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GEOMETRIC CONSEQUENCES OF ALGEBRAIC CONDITIONS ON CURVATURE OPERATORS

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Memoria realizada no Departamento de Xeometría e Topoloxía da Facultade de Matemáticas, baixo a dirección dos profesores Eduardo García Río e Ramón Vázquez Lorenzo, para obter o Grao de Doutor en Ciencias Matemáticas pola Universidade de Santiago de Compostela.

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Introduction

As a Riemannian invariant, the curvature and its derivatives are the most natural algebraic invariants which stem from the connection. Therefore, this suggests that the curvature encodes a lot of information of the geometry of a Riemannian manifold. These considerations show that the curvature is a fundamental concept in differential geometry, nevertheless the role played by this important tensor is not yet completely understood.

The main purpose of this thesis is to obtain geometric consequences from algebraic conditions on the curvature tensor. Usually, we will impose these conditions on operators associated to the curvature tensor, since the curvature tensor itself is hard to handle. Generally we work in the broad setting of pseudo-Riemannian manifolds; however, in some chapters or sections we will restrict our analysis to positive definite metrics.

The scheme of this memoir goes as follows. A preliminary chapter, Chapter 1, is presented with the purpose of establishing the main definitions and some basic results on the subject that will be needed later on. Now, the main body of the thesis is divided into four different parts. On the one hand, although they are all closely related, since all of them focus on the task of describing the geometry of manifolds with a given algebraic condition on the curvature, they are essentially independent. On the other hand, the order that we have chosen is a natural one and we find some part motivating the next one.

Part I deals with some aspects of the Osserman problem. Since the Osserman conjecture was stated in [145] for Riemannian manifolds, many related problems have been solved as well as the conjecture itself (except for dimension 16) [51, 52, 135, 136]. An illustrative example is the classification of Osserman manifolds in Lorentzian signature, which shows that they are of constant sectional curvature [16, 75]. On the other hand, interesting questions have arisen during last years that remain still unanswered. Four-dimensional manifolds of neutral signature appear as the simplest unsolved case to study. Therefore, we devote Chapters 2, 3 and 4 to the study of Osserman manifolds in signature $(2, 2)$. We first relate the eigenvalue structure of the Jacobi operator of an Osserman algebraic curvature tensor with the eigenvalue structure of the self-dual (or anti-self-dual) Weyl operator. Thus, the four possibilities (depending on the eigenvalues and the Jordan normal form) are in one to one correspondence between the two operators. Next we use Walker metrics to provide examples which realize all those four types for the conformal Jacobi operator; this is, we show the existence of conformally Osserman metrics realizing all algebraic possibilities.

Although the conformally Osserman property is local, we are also interested in global properties such as completeness. Hence, we study geodesic completeness in several Osserman and conformally Osserman examples, all of them being Walker manifolds. Another global topological condition such as compactness will allow us to better understand the Jordan Osserman condition in signature $(2,2)$. The following theorem will be proved in Chapter 4:

Theorem 4.1.3. *Let (M,g) be a compact Jordan-Osserman manifold with metric of signature $(2,2)$. Then (M,g) has either constant sectional curvature or nilpotent Jacobi operators.*

In the last chapter of Part I we study the conformally Osserman and the Osserman condition in manifolds with a given structure, namely a warped product structure. In summary, we see that the warped product structure is so rigid that the conformally Osserman condition is equivalent to local conformal flatness for a Riemannian warped product. On the other hand, this is not the case in higher signature and a explicit example will be given. These considerations, together with the fact that the Osserman condition is equivalent to the conformally Osserman condition and the Einstein property, lead us to a better understanding of Osserman manifolds whose underlying structure is that of a warped or a twisted product.

Motivated by results in Chapter 5, we develop in Part II an extensive analysis of locally conformally flat manifolds whose metric structure is that of a warped product. Using the fact that every warped product is in the conformal class of a direct product, in Chapter 6 we characterize local conformal flatness:

Theorem 6.1.2. *Let $(M,g) = B \times_f F$ be a pseudo-Riemannian warped product. Then the following hold:*

(i) *If $\dim B = 1$, then (M,g) is locally conformally flat if and only if (F, g_F) is a space of constant curvature.*

(ii) *If $\dim B > 1$ and $\dim F > 1$, then (M,g) is locally conformally flat if and only if*

  (ii.a) *(F, g_F) is a space of constant curvature K^F.*

  (ii.b) *The function $f : B \to \mathbb{R}^+$ defines a global conformal deformation on B such that $(B, \frac{1}{f^2} g_B)$ is a space of constant curvature $\tilde{K}_B = -K^F$.*

(iii) *If $\dim F = 1$, then (M,g) is locally conformally flat if and only if the function $f : B \to \mathbb{R}^+$ defines a conformal deformation on B such that $(B, \frac{1}{f^2} g_B)$ is a space of constant curvature.*

Due to the fact that the domain of the warping function is the base manifold B, the geometric properties of B strongly influence the geometry of the whole manifold $B \times_f F$.

Thus, special attention is paid to warped products with base a model space (Euclidean, hyperbolic or spherical). Furthermore, we show that global properties on the base, such as compactness or geodesic completeness, have interesting consequences.

There are several ways of generalizing the warped structure of a warped product manifold. One is by enlarging the domain of the warping function, thus getting a twisted product. It was shown in [68] that an Einstein twisted product is indeed a warped product (except if the fiber has dimension one). We show that a locally conformally flat twisted product is also a warped one if the dimension of their factors is greater than one. Therefore we may apply the previous study to these kind of manifolds.

A different way of generalizing warped products is by adding new fibers with their corresponding warping functions, thus getting the so-called multiply warped structure. In Chapter 7 we study local conformal flatness on manifolds with this structure; we distinguish two cases depending on the dimension of the base as in Theorems 7.2.6 and 7.3.1 and Remark 7.3.4. An unexpected consequence of both analysis is the following restriction on the number and the geometry of the fibers:

> *Let $B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ be a locally conformally flat multiply warped product. Then:*
>
> - *the number of fibers is less or equal than $\dim B + 2$,*
> - *all the fibers are of constant sectional curvature,*
> - *the sign of the curvature of the fibers depends on the signature of the base; thus, if the signature is Riemannian, there is at most one fiber of negative curvature.*

Similarly, we work out some interesting results on manifolds with this structure which have constant sectional curvature.

Locally conformally flat manifolds are far from being completely classified. However, some results are known if the Ricci curvature is positive. For instance, a complete simply connected locally conformally flat manifold of nonnegative Ricci curvature is in the conformal class of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{R} \times \mathbb{S}^{n-1}$. In Chapter 8 we take advantage of the results in Chapters 6 and 7 to construct new examples of complete locally conformally flat manifolds of negative (Ricci) curvature.

Since many of the cosmological models describing the Universe are multiply warped products, both isotropic and anisotropic, one uses the characterizations given previously to attain a better understanding of the geometry of these models. These applications are also described in Chapter 8.

Recall here that the main objective when studying Osserman-like problems is to obtain geometric consequences of the constancy of the eigenvalues of a certain operator which is closely related to the curvature tensor. In Part III we turn our attention to the task of classifying manifolds with commuting curvature operators. Thus, the core of our study here is not the eigenvalues of a curvature operator but the corresponding eigenspaces, that we analyze by means of commutativity properties. For example, the aim of Chapter 9 is to prove Theorem 9.3.1, that we rephrase as follows:

> *Let (M, g) be a Riemannian manifold of dimension n. Then the Jacobi operators commute for arbitrary directions, i.e.*
>
> $$\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x) \qquad \text{for any} \quad x, y,$$
>
> *if and only if (M, g) is flat; and if $n \geq 3$, the Jacobi operators commute for orthogonal directions, i.e.*
>
> $$\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x) \qquad \text{for any} \quad x \perp y,$$
>
> *if and only if (M, g) has constant sectional curvature.*

In Chapter 10 we obtain some partial results in this direction for higher signature manifolds, where an analogous characterization does not hold. For instance, we show that for dimension lower than 14, the condition $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ is equivalent to $\mathcal{J}(x)\mathcal{J}(y) = 0$ for any x, y. The example of minimal dimension which does not verify this equivalence occurs in dimension 14; we deepen into his geometric properties in Chapter 10, too.

Most of the conditions that one can study over the Jacobi operator can also be applied to the skew-symmetric curvature operator. Indeed, after the Osserman problem called the attention of the mathematics community, Ivanov-Petrova manifolds arose as a parallel problem motivated by the Osserman one and the behavior of the curvature tensor along unit circles. Hence, in Chapter 11 we study algebraic models whose skew-symmetric curvature operators commute and we give an algebraic classification of indecomposable algebraic models with definite signature. Moreover, we give several geometric examples which suggest that a geometric classification is much more complicated.

Let π be a k-plane on the tangent space of a point p. For $\{e_1, \dots, e_k\}$ a basis of π, one defines the Jacobi operator of order k by

$$\mathcal{J}(\pi) := \sum_{i=1}^{k} \mathcal{J}(e_i)\,.$$

As an Osserman-like problem, Gilkey [82] characterized Riemannian manifolds whose Jacobi operators of order k have constant eigenvalues, showing that for $2 \leq k \leq n-2$ they are manifolds of constant sectional curvature. For $k = 1$ or $k = n-1$ this is the Osserman condition. Let (M, g, J) be an almost Hermitian manifold. A natural generalization of the Jacobi operator to the complex setting is giving by

$$\mathcal{J}(\pi_x) = \mathcal{J}(x) + \mathcal{J}(Jx),$$

where $\pi_x = \operatorname{Span}\{x, Jx\}$. One of the features that makes the usual Jacobi operator so special is that it completely determines the curvature tensor. Thus, it looks natural to wonder if this is true for the complex Jacobi operator we have just defined. The first task in Part IV is to answer this question and to understand when the complex Jacobi operator determines the curvature tensor of an almost Hermitian manifold. This is the purpose of Chapter 12. There we will show in Theorem 12.1.3 that:

> *For (M, g, J) a Hermitian or a nearly Kähler manifold, the complex Jacobi operator completely determines the curvature tensor.*

The above is not true in the general almost Hermitian setting (see Theorem 12.1.2 for an explicit example). Some results concerning almost Kähler manifolds or dealing with conformally equivalent manifolds are also obtained.

Afterwards, we begin the study of complex Osserman manifolds, defined in a natural fashion as those almost Hermitian manifolds whose complex Jacobi operators have constant eigenvalues. In Chapter 13 we obtain some general results, thus showing that these kind of manifolds are Einstein and, moreover, that the eigenvalue structure of their complex Jacobi operators is controlled as follows (Theorem 13.1.7):

> *Let (M, g, J) be a complex Osserman almost Hermitian manifold of dimension n which verifies the compatibility condition $J^*R = R$ and such that the Jacobi operator determines the curvature tensor. Then the eigenvalues of the complex Jacobi operator verify the following:*
>
> 1. *If $n \equiv 2 \pmod 4$, there are 2 eigenvalues with multiplicities $(n-2, 2)$.*
> 2. *If $n \equiv 0 \pmod 4$, then one of the following holds:*
>   (a) *There are 2 eigenvalues with multiplicities $(n-2, 2)$.*
>   (b) *There are 2 eigenvalues with multiplicities $(n-4, 4)$.*
>   (c) *There are 3 eigenvalues with multiplicities $(n-4, 2, 2)$.*

We also use Clifford families to construct examples, mainly at the algebraic level, and show that all these possibilities occur. Furthermore, we delve into the structure of algebraic curvature tensors given by Clifford families to analyze which of them are complex Osserman.

In the last chapter we concentrate on Kähler manifolds. The fact that the complex structure is parallel has geometric consequences which allow us to obtain some partial results when the complex Jacobi operators have 2 eigenvalues of multiplicities $(n-2, 2)$. The main result of the chapter is the complete classification of complex Osserman Kähler manifolds in dimension 4:

Theorem 14.2.9. *Let $\mathcal{M} = (M, g, J)$ be a 4-dimensional Kähler manifold. Then $\mathcal{M}$ is pointwise complex Osserman if and only if it is of constant holomorphic sectional curvature.*

Contents

Chapter 1

Preliminaries

This first chapter of the thesis is devoted to establishing basic notation as well as to recalling some fundamental results we will use in subsequent chapters. Thus, in the first sections we settle the framework for our investigation; afterwards, we define the specific concepts which are in the focus of our work. Primary properties will be enumerated without a proof, as our purpose is just to recall some well known facts we will need later.

We may refer to [139] for most of the content of this chapter. However, any introduction to pseudo-Riemannian Geometry can be helpful, for example we cite monographs [113, 122, 153, 157], and more specific books as [77, 84, 85]. More concrete results which may not appear on a introductory course will be cited appropriately.

1.1 General context: pseudo-Riemannian manifolds

In this section we fix the context we are going to work in and establish the more general setup. There are also some conventions we adopt here that will remain valid throughout this memoir. The principal object of interest in our study are pseudo-Riemannian manifolds. This is, we consider an n-dimensional manifold M endowed with a metric g of signature (p, q) to form the pair $\mathcal{M} = (M, g)$.

As a matter of notation, let TM be the tangent bundle and T_pM be the tangent space at the point $p \in M$. Let $\mathfrak{X}(M)$ be the space of all tangent vector fields on M. As a general rule, we will denote by capital letters X, Y, Z, U, V, W tangent vector fields and by small letters x, y, z, u, v, w tangent vectors at a particular point of the manifold.

Let v be a non-zero vector in T_pM. We say x is *timelike* if $g(x, x) < 0$, *spacelike* if $g(x, x) > 0$ and *null* if $g(x, x) = 0$. For unit vectors, we let $\varepsilon_x = g(x, x)$. The corresponding pseudo-sphere in T_pM is

$$S_p(\mathcal{M}) := \{v \in T_pM : |g(v, v)| = 1\}.$$

The corresponding bundle in TM is then $S(\mathcal{M}) := \bigcup_{p\in M} S_p(\mathcal{M})$. When we restrict the character to timelike or spacelike we use notation $S_p^{\pm}(\mathcal{M}) := \{v \in T_pM : g(v, v) = \pm 1\}$

for the pseudo-spheres and $S^{\pm}(\mathcal{M}) := \bigcup_{p\in M} S^{\pm}_p(\mathcal{M})$ for the corresponding subbundles.

The Levi-Civita connection on $\mathcal{M}$ is denoted by ∇. Its characterization is given by the Koszul formula as follows:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad + g(X,[Z,Y]) + g(Y,[Z,X]) + g(Z,[X,Y]), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

We also denote by ∇ the *gradient* operator on $\mathcal{M}$; note that the *gradient* of a function $f : (M, g) \to \mathbb{R}$ is determined by $g(\nabla f, X) = X(f)$. The *divergence* of a vector field X is defined as $div\, X = trace\, \nabla X$. The *Hessian tensor* h_f of a real-valued function $f : M \to \mathbb{R}$ is defined by $h_f(X) = \nabla_X \nabla f$, where $X \in \mathfrak{X}(M)$. Also the symmetric $(0,2)$-tensor field H_f defined by $H_f(X,Y) = g(h_f(X), Y)$ is called the *Hessian form* of f. We define the *Laplacian* Δf by $\Delta f = \text{trace}\ h_f = div\, \nabla f$.

We define the *curvature operator* $\mathcal{R}$ as

$$\mathcal{R}(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

from where the $(0,4)$-*curvature tensor* is given by

$$R(X,Y,Z,V) = g(\mathcal{R}(X,Y)Z, V).$$

The curvature tensor verifies the following symmetries:

$$R(X,Y,Z,V) = -R(Y,X,Z,V) = R(Z,V,X,Y), \tag{1.1}$$

$$R(X,Y,Z,V) + R(Y,Z,X,V) + R(Z,X,Y,Z) = 0. \tag{1.2}$$

Note that expression (1.2) is the *First Bianchi Identity.* Also, the curvature tensor verifies the following differential identity:

$$(\nabla_X R)(Y,Z,U,V) + (\nabla_Y R)(Z,X,U,V) + (\nabla_Z R)(X,Y,U,V) = 0,$$

which is usually known as *Second Bianchi Identity.*

1.2 Algebraic preliminaries

When studying a problem in pseudo-Riemannian geometry, it is sometimes useful to think of it, if possible, from an algebraic point of view. This is, one may study the condition in the tangent space of an arbitrary point in the manifold, so that one works in a vector space. We will see that this way of proceeding is often a helpful tool. In this section we set terminology in a purely algebraic context.

Let V be an n-dimensional real vector space endowed with a non-degenerate inner product $\langle \cdot, \cdot \rangle$ of signature (p,q). To be consistent with the notation established in the previous section, we use $x, y, z, ...$ to denote vectors in V. Thus, for given $(V, \langle\cdot,\cdot\rangle)$ we may also define the pseudo-spheres $S^{\pm}(V, \langle\cdot,\cdot\rangle) = \{x \in V : \langle x, x\rangle = \pm 1\}$ or

$$S(V, \langle\cdot,\cdot\rangle) = \{x \in V : |\langle x, x\rangle| = 1\}.$$

A $(0, 4)$-tensor A is said to be an *algebraic curvature tensor* if it verifies the symmetries of Equations (1.1) and (1.2). Thus, an algebraic model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is the triple consisting of the vector space V, the inner product $\langle\cdot,\cdot\rangle$ and the algebraic curvature tensor A. When we refer to the curvature operator we use calligraphic notation $\mathcal{A}$. Henceforth we will use sometimes the notation R for both the curvature tensor of a manifold and the algebraic curvature tensor of an algebraic model; the meaning will be clear from context.

1.3 Curvature decomposition: the Weyl tensor

In this section we introduce basic operators associated to the curvature tensor. Since all these concepts are defined pointwise, we may think of them in a point p of a pseudo-Riemannian manifold $\mathcal{M}$. Consequently these definitions are automatically translated to the purely algebraic setting of a model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$.

For a given basis $\{e_1, \ldots, e_n\}$ of vector fields, set $g_{ij} = g(e_i, e_j)$ and let (g^{ij}) denote the inverse matrix. Then the associated Ricci tensor ρ_R and scalar curvature τ_R are given by

$$\rho_R(x, y) = \sum_{i,j=1}^{n} g^{ij} R(x, e_i, y, e_j), \qquad \tau_R = \sum_{i,j=1}^{n} g^{ij} \rho_R(e_i, e_j). \tag{1.3}$$

Also, recall that the sectional curvature K of a non-degenerate plane $\pi = \text{Span}\,\{x, y\}$ in T_pM is given by $K(\pi) = \frac{R(x,y,x,y)}{g(x,x)g(y,y)-g(x,y)^2}$. Moreover, a manifold has constant sectional curvature κ if and only if its curvature tensor is given by

$$R(x, y, z, w) = \kappa\{g(x, z)g(y, w) - g(y, z)g(x, w)\}.$$

This curvature tensor will play an important role in subsequent chapters, so we denote by $R_0(x, y, z, w) = g(x, z)g(y, w) - g(y, z)g(x, w)$ the algebraic curvature tensor of constant sectional curvature $+1$.

Using notation established above, the Weyl tensor W_R is defined by

$$\begin{aligned} W_R(x, y, z, v) \;=\;& R(x, y, z, v) + \tfrac{\tau_R}{(n-1)(n-2)}\{g(x, z)g(y, v) - g(y, z)g(x, v)\} \\ & -\tfrac{1}{n-2}\{\rho_R(x, z)g(y, v) - \rho_R(y, z)g(x, v) \\ & +\rho_R(y, v)g(x, z) - \rho_R(x, v)g(y, z)\}, \end{aligned} \tag{1.4}$$

for all $x, y, z, v \in T_pM$. We will avoid subindexes when it is clear from the context the curvature tensor we are working with. Moreover, sometimes it will be convenient to use subscript notation for the components of each tensor on the corresponding basis; thus, for example, $\rho_{ij} = \rho(e_i, e_j)$, $R_{ijkl} = R(e_i, e_j, e_k, e_l)$, ...

Let us introduce the following useful notation.

Definition 1.3.1 Let B, D be $(0,2)$-tensors. The *Kulkarni-Nomizu* product is defined as follows:

$$\begin{aligned}(B \bullet D)\,(X,Y,Z,V) \;&:=\; A(X,Z)B(Y,V) + A(Y,V)B(X,Z)\\ &\quad -A(X,V)B(Y,Z) - A(Y,Z)B(X,V),\end{aligned}$$

for any $X, Y, Z, V \in \mathfrak{X}(M)$.

Remark 1.3.1 Given any two symmetric $(0,2)$-tensors B, D, the Kulkarni-Nomizu product $B \bullet D$ gives an algebraic curvature tensor. In particular note that $g \bullet g = 2R_0$. Furthermore, the space of all algebraic curvature tensors is spanned by elements of the form $B \bullet B$ (see [59], [70] and [84] for three different proofs).

The next theorem provides a meaningful decomposition of the curvature tensor.

Theorem 1.3.2 [113] *An algebraic curvature tensor A decomposes as $A = \mathfrak{U} + \mathfrak{Z} + W$, where*

$$\mathfrak{U} = \frac{\tau}{2n(n-1)}\, g \bullet g, \quad \mathfrak{Z} = \frac{1}{n-2}\left(\rho - \frac{\tau}{n} g\right) \bullet g, \quad \text{and} \quad W = R - \mathfrak{U} - \mathfrak{Z} = R - C \bullet g,$$

where $C = \frac{1}{n-2}\left(\rho - \frac{\tau}{2(n-1)} g\right)$ is the Schouten tensor.

Remark 1.3.3 One can reinterpret curvature tensors as lying in the space of 2-forms $\Lambda(V)$. Under this terminology, components $\mathfrak{U}$, $\mathfrak{Z}$ and W given in Theorem 1.3.2 are orthogonal with respect to the metric

$$\langle B, D\rangle := \operatorname{tr}(B \circ D),$$

for B, D operators in $\Lambda(V)$. Moreover, each of the components $\mathfrak{U}$, $\mathfrak{Z}$ and W of the decomposition above has a geometrical meaning:

- The component $\mathfrak{U}$ is the orthogonal projection on the space of constant sectional curvature tensors.
- A pseudo-Riemannian manifold $\mathcal{M}$ is said to be *Einstein* if its Ricci tensor is a scalar multiple of the metric. In such a case one has $\rho = \frac{\tau}{n} g$. Therefore, one has that the Einstein condition is equivalent to $\mathfrak{Z} = 0$.
- A pseudo-Riemannian manifold $\mathcal{M} = (M, g)$ is said to be *locally conformally flat* if for every point $p \in M$ there exists an open neighborhood U, $p \in U$, and a conformal change e^σ, $\sigma : U \longrightarrow \mathbb{R}$, such that $g = e^\sigma g_0$ where g_0 is the Euclidean metric.

  Locally conformally flat manifolds are characterized by means of properties of tensors associated to the curvature depending on the dimension of the manifold. Thus, 3-dimensional manifolds are locally conformally flat if and only if the Schouten tensor is Codazzi, i.e. totally symmetric

  $$(\nabla_X C)(Y,Z) = (\nabla_Y C)(X,Z).$$

In dimension $n \geq 4$, a pseudo-Riemannian manifold is locally conformally flat if and only if its Weyl tensor vanishes, i.e. $W = 0$.

1.3.1 Self-duality and anti-self-duality in dimension 4

Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a 4-dimensional algebraic model, let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis in V and let $\{e^1, e^2, e^3, e^4\}$ be its associated dual. A specific feature of dimension 4 comes from the properties of the Hodge star operator, which acts on the space of 2-forms $\Lambda = \text{Span}\{e^i \wedge e^j : i, j \in \{1, 2, 3, 4\}, i < j\}$ in the following way

$$e^i \wedge e^j \wedge \star(e^k \wedge e^l) = (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k)\, \varepsilon_i \varepsilon_j\, e^1 \wedge e^2 \wedge e^3 \wedge e^4,$$

where $\epsilon_i = g(e_i, e_i)$. Since $\star^2 = Id$ for any inner product of definite or neutral signature, the Hodge star operator induces a splitting $\Lambda = \Lambda^+ \oplus \Lambda^-$, where Λ^+ and Λ^- denote the spaces of self-dual and anti-self-dual 2-forms

$$\Lambda^+ = \{\alpha \in \Lambda : \star\alpha = \alpha\}, \qquad \Lambda^- = \{\alpha \in \Lambda : \star\alpha = -\alpha\}.$$

Next put $W_A^\pm$ for the restriction of the Weyl tensor W_A to the spaces $\Lambda^\pm$. Then A is said to be a *self-dual* (resp. *anti-self-dual*) algebraic curvature tensor if $W_A^- = 0$ (resp. $W_A^+ = 0$). Finally observe that the induced inner products on $\Lambda^\pm$ are positive definite if g is Riemannian, but they are Lorentzian if g is of neutral signature.

An orthonormal basis for the self-dual and anti-self-dual space is given by

$$\begin{aligned}\Lambda^\pm = \text{Span}\, \big\{ E_1^\pm = (e^1 \wedge e^2 \pm \epsilon_3\epsilon_4 e^3 \wedge e^4)/\sqrt{2}, E_2^\pm = (e^1 \wedge e^3 \mp \epsilon_2\epsilon_4 e^2 \wedge e^4)/\sqrt{2},\\ E_3^\pm = (e^1 \wedge e^4 \pm \epsilon_2\epsilon_3 e^2 \wedge e^3)/\sqrt{2} \big\}\,.\end{aligned} \tag{1.5}$$

Henceforth we will use these two bases when coordinates in the self-dual or anti-self-dual space are needed, except where indicated explicitly to the contrary.

1.4 Curvature operators

Since the whole curvature tensor is very difficult to handle, one often focuses on the study of properties of simpler objects which nevertheless determine the whole curvature tensor. In this section we recall the definitions as well as some basic features of such operators. Although some of the definitions are given in an algebraic context, they automatically translate to the geometrical setting.

1.4.1 The Jacobi operator and the Osserman condition

Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be an algebraic model. The *Jacobi operator* $\mathcal{J}_A$ associated to A is the self-adjoint map on V defined by

$$\mathcal{J}_A(x)y = \mathcal{A}(x, y)x.$$

As a matter of notation, sometimes it will be convenient to avoid the subindex A, if it is clear from the context, or express the Jacobi operator as $\mathcal{J}_x := \mathcal{J}(x)$.

$\mathcal{V}$ is said to be *spacelike* (resp., *timelike*) *Osserman* if the eigenvalues of $\mathcal{J}_A$ are constant on $S^+(V, \langle\cdot,\cdot\rangle)$ (resp., on $S^-(V, \langle\cdot,\cdot\rangle)$). Now, observe that the results of [76, 84] show the equivalence between spacelike and timelike Osserman conditions. In fact if $p > 0$ and if $q > 0$, one only needs to assume that the spectrum of $\mathcal{J}$ is bounded on $S^+(\mathcal{V})$ or on $S^-(\mathcal{V})$ to ensure $\mathcal{V}$ is pointwise Osserman (see [15]). Therefore, from now on we just refer to $\mathcal{V}$ as Osserman or to A as an Osserman algebraic curvature tensor if any of these conditions is satisfied. Note that the eigenvalues of the Jacobi operators $\mathcal{J}_A(x)$ change the sign when moving from timelike to spacelike directions, nevertheless they remain constant on each of $S^-(V, \langle\cdot,\cdot\rangle)$ and $S^+(V, \langle\cdot,\cdot\rangle)$.

Although we have first defined the Osserman condition in the algebraic setting, it was historically introduced in the differentiable setting, where it is a little bit more tricky since there is a distinction between the global and the pointwise condition. More specifically, $\mathcal{M}$ is said to be *globally Osserman* if the eigenvalues of $\mathcal{J}_R(X)$ do not depend on the vector field $X \in \mathfrak{X}(M)$. On the other hand, $\mathcal{M}$ is *pointwise Osserman* if the eigenvalues of $\mathcal{J}_R(x)$ do not depend on $x \in T_pM$, but they may change from point to point.

The fact that the local isometries of any locally two-point homogeneous space (i.e. a flat space or a space locally isometric to a rank 1-symmetric space or their non compact duals), act transitively on the unit pseudo-sphere bundles implies that any locally two-point homogeneous space is Osserman. This terminology of Osserman manifolds is motivated by the paper [145] where R. Osserman conjectured that a globally Osserman Riemannian manifold is two-point-homogeneous. This was proved by Chi in dimension 4 and in any dimension which is not a multiple of 4 (i.e., $2k+1$ and $4k+2$ for $k = 1, 2, 3, \ldots$), see [51] and also [52], and by Nikolayevsky in dimension $4k$ with $4k \neq 16$ (see [135, 136]), following a two step process suggested in [89]. There is a similar classification result in the Lorentzian setting. It is known [16, 75] that any Osserman Lorentzian manifold has constant sectional curvature. In the higher signature setting, such classification results fail. There are, for instance, Osserman manifolds that are not even locally affine homogeneous [64, 86]. As a stronger condition *Jordan-Osserman* manifolds are those Osserman manifolds whose Jacobi operator has not only constant eigenvalues but also constant Jordan normal form.

Later on, the constancy of eigenvalues of several operators such as the higher order Jacobi operator, the skew-symmetric curvature operator or the Szabó operator were studied. We recommend [77], [84] and [85] for nice expositions in the field.

1.4.2 The conformally Osserman condition

From an algebraic point of view, the concept of Osserman conformality can be thought of as a specialization of the Osserman condition to the Weyl tensor, consequently it appeared later historically. Since the Weyl tensor is an algebraic curvature tensor we may consider the associated Jacobi operator $\mathcal{J}_W$. Following [18] we refer to it as the *conformal Jacobi* *operator* and $\mathcal{M}$ is said to be *conformally Osserman* if (T_pM, g_p, W_p) is Osserman at each point $p \in M$. We shall make clear that the eigenvalues of $\mathcal{J}_W$ are allowed to change from point to point. An important feature of Osserman conformality is that it is conformally invariant as was proved in [20]. If the metric is not definite, one has to be careful since spacelike and timelike directions provide different eigenvalues; however, since Osserman conformality is a particularization of the Osserman property, it inherits its properties and spacelike conformally Osserman is equivalent to timelike conformally Osserman.

The following result, first proved in [20], relates Osserman condition with Osserman conformality, thereby improving the understanding of the role played by these two concepts in the description of the geometry of the manifold.

Theorem 1.4.1 *Let $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A)$ be a model. Then $\mathcal{V}$ is Osserman if and only if $\mathcal{V}$ is Einstein and conformally Osserman.*

Proof. First recall that if A is Osserman then it is necessarily Einstein. Therefore we shall show that if A is Einstein the eigenvalues of $\mathcal{J}$ are constant if and only if the eigenvalues of $\mathcal{J}_W$ are constant. Suppose A is Einstein, then by Theorem 1.3.2 the curvature tensor is written as

$$A = \frac{\tau}{n(n-1)} A_0 + W. \tag{1.6}$$

This equality shows $\mathcal{J}(x) = \frac{\tau}{n(n-1)} Id + \mathcal{J}_W(x)$ on $x^\perp$ and hence the result follows. □

The previous observation extends automatically to the geometric context of pointwise Osserman manifolds, thus showing that a pseudo-Riemannian manifold $\mathcal{M}$ is pointwise Osserman if and only if it is Einstein and conformally Osserman.

Let $\mathcal{M} = (M, g)$ be a pseudo-Riemannian manifold. For a function $\phi : M \to \mathbb{R}^+$, define the *Schwarzian tensor* as

$$B(\phi) = H_\phi - d\phi \otimes d\phi - \frac{1}{n}\{\Delta\phi - \|\nabla\phi\|^2\}g,$$

which is symmetric and traceless. The equation $B(\phi) = 0$ is called the *Möbius equation* and characterizes conformal changes which preserve the Ricci eigenspaces. Thus, a conformal transformation ϕ preserves the Einstein property if and only if $B(\phi) = 0$. In such a case we say that ϕ is a *Möbius transformation*.

Note, as a consequence of Theorem 1.4.1, that the Osserman condition is preserved by a conformal transformation if and only if it preserves the Einstein property, this is, if and only if it is a Möbius transformation. Let $\mathcal{M}$ be an Osserman pseudo-Riemannian manifold. Then every manifold in its conformal class is conformally Osserman. Nevertheless, it is not true in general that a conformally Osserman manifold is in the conformal class of an Osserman manifold. Examples of this fact will be explicitly given in Chapter 3. Observe that, as a consequence of Theorem 1.4.1, a conformally Osserman manifold is in the conformal class of an Osserman manifold if and only if there exists an Einstein manifold in its conformal class.

1.4.3 The skew-symmetric curvature operator

Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be an algebraic model. For vectors x, y, the *skew-symmetric curvature operator* $\mathcal{A}(x, y)$ is given by

$$\langle \mathcal{A}(x, y)z, w\rangle = A(x, y, z, w),$$

where here we emphasize the role played by arguments x and y, with a skew-symmetric behavior.

If $\{e_1, e_2\}$ is an oriented orthonormal basis for an oriented non-degenerate 2-plane π, the skew-symmetric curvature operator $\mathcal{A}_\pi = \mathcal{A}(\pi)$ is given by

$$\mathcal{A}_\pi z = \mathcal{A}(\pi)z := \mathcal{A}(e_1, e_2)z\,.$$

This definition is independent of the particular oriented orthonormal basis chosen. Note that the skew-symmetric curvature operator is skew-adjoint, in contrast to the Jacobi operator which is self-adjoint. This justifies in many cases the different properties that both operators present in certain instances.

1.4.4 The higher order Jacobi operators

Here, we follow the discussion of Stanilov and Videv [160] to define a *higher order Jacobi operator* as follows. Let $\{e_1, ..., e_k\}$ be an orthonormal basis for a k-plane Π. Set

$$\mathcal{J}(\Pi) = \sum_{i=1}^{k} \epsilon_i \mathcal{J}(e_i)\,;$$

this is independent of the particular orthonormal basis chosen and, roughly speaking, it measures the average of the Jacobi operators in the k-plane Π. Indeed, if $\mathcal{M}$ is Riemannian,

$$\mathcal{J}(\Pi) = c(k) \int_{x\in S(\Pi,\langle\cdot,\cdot\rangle)} \mathcal{J}(x)dx,$$

where $c(k)$ is a suitably chosen normalizing constant. If $k = 1$, one recovers the ordinary Jacobi operator. Furthermore, if $k = n$ then $\rho := \mathcal{J}(V)$ is the *Ricci operator*; thus suggesting that the higher order Jacobi operator can also be thought of as a generalization of the Ricci operator to lower dimensional subspaces.

One says that a model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is *k-Osserman* if the eigenvalues of $\mathcal{J}(\Pi)$ are constant on the Grassmannian $Gr_k(V)$ of k-planes.

Gilkey proved in [82] that the geometry of a Riemannian k-Osserman model is very rigid:

- If $k = 1$ or $k = n - 1$, then $\mathcal{V}$ is k-Osserman if and only if it is Osserman.

- If $2 \leq k \leq n-2$, then $\mathcal{V}$ is k-Osserman if and only if there exists a constant c such that $A = cA_0$ or $A = cA_J$, where J is a complex structure. Geometric consequences of this algebraic classification are then obtained by using the Second Bianchi Identity, showing that a Riemannian manifold $\mathcal{M}$ is k-Osserman if and only if it has constant sectional curvature.

1.4.5 The complex Jacobi operator and the complex Osserman condition

Let us consider an algebraic model $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A)$, in which we have a complex structure. We give the following definition.

Definition 1.4.1 *Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Let A be an algebraic curvature tensor on V and let J be a Hermitian complex structure on $(V, \langle \cdot, \cdot \rangle)$. Then we say the quadruple $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A, J)$ is a* complex model.

Let $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A, J)$ be a complex model. If $\pi \in \mathbb{CP}(V, J)$, let $S(\pi)$ be the set of unit vectors in π. Let $x \in S(\pi)$ for $\pi \in \mathbb{CP}(V, J)$. Denote $\pi_x := \text{Span}\{x, Jx\}$. For any $x \in S(\pi)$ we define

- the *holomorphic sectional curvature* $Q(\pi_x)$:

$$Q(\pi_x) := A(x, Jx, x, Jx),$$

- the *complex Jacobi operator* $\mathcal{J}(\pi_x)$:

$$\mathcal{J}(\pi_x) := \mathcal{J}(x) + \mathcal{J}(Jx),$$

- and the *complex skew-symmetric curvature operator* $\mathcal{R}(\pi_x)$:

$$\mathcal{R}(\pi_x) := \mathcal{R}(x, Jx).$$

Note that $Q(\pi_x)$, $\mathcal{J}(\pi_x)$ and $\mathcal{R}(\pi_x)$ do not depend on the choice of x. These definitions are translated to the geometric setting for almost Hermitian manifolds in a natural way.

In order to define the complex Osserman concept, instead of the 2-Osserman condition where the eigenvalues are constant on the Grassmannian of 2-planes, we consider a natural weaker condition with constant eigenvalues on the space of holomorphic planes $\mathbb{CP}(V, J)$. Thus, we give the following definition.

Definition 1.4.2 *Let $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A, J)$ be a complex model. We say that $\mathcal{V}$ is* complex Osserman *if*

1. *J and A are compatible, i.e. $J^*A = A$.*
2. *The eigenvalues of $\mathcal{J}_A(\pi_x)$ are constant on $\mathbb{CP}(V, J)$.*

We shall also sometimes simply say that A is complex Osserman *in this situation.*

Note that we impose an extra condition of compatibility between the curvature and the almost complex structure. Although it may seem a bit artificial, we will see in Chapter 12 that it is indeed very natural. Moreover, some important families of almost Hermitian manifolds satisfy that condition.

In Part IV we will study the complex Jacobi operator and we will give some partial results on the classification of complex Osserman models and manifolds. Furthermore, we will give a complete classification of complex Osserman Kähler manifolds in dimension 4.

1.4.6 Commutativity of operators associated to the curvature tensor

In Part III of this memoir we study how some commutativity properties of natural operators associated to the curvature tensor influence the geometry of the manifold. A seminal paper on the subject is due to Tsankov [167], who first studied these questions for Riemannian hypersurfaces. In the present section we establish some notation we will use later and describe a few basic properties. We will use subscripts more than usual in this section with the purpose of simplifying notation.

Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be an algebraic model. The Jacobi operator $\mathcal{J}(x)$ is quadratic in x and can be polarized to define a bilinear operator $\mathcal{J}_{xy} = \mathcal{J}(x, y)$ by setting

$$\mathcal{J}_{xy}z = \mathcal{J}(x, y)z := \frac{1}{2}\{\mathcal{A}(x, z)y + \mathcal{A}(y, z)x\}\,.$$

The following identities are immediate consequences of the previous definition:

$$\begin{aligned} &\mathcal{J}_x = \mathcal{J}_{xx}, \qquad \mathcal{J}_{xy}y = -\tfrac{1}{2}\mathcal{J}_y x, \\ &\mathcal{J}_{\cos\theta x+\sin\theta y} = \cos^2\theta\mathcal{J}_x + 2\cos\theta\sin\theta\mathcal{J}_{xy} + \sin^2\theta\mathcal{J}_y\,. \end{aligned} \tag{1.7}$$

It is convenient to set from the beginning the notation we are going to work with. We establish the main specific terminology in next definition; we first define concepts in the algebraic context to later extend them to the differentiable setting.

Definition 1.4.3 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be an algebraic model. We say that*

1. *$\mathcal{V}$ is* Jacobi Tsankov *if $\mathcal{J}_x\mathcal{J}_y = \mathcal{J}_y\mathcal{J}_x$ for all $x, y \in V$,*
2. *$\mathcal{V}$ is* orthogonally Jacobi Tsankov *if $\mathcal{J}_x\mathcal{J}_y = \mathcal{J}_y\mathcal{J}_x$ for all $x, y \in V$ with $x \perp y$,*
3. *$\mathcal{V}$ is* skew Tsankov *if $\mathcal{A}_\pi\mathcal{A}_\sigma = \mathcal{A}_\sigma\mathcal{A}_\pi$ for all non-degenerate oriented 2-planes π and σ,*
4. *$\mathcal{V}$ is* orthogonally skew Tsankov *if $\mathcal{A}_\pi\mathcal{A}_\sigma = \mathcal{A}_\sigma\mathcal{A}_\pi$ for all non-degenerate oriented 2-planes π and σ with $\pi \perp \sigma$.*

In order to translate all these concepts to the differentiable setting, let $\mathcal{M} = (M, g)$ be a pseudo-Riemannian manifold. Then $\mathcal{M}$ is said to have one of the properties discussed above if and only if the associated model $\mathcal{V}(\mathcal{M}, p) = (T_pM, g_p, R_p)$ has such a property for all points $p \in M$.

1.5 Manifolds with special structure

The existence of nontrivial solutions of the Möbius equation influences the local structure of the manifold, which must be a warped product (see [67, 115, 119, 144]). Moreover, local twisted products are the underlying structure of manifolds admitting Codazzi tensors in many cases. Therefore twisted and warped product decompositions are natural structures to be analyzed when considering conformal properties.

1.5.1 Warped product structures

A simple geometric structure is that of a direct product manifold, this is, a manifold $\mathcal{M} = (M, g)$ which decomposes as $\mathcal{M} = (B \times F, g_B \oplus g_F)$. Note that these manifolds are reducible at every point. Now, we are going to slightly modify the metric of one of these factors to get a much more interesting structure.

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and let $f : B \longrightarrow \mathbb{R}^+$ be a positive function on B. The product manifold $M = B \times F$ endowed with the metric

$$g = g_B \oplus f^2 g_F$$

is called a *warped product*; B is called the *base*, F is called the *fiber* and f is called the *warping function*. We denote by $B \times_f F$ the warped product with base B, fiber F and warping function f.

The following results provide a basic description of the geometry of a manifold with warped product structure.

Lemma 1.5.1 [139] *Let $\mathcal{M} = B \times_f F$ be a warped product. Let $X, Y \in \mathfrak{X}(B)$ and $U, V \in \mathfrak{X}(F)$, the Levi-Civita connection is given by:*

(i) $\nabla_X Y$ *is the lift of* $\nabla^B_X Y$,

(ii) $\nabla_X U = \nabla_U X = \frac{X(f)}{f} U$,

(iii) $nor(\nabla_U V) = II(U, V) = -\frac{\langle U,V \rangle}{f} \nabla f$,

(iv) $tan(\nabla_U V)$ *is the lift of* $\nabla^F_U V$,

where ∇^B and ∇^F denote the Levi-Civita connections of (B, g_B) and (F, g_F), respectively, and II is the second fundamental form with respect to F.

Let $\mathcal{M} = B \times_f F$ be a warped product. Denote by $\mathfrak{L}_B$ and $\mathfrak{L}_F$ the canonic foliations associated to the submanifolds determined by B and F. Note from Lemma 1.5.1 that $\mathfrak{L}_B$ is totally geodesic (i.e. the second fundamental form vanishes $II_F = 0$), whereas $\mathfrak{L}_F$ is spheric (i.e. $II_F = Zg_F$ for a Z orthogonal to $\mathfrak{L}_F$ and parallel in the normal bundle of $\mathfrak{L}_F$ [147]). The next result shows this is a characteristic fact of warped products among product spaces for orthogonal foliations:

Theorem 1.5.2 [147] *Let g be a metric defined on $B \times F$. If $\mathfrak{L}_B$ and $\mathfrak{L}_F$ intersect orthogonally, then g is a warped product if and only if $\mathfrak{L}_B$ is totally geodesic and $\mathfrak{L}_F$ is spheric.*

Warped products were introduced in [13] as a tool to construct Riemannian manifolds with non positive curvature. From that point on they were extensively studied, moreover they found one of the main motivations in Physics, where many models present this structure. Specially, many cosmological models are warped or multiply warped products.

The curvature of a warped product is described in next lemma.

Lemma 1.5.3 [139] *Let $\mathcal{M} = B \times_f F$ be a warped product. Let $X, Y, Z \in \mathfrak{X}(B)$ and let $U, V, W \in \mathfrak{X}(F)$. The curvature tensor R is given by:*

(i) $R(X,Y)Z$ *is the lift of* $R^B(X,Y)Z$ *on* B,

(ii) $R(U,X)Y = \frac{H_f(X,Y)}{f} U$,

(iii) $R(X,Y)U = R(U,V)X = 0$,

(iv) $R(X,U)V = \frac{\langle U,V\rangle}{f} \nabla_X(\nabla f)$,

(v) $R(U,V)W = R^F(U,V)W - \frac{\langle \nabla f, \nabla f\rangle}{f^2} (\langle U,W\rangle V - \langle V,W\rangle U)$.

From the expressions of the curvature, one easily obtains the expressions for the Ricci tensor.

Corollary 1.5.4 [139] *Let $\mathcal{M} = B \times_f F$ be a warped product with $d = dim\, F > 1$, $X, Y \in \mathfrak{X}(B)$ and $U, V \in \mathfrak{X}(F)$, then:*

(i) $\rho(X,Y) = \rho^B(X,Y) - \frac{d}{f} H_f(X,Y)$,

(ii) $\rho(X,U) = 0$,

(iii) $\rho(U,V) = \rho^F(U,V) - \langle U,V\rangle \left(\frac{\Delta f}{f} + (d-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2}\right)$.

From the previous corollary we deduce the expression for the scalar curvature of a warped product:

Corollary 1.5.5 *Let $\mathcal{M} = B \times_f F$ be a warped product, then the scalar curvature is given by:*

$$\tau = \tau^B + \frac{\tau^F}{f^2} - 2d\frac{\Delta f}{f} - d(d-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2},$$

where $d = dim\, F$.

As we said before, warped products were used to provide examples of Riemannian manifolds with negative curvature. The following result characterizes such manifolds:

Theorem 1.5.6 [13] *A Riemannian warped product $B \times_f F$ has negative sectional curvature $K < 0$ if and only if the following conditions hold:*

(i) $dim\, B = 1$ *or* $K^B < 0$,

(ii) f *is strictly convex,*

(iii) *(a)* $dim\, F = 1$, *or*
(b) $K^F < 0$ *if* f *has a minimum;* $K^F \leq 0$ *if* f *does not have a minimum.*

The warped product structure may be generalized in two different directions, namely, by adding an arbitrary number of fibers to get a *multiply warped product* or by extending the domain of the warping function to the whole manifold, thus obtaining a *twisted product.*

Let (B, g_B), (F_1, g_1), ..., (F_k, g_k) be pseudo-Riemannian manifolds. The product manifold $M = B \times F_1 \times \ldots \times F_k$ equipped with the metric

$$g = g_B \oplus f_1^2 g_1 \oplus \ldots \oplus f_k^2 g_k,$$

where $f_1, \ldots, f_k : B \longrightarrow \mathbb{R}^+$ are positive functions on B, is called a *multiply warped product.* B is the *base*, $F_1, \ldots, F_k$ are the *fibers* and $f_1, \ldots, f_k$ are referred to as the *warping functions.* We will denote a multiply warped product manifold as above by $B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$.

Since multiply warped products are obtained from warped products by adding new fibers, these multiple structures preserve the geometry of warped products. Moreover, a multiply warped product $B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ can be thought of as a warped product with base $B \times_{f_1} F_1 \times \ldots \times_{f_{k-1}} F_{k-1}$ and fiber F_k. Thus, the components of the curvature for a multiply warped product $B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ are the same as those of a warped product, and hence they are given by Lemma 1.5.3, except for the following additional components:

(vi) $R(U_j, U_i)V_i = \frac{\langle U_i, V_i \rangle}{f_i f_j} \langle \nabla f_i, \nabla f_j \rangle U_j$,

(vii) $R(U_i, V_i)W_i = R^{F_i}(U_i, V_i)W_i - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} (\langle U_i, W_i \rangle V_i - \langle V_i, W_i \rangle U_i)$,

where $U_i, V_i, W_i \in \mathfrak{X}(F_i)$.

1.5.2 Twisted product structures

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and let $f : B \times F \longrightarrow \mathbb{R}^+$ be a positive function on $B \times F$. The product manifold $M = B \times F$ endowed with the metric

$$g = g_B \oplus f^2 g_F$$

is called a *twisted product*. The same terminology of *base* and *fiber* applies in this case, whereas f is the *twisting function* (sometimes it is also referred to as the warping function).

Twisted products were introduced in [49] and systematically studied by Ponge and Reckziegel in [147]. In this subsection we are going to describe the geometry of a twisted product as we have done previously for a warped one. For simplicity in some of the expressions, we use $\xi = Log\, f$ instead of f.

Lemma 1.5.7 [67] *Let $\mathcal{M} = B \times_f F$ be a twisted product, $X, Y \in \mathfrak{X}(B)$ and $U, V \in \mathfrak{X}(F)$. The Levi-Civita connection of $\mathcal{M}$ behaves as follows*

(i) $\nabla_X Y$ *is the lift of* $\nabla^B_X Y$,

(ii) $\nabla_X U = X(\xi)U$,

(iii) $\nabla_U V = \nabla^F_U V + U(\xi)V + V(\xi)U - g(U, V)\nabla\xi$.

From previous lemma it follows that the canonical foliations with respect to B and F, that we denote by $\mathfrak{L}_B$ and $\mathfrak{L}_F$, are totally geodesic and totally umbilic (i.e. the second fundamental form can be written as $II_F = g_F Z$ for a vector field Z orthogonal to $\mathfrak{L}_F$), respectively. As for warped products, this fact also characterizes twisted products for orthogonal foliations.

Lemma 1.5.8 [147] *Let g be a metric defined on $B \times F$. If $\mathfrak{L}_B$ and $\mathfrak{L}_F$ intersect orthogonally, then g is a twisted product if and only if $\mathfrak{L}_B$ is totally geodesic and $\mathfrak{L}_F$ is totally umbilic.*

Next lemma gives the expression of the curvature tensor for a twisted product, which can be obtained from the expressions of the Levi-Civita connection in Lemma 1.5.7.

Lemma 1.5.9 [67] *Let $\mathcal{M} = B \times_f F$ be a twisted product. The curvature tensor is given by:*

(i) $R(X, Y)Z = R^B(X, Y)Z$,

(ii) $R(U, X)Y = (H_\xi(X, Y) + X(\xi)Y(\xi))U$,

(iii) $R(X, U)V = g(U, V)(X(\xi)\nabla\xi + h_\xi(X)) - XV(\xi)U$,

(iv) $R(U, V)X = XV(\xi)U - XU(\xi)V$,

$$\begin{aligned}(v)\ R(U,V)W = R^F(U,V)W &+ g(\nabla\xi,\nabla\xi)(g(U,W)V - g(V,W)U)\\ &+ H_\xi(V,W)U - H_\xi(U,W)V\\ &+ g(V,W)h_\xi(U) - g(U,W)h_\xi(V)\\ &+ W(\xi)(V(\xi)U - U(\xi)V)\\ &+ (U(\xi)g(V,W) - V(\xi)g(U,W))\nabla\xi\,,\end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(B)$ *and* $U, V, W \in \mathfrak{X}(F)$.

The Ricci tensor can be obtained by contraction.

Lemma 1.5.10 [67] *Let* $\mathcal{M} = B \times_f F$ *be a twisted product. For* $X, Y \in \mathfrak{X}(B)$ *and* $V, W \in \mathfrak{X}(F)$ *we have*

(i) $\rho(X,Y) = \rho^B(X,Y) - d(X(\xi)Y(\xi) + H_\xi(X,Y))$,

(ii) $\rho(X,V) = (1-d)XV(\xi)$,

(iii) $$\begin{aligned}\rho(V,W) = \rho^F(V,W) &+ (2-d)(V(\xi)W(\xi) + H_\xi(V,W))\\ &+ (d-2)g(V,W)g(\nabla\xi,\nabla\xi) - g(V,W)\Delta\xi\,,\end{aligned}$$

where $d = dim\, F$.

From the definitions above and, moreover, taking into account the geometry of warped and twisted products, it seems that the twisted product structure is much more flexible than the warped one. Indeed, if B restricts to a point p, the manifold $\{p\} \times_f F$ is nothing but a homothety on g_F for a warped product, but a conformal deformation of g_F for a twisted one. Nevertheless, we find in the literature that under certain circumstances twisted products reduce to warped products, as next results show.

Theorem 1.5.11 [68] *Let* $B \times_f F$ *be a twisted product with* $dim\, F > 1$. *Then* $\rho(X,V) = 0$ *for all* X, V, *with* X *tangent to* B *and* V *tangent to* F, *if and only if* $B \times_f F$ *may be expressed as a warped product* $B \times_{\tilde{f}} F$ *of* (B, g_B) *and* $(F, \widetilde{g_F})$, *where* $\widetilde{g_F}$ *is a metric tensor conformally equivalent to* g_F.

Corollary 1.5.12 [68] *Let* $\mathcal{M} = B \times_f F$, *with* $dim\, F > 1$, *be a twisted product. Then if* $\mathcal{M}$ *is Einstein, it may indeed be expressed as a warped product.*

In Section 6.3 we will give a result following this philosophy, that is, by means of conditions on the curvature it ensures that a twisted product can in fact be written as a warped one.

