex structures on each $E_\lambda(\cdot)$. More precisely, all of them are necessarily complex or there are exactly $(\tau-1)/2$ complex structures and $(\tau+1)/2$ paracomplex structures on each $E_\lambda(\cdot)$ contained in $E_\lambda(\xi)^\perp$. Now, the anticommutative complex structures determine a Clifford module structure on $E_\lambda(\cdot)$. At this point we recall the following result from Steenrod on the existence of Clifford module structures.

Theorem 3.1 [26] *Let $m = 2^r \cdot m_0$, with m_0 odd.*

(i) *V^m admits a Cliff(ν)-module structure if and only if $\nu \leq \nu(r)$,*

(ii) *TS^{m-1} admits a q-dimensional distribution, for $2q \leq m-1$, if and only if $q \leq \nu(r)$,*

where ν is given by $\nu(i+4) = \nu(i) + 8$ and $\nu(i) = 2^i - 1$ for $i = 0, 1, 2, 3$.

By making use of previous theorem, it follows that the multiplicity of the eigenvalue λ must be 1, 3, 7 or 15, with the following restrictions on the dimension of the manifold and on the signature of the metric:

Theorem 3.2 [7] *Let (M, g) be an special Osserman manifold. Then one of the following conditions holds:*

(i) *$\tau = 1$ and M is a $2n$-dimensional manifold with metric of signature (n, n) or $(2p, 2q)$, for some $p, q \geq 0$,*

(ii) *$\tau = 3$ and M is a $4n$-dimensional manifold with metric of signature $(2n, 2n)$ or $(4p, 4q)$, for some $p, q \geq 0$,*

(iii) *$\tau = 7$ and M is a 16-dimensional manifold with metric of signature $(8, 8)$, $(16, 0)$ or $(0, 16)$, or*

(iv) *$\tau = 15$ and M is a 32-dimensional manifold with metric of signature $(16, 16)$,*

where τ denotes the multiplicity of the distinguished eigenvalue λ.

In the cases (i) and (ii) in the previous theorem, we explicitly determine the form of the curvature tensors which may occur at a point of the manifold. To do this, we recall that a curvaturelike function $\tilde{F}$ on a vector space V is a $(0,4)$-tensor field on V satisfying the following properties:

$$\tilde{F}(x, y, z, w) = -\tilde{F}(y, x, z, w) = -\tilde{F}(x, y, w, z),$$

$$\tilde{F}(x, y, z, w) = \tilde{F}(z, w, x, y),$$

$$\tilde{F}(x, y, z, w) + \tilde{F}(y, z, x, w) + \tilde{F}(z, x, y, w) = 0,$$

for all vectors x, y, z, $w \in V$. Moreover, if $\langle\, ,\, \rangle$ denotes an inner product on V, then there exists an unique tensor field of type $(1,3)$ on V, which we denote by F, satisfying $\tilde{F}(x,y,z,w) = \langle F(x,y)z, w\rangle$ for all vectors x, y, z, $w \in V$.

In particular, associated with the inner product $\langle\, ,\, \rangle$, the curvaturelike tensor R^0 is defined by

$$R^0(x,y)z = \langle y,z\rangle x - \langle x,z\rangle y,$$

and if J is a complex (resp., paracomplex) structure such that $(V, \langle\, ,\, \rangle, J)$ is a Hermitian (resp., para–Hermitian) vector space, then the curvaturelike tensor R^J is defined by

$$R^J(x,y)z = \langle Jx,z\rangle Jy - \langle Jy,z\rangle Jx + 2\langle Jx,y\rangle Jz.$$

The following theorem shows that when the dimension is different from 16 and 32, the curvature tensor of an special Osserman manifold can be written, at each point, as a linear combination of R^0 and the tensors R^{J_i} associated with certain structures defined on the tangent space to the manifold at that point.

Theorem 3.3 *Let (M,g) be an special Osserman manifold with dimension different from* 16 *and* 32. *Then, at each point p of the manifold one of the following conditions holds:*

(i) *There exists a complex structure J such that (g,J) defines a Hermitian structure on T_pM and the curvature tensor R takes the form*

$$R = \mu R^0 - \frac{\lambda-\mu}{3}R^J.$$

(ii) *There exists a paracomplex structure J such that (g,J) defines a para–Hermitian structure on T_pM and the curvature tensor R takes the form*

$$R = \mu R^0 + \frac{\lambda-\mu}{3}R^J.$$

(iii) *There exists a quaternionic structure V such that (g,V) defines a Hermitian quaternionic structure on T_pM and the curvature tensor R takes the form*

$$R = \mu R^0 - \frac{\lambda-\mu}{3}\sum_{i=1}^{3} R^{J_i},$$

where $\{J_1, J_2, J_3\}$ is an adapted basis for V.

(iv) *There exists a paraquaternionic structure V such that (g,V) defines a Hermitian paraquaternionic structure on T_pM and the curvature tensor R takes the form*

$$R = \mu R^0 + \frac{\lambda-\mu}{3}\sum_{i=1}^{3} \sigma_i R^{J_i},$$

where $\{J_1, J_2, J_3\}$ is an adapted basis for V and $J_i^2 = \sigma_i Id$, $i = 1,2,3$, $(\sigma_1 = -1,\ \sigma_2 = \sigma_3 = 1)$.

3.2 Local classification

By means of the repeated use of the second Bianchi identity, we have obtained in [7] the local classification of those special Osserman manifolds of dimension different from 16 and 32. More precisely, one has the following

Theorem 3.4 [7] *Let (M, g) be an special Osserman manifold. If the dimension of M is different from 16 and 32, then (M, g) is locally isometric to one of the following*

(a) *an indefinite Kähler manifold of constant holomorphic sectional curvature with metric of signature $(2p, 2q)$, $p, q \geq 0$,*

(b) *a para–Kähler manifold of constant paraholomorphic sectional curvature with metric of signature (n, n),*

(c) *an indefinite quaternionic Kähler manifold of constant quaternionic sectional curvature with metric of signature $(4p, 4q)$, $p, q \geq 0$, or*

(d) *a paraquaternionic Kähler manifold of constant paraquaternionic sectional curvature with metric of signature $(2n, 2n)$.*

The remaining dimensions correspond to the cases of multiplicity $\tau = 7$ and $\tau = 15$. In these cases, we have only two subspaces $E_\lambda(\cdot)$ in the local decomposition of TM, $E_\lambda(X_0)$ and $E_\lambda(Y_0)$. On any of these subspaces, for example $E_\lambda(Y_0)$, we define a product in the following way:

$$(3.6) \qquad \begin{cases} Y_0 \cdot Y_i = Y_i \cdot Y_0 = Y_i, \quad i = 0, 1, \ldots; \tau, \\ \\ Y_i \cdot Y_j = \phi_i \phi_j Y_0, \quad i, j = 1, \ldots, \tau, \end{cases}$$

where $\{Y_0, Y_1, \ldots, Y_\tau\}$ is the basis of $E_\lambda(Y_0)$ determined by $Y_i = \phi_i Y_0$, ϕ_i being the structures defined in (3.5). This product equips $E_\lambda(X_0)$ with an structure of algebra, which is isomorphic to the octaves $\mathbb{O}$ or the split octaves $\mathbb{O}'$, if $\tau = 7$, and to the product of the octaves and the paracomplex numbers $\mathbb{O} \otimes \mathbb{C}'$, if $\tau = 15$. After constructing an standard basis for any of these three algebras, and determining the components of the curvature of the manifold in that basis, we obtain the following.

Theorem 3.5 *Let (M, g) be an special Osserman manifold and suppose that the multiplicity of the distinguished eigenvalue λ is $\tau = 7$ or $\tau = 15$. Then, M is locally symmetric.*

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ON THE VOLUME FUNCTIONAL IN THE MANIFOLD OF UNIT VECTOR FIELDS

O. Gil-Medrano

Abstract.– The paper contains a description of the manifold of unit vector fields and of some functionals on it, with special emphasis on the first variation of the volume and on the notion of critical vector field. It also includes some results of the author's work with E. Llinares-Fuster [5], concerning the case of spaces of constant curvature and that of Killing vector fields on a general manifold.

1 Introduction

The contents of this paper correspond roughly to a talk given at the Workshop on Recent Topics on Differential Geometry. It is devoted to report some results concerning unit vector fields in a Riemannian manifold, specially those that are critical for the volume functional. Let us start by a brief survey of some previous results that motivated our interest in the subjet.

Let M be a Riemannian manifold such that the set $\mathcal{X}^1(M)$ of unit vector fields is not empty. In [8], where M is also assumed to be compact, oriented and boundaryless, the volume of an element $V \in \mathcal{X}^1(M)$ was defined to be the volume of the submanifold $V(M)$ of the unit tangent bundle equipped with the restriction of the Sasaki metric. There is a trivial absolute minimum of the volume functional when unit parallel vector fields exist, but that will usually not be possible, since such a vector field will determine two mutually orthogonal complementary totally geodesic foliations.

On a round unit odd-dimensional sphere, Gluck and Ziller ([8]) considered Hopf vector fields as the candidates for this absolute minimum and showed that it is the case for the 3-sphere. Their method of calibrated geometries cannot be applied to higher dimensional spheres and in fact, Johnson ([11]) showed that

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the Hopf vector fields on S^5 are not local minima of the volume. He used direct methods to show that for every deformation of a Hopf vector field the first variation vanishes but that there are deformations on which the second variation is negative. In both papers the results are derived using the specific properties of Hopf vector fields and spheres.

In the author's work with E. Llinares-Fuster [5] the volume has been considered as a functional on the manifold of unit vector fields on a general M. The study of the first variation has suggested a notion of critical unit vector field valid for a general manifold not assumed to be neither compact nor boundaryless.

That the study of geometrical properties defined by this variational procedure deserves some attention is an idea deeply established on the basis of remarkable examples: Ricci-flat or Eintein metrics, harmonic maps, minimal or constant mean curvature immersions, among others can be described in this way. Moreover, it is worth noting that as suggested by results in [15], [12] and [13] the minimun volume of vector fields in a compact manifold is very likely attained by singular vector fields and so properly defined in an open submanifold.

This paper is divided into four sections. In the second one, we describe the manifold of unit vector fields and some functionals on it; we put special emphasis on the study of the first variation of the volume. Section Three contains some examples of critical unit vector field on manifolds of constant curvature and we discuss the case of Killing vector fields in the last section. The material of these two sections is included in [5] where the reader will find explicit computations and proofs.

2 First variation of the volume functional

If (M, g) is a smooth, connected, closed Riemannian manifold the set $\mathcal{X}^1(M)$ of all smooth unitary vector fields, if nonempty, can be endowed with a structure of Fréchet manifold, compatible with its C^∞-topology, such that each $V \in \mathcal{X}^1(M)$ is contained in a chart modelled in $\mathcal{H}^V$, the space of smooth vector fields in the horizontal distribution determined by V and the metric. Namely we have

Proposition 1 *If M is closed $\mathcal{X}^1(M)$ is a Fréchet submanifold of the Fréchet space $\mathcal{X}(M)$. If M is noncompact $\mathcal{X}^1(M)$ is a LF-submanifold of the LF-manifold $\mathcal{X}(M)$.*

Proof. Let us assume that M is closed, then adapted charts can be constructed as follows: for each $V \in \mathcal{X}^1(M)$ take the decomposition $\mathcal{X}(M) = \mathcal{H}^V \oplus \mathcal{V}$ where $\mathcal{V}$ is the 1-dimensional distribution spanned by V, with projection maps h and v, respectively. It is easy to see that the map $\varphi_V(X) = hX + (1 - g(X, X))V$, when resticted to the open neighbourhood of V $\mathcal{U}_V = \{X \in \mathcal{X}(M) \ ; \ g(X, V) > 0\}$, is injective and verifies that $X \in \mathcal{X}^1(M) \cap \mathcal{U}_V$ if and only if $\varphi_V(X) \in \mathcal{H}^V \cap \varphi_V(\mathcal{U}_V)$.

If M is noncompact, by a procedure similar to that described in [6] for spaces of smooth covariant tensor fields, $\mathcal{X}(M)$ admits a structure of manifold modelled on the space of vector fields with compact support, $\mathcal{X}_c(M)$, with a topology obtained as the inductive limit of Fréchet spaces. $\mathcal{X}_c(M)$ is a complete LF-space and consequently a convenient vector space in the sense of [14] pg. 297. For each $V \in \mathcal{X}^1(M)$ the decomposition $\mathcal{X}_c(M) = \mathcal{H}_c^V \oplus \mathcal{V}_c$ holds and, since $\mathcal{X}_c(M)$ is the disjoint union of subsets of the form $X + \mathcal{X}_c(M)$, with $X \in \mathcal{X}_c(M)$, the chart φ_V described above, when resticted to the open neighbourhood of V $\mathcal{U}_V = \{X \in V + \mathcal{X}_c(M) \ ; \ g(X,V) > 0\}$, has the properties needed to provide an adapted chart. Therefore, the subset $\mathcal{X}^1(M)$ is an LF-submanifold of $\mathcal{X}(M)$.

For a closed M, the volume $F(V)$ of an element $V \in \mathcal{X}^1(M)$ (see [8]) is defined to be the volume of $V(M)$ when in T^1M the usual metric g^S, defined by Sasaki, is considered. It can be described as follows:

For a given vector field V its covariant derivative, ∇V, is a $(1,1)$-tensor field and we can construct the symmetric $(1,1)$-tensor field $L_V = \mathrm{Id} + (\nabla V)^t \circ \nabla V$; here $(\nabla V)^t = g^{-1} \circ (\nabla V)^* \circ g$ and then for $X, Y \in \mathcal{X}(M)$

$$g(L_V(X), Y) = g(X,Y) + g(\nabla_X V, \nabla_Y V).$$

A map $f : \mathcal{X}^1(M) \to C^\infty(M)$ can so be defined as $f(V) = \sqrt{\det L_V}$ and then the volume functional $F : \mathcal{X}^1(M) \to \mathbb{R}$ is given by $F(V) = \int_M f(V) dv$ where dv is the density on M defined by g.

Since the metric on M induced by the immersion $V : M \to (T^1M, g^S)$ is given by $h(X,Y) = g(L_V(X), Y)$,the volume of V is the volume of the Riemannian manifold (M,h), that is to say the volume of the immersion V.

By developping the determinant of L_V, one can see that for every vector field $F(V) \geq \mathrm{vol}(M,g)$ with equality only if V is parallel or, equivalently, if and only if the map $V : (M,g) \to (T^1M, g^S)$ is an isometric embedding.

The first variation of F has been computed in [5].

Proposition 2 ([5]) *Let $V \in \mathcal{X}^1(M)$ be a unitary vector field and let $A \in T_V\mathcal{X}^1(M) = \mathcal{H}^V$ be a tangent vector. The tangent map to F at V acting on A is given by*

$$(T_V F)(A) = \int_M (T_V f)(A) dv$$

and $(T_V f)(A) = tr(K_V \circ \nabla A)$, where $K_V = f(V) L_V^{-1} \circ (\nabla V)^t$.

Now, it is easy to see that

$$\mathrm{tr}(K_V \circ \nabla A) = -\omega_V(A) - \delta \alpha_{VA},$$

where $\omega_V = (C_1^1 \nabla K_V)$, δ represents the divergence operator of g and $\alpha_{VA} = g(K_V(A))$. Then

Proposition 3 ([5]) *Let $V \in \mathcal{X}^1(M)$ be a unitary vector field and let $A \in T_V\mathcal{X}^1(M) = \mathcal{H}^V$ be a tangent vector. The tangent map to F at V acting on A can be written as*

$$(T_V F)(A) = -\int_M \omega_V(A)dv.$$

This result suggest the following definition valid for a general M, not assumed to be neither compact nor boundaryless.

Definition 1 ([5]) *A unitary vector field $V \in \mathcal{X}^1(M)$ is said to be critical if and only if the 1-form ω_V annihilates $\mathcal{H}^V$, or equivalently, if and only if the vector field X_V, given by $\omega_V = g(X_V)$, is in the distribution $\mathcal{V}$ determined by V.*

The condition above means that $(T_V f)(A) = -\delta\alpha_{VA}$ and so, if M is closed it is equivalent to V being a critical point of F and if M is compact with nonempty boundary then V is a critical vector field if and only if for each $A \in \mathcal{H}^V$

$$(T_V F)(A) = \int_{\partial M} f(V)g(A, (\nabla V \circ L_V^{-1})(\eta))d\overline{v}.$$

If M is noncompact but $V \in \mathcal{X}^1(M)$ has finite volume, the funcional F can be defined in $\mathcal{X}^1(M) \cap V + \mathcal{X}_c(M)$ and then the tangent map $(T_V F) : \mathcal{H}_c^V \to I\!R$ vanishes if and only if V is a critical vector field.

The energy $E(V)$ of a unit vector field V, in a closed manifold, can be defined as the energy of the map $V : (M, g) \to (T^1M, g^S)$. In [17], Wood has shown that the corresponding functional is given by

$$2E(V) = \int_M |\nabla V|^2 dv + n\mathrm{vol}(M, g).$$

The energy of V is, up to constants, also known as the bending $B(V)$ of V (see [2] and [16]) that is given by

$$(n-1)\mathrm{vol}(S^n)B(V) = \int_M |\nabla V|^2 dv.$$

A critical point of E is called a harmonic section of the unit tangent bundle and can be characterized as follows:

Proposition 4 ([17]) *The unit vector field V is a harmonic section of the unit tangent bundle if and only if $\nabla^*\nabla V = |\nabla V|^2 V$, where ∇^* represents the formal adjoint of ∇.*

It is worth noting that harmonic sections of the unit tangent bundle are substantially different from vector fields associated by the metric to harmonic one forms.

3 The case of constant curvature spaces

It will be useful for the sequel to write the condition for a unit vector field to be critical using adapted orthonormal references.

For $V \in \mathcal{X}^1(M)$, let $\{E_i\}_{i=1}^n$ be a local orthonormal reference such that $E_n = V$ and $E_i \in \mathcal{H}^V$ for $i \in \{1, \ldots, n-1\}$.

If we denote $G_{ij}^k = g(\nabla_{E_i} E_j, E_k)$ then $(\nabla V)_i^j = G_{in}^j$, using the fact that $G_{ij}^k = -G_{ik}^j$, we obtain that $(\nabla V)_i^n = 0$ and for the nonvanishing components of ∇V we will use the notation $(\nabla V)_i^j = -H_i^j$, $(\nabla V)_n^j = A^j$, with $i, j \in \{1, \ldots, n-1\}$; the symmetric part of H is the second fundamental form of the distribution $\mathcal{H}^V$ and $\nabla_V V = \sum_j A^j E_j$. It is easy to see that if R denotes the curvature tensor then

Lemma 1 ([5])*For any unit vector field in a Riemannian manifold*

$$E_j(G_{ik}^r) = E_i(G_{jk}^r) + R_{ijkr} + \sum_{l=1}^{n} \{G_{jk}^l G_{il}^r - G_{ik}^l G_{jl}^r - G_{ij}^l G_{lk}^r + G_{ji}^l G_{lk}^r\}$$

and in particular for $i, j, k \in \{1, \ldots, n-1\}$

$$E_j(H_i^k) = E_i(H_j^k) + R_{ijkn} + A^k(H_i^j - H_j^i) + \sum_{l=1}^{n-1} \{G_{il}^k H_j^l - G_{jl}^k H_i^l - G_{ij}^l H_l^k + G_{ji}^l H_l^k\}.$$

Now, the components of L_V are $L_i^j = \delta_i^j + \sum_k (\nabla V)_i^k (\nabla V)_j^k$ and those of K_V are $K_i^j = f \sum_k (\nabla V)_k^i (L_V^{-1})_k^j$, in particular $K_n^j = 0$. Since the 1-form ω_V can be computed as

$$\omega_V(X) = \sum_j g((\nabla_{E_j} K_V)(X), E_j),$$

it is easy to see that $\omega_i = \sum_j \{E_j(K_i^j) + \sum_k (G_{jk}^j K_i^k - G_{ji}^k K_k^j)\}$ and we have proved

Proposition 5 ([5]) *A unitary vector field is critical if and only if*

$$\sum_j \{E_j(K_i^j) + \sum_k (G_{jk}^j K_i^k - G_{ji}^k K_k^j)\} = 0,$$

for all $i \in \{1, \ldots, n-1\}$.

First example is that of a vector field defined only on the non complete manifold $M = S^n - \{-p_0\}$. In [15], Pedersen has constructed for any dimension unit vector fields, of exceptionally small volume, defined on the sphere minus one point. In particular, for S^{2m+1} with $m \geq 2$ their volume is lower than the volume of Hopf vector fields and she conjectured that this value is the not attained infimum of the volume of unit vector fields in this case.

For $v_0 \in T_{p_0} S^n$, a vector field V on M is defined in [15] by taking $V(p)$ as the element of $T_p S^n$ obtained by parallel translating v_0 along the great circle of S^n

passing through p_0 and p. To compute the tensor field involved in the condition for a vector field to be critical it will be useful to have the explicit expression of V; to do so, we can assume, without lost of generality, that $p_0 = N = (0, \dots, 1)$ and so $M = S^n - \{S\}$ with $S = (0, \dots, -1)$.

Lemma 2 ([5]) *Let $v_0 \in T_N S^n$ be a unit vector, the corresponding vector field V in M is given by*

$$V(p) = < v_o, \overline{p} > (h(p)\overline{p} - \frac{\partial}{\partial r^{n+1}}_{|p}) + v_0,$$

where $\overline{p} = \sum_{k=1}^{n} p_k \frac{\partial}{\partial r^k}_{|p}$ and $h(p) = -(1 + p_{n+1})^{-1}$.

Using this expression it is possible to compute explicitly all the terms involved in the condition of Proposition 5 in order to obtain

Theorem 1 ([5]) *Any vector field defined as above is critical.*

The proof is long and thecnical and show us that, although the condition for a vector field to be critical is easy to state, the 1-form ω_V is difficult to compute, in practice, if no additional assumptions on V are made. Since it is known that Hopf vector fields on spheres are critical (see [11]), in order to decide which conditions on V are the most convenient, it would be useful to have them as a model. We can remember, for instance, that they are unit Killing vector fields or, equivalently, that they define a totally geodesic foliation with bundle-like metric. For that kind of vector fields the map $f(V)$ is completely determined by the curvature:

Proposition 6 ([5]) *If V is a unit Killing vector field then $(\nabla V)^t \circ \nabla V = R(V, \ ,V)$.*

In particular, if M has constant curvature the only relevant components of K are equal, up to a constant factor, to those of H and then one can use Lemma 1 and Proposition 5 to obtain the following result

Proposition 7 ([5]) *Let M be a manifold of constant sectional curvature k. Every unit Killing vector field is critical. Moreover, $f(V) = (k+1)^{\frac{n-1}{2}}$ and $F(V) = (k+1)^{\frac{n-1}{2}} vol(M)$.*

It is worth noting that if a Riemannian manifold of constant curvature k admits a distribution $\mathcal{V}$ such that $\mathcal{V}$ and $\mathcal{H}$ have both vanishing second fundamental form, then $k \geq 0$ and $k = 0$ would implie both distributions being involutive; therefore the manifold would be, locally, a Riemannian product ([3], [7]). On the other hand, the existence of a unit Killing vector field in a manifold of positive curvature implies that the dimension must be odd (see [9]). Consequently, apart from the trivial case of parallel vector fields, the hypotheses of proposition above imply $k > 0$ and $n = 2m + 1$.

In view of the corresponding results in [8] and [11], Proposition 7 provides essentially new information only in the case where the manifold is not complete. In fact, if we assume, moreover, M to be complete then M should be a quotient of S^{2m+1} and, accordingly to [10], the lift of the vector field must be a Hopf vector field. Since it was known that the Hopf vector fields on S^{2m+1} have volume $2^m \mathrm{vol}(S^{2m+1})$ ([8]) and that they are critical ([11]), proposition above could be seen as a slight generalization and an extremely simple new proof of these results. The merit is, however, that as the method is purely local, no completeness assumption is needed.

4 The case of Killing vector fields

In a general manifold not every unit Killing vector field is critical but the following result provides an equivalent condition where only first order derivatives of V appear.

Theorem 2 ([5]) *Let V be a unit Killing vector field then V is critical if and only if the 1-form $\tilde{\rho}_V$ annihilates $\mathcal{H}^V$ where $\tilde{\rho}_V(X)$ is defined as the trace of the $(1,1)$-tensor field that maps each Z to*

$$R((L_V^{-1} \circ \nabla V)(X), (L_V^{-1} \circ \nabla V)(Z), V) + R((\nabla V \circ L_V^{-1} \circ \nabla V + Id)(X), L_V^{-1}(Z), V).$$

It is commonly said concerning this problem that best organized vector fields are rewarded by small volume; we can see now that even for a well organized vector field it is necessary, at least, to be well adapted to the ambient.

This curvature condition is trivially satified when $\nabla V = 0$ and when M has constant curvature. It is known that if V is the characteristic vector field of a Sasakian manifold it is a unit Killing vector field and that for all vector fields X, Y the curvature satisfies $R(X,Y,V) = g(Y,V)X - g(X,V)Y$; in fact, every unit Killing vector field on an odd-dimensional manifold with this curvature property is the characteristic vector field of a Sasakian manifold (see [1], pg.75). Consequently we have the following

Corollary 1 ([5]) *The characteristic vector field of a Sasakian manifold is critical.*

In the three cases mentioned above, the vector field V satisfies the curvature condition $R(X,Y,V) = 0$ for all $X, Y \in \mathcal{H}^V$; examples of unit Killing vector fields with this property, and therefore critical, can be found (see [9]) on a compact quotient of the generalized Heisenberg group where no Sasakian structures exists ([4]).

The curvature condition $\tilde{\rho}_V(\mathcal{H}^V) = \{0\}$ is, of course, more general and, unlike the constant curvature case, the existence of such a vector field, in general, does not implie any restriction on the dimension and, only sectional curvatures of planes containing V have to be nonnegative.

For three and four dimensional manifolds it is shown in [5] that the condition is equivalent to $\rho_V(\mathcal{H}^V) = \{0\}$ where ρ_V is the 1-form related to the Ricci tensor ρ by $\rho_V(X) = \rho(X, V)$. The proof makes use of the hypothesis on the dimension in a very specific form, so one does not necessarily expect to obtain the same result in other dimensions. A better understanding of the condition $\tilde{\rho}_V(\mathcal{H}^V) = \{0\}$, of the theorem, is by the moment an open question.