1.6 Completeness and Ricci blow-up

We say that $\mathcal{M}$ is *geodesically complete* if all geodesics exist for all time. In the Riemannian setting, the Hopf-Rinow Theorem provides a useful tool to guarantee completeness. However, in higher signature it is generally harder to check wether or not a manifold is geodesically complete. Since our task in several occasions will be to study completeness of higher signatured manifolds, we will check whether there is blow up of operators associated to the curvature along geodesics. We say that $\mathcal{M}$ exhibits *Ricci blowup* if there exists a geodesic γ defined for $t \in [0, T)$ with $T < \infty$ and such that $\lim_{t\to T} |\rho(\dot\gamma, \dot\gamma)| = \infty$. If $\mathcal{M}$ exhibits Ricci blowup, then it is geodesically incomplete and it can not be isometrically embedded in a geodesically complete manifold of the same dimension. In the remaining of this section we present a couple of criteria for geodesic completeness.

Complete Riemannian twisted products

The following result is due to Bishop and O'Neill [13]:

Theorem 1.6.1 *Let $\mathcal{M} = B \times_f F$ be a Riemannian warped product. $\mathcal{M}$ is complete if and only if B and F are complete.*

An analogous result of Theorem 1.6.1 does not hold for twisted products. However we may force the condition on the fiber to get a simple criterion by means of sufficient conditions for a twisted product to be complete.

Theorem 1.6.2 *Let $\mathcal{M} = B \times_f F$ be a Riemannian twisted product with compact fiber. Then $\mathcal{M}$ is complete if and only if B is complete.*

Proof. Assume $\mathcal{M}$ is complete. Then, since B inherits the geometry of $\mathcal{M}$ (i.e., its canonic foliation $\mathfrak{L}_B$ is totally geodesic), B is also complete. Now we prove the opposite implication. Since $\mathcal{M}$ is Riemannian we use Hopf-Rinow Theorem. Let $\{(p_i, q_i)\}_i$ be a Cauchy sequence on $B \times_f F$. Obviously $\{p_i\}$ is a Cauchy sequence on B so, since B is complete, it converges in B. Furthermore, there exists a compact $K \subset B$ which contains the sequence $\{p_i\}_{i>k}$ for a certain k. Consider the product $K \times F$, since K and F are compact, the product also is. The twisting function f attains its maximum and minimum in $K \times F$, hence $0 < c \leq f \leq d$ for some $c, d \in \mathbb{R}$. This bound implies $d(q_i, q_j) \leq (1/c) d((p_i, q_i), (p_j, q_j))$, so $\{q_i\}$ is a Cauchy sequence on F. F is compact, so it is complete and $\{q_i\}$ converges. Hence $\{(p_i, q_i)\}$ also converges. $\square$

Although this result is based on Hopf–Rinow Theorem, different tools are used to get interesting results in Lorentzian signature. See, for instance, [152] for examples of incomplete metrics on the Lorentzian torus.

Generalized plane wave manifolds

Here we define a large family of manifolds that has been extensively studied and that contains some of the examples we will study later. We refer to [85] for more information on this kind of manifolds.

Definition 1.6.1 *Let* $(x_1, \ldots, x_n)$ *be the usual coordinates in* $\mathbb{R}^n$*. A pseudo-Riemannian manifold* $(\mathbb{R}^n, g)$ *is said to be a* generalized plane wave manifold *if the Levi-Civita connection verifies*

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{k > max\{i,j\}} \Gamma_{ij}^k(x_1, \ldots, x_{k-1}) \partial_{x_k}\,, \tag{1.8}$$

where Γ_{ij}^k *denotes the Christoffel symbols.*

Generalized plane wave manifolds have several interesting properties. In the following theorem we mention just a few (see [85] for a broad exposition).

Theorem 1.6.3 [85] *Let* $\mathcal{M}$ *be a generalized plane wave manifold. Then*

- $\mathcal{M}$ *is geodesically complete,*
- $\mathcal{M}$ *is nilpotent Osserman, nilpotent k-Osserman and Ricci flat,*
- *the scalar Weyl invariants of* $\mathcal{M}$ *vanish.*

There are other kind of plane waves such as plane fronted waves or pp-waves which are interesting from a physical point of view, however they do not necessarily correspond to generalized plane wave manifolds. See, for instance, [45], where necessary and sufficient conditions for a plane fronted wave to be geodesically complete are given.

Part I

Conformally Osserman manifolds

The fact that the metric is definite in the Riemannian case, implies that the Jacobi operator is diagonalizable. This is not true in higher signature and makes the Osserman problem less tractable in general. Proceeding as in [16] or [75], if the signature is Lorentzian, then $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is Osserman if and only if the sectional curvature is constant, and thus the space of Osserman algebraic curvature tensors reduces to Span $\{A_0\}$. The same result holds for Osserman algebraic curvature tensors in an odd-dimensional Riemannian space [51], [89].

If $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is an Osserman model of arbitrary signature, then the Ricci tensor ρ satisfies $\rho = \frac{\tau}{n}\langle\cdot,\cdot\rangle$. This shows that any Osserman algebraic curvature tensor in a 3-dimensional vector space V has constant sectional curvature. Therefore the first nontrivial case is that of dim $V = 4$.

These two previous considerations suggest that the first meaningful case in the study of Osserman manifolds is that of 4-dimensional metrics of neutral signature. This has been extensively done during last years, although a good understanding of the subject has not been attained so far.

Despite the fact that in the higher signature setting some of the two-point homogeneous spaces can be recognized by some Osserman-like properties ([17], [24]), a remarkable fact is the existence of many nonsymmetric and even not locally homogeneous Osserman pseudo-Riemannian metrics. A two-step strategy has been followed so far in the study of Osserman manifolds [89]. The first step consists in the determination of the Osserman algebraic curvature tensors, which is closely related to the existence of certain Clifford structures ([24], [51], [135], [136]). Secondly, the objective is to classify the manifolds with such a structure as Nikolayevsky did in [134]. Therefore, a fundamental aspect towards an understanding of Osserman metrics is to determine the solution of the Osserman problem at a purely algebraic level. Here, it is worth emphasizing the existence of many Osserman algebraic curvature tensors which are not geometrically realizable ([17], [81]). For a more complete exposition on the subject see [77] and [84].

In Chapter 2 we recall some facts about Riemannian Osserman models and study the algebraic structure of Osserman algebraic curvature tensors in neutral signature.

Recently, the Osserman problem has evolved into related questions as those on the constancy of eigenvalues of the Weyl conformal Jacobi operators [18], [19], [20]. Thus, in Chapter 3 we study conformally Osserman manifolds in signature $(2, 2)$. We use Walker manifolds to provide a broad variety of examples, showing that all possible algebraic structures, given in Chapter 2, are geometrically realizable for the conformal Jacobi operator. Here, it is worth mentioning that we do not have this realizability for the usual Jacobi operator and Osserman manifolds (see [17]).

In Chapter 4 we turn our attention to global properties of Osserman and conformally Osserman manifolds in dimension 4 and neutral signature. Firstly, we characterize those which are both compact and Jordan Osserman. Secondly, we answer questions of geodesic completeness for Osserman and conformally Osserman examples.

We conclude Part I in Chapter 5, which is devoted to the analysis of the Osserman condition on manifolds whose metric has the structure of a warped product. We characterize Osserman Riemannian warped products by showing that they have constant sectional curvature. Some conclusions are also obtained for twisted products; however, we shall say that the dimensions of the base and the fiber of the product play an important role here. A consequence of these characterizations is that a conformally Osserman warped product is locally conformally flat. This fact motivates the study, in Part II, of locally conformally flat manifolds with warped product structure. Therefore, Chapter 5 acts as a link between Parts I and II.

Chapter 2

Algebraic structure of 4-dimensional Osserman manifolds

In this chapter we study neutral signature Osserman algebraic models. Results given here will be used in subsequent chapters, specially in the study of conformally Osserman 4-dimensional manifolds in Chapter 3.

First we recall some facts about Osserman models and manifolds in the Riemannian setting. Afterwards we concentrate on neutral signature and investigate the algebraic structure of Jacobi operators in Osserman models, based on previous works [17] and [77]. We then relate the Jordan normal form of the Jacobi operator with that of the Weyl self-dual operator, giving the explicit terms of the curvature and the matrix components of each operator.

2.1 The Osserman condition in the Riemannian setting

The Osserman conjecture is totally solved in dimension 4 for Riemannian manifolds in an affirmative way, as next theorem shows.

Theorem 2.1.1 [51] *Let $\mathcal{M} = (M, g)$ be a 4-dimensional Riemannian globally Osserman manifold. Then $\mathcal{M}$ is flat or locally a rank one symmetric space.*

Nevertheless, there are several peculiarities in dimension 4 that appear when studying Riemannian Osserman manifolds and that will be useful when passing to the higher signature setting; we recall some of them in this section.

The next result gives a nice characterization of Osserman models.

Theorem 2.1.2 [89] *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a 4-dimensional Riemannian algebraic model. The following assertions are equivalent:*

1. $\mathcal{V}$ is Osserman,

2. $\mathcal{V}$ *is Einstein and self-dual or anti-self-dual,*

3. $\mathcal{V}$ *is* 2*-stein (i.e.* $\langle x, x\rangle^{-2}\mathrm{tr}\{\mathcal{J}(x)^2\}$ *is independent of* $x \in V$*).*

Remark 2.1.3 Although the pointwise and global Osserman conditions are in many cases equivalent, there exist examples of 4-dimensional Riemannian pointwise Osserman manifolds which are not globally Osserman (see [89] and [142]). Furthermore and more concretely, it follows from the work of Nikolayevsky that, with the exception of dimension sixteen which is still open, pointwise Osserman metrics which are not Osserman only exist in dimension four.

2.2 The Osserman condition in signature $(--++)$

The Osserman problem is solved for Riemannian and Lorentzian models in dimension 4, hence we deal with neutral signature in the remaining of this chapter.

Remark 2.2.1 Since $\mathcal{J}(x)x = 0$, one often restricts the Jacobi operator $\mathcal{J}(x)$ to $x^\perp$. Moreover, since for any spacelike or timelike vector x the inner product induced on $x^\perp$ is of Lorentzian signature, the Jacobi operators are not completely determined by their eigenvalues, but by the Jordan normal form. Thus, we have the following four possibilities:

$$\underset{\textit{Type Ia}}{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}}, \quad \underset{\textit{Type Ib}}{\begin{pmatrix} \gamma & -\beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \alpha \end{pmatrix}}, \quad \underset{\textit{Type II}}{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{pmatrix}}, \quad \underset{\textit{Type III}}{\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}}.$$

Observe that type Ia corresponds to diagonalizable Jacobi operators and type Ib corresponds to Jacobi operators with a complex eigenvalue, while type II and type III correspond to a double and a triple root of the minimal polynomial of the Jacobi operators, respectively.

Subsequent results provide an algebraic description of Osserman algebraic curvature tensors in neutral signature, characterizing the different possibilities for the structure of the Jacobi operators. They are essentially obtained in the same way as in [17], [77].

Lemma 2.2.2 *Let* $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ *be an algebraic model with* $\langle\cdot,\cdot\rangle$ *of signature* $(2,2)$ *and let* $\{e_1(-), e_2(-), e_3(+), e_4(+)\}$ *be an orthonormal basis. The following assertions are equivalent:*

(i) $\mathcal{V}$ *is Osserman and the Jacobi operator* $\mathcal{J}(e_1)$ *is diagonalizable with eigenvalues* (α, β, γ)*:*

$$\mathcal{J}(e_1) = \langle e_1, e_1\rangle \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

(ii) The curvature tensor is given by

$$\begin{array}{ll} A_{1212}=A_{3434}=\alpha, & A_{1234}=\frac{2\alpha-\beta-\gamma}{3}, \\ A_{1313}=A_{2424}=-\beta, & A_{1324}=-\frac{-\alpha+2\beta-\gamma}{3}, \\ A_{1414}=A_{2323}=-\gamma, & A_{1423}=\frac{-\alpha-\beta+2\gamma}{3}. \end{array}$$

(iii) The self-dual and anti-self-dual Weyl operators satisfy $W^-=0$ *and*

$$\begin{aligned} W^+ &= \frac{2}{3}\begin{pmatrix} 2\alpha-\beta-\gamma & 0 & 0 \\ 0 & -\alpha+2\beta-\gamma & 0 \\ 0 & 0 & -\alpha-\beta+2\gamma \end{pmatrix} \\ &= \begin{pmatrix} 2\alpha-\frac{\tau}{6} & 0 & 0 \\ 0 & 2\beta-\frac{\tau}{6} & 0 \\ 0 & 0 & -2(\alpha+\beta)+\frac{\tau}{3} \end{pmatrix}. \end{aligned}$$

Lemma 2.2.3 *Let* $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,A)$ *be an algebraic model with* $\langle\cdot,\cdot\rangle$ *of signature* $(2,2)$ *and let* $\{e_1(-),e_2(-),e_3(+),e_4(+)\}$ *be an orthonormal basis. The following assertions are equivalent:*

(i) $\mathcal{V}$ *is Osserman and the Jacobi operator* $\mathcal{J}(e_1)$ *has one real and two complex eigenvalues* $(\alpha,\gamma\pm\beta\sqrt{-1})$*:*

$$\mathcal{J}(e_1)=\langle e_1,e_1\rangle\begin{pmatrix} \gamma & -\beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

(ii) The curvature tensor is given by

$$\begin{array}{ll} A_{1212}=-A_{1313}=-A_{2424}=A_{3434}=\gamma\,, & A_{1414}=A_{2323}=-\alpha\,, \\ A_{1213}=A_{1224}=A_{1334}=A_{2434}=-\beta\,, & \\ A_{1234}=-A_{1324}=-\frac{\alpha-\gamma}{3}\,, & A_{1423}=\frac{2(\alpha-\gamma)}{3}\,. \end{array}$$

(iii) The self-dual and anti-self-dual Weyl operators satisfy $W^-=0$ *and*

$$W^+=\begin{pmatrix} -\frac{2}{3}(\alpha-\gamma) & -2\beta & 0 \\ 2\beta & -\frac{2}{3}(\alpha-\gamma) & 0 \\ 0 & 0 & \frac{4}{3}(\alpha-\gamma) \end{pmatrix}.$$

Lemma 2.2.4 *Let* $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,A)$ *be an algebraic model with* $\langle\cdot,\cdot\rangle$ *of signature* $(2,2)$ *and let* $\{e_1(-),e_2(-),e_3(+),e_4(+)\}$ *be an orthonormal basis. The following assertions are equivalent:*

(i) $\mathcal{V}$ *is Osserman, the Jacobi operator* $\mathcal{J}(e_1)$ *has one simple and one double eigenvalue* (α, β, β) *and the following associated matrix*

$$\mathcal{J}(e_1) = \langle e_1, e_1 \rangle \begin{pmatrix} \beta + \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \beta - \frac{1}{2} & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

(ii) The curvature tensor is given by

$$\begin{array}{ll} A_{1212} = A_{3434} = \beta + \frac{1}{2}\,, & A_{1234} = -\frac{\alpha-\beta}{3} + \frac{1}{2}\,, \\ A_{1313} = A_{2424} = -\beta + \frac{1}{2}\,, & A_{1324} = \frac{\alpha-\beta}{3} + \frac{1}{2}\,, \\ A_{1414} = A_{2323} = -\alpha\,, & A_{1423} = \frac{2(\alpha-\beta)}{3}\,, \\ A_{1213} = A_{1224} = A_{1334} = A_{2434} = -\frac{1}{2}\,. & \end{array}$$

(iii) The self-dual and anti-self-dual Weyl operators satisfy $W^- = 0$ *and*

$$W^+ = \begin{pmatrix} -\frac{2(\alpha-\beta)}{3} + 1 & -1 & 0 \\ 1 & -\frac{2(\alpha-\beta)}{3} - 1 & 0 \\ 0 & 0 & \frac{4(\alpha-\beta)}{3} \end{pmatrix}.$$

Lemma 2.2.5 *Let* $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ *be an algebraic model with* $\langle\cdot,\cdot\rangle$ *of signature* $(2,2)$ *and let* $\{e_1(-), e_2(-), e_3(+), e_4(+)\}$ *be an orthonormal basis. The following assertions are equivalent:*

(i) $\mathcal{V}$ *is Osserman, the Jacobi operator* $\mathcal{J}(e_1)$ *has one triple eigenvalue* (α, α, α) *and the following associated matrix*

$$\mathcal{J}(e_1) = \langle e_1, e_1 \rangle \begin{pmatrix} \alpha & 0 & \frac{1}{\sqrt{2}} \\ 0 & \alpha & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \alpha \end{pmatrix}.$$

(ii) The curvature tensor is given by

$$\begin{array}{l} A_{1212} = -A_{1313} = -A_{1414} = -A_{2323} = -A_{2424} = A_{3434} = \alpha\,, \\ A_{1214} = -A_{1223} = -A_{1314} = A_{1323} = -A_{1424} = A_{1434} = A_{2324} = -A_{2334} = \frac{1}{\sqrt{2}}\,. \end{array}$$

(iii) The self-dual and anti-self-dual Weyl operators satisfy $W^- = 0$ *and*

$$W^+ = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}.$$

As a consequence of Lemmas 2.2.2, 2.2.3, 2.2.4 and 2.2.5 one has the following result.

Theorem 2.2.6 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be an algebraic model in signature* $(2,2)$. *Then $\mathcal{V}$ is Osserman if and only if it is Jordan-Osserman.*

Proof. Assume $\mathcal{V}$ is Osserman. Let $e_1 \in S^-(V, \langle\cdot,\cdot\rangle)$. Then $\mathcal{J}(e_1)$ is of one of the types in Remark 2.2.1 and there exist orthonormal vectors $e_2(-), e_3(+), e_4(+)$ so that the curvature tensor is given by Lemmas 2.2.2–2.2.5. Now, it is straightforward to show that such a curvature tensor has all the Jacobi operators of the same type and, therefore, $\mathcal{V}$ is Jordan-Osserman. □

Remark 2.2.7 The four types of the Jacobi operators characterized in Lemmas 2.2.2–2.2.5 may be given in terms of the self-dual operators as well. Thus, for an orthonormal basis $\{e_1(-), e_2(-), e_3(+), e_4(+)\}$, one obtains the following:

1. Type Ia:
   For an algebraic curvature tensor given by

$$\begin{array}{ll} A_{1212} = A_{3434} = \frac{\alpha}{2} + \frac{\tau}{12}\,, & A_{1234} = \frac{\alpha}{2}\,, \\ A_{1313} = A_{2424} = -\frac{\beta}{2} - \frac{\tau}{12}\,, & A_{1324} = -\frac{\beta}{2}\,, \\ A_{1414} = A_{2323} = \frac{\alpha+\beta}{2} - \frac{\tau}{12}\,, & A_{1423} = -\frac{\alpha+\beta}{2}\,, \end{array}$$

   the self-dual and anti-self-dual Weyl operators satisfy

$$W^+ = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{with} \quad \alpha+\beta+\gamma = 0 \quad \text{and} \quad W^- = 0.$$

   The Jacobi operator in the direction of e_1 is given by:

$$\mathcal{J}(e_1) = -\begin{pmatrix} \hat\alpha & 0 & 0 \\ 0 & \hat\beta & 0 \\ 0 & 0 & \hat\gamma \end{pmatrix}, \text{ where } \hat\alpha = \frac{\alpha}{2} + \frac{\tau}{12}\,,\ \hat\beta = \frac{\beta}{2} + \frac{\tau}{12}\,,\ \hat\gamma = -\frac{\alpha+\beta}{2} + \frac{\tau}{12}\,.$$

2. Type Ib:
   For an algebraic curvature tensor given by

$$\begin{array}{l} A_{1212} = -A_{1313} = -A_{2424} = A_{3434} = -\frac{\alpha}{4} + \frac{\tau}{12}\,, \\ A_{1414} = A_{2323} = -\frac{\alpha}{2} - \frac{\tau}{12}\,, \\ A_{1213} = A_{1224} = A_{1334} = A_{2434} = -\frac{\beta}{2}\,,\ A_{1234} = -A_{1324} = -\frac{\alpha}{4}\,,\ A_{1423} = \frac{\alpha}{2}\,, \end{array}$$

   the self-dual and anti-self-dual Weyl operators satisfy

$$W^+ = \begin{pmatrix} \gamma & -\beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad \text{with } \beta \neq 0,\ \alpha + 2\gamma = 0 \text{ and } W^- = 0.$$

The Jacobi operator in the direction of e_1 is given by:

$$\mathcal{J}(e_1) = -\begin{pmatrix} \hat\gamma & -\hat\beta & 0 \\ \hat\beta & \hat\gamma & 0 \\ 0 & 0 & \hat\alpha \end{pmatrix}$$

with

$$\hat\alpha = \frac{\alpha}{2} + \frac{\tau}{12}, \ \hat\gamma \pm \hat\beta\sqrt{-1} = \left(-\frac{\alpha}{4} + \frac{\tau}{12}\right) \pm \frac{\beta}{2}\sqrt{-1}.$$

3. Type II:
For an algebraic curvature tensor given by

$$\begin{aligned} &A_{1212} = A_{3434} = -\tfrac{\alpha}{4} + \tfrac{1}{4} + \tfrac{\tau}{12}, \\ &A_{1313} = A_{2424} = \tfrac{\alpha}{4} + \tfrac{1}{4} - \tfrac{\tau}{12}, \\ &A_{1414} = A_{2323} = -\tfrac{\alpha}{2} - \tfrac{\tau}{12}, \\ &A_{1213} = A_{1224} = A_{1334} = A_{2434} = -\tfrac{1}{4}, \\ &A_{1234} = -\tfrac{\alpha}{4} + \tfrac{1}{4}, A_{1324} = \tfrac{\alpha}{4} + \tfrac{1}{4}, A_{1423} = \tfrac{\alpha}{2}, \end{aligned}$$

the self-dual and anti-self-dual Weyl operators satisfy

$$W^+ = \begin{pmatrix} \beta + \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \beta - \frac{1}{2} & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad \text{with } \alpha + 2\beta = 0 \ \text{ and } \ W^- = 0.$$

The Jacobi operator in the direction of e_1 is given by:

$$\mathcal{J}(e_1) = -\begin{pmatrix} \hat\beta + \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & \hat\beta - \frac{1}{4} & 0 \\ 0 & 0 & \hat\alpha \end{pmatrix},$$

with

$$\hat\alpha = \frac{\alpha}{2} + \frac{\tau}{12}, \ \hat\beta = -\frac{\alpha}{4} + \frac{\tau}{12}.$$

4. Type III:
For an algebraic curvature tensor given by

$$\begin{aligned} &A_{1212} = -A_{1313} = -A_{1414} = -A_{2323} = -A_{2424} = A_{3434} = \tfrac{\tau}{12}, \\ &A_{1214} = -A_{1223} = -A_{1314} = A_{1323} = -A_{1424} = A_{1434} = A_{2324} = -A_{2334} = \tfrac{1}{2\sqrt{2}}, \end{aligned}$$

the self-dual and anti-self-dual Weyl operators satisfy

$$W^+ = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad \text{and} \quad W^- = 0.$$

The Jacobi operator in the direction of e_1 is given by:

$$\mathcal{J}(e_1) = -\begin{pmatrix} \hat{\alpha} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & \hat{\alpha} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \hat{\alpha} \end{pmatrix} \quad \text{with } \hat{\alpha} = \frac{\tau}{12}\,.$$

Lemmas 2.2.2, 2.2.3, 2.2.4, 2.2.5 and Remark 2.2.7 classify all possible types of Osserman algebraic curvature tensors. Moreover, they relate the eigenvalues of the Jacobi operator with the eigenvalues of the self-dual conformal operator. The following is an analogue of Theorem 2.1.2 in the neutral signature setting.

Theorem 2.2.8 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be an algebraic model of signature $(2,2)$. Then, $\mathcal{V}$ is Osserman if and only if it is self-dual (or anti-self-dual) and $\rho = \frac{\tau}{4}\langle\cdot,\cdot\rangle$. Moreover, the types of the self-dual curvature W^+ (resp., anti-self-dual curvature W^-) are in one to one correspondence with the different types of the Jacobi operators in Remark 2.2.1.*

Proof. Note that any algebraic curvature tensor A can be interpreted as an endomorphism in the space of 2-forms Λ. Moreover, from Theorem 1.3.2 and since V is a 4-dimensional vector space, A decomposes as

$$A = \frac{\tau}{12} Id_\Lambda + \mathfrak{Z}_\Lambda + \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix}, \tag{2.1}$$

where $\mathfrak{Z}_\Lambda$ denotes the trace-free Ricci tensor as an endomorphism in Λ. Furthermore, in order to express the self-dual and anti-self-dual curvature tensors, a basis of the space of 2-forms is constructed as follows: let $\{e_i\}$ be an orthonormal basis of V and consider the induced basis on the spaces of self-dual and anti-self-dual 2-forms $\Lambda^\pm = \text{Span}\ \{E_1^\pm, E_2^\pm, E_3^\pm\}$ given in (1.5). Note that

$$\langle E_1^\pm, E_1^\pm\rangle = \varepsilon_1\varepsilon_2\,, \langle E_2^\pm, E_2^\pm\rangle = \varepsilon_1\varepsilon_3\,, \langle E_3^\pm, E_3^\pm\rangle = \varepsilon_1\varepsilon_4\,.$$

With respect to the basis $\{E_1^\pm, E_2^\pm, E_3^\pm\}$, the self-dual and the anti-self-dual Weyl operators $W_A^\pm : \Lambda^\pm \longrightarrow \Lambda^\pm$ have the following matrix form:

$$W_A^\pm = \begin{pmatrix} \varepsilon_1\varepsilon_2(W_A^\pm)_{11} & \varepsilon_1\varepsilon_2(W_A^\pm)_{12} & \varepsilon_1\varepsilon_2(W_A^\pm)_{13} \\ \varepsilon_1\varepsilon_3(W_A^\pm)_{12} & \varepsilon_1\varepsilon_3(W_A^\pm)_{22} & \varepsilon_1\varepsilon_3(W_A^\pm)_{23} \\ \varepsilon_1\varepsilon_4(W_A^\pm)_{13} & \varepsilon_1\varepsilon_4(W_A^\pm)_{23} & \varepsilon_1\varepsilon_4(W_A^\pm)_{33} \end{pmatrix}. \tag{2.2}$$

Now we proceed as in [77]. If A is an Osserman algebraic curvature tensor, then specialize the orthonormal basis $\{e_i\}$ above as given in Lemmas 2.2.2, 2.2.3, 2.2.4 or 2.2.5 to get, after a straightforward calculation, that $\rho = \frac{\tau}{4}\langle\cdot,\cdot\rangle$ and W^- or W^+ vanishes. Moreover, the structure of the self-dual or anti-self-dual part of the Weyl tensor corresponds to the structure of the Jacobi operators. Conversely, if A is assumed to be self-dual and $\rho = \frac{\tau}{4}\langle\cdot,\cdot\rangle$, then (2.1) becomes

$$A = \frac{\tau}{12}\, Id_\Lambda + \begin{pmatrix} W^+ & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

from where it follows that A is Osserman proceeding as in [3], [77] or [89]. □

Chapter 3

Differentiable structure of four-dimensional conformally Osserman manifolds

The purpose of this chapter is to investigate the geometry of 4-dimensional conformally Osserman pseudo-Riemannian manifolds. Special attention is paid to the construction of a large number of examples of conformally Osserman metrics. As a consequence we obtain that the eigenvalue structure of the conformal Jacobi operators is much richer than the corresponding one for the usual Jacobi operators. In that sense, two aspects should be emphasized as concerns those examples:

- *There exist conformally Osserman manifolds which are not in the conformal class of any Osserman metric.*
- *All the possible algebraic structures of the conformal Jacobi operators in Remark 2.2.1 can be realized at the differentiable level.*

First note that, although the algebraic study we carried out in previous chapter dealt with the Osserman condition, and not with the conformally Osserman itself, in this chapter we apply all those results to the Weyl tensor and we take advantage of them to analyze conformally Osserman manifolds. In Sections 3.1-3.4 we present results in [33] including some new examples, whereas results of Section 3.5 are collected in [40].

3.1 Conformally Osserman manifolds and (anti-) self-dual structures

The following characterization was established in the Riemannian setting in [19].

Theorem 3.1.1 [19] *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian model with dim $V = 4$. $\mathcal{V}$ is conformally Osserman if and only if $\mathcal{V}$ is self-dual or anti-self-dual.*

In the following result we use Theorem 2.2.8 to see that an analogous characterization of 4-dimensional conformally Osserman manifolds also holds in neutral signature.

Theorem 3.1.2 *Let $\mathcal{M} = (M, g)$ be a 4-dimensional pseudo-Riemannian manifold. Then $\mathcal{M}$ is conformally Osserman if and only if there is a choice of orientation for M such that it is self-dual or anti-self-dual. Moreover, there is a one to one correspondence between the Jordan normal form of $\mathcal{J}_W$ and $W^{\pm}$.*

Proof. Since the Weyl tensor is trace-free, $\rho_W = 0$ and $\tau_W = 0$, from where it follows that the algebraic Weyl curvature tensor W_W constructed by (1.4) from the Weyl tensor W, coincides with the Weyl tensor, i.e. $W_W = W$. Now the result follows immediately from Theorem 2.2.8. □

As an immediate application of previous Theorem, examples of conformally Osserman $(--++)$-metrics are obtained from the work of Dunajski [62]. Our purpose in the remainder of this chapter is to use Theorem 3.1.2 to construct new examples of conformally Osserman metrics.

3.2 Walker metrics

A specific feature of strictly pseudo-Riemannian metrics is related to the local reductibility/decomposability of such structures as shown by Wu in [172]. There are many striking differences between the Riemannian and pseudo-Riemannian situations which come from the existence of parallel degenerate distributions, which do not lead to local decompositions of the manifold in the indefinite setting. Among such metrics, an interesting family was investigated by Walker [169]: pseudo-Riemannian manifolds admitting a parallel degenerate distribution of maximal rank, which include some metrics on tangent (for example some complete lifts) and cotangent (for example Riemann extensions) bundles as special cases [76]. One says that a pseudo-Riemannian manifold $\mathcal{M}$ of signature $(2, 2)$ is a *Walker manifold* if it admits a parallel totally isotropic 2-plane field. It was shown by Walker that any such 4-dimensional metric can be locally expressed in adapted coordinates $(x_1, \dots, x_4)$ as manifolds in next definition.

Definition 3.2.1 *$\mathcal{M}$ is called a Walker manifold if $\mathcal{M}$ is a neutral signature 4-dimensional manifold with metric tensor expressed in local coordinates as*

$$g(x_1, x_2, x_3, x_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x_1, x_2, x_3, x_4) & c(x_1, x_2, x_3, x_4) \\ 0 & 1 & c(x_1, x_2, x_3, x_4) & b(x_1, x_2, x_3, x_4) \end{pmatrix},$$

for arbitrary functions a, b, c.

During last years, a broad analysis of the geometry of Walker metrics has been developed (see, for instance, [47], [60], [61], [128]); these appear in several different contexts.

3.2.1 Self-duality and anti-self-duality in general Walker metrics

In this subsection we summarize results obtained in [61] giving the expression of the self–dual and the anti–self–dual Weyl tensors for a Walker metric g as in Definition 3.2.1. The components will be given with respect to the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ where

$$\begin{aligned} e_1 &= \frac{1}{2}(1-a)\partial_1 + \partial_3, & e_2 &= -c\,\partial_1 + \frac{1}{2}(1-b)\partial_2 + \partial_4, \\ e_3 &= -\frac{1}{2}(1+a)\partial_1 + \partial_3, & e_4 &= -c\,\partial_1 - \frac{1}{2}(1+b)\partial_2 + \partial_4. \end{aligned} \tag{3.1}$$

The following lemma gives the expressions of the self-dual and the anti-self-dual components of the Weyl tensor.

Lemma 3.2.1 *With respect to the basis induced in Λ^- by (1.5) and the basis (3.1), the components of W^- are given by*

$$\begin{aligned} W^-_{11} &= -\frac{1}{12}(a_{11} + 3a_{22} + 3b_{11} + b_{22} - 4c_{12}), & W^-_{22} &= -\frac{1}{6}(a_{11} + b_{22} - 4c_{12}), \\ W^-_{33} &= \frac{1}{12}(a_{11} - 3a_{22} - 3b_{11} + b_{22} - 4c_{12}), & W^-_{12} &= \frac{1}{4}(a_{12} + b_{12} - c_{11} - c_{22}), \\ W^-_{13} &= \frac{1}{4}(a_{22} - b_{11}), & W^-_{23} &= -\frac{1}{4}(a_{12} - b_{12} + c_{11} - c_{22}). \end{aligned}$$

All the components of W^+ can be written in terms of W^+_{11}, W^+_{12} and the scalar curvature as follows:

$$W^+_{22} = -\frac{\tau}{6}, \qquad W^+_{33} = W^+_{11} + \frac{\tau}{6}, \qquad W^+_{13} = W^+_{11} + \frac{\tau}{12}, \qquad W^+_{23} = W^+_{12}.$$

The expressions for W^+_{11} and W^+_{12} are

$$\begin{aligned} W^+_{11} = \frac{1}{12}\Big(& 6ca_1b_2 - 6a_1b_3 - 6ba_1c_2 + 12a_1c_4 - 6ca_2b_1 + 6a_2b_4 + 6ba_2c_1 + 6a_3b_1 - 6a_4b_2 \\ & - 12a_4c_1 + 6ab_1c_2 - 6ab_2c_1 + 12b_2c_3 - 12b_3c_2 - a_{11} - 12c^2a_{11} - 12bca_{12} \\ & + 24ca_{14} - 3b^2a_{22} + 12ba_{24} - 12a_{44} - 3a^2b_{11} + 12ab_{13} - b_{22} - 12b_{33} \\ & + 12acc_{11} - 2c_{12} + 6abc_{12} - 24cc_{13} - 12ac_{14} - 12bc_{23} + 24c_{34}\Big), \end{aligned}$$

$$W^+_{12} = \frac{1}{4}(-2ca_{11} - ba_{12} + 2a_{14} + ab_{12} - 2b_{23} + ac_{11} - 2cc_{12} - 2c_{13} - bc_{22} + 2c_{24}).$$

Remark 3.2.2 Note, from Lemma 3.2.1, that

$$W^+ = \begin{pmatrix} W^+_{11} & W^+_{12} & W^+_{11} + \frac{\tau}{12} \\ -W^+_{12} & \frac{\tau}{6} & -W^+_{12} \\ -(W^+_{11} + \frac{\tau}{12}) & -W^+_{12} & -(W^+_{11} + \frac{\tau}{6}) \end{pmatrix},$$

and, as a consequence, the eigenvalues of W^+ are $\{\tau/6, -\tau/12, -\tau/12\}$. Since the induced metric on Λ^2_+ has Lorentzian signature, the structure of W^+ is determined by its Jordan normal form, which may correspond to type Ia, type II or type III, depending on whether W^+ is diagonalizable or not. A straightforward calculation shows that

$$\left(W^+ - \frac{\tau}{6} Id\right) \cdot \left(W^+ + \frac{\tau}{12} Id\right) = \frac{\tau^2 + 12\tau W^+_{11} + 48\left(W^+_{12}\right)^2}{48} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

from where we have the following:

(i) If $\tau \neq 0$, we have that W^+ has non–zero eigenvalues $\{\tau/6, -\tau/12, -\tau/12\}$ and the equality $\tau^2 + 12\tau W^+_{11} + 48\left(W^+_{12}\right)^2 = 0$ is the necessary and sufficient condition for the diagonalizability of W^+. If the above equation does not hold, then $-\tau/12$ is a double root of the minimal polynomial of W^+.

(ii) If $\tau = 0$, then W^+ vanishes if and only if $W^+_{11} = W^+_{12} = 0$ and moreover

1. W^+ is two–step nilpotent if and only if $W^+_{11} \neq 0$ and $W^+_{12} = 0$,
2. W^+ is three–step nilpotent if and only if $W^+_{12} \neq 0$.

Note from the eigenvalues of W^+ that, if the metric is anti-self-dual then necessarily $\tau = 0$. This shows, in particular, that an Osserman anti-self-dual Walker metric is Ricci flat, since Osserman implies Einstein.

Even if the expressions for the eigenvalues of the self-dual operator are quite simple, this is not the case for the eigenvalues of the anti-self-dual operator. Thus, they are not easy to compute in a general frame and that makes it hard to obtain specific conclusions on the Jordan structure of the Jacobi operators. For this reason we shall restrict in Section 3.3 to a simpler family of Walker metrics; as this provides all the examples we are looking for.

3.2.2 Explicit form of self-dual Walker metrics

To motivate our later study, we recall in this section some results in [61] concerning self-dual Walker metrics. More specifically, a complete description of self-dual Walker metrics is given by integrating the partial differential equations obtained from Lemma 3.2.1.

Theorem 3.2.3 [61] *A Walker metric g is self-dual if and only if the defining functions a, b and c are given by*

$$a = x_1^3\mathcal{A} + x_1^2\mathcal{B} + x_1^2 x_2\mathcal{C} + x_1x_2\mathcal{D} + x_1P + x_2Q + \xi,$$

$$b = x_2^3\mathcal{C} + x_2^2\mathcal{E} + x_1x_2^2\mathcal{A} + x_1x_2\mathcal{F} + x_1S + x_2T + \eta,$$

$$c = \frac{1}{2}x_1^2\mathcal{F} + \frac{1}{2}x_2^2\mathcal{D} + x_1^2x_2\mathcal{A} + x_1x_2^2\mathcal{C} + \frac{1}{2}x_1x_2(\mathcal{B}+\mathcal{E}) + x_1U + x_2V + \gamma,$$

where P, Q, S, T, U, V, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, ξ, η and γ are functions depending on the coordinates (x_3, x_4).

3.2.3 Conformally Osserman metrics with type II conformal Jacobi operators

Theorem 3.2.3 provides a useful characterization of self-dual Walker metrics. Use that notation to compute the scalar curvature of a self-dual Walker metric and obtain

$$\tau = 3(4x_1\mathcal{A} + \mathcal{B} + 4x_2\mathcal{C} + \mathcal{E}). \tag{3.2}$$

Now, taking into account Remark 3.2.2 and due to the fact that the type of the Jacobi operator coincides with the type of the self-dual Weyl operator (as we have seen in Chapter 2), one may use characterization of Theorem 3.2.3 to construct explicit examples of conformally Osserman metrics whose conformal Jacobi operators are of type II. We will use these considerations in Section 3.4 to construct an explicit example which is not in the conformal class of an Osserman metric.

3.2.4 Strict Walker manifolds

We have seen in this section that Walker manifolds are those neutral signature manifolds admitting a 2-dimensional parallel degenerate distribution. When, moreover, the manifold admits two orthogonal parallel null vector fields, it is called a *strict Walker manifold.* In such a case there are local coordinates where the metric is given by:

$$g(x_1,x_2,x_3,x_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x_3,x_4) & c(x_3,x_4) \\ 0 & 1 & c(x_3,x_4) & b(x_3,x_4) \end{pmatrix}, \tag{3.3}$$

for arbitrary functions a, b, c [169].

Remark 3.2.4 Theorem 3.2.3 implies that any strict Walker metric is self–dual. Moreover, one may compute the Ricci tensor to check that any strict Walker metric is Ricci flat (see [61]). Therefore, any strict Walker metric is Osserman. Furthermore, Remark 3.2.2 shows that the Jacobi operators are either identically zero or two–step nilpotent (depending on whether $W^+_{11} = 2c_{34} - a_{44} - b_{33}$ vanishes or not).

We will refer to strict Walker metrics later on in Chapter 4 and will see that they have some nice global properties; they are, for example, geodesically complete.

3.3 A particular family of Walker metrics

As we have said in the introduction, the purpose of this chapter is to provide examples of Osserman and conformally Osserman manifolds which realize different kind of algebraic structures. We restrict therefore to the following more tractable case:

$$g(x_1,x_2,x_3,x_4)=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & c(x_1,x_2,x_3,x_4)\\ 0 & 1 & c(x_1,x_2,x_3,x_4) & 0 \end{pmatrix}. \tag{3.4}$$

This family of metrics will suffice to provide examples corresponding to all the possibilities in Remark 2.2.1, as we will see in Remark 3.4.1.

We begin our study of the geometry of metrics (3.4) by determining their curvature tensor, Ricci tensor and scalar curvature. Then we analyze the self-dual and the anti-self-dual properties.

As a matter of notation, in what follows we write $h_i=\frac{\partial h}{\partial x_i}$, $h_{ij}=\frac{\partial h}{\partial x_i\partial x_j}$ for any function $h(x_1,\dots,x_4)$, and $\partial_i=\frac{\partial}{\partial x_i}$ $(i,j=1,\dots,4)$. After a long but straightforward calculation we get that the non vanishing components of the $(0,4)$-curvature tensor are

$$\begin{array}{rclrcl} R_{1314} &=& -\frac12\,c_{11}, & R_{2324} &=& -\frac12\,c_{22},\\ R_{1324} &=& -\frac12\,c_{12}, & R_{2334} &=& \frac14\left\{c_2^2-2\,c_{23}\right\},\\ R_{1334} &=& \frac14\left\{c_1\,c_2-2\,c_{13}\right\}, & R_{2434} &=& \frac14\left\{-c_1\,c_2+2\,c_{24}\right\},\\ R_{1423} &=& -\frac12\,c_{12}, & R_{3434} &=& \frac12\left\{-c\,c_1\,c_2+2\,c_{34}\right\},\\ R_{1434} &=& \frac14\left\{-c_1^2+2\,c_{14}\right\}. & & & \end{array} \tag{3.5}$$

From (3.5) we obtain that the non vanishing components of the Ricci tensor are

$$\begin{array}{rclrcl} \rho_{13} &=& \frac12\,c_{12}, & \rho_{33} &=& \frac12\left\{-c_2^2+2\,c_{23}\right\},\\ \rho_{14} &=& \frac12\,c_{11}, & \rho_{34} &=& \frac12\left\{c_1\,c_2+2\,c\,c_{12}-c_{13}-c_{24}\right\},\\ \rho_{23} &=& \frac12\,c_{22}, & \rho_{44} &=& \frac12\left\{-c_1^2+2\,c_{14}\right\},\\ \rho_{24} &=& \frac12\,c_{12}, & & & \end{array} \tag{3.6}$$

and the scalar curvature is given by

$$\tau=2\,c_{12}. \tag{3.7}$$

Let $\{e_1,e_2,e_3,e_4\}$ be the orthonormal basis given by

$$\begin{array}{rclrcl} e_1 &=& \frac12\,\partial_1+\partial_3, & e_2 &=& -c\,\partial_1+\frac12\,\partial_2+\partial_4,\\ e_3 &=& -\frac12\,\partial_1+\partial_3, & e_4 &=& -c\,\partial_1-\frac12\,\partial_2+\partial_4. \end{array} \tag{3.8}$$

Note that e_1 and e_2 are spacelike, while e_3 and e_4 are timelike. Now, from Lemma 3.2.1, the non vanishing components of W^- and W^+ in this basis are given by

$$\begin{array}{rclrcl} W^-_{11} &=& \frac13\,c_{12}, & W^-_{12} &=& -\frac14\left(c_{11}+c_{22}\right),\\ W^-_{22} &=& \frac23\,c_{12}, & W^-_{23} &=& -\frac14\left(c_{11}-c_{22}\right),\\ W^-_{33} &=& -\frac13\,c_{12}, & & & \end{array} \tag{3.9}$$

and

$$
(3.10)\quad
\begin{aligned}
W^+_{11} &= -\tfrac{1}{6}\,c_{12} - 2\,cc_{13} + 2\,c_{34}, & W^+_{12} &= W^+_{23} = -\tfrac{1}{2}\,(cc_{12} + c_{13} - c_{24}),\\
W^+_{22} &= -\tfrac{1}{3}\,c_{12}, & W^+_{13} &= -2(c\,c_{13} - c_{34}),\\
W^+_{33} &= \tfrac{1}{6}\,c_{12} - 2c\,c_{13} + 2c_{34}.
\end{aligned}
$$

3.3.1 Characterization of (anti-) self-duality

We now use the calculations performed in the previous section to characterize self-dual metrics and to give information about the algebraic structure of the self-dual operator by means of the characteristic polynomial p_λ and the minimal polynomial n_λ.

Theorem 3.3.1 *The Walker metric of Equation* (3.4) *is self-dual if and only if the defining function* $c(x_1, x_2, x_3, x_4)$ *has the following form*

$$(3.11)\qquad c(x_1, x_2, x_3, x_4) = x_1\,P(x_3, x_4) + x_2\,Q(x_3, x_4) + S(x_3, x_4)\,,$$

for any functions $P(x_3, x_4)$, $Q(x_3, x_4)$ *and* $S(x_3, x_4)$. *Moreover, in such a case, the characteristic polynomial of* W^+ *reduces to* $p_\lambda(W^+) = -\lambda^3$, *and the minimal polynomial is characterized as follows:*

(i) $n_\lambda(W^+) = \lambda^3$ *at those points where* $c_{13} - c_{24} \neq 0$.

(ii) $n_\lambda(W^+) = \lambda$ *or* $n_\lambda(W^+) = \lambda^2$ *at those points where* $c_{13} - c_{24} = 0$, *depending on whether the function* $\mathfrak{F} := c\,c_{13} - c_{34}$ *vanishes or not.*

Proof. Note from (3.9) that a Walker metric (3.4) is self-dual if and only if the defining function $c(x_1, x_2, x_3, x_4)$ satisfies

$$(3.12)\qquad c_{11} = c_{22} = c_{12} = 0.$$

First, $c_{12} = 0$ implies that $c(x_1, x_2, x_3, x_4) = \bar{c}(x_1, x_3, x_4) + \hat{c}(x_2, x_3, x_4)$. But $c_{11} = \bar{c}_{11} = 0$ and $c_{22} = \hat{c}_{22} = 0$, so it follows that

$$
\begin{aligned}
\bar{c}(x_1, x_3, x_4) &= x_1\,P(x_3, x_4) + \bar{S}(x_3, x_4),\\
\hat{c}(x_2, x_3, x_4) &= x_2\,Q(x_3, x_4) + \hat{S}(x_3, x_4).
\end{aligned}
$$

Therefore $c(x_1, x_2, x_3, x_4) = x_1\,P(x_3, x_4) + x_2\,Q(x_3, x_4) + S(x_3, x_4)$, with $S = \bar{S} + \hat{S}$, which shows (3.11).

On the other hand, the characterization of W^+ given by (3.10) lets us determine the characteristic polynomial of W^+, to be

$$(3.13)\qquad p_\lambda(W^+) = \frac{-1}{108}\left\{108\,\lambda^3 - 9\,c_{12}^2\,\lambda - c_{12}^3\right\}.$$

Now, if the metric is self-dual, then $c_{12} = 0$ and therefore (3.13) reduces to $p_\lambda(W^+) = -\lambda^3$. Finally, it is straightforward to show that under the conditions given in (3.12) one has

$$W^+ = \begin{pmatrix} -2\mathfrak{F} & -\frac{1}{2}(c_{13} - c_{24}) & -2\mathfrak{F} \\ \frac{1}{2}(c_{13} - c_{24}) & 0 & \frac{1}{2}(c_{13} - c_{24}) \\ 2\mathfrak{F} & \frac{1}{2}(c_{13} - c_{24}) & 2\mathfrak{F} \end{pmatrix}.$$

The square of this matrix is

$$\left(W^+\right)^2 = \frac{1}{4}(c_{13} - c_{24})^2 \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

from where the characterization of the minimal polynomial of W^+ follows. □

Anti-self-dual metrics are characterized proceeding in an analogous way in the next theorem.

Theorem 3.3.2 *A Walker metric* (3.4) *is anti-self-dual if and only if the defining function* $c(x_1, x_2, x_3, x_4)$ *has the following form*

$$\begin{aligned} c(x_1, x_2, x_3, x_4) &= x_1\, P(x_3, x_4) + x_2\, Q(x_3, x_4) + S(x_3, x_4) \\ &\quad + \xi(x_1, x_4) + \eta(x_2, x_3) \end{aligned} \tag{3.14}$$

for any functions $\xi(x_1, x_4)$ *and* $\eta(x_2, x_3)$*, and functions* $P(x_3, x_4)$*,* $Q(x_3, x_4)$ *and* $S(x_3, x_4)$ *satisfying*

$$P_3 - Q_4 = 0, \qquad c\, P_3 - x_1\, P_{34} - x_2\, P_{33} - S_{34} = 0.$$

Moreover, in such a case, W^- *has eigenvalues* $(0, \pm\frac{1}{2}(-c_{11}\, c_{22})^{\frac{1}{2}})$ *and the minimal polynomial is characterized by the following:*

(i) $n_\lambda(W^-) = -p_\lambda(W^-)$ *if there are three different eigenvalues (i.e.,* $c_{11}\, c_{22} \neq 0$*).*

(ii) $n_\lambda(W^-) = \lambda$ *or* $n_\lambda(W^-) = \lambda^3$ *depending on whether* c_{11} *and* c_{22} *vanish simultaneously or not, if zero is the unique eigenvalue (i.e.,* $c_{11}\, c_{22} = 0$*).*

Proof. First note that, from (3.10), the anti-self-duality is equivalent to

$$c_{12} = 0, \qquad c_{13} - c_{24} = 0, \qquad c\, c_{13} - c_{34} = 0. \tag{3.15}$$

Since $c_{12} = 0$, we have $c(x_1, x_2, x_3, x_4) = \bar{c}(x_1, x_3, x_4) + \hat{c}(x_2, x_3, x_4)$. Now, from the condition $c_{13} - c_{24} = \bar{c}_{13} - \hat{c}_{24} = 0$, we obtain

$$\bar{c}(x_1, x_3, x_4) = x_1 P(x_3, x_4) + \xi(x_1, x_4) + \bar{S}(x_3, x_4),$$

$$\hat{c}(x_2, x_3, x_4) = x_2 Q(x_3, x_4) + \eta(x_2, x_3) + \hat{S}(x_3, x_4),$$

where $P_3 - Q_4 = 0$. Then, putting $S = \bar{S} + \hat{S}$, from $c\,c_{13} - c_{34} = 0$ we obtain that $c\,P_3 - x_1\,P_{34} - x_2\,Q_{34} - S_{34} = 0$, which shows the first part of the result since $P_3 - Q_4 = 0$ implies that $Q_{34} = P_{33}$.

Now we analyze the characteristic and the minimal polynomials of W^-. From (3.9) we get the general expression of the characteristic polynomial of W^-, which is given by

$$p_\lambda(W^-) = \frac{-1}{108}\left\{108\,\lambda^3 + 9\left(3\,c_{11}\,c_{22} - 4\,c_{12}^2\right)\lambda - c_{12}\left(9\,c_{11}\,c_{22} - 8\,c_{12}^2\right)\right\}. \tag{3.16}$$

If the manifold is anti-self-dual then $c_{12} = 0$ and this lets us reduce (3.16) to the expression $p_\lambda(W^-) = \frac{-1}{4}\left\{4\,\lambda^3 + c_{11}\,c_{22}\,\lambda\right\}$. Now, if $c_{11}\,c_{22} \neq 0$ then there are three distinct eigenvalues and therefore $n_\lambda(W^-) = -p_\lambda(W^-)$. On the other hand, it is straightforward to show that under the conditions given in (3.15) we have

$$W^- = \frac{1}{4}\begin{pmatrix} 0 & -(c_{11}+c_{22}) & 0 \\ c_{11}+c_{22} & 0 & c_{11}-c_{22} \\ 0 & c_{11}-c_{22} & 0 \end{pmatrix}.$$

Now we compute the square of W^- to obtain

$$\left(W^-\right)^2 = \frac{1}{16}\begin{pmatrix} -(c_{11}+c_{22})^2 & 0 & -(c_{11}^2-c_{22}^2) \\ 0 & -4c_{11}c_{22} & 0 \\ c_{11}^2-c_{22}^2 & 0 & (c_{11}-c_{22})^2 \end{pmatrix},$$

which shows the different possibilities for the minimal polynomial when $p_\lambda(W^-) = -\lambda^3$, this is, when $c_{11}\,c_{22} = 0$. □

3.3.2 Characterization of Osserman manifolds

The main result of this section gives necessary and sufficient conditions for a Walker metric as in (3.4) to be Osserman. This is a first step to later, in Section 3.4, construct examples of conformally Osserman Walker metrics which are not in the conformal class of any Osserman manifold.

The following is a technical lemma we need to characterize the Osserman condition in this setting [42].