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GEODESIC SPHERES, TUBES AND SPACE FORMS

J. Gillard and L. Vanhecke

1 Introduction

It is well-known (see [13]) that for a Riemannian manifold (M^n, g) of dimension $n \geq 3$, the small geodesic spheres are *(locally) symmetric* if and only if the manifold M is a real space form. Since local symmetry is a rather strong geometric property, it is natural to look for the class of manifolds all of whose small geodesic spheres satisfy some weaker condition. In fact, this problem fits within the following more general project: *suppose we have a Riemannian manifold (M, g) considered as the ambient space of a family of geometrical objects. Study the mutual influence between the geometry of the ambient space and the geometry of the family of objects in that space.* (See also [10] and [11] in these proceedings.) Here, we will take as objects either small geodesic spheres or small geodesic tubes (i.e., tubes about geodesics). With respect to the geometry, we will mainly focus on curvature properties.

Now, taking *local homogeneity* instead of local symmetry, we could try to classify the Riemannian manifolds all of whose geodesic spheres are locally homogeneous. From [2, Proposition 3] we know that if the sufficiently small geodesic spheres are locally homogeneous, the ambient manifold is a harmonic globally Osserman space. Recall that *harmonic spaces* are characterized for example by the property that their small geodesic spheres have constant scalar curvature (when $\dim M \geq 3$), whereas a space is called *globally Osserman* if the Jacobi operators $R_v := R(., v)v$ (with v an arbitrary unit vector) have globally constant eigenvalues. The Osserman conjecture now states that every globally Osserman space is locally isometric to a two-point homogeneous space. So, if this conjecture really holds, it would already imply that the manifolds with locally homogeneous small geodesic spheres are the two-point homogeneous spaces (up to local isometry). Furthermore, there was also the conjecture of Lichnerowicz which said that every harmonic space is locally isometric to a two-point homogeneous space. This is

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true for $\dim M \leq 4$ and for compact manifolds with finite fundamental group [26]. On the other hand, counterexamples in the non-compact case are given by the Damek-Ricci spaces [14]. These are the only known counterexamples and they are globally Osserman spaces if and only if they are symmetric [4]. So, it is worthwhile to determine which of the globally Osserman spaces are harmonic. Because of the results about the Osserman conjecture, only the cases where $\dim M = 4k, k > 1$ remain open. For all the other cases we obtain, up to local isometry, the two-point homogeneous spaces. See [15] for more details. Conversely, it is clear that two-point homogeneous spaces have homogeneous geodesic spheres. Finally, we remark that all this remains true when local homogeneity is replaced by the weaker condition of curvature homogeneity [2].

The next obvious step is to consider geometrical properties that are stronger than local homogeneity but still weaker than local symmetry.

Ready candidates are in the first place the weak symmetry and the g.o. property, the latter being weaker than the first (see [1] and [3] for a proof of this). Recall that a space is *weakly symmetric* [6] if for any two points there exists an isometry of the space interchanging the two points. A *g.o. space* [23] is defined as a Riemannian manifold such that every geodesic is an orbit of a one-parameter group of isometries. G.o. spaces and weakly symmetric spaces are homogeneous.

Other candidates are spaces having the *C-property* [5] (that is, for every geodesic γ, the eigenvalues of the Jacobi operator R_γ are constant along γ) and *D'Atri spaces*, that is, the local geodesic symmetries are volume-preserving up to sign (see [22] for a survey). The C-spaces, and also the D'Atri spaces, form a class which is strictly broader than that of the g.o. spaces [4]. But, in contrast, it is still an intriguing open problem whether they are in general locally homogeneous or not. We refer to [22], [29] for a survey on this and other related problems.

Nevertheless, prescribing one of those four properties to all small geodesic spheres makes that the ambient space has to be locally isometric to a two-point homogeneous space. The converse also holds: all small geodesic spheres in a two-point homogeneous space are weakly-symmetric, g.o., C- and D'Atri spaces. See [1], [2] and [6] for details and further references. Since all of these spaces have a cyclic-parallel Ricci tensor, it is interesting to mention the following unifying fundamental result [13, Theorem 12.8]:

Proposition 1.1 *A Riemannian manifold (M^n, g) (with $n \geq 3$) is locally isometric to a two-point homogeneous space if and only if the Ricci tensor of any small geodesic sphere is cyclic-parallel (or equivalently, is a Killing tensor).*

Another property weaker than local symmetry is that of *semi-symmetry*. This is a pointwise algebraic condition saying that at each point of the manifold, the curvature tensor is the same as that of some symmetric space. This "model" space may vary with the point. See [9] for a comprehensive treatment. So, it is natural to ask

Question 1.2 *Which are the Riemannian manifolds all of whose small geodesic spheres are semi-symmetric ?*

Obviously, this class of manifolds must contain the real space forms, but it is not clear in advance whether it is larger or not. In [8], this question is solved by considering the notion of *Ricci-semi-symmetry*, that is, the condition $R_{XY} \cdot \rho = 0$ holds for arbitrary vectors X, Y, the curvature operator acting as a derivation. This last condition is a consequence of $R_{XY} \cdot R = 0$ which defines semi-symmetric spaces in an analytic way. Hence, semi-symmetry implies Ricci-semi-symmetry. The extrinsic analogon of this notion is provided by that of *semi-parallelity* [12], which is defined analytically by $R_{XY} \cdot \tilde{\sigma} = 0$, where $\tilde{\sigma}$ denotes the second fundamental form of the hypersurface.

Taking into account the geodesic tubes as well, we now have the following

Question 1.3 *Which are the Riemannian manifolds all of whose geodesic spheres or geodesic tubes are Ricci-semi-symmetric or semi-parallel ?*

The aim of this short survey is to summarize the answers to this question (Section 3). Furthermore, we consider related problems, in the sense that we add some structure to the Riemannian manifold and then study conditions of Ricci-semi-symmetric or semi-parallel type adapted to the geometry determined by that additional structure. More specifically, in Sections 4, 5, 6, we look at Kähler, quaternionic Kähler and Sasakian spaces. In all these cases we find characterizations for the corresponding space forms.

2 Preliminaries

Let (M^n, g) be an n-dimensional, smooth, connected Riemannian manifold. Denote by ∇ its Levi Civita connection and by R and ρ the corresponding Riemann curvature and Ricci tensor, respectively. We use the following sign convention for R:

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all tangent vector fields X, Y on M.

We denote by $G_m(r)$ the *geodesic sphere* with center m and radius r, that is,

$$G_m(r) = \{p \in M \mid d(m,p) = r\}.$$

It is always supposed that $r < i(m)$, the injectivity radius at the point $m \in M$. Because of this, $G_m(r)$ is a hypersurface of M and these geodesic spheres are frequently called *small* geodesic spheres. Note also that $G_m(r) = \exp_m(S^{n-1}(r))$, where $S^{n-1}(r) = \{x \in T_mM \mid \|x\| = r\}$ is the sphere of radius r in the tangent space to M at m. So, for any point $p \in G_m(r)$, there exists a unique unit vector $u \in T_mM$ such that $p = \exp_m(ru)$. This vector determines a unique unit speed geodesic γ defined by $\gamma(t) = \exp_m(tu)$. This geodesic connects the point p on the sphere with the center m and is called a *geodesic ray*. Because of the Gauss Lemma, tangent vectors to $G_m(r)$ at p are just tangent vectors to M at p, orthogonal to the geodesic ray γ.

A *geodesic tube* of radius r, denoted by $P_\sigma(r)$, is a tube about a geodesic $\sigma : [a,b] \to M$ (the axial curve of the tube). The geodesic tubes we consider are also frequently referred to as *small* geodesic tubes since the radius r is always supposed to be smaller than the distance from σ to its nearest focal point. For any point $p \in P_\sigma(r)$, there exists a unique point m on the axial curve σ and a unique unit tangent vector u to M orthogonal to σ at m such that $p = \exp_m(ru)$. As above, this vector u determines a *geodesic ray* γ and this ray cuts the axial curve σ orthogonally at m. Also here, tangent vectors to the tube $P_\sigma(r)$ are tangent vectors to the manifold M at p which are orthogonal to the geodesic ray γ. We refer to [13], [19], [28] for a detailed treatment of the geometry of geodesic spheres and geodesic tubes and also for further references.

3 General Riemannian manifolds

As already mentioned, a Riemannian manifold is said to be *Ricci-semi-symmetric* if the contracted semi-symmetry condition $R_{XY} \cdot \rho = 0$ is satisfied for arbitrary vectors X, Y. Here, R acts as a derivation, that is, $(R_{XY} \cdot T)(Z,W) = -T(R_{XY}Z, W) - T(Z, R_{XY}W)$ where T is a $(0,2)$-tensor. A hypersurface $\tilde{M}$ of a Riemannian manifold is *semi-parallel* if it satisfies $\tilde{R}_{XY} \cdot \tilde{\sigma} = 0$ for arbitrary tangent vectors to $\tilde{M}$. Here, $\tilde{\sigma}$ denotes the second fundamental form of the hypersurface.

Prescribing one of these conditions to the geodesic spheres or tubes of a Riemannian manifold influences the geometry of the ambient manifold in a considerable way, except for some low-dimensions where they become meaningless. This is obviously so for both notions when the ambient manifold is two-dimensional. If it has dimension three, the hypersurfaces are two-dimensional and hence, always semi-symmetric. So, in this case also Ricci-semi-symmetry becomes irrelevant.

A summary of the results, proved in [8], is given by

Theorem 3.1 *Let (M^n, g) be an n-dimensional Riemannian manifold. Then the following statements are equivalent:*

(a) *all geodesic spheres are locally symmetric ($n \geq 3$);*

(b) *all geodesic spheres are semi-symmetric ($n \geq 4$);*

(c) *all geodesic spheres are Ricci-semi-symmetric ($n \geq 4$);*

(d) *all geodesic spheres are semi-parallel ($n \geq 3$);*

(e) *(M, g) is a real space form.*

This also holds for geodesic tubes.

Moreover, we remark that for geodesic tubes and $n \geq 4$, it is sufficient to take X, Y in the defining conditions above as so-called *horizontal* tangent vectors to the tubes, that is, in addition to being orthogonal to the geodesic ray γ, they are also orthogonal to the parallel translate along γ of the velocity vector of the axial curve σ at the point where it cuts γ. In terms of the notations in Section 2, this means that the vector $X \in T_pP_\sigma(r)$ has to be orthogonal to the parallel translate along γ of $\dot{\sigma}(m)$.

Complete proofs of these results can be found in [8]. Here, we will only sketch the general idea of the method and techniques. Modulo some adaptations specific to the additional geometric structure, this idea of proof is also relevant for the next sections where we consider the space forms in the framework of Kähler, quaternionic and Sasakian geometry.

Proof. Suppose that the properties of the geodesic spheres are imposed and that we want to derive curvature information about the ambient manifold. A way to do this, is by letting the geodesic spheres shrink towards their center point. As such, the information given by the properties at points of the sphere is transferred to information about the manifold at the center point of the sphere. More specifically, we know that $\tilde{R} \cdot \tilde{\rho} = 0 = \tilde{R} \cdot \tilde{\sigma}$ at arbitrary points $p \in G_m(r)$. Considering m as the limit point of $G_m(r)$ for $r \to 0$, we want to obtain information on R at m. This limiting process is expressed analytically by using series expansions of $\tilde{R}$, $\tilde{\rho}$, $\tilde{\sigma}$ in terms of the radius r, with coefficients related to the curvature of the ambient manifold at m. In fact, the expansions are given in [13] for the components $\tilde{R}_{abcd}(p)$, $\tilde{\rho}_{ab}(p)$, $\tilde{\sigma}_{ab}(p)$ with respect to a suitable parallel orthonormal frame field $\{E_a(r) \mid a = 1, \ldots, n\}$ along the geodesic ray γ connecting m with p. Suitable in this context means that we take $E_1 = \gamma'$, because in this way $T_pG_m(r)$ is simply spanned by $\{E_2, \ldots, E_n\}(r)$. By means of this, we compute the series expansions of $(\tilde{R}_{ab} \cdot \tilde{\rho})_{cd}$ and $(\tilde{R}_{ab} \cdot \tilde{\sigma})_{cd}$ up to some order. Then, we know that the terms of this expansion vanish and from this we obtain curvature information about the ambient manifold at the point m. Making specific choices for a, b, c, d and manipulating the obtained expressions (e.g., polarizing, contracting) simplifies them and makes it possible to use known characterizations of space forms to prove the required results.

For the geodesic tube case, the idea of proof is similar. The main difference is that here the direction of the axial curve plays an important role. Indeed, the parallel orthonormal frame field along the geodesic ray γ between m and p (see Section 2 for the notations) is taken with $E_2 = \gamma'$ and $E_1 =$ the parallel translate of $\dot{\sigma}(m)$ along γ. Clearly, this gives rise to the notion of horizontal tangent vectors to the tubes. These are spanned by $\{E_3, \ldots, E_n\}(r)$.

Conversely, suppose that we know that the ambient manifold is some kind of space form. Then we have complete expressions for the curvature tensor R of the manifold and we can compute the Jacobi vector fields along geodesics. From this we calculate the second fundamental form $\tilde{\sigma}$ of the spheres and tubes (see [28]).

The Gauss equation then yields an explicit expression for $\tilde{R}$ and contracting this, for $\tilde{\rho}$. So, combining these, we obtain explicit expressions for $\tilde{R}\cdot\tilde{\rho}$ and $\tilde{R}\cdot\tilde{\sigma}$ and then we just verify the Ricci-semi-symmetry and semi-parallelity conditions. ∎

4 Kähler manifolds

Let (M^n, g, J) be a *Kähler manifold*, that is, J is a $(1,1)$-tensor field on M such that

(1) $$J^2 = -I,\ g(JX, JY) = g(X,Y),\ \nabla J = 0$$

for all tangent vector fields X, Y on M. A Kähler manifold is said to be a *complex space form* if it has constant *holomorphic sectional curvature* $K(u, Ju)$. (See [30, Chapter III] for more details.)

As indicated in the Introduction, the project is to investigate also the meaning of the Ricci-semi-symmetry and semi-parallelity conditions for this type of manifolds. First of all, we state: which are the Kähler manifolds all of whose geodesic spheres or tubes are Ricci-semi-symmetric or semi-parallel ? But, in view of the results in the previous section, these must have constant sectional curvature and then it is known [30, Proposition 4.3] that they are flat. This means that Ricci-semi-symmetry and semi-parallelity as such are too strong conditions for Kähler manifolds. So, we look for slightly modified versions. A natural way to do this, is to consider the expressions

(2) $$\tilde{R}_{XY}\cdot\tilde{\rho},$$

(3) $$\dot{\tilde{R}}_{XY}\cdot\tilde{\sigma}$$

for specific directions of the vectors X, Y, possibly at special points of the sphere or tube.

To get an idea of the modifications we shall make, we compute (2) and (3) for complex space forms. This is done as explained in the second part of the proof in Section 3. First, we obtain that the second fundamental form of a geodesic sphere in a complex space form looks like

$$\tilde{\sigma} = \lambda\, g + \mu\, \eta \otimes \eta,$$

where λ, μ are radial functions and η is a one-form on the sphere, metrically related to the Kähler structure by $\eta(X) = g(X, J\gamma'_{|p})$ for tangent vectors X at p and γ the geodesic ray leading to p. Moreover, for geodesic *tubes*, this formula holds at so-called *special points*. These are points for which the geodesic ray γ is "J-related" to the axial curve σ. More specifically and with the notations of Section 2, this means that $\gamma'_{|m} = J\dot{\sigma}_{|m}$. Furthermore and both for geodesic spheres and tubes, (2) and (3) vanish if the vectors X, Y, tangent to the sphere or tube (at special points), belong to the kernel of η, that is, they are orthogonal to $\gamma'_{|p}$ and its J-related vector $J\gamma'_{|p}$. We call them *horizontal* tangent vectors. Remark that for tubes at special points, this notion corresponds to the one introduced in Section 3 for general Riemannian manifolds.

Summarizing, we have that for geodesic spheres in a complex space form the expressions (2) and (3) vanish for horizontal tangent vectors X, Y to the spheres. Geodesic spheres having these properties are called respectively *horizontally Ricci-semi-symmetric* and *horizontally semi-parallel*. For geodesic tubes, this is true *at special points*.

The converse is answered by the following (see [16])

Theorem 4.1 *Let (M^n, g, J), $n \geq 4$, be a Kähler manifold. Then the following statements are all equivalent:*

(a) *all geodesic spheres are horizontally Ricci-semi-symmetric;*

(b) *all geodesic spheres are horizontally semi-parallel;*

(c) *(M, g) is a complex space form.*

This also holds for geodesic tubes at special points.

For detailed proofs, we refer to [16]. Concerning the frame field $\{E_a(r)\}_{a=1}^n$ mentioned in the sketch of proof in Section 3, this is chosen such that the notions of horizontality and special points can be expressed in a natural way with respect to it.

5 Quaternionic Kähler manifolds

Now, suppose that (M, g) is a quaternionic Kähler manifold [21], that is, there exists a three-dimensional bundle V of tensors of type $(1,1)$ over M such that locally the bundle V has a basis of almost Hermitian structures $\{J_0, J_1, J_2\}$ satisfying

$$J_i J_j = -J_j J_i = J_k, \tag{4}$$

$$\begin{aligned} \nabla_X J_0 &= r(X)J_1 - q(X)J_2, \\ \nabla_X J_1 &= -r(X)J_0 + p(X)J_2, \\ \nabla_X J_0 &= q(X)J_0 - p(X)J_1, \end{aligned} \tag{5}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and p, q, r are local one-forms. Such a basis is called *adapted*. It follows that $\dim M = 4m$. As is well-known, for $n \geq 8$, M is an Einstein manifold [21]. A quaternionic Kähler manifold is a *quaternionic space form* if it has constant *Q-sectional curvature*, that is, the sectional curvature $K(Y,Z)$ with $Y,Z \in Q(X) = \text{span}\{X, J_0X, J_1X, J_2X\}$ is constant for all tangent vectors X,Y,Z. For more details, we refer to [30, Chapter III].

Because of similar reasons as in the previous Section, the notions of Ricci-semi-symmetry and semi-parallelity are too strong and need to be modified. In fact, we introduce analogous notions of horizontal vectors and special points.

First, for quaternionic space forms, the second fundamental form for geodesic spheres may be written as

$$\tilde{\sigma} = \lambda\, g + \sum_{i=0}^{2} \nu_i\, \eta_i \otimes \eta_i, \tag{6}$$

where λ, ν_i $(i = 0,1,2)$ are radial functions and the η_i are one-forms on the spheres, defined by $\eta_i(X) = g(X, J_i\gamma'_{|p})$ for tangent vectors X at p and γ the geodesic ray leading towards p. For geodesic *tubes*, all of this also holds at *special points*. The definition here is quite similar to that in the Kähler case; it are are points for which the geodesic ray γ is J-related to the axial curve σ for some $J \in V(m)$. A tangent vector to a sphere or tube is called *horizontal* if it belongs to the kernel of η_1, η_2 and η_3, that is, the tangent vector has to be orthogonal to $J\gamma'_{|p}$ for all $J \in V(p)$.

With these definitions, we have (see [17])

Theorem 5.1 *Let (M^n, g, V), $n \geq 8$, be a quaternionic Kähler manifold. Then the following statements are all equivalent:*

(a) *all geodesic spheres are horizontally Ricci-semi-symmetric;*

(b) *all geodesic spheres are horizontally semi-parallel;*

(c) *(M,g) is a quaternionic space form.*

The coefficients ν_i in the formula of the second fundamental form for geodesic spheres are in fact equal to one another. Since this is not the case for geodesic tubes, the theorem becomes more technical. Here, we have

Theorem 5.2 *Let (M^n, g, V), $n \geq 8$, be a quaternionic Kähler manifold. Then M is a quaternionic space form if and only if its geodesic tubes satisfy one of the conditions*

$$(\tilde{R}_{XY} \cdot \tilde{\sigma})_{ZW} = 0 \quad \textit{or} \quad (\tilde{R}_{XY} \cdot \tilde{\rho})_{ZW} = 0$$

for all horizontal tangent vectors X,Y,Z and every tangent vector W to the tubes at any special point.

6 Sasakian manifolds

Let $(M^n, g, \varphi, \xi, \eta)$ be a Sasakian manifold. A characteristic property for this type of manifolds is that they admit a unit Killing vector field ξ such that the Riemann curvature tensor satisfies the condition

$$R_{XY}\xi = \eta(X)Y - \eta(Y)X$$

for all vectors tangent to M. Here, η denotes the metric dual one-form of ξ defined by $\eta(X) = g(X, \xi)$. The $(1,1)$-structure tensor φ is defined by $\varphi = -\nabla\xi$. In the Sasakian context, a vector is called *horizontal* if it belongs to the kernel of η. So, they are orthogonal to the characteristic vector field ξ. Since this vector field is Killing, its integral curves are geodesics, called *ξ-geodesics*. They also determine the fibers of a local fibration of the Sasakian manifold over Kählerian base spaces [24]. Further, a geodesic which is orthogonal to ξ at one of its points, stays orthogonal to ξ at all of its points. Such geodesics are called *φ-geodesics* or *horizontal geodesics*. A Sasakian manifold is said to be a *Sasakian space form* if it has constant *φ-sectional curvature* $K(u, \varphi u)$, u being a horizontal unit vector. For more details and background, we refer to [7], [30, Chapter V].

It is known [25], [27] that a locally symmetric Sasakian manifold has constant sectional curvature 1. So, if a Sasakian manifold has constant sectional curvature, then it is automatically equal to 1. Therefore, also here the notions of Ricci-semi-symmetry and semi-parallelity in their original form are too strong.

First, we introduce the special points at which the conditions will be considered. (See [18].) For geodesic spheres, these are the intersection points with φ-geodesic rays, called *φ-geodesic points*. Depending on the direction of the axial curve σ of the geodesic tube, we have the following: if σ is a φ-geodesic, then the tube is called a *φ-geodesic tube* and for these tubes we consider so-called *φ-special points*. This notion is defined analogously to that of special points in the Kähler case, in the sense that here the geodesic ray leading to the φ-special point, has to be "φ-related" to the axial curve. If the axial curve is a ξ-geodesic, then the tube is called a *ξ-geodesic tube* and here arbitrary points are considered. But, since in this case any geodesic ray is a φ-geodesic, they are in fact φ-geodesic points.

Now, if the ambient manifold is a Sasakian space form of dimension $n \geq 5$, the second fundamental form of geodesic spheres or tubes at the respective points as introduced above, is explicitly known and can be written in the form

$$\tilde{\sigma} = \alpha\, g + \beta\, \eta \otimes \eta + \delta\, \nu \otimes \nu + \epsilon_1\, \eta \otimes \nu + \epsilon_2\, \nu \otimes \eta, \tag{7}$$

where ν is the $(0,1)$-tensor on the sphere or tube defined by $\nu(X) = g(X, \varphi\gamma'(r))$ and $\alpha, \beta, \delta, \epsilon_1, \epsilon_2$ depend only on the radius r.

By means of this, it is straightforward to compute (2) and (3) explicitly and then it follows easily that these expressions vanish for horizontal vectors X, Y that belong also to the kernel of ν. These tangent vectors to the sphere or tube are called *strictly horizontal.* Geometrically, they are tangent vectors of the manifold orthogonal to $\{\gamma', \varphi\gamma', \xi\}$ at the point considered on the sphere or tube.

The results in the Sasakian case are now summarized by the following

Theorem 6.1 *Let $(M^n, g, \varphi, \xi, \eta)$, $n \geq 5$, be a Sasakian manifold. Then the following statements are equivalent:*

(a) *all geodesic spheres are strictly horizontally Ricci-semi-symmetric at φ-geodesic points;*

(b) *all geodesic spheres are strictly horizontally semi-parallel at φ-geodesic points;*

(c) *all φ-geodesic tubes are strictly horizontally Ricci-semi-symmetric at φ-special points;*

(d) *all φ-geodesic tubes are strictly horizontally semi-parallel at φ-special points;*

(e) *all ξ-geodesic tubes are strictly horizontally Ricci-semi-symmetric;*

(f) *all ξ-geodesic tubes are strictly horizontally semi-parallel;*

(g) *(M, g) is a Sasakian space form.*

Complete proofs are given in [18].

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LOCAL HOMOGENEITY IN FLOW GEOMETRY

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1 Introduction

Riemannian manifolds equipped with a transitive pseudogroup of local isometries are called *locally homogeneous spaces.* Clearly, such spaces have the property that the volumes of sufficiently small geodesic spheres or balls only depend on the radius. Riemannian manifolds having this property are called *ball-homogeneous spaces* [25]. It is still an intriguing open problem whether, in general, ball-homogeneous Riemannian manifolds are locally homogeneous or not. We refer to [36] for a survey and to [9], [10], [11], [16] for a list of partial results, obtained recently, about this natural but difficult problem.

A Riemannian manifold is said to be *harmonic* if all sufficiently small geodesic spheres are hypersurfaces of constant mean curvature [5], [12], [29], [34] and a *D'Atri space* if its local geodesic symmetries are volume-preserving (up to sign) (see [24] for a survey). All harmonic spaces are D'Atri manifolds and all D'Atri spaces are ball-homogeneous [21]. The known examples of these classes of spaces are all locally homogeneous. Nevertheless, the converse problems have been formulated but also only partial results are known. We refer to [23] where it is shown that three-dimensional D'Atri spaces are indeed locally homogeneous and to [13], [16] where partial answers are given for the four- and five-dimensional cases.

Another class of Riemannian manifolds where local homogeneity is also an open problem it is that of the *C-spaces.* The C-spaces are introduced in [2] (see also [1]) as Riemannian manifolds such that the eigenvalues of the Jacobi operator are constant along each geodesic. Although their geometry shares some properties with that of the D'Atri spaces (in particular, both classes coincide in dimension two and three) they do not coincide in general. We note that the so-called Damek-Ricci spaces [14] provide examples of D'Atri spaces which are not C-spaces (see [1]). For more details and references about C-spaces we refer to [1], [2], [3].

The classification of C-, D'Atri and ball-homogeneous spaces in the framework of Sasakian geometry has been treated in [16] where in all the cases considered one obtained again local homogeneity. Moreover, the authors proved that Sasakian harmonic manifolds are spaces of constant curvature 1.