Lemma 3.3.3 *Let $\mathcal{O}$ be an open connected subset of $\mathbb{R}^4$. Let $P, Q \in C^\infty(\mathcal{O})$ be functions only of (x_3, x_4). Then the following conditions are equivalent:*

1. *$P^2 = 2P_4$, $Q^2 = 2Q_3$, and $P_3 = Q_4 = \frac{1}{2}PQ$.*
2. *$P^2 = 2P_4$, $Q^2 = 2Q_3$, and $PQ = P_3 + Q_4$.*
3. *There exist $(a_0, a_3, a_4) \in \mathbb{R}^3 - \{0\}$ so that $P = -2a_4(a_0 + a_3x_3 + a_4x_4)^{-1}$ and $Q = -2a_3(a_0 + a_3x_3 + a_4x_4)^{-1}$.*

Proof. It is clear that (1) implies (2); a direct computation shows that (3) implies (1). We must therefore show that (2) implies (3). Let P and Q satisfy the properties given in Assertion (2). Set

$$\mathcal{O}_P := \{(x_1, x_2, x_3, x_4) \in \mathcal{O} : P(x_3, x_4) \neq 0\},$$
$$\mathcal{O}_Q := \{(x_1, x_2, x_3, x_4) \in \mathcal{O} : Q(x_3, x_4) \neq 0\}.$$

We suppose first that $\mathcal{O}_P \cap \mathcal{O}_Q$ is non-empty. Let B be a closed ball $\mathbb{R}^4$ with non-empty interior which is contained in $\mathcal{O}$ and which has $\mathrm{int(B)} \subset \mathcal{O}_\mathrm{P} \cap \mathcal{O}_\mathrm{Q}$. We integrate the equation $P^2 = 2P_4$ on $int(B)$ to express

$$P(x_3, x_4) = -2(\xi(x_3) + x_4)^{-1} \quad \text{on} \quad \mathrm{int}(B)\,. \tag{3.17}$$

We use the relation $PQ = P_3 + Q_4$ to conclude

$$-2(\xi(x_3) + x_4)^{-1} Q = 2\dot{\xi}(x_3)(\xi(x_3) + x_4)^{-2} + Q_4(x_3, x_4)$$

which can be written in the form $\{Q(x_3, x_4)(\xi(x_3) + x_4)^2\}_4 = -2\dot{\xi}(x_3)$ and thus

$$Q(x_3, x_4) = \{\phi(x_3) - 2\dot{\xi}(x_3)x_4\}(\xi(x_3) + x_4)^{-2}\,. \tag{3.18}$$

We set $Q^2 = 2Q_3$ and clear denominators to obtain the relation:

$$\begin{aligned} \{\phi(x_3) - 2\dot{\xi}(x_3)x_4\}^2 &= 2\{\dot{\phi}(x_3) - 2\ddot{\xi}(x_3)x_4\}(\xi(x_3) + x_4)^2 \\ &\quad -2\{\phi(x_3) - 2\dot{\xi}(x_3)x_4\}2\dot{\xi}(x_3)(\xi(x_3) + x_4)\,. \end{aligned} \tag{3.19}$$

Setting the coefficient of $x_4^3 = 0$ then yields $\ddot{\xi}(x_3) = 0$ so $\xi = \alpha_0 + \alpha_1 x_3$ and Equation (3.19) becomes:

$$\begin{aligned} \{\phi(x_3) - 2\alpha_1 x_4\}^2 &= 2\dot{\phi}(x_3)(\alpha_0 + \alpha_1 x_3 + x_4)^2 \\ &\quad -4\alpha_1(\phi(x_3) - 2\alpha_1 x_4)(\alpha_0 + \alpha_1 x_3 + x_4)\,. \end{aligned} \tag{3.20}$$

Examining the coefficient of x_4^2 in Equation (3.20) shows that $\dot{\phi}(x_3) = -2\alpha_1^2$, hence we have $\phi(x_3) = \beta_0 - 2\alpha_1^2 x_3$. Equation (3.20) then further simplifies to become:

$$\begin{aligned} (\beta_0 - 2\alpha_1^2 x_3 - 2\alpha_1 x_4)^2 &= -4\alpha_1^2(\alpha_0 + \alpha_1 x_3 + x_4)^2 \\ &\quad -4\alpha_1(\beta_0 - 2\alpha_1^2 x_3 - 2\alpha_1 x_4)(\alpha_0 + \alpha_1 x_3 + x_4)\,. \end{aligned} \tag{3.21}$$

This leads to the relation $\beta_0^2 = -4\alpha_1^2\alpha_0^2 - 4\beta_0\alpha_1\alpha_0$ which implies $\beta_0 = -2\alpha_1\alpha_0$. Equations (3.17) and (3.18) now yield

$$\begin{aligned} P(x_3, x_4) &= -2(\alpha_0 + \alpha_1 x_3 + x_4)^{-1}, \\ Q(x_3, x_4) &= -2\alpha_1(\alpha_0 + \alpha_1 x_3 + x_4)^{-1}. \end{aligned} \tag{3.22}$$

By continuity, Equation (3.22) holds on the closed ball B and in particular P and Q do not vanish on B. Consequently $\mathcal{O}_P \cap \mathcal{O}_Q$ is a closed set so as $\mathcal{O}$ is connected, we may conclude $\mathcal{O} = \mathcal{O}_P = \mathcal{O}_Q$. Analytic continuation now shows P and Q are given by Equation (3.22) on all of $\mathcal{O}$ and thus Assertion (3) holds.

We therefore assume $\mathcal{O}_P \cap \mathcal{O}_Q$ is empty. If $\mathcal{O}_P$ and $\mathcal{O}_Q$ are both empty, then $P = Q = 0$ and we may take $(a_0, a_3, a_4) = (1, 0, 0)$ to obtain a representation of the form given in (3). We therefore assume $\mathcal{O}_Q$ is non-empty; the case $\mathcal{O}_P$ is non-empty is handled similarly. Let B be a closed ball with non-empty interior which is contained in $\mathcal{O}$ and which satisfies $int(B) \subset \mathcal{O}_Q$. We integrate the equation $Q^2 = 2Q_3$ to express

$$Q = -2\{x_3 + \eta(x_4)\}^{-1} \quad \text{on} \quad int(B)\,.$$

Since $PQ = P_3 + Q_4$ and since $P = 0$ on $int(B)$, we have $\dot{\eta} = 0$ and hence

$$Q = -2(x_3 + a)^{-1} \quad \text{on} \quad int(B)\,. \tag{3.23}$$

Again, by continuity, this representation holds on all of B and thus Q is non-zero on B. Thus $\mathcal{O}_Q$ is closed and consequently $\mathcal{O} = \mathcal{O}_Q$. Thus Equation (3.23) holds on all of $\mathcal{O}$ and $P = 0$ on all of $\mathcal{O}$. This again obtains a representation for P and Q of the form given in Assertion (3). □

Remark 3.3.4 *If $\lambda \neq 0$, then $(\lambda a_0, \lambda a_3, \lambda a_4)$ and (a_0, a_3, a_4) determine the same functions P and Q. Thus we may regard (a_0, a_3, a_4) as belonging to the real projective space $\mathbb{RP}^2 := \{\mathbb{R}^3 - \{0\}\}/\{\mathbb{R} - \{0\}\}$. If $a_4 = 0$, then $P = 0$; if $a_3 = 0$, then $Q = 0$.*

Now, we are ready to give the following characterization.

Theorem 3.3.5 *A Walker metric* (3.4) *is Osserman if and only if the defining function $c(x_1, x_2, x_3, x_4)$ has the following form*

$$c(x_1, x_2, x_3, x_4) = x_1 P(x_3, x_4) + x_2 Q(x_3, x_4) + S(x_3, x_4)\,,$$

where

$$\begin{aligned} P(x_3, x_4) &= \tfrac{-2a_4}{a_0 + a_3 x_3 + a_4 x_4}\,, \\ Q(x_3, x_4) &= \tfrac{-2a_3}{a_0 + a_3 x_3 + a_4 x_4}\,. \end{aligned}$$

Proof. Recall from Theorem 2.2.8 that a metric (3.4) is Osserman at a given point if and only if it is Einstein and self-dual or anti-self-dual at that point. Note from (3.6) and (3.7) that a metric (3.4) is Einstein if and only if

$$c_{11} = c_{22} = 0, \quad c_1^2 = 2\,c_{14}, \quad c_2^2 = 2\,c_{23}, \quad c_1\,c_2 - c_{13} - c_{24} + c\,c_{12} = 0. \tag{3.24}$$

On the other hand, the metric must be self-dual or anti-self-dual and, in both cases, $c_{12} = 0$ (see the proofs of Theorems 3.3.1 and 3.3.2). Hence $c_{11} = c_{22} = c_{12} = 0$, which is nothing but (3.12), and thus the Osserman condition implies that $c(x_1, x_2, x_3, x_4)$ must be of the form

$$c(x_1, x_2, x_3, x_4) = x_1 P(x_3, x_4) + x_2 Q(x_3, x_4) + S(x_3, x_4), \tag{3.25}$$

for some functions $P(x_3, x_4)$, $Q(x_3, x_4)$ and $S(x_3, x_4)$ satisfying

$$P^2 = 2P_4, \quad Q^2 = 2Q_3, \quad PQ = P_3 + Q_4. \tag{3.26}$$

Now the result follows from Lemma 3.3.3. □

Remark 3.3.6 Note that expressions (3.24) by themselves are sufficient to guarantee that the manifold is not only Einstein but Osserman. Indeed, from $c_{11} = c_{22} = 0$ we get that c may be written as

$$c(x_1, x_2, x_3, x_4) = x_1 P(x_3, x_4) + x_2 Q(x_3, x_4) + x_1 x_2 R(x_3, x_4) + S(x_3, x_4).$$

Now use equation $c_1 c_2 - c_{13} - c_{24} + c\, c_{12} = 0$, which is a polynomial in the variables x_1 and x_2, to see that the coefficient of $x_1 x_2$ is $2R(x_1, x_2)$ and necessarily vanishes. Hence c has the form given in (3.25) for P and Q satisfying conditions in (3.26). Moreover, these conditions imply that $\rho = 0$.

To sum up, if $\mathcal{M}$ has a metric of the form given in (3.4), the following conditions are equivalent:

1. $\mathcal{M}$ is Osserman,
2. $\mathcal{M}$ is Einstein,
3. $\rho = 0$.

Remark 3.3.7 In order to analyze the Jordan normal form of the Jacobi operators corresponding to the Osserman metrics in the previous theorem we do the following calculation. Let $x = \sum_{i=1}^{4} \alpha_i \partial_i$ be any non null tangent vector. Then the Jacobi operator $\mathcal{J}_R(x)$ takes the form

$$\mathcal{J}_R(x) = \delta \begin{pmatrix} 0 & 0 & \alpha_4^2 & -\alpha_3 \alpha_4 \\ 0 & 0 & -\alpha_3 \alpha_4 & \alpha_3^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\delta = \frac{-2a_3 a_4 S(x_3, x_4) + (a_0 + a_3 x_3 + a_4 x_4)^2 S_{34}(x_3, x_4)}{(a_0 + a_3 x_3 + a_4 x_4)^2}.$$

This shows that all Osserman metrics in Theorem 3.3.5 have nilpotent Jacobi operators and hence all the eigenvalues are zero. Moreover, in order to decide the degree of nilpotency, note that $J_R(x)^2 = 0$ for any of these metrics, so the minimal polynomial is λ^2 in all the cases except when $\delta = 0$ that $J_R(x) = 0$.

3.4 Conformally Osserman examples

We now use the results of the previous sections to construct explicit examples of conformally Osserman metrics. Special emphasis will be made on those examples being conformally Osserman but neither Osserman nor in the conformal class of an Osserman metric.

3.4.1 Examples with globally constant Jordan normal form

Next we will give some simple examples of conformally Osserman metrics corresponding to the different possibilities in Remark 2.2.1. Note that none of them is Osserman as an application of Theorem 3.3.5. Further, observe that although the conformally Osserman property does not require the global constancy of the eigenvalues of the conformal Jacobi operators (in order to be a conformal property), that is indeed the case in the examples below.

The following examples are conformally Osserman:

(Ia) For the special choice of $c(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2$ the characteristic polynomial of $\mathcal{J}_W(x)$ is given by $p_\lambda(\mathcal{J}_W(x)) = \lambda^4 - \frac{1}{4}\lambda^2$ and, therefore, its eigenvalues are $(0, 0, \frac{1}{2}, -\frac{1}{2})$.

(Ib) For the special choice of $c(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2$ the characteristic polynomial of $\mathcal{J}_W(x)$ is given by $p_\lambda(\mathcal{J}_W(x)) = \lambda^4 + \frac{1}{4}\lambda^2$ and, therefore, its eigenvalues are $(0, 0, \frac{1}{2}i, -\frac{1}{2}i)$.

(II) For the special choice of $c(x_1, x_2, x_3, x_4) = x_1 x_4 + x_3 x_4$ the characteristic polynomial of $\mathcal{J}_W(x)$ is given by $p_\lambda(\mathcal{J}_W(x)) = \lambda^4$ and its minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^2$.

(III) For the special choice of $c(x_1, x_2, x_3, x_4) = x_1^2$ the characteristic polynomial of $\mathcal{J}_W(x)$ is given by $p_\lambda(\mathcal{J}_W(x)) = \lambda^4$ and its minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^3$.

The following table summarizes information about the examples above.

$c(x_1, x_2, x_3, x_4) =$	Type	Characteristic polynomial of $\mathcal{J}_W$	Eigenvalues of $\mathcal{J}_W$	Minimal polynomial of $\mathcal{J}_W$
$x_1^2 - x_2^2$	Ia	$\lambda^4 - \frac{1}{4}\lambda^2$	$0,\ 0,\ \frac{1}{2},\ -\frac{1}{2}$	$\lambda^2(\lambda - \frac{1}{2})(\lambda + \frac{1}{2})$
$x_1^2 + x_2^2$	Ib	$\lambda^4 + \frac{1}{4}\lambda^2$	$0,\ 0,\ \frac{1}{2}i,\ -\frac{1}{2}i$	$\lambda^2(\lambda - \frac{1}{2}i)(\lambda + \frac{1}{2}i)$
$x_1x_4 + x_3x_4$	II	λ^4	0, 0, 0, 0	λ^2
x_1^2	III	λ^4	0, 0, 0, 0	λ^3

Remark 3.4.1 Recall that a metric $\hat{g}$ is in the conformal class of a metric g if and only if $\hat{g} = \frac{1}{\Psi^2} g$ for some positive function Ψ. Let $\hat{g} = \frac{1}{\Psi^2} g$ for a Walker metric g as in (3.4). A long but straightforward calculation from the components $\hat{\rho}_{11}$, $\hat{\rho}_{12}$ and $\hat{\rho}_{22}$ of the Ricci tensor of $\hat{g}$ shows that if $\hat{g}$ is Einstein, then the conformal factor must be of the form

$$\Psi = x_1\phi(x_3, x_4) + x_2\psi(x_3, x_4) + \xi(x_3, x_4)\,,$$

for some functions ϕ, ψ and ξ. Now, assuming g to be any of the metrics (Ia), (Ib) or (III) above, it follows from the component $\hat{\rho}_{14}$ that both functions ϕ and ψ must vanish, and thus $\Psi = \xi(x_3, x_4)$. Moreover a conformal deformation $\hat{g} = \frac{1}{\Psi^2} g$ of a metric (3.4) with $\Psi = \xi(x_3, x_4)$ is Einstein if and only if

$$
\begin{aligned}
&c_{11} = c_{22} = 0, \quad c_1^2 = 2\,c_{14} + 4\,\tfrac{\xi_{44}}{\xi}, \quad c_2^2 = 2\,c_{23} + 4\,\tfrac{\xi_{33}}{\xi}, \\
&\xi\,c_1\,c_2 - \xi\,(c_{13} + c_{24} - c\,c_{12}) + 2\,\xi_3\,c_1 + 2\,\xi_4\,c_2 + 4\,\xi_{34} = 0.
\end{aligned} \tag{3.27}
$$

As a consequence, we obtain the nonexistence of Osserman metrics in the conformal class of the conformally Osserman metrics (Ia), (Ib) and (III) above.

3.4.2 Examples with eigenvalue structure changing from point to point

All the examples in previous section are conformally Osserman in a global sense, this is, the eigenvalues of the conformal Jacobi operator are independent of the point. Here we give some examples showing that the Jordan normal form of the conformal Jacobi operator may change from point to point and, moreover, that the eigenvalues of the conformal Jacobi operators may also change (even from real to complex eigenvalues).

Examples with nilpotent conformal Jacobi operator

1. The characteristic polynomial is $p_\lambda(\mathcal{J}_W(x)) = \lambda^4$ while the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^3$, λ^2 or λ, depending on the point considered. For the special choice of $c(x_1, x_2, x_3, x_4) = x_2\,x_4^2 + x_3^2\,x_4$, the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^3$ at any point with $x_4 \neq 0$, $n_\lambda(\mathcal{J}_W(x)) = \lambda^2$ at those points with $x_4 = 0$, $x_3 \neq 0$ and $n_\lambda(\mathcal{J}_W(x)) = \lambda$ at points with $x_3 = x_4 = 0$.

2. The characteristic polynomial is $p_\lambda(\mathcal{J}_W(x)) = \lambda^4$ while the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^3$ or λ^2, depending on the point considered. For the special choice of $c(x_1, x_2, x_3, x_4) = x_2\,x_4^2 + x_3\,x_4$, the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^3$ at any point with $x_4 \neq 0$ and $n_\lambda(\mathcal{J}_W(x)) = \lambda^2$ at those points with $x_4 = 0$.

3. The characteristic polynomial is $p_\lambda(\mathcal{J}_W(x)) = \lambda^4$ while the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^3$ or λ, depending on the point considered. For the special choice of $c(x_1, x_2, x_3, x_4) = x_1\,x_3^2$, the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^3$ at any point with $x_3 \neq 0$ and $n_\lambda(\mathcal{J}_W(x)) = \lambda$ at those points with $x_3 = 0$.

4. The characteristic polynomial is $p_\lambda(\mathcal{J}_W(x)) = \lambda^4$ while the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^2$ or λ, depending on the point considered. For the special choice of $c(x_1, x_2, x_3, x_4) = x_1\,x_3 + x_2\,x_4$, the minimal polynomial is $n_\lambda(\mathcal{J}_W(x)) = \lambda^2$ at any point with $x_1\,x_3 + x_2\,x_4 \neq 0$ and $n_\lambda(\mathcal{J}_W(x)) = \lambda$ at those points with $x_1\,x_3 + x_2\,x_4 = 0$.

In the following table we can see that previous examples cover all the possibilities of variation among the different algebraic structures of Remark 2.2.1 with real eigenvalues.

$c(x_1, x_2, x_3, x_4) =$	Characteristic polynomial of $\mathcal{J}_W$	Where?	Type	Minimal polynomial of $\mathcal{J}_W$
$x_2 x_4^2 + x_3^2 x_4$	λ^4	$x_4 \neq 0$	III	λ^3
		$x_4 = 0, x_3 \neq 0$	II	λ^2
		$x_3 = x_4 = 0$	Ia	λ
$x_2 x_4^2 + x_3 x_4$	λ^4	$x_4 \neq 0$	III	λ^3
		$x_4 = 0$	II	λ^2
		—	—	—
$x_1 x_3^2$	λ^4	$x_3 \neq 0$	III	λ^3
		—	—	—
		$x_3 = 0$	Ia	λ
$x_1 x_3 + x_2 x_4$	λ^4	—	—	—
		$x_1 x_3 + x_2 x_4 \neq 0$	II	λ^2
		$x_1 x_3 + x_2 x_4 = 0$	Ia	λ

Scheme of examples with globally constant eigenvalues but changing Jordan normal form.

Type II non-nilpotent conformal Jacobi operators

We use the results of Subsection 3.2.3 to construct examples with non-nilpotent conformal Jacobi operators of type II. Recall from Remark 3.2.2 that the eigenvalues of W^+ are given by $(\tau/6, -\tau/12, -\tau/12)$. Hence, from expression (3.2), one chooses $\mathcal{A} = \mathcal{C} = 0$ and $\mathcal{B} + \mathcal{E}$ constant, so that the scalar curvature is constant, to get constant eigenvalues of the conformal Jacobi operator. Now, the Jordan normal form depends on the values of τ, W_{11}^+ and W_{22}^+, but it is always of type Ia or II (see Remark 3.2.2). To illustrate this procedure, we give here an explicit example which is not in the conformal class of an Osserman manifold.

Let $\mathcal{M}$ be a Walker metric as in Definition 3.2.1 with

$$a(x_1, x_2, x_3, x_4) = x_1^2 + x_1 x_4\,, \quad b(x_1, x_2, x_3, x_4) = x_2^2\,, \quad c(x_1, x_2, x_3, x_4) = x_1 x_2\,.$$

Hence the eigenvalues of the conformal Jacobi operator are $(0, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{4})$ everywhere. Moreover, the main feature of this example is that it is of type II in the open subset $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 6x_1 x_2 \neq 1\}$ and type Ia in $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 6x_1 x_2 = 1\}$. Moreover, it is not in the conformal class of an Einstein metric. Indeed, for a conformal deformation $\hat{g} = \frac{1}{\Psi^2} g$ we proceed as in Remark 3.4.1. A lengthy calculation of the components $\hat{\rho}_{12}$, $\hat{\rho}_{11}$ and $\hat{\rho}_{22}$ shows that

$$\begin{aligned} \Psi &= x_1 \phi_1(x_3, x_4) + \psi(x_3, x_4), \text{ or} \\ \Psi &= x_2 \phi_2(x_3, x_4) + \psi(x_3, x_4). \end{aligned}$$

Now, computing $\hat\rho_{14}$ and $\hat\rho_{23}$ we derive that $\Psi = \psi(x_3, x_4)$. Then, after computing $\hat\rho_{33}$ and $\hat\rho_{44}$, one gets that Ψ must be constant and $\hat g$ is not Einstein.

One may also choose values of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{E}$ so that τ is not constant and, thus, build examples with eigenvalues of the conformal Jacobi operator changing from point to point.

Examples with changing eigenvalues

Next we provide some examples of metrics with the form of (3.4), which are of type Ia or Ib and with eigenvalues of their conformal Jacobi operators changing from point to point.

1. The conformal Jacobi operators have real constant eigenvalues at each point, but changing from point to point. For the choice of $c(x_1, x_2, x_3, x_4) = x_1^4 + x_1^2 - x_2^4 - x_2^2$ we have $p_\lambda(\mathcal{J}_W(x)) = \lambda^4 - \frac{1}{4}(6\,x_1^2+1)(6x_2^2+1)\,\lambda^2$ and, therefore, its eigenvalues are $\left(0, 0, \frac{1}{2}\,\left((6\,x_1^2+1)(6x_2^2+1)\right)^{\frac{1}{2}}, -\frac{1}{2}\,\left((6\,x_1^2+1)(6x_2^2+1)\right)^{\frac{1}{2}}\right)$.
2. The conformal Jacobi operators have imaginary constant eigenvalues at each point, but changing from point to point. For the choice of $c(x_1, x_2, x_3, x_4) = x_1^4+x_1^2+x_2^4+x_2^2$ we have $p_\lambda(\mathcal{J}_W(x)) = \lambda^4 + \frac{1}{4}(6\,x_1^2+1)(6x_2^2+1)\,\lambda^2$ and, therefore, its eigenvalues are $\left(0, 0, \frac{1}{2}\,\left((6\,x_1^2+1)(6x_2^2+1)\right)^{\frac{1}{2}}\,i, -\frac{1}{2}\,\left((6\,x_1^2+1)(6x_2^2+1)\right)^{\frac{1}{2}}\,i\right)$.
3. The conformal Jacobi operators have real or imaginary constant eigenvalues depending on the considered point. For the special choice of $c(x_1, x_2, x_3, x_4) = x_1^3 - x_2^3$ the characteristic polynomial is $p_\lambda(\mathcal{J}_W(x)) = \lambda^4 - \frac{9}{4}\,x_1\,x_2\,\lambda^2$ and, therefore, its eigenvalues are $\left(0, 0, \frac{3}{2}\,(x_1\,x_2)^{\frac{1}{2}}, -\frac{3}{2}\,(x_1\,x_2)^{\frac{1}{2}}\right)$.

We summarize our results in the following table. It shows diagonalizable examples with changing real, imaginary or alternating real and imaginary eigenvalues.

$c(x_1, x_2, x_3, x_4) =$	Eigenvalues
$x_1^4 + x_1^2 - x_2^4 - x_2^2$	$0,\ 0,\ \pm\frac{1}{2}\,\left((6\,x_1^2+1)(6x_2^2+1)\right)^{\frac{1}{2}}$
$x_1^4 + x_1^2 + x_2^4 + x_2^2$	$0,\ 0,\ \pm\frac{1}{2}\,\left((6\,x_1^2+1)(6x_2^2+1)\right)^{\frac{1}{2}}\,i$
$x_1^3 - x_2^3$	$0,\ 0,\ \pm\frac{3}{2}\,(x_1\,x_2)^{\frac{1}{2}}$

Examples with changing eigenvalues from point to point.

3.5 Curvature homogeneity

A pseudo-Riemannian manifold $\mathcal{M} = (M, g)$ is said to be *curvature homogeneous* if for any $p, q \in M$ there is an isometry $\phi : T_pM \to T_qM$ such that $\phi^* R_q = R_p$. Roughly speaking, the curvature tensor of a curvature homogeneous manifold looks the same at every point. Thus, homogeneous manifolds are necessarily curvature homogeneous. In the following result we analyze examples of the previous section [40].

Theorem 3.5.1 *The manifolds of Section 3.4 are not curvature homogeneous.*

Proof. First of all note that the Ricci and the (anti)-self-dual operators of a curvature homogeneous manifold have constant Jordan normal form. Hence, we may use the description given in Section 3.4.2 to derive the result for examples given there. However we analyze all the cases separately giving some more information on their geometry.

$\underline{c = x_1^2 - x_2^2}$: The square of the Ricci operator is given by

$$\rho^2 = \begin{pmatrix} -1 & 0 & 4x_1x_2 & 4(x_1^2 - x_2^2) \\ 0 & -1 & 4(x_1^2 - x_2^2) & -4x_1x_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

thus ρ^2 is a multiple of the identity at a point p if and only if $x_1 = x_2 = 0$ and $\mathcal{M}$ is not curvature homogeneous.

$\underline{c = x_1^2 + x_2^2}$: Again we compute the square of the Ricci operator to obtain

$$\rho^2 = \begin{pmatrix} 1 & 0 & 4x_1x_2 & -4(x_2^1 - x_2^2) \\ 0 & 1 & -4(x_1^2 + x_2^2) & 4x_1x_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

hence, again ρ^2 is a multiple of the identity at a point p if and only if $x_1 = x_2 = 0$ and $\mathcal{M}$ is not curvature homogeneous.

$\underline{c = x_1x_4 + x_3x_4}$: The Ricci operator is given by

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - \frac{1}{2}x_4^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\mathcal{M}$ is Ricci flat if and only if $x_4^2 = 2$ and $\mathcal{M}$ is not curvature homogeneous.

$\underline{c = x_1^2}$: This case is surprisingly difficult to treat. We work in somewhat greater generality and let $c = c(x_1)$. From Equations (3.5) we specialize the curvature tensor to obtain that the only nonnull components are:

$$R_{1314} = -\tfrac{1}{2}\, c_{11}, \qquad R_{1434} = -\tfrac{1}{4}\, c_1^2.$$

It is convenient to work in an algebraic context. We consider a model $\mathcal{V}$. Let $\{e_1, e_2, e_3, e_4\}$ be a basis for $\mathbb{R}^4$. Let $\mathcal{V}_{\alpha,\beta,\gamma} = (\mathbb{R}^4, g_\alpha, A_{\beta,\gamma})$ where the non-zero entries of g_α and $A_{\beta,\gamma}$ are, up to the usual $\mathbb{Z}_2$ symmetries, given by

$$g_a(e_1, e_3) = 1, \quad g_a(e_2, e_4) = 1, \quad g_\alpha(e_3, e_4) = \alpha,$$
$$A_{\beta,\gamma}(e_1, e_3, e_1, e_4) = -\gamma, \quad A_{b,c}(e_1, e_4, e_3, e_4) = -\beta^2\,.$$

Let $\mathcal{V}(\mathcal{M},p)=(\mathbb{R}^4,g_p,R_p)$. It is then immediate that

$$\mathcal{V}(\mathcal{M},p)=\mathcal{V}_{c,\frac{1}{2}c_1,\frac{1}{2}c_{11}}\,.$$

Lemma 3.5.2

1. *$\mathcal{V}_{\alpha,\beta,\gamma}$ is isomorphic to $\mathcal{V}_{0,\beta,\gamma}$.*
2. *$\mathcal{V}_{0,\beta,\gamma}$ is isomorphic to $\mathcal{V}_{0,1,1}$ if $\beta\neq 0$ and if $\gamma\neq 0$.*
3. *$\mathcal{V}_{0,0,\gamma}$ is isomorphic to $\mathcal{V}_{0,0,1}$ if $\gamma\neq 0$.*
4. *$\mathcal{V}_{0,\beta,0}$ is isomorphic to $\mathcal{V}_{0,1,0}$ if $\beta\neq 0$.*

Proof. Set $f_1:=e_1$, $f_2:=e_2$, $f_3:=e_3-ae_2$, and $f_4:=e_4$. We prove Assertion (1) by computing:

$$\begin{aligned}
&g_a(f_1,f_3)=g_a(f_2,f_4)=1,\\
&g_a(f_3,f_4)=g(e_3,e_4)-ag(e_2,e_4)=0,\\
&A_{b,c}(f_1,f_4,f_3,f_4)=A_{b,c}(e_1,e_4,e_3,e_4)=-\beta^2,\\
&A_{b,c}(f_1,f_3,f_1,f_4)=A_{b,c}(e_1,e_3,e_1,e_4)=-\gamma\,.
\end{aligned}$$

Clear the previous notation. Set $f_1:=\varepsilon_1e_1$, $f_2:=\varepsilon_2e_2$, $f_3:=\varepsilon_1^{-1}e_3$, and $f_4:=\varepsilon_2^{-1}e_4$. This is still a hyperbolic basis. We compute:

$$A_{\beta,\gamma}(f_1,f_4,f_3,f_4)=-\varepsilon_2^{-2}\beta^2\quad\text{and}\quad A_{\beta,\gamma}(f_1,f_3,f_1,f_4)=-\varepsilon_1\varepsilon_2^{-1}\gamma\,.$$

If $\beta,\gamma\neq 0$, we take $\varepsilon_2=\beta$ and $\varepsilon_1=\beta\gamma^{-1}$ to establish Assertion (2). If $\beta=0$ and $\gamma\neq 0$, we take $\varepsilon_1=1$ and $\varepsilon_2=\gamma$ to establish Assertion (3). If $\beta\neq 0$ and $\gamma=0$, we take $\varepsilon_2=\beta$ and $\varepsilon_1=1$ to establish Assertion (4). □

Set $\mathcal{V}_{\beta,\gamma}:=\mathcal{V}_{0,\beta,\gamma}$. The analysis performed above shows that we have at most 4 different models. In fact, these models are all distinct.

Lemma 3.5.3 *The models $\{\mathcal{V}_{0,0},\mathcal{V}_{1,1},\mathcal{V}_{0,1},\mathcal{V}_{1,0}\}$ are pairwise non-isomorphic.*

Proof. As $\mathcal{V}_{0,0}$ is flat, it is distinct from the other models. If $\mathcal{V}$ is a model, define:

$$\begin{aligned}
&\ker(\mathcal{V}):=\{\xi:\mathcal{A}(\xi,\xi_1,\xi_2,\xi_3)=0\ \forall\ \xi_1,\xi_2,\xi_3\in\mathbb{R}^4\},\\
&\text{Range}\,(\mathcal{V})=\text{Span}\,\{\mathcal{A}(\xi_1,\xi_2)\xi_3\ \forall\ \xi_1,\xi_2,\xi_3\in\mathbb{R}\}\,.
\end{aligned}$$

It is then immediate that

$$\ker(\mathcal{V}_{0,1})=\ker(\mathcal{V}_{1,0})=\ker(\mathcal{V}_{1,1})=\text{Span}\,\{e_2\}\,. \tag{3.28}$$

Moreover, note that

$$\begin{aligned} &\mathrm{Range}\,(\mathcal{V}_{\beta,\gamma}) = \mathrm{Span}\,\{\gamma e_1, \gamma e_2, \gamma e_3, \beta^2 e_1, \beta^2 e_2, \beta^2 e_3\} \quad \text{so} \\ &\mathrm{Range}\,(\mathcal{V}_{1,1}) = \mathrm{Range}\,(\mathcal{V}_{1,0}) = \mathrm{Range}\,(\mathcal{V}_{0,1}) = \mathrm{Span}\,\{e_1, e_2, e_3\}\,. \end{aligned} \tag{3.29}$$

Let $(\beta,\gamma) \neq (0,0)$. Suppose there exists an isometry T of $\mathbb{R}^4$ so $T^*\mathcal{V}_{\bar\beta,\bar\gamma} = \mathcal{V}_{\beta,\gamma}$; necessarily $(\bar\beta,\bar\gamma) \neq (0,0)$ as well. Equations (3.28) and (3.29) yield:

$$\begin{aligned} &T\mathrm{Span}\,\{e_2\} = T\ker\{\mathcal{V}_{\beta,\gamma}\} = \ker\{\mathcal{V}_{\bar\beta,\bar\gamma}\} = \mathrm{Span}\,\{e_2\}, \\ &T\mathrm{Span}\,\{e_4\} = T\mathrm{Span}\,\{e_1, e_2, e_3\}^\perp = T\mathrm{Range}\,\{\mathcal{V}_{\beta,\gamma}\}^\perp \\ &\qquad = \mathrm{Range}\,\{\mathcal{V}_{\bar\beta,\bar\gamma}\}^\perp = \mathrm{Span}\,\{e_1, e_2, e_3\}^\perp = \mathrm{Span}\,\{e_4\}, \\ &T\mathrm{Span}\,\{e_1, e_3\} = T\mathrm{Span}\,\{e_2, e_4\}^\perp = \mathrm{Span}\,\{e_2, e_4\}^\perp = \mathrm{Span}\,\{e_1, e_3\}\,. \end{aligned}$$

Consequently we have

$$Te_1 = t_{11}e_1 + t_{13}e_3,\ Te_2 = \varepsilon_2 e_2,\ Te_3 = e_{31}e_1 + e_{33}e_3,\ Te_4 = \varepsilon_2^{-1}e_4\,.$$

Since $Te_1 \perp Te_1$ and $Te_3 \perp Te_3$, we have $T = U$ or $T = V$ where

$$\begin{aligned} &Ue_1 = \varepsilon_1 e_1,\ Ue_2 = \varepsilon_2 e_2,\ Ue_3 = \varepsilon_1^{-1}e_3,\ Ue_4 = \varepsilon_2^{-1}e_4, \\ &Ve_1 = \varepsilon_1 e_3,\ Ve_2 = \varepsilon_2 e_2,\ Ve_3 = \varepsilon_1^{-1}e_1,\ Ve_4 = \varepsilon_2^{-1}e_4\,. \end{aligned}$$

We have

$$\begin{aligned} &U^*(e^1 \otimes e^3 \otimes e^1 \otimes e^4) = \delta_1 e^1 \otimes e^3 \otimes e^1 \otimes e^4, \\ &U^*(e^1 \otimes e^4 \otimes e^3 \otimes e^4) = \delta_2 e^1 \otimes e^4 \otimes e^3 \otimes e^4, \\ &V^*(e^1 \otimes e^3 \otimes e^1 \otimes e^4) = \delta_1 e^3 \otimes e^1 \otimes e^3 \otimes e^4, \\ &V^*(e^1 \otimes e^4 \otimes e^3 \otimes e^4) = \delta_2 e^3 \otimes e^4 \otimes e^1 \otimes e^4\,. \end{aligned}$$

Thus such a map can not intertwine different elements of $\{\mathcal{V}_{0,1}, \mathcal{V}_{1,0}, \mathcal{V}_{1,1}\}$. □

In the example in question, c vanishes precisely when $x_1 = 0$. The analysis performed above shows $\mathcal{M}$ is not curvature homogeneous; however it is curvature homogeneous on the hyperplane complement.

<u>$c = x_2x_4^2 + x_3^2x_4$</u>: We compute the Ricci operator

$$\rho = \begin{pmatrix} 0 & 0 & -\frac{1}{2}x_4^4 & -x_4 \\ 0 & 0 & -x_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\mathcal{M}$ is Ricci flat at p if and only if $x_4 = 0$, so $\mathcal{M}$ is not curvature homogeneous.

$\underline{c = x_2 x_4^2 + x_3 x_4}$: Again the Ricci operator

$$\rho = \begin{pmatrix} 0 & 0 & -\frac{1}{2}x_4^4 & -x_4 \\ 0 & 0 & -x_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

shows $\mathcal{M}$ is not curvature homogeneous, since it is Ricci flat at p if and only if $x_4 = 0$.

$\underline{c = x_1 x_3^2}$: Now, the Ricci operator

$$\rho = \begin{pmatrix} 0 & 0 & 0 & -x_3 \\ 0 & 0 & -x_3 & -\frac{1}{2}x_3^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

has all its components zero at p if and only if $x_3 = 0$ and hence $\mathcal{M}$ is not curvature homogeneous.

$\underline{c = x_1 x_3 + x_2 x_4}$: Let R_Λ be the induced map from Λ to Λ. Relative to a suitably chosen basis, one has:

$$R_\Lambda^2 = \begin{pmatrix} -1 + x_3 x_4 & 0 & -1 + x_3 x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 - x_3 x_4 & 0 & 1 - x_3 x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0. \end{pmatrix}.$$

Thus $R_\Lambda^2 = 0$ if and only if $x_3 x_4 = 1$ and $\mathcal{M}$ is not curvature homogeneous.

Remark 3.5.4 Derdzinski [57] showed a 4-dimensional Riemannian manifold is curvature homogeneous if and only if R_Λ has constant eigenvalues; furthermore if such a manifold is Einstein, then it is locally symmetric. This fails in signature $(2, 2)$. We have computed that R_Λ has constant eigenvalues $(0, 0, 0, 0, 0, 0)$ in example with $c = x_1 x_3 + x_2 x_4$ but that the Jordan normal form of R_Λ varies with the point in question. Thus these manifolds are not curvature homogeneous. This is not surprising; it is known that the eigenvalue structure alone is not a sufficient determinant; one must also consider the Jordan normal form. We refer to [58] for additional results in the $(2, 2)$ setting.

$\underline{a = x_1^2 + x_1 x_4,\ b = x_2^2 \text{ and } c = x_1 x_2}$: We have shown that the conformal Jacobi operator is of type II if $6x_1 x_2 \neq 0$ but of type I if $6x_1 x_2 = 0$. Hence the self-dual tensor also changes its type and the example is not curvature homogeneous.

Let $\underline{c = x_1^4 + x_1^2 - x_2^4 - x_2^2}$, $\underline{c = x_1^4 + x_1^2 + x_2^4 + x_2^2}$ or $\underline{c = x_1^3 - x_2^3}$. Then we have seen in Section 3.4.2 that $\mathcal{M}$ is conformally Osserman but the eigenvalues change from point to point. Therefore $\mathcal{M}$ is not curvature homogeneous. □

Chapter 4

Some global results on manifolds with signature $(- - ++)$

The local to global study of geometric properties has been particularly developed in Riemannian geometry. In contrast, in areas such as Lorentzian or more generally pseudo-Riemannian geometry, little is known about global properties of the geometry. In most results, the sign of the curvature (sectional curvature, Ricci curvature, etc.) plays a fundamental role, like for instance the celebrated Calabi-Markus Theorem [44]. However, such assumptions have no sense in the case of pseudo-Riemannian metrics of neutral signature (ν, ν), since changing the sign of the metric takes one from positive to negative curvature. The simplest case to be considered is that of 4-dimensional $(- - ++)$-metrics, where some generalizations of the Hitchin-Thorpe inequality for Einstein metrics have been developed by Law [120] and Law-Matsushita [121].

4.1 Compact Osserman neutral signature manifolds

We begin the global study of Osserman manifolds by analyzing those which are compact. More concretely, the purpose of this section is to show that a compact Jordan-Osserman 4-dimensional manifold with metric of neutral signature, necessarily has constant sectional curvature or is Ricci flat. The results in this section are collected in [43].

4.1.1 Basic facts

Recall that, if we consider the curvature tensor R as an endomorphism of $\Lambda(M)$, we can decompose it as

$$R = \frac{\tau}{12}\, Id_\Lambda + \mathfrak{Z}_\Lambda + W^+ + W^-, \tag{4.1}$$

where $\mathfrak{Z}_\Lambda$ denotes the traceless Ricci tensor, $\mathfrak{Z}_\Lambda(X, Y) = \rho(X, Y) - \frac{\tau}{4}\, g(X, Y)$ (see Theorem 1.3.2).

A 4-dimensional metric is Einstein if and only if the decomposition (4.1) becomes $R = \frac{\tau}{12}\, Id_\Lambda + W^+ + W^-$. In such a case, the Euler characteristic $\chi[M]$ and the Hirzebruch signature $\Upsilon[M]$ can be expressed as follows [120]:

$$\chi[M] = -\frac{1}{8\pi^2}\int_M \{\mathrm{tr}[(W^+)^2] + \mathrm{tr}[(W^-)^2] + \frac{\tau^2}{24}\}v,$$
(4.2)
$$\Upsilon[M] = \frac{2}{3}\frac{1}{8\pi^2}\int_M \{\mathrm{tr}[(W^+)^2] - \mathrm{tr}[(W^-)^2]\}v.$$

Observe from (4.2) that the Euler characteristic of any compact Einstein $(--++)$–metric is nonpositive, provided that $W^\pm$ are not of type Ib (i.e. provided they have real eigenvalues).

It is a fundamental fact that an orientable 4-dimensional manifold with a field of 2-planes (equivalently, a 2-dimensional distribution) admits two kinds of almost complex structures with opposite orientations. Since an indefinite manifold of metric signature $(--++)$ admits a field of 2-planes, it is necessarily an almost Hermitian manifold.

Fact 4.1.1 [127] *An orientable* 4*-dimensional manifold admits a* $(--++)$*–metric if and only if it satisfies a pair of Wu's conditions as follows*

$$\begin{aligned} c_1^2[M] &= 3\Upsilon[M] + 2\chi[M], \\ c_1^2[-M] &= 3\Upsilon[-M] + 2\chi[-M] = -3\Upsilon[M] + 2\chi[M], \end{aligned}$$

where $-M$ *stands for* M *with the opposite orientation.*

Hence, it follows from (4.2) that the first Chern number $c_1^2[M]$ and the first opposite Chern number $c_1^2[-M]$ are given by

$$c_1^2[M] = -\frac{1}{2\pi^2}\int_M \{\mathrm{tr}[(W^-)^2] + \frac{\tau^2}{48}\}v,$$
(4.3)
$$c_1^2[-M] = -\frac{1}{2\pi^2}\int_M \{\mathrm{tr}[(W^+)^2] + \frac{\tau^2}{48}\}v.$$

Remark 4.1.2 The fundamental form of an indefinite almost Hermitian structure defines a smooth section of Λ^- of constant norm 2 and conversely, any smooth section Ω of Λ^- of constant norm 2 is the fundamental form of an indefinite almost Hermitian structure. The fact that the inner product on $\Lambda^\pm$ induced by a $(--++)$-metric is of signature $(--+)$ shows the existence of two almost complex structures with opposite orientations just considering the 1-dimensional spacelike subspaces of $\Lambda^\pm$, from where Fact 4.1.1 can also be obtained.

4.1.2 Main results

The following result, which is central in this section, shows that compactness is a strong restriction when dealing with Osserman metrics.

Theorem 4.1.3 *Let $\mathcal{M} = (M, g)$ be a compact Jordan-Osserman manifold with metric of signature $(--++)$. Then $\mathcal{M}$ has constant sectional curvature or the Jacobi operators are nilpotent.*

Proof. Recall from Theorem 2.2.8 that an algebraic curvature tensor is Osserman if and only if it is Einstein and self-dual or anti-self-dual. We assume in what follows that $W^- = 0$ (a completely analogous analysis will prove the anti-self-dual case). Hence, the self-dual part of the Weyl conformal curvature tensor corresponds to one of the Jordan canonical forms Ia, Ib, II or III in Remark 2.2.1.

Also recall that (g, P) is said to be an almost paraHermitian structure if $P^2 = Id$ and $g(PX, PY) = -g(X, Y)$ for all vector fields X, Y on M. Then the fundamental form $\Omega_P(X, Y) = g(PX, Y)$ defines a section of Λ^+ of constant norm -2. Conversely, any smooth section Ω in Λ^+ of constant norm -2 is the fundamental form of an almost paraHermitian structure (see also [102], where the inner product on Λ is taken with the opposite sign of ours).

Assume W^+ is of type Ib. Then Lemma 2.2.3 implies $\ker(W^+ + \frac{4}{3}(\alpha - \gamma)Id)$ is 1-dimensional and, moreover, it is timelike since otherwise the self-adjoint operator W^+ would diagonalize. The same occurs if we assume W^+ is of type II. In fact, if $\alpha \neq \beta$, then the eigenspace corresponding to $\frac{4}{3}(\alpha - \beta)$ defines an almost paraHermitian structure on $\mathcal{M}$. Next, assume $\alpha = \beta = 0$, then W^+ is 2-step nilpotent and $\mathrm{Im}(W^+)$ is 1-dimensional and has an induced degenerate inner product. Therefore the restriction of the metric to $\mathrm{Im}(W^+)$ defines a 1-dimensional null subspace. Recall from Remark 4.1.2 that unit sections $\Omega^+_\pm$ of $\Lambda^\pm$ of positive norm (equivalently, almost complex structures inducing opposite orientations) exist on M. Hence, for any null section $\Omega^0_\pm$ of $\Lambda^\pm$, define a 1-dimensional unit timelike section of $\Lambda^\pm$ as $\Omega^-_\pm = \frac{1}{\langle\Omega^0_\pm, \Omega^+_\pm\rangle}\Omega^0_\pm - \langle\Omega^0_\pm, \Omega^+_\pm\rangle\Omega^+_\pm$, which is an almost paraHermitian structure with fundamental form $\sqrt{2}\Omega^-_\pm$. Finally, assume W^+ to be of type III. Then $(W^+)^2$ is 2-step nilpotent, and the result follows as in type II above.

Further, observe that the existence of an almost paraHermitian structure (g, P) is an equivalent condition to the existence of an almost anti-Hermitian structure (g, J) (i.e., $J^2 = -Id$, $g(JX, JY) = -g(X, Y)$ for all vector fields X, Y on M). Indeed, let (g, P) be an almost paraHermitian structure and let h be an arbitrary Riemannian metric on M such that $h(PX, PY) = h(X, Y)$ for all X, Y. Define an almost product structure Q by $h(QX, Y) = g(X, Y)$ and put $J = -PQ$. A straightforward calculation shows that J is an almost complex structure on $\mathcal{M}$ and moreover

$$g(JX, JY) = g(PQX, PQY) = -g(QX, QY) = -g(X, Y),$$

which shows that (g, J) is almost anti-Hermitian. Moreover a straightforward calculation shows that J is h-orthogonal and the fundamental forms Ω_P and $\Omega^h_J(X, Y) = h(JX, Y)$

coincide with each other. Hence, both P and J induce the same orientation on M (see [23], [25] for more information on anti-Hermitian and anti-Kähler structures).

Chern classes of almost complex manifolds with anti-Hermitian metrics were studied in [23], [25], showing that the existence of such structures is a much more restrictive condition than that of almost Hermitian ones, as it is shown in the following assertion.

Fact 4.1.4 [23] *Let (M, g, J) be an almost anti-Hermitian manifold. Then all the odd Chern classes vanish (i.e., $c_{2k+1}[M] = 0$, for all k).*

Then, since we are assuming $W^- = 0$, (4.3) shows that

$$c_1^2[M] = -\frac{1}{2\pi^2}\int_M \{\mathrm{tr}[(W^-)^2] + \frac{\tau^2}{48}\}v = -\frac{1}{2\pi^2}\int_M \frac{\tau^2}{48}v$$

and hence $\tau = 0$, which proves that $\mathcal{M}$ is Ricci flat.

Finally assume W^+ is of type Ia. A complete solution for the Osserman problem is known in this case: either it is a space of constant sectional curvature, an indefinite Kähler manifold of constant holomorphic sectional curvature or a paraKähler manifold of constant paraholomorphic sectional curvature [17]. Observe that in the last two cases there are exactly two-distinct eigenvalues of the Jacobi operators in a ratio $1 : \frac{1}{4}$. If $\mathcal{M}$ is a paracomplex space form, then it admits an almost anti-Hermitian structure and hence $c_1[M] = 0$, which implies that $\tau = 0$ and hence $\mathcal{M}$ is flat in the compact case. The situation for the indefinite complex space forms is somehow different, since they are anti-self-dual (instead of self-dual as it occurs in the positive definite case), with

$$W^- = \frac{\tau}{12}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and thus

$$\chi[M] = -\frac{1}{8\pi^2}\int_M \{\mathrm{tr}[(W^+)^2] + \mathrm{tr}[(W^-)^2] + \frac{\tau^2}{24}\}v = -\frac{1}{8\pi^2}\int_M \frac{\tau^2}{12}v,$$

which shows that $\chi[M] \leq 0$, with equality if and only if $\mathcal{M}$ is flat. Kähler-Einstein metrics have been investigated by Petean [146], showing that the possible non Ricci flat ones should occur on minimal ruled surfaces over curves of genus $\mathfrak{g} \geq 2$, or they should realize on minimal surfaces of class VII_0 (see [9]). Now, since any VII_0-surface has nonnegative Euler characteristic, any such surface supports a Kähler Osserman metric if and only if $\chi[M] = 0$, and the metric is flat. Next, assume M to be a minimal ruled surface over a curve of genus $\mathfrak{g} \geq 2$. Then the Chern numbers $c_1^2[M]$ and $c_2[M]$ satisfy

$$c_1^2[M] = 8(1 - \mathfrak{g}), \qquad c_2[M] = 4(1 - \mathfrak{g}).$$

Hence, the Hirzebruch signature $\Upsilon[M] = \frac{1}{3}\left(c_1^2[M] - 2c_2[M]\right)$ vanishes identically, and thus

$$\Upsilon[M] = \frac{1}{12\pi^2}\int_M \{\mathrm{tr}[(W^+)^2] - \mathrm{tr}[(W^-)^2]\}v = -\frac{1}{12\pi^2}\int_M \frac{\tau^2}{24}v,$$

which shows that $\tau = 0$, and hence M is flat. □

Further observe that the Jordan normal form of the Jacobi operators may change from point to point in an Osserman manifold. We have the following result.

Theorem 4.1.5 *Let $\mathcal{M} = (M, g)$ be a compact Osserman manifold with metric of signature $(--++)$. Then $\Upsilon[M] = 0$ and the Jacobi operators have only one eigenvalue (which may be a single, double or triple root of the minimal polynomial). Moreover $\chi[M] \leq 0$ and $\chi[M] = 0$ if and only if the Jacobi operators are nilpotent.*

Proof. First of all, note that there does not exist any 4-dimensional Osserman metric whose Jacobi operators have three distinct real eigenvalues. Moreover, the same holds true as concerns Osserman metrics with complex eigenvalues for the Jacobi operators [17]. Hence, we will show the non-existence of compact Osserman metrics with two distinct real eigenvalues. Let α and β denote the constant eigenvalues of the Jacobi operators, the later assumed to be of multiplicity two, and let $E_\alpha(X) = \langle X\rangle \oplus \ker(R_X - \alpha\langle X, X\rangle Id)$ be the two dimensional subspace spanned by X and the eigenvector corresponding to the distinguished eigenvalue α.

Observe that, for any non null vector X, the restriction of the metric tensor to $E_\alpha(X)$ must be non degenerate, and thus either definite (of signature $(++)$ or $(--)$) or indefinite (of Lorentzian signature $(-+)$). Furthermore, the signature type of the E_α's cannot change from definite to indefinite, since in such a case it would pass through a degenerate situation. Next, we analyze the two different possibilities.

If the induced metric on the E_α's is definite, then $\ker(R_X - \beta\langle X, X\rangle Id)$ is also definite, and thus the Jacobi operators are diagonalizable. This shows that $\mathcal{M}$ is locally an indefinite complex space form [17], and thus such metrics do not exist in the compact case as it was shown in the proof of Theorem 4.1.3. Now, assume the induced metric on the E_α's is indefinite. Then, $\ker(R_X - \alpha\langle X, X\rangle Id)$ defines an almost paracomplex structure P on M which makes (M, g, P) an almost paraHermitian manifold. Hence $c_1^2[M] = 0$, which proves $\mathcal{M}$ is Ricci flat, since $\tau = 0$, just by proceeding as in the proof of Theorem 4.1.3. Finally observe that the eigenvalues α and β are in a ratio $1 : \frac{1}{4}$, as it was proven in [17] for the diagonalizable case (it also follows as a consequence of the local description of the non diagonalizable case given in [61]). Hence, $\alpha = 4\beta = 0$, which is a contradiction. □

Remark 4.1.6 It follows as a scholium to the proof given above that a compact Osserman 4-dimensional manifold $\mathcal{M}$ with metric of signature $(--++)$ has $\Upsilon[M] = 0$ and $\chi[M] \leq 0$, with equality if and only if the Jacobi operators are nilpotent.

Remark 4.1.7 The following are consequences of previous work of Petean [146]:

- There exist Osserman metrics whose Jacobi operators change from zero to 2-step nilpotent on tori.