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Sasakian manifolds are endowed with a unit Killing vector field. In a series of papers, M. C. González-Dávila and the authors have studied the geometry of Riemannian manifolds equipped with such a vector field, generalizing in this way many aspects of Sasakian geometry to what they called *flow geometry*. See [17], [18] for the basic material, for some local and global classification results and for a lot of examples.

The aim of this note is to give a brief survey of some aspects of the work done by the authors relating to local homogeneity of the ball-homogeneous, harmonic, C- and D'Atri spaces in the framework of flow geometry. We refer to [19], [20] for the complete details. First, in Section 2, we collect some needed definitions and basic material about flow geometry. In Section 3, we sketch a proof of the fact that any harmonic (M,g) equipped with a unit Killing vector field whose flow is normal (see Section 2 for the definition) is a space of non-negative constant sectional curvature. In the subsequent sections we consider the ball-homogeneity, C- and D'Atri conditions in this framework and in particular for dimensions not greater than five. It turns out that also here we find only locally homogeneous examples.

2 Preliminaries

Let (M,g) be an n-dimensional, connected, smooth Riemannian manifold with $n \geq 2$. Furthermore, let ∇ denote the Levi Civita connection of (M,g) and R the corresponding Riemannian curvature tensor with the sign convention

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M. ρ and τ denote the Ricci tensor and the scalar curvature, respectively.

A tangentially oriented foliation of dimension one on (M,g) is called a *flow*. The leaves of this foliation are the integral curves of a non-singular vector field on M and hence, after normalization, a flow is also given by a unit vector field. In particular, a non-singular Killing vector field defines a *Riemannian flow* and such a flow is said to be an *isometric* flow. See [32] for more information.

In this paper, we consider and denote by $\mathfrak{F}_\xi$ an isometric flow generated by a *unit* Killing vector field ξ. The flow lines of $\mathfrak{F}_\xi$ are geodesics and moreover, a geodesic which is orthogonal to ξ at one of its points, is orthogonal to it at all of its points. Such geodesics are called *transversal* or *horizontal* geodesics.

$\mathfrak{F}_\xi$ determines locally a Riemannian submersion. For each $m \in (M,g)$, let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the mapping $\pi : \mathcal{U} \to \tilde{\mathcal{U}} = \mathcal{U}/\xi$ is a submersion. Furthermore, let $\tilde{g}$ denote the induced metric on $\tilde{\mathcal{U}}$ given by

$$\left(\tilde{g}(\tilde{X},\tilde{Y})\right)^* = g(\tilde{X}^*,\tilde{Y}^*)$$

for $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\tilde{\mathcal{U}})$ and where $\tilde{X}^*, \tilde{Y}^*$ denote the horizontal lifts of $\tilde{X}, \tilde{Y}$ with respect to the $(n-1)$-dimensional horizontal distribution on $\mathcal{U}$ determined by $\eta = 0$, η being the dual one-form of ξ with respect to g. Then $\pi : (\mathcal{U}, g_{|\mathcal{U}}) \to (\tilde{\mathcal{U}}, \tilde{g})$ is a Riemannian submersion. Now, we shall use O'Neill's integrability tensor A [28]. (See also [5].) Then we have

$$\begin{aligned} A_U\xi &= \nabla_U\xi \quad &, \quad A_\xi U = 0, \\ A_XY &= (\nabla_XY)^{\mathcal{V}} = -A_YX \quad &, \quad g(A_XY,\xi) = -g(A_X\xi, Y) \end{aligned}$$

for $U \in \mathfrak{X}(M)$, and for horizontal vector fields X, Y. Here, $\mathcal{V}$ denotes the vertical component.

Next, put $HU = -A_U\xi$ and define the $(0,2)$-tensor field h by $h(U,V) = g(HU,V)$, for all $U, V \in \mathfrak{X}(M)$. Then h is skew-symmetric because ξ is a Killing vector field. Moreover, we have

$$A_XY = h(X,Y)\xi = \frac{1}{2}\eta([X,Y])\xi.$$

So, we obtain $h = -d\eta$. Note that $A = 0$, or equivalently $h = 0$, if and only if the horizontal distribution is integrable. In this case, since the flow lines are geodesics, (M, g) is locally a product of an $(n-1)$-dimensional manifold and a curve.

The Levi Civita connection $\tilde{\nabla}$ on $(\tilde{\mathcal{U}}, \tilde{g})$ is determined by

$$\nabla_{\tilde{X}^*}\tilde{Y}^* = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})^* + h(\tilde{X}^*, \tilde{Y}^*)\xi \tag{2.1}$$

for $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\tilde{\mathcal{U}})$ and the curvature tensor R of (M, g) satisfies

$$R(X,\xi,Y,\xi) = g(HX,HY) = -g(H^2X,Y) \tag{2.2}$$

for all horizontal vector fields X, Y. Here we use the notation $R(X,Y,Z,W) = g(R_{XY}Z, W)$. It follows that the *$\xi$-sectional curvature* $K(X,\xi)$ of the two-plane spanned by X and ξ is non-negative for all horizontal X and since $H\xi = 0$, $K(X,\xi) = 0$ holds for all horizontal X if and only if $h = 0$. Moreover, $K(X,\xi)$ is strictly positive for all X if and only if H is of maximal rank $n-1$ in which case n is necessarily odd. Then η is a contact form and the flow $\mathfrak{F}_\xi$ is called a *contact flow.*

In what follows, we shall consider a special kind of flow $\mathfrak{F}_\xi$ which appears naturally in this framework. $\mathfrak{F}_\xi$ is said to be a *normal flow* [17] if, for all horizontal X, Y, the curvature transformations R_{XY} leave the horizontal subspaces invariant, or equivalently, $R(X,Y,X,\xi) = 0$. Here, we note that a Sasakian manifold is a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that $K(X,\xi) = 1$ for all horizontal X (see [6] for more details). Moreover, if the ξ-sectional curvature is a non-vanishing constant $k = c^2$, then $H^2X = -kX$ for horizontal vectors and $(M, c^2g, c^{-1}H, c^{-1}\xi, c\eta)$ is a Sasakian manifold.

$\mathfrak{F}_\xi$ is normal if and only if

$$(\nabla_U H)V = g(HU, HV)\xi + \eta(V)H^2U \tag{2.3}$$

for all $U, V \in \mathfrak{X}(M)$. In this case the curvature tensor satisfies the following identities [17]:

$$\begin{cases} R_{UV}\xi &= \eta(V)H^2U - \eta(U)H^2V, \\ R_{U\xi}V &= g(HU, HV)\xi + \eta(V)H^2U \end{cases} \tag{2.4}$$

and

$$\begin{aligned} R_{HUV}W + R_{UHV}W &= g(HV, W)H^2U - g(HU, W)H^2V \\ &\quad -g(H^2U, W)HV + g(H^2V, W)HU \\ &\quad +\eta(V)R_{HU\xi}W - \eta(U)R_{HV\xi}W \end{aligned} \tag{2.5}$$

for all $U, V, W \in \mathfrak{X}(M)$. Hence, using also (2.1), it follows that the curvature tensors of ∇ and $\tilde{\nabla}$ are related by

$$\begin{aligned} (\tilde{R}_{\tilde{X}\tilde{Y}}\tilde{Z})^* &= R_{\tilde{X}^*\tilde{Y}^*}\tilde{Z}^* - g(H\tilde{Y}^*, \tilde{Z}^*)H\tilde{X}^* \\ &\quad +g(H\tilde{X}^*, \tilde{Z}^*)H\tilde{Y}^* + 2g(H\tilde{X}^*, \tilde{Y}^*)H\tilde{Z}^* \end{aligned} \tag{2.6}$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\tilde{\mathcal{U}})$. This yields

$$(\tilde{\rho}(\tilde{X}, \tilde{Y}))^* = \rho(\tilde{X}^*, \tilde{Y}^*) + 2g(H\tilde{X}^*, H\tilde{Y}^*), \tag{2.7}$$

$$\tilde{\tau}^* = \tau + \rho(\xi, \xi). \tag{2.8}$$

Moreover, $\rho(X, \xi) = 0$ for each horizontal X. Using (2.7), we get

$$\left((\tilde{\nabla}_{\tilde{X}}\tilde{\rho})(\tilde{Y}, \tilde{Z})\right)^* = (\nabla_{\tilde{X}^*}\rho)(\tilde{Y}^*, \tilde{Z}^*). \tag{2.9}$$

ρ is said to be *η-parallel* if it satisfies the condition $(\nabla_X\rho)(Y, Z) = 0$ for all horizontal X, Y, Z. It follows from (2.9) that ρ is η-parallel if and only if $\tilde{\rho}$ is parallel on each base space $(\tilde{\mathcal{U}}, \tilde{g})$.

Now, on $\tilde{\mathcal{U}}$ we consider the (1,1)-tensor field $\tilde{H}$ defined by $\tilde{H}\tilde{X} = \pi_*H\tilde{X}^*$. Then, $\mathfrak{F}_\xi$ is normal if and only if $\tilde{\nabla}\tilde{H} = 0$. Furthermore, in that case, from (2.5) and (2.6), we have on $\tilde{\mathcal{U}}$:

$$\tilde{R}_{\tilde{H}\tilde{X}\tilde{Y}} = \tilde{R}_{\tilde{H}\tilde{Y}\tilde{X}}. \tag{2.10}$$

Hence, it follows that

$$\rho_{Hxy} + \rho_{Hyx} = 0 \tag{2.11}$$

for all tangent vectors x, y of M, or equivalently, by using (2.7), that $\tilde{\rho}_{\tilde{H}\tilde{x}\tilde{y}}+\tilde{\rho}_{\tilde{H}\tilde{y}\tilde{x}} = 0$ for all tangent vectors $\tilde{x}, \tilde{y}$ on each base space $\tilde{\mathcal{U}}$.

Furthermore, we have

Proposition 2.1 [20] *If (M, g) is a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$, then tr H^{2k} is a constant for each $k \geq 1$.*

From this it follows (see [20]):

Corollary 2.2 *Let (M, g) be a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$. If the ξ-sectional curvature is pointwise constant, then it is globally constant.*

Corollary 2.3 *On a Riemannian manifold (M, g) equipped with a normal flow $\mathfrak{F}_\xi$ the rank of H is constant.*

Now, we collect some facts about locally Killing-transversally symmetric spaces. Let $m \in (M, g)$ and let $\sigma = \sigma_m : [-\delta, \delta] \to M$ be a geodesic flow line through $m = \sigma(0)$ where δ is sufficiently small. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of m is said to be a *(local) reflection with respect to* σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ and σ, we have $(s_m \circ \gamma)(s) = \gamma(-s)$ for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length of γ. A Riemannian manifold (M, g) equipped with a flow $\mathfrak{F}_\xi$ such that each local reflection s_m is an isometry, is called a *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*). In that case, $\mathfrak{F}_\xi$ is necessarily normal. These spaces may be characterized as follows.

Proposition 2.4 [17] *$(M, g, \mathfrak{F}_\xi)$ is a locally* KTS-*space if and only if the flow $\mathfrak{F}_\xi$ is normal and moreover,*

$$(\nabla_X R)(X, Y, X, Y) = 0$$

for all horizontal X, Y.

Proposition 2.5 [17] *Let $\mathfrak{F}_\xi$ be a normal flow on (M, g). Then $(M, g, \mathfrak{F}_\xi)$ is a locally* KTS-*space if and only if each base space $\tilde{\mathcal{U}}$ of a local Riemannian submersion $\pi: \mathcal{U} \to \tilde{\mathcal{U}} = \mathcal{U}/\xi$ is a locally symmetric space.*

Hence, $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is a normal flow which is transversally modelled on a locally symmetric space or equivalently, according to the terminology used in [33], $\mathfrak{F}_\xi$ is a normal transversally symmetric foliation.

An important class of locally KTS-spaces, which have motived the study of these spaces, is that of the *locally φ-symmetric spaces* [31]. A locally φ-symmetric space is a Sasakian manifold which is a locally KTS-space with respect to the characteristic vector field of the contact metric structure (see [7] and [8] for more details and references).

Furthermore, when the isometric flow $\mathfrak{F}_\xi$ of a locally KTS-space $(M, g, \mathfrak{F}_\xi)$ is complete and the local reflections with respect to the flow lines can be extended to global isometries, $(M, g, \mathfrak{F}_\xi)$ is said to be a *(globally) Killing-transversally symmetric space* (briefly, a KTS-*space*). A complete, simply connected locally KTS-space is a KTS-space. Moreover, any simply connected KTS-space is a naturally reductive homogeneous space. (See [18] for more information about these spaces.)

3 2-stein spaces and harmonicity

An Einstein manifold, that is, $\rho = \lambda g$, $\lambda = \frac{\tau}{n}$, is said to be a *2-stein space* if

$$\sum_{a,b=1}^{n} R^2_{xaxb} = \mu g(x,x)^2 \tag{3.1}$$

for any tangent vector x at m and all $m \in M$. Here, $R_{xaxb} = g(R_{xa}x, b)$ and $\{e_a, a = 1, \ldots, n\}$ is an arbitrary orthonormal basis of the tangent space T_mM. In this case we have

$$\mu = \frac{1}{n(n+2)}\left(\frac{3}{2}\|R\|^2 + \|\rho\|^2\right) \tag{3.2}$$

(see for example [4], [12]).

As mentioned already, an (M, g) is said to be a *harmonic manifold* if all geodesic spheres of sufficiently small radius are constant mean curvature hypersurfaces. Any harmonic manifold is a 2-stein space [4], [12]. We start with the following theorem.

Theorem 3.1 [19] *Let (M, g), $\dim M \geq 3$, be a 2-stein space equipped with a non-vanishing vector field ξ such that the sectional curvature of the two-planes containing ξ is pointwise constant. Then (M, g) is a space of constant curvature.*

Since a 2-dimensional harmonic space has constant curvature (see, for example, [4], [12], [35]), we get at once

Corollary 3.2 *A harmonic space equipped with a non-vanishing vector field ξ such that the ξ-sectional curvature is pointwise constant, is a space of constant curvature.*

Now, we state the following result. We always suppose $\dim M \geq 3$.

Theorem 3.3 [19] *Let (M,g) be a Riemannian manifold equipped with a normal flow. If (M,g) is a 2-stein space, then it is a space of (non-negative) constant sectional curvature.*

From this result we then get at once

Corollary 3.4 *A harmonic manifold which is equipped with a normal flow is a space of (non-negative) constant sectional curvature.*

Before giving a sketch of the proof of Theorem 3.3 we first mention

Lemma 3.5 [19] *Let (M,g) be an Einstein manifold equipped with a normal flow $\mathfrak{F}_\xi$ and let $\pi : \mathcal{U} \to \tilde{\mathcal{U}} = \mathcal{U}/\xi$ be a local Riemannian submersion determined by $\mathfrak{F}_\xi$. If $\tilde{\mathcal{U}}$ is locally irreducible, then $\tilde{\mathcal{U}}$ is an Einstein manifold and the ξ-sectional curvature is constant on $\mathcal{U}$.*

Proof of Theorem 3.3. We shall prove that the 2-stein space (M,g), equipped with a normal flow $\mathfrak{F}_\xi$, has pointwise constant ξ-sectional curvature. Then the result follows at once from Theorem 3.1.

So, assume the contrary, that is, suppose that the ξ-sectional curvature is not pointwise constant. Then there exists a point m in M such that the ξ-sectional curvature at m is not constant. In this case it follows from Lemma 3.5 that there exists a small open neighborhood $\mathcal{U}$ of m such that $\tilde{\mathcal{U}} = \mathcal{U}/\xi$ is reducible and we may write $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_1 \times \ldots \times \tilde{\mathcal{U}}_r$ where $\tilde{\mathcal{U}}_i$ is an Einstein space for each $i = 1,\ldots,r$. Put $\dim \tilde{\mathcal{U}}_i = n_i$ and denote by $\tilde{\tau}_i$, $i = 1,\ldots,r$, the scalar curvature of $\tilde{\mathcal{U}}_i$. Then $\sum_{i=1}^{r} n_i = n-1$ and $\sum_{i=1}^{r} \tilde{\tau}_i = \tilde{\tau}$. Moreover, we may assume that $\frac{\tilde{\tau}_i}{n_i} \neq \frac{\tilde{\tau}_j}{n_j}$ for $i \neq j$. Applying (2.7), we get

$$2c_i^2 = \frac{\tilde{\tau}_i}{n_i} - \frac{\tau}{n}, \quad i = 1,\ldots,r, \tag{3.3}$$

where c_i^2 is the ξ-sectional curvature $K(\tilde{X}_i^*, \xi)$ for all $\tilde{X}_i^* \in \mathcal{U}_i$. Note that, because of our assumption, we have $c_i^2 \neq c_j^2$, $i \neq j$.

Next, let u be an arbitrary unit *horizontal* vector at $m \in \mathcal{U}$ and denote its projection on $\tilde{\mathcal{U}}$ also by u. Let $\{e_i, i = 1,\ldots,n\}$ be an orthonormal basis of T_mM such that $e_n = \xi$. From (2.6) we get

$$\mu = \sum_{a,b=1}^{n-1} \tilde{R}_{uaub}^2 - 6\tilde{R}_{u\tilde{H}uu\tilde{H}u} + 10\|\tilde{H}u\|^4. \tag{3.4}$$

Since this expression is independent of u, we can take u tangent to $\tilde{\mathcal{U}}_i$ and also tangent to $\tilde{\mathcal{U}}_j$, for $i,j \in \{1,\ldots,r\}$, $i \neq j$, to obtain

$$\mu = 10c_i^2c_j^2. \tag{3.5}$$

From this it follows, since the ξ-sectional curvature is not constant at m, that $\tilde{\mathcal{U}}$ has exactly two factors. So, put $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_1 \times \tilde{\mathcal{U}}_2$ and let $\tilde{R}_1$, respectively $\tilde{R}_2$, denote the Riemann curvature tensor of $\tilde{\mathcal{U}}_1$, respectively $\tilde{\mathcal{U}}_2$. At $\tilde{m} = \pi(m) \in \tilde{\mathcal{U}}$ we choose an orthonormal basis $\{e_i, i = 1, \ldots, n-1\}$ such that $e_1, \ldots, e_{n_1}$ span $T_{\tilde{m}}\tilde{\mathcal{U}}_1$ and $e_{n_1+1}, \ldots, e_{n-1}$ span $T_{\tilde{m}}\tilde{\mathcal{U}}_2$. Now, let u_1 be a unit vector of $T_{\tilde{m}}\tilde{\mathcal{U}}_1$. Then we have from [12], [22], using the first Bianchi identity and (2.10), that

$$\begin{aligned}
\int_{S^{n_1-1}(1)} \sum_{a,b=1}^{n_1} \tilde{R}^2_{1u_1au_1b}du_1 &= \frac{C_{n_1-1}}{n_1(n_1+2)}\left(\frac{3}{2}\|\tilde{R}_1\|^2 + \frac{\tilde{\tau}_1^2}{n_1}\right),\\
\int_{S^{n_1-1}(1)} \tilde{R}_{1u_1\tilde{H}u_1u_1\tilde{H}u_1}du_1 &= \frac{4C_{n_1-1}}{n_1(n_1+2)}c_1^2\tilde{\tau}_1,\\
\int_{S^{n_1-1}(1)} \|\tilde{H}u_1\|^4du_1 &= \frac{C_{n_1-1}}{n_1(n_1+2)}\left\{(\text{tr } \tilde{H}_1^2)^2 + 2\text{tr } \tilde{H}_1^4\right\} = c_1^4C_{n_1-1}
\end{aligned}$$

where C_{n_1-1} denotes the volume of the unit sphere $S^{n_1-1}(1)$ in $\mathbb{R}^{n_1}$. We can do the same for a unit vector $u_2 \in T_{\tilde{m}}\tilde{\mathcal{U}}_2$. Then, summing up the expressions obtained by the integration of (3.4) for $u = u_1$ and $u = u_2$ and making some straightforward calculations, we get

$$\frac{3}{2}\|\tilde{R}\|^2 + (n+11)\left(\frac{\tau}{n}\right)^2 - 22(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) = (n^2-21)\mu. \tag{3.6}$$

Next, we express $\|\tilde{R}\|^2$ in terms of $\|R\|^2$. Using (2.6), the first Bianchi identity and (2.10), it follows

$$\|\tilde{R}\|^2 = \|R\|^2 + 12(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) - 6\left(\frac{\tau}{n}\right)^2 - 10\mu.$$

From this we see that (3.6) may be written as

$$(n+3)\mu + \left(\frac{\tau}{n}\right)^2 - 2(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) = 0. \tag{3.7}$$

Since, from (2.2), we have that tr $H^2 = -\tau/n$ and tr $H^4 = \mu$, it follows using also (3.3) that

$$\frac{\tilde{\tau}_1}{n_1} = (n_1+2)c_1^2 + n_2c_2^2 \quad , \quad \frac{\tilde{\tau}_2}{n_2} = n_1c_1^2 + (n_2+2)c_2^2$$

and with this, (3.7) yields $n_1n_2(c_1^2 - c_2^2)^2 = 0$. Hence, we have that $c_1^2 = c_2^2$ which contradicts the hypothesis that the ξ-sectional curvature is not constant at m. This completes the proof of the theorem. ∎

4 C- and D'Atri spaces

A Riemannian manifold is said to be a *D'Atri space* if all local geodesic symmetries are volume-preserving (up to sign), or equivalently, are divergence-preserving. Let $\theta_m = \left(\det g_{ij}\right)^{1/2}$ be the volume density function of $\exp_m$ with respect to normal coordinates centered at m. Then, (M, g) is a D'Atri space if and only if $\theta_m(\exp_m(ru)) = \theta_m(\exp_m(-ru))$ for any unit vector $u \in T_mM$, all sufficiently small $r > 0$ and all $m \in M$. Since

$$\theta_m(\exp_m(ru)) = 1 - \frac{r^2}{6}\rho_{uu}(m) - \frac{r^3}{12}(\nabla_u\rho_{uu})(m) + O(r^4)$$

(see, for example, [22]) it follows that for a D'Atri space $\nabla_u\rho_{uu} = 0$ always holds, that is, ρ is a Killing tensor or equivalently, ρ is cyclic-parallel. This implies that the scalar curvature τ is constant and furthermore, (M, g) is analytic in normal coordinates. For further information, see [24].

Next, (M, g) is said to be a *C-space* if for any geodesic γ the eigenvalues of the Jacobi operator $R_\gamma := R_{\gamma'}.\gamma'$ are constant along γ. This is equivalent to the condition that tr R_γ^k is constant along γ for all $k \in \mathbb{N}$. For $k = 1$, this yields that, again, ρ is cyclic-parallel.

Returning to flow geometry, we have

Proposition 4.1 [20] *Let (M, g) be a Riemannian manifold equipped with a normal contact flow $\mathfrak{F}_\xi$ such that $\nabla_x\rho_{xx} = 0$ for all $x \in T_mM$ and all $m \in M$. Then ρ is η-parallel.*

Hence, we have

Corollary 4.2 *Any D'Atri or C-space (M, g) equipped with a normal contact flow $\mathfrak{F}_\xi$ has η-parallel Ricci tensor.*

Furthermore, when the contact condition is deleted in Proposition 4.1, we get

$$\tilde{\nabla}_{\tilde{H}\tilde{u}}\tilde{\rho}_{\tilde{v}\tilde{w}} = 0 \tag{4.1}$$

for all vectors $\tilde{u}, \tilde{v}, \tilde{w}$ on $\tilde{\mathcal{U}}$.

Next, it follows easily from the definitions of a D'Atri and a C-space that, if (M, g) is locally a Riemannian product $(M_1, g_1) \times \ldots \times (M_r, g_r)$, then it is a D'Atri or a C-space if and only if each factor (M_i, g_i) is such a space. From this remark it then follows at once:

Proposition 4.3 *Let (M, g) be an n-dimensional Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that* rank $H = 0$. *Then (M, g) is a D'Atri space (respectively, a C-space) if and only if (M, g) is locally a product of an $(n-1)$-dimensional D'Atri space (respectively, a C-space) and a curve.*

We note that a locally KTS-space is locally homogeneous and moreover, it is equipped with a naturally reductive structure [17]. Manifolds endowed with such a structure are necessarily D'Atri spaces [15], [24] and C-spaces [2]. Hence, locally KTS-spaces are automatically D'Atri spaces and C-spaces. Next, we shall treat, for dimensions ≤ 5, D'Atri and C-spaces which are equipped with a normal flow and in particular, we will be interested in the question whether such spaces are locally KTS-spaces or not. This question has been studied in [16] for the class of Sasakian manifolds.

Since a two-dimensional manifold is equipped with a normal flow if and only if it is locally flat, we shall restrict to the cases where $\dim M \in \{3, 4, 5\}$.

4.1 The three-dimensional case

We may restrict to the class of D'Atri spaces since the class of three-dimensional C-spaces coincides with that formed by the D'Atri spaces [2]. First, we have

Proposition 4.4 *A three-dimensional manifold equipped with a normal flow is either homothetic to a Sasakian manifold or locally a product of a two-dimensional manifold and a curve.*

Taking into account Proposition 4.3, Proposition 4.4 and Watanabe's result [37] for Sasakian spaces (or (2.7) and Proposition 2.5), we get

Theorem 4.5 [20] *A three-dimensional manifold equipped with a normal flow is a locally* KTS*-space if and only if it has constant scalar curvature.*

Hence, we have

Corollary 4.6 *Any three-dimensional manifold equipped with a normal flow is a D'Atri space if and only if it is a locally* KTS*-space.*

Using the classification result of [17], we get

Proposition 4.7 *A three-dimensional complete, simply connected (M, g) equipped with a normal flow is a D'Atri space if and only if it is one of the following spaces:*

- (i) *the Riemannian symmetric spaces $S^3, E^3, S^2 \times E^1, H^2 \times E^1$;*
- (ii) *$SU(2)$ and the universal covering of $SL(2, \mathbb{R})$, both with suitable left invariant metrics;*
- (iii) *the three-dimensional Heisenberg group with any left-invariant metric.*

If (M, g) is not complete or not simply connected, then it is locally isometric to one of these spaces.

The classification of three-dimensional D'Atri spaces is given in [23] and it turns out (see also [17]) that all non-symmetric examples are equipped with a normal contact flow. These are the examples given in (ii) and (iii) of Proposition 4.7.

4.2 The four-dimensional case

First, we note that we do not yet know a complete classification of four-dimensional D'Atri and C-spaces and even we do not know if all these spaces are locally homogeneous. Nevertheless, for manifolds equipped with a normal flow, we have a complete and positive answer.

Theorem 4.8 *Any four-dimensional D'Atri space (respectively,* C*-space) equipped with a normal flow is a locally* KTS*-space or locally a product of a three-dimensional D'Atri space (respectively,* C*-space) and a curve.*

Proof. Corollary 2.3 implies that we have only to consider the two following cases: rank $H = 0$ or rank $H = 2$.

First, when rank $H = 0$, then the result follows from Proposition 4.3.

Next, let rank $H = 2$, that is, Ker $\tilde{H}$ is a one-dimensional subspace at each point of any base space $\tilde{\mathcal{U}}$. Then (4.1), the cyclic-parallel condition on $\tilde{\mathcal{U}}$ and (2.11) yield that $\tilde{\rho}$ is parallel. Hence, each base space $(\tilde{\mathcal{U}}, \tilde{g})$ is locally symmetric. The required result now follows from Proposition 2.5. ∎

From this, we have

Corollary 4.9 *Let (M, g) be a simply connected, complete four-dimensional manifold equipped with a normal flow. Then (M, g) is a naturally reductive Riemannian manifold if and only if it is a D'Atri space or a* C*-space, respectively.*

Using the classification given in [26], we then obtain

Corollary 4.10 *Any four-dimensional complete and simply connected Riemannian manifold equipped with a normal flow is a D'Atri or* C*-space if and only if it is one of the following spaces:*

(i) *the Riemannian symmetric spaces E^4, $S^3 \times E^1$, $S^2 \times E^2$, $H^3 \times E^1$, $H^2 \times E^2$;*

(ii) *the product of E^1 with one of the spaces given in Proposition* 4.7, (ii) *and* (iii).

If (M, g) is not complete or not simply connected, then it is locally isometric to one of these spaces.

4.3 The five-dimensional case

In this case we have weaker results. Also here, we do not yet have a complete classification of D'Atri and C-spaces but we have a classification of five-dimensional naturally reductive spaces [24], [27]. Furthermore, in the Sasakian case, we have [16]

Proposition 4.11 *Let $(M, g, \varphi, \xi, \eta)$ be a five-dimensional Sasakian manifold. Then the following statements are equivalent:*

(i) *$(M, g, \varphi, \xi, \eta)$ is a locally φ-symmetric space;*

(ii) *(M, g) is a D'Atri space;*

(iii) *(M, g) is a* C*-space.*

This result is useful in the proof of

Theorem 4.12 [20] *Let (M, g) be a five-dimensional D'Atri space (respectively,* C*-space) equipped with a normal flow $\mathfrak{F}_\xi$. Then it is a locally* KTS*-space or locally a product of a four-dimensional D'Atri space (respectively,* C*-space) and a curve.*

Proof. First, if rank $H = 0$, then the result follows from Proposition 4.3. When rank $H = 2$, we obtain from (2.10), using the normality of the flow, (2.11) and (4.1) that $(\tilde{\mathcal{U}}, \tilde{g})$ is Ricci-parallel. Also, when rank $H = 4$, that is, $\mathfrak{F}_\xi$ is a contact flow, it follows from Corollary 4.2 that each $(\tilde{\mathcal{U}}, \tilde{g})$ is Ricci-parallel. Then, in both cases, if $(\tilde{\mathcal{U}}, \tilde{g})$ is locally reducible, it is locally symmetric and if $(\tilde{\mathcal{U}}, \tilde{g})$ is locally irreducible, it is an Einstein space.

Now, taking into account that on four-dimensional Einstein manifolds the sectional curvature of a two-plane is equal to that of the orthogonal plane and using Proposition 4.11, we have that $(\tilde{\mathcal{U}}, \tilde{g})$ is a curvature-homogeneous Einstein space. It then follows that $(\tilde{\mathcal{U}}, \tilde{g})$ is again locally symmetric [30]. Finally, we have, from Proposition 2.5 that (M, g) is a locally KTS-space. ∎

From this result, we then get

Corollary 4.13 *Let (M, g) be a five-dimensional Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that rank $H \neq 0$ (in particular, $\mathfrak{F}_\xi$ is a contact flow). Then (M, g) is a D'Atri space or a* C*-space, respectively, if and only if it is a locally* KTS*-space.*

Corollary 4.14 *Let (M, g) be a five-dimensional complete, simply connected Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that rank $H \neq 0$. If (M, g) is a D'Atri space or a* C*-space, then (M, g) is a naturally reductive space and conversely.*

5 Ball-homogeneous spaces

We recall that a Riemannian manifold (M, g) is said to be ball-homogeneous if the volumes of small geodesic spheres or balls are independent of the center, that is, only depend on the radius. Since no examples are known which are not locally homogeneous, it is natural to study the question whether all ball-homogeneous are locally homogeneous or not. We shall now give a positive answer for some cases in flow geometry.

First, we note that ball-homogeneity implies that the scalar curvature must be constant on the connected manifold (M, g) [22]. From this we deduce, using Theorem 4.5,

Theorem 5.1 *Let (M, g) be a three-dimensional manifold equipped with a normal flow. Then (M, g) is ball-homogeneous if and only if it is a locally* KTS*-space.*

This result extends a similar one in Sasakian geometry. (See, for example, [37].) Also, in [16], the following result is derived:

Proposition 5.2 *Let $(M, g, \varphi, \xi, \eta)$ be a five-dimensional Sasakian space. Then it is locally φ-symmetric if and only if it is ball-homogeneous and η-parallel.*

Using this, the results of Section 4, the proof of Theorem 4.12 and the fact that any D'Atri space is ball-homogeneous [21], we now have the following extension:

Theorem 5.3 *Let (M, g) be a five-dimensional Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that rank $H \neq 0$ (in particular, $\mathfrak{F}_\xi$ is a contact flow). Then the following statements are equivalent:*

(i) $(M, g, \mathfrak{F}_\xi)$ *is a locally* KTS*-space;*

(ii) (M, g) *is a ball-homogeneous space with η-parallel Ricci tensor;*

(iii) (M, g) *is a D'Atri space;*

(iv) (M, g) *is a* C*-space.*

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ON CURVATURE HOMOGENEOUS SPACES

Oldřich Kowalski

Abstract.– We give a short survey about the theory of curvature homogeneous spaces (of arbitrary order) which was initiated by I. M. Singer in 1960.

We start with the basic definitions.

Definition 1 Let (M,g), $(\overline{M},\overline{g})$ be two Riemannian manifolds (of the same dimension n and of class C^∞) and let $p \in M$, $\overline{p} \in \overline{M}$ be two points. We say that these manifolds have *contact of order* s (≥ 0) *at the pair of points* $(p,\overline{p})$ if there are two systems of local coordinates $(u^1,\dots,u^n)$, $(\overline{u}^1,\dots,\overline{u}^n)$ in the neighborhoods of the points p and $\overline{p}$ respectively such that

$$u^i(p) = \overline{u}^i(\overline{p}) = 0 \qquad \text{for} \quad i = 1,\dots,n \tag{1}$$

and

$$(\partial^r g_{ij}/\partial u^{i_1}\cdots\partial u^{i_r})(p) = (\partial^r \overline{g}_{ij}/\partial \overline{u}^{i_1}\cdots\partial \overline{u}^{i_r})(\overline{p}) \tag{2}$$

hold for all $r \in \{0,1,\dots,s\}$ and all $i,j,i_1,\dots,i_r \in \{1,\dots,n\}$.

Let now ∇, $\overline{\nabla}$, R, $\overline{R}$ denote the corresponding Levi-Civita connections and the corresponding Riemannian curvature tensors, respectively. For the sake of simplicity, the components of the tensors $\nabla^r R$ and $\overline{\nabla}^r\overline{R}$ respectively with respect to some vector basis will be marked only by the corresponding indices.

Definition 2 We say that two Riemannian manifolds (M,g) and $(\overline{M},\overline{g})$, have *the same curvature up to order* s *at the pair of points* $(p,\overline{p})$ if there is an orthonormal basis $(e_1,\dots,e_n)$ for g in T_pM and an orthonormal basis $(\overline{e}_1,\dots,\overline{e}_n)$ for $\overline{g}$ in $T_{\overline{p}}\overline{M}$ such that

$$\left(\nabla^r_{i_1,\dots,i_r}R\right)_{ijkl} = \left(\overline{\nabla}^r_{i_1,\dots,i_r}\overline{R}\right)_{ijkl} \tag{3}$$

hold for all $r \in \{0,1,\dots,s\}$ and all $i,j,k,l,i_1,\dots,i_r \in \{1,\dots,n\}$. Using the

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classical formulas in Riemannian geometry and the power-series expansions for the metric tensor in normal coordinates (see [7], [8], [9], [15]), we can prove that the definitions 1 and 2 are equivalent in some sense:

Proposition 1 *Two Riemannian manifolds (M, g), $(\overline{M}, \overline{g})$ have contact of order $s+2$ at the pair of points $(p, \overline{p})$ if and only if they have the same curvature up to order s at $(p, \overline{p})$.*

In particular, for $s = 0$, we can use the well-known formula

$$g_{ij} = \delta_{ij} - \tfrac{1}{3} \sum_{k,l} (R_{ikjl})_p x^k x^l + (\text{higher order terms}), \tag{4}$$

where $(x^1, \ldots, x^n)$ is a system of normal coordinates centered at p.

We continue with the homogeneity concepts:

Definition 3 A Riemannian manifold (M, g) is said to be *contact homogeneous of order s* if, for every two points $p, q \in M$, the manifold (M, g) has contact of order s with itself at the pair of points (p, q).

The following definition is essentially due to I. M. Singer (see [31], [25] and [2]).

Definition 4 A Riemannian manifold (M, g) is said to be *curvature homogeneous up to order s* if, for every two points $p, q \in M$, there is a linear isometry $\varphi: T_pM \to T_qM$ such that

$$\varphi^* R_q = R_p, \ldots, \quad \varphi^*(\nabla^s R)_q = (\nabla^s R)_p. \tag{5}$$

As a direct consequence of Proposition 1, we have now the following

Proposition 2 *A Riemannian manifold (M, g) is curvature homogeneous up to order s if and only if it is contact homogeneous of order $s+2$.*

Obviously, a locally homogeneous Riemannian manifold (M, g) is curvature homogeneous, and also contact homogeneous, in each finite order. We also see that *every* Riemannian manifold is contact homogeneous of order 1. (One can just use normal coordinate systems at two different points and the formula (4).) The contact homogeneous spaces of order 2 are the same as curvature homogeneous spaces in the sense of I. M. Singer. One of the basic problems in [31] was the question if there are any *curvature homogeneous spaces which are not locally homogeneous.* The first example was constructed in 1973 by K. Sekigawa ([27], see also [28]). Presently, the theory of curvature homogeneous spaces is well-developed: see [21], [22], [36], [1] and especially the survey article in [2]. A complete classification is known in dimension $n = 3$ (see [12], [18], [23] and the next pages).

The main result of [31] can be now presented in the following way: let (M, g) be a connected Riemannian manifold of dimension n (and class C^∞). For each

$p \in M$ and $s = 0, 1, 2, \dots$ define the Lie algebra

$$(6) \qquad \mathfrak{g}(p;s) = \{A \in \mathfrak{gl}(T_pM) | \, A \cdot g_p = A \cdot R_p = \dots = A \cdot (\nabla^s R)_p = 0\},$$

where A acts as a derivation on the tensor algebra of T_pM. Now, denote by $k(M,p)$ the least integer for which the sequence

$$(7) \qquad \mathfrak{g}(p;0) \supseteq \mathfrak{g}(p;1) \supseteq \dots \supseteq \mathfrak{g}(p;s) \supseteq \dots$$

stabilizes. Obviously $k(M,p) < \frac{n(n-1)}{2}$. Let us denote by k_M the maximum of all numbers $k(M,p)$, $p \in M$, and by k_n the maximum of all numbers k_M for all manifolds (M,g) of dimension n. We usually call k_n *the Singer number for the dimension* n.

By translating the main theorem from [31] (see also [25]) in our language we obtain

Theorem 1 *Let (M,g) be a Riemannian manifold of dimension n which is contact homogeneous up to order $k_n + 3$. Then (M,g) is locally homogeneous.*

In [29] and [30] the following results have been (*in fact*) proved:

Theorem 2 *Let (M,g) be a Riemannian manifold of dimension* 3 *or* 4. *If (M,g) is contact homogeneous of order* 3, *then it is locally homogeneous.*

The existence of contact homogeneous Riemannian manifolds of order 3 which are not locally homogeneous (for the dimensions $n \geq 5$) remains an interesting and rather difficult open problem.

Recently, P. Bueken and L. Vanhecke [3] have proved that the situation is completely different in the Lorentzian case. They found a 3-dimensional Lorentzian space which is contact homogeneous of order 3 but not locally homogeneous. (Here the concept of contact can be easily generalized.)

B. Opozda [26] studied the curvature homogeneity in the affine differential geometry. She gave the first example of a 2-dimensional affine manifold which is curvature homogeneous up to order 1 but not locally homogeneous. (Of course, our definition of contact homogeneity does not work in the affine geometry.)

For more result about the Riemannian "contact" geometry see [17].

In the next part we shall concentrate on the main classification results about curvature homogeneous Riemannian manifolds (of order 0). But before, we start with one more definition and examples.

Definition 5 Let $(\overline{M}, \overline{g})$ be a homogeneous Riemannian manifold and (M,g) a Riemannian manifold. We say that $(\overline{M}, \overline{g})$ is a *homogeneous model* of (M,g) (or, that (M,g) *has the same curvature as* $(\overline{M}, \overline{g})$) if, for a fixed $o \in \overline{M}$ and each point $p \in M$, the manifolds $(\overline{M}, \overline{g})$ and (M,g) have the same curvature (of order 0) at the pair of points $(o,\, p)$.

Obviously, each Riemannian manifold possessing a homogeneous model is curvature homogeneous. The converse does not hold, as we shall see later.

The first example of curvature homogeneous space which is not locally homogeneous was constructed, as already mentioned, by K. Sekigawa in 1973. In a more convenient local coordinate system, it can be written in the following form:

The underlying Riemannian manifold M is the Cartesian space $R^3[w,x,y]$. The Riemannian metric g on R^3 is given by

$$g = \sum_{i=1}^{3} (\omega^i)^2,$$

where the orthonormal coframe $(\omega^1, \omega^2, \omega^3)$ of Pfaffian forms is given by

$$\omega^1 = (a\mathrm{e}^{\lambda y} + b\mathrm{e}^{-\lambda y})\mathrm{d}w, \quad \omega^2 = \mathrm{d}x - y\mathrm{d}w, \quad \omega^3 = \mathrm{d}y + x\mathrm{d}w, \quad ab \neq 0.$$

It can be shown that (R^3, g) is locally nonhomogeneous, locally irreducible and if a, b are positive, then it is complete. Moreover, (M,g) has the symmetric space $H^2(-\lambda^2) \times R$ as a homogeneous model. Hence it is curvature homogeneous.

F. Tricerri, L. Vanhecke and the present author constructed in [18] the following generalization of the Sekigawa's example: Let $U \subset R^n[w, x^1, \ldots, x^{n-1}]$, $n \geq 3$, be an open subset. On U we define a Riemannian metric

$$g = \sum_{i=0}^{n-1} (\omega^i)^2$$

where

$$\omega^0 = f(w, x^1)\mathrm{d}w, \qquad \omega^i = \mathrm{d}x^i + \Big(\sum_{j=1}^{n-1} D^i_j(w) x^j\Big)\mathrm{d}w \qquad (i = 1, \ldots, n-1).$$

Here $D^i_j(w)$ are arbitrary smooth functions such that

$$D^i_j(w) + D^j_i(w) = 0 \qquad \text{for all } i,\ j.$$

The function $f(w,x)$ is specified as follows:

A) If $f(w,x) = a(w)\mathrm{e}^{\lambda x} + b(w)\mathrm{e}^{-\lambda x}$, $a(w)$, $b(w)$ arbitrary, $f(w,x) \neq 0$, then (U,g) is curvature homogeneous with the homogeneous model $H^2(-\lambda^2) \times R^{n-2}$.

B) If $f(w,x) = a(w)\cos\lambda x + b(w)\sin\lambda x$, $a(w)$, $b(w)$ arbitrary, $f(w,x) \neq 0$, then (U,g) is curvature homogeneous with the homogeneous model $S^2(\lambda^2) \times R^{n-2}$.

If $a(w)b(w) \neq 0$ and $D^1_2(w) \neq 0$, the space (U,g) is not locally homogeneous. In [21] the authors also prove that, in the case A, if $U = R^n$, $a(w) > a > 0$,

$b(w) > b > 0$ and $D^i_j(w)$ are bounded, then the metric g is complete. To the contrary, a metric of type B is never complete. For a "generic" choice of the functions $D^i_j(w)$, the space is locally irreducible (see [21] for more details).

These examples give rise to the following natural problems:

Problem 1 Characterize all Riemannian symmetric spaces (connected and simply connected) which are homogeneous models of locally nonhomogeneous curvature homogeneous spaces.

Problem 2 Calculate all locally nonhomogeneous curvature homogeneous spaces having the same curvature as a given Riemannian symmetric space.

Problem 1 was first attacked by F. Tricerri and L. Vanhecke in [35], [36]. They eliminated most of the symmetric spaces:

Theorem 3 *Let (M, g) be a (curvature homogeneous) Riemannian manifold whose homogeneous model is a simply connected Riemannian symmetric space $(\overline{M}, \overline{g})$ without Euclidean factor. Then (M, g) is locally symmetric and locally isometric to $(\overline{M}, \overline{g})$.*

The final solution of Problem 1 was found later by F. Tricerri, L. Vanhecke and the present author in [21].

Theorem 4 *Let $(\overline{M}, \overline{g})$ be a simply connected Riemannian symmetric space with the de Rham decomposition $\overline{M} = \overline{M}_0 \times \overline{M}_1 \times \cdots \times \overline{M}_r$. Then $(\overline{M}, \overline{g})$ is a homogeneous model of a locally nonhomogeneous Riemannian manifold if and only if the de Rham decomposition contains a product of the form $H^2(-\lambda^2) \times R^k$, or of the form $S^2(\lambda^2) \times R^k$, where $k \geq 1$.*

Corollary 5 *Let (M, g) be a locally nonhomogeneous and locally irreducible curvature homogeneous manifold with a simply connected symmetric model space $(\overline{M}, \overline{g})$. Then $(\overline{M}, \overline{g})$ is either $H^2(-\lambda^2) \times R^k$ or $S^2(\lambda^2) \times R^k$, where $k \geq 1$.*

Let us remark that, for deriving Theorem 4, one needs some structural formulas by Z. Szabó [33].

Now, E. Boeckx, L. Vanhecke and the present author gave the definitive solution of Problem 2. The method used here is to find an explicit general solution of a system of nonlinear PDE of the second order.

Theorem 6 ([1]) *Let (M, g) be a curvature homogeneous space with a symmetric model $H^2(-\lambda^2) \times R^k$, or $S^2(\lambda^2) \times R^k$ $(k \geq 1)$, respectively. Then, in a neighborhood of each generic point $p \in M$, (M, g) is locally isometric to a generalized Sekigawa example of type A), or B), respectively.*

Next, we shall concentrate ourselves on 3-dimensional curvature homogeneous Riemannian manifolds. Here a satisfactory classification has been also found. As well-known, in dimension three the curvature tensor is uniquely determined by the Ricci tensor. Hence we obtain easily

Proposition 7 *A 3-dimensional Riemannian manifold (M, g) is curvature homogeneous if and only if the principal Ricci curvatures of (M, g) are constant.*

Now, if all principal Ricci curvatures are equal, one obtains a space of constant curvature. Thus, only the following cases are of interest (and of very different nature):

Type I. Two of the Ricci eigenvalues are equal; $\varrho_1 = \varrho_2 \neq \varrho_3$.
Type II. All three Ricci eigenvalues ϱ_1, ϱ_2, ϱ_3 are distinct.

We shall now give more details about the solution in case I and II.

I. Let (M, g) be a 3-dimensional curvature homogeneous manifold of type I. One can derive a system of nine nonlinear partial differential equations of 2nd order (see [11] and [12] for more details). If $\varrho_3 = 0$, we obtain just generalized Sekigawa examples in dimension 3. If $\varrho_3 < 0$, say $\varrho_3 = -2\lambda^2$, then we obtain the following reduction of the problem: each space (M, g) has locally the form (U, g), where $U \subset R^3[w, x, y]$ is an open subset and $g = \sum_{i=1}^{3} (\omega^i)^2$, where

$$\omega^1 = \frac{1}{p} e^{-\lambda y} dw, \qquad \omega^2 = p e^{\lambda y} dx + (r e^{\lambda y} + s e^{-\lambda y}) dw, \qquad \omega^3 = dy + H dw,$$

and p, r, s, H are functions of w, x only satisfying the following system of PDE:

$$(A1) \qquad \left[\frac{1}{p^2} + s^2\right]'_x = 0,$$

$$(A2) \qquad H'_x = 2\lambda p s,$$

$$(A3) \qquad p'_w - r'_x - \lambda p H = 0,$$

$$(A4) \qquad -(p s'_x)'_w + (r s'_x)'_x = -\varrho_1.$$

The following interesting explicit examples are known:

Example 1 $p = p(w)$, $s = s(w)$, $H = 2\lambda p(w) s(w) x + \varphi(w)$, $r = -\lambda^2 p^2(w) s(w) x^2 + \big(p'(w) - \lambda p(w)\varphi(w)\big) x + \psi(w)$, where $p(w)$, $s(w)$, $\varphi(w)$ and $\psi(w)$ are arbitrary smooth functions. The corresponding Ricci eigenvalues are $\varrho_1 = \varrho_2 = 0, \varrho_3 = -2\lambda^2$.

Example 2 $p = \sqrt{1 + x^2}$, $s = \frac{x}{\sqrt{1+x^2}}$, $H = \lambda(x^2 + \frac{1}{4})$, $r = -\frac{\lambda^2}{4} x (1 + x^2)^{3/2}$. The corresponding Ricci eigenvalues are $\varrho_1 = \varrho_2 = \frac{1}{4}\lambda^2$, $\varrho_3 = -2\lambda^2$.

Example 3 $p = \sqrt{1 + x^4}$, $s = \frac{x^2}{\sqrt{1+x^4}}$, $H = \frac{2}{3}\lambda x^3$, $r = -\frac{1}{9}\lambda^2 (1 + x^4)^{3/2}$. The corresponding Ricci eigenvalues are $\varrho_1 = \varrho_2 = \frac{2}{9}\lambda^2$, $\varrho_3 = -2\lambda^2$.

These are the only known explicit examples! Example 1 is of special interest because these spaces appear in other parts of differential geometry:

a) The spaces from Example 1 have the property that the eigenvalues of some specific curvature operator (which is different from the Jacobi operator) are constant along curves with unit curvature and zero torsion (see [10]).

b) E. Boeckx and L. Vanhecke [3] proved that the unit sphere bundle of a 3-dimensional Riemannian manifold (M, g) (equipped with the induced Sasaki metric) has constant scalar curvature if and only if either (M, g) is a space of constant curvature or (M, g) is curvature homogeneous and $\varrho_1 = \varrho_2 = 0$. Hence the spaces from Example 1 are explicit and locally nonhomogeneous examples of such a situation.

Examples 2 and 3 are interesting for a different reason: these are the first explicit examples of 3-dimensional curvature homogeneous manifolds *without homogeneous models*. Indeed, according to [24], the signature of the Ricci tensor of a 3-dimensional homogeneous Riemannian manifold cannot be $(+,+,-)$ which is the case in our examples. (See also more details later.)

As concerns the case $\varrho_1 = \varrho_2 \neq \varrho_3 > 0$, here the processing of the basic system of PDE is much more difficult and no explicit examples are known. Yet, for all three possibilities ($\varrho_3 = 0$, $\varrho_3 < 0$ and $\varrho_3 > 0$) we have the following existence theorem:

Theorem 8 *The local isometry classes of curvature homogeneous manifolds of type I always depend on two arbitrary functions of one variable.*

An analogous classification in the three-dimensional Lorentzian geometry belongs to P. Bueken [4].

II. The first examples of locally nonhomogeneous spaces of Type II were given by K. Yamato [39].

Example. Let ϱ_1, ϱ_2, ϱ_3 be distinct constants and put

$$A = \frac{\varrho_1 + \varrho_2 - \varrho_3}{2}, \qquad B = \frac{\varrho_1 - \varrho_3}{\varrho_3 - \varrho_2}, \qquad C = -\frac{(\varrho_1 + \varrho_2)(\varrho_3 - \varrho_2)^2}{(\varrho_2 - \varrho_1)^2}.$$

If $A > 0$, $C > 0$, $A + BC > 0$, then a complete metric g can be given on $R^3[x, y, w]$ in the following way: we choose the metric in the form $g = \sum_{i=1}^{3} (\omega^i)^2$, where

$$\omega^1 = \mathrm{d}x + P(x, y, w)\mathrm{d}w, \qquad \omega^2 = \mathrm{d}y + Q(x, y, w)\mathrm{d}w, \qquad \omega^3 = \mathrm{d}w.$$

Define functions $\alpha(w)$, $\beta(w)$, $\gamma(w)$ as follows:

$$\alpha(w) = \sqrt{C}(\mathrm{e}^{2G(w)} - 1)\big/(\mathrm{e}^{2G(w)} + 1), \quad \text{where } G(w) = \sqrt{C}(B+1)w, \quad \text{and}$$

$$\beta(w) = \sqrt{A + B\alpha^2(w)}, \qquad \gamma(w) = \frac{\varrho_1 + (B+1)C}{2\beta(w)}.$$

Then we put

$$P = -\alpha(w)x + \big[\beta(w) + \gamma(w)\big]y, \quad Q = \big[\beta(w) - \gamma(w)\big]x - B\alpha(w)y.$$

The corresponding Riemannian manifold (R^3, g) has constant principal Ricci curvatures ϱ_1, ϱ_2, ϱ_3 and it is not locally homogeneous. Moreover, it is *complete*.