- Compact indefinite Kähler Osserman 4- dimensional manifolds are Ricci flat, and thus they are a complex torus, a hyperelliptic surface, or a primary Kodaira surface. Indeed, since $\Upsilon[M] = b^+ - b^- = 0$, it follows that $\chi[M] = 2(1 - b_1 + b^-)$, and thus (since $\chi[M] \leq 0$) $b_1 \geq 1 + b^-$ and $b_1 = 1 + b^-$ if and only if $\chi[M] = 0$. Now, an indefinite Kähler surface satisfies $H^2(M; \mathbb{R}) \neq 0$, from where $b^- = b^+ \geq 1$, and thus $b_1 \geq 2$. This shows that no minimal surface of class VII_0 is Kähler Osserman.

4.2 Completeness of Osserman and conformally Osserman Walker metrics

Our purpose in this section is to study geodesic completeness on some Walker manifolds with special features. We will relate the Osserman condition and the conformally Osserman condition, which are purely algebraic conditions, to the global geometry of the manifold by answering questions on geodesic completeness.

We will focus on a family of Walker manifolds of signature $(2, 2)$ whose Jacobi operator has eigenvalues $(0, 4k, k, k)$ but whose Jacobi operator is not diagonalizable [60], as well as Osserman and conformally Osserman examples studied in Chapter 3. Some of these manifolds are geodesically complete and others exhibit Ricci blowup. In this section we present results from [40].

4.2.1 Geodesic equations

In order to study geodesic completeness, a first step is to obtain the corresponding geodesic equations. Thus, we begin by computing the geodesic equations for a Walker metric in general, since all the examples we are going to study are of this kind.

Lemma 4.2.1 *Let $\mathcal{M}$ be a Walker metric as in Definition 3.2.1. Let $a_k := \partial_k a$, $b_k := \partial_k b$ and $c_k := \partial_k c$. Then the geodesic equations for $\mathcal{M}$ are given by:*

$$\begin{aligned} 0 \quad = \quad & \ddot{x}_1 + \dot{x}_1\dot{x}_3 a_1 + \dot{x}_1\dot{x}_4 c_1 + \dot{x}_2\dot{x}_3 a_2 + \dot{x}_2\dot{x}_4 c_2 \\ & +\tfrac{1}{2}\dot{x}_3\dot{x}_3(a_3 + c a_2 + a a_1) \\ & +\dot{x}_3\dot{x}_4(a_4 + c c_2 + a c_1) \\ & +\tfrac{1}{2}\dot{x}_4\dot{x}_4(2c_4 - b_3 + c b_2 + a b_1), \end{aligned}$$

$$\begin{aligned}
0 &= \ddot{x}_2 + \dot{x}_1\dot{x}_3 c_1 + \dot{x}_1\dot{x}_4 b_1 + \dot{x}_2\dot{x}_3 c_2 + \dot{x}_2\dot{x}_4 b_2 \\
&\quad + \tfrac{1}{2}\dot{x}_3\dot{x}_3(2c_3 - a_4 + ba_2 + ca_1) \\
&\quad + \dot{x}_3\dot{x}_4(b_3 + bc_2 + cc_1) \\
&\quad + \tfrac{1}{2}\dot{x}_4\dot{x}_4(b_4 + bb_2 + cb_1), \\
0 &= \ddot{x}_3 - \tfrac{1}{2}\dot{x}_3\dot{x}_3 a_1 - \dot{x}_3\dot{x}_4 c_1 - \tfrac{1}{2}\dot{x}_4\dot{x}_4 b_1, \\
0 &= \ddot{x}_4 - \tfrac{1}{2}\dot{x}_3\dot{x}_3 a_2 - \dot{x}_3\dot{x}_4 c_2 - \tfrac{1}{2}\dot{x}_4\dot{x}_4 b_2.
\end{aligned}$$

Proof. We use the discussion in [78] to determine the Christoffel symbols; the Lemma then follows:

$$\begin{aligned}
\nabla_{\partial_1}\partial_3 &= \tfrac{1}{2}a_1\partial_1 + \tfrac{1}{2}c_1\partial_2, & \nabla_{\partial_1}\partial_4 &= \tfrac{1}{2}c_1\partial_1 + \tfrac{1}{2}b_1\partial_2, \\
\nabla_{\partial_2}\partial_3 &= \tfrac{1}{2}a_2\partial_1 + \tfrac{1}{2}c_2\partial_2, & \nabla_{\partial_2}\partial_4 &= \tfrac{1}{2}c_2\partial_1 + \tfrac{1}{2}b_2\partial_2,
\end{aligned}$$

$$\begin{aligned}
\nabla_{\partial_3}\partial_3 &= \tfrac{1}{2}(2c_3 - a_4 + ba_2 + ca_1)\partial_2 + \tfrac{1}{2}(a_3 + ca_2 + aa_1)\partial_1 - \tfrac{1}{2}a_1\partial_3 - \tfrac{1}{2}a_2\partial_4, \\
\nabla_{\partial_3}\partial_4 &= \tfrac{1}{2}(a_4 + cc_2 + ac_1)\partial_1 + \tfrac{1}{2}(b_3 + bc_2 + cc_1)\partial_2 - \tfrac{1}{2}c_1\partial_3 - \tfrac{1}{2}c_2\partial_4, \\
\nabla_{\partial_4}\partial_4 &= \tfrac{1}{2}(2c_4 - b_3 + cb_2 + ab_1)\partial_1 + \tfrac{1}{2}(b_4 + bb_2 + cb_1)\partial_2 - \tfrac{1}{2}b_1\partial_3 - \tfrac{1}{2}b_2\partial_4.
\end{aligned} \tag{4.4}$$

□

Now, we specialize Lemma 4.2.1 to the examples (3.4), since we are going to study them in subsequent sections.

Lemma 4.2.2 *Let $\mathcal{M}$ be a Walker metric as in (3.4). Then the geodesic equations are:*

$$\begin{aligned}
0 &= \ddot{x}_1 + \dot{x}_1\dot{x}_4 c_1 + \dot{x}_2\dot{x}_4 c_2 + \dot{x}_3\dot{x}_4 cc_2 + \dot{x}_4\dot{x}_4 c_4, \\
0 &= \ddot{x}_2 + \dot{x}_1\dot{x}_3 c_1 + \dot{x}_2\dot{x}_3 c_2 + \dot{x}_3\dot{x}_3 c_3 + \dot{x}_3\dot{x}_4 cc_1, \\
0 &= \ddot{x}_3 - \dot{x}_3\dot{x}_4 c_1, \\
0 &= \ddot{x}_4 - \dot{x}_3\dot{x}_4 c_2.
\end{aligned}$$

4.2.2 Completeness of strict Walker manifolds

The importance of strict Walker metrics is emphasized by the fact that they are geodesically complete, as next result shows.

Theorem 4.2.3 *Let $\mathcal{M}$ be a strict Walker manifold. Then $\mathcal{M}$ is geodesically complete.*

Proof. By Lemma 4.2.1, the geodesic equations for a strict Walker manifold as in (3.3) are:

$$\begin{aligned}
0 &= \ddot{x}_1 + \tfrac{1}{2}\dot{x}_3\dot{x}_3 a_3 + \tfrac{1}{2}\dot{x}_4\dot{x}_4(2c_4 - b_3) + \dot{x}_3\dot{x}_4 a_4, \\
0 &= \ddot{x}_2 + \tfrac{1}{2}\dot{x}_3\dot{x}_3(2c_3 - a_4) + \tfrac{1}{2}\dot{x}_4\dot{x}_4 b_4 + \dot{x}_3\dot{x}_4 b_3, \\
0 &= \ddot{x}_3, \\
0 &= \ddot{x}_4\,.
\end{aligned}$$

The final two equations can be solved to yield an affine solution, thus, let $x_3 = \alpha + \beta\, t$ and $x_4 = \gamma + \delta\, t$. The first two equations then have the form $\ddot{x}_1 = f_1(t)$ and $\ddot{x}_2 = f_2(t)$ which can be solved. Thus geodesics extend for infinite time and $\mathcal{M}$ is geodesically complete. □

Remark 4.2.4 Note that strict Walker manifolds verify identity (1.8), therefore they are generalized plane wave manifolds. This means, in particular, that they have, among many other interesting properties, vanishing scalar Weyl invariants and nilpotent Jacobi operators. Furthermore, they all are geodesically complete as Theorem 1.6.3 states.

4.2.3 Technical ODEs lemma

In the remaining of this chapter we are going to study completeness of some particular Walker metrics. However, the geometry of such examples is more complicated than strict Walker metrics, so we use different techniques. The following technical result from [40] will be essential for our purpose.

Lemma 4.2.5 *Let $f(t)$ satisfy $\ddot{f}(t) = \Xi(\dot{f}, f)$ with $f(0) = 1$ and $\dot{f}(0) = 1$ and maximal domain $[0, T)$. Assume $\Xi(x, y) \geq \varepsilon x^\alpha y^\beta$ for $x \geq 1$ and $y \geq 1$ where $2\alpha + \beta \geq 3$ and $\varepsilon > 0$. Then $T < \infty$, $\lim_{t\to T} \ddot{f}(t) = \infty$, and $\lim_{t\to T} \dot{f}(t) = \infty$.*

Proof. Since $\ddot{f}$ is positive, f and $\dot{f}$ are monotonically increasing. Suppose $T < \infty$ but $\lim_{t\to T} \dot{f} < \infty$. Then $\dot{f}$ is bounded and hence f is bounded as well. Thus $\lim_{t\to T} \dot{f} = \dot{f}_T$ and $\lim_{t\to T} f = f_T$ exist and are finite. The fundamental theorem of ODEs shows $[0, T)$ is not the maximal domain of the function f. Thus if T is finite, $\lim_{t\to T} \dot{f}(t) = \infty$. Hence $\lim_{t\to T} \ddot{f}(t) = \infty$ as well and the Lemma holds.

To complete the proof, we suppose that $T = \infty$ and argue for a contradiction. Without loss of generality, we assume $\varepsilon < 1$. Let $t_1 := 0$ and let $t_{n+1} := t_n + \frac{3}{\varepsilon n^2}$. We wish to show $f(t_n) \geq n$ and $\dot{f}(t_n) \geq n^2$. As this holds for $n = 1$, we proceed by induction on n. Because $\dot{f}$ is monotonically increasing, $f(t_{n+1}) \geq f(t_n) + \dot{f}(t_n)\frac{3}{\varepsilon n^2} \geq n + \frac{3n^2}{\varepsilon n^2} \geq n + 1$. To prove the second estimate, we use the mean value theorem to find s with $s \in [t_n, t_{n+1}]$ so $\dot{f}(t_{n+1}) = \dot{f}(t_n) + \frac{3}{\varepsilon n^2}\ddot{f}(s)$. We may estimate that:

$$\ddot{f}(s) \;=\; \Xi(\dot{f}(s), f(s)) \geq \varepsilon \dot{f}(s)^\alpha f(s)^\beta \geq \varepsilon \dot{f}(t_n)^\alpha f(t_n)^\beta \geq \varepsilon\, n^{2\alpha+\beta} \geq \varepsilon\, n^3\,.$$

Consequently $\dot{f}(t_{n+1}) \geq n^2 + \frac{3}{\varepsilon n^2}\varepsilon n^3 \geq n^2 + 3n \geq (n+1)^2$. The desired contradiction follows as $\lim_{n\to\infty} t_n < \infty$. □

4.2.4 Conformally Osserman examples

We study the global geometry of the conformally Osserman examples given in previous chapter:

Theorem 4.2.6 *Among the explicit examples given in Section 3.4, we have:*

(i) the manifold given by the warping function $c = x_1x_4 + x_3x_4$ *is geodesically complete,*

(ii) all other examples present Ricci blowup, therefore they cannot be embedded isometrically in a 4-dimensional geodesically complete manifold.

Proof. Suppose first that for $x_1 \geq 1$ one has

$$c_1 = p(x_1) \geq x_1, \quad c_{11} \geq 1, \quad c_4 = 0\,.$$

This is the case for the warping functions $c = x_1^2 - x_2^2$, $c = x_1^2 + x_2^2$, $c = x_1^2$, $c = x_1^3 - x_2^3$, $c = x_1^4 + x_1^2 - x_2^4 - x_2^2$ and $c = x_1^4 + x_1^2 + x_2^4 + x_2^2$. We set $x_2(t) = 0$, $x_3(t) = 0$ and $x_4(t) = -t$. The geodesic equations given in Lemma 4.2.2 then become

$$\ddot{x}_1 - \dot{x}_1 p(x_1) = 0, \quad \ddot{x}_2 = 0, \quad \ddot{x}_3 = 0, \quad \ddot{x}_4 = 0\,.$$

This yields a consistent set of equations with $\ddot{x}_1 = \dot{x}_1 p(x_1)$. By Lemma 4.2.5, we have that $\lim_{t\to T} \dot{x}_1(t) = \infty$ for T finite. By Lemma 4.2.2,

$$\rho(\dot\gamma, \dot\gamma) \;=\; \dot{x}_1\dot{x}_1\rho_{11} + 2\dot{x}_1\dot{x}_4\rho_{14} + \dot{x}_4\dot{x}_4\rho_{44} = -\dot{x}_1c_{11} - \tfrac{1}{2}c_1^2\,.$$

As $c_{11} \geq 1$, $\lim_{t\to T}\rho(\dot\gamma,\dot\gamma) = -\infty$ and these manifolds exhibit Ricci blowup.

Let $c = x_1x_3 + x_2x_4$. The geodesic equations are:

$$\begin{aligned}
0 &= \ddot{x}_1 + \dot{x}_1\dot{x}_4x_3 + \dot{x}_2\dot{x}_4x_4 + \dot{x}_3\dot{x}_4(x_1x_3 + x_2x_4)x_4 + \dot{x}_4\dot{x}_4x_2,\\
0 &= \ddot{x}_2 + \dot{x}_1\dot{x}_3x_3 + \dot{x}_2\dot{x}_3x_4 + \dot{x}_3\dot{x}_3x_1 + \dot{x}_3\dot{x}_4(x_1x_3 + x_2x_4)x_3,\\
0 &= \ddot{x}_3 - \dot{x}_3\dot{x}_4x_3,\\
0 &= \ddot{x}_4 - \dot{x}_3\dot{x}_4x_4.
\end{aligned}$$

We start with initial conditions $x_3(0) = \dot{x}_3(1) = x_4(1) = \dot{x}_4(1) = 1$. Symmetry implies that $x_3(t) = x_4(t) = h(t)$ where h satisfies $\ddot{h}(t) = \dot{h}(t)\,\dot{h}(t)\,h(t)$. Lemma 4.2.5 now shows $\dot{h} \to \infty$ at finite time so $\mathcal{M}$ is incomplete. Furthermore, we use equations (3.6) to see that $\rho_{ij} = 0$ for $i, j \neq 3, 4$ and thereby show $\mathcal{M}$ exhibits Ricci blowup by computing:

$$\begin{aligned}
\rho(\dot\gamma,\dot\gamma) &= \dot{x}_3^2\rho_{33} + \dot{x}_4^2\rho_{44} + 2\dot{x}_3\dot{x}_4\rho_{34} = \dot{x}_3^2\{\rho_{33} + \rho_{44} + 2\rho_{34}\}\\
&= \dot{x}_3^2\{-\frac{1}{2}x_4^2 - \frac{1}{2}x_3^2 + x_3x_4 - 2\} = -2\dot{x}_3^2\,.
\end{aligned}$$

Let $c = x_1x_3^2$ be the warping function. The final two geodesic equations become $\ddot{x}_3 = \dot{x}_3\dot{x}_4x_3^2$ and $\ddot{x}_4 = 0$. Setting $x_4 = t$ then yields the equation $\ddot{x}_3 = \dot{x}_3x_3^2$ and, by Lemma 4.2.5, $\dot{x}_3(t) \to \infty$ as $t \to T$ for $T < \infty$. All the components of the Ricci tensor vanish except ρ_{34} and ρ_{44}. Since $x_3(t) \geq 1$, we show $\mathcal{M}$ exhibits Ricci blowup by computing:

$$\lim_{t\to T} \rho(\dot\gamma,\dot\gamma) = \lim_{t\to T}\{-2\dot x_3\dot x_4 x_3 - \tfrac{1}{2}\dot x_4\dot x_4 x_3^4\} \leq \lim_{t\to T}\{-2\dot x_3\} = -\infty\,.$$

Suppose that $c = x_2x_4^2 + x_3^2x_4$ or that $c = x_2x_4^2 + x_3x_4$ are the warping functions. The final two geodesic equations become $0 = \ddot x_3$ and $\ddot x_4 = \dot x_3\dot x_4 x_4^2$. We take $x_3 = t$ so $\ddot x_4 = \dot x_4 x_4^2$. Let $x_4(0) = \dot x_4(0) = 1$. Thus $\lim_{t\to T}\dot x_4 = \infty$ at some finite time and $x_4(t) \geq 1$ for all t. Only ρ_{33} and ρ_{34} are non-zero. We show $\mathcal{M}$ exhibits Ricci blowup by computing

$$\lim_{t\to T} \rho(\dot\gamma,\dot\gamma) = \lim_{t\to T}\left\{-\tfrac{1}{2}\dot x_3\dot x_3 x_4^4 - 2\dot x_3\dot x_4 x_4\right\} \leq \lim_{t\to T}\{-2\dot x_4\} = -\infty\,.$$

Finally let $c = x_1x_4 + x_3x_4$. The geodesic equations in the last two variables are $\ddot x_3 - \dot x_3\dot x_4 x_4 = 0$ and $\ddot x_4 = 0$ so

$$x_4 = \alpha + \beta\, t \quad \text{and} \quad \dot x_3 = c\, e^{\beta(\alpha\, t + \beta\frac{1}{2}\, t^2)}\,.$$

We integrate this equation to determine x_3. As the equation for $\ddot x_2$ takes the form $\ddot x_2 + F(x_1, x_3, x_4, \dot x_1, \dot x_3, \dot x_4) = 0$, it poses no difficulty and the only task is to determine x_1. The equation for x_1 takes the following form:

$$\ddot x_1 + \dot x_1\dot x_4 x_4 + \dot x_4\dot x_4 x_1 + \dot x_4\dot x_4 x_3 = 0\,.$$

By rescaling the geodesic parameter, we see that there are really only two cases to be considered. These are $x_4 = \alpha$ and $x_4 = t$. If $x_4 = \alpha$, we get the equation $\ddot x_1 = 0$ which has linear solutions. If $x_4 = t$, we get the equation

$$\ddot x_1 + t\,\dot x_1 + x_1 = \psi(t)$$

for suitably chosen ψ. We set $x_1 := fe^{-\frac{1}{2}\,t^2}$ to reduce the order of the equation, thus

$$\begin{aligned}\dot x_1 &= (\dot f - tf)e^{-\frac{1}{2}\,t^2},\\ \ddot x_1 &= (\ddot f - 2\,t\dot f + t^2 f - f)e^{-\frac{1}{2}\,t^2},\end{aligned}$$

and then

$$\begin{aligned}\ddot x_1 + t\dot x_1 + x_1 &= (\ddot f - 2\,t\dot f + t^2 f - f + t\dot f - t^2 f + f)e^{-\frac{1}{2}t^2}\\ &= (\ddot f - t\dot f)e^{-\frac{1}{2}\,t^2} = \psi(t)\,.\end{aligned}$$

Setting $f_1 := \dot f$ then leads to an equation of the form $\dot f_1 - tf_1 = \psi_1(t)$ for suitably chosen α_1. Setting $f_1 = f_2\, e^{\frac{1}{2}\,t^2}$ then yields

$$\dot f_1 - tf_1 = (\dot f_2 + tf_2 - tf_2)e^{\frac{1}{2}t^2} = \psi_1(t)\,,$$

which leads to the equation $\dot f_2 = \psi(t)$. This equation can be solved for all time; hence it follows that $\mathcal{M}$ is geodesically complete. □

4.2.5 Non-diagonalizable Jacobi operators

We have said before that in signature $(2,2)$ the eigenvalue structure of the Jacobi operator does not determine the operator up to conjugacy; one must instead consider the Jordan normal form. We know from Remark 4.2.4 that any strict Walker manifold of signature $(2,2)$ is nilpotent Osserman. However there are Walker manifolds of signature $(2,2)$ which are Osserman but not nilpotent and whose Jacobi operators are not diagonalizable [60], more specifically, the Jacobi operators are of type II with non-null eigenvalues at some points, as next result shows.

Theorem 4.2.7 [60] *Let $\mathcal{M}$ be a Walker metric (Definition 3.2.1) with*

$$a = 4kx_1^2 - \tfrac{1}{4k}f(x_4)^2, \quad b = 4kx_2^2,$$
$$c = 4kx_1x_2 + x_2 f(x_4) - \tfrac{1}{4k}\dot f(x_4),$$

where $f = f(x_4)$ is non-constant and $k \neq 0$. Then $\mathcal{M}$ is Osserman with eigenvalues $(0, 4k, k, k)$ and the Jacobi operators are diagonalizable at a given point p *if and only if*

$$24\,kf(x_4)\dot f(x_4)x_2 - 12\,k\ddot f(x_4)x_1 + 3f(x_4)\ddot f(x_4) + 4\dot f(x_4)^2 = 0.$$

We continue the study of geodesic completeness; the next theorem shows these manifolds are geodesically incomplete.

Theorem 4.2.8 *Let $\mathcal{M}$ be as in Theorem 4.2.7. Then $\mathcal{M}$ is geodesically incomplete and can not be embedded isometrically in a geodesically complete manifold.*

Proof. Since f is non-constant, we may choose ξ_4 so $f(\xi_4) \neq 0$ and $\dot f(\xi_4) \neq 0$. Choose ξ_1 so $16k^2\xi_1^2 = f(\xi_4)^2$; normalize the choice of sign so $k\,\xi_1 > 0$. As an ansatz, we set $x_1 = \xi_1$, $x_2 = 0$, and $x_4 = \xi_4$ to be constant. This implies $a = 0$. The geodesic equations in $\ddot x_1$, $\ddot x_2$, and $\ddot x_4$ given by Lemma 4.2.1 then become $\ddot x_1 = \ddot x_2 = \ddot x_4 = 0$, which are satisfied. The remaining geodesic equation is $0 = \ddot x_3 - 4\,k\,\xi_1\dot x_3\dot x_3$. We can solve this equation by setting

$$\begin{aligned} x_3 &= -\tfrac{1}{4k\xi_1}\ln(1-t),\\ \dot x_3 &= \tfrac{1}{4k\xi_1}(1-t)^{-1},\\ \ddot x_3 &= \tfrac{1}{4k\xi_1}(1-t)^{-2} = 4\,k\,\xi_1\dot x_3\dot x_3\,. \end{aligned}$$

This is defined for $t \in (-\infty, 1)$ and we have $\lim_{t\to 1} 4\,k\,\xi_1 x_3 = \infty$. In particular, $\mathcal{M}$ is geodesically incomplete.

Since $\mathcal{M}$ is Einstein, it does not exhibit Ricci blowup. Instead we use a different argument to show $\mathcal{M}$ is essentially incomplete. Let $\{e_1, e_2, e_3, e_4\}$ be a parallel frame along γ with $e_i(0) = \partial_i$. From Equation (4.4):

$$\begin{aligned} \nabla_{\partial_3}\partial_1 &= 64k\xi_1\partial_1, & \nabla_{\partial_3}\partial_2 &= 4k\xi_1\partial_2,\\ \nabla_{\partial_3}\partial_3 &= -4k\xi_1\partial_3, & \nabla_{\partial_3}\partial_4 &= -2\xi_1\dot f(\xi_4)\partial_1 - 4k\xi_1\partial_4\,. \end{aligned}$$

Consequently

$$\begin{aligned} e_1(x_3) &= e^{-4k\xi_1 x_3}\partial_1, \\ e_2(x_3) &= e^{-4k\xi_1 x_3}\partial_2, \\ e_3(x_3) &= e^{4k\xi_1 x_3}\partial_3, \\ e_4(x_3) &= \frac{1}{4k}\dot f(\xi_4)(e^{4k\xi_1 x_3} - e^{-4k\xi_1 x_3})\partial_1 + e^{4k\xi_1 x_3}\partial_4\,. \end{aligned}$$

Since $R(\partial_1, \partial_3, \partial_3, \partial_4) = 0$, $R(\partial_1, \partial_3, \partial_3, \partial_1) = 4k$, $\dot f(\xi_4) \neq 0$, and since $4k\xi_1 x_3(t) \to \infty$ as $t \to 1$, $\mathcal{M}$ is seen to be essentially incomplete as:

$$\begin{aligned} \lim_{t\to 1} R(e_1, e_3, e_3, e_4) &= \lim_{t\to 1}\{\tfrac{1}{4k}\dot f(\xi_4)(e^{4k\xi_1 x_3} - e^{-4k\xi_1 x_3})\}e^{4k\xi_1 x_3}4k \\ &= \lim_{t\to 1}\dot f(\xi_4)(e^{8k\xi_1 x_3} - 1) \;=\; \pm\infty\,. \end{aligned}$$

□

Chapter 5

Conformally Osserman warped product metrics

In this chapter we study Osserman conformality over manifolds with a special structure, more specifically with a warped or a twisted product structure. We have already mentioned how these structures have been useful to find examples of negative curvature manifolds. Also, they naturally appear in many different contexts, so a deeper study is worthwhile here. In this chapter we show that a Riemannian warped product and most of the twisted ones (those whose factors have dimension greater than one) are conformally Osserman if and only if they are locally conformally flat. This fact will lead to Part II, since it motivates the study of locally conformally flat warped products.

5.1 Four-dimensional warped products

When studying the Osserman problem, manifolds of dimension 4 manifest a quite peculiar behavior. This is pointed out when studying Osserman conformality as well. On the one hand dimension 4 is the minimal dimension where the Weyl tensor plays a significant role, since in dimension 3 all the information of the curvature is encoded in the Ricci tensor. On the other hand, we will see below some behavior which is unique to dimension 4 and which is related to the fact that in that dimension the Hodge star operator is idempotent for even signature.

We begin this chapter by concentrating on 4-dimensional products. The main reason to do this is not only that we are going to find slightly different features in dimension 4, but also that the tools we make use of are different from those we use in the general case. In this section we summarize results in [27].

The characterization given in Theorem 3.1.2 is very useful in our study of warped products due to the following result:

Theorem 5.1.1 *Let (M, g) be a pseudo-Riemannian manifold of dimension* 4. *If (M, g)*

decomposes as a warped product $\mathcal{M} = B \times_f F$ or as a twisted product $\mathcal{M} = B \times_f F$ with $\dim B = 1$ or $\dim F = 1$, then the following assertions are equivalent:

1. *$\mathcal{M}$ is self-dual,*
2. *$\mathcal{M}$ is anti-self-dual,*
3. *$\mathcal{M}$ is locally conformally flat.*

Proof. Since $\mathcal{M}$ is locally conformally flat if and only if it is self-dual and anti-self-dual, it suffices to show that (1) and (2) are equivalent. Since the signature is arbitrary, there are three cases to be analyzed: Riemannian $(++++)$, Lorentzian $(-+++)$ or neutral $(++--)$. The Lorentzian case follows directly from [20], where it is proven that a Lorentzian manifold is conformally Osserman if and only if it is locally conformally flat.

A warped product $B \times_f F$ is in the conformal class of $(B \times F, 1/f^2 g_B \oplus g_F)$, which is a direct product. Thus, since the three properties under consideration are conformal invariants, it suffices for our purpose to restrict our analysis to direct products. Therefore, we are going to compute the components of the self-dual and anti-self-dual tensors of the corresponding direct product and check that they simultaneously vanish.

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of the tangent space and let $\{e^1, e^2, e^3, e^4\}$ be the corresponding dual basis. Consider the orthonormal basis for the self-dual and anti-self-dual spaces given in (1.5):

$$\Lambda^\pm = \mathrm{Span}\,\{E_1^\pm = (e^1 \wedge e^2 \pm \epsilon_3\epsilon_4 e^3 \wedge e^4)/\sqrt{2}, E_2^\pm = (e^1 \wedge e^3 \mp \epsilon_2\epsilon_4 e^2 \wedge e^4)/\sqrt{2},$$
$$E_3^\pm = (e^1 \wedge e^4 \pm \epsilon_2\epsilon_3 e^2 \wedge e^3)/\sqrt{2}\},$$

where $\epsilon_i = g(e_i, e_i)$. The diagonal terms of the self-dual and anti-self-dual matrix associated to these basis are given by:

$$\begin{aligned} 2W_{11}^+ &= W_{1212} + W_{3434} + 2\epsilon_3\epsilon_4 W_{1234}, \\ 2W_{11}^- &= W_{1212} + W_{3434} - 2\epsilon_3\epsilon_4 W_{1234}, \\ 2W_{22}^+ &= W_{1313} + W_{2424} - 2\epsilon_2\epsilon_4 W_{1324}, \\ 2W_{22}^- &= W_{1313} + W_{2424} + 2\epsilon_2\epsilon_4 W_{1324}, \\ 2W_{33}^+ &= W_{1414} + W_{2323} + 2\epsilon_2\epsilon_3 W_{1423}, \\ 2W_{33}^- &= W_{1414} + W_{2323} - 2\epsilon_2\epsilon_3 W_{1423}, \end{aligned} \tag{5.1}$$

but for a warped product $W_{1234} = W_{1324} = W_{1423} = 0$, so $W_{11}^+ = W_{11}^-$, $W_{22}^+ = W_{22}^-$ and $W_{33}^+ = W_{33}^-$. The remaining terms are:

$$\begin{aligned} 2W_{12}^+ &= W_{1213} - \epsilon_2\epsilon_4 W_{1224} + \epsilon_3\epsilon_4 W_{3413} - \epsilon_2\epsilon_3 W_{3424}, \\ 2W_{12}^- &= W_{1213} + \epsilon_2\epsilon_4 W_{1224} - \epsilon_3\epsilon_4 W_{3413} - \epsilon_2\epsilon_3 W_{3424}, \\ 2W_{13}^+ &= W_{1214} + \epsilon_2\epsilon_3 W_{1223} + \epsilon_3\epsilon_4 W_{3414} + \epsilon_2\epsilon_4 W_{3423}, \\ 2W_{13}^- &= W_{1214} - \epsilon_2\epsilon_3 W_{1223} - \epsilon_3\epsilon_4 W_{3414} + \epsilon_2\epsilon_4 W_{3423}, \\ 2W_{23}^+ &= W_{1314} + \epsilon_2\epsilon_3 W_{1323} - \epsilon_2\epsilon_4 W_{2414} - \epsilon_3\epsilon_4 W_{2423}, \\ 2W_{23}^- &= W_{1314} - \epsilon_2\epsilon_3 W_{1323} + \epsilon_2\epsilon_4 W_{2414} - \epsilon_3\epsilon_4 W_{2423}. \end{aligned} \tag{5.2}$$

Suppose *dim* $B = 1$ and *dim* $F = 3$. Let $e_1 \in \mathfrak{X}(B)$ and $e_2, e_3, e_4 \in \mathfrak{X}(F)$. Then

$$\begin{aligned} W_{1224} &= W_{3413} = 0, \\ W_{1223} &= W_{3414} = 0, \\ W_{1323} &= W_{2414} = 0, \end{aligned}$$

hence $W_{12}^+ = W_{12}^-$, $W_{13}^+ = W_{13}^-$ and $W_{23}^+ = W_{23}^-$.

Suppose *dim* $B = 2$ and *dim* $F = 2$. Let $e_1, e_2 \in \mathfrak{X}(B)$ and $e_3, e_4 \in \mathfrak{X}(F)$. Then

$$\begin{aligned} W_{1213} = W_{1224} = W_{3413} = W_{3424} = 0, \\ W_{1214} = W_{1223} = W_{3414} = W_{3423} = 0, \\ W_{1314} = -\tfrac{\epsilon_1}{2}\rho_{34}, W_{1323} = -\tfrac{\epsilon_3}{2}\rho_{12}, W_{2414} = -\tfrac{\epsilon_4}{2}\rho_{12}, W_{2423} = -\tfrac{\epsilon_2}{2}\rho_{34}. \end{aligned}$$

Hence $W_{12}^+ = W_{12}^- = W_{13}^+ = W_{13}^- = 0$. Now, since the signature is Riemannian or neutral, we have $\epsilon_1 = \epsilon_2\epsilon_3\epsilon_4$, therefore

$$W_{23}^+ = -\frac{1}{4}(\epsilon_1 - \epsilon_2\epsilon_3\epsilon_4)\rho_{34} = 0 = W_{23}^-.$$

Suppose *dim* $B = 3$ and *dim* $F = 1$. Let $e_1, e_2, e_3 \in \mathfrak{X}(B)$ and $e_4 \in \mathfrak{X}(F)$. Then

$$\begin{aligned} W_{1224} &= W_{3413} = 0, \\ W_{1214} &= W_{3423} = 0, \\ W_{1314} &= W_{2423} = 0, \end{aligned}$$

so $W_{12}^+ = W_{12}^-$, $W_{13}^+ = -W_{13}^-$ and $W_{23}^+ = -W_{23}^-$. Note that these relations still hold if we consider a twisted product $B \times_f F$ with *dim* $F = 1$. For a twisted product $B \times_f F$ with *dim* $B = 1$ we consider a conformal change by $1/f^2$ to get $(F \times B, g_F \oplus 1/f^2 g_B)$, so that the dimension of the fiber is 1. Again the result follows since the conditions under consideration are conformally invariant.

The relations among the self-dual and anti-self-dual components show that in all cases the self-dual and anti-self-dual operators simultaneously vanish. Hence the result follows.□

The following is an immediate corollary of Theorems 3.1.1, 3.1.2 and 5.1.1:

Theorem 5.1.2 *Let $\mathcal{M} = B \times_f F$ be a 4-dimensional pseudo-Riemannian warped product or a twisted product $\mathcal{M} = B \times_f F$ with 1-dimensional base or 1-dimensional fiber. Then $\mathcal{M}$ is conformally Osserman if and only if it is locally conformally flat.*

One may wonder if this result holds in general for a 4-dimensional twisted product. This is not the case, as next example shows.

Example 5.1.3 Consider the following twisted product:

$$\mathcal{M} = \mathbb{R}^2 \times_f \mathbb{R}^2, \quad \text{with } f(x_1, x_2, x_3, x_4) = e^{x_1 x_3 - x_2 x_4}.$$

A straightforward calculation shows that

$$W^+ = \begin{pmatrix} 0 & 0 & \frac{1}{2}(1 + e^{x_1 x_3 - x_2 x_4}) \\ 0 & 0 & 0 \\ \frac{1}{2}(1 + e^{x_1 x_3 - x_2 x_4}) & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad W^- = 0.$$

Hence $\mathcal{M}$ is self-dual but it is not anti-self-dual and, consequently, it is conformally Osserman but not locally conformally flat.

The following result shows the non-existence of compact (anti-)self-dual twisted products which are not locally conformally flat.

Theorem 5.1.4 *Let $\mathcal{M} = B \times_f F$ be a compact Riemannian twisted product such that dim $B, F = 2$. Then $B \times_f F$ is conformally Osserman if and only if it is locally conformally flat. Moreover, in such a case it is actually a warped product.*

Proof. Since $\mathcal{M}$ is conformally Osserman, it is self-dual or anti-self-dual. Suppose $\mathcal{M}$ is self-dual (reversing the orientation interchanges the roles of self-dual and anti-self-dual). Since B and F are 2-dimensional and oriented, let J^B and J^F be orthogonal complex structures on B and F, respectively. Then $\mathcal{M}$ is Hermitian and opposite just considering the complex structure $J^B \oplus \pm J^F$. Then if $\mathcal{M}$ is self-dual, results in [6] show that $\mathcal{M}$ is locally conformally flat or conformally equivalent to $\mathbb{CP}^2$ with the Fubini-Study metric or to a compact quotient of the unit ball in $\mathbb{C}^2$ with the Bergman metric. Hence, it follows that $\mathcal{M}$ is locally conformally flat or a locally conformally Kähler manifold. Assume J is locally conformally Kähler. Let X be a vector on the base and U a vector on the fiber. Hence $\{X, JX, U, JU\}$ is an adapted basis for the product. Since $(B \times_f F, J)$ is locally conformally Kähler, then there exists ψ such that $(B \times F, \psi\, g^B \oplus \psi\, f^2 g^F)$ is Kähler on a suitable open set (note that this is a doubly twisted product). Then

$$\begin{aligned}
(\nabla_X J)U &= \nabla_X(JU) - J\nabla_X U \\
&= JU(\ln \psi)X + X(\ln \psi\, f^2)JU - U(\ln \psi)JX - X(\ln \psi\, f^2)JU \\
&= JU(\ln \psi)X - U(\ln \psi)JX, \\
(\nabla_U J)X &= \nabla_U(JX) - J\nabla_U X \\
&= JX(\ln \psi\, f^2)U + U(\ln \psi)JX - X(\ln \psi\, f^2)JU - U(\ln \psi)JX \\
&= JX(\ln \psi\, f^2)U - X(\ln \psi\, f^2)JU,
\end{aligned}$$

from where $U(\ln \psi) = 0$, $JU(\ln \psi) = 0$, $X(\ln \psi f^2) = 0$ and $JX(\ln \psi f^2) = 0$. This implies that ψ is constant over F and one proceeds in an analogous way to show that ψf^2 is constant over B too. Hence, f may be decomposed as a product $f = f_B f_F$ such that f_B is constant on F and f_F is constant on B, so $B \times_f F$ is indeed a warped product and by Theorem 5.1.2 it is locally conformally flat. □

5.2 Conformally Osserman warped products in higher dimension

Motivated by the results of previous section, one may wonder if an arbitrary conformally Osserman warped product is locally conformally flat. This question makes no sense in Lorentzian signature, since we already know that any conformally Osserman Lorentzian manifold is locally conformally flat [20]. This section is devoted to giving a positive answer to this question in the Riemannian setting and give a negative answer, by means of a counterexample, in the higher signature context.

We begin by analyzing the Riemannian setting. Since every warped product is in the conformal class of a direct product and Osserman conformality is a conformally invariant property, it suffices to study when a direct product is conformally Osserman. The warped product characterization will be obtained as a consequence of the following theorem.

Theorem 5.2.1 *Let $\mathcal{M} = (M, g)$ be a conformally Osserman Riemannian manifold which decomposes as a direct product $\mathcal{M} = (B \times F, g_B \oplus g_F)$. Then $W^M = 0$.*

Proof. Let $p \in M$ be an arbitrary point. Denote $b = dim\, B$ and $d = dim\, F$. On T_pM we can choose an orthonormal basis $\{e_1, \dots, e_b, f_1, \dots, f_d\}$, with $\{e_i\} \subset T_p^B M$ and $\{f_i\} \subset T_p^F M$, which diagonalizes the Ricci tensor.

Note that $\mathcal{J}_W(e_i)e_j, \mathcal{J}_W(f_i)e_j \in T_p^B M$ and $\mathcal{J}_W(f_i)f_j, \mathcal{J}_W(e_i)f_j \in T_p^F M$. Indeed, from the expressions of the curvature in Lemma 1.5.3 we may compute:

$$\begin{aligned} \mathcal{J}_W(e_i)e_j &= \mathcal{J}_R(e_i)e_j - \tfrac{1}{n-2}\left(\rho(e_i,e_i) + \rho(e_j,e_j) - \tfrac{\tau}{n-1}\right)e_j, \\ \mathcal{J}_W(e_i)f_j &= -\tfrac{1}{n-2}\left(\rho(e_i,e_i) + \rho(f_j,f_j) - \tfrac{\tau}{n-1}\right)f_j, \end{aligned} \tag{5.3}$$

and, therefore, f_j is an eigenvector for every $\mathcal{J}_W(e_i)$ (analogously, e_i is an eigenvector for every $\mathcal{J}_W(f_j)$). Also notice that mixed terms of the Weyl tensor vanish, that is, $W(e_i, f_j)e_k = 0$ and $W(f_i, e_j)f_k = 0$ if $i \neq k$.

Suppose $\mathcal{J}_W(e_1)f_j = \lambda\, f_j$, $\mathcal{J}_W(e_1)f_k = \mu\, f_k$ with $j \neq k$. Recall the Rakić duality principle [150] (see also [84]): let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be an Osserman algebraic model with $\langle\cdot,\cdot\rangle$ positive definite; then $\mathcal{J}_A(x)y = \lambda\, y \Leftrightarrow \mathcal{J}_A(y)x = \lambda\, x$. We now use the Rakić duality principle to compute

$$\begin{aligned} \mathcal{J}_W(\cos\theta f_j + \sin\theta f_k)e_1 &= \cos^2\theta\,\mathcal{J}_W(f_j)e_1 + \sin^2\theta\,\mathcal{J}_W(f_k)e_1 \\ &\quad + \cos\theta\sin\theta(W(f_j,e_1)f_k + W(f_k,e_1)f_j) \\ &= \cos^2\theta\,\mathcal{J}_W(f_j)e_1 + \sin^2\theta\,\mathcal{J}_W(f_k)e_1 \\ &= \left(\cos^2\theta\lambda + \sin^2\theta\mu\right)e_1. \end{aligned}$$

This shows e_1 is an eigenvector for $\mathcal{J}_W(\cos\theta f_j + \sin\theta f_k)$ associated to the eigenvalue $\left(\cos^2\theta\lambda + \sin^2\theta\mu\right)$; but, since the eigenvalues are constant, we conclude that $\lambda = \mu$. By repeating this argument, we may show that all the eigenvalues of $\mathcal{J}_W(e_1)$ associated to eigenvectors f_j in $T_p^F M$ are equal.

Next, we are going to show that all the eigenvalues of $\mathcal{J}_W(e_1)$ are indeed equal. On the contrary, suppose there exists a unitary vector $x \in T_p^B M$ such that $\mathcal{J}_W(e_1)x = \nu x$. Then

$$\begin{aligned}\mathcal{J}_W(\cos\theta e_1 + \sin\theta f_1)x &= \cos^2\theta \mathcal{J}_W(e_1)x + \sin^2\theta \mathcal{J}_W(f_1)x \\ &\quad + \cos\theta\sin\theta\,(W(e_1,x)f_1 + W(f_1,x)e_1) \\ &= \left(\cos^2\theta\nu + \sin^2\theta\lambda\right)x.\end{aligned}$$

Consequently $\lambda = \nu$ and all the eigenvalues are equal. Since the trace of $\mathcal{J}_W(\cdot)$ is zero, we conclude that all the eigenvalues of $\mathcal{J}_W(\cdot)$ vanish and thus the Weyl tensor is zero. □

Corollary 5.2.2 *Let $\mathcal{M} = B \times_f F$ be a Riemannian manifold with a warped product structure. Then $\mathcal{M}$ is conformally Osserman if and only if it is locally conformally flat.*

Proof. The property of being conformally Osserman is conformally invariant. Then $(M, g) = (B \times F, g_B \oplus f^2 g_F)$ is conformally Osserman if and only if so is

$$(M, g') = (B \times F, \frac{1}{f^2} g_B \oplus g_F),$$

which is a direct product. We apply Theorem 5.2.1 to complete the proof. □

Previous results, together with the complete characterization in [20] of conformally Osserman Lorentzian manifolds and results in the previous section concerning 4-dimensional warped products of neutral signature, show the equivalence between Osserman conformality and local conformal flatness in several general contexts. Nonetheless, this is not a general feature and strongly depends on the conditions such as metric structure, signature,.... The next example shows that this sort of results can not be extended to higher signature direct products in dimensions greater than 4 and, consequently, the result also fails for warped products.

Example 5.2.3 *Let $\mathcal{M} = \mathcal{B} \times \mathbb{R}^p_\nu$, where $\mathcal{B} = (\mathbb{R}^4, g_\mathcal{B})$ is a strict Walker manifold with metric of the following form*

$$g_\mathcal{B}(x_1,x_2,x_3,x_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & c(x_3,x_4) \\ 0 & 1 & c(x_3,x_4) & 0 \end{pmatrix},$$

where x_1, x_2, x_3, x_4 are coordinates in $\mathcal{B}$. The signature of $\mathcal{B}$ is $(2,2)$ but $\mathbb{R}^p$ is endowed with an Euclidian metric of arbitrary signature. Then the only non-null component of the curvature, up to the usual symmetries, is

$$R(\partial_3, \partial_4, \partial_3, \partial_4) = c_{34},$$

so, since $\mathcal{M}$ is Ricci flat,

$$W(\partial_3, \partial_4, \partial_3, \partial_4) = c_{34}.$$

Then $\mathcal{J}_W$ has constant eigenvalues $(0, \ldots, 0)$ everywhere and hence $\mathcal{M}$ is conformally Osserman with nilpotent conformal Jacobi operator; however $\mathcal{M}$ is not locally conformally flat unless $\partial_3\partial_4 c(x_3, x_4) = 0$.

5.3 Conformally Osserman twisted products

From the results of the last section, we know that a Riemannian conformally Osserman warped product is locally conformally flat and we will see in Theorem 6.3.1 that a locally conformally flat twisted product with factors of dimension greater than one may be expressed as a warped product. We may then wonder if the condition of Osserman conformality itself implies that a twisted product (with factors of dimension greater than one) may be expressed as a warped product. The answer to this question is positive for odd dimensions, since in this case the conformally Osserman condition is equivalent to local conformal flatness (see [20]). Nevertheless, this is not true in general. Example 5.1.3 is a counterexample in the lowest possible dimension; it is self-dual but it is not anti-self-dual and, in conclusion, it is conformally Osserman but not locally conformally flat. However, manifolds with twisted product structure and fiber or base of dimension 1 behave quite differently.

Theorem 5.3.1 *A twisted product $\mathcal{M} = B \times_f F$, with dim $B = 1$ or $\dim F = 1$, is conformally Osserman if and only if it is locally conformally flat.*

Proof. If *dim* $B = 1$, $\mathcal{M}$ is in the conformal class of $F \times_{1/f} B$ with *dim* $B = 1$, so we may suppose without loss of generality that *dim* $F = 1$. Let $p \in M$ be an arbitrary point. Let v be a unit vector on the fiber F. Now consider $\mathcal{J}_W(v)$ and take a basis of orthonormal eigenvectors $\{e_1, \ldots, e_{n-1}\}$ tangent to the base of the twisted product. Denote by $\{\lambda_1, \ldots, \lambda_{n-1}, 0\}$ the eigenvalues associated to the orthonormal basis $\{e_1, \ldots, e_{n-1}, v\}$. Note that for $i \neq j$,

$$W(e_i, v, e_j, v) = \langle \mathcal{J}_W(v)e_i, e_j\rangle = 0,$$

and, for any i, j, k with $i \neq j$, $j \neq k$, $k \neq i$,

$$\begin{aligned} W(e_i, v, e_j, e_k) &= W(e_j, e_k, e_i, v) \\ &= R(e_j, e_k, e_i, v) - \tfrac{1}{n-2}\big(\rho(e_j, e_i)\langle e_k, v\rangle + \rho(e_k, v)\langle e_j, e_i\rangle \\ &\qquad -\rho(e_j, v)\langle e_k, e_i\rangle - \rho(e_k, e_i)\langle e_j, v\rangle \\ &\qquad -\tfrac{\tau}{n-1}\langle e_j, e_i\rangle\langle e_k, v\rangle - \langle e_j, v\rangle\langle e_i, e_k\rangle\big) \\ &= 0. \end{aligned}$$

Also note that $W(e_i, v, e_j, e_i) = 0$. Hence $W(e_i, v)e_j = 0$ for $i \neq j$ and $\mathcal{J}_W(e_i)v = \lambda_i v$. Then

$$\begin{aligned}\mathcal{J}_W(\cos\theta e_i + \sin\theta e_j)v &= \cos^2\theta \mathcal{J}_W(e_i)v + \sin^2\theta \mathcal{J}_W(e_j)v \\ &\quad + \cos\theta \sin\theta (W(e_i, v)e_j + W(e_j, v)e_i) \\ &= (\cos^2\theta \lambda_i + \sin^2\theta \lambda_j)v.\end{aligned}$$

Since M is conformally Osserman, the eigenvalues of the conformal Jacobi operator are constant. Therefore $\lambda_i = \lambda_j$. But, since $\mathrm{tr}(\mathcal{J}_W(\cdot)) = 0$, all the eigenvalues are zero and thus the Weyl tensor vanishes. Hence M is locally conformally flat. □

5.3.1 Osserman twisted products

In this section we take advantage of Theorem 1.4.1 to give a complete characterization of Osserman twisted products in the Riemannian setting.

Theorem 5.3.2 *Let $\mathcal{M} = B \times_f F$ be a Riemannian manifold with local structure of a twisted product. Then $\mathcal{M}$ is pointwise Osserman if and only if it is a space of constant sectional curvature. Moreover, B also has constant sectional curvature.*

Proof. Suppose $\mathcal{M}$ is Osserman. Hence, by Theorem 1.4.1, (M, g) is Einstein and conformally Osserman. We may now apply Theorem 1.5.12 and Corollary 5.2.2 if $dim\, F > 1$ or Theorem 5.3.1 if $dim\, F = 1$, to show that $\mathcal{M}$ is locally conformally flat. Since $\mathcal{M}$ is also Einstein, it has constant sectional curvature. The converse is immediate. Furthermore, $\mathcal{M}$ has constant sectional curvature implies B also has constant sectional curvature since $R^B = R$ on the subbundle tangent to B. □

The relation between Osserman manifolds and two-point-homogeneous manifolds gives us the following consequence of Theorem 5.3.2.

Corollary 5.3.3 *Neither $\mathbb{C}P^n$ nor its negative curvature dual may be decomposed as a twisted product.*

Proof. Every two-point-homogeneous space is Osserman, in particular $\mathbb{C}P^n$ and its negative curvature dual are Osserman. The result is now immediate from Theorem 5.3.2. □

Open problems

Although we have done some progress in understanding the structure of Osserman and conformally Osserman manifolds, there are still open questions of interest:

1. In Chapter 3 we found examples whose conformal Jacobi operators realize all possible normal Jordan forms. In order to do that, we have chosen Walker metrics of a specific form, namely as in expression (3.4). However, a complete classification of conformally Osserman Walker metrics is far from available.

2. We have shown in Chapter 4 that compactness is a strong condition when dealing with Jordan Osserman manifolds. When we weaken the Jordan Osserman condition to the Osserman one, the situation is not so well understood. Thus, a better understanding of the implications of Theorem 4.1.5 is needed.

3. We have studied completeness of many examples with algebraic properties in Chapter 4, thus relating in some sense the pointwise geometry of the manifold with the global one. It would be interesting to find relationships between pointwise conditions and completeness for Walker manifolds.

4. The higher signature setting is very often much more difficult to handle than the Riemannian one. A better understanding of twisted and warped products of high signature is required to improve the general results of Chapter 5.

Part II

Locally conformally flat manifolds with warped product structures

Locally conformally flat structures on Riemannian manifolds are the natural generalizations of isothermal coordinate systems, which are available on Riemann surfaces. However, not every higher dimensional Riemannian manifold admits a locally conformally flat structure, and it is difficult to provide a classification of such manifolds; this is still an open problem. For example, M is locally conformally flat implies all the Pontrjagin numbers of M vanish; thus $\mathbb{C}P^2$ is not locally conformally flat. More generally, some classification results are known under suitable topological conditions: a compact simply connected locally conformally flat manifold must be an Euclidean sphere (cf. [116], [156]). Also some classification results have been obtained under different geometrical assumptions like being locally symmetric or locally isometric to a product (cf. [119], [175]). Complete locally conformally flat manifolds with nonnegative Ricci curvature have been studied by several authors, showing that their universal cover is in the conformal class of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{R} \times \mathbb{S}^{n-1}$, where $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$ are spheres of constant sectional curvature [176]. Such conformal equivalence can be specialized to isometric equivalence under some extra assumptions on the scalar curvature and the sign of the Ricci curvatures [50], [162] (see also [46]). In spite of the results on locally conformally flat manifolds of nonnegative curvature, there is a lack of information as concerns negative curvature. One of the purpose in this part is to construct new examples of complete locally conformally flat Riemannian manifolds with nonpositive curvature.

Since the seminal work of Bishop and O'Neill [13], warped products have been seen to be a powerful tool for constructing manifolds of nonpositive curvature (see also [12]). Therefore, our aim is to investigate the existence of locally conformally flat structures on manifolds equipped with a warped product structure, or more generally on multiply warped spaces, as being a natural generalization of warped products (cf. for example [164] and the references therein). Other generalizations of warped product structures, like twisted or multiply quasi-warped (cf. [130], [147], [164]) are not of interest for our purposes, since they reduce to warped and multiply warped spaces, respectively, provided that they are locally conformally flat. Another motivation for the consideration of locally conformally flat structures on manifolds equipped with a warped product metric comes from the fact that the Schouten tensor is Codazzi for any locally conformally flat manifold. Moreover, although the local structure of Codazzi tensors is not yet completely understood, they lead to warped product decompositions of the manifold in many cases [14], [164].

Locally conformally flat multiply warped spaces are investigated in Chapter 7. Our approach relies on the fact that any multiply warped space is in the conformal class of a suitable product, a fact previously observed for warped product metrics [119]; this has several implications for the geometry of the fibers and the base of the multiply warped space. A local description of locally conformally flat spaces with the underlying structure of a multiply warped product is then obtained from the fact that any warping function must define a global conformal transformation on the base which makes it of constant sectional curvature. Then the case of n-dimensional base with $n \geq 2$ reduces to the existence of nontrivial solutions of some Obata's type equations on the base (sometimes called concircular transformations [114], [163]) together with some compatibility conditions among the different warping functions. This analysis is carried out in Section 7.3. Conditions become much weaker when the base is assumed to be 1-dimensional as shown in Section 7.2, in accordance with Roberston-Walker type metrics, which are locally conformally flat independently of the warping function. Some global consequences are obtained in Section 7.4, where locally conformally flat warped product manifolds with complete base of constant curvature are classified as well as those being multiply warped if the base is further assumed to be simply connected.