The present author and F. Prüfer [18] extended this example to *all* prescribed constant Ricci eigenvalues without limitation. (Nevertheless, if the inequalities above are not satisfied for some numeration of the Ricci eigenvalues, then one cannot prove completeness. We even conjecture that, in such a case, a complete metric does not exist on any underlying 3-manifold.) For the construction in the general case, we mention first an easy modification of the Yamato example [13] which covers "almost all" cases. Here the function $\alpha(w)$ is given by $\sqrt{|C|}\mathrm{tg}(G(w))$ for $C < 0$ and by $1/((B+1)w)$ for $C = 0$; the rest of the formulas remain unchanged. Yet, passing from "almost all" to *all* possibilities is nontrivial and requires a detailed analysis of the basic system of partial differential equations for the problem. We shall not write down this rather complicated system here. Instead we describe a simple explicit solution (see [18] and [2]).

Theorem 9 *For every choice of real numbers $\varrho_1 > \varrho_2 > \varrho_3$ there exist an explicit Riemannian metric g on $R^3[w,x,y]$ with the constant Ricci eigenvalues ϱ_1, ϱ_2, ϱ_3 and not locally homogeneous.*

Construction

Define first $\lambda_i = (\varrho_1 + \varrho_2 + \varrho_3) - \varrho_i \quad (i = 1, 2, 3), \qquad B = \dfrac{\varrho_1 - \varrho_3}{\varrho_3 - \varrho_2} \neq 0.$

Then $B + 1 = \dfrac{\varrho_1 - \varrho_2}{\varrho_3 - \varrho_2} < 0.$

Define smooth functions $\varphi_1(w)$, $\varphi_2(w)$, $\varphi_3(w)$ on R as follows:

a) $\varphi_1(w)$ is an arbitrary smooth function such that
$\varphi_1'(w) \neq 0$ and $\left(\varphi_1(w)\right)^2 > \max\left\{0,\ \lambda_2,\ \frac{(B+2)((B+1)\lambda_3+\lambda_2)}{B^2}\right\}$,

b) $\varphi_2(w) > 0$ is calculated from the algebraic equation $(B+1)(\varphi_2)^2 + (\varphi_1)^2 = \lambda_2$,

c) $\varphi_3(w)$ is calculated from the algebraic equation $-B\varphi_1\varphi_3 = (B+1)\lambda_3 + \lambda_2$.

Then define the metric $g = \sum_{i=1}^3 (\omega^i)^2$ on $R^3[w, x, y,]$ by

$$\omega^1 = A\mathrm{d}w + \mathrm{d}x, \quad \omega^2 = C\mathrm{d}w, \quad \omega^3 = \mathrm{d}y + G\mathrm{d}w,$$

where

$$C = -\varphi_1'(w)\Big/\Big[\varphi_2(B\varphi_1 + (B+2)\varphi_3)\Big] \neq 0, \qquad A = C(\varphi_3 - \varphi_1)y - C\varphi_2 x,$$
$$G = (B+1)C\varphi_2 y - C(\varphi_1 + \varphi_3)x.$$

The correctness of this construction is easily checked. To check that the Ricci eigenvalues of (R^3, g) are equal to the prescribed constants is more delicate and the reader is advised to see the paper [18]. The motivation by the Yamato example is obvious.

In [19] the authors have classified all Riemannian manifolds of type II satisfying some additional geometric conditions. One part of this classification is formed by "generalized Yamato examples", a broader family containing the examples described above. A new family of explicit solutions was also found which is not of Yamato type.

The following problem remained open up to recently: "how many" local isometry classes exists of curvature homogeneous manifolds of type II? First A. Spiro and F. Tricerri [32] have proved that these local isometry classes depend on a infinite number of parameters. Finally, the present author and Z. Vlášek [23] gave a definitive answer in the following theorem (which is analogous to Theorem 8):

Theorem 10 *The local isometry classes of curvature homogeneous manifolds of type II always depend on three arbitrary functions of two variables.*

The method of the proof is a computer-aided manipulation with a complicated system of 18 nonlinear PDE for 12 unknown functions of 3 variables and some specific modification of the Cauchy-Kowalewski theorem.

We see that the family of Riemannian metrics with prescribed distinct constant Ricci eigenvalues is much bigger than that in the case when the prescribed Ricci eigenvalues are not distinct. This explains why explicit solutions always exist in case II whereas only exceptionally in case I.

One can also solve the following problem: in which cases the corresponding curvature homogeneous Riemannian manifold possesses a homogeneous model? The following result belongs to the present author and S. Nikčević [16]:

Theorem 11 *Let (M, g) be a curvature homogeneous Riemannian manifold of dimension 3. Then (M, g) has a homogeneous model if and only if the principal Ricci curvatures $\varrho_1 \geq \varrho_2 \geq \varrho_3$ of (M, g) satisfy the following conditions:*

(a) *The Ricci form does not have the signature $(+,+,-)$ or $(+,0,-)$,*

(b) *for the Ricci signature $(+,+,0)$ one has $\varrho_1 = \varrho_2$,*

(c) *for the Ricci signature $(-,-,-)$ or $(0,-,-)$ one has either $\varrho_1 = \varrho_2 = \varrho_3$ or $\frac{(\varrho_1)^2+(\varrho_3)^2}{\varrho_1+\varrho_3} \leq \varrho_2 < \frac{\varrho_1+\varrho_3}{2}$.*

The proof is based on the paper by J. Milnor [24] and one result by K. Sekigawa [29]. In particular, one can see that the original Yamato examples with complete metrics always have a homogeneous model (see [22]). Hence the following open problem may be of interest:

Problem 3 Decide if there exists a *complete* 3-dimensional curvature homogeneous Riemannian manifold without a homogeneous model.

(In dimension four, such an example was, in fact, constructed by K. Tsukada, see [38] and also [22].)

The most important open question for the 3-dimensional case seems to be the following:

Problem 4 Decide if there exists a *compact* 3-dimensional curvature homogeneous Riemannian manifold which is locally nonhomogeneous.

If the answer to this problem is negative, then the so-called *Gromov conjecture* holds in dimension 3. (The Gromov conjecture, in the most general setting, says the following: if M is a compact manifold, then the global isometry classes of all curvature homogeneous Riemannian metrics on M depend on at most finite number of parameters.) Up to now, only special negative results are known, for example that by K. Yamato [39]:

Theorem 12 *Suppose that $\varrho_1 = \varrho_2 \neq \varrho_3$ are constant on a connected compact Riemannian manifold (M, g). If $\varrho_1 \geq 0$ or $\varrho_3 \leq 0$, then M is locally homogeneous.*

For another partial answer to the Gromov conjecture see [37].

Finally, let us remark that Problem 4 has a positive answer in specific higher dimensions; yet the corresponding examples are rather isolated (see [6]). Thus the Gromov conjecture in higher dimensions remains undecided, too.

We can summarize the existence results about 3-dimensional curvature homogeneous spaces in the following alternative form: let $(V, <,>)$ be a vector space with a (positive) scalar product. Let K be an *algebraic curvature tensor* of Riemannian type on V, i.e., an element $K \in \Lambda^2 V^* \otimes \Lambda^2 V^*$ where K satisfies the usual algebraic identities of a Riemannian curvature tensor, including the first Bianchi identity. We have

Theorem 13 *For each algebraic curvature tensor K on a 3-dimensional vector space $(V, <,>)$ there is a curvature homogeneous Riemannian manifold (M, g) with the typical curvature tensor K. If K is not of constant sectional curvature, then a locally nonhomogeneous (M, g) always exists.*

More precisely, for each $m \in M$ there is a linear isometry $\varphi_m \colon T_m M \to V$ which maps the curvature tensor R_m into the given algebraic curvature tensor K. This result follows immediately from the previous results and from the fact that, in dimension three, an algebraic curvature tensor is uniquely determined, up to an isomorphismus, by the Ricci eigenvalues.

Let us remark, that due to Theorem 11, the first statement of Theorem 13 is not more valid if we replace the words "curvature homogeneous" by the words "locally homogeneous".

Theorem 13 is not more true in dimension four (see [20]). A new open direction in the theory of curvature homogeneous spaces is the study of so-called *isocurved deformations*. See [34] and [14] for more details.

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k–HAMILTONIAN SYSTEMS

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Abstract.– We introduce a geometrical formalism which extends the symplectic and cosymplectic ones, and permits to derive in a global setting the field equations for order one classical field theories.

1 Introduction

As is well known, symplectic manifolds are the natural setting for classical mechanics. In its time-dependent version, cosymplectic manifolds provide a natural arena to derive the equations of motion in a geometrical way [1, 6, 25].

In the last four decades there have been many attempts to extend this symplectic setting for classical field theories. There was very succesfull a geometrical formalism in terms of jet manifolds by using the so-called multisymplectic structure (see [9, 10, 11, 16, 17, 18, 33], and more recently [4, 5, 7, 8, 12, 13, 15, 19, 20, 27, 28, 31, 32], see also [3, 26]). A different approach were also suggested [2, 14, 21, 22, 29, 30] taking into account the geometry of the tangent bundles of k-covelocities. This led to the development of the so-called k-symplectic structures. This structure provides in fact a natural framework for classical field theories, however this formalism does not include the case of theories which depend explicitly on the independent parameters. In order to extent the theory for this case we have recently introduced the notion of k-cosymplectic structure [23]. The purpose of the present paper is to give a succint account of our recent results on this direction contained in our papers [23, 24].

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2 Almost k-cosymplectic structures

In [23] we have introduced the following definition.

Definition 2.1 Let M be a differentiable manifold of dimension $k(n+1)+n$. A family $(\eta_i, \omega_i, V; 1 \leq i \leq k)$ where each η_i is a 1-form, each ω_i is a 2-form and V is an nk-dimensional distribution on M such that:

1. $\eta_1 \wedge \ldots \wedge \eta_k \neq 0$,
2. $\dim(\ker \omega_1 \cap \ldots \cap \ker \omega_k) = k$,
3. $\ker \eta_1 \cap \ldots \cap \ker \eta_k \cap \ker \omega_1 \cap \ldots \cap \ker \omega_k = \{0\}$,
4. $\eta_{i_{|V}} = 0, \qquad \omega_{i_{|V \times V}} = 0, \qquad (1 \leq i \leq k)$

will be called an *almost k-cosymplectic structure*, and the manifold M an *almost k-cosymplectic manifold.*

From the conditions of Definition 2.1 we deduce that there exist k vector fields $\xi_1, \ldots, \xi_k$ on M satisfying

(1) $$\iota_{\xi_i}\eta_j = \delta_{ij}, \qquad \iota_{\xi_i}\omega_j = 0,$$

with $1 \leq i, j \leq k$, and wich will be called the *Reeb vector fields* associated to the almost k-cosymplectic structure.

In particular, if $k = 1$, then $\dim M = 2n+1$, and (η, ω) is an almost cosymplectic structure on M [1, 6], and (η, ω, V) is an almost stable cotangent structure on M [25].

The above definition was motivated by the existence of a canonical k-almost cosymplectic structure on the jet bundle $J^1(Q, \mathbb{R})$. Let $\pi : \mathbb{R}^k \times Q \longrightarrow Q$ be a trivial fibred manifold and denote by $J^1\pi$ the manifold of 1-jets of local sections of π; $J^1\pi$ is a vector bundle over Q with standard fibre $\mathbb{R}^k \times \mathbb{R}^{nk}$, where $\dim Q = n$. We have a canonical identification $J^1\pi \cong \mathbb{R}^k \times (T^1_k)^*Q$. If (x^α) are local coordinates on Q, then $(t^i, x^\alpha, x^i_\alpha)$, $1 \leq i \leq k, 1 \leq \alpha \leq n$, is a local coordinate system on $\mathbb{R}^k \times (T^1_k)^*Q$. The family $((\eta_0)_i, (\omega_0)_i, V_0)$ of 1-forms $(\eta_0)_i$, 2-forms $(\omega_0)_i$ and a distribution V_0 given by

$$(\eta_0)_i = dt^i, \qquad (\omega_0)_i = dx^\alpha \wedge dx^i_\alpha, \qquad V_0 = < \frac{\partial}{\partial x^1_\alpha}, \ldots, \frac{\partial}{\partial x^k_\alpha} >$$

is an almost k-cosymplectic structure on $J^1(Q, \mathbb{R})$.

In [23] we have proved that a manifold M of dimension $k(n+1)+n$ admits an almost k-cosymplectic structure if and only if the structure group of its tangent bundle is reducible to the group G of matrices of the form

$$\begin{pmatrix} I_k & 0 & 0 & \ldots & 0 \\ 0 & A & 0 & \ldots & 0 \\ 0 & B_1 & C & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & B_k & 0 & \ldots & C \end{pmatrix}$$

with $B_i^t A = A^t B_i$ and $C = (A^{-1})^t$ for all $1 \leq i \leq k$.

Definition 2.2 Let M be a $k(n+1)+n$-dimensional manifold with an almost k-cosymplectic structure (η_i, ω_i, V). We say that (η_i, ω_i, V) is *integrable* if the corresponding G-structure is integrable, and in such a case it is called k-cosymplectic.

Therefore, an almost k-cosymplectic structure (η_i, ω_i, V) on a manifold M is integrable if around each point of M there exist local coordinates $(s^i,\ x^\alpha,\ x^i_\alpha;\ 1 \leq i \leq k,\ 1 \leq \alpha \leq n)$ such that

$$(2) \qquad \eta_i = ds^i, \qquad \omega_i = dx^\alpha \wedge dx^i_\alpha, \qquad V = < \frac{\partial}{\partial x^1_\alpha}, \dots, \frac{\partial}{\partial x^k_\alpha} >_{\alpha=1,\dots,n} .$$

Such a coordinate functions will be called *Darboux* or *canonical coordinates.*

In Darboux coordinates, the Reeb vector fields are written as $\xi_i = \dfrac{\partial}{\partial t^i}$.

The integrability of a k-cosymplectic structure is characterized in the following

Theorem 2.3 [23] *An almost k-cosymplectic structure (η_i, ω_i, V) on a manifold M is integrable if and only if the following conditions are satisfied:*

$$(3) \qquad d\eta_i = 0, \qquad d\omega_i = 0, \qquad [V, V] \subset V, \qquad (1 \leq i \leq k).$$

Remark 2.4 When $k = 1$ the condition $[\xi, V] \subset V$ is a necessary condition for the integrability of the almost stable cotangent structure (η, ω, V), and it is an independent condition of the other ones (see [25]). For $k > 1$, we have the following

Proposition 2.5 [23] *If M is an almost k-cosymplectic manifold with $k > 1$, and $\xi_1, \dots, \xi_k$ are the Reeb vector fields associated to the almost k-cosymplectic structure, then:*

$$(d\eta_j = 0, d\omega_j = 0,\ \forall j,\ 1 \leq j \leq k) \Rightarrow [\xi_i, V] \subset V,\ \forall j,\ 1 \leq j \leq k.$$

3 k-vector fields

Let M be an arbitrary manifold and $\tau^k : T^1_k M \longrightarrow M$ its tangent bundle of k-velocities. Let us recall that $T^1_k M$ is the manifold of 1-jets at $0 \in \mathbb{R}^k$ of mappings from R^k into M.

Definition 3.1 A section $s : M \longrightarrow T^1_k M$ of the projection τ^k will be called a *k-vector field* on M.

Obviously, T^1_kM may be canonically identified with the Whitney sum of k copies of TM, say

$$T^1_kM \equiv TM \oplus \ldots \oplus TM.$$

Hence, a k-vector field s defines k vector fields $X_1, \ldots, X_k$ on M, so that, in what follows, we will use the notation $(X_1, \ldots, X_k)$ for s.

Definition 3.2 A mapping $\sigma : U \subset \mathbb{R}^k \to M$ defined on some open neighborhood of $0 \in \mathbb{R}^k$ will be called an *integral section* of a k-vector field $(X_1, \ldots, X_k)$ passing through a point $x \in M$ if and only if

$$\sigma(0) = x, \qquad \sigma_*(t)(\frac{\partial}{\partial t^i}) = X_i(\sigma(t)) \text{ for all } t \in U.$$

We say that a k-vector field $(X_1, \ldots, X_k)$ on M is *integrable* if there is an integral section passing trough each point of M.

Let us remark that if σ is an integral section of a k-vector field $(X_1, \ldots, X_k)$ then each curve on M defined by $\sigma_i = \sigma \circ J_i$ where $J_i : \mathbb{R} \to \mathbb{R}^k$ is the natural inclusion $J_i(t) = (0, \ldots, t, \ldots, 0)$, is an integral curve of the vector field X_i on M, with $1 \leq i \leq k$.

4 Hamiltonian systems on k-cosymplectic manifolds

Now, we introduce the dynamics on a k-cosymplectic manifold M with k-cosymplectic structure $(\eta_1, \ldots, \eta_k, \omega_1, \ldots, \omega_k, V)$. We define two vector bundle morphisms $\Omega^\flat$ and $\Omega^\sharp$ as follows:

$$\begin{array}{rcl} \Omega^\flat : \; TM & \longrightarrow & (T^1_k)^*M \\ X & \longrightarrow & \Omega^\flat(X) = (\iota_X\omega_1 + \eta_1(X)\eta_1, \ldots, \iota_X\omega_k + \eta_k(X)\eta_k) \end{array}$$

and

$$\begin{array}{rccl} \Omega^\sharp : & T^1_kM & \longrightarrow & T^*M \\ & (X_1, \ldots, X_k) & \longrightarrow & \Omega^\sharp(X_1, \ldots, X_k) \end{array}$$

such that

$$\begin{aligned} \Omega^\sharp(X_1, \ldots, X_k)(Y) &= \operatorname{trace}((\Omega^\flat(X_j))_i(Y)) \\ &= \sum_{i=1}^k (\Omega^\flat(X_i))_i(Y) \\ &= \sum_{i=1}^k (\omega_i(X_i, Y) + \eta_i(X_i)\eta_i(Y)), \end{aligned}$$

for all $Y \in TM$. The above morphisms induce two morphism of $C^\infty(M)$-modules between the corresponding spaces of sections.

If $(s^i,\ x^\alpha,\ x^i_\alpha;\ 1 \le i \le k,\ 1 \le \alpha \le n)$ are Darboux coordinates for the k-cosymplectic structure (η_i, ω_i, V), and if the k-vector field $(X_1, \ldots, X_k)$ is expressed with respect to this system by

$$X_i = (X_i)^j \frac{\partial}{\partial s^j} + (X_i)^\alpha \frac{\partial}{\partial x^\alpha} + (X_i)^j_\alpha \frac{\partial}{\partial x^i_\alpha}$$

then

$$\Omega^\sharp(X_1, \ldots, X_k) = \sum_i (X_i)^i ds^i - \sum_{i,\alpha} (X_i)^i_\alpha dx^\alpha + \sum_{i,\alpha} (X_i)^\alpha dx^i_\alpha. \tag{4}$$

Let $H : M \longrightarrow \mathbb{R}$ be a function on M. If $(X_1, \ldots, X_k)$ is a k-vector field on M, the equations:

$$\begin{aligned} \eta_i(X_j) &= \delta_{ij}, \qquad \forall i, j \\ \Omega^\sharp(X_1, \ldots, X_k) &= dH + \sum_{i=1}^k (1 - \xi_i(H))\eta_i, \end{aligned} \tag{5}$$

would imply

$$(X_i)^j = \delta_{ij}, \quad \frac{\partial H}{\partial x^\alpha} = -\sum_{i=1}^k (X_i)^i_\alpha, \quad \frac{\partial H}{\partial x^i_\alpha} = (X_i)^\alpha. \tag{6}$$

From these local conditions we can define, in a neigbourhood of each point $x \in M$, a k-vector field satisfying (6). For example we can put

$$(X_i)^j = \delta_{ij}, \quad (X_1)^1_\alpha = \frac{\partial H}{\partial x^\alpha}, \quad (X_i)^j_\alpha = 0, \text{for } i \neq 1 \neq j, \quad (X_i)^\alpha = \frac{\partial H}{\partial x^i_\alpha}.$$

Now one can construct a global k-vector field which is a solution of (6) by using a partition of the unity.

Remark 4.1 Equations (6) have not, in general, a unique solution. In fact, if we denote by $\mathcal{M}_k(C^\infty(M))$ the space of matrices of order k whose entries are functions on M, and we define the map

$$\begin{array}{rccc} \eta^\sharp : & T^1_k M & \longrightarrow & \mathcal{M}_k(C^\infty(M)) \\ & (X_1, \ldots, X_k) & \longrightarrow & (\eta_i(X_j)), \end{array}$$

the solutions of (6) are given by $(X_1, \ldots, X_k) + (\ker \Omega^\sharp \cap \ker \eta^\sharp)$, where $(X_1, \ldots, X_k)$ is a particular solution.

Definition 4.2 Any k-vector field $(X_1, \ldots, X_k)$ on M such that

$$\eta_i(X_j) = \delta_{ij},$$
$$\Omega^\sharp(X_1, \ldots, X_k) = dH + \sum_{i=1}^{k}(1 - \xi_i(H))\eta_i,$$

for all $1 \leq i, j \leq k$, will be called an *evolution k-vector field* on M associated with the hamiltonian function H.

Let $(X_1, \ldots, X_k)$ be an evolution k-vector field associated to H, and assume that it is integrable. Let

$$\begin{array}{rccl} \sigma : & \mathbb{R}^k & \longrightarrow & M \\ & (t^i) & \longrightarrow & (\sigma^j(t^i), \sigma^\alpha(t^i), \sigma^j_\alpha(t^i)), \end{array}$$

be an integral section of $(X_1, \ldots, X_k)$; then, we have

$$\frac{\partial \sigma^j}{\partial t^i} = \delta_{ij}, \qquad \frac{\partial \sigma^\alpha}{\partial t^i} = (X_i)^\alpha, \qquad \frac{\partial \sigma^j_\alpha}{\partial t^i} = (X_i)^j_\alpha,$$

for all $1 \leq i, j \leq k$ and $1 \leq \alpha \leq n$. Therefore, Equations (6) give

$$\frac{\partial H}{\partial x^\alpha} = -\sum_{i=1}^{k} \frac{\partial \sigma^i_\alpha}{\partial t^i},$$
$$\frac{\partial H}{\partial x^i_\alpha} = \frac{\partial \sigma^\alpha}{\partial t^i},$$

with $1 \leq i \leq k$ and $1 \leq \alpha \leq n$, which are the field equations for H.

Remark 4.3 Let $(X_1, \ldots, X_k)$ be an evolution k-vector field. Since $\eta_i(X_j) = \delta_{ij}$, it follows that the vector fields $X_1, \ldots, X_k$ on M are linearly independent.

5 Lagrangian systems on k-cosymplectic manifolds

In this section we will apply the precedent results to give a geometrical description for the generalized Euler-Lagrange equations.

A Lagrangian $L = L(t^i, q^\alpha, v^\alpha_i)$, $1 \leq i \leq k, 1 \leq \alpha \leq n$ can be interpreted as a function $L : J^1(\mathbb{R}^k, Q) \longrightarrow \mathbb{R}$, defined on the vector bundle $J^1(\mathbb{R}, Q) \cong \mathbb{R}^k \times T^1_k Q$, where Q has dimension n. Indeed, $(t^i, q^\alpha, v^\alpha_i)$ are bundle coordinates on $J^1(\mathbb{R}^k, Q)$.

Given L, one constructs the Legendre transformation

$$\mathcal{F}L : \mathbb{R}^k \times T^1_k Q \longrightarrow \mathbb{R}^k \times (T^1_k)^* Q$$

locally defined by

$$(t^i, q^\alpha, v^\alpha_i) \longrightarrow (t^i, q^\alpha, \frac{\partial L}{\partial v^\alpha_i}).$$

Definition 5.1 A Lagrangian function $L : \mathbb{R}^k \times T^1_k Q \longrightarrow \mathbb{R}$ is said to be *regular* (resp. *hiperregular*) if the corresponding Legendre mapping $\mathcal{F}L$ is a local (resp. global) diffeomorphism.

Given a Lagrangian $L : \mathbb{R}^k \times T^1_k Q \longrightarrow \mathbb{R}$ we put

$$(\omega_L)_i = \mathcal{F}L^*(\omega_0)_i, \ 1 \leq i \leq k.$$

We obtain the following characterization of the regularity of a Lagrangian L:

Proposition 5.2 [24] *$L : \mathbb{R}^k \times T^1_k Q \longrightarrow \mathbb{R}$ is a regular Lagrangian if and only if $(dt_i, (\omega_L)_i, W_0)$ is a k-cosymplectic structure on $\mathbb{R}^k \times T^1_k Q$, where W_0 is the vertical distribution corresponding to the canonical fibration $\mathbb{R}^k \times T^1_k Q \longrightarrow \mathbb{R}^k \times Q$.*

Assume that $L : \mathbb{R}^k \times T^1_k Q \longrightarrow \mathbb{R}$ is regular, and let $(\xi_L)_1, \ldots, (\xi_L)_k$ be the Reeb vector fields associated with the k-cosymplectic structure $(dt_i, (\omega_L)_i, W_0)$. We denote by

$$\Omega^\sharp_L : T^1_k(\mathbb{R}^k \times T^1_k Q) \longrightarrow T^*(\mathbb{R}^k \times T^1_k Q)$$

the corresponding morphism, and we consider the following equations:

$$(7) \qquad \begin{aligned} &\eta_i(X_j) = \delta_{ij}, \\ &\Omega^\sharp_L(X_1, \ldots, X_k) = dE_L + \sum_{i=1}^k (1 - (\xi_L)_i(E_L))\eta_i, \end{aligned}$$

where $E_L = C(L) - L$, being C the canonical vector field of the vector bundle $\mathbb{R}^k \times T^1_k Q \longrightarrow \mathbb{R}^k \times Q$.

We deduce that

$$(8) \qquad X_i(t, q^\alpha, v^\alpha_i) = \frac{\partial}{\partial t^i} + v^\alpha_i \frac{\partial}{\partial q^\alpha} + (X_i)^\alpha_j \frac{\partial}{\partial v^\alpha_j},$$

where the functions $(X_i)^\alpha_j$ satisfy the following equations:

$$(9) \quad \sum_i \frac{\partial^2 L}{\partial t^i \partial v^\alpha_i} + \sum_{i,\beta} (v^\beta_i \frac{\partial^2 L}{\partial q^\beta \partial v^\alpha_i} + \sum_j (X_i)^\beta_j \frac{\partial^2 L}{\partial v^\beta_j \partial v^\alpha_i}) = \frac{\partial L}{\partial q^\alpha} \quad 1 \leq \alpha \leq n.$$

Since L is regular, these equations leads us to construct local solutions of (7) in a neigborhood of each point in $\mathbb{R}^k \times T^1_k Q$. Using a partition of the unity one can easily obtain a global solution of (7).