Applications of the results in Section 7.3 have already been found by R. Tojeiro in the study of conformal immersions into the Euclidean space [165]. Other applications are obtained in Chapter 8, where a complete description of multiply warped products of constant curvature is obtained (Section 8.1). The already announced examples of complete locally conformally flat manifolds of negative (Ricci) curvature are explicitly given in Section 8.2. Warped spaces with base the hyperbolic space are of main interest, providing of some new examples of complete locally conformally flat manifolds with nonpositive sectional curvature, and with nonpositive Ricci curvatures but no sign on the sectional curvature.

Finally, some cosmological models are discussed in Section 8.3 with special attention to multidimensional models.

Chapter 6

Warped products and local conformal flatness

Recall that a pseudo-Riemannian manifold (M, g) is said to be locally conformally flat if and only if every point in M admits a coordinate neighborhood $\mathfrak{U}$ which is conformal to the pseudo-Euclidean space $\mathbb{R}^n_\nu$, i.e. there is a diffeomorphism $\Phi : V \subset \mathbb{R}^n_\nu \to \mathfrak{U}$ and a positive function Ψ such that $\Phi^* g = \Psi^2 g_{\mathbb{R}^n_\nu}$. Note that any surface is locally conformally flat, but not every higher dimensional pseudo-Riemannian manifold admits a locally conformally flat structure. Necessary and sufficient conditions for the existence of such a structure are given by the nullity of the Weyl tensor (if dim $M \geq 4$) and the fact that the Schouten tensor is a Codazzi tensor in dimension 3.

Locally conformally flat warped product spaces have been investigated by several authors (see, for example [79], [80], [138]) who obtained necessary and sufficient conditions for $M = B \times_f F$ to be locally conformally flat in terms of both the curvatures of the base (B, g_B) and the fiber (F, g_F) and some PDE's involving the warping function f. Those results have been obtained directly as necessary and sufficient conditions for the Weyl tensor on $B \times_f F$ to vanish.

The results we present in this chapter are specialized in [30] for Lorentzian manifolds.

6.1 Locally conformally flat warped products: local structure

In the present section we characterize locally conformally flat warped products by establishing a criterion which describes the geometry of the base and the fiber of the product and which imposes some strong restrictions to the warping function. First recall the following result due to Yau [175](for an alternative proof see [28]).

Theorem 6.1.1 [175] *Let $\mathcal{M} = B \times F$ be a pseudo-Riemannian direct product. Then $\mathcal{M}$ is locally conformally flat if and only if (B, g_B) and (F, g_F) have constant sectional* *curvature and if one of the following holds:*

a) $\dim B = 1$ *or* $\dim F = 1$.

b) $\dim B \geq 2$, $\dim F \geq 2$ *and* $K^B = -K^F$.

The next theorem uses the fact that every warped product is in the conformal class of a direct product to take advantage of the previous theorem and to characterize locally conformally flat warped product metrics.

Theorem 6.1.2 *Let* $\mathcal{M} = B \times_f F$ *be a pseudo-Riemannian warped product. Then the following hold:*

(i) *If* $\dim B = 1$, *then* $\mathcal{M}$ *is locally conformally flat if and only if* (F, g_F) *is a space of constant curvature.*

(ii) *If* $\dim B \geq 2$ *and* $\dim F \geq 2$, *then* $\mathcal{M}$ *is locally conformally flat if and only if*

 (ii.a) (F, g_F) *is a space of constant curvature* K^F.

 (ii.b) *The function* $f : B \to \mathbb{R}^+$ *defines a global conformal deformation on* B *such that* $(B, \frac{1}{f^2} g_B)$ *is a space of constant curvature* $\tilde{K}^B = -K^F$.

(iii) *If* $\dim F = 1$, *then* $\mathcal{M}$ *is locally conformally flat if and only if the function* $f : B \to \mathbb{R}^+$ *defines a conformal deformation on* B *such that* $(B, \frac{1}{f^2} g_B)$ *is a space of constant curvature.*

Proof. Let $g = g_B \oplus f^2 g_F$ be a warped product metric on $M = B \times F$. Apply on $\mathcal{M}$ the conformal transformation given by $1/f^2$ to write $g = f^2 \left(\frac{1}{f^2} g_B \oplus g_F \right)$. Since local conformal flatness is a conformally invariant property, g is a locally conformally flat metric on $\mathcal{M}$ if and only if so is $\tilde{g} = \frac{1}{f^2} g_B \oplus g_F$. Since f is defined on B, $\tilde{g}$ is the product metric of $(B, \frac{1}{f^2} g_B)$ and (F, g_F). The result now follows from Theorem 6.1.1. □

Note that Theorem 6.1.2 enables us to understand the geometrical meaning of the PDE's involving the warping function in [79] and [138].

An immediate consequence of Theorem 6.1.2 is the following result, which shows there are several strong restrictions on the base and the fiber that must be satisfied for a warped product to be locally conformally flat.

Corollary 6.1.3 *Let* $\mathcal{M} = B \times_f F$ *be a locally conformally flat pseudo-Riemannian warped product. Then* (B, g_B) *is locally conformally flat and* (F, g_F) *is of constant sectional curvature.*

Remark 6.1.4 As a consequence of previous results, the local structure of locally conformally flat warped product metrics can be given as follows as a local converse to Corollary 6.1.3.

Let (B, g_B) be a locally conformally flat manifold and (F, g_F) a space of constant sectional curvature. Then there exist locally defined warping functions $f_{\mathfrak{U}} : \mathfrak{U} \subset B \to \mathbb{R}^+$ such that the warped product manifold $\mathfrak{U} \times_{f_{\mathfrak{U}}} F$ is locally conformally flat.

Indeed, note that $f_{\mathfrak{U}}$ is defined in terms of the local conformal factor and some appropriately chosen Möbius transformations in order to adjust the sectional curvatures of $(\mathfrak{U}, \widetilde{g})$, where $\widetilde{g} = \frac{1}{f_{\mathfrak{U}}} g_B$, so that $\widetilde{K}^{\mathfrak{U}} = -K^F$.

Observe that the vanishing of the Weyl tensor is a local property. In fact, $\mathcal{M} = B \times_f F$ is locally conformally flat if and only if for each point $p \in M$ there exist functions Ψ defined in a neighborhood of p such that the restriction of $\Psi^2 \cdot g$ to the defining neighborhood is flat. Since all spaces of constant curvature are locally conformally flat, we have plenty of functions Ψ as before in any locally conformally flat manifold. However, the implication in Theorem 6.1.2 is on the whole base B, since the warping function is globally defined. Therefore, two basic problems arise from the above: when is it possible to extend the locally defined warping functions to the whole base B? and, in such case, is there any kind of uniqueness on the warping functions?.

Both problems will be considered in what follows under some additional assumptions (Sections 6.1.1 and 6.2) in the Riemannian setting.

As an application of Theorem 6.1.2-(iii), a 4-dimensional static spacetime $B \times_f \mathbb{R}$ is locally conformally flat if and only if $(B, \frac{1}{f^2} g_B)$ is Einstein. Therefore, any conformal Einstein 3-dimensional manifold gives rise to a suitably defined locally conformally flat static spacetime. (See [112], [129] and the references therein for more information on conformal Einstein equations). On the other hand, Friedman-Robertson-Walker spacetimes $\mathbb{R} \times_f F$ are locally conformally flat for any fiber F of constant curvature, independently of the warping function.

Remark 6.1.5 As a generalization of the warped product structure, one can consider the product manifold $\mathcal{M} = B \times F$ with metric tensor $g = b^2 g_B \oplus f^2 g_F$, where f and b only depend on the points of B and F, respectively. This is said to be a *doubly warped product* and we denote it by $B_b \times {}_f F$. Now, proceeding as above, one can express the doubly warped metric g as follows,

$$g = b^2 g_B \oplus f^2 g_F = b^2 \left(g_B \oplus f^2 (\frac{1}{b^2} g_F) \right) = b^2 f^2 \left((\frac{1}{f^2} g_B) \oplus (\frac{1}{b^2} g_F) \right).$$

This shows that the doubly warped product is in the conformal class of a certain warped product, moreover it is also in the conformal class of a direct product. Thus, Theorem 6.1.2 applies to show that $B_b \times {}_f F$ is locally conformally flat if and only if $(B, \frac{1}{f^2} g_B)$ and $(F, \frac{1}{b^2} g_F)$ have constant sectional curvatures, which are necessarily of opposite sign if both B and F have dimension greater than one. Again, this interpretation gives a geometrical meaning to the PDE's systems in [80].

6.1.1 Warped products with base of constant sectional curvature

The local existence of warping functions making $(\mathfrak{U} \subset B) \times_{f_{\mathfrak{U}}} F$ locally conformally flat follows from Remark 6.1.4. The global existence of such functions is the object of our study in this section. First, we assume the dimension of B is strictly greater than 1 ($dim\, B > 1$), otherwise any globally defined warping function f makes the manifold locally conformally flat. Furthermore, in order to understand the simplest cases, we consider warped products with base (B, g_B) of constant sectional curvature and dim $B = n$. Our first result is the following analytical characterization of the warping function.

Lemma 6.1.6 *Let $\mathcal{M} = B \times_f \mathbb{R}$ be a locally conformally flat warped product. Then (B, g_B) has constant sectional curvature if and only if the warping function f defines a solution $\phi = -\ln f$ of the Möbius equation $B_{g_B}(\phi) = 0$, where*

$$B_{g_B}(\phi) = H_\phi - d\phi \otimes d\phi - \frac{1}{n}\{\Delta\phi - \|\nabla\phi\|^2\} g_B. \tag{6.1}$$

Moreover, for any base (B, g_B) of constant sectional curvature and any positive solution ϕ of (6.1)*, the warped product $B \times_f \mathbb{R}$ is locally conformally flat.*

Proof. Since (B, g_B) is locally conformally flat by Corollary 6.1.3, then it is of constant sectional curvature if and only if it is Einstein. On the other hand, $(B, \frac{1}{f^2} g_B)$ is a space of constant sectional curvature by Theorem 6.1.2, and thus Einstein. Therefore (B, g_B) is of constant sectional curvature if and only if the conformal deformation $g_B \mapsto \frac{1}{f^2} g_B$ preserves the Einstein property, which occurs if and only if f is a solution of the Möbius equation [144]. Now, the second part of the lemma follows from Theorem 6.1.2 and the considerations above. □

It is shown in [144] that, if $\phi : B \to \mathbb{R}$ satisfies the Möbius equation on (B, g_B) then, since $H_{exp(\phi)} = exp(\phi)\{H_\phi + d\phi \otimes d\phi\}$, the function $\psi = exp(-\phi)$ satisfies $H_\psi = \frac{1}{n}\Delta\psi g_B$ on (B, g_B). Henceforth we will refer to $H_\psi = \frac{1}{n}\Delta\psi g_B$ as the *linearized Möbius equation* on (B, g_B).

We use the previous lemma to obtain the following description of the possible warping functions on a locally conformally flat warped product with base a complete and simply connected Riemannian space of constant sectional curvature.

Theorem 6.1.7 *Let (B, g_B) be a complete and simply connected Riemannian manifold of constant sectional curvature. Then the warped product $B \times_f \mathbb{R}$ is locally conformally flat if and only if one of the following occurs:*

(i) $B \equiv \mathbb{R}^n$, where $\mathbb{R}^n$ denotes the Euclidean space, then all possible warping functions are given by

$$f(\overrightarrow{\mathbf{x}}) = a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}}\rangle + c, \quad \overrightarrow{\mathbf{b}} \in \mathbb{R}^n,\ a, c \in \mathbb{R}, \tag{6.2}$$

where the coefficients a, $\overrightarrow{\mathbf{b}}$ and c satisfy $4ac - \|\overrightarrow{\mathbf{b}}\|^2 > 0$, $a > 0$ and $\langle\, ,\, \rangle$ is the Euclidean inner product in $\mathbb{R}^n$.

(ii) $B \equiv \mathbb{H}^n$*, where* $\mathbb{H}^n$ *denotes the Poincaré half-space model of the hyperbolic geometry, then all possible warping functions are given by*

$$f(\overrightarrow{\mathbf{x}}) = \frac{a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}}\rangle + c}{x_n}, \qquad \overrightarrow{\mathbf{b}} \in \mathbb{R}^n, \ a, c \in \mathbb{R}, \tag{6.3}$$

where the coefficients $a > 0$*,* $\overrightarrow{\mathbf{b}}$ *and* c *satisfy either of*

$$\begin{array}{ll} (ii.1) & 4ac - \|\overrightarrow{\mathbf{b}}\|^2 > 0, \ or \\ (ii.2) & 4ac - (b_1^2 + b_2^2 + \cdots + b_{n-1}^2) \geq 0 \ and \ b_n \geq 0. \end{array}$$

(iii) $B \equiv \mathbb{S}^n$*, where* $\mathbb{S}^n$ *denotes the Euclidean sphere, then all possible warping functions are given by*

$$f = -\frac{n-1}{\tau}\psi + \kappa \tag{6.4}$$

where ψ *is an eigenfunction for the largest eigenvalue of the Laplacian, i.e. it satisfies* $\Delta\psi = -\frac{\tau}{n-1}\psi$*, and* κ *is a real constant making* f *positive.*

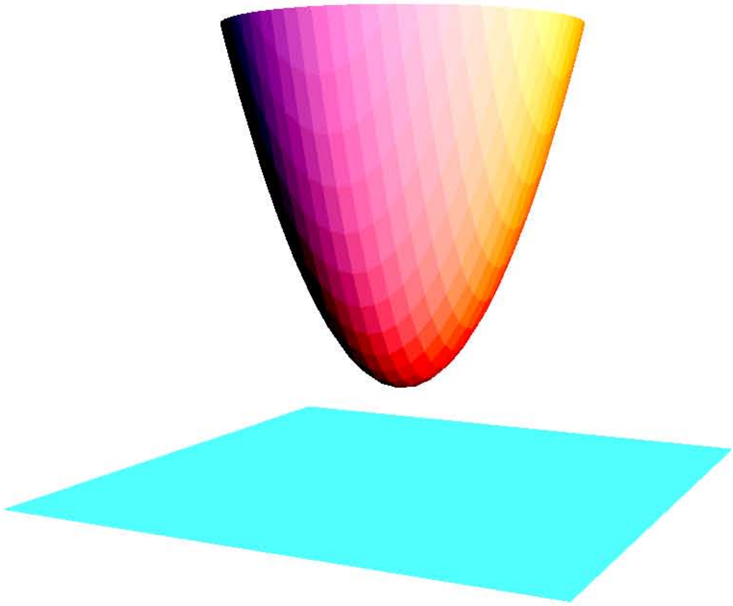
Proof. We analyze each case above separately.

(i) It follows from Lemma 6.1.6 that a function $f : \mathbb{R}^n \to \mathbb{R}$ defines a locally conformally flat warped product $\mathbb{R}^n \times_f \mathbb{R}$ if and only if $\phi = -\ln f$ is a solution of the Möbius equation on $(\mathbb{R}^n, g_0)$. By considering $u = exp(-\phi)$, one gets that f is indeed a positive solution of the linearized Möbius equation $H_f = \frac{1}{n}\Delta f g_0$.

Now it follows easily from the linearized Möbius equation that any such function f must be of the form

$$f(\overrightarrow{\mathbf{x}}) = a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}}\rangle + c, \qquad a, c \in \mathbb{R}, \overrightarrow{\mathbf{b}} \in \mathbb{R}^n.$$

Moreover, since f is required to be strictly positive in order to be a warping function, we have the following. First observe that $a > 0$, otherwise f is negative outside a compact set.



The warping function does not intersect the plane $x_{s+1} = 0$.

Now, assuming $a > 0$, $f(\overrightarrow{\mathbf{x}}) = a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}}\rangle + c$ has its minimum at $\overrightarrow{\mathbf{x}} = -\frac{1}{2a}\overrightarrow{\mathbf{b}}$ and $f(-\frac{1}{2a}\overrightarrow{\mathbf{b}}) = -\frac{1}{4a}\|\overrightarrow{\mathbf{b}}\|^2 + c$, which has to be positive, from where (i) is obtained.

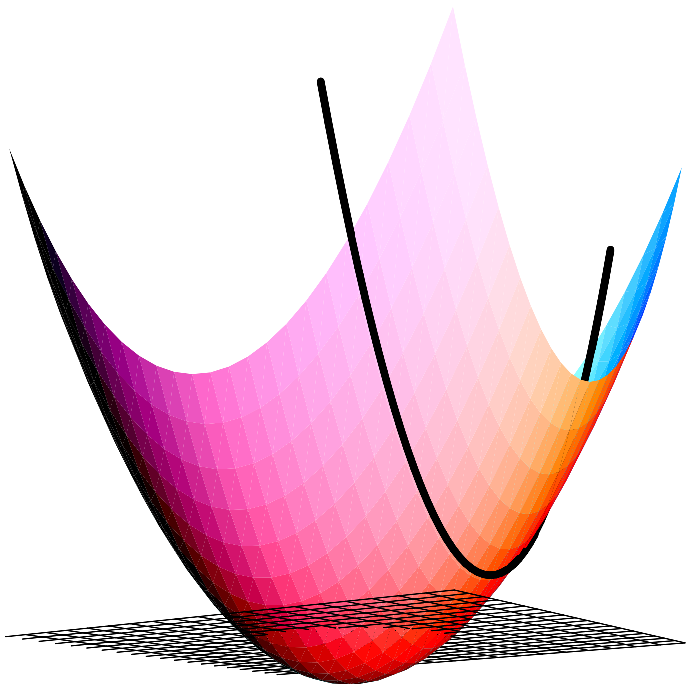
(ii) Let $(\mathfrak{U}^n, g_0)$ denote the Euclidean upper-half-space and consider the Poincaré half-space model $(\mathbb{H}^n, g_{\mathbb{H}})$ as a conformal deformation $g_{\mathbb{H}} = \vartheta^* g_0 = \frac{1}{x_n^2} g_0$ of the Euclidean metric. Let φ be a Möbius transformation from the hyperbolic space $(\mathbb{H}^n, g_{\mathbb{H}})$ into $(\mathfrak{U}^n, \frac{1}{f^2} g_{\mathbb{H}})$ and consider the following diagram:

$$\begin{array}{ccc} (\mathbb{H}^n, g_{\mathbb{H}} = \frac{1}{x_n^2} g_0) & \stackrel{\varphi}{\longrightarrow} & (\mathfrak{U}^n, \frac{1}{f^2} g_{\mathbb{H}}) \\ \vartheta \downarrow & \nearrow & \\ (\mathfrak{U}^n, g_0) & & \end{array}$$

Note that the set of conformal transformations forms a group with composition; moreover the Möbius transformations are a proper subgroup [144]. Further, since ϑ is a Möbius transformation, then φ is a Möbius transformation if and only if so is $\varphi \circ \vartheta^{-1}$, which is defined in $(\mathfrak{U}^n, g_0)$. Now, it follows from *(i)* that $\frac{1}{f^2} g_{\mathbb{H}} = \frac{1}{(a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}}\rangle + c)^2} g_0$, from where it follows that

$$f(\overrightarrow{\mathbf{x}}) = \frac{a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}}\rangle + c}{x_n}.$$

Since $x_n > 0$ we only have to consider the numerator in the above definition of f in order to analyze its positivity. Now note that $a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}} \rangle + c$ defines a convex paraboloid (considered as a function in $\mathbb{R}^n$) which is positive on $\mathfrak{U}$ if and only if $a > 0$ and it does not intersect the domain $\mathfrak{U}$. Hence two different possibilities may occur.



The warping function may be positive even if the paraboloid is negative in points where $x_s < 0$.

The minimum of the paraboloid is positive and the required condition is $4ac - \|\overrightarrow{\mathbf{b}}\|^2 > 0$, which shows $(ii.1)$ as in previous case (i). The other possibility occurs if the minimum is nonpositive but it is realized on the down-half-space (i.e., $\frac{-b_n}{2a} \leq 0$). In this case the desired condition is obtained by the positivity of the intersection of the paraboloid defined by $a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}} \rangle + c$ and the hyperplane $x_n = 0$, which gives $4ac - (b_1^2 + b_2^2 + \cdots + b_{n-1}^2) \geq 0$, thus showing $(ii.2)$.

(iii) As in the previous cases, it follows from Lemma 6.1.6 that a function $f : \mathbb{S}^n \to \mathbb{R}$ defines a locally conformally flat warped product $\mathbb{S}^n \times_f \mathbb{R}$ if and only if $\phi = -\ln f$ is a solution of the Möbius equation on $(\mathbb{S}^n, g_{\mathbb{S}^n})$. By considering $u = exp(-\phi)$, one gets that f is indeed a positive solution of the linearized Möbius equation and thus $H_f = \frac{1}{n} \Delta f g_{\mathbb{S}^n}$.

Next, consider the gradient vector field ∇f on $\mathbb{S}^n$. Since $(\mathcal{L}_{\nabla f} g)(Y, Z) = 2H_f(Y, Z)$, it follows from the linearized Möbius equation that ∇f is a conformal vector field on $\mathbb{S}^n$, and then

$$-\Delta(div \nabla f) = \frac{\tau}{n-1} div(\nabla f), \tag{6.5}$$

where τ denotes the scalar curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$ (cf. [173]). Hence $\Delta(\Delta f) = -\frac{\tau}{n-1}\Delta f$. This shows that Δf is an eigenfunction for the first non-trivial eigenvalue $\lambda_1 = -\frac{\tau}{n-1}$ of the Laplacian in $(\mathbb{S}^n, g_{\mathbb{S}^n})$. Further, it also follows from (6.5) that $\Delta(\Delta f + \frac{\tau}{n-1} f) = 0$ and thus that $\Delta f + \frac{\tau}{n-1} f$ is a constant function. Therefore, if ψ denotes a λ_1-eigenfunction for the Laplacian in $(\mathbb{S}^n, g_{\mathbb{S}^n})$, then the desired warping functions are given by $f = -\frac{n-1}{\tau}\psi + \kappa$, for some constant κ which makes f positive. □

Remark 6.1.8 The eigenspace corresponding to the largest eigenvalue $\lambda_1 = -\frac{\tau}{n-1}$ of the Laplacian in $\mathbb{S}^n$ is generated by the restriction to $\mathbb{S}^n$ of the homogeneous polynomials of degree one in $\mathbb{R}^{n+1}$ (see for example [10, Chap. 3]). Thus any such λ_1-eigenfunction is the restriction to $\mathbb{S}^n$ of some Ψ defined on $\mathbb{R}^{n+1}$ by $\Psi(\vec{\mathbf{x}}) = \langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle$ for $\mathbf{0} \neq \vec{\mathbf{a}} \in \mathbb{R}^{n+1}$.

Remark 6.1.9 The conformal deformations of the model spaces defined by the different functions (6.2), (6.3) and (6.4) in previous theorem affect the sectional curvature as follows:

(i) The sectional curvature of $(\mathbb{R}^n, \frac{1}{f^2} g_0)$ is constant $4ac - \|\vec{\mathbf{b}}\|^2 > 0$, where f is defined by (6.2).

(ii) The sectional curvature of $(\mathbb{H}^n, \frac{1}{f^2} g_{\mathbb{H}^n})$ is constant $4ac - \|\vec{\mathbf{b}}\|^2$, where f is given by (6.3).

(iii) The sectional curvature of $(\mathbb{S}^n, \frac{1}{f^2} g_{\mathbb{S}^n})$ is constant $\kappa^2 - \frac{(n-1)^2}{\tau^2}\|\vec{\mathbf{a}}\|^2$, where f is given by (6.4) and $\psi = \Psi\,|_{\mathbb{S}^n}$ is the restriction to $\mathbb{S}^n$ of $\Psi(\vec{\mathbf{x}}) = \langle \vec{\mathbf{a}}, \vec{\mathbf{x}} \rangle$. Note here that $\kappa^2 - \frac{(n-1)^2}{\tau^2}\|\vec{\mathbf{a}}\|^2$ is positive, since κ is greater than $\frac{n-1}{\tau}\|\vec{\mathbf{a}}\|$ in order to make f positive.

It follows from Theorem 6.1.2 that the sectional curvature of the modified base and the fiber must have opposite signs. Hence, given one of the bases above, there are restrictions on the sign of the curvature of the fiber as in the table below.

		Fibers				**Curvature of**
Base	**Warping function**	$\mathbb{R}$	$\mathbb{R}^d$	$\mathbb{S}^d$	$\mathbb{H}^d$	$(B, \frac{1}{f^2} g_B)$
$\mathbb{R}^s$	$f(\vec{\mathbf{x}}) = a\|\vec{\mathbf{x}}\|^2 + \langle \vec{\mathbf{b}}, \vec{\mathbf{x}} \rangle + c$	✓	×	×	✓	$4\,a\,c - \|\vec{\mathbf{b}}\|^2$
$\mathbb{S}^s$	$f(\vec{\mathbf{x}}) = -\frac{s-1}{\tau}\langle \vec{\mathbf{b}}, \vec{\mathbf{x}} \rangle + C$	✓	×	×	✓	$C^2 - \frac{(s-1)^2}{\tau^2}\|\vec{\mathbf{b}}\|^2$
$\mathbb{H}^s$	$f(\vec{\mathbf{x}}) = \frac{a\|\vec{\mathbf{x}}\|^2 + \langle \vec{\mathbf{b}}, \vec{\mathbf{x}} \rangle + c}{x_s}$	✓	✓	✓	✓	$4\,a\,c - \|\vec{\mathbf{b}}\|^2$

Scheme of the possible combinations among the bases and the fibers.

6.2 Locally conformally flat warped products: global structure

For a given manifold (B, g_B), to be the base of two different locally conformally flat warped products is a very restrictive fact. This is shown in the next results, which give some partial answers to the uniqueness of the warping function of a locally conformally flat warped product under some global conditions.

Theorem 6.2.1 *Let (B, g_B) be a compact Riemannian manifold admitting two-distinct (i.e., $f \neq c\hat{f}$) warping functions such that both $B \times_f F$ and $B \times_{\hat{f}} F$ are locally conformally flat manifolds for a certain fiber F. Then (B, g_B) is conformal to the Euclidean sphere by means of the conformal deformation $\frac{1}{f^2} g_B$ and $\hat{f}/f$ is one of the functions in Theorem 6.1.7-(iii).*

Proof. Let f and $\hat{f}$ be warping functions such that $B \times_f F$ and $B \times_{\hat{f}} F$ are locally conformally flat warped products. Then $Id : (B, \frac{1}{f^2} g_B) \to (B, \frac{1}{\hat{f}^2} g_B)$ is a conformal transformation between two compact spaces of constant sectional curvature. Proceeding as in case (iii) of Theorem 6.1.7, it follows that $\hat{f}/f$ is a solution of the linearized Möbius equation on $(B, \frac{1}{f^2} g_B)$

$$H_{\hat{f}/f} = \frac{1}{n} \Delta(\hat{f}/f) g$$

and thus $\nabla(\hat{f}/f)$ is a conformal vector field on $(B, \frac{1}{f^2} g_B)$. Note that τ is the scalar curvature of $(B, \frac{1}{f^2} g_B)$ and it is constant, since $(B, \frac{1}{f^2} g_B)$ has constant sectional curvature. Then it follows from (6.5) that $\Delta(\Delta(\hat{f}/f) + \frac{\tau}{n-1} \hat{f}/f) = 0$ and thus $\Delta(\hat{f}/f) + \frac{\tau}{n-1} \hat{f}/f$ is a constant function on $(B, \frac{1}{f^2} g_B)$.

Next, using the linearized Möbius equation, we compute the Hessian of $\Delta(\hat{f}/f)$ to see that

$$H_{\Delta(\hat{f}/f)} + \frac{\tau}{n(n-1)} \Delta(\hat{f}/f) g = 0. \tag{6.6}$$

Now, if $\tau > 0$, then (6.6) is Obata's equation for $\Delta(\hat{f}/f)$. Since $f \neq c\hat{f}$ and for some constant C, $\Delta(\hat{f}/f) = -\frac{\tau}{n-1} \hat{f}/f + C$, then it follows that $\Delta(\hat{f}/f)$ is not constant and hence $(B, \frac{1}{f^2} g_B)$ is isometric to an Euclidean sphere [137].

Finally note that $\tau \leq 0$ leads to a contradiction as follows. Indeed, if $\tau < 0$, then the existence of a nonconstant solution $\Delta(\hat{f}/f)$ of (6.6) is characteristic of warped product structures $\mathbb{R} \times_\xi N$, where N is a complete Riemannian manifold and the warping function satisfies $\xi'' + \frac{\tau}{n(n-1)} \xi = 0$, $\xi > 0$ among complete Riemannian manifolds [107, Thm. C, D]. This contradicts the assumption that B was compact. The case $\tau = 0$ implies $\Delta(\hat{f}/f)$ is constant and thus it follows from the linearized Möbius equation that $H_{\hat{f}/f} = \frac{\sigma}{n} g$ for some constant σ. This shows that $\hat{f}/f$ is a special concircular function on $(B, \frac{1}{f^2} g_B)$ and hence it follows from [163, Thm. 2] that $(B, \frac{1}{f^2}g_B)$ is isometric to the Euclidean space, which is also a contradiction. □

Remark 6.2.2 The previous theorem shows, in particular, the nonexistence of nonconstant globally defined warping functions on the flat tori $\mathbb{T}^n$ which make $\mathbb{T}^n \times_f F$ locally conformally flat (where F is an arbitrary pseudo-Riemannian manifold). However note that such warping functions always exist locally (see Remark 6.1.4).

The result of Theorem 6.2.1 can be extended to warped products with a noncompact base under some alternative geometrical assumptions such as completeness.

Theorem 6.2.3 *Let $\mathcal{M} = B \times_f F$ be a locally conformally flat warped product with $(B, \frac{1}{f^2}g_B)$ geodesically complete. If there exists a warping function $\hat{f} \neq cf$ on B such that $B \times_{\hat{f}} F$ is also locally conformally flat, then one of the following holds*

1. *$(B, \frac{1}{f^2}g_B)$ is a complete and simply connected space of constant sectional curvature and the warping function $\hat{f}/f$ is given by Theorem 6.1.7, or otherwise*
2. *$(B, \frac{1}{f^2}g_B)$ is a warped product $\mathbb{R} \times_{\alpha \exp(\alpha t+\beta)} N$, where (N, g_N) is a complete flat Riemannian manifold and the warping functions satisfy*

$$\hat{f}/f = \exp(\alpha t + \beta) + \kappa$$

for some real constants $\alpha > 0$, β, $\kappa \geq 0$, where $\alpha^2 = -\frac{\tau^B}{n(n-1)}$ and τ^B denotes the scalar curvature of $(B, \frac{1}{f^2}g_B)$.

Proof. Let $f \neq c\hat{f}$ be two warping functions on B such that $B \times_{\hat{f}} F$ and $B \times_f F$ are locally conformally flat. Then, by Theorem 6.1.2 it follows that $\frac{1}{\hat{f}^2}g_B$ and $\frac{1}{f^2}g_B$ are constant sectional curvature metrics on B. Thus, the identity map is a conformal diffeomorphism between two Einstein metrics on B. Since $(B, \frac{1}{f^2}g_B)$ is assumed to be complete and $\hat{f} \neq cf$, then either $(B, \frac{1}{f^2}g_B)$ is a complete and simply connected space of constant sectional curvature (and then isometric to one of the model spaces in Theorem 6.1.7), or $(B, \frac{1}{f^2}g_B)$ is a warped product $\mathbb{R} \times_{\alpha \exp(\alpha t+\beta)} N$, where (N, g_N) is a complete Ricci flat $(n-1)$-dimensional Riemannian manifold and the warping functions satisfy $\hat{f}/f = \exp(\alpha t + \beta) + \kappa$, for some constants α, β, κ [115]. Now, by Corollary 6.1.3 it follows that $\mathbb{R} \times_{\alpha \exp(\alpha t+\beta)} N$ is locally conformally flat and thus (N, g_N) is of constant sectional curvature by Theorem 6.1.2, which shows that (N, g_N) is flat. Moreover, it follows after some standard calculations that the scalar curvature of $(B, \frac{1}{f^2}g_B)$ is $\tau^B = -n(n-1)\alpha^2$, which completes the proof of the result. □

6.3 Locally conformally flat twisted products

In the same spirit as Corollary 1.5.12, the following result shows that the local conformal flatness condition reduces twisted to warped products under some general assumptions.

Theorem 6.3.1 *Let $\mathcal{M} = B \times_f F$ be a twisted product with $dim\, B \geq 2$ and $dim\, F \geq 2$. If the Weyl tensor is zero, then $\mathcal{M}$ may be written as a warped product.*

Proof. Let X and V denote two non null vector fields on B and F, respectively. Since $\dim B \geq 2$, we choose a non null vector field Y on B which belongs to the space orthogonal to X; $Y \in \langle X \rangle^{\perp}$. Then

$$
\begin{aligned}
W(Y,X,Y,V) &= R(Y,X,Y,V) \\
&\quad + \tfrac{\tau}{(n-1)(n-2)}\{\langle Y,Y\rangle\langle X,V\rangle - \langle X,Y\rangle\langle Y,V\rangle\} \\
&\quad - \tfrac{1}{n-2}\{\rho(Y,Y)\langle X,V\rangle - \rho(X,Y)\langle Y,V\rangle \\
&\qquad + \rho(X,V)\langle Y,Y\rangle - \rho(Y,V)\langle X,Y\rangle\} \\
&= -\tfrac{1}{n-2} g(Y,Y)\rho(X,V).
\end{aligned} \tag{6.7}
$$

If $W = 0$, then necessarily $\rho(X,V) = 0$. The result now follows from Theorem 1.5.11. □

Thus, the previous result reduces the study of locally conformally flat twisted products to warped products and twisted products with one factor of dimension 1.

Remark 6.3.2 As a scholium to Theorem 6.1.2, for the metric tensor of any twisted product $B \times_f F$, one can write

$$g_B \oplus f^2 g_F = f^2 \left(\frac{1}{f^2} g_B \oplus g_F \right).$$

This shows that $B \times_f F$ is locally conformally flat if and only if so is $F \times_{\frac{1}{f}} B$. But this time $g_F \oplus \frac{1}{f^2} g_B$ is still a twisted product metric, since the twisting function f depends both on B and F. The fact that f decomposes as a product of two functions f_B and f_F defined on B and F, respectively, if and only if so does $\frac{1}{f}$, comes up from the following expression

$$XU(\log f) = -XU\left(\log \frac{1}{f}\right),$$

for all vector fields X, U on B and F, respectively.

The duality condition in the previous remark justifies the symmetry on the conditions $dim\, B \geq 2$, $dim\, F \geq 2$ in Theorem 6.3.1. Now, in order to show the necessity of these restrictions, we construct some simple examples showing the necessity of the assumptions on $\dim B \geq 2$ and $\dim F \geq 2$ as follows. Let $I \subset \mathbb{R}$ be an open interval and take $U \subset \mathbb{R}^3$

an open set such that $f(t,x,y,z) = \frac{1}{t+z}$ is positive on $I \times U$. After a long calculation one gets that the twisted product manifold $I \times_f U$ has vanishing Weyl tensor, hence it is locally conformally flat. Moreover, note that $I \times_f U$ cannot be reduced to a warped product structure since

$$\rho\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial z}\right) = \frac{-2}{(t+z)^2} \neq 0;$$

this should vanish, of course, for a warped product.

Locally conformally flat twisted product metrics with 1-dimensional fiber are easily obtained from Remark 6.3.2 by considering $U \times_{\frac{1}{f}} I$.

In the remainder of this section we characterize local conformal flatness in twisted products of the form $B \times_f \mathbb{R}$. First, recall the following expressions for the curvature of such twisted products, obtained by specializing Lemmas 1.5.9 and 1.5.10.

Lemma 6.3.3 *Let $B \times_f \mathbb{R}$ be a twisted product. Let X, Y, Z, T be vectors tangent to the base B and let U be a vector tangent to the fiber. Then the curvature tensor is given by:*

$$R(X,Y,Z,T) = R^B(X,Y,Z,T), \qquad R(X,U,Y,U) = -\frac{H_f(X,Y)}{f}\langle U,U\rangle,$$

the Ricci tensor is given by:

$$\rho(X,Y) = \rho^B(X,Y) - \frac{H_f(X,Y)}{f}, \qquad \rho(X,U) = 0, \qquad \rho(U,U) = -\frac{\Delta^B f}{f},$$

and the scalar curvature is given by:

$$\tau = \tau^B - \frac{2\Delta^B f}{f}.$$

Now we have the following results.

Theorem 6.3.4 *Let $\mathcal{M} = B \times_f \mathbb{R}$ be a twisted product with $\dim B \geq 4$. Then the following assertions are equivalent:*

(1) $\mathcal{M}$ is locally conformally flat.

(2) B is locally conformally flat. Furthermore for any orthonormal vectors X, Y, Z tangent to B, the following equation holds

$$K_{XY} - K_{XZ} = \frac{H_f(Z,Z)}{f} - \frac{H_f(Y,Y)}{f}. \tag{6.8}$$

(3) For any orthonormal vectors X, Y tangent to B, the following equation holds

$$K_{XY} = \frac{\tau^B}{(n-1)(n-2)} - \frac{H_f(X,X)}{f} - \frac{H_f(Y,Y)}{f} + \frac{2\Delta^B f}{(n-1)f}. \tag{6.9}$$

Proof. We first establish the equivalence of (1) and (2). Suppose that $\mathcal{M}$ is locally conformally flat or, equivalently, that $K_{EF} + K_{GH} = K_{EG} + K_{FH}$ for any orthogonal vectors $E, F, G, H \in \mathfrak{X}(\mathcal{M})$ (see [117]). Then, from Lemma 6.3.3 it follows that B is locally conformally flat and

$$K_{XY} - \frac{H_f(Z,Z)}{f} = K_{XZ} - \frac{H_f(Y,Y)}{f},$$

for X, Y, Z vectors tangent to B, from where (2) follows. Conversely, this also shows that $K_{EF}+K_{GH} = K_{EG}+K_{FH}$ for any orthogonal vectors E, F, G, H in $\mathcal{M}$, so (1) also follows from (2).

We now show that (1) implies (3). Suppose M is locally conformally flat, then the Weyl tensor vanishes, so

$$\begin{aligned} W(X,Y,X,Y) &= K_{XY} - \tfrac{1}{n-2}\left(\rho(X,X) + \rho(Y,Y) - \tfrac{\tau}{n-1}\right) \\ &= K_{XY} - \tfrac{1}{n-2}\Big(\rho^B(X,X) + \rho^B(Y,Y) - \tfrac{\tau^B}{n-1} \\ &\qquad - \tfrac{H_f(X,X)}{f} - \tfrac{H_f(Y,Y)}{f} + \tfrac{2\Delta^B f}{f}\Big) \\ &= 0. \end{aligned} \tag{6.10}$$

We replace Y by elements of an orthonormal basis and sum to get

$$\rho^B(X,X) = \frac{\tau^B}{n-1} - (n-3)\frac{H_f(X,X)}{f} + \frac{n-3}{n-1}\frac{\Delta^B f}{f}. \tag{6.11}$$

Now, substituting the Ricci terms in (6.10) we get

$$K_{XY} = \frac{\tau^B}{(n-1)(n-2)} - \frac{H_f(X,X)}{f} - \frac{H_f(Y,Y)}{f} + \frac{2\Delta^B f}{(n-1)f},$$

which is the expression in (3).

Finally, we show (3) implies (2). We use (6.9) to show that (6.8) is met and also that $K_{XY} + K_{ZT} = K_{XZ} + K_{YT}$ for X, Y, Z, T orthogonal vectors tangent to B, hence (2) follows. □

Remark 6.3.5 Note from item (2) in Theorem 6.3.4 that, if B is locally conformally flat, then there exists locally a conformal factor ψ such that $g^B = \psi^2 g^0$, where g^0 is the usual Euclidean metric. Thus, one may make a conformal transformation of $B \times_f \mathbb{R}$ with conformal factor $1/\psi^2$ to get

$$g^0 \oplus \frac{f^2}{\psi^2} g^0.$$

Since this new twisted product is also locally conformally flat, we obtain from (6.8) that for any X, Y tangent to the base

$$H^\phi(X,X) = H^\phi(Y,Y),$$

where $\phi = \frac{f}{\psi}$ and H is the Hessian with respect to the Euclidean metric.

From Theorem 6.3.4, the following relations among properties of a twisted product $B \times_f \mathbb{R}$ follow.

Corollary 6.3.6 *Let $\mathcal{M} = B \times_f \mathbb{R}$ be a twisted product of dimension $n \geq 4$. Then any two of the following assertions imply the third one:*

(a) M is locally conformally flat.

(b) B has constant sectional curvature.

(c) $H_f^B = \lambda\, g^B$.

Proof. $(a) + (b) \Rightarrow (c)$. If M is locally conformally flat, by Theorem 6.3.4, equation (6.8) holds, and since the sectional curvature is constant we have $H_f(X, X) = H_f(Y, Y)$ for any orthonormal vectors X, Y, which implies (c).

$(a) + (c) \Rightarrow (b)$. Again, by equation (6.8) item (b) follows trivially.

$(b) + (c) \Rightarrow (a)$. Note that (b) and (c) imply (2) of Theorem 6.3.4, so (a) holds. □

Chapter 7

Multiply warped products and local conformal flatness

In general, a multiply warped product takes the form

$$B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k.$$

We may make this decomposition slightly more canonical by assuming, without loss of generality, that f_i is not a constant multiple of f_j for $i \neq j$. The order of the fibers plays no role in our discussion. Henceforth for the remainder of Chapter 7 we set $s = dim\, B$ and $d_i = dim\, F_i$.

The local study of multiply warped products we present in this chapter is developed in general for arbitrary signature. Nonetheless, several global results we present in Section 7.4 need some restrictions on the signature of the base. The Riemannian versions of most of the results in this chapter are collected in [34].

7.1 Basic results

The following lemmas give some useful expressions that we derived from Lemma 1.5.3 and that we will need further on.

Lemma 7.1.1 *Let $\mathcal{M} = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a pseudo-Riemannian multiply warped product, $X, Y \in \mathfrak{X}(B)$ and $U_i, V_i \in \mathfrak{X}(F_i)$, and let $d_i = dim\, F_i$ for $i = 1, \ldots, k$. Then the Ricci tensor is given by:*

(i) $\rho(X,Y) = \rho^B(X,Y) - \sum_{i=1}^k \frac{H_{f_i}(X,Y)}{f_i} d_i$.

(ii) $\rho(X, V_i) = 0$, *for all* $i = 1, \ldots, k$.

(iii) $\rho(U_i, V_i) = \rho^{F_i}(U_i, V_i) - \langle U_i, V_i \rangle \left(\frac{\Delta f_i}{f_i} + (d_i - 1)\frac{\|\nabla f_i\|^2}{f_i^2} + \sum_{j \neq i} d_j \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} \right)$ *for all* $i = 1, \ldots, k$.

(iv) $\rho(V_i, V_j) = 0$*, if* $i \neq j$*.*

And the scalar curvature has the following expression:

$$\tau = \tau^B + \sum_i \frac{1}{f_i^2}\tau^{F_i} - 2\sum_i d_i \frac{\Delta f_i}{f_i} - \sum_i d_i(d_i-1)\frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} - \sum_i \sum_{j\neq i} d_i d_j \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j}\,. \tag{7.1}$$

For the sectional curvature we will need expressions given in the following Lemma.

Lemma 7.1.2 *Let* $\mathcal{M} = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ *be a multiply warped product,* $X, Y \in \mathfrak{X}(B)$ *and* $U_i, V_i \in \mathfrak{X}(F_i)$*. Then we have the following expressions for the sectional curvature:*

$$\begin{aligned} K_{XY} &= K^B_{XY}, \\ K_{XU_i} &= -\frac{H_{f_i}(X,X)}{f_i\|X\|^2}, \\ K_{U_iV_i} &= \frac{1}{f_i^2}K^{F_i}_{U_iV_i} - \frac{\|\nabla f_i\|^2}{f_i^2}, \\ K_{U_iV_j} &= -\frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j},\ i \neq j, \end{aligned}$$

where K^B *and* K^{F_i} *denote the sectional curvatures on the base* B *and the fiber* F_i*, respectively.*

Recall from the previous chapter that a non flat locally decomposable pseudo-Riemannian manifold is locally conformally flat if and only if it is locally equivalent to the product of an interval and a space of constant sectional curvature $N(c) \times \mathbb{R}$ or to the product of two spaces of constant opposite sectional curvature $N_1(c) \times N_2(-c)$ [119], [175].

We begin by extending Corollary 6.1.3 to the multiply warped product context.

Lemma 7.1.3 *Let* $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ *be a locally conformally flat multiply warped product. Then*

(i) (B, g_B) *is locally conformally flat,*

(ii) (F_i, g_i) *is a space of constant sectional curvature for all* $i = 1, \ldots k$*, provided that* $d_i \geq 2$*.*

Proof. For any $i = 1, \ldots, k$, write the multiply warped metric as

$$g = f_i^2\left(\frac{1}{f_i^2}g_B \oplus \frac{f_1^2}{f_i^2}g_1 \oplus \ldots \oplus g_i \oplus \ldots \oplus \frac{f_k^2}{f_i^2}g_k\right).$$

Since $f_i : B \to \mathbb{R}^+$, the above expression shows that g is in the conformal class of a suitable product metric tensor. Hence, the multiply warped metric is locally conformally flat if and only if so is the product metric of (F_i, g_i) and the multiply warped product

$\tilde{B} \times_{\frac{f_1}{f_i}} F_1 \times \ldots \times \widehat{F_i} \times \ldots \times_{\frac{f_k}{f_i}} F_k$ with base $\tilde{B} \equiv (B, \frac{1}{f_i^2} g_B)$. This shows that either $dim\, F_i = 1$ or, as a consequence of Theorem 6.1.1, (F_i, g_i) is of constant sectional curvature, and moreover that $\tilde{B} \times_{\frac{f_1}{f_i}} F_1 \times \ldots \times \widehat{F_i} \times \ldots \times_{\frac{f_k}{f_i}} F_k$ is of constant sectional curvature too. The result is obtained by repeating this argument with every fiber. □

Remark 7.1.4 Note from the previous proof that if $B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ is locally conformally flat, then so is $B \times_{f_1} F_1 \times \ldots \times_{f_{k-1}} F_{k-1}$.

7.2 Multiply warped products with one-dimensional base

This section is devoted to the study of multiply warped spaces with 1-dimensional base, this is

$$\mathcal{M} = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k, \quad \text{where } I \subset \mathbb{R}, \tag{7.2}$$

and the fibers $F_1, \ldots, F_k$ are arbitrary pseudo-Riemannian manifolds. Note that, although we assume the base to be of positive signature, the analysis for negative signature is analogous.

Firstly we describe all multiply warped products of constant sectional curvature. Then, such classification will be used to obtain a complete description of those which are locally conformally flat. These results are exposed in [29] and [31].

7.2.1 Multiply warped products of constant sectional curvature

In this section we obtain a complete description of multiply warped products with 1-dimensional base and constant sectional curvature. This will be useful to later describe locally conformally flat multiply warped products.

Firstly, specialize Lemma 7.1.2 to obtain the expressions for the sectional curvature of a multiply warped space of the form $\mathcal{M} = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$:

$$\begin{aligned} K_{\partial_t U_i} &= \frac{f_i''}{f_i}, \\ K_{U_i V_i} &= \frac{1}{f_i^2}\left(K^{F_i}_{U_i V_i} + (f_i')^2\right), \\ K_{U_i U_j} &= \frac{f_i' f_j'}{f_i f_j}. \end{aligned} \tag{7.3}$$

The next theorem was obtained previously by Mignemi and Schmidt in [131]; for the sake of completeness we provide a proof which differs a bit from that of [131]. First of all note that, as a consequence of (7.3), if $M = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a space of constant sectional curvature, then each fiber F_i must be of constant sectional curvature.

Theorem 7.2.1 *Let $\mathcal{M} = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a Riemannian multiply warped product with $dim\, I = 1$. Then $\mathcal{M}$ has constant sectional curvature K if and only if one of the following holds:*

(i) $\mathcal{M} = I \times_{\alpha_1} F_1$ *or* $\mathcal{M} = I \times_{\alpha_1} F_1 \times_{\alpha_2} F_2$*, with warping functions given by*

$$\alpha_i(t) = a_i\, t + b_i, \qquad i = 1, 2.$$

Moreover, the fibers (F_i, g_i) *are necessarily of constant sectional curvature* $K^{F_i} = a_i^2$*, provided that* $\dim F_i \geq 2$ $(i = 1, 2)$*, and the warping functions satisfy the compatibility condition* $a_1 a_2 = 0$ *in the case of two fibers (this is, one of the warping functions is constant). In this case* $K = 0$*.*

(ii) $\mathcal{M} = I \times_{\beta_1} F_1$ *or* $\mathcal{M} = I \times_{\beta_1} F_1 \times_{\beta_2} F_2$*, with warping functions given by*

$$\beta_i(t) = a_i \sin ct + b_i \cos ct, \qquad i = 1, 2.$$

Moreover, the fibers (F_i, g_i) *are of constant sectional curvature* $K^{F_i} = c^2(a_i^2 + b_i^2)$*, provided that* $\dim F_i \geq 2$ $(i = 1, 2)$*, and the warping functions satisfy the compatibility condition* $a_1 a_2 + b_1 b_2 = 0$ *in the case of two fibers. The value of the sectional curvature is* $K = c^2$*.*

(iii) $\mathcal{M} = I \times_{\gamma_1} F_1$ *or* $\mathcal{M} = I \times_{\gamma_1} F_1 \times_{\gamma_2} F_2$*, with warping functions given by*

$$\gamma_i(t) = a_i \sinh ct + b_i \cosh ct, \qquad i = 1, 2.$$

Moreover, the fibers (F_i, g_i) *are of constant sectional curvature* $K^{F_i} = c^2(a_i^2 - b_i^2)$*, provided that* $\dim F_i \geq 2$ $(i = 1, 2)$*, and the warping functions satisfy the compatibility condition* $a_1 a_2 - b_1 b_2 = 0$ *in the case of two fibers. The sectional curvature is* $K = -c^2$*.*

Proof. Assume $\mathcal{M}$ has constant sectional curvature K. Then conditions in (7.3) can be rewritten in a simpler way as follows:

(i) $$f_i''(t) + K\, f_i(t) = 0,$$

(ii) $$f_i'(t)^2 + K\, f_i(t)^2 = K^{F_i},$$

(iii) $$f_i'(t) f_j'(t) + K\, f_i(t) f_j(t) = 0.$$

Now observe that equations (i) only depend on the warping functions and the constant value of the sectional curvature. Therefore, such equations give us the general form of the warps. Thus, depending on the sign of K, one gets:

$$\begin{array}{ll} K = 0: & \alpha_i(t) = a_i\, t + b_i, \\ K = c^2: & \beta_i(t) = a_i \sin ct + b_i \cos ct, \\ K = -c^2: & \gamma_i(t) = a_i \sinh ct + b_i \cosh ct. \end{array} \tag{7.4}$$

Equations (ii) come from the sectional curvature of a plane generated by two vectors in the same fiber, so they express the compatibility between the fiber and the corresponding warping function when $\dim F_i \geq 2$. We use the expressions from (7.4) to obtain

$$\begin{aligned} K = 0 : \qquad & K^{F_i} = a_i^2, \\ K = c^2 : \qquad & K^{F_i} = c^2(a_i^2 + b_i^2), \\ K = -c^2 : \qquad & K^{F_i} = c^2(a_i^2 - b_i^2). \end{aligned} \tag{7.5}$$

Finally note that equations (i) and (ii) completely determine the structure of a warped product of constant sectional curvature, but if M has more than one fiber then equations (iii), which correspond to the sectional curvature of a plane generated by vectors in different fibers, must also be considered. Thus, equations (iii) can be interpreted as a compatibility condition among the different warping functions. Hence, as one can expect, it provides a bound in the possible number of fibers arguing as follows: if one of the warping functions is constant, suppose without loss of generality that f_1 is constant, then from (iii) we have $Kf_1f_2 = 0$ and necessarily $K = 0$. Now, for f_2 and f_3 we have $f_2'f_3' = 0$ so f_2 or f_3 has to be constant and hence a multiple of f_1 which contradicts the assumption that no warping function is a multiple of any other; therefore only two fibers may occur. On the other hand, suppose that there are three different warping functions f_1, f_2 and f_3, none of them constant. Then from (iii) we get

$$\frac{f_1'}{f_1} = -K\,\frac{f_2}{f_2'} = \frac{f_3'}{f_3}. \tag{7.6}$$

Now it follows that $f_3 = \kappa f_1$ for some constant κ, which is a contradiction since warping functions are assumed to be different. Therefore the maximum number of fibers is exactly two and, if this is the case, the restrictions at (iii) provide the following constraints

$$\begin{aligned} K = 0 : \quad & a_1a_2 = 0, \\ K = c^2 : \quad & a_1a_2 + b_1b_2 = 0, \\ K = -c^2 : \quad & a_1a_2 - b_1b_2 = 0, \end{aligned} \tag{7.7}$$

which completes the proof. □

Remark 7.2.2 A connected 1-dimensional manifold is diffeomorphic to a interval in $\mathbb{R}$ or the circle S^1. Assume the manifold $\mathcal{M} = S^1 \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ has constant sectional curvature. Then, since S^1 is compact, each function f_i' attains its maximum in S^1 and there exist t_i so that $f_i''(t_i) = 0$. Observe from equations (i) in the proof of Theorem 7.2.1 that this implies $K = 0$ and hence $f_i''(t) = 0$ for all $t \in S^1$. Therefore $f_i(t) = a_i\,t + b_i$. But this functions are not defined on S^1. Hence we conclude that there are no multiply warped products of constant sectional curvature with compact 1-dimensional base.