Now, let $(X_1, \ldots, X_k)$ be a solution of (7), that is, an evolution k–vector field associated to E_L, and let us assume that it is integrable. If

$$\begin{aligned} \sigma : \ \mathbb{R}^k &\longrightarrow \mathbb{R}^k \times T^1_k Q \\ (t^i) &\longrightarrow (\sigma^j(t^i), \sigma^\alpha(t^i), \sigma^\alpha_j(t^i)), \end{aligned}$$

is an integrable section, from Definition 3.2 and (8) we deduce that

$$\frac{\partial \sigma^j}{\partial t^i} = \delta_{ij}, \qquad \frac{\partial \sigma^\alpha}{\partial t^i} = v_i^\alpha, \qquad \frac{\partial^2 \sigma^\alpha}{\partial t^i \partial t^j} = (X_i)_j^\alpha.$$

Replacing in (9) we obtain

$$(10) \quad \sum_i \frac{\partial^2 L}{\partial t^i \partial v_i^\alpha} + \sum_{i,\beta} \left(\frac{\partial \sigma^\beta}{\partial t^i} \frac{\partial^2 L}{\partial v^\beta \partial v_i^\alpha} + \sum_j \frac{\partial^2 \sigma^\beta}{\partial t^i \partial t^j} \frac{\partial^2 L}{\partial v_j^\beta \partial v_i^\alpha} \right) = \frac{\partial L}{\partial v^\alpha}, \quad 1 \leq \alpha \leq n$$

Therefore, the projection of σ onto Q is a solution of the generalized Euler-Lagrange equations

$$(11) \qquad \sum_{i=1}^k \frac{\partial}{\partial t^i} \left(\frac{\partial L}{\partial v_i^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} = 0, \qquad v_i^\alpha = \frac{\partial q^\alpha}{\partial t^i}.$$

Therefore, (7) may be considered as an geometric version of the Euler-Lagrange field equations.

Example 5.3 Let us consider the equation of a scalar field ψ (gravitacional field, for instance) [3]:

$$\sqrt{-g} F'(\psi) = \sqrt{-g} m^2 \psi - \sum_{i,j} \frac{\partial^2}{\partial t^i \partial t^j} (\psi g^{ij}),$$

where m is the mass of the particle, g is a metric of signature $(-+++)$ and $F(\psi)$ is a scalar function such that $F(\psi) - (1/2)\, m^2 \psi^2$ is the potential energy of the particle.

We shall use the above formalism to obtain an geometric version for these equations. To do this, we consider the manifold $\mathbb{R}^4 \times T_4^1 \mathbb{R}$ with coordinates $(t^i, q, v_i; 1 \leq i \leq 4)$ and the Lagrangian $L(t^i, q, v_i)$ given by

$$L = \sqrt{-g} (F(q) - \frac{1}{2} m^2 q^2 + \frac{1}{2} g^{ij} v_i v_j).$$

Since L is regular we have a 4-cosymplectic structure $(\eta_i, (\omega_L)_i, W_0; 1 \leq i \leq 4)$ on $\mathbb{R}^4 \times T_4^1 \mathbb{R}$ determined by L. Then we consider the equations:

$$(12) \qquad \begin{aligned} &\eta^i(X_j) = \delta_{ij}, \\ &\Omega_L^\sharp(X_1, X_2, X_3, X_4) = dE_L + \sum_{i=1}^4 (1 - (\xi_L)_i(E_L))\eta_i\,, \end{aligned}$$

where $((\xi_L)_i; 1 \leq i \leq 4)$ are the corresponding Reeb vector fields and (X_1, X_2, X_3, X_4) is a 4-vector field on $\mathbb{R}^4 \times T_4^1 \mathbb{R}$. Let $\sigma : \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \times T_4^1 \mathbb{R}$, $\sigma(t) = (t, \psi(t), \psi_i(t))$, be an integral section of (X_1, X_2, X_3, X_4). Then $\psi(t)$ is a solution of the scalar field equation.

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ISOTHERMIC SURFACES IN EUCLIDEAN SPACE

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Abstract.– The aim of this paper is to give a brief survey of some aspects of the theory of isothermic surfaces in Euclidean space. We shall be particularly concerned with the Christoffel transforms and with the deformation problem of surfaces with respect to the pseudogroup of all orientation-preserving conformal transformations.

1 Introduction

The purpose of the present paper is to give an introduction to the theory of isothermic surfaces. The subject was considered in the latter part of the 19th century as is evidenced by the works of Christoffel, Darboux, Bianchi, Calapso, Blaschke, Thomsen and Vessiot [Chr], [D1], [D2], [B1], [B2], [B3], [B4], [Ca1], [Ca2], [Bl], [T], [V1], [V2]. It was A.Bobenko [Bob] in 1992 who considered again isothermic surfaces in the contest of integrable systems (see also [Ci1]-[Cie6] and [CGS]). In recent years,new ideas, such as the notion of a curved flat introduced by Ferus and Pedit [FP1], [FP2], have become available and have led to a rapid development of the subject. Quite recently, isothermic surfaces and the quaternionic calculus [Ka], [W], [HJ2] have been used by Kamberov, Pedit and Pinkall [KPP] to show the existence of new Bonnet pairs (see also the original paper of O. Bonnet [Bon] and references [Car],[Che],[S]). Since we have a restricted amount of space, we shall consider only two topics: Christoffel transforms and deformation of surfaces in conformal geometry. In the §2 we start with the definitions and with a description of the basic examples. The definition of isothermic surface that we use in this paper is a generalization of the classical one and it is essentially due to Kamberov, Pedit, Pinkall and Hertrich-Jeromin. In §3 we examine Christoffel transforms of isothermic surfaces and, in the last section, we give a brief survey of some results of the work done by the

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author relating the geometry of isothermic surfaces with the general theory of the deformation of submanifolds in homogenous spaces. We refer to [M] for the complete details, we also refer to [G], [J1] and [J2] for more information on the theory of the deformation.

2 Isothermic Surfaces

2.1 Basic Definitions

Let (M,g) be a 3-dimensional oriented Riemannian manifold and S be an oriented surface and suppose we are given a smooth immersion $f : S \to (M,g)$. We denote by I the first fundamental form and by II the second fundamental form. It follows from the existence theorem of isothermal coördinates [CHW] that S possesses a unique complex structure compatible with the given orientation and with the conformal structure defined by the first fundamental form. The defining property of this complex structure is that, given any positive orthonormal coframig (α^1, α^2), defined on an open subset $U \subset S$, the complex-valued exterior differential one form $\alpha^1 + i\alpha^2$ is of type $(1,0)$. We then regard S as a Riemann surfaces with this complex structure and we will denote by $\overline{S}$ the same surface but with the opposite complex structure. We decompose the second fundamental form into the $(2,0)$, $(1,1)$ and $(0,2)$ parts :

$$II = II^{(2,0)} + II^{(1,1)} + II^{(0,2)}. \tag{2.1}$$

If $\zeta = \alpha^1 + i\alpha^2$ is a complex coframing of type $(1,0)$ on the open subset $U \subset S$ and if

$$II = h_{11}(\alpha^1)^2 + 2h_{12}\alpha^1\alpha^2 + h_{22}(\alpha^2)^2, \tag{2.2}$$

we then have

$$II^{(2,0)} = \frac{1}{4}[(h_{11} - h_{22}) - 2ih_{12}](\zeta)^2, \tag{2.3}$$

$$II^{(1,1)} = \frac{1}{2}(h_{11} + h_{22})\zeta\overline{\zeta}. \tag{2.4}$$

The $(0,2)$ part $II^{(0,2)}$ is the complex coniugate of $II^{(2,0)}$.

Definition 2.1 The (2,0) part $II^{(2,0)}$ of the second fundamental form is called the Hopf differential of the isometric immersion $f : (S, I) \to (M,g)$.

Definition 2.2 The oriented surface $f : S \to (M,g)$ is called isothermic if there exist a non-zero holomorphic quadratic differential Q and a smooth real-valued function $m : S \to \mathbb{R}$ such that $II^{(2,0)} = mQ$. We say that Q is a polarization of (S, f).

Remark 2.1 If the isothermic immersion is not totally umbilic, then the polarization is uniquely determined, up to a non-zero constant multiple. If the immersion is totally umbilic and if there exist a non-zero holomorphic quadratic differential Q on S, then (S, f) is isothermic and Q is a polarization.

Definition 2.3 Let (S, f) be an isothermic surface in (M, g) and let Q be a polarization. Denote by $\{Q\}$ the zero set of Q and let $S_o = S - \{Q\}$. A complex parameter $z : U \subset S_o \to \mathbb{C}$ is said to be a principal isothermic chart if $Q|_U = (dz)^2$.

Remark 2.2 Around each point of S_o there exists a principal isothermic chart (U, z). If we write $z = x + iy$, where x and y are real functions, then

$$(2.5) \qquad \begin{aligned} I &= E(dx)^2 + G(dy)^2, \\ II &= e(dx)^2 + g(dy)^2. \end{aligned}$$

This means that (x, y) is a principal coordinate system which is isothermic for the first fundamental form.

Given a polarization Q we write $II^{(2,0)} = mQ$, where m is a real valued function. If (U, z) is any principal isothermic chart, then $I|_U = \lambda^2 dz\overline{dz}$, where λ is a positive function. Since $2m\lambda^{-1}$ does not depend on the choice of the principal isothermic chart, there exist a global smooth function $\Phi : X_o \to \mathbb{R}$ such that $\Phi|_U = 2m\lambda^{-1}$.

Definition 2.4 The real-valued function Φ is called the Calapso potential of the isothermic surface $f : S \to (M, g)$ with respect to the polarization Q.

Remark 2.3 If we replace Q by another polarization $Q' = cQ$, $c \in \mathbb{R}$, then $\Phi' = c\Phi$. It follows from the definition that the Calapso potential of a totally umbilical isothermic surface vanishes identically.

2.2 Conformal Invariance

Replacing g by $g_r = r^2 g$,where r is a positive function, then the first and second fundamental forms I_r and II_r of $f : S \to (M, g_r)$ are

$$(2.6) \qquad I_r = (f^*(r))^2 I_r,$$

$$(2.7) \qquad II_r = f^*(r) II + sI.$$

The complex structure on S is compatible with I_r and

$$(2.8) \qquad II_r^{(2,0)} = f^*(r) II^{(2,0)}.$$

This implies that, if $f : S \to (M, g)$ is isothermic and if Q is a polarization, then $f : S \to (M, g_r)$ is isothermic and it is polarized by the same quadratic differential. Thus, the notion of isothermic surface is invariant for the action of the conformal group of (M, g). This is relevant if (M, g) has a large group of conformal transformations, as in the case of a 3-dimensional Riemannian space form. We shall breafly discuss the case of the Euclidean space and we will refer the reader to [Ko] and [Ce] for more details.

We embed $\mathbb{R}^3$ into the 4-dimensional real projective space $\mathbb{P}^4$

$$x = (x^1, x^2, x^3) \to [(1, x^1, x^2, x^3, \frac{1}{2}|x|^2)].$$

If $y^0, \dots, y^4$ are homogeneous coordinates on $\mathbb{P}^4$,then the image of $\mathbb{R}^3$ in $\mathbb{P}^4$ is the quadric

$$-2y^0y^4 + (y^1)^2 + (y^2)^2 + (y^3)^2 = 0$$

minus the "point at infinity" $[(1, 0, 0, 0, 0)]$. Let G be the connected component of the identity of the pseudo-orthogonal Lie group of linear transformations leaving the quadratic form $-2y^0y^4+(y^1)^2+(y^2)^2+(y^3)^2$ invariant. Then G can be viewed as a transitive pseudogroup of transformations acting on $\mathbb{R}^3$. Since it is generated by rigid motions, dilations and inversions, G is the conformal pseudogroup of the Euclidean space. We can state the following

Proposition 2.1 *If $f : S \to \mathbb{R}^3$ is an isothermic immersion and if $A \in G$ is an orientation-preserving conformal transformation, then $L_A \circ f : S \to \mathbb{R}^3$ is an isothermic immersion.*

Consider the embedding

$$F : x \in \mathbb{R}^3 \to \left(\frac{|x|^2 - 1}{|x|^2 + 1}, \frac{2x}{|x|^2 + 1} \right) \in S^3. \tag{2.9}$$

This is the inverse of the stereographic projection from the north pole $N = (1, 0, 0, 0) \in S^3$. Denote by g_{S^3} and by $g_{\mathbb{R}^3}$ the standard Riemannian metrics on S^3 and on $\mathbb{R}^3$ respectively. Then

$$F^*(g_{S^3}) = \frac{4}{(1 + |x|^2)^2} g_{\mathbb{R}^3}.$$

At this point we can assert the following

Proposition 2.2 *Suppose that $f : S \to S^3$ is an isothermic immersion such that $N \notin f(S)$. Then $F^{-1} of : S \to \mathbb{R}^3$ is an isothermic surface in Euclidean space.*

Let $\mathbb{H}^3 \subset \mathbb{R}^4$ be the hyperbolic space

$$(y^1)^2 + (y^2)^2 + (y^3)^2 - (y^4)^2 = -1, \quad 1 \leq y^4$$

endowed with the Riemannian metric

$$g_{\mathbb{H}^3} = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 - (dy^4)^2$$

of constant sectional curvature -1. We set $B = \{x \in \mathbb{R}^3 / |x|^2 < 2\}$ and we consider the diffeomorphism

$$G : x \in B \to \left(\frac{2\sqrt{2}x}{2-|x|^2}, \frac{2+|x|^2}{2-|x|^2} \right) \in \mathbb{H}^3. \tag{2.10}$$

We then have

$$G^*(g_{\mathbb{H}^3}) = \frac{8}{2-|x|^2} g_{\mathbb{R}^3}.$$

Proposition 2.3 *Let $f : S \to \mathbb{H}^3$ be an isothermic surface in Hyperbolic space. Then $G^{-1} o f : S \to B \subset \mathbb{R}^3$ is an isothermic surface in Euclidean space.*

2.3 Examples

We give a brief review of the fundamental examples of isothermic surfaces in Euclidean space.

2.3.1 Surfaces of revolution

Let γ be a smooth parametrized curve in the (x, z) plane of $\mathbb{R}^3$

$$\gamma(t) = {}^t(a(t), 0, b(t)),$$

where $t \in (-\varepsilon, \varepsilon)$ and $a(t) > 0$. The surface obtained by revolving γ around the z-axis is parametrized by $f : (-\varepsilon, \varepsilon) \times \mathbb{R} \to \mathbb{R}^3$

$$f(x, y) = {}^t(a(x)\cos(y), a(x)\sin(y), b(x)).$$

We then have

$$I = w^2(dx)^2 + a^2(dy)^2,$$

and

$$II = \frac{(a'g'' - a''b')}{w}(dx)^2 + \frac{b'a}{w}(dy)^2,$$

where $w^2 = (a')^2 + (b')^2$.

The complex parameter $z : (-\varepsilon'\varepsilon) \times \mathbb{R} \to \mathbb{C}$ is defined by $dz = a^{-1}wdx + idy$ and the Hopf differential is given by

$$II^{(2,0)} = a^2\left(\frac{a'b'' - a''b'}{w^3} - \frac{b'}{wa}\right)(dz)^2.$$

This shows that surfaces of revolution are isothermic and that $(dz)^2$ is a polarization. Note that any other surface obtained from a surface of revolution by a Möbius transformation is isothermic.

2.3.2 Cones

Let $\gamma : (-\varepsilon, \varepsilon) \to S^2$ be a smooth curve in the unit sphere $S^2 \subset \mathbb{R}^3$ parametrized by the arclength and let S be the positive cone with vertex in the origin and directrix curve γ. The surface is parametrized by

$$f : (s,r) \in (-\varepsilon,\varepsilon) \times \mathbb{R}^+ \to r\gamma(s) \in \mathbb{R}^3.$$

If we denote by $k(s)$ the curvature of γ, we get

$$I = (dr)^2 + r^2(ds)^2$$

and

$$II = rk(ds)^2.$$

Then, $z = log(r) + is$ is a complex parameter and the Hopf differential is

$$II^{(2,0)} = -\frac{kr}{4}(dz)^2.$$

This shows that S is an isothermic and that $(dz)^2$ is a polarization.

2.3.3 Cylinders

Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a curve in the (x,y) plane and assume that γ is parametrized by the arclength. Let $S \subset \mathbb{R}^3$ be the cylinder with directrix curve γ and generating lines perpendicular to the (x,y) plane. Then, S is parametrized by

$$f : (s,t) \in (-\varepsilon,\varepsilon) \times \mathbb{R} \to \gamma(s) + t\varepsilon_3 \in \mathbb{R}^3.$$

Thus, $z = s - it$ is a complex parameter and $II^{(2,0)} = k(dz)^2$, where k is the curvature of the directrix curve. We then conclude that S is an isothermic surface polarized by $(dz)^2$.

2.3.4 Surfaces with constant mean curvature in 3-dimensional space forms

We will denote by M_ε, $\varepsilon = -1, 0, 1$ the simply connected space of constant sectional curvature $\varepsilon\Big(M_{-1} = \mathbb{H}^3,\ M_0 = \mathbb{R}^3,\ M_1 = S^3\Big)$ and we denote by G_ε the group of orientation-preserving isometries of $M_\varepsilon\Big(G_{-1} = SO(3,1),\ G_1 = SO(4),\ G_0 = \mathbb{E}(3)\Big)$.

Proposition 2.4 [Ho] *The Hopf differential of an oriented surface $f : S \to M_\varepsilon$ is holomorphic if and only if (S, f) has constant mean curvature.*

Proof. Since the arguments are local we will assume the existence of a global first order frame field $A : S \to G_\varepsilon$ and we will denote by $\eta = A^{-1}dA$ the Maurer-Cartan form of this frame field. We then have

$$\eta = \begin{pmatrix} 0 & -\varepsilon\eta_0^1 & -\varepsilon\eta_0^2 & 0 \\ \eta_0^1 & 0 & \eta_1^2 & -\eta_1^3 \\ \eta_0^2 & \eta_1^2 & 0 & -\eta_2^3 \\ 0 & \eta_1^3 & \eta_2^3 & 0 \end{pmatrix},$$

where (η_0^1, η_0^2) is a positive-oriented orthonormal coframing. Thus $\zeta = \eta_0^1 + i\eta_0^2$ is a nowhere vanishing form of type $(1,0)$. We then write

$$\eta_1^3 = h_{11}\eta_0^1 + h_{12}\eta_0^2,$$
$$\eta_2^3 = h_{12}\eta_0^1 + h_{22}\eta_0^2,$$

so that

$$II^{(2,0)} = \frac{1}{4}[(h_{11} - h_{22}) - 2ih_{12}](\zeta)^2.$$

Set $C = (h_{11} - h_{22}) - 2ih_{12}$ and observe that (cfr.[Br]) $II^{(2,0)}$ is holomorphic quadratic differential if and only if

$$(dC - 2iC\eta_1^2) \wedge \zeta = 0.$$

We have

$$\alpha = \eta_1^3 - i\eta_2^3 = \frac{1}{2}C\zeta + H\overline{\zeta},$$

where H denote the mean curvature. By the structure equations we get $d\zeta = -i\eta_1^2 \wedge \zeta$ and $d\alpha = i\eta_1^2 \wedge \alpha$. We then obtain

$$\frac{1}{2}(dC - 2iC\eta_1^2) \wedge \zeta + dH \wedge \overline{\zeta} = 0.$$

Therefore $dH = 0$ if and only if $\frac{1}{2}(dC - 2iC\eta_1^2) \wedge \zeta = 0$. □

Proposition 2.5 *Every non totally umbilic constant mean curvature surface in a 3-dimensional space form is isothermic and its Hopf differential is a polarization.*

From Proposition 2.2 and Proposition 2.3 we get

Corollary 2.1 *Consider a constant mean curvature surface $f : S \to S^3$ and assume that (S, f) is not totally umbilic and that $N \notin f(S)$. Then $F^{-1} \circ f : S \to \mathbb{R}^3$ is an isothermic surface polarized by the Hopf differential of (S, f).*

and

Corollary 2.2 *Let $f : S \to \mathbb{H}^3$ be a constant mean curvature surface and assume that (S, f) is not totally umbilic. Then $G^{-1} of : S \to B \subset \mathbb{R}^3$ is an isothermic surface polarized by the Hopf differential of (S, f).*

3 Christoffel transforms of isothermic surfaces in Euclidean space

3.1 Christoffel transforms

Consider an isothermic surface $f : S \to \mathbb{R}^3$ endowed with a polarization Q and let $(f; e_1, e_2, e_3)$ be any field of orthonormal frames so that e_3 is the unit normal vector compatible with the orientation on S and (e_1, e_2) is a positive oriented orthonormal frame field along the surface. We denote by η_0^i, η_j^i the Maurer-Cartan forms of this framing. We have then

$$df = \eta_0^1 e_1 + \eta_0^2 e_2,$$

and

$$\begin{aligned} de_1 &= +\eta_1^2 e_2 + \eta_1^3 e_3, \\ de_2 &= -\eta_1^2 e_1 + \eta_2^3 e_3, \\ de_3 &= -\eta_1^3 e_1 - \eta_2^3 e_2. \end{aligned} \tag{3.1}$$

Thus, (η_0^2, η_0^2) is a positive-oriented orthonormal coframing and $\zeta = \eta_0^1 + i\eta_0^2$ is a nowhere vanishing $(1,0)$ form on U. We set $Q = w(\zeta)^2$, where $w : U \to \mathbb{C}$ is a complex valued function such that

$$(dw - 2iw\eta_1^2) \wedge \zeta = 0. \tag{3.2}$$

We set $\tilde{\zeta} = \overline{w\zeta}$ and we define real-valued 1-forms $\tilde{\eta}_0^1$ and $\tilde{\eta}_0^2$ by $\tilde{\zeta} = \tilde{\eta}_0^1 + i\tilde{\eta}_0^2$. Since $\tilde{\eta}_0^1 e_1 + \tilde{\eta}_0^2 e_2$ is independent on the choice of the first order framing, there exists a $\mathbb{R}^3$-valued 1-form $\tilde{\phi}$ on S such that

$$\tilde{\phi}|_U = \tilde{\eta}_0^1 e_1 + \tilde{\eta}_0^2 e_2.$$

Definition 3.1 We say that $\tilde{\phi}$ is the infinitesimal Christoffel transform of the isothermic surface (S, f) with respect to the polarization Q.

Remark 3.1 If we replace Q by $cQ, c \in \mathbb{R}$, then the infinitesimal Christoffel transform of the isothermic surface with respect to cQ is $c\tilde{\phi}$.

Proposition 3.1 *The infinitesimal Christoffel transform of an isothermic surface is closed.*

Proof. Without loss of generality we may assume the existence of a global first order frame field $(f; e_1, e_2, e_3)$ and that the polarization Q is nowhere vanishing. We then have $Q = w(\zeta)^2$ and $w = \rho e^{i\theta}$, where ρ and θ are real-valued functions. By the structure equations we get $d\zeta = -i\eta_1^2 \wedge \zeta$ and, by (3.2), we obtain

$$\begin{aligned} d\tilde{\eta}_0^1 &= +\eta_1^2 \wedge \tilde{\eta}_0^2, \\ d\tilde{\eta}_0^2 &= -\eta_1^2 \wedge \tilde{\eta}_0^1. \end{aligned} \tag{3.3}$$

We put

$$\begin{aligned}\eta_1^3 &= h_{11}\eta_0^1 + h_{12}\eta_0^2,\\ \eta_2^3 &= h_{12}\eta_0^1 + h_{22}\eta_0^2,\end{aligned}$$

so that

$$4II^{(2,0)} = [(h_{11} - h_{22}) - 2ih_{12}](\zeta)^2.$$

Since the surface is isothermic, there exists $r : S \longrightarrow \mathbb{R}$ such that

$$[(h_{11} - h_{22}) - 2ih_{12}] = rw.$$

Then

$$h_{11} - h_{22} = r\rho\cos\theta,$$

and

$$2h_{12} = r\rho\sin\theta.$$

On the other hand we have

$$\begin{aligned}\tilde{\eta}_0^1 &= \rho\cos\theta\eta_0^1 + \rho\sin\theta\eta_0^2,\\ \tilde{\eta}_0^2 &= \rho\sin\theta\eta_0^1 - \rho\cos\theta\eta_0^2.\end{aligned}$$

We then obtain

$$\tilde{\eta}_0^1 \wedge \eta_1^3 + \tilde{\eta}_0^2 \wedge \eta_2^3 = 0. \tag{3.4}$$

Combining (3.1) with (3.3) and (3.4) we get the required result. □

Definition 3.2 Let $f : S \longrightarrow \mathbb{R}^3$ be an isothermic surface endowed with a polarization Q. We say that $\tilde{f} : S \longrightarrow \mathbb{R}^3$ is a Christoffel transform of (S, f, Q) if $d\tilde{f} = \tilde{\phi}$.

When we consider the Christoffel transform it is convenient to put on S the opposite complex structure and hence we adopt the notation $\tilde{f} : \overline{S} \longrightarrow \mathbb{R}^3$. If we assume S simply connected, then the Christoffel transform does exist and it is uniquely determined (up to translations) by the polarization. The polarization Q, the Hopf differentials, the mean curvatures of $f : S_o \longrightarrow \mathbb{R}^3$ and $\tilde{f} : S_o \longrightarrow \mathbb{R}^3$ are related by

$$\begin{aligned}2II^{(2,0)} &= \tilde{H}Q,\\ 2\tilde{II}^{(2,0)} &= H\overline{Q}.\end{aligned} \tag{3.5}$$

This implies

Proposition 3.2 *The Christoffel transform $\tilde{f} : \overline{S} \longrightarrow \mathbb{R}^3$ of a polarized isothermic immersion $f : S \longrightarrow \mathbb{R}^3$ is isothermic and $\overline{Q}$ is a polarization.*

3.2 Examples

3.2.1 Christoffel transforms of minimal and totally umbilical surfaces

Proposition 3.3 *Let $f : S \to \mathbb{R}^3$ be a minimal surface and assume that S is simply connected and that $II^{(2,0)} \neq 0$. The Christoffel transform $\tilde{f} : \overline{S} \to \mathbb{R}^3$ with respect to the polarization $Q = -2II^{(2,0)}$ is (up to translations) the Gauss map of (S, f) and the orientation of $\overline{S}$ is given by the outward unit normals to S^2.*

Proof. Let $(f; e_i)$ be a first order frame field along (S, f). Then the polarization Q is given by

$$Q = -\frac{1}{2}\Big((h_{11} - h_{22}) - ih_{12}\Big)(\zeta)^2 = -\Big(h_{11} - ih_{12}\Big)\Big(\zeta\Big)^2.$$

We thus have $\tilde{\eta}_0^1 = -\eta_1^3$ and $\tilde{\eta}_0^2 = -\eta_2^3$. This implies

$$\tilde{\phi} = -\eta_1^3 e_1 - \eta_2^3 e_2 = de_3.$$

Which gives the required result. □

Proposition 3.4 *Let $f : S \to \mathbb{R}^3$ be a simply connected totally umbilical immersion and let Q be a holomorphic quadratic differential. Then, the Christoffel transform $\tilde{f} : \overline{S} \to \mathbb{R}^3$ is a conformal, branched minimal immersion with Hopf differential $-\frac{1}{2}\overline{Q}$ and (S, f) is the Gauss map of $(\overline{S}_o, \tilde{f})$.*

Proof. Let $(f; e_i)$ be a first order frame field along f and let η_0^i, η_j^i be the Maurer-Cartan forms. If $\zeta = \eta_0^1 + i\eta_0^2$ and if $Q = w(\zeta)^2$, then

$$d\tilde{f} = \tilde{\eta}_0^1 e_1 + \tilde{\eta}_0^2 e_2, \tag{3.6}$$

where $\tilde{\eta}_0^1 + i\tilde{\eta}_0^2 = \overline{g}(\eta_0^1 - i\eta_0^2)$. On S_o we can write $w = re^{i\theta}$ for some real-valued functions r and θ. Thus, $(\tilde{f}; e_i)$ is a first order frame field along $(\overline{S}, \tilde{f})$ and the Maurer-Cartan forms of this framing are given by

$$\begin{aligned} \eta_0^1 &= +\frac{\cos\theta}{r}\tilde{\eta}_0^1 - \frac{\sin\theta}{r}\tilde{\eta}_0^2, \\ \eta_0^2 &= -\frac{\sin\theta}{r}\tilde{\eta}_0^1 - \frac{\cos\theta}{r}\tilde{\eta}_0^2 \end{aligned}$$

and by

$$\tilde{\eta}_0^3 = 0, \quad \tilde{\eta}_1^3 = -\eta_0^1, \quad \tilde{\eta}_2^3 = -\eta_0^2.$$

We then deduce that

$$\begin{aligned} \tilde{\eta}_1^3 &= -\frac{\cos\theta}{r}\tilde{\eta}_0^1 + \frac{\sin\theta}{r}\tilde{\eta}_0^2, \\ \tilde{\eta}_2^3 &= +\frac{\sin\theta}{r}\tilde{\eta}_0^1 + \frac{\cos\theta}{r}\tilde{\eta}_0^2. \end{aligned}$$

Then $(\overline{S}_o, \tilde{f})$ is a minimal immersion with Hopf differential

$$\tilde{II}^{(2,0)} = -\frac{1}{2}\left(\frac{\cos\theta}{r} + i\frac{\sin\theta}{r}\right)(\tilde{\eta}_0^1 + i\tilde{\eta}_0^2)^2 = -\frac{1}{2}\overline{w}(\zeta)^2 = -\frac{1}{2}\overline{Q}.$$

We also have $de_3 = df$ and this implies that f is the Gauss map of $(\overline{S}, \tilde{f})$. □

Remark 3.2 Consider the parametric equation of the unit sphere

$$(3.7)\quad x = (u,v) \in \mathbb{R}^2 \to \frac{1}{1+|x|^2}(2u, 2v, |x|^2 - 1) \in S^2 - \{(0,0,1)\} \subset \mathbb{R}^3.$$

We put on S^2 the orientation determined by the outward unit normals so that the induced complex structure is defined by the complex parameter $\overline{z} = u - iv$. A holomorphic quadratic differential on the surface is then given by $\overline{F}(d\overline{z})^2$, where F is a holomorphic funtion of the complex variable $z = u + iv$. Then, the Christoffel transform of (3.7) with respect to the polarization $\overline{F}(d\overline{z})^2$ is the Enneper-Weiesrtrass representation of a minimal surface determined by the holomorphic function F, that is

$$\tilde{f}(z) = \frac{1}{4}\Re\left(\int F(z)(1-z^2)dz, -i\int F(z)(1+z^2)dz, 2\int F(z)zdz\right).$$

3.2.2 Christoffel transform of constant mean curvature surfaces in Euclidean space

Let $f : S \to \mathbb{R}^3$ be surface of constant mean curvature $H \neq 0$,let $n : S \to S^2$ be the field of unit normals compatible with the orientation on S and assume that S is simply connected.

Proposition 3.5 *The Christoffel transform of (S, f) with respect to the polarization $Q = -\frac{4}{H}II^{(2,0)}$ is (up to translations) the parallel surface $\tilde{f} = f + \frac{1}{H}n$. Notice that $\tilde{f} : S_o \to \mathbb{R}^3$ is a surface of constant mean curvature $\tilde{H} = -H$.*

Proof. Since the arguments are local, we assume the existence of a principal first order frame field $(f; e_i)$, so that $e_3 = n$. We then have

$$Q = -\frac{h_{11} - h_{22}}{h_{11} + h_{22}}(\zeta)^2.$$

This implies that

$$\tilde{\eta}_0^1 = -\frac{h_{11} - h_{22}}{h_{11} + h22}\eta_0^1,$$

$$\tilde{\eta}_0^2 = +\frac{h_{11} - h_{22}}{h_{11} + h_{22}}\eta_0^2.$$

Therefore,the Christoffel transform of (S, f) satisfies

$$d\tilde{f} = -\frac{h_{11} - h_{22}}{h_{11} + h22}\eta_0^1 e_1 + \frac{h_{11} - h_{22}}{h_{11} + h_{22}}\eta_0^2 e_2 = d\left(f + \frac{1}{H}n\right).$$

This gives the required result. □

3.3 Conformal transformations by parallel planes

Let $f : S \to \mathbb{R}^3$ and $f' : S' \to \mathbb{R}^3$ be two oriented surfaces and let n and n' be the corresponding unit normal vector fields.

Definition 3.3 A conformal transformation by parallel planes is a conformal diffeomorphism $C : S \to S'$ such that $n(p) = n'[C(p)]$, for every $p \in S$.

Remark 3.3 If (S, f) is an isothermic surface endowed with a nowhere vanishing polarization Q, then the corresponding Christoffel transform $(\overline{S}, \tilde{f})$ is a conformal tansformation by parallel planes. In this case $C = id_S : S \to \overline{S}$ is orientation-reversing.

Remark 3.4 (S, f) admits a non trivial orientation-preserving conformal transform by parallel planes if and only if it is minimal [D1].

Proposition 3.6 *Let $C : (S, f) \to (S', f')$ be an orientation-reversing conformal transformation by parallel planes. Then, (S, f) is an isothermic surface and there exists a nowhere vanishing polarization Q such that $(\overline{S}, f'oC)$ is a Christoffel transform of (S, f) with respect to Q.*

Proof. Without loss of generality we may assume $S' = \overline{S}$ and $C = id_S$. Take a first order frame field $(f; e_i)$ along (S, f). Then, $(f'; e_i)$ is a first order framing along $(\overline{S}, f')$ and

$$df = \eta_0^1 e_1 + \eta_0^2 e_2,$$

$$df' = \tilde{\eta}_0^1 e_1 + \tilde{\eta}_0^2 e_2.$$

We have that $\zeta = \eta_0^1 + i\eta_0^2$ is a $(1,0)$ coframing on S and that $\tilde{\zeta} = \tilde{\eta}_0^1 + i\tilde{\eta}_0^2$ is a coframing of type $(0,1)$.Thus, there exist a nowhere vanishing complex-valued function w such taht $\tilde{\zeta} = \overline{w\zeta}$. Since the quadratic differential $w(\zeta)^2$ does not depend on the choice of the framing there exists a nowhere vanishing quadratic differential Q of type $(2,0)$ on S such that $Q|_U = w(\zeta)^2$, for every first order framing $(f; e_i)$. Using the structure equations it follows that Q is holomorphic and that $(\overline{S}, f')$ is a Christoffel transform of (S, f) with respect to Q. □

4 Conformal deformation of surfaces in Euclidean space

In this section we will state some results relating isothermic surfaces with the general theory of deformation of submanifolds in homogeneous spaces.

4.1 Deformation of submanifolds

Definition 4.1 Assume that the Lie group H acts as a pseudogroup of transformations on M and let $f : N \to M$ and $\tilde{f} : \tilde{N} \to M$ be two submanifolds of the same dimension. We say that (N, f) and $(\tilde{f}, \tilde{N})$ are kth order H-deformations each of the other if there exists a diffeomorphism $F : N \to \tilde{N}$ and a smooth map $B : N \to H$ such that, (N, f) and $(N, L_{B(p)}\tilde{f}oF)$ have the same k-th order jets at p, for every $p \in N$. We then say that $F : (N, f) \to \left(\tilde{N}, \tilde{f}\right)$ is a k-th order *H-deformation.* The deformation F is trivial if $B : N \to H$ can be chosen to be constant.

Definition 4.2 A submanifold (N, f) is *H-deformable* of order k if it admits non-trivial k-th order deformations with respect to H. Otherwise we say that (N, f) is H-rigid at order k.

4.1.1 Example

Two submanifolds of the same dimension in the Euclidean space $\mathbb{R}^n$ are first order deformations each of the other with respect to the action of the group of rigid motions if and only if they are isometric. Each submanifold in $\mathbb{R}^n$ is rigid at the second order.

4.2 Conformal deformation of surfaces in Euclidean space

On the Euclidean space $\mathbb{R}^3$ we consider the pseudogroup G of all orientation-preserving conformal transformations, this is a 10-dimensional connected Lie group isomorphic to the connected componenet of the identity of $SO(4,1)$ (see section 2.2). From now on we shall refer to deformation with respect to G as *conformal deformation.* The essential results are as follows (see ref.[M]):

Proposition 4.1 *Two oriented surfaces (S, f) and $(\tilde{S}, \tilde{f})$ in $\mathbb{R}^3$ are first order conformal deformations each of the other if and only if S and $\tilde{S}$ are biholomorphic each to the other and every biholomorphic map $S \to \tilde{S}$ is a first order deformation.*

Proposition 4.2 *Let S be a simply connected, oriented, 2-dimensional manifold and let $f : S \to \mathbb{R}^3$ be a smooth immersion without umbilical points. Then (S, f) possesses not trivial second order conformal deformations if and only if (S, f) is isothermic. Moreover, (S, f) and $(S, \tilde{f})$ are second order conformal deformations each of the other if and only if they are polarized by the same holomorphic quadratic differential and the corresponding Calapso potentials coincide.*

Remark 4.1 Given an umbilic free simply conected isothermic surface (S, f) in Euclidean space then the second order conformal deformations of the surface depend (up to reparametrizations and conformal transformations) on one arbitrary real parameter $m \in \mathbb{R}$.

Proposition 4.3 *Each oriented surface in Euclidean space is rigid with respect to third order conformal deformations.*

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THE GEOMETRY OF L-ISOTHERMIC SURFACES

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Abstract.– The subject of Laguerre invariants of surfaces in Euclidean space is considered. Some aspects of the theory of L-isothermic surfaces in relation with the problem of deformation are reviewed, and some examples are discussed.

1 Introduction

While the subject of conformal invariants of submanifolds has received much attention since the latter part of the 19th century, the subject of Lie and Laguerre invariants of submanifolds has been less studied. Considerable progress on the subject was made by Blaschke and Thomsen [Bl1], [Bl2], [Bl3], [Bl4] in the 1920s. In recent years, researchers such as Cecil, Chern, Pinkall, Thorbergsson et al., took up Lie geometry again in connection with the study of Dupin hypersurfaces [Ce]. We became interested in Laguerre geometry when studying the variational problem Laguerre invariant area element

$$\left(H^2 - K\right) K^{-1} dA,$$

where H and K are the mean and Gaussian curvatures of the immersion, and dA is the induced area element (cf. [MN2]). This is the analogue in Laguerre geometry of the so-called Willmore variational problem.

In this expository article, we review the basic concepts of surface theory in Laguerre geometry and report on the class of L-isothermic surfaces focusing in particular on their relation with the theory of deformation in homogeneous spaces. We shall discuss some examples such as surfaces with plane lines of curvature, molding surfaces and L-minimal surfaces with plane lines of curvature. For more information on the analogous circle of ideas in conformal geometry, we refer to the article of E. Musso in these proceedings.

The results presented here were obtained in collaboration with E. Musso. The detailed proofs will appear elsewhere.

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2 Generalized surfaces and Legendre surfaces in $T_1\mathbb{E}^3$

Let $\mathcal{F} \to \mathbb{E}^3$ be the $SO(3)$–bundle of oriented orthonormal frames of Euclidean space $\mathbb{E}^3$ with canonical forms θ^i, and Levi-Civita forms $\theta^\alpha_\beta = -\theta^\beta_\alpha$, $1 \le \alpha, \beta, \gamma \le 3$, defining a parallelization of $\mathcal{F}$. The structure equations are

$$d\theta^\alpha = -\theta^\alpha_\beta \wedge \theta^\beta, \quad d\theta^\alpha_\beta = -\theta^\alpha_\gamma \wedge \theta^\gamma_\beta.$$

Let $T_1\mathbb{E}^3 \cong \mathcal{F}/SO(2)$ be the unit tangent bundle of $\mathbb{E}^3$ and $\pi : \mathcal{F} \to T_1\mathbb{E}^3$ be the projection $\pi(p; e_1, e_2, e_3) = (p; e_1)$. The forms θ^1, $d\theta^1$, and $\theta^2 \wedge \theta^3$ descend to $T_1\mathbb{E}^3$.

If $f : M \mapsto \mathbb{E}^3$ is any immersed oriented surface, then there is a lift (the *Gauss lift*) $F : M \to T_1\mathbb{E}^3$ given by $F(p) = (f(p), n(p))$, where $n(p)$ is the oriented unit normal to f at p. If $(f; n, e_2, e_3) = (f; e) : \mathbb{E}^3 \to \mathcal{F}$ is a local Darboux frame field along f, then $F = \pi \circ (f; e)$. Thus

$$F^*\theta^1 = (f; e)^*\theta^1 = 0,$$
$$F^*(\theta^2 \wedge \theta^3) = (f; e)^*(\theta^2 \wedge \theta^3) = dA, \text{ area element of the induced metric.}$$

Conversely, an immersion $F : M \to T_1\mathbb{E}^3$ is the Gauss lift of an immersion $f : M \to \mathbb{E}^3$ if

$$F^*\theta^1 = 0, \quad F^*(\theta^2 \wedge \theta^3) \neq 0.$$

If F satisfies only $F^*\theta^1 = 0$, it is called a *generalized surface*, because the Euclidean projection f may have singularities.

On $T_1\mathbb{E}^3$,

$$\theta^1 \wedge (d\theta^1)^2 \neq 0$$

at every point. Thus θ^1 defines a *contact structure.* Integral submanifolds of the contact distribution have maximal dimension 2, and immersed 2-dimensional submanifolds are called *Legendre surfaces* of the contact structure. Thus generalized surfaces are Legendre surfaces.

3 $T_1\mathbb{E}^3$ as a homogeneous space of the Poincaré group

Let $\mathbb{R}^4_1$ denote Minkowski 4-space with its structure of affine vector space and a translation invariant Lorentz scalar product $\langle\ ,\ \rangle$ which takes the form

$$\langle v, w\rangle = -(v^1w^4 + v^4w^1) + v^2w^2 + v^3w^3 = g_{ij}v^iw^j, g_{ij} = g_{ji}$$

with respect to the standard basis $e_1, \ldots, e_4$. We use the index ranges $1 \le i, j, h, k \le 4$, $1 \le \alpha, \beta \le 3$, and the summation convention. We fix a space orientation by requiring that the standard basis is positive, and fix a time orientation by saying that a timelike or lightlike vector v is positive if $\langle v, e_1 + e_4\rangle < 0$. The corresponding positive lightcone is given by

$$\mathcal{L}^+ = \left\{v \in \mathbb{R}^4_1 : \langle v, v\rangle = 0,\ \langle v, e_1 + e_4\rangle < 0\right\}.$$

The (restricted) *Poincaré group* L is the group of isometries of $\mathbb{R}^4_1$ which preserve the given space and time orientations. It is isomorphic to the semidirect product $L = \mathbb{R}^4 \rtimes G$, where G consists of elements $a = (a^i_j) \in GL(4, \mathbb{R})$ such that

$$\det a = 1, \quad g_{hk} a^h_i a^k_j = g_{ij}, \quad \langle ae_1 + ae_4, e_1 + e_4\rangle > 0.$$

Any time-oriented isotropic line in $\mathbb{R}^4_1$ may be realized as $x + tv$ where $x \in \mathbb{R}^4_1$, $v \in \mathcal{L}^+$ and t ranges over $\mathbb{R}$, and will be denoted by $[x, v]$. The set of all time-oriented isotropic lines in $\mathbb{R}^4_1$ forms a five-dimensional smooth manifold, which we denote by

$$\Lambda = \left\{[x, v] : x \in \mathbb{R}^4_1, v \in \mathcal{L}^+\right\}.$$

The group L induces an action on Λ which is transitive:

$$L \times \Lambda \to \Lambda, \quad ((x, a), [y, v]) \mapsto [x + ay, av].$$

We establish a bijective correspondence between the points of $T_1\mathbb{E}^3$ and Λ by the map

$$(p, n) \in T_1\mathbb{E}^3 \mapsto [x(p), v(n, p)] \in \Lambda, \tag{1}$$

where

$$x(p) = {}^t\!\left(\frac{p^1}{\sqrt{2}}, p^2, p^3, \frac{-p^1}{\sqrt{2}}\right), \quad v(n, p) = {}^t\!\left(\frac{1 + n^1}{2}, \frac{n^2}{\sqrt{2}}, \frac{n^3}{\sqrt{2}}, \frac{1 - n^1}{2}\right).$$

4 Laguerre geometry in Euclidean space

The points in $\mathbb{R}^4_1$ are in bijective correspondence with the set of all oriented spheres and point spheres in $\mathbb{E}^3$. The orientation of a sphere is designated by giving a sign to its radius, while point spheres have no orientation. The oriented sphere $\sigma(p, r)$ with center $p = (p^1, p^2, p^3)$ and signed radius $r \in \mathbb{R}$ corresponds to the point in $\mathbb{R}^4_1$ given by

$${}^t\!\left(\frac{r + p^1}{\sqrt{2}}, p^2, p^3, \frac{r - p^1}{\sqrt{2}}\right),$$

and the point sphere $\{p\}$ corresponds to the point

$$x(p) = {}^t\!\left(\frac{p^1}{\sqrt{2}}, p^2, p^3, \frac{-p^1}{\sqrt{2}}\right).$$

Instead, the oriented plane $\pi(n,p)$ through p and orthogonal to $n = (n^1, n^2, n^3) \in S^2 \subset \mathbb{E}^3$ is identified with the hyperplane (isotropic hyperplane) through the point $x(p) \in \mathbb{R}^4_1$ with isotropic normal vector

$$v(n,p) = {}^t\left(\frac{1+n^1}{2}, \frac{n^2}{\sqrt{2}}, \frac{n^3}{\sqrt{2}}, \frac{1-n^1}{2}\right).$$

Oriented contact of spheres and planes corresponds to incidence of points and isotropic hyperplanes in $\mathbb{R}^4_1$. Two oriented spheres corresponding to x and y in $\mathbb{R}^4_1$ are in oriented contact if and only if $\langle x-y, x-y\rangle = 0$. The isotropic hyperplane with normal vector $v = x - y$ corresponds to the common tangent plane to the spheres x, y in $\mathbb{E}^3$. Geometrically, this means that the points on a time-oriented isotropic line in $\mathbb{R}^4_1$ correspond to a parabolic pencil of oriented spheres in oriented contact at a certain point (p,n) in the unit tangent bundle of $\mathbb{E}^3$.

A *Laguerre transformation* is a contact transformation induced by a transformation in the group L. In terms of $\mathbb{E}^3$, a Laguerre transformation is a map on the space of oriented spheres (including point spheres) and oriented planes which preserves oriented contact and takes plane to planes.

5 Moving frames in Laguerre geometry

We now introduce the method of moving frames in the context of Laguerre geometry.

By a *Laguerre frame* $(x; a_1, \dots, a_4)$ is meant a position vector $x \in \mathbb{R}^4_1$ and an oriented basis $(a_1, \dots, a_4)$ of $\mathbb{R}^4_1$, such that

$$\langle a_i, a_j\rangle = g_{ij}, \quad a_1, a_4 \in \mathcal{L}^+.$$

L acts simply transitively on Laguerre frames and the manifold of all such frames may be identified, up to the choice of a reference frame, with L. For any $(x,a) \in L$, we regard x and $a_i = ae_i$ as $\mathbb{R}^4$-valued functions. There are unique 1-forms ω^i and ω^i_j such that

(2) $$dx = \omega^i a_i, \quad da_i = \omega^i_j a_j.$$

Exterior differentiation of (2) yields

(3) $$d\omega^i = -\omega^i_j \wedge \omega^j, \quad d\omega^i_j = -\omega^i_k \wedge \omega^k_j.$$

(3) are the *structure equations* of L.

The transitive action of L on Λ defines a principal L_0-bundle

$$\pi_L : L \to \Lambda = L/L_0, A \mapsto A[0, e_1] = [x, a_1].$$

Definition. A *Laguerre frame field* in Λ is a local smooth section $A = (x,a)$ of π_L.

Remark. By the structure equations of L, the 1-form $-\langle dx, a_1\rangle$ defines a L-invariant contact distribution on Λ. By (1), this contact structure coincides with the contact structure on $T_1\mathbb{E}^3$ defined by θ^1.

Definition. A (local) *Laguerre frame field along a Legendre surface* $F : M \to \Lambda$ is a smooth map $A = (x, a) : \mathcal{U} \to L$ defined on an open subset $\mathcal{U} \subset M$, such that $\pi_L(x, a) = [x, a_1] = F$.

For any Laguerre frame field $A : \mathcal{U} \to L$ we let $\alpha = (\alpha^i, j) = \left(A^*(\omega^i), A^*(\omega^i_j)\right)$ be its connection form. We then have

$$\alpha^4 = 0, \quad 1 \wedge 1 \neq 0.$$

Any other Laguerre frame field $\hat{A}$ on $\mathcal{U}$ is given by $\hat{A} = AX(d; b; x)$, where $X = X(d; b; x) : \mathcal{U} \to L_0$ is a smooth map, and $\hat{\alpha}$ and α are related by $\hat{\alpha} = X^{-1}\alpha X + X^{-1}dX$.

Definition. Two Legendre surfaces (M, F) and (M', F') are said to be *L-equivalent* if there exists a diffeomorphism $\varphi : M \to M'$ and $A \in L$ such that $F' \circ \varphi = AF$. In particular, two oriented immersed surfaces in $\mathbb{E}^3$ are L-equivalent if their Gauss lifts are L-equivalent. In the case of Laguerre equivalent immersions, there is no loss of generality in replacing F' by $F' \circ \varphi$, so that $M = M'$ and $F' = AF$. If B is a Laguerre frame alonf F, then AB is a Laguerre frame along F' with no change in the pull-back of the Maurer-Cartan forms.

Remark. *The method of moving frames.*

The idea of the method of moving frames [Car],[Gr],[Je] is to associate to an immersion in a homogeneous space L/L_0 a lift of the immersion to L. One expects the pullback of the left-invariant Maurer-Cartan form on L to give a complete set of differential invariants for the immersion. These generalize velocity, curvature, and torsion for curves in Euclidean space $\mathbb{E}^3$, which are differential invariants of order 1, 2 and 3 respectively. The problem is that the lifts are not unique. One uses algebraic or geometric criteria to reduce the choices of these lifts to something unique or invariant in some sense. If the reduction is made in terms of conditions imposed on the components of the Maurer-Cartan form, then the same conditions hold for any L-equivalent (congruent) surface, namely for the frame obtained by the congruence. The principal difficulty in the general theory lies in the phenomenon of degeneracy, exemplified by the undefinability of torsion of curves in Euclidean space at points where the curvature vanishes. Some restriction on the class of immersions to be considered is thus necessary. In our situation, the natural class turns out to be that of immersions with nor parabolic nor umbilic points.

Proposition 5.1 [MN2] *Let $f : M \to \mathbb{E}^3$ be an oriented immersion with nor parabolic nor umbilic points. Then there exists a canonical frame (normal frame) $A : \mathcal{U} \subset M \to L$ along the Gauss lift of f such that $1 \wedge 1 \neq 0$ and such that*

$$\alpha = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2q_2\alpha_1^2 + 2q_1\alpha_1^3 & p_1\alpha_1^2 - p_2\alpha_1^3 & p_2\alpha_1^2 - p_3\alpha_1^3 & 0 \\ \alpha_1^2 & 0 & -q_1\alpha_1^2 + q_2\alpha_1^3 & p_1\alpha_1^2 - p_2\alpha_1^3 \\ \alpha_1^3 & q_1\alpha_1^2 - q_2\alpha_1^3 & 0 & p_2\alpha_1^2 - p_3\alpha_1^3 \\ 0 & \alpha_1^2 & \alpha_1^3 & 0 \end{pmatrix},$$

where p_1, p_2, p_3 and q_1, q_2 are real-valued smooth functions.

Remark.