Remark 7.2.3 Although Theorem 7.2.1 is restricted to Riemannian signature, one extends that result for arbitrary signature as follows. First of all, note that the signature of the fibers plays no role in previous classification. Secondly, the change of sign in the signature of the base influences equations (i), (ii), (iii), (7.4), (7.5), (7.6) and (7.7). In summary, one obtains that

> $\mathcal{M} = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ with g_I of negative signature has constant sectional curvature K if and only if $k \leq 2$ and one of the following holds:
>
> (i) $K = 0$, then $\mathcal{M} = I \times_{\alpha_1} F_1$ or $\mathcal{M} = I \times_{\alpha_1} F_1 \times_{\alpha_2} F_2$, with warping functions given by
>
> $$\alpha_i(t) = a_i\, t + b_i, \qquad i = 1, 2.$$
>
> Moreover, the fibers (F_i, g_i) are necessarily of constant sectional curvature $K^{F_i} = -a_i^2$, provided that $\dim F_i \geq 2$ $(i = 1, 2)$, and the warping functions satisfy the compatibility condition $a_1 a_2 = 0$ in the case of two fibers (this is, one of the warping functions is constant).
>
> (ii) $K = c^2$, then $\mathcal{M} = I \times_{\beta_1} F_1$ or $\mathcal{M} = I \times_{\beta_1} F_1 \times_{\beta_2} F_2$, with warping functions given by
>
> $$\beta_i(t) = a_i \sinh ct + b_i \cosh ct, \qquad i = 1, 2.$$
>
> Moreover, the fibers (F_i, g_i) are necessarily of constant sectional curvature $K^{F_i} = c^2(-a_i^2 + b_i^2)$, provided that $\dim F_i \geq 2$ $(i = 1, 2)$, and the warping functions satisfy the compatibility condition $a_1 a_2 - b_1 b_2 = 0$ in the case of two fibers.
>
> (iii) $K = -c^2$, then $\mathcal{M} = I \times_{\gamma_1} F_1$ or $\mathcal{M} = I \times_{\gamma_1} F_1 \times_{\gamma_2} F_2$, with warping functions given by
>
> $$\gamma_i(t) = a_i \sin ct + b_i \cos ct, \qquad i = 1, 2.$$
>
> Moreover, the fibers (F_i, g_i) are necessarily of constant sectional curvature $K^{F_i} = c^2(-a_i^2 - b_i^2)$, provided that $\dim F_i \geq 2$ $(i = 1, 2)$, and the warping functions satisfy the compatibility condition $a_1 a_2 + b_1 b_2 = 0$ in the case of two fibers.

Observe that the change of signature on the base switches the role of positive and negative curvature with respect to the warping functions.

	$dt^2 \oplus f_1^2 g_1 \oplus f_2^2 g_2$		$-dt^2 \oplus f_1^2 g_1 \oplus f_2^2 g_2$	
Curvature	Warping functions	Compatibility conditions	Warping functions	Compatibility conditions
$K = 0$	$a_i t + b_i$	$a_1 a_2 = 0$, $K^{F_i} = a_i^2$	$a_i t + b_i$	$a_1 a_2 = 0$, $K^{F_i} = -a_i^2$
$K = c^2$	$a_i \sin ct + b_i \cos ct$	$a_1 a_2 + b_1 b_2 = 0$, $K^{F_i} = c^2(a_i^2 + b_i^2)$	$a_i \sinh ct + b_i \cosh ct$	$a_1 a_2 - b_1 b_2 = 0$, $K^{F_i} = c^2(-a_i^2 + b_i^2)$
$K = -c^2$	$a_i \sinh ct + b_i \cosh ct$	$a_1 a_2 - b_1 b_2 = 0$, $K^{F_i} = c^2(a_i^2 - b_i^2)$	$a_i \sin ct + b_i \cos ct$	$a_1 a_2 + b_1 b_2 = 0$, $K^{F_i} = c^2(-a_i^2 - b_i^2)$

The warping functions and the compatibility conditions depend on the signature of the base.

Remark 7.2.4 An important consequence of Theorem 7.2.1 is that a manifold of constant sectional curvature with local structure $I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ cannot have more than two fibers.

7.2.2 Locally conformally flat multiply warped products

The next result is the first step in generalizing Theorem 6.1.2-(i) to the class of multiply warped spaces; it gives an upper bound on the number of different fibers of the multiply product space. Although this result can be derived directly from Theorem 7.2.1, we give an independent proof which is self-contained.

Theorem 7.2.5 *Let $\mathcal{M} = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a locally conformally flat multiply warped product. Then $k \leq 3$.*

Proof. Consider the metric g written as follows

$$\begin{aligned} g &= dt^2 \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k \\ &= f_k^2 \left(\left(\tfrac{1}{f_k}\right)^2 dt^2 \oplus \left(\tfrac{f_1}{f_k}\right)^2 g_1 \oplus \cdots \oplus \left(\tfrac{f_{k-1}}{f_k}\right)^2 g_{k-1} \oplus g_k \right). \end{aligned} \tag{7.8}$$

Although we are implicitly assuming the base is Riemannian, the proof is totally analogous if the signature is negative. Expression (7.8) shows g is in the conformal class of the product metric

$$\left(\frac{1}{f_k^2} dt^2 \oplus \left(\frac{f_1}{f_k}\right)^2 g_1 \oplus \cdots \oplus \left(\frac{f_{k-1}}{f_k}\right)^2 g_{k-1} \right) \oplus g_k, \tag{7.9}$$

which is also locally conformally flat. Therefore, Theorem 6.1.1 implies the constancy of the sectional curvature of the fiber (F_k, g_k) (provided that $\dim F_k \geq 2$) and moreover that the metric

$$\frac{1}{f_k^2} dt^2 \oplus \left(\frac{f_1}{f_k}\right)^2 g_1 \oplus \cdots \oplus \left(\frac{f_{k-1}}{f_k}\right)^2 g_{k-1} \tag{7.10}$$

also has constant sectional curvature. Hence, it follows that $I \times_{f_1} F_1 \times \cdots \times_{f_{k-1}} F_{k-1}$ is also locally conformally flat.

Rescaling the metric on I by means of $\widetilde{t} = \int \frac{1}{f_k(t)}$, the metric tensor (7.10) defines the following multiply warped product

$$I \times_{\widetilde{f}_1} F_1 \times \cdots \times_{\widetilde{f}_{k-1}} F_{k-1}, \tag{7.11}$$

where $\widetilde{f}_i(\widetilde{t}) = \frac{f_i(\widetilde{t})}{f_k(\widetilde{t})}$; recall that it has constant sectional curvature. Now it follows from the curvature identities at (7.3) that

$$\frac{\widetilde{f}_1'\widetilde{f}_2'}{\widetilde{f}_1\widetilde{f}_2} = K_{V_1W_2} = \cdots = K_{V_1W_{k-1}} = \frac{\widetilde{f}_1'\widetilde{f}_{k-1}'}{\widetilde{f}_1\widetilde{f}_{k-1}}, \tag{7.12}$$

for vector fields $V_i, W_i \in \mathfrak{X}(F_i)$. Hence there are two possibilities: either $\widetilde{f}_1$ is constant or

$$\frac{\widetilde{f}_2'}{\widetilde{f}_2} = \cdots = \frac{\widetilde{f}_{k-1}'}{\widetilde{f}_{k-1}}. \tag{7.13}$$

First suppose $\widetilde{f}_1$ is constant, then $f_1 = \kappa_{1k} f_k$ for some constant κ_{1k}. But this contradicts the assumption that no warping function is multiple of any other. Therefore (7.13) holds and hence $\widetilde{f}_2 = \kappa_{23}\widetilde{f}_3 = \cdots = \kappa_{2(k-1)}\widetilde{f}_{k-1}$ for some constants κ_{ij}. Now, considering again the same assumption, the maximum number of different functions between $\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_{k-1}$ is two. Hence, there cannot be more than three different warping functions among $f_1, \ldots, f_k$. This proves the result. □

The following is a characterizing result which describes the warping functions for each possible number of fibers.

Theorem 7.2.6 *Let $\mathcal{M} = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a pseudo-Riemannian multiply warped product with 1-dimensional base. Then $\mathcal{M}$ is locally conformally flat if and only if, up to a reparametrization on the base, one of the following holds:*

(i) $\mathcal{M} = I \times_f F$ is a warped product with fiber F of constant sectional curvature (provided that dim $F \geq 2$) and any (positive) warping function f.

(ii) $\mathcal{M} = I \times_{f_1} F_1 \times_{f_2} F_2$ is a multiply warped product with two fibers of constant sectional curvature (provided that dim $F_i \geq 2$, $i = 1, 2$) and warping functions

$$f_1 = (\xi \circ f)\frac{1}{f'}, \qquad f_2 = \frac{1}{f'},$$

where f is a strictly increasing function and ξ is a warping function making $I \times_\xi F_1$ of constant sectional curvature (cf. Theorem 7.2.1) and $(\xi \circ f) > 0$.

(iii) $\mathcal{M} = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ is a multiply warped product with three fibers of constant sectional curvature (provided that dim $F_i \geq 2$) and warping functions

$$f_1 = (\xi_1 \circ f)\frac{1}{f'}, \qquad f_2 = (\xi_2 \circ f)\frac{1}{f'}, \qquad f_3 = \frac{1}{f'},$$

where f is a strictly increasing function and ξ_i are warping functions which make $I \times_{\xi_1} F_1 \times_{\xi_2} F_2$ of constant sectional curvature (cf. Theorem 7.2.1) and such that $(\xi_i \circ f) > 0$, $i = 1, 2$.

Proof. By Theorem 7.2.5, we know that $k \leq 3$ and that, in any case, the fibers have constant sectional curvature (provided that their dimension is greater or equal than 2). Moreover, if $\mathcal{M}$ is a warped product ($k = 1$), then such condition is, indeed, sufficient by Theorem 6.1.2. This proves (i). Now, if there are two fibers ($\mathcal{M} = I \times_{f_1} F_1 \times_{f_2} F_2$) then

$$\frac{1}{f_2^2} dt^2 \oplus \frac{f_1^2}{f_2^2} g_{F_1} \tag{7.14}$$

has constant sectional curvature. (Note that we are again assuming the base has positive signature, if it does not the proof is essentially the same, so we restrict here to the Riemannian case). But since f_2 is strictly positive, we can introduce a reparametrization on I by $\widetilde{t} = \int \frac{1}{f_2(t)}$, and thus (7.14) leads to

$$d\widetilde{t}^2 \oplus \xi(\widetilde{t})^2 g_{F_1}, \tag{7.15}$$

which is a warped product with constant sectional curvature, where $\xi(\widetilde{t}) = \frac{f_1(t)}{f_2(t)}$. This proves (ii).

Finally, if $k = 3$ ($M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$), then

$$\frac{1}{f_3^2} dt^2 \oplus \frac{f_1^2}{f_3^2} g_{F_1} \oplus \frac{f_2^2}{f_3^2} g_{F_2} \tag{7.16}$$

has constant sectional curvature, and proceeding as above we get that

$$d\widetilde{t}^2 + \xi_1(\widetilde{t})^2 g_{F_1} + \xi_2(\widetilde{t})^2 g_{F_2} \tag{7.17}$$

has constant sectional curvature, with $\xi_1(\widetilde{t}) = \frac{f_1(t)}{f_3(t)}$ and $\xi_2(\widetilde{t}) = \frac{f_2(t)}{f_3(t)}$ ($\widetilde{t} = \int \frac{1}{f_3(t)}$). Thus, (iii) is obtained. □

Remark 7.2.7 Note that Robertson-Walker spacetimes are locally conformally flat, independently of the warping function. Previous theorem shows that the warping functions of a multidimensional model $M \equiv I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ are very specific (see also Theorem 7.2.1). However we still have essentially no restrictions for one of the warping functions, in the same way as for Robertson-Walker metrics.

Remark 7.2.8 Let $\mathcal{M} = S^1 \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a locally conformally flat multiply warped product. Using notation of the proof of Theorem 7.2.6 we obtain that

$$d\widetilde{t}^2 + \xi_1(\widetilde{t})^2 g_{F_1} + \ldots + \xi_{k-1}(\widetilde{t})^2 g_{F_{k-1}}$$

has constant sectional curvature. Hence, from Remark 7.2.2 necessarily $k \leq 1$. So $\mathcal{M}$ is a warped product $S^1 \times_f F$ with arbitrary warping function. Therefore, a multiply warped product with compact 1-dimensional base is locally conformally flat if and only if it is a warped product.

7.2.3 Global aspects

First of all recall from [13] that a Riemannian warped product metric is complete if and only if both the base and the fiber are complete. This result can be easily extended to the more general class of multiply warped products to ensure that a multiply warped product is complete if and only if the base and all the fibers are complete, independently of the warping functions (which are restricted by their positivity on the whole base).

- It follows from Theorem 7.2.6 that any locally conformally flat multiply warped product of type (7.2) derives from a suitable space of constant sectional curvature as in Theorem 7.2.1. It is important to emphasize here that complete locally conformally flat metrics can be constructed from non necessarily complete metrics of constant sectional curvature. For instance, the multiply warped space $I \times_{h_1} \mathbb{S}^{d_1} \times_{h_2} \mathbb{S}^{d_2}$ with warping functions

$$h_1(t) = \frac{1}{\sqrt{2}} \sin(t) + \frac{1}{\sqrt{2}} \cos(t), \qquad h_2(t) = \frac{1}{\sqrt{2}} \sin(t) - \frac{1}{\sqrt{2}} \cos(t),$$

  is an incomplete manifold of constant sectional curvature $K = 1$. However, by making an appropriate choice $f(t) = \frac{3\pi}{8} + \frac{1}{4} arctg(t)$ and using Theorem 7.2.6, we get that

$$\mathbb{R} \times_{f_1} \mathbb{S}^{d_1} \times_{f_2} \mathbb{S}^{d_2} \times_{f_3} \mathbb{H}^{d_3}$$

  is a complete locally conformally flat space with warping functions

$$\begin{aligned} f_1(t) &= \{\tfrac{1}{\sqrt{2}} \sin(\tfrac{3\pi}{8} + \tfrac{1}{4} \arctan(t)) + \tfrac{1}{\sqrt{2}} \cos(\tfrac{3\pi}{8} + \tfrac{1}{4} \arctan(t))\}4(1+t^2), \\ f_2(t) &= \{\tfrac{1}{\sqrt{2}} \sin(\tfrac{3\pi}{8} + \tfrac{1}{4} \arctan(t)) - \tfrac{1}{\sqrt{2}} \cos(\tfrac{3\pi}{8} + \tfrac{1}{4} \arctan(t))\}4(1+t^2), \\ f_3(t) &= 4(1+t^2). \end{aligned}$$

Moreover, in order to obtain new examples of complete locally conformally flat manifolds of nonpositive curvature by using warped and multiply warped product metrics, there are some facts to be considered:

- Clearly any space of constant sectional curvature is locally conformally flat, and thus it follows from Theorem 7.2.1-(iii) the existence of complete warped products $\mathbb{R} \times_f F$ of constant negative sectional curvature. However it follows from the compatibility conditions at Theorem 7.2.1 the nonexistence of complete multiply warped manifolds $I \times_{f_1} F_1 \times_{f_2} F_2$ of constant curvature.

Remark 7.2.9 The following conditions, which are obtained in a similar way as in [13] by proceeding from the expressions in (7.3), guarantee that a multiply warped product has nonpositive sectional curvature:

(a) Any fiber F_i with dimension $d_i \geq 2$ is of nonpositive sectional curvature.

(b) Warping functions are convex (i.e., f_i'' is non negative).

(c) All the warping functions are increasing or decreasing functions, $f_i' \geq 0$ or $f_i' \leq 0\ \forall i$.

Moreover, conditions (a)–(c) are necessary if the base is complete (cf. [13]). Thus (a)–(c) are equivalent conditions to nonpositive sectional curvature in a complete multiply warped product of type (7.2).

- New examples of complete locally conformally flat manifolds of nonpositive curvature can now be constructed by using a multiply warped structure. For example, consider the multiply warped product

$$\mathbb{R} \times_{f_1} \mathbb{R} \times_{f_2} \mathbb{H}^2$$

with warping functions:

$$f_1(t) = \left(\sin(\tfrac{\pi}{4} + \tfrac{1}{2}\arctan(t)) + \cos(\tfrac{\pi}{4} + \tfrac{1}{2}\arctan(t))\right) 2(1+t^2)$$
$$f_2(t) = 2(1+t^2).$$

It follows from Theorem 7.2.6 and Remark 7.2.9 that this is a complete locally conformally flat manifold with nonpositive sectional curvature.

Alternatively, note that another way of checking the nonpositiveness of the sectional curvature in the above example (and, in general, for any multiply warped product with base of constant sectional curvature) consists in testing the sectional curvature with respect to each pair of vector fields in an orthogonal frame adapted to the product structure. This follows from the expressions of the curvature in Section 1.5.1, which guarantee that the sign of all sectional curvatures is nonpositive if it is nonpositive for pairs of vectors in such an orthogonal frame.

7.3 Locally conformally flat multiply warped products with base of dimension $s \geq 2$: local structure

The fibers of any locally conformally flat multiply warped space have constant curvature; however, such necessary condition does not suffice for the local conformal flatness of the product space, which strongly depends on the action of the warping functions. The purpose of this section is to obtain a local description of such warping functions. As a consequence, we will show the existence of some limitations on the number of fibers of a locally conformally flat multiply warped space and also on their geometries. We begin by considering the situation where the base (B, g_B) has constant sectional curvature, in such a case necessary and sufficient conditions for local conformal flatness are given by the following result.

Theorem 7.3.1 *Let $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ be a multiply warped product with base of dimension $s \geq 2$ and constant sectional curvature. Then M is locally conformally flat if and only if the warping functions satisfy*

$$H_{f_i} = \frac{\Delta f_i}{s} g_B, \tag{7.18}$$

$$\frac{\Delta f_i}{f_i} + \frac{\Delta f_j}{f_j} = s\frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} - sK^B, \qquad i \neq j. \tag{7.19}$$

$$K^{F_i} = \|\nabla f_i\|^2 - \frac{2}{s} f_i \Delta f_i - f_i^2 K^B, \qquad \textit{whenever } \ d_i \geq 2. \tag{7.20}$$

Proof. First of all, note that condition (7.18) is equivalent to the constancy of the sectional curvature of the base of a locally conformally flat multiply warped space. Since (B, g_B) is locally conformally flat and $(B, \frac{1}{f_i^2} g_B)$ is a space of constant sectional curvature by Lemma 7.1.3, we have that (B, g_B) is of constant sectional curvature if and only if the conformal deformation $\frac{1}{f_i^2} g_B \mapsto g_B$ preserves the unique eigenspace of the Ricci tensor, which occurs if and only if f is a solution of the Möbius equation; hence (7.18) follows (cf. [114], [144]).

In order to show that (7.19) and (7.20) hold, we note that if $\mathcal{M}$ is locally conformally flat, then it follows from Remark 7.1.4 that the warped product space $B \times_{f_i} F_i$ is also locally conformally flat, and thus the associated Weyl tensors vanish, for all $i = 1, \ldots, k$. We use the equation $H_{f_i} = \frac{\Delta f_i}{s} g_B$ to compute:

$$\begin{aligned}
W(X,Y,X,Y) &= R^B(X,Y,X,Y) + \tfrac{\varepsilon_X \varepsilon_Y \tau}{(s+d_i-1)(s+d_i-2)} \\
&\quad - \tfrac{1}{s+d_i-2} \{\varepsilon_Y \rho(X,X) + \varepsilon_X \rho(Y,Y)\} \\
&= \varepsilon_X \varepsilon_Y K^B + \tfrac{\varepsilon_X \varepsilon_Y}{(s+d_i-1)(s+d_i-2)} \Big\{ s(s-1)K^B + \tfrac{d_i(d_i-1)K^{F_i}}{f_i^2} \\
&\quad - 2d_i \tfrac{\Delta f_i}{f_i} - d_i(d_i-1)\tfrac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} \Big\} - \tfrac{2\varepsilon_X \varepsilon_Y}{s+d_i-2} \Big\{ (s-1)K^B - \tfrac{d_i}{s}\tfrac{\Delta f_i}{f_i} \Big\}
\end{aligned}$$

reordering we get

$$\begin{aligned}
&= \varepsilon_X \varepsilon_Y \Big(K^B + \tfrac{s(s-1)}{(s+d_i-1)(s+d_i-2)} K^B + \tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} \tfrac{K^{F_i}}{f_i^2} \\
&\quad - \tfrac{2d_i}{(s+d_i-1)(s+d_i-2)} \tfrac{\Delta f_i}{f_i} - \tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} \tfrac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} \\
&\quad - \tfrac{2(s-1)}{s+d_i-2} K^B + \tfrac{2d_i}{(s+d_i-2)s} \tfrac{\Delta f_i}{f_i} \Big) \\
&= \varepsilon_X \varepsilon_Y \Big(K^B \Big\{ 1 + \tfrac{s(s-1)}{(s+d_i-1)(s+d_i-2)} - \tfrac{2(s-1)}{s+d_i-2} \Big\} \\
&\quad + \tfrac{\Delta f_i}{f_i} \Big\{ \tfrac{2d_i}{(s+d_i-2)s} - \tfrac{2d_i}{(s+d_i-1)(s+d_i-2)} \Big\} \\
&\quad + \tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} \tfrac{K^{F_i}}{f_i^2} - \tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} \tfrac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} \Big)
\end{aligned}$$

and simplifying

$$\begin{aligned}
&= \varepsilon_X\varepsilon_Y \left(\tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} K^B + \tfrac{2d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)s} \tfrac{\Delta f_i}{f_i} \right.\\
&\qquad \left. + \tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} \tfrac{K^{F_i}}{f_i^2} - \tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} \tfrac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} \right)\\
&= \varepsilon_X\varepsilon_Y \left(\tfrac{d_i(d_i-1)}{(s+d_i-1)(s+d_i-2)} \left\{ K^B + \tfrac{2}{s}\tfrac{\Delta f_i}{f_i} + \tfrac{K^{F_i}}{f_i^2} - \tfrac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} \right\} \right)
\end{aligned}$$

for all orthogonal unitary vector fields $X, Y \in \mathfrak{X}(B)$; hence equations (7.20) hold. We proceed in an analogous way to show the necessity of (7.19); we now consider the multiply warped space $B \times_{f_i} F_i \times_{f_j} F_j$, which is also locally conformally flat for all $i \neq j \in \{1, \ldots, k\}$. We use Lemma 7.1.1 and also (7.20) which was previously obtained; thus, we compute

$$\begin{aligned}
W(X,Y,X,Y) &= R^B(X,Y,X,Y) + \tfrac{\varepsilon_X\varepsilon_Y\,\tau}{(s+d_i+d_j-1)(s+d_i+d_j-2)}\\
&\qquad - \tfrac{1}{s+d_i+d_j-2} \{\varepsilon_Y \rho(X,X) + \varepsilon_X \rho(Y,Y)\}\\
&= \varepsilon_X\varepsilon_Y\, K^B\\
&\quad + \tfrac{\varepsilon_X\varepsilon_Y}{(s+d_i+d_j-1)(s+d_i+d_j-2)} \Big\{ \{s(s-1) - d_i(d_i-1) - d_j(d_j-1)\} K^B\\
&\qquad - \tfrac{2d_i(s+d_i-1)}{s}\tfrac{\Delta f_i}{f_i} - \tfrac{2d_j(s+d_j-1)}{s}\tfrac{\Delta f_j}{f_j} - 2d_i d_j \tfrac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} \Big\}\\
&\qquad - \tfrac{2\varepsilon_X\varepsilon_Y}{s+d_i+d_j-2} \left\{ (s-1)K^B - \tfrac{d_i}{s}\tfrac{\Delta f_i}{f_i} - \tfrac{d_j}{s}\tfrac{\Delta f_j}{f_j} \right\}
\end{aligned}$$

ordering things slightly

$$\begin{aligned}
&= \varepsilon_X\varepsilon_Y \left(K^B + \tfrac{s(s-1)-d_i(d_i-1)-d_j(d_j-1)}{(s+d_i+d_j-1)(s+d_i+d_j-2)} K^B \right.\\
&\qquad - \tfrac{2d_i(s+d_i-1)}{(s+d_i+d_j-1)(s+d_i+d_j-2)s}\tfrac{\Delta f_i}{f_i} - \tfrac{2d_j(s+d_j-1)}{(s+d_i+d_j-1)(s+d_i+d_j-2)s}\tfrac{\Delta f_j}{f_j}\\
&\qquad - \tfrac{2d_i d_j}{(s+d_i+d_j-1)(s+d_i+d_j-2)}\tfrac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} - \tfrac{2(s-1)}{s+d_i+d_j-2} K^B\\
&\qquad \left. + \tfrac{2d_i}{(s+d_i+d_j-2)s}\tfrac{\Delta f_i}{f_i} + \tfrac{2d_j}{(s+d_i+d_j-2)s}\tfrac{\Delta f_j}{f_j} \right)\\
&= \varepsilon_X\varepsilon_Y \left(K^B \left\{ 1 + \tfrac{s(s-1)-d_i(d_i-1)-d_j(d_j-1)}{(s+d_i+d_j-1)(s+d_i+d_j-2)} - \tfrac{2(s-1)}{s+d_i+d_j-2} \right\} \right.\\
&\qquad + \tfrac{\Delta f_i}{f_i} \left\{ \tfrac{2d_i}{(s+d_i+d_j-2)s} - \tfrac{2d_i(s+d_i-1)}{(s+d_i+d_j-1)(s+d_i+d_j-2)s} \right\}\\
&\qquad + \tfrac{\Delta f_j}{f_j} \left\{ \tfrac{2d_j}{(s+d_i+d_j-2)s} - \tfrac{2d_j(s+d_j-1)}{(s+d_i+d_j-1)(s+d_i+d_j-2)s} \right\}\\
&\qquad \left. - \tfrac{2d_i d_j}{(s+d_i+d_j-1)(s+d_i+d_j-2)}\tfrac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} \right)
\end{aligned}$$

simplifying

$$
\begin{aligned}
&= \varepsilon_X\varepsilon_Y\left(\tfrac{2d_id_j}{(s+d_i+d_j-1)(s+d_i+d_j-2)}K^B + \tfrac{2d_id_j}{(s+d_i+d_j-1)(s+d_i+d_j-2)s}\tfrac{\Delta f_i}{f_i}\right.\\
&\qquad\left.+\tfrac{2d_id_j}{(s+d_i+d_j-1)(s+d_i+d_j-2)s}\tfrac{\Delta f_j}{f_j} - \tfrac{2d_id_j}{(s+d_i+d_j-1)(s+d_i+d_j-2)}\tfrac{\langle\nabla f_i,\nabla f_j\rangle}{f_if_j}\right)\\
&= \varepsilon_X\varepsilon_Y\left(\tfrac{2d_id_j}{(s+d_i+d_j-1)(s+d_i+d_j-2)}\left\{K^B + \tfrac{1}{s}\tfrac{\Delta f_i}{f_i} + \tfrac{1}{s}\tfrac{\Delta f_j}{f_j} - \tfrac{\langle\nabla f_i,\nabla f_j\rangle}{f_if_j}\right\}\right)
\end{aligned}
$$

for all orthogonal unitary vectors $X, Y \in \mathfrak{X}(B)$. Hence (7.19) follows.

Next we will show that conditions (7.18)–(7.20) are indeed sufficient for $\mathcal{M}$ to be locally conformally flat. First of all note that the "a priori" nonzero components of the Weyl tensor in a local orthonormal frame $\{X, Y, \dots, U_1, V_1, \dots, U_i, V_i, \dots\}$ with $X, Y, \dots \in \mathfrak{X}(B)$ and $U_i, V_i, \dots \in \mathfrak{X}(F_i)$ are those given by $W(X,Y,X,Y)$, $W(X,U_i,X,U_i)$, $W(U_i,U_j,U_i,U_j)$ and $W(U_i,V_i,U_i,V_i)$. Now, a long but straightforward calculation from expressions in Lemma 7.1.1, using that the warping functions satisfy $H_{f_i} = \frac{\Delta f_i}{s}g_B$, shows that

$$
\begin{aligned}
W(X,Y,X,Y) &= \varepsilon_X\varepsilon_Y\left(\textstyle\sum_i \tfrac{d_i(d_i-1)}{(n-1)(n-2)}\left\{K^B - \tfrac{\langle\nabla f_i,\nabla f_i\rangle}{f_i^2} + \tfrac{K^{F_i}}{f_i^2} + \tfrac{2\Delta f_i}{sf_i}\right\}\right.\\
&\qquad\left.+\textstyle\sum_i\sum_{j\neq i}\tfrac{d_id_j}{(n-1)(n-2)}\left\{K^B - \tfrac{\langle\nabla f_i,\nabla f_j\rangle}{f_if_j} + \tfrac{\Delta f_i}{sf_i} + \tfrac{\Delta f_j}{sf_j}\right\}\right),
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(B)$. Also, for $X \in \mathfrak{X}(B)$ and $U_a \in \mathfrak{X}(F_a)$, one has

$$
\begin{aligned}
W(X,U_a,X,U_a) &= \varepsilon_X\varepsilon_{U_a}\left(\textstyle\sum_i \tfrac{d_i(d_i-1)}{(n-1)(n-2)}\left\{K^B - \tfrac{\langle\nabla f_i,\nabla f_i\rangle}{f_i^2} + \tfrac{K^{F_i}}{f_i^2} + \tfrac{2\Delta f_i}{sf_i}\right\}\right.\\
&\qquad+\textstyle\sum_i\sum_{j\neq i}\tfrac{d_id_j}{(n-1)(n-2)}\left\{K^B - \tfrac{\langle\nabla f_i,\nabla f_j\rangle}{f_if_j} + \tfrac{\Delta f_i}{sf_i} + \tfrac{\Delta f_j}{sf_j}\right\}\\
&\qquad+\textstyle\sum_{i\neq a}\tfrac{d_i}{n-2}\left\{\tfrac{\langle\nabla f_a,\nabla f_i\rangle}{f_af_i} - \tfrac{\Delta f_a}{sf_a} - \tfrac{\Delta f_i}{sf_i} - K^B\right\}\\
&\qquad\left.+\tfrac{d_a-1}{n-2}\left\{\tfrac{\langle\nabla f_a,\nabla f_a\rangle}{f_a^2} - \tfrac{K^{F_a}}{f_a^2} - \tfrac{2\Delta f_a}{sf_a} - K^B\right\}\right).
\end{aligned}
$$

Next, put $U_a \in \mathfrak{X}(F_a)$ and $U_b \in \mathfrak{X}(F_b)$, $(a \neq b)$ to get

$$
\begin{aligned}
W(U_a,U_b,U_a,U_b) = \varepsilon_{U_a}\varepsilon_{U_b} \Big(& \textstyle\sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)} \Big\{ K^B - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i} \Big\} \\
& + \textstyle\sum_i \sum_{j \neq i} \frac{d_i d_j}{(n-1)(n-2)} \Big\{ K^B - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j} \Big\} \\
& + \textstyle\sum_{i \neq a} \frac{d_i}{n-2} \Big\{ \frac{\langle \nabla f_a, \nabla f_i \rangle}{f_a f_i} - \frac{\Delta f_a}{s f_a} - \frac{\Delta f_i}{s f_i} - K^B \Big\} \\
& + \textstyle\sum_{i \neq b} \frac{d_i}{n-2} \Big\{ \frac{\langle \nabla f_b, \nabla f_i \rangle}{f_b f_i} - \frac{\Delta f_b}{s f_b} - \frac{\Delta f_i}{s f_i} - K^B \Big\} \\
& + \frac{d_a-1}{n-2} \Big\{ \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{s f_a} - K^B \Big\} \\
& + \frac{d_b-1}{n-2} \Big\{ \frac{\langle \nabla f_b, \nabla f_b \rangle}{f_b^2} - \frac{K^{F_b}}{f_b^2} - \frac{2\Delta f_b}{s f_b} - K^B \Big\} \\
& + \Big\{ K^B - \frac{\langle \nabla f_a, \nabla f_b \rangle}{f_a f_b} + \frac{\Delta f_a}{s f_a} + \frac{\Delta f_b}{s f_b} \Big\} \Big),
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
W(U_a,V_a,U_a,V_a) = \varepsilon_{U_a}\varepsilon_{V_a} \Big(& \textstyle\sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)} \Big\{ K^B - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i} \Big\} \\
& + \textstyle\sum_i \sum_{j \neq i} \frac{d_i d_j}{(n-1)(n-2)} \Big\{ K^B - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j} \Big\} \\
& + \textstyle\sum_{i \neq a} \frac{2d_i}{n-2} \Big\{ \frac{\langle \nabla f_a, \nabla f_i \rangle}{f_a f_i} - \frac{\Delta f_a}{s f_a} - \frac{\Delta f_i}{s f_i} - K^B \Big\} \\
& + \frac{2(d_a-1)}{n-2} \Big\{ \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{s f_a} - K^B \Big\} \\
& + \Big\{ K^B - \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} + \frac{K^{F_a}}{f_a^2} + \frac{2\Delta f_a}{s f_a} \Big\} \Big),
\end{aligned}
$$

for all $U_a, V_a \in \mathfrak{X}(F_a)$.

Now, it follows from the expressions above that, once we have assumed (7.18), the compatibility conditions (7.19) and (7.20) suffice to show the local conformal flatness of the multiply warped space $\mathcal{M}$. □

Considering the special case of B being an open subset of the Euclidean space, Theorem 7.3.1 reads as follows.

Theorem 7.3.2 *Let $\mathcal{M} = \mathfrak{U}^s \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ be a multiply warped space with $\mathfrak{U}^s \subset \mathbb{R}^s_\nu$, $s \geq 2$. Then M is locally conformally flat if and only if the warping functions satisfy*

$$f_i(\overrightarrow{\mathbf{x}}) = a_i \|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}_i, \overrightarrow{\mathbf{x}} \rangle + c_i, \tag{7.21}$$

for all $\overrightarrow{\mathbf{x}} \in \mathfrak{U}^s$, where $a_i > 0$, $c_i \in \mathbb{R}$ and $\overrightarrow{\mathbf{b}}_i \in \mathbb{R}^s$. Moreover the warping functions are compatible in the sense that

$$\langle \overrightarrow{\mathbf{b}}_i, \overrightarrow{\mathbf{b}}_j \rangle = 2(a_i c_j + a_j c_i), \quad \textit{for any } i \neq j, \tag{7.22}$$

and the sectional curvature of each fiber F_i with $dimF_i \geq 2$ is given by

$$K^{F_i} = \|\overrightarrow{\mathbf{b}}_i\|^2 - 4\,a_i c_i, \quad i,j = 1,\ldots,k. \tag{7.23}$$

Proof. It follows from [144] that the solutions of the Möbius equation in the Euclidean space are given by $f_i(\overrightarrow{\mathbf{x}}) = a_i\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}_i, \overrightarrow{\mathbf{x}}\rangle + c_i$ for some $a_i, c_i \in \mathbb{R}$, $\overrightarrow{\mathbf{b}}_i \in \mathbb{R}^s$. Then (7.22) and (7.23) follow from (7.19) and (7.20), respectively, by using the expression given in (7.21) for the warping functions. □

As an application of Theorem 7.3.2, the following Remark gives the local structure of locally conformally flat multiply warped products.

Remark 7.3.3 Note from Lemma 7.1.3 that the base of an arbitrary locally conformally flat warped product is necessarily locally conformally flat. Hence, there exist local coordinates such that $g_B = \Psi^2 g_{\mathfrak{U}^s}$. Observe that, when using such coordinates, the multiply warped metric may be written as

$$g_B \oplus f_1^2 g_1 \oplus \ldots \oplus f_k^2 g_k = \Psi^2 \left(g_{\mathfrak{U}^s} \oplus (\frac{f_1}{\Psi})^2 g_1 \oplus \ldots \oplus (\frac{f_k}{\Psi})^2 g_k \right).$$

Therefore the multiply warped product $g_B \oplus f_1^2 g_1 \oplus \ldots \oplus f_k^2 g_k$ is locally conformally flat if and only if so is $g_{\mathfrak{U}^s} \oplus (\frac{f_1}{\Psi})^2 g_1 \oplus \ldots \oplus (\frac{f_k}{\Psi})^2 g_k$. Hence the warping functions are locally determined by Theorem 7.3.1 up to a conformal factor Ψ, since the warping functions are given by $f_i(\overrightarrow{\mathbf{x}}) = (a_i\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}_i, \overrightarrow{\mathbf{x}}\rangle + c_i)\Psi$ for all $i = 1,\ldots,k$, in local coordinates where $g_B = \Psi^2 g_{\mathfrak{U}^s}$.

We summarize in the next Remark the main consequences one gets from Theorem 7.3.2 and from Remark 7.3.3.

Remark 7.3.4 Since any warping function of a locally conformally flat multiply warped space $\mathcal{M} = \mathfrak{U}^s \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ is completely determined by scalars $a_i, c_i \in \mathbb{R}$ and vectors $\overrightarrow{\mathbf{b}}_i = (b_{i1}, \ldots, b_{is}) \in \mathbb{R}^s$, consider the vectors $\overrightarrow{\xi}_i = (b_{i1}, \ldots, b_{is}, a_i, c_i)$ in $\mathbb{R}^{s+2}$. Next, define an inner product of signature $(\nu + 1,\, s - \nu + 1)$ in $\mathbb{R}^{s+2}$ by

$$\left(\begin{array}{cccccc|cc}
-1 & & & & & & & \\
& \overset{\nu}{\ddots} & & & & & & \\
& & -1 & & & & & \\
& & & 1 & & & & \\
& & & & \overset{s-\nu}{\ddots} & & & \\
& & & & & 1 & & \\
\hline
& & & & & & 0 & -2 \\
& & & & & & -2 & 0
\end{array}\right)$$

and note that equations (7.22) and (7.23) at Theorem 7.3.2 are interpreted in terms of the orthogonality $\overrightarrow{\xi}_i \perp \overrightarrow{\xi}_j$ (for all $i \neq j$) and $K^{F_i} = \|\overrightarrow{\xi}_i\|^2$ (whenever $d_i \geq 2$), respectively. Hence the following follows from Remark 7.3.3.

- Let $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ be a locally conformally flat multiply warped product with (B, g^B) of signature $(\nu, s-\nu)$. Then $\mathcal{M}$ has at most $s+2$ different fibers.

  Furthermore, the sectional curvatures of the fibers (F_i, g_i) with $d_i \geq 2$ have the following restrictions:

  (1) there are at most $\nu + 1$ fibers of negative curvature,

  (2) there are at most $s - \nu + 1$ fibers of positive curvature.

- For any locally conformally flat manifold (B^s, g_B), there exist $(s+2)$ locally defined warping functions $f_i : \mathfrak{U} \subset B \to \mathbb{R}^+$ and (F_i, g_i) spaces of constant curvature such that $\mathcal{M} = \mathfrak{U} \times_{f_1} F_1 \times \ldots \times_{f_{s+2}} F_{s+2}$ is locally conformally flat.

  The following simple example shows that the bound in the number of fibers is sharp. Let $U^s \times_{f_1} \mathbb{R} \times \cdots \times_{f_{s+2}} \mathbb{R}$ be a multiply warped product where the base is the space $U^s = \{\overrightarrow{\mathbf{x}} = (x_1, \ldots, x_s) \in \mathbb{R}^s : x_i > 1, \forall i = 1, \ldots, s\}$ and the warping functions are given by (we denote by $\leftrightarrow$ the correspondence between each function and its representation in $\mathbb{R}^{s+2}$)

$$\begin{array}{rclcrcl}
\overrightarrow{\xi}_1 & = & (1,0,\ldots,0,0,0) & \leftrightarrow & f_1(\overrightarrow{\mathbf{x}}) & = & x_1, \\
\overrightarrow{\xi}_2 & = & (0,1,\ldots,0,0,0) & \leftrightarrow & f_2(\overrightarrow{\mathbf{x}}) & = & x_2, \\
\vdots & = & \vdots & \leftrightarrow & \vdots & = & \vdots \\
\overrightarrow{\xi}_s & = & (0,0,\ldots,1,0,0) & \leftrightarrow & f_s(\overrightarrow{\mathbf{x}}) & = & x_s, \\
\overrightarrow{\xi}_{s+1} & = & (0,0,\ldots,0,1,1) & \leftrightarrow & f_{s+1}(\overrightarrow{\mathbf{x}}) & = & \|\overrightarrow{\mathbf{x}}\|^2 + 1, \\
\overrightarrow{\xi}_{s+2} & = & (0,0,\ldots,0,1,-1) & \leftrightarrow & f_{s+2}(\overrightarrow{\mathbf{x}}) & = & \|\overrightarrow{\mathbf{x}}\|^2 - 1.
\end{array}$$

7.4 Locally conformally flat multiply warped products with base of dimension $s \geq 2$: global structure

In previous sections of this chapter we have developed a thorough analysis of the local geometry of locally conformally flat multiply warped products. We devote this section to the study of some global aspects, with special attention to Riemannian multiply warped products.

Remark 7.4.1 Let (B, g_B) be a geodesically complete strictly pseudo-Riemannian manifold of constant sectional curvature. Then, if $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ is locally conformally flat, it reduces to a product $\mathcal{M} = B \times F$, where F is 1-dimensional or a space of constant curvature opposite to that of B. Indeed, for such a manifold, any warping function f_i must satisfy (7.18), i.e. $H_{f_i} = \frac{\Delta f_i}{s} g_B$ and thus along any null geodesic $\gamma(t)$ one has

$$\frac{d^2}{dt^2}(f_i(\gamma(t)) = \frac{d}{dt} g_B(\nabla f_i, \gamma') = g_B\left(\frac{\Delta f_i}{s}\gamma', \gamma'\right) = \frac{\Delta f_i}{s} g_B\left(\gamma', \gamma'\right) \equiv 0,$$

which shows that $f_i(\gamma(t))$ is linear on t and thus f_i cannot be positive if B is null geodesically complete unless it is constant.

The existence of nontrivial globally defined solutions of (7.18) on the constant curvature model spaces has significant geometrical consequences, as it is shown in [114]. These ideas lead to the following result.

Theorem 7.4.2 *Let $\mathcal{M} = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a locally conformally flat multiply warped product space with complete and simply connected Riemannian base (B, g_B) of constant sectional curvature. Then, if $k \geq 2$, B is isometric to the hyperbolic space $\mathbb{H}^s$. Moreover, for each $k \leq s+2$ there exist locally conformally flat multiply warped spaces $\mathcal{M} = \mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$.*

Proof. First of all, note that $B \times_{f_i} F_i$ is locally conformally flat, as it is shown in Remark 7.1.4. Then we apply Theorem 6.1.7 to obtain the form of the possible warping functions. Now, the way we proceed is showing that a locally conformally flat multiply warped space whose base is the Euclidean space or the sphere reduces to a warped product. Therefore, an analysis of the curvature of the induced metrics $(B, \frac{1}{f^2} g_B)$ is needed.

Since $\mathcal{M}$ is locally conformally flat, so is $\mathcal{M}_{ij} = B \times_{f_i} F_i \times_{f_j} F_j$, whose metric tensor can be expressed as $g_{M_{ij}} = f_j^2 (\frac{1}{f_j^2} g_B \oplus \frac{1}{f_j^2} f_i^2 g_i \oplus g_j)$. Hence $M_{\widehat{j}\,i} = B \times F_i$ equipped with the metric $\frac{1}{f_j^2} g_B \oplus \frac{1}{f_j^2} f_i^2 g_i$ has constant sectional curvature $K^{M_{\widehat{j}\,i}}$. Since $M_{\widehat{j}\,i}$ can be viewed as a warped product, it follows from Lemma 1.5.3 that

$$K^{M_{\widehat{j}\,i}}_{XU} = -\frac{f_j^3}{f_i} \widehat{H}_{f_i/f_j}(X, X)$$

for all unit vectors $X \in \mathfrak{X}(B)$, $U \in \mathfrak{X}(F_i)$, where $\widehat{H}_{f_i/f_j}$ denotes the Hessian of $\frac{f_i}{f_j}$ with respect to the conformal metric $\frac{1}{f_j^2} g_B$. Now, since (cf. [74])

$$\widehat{H}_{f_i/f_j} = \frac{1}{f_j}\{H_{f_i} - \frac{f_i}{f_j} H_{f_j} - \frac{1}{f_j} g_B(\nabla f_j, \nabla f_i) g_B + \frac{f_i}{f_j^2} g_B(\nabla f_j, \nabla f_j) g_B\},$$

one gets

$$-K^{M_{\widehat{j}\,i}} \frac{f_i}{f_j} g_B = f_j H_{f_i} - f_i H_{f_j} - g_B(\nabla f_j, \nabla f_i) g_B + \frac{f_i}{f_j} g_B(\nabla f_j, \nabla f_j) g_B. \tag{7.24}$$

Proceeding in an analogous way and expressing the metric tensor of $\mathcal{M}_{ij} = B \times_{f_i} F_i \times_{f_j} F_j$ as $g_{M_{ij}} = f_i^2 (\frac{1}{f_i^2} g_B \oplus \frac{1}{f_i^2} f_j^2 g_j \oplus g_i)$, one also has

$$-K^{M_{\widehat{i}\,j}} \frac{f_j}{f_i} g_B = f_i H_{f_j} - f_j H_{f_i} - g_B(\nabla f_i, \nabla f_j) g_B + \frac{f_j}{f_i} g_B(\nabla f_i, \nabla f_i) g_B. \tag{7.25}$$

Now it follows from (7.24) and (7.25) that

$$-K^{M_{\widehat{j}\,i}} f_i^2 - K^{M_{\widehat{i}\,j}} f_j^2 = \| f_j \nabla f_i - f_i \nabla f_j \|^2. \tag{7.26}$$

As an immediate application of the previous relation we have that *if the multiply warped space* $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ *is locally conformally flat, then the (constant) sectional curvature of* $(B, \frac{1}{f_i^2} g_B)$ *cannot be nonnegative for two different warping functions.* Indeed, if $\|f_j \nabla f_i - f_i \nabla f_j\|^2 = 0$, then $\nabla \ln(\frac{f_i}{f_j}) = 0$, from where it follows that f_i is a multiple of f_j, which is not possible. This shows the nonexistence of nontrivial locally conformally flat multiply warped metrics with base the Euclidean space or the sphere.

Finally, in order to show the existence of complete locally conformally flat multiply warped products with base $\mathbb{H}^s$ and the maximum number of fibers, just consider the following set of functions

$$\begin{aligned}
\overline{f}_1(\overrightarrow{\mathbf{x}}) &= \tfrac{s+4}{4}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + \cdots + x_{s-1} + (s+2)x_s + s + 1, \\
\overline{f}_2(\overrightarrow{\mathbf{x}}) &= \tfrac{s+4}{4}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + \cdots + x_{s-1} + s x_s + s - 1, \\
\overline{f}_3(\overrightarrow{\mathbf{x}}) &= \|\overrightarrow{\mathbf{x}}\|^2 + 3x_s + 2, \\
\overline{f}_4(\overrightarrow{\mathbf{x}}) &= \tfrac{1}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_{s-1} + 2x_s + 2, \\
\overline{f}_5(\overrightarrow{\mathbf{x}}) &= \tfrac{1}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_{s-2} + 2x_s + 2, \\
&\vdots \\
\overline{f}_{s+2}(\overrightarrow{\mathbf{x}}) &= \tfrac{1}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 2x_s + 2.
\end{aligned}$$

Note that all the functions above are positive on the hyperbolic space and, furthermore, they satisfy the compatibility conditions in Theorem 7.3.2. Hence, proceeding as in Remark 7.3.3, one has that $f_i(\overrightarrow{\mathbf{x}}) = \frac{\overline{f}_i(\overrightarrow{\mathbf{x}})}{x_s}$ are positive warping functions on $\mathbb{H}^s$. In this way one defines a locally conformally flat multiply warped space either for 1-dimensional fibers or higher dimensional fibers of suitable constant curvature as in Remark 7.3.4. □

Theorem 7.4.3 *If* $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ *is a Riemannian locally conformally flat multiply warped product with compact base* B*, then* $k = 1$.

Proof. Indeed, let f_i, f_j be two distinct warping functions. Then, proceeding as in Lemma 7.1.3, $(B, \frac{1}{f_i^2} g_B)$ and $(B, \frac{1}{f_j^2} g_B)$ are of constant sectional curvature. Since $\frac{f_i}{f_j}$ is not constant it follows that $(B, \frac{1}{f_i^2} g_B)$ and $(B, \frac{1}{f_j^2} g_B)$ are conformal metrics of constant curvature, and thus Euclidean spheres [114]. Hence, proceeding as in the proof of Theorem 7.4.2, one obtains that there is only one admissible fiber. □

Chapter 8

Applications

8.1 Warped products of constant sectional curvature

This section is devoted to the study of warped and multiply warped product spaces with constant sectional curvature. Note that a pseudo-Riemannian manifold of dimension greater than 4 is a space of constant sectional curvature if and only if it is Einstein and locally conformally flat. Since we have already studied local conformal flatness in Chapters 6 and 7, the results we obtain in the present section are those corresponding to impose the additional condition of being Einstein.

Let s be the dimension of the base. We shall study the cases $s = 1$ and $s \geq 2$ separately. The case of 1-dimensional base has been already considered in Section 7.2.1, hence from now on we restrict to the case of s-dimensional base with $s \geq 2$. Warped products are going to be treated as a particular case of multiply warped ones. We assume all along this chapter that each warping function is neither a constant nor a constant multiple of any other warping function.

8.1.1 Multiply warped products with base of dimension $s \geq 2$

In this section we describe the local structure of multiply warped product manifolds with base of dimension greater than one which are spaces of constant curvature. We determine the warping functions as well as the corresponding restrictions on the geometry of the fibers. First of all, recall from Lemma 7.1.2 that the sectional curvature of a multiply warped product $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$, depending on the chosen plane, has the following expressions:

$$
\begin{array}{rcl}
K_{XY} & = & K^B_{XY}, \\
K_{XU_i} & = & -\frac{H_{f_i}(X,X)}{f_i\langle X,X\rangle}, \\
K_{U_iV_i} & = & \frac{1}{f_i^2}K^{F_i}_{U_iV_i} - \frac{\langle \nabla f_i,\nabla f_i\rangle}{f_i^2}, \\
K_{U_iU_j} & = & -\frac{\langle \nabla f_i,\nabla f_j\rangle}{f_if_j}, \qquad i\neq j,
\end{array}
\tag{8.1}
$$

where, as usual, we use notation $X, Y \in \mathfrak{X}(B)$ and U_i, $V_i \in \mathfrak{X}(F_i)$. Also, recall that the non-vanishing components of the Ricci tensor are given by

$$
\begin{array}{l}
\rho(X,Y) = \rho^B(X,Y) - \sum_i d_i \frac{H_{f_i}(X,Y)}{f_i}, \\
\rho(U_i,V_i) = \rho^{F_i}(U_i,V_i) - \langle U_i,V_i\rangle \left\{ \frac{\Delta f_i}{f_i} + (d_i-1)\frac{\langle \nabla f_i,\nabla f_i\rangle}{f_i^2} + \sum_{j\neq i} d_j \frac{\langle \nabla f_i,\nabla f_j\rangle}{f_if_j} \right\},
\end{array}
\tag{8.2}
$$

where $d_i = dim\, F_i$.

Remark 8.1.1 Note from (8.1) that if $\mathcal{M}$ is a space of constant curvature κ, then each fiber F_i must also have constant sectional curvature and, moreover, the base B has constant sectional curvature equal to κ. Therefore, throughout this section we always assume that the base and the fibers of the multiply warped space are of constant sectional curvature.

Now, using Theorem 7.3.1, the following result is obtained.

Theorem 8.1.2 *Let $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ be a multiply warped product with $B, F_1, \ldots, F_k$ of constant sectional curvature, $s = dim\, B \geq 2$ and $K^B = \kappa$. Then $\mathcal{M}$ is a space of constant curvature ($K = \kappa$) if and only if the following conditions hold:*

(i) $H_{f_i} = \frac{\Delta f_i}{d} g_B$,

(ii) $\Delta f_i = -s\,\kappa f_i$,

(iii) $\langle \nabla f_i, \nabla f_j\rangle = -\kappa f_i f_j, \quad i \neq j$,

(iv) $\langle \nabla f_i, \nabla f_i\rangle = -\kappa f_i^2 + K^{F_i}$, *whenever* $d_i \geq 2$,

where $i, j = 1, \ldots, k$.