1. p_1, p_2, p_3, q_1, q_2 are the *invariant functions* and $(1, 1)$ the *normal coframing* of the Legendre surface $F = (f, n)$ with respect to the normal frame field A. The Maurer-Cartan compatibility condition yields the structure equations of the surface:

$$d1 = -q_1 1 \wedge 1, \quad d0 = -q_2 1 \wedge 1,$$

$$\begin{aligned} dq_1 \wedge 1 - dq_2 \wedge 1 &= (p_1 - p_3 + q_1{}^2 + q_2{}^2) 1 \wedge 1, \\ dq_1 \wedge 1 + dq_2 \wedge 1 &= -p_2 1 \wedge 1, \\ dp_1 \wedge 1 - dp_2 \wedge 1 &= (3q_1p_1 + 4q_2p_2 - q_1p_3) 1 \wedge 1, \\ dp_2 \wedge 1 - dp_3 \wedge 1 &= (3q_2p_3 + 4q_1p_2 - q_2p_1) 1 \wedge 1. \end{aligned}$$

2. The normal frame A is defined up to the action of a subgroup of L_0 which is isomorphic to $\mathbb{Z}_2$.

6 Euclidean geometry as a subgeometry of Laguerre geometry

Euclidean space $\mathbb{E}^3$ is identified with the hyperplane $\langle x, e_1 + e_4 \rangle = 0$ (Euclidean hyperplane) in $\mathbb{R}^4_1$, and Euclidean motions correspond to the elements of L which leave the Euclidean hyperplane invariant. Explicitly, for any $(p, a) = (p, (a^\alpha_\beta)) \in \mathbb{E}(3) = \mathbb{R}^3 \rtimes SO(3)$, we define the corresponding element in L by

$$(4) \qquad \begin{pmatrix} \frac{p^1}{\sqrt{2}} \\ p^2 \\ p^3 \\ \frac{-p^1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1 + a_1^1}{2} & \frac{a_2^1}{\sqrt{2}} & \frac{a_3^1}{\sqrt{2}} & \frac{1 - a_1^1}{2} \\ \frac{a_1^2}{\sqrt{2}} & a_2^2 & a_3^2 & \frac{-a_1^2}{\sqrt{2}} \\ \frac{a_1^3}{\sqrt{2}} & a_2^3 & a_3^3 & \frac{-a_1^3}{\sqrt{2}} \\ \frac{1 - a_1^1}{2} & \frac{-a_2^1}{\sqrt{2}} & \frac{-a_3^1}{\sqrt{2}} & \frac{1 + a_1^1}{2} \end{pmatrix}.$$

Let $f : M \to \mathbb{E}^3$ be an oriented immersion in Euclidean space with no parabolic points and let n be a unit normal field on the immersion. If $(f; n, e_2, e_3) : \mathcal{U} \to \mathbb{E}(3)$ is a local Darboux frame along f, we have

$$df = \theta^2 e_2 + \theta^3 e_3, \quad dn = \theta_1^2 e_2 + \theta_1^3 e_3, \quad de_2 = \theta_2^3 e_3 - \theta_1^2 n, \quad de_3 = -\theta_2^3 e_2 - \theta_1^3 n,$$

where (θ^2, θ^3) is the dual coframe of (e_2, e_3) on M and $\theta_1^2 = h_{11}\theta^2 + h_{12}\theta^3$ and $\theta_1^3 = h_{12}\theta^2 + h_{22}\theta^2$ with $h_{11}h_{22} - h_{12}^2 > 0$. Then the Maurer–Cartan form $(\varphi^i, (\varphi^i_j))$ of the Laguerre frame corresponding to $(f; n, e_2, e_3)$ is

$$
(5) \qquad \varphi = \left(\begin{array}{c} 0 \\ \theta^2 \\ \theta^3 \\ 0 \end{array} , \begin{array}{cccc} 0 & -\frac{\theta_1^2}{\sqrt{2}} & -\frac{\theta_1^3}{\sqrt{2}} & 0 \\ \frac{\theta_1^2}{\sqrt{2}} & 0 & -\theta_2^3 & -\frac{\theta_1^2}{\sqrt{2}} \\ \frac{\theta_1^3}{\sqrt{2}} & \theta_2^3 & 0 & -\frac{\theta_1^3}{\sqrt{2}} \\ 0 & \frac{\theta_1^2}{\sqrt{2}} & \frac{\theta_1^3}{\sqrt{2}} & 0 \end{array} \right).
$$

7 L-isothermic surfaces

Let $f : M \to \mathbb{E}^3$ be an immersion of a surface M in Euclidean space with no parabolic points. Let consider on M the unique complex structure compatible with the given orientation and the conformal structure defined by the third fundamental form $\mathrm{III} = dn \cdot dn$ of the immersion. Accordingly, the second fundamental form II decomposes into bidegrees:

$$\mathrm{II} = \mathrm{II}^{(2,0)} + \mathrm{II}^{(1,1)} + \mathrm{II}^{(0,2)}.$$

$\mathrm{II}^{(2,0)}$ is a globally defined $(2,0)$ symmetric bilinear form on M. If $(f; n, e_2, e_3)$ is a Darboux frame on $\mathcal{U} \subset M$ along f, the complex structure induced on M is defined on $\mathcal{U}$ by $\varphi = \theta_1^2 + i\theta_1^3$, and

$$\mathrm{II}^{(2,0)} = \frac{1}{4}[(h_{11} - h_{22}) - 2ih_{12}]\varphi\varphi.$$

Definition. $f : M \to \mathbb{E}^3$ is said to be *L-isothermic* if there exists a non-zero holomorphic differential Q and a smooth real-valued function $\mu : M \to \mathbb{R}$ such that $\mathrm{II}^{(2,0)} = \mu Q$. Q is said to be a *polarization* of f.

Remark.
1. The notion of an L-isothermic surface is invariant under Laguerre transformations. By (5), an easy calculation shows that under a Laguerre transformation II is taken into the conformal class of II modulo III.
2. The classical notion of an L-isothermic surface is that $f : M \to \mathbb{E}^3$ admits local curvature line coordinates which are conformal for the third fundamental form away from umbilic points.
3. If the isothermic immersion is not totally umbilic, the polarization is uniquely defined up to a non-zero constant. If the immersion is totally umbilic and there is a non-zero holomorphic quadratic differential Q on M, then (f, M) is L-isothermic and Q is a polarization.

Definition. Let $f : M \to \mathbb{E}^3$ be an L-isothermic surface in $\mathbb{E}^3$ and let Q be a polarization. Let $\{Q\}$ denote the zero set of Q and $M_0 = M - \{Q\}$. A local complex coordinate $z : \mathcal{U} \subset M_0 \to \mathbb{C}$ is said to be an *adapted chart* if $Q_{|\mathcal{U}} = (dz)^2$.

If $(\mathcal{U}, z)$ is an adapted chart, then $\mathrm{III}_{|\mathcal{U}} = (\lambda)^2 dz d\bar{z}$, where λ is a positive function. We can then define a global smooth function $\Phi : M_0 \to \mathbb{R}$ such that $\Phi_{|\mathcal{U}} = 2\mu\lambda^{-1}$. Φ is called the *Blaschke potential* of the L-isothermic surface with respect to the polarization Q.

Around each point of M_0 we can write $z = u + iv$ and consider a local parametrization $x = z^{-1} : \Sigma \to M$, for some open connected domain Σ. The Maurer-Cartan form $(\theta^\alpha, (\theta^\alpha_\beta))$ of the Gaussian framing of x can be written as

$$\begin{aligned} &\theta^1 = 0, \qquad \theta^2 = Edu, \qquad \theta^3 = Gdv, \\ &\theta^2_1 = Ldu, \qquad \theta^3_1 = Ndv, \end{aligned} \tag{6}$$

where E, G, L and N are nonvanishing smooth functions and

$$\theta^3_2 = -\frac{1}{N} L_v du + \frac{1}{L} N_u dv. \tag{7}$$

It is easy to prove

(8) An immersed surface $M \subset \mathbb{E}^3$ is L-isothermic if and only if, for any adapted chart $(\mathcal{U}, z)$

$$(\log \frac{L}{N})_{uv} = 0$$

on M_0.

An L-isothermic surface can locally be parametrized by two-parameter maps $x : \Sigma \subset \mathbb{R}^2 \to \mathbb{E}^3$, such that Σ is a connected, simply connected open subset with coordinates (u, v), that the tangent vector fields x_u and x_v are along principal directions, and that the Gauss map

$$n = \frac{x_u \wedge x_v}{\|x_u \wedge x_v\|} : \mathcal{U} \to S^2 \subset \mathbb{E}^3$$

is holomorphic. Let $(x, e_1, e_2, e_3) : \Sigma \to \mathbb{E}(3)$ be the Darboux frame field along x defined by $e_1 = n$, $e_2 = \frac{x_u}{\|x_u\|}$, $e_3 = \frac{x_v}{\|x_v\|}$. The corresponding connection form $(\theta^\alpha, (\theta^\alpha_\beta))$ can be written as

$$\begin{aligned} &\theta^1 = 0, \qquad \theta^2 = Edu, \qquad \theta^3 = Gdv, \\ &\theta^2_1 = e^U du, \qquad \theta^3_1 = e^U dv, \qquad \theta^3_2 = -U_v du + U_u dv, \end{aligned}$$

where U, E and G are smooth functions and $EG \neq 0$ at each point. We then have

$$\begin{aligned} dx &= Edue_2 + Gdve_3, \\ dn &= e^U(due_2 + dve_3), \\ de_2 &= -e^U dun + (-U_v du + U_u dv)e_3, \\ de_3 &= -e^U dvn - (-U_v du + U_u dv)e_2, \end{aligned}$$

and hence

$$\Delta U = -e^{2U}, \tag{9}$$

$$E_v = GU_v, \quad G_u = EU_u. \tag{10}$$

The *Blaschke potential* of (Σ, x) takes the form

$$\Phi = \frac{1}{2}(E - G). \tag{11}$$

Equations (9) and (10) imply

$$\Phi_{uv} = -\Phi\left(U_{uv} - U_u U_v\right), \tag{12}$$

$$\Delta\left(U_{uv} - U_u U_v\right) = 0, \tag{13}$$

from which we deduce that the Blaschke potential is a solution of the fourth order nonlinear PDE[1]

$$\Delta\left(\frac{\Phi_{uv}}{\Phi}\right) = 0 \quad (\textit{Blaschke differential equation}). \tag{14}$$

Conversely, let $\Phi = e^{\psi} : \Sigma \to \mathbb{R}$ be a Blaschke potential, and m be a harmonic conjugate to $\Phi_{uv}\Phi^{-1}$. Define p_1 and p_3 by

$$p_1 = -e^{-2\psi}\left(\psi_{uu} + \tfrac{1}{2}(\psi_u)^2 - \tfrac{1}{2}(\psi_v)^2 - m\right),$$
$$p_3 = -e^{-2\psi}\left(\psi_{vv} - \tfrac{1}{2}(\psi_u)^2 + \tfrac{1}{2}(\psi_v)^2 + m\right).$$

The exterior differential form α given by

$$\begin{pmatrix} 0 & d\log\Phi^2 & p_1\Phi du & p_3\Phi dv & 0 \\ \Phi du & \Phi du & 0 & -(\Phi_v du - \Phi_u dv)/\Phi & p_1\Phi du \\ -\Phi dv & \Phi dv & (\Phi_v du - \Phi_u dv)/\Phi & 0 & p_3\Phi dv \\ 0 & 0 & \Phi du & \Phi dv & -d\log\Phi^2 \end{pmatrix}$$

takes values in the Lie algebra of L and satisfies the Maurer-Cartan equation $d\alpha = -\alpha\wedge\alpha$. This implies that there exists a Legendre surface $F : \Sigma \to \Lambda$ whose normal frame field $A : \Sigma \to L$ satisfies $dA = A\alpha$. The map F is unique up to the action of L. Moreover, the Euclidean projection $x : \Sigma \to \mathbb{E}^3$ of F is an L-isothermic parametrization away from the singular locus with Blaschke potential Φ.

Remark. Observe that local L-isothermic surfaces can be obtained from an infinite dimenional integrable system. They depend on four functions in one variables (cf. [MN3], [MN4]).

[1] In the classical terminology, this is an equation with *equal invariants*. They were extensively studied by Darboux [Da]; see also Bianchi [Bi].

8 L-isothermic surfaces and the deformation problem

The notion of an L-isothermic surface is related to the deformation theory of submanifolds in homogeneous spaces [Gr], [Je].

Definition. Two immersions $F, F' : N \to G/K$ of a smooth manifold N into a homogeneous space G/K are G-deformations of each other *of order* k if there is a smooth map $A : N \to G$ such that F anf $A(x)F'$ agree up to order k at x, for each $x \in N$, i.e., they have the same jets of order k at x. An immersion is *deformable* of order k if it admits non-trivial deformations of order k; otherwise is said *rigid* of order k.

Proposition 8.1 [MN2], [MN3] *A Legendre surface $F : M \to \Lambda$ is deformable (of order two) if and only if the invariant function $p_2 = 0$. Moreover, non-trivial deformations of F arise in a 1-parameter family.*

Remark.
1. Any Legendre surface in Λ is deformable of order 1 and rigid of order 3.
2. The Gauss lift of an L-isothermic surface has $p_2 = 0$.

The latter condition is characteristic for L-isothermic surfaces. In fact,

Proposition 8.2 [MN2], [MN3] *A Legendre immersion $F : M \to \Lambda$ is deformable if and only if it is the lift of an L-isothermic surface in $\mathbb{E}^3$. The lifts of two isothermic surfaces are deformations of each other if and only if they have the same potential Φ. Hence, from a solution of the Blaschke equation we can construct an invariant frame of an L-isothermic surface and all its deformations.*

9 Example 1: surfaces with plane lines of curvature

A remarkable class of L-isothermic surfaces is provided by the surfaces with plane lines of curvature in both systems.

Proposition 9.1 [Bl4], [MN5] *An immersed surface $M \subset \mathbb{E}^3$ with nor umbilic nor parabolic points and plane lines of curvature in both systems is L-isothermic. Moreover, the Blaschke potential with respect to a polarization Q is a solution of the one dimensional wave equation*

$$\Phi_{uv} = 0, \tag{15}$$

where $z = u + iv$ is an adapted chart.

Proof. Let $x = z^{-1} : \Sigma \to M$ be a principal parametrization of M. Define the mapping

$$m = e_2 \times (e_2)_u : \Sigma \to \mathbb{E}^3.$$

The curvature lines $v = \text{const.}$ are plane if and only if $m \times m_u = 0$. This reduces to

$$LNL_{uv} - LN_uL_v - NL_uL_v = 0. \tag{16}$$

Similarly, $u = \text{const.}$ is plane if and only if

$$LNN_{uv} - NN_uL_v - LN_uN_v = 0. \tag{17}$$

We then obtain

$$(\log \frac{L}{N})_{uv} = 0,$$

and hence M is L-isothermic by (8). Next, if $x : \Sigma \to \mathbb{E}^3$ is a local parametrization for M, $L = N = e^U$. Thus (16) and (17) reduce to $U_{uv} = U_uU_v$. By (12), equation (15) follows. □

Remark.

1. For a polarized surface with plane lines of curvature m is a constant which depends on the polarization Q. The conditions $m \neq 0$ and $m = 0$ are independent of the polarization.
2. Two surfaces with plane lines of curvature are Laguerre equivalent if and only if they have the same Blaschke potential Φ and the same m.
3. If $\Phi_u = 0$, or else $\Phi_v = 0$, then the surface is L-equivalent to a canal surface, i.e., an envelope of a 1-parameter family of spheres. In this case $m \neq 0$ if and only if the surface is L-equivalent to a surface of revolution.
4. $\Phi = \text{const.} \neq 0$ if and only if the surface is L-equivalent to a Dupin surface.
5. For more details on surfaces with plane lines of curvature we refer to [MN5].

10 Example 2: molding surfaces [B], [BCG], [Da], [Ei]

A special class of surfaces with plane lines of curvature and hence of L-isothermic surfaces is given by molding surfaces, which we shall discuss in this paragraph. A *molding surface* is described as follows: take a plane curve b (directrix curve) and a curve a on one of its normal planes (the profile). Then, the surface is generated by the curve a as its plane moves remaining normal to b. We shall denote by $X_{a,b}$ the molding surface with directrix curve b and profile a.

Assume that b is a curve in the (x, z) plane represented by means of

$$x_\beta : v \in I' \subset \mathbb{R} \mapsto \left(2\int \beta(v)\cos(v)dv, 0, -2\int \beta(v)\sin(v)dv\right) \in \mathbb{E}^3, \tag{18}$$

where $\beta : I' \to \mathbb{R}$ is a smooth function. Although x_β is not necessarily an immersion, we may define tangent and normal vector fields

$$t_\beta : v \in I' \mapsto (\cos(v), 0, -\sin(v)), \tag{19}$$

$$n_\beta : v \in I' \mapsto (\sin(v), 0, \cos(v)) \tag{20}$$

Remark. If $\beta(v) \neq 0$, for each $v \in I'$, then b is an immersed curve; the measure of the arc of the curve is $ds_\beta = 2|\beta(v)|dv$, and the curvature of b at $x_\beta(v)$ is $\kappa_\beta(v) = -\frac{1}{2\beta(v)}$.

Let $(\epsilon_1, \epsilon_2, \epsilon_3)$ denote the standard basis of $\mathbb{R}^3$ and let

$$x_a : u \in I \mapsto (0, \mu_a(u), \nu_a(u)) \tag{21}$$

be an analytic representation of the profile. Then

$$x_{a,\beta} : (u,v) \in \mathcal{U} \mapsto x_\beta(v) + \nu_a(u)n_\beta(v) + \mu_a(u)\epsilon_2$$

is a parametric equation of the molding surface $X_{a,b}$. The mapping $x_{a,\beta}$ is an immersion away from

$$\mathcal{U}'_{a,\beta} = \{(u,v) \in I \times I' : \nu_a(u) + 2\beta(v) = 0\}.$$

Let consider the curve b' defined by

$$x'_\beta(v) = x_\beta(v) - 2\beta(v)n_\beta(v). \tag{22}$$

Since

$$\frac{dx'_\beta}{dv} = -2\frac{d\beta}{dv}n_\beta(v),$$

we see that $x'_\beta(v) - x_\beta(v)$ is always normal to b at the point $x_\beta(v)$ and is tangent to b' at $x'_\beta(v)$, for each $v \in I'$. Thus b' is the evolute of b.

Let t'_β the unit tangent vector field along b'. Then

$$x_{a,\beta} : (u,v) \in \mathcal{U} \mapsto x'_\beta(v) + (\nu_a(u) + 2\beta(v))t'_\beta + \mu_a(u)\epsilon_2. \tag{23}$$

This implies that $X_{a,b}$ is the surface generated by the profile a whose plane rolls without slipping over a cylinder having b' as right section.

Proposition 10.1 [B], [Ei] *Molding surfaces have plane lines of curvature in both systems.*

Let x_a be the arclenght parametric representation of the profile. Then the Gaussian curvature $K_{a,b}$ of $X_{a,b}$ vanishes at $x_{a,\beta}(u,v)$ if and only if

$$\kappa_a \frac{d\mu_a}{du}_{|u} = 0. \tag{24}$$

Thus if $K_{a,b}$ is nowhere vanishing, we may assume that μ_a is a decreasing function, and the parametric equations of the surface can be recast in the following form:

$$\text{(25)}\qquad \begin{aligned} x_{a,\beta}(u,v) &= \nu_a(u)\sin(v) + 2\textstyle\int \beta(v)\cos(v)dv,\\ y_{a,\beta}(u,v) &= -\textstyle\int \sqrt{1-\left(\frac{d\nu_a}{du}\right)^2}du,\\ z_{a,\beta}(u,v) &= \nu_a(u)\cos(v) - 2\textstyle\int \beta(v)\sin(v)dv. \end{aligned}$$

This is the classical analytic representation of a molding surface with nor umbilic nor parabolic points ([Ei],[B]) and is referred to as the *Bour parametric equation* of $X_{a,b}$.

With the above assumptions, we have

$$\frac{d\mu_a}{du} \in [-1,0), \quad \frac{d\nu_a}{du} \in (-1,1).$$

Therefore there exists a smooth function $r_a : I \to \mathbb{R}$ such that

$$\text{(26)}\qquad \frac{d\nu_a}{du}_{|u} = -\tanh r_a(u),$$

for all $u \in I$. Then

$$\text{(27)}\qquad \frac{d\mu_a}{du}_{|u} = -\frac{1}{\cos r_a(u)}, \quad \frac{dr_a}{du} = \kappa_a(u)\cosh r_a(u).$$

Let I_α denote the image of r_a and consider the parametrization $x_{\alpha,\beta}(u,v) = x_{a,\beta}({r_a}^{-1}(u),v)$. The profile is then parametrized by

$$x_\alpha : u \in I_\alpha \mapsto (0, \mu_\alpha(u), \nu_\alpha(u)),$$

where $\mu_\alpha(u) = \mu_a({r_a}^{-1}(u))$, $\nu_\alpha(u) = \nu_a({r_a}^{-1}(u))$. We then set $\kappa_\alpha(u) = \kappa_a({r_a}^{-1}(u))$.

From (26) and (27) we have

$$\text{(28)}\qquad d\nu_\alpha = \sinh(u)d\mu_\alpha.$$

By (27), we have

$$\text{(29)}\qquad \frac{d\mu_\alpha}{du} = -\frac{1}{\kappa_\alpha(u)\cosh(u)}.$$

Combining (28) and (29) we obtain

$$\text{(30)}\qquad \frac{d}{du}(\cosh(u)\nu_\alpha) = -2\sinh(u)\alpha(u)$$

where α is the function defined by

$$\text{(31)}\qquad 2\alpha(u) = \frac{1}{\kappa_\alpha(u)\cosh(u)} - \nu_\alpha(u).$$

We then have

$$\mu_\alpha(u) = -\frac{2}{\cosh(u)}\left(\cosh(u)\int \alpha(u)\cosh(u)du - \sinh(u)\int \alpha(u)\sinh(u)du\right).$$

and

$$\nu_\alpha(u) = -\frac{2}{\cosh(u)}\int \alpha(u)\sinh(u)du.$$

Consequently, $x_{\alpha,\beta}(u,v)$ has the following expression in terms of the two function α and β:

$$\begin{aligned}
x_{\alpha,\beta}(u,v) &= -\frac{2\sin(v)}{\cosh(u)}\int \alpha(u)\sinh(u)du + 2\int \beta(v)\cos(v)dv,\\
y_{\alpha,\beta}(u,v) &= -\frac{2}{\cosh(u)}\left(\cosh(u)\int \alpha(u)\cosh(u)du - \sinh(u)\int \alpha(u)\sinh(u)du\right),\\
z_{\alpha,\beta}(u,v) &= -\frac{2\cos(v)}{\cosh(u)}\int \alpha(u)\sinh(u)du - 2\int \beta(v)\sin(v)dv.
\end{aligned}$$

This will be referred to as the *Blaschke parametric equation* of the molding surface $X_{a,b}$. α and β are said the *potential functions* of the surface.

We have

Proposition 10.2 [MN5] *The Blaschke representation $x_{\alpha,\beta} : I_\alpha \times I' \to \mathbb{E}^3$ of a molding surface with nor umbilic nor parabolic points is an L-isothermic immersion. The Euclidean invariants of the map are given by*

$$\begin{aligned}
E_{\alpha,\beta}(u,v) &= \nu_\alpha(u) + 2\alpha(u),\\
G_{\alpha,\beta}(u,v) &= \nu_\alpha(u) + 2\beta(v),\\
U(u,v) &= -\log\cosh(u).
\end{aligned} \tag{32}$$

The Blaschke potential of the immersion is

$$\Phi_{\alpha,\beta}(u,v) = \alpha(u) - \beta(v).$$

Proof. Consider the Gauss framing $(x_{a,\beta}; e_1, e_2, e_3)$ of $x_{a,\beta}$, where

$$e_1 = -\frac{d\mu_a}{du}_{|u} n_\beta(v) + \frac{d\nu_a}{du}_{|u}\epsilon_2, \quad e_2 = \frac{d\nu_a}{du}_{|u} n_\beta(v) + \frac{d\mu_a}{du}_{|u}\epsilon_2, \quad e_3 = t_\beta(v).$$

The Gauss framing $(x_{\alpha,\beta}; e'_1, e'_2, e'_3)$ of $x_{\alpha,\beta}$ is such that

$$\begin{aligned}
e'_1 &= \frac{1}{\cosh(u)} n_\beta(v) - \tanh(u)\epsilon_2,\\
e'_2 &= -\tanh(u) n_\beta(v) - \frac{1}{\cosh(u)}\epsilon_2,\\
e'_3 &= t_\beta(v).
\end{aligned} \tag{33}$$

For the corresponding Maurer-Cartan form $(\theta'^{\alpha}, (\theta'^{\alpha}_{\beta}))$ we compute, by (33),

$$\theta'^1_0 = 0, \qquad \theta'^2_0(2\alpha(u) + \nu_\alpha)du, \qquad \theta'^3_0 = (\nu_\alpha + 2\beta(v))dv,$$
$$\theta'^2_1 = \frac{1}{\cosh(u)}du, \qquad \theta'^3_1 = \frac{1}{\cosh(u)}du,$$

and this gives the required result. □

Remark.

1. Note that the spherical representation of $x_{\alpha,\beta}$ is given by

$$n(u,v) = \left(\frac{\sin(v)}{\cosh(u)}, -\frac{\sinh(u)}{\cosh(u)}, \frac{\cos(v)}{\cosh(u)}\right),$$

for each $(u, v) \in I_\alpha \times I'$.

2. It is clear that, if we assign two functions $\alpha(u) : I \to \mathbb{R}$ and $\beta(v) : I' \to \mathbb{R}$ and define $x_{\alpha,\beta}$ by means of the corresponding Blaschke parametric equations, we obtain an isothermic representation of a molding surface with potential functions α and β.

3. Any surface with plane lines of curvature and $m \neq 0$ is Laguerre equivalent to a molding surface [MN5].

4. As a limiting case of the molding surfaces we obtain surfaces of revolution.

11 Other examples: minimal surfaces and L-minimal surfaces with plane curvature lines

Examples of L-isothermic surfaces clearly include minimal surfaces. These form a special class of Laguerre minimal surfaces (L-minimal surfaces) which are also L-isothermic. These surfaces are defined by a family of integrable nonlinear second-order PDE's:

$$\Delta \log \Phi = c\Phi^{-2},$$

c a constant, whose solutions are automatically solutions of the Blaschke equation (14) and the Euler-Lagrange equation

$$\Delta^{\mathrm{III}}\left(\frac{H}{K}\right) = 0$$

of the variational problem for the invariant area element $(H^2 - K)\,K^{-1}dA$. Here Δ^{III} denotes the Laplace-Beltrami operator with respect to the third fundamental form of the surface. Only the sign of the constant is relevant and the study of L-minimal isothermic surfaces is reduced to the study of the three cases corresponding to $c = 1, -1, 0$. The case $c = 1$ defines minimal surfaces in Euclidean space. Geometrically, these solution surfaces are characterized by having degenerate central sphere congruences (cf. [Me], [MN2], [MN4]).

We refer to [MN5] for an explicit description of L-minimal surfaces with plane lines of curvature.

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