Proof. Assume $\mathcal{M}$ has constant sectional curvature κ. Then $(i) - (iv)$ follow directly from (8.1). Conversely, if we assume that conditions $(i)-(iv)$ hold, then one easily checks that $\mathcal{M}$ is locally conformally flat, since (7.18) – (7.20) in Theorem 7.3.1 hold. But from (8.2) one also obtains $\mathcal{M}$ is Einstein. Hence $\mathcal{M}$ is of constant sectional curvature. □

We have seen in Remark 8.1.1 that, for a multiply warped product $\mathcal{M}$ of constant curvature κ, the base B must also have constant curvature κ. Thus, in analyzing the local structure of $\mathcal{M}$, we may consider an open subset $\mathfrak{U} \subset \mathbb{R}^s_\nu$ endowed with the metric

$$g_\kappa(x_1,\ldots,x_s) = \frac{1}{\left(1+\frac{\kappa}{4}(-x_1^2-\cdots-x_\nu^2+x_{\nu+1}^2\cdots+x_s^2)\right)^2}\begin{pmatrix} -1 & & & & & \\ & \ddots^{\,\nu} & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots^{\,s-\nu} & \\ & & & & & 1 \end{pmatrix}, \tag{8.3}$$

where $(x_1,\ldots,x_s)$ denote the usual coordinates (see [171, Th. 2.4.11]). Note that the metric above is a conformal deformation of the pseudo-Euclidean metric g_0 of signature $(\nu, s-\nu)$ on $\mathbb{R}^s_\nu$, $g_\kappa = \frac{1}{\phi^2} g_0$, where the conformal factor is given by

$$\phi(x_1,\ldots,x_s) = 1 + \frac{\kappa}{4}(-x_1^2-\cdots-x_\nu^2+x_{\nu+1}^2+\cdots+x_s^2). \tag{8.4}$$

Now we use these considerations to obtain the following result.

Theorem 8.1.3 *Let $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ be a multiply warped product with $B, F_1, \ldots, F_k$ of constant sectional curvature, $s = \dim B \geq 2$ and $K^B = \kappa$. Then $\mathcal{M}$ is a space of constant sectional curvature ($K = \kappa$) if and only if the warping functions have locally the expression*

$$f_i(\overrightarrow{\mathbf{x}}) = \frac{\frac{-\kappa\, c_i}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle + \langle \overrightarrow{\mathbf{b_i}}, \overrightarrow{\mathbf{x}}\rangle + c_i}{1+\frac{\kappa}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle}, \quad c_i \in \mathbb{R}, \quad \overrightarrow{\mathbf{b_i}} \in \mathbb{R}^s,$$

with

(i) $\langle \overrightarrow{\mathbf{b_i}}, \overrightarrow{\mathbf{b_j}}\rangle + \kappa\, c_i\, c_j = 0, \quad i \neq j,$

(ii) $\langle \overrightarrow{\mathbf{b_i}}, \overrightarrow{\mathbf{b_i}}\rangle + \kappa\, c_i^2 = K^{F_i}, \quad$ *if* $\dim F_i \geq 2$,

for all $i, j = 1, \ldots, k$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in the pseudo-Euclidean space $\mathbb{R}^s_\nu$.

Proof. Since the base B is a space of constant curvature κ, we can assume that $g_B = g_\kappa$ (given by (8.3)) on an open subset $\mathfrak{U} \subset \mathbb{R}^s$. In what follows we use a two-step process: first, we determine the solutions of the Möbius equation $H_u = \frac{\Delta u}{s} g_\kappa$ in $(\mathfrak{U}, g_\kappa)$ and, secondly, we compute the Laplacian and the gradient of such solutions with respect to g_κ to use Theorem 8.1.2.

As it is well-known, solutions of $H_u = \frac{\Delta u}{s} g_0$ in $(\mathfrak{U}, g_0)$ are those of the following form:

$$\hat{f}(\overrightarrow{\mathbf{x}}) = a\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}}\rangle + c, \tag{8.5}$$

for all $a, c \in \mathbb{R}$ and $\overrightarrow{\mathbf{b}} \in \mathbb{R}^s$. As a consequence, we determine locally the scaling functions f_i (they must be solutions of the Möbius equation in $(\mathfrak{U}, g_\kappa)$), which are of the form

$$f_i(\overrightarrow{\mathbf{x}}) = \frac{a_i\langle\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle + \langle\overrightarrow{\mathbf{b_i}}, \overrightarrow{\mathbf{x}}\rangle + c_i}{1 + \frac{\kappa}{4}\langle\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle}, \tag{8.6}$$

for all $a_i, c_i \in \mathbb{R}$ and $\overrightarrow{\mathbf{b_i}} \in \mathbb{R}^s$, $i = 1, \ldots, k$. Now, using [74, Lemma 6.1.1], a long but straightforward calculation lets us compute the Laplacian and the gradient with respect to g_κ of a solution (8.6), which are given by

$$\begin{aligned}
\Delta^{g_\kappa} f_i &= -s\kappa f_i + 2\,s\left(a_i + \tfrac{\kappa c_i}{4}\right), \\
g_\kappa(\nabla^{g_\kappa} f_i, \nabla^{g_\kappa} f_j) &= -\kappa f_i f_j + \langle\overrightarrow{\mathbf{b_i}}, \overrightarrow{\mathbf{b_j}}\rangle + \kappa c_i c_j \\
&\quad +2\left(a_i + \frac{\kappa c_i}{4}\right)\frac{2a_j\langle\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle + \langle\overrightarrow{\mathbf{b_j}}, \overrightarrow{\mathbf{x}}\rangle}{1 + \frac{\kappa}{4}\langle\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle} \\
&\quad -2\left(a_j + \frac{\kappa c_j}{4}\right)\frac{\frac{\kappa c_i}{2}\langle\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle - \langle\overrightarrow{\mathbf{b_i}}, \overrightarrow{\mathbf{x}}\rangle}{1 + \frac{\kappa}{4}\langle\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}}\rangle},
\end{aligned}$$

where $i, j = 1, \ldots, k$. Then, from Theorem 8.1.2 we have

$$\begin{aligned}
&\Delta^{g_\kappa} f_i = -s\kappa f_i, \\
&g_\kappa\left(\nabla^{g_\kappa} f_i, \nabla^{g_\kappa} f_j\right) = -\kappa f_i f_j, \quad i \neq j, \\
&g_\kappa\left(\nabla^{g_\kappa} f_i, \nabla^{g_\kappa} f_i\right) = -\kappa f_i^2 + K^{F_i}, \quad \text{if } dim\, F_i \geq 2,
\end{aligned}$$

for $i, j = 1, \ldots, k$, and hence $a_i + \frac{\kappa c_i}{4} = 0$ $(i = 1, \ldots, k)$, from where the result is obtained.□

Remark 8.1.4 The local description of multiply warped product spaces of constant curvature in the previous theorem leads to some restrictions on the number and the geometry of the fibers. Consider vectors $\overrightarrow{\xi_{\mathbf{i}}} = (\overrightarrow{\mathbf{b_i}}, c_i) = (b_{i1}, \ldots, b_{ic}, c_i)$ in $\mathbb{R}^{s+1}$ endowed with the symmetric bilinear form given by

$$\left(\begin{array}{cccccc|c}
-1 & & & & & & \\
 & \ddots^{\,\nu} & & & & & \\
 & & -1 & & & & \\
 & & & 1 & & & \\
 & & & & \ddots^{\,s-\nu} & & \\
 & & & & & 1 & \\
\hline
 & & & & & & \kappa
\end{array}\right). \tag{8.7}$$

Then, conditions (i) and (ii) in Theorem 8.1.3 mean that the vectors $\overrightarrow{\xi_{\mathbf{i}}}$, $i = 1, \ldots, k$, must be orthogonal to each other and, if $dim\, F_i \geq 2$, then the associated vector $\overrightarrow{\xi_{\mathbf{i}}}$ must satisfy $\langle\overrightarrow{\xi_{\mathbf{i}}}, \overrightarrow{\xi_{\mathbf{i}}}\rangle = K^{F_i}$. Moreover, it follows that (8.7) is a scalar product in $\mathbb{R}^{s+1}$ of index $(-1, \overset{\nu}{\ldots}, -1, 1, \overset{s-\nu}{\ldots}, 1, \frac{\kappa}{|\kappa|})$ if $\kappa \neq 0$ and $(-1, \overset{\nu}{\ldots}, -1, 1, \overset{s-\nu}{\ldots}, 1, 0)$ if $\kappa = 0$. Now, if $\kappa = 0$, Theorem 8.1.3 implies that the scaling functions are polynomials of degree one, $f_i(\overrightarrow{\mathbf{x}}) = \langle \overrightarrow{\mathbf{b_i}}, \overrightarrow{\mathbf{x}} \rangle + c_i$, and the corresponding vectors $\overrightarrow{\xi_\mathbf{i}}$ are orthogonal if and only if so are the vectors $\overrightarrow{\mathbf{b_i}}$. This shows that no more than s fibers may exist if $\kappa = 0$. (Recall that warping functions are assumed to be nonconstant, and thus the case $\overrightarrow{\xi} = (\overrightarrow{\mathbf{0}}, c)$ is excluded). Therefore, proceeding as in Remark 7.3.4, we conclude:

1. A multiply warped product $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ of constant sectional curvature κ, with a $(s \geq 2)$-dimensional base has, at most, s fibers if $\kappa = 0$, and at most $s + 1$ if $\kappa \neq 0$.

2. The sectional curvatures of the fibers F_i of a multiply warped product of constant curvature and with base of signature $(\nu, s - \nu)$ are as follows:

   (a) If $\kappa > 0$, then no more than $s - \nu + 1$ fibers of dimension ≥ 2 and positive sectional curvature are admissible and no more than ν fibers of dimension ≥ 2 and negative sectional curvature may occur.

   (b) If $\kappa < 0$, then no more than $s - \nu$ fibers of dimension ≥ 2 and positive sectional curvature are admissible and no more than ν fibers of dimension ≥ 2 and negative sectional curvature may occur.

   (c) If $\kappa = 0$, then no more than $s - \nu$ fibers of dimension ≥ 2 and positive sectional curvature are admissible and no more than $\nu + 1$ fibers of dimension ≥ 2 and negative sectional curvature may occur.

3. For any $(s \geq 2)$-dimensional base (B, g_B) of constant curvature κ, there exist $s + 1$ (if $\kappa \neq 0$) or s (if $\kappa = 0$) locally defined scaling functions $f_i : \mathfrak{U} \subset B \to \mathbb{R}^+$ and (F_i, g_{F_i}) spaces of constant curvature so that $M = \mathfrak{U} \times_{f_1} M_1 \times \cdots \times_{f_\nu} M_\nu$ has constant sectional curvature.

8.2 Complete locally conformally flat Riemannian manifolds of nonpositive curvature

In this section we are going to make use of the tools developed in previous chapters to work out some examples of complete Riemannian manifolds with nonpositive curvature. Recall that complete locally conformally flat manifolds of nonnegative Ricci curvature are well understood, but there is a lack of information concerning negative curvature. The examples we present in this section are collected in [34].

Pseudo-Riemannian manifolds with indefinite metric and with the sectional curvature bounded from above or from below for all non-degenerate planes (respectively, on the Ricci curvature) are necessarily of constant curvature [118] (respectively, Einstein [55]). Henceforth we restrict to the Riemannian setting, so we will use results obtained in previous chapters in a more specific context.

8.2.1 Warped products of nonpositive sectional curvature

As we have mentioned previously, warped products are a basic tool in constructing manifolds of nonpositive curvature, which are complete if and only if both factors are complete Riemannian manifolds [13].

Let $\mathcal{M} = B \times_f F$ be a locally conformally flat Riemannian warped product with base B a model space $\mathbb{R}^s$, $\mathbb{S}^s$ or $\mathbb{H}^s$. If $\mathcal{M}$ is assumed to be complete, we have the following consequences of results in Chapter 6 and expressions (8.1):

(i) If $B \equiv \mathbb{R}^s$, then $\mathbb{R}^s \times_f F$ has nonpositive sectional curvature for any warping function f as in Theorem 6.1.7.

(ii) If $B \equiv \mathbb{S}^s$, then no locally conformally flat warped product $\mathbb{S}^s \times_f F$ may be of nonpositive sectional curvature.

Furthermore, *there is no locally conformally flat warped product with compact base and nonpositive sectional curvature, unless it is a direct product.* Indeed, it follows from (8.1) that the warping function of a warped product $B \times_f F$ of nonpositive sectional curvature satisfies $H_f \geq 0$, and thus $\Delta f \geq 0$. Now, since B is compact without boundary and $\dim B \geq 2$, it follows that $\Delta f = 0$, and thus f is constant.

(iii) If $B \equiv \mathbb{H}^s$, then a necessary and sufficient condition for a locally conformally flat warped product $\mathbb{H}^s \times_f F$ to be of nonpositive sectional curvature is given in terms of the warping function $f(\overrightarrow{\mathbf{x}}) = \frac{a\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{x}} \rangle + c}{x_s}$ by $f \geq 2b_s$, whenever $\dim F \geq 2$.

We finish with some simple examples which illustrate the previous situation.

- Let $\mathcal{M}$ be the product manifold $M = \mathbb{H}^2 \times \mathbb{H}^2$ equipped with the warped metric defined by the warping function

$$f(\overrightarrow{\mathbf{x}}) = \frac{\frac{1}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_2 + 1}{x_2}.$$

- Let $\mathcal{M}$ be the product manifold $M = \mathbb{H}^2 \times \mathbb{R}^2$ equipped with the warped metric defined by the warping function

$$f(\overrightarrow{\mathbf{x}}) = \frac{\frac{1}{4}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 1}{x_2}.$$

After some straightforward calculations we get that the sectional curvature is nonpositive in any of these particular examples.

Proceeding in the same way as in [13] one gets that a multiply warped product manifold $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ is complete if and only if the base and all the fibers are complete. Moreover, in such a case, the sectional curvature is nonpositive if and only if the following three conditions hold:

(a) The sectional curvatures of the base and the fibers are nonpositive, i.e. $K^B \leq 0$ and $K^{F_i} \leq 0$.

(b) The warping functions are convex, i.e., H_{f_i} is positive semi-definite.

(c) We have that $\langle \nabla f_i, \nabla f_j \rangle \geq 0$, for all $i \neq j$.

Note that condition (a) may be omitted whenever the base or the corresponding fiber are 1-dimensional.

Let $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ be a complete locally conformally flat multiply warped space with $k \geq 2$ and with simply connected base of constant curvature. Then $\mathcal{M}$ is of nonpositive sectional curvature if and only if $B \equiv \mathbb{H}^s$ and the warping functions $f_i(\overrightarrow{\mathbf{x}}) = \frac{a_i \|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}_i}, \overrightarrow{\mathbf{x}} \rangle + c_i}{x_s}$ satisfy

(1) $f_i \geq 2\, b_{is}$ (where b_{is} is the s-th component of b_i), whenever $d_i \geq 2$, and

(2) $1 \geq \frac{b_{is}}{f_i} + \frac{b_{js}}{f_j}$ for all $i \neq j$.

Here are a couple of simple examples illustrating the previous situation.

(a) Let $\mathcal{M}$ be the product manifold $M = \mathbb{H}^2 \times F_1^d \times F_2$ equipped with the multiply warped metric tensor defined by the warping functions

$$f_1(\overrightarrow{\mathbf{x}}) = \kappa \frac{\frac{1}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_2 + 1}{x_2}, \qquad f_2(\overrightarrow{\mathbf{x}}) = \frac{\frac{1}{4}\|\overrightarrow{\mathbf{x}}\|^2 + x_2 + \frac{1}{2}}{x_2},$$

where F_2 is 1-dimensional and F_1 is either 1-dimensional or of negative sectional curvature $K^{F_1} = -\kappa^2$.

(b) Let $\mathcal{M}$ be the product manifold $M = \mathbb{H}^2 \times {F_1}^d \times F_2$ equipped with the multiply warped metric tensor defined by the warping functions

$$f_1(\overrightarrow{\mathbf{x}}) = \frac{\frac{1}{4}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 1}{x_2}, \qquad f_2(\overrightarrow{\mathbf{x}}) = \frac{\frac{1}{2}\|\overrightarrow{\mathbf{x}}\|^2 + 2x_1 + x_2 + 2}{x_2},$$

where F_2 is 1-dimensional and F_1 is either 1-dimensional or flat.

Furthermore, if $\mathcal{M} = B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ is an n-dimensional locally conformally flat multiply warped space with base of constant sectional curvature, then it follows from (7.18)–(7.20) that $\mathcal{M}$ has at most $k+1$ different Ricci curvatures given by

$$\begin{aligned} \lambda_B &= (s-1)K^B - \tfrac{1}{s}\textstyle\sum_i d_i \frac{\Delta f_i}{f_i}, \\ \lambda_{F_a} &= (s-1)K^B - \tfrac{1}{s}\textstyle\sum_i d_i \frac{\Delta f_i}{f_i} - (n-2)\left(K^B + \tfrac{1}{s}\frac{\Delta f_a}{f_a}\right). \end{aligned} \tag{8.8}$$

Now a straightforward calculation shows that examples (a) above have exactly three different Ricci curvatures but only two different Ricci curvatures occur in case (b).

Remark 8.2.1 Observe that the base and the fibers of a multiply warped product play completely different roles. For instance, if $\mathcal{M}$ is a warped product with compact base and nonpositive sectional curvature, then it follows from (8.1) that the warping function satisfies $H_f \geq 0$, and thus f is constant, which shows that $\mathcal{M}$ must be a direct product. In opposition, one can easily construct examples of locally conformally flat multiply warped spaces of nonpositive sectional curvature with compact fibers.

In addition to examples (b) above, the metrics in Theorem 6.2.3–2 can also be viewed as multiply warped metrics with 1-dimensional base. A straightforward calculation shows that $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N \times_{\frac{\alpha}{\beta}e^{\beta t+\gamma}+c} F$ has nonpositive sectional curvature if and only if F is 1-dimensional. Also note that both N and F can be chosen to be compact. Moreover, $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N \times_{\frac{\alpha}{\beta}e^{\beta t+\gamma}+c} F$ has three distinct Ricci curvatures and thus it is not isometric to example (b) above where only two distinct Ricci curvatures occur.

8.2.2 Warped products of nonpositive Ricci curvature

The Ricci tensor of a locally conformally flat manifold completely determines its curvature tensor. Nonetheless, nonpositive curvature and nonpositive Ricci curvature are not equivalent conditions. We now construct examples with nonpositive Ricci curvature but not necessarily nonpositive sectional curvature.

The following examples are complete locally conformally flat Riemannian manifolds with nonpositive Ricci operator, and are built using only one fiber, this is, they are warped products.

- $\mathbb{H}^2 \times_f \mathbb{H}^2$ with warping function
$$f(\overrightarrow{\mathbf{x}}) = \frac{\frac{3}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 4x_2 + 3}{x_2}.$$
The eigenvalues (λ_1, λ_2) of the Ricci operator have multiplicities $(2, 2)$ and are always negative:
$$\lambda_1 = -\frac{9\|\overrightarrow{\mathbf{x}}\|^2 + 6x_1 + 8x_2 + 18}{2x_2 f(\overrightarrow{\mathbf{x}})}, \qquad \lambda_2 = -\frac{9\|\overrightarrow{\mathbf{x}}\|^2 + 6x_1 - 8x_2 + 18}{2x_2 f(\overrightarrow{\mathbf{x}})}.$$
- $\mathbb{H}^2 \times_f \mathbb{S}^2$ with warping function
$$f(\overrightarrow{\mathbf{x}}) = \frac{\|\overrightarrow{\mathbf{x}}\|^2 + 3x_2 + 2}{x_2}.$$
Again, the eigenvalues (λ_1, λ_2) of the Ricci operator have multiplicities $(2, 2)$ and are negative:
$$\lambda_1 = -\frac{3(\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 2}{x_2 f(\overrightarrow{\mathbf{x}})}, \qquad \lambda_2 = -\frac{3(\|\overrightarrow{\mathbf{x}}\|^2 - x_1 + 2}{x_2 f(\overrightarrow{\mathbf{x}})}.$$

Moreover, note that the sectional curvature of $\mathbb{H}^2 \times_f \mathbb{S}^2$ has no sign, since

$$K_{\partial_{y_1}\partial_{y_2}} = -\frac{\|\overrightarrow{\mathbf{x}}\|^2 - 3x_2 + 2}{\|\overrightarrow{\mathbf{x}}\|^2 + 3x_2 + 2},$$

which changes the sign depending on the point under consideration.

Remark 8.2.2 As an immediate application of (8.8), an n-dimensional locally conformally flat multiply warped space $\mathcal{M} = \mathbb{H}^s \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$, $(s \geq 2)$ has nonpositive Ricci curvature if and only if the warping functions $f_i(\overrightarrow{\mathbf{x}}) = \frac{a_i\|\overrightarrow{\mathbf{x}}\|^2 + \langle \overrightarrow{\mathbf{b}_i}, \overrightarrow{\mathbf{x}}\rangle + c_i}{x_s}$ satisfy

(i) $\sum_i d_i \frac{b_{is}}{f_i} \leq n-1$, for all $i = 1, \ldots, k$, and

(ii) $(n-2)\frac{b_{is}}{f_i} + \sum_j d_j \frac{b_{js}}{f_j} \leq n-1$, for all $i \neq j \in \{1, \ldots, k\}$.

Finally, note that simple examples of complete locally conformally flat manifolds with nonpositive Ricci curvature are

$$\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2 \times_{f_3} \mathbb{S}^2 \times_{f_4} \mathbb{H}^2$$

with warping functions

$$f_1(\overrightarrow{\mathbf{x}}) = \frac{\frac{3}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 4x_2 + 3}{x_2}, \qquad f_2(\overrightarrow{\mathbf{x}}) = \frac{\|\overrightarrow{\mathbf{x}}\|^2 + 3x_2 + 2}{x_2},$$

$$f_3(\overrightarrow{\mathbf{x}}) = \frac{\frac{1}{2}\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 2x_2 + 2}{x_2}, \qquad f_4(\overrightarrow{\mathbf{x}}) = \frac{\|\overrightarrow{\mathbf{x}}\|^2 + x_1 + 2x_2 + 1}{x_2}.$$

Moreover notice that the same conclusions hold for the spaces $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2 \times_{f_3} \mathbb{S}^2$, $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2$ and the warped product $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2$. Also note from (7.26) that if the manifold $\mathbb{H}^s \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ is a locally conformally flat space of nonpositive sectional curvature then there is at most one fiber F_a with dim $F_a \geq 2$. Furthermore F_a must be of nonpositive sectional curvature; this shows that none of the examples above has nonpositive sectional curvature.

8.3 Locally conformally flat cosmological models

In the standard Friedmann-Robertson-Walker (FRW) cosmology, the isotropy of the space-time is reflected on a warped product structure with a fiber of constant sectional curvature, but no restrictions are needed on the warping function to ensure that the resulting space-time is locally conformally flat. On the other hand multidimensional cosmological models have attracted a lot of attention during the last years by constructing mathematical models of an anisotropic universe (see, for example, [48], [66], [105], [106] and [108]). The prototype of such cosmological models is the Kasner metric, where each restspace hypersurface is a flat 3-dimensional space [101]. The underlying structure of those models is given by a multiply warped product of several spaces where the warping functions depend only on time.

Since many physical/mathematical properties of the spacetime are invariant under conformal transformations and local conformal flatness is a characteristic of Friedmann-Robertson-Walker cosmological models, our purpose in this section is to apply results in Chapter 7 and in Section 8.1 to discuss some generalizations of FRW spacetimes with special attention to that property. Moreover the mathematical consideration of multidimensional cosmological models as a basic structure for constructing locally conformally flat solutions of the Einstein equations is motivated by the fact that the Schouten tensor of any locally conformally flat manifold is a Codazzi tensor. The Einstein equations show that the eigenvalue structure of the stress-energy tensor determines the eigenvalue structure of the Schouten tensor, which leads to warped and multiply warped product decompositions of the spacetime, the mathematical structure behind multidimensional models, in many cases (see for example [14] and [164] for more information on Codazzi tensors).

A curvature/physical characterization of local conformal flatness in terms of the Jacobi operators corresponding to light directions can be given as follows. For a null vector $u \in T_pM$, define the nondegenerate normal space $\overline{u}^{\perp} = u^{\perp}/\mathrm{Span}\,\{u\}$, where $u^{\perp}$ is the orthogonal space to $\mathrm{Span}\,\{u\}$, and define the induced inner product on $\overline{u}^{\perp}$ by $\overline{g}(\overline{x}, \overline{y}) = g(x, y)$, where $x, y \in u^{\perp}$ with $\pi(x) = \overline{x}$, $\pi(y) = \overline{y}$ and $\pi : u^{\perp} \mapsto \overline{u}^{\perp}$ is the canonical projection. The Jacobi operator $\overline{R}_u : \overline{u}^{\perp} \mapsto \overline{u}^{\perp}$ is given by $\overline{R}_u\overline{x} = \pi\,(R(u, x)u)$, where $x \in u^{\perp}$ with $\pi(x) = \overline{x}$ and R is the curvature tensor. A null vector $u \in T_pM$ is called isotropic if $\overline{R}_u = c_u\overline{\mathrm{id}}$, where $c_u \in \mathbb{R}$ and (M, g) is called *null isotropic* if every null vector field is isotropic. Physically, null isotropy corresponds to the situation of an observer in a cosmological circumstance, who observes that the density of light is (ideally) locally uniform on his celestial sphere (null isotropy) rather than it is globally uniform on his celestial sphere (which corresponds to infinitesimal null isotropy as discussed in [99], [111]). Now, it follows from [73, Thm. 3.2] that a Lorentzian manifold is locally conformally flat if and only if it is null isotropic. Thus local conformal flatness represents an intermediate condition between isotropy and anisotropy of the spacetime.

This section is based on results from [29] and [32].

8.3.1 Some considerations on spacetimes and models

Among anisotropic cosmological models, the Kasner metric represents one of the simplest solutions of the Einstein equations. Such solution is defined on a manifold of the form $\mathcal{M} = \mathbb{R} \times F_1 \times F_2 \times F_3$, where each F_i is either $\mathbb{R}$ or $\mathbb{S}^1$. Natural generalizations of Kasner solution have been extensively studied during last years (see for example [104], [106], [123] and [125]). Similarly, most of multidimensional cosmological models are based on a manifold

$$\mathcal{M} = \mathbb{R} \times F_1 \times \cdots \times F_k, \tag{8.9}$$

where the Riemannian manifolds F_i are assumed to be of constant sectional curvature (or more generally, Einstein spaces) and the metric tensor is given by

$$g = -e^{2\gamma(t)}dt^2 + \sum_{i=1}^{k} e^{2\phi^i(t)} g_i, \tag{8.10}$$

where g_i is the metric tensor of the factor F_i, of dimension d_i. Thus, in what follows we restrict to multiply warped spaces with 1-dimensional base whose fibers are all Riemannian manifolds. Therefore, we consider a Lorentzian manifold $\mathcal{M}$ with the underlying structure of a multiply warped product space of the form

$$\mathcal{M} = I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k \qquad (I \subset \mathbb{R}), \tag{8.11}$$

with metric tensor

$$g = -dt^2 + f_1^2 g_1 + \cdots + f_k^2 g_k, \tag{8.12}$$

where $g_1, \ldots, g_k$ are Riemannian metrics on $F_1, \ldots, F_k$, respectively.

Remark 8.3.1 The following consequences are obtained from Theorems 7.2.1 and 7.2.6:

- Restrictions on the number of fibers:
  - *No more than two fibers are admissible for a space $I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ to be of constant sectional curvature,*
  - *No more than three fibers are admissible for a space $I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ to be locally conformally flat.*
- Restrictions on the curvature of the fibers:

  If $\mathcal{M} = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ is locally conformally flat, then the fibers have constant sectional curvature and no more than one d-dimensional fiber with $d \geq 2$ may be of nonnegative curvature.
- Restrictions on the warping functions:

  If $\mathcal{M} = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ is locally conformally flat, then only one warping function is free of constraints. The other two warping functions are explicitly given, up to a reparametrization of time.

We have just described a multidimensional generalization of Friedmann-Robertson-Walker models with the underlying structure of a multiply warped product (with 1-dimensional base) $I \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$. Observe that the generalized Friedmann-Robertson-Walker spacetimes obey the Einstein equations for a stress-energy tensor which is not necessarily a perfect fluid.

Now, we exhibit some specific examples of locally conformally flat cosmological models whose underlying structure corresponds to a multiply warped product metric.

Bianchi type-I homogeneous spacetimes

Bianchi type-I universes are the simplest anisotropic cosmological models. They generalize the Kasner universe and the Heckmann-Schucking solution (cf. [100], [109], [123]). Recall that the general form of a Bianchi type-I metric is as follows

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2. \tag{8.13}$$

Then Theorems 7.2.1 and 7.2.6 show that, after some reparametrization of time, a metric (8.13) is locally conformally flat if and only if one of the following holds

$$\begin{aligned} a(t) &= (a_1 f(t) + b_1)\tfrac{1}{f'(t)}, \\ b(t) &= (a_2 f(t) + b_2)\tfrac{1}{f'(t)}, \qquad \text{with } a_1 a_2 = 0, \\ c(t) &= \tfrac{1}{f'(t)}, \end{aligned} \tag{8.14}$$

$$\begin{aligned} a(t) &= (a_1 \sinh f(t) + b_1 \cosh f(t))\tfrac{1}{f'(t)}, \\ b(t) &= (a_2 \sinh f(t) + b_2 \cosh f(t))\tfrac{1}{f'(t)}, \qquad \text{with } a_1 a_2 - b_1 b_2 = 0, \\ c(t) &= \tfrac{1}{f'(t)}, \end{aligned} \tag{8.15}$$

$$\begin{aligned} a(t) &= (a_1 \sin f(t) + b_1 \cos f(t))\tfrac{1}{f'(t)}, \\ b(t) &= (a_2 \sin f(t) + b_2 \cos f(t))\tfrac{1}{f'(t)}, \qquad \text{with } a_1 a_2 + b_1 b_2 = 0, \\ c(t) &= \tfrac{1}{f'(t)}. \end{aligned} \tag{8.16}$$

Multidimensional perfect-fluid type solutions

Multidimensional solutions of the Einstein equations for a perfect fluid energy-momentum tensor have been considered in [104], [125]. The starting point in those discussions is a metric (8.10) on a manifold (8.9), where the fibers F_i are Ricci flat Riemannian manifolds. Now, if the manifold is locally conformally flat, as a consequence of Theorem 7.2.5 any such d-dimensional fiber with $d \geq 2$ is of constant curvature and hence flat. Moreover, as a consequence of Remark 8.3.1, no more than three fibers may occur and moreover only one d-dimensional fiber (with $d \geq 2$) is admissible.

Brane cosmologies with anisotropic bulk

Recall that the bulk metric has the form

$$ds^2 = -e^{2A_0(t,w)} dt^2 + \sum_{i=1}^{3} e^{2A_i(t,w)} (dx^i)^2 + dw^2, \tag{8.17}$$

where the x^i coordinates span the three spatial dimensions and w is the coordinate of the extra dimension. Clearly bulk metrics do not correspond to multiply warped products with

1-dimensional base, but they correspond to a multiply warped product metric where t and w provide coordinates on a two dimensional base. Further, assuming that the anisotropy depends only on time and not on the extra dimension, (8.17) becomes [65]

$$ds^2 = e^{2\mathcal{A}(w)} \left[-e^{2\alpha_0(t)}dt^2 + \sum_{i=1}^{3} e^{2\alpha_i(t)}(dx^i)^2 \right] + dw^2, \tag{8.18}$$

which is locally conformally flat if and only if the multiply warped metric

$$ds^2 = -dt^2 + \sum_{i=1}^{3} e^{2(\alpha_i-\alpha_0)(t)}(dx^i)^2 + e^{-2\alpha_0(t)}(e^{-\mathcal{A}(w)}dw)^2$$

is locally conformally flat.

Now, it follows from Theorem 7.2.6 that a metric (8.18) cannot be locally conformally flat unless there is a reduction on the number of different warping functions (note that it corresponds to a multiply warped product with four different fibers) and the corresponding warping functions are obtained from (8.14), (8.15), (8.16).

BTZ black hole models

Many spacetime models can be written in the form of a multiply warped metric. For instance the Schwarzschild metric or the BTZ black hole model which can be described by

$$ds^2 = N^2dt^2 - N^{-2}dr^2 + r^2d\phi^2, \tag{8.19}$$

where $d\phi^2$ is the line element on the sphere and where $N^2 = m - \frac{r^2}{l^2}$. Moreover, by introducing a new coordinate μ, the metric (8.19) becomes [101]

$$ds^2 = -d\mu^2 + f_1(\mu)^2dt^2 + f_2(\mu)^2d\phi^2, \tag{8.20}$$

and a straightforward calculation shows that it is locally conformally flat if and only if

$$\frac{f_1'f_2' - f_2f_1''}{f_1} = \frac{1 + f_2'f_2' - f_2f_2''}{f_2}.$$

8.3.2 Multidimensional cosmological models with higher dimensional external spacetime

Although the large scale of the observable part of our present time Universe is well described by the 4-dimensional Friedmann-Robertson-Walker model, it is possible that spacetime at Planck distances might have some extra dimensions. The multidimensionality of our Universe is one of the most intriguing assumptions in modern physics and a natural ingredient in different theories like string theory and some recent generalizations. Then a process should exist leading from all dimensions on the same scale to the actual stage of the Universe, where we have only four external dimensions and all internal spaces have to be contracted to sufficiently small scales making them unobservable. Hence, it is natural to generalize the FRW model to multidimensional cosmological models (MCMs) with spacetime manifold

$$\mathcal{M} = B \times F_1 \times \cdots \times F_k \tag{8.21}$$

and with decomposed metric

$$g = g_B + \sum_{i=1}^{k} e^{2\phi^i(x)} g_i \tag{8.22}$$

where x are some coordinates on the s-dimensional spacetime B and g_i are the metrics on the internal spaces F_i $(i = 1, \dots, k)$ (see, for example, [93], [94], [95], [96], [97], [98], [148], [149]).

In what follows we restrict ourselves to multiply warped spaces with a higher-dimensional Lorentzian external spacetime B $(dim\, B = b \geq 2)$ whose internal spaces F_i are all Riemannian manifolds.

Remark 8.3.2 We use Remarks 7.3.4 and 8.1.4 to give some restrictions on multidimensional cosmological models of the form $\mathcal{M} = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$, where B is the (Lorentzian) external spacetime ($b \geq 2$), and with Riemannian internal spaces F_i.

- Restrictions on the number of fibers:
  - *No more than $b + 2$ different internal spaces may occur if the MCM is locally conformally flat.*
  - *No more than b different internal spaces may occur if the MCM is flat, and no more than $b + 1$ are allowed if the curvature is constant but nonflat.*
- Restrictions on the curvature of the fibers:

  If $\mathcal{M}$ is locally conformally flat, then all internal spaces of dimension ≥ 2 are necessarily of constant curvature and moreover, no more than two may be of nonpositive curvature.

Remark 8.3.3 It is well known that in multidimensional cosmological models the internal spaces should be compact and small enough to make them unobservable at the present time. Note from previous remark that examples of locally conformally flat models of the form $\mathcal{M} = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ can be constructed in such a way that the internal spaces (F_i, g_i) are compact and of constant curvature. Moreover the size of the compact internal spaces can be specialized to be as small as desired just specializing the scaling functions.

8.3.3 Locally conformally flat MCMs with compact internal spaces

Let $\mathfrak{U}$ be an open subset in the Minkowskian space $\mathbb{L}^b$. Then

$$\mathcal{M} = \mathfrak{U} \times_{f_1} \mathbb{S}^{d_1} \times \cdots \times_{f_b} \mathbb{S}^{d_b} \times_{f_{b+1}} \mathbb{H}^{d_{b+1}} \times_{f_{b+2}} \mathbb{H}^{d_{b+2}}$$

with scaling functions

$$\begin{aligned} f_i(\overrightarrow{\mathbf{x}}) &= x_{i+1}, \qquad i = 1, \ldots, b-1, \\ f_b(\overrightarrow{\mathbf{x}}) &= 1 - \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle, \\ f_{b+1}(\overrightarrow{\mathbf{x}}) &= x_1, \\ f_{b+2}(\overrightarrow{\mathbf{x}}) &= 1 + \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle \end{aligned}$$

is a locally conformally flat MCM.

Observe that the scaling functions do not depend on the particular choice of internal spaces. In fact, the internal spaces influence on the scaling functions relies only on the value of their constant sectional curvature (just consider the scaling functions above multiplied by $\alpha_i > 0$ to obtain internal spaces of constant sectional curvature $\pm\alpha_i^2$). Therefore examples of locally conformally flat models with compact internal spaces and the maximum number of fibers are easily derived from previous expressions by considering spheres, Euclidean tori or compact hyperbolic manifolds as compact model spaces of constant sectional curvature (see [151] for constructions of compact hyperbolic manifolds).

MCMs of constant sectional curvature

MCMs of any constant sectional curvature can be explicitly given as follows. Let $\mathfrak{U}$ be an open subset in $(\mathbb{R}^b, g_\kappa)$ for any κ (see (8.3)). Examples with the maximum number of internal spaces and constant curvature $\kappa = 0$ are given by

$$\mathcal{M} = \mathfrak{U} \times_{f_1} \mathbb{H}^{d_1} \times_{f_2} \mathbb{S}^{d_2} \times \cdots \times_{f_b} \mathbb{S}^{d_b}$$

where the scaling functions are defined by

$$f_i(\overrightarrow{\mathbf{x}}) = x_i, \qquad i = 1, \ldots, b.$$

In case of nonzero constant curvature, $b+1$ internal spaces can be considered as follows:

$$\mathcal{M} = \mathfrak{U} \times_{f_1} \mathbb{H}^{d_1} \times_{f_2} \mathbb{S}^{d_2} \times \cdots \times_{f_b} \mathbb{S}^{d_b} \times_{f_{b+1}} \mathbb{F}^{d_{b+1}},$$

where $\mathbb{F} = \mathbb{S}$ if $\kappa > 0$ or $\mathbb{F} = \mathbb{H}$ if $\kappa < 0$, and with scaling functions given by

$$\begin{aligned} f_i(\overrightarrow{\mathbf{x}}) &= \frac{1}{1+\frac{\kappa}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle} x_i, \qquad i = 1, \ldots, b, \\ f_{b+1}(\overrightarrow{\mathbf{x}}) &= \frac{1}{1+\frac{\kappa}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle} \left(\frac{1}{\sqrt{|\kappa|}} - \frac{\kappa}{4\sqrt{|\kappa|}} \langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle \right). \end{aligned}$$

In an analogous way as it was done for conformal flatness, internal spaces can be taken of arbitrary constant curvature $\pm\alpha_i^2$, by multiplying the corresponding scaling function by $\alpha_i > 0$.

MCMs with FRW external spacetime

Here we construct locally conformally flat MCMs with external spacetime an open subset $\mathfrak{U}$ in a Friedmann-Robertson-Walker spacetime $\mathbb{R} \times_\varphi \mathbb{R}^{b-1}$ or $\mathbb{R} \times_\varphi \mathbb{H}^{b-1}$. In what follows we take coordinates $(t, x_1, \ldots, x_{b-1})$ on $\mathfrak{U}$. If $\mathfrak{U} \subset \mathbb{R} \times_\varphi \mathbb{R}^{b-1}$, then

$$\mathcal{M} = \mathfrak{U} \times_{f_1} \mathbb{S}^{d_1} \times \cdots \times_{f_{b-1}} \mathbb{S}^{d_{b-1}} \times_{f_b} \mathbb{H}^{d_b}$$

where

$$\begin{aligned} f_i(t, \overrightarrow{\mathbf{x}}) &= x_i \varphi, \qquad i = 1, \ldots, b-1, \\ f_b(t, \overrightarrow{\mathbf{x}}) &= (1 + \textstyle\int \varphi^{-1}) \cdot \varphi \end{aligned}$$

is a locally conformally flat MCM. In particular, the MCM

$$\mathcal{M} = \mathfrak{U} \times_{f_1} \mathbb{S}^{d_1}\left(\frac{1}{\alpha_1}\right) \times \cdots \times_{f_{b-1}} \mathbb{S}^{d_{b-1}}\left(\frac{1}{\alpha_{b-1}}\right),$$

with the above scaling functions multiplied by $\alpha_i > 0$, is locally conformally flat with compact internal spaces which can be made as small as desired.

On the other hand if $\mathfrak{U} \subset \mathbb{R} \times_\varphi \mathbb{H}^{b-1}$, then we take

$$\mathcal{M} = \mathfrak{U} \times_{f_1} \mathbb{S}^{d_1} \times \cdots \times_{f_{b-2}} \mathbb{S}^{d_{b-2}} \times_{f_{b-1}} \mathbb{R}^{d_{b-1}}$$

with scaling functions

$$\begin{aligned} f_i(t, \overrightarrow{\mathbf{x}}) &= \tfrac{x_i \varphi}{x_{b-1}}, \qquad i = 1, \ldots, b-2, \\ f_{b-1}(t, \overrightarrow{\mathbf{x}}) &= \tfrac{\varphi}{x_{b-1}}. \end{aligned}$$

This is a locally conformally flat MCM. Once again we can multiply the scaling functions above by $\alpha_i > 0$ to obtain a locally conformally flat MCM

$$\mathcal{M} = \mathfrak{U} \times_{f_1} \mathbb{S}^{d_1}\left(\frac{1}{\alpha_1}\right) \times \cdots \times_{f_{b-2}} \mathbb{S}^{d_{b-2}}\left(\frac{1}{\alpha_{b-2}}\right)$$

with compact internal spaces which can be made as small as desired.

5D solutions of the vacuum Einstein field equations

Next we analyze the local conformal flatness of two families of 5-dimensional metrics which embed 4-dimensional FRW models with flat, spherical or hyperbolic spatial sections. When these metrics satisfy the 5-dimensional vacuum field equations, $\rho_{AB} = 0$, they are suitable manifolds for the space-time-matter theory, which proposes that our Universe is an embedded 4-dimensional surface in a vacuum 5-dimensional manifold. In such a case, since the manifold is Ricci flat, the local conformal flatness implies that the whole curvature tensor vanishes. As well as the study of that case, we will also characterize the local conformal flatness of these families of metrics when the 5-dimensional vacuum field equations are not required.

The Liu-Mashhoon-Wesson metric

Consider the Liu-Mashhoon-Wesson (LMW) metric [157] in the form

$$ds^2_{LMW} = -\frac{a_t^2(t,l)}{\mu^2(t)}dt^2 + dl^2 + a^2(t,l)d\sigma^2_{(K,3)}, \tag{8.23}$$

where a and μ are undetermined functions and $d\sigma^2_{(K,3)}$ is a 3-dimensional metric of a unit sphere, plane or hyperboloid for $K = +1$, 0 or -1, respectively,

$$d\sigma^2_{(K,3)} = d\psi^2 + S_K^2(\psi)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{8.24}$$

where

$$S_K(\psi) = \begin{cases} \sin\psi, & \text{K=+1,} \\ \psi, & \text{K=0,} \\ \sinh\psi, & \text{K=-1.} \end{cases}$$

The LMW metric (8.23) satisfies one component of the vacuum field equations, and in this general form it is locally conformally flat if and only if the functions $a(t,l)$ and $\mu(t)$ satisfy

$$a_t(K + \mu^2 - a_l^2) + a(-\mu\mu' + a_t a_{ll} + a_l a_{tl}) - a^2 a_{tll} = 0.$$

Note that, in this general setting, one can find nontrivial solutions of the above equation. For instance, choosing

$$a(t,l) = e^{t+l}, \qquad \mu(t) = \left(e^{2t+K} - K\right)^{\frac{1}{2}},$$

the corresponding MCM is locally conformally flat and of nonconstant curvature. To ensure that all the vacuum field equations are satisfied, one may take the function $a(t,l)$ to be given by

$$a^2(t,l) = (\mu^2(t) + K)l^2 + 2k(t)l + \frac{k^2(t) + \mathcal{K}}{\mu^2(t) + K},$$

where $\mathcal{K}$ is an integration constant and $\mu(t)$ and $k(t)$ are completely arbitrary functions of time, with the unique constraint of $\mathcal{K} \leq a^2(t,l)(\mu^2(t) + K)$ (cf. [157]). In this particular case, we know that the MCM is locally conformally flat if and only if it is flat, and a straightforward calculation shows that this occurs if and only the integration constant $\mathcal{K}$ vanishes.

The Fukui-Seahra-Wesson metric

Proceeding as in the previous example, the Fukui-Seahra-Wesson (FSW) metric [72],

$$ds^2_{FSW} = -d\tau^2 + \frac{b_\omega^2(\tau,\omega)}{\zeta^2(\omega)}d\omega^2 + b^2(\tau,\omega)d\sigma^2_{(K,3)}, \tag{8.25}$$

is locally conformally flat if and only if the functions b and ζ satisfy

$$b_\omega(-K+\zeta^2-b_\tau^2)+b(-\zeta\zeta'+b_\omega b_{\tau\tau}+b_\tau b_{\tau\omega})-b^2 b_{\tau\tau\omega}=0.$$

Again, in this general setting, one can find nontrivial solutions. For example, if we choose

$$b(\tau,\omega)=e^{\tau+\omega}, \qquad \zeta(\omega)=\left(e^{2\omega+K}+K\right)^{\frac{1}{2}},$$

then the corresponding MCM is locally conformally flat with nonconstant curvature. However, to ensure that the vacuum field equations are satisfied, the function $b(\tau,\omega)$ must have the following form

$$b^2(\tau,\omega)=(\zeta^2(\omega)-K)\tau^2+2\chi(\omega)\tau+\frac{\chi^2(\omega)-\mathcal{K}}{\zeta^2(\omega)-K}.$$

Here $\mathcal{K}$ is an integration constant and $\zeta(\omega)$ and $\chi(\omega)$ are arbitrary functions (see [157]). In this case, again the MCM is locally conformally if and only if it is flat, and this occurs if and only $\mathcal{K}$ vanishes.

Generalized Kasner-AdS spacetimes

A n-dimensional MCM described by the metric

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2 d\sigma^2_{(K,n-2)} \tag{8.26}$$

with $f(r)$ a positive function and $d\sigma^2_{(K,n-2)}$ as in (8.24) but for the $(n-2)$-dimensional case is called a generalized Kasner-AdS spacetime. First, a straightforward calculation shows that (8.26) is locally conformally flat if and only if f satisfies $r^2f''-2rf'+2f=2K$. Metrics (8.26) appear in the study of solutions of the Einstein-Maxwell equation in a higher-dimensional background with a cosmological constant Λ. This equation allows a 3-family of static black hole solutions (parameterized by the constant K), whose gravitational field is given by (8.26) with $f(r)=K-\frac{\Lambda}{3}r^2-Mr^{3-n}+Q^2r^{2(3-n)}$, $n\geq 4$, and where M and Q denote the mass and the charge parameters, respectively (we refer to [69], [126] and the references therein). In such a setting, it can be shown that the MCM is locally conformally flat if and only if $M=Q=0$ (which is always the case for $n=4$), and this occurs if and only if the MCM has constant curvature $\frac{\Lambda}{3}$.

Open problems

The following open problems arise from the discussion in Part II.

- As a consequence of the work of Lokhamp [124], any manifold of dimension $n \geq 3$ admits a complete metric of negative Ricci curvature. Moreover, this is bounded between two negative constants (which depend on the dimension). This result shows one cannot draw conclusions from negative Ricci curvature; however, classifying locally conformally flat manifolds of negative Ricci curvature is desirable. Such a classification would be also interesting under some weaker assumptions as 2 nonpositive curvatures.

- Another direction of interest for future research is the study of constant mean curvature hypersurfaces of locally conformally flat multiply warped products, especially in Lorentzian signature.

- In Riemannian signature, completeness of multiply warped products is characterized in terms of the completeness of the base and the fibers. However, we do not have a similar result if the metric is not definite. Therefore, a systematic analysis of complete multiply warped products in higher signature appears to be an interesting line of investigation. Moreover, in Lorentzian signature this should provide a better understanding of cosmological models.

Part III

Commutativity of operators associated to the curvature tensor: Tsankov manifolds

During last years, there has been a tendency to classify certain manifolds by means of properties of operators associated to the curvature tensor. Among the different projects following this line, Osserman-like problems received special attention. While Osserman conditions mainly concern the eigenvalue structure of curvature operators, not much attention has been paid to the investigation of commutativity properties of such operators. There are, however, some exceptions, for example the Ricci-semi-symmetric condition ($R(x, y) \cdot \rho = 0$, where R acts as a derivation) is equivalent to the commutativity of the skew-symmetric curvature and the Ricci operator. Even more, some commuting properties of geometrical operators have been investigated in connection with submanifolds theory. For instance, submanifolds with flat normal bundle are those with commuting shape operators and curvature-adapted hypersurfaces are those whose normal Jacobi operator commutes with the shape operator. Umbilic hypersurfaces and all hypersurfaces in real space forms are automatically curvature-adapted. For complex and quaternionic space forms, curvature-adapted hypersurfaces are, respectively, Hopf hypersurfaces and hypersurfaces for which the quaternionic distribution is invariant under the action of the shape operator [11].

There are also previous works which are closely related with this part of the memory, such as [143], where the problem of classifying Riemannian manifolds whose Jacobi operators commute has been posted. We use the word 'Tsankov' in this context owing to the seminal paper of Y. Tsankov [167] where hypersurfaces with these sort of properties are studied. Let L denote the second fundamental form of a hypersurface $\mathcal{M}$ in $\mathbb{R}^{n+1}$ and let $\{\lambda_1, ..., \lambda_n\}$ be the eigenvalues of the associated shape operator. The following two theorems inspired many of the results in the following chapters [167]:

A hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$, $n \geq 3$, satisfies the relation

$$\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$$

for all orthonormal $x, y \in T_pM$ and for all $p \in M$ if and only if either $\lambda_1 = ... = \lambda_n$ or $\lambda_1 = ... = \lambda_{n-1} = 0$.

A similar result also relates commutativity properties of the skew-symmetric curvature operator to the underlying geometry of the hypersurfaces [167].

A hypersurface in $\mathcal{M} \subset \mathbb{R}^{n+1}$ satisfies the relation

$$R(x, y)R(z, w) = R(z, w)R(x, y)$$

for all orthonormal vectors $x, y, z, w \in T_pM$ and for all $p \in M$ if and only if $|\lambda_1| = ... = |\lambda_n|$, or $\lambda_1 = ... = \lambda_{n-1} = 0$ and $\lambda_n \neq 0$, or $\lambda_1 = ... = \lambda_{n-2} = 0$, and $\lambda_{n-1} \neq 0$ and $\lambda_n \neq 0$.

Thus, motivated by this paper we began a systematic study of manifolds whose Jacobi operators commute (*Jacobi Tsankov*) and of manifolds whose skew-symmetric curvature operators commute (*skew-Tsankov*). This study is far from being complete; indeed, there are many open questions and more progress is being done at the present. Moreover, new questions emerge and the commutativity of different operators is under the consideration of several authors interested on the field (see, for example [88], [103], [161]).

We devote the first chapter of this part to give a complete characterization of Jacobi Tsankov and orthogonally Jacobi Tsankov manifolds in Riemannian signature. More precisely, we show that a Riemannian manifold is Jacobi Tsankov if and only if it is flat and it is orthogonally Jacobi Tsankov if and only if it has constant sectional curvature, thus solving the problem stated in [143].

Chapter 10 deals with Jacobi Tsankov manifolds in higher signature. A complete classification is obtained in Lorentzian signature, showing that Lorentzian Jacobi Tsankov manifolds are necessarily flat. However, the situation is much more complicated in higher signature and, although a classification is far from an end, we present some interesting results. Some of them deal with the strongest condition $\mathcal{J}(x)\mathcal{J}(y) = 0$ for all x, y which is equivalent, as we will show, to the Jacobi Tsankov one in dimension lower than 14. In this chapter we will construct a 14-dimensional counterexample and study some geometrical realizations with different properties.

Then we turn our attention to skew Tsankov manifolds. In Chapter 11 we identify algebraic curvature tensors which are skew Tsankov in the Riemannian setting. In the remaining of the chapter we give a wide variety of examples of skew Tsankov manifolds, thus suggesting that, even if we have a complete classification at the algebraic level, a classification at the differentiable level is broadly open.

Chapter 9

Riemannian Jacobi Tsankov manifolds

In this chapter we concentrate on the study of commutativity properties for the Jacobi operator in the Riemannian setting. This chapter is based on the results in [35].

Recall that an algebraic model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is said to have *constant sectional curvature* κ if and only if $A = \kappa A_0$ where $A_0(x,y,z,w) = \langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle$. Let J be a Hermitian almost complex structure on V; note that J exists if and only if $dim\, V$ is even. Now define an algebraic curvature tensor A_J in the following way:

$$A_J(x,y,z,w) := \langle Jx,z\rangle\langle Jy,w\rangle - \langle Jy,z\rangle\langle Jx,w\rangle + 2\langle Jx,y\rangle\langle Jz,w\rangle\,. \tag{9.1}$$

These two algebraic curvature tensors will play an important role along the study developed in this Chapter. Let $x \in S(V)$. One computes the Jacobi operator in the direction of x for these algebraic curvature tensors to obtain:

$$\begin{aligned} \mathcal{J}_{A_0}(x)y &= \begin{cases} y & \text{if } y \perp x, \\ 0 & \text{if } y \in \operatorname{Span}\{x\}, \end{cases} \\ \mathcal{J}_{A_J}(x)y &= 3\langle y, Jx\rangle Jx\,. \end{aligned} \tag{9.2}$$

As a matter of notation, we adopt the conventions established in Section 1.4.6. Recall that $\mathcal{J}_x := \mathcal{J}(x)$ and $\mathcal{J}_{xy}z := \frac{1}{2}(\mathcal{A}(x,z)y + \mathcal{A}(y,z)x)$.

9.1 Riemannian Jacobi Tsankov algebraic models

As a first step on the classification of Tsankov manifolds, the following result shows that a Riemannian Jacobi Tsankov model has zero curvature. Although this result is a consequence of a more general result we show in Chapter 10 for arbitrary signature, here we present a specific proof for the Riemannian case.

Theorem 9.1.1 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian model. Then $\mathcal{V}$ is Jacobi Tsankov if and only if $A = 0$.*

Proof. Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian Jacobi Tsankov model. The Jacobi operators $\{\mathcal{J}_x\}_{x\in V}$ form a commuting family of self-adjoint operators. Such a family can be simultaneously diagonalized; there is an orthogonal direct sum decomposition:

$$V = \oplus_\lambda E_\lambda \quad \text{where} \quad \mathcal{J}_x\xi = \lambda(x)\xi \ \forall\ \xi \in E_\lambda,\ \forall\ x \in V\,.$$

Fix a unit vector $\eta_\lambda \in E_\lambda$. Then $\lambda(x) = \langle \mathcal{J}_x\eta_\lambda, \eta_\lambda\rangle$. Consequently the functions $x \to \lambda(x)$ are continuous functions of x.

Choose $\xi \in V$ and decompose $\xi = \sum_\lambda \xi_\lambda$ for $\xi_\lambda \in E_\lambda$. Let

$$\mathcal{O} := \{\xi \in V : \xi_\lambda \neq 0\ \forall\ \lambda\}\,.$$

Then $\mathcal{O}$ is the complement of a finite number of hyperplanes and hence it is a dense open subset of V. Let $\xi \in \mathcal{O}$. One then has:

$$0 = \mathcal{J}_\xi\xi = \sum_\lambda \lambda(\xi)\xi_\lambda\,.$$

Since the $\{\xi_\lambda\}$ are linearly independent, this implies $\lambda(\xi) = 0$ for all λ. As $\lambda(\cdot)$ vanishes on $\mathcal{O}$ which is an open dense subset of V, $\lambda(\cdot)$ vanishes identically. Thus $\mathcal{J}_x = 0$ for all $x \in V$ and $A = 0$. □

9.2 Orthogonally Jacobi Tsankov algebraic models

We devote this section to the classification of orthogonally Jacobi Tsankov algebraic models. As we have seen in the previous section, Jacobi Tsankov models are necessarily of zero curvature; however, note that the orthogonally Jacobi Tsankov condition is weaker and a deeper analysis is needed.

The following remark gives a couple of examples of orthogonally Jacobi Tsankov models.

Remark 9.2.1 Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian model. Consider the algebraic curvature tensors of constant sectional curvature +1, that is, $A = A_0$. Then apply Equation (9.2) for $x, y \in S(V)$ with $x \perp y$ to obtain

$$\begin{aligned} \mathcal{J}_x\mathcal{J}_y z &= \mathcal{J}_x \left\{ \begin{array}{lll} z & \text{if} & z \perp y, \\ 0 & \text{if} & z \in \operatorname{Span}\{y\}, \end{array} \right\} \\ &= \left\{ \begin{array}{lll} z & \text{if} & z \perp x, y, \\ 0 & \text{if} & z \in \operatorname{Span}\{x, y\}\,. \end{array} \right. \end{aligned}$$

This is symmetric in the roles of x and y and thus $\mathcal{J}_x\mathcal{J}_y = \mathcal{J}_y\mathcal{J}_x$, so A_0 is orthogonally Jacobi Tsankov.

Consider now $A = A_J$ for some Hermitian almost complex structure J on V. Then $\langle Jx, Jy\rangle = \langle x, y\rangle = 0$ so

$$\mathcal{J}_x\mathcal{J}_y z = 3\mathcal{J}_x\{\langle Jy, z\rangle Jy\} = 9\langle Jy, z\rangle\langle Jy, Jx\rangle Jx = 0\,.$$

Again, this is symmetric in the roles of x and y so A_J is orthogonally Jacobi Tsankov.

The remainder of Section 9.2 is devoted to the proof of the following Theorem, which shows that examples given in Remark 9.2.1 are the only models which are orthogonally Jacobi Tsankov. Thus we have the following result:

Theorem 9.2.2 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian model. Then $\mathcal{V}$ is orthogonally Jacobi Tsankov if and only if either $\mathcal{V}$ has constant sectional curvature or there is a Hermitian almost complex structure J so that A is a multiple of A_J.*

In order to prove such a result we need several technical lemmas which classify models depending on the rank of the Jacobi operators.

Let

$$r(x) := \operatorname{Rank}\{\mathcal{J}_x\}\,.$$

Since $\mathcal{J}_x x = 0$, $r(x) \le n - 1$ for any $x \in V$.

Lemma 9.2.3 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian orthogonally Jacobi Tsankov model of dimension n. If there exists $x \in V$ with $r(x) = n - 1$, then A has constant sectional curvature.*

Proof. Let $\mathcal{O} := \{x \in V : r(x) = n - 1\}$. We suppose $\mathcal{O}$ is non-empty. We wish to show $\mathcal{O}$ is an open and dense subset of V. Let $x \in \mathcal{O}$. Let $\mathcal{B} := \{e_1, ..., e_{n-1}\}$ be an orthonormal basis for $x^\perp = \operatorname{Range}(\mathcal{J}_x)$. Let $\mathcal{J}_{ij}(z) := \langle \mathcal{J}_z e_i, e_j\rangle$. Set

$$p_{\mathcal{B}}(z) := \det(\mathcal{J}_{ij})(z)\,.$$

Then $p_{\mathcal{B}}$ is polynomial in z and by hypothesis $p_{\mathcal{B}}(x) \neq 0$. Thus

$$\mathcal{O}_{\mathcal{B}} := \{z : p_{\mathcal{B}}(z) \neq 0\}$$

is a non-empty open dense subset of V which contains x. If $y \in \mathcal{O}_{\mathcal{B}}$, then necessarily $r(y) \ge n - 1$. Since $r(y) \le n - 1$, $r(y) = n - 1$. This shows that $\mathcal{O}_{\mathcal{B}} \subset \mathcal{O}$ and $\mathcal{O}_{\mathcal{B}}$ is a neighborhood of x; since x was arbitrary, $\mathcal{O}$ is open. Since $\mathcal{O}$ contains a dense subset, this shows, as desired, that $\mathcal{O}$ is an open dense subset of V.

Since $\mathcal{V}$ has constant sectional curvature if $n = 2$, we suppose $n \ge 3$. Let $x \in \mathcal{O}$ and let $y \in x^\perp$. Then

$$\mathcal{J}_x\mathcal{J}_y x = \mathcal{J}_y\mathcal{J}_x x = 0 \quad \text{so} \quad \langle \mathcal{J}_y x, \mathcal{J}_x z\rangle = 0 \quad \text{for all} \quad z\,.$$

As Range $(\mathcal{J}_x) = x^\perp$, we have $\langle \mathcal{J}_y x, z\rangle = 0$ if $z \perp x$. Thus

$$A(y,x,y,z) = 0 \quad \text{if} \quad x \in \mathcal{O},\ z \perp x,\ y \perp x\,. \tag{9.3}$$

Since $\mathcal{O}$ is dense, Equation (9.3) holds for all $x \in V$. Thus if $\{e_i\}$ is an orthonormal basis for V and if $\{i,j,k\}$ are distinct indices,

$$A(e_j,e_i,e_j,e_k) = 0 \quad \text{for} \quad i,j,k \quad \text{distinct}\,.$$

Suppose that ℓ is a fourth distinct index; this can not happen, of course, if $n = 3$. Polarization yields

$$A(e_j,e_i,e_\ell,e_k) + A(e_\ell,e_i,e_j,e_k) = 0 \quad \text{for} \quad i,j,k,\ell \quad \text{distinct}\,.$$

The previous relation together with the first Bianchi identity and the other curvature identities show that:

$$\begin{aligned} 0 &= A(e_i,e_j,e_k,e_\ell) + A(e_i,e_k,e_\ell,e_j) + A(e_i,e_\ell,e_j,e_k) \\ &= A(e_i,e_j,e_k,e_\ell) - A(e_i,e_j,e_\ell,e_k) - A(e_i,e_\ell,e_k,e_j) \\ &= A(e_i,e_j,e_k,e_\ell) + A(e_i,e_j,e_k,e_\ell) + A(e_i,e_j,e_k,e_\ell) \\ &= 3A(e_i,e_j,e_k,e_\ell) \quad \text{for} \quad i,j,k,\ell \quad \text{distinct}\,. \end{aligned}$$

Thus the only non-zero curvatures are $A(e_i,e_j,e_i,e_j) = c_{ij}$ for $i \neq j$. Consider the new basis

$$e_\nu(\theta) := \begin{cases} \cos\theta e_i + \sin\theta e_j & \text{if} \quad \nu = i, \\ -\sin\theta e_i + \cos\theta e_j & \text{if} \quad \nu = j, \\ e_\nu & \text{if} \quad \nu \neq i,j\,. \end{cases}$$

If i, j and k are distinct indices, then:

$$0 = A(e_i(\theta),e_k,e_j(\theta),e_k) = \cos\theta\sin\theta\{-c_{ik} + c_{jk}\}\,.$$

It now follows that $c_{ik} = c_{jk}$ for i,j,k distinct and, consequently, A has constant sectional curvature. □

In light of Lemma 9.2.3, we may suppose that $r(x) < n-1$ for all x henceforth. Thus, in particular, given x, we can always choose y so $x \perp y$ and $\mathcal{J}_x y = 0$.

Recall from Section 1.4.6 the following notation:

$$\mathcal{J}_{xy}z = \mathcal{J}(x,y)z := \frac{1}{2}\{\mathcal{A}(x,z)y + \mathcal{A}(y,z)x\}.$$

Lemma 9.2.4 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian orthogonally Jacobi Tsankov algebraic model of dimension n. Assume $r(x) < n-1$ for all $x \in V$. Let $x \in S(V)$. Choose $y \in S(V)$ so $y \perp x$ and so $\mathcal{J}_x y = 0$. Then:*

(1) $\mathcal{J}_y x = 0$ *and* $\mathcal{J}_x\mathcal{J}_y = 0$.

(2) $0 = \mathcal{J}_y^2 + \mathcal{J}_x^2 - 4\mathcal{J}_{xy}^2$, $\mathcal{J}_{xy}\mathcal{J}_x = \mathcal{J}_y\mathcal{J}_{xy}$ *and* $\mathcal{J}_x\mathcal{J}_{xy} = \mathcal{J}_{xy}\mathcal{J}_y$.

(3) Let $\{x, z_1, z_2\}$ *be an orthonormal set. Suppose that* $\mathcal{J}_x z_1 = \lambda_1 z_1$ *and that* $\mathcal{J}_x z_2 = \lambda_2 z_2$ *where* $\lambda_1 \neq \lambda_2$*. Then* $\mathcal{J}_{z_1} z_2 = 0$.

(4) Let $\Xi = diag(\lambda_1, \dots, \lambda_r)$ *where* λ_i *are the non-zero eigenvalues of* $\mathcal{J}_x$*, repeated according to multiplicity. We can choose an orthonormal basis for* V *so that*

$$\mathcal{J}_x = \begin{pmatrix} \Xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Xi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_{xy} = \frac{1}{2}\begin{pmatrix} 0 & \Xi & 0 \\ \Xi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Suppose that $\mathcal{J}_x y = 0$. By Equation (1.7),

$$\begin{aligned}
&\{\mathcal{J}_{\cos\theta x+\sin\theta y}\mathcal{J}_{-\sin\theta x+\cos\theta y}\}y = \\
&\quad = \mathcal{J}_{\cos\theta x+\sin\theta y}\{\sin^2\theta\mathcal{J}_x - 2\sin\theta\cos\theta\mathcal{J}_{xy} + \cos^2\theta\mathcal{J}_y\}y \\
&\quad = \{\cos^2\theta\mathcal{J}_x + 2\sin\theta\cos\theta\mathcal{J}_{xy} + \sin^2\theta\mathcal{J}_y\}\{\sin\theta\cos\theta\mathcal{J}_y x\} \\
&\quad = 2\sin^2\theta\cos^2\theta\mathcal{J}_{xy}\mathcal{J}_y x + \sin^3\theta\cos\theta\mathcal{J}_y\mathcal{J}_y x,
\end{aligned} \tag{9.4}$$

and

$$\begin{aligned}
&\{\mathcal{J}_{-\sin\theta x+\cos\theta y}\mathcal{J}_{\cos\theta x+\sin\theta y}\}y = \\
&\quad = \mathcal{J}_{-\sin\theta x+\cos\theta y}\{\cos^2\theta\mathcal{J}_x + 2\cos\theta\sin\theta\mathcal{J}_{xy} + \sin^2\theta\mathcal{J}_y\}y \\
&\quad = \{\sin^2\theta\mathcal{J}_x - 2\cos\theta\sin\theta\mathcal{J}_{xy} + \cos^2\theta\mathcal{J}_y\}\{-\cos\theta\sin\theta\mathcal{J}_y x\} \\
&\quad = 2\cos^2\theta\sin^2\theta\mathcal{J}_{xy}\mathcal{J}_y x - \cos^3\theta\sin\theta\mathcal{J}_y\mathcal{J}_y x.
\end{aligned} \tag{9.5}$$

Subtracting Equation (9.5) from Equation (9.4) yields $\sin\theta\cos\theta\mathcal{J}_y^2 x = 0$, which shows $\mathcal{J}_y^2 x = 0$. Since $\mathcal{J}_y$ is self-adjoint and the metric is definite, $\mathcal{J}_y x = 0$.

We have $\mathcal{J}_x\mathcal{J}_y y = 0$ and $\mathcal{J}_x\mathcal{J}_y x = 0$. Let $z \perp \{x, y\}$. To complete the proof of Assertion (1), we must show $\mathcal{J}_x\mathcal{J}_y z = 0$. We compute:

$$\begin{aligned}
0 &= \mathcal{J}_{\cos\theta x+\sin\theta z}\mathcal{J}_y x = \mathcal{J}_y\mathcal{J}_{\cos\theta x+\sin\theta z} x \\
&= \mathcal{J}_y\{\cos^2\theta\mathcal{J}_x + 2\cos\theta\sin\theta\mathcal{J}_{xz} + \sin^2\theta\mathcal{J}_z\}x \\
&= -\cos\theta\sin\theta\mathcal{J}_y\mathcal{J}_x z + \sin^2\theta\mathcal{J}_z\mathcal{J}_y x \\
&= -\cos\theta\sin\theta\mathcal{J}_y\mathcal{J}_x z.
\end{aligned}$$

We now prove Assertion (2). Since $\mathcal{J}_{xy} x = -\frac{1}{2}\mathcal{J}_x y = 0$ and $\mathcal{J}_{xy} y = -\frac{1}{2}\mathcal{J}_y x = 0$, we have

$$\mathcal{J}_{xy}\{-\sin\theta x + \cos\theta y\} = 0,$$

so

$$\mathcal{J}_{\cos\theta x+\sin\theta y}\{-\sin\theta x+\cos\theta y\}=0\,.$$

Thus applying Assertion (1) to the pair $\{\cos\theta x+\sin\theta y,-\sin\theta x+\cos\theta y\}$ permits us to derive Assertion (2) from the following identity:

$$\begin{aligned}
0 &= \mathcal{J}_{\cos\theta x+\sin\theta y}\mathcal{J}_{-\sin\theta x+\cos\theta y}\\
&= \{\cos^2\theta\mathcal{J}_x+2\sin\theta\cos\theta\mathcal{J}_{xy}+\sin^2\theta\mathcal{J}_y\}\,\{\sin^2\theta\mathcal{J}_x-2\sin\theta\cos\theta\mathcal{J}_{xy}+\cos^2\theta\mathcal{J}_y\}\\
&= \cos^2\theta\sin^2\theta\{\mathcal{J}_y^2+\mathcal{J}_x^2-4\mathcal{J}_{xy}^2\}+2\sin^3\theta\cos\theta\{\mathcal{J}_{xy}\mathcal{J}_x-\mathcal{J}_y\mathcal{J}_{xy}\}\\
&\quad+2\sin\theta\cos^3\theta\{\mathcal{J}_{xy}\mathcal{J}_y-\mathcal{J}_x\mathcal{J}_{xy}\}\,.
\end{aligned}$$

Let $\{x,z_1,z_2\}$ be an orthonormal set with $\mathcal{J}_xz_i=\lambda_iz_i$ where $\lambda_1\neq\lambda_2$. To prove Assertion (3), we compute

$$\begin{aligned}
\mathcal{J}_x\mathcal{J}_{\cos\theta z_1+\sin\theta z_2}z_1 &= \mathcal{J}_x\{2\cos\theta\sin\theta\mathcal{J}_{z_1z_2}+\sin^2\theta\mathcal{J}_{z_2}\}z_1\\
&= \mathcal{J}_x\{-\cos\theta\sin\theta\mathcal{J}_{z_1}z_2+\sin^2\theta\mathcal{J}_{z_2}z_1\}\\
&= -\lambda_2\cos\theta\sin\theta\mathcal{J}_{z_1}z_2+\lambda_1\sin^2\theta\mathcal{J}_{z_2}z_1,\\
\mathcal{J}_{\cos\theta z_1+\sin\theta z_2}\mathcal{J}_xz_1 &= \lambda_1\{2\cos\theta\sin\theta\mathcal{J}_{z_1z_2}+\sin^2\theta\mathcal{J}_{z_2}\}z_1\\
&= -\lambda_1\cos\theta\sin\theta\mathcal{J}_{z_1}z_2+\lambda_1\sin^2\theta\mathcal{J}_{z_2}z_1.
\end{aligned}$$

Assertion (3) now follows since we have

$$\lambda_2\mathcal{J}_{z_1}z_2=\lambda_1\mathcal{J}_{z_1}z_2\,.$$

To prove Assertion (4), choose an orthonormal basis $\{e_1,...,e_r\}$ for Range$\,(\mathcal{J}_x)$ so

$$\mathcal{J}_xe_i=\lambda_ie_i\quad\text{for}\quad\lambda_i\neq0\,.$$

We then have $\mathcal{J}_ye_i=0$ and thus $4\mathcal{J}_{xy}^2e_i=\lambda_i^2e_i$. Define:

$$f_i:=2\lambda_i^{-1}\mathcal{J}_{xy}e_i\,.$$

The collection $\{f_1,...,f_r\}$ is an orthonormal set since:

$$\langle f_i,f_j\rangle=4\lambda_i^{-1}\lambda_j^{-1}\langle\mathcal{J}_{xy}e_i,\mathcal{J}_{xy}e_j\rangle=4\lambda_i^{-1}\lambda_j^{-1}\langle\mathcal{J}_{xy}^2e_i,e_j\rangle=\delta_{ij}\,.$$

Furthermore, $f_i\in\ker(\mathcal{J}_x)=\text{Range}\,(\mathcal{J}_x)^\perp$ because:

$$\mathcal{J}_xf_i=2\lambda_i^{-1}\mathcal{J}_x\mathcal{J}_{xy}e_i=2\lambda_i^{-1}\mathcal{J}_{xy}\mathcal{J}_ye_i=0\,.$$

This shows $\{e_1,\dots,e_r,f_1,\dots,f_r\}$ is an orthonormal set. We set

$$\Xi=diag(\lambda_1,\dots,\lambda_r)\,.$$

Since $\mathcal{J}_y\mathcal{J}_{xy} = \mathcal{J}_{xy}\mathcal{J}_x$, we have $\mathcal{J}_y f_i = \lambda_i f_i$. Note that

$$\mathcal{J}_{xy}e_i = \tfrac{1}{2}\lambda_i f_i \quad \text{and} \quad \mathcal{J}_{xy}f_i = 2\lambda_i^{-1}\mathcal{J}_{xy}^2 e_i = \tfrac{1}{2}\lambda_i e_i\,.$$

On the subspace $\operatorname{Span}\{e_1,\ldots,e_r,f_1,\ldots,f_r\}$ one has that

$$\mathcal{J}_x = \begin{pmatrix} \Xi & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 \\ 0 & \Xi \end{pmatrix}, \quad \mathcal{J}_{xy} = \begin{pmatrix} 0 & \frac{1}{2}\Xi \\ \frac{1}{2}\Xi & 0 \end{pmatrix}.$$

On the other hand, it is clear that $\mathcal{J}_x = \mathcal{J}_y = 0$ on $\{\operatorname{Range}(\mathcal{J}_x) \oplus \operatorname{Range}(\mathcal{J}_y)\}^\perp$. Now, if $\xi \in \{\operatorname{Range}(\mathcal{J}_x) \oplus \operatorname{Range}(\mathcal{J}_y)\}^\perp$, then $\mathcal{J}_x\xi = \mathcal{J}_y\xi = 0$ so Assertion (2) yields

$$\langle \mathcal{J}_{xy}\xi, \mathcal{J}_{xy}\xi\rangle = \langle \mathcal{J}_{xy}^2\xi, \xi\rangle = \tfrac{1}{4}\langle(\mathcal{J}_x^2 + \mathcal{J}_y^2)\xi, \xi\rangle = 0\,.$$

This shows $\mathcal{J}_{xy}\xi = 0$ as well and gives the desired decomposition. □

We continue our study. Let

$$R(x) := \operatorname{Span}\{x\} \oplus \operatorname{Range}(\mathcal{J}_x)\,.$$

Lemma 9.2.5 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian orthogonally Jacobi Tsankov model of dimension n. Assume that $r(x) < n-1$ for all x. Let $x \in S(V)$. If $w \in S(R(x))$, then:*

1. $\operatorname{Range}(\mathcal{J}_w) \subset R(x)$ *and* $\mathcal{J}_w$ *vanishes on* $R(x)^\perp$.
2. $\mathcal{J}_w$ *is similar to* $\mathcal{J}_x$.
3. $\mathcal{J}_x$ *has at most one non-zero eigenvalue.*

Proof. Fix $w \in S(R(x))$. Expand $w = a_0x + \sum a_iw_i$ where we have $\mathcal{J}_xw_i = \lambda_iw_i$ for $\lambda_i \neq 0$. Let $y \in R(x)^\perp \cap S(V)$. We apply Lemma 9.2.4. As $y \perp \operatorname{Range}(\mathcal{J}_x)$, $\mathcal{J}_xy = 0$ so $\mathcal{J}_yx = 0$. Furthermore since $\mathcal{J}_xy = 0$, since $\mathcal{J}_xw_i = \lambda_iw_i$, and since $\lambda_i \neq 0$, $\mathcal{J}_yw_i = 0$. Thus $\mathcal{J}_yw = 0$ and consequently $\mathcal{J}_wy = 0$ for all $y \in R(x)^\perp$. Thus

$$\operatorname{Range}(\mathcal{J}_w) \subset R(x) \quad \text{and} \quad \mathcal{J}_w = 0 \quad \text{on} \quad R(x)^\perp\,.$$

This proves Assertion (1). Furthermore $\mathcal{J}_xy = 0$ and $\mathcal{J}_wy = 0$ implies $\mathcal{J}_x$ is similar to $\mathcal{J}_y$ and $\mathcal{J}_w$ is similar to $\mathcal{J}_y$. This establishes Assertion (2).

To show that Assertion (3) is true, we apply Assertion (2) to see

$$\operatorname{Rank}(\mathcal{J}_w) = \operatorname{Rank}(\mathcal{J}_w|_{R(x)}) = \dim(R(x)) - 1 = r(x)\,.$$

Suppose $\mathcal{J}_x$ has two distinct non-zero eigenvalues $\lambda_i \neq \lambda_j$ for some $i < j$. Then $\mathcal{J}_{w_i}w_j = 0$. Since $\mathcal{J}_{w_i}w_i = 0$, we would have $\operatorname{Rank}\{\mathcal{J}_{w_i}\} \leq r(x) - 1$ which is false. Thus $\mathcal{J}_x = \lambda\, Id$ on $\operatorname{Range}(\mathcal{J}_x)$. □

Lemma 9.2.6 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian orthogonally Jacobi Tsankov model of dimension n where $A \neq 0$. Assume that $r(x) < n-1$ for all x. Then $\mathcal{J}$ has only one non-zero eigenvalue λ on $S(V)$, that λ has multiplicity 1 and, moreover, $\mathcal{V}$ is Osserman.*

Proof. Choose $y \in S(x^\perp)$ with $\mathcal{J}_x y = 0$. Let $e_0 = x$ and $f_0 = y$. Let λ be the non-zero eigenvalue for $\mathcal{J}_x$ and let $r = r(x)$. Complete $\{e_0, f_0\}$ to an orthonormal basis $\{e_0, ..., e_r, f_0, ..., f_r, g_1, ..., g_\ell\}$ for V so that

$$\begin{array}{lll} \mathcal{J}_{e_0} e_j = \lambda(1-\delta_{0j}) e_j, & \mathcal{J}_{e_0} f_j = 0, & \mathcal{J}_{e_0} g_k = 0, \\ \mathcal{J}_{f_0} f_j = \lambda(1-\delta_{0j}) f_j, & \mathcal{J}_{f_0} e_j = 0, & \mathcal{J}_{f_0} g_k = 0, \\ \mathcal{J}_{e_0 f_0} e_j = \frac{1}{2}\lambda(1-\delta_{0j}) f_j, & \mathcal{J}_{e_0 f_0} f_j = \frac{1}{2}\lambda(1-\delta_{0j}) e_j, & \mathcal{J}_{e_0 f_0} g_k = 0\,. \end{array}$$

As $\mathcal{J}_{e_1}$ preserves $\operatorname{Span}\{e_0, ..., e_r\} = R(e_0)$, as λ is an eigenvalue of multiplicity r for $\mathcal{J}_{e_1}$ on $R(e_0)$, that $\mathcal{J}_{e_1}$ vanishes on $R(e_0)^\perp$, and as $\mathcal{J}_{e_1} e_1 = 0$,

$$\mathcal{J}_{e_1} e_j = \lambda(1-\delta_{1j}) e_j, \quad \mathcal{J}_{e_1} f_j = 0, \quad \mathcal{J}_{e_1} g_k = 0\,.$$

Let $\xi = \frac{1}{\sqrt{2}}(e_0 + f_0)$. Then $\mathcal{J}_\xi = \frac{1}{2}\{\mathcal{J}_{e_0} + \mathcal{J}_{f_0} + 2\mathcal{J}_{e_0 f_0}\}$. We show $r = 1$ by deriving the following contradiction:

$$\begin{array}{ll} \mathcal{J}_\xi e_2 = \frac{1}{2}\lambda(e_2 + f_2), & \mathcal{J}_\xi f_2 = \frac{1}{2}\lambda(e_2 + f_2), \\ \mathcal{J}_\xi \mathcal{J}_{e_1} f_2 = 0, & \mathcal{J}_{e_1} \mathcal{J}_\xi f_2 = \frac{1}{2}\lambda^2 e_2\,. \end{array}$$

Fix $e \in S(V)$. Consider the 2-plane

$$\pi := \operatorname{Span}\{e, \operatorname{Range}(\mathcal{J}_e)\}\,.$$

Decompose $x \in S(V)$ in the form $x = \cos\theta e_1 + \sin\theta f_1$ for $\theta \in [0, \frac{\pi}{2}]$, $e_1 \in S(\pi)$ and $f_1 \in S(\pi^\perp)$; e_1 is not unique if $\theta = \frac{\pi}{2}$ and f_1 is not unique if $\theta = 0$. As $e_1 \in \pi$, $\operatorname{Range}(\mathcal{J}_{e_1}) \subset \pi$ so

$$\pi = \operatorname{Span}\{e_1, \operatorname{Range}(\mathcal{J}_{e_1})\}\,.$$

Since $f_1 \perp \pi$, Lemma 9.2.4 pertains. As $\mathcal{J}_{f_1}$ is similar to $\mathcal{J}_{e_1}$ and also to $\mathcal{J}_e$, we have $\lambda(e_1) = \lambda(f_1) = \lambda(e)$. By Lemma 9.2.4, we can extend $\{e_1, f_1\}$ to an orthonormal set $\{e_1, e_2, f_1, f_2\}$ so that

$$\begin{array}{lll} \mathcal{J}_{e_1} e_2 = \lambda(e) e_2, & \mathcal{J}_{f_1} e_2 = 0, & \mathcal{J}_{e_1 f_1} e_2 = \frac{1}{2}\lambda(e) f_2, \\ \mathcal{J}_{e_1} f_2 = 0, & \mathcal{J}_{f_1} f_2 = \lambda(e) f_2, & \mathcal{J}_{e_1 f_1} f_2 = \frac{1}{2}\lambda(e) e_2\,. \end{array}$$

We may now compute

$$\begin{aligned} \mathcal{J}_x(\cos\theta e_2 + \sin\theta f_2) &= \{\cos^2\theta \mathcal{J}_{e_1} + 2\cos\theta\sin\theta \mathcal{J}_{e_1 f_1} + \sin^2\theta \mathcal{J}_{f_1}\} \\ &\quad \cdot\{\cos\theta e_2 + \sin\theta f_2\} \\ &= \lambda(e)(\cos^3\theta e_2 + \cos^2\theta\sin\theta f_2 + \cos\theta\sin^2\theta e_2 + \sin^3\theta f_2) \\ &= \lambda(e)(\cos\theta e_2 + \sin\theta f_2)\,. \end{aligned}$$

This implies that $\lambda(x) = \lambda(e)$ and, hence, $\mathcal{V}$ is Osserman. □

Now we are ready to prove the complete classification in the algebraic setting.

Proof of Theorem 9.2.2. Suppose $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is an orthogonally Jacobi Tsankov Riemannian model. Suppose that $A \neq 0$. If $r(x) = n - 1$ for any vector $x \in S(V)$, then A has constant sectional curvature by Lemma 9.2.3. On the other hand, if $r(x) < n - 1$ for every point $x \in S(V)$, then A is Osserman and $\text{Rank}\{\mathcal{J}_x\} = 1$ for every $x \in S(V)$ by Lemma 9.2.6. Now, from [51] there exists some Hermitian almost complex structure J such that $A = \frac{\lambda}{3}A_J$, where λ is the only nonzero eigenvalue. □

9.3 Riemannian Jacobi Tsankov manifolds

As we have done before, when studying a geometric problem, we first study the corresponding algebraic condition and then solve the geometric problem. Here we use the results in previous sections to derive the geometric classification of Riemannian Jacobi Tsankov manifolds.

Theorem 9.3.1 *Let $\mathcal{M}$ be a connected Riemannian manifold of dimension n. The following assertions hold:*

1. *$\mathcal{M}$ is Jacobi Tsankov if and only if $\mathcal{M}$ is flat.*
2. *Let $n \geq 3$. $\mathcal{M}$ is orthogonally Jacobi Tsankov if and only if $\mathcal{M}$ has constant sectional curvature.*

Proof. Let $\mathcal{M} = (M, g)$ be a connected Riemannian manifold. We shall apply Theorem 9.1.1 to prove Theorem 9.3.1. Suppose first that $\mathcal{M}$ is Jacobi Tsankov. Then $R = 0$ so $\mathcal{M}$ is flat.

Suppose $\mathcal{M}$ is orthogonally Jacobi Tsankov and $n \geq 3$. Let $\mathcal{O}$ be the open subset of points $p \in M$ so that there exists a unit tangent vector $x(p)$ with $r(x(p)) = n - 1$. Then $g|_{\mathcal{O}}$ has constant sectional curvature. Thus $R_P = \kappa_P R_0$ on $\mathcal{O}$. Since $n \geq 3$, the multiple κ_P is locally constant. Thus $R = \kappa R_0$ on the closure of $\mathcal{O}$. Therefore $\mathcal{O}$ is an open and closed subset of M and hence all of M.

If $\mathcal{M}$ does not have constant sectional curvature, $r(x) < n - 1$ for all $x \in S(\mathcal{M})$. Hence $r(x) = 1$ for all $x \in S(\mathcal{M})$. Furthermore n is even and there is an almost complex structure $J(q)$ defined on T_q for every $q \in M$ so that $R = \lambda R_J$. In addition, the almost complex structure is uniquely determined up to sign. After a bit of technical fuss, one can see that J can be chosen to vary smoothly with q, at least locally; global questions are irrelevant to our argument. The metric in question is Einstein and thus $\rho(x, x) = 3\lambda(x)$ is constant. Thus $\mathcal{M}$ is globally Osserman. This possibility is ruled out by work of Chi [51] (one may also use work of Tricerri and Vanhecke [166] to conclude this cannot happen).□

9.4 Conformal aspect of Tsankov theory

Most of the conditions studied for the Jacobi operator automatically translate to the conformal Jacobi operator. Thus, in the Tsankov context one says that a manifold $\mathcal{M}$ of dimension $n \geq 4$ is *conformal Jacobi Tsankov* if the conformal Jacobi operators commute, this is,

$$\mathcal{J}_W(x)\mathcal{J}_W(y) = \mathcal{J}_W(y)\mathcal{J}_W(x) \quad \text{for all } x, y,$$

or *orthogonally conformal Jacobi Tsankov* if

$$\mathcal{J}_W(x)\mathcal{J}_W(y) = \mathcal{J}_W(y)\mathcal{J}_W(x) \quad \text{for all } x \perp y.$$

One may consider the model (T_pM, g_p, W_p), where W is the Weyl tensor, and use Theorem 9.1.1 to show that a Riemannian manifold is conformal Jacobi Tsankov if and only if $W = 0$, this is, it is locally conformally flat. What is somewhat surprising is the fact that the same classification result holds for orthogonally conformal Jacobi Tsankov Riemannian manifolds, as next theorem shows.

Theorem 9.4.1 *Let $\mathcal{M}$ be a connected Riemannian manifold of dimension n. Let $n \geq 4$. $\mathcal{M}$ is orthogonally conformal Jacobi Tsankov if and only if $\mathcal{M}$ is locally conformally flat.*

Proof. If $\mathcal{M}$ is locally conformally flat, then $W = 0$ and $\mathcal{M}$ is orthogonally Jacobi Tsankov. To prove the converse implication choose a point $p \in M$. Consider the algebraic model (T_pM, g_p, W_p) and apply Theorem 9.2.2 to obtain that $W_p = \kappa A_0$ or $W_p = \lambda A_J$ for a Hermitian almost complex structure J on T_pM. In both cases, since $\mathcal{J}_W$ is trace free, we obtain $\mathcal{J}_W = 0$ in p, and then $W_p = 0$. Since this holds for every $p \in M$, we conclude $W = 0$ and hence $\mathcal{M}$ is locally conformally flat. $\square$

Chapter 10

Tsankov models in the pseudo-Riemannian setting

In the previous chapter we studied Jacobi Tsankov and orthogonally Jacobi Tsankov models and manifolds in the Riemannian setting. All the results there, strongly depended on the fact that the metric was definite. When passing to indefinite signature, the situation is much more complicated. Due to this complexity we are going to concentrate on the study of the Jacobi Tsankov condition, which is more restrictive than the orthogonally Jacobi Tsankov one. Although we showed that in Riemannian signature Jacobi Tsankov models have zero curvature, this is not the case in higher signature and, except for Lorentzian signature where this result also holds, the problem is more difficult to handle and more possibilities exist. We present here results in [36] and [38].

10.1 Jacobi Tsankov models and manifolds

We begin this section by studying Jacobi Tsankov algebraic models. Although we have already classified Riemannian models being Jacobi Tsankov, we overlap those results with the following as the new approach provides a different treatment that we consider of interest.

Theorem 10.1.1 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Jacobi Tsankov algebraic model. Then:*

1. $\mathcal{J}_x^2 = 0$ *for all* $x \in V$.
2. $\mathcal{V}$ *is Osserman.*
3. *If* $\mathcal{V}$ *is Riemannian or Lorentzian, then* $A = 0$.

Proof. Equations (1.7) are the key in our discussion. Suppose $\mathcal{V}$ is a Jacobi Tsankov model of signature (p, q). Recall the notation $\mathcal{J}_{xy}z := \frac{1}{2}\{A(x,z)y + A(y,z)x\}$ that we introduced in Section 1.4.6. Polarizing the identity $\mathcal{J}_x\mathcal{J}_y = \mathcal{J}_y\mathcal{J}_x$ yields:

$$\mathcal{J}_{x_1x_2}\mathcal{J}_{x_3x_4} = \mathcal{J}_{x_3x_4}\mathcal{J}_{x_1x_2} \quad \text{for all} \quad x_1, x_2, x_3, x_4 \in V\,.$$

We have $\mathcal{J}_x x = 0$. Equation (1.7) yields Assertion (1) of Theorem 10.1.1 since

$$0 = \mathcal{J}_{xy}\mathcal{J}_{xx}x = \mathcal{J}_{xx}\mathcal{J}_{xy}x = -\tfrac{1}{2}\mathcal{J}_x\mathcal{J}_x y\,.$$

Since the Jacobi operator is nilpotent, $\{0\}$ is the only eigenvalue of $\mathcal{J}$. This shows that A is Osserman, so Assertion (2) holds.

If $p = 0$, then $\mathcal{J}_x$ is diagonalizable. Consequently, $\mathcal{J}_x^2 = 0$ implies $\mathcal{J}_x = 0$ for all x and hence $A = 0$. If $p = 1$, then A is Osserman implies A has constant sectional curvature κ [16, 75]. Since $\mathcal{J}_x^2 = 0$, $\kappa = 0$ which again implies $A = 0$. This shows Assertion (3). □

In fact, it is possible to work in a slightly more general setting. Following [21], one says that $\mathcal{C}$ is a *generalized curvature operator* if it has the symmetries of the curvature operator which is defined by a torsion free connection. This means that one imposes the symmetries:

$$\begin{aligned} &\mathcal{C}(x,y)z = -\mathcal{C}(y,x)z,\\ &\mathcal{C}(x,y)z + \mathcal{C}(y,z)x + \mathcal{C}(z,x)y = 0. \end{aligned}$$

The proof given above then generalizes immediately to yield:

Corollary 10.1.2 *If $\mathcal{C}$ is a generalized curvature operator on V which is Jacobi Tsankov, then $\mathcal{J}_C$ is Osserman and $\mathcal{J}_C(x)^2 = 0$ for all $x \in V$.*

It has been recently shown in [87] that the Jacobi Tsankov condition is equivalent to the socalled mixed Tsankov condition (i.e. $\mathcal{J}(x_1)A(x_2,x_3) = A(x_2,x_3)\mathcal{J}(x_1)$ for all x_1, x_2, x_3).

Theorem 10.1.1 also has the following geometrical consequence:

Corollary 10.1.3 *Let $\mathcal{M}$ be a Jacobi Tsankov pseudo-Riemannian manifold of signature (p, q). Then $\mathcal{M}$ is nilpotent Osserman. If $p = 0$ or if $p = 1$, then $\mathcal{M}$ is flat.*

10.1.1 Models with two-step nilpotent Jacobi operators which are not Jacobi Tsankov

In view of Theorem 10.1.1, one might conjecture that the condition $\mathcal{J}_x^2 = 0$ for all $x \in V$ is sufficient to imply $\mathcal{V}$ is Jacobi Tsankov. Actually, next result shows this is not the case.

Theorem 10.1.4 *There exists a model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ which is not Jacobi Tsankov but which has $\mathcal{J}_x^2 = 0$ for all $x \in V$.*

Let ϕ be a skew-symmetric endomorphism of V. We generalize the construction given in (9.1) by defining the following algebraic curvature tensor:

$$A_\phi(x,y,z,w) := \langle\phi x,z\rangle\langle\phi y,w\rangle - \langle\phi y,z\rangle\langle\phi x,w\rangle + 2\langle\phi x,y\rangle\langle\phi z,w\rangle.$$

Recall from Equation (9.2) that the associated Jacobi operator $\mathcal{J}_\phi$ is given by:

$$\mathcal{J}_\phi(x)y = 3\langle\phi x,y\rangle\phi x\,.$$

Let $\mathbb{R}^{(p,q)}$ denote the Euclidean space with a metric of signature (p,q). Theorem 10.1.4 is a consequence of the following result:

Lemma 10.1.5 *There exist skew-symmetric endomorphisms $\{\Phi,\Psi\}$ of $\mathbb{R}^{(\ell,\ell)}$ for some ℓ so that $\Phi^2=\Psi^2=0$, $\Phi\Psi+\Psi\Phi=0$ and $\Phi\Psi\neq 0$. Set $\mathcal{V} := (\mathbb{R}^{(\ell,\ell)},\langle\cdot,\cdot\rangle,A)$ where $A=\frac{1}{3}\{A_\Phi+A_\Psi\}$. Then $\mathcal{J}_A(x)^2=0$ for all x and $\mathcal{V}$ is not Jacobi Tsankov.*

Proof. There exists (see [84, 85]) a collection of skew-symmetric endomorphisms of $\mathbb{R}^{(\ell,\ell)}$ so that

$$\phi_1^2=\phi_2^2=Id,\quad \phi_3^2=\phi_4^2=-Id,\quad \phi_i\phi_j+\phi_j\phi_i=0 \text{ for } i\neq j\,.$$

One may then set $\Phi=\phi_1+\phi_3$ and $\Psi=\phi_2+\phi_4$ to construct skew-adjoint endomorphisms so

$$\Phi^2=\Psi^2=0,\quad \Phi\Psi+\Psi\Phi=0\,.$$

Suppose that $\Phi\Psi=0$. We argue for a contradiction. Conjugating the identity

$$(\phi_1+\phi_3)(\phi_2+\phi_4)=0$$

by ϕ_1 yields

$$0=(-\phi_1+\phi_3)(\phi_2+\phi_4)\,.$$

Adding these two equations yields $\phi_3(\phi_2+\phi_4)=0$. Multiplying by ϕ_3 implies $\phi_2+\phi_4=0$. Conjugating this identity by ϕ_2 yields $\phi_2-\phi_4=0$ and thus $\phi_2=0$. This is not possible. Thus $\Phi\Psi\neq 0$. Consider the algebraic curvature tensor A. One computes:

$$\begin{aligned}
\mathcal{J}_x y &= \langle y,\Phi x\rangle\Phi x + \langle y,\Psi x\rangle\Psi x,\\
\mathcal{J}_u\mathcal{J}_v y &= \langle y,\Phi v\rangle\langle\Phi v,\Phi u\rangle\Phi u + \langle y,\Phi v\rangle\langle\Phi v,\Psi u\rangle\Psi u\\
&\quad +\langle y,\Psi v\rangle\langle\Psi v,\Phi u\rangle\Phi u + \langle y,\Psi v\rangle\langle\Psi v,\Psi u\rangle\Psi u\\
&= \langle y,\Phi v\rangle\langle\Phi v,\Psi u\rangle\Psi u + \langle y,\Psi v\rangle\langle\Psi v,\Phi u\rangle\Phi u\,.
\end{aligned}$$

Since $\langle\Phi x,\Psi x\rangle = -\langle\Psi\Phi x,x\rangle = \langle\Phi\Psi x,x\rangle = -\langle\Psi x,\Phi x\rangle$, we have, as desired,

$$\mathcal{J}_x\mathcal{J}_x=0\,.$$

Choose u so $\Psi\Phi u \neq 0$. Set $y = \Phi u$. Choose v so $\langle \Phi u, \Psi v\rangle \neq 0$. Then:

$$\begin{aligned}
\mathcal{J}_u\mathcal{J}_v y &= \langle \Phi u, \Phi v\rangle\langle \Phi v, \Psi u\rangle\Psi u + \langle \Phi u, \Psi v\rangle\langle \Psi v, \Phi u\rangle\Phi u\\
&= \langle \Phi u, \Psi v\rangle^2\Phi u \neq 0,\\
\mathcal{J}_v\mathcal{J}_u y &= \langle \Phi u, \Phi u\rangle\langle \Phi u, \Psi v\rangle\Psi v + \langle \Phi u, \Psi u\rangle\langle \Psi u, \Phi v\rangle\Phi v\\
&= 0\,.
\end{aligned}$$

Then $\mathcal{J}_u\mathcal{J}_v y \neq 0$ while $\mathcal{J}_v\mathcal{J}_u y = 0$. Consequently $\mathcal{V}$ is not a Jacobi Tsankov model. □

10.2 Models with $\mathcal{J}_x\mathcal{J}_y = 0\ \forall\ x, y \in V$

It is obvious that if $\mathcal{J}_x\mathcal{J}_y = 0$ for all $x, y \in V$, then necessarily $\mathcal{V}$ is Jacobi Tsankov. However, what is somehow surprising is that the converse holds in low dimensions, as next result shows.

Theorem 10.2.1 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a Jacobi Tsankov model of dimension n. If $n \leq 13$, then $\mathcal{J}_x\mathcal{J}_y = 0$ for all $x, y \in V$.*

Theorem 10.2.1 will follow as an immediate consequence of the following lemma which also motivates the example we are going to study in Section 10.3.

Lemma 10.2.2 *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A)$ be a Jacobi Tsankov model. Suppose that there exist $x, y \in V$ so that $\mathcal{J}_x\mathcal{J}_y \neq 0$.*

1. *There exists $w \in V$ such that $\langle \mathcal{J}_x\mathcal{J}_y w, w\rangle = \langle \mathcal{J}_y\mathcal{J}_w x, x\rangle = \langle \mathcal{J}_w\mathcal{J}_x y, y\rangle \neq 0$.*
2. *Set*

$$\begin{array}{lllll}
e_1 := w, & e_2 := \mathcal{J}_x\mathcal{J}_y w, & e_3 := \mathcal{J}_x w, & e_4 := \mathcal{J}_y w, & e_5 := \mathcal{J}_{xy} w,\\
f_1 := x, & f_2 := \mathcal{J}_y\mathcal{J}_w x, & f_3 := \mathcal{J}_y x, & f_4 := \mathcal{J}_w x, & f_5 := \mathcal{J}_{yw} x,\\
g_1 := y, & g_2 := \mathcal{J}_w\mathcal{J}_x y, & g_3 := \mathcal{J}_w y, & g_4 := \mathcal{J}_x y, & g_5 := \mathcal{J}_{wx} y,
\end{array}$$

then $S := \{e_1, \ldots, e_5, f_1, \ldots, f_5, g_1, \ldots, g_4\}$ is a linearly independent set.

3. $e_5 + f_5 + g_5 = 0$.
4. $\dim(V) \geq 14$.

Proof. Choose w and f so $\langle \mathcal{J}_x\mathcal{J}_y w, f\rangle \neq 0$. Set $w(\varepsilon) := w + \varepsilon f$. Then

$$\begin{aligned}
p(\varepsilon) :=&\ \langle w(\varepsilon), \mathcal{J}_x\mathcal{J}_y w(\varepsilon)\rangle\\
=&\ \langle w, \mathcal{J}_x\mathcal{J}_y w\rangle + 2\varepsilon\langle \mathcal{J}_x\mathcal{J}_y w, f\rangle + \varepsilon^2\langle \mathcal{J}_x\mathcal{J}_y f, f\rangle\,.
\end{aligned}$$

As $\langle \mathcal{J}_x\mathcal{J}_y w, f\rangle \neq 0$, $p(\varepsilon)$ is a non-trivial polynomial in ε. Thus it is non-zero for a suitable choice of ε. Thus after replacing w by $w(\varepsilon)$ for suitably chosen ε, we see that there is $w \in V$ with $\langle w, \mathcal{J}_x\mathcal{J}_y w\rangle \neq 0$. Applying Equation (1.7) yields:

$$\begin{aligned}\langle \mathcal{J}_y\mathcal{J}_w x, x\rangle &= -2\langle \mathcal{J}_y\mathcal{J}_{wx} w, x\rangle = -2\langle \mathcal{J}_y w, \mathcal{J}_{wx} x\rangle\\ &= \langle \mathcal{J}_y w, \mathcal{J}_x w\rangle = \langle \mathcal{J}_x\mathcal{J}_y w, w\rangle\,.\end{aligned}$$

Similarly, $\langle \mathcal{J}_w\mathcal{J}_x y, y\rangle = \langle \mathcal{J}_x\mathcal{J}_y w, w\rangle$ and Assertion (1) follows.

Because $\mathcal{J}_{x+\varepsilon y}\mathcal{J}_{x+\varepsilon y} = 0$ for every $\varepsilon \in \mathbb{R}$ and because $\mathcal{V}$ is Jacobi Tsankov, we have the following relations:

$$\begin{array}{lll}\mathcal{J}_x^2 = 0, & \mathcal{J}_y^2 = 0, & \mathcal{J}_x\mathcal{J}_y = \mathcal{J}_y\mathcal{J}_x,\\ \mathcal{J}_x\mathcal{J}_{xy} = \mathcal{J}_{xy}\mathcal{J}_x = 0, & \mathcal{J}_y\mathcal{J}_{xy} = \mathcal{J}_{xy}\mathcal{J}_y = 0, & \mathcal{J}_{xy}^2 = -\frac{1}{2}\mathcal{J}_x\mathcal{J}_y\,.\end{array}$$

We have $\mathcal{J}_w\mathcal{J}_y x \neq 0$ and $\mathcal{J}_w\mathcal{J}_x y \neq 0$ by Assertion (1). To prove Assertion (2), suppose there is a non-trivial dependence relation among the elements of the set S:

$$\begin{aligned} 0 &= \sum_{i=1}^{5}\{a_i e_i + b_i f_i + c_i g_i\}\\ &= a_1 w + a_2\mathcal{J}_x\mathcal{J}_y w + a_3\mathcal{J}_x w + a_4\mathcal{J}_y w + a_5\mathcal{J}_{xy} w\\ &\quad + b_1 x + b_2\mathcal{J}_y\mathcal{J}_w x + b_3\mathcal{J}_y x + b_4\mathcal{J}_w x + b_5\mathcal{J}_{yw} x\\ &\quad + c_1 y + c_2\mathcal{J}_w\mathcal{J}_x y + c_3\mathcal{J}_w y + c_4\mathcal{J}_x y + c_5\mathcal{J}_{wx} y\,,\end{aligned} \tag{10.1}$$

where, since $g_5 \notin S$, we suppose $c_5 = 0$.

We can apply $\mathcal{J}_x\mathcal{J}_y$ to Equation (10.1) to see $a_1\mathcal{J}_x\mathcal{J}_y w = 0$. Since, by Assertion (1), $\mathcal{J}_x\mathcal{J}_y w \neq 0$, $a_1 = 0$. Similarly $b_1 = c_1 = 0$. If we now apply $\mathcal{J}_x$ to Equation (10.1), we see

$$a_4\mathcal{J}_x\mathcal{J}_y w + c_3\mathcal{J}_x\mathcal{J}_w y = 0$$

so

$$0 = \langle a_4\mathcal{J}_x\mathcal{J}_y w + c_3\mathcal{J}_x\mathcal{J}_w y, w\rangle = a_4\langle \mathcal{J}_x\mathcal{J}_y w, w\rangle\,.$$

By Assertion (1), $a_4 = 0$. Similarly, $a_3 = b_3 = b_4 = c_3 = c_4 = 0$. Thus Equation (10.1) simplifies to become

$$0 = a_2\mathcal{J}_x\mathcal{J}_y w + a_5\mathcal{J}_{xy} w + b_2\mathcal{J}_y\mathcal{J}_w x + b_5\mathcal{J}_{yw} x + c_2\mathcal{J}_w\mathcal{J}_x y + c_5\mathcal{J}_{wx} y\,.$$

Applying $\mathcal{J}_{xy}$ then yields

$$\begin{aligned} 0 &= a_5\mathcal{J}_{xy}^2 w + b_5\mathcal{J}_{xy}\mathcal{J}_{yw} x + c_5\mathcal{J}_{xy}\mathcal{J}_{wx} y\\ &= (a_5\mathcal{J}_{xy}^2 + \tfrac{1}{4}(b_5 + c_5)\mathcal{J}_x\mathcal{J}_y) w\\ &= (a_5 - \tfrac{1}{2}(b_5 + c_5))\mathcal{J}_{xy}^2 w\,.\end{aligned}$$

This shows $a_5 = \frac{1}{2}(b_5 + c_5)$; since a_5, b_5, and c_5 play symmetric roles, we obtain that $a_5 = b_5 = c_5$. Since $c_5 = 0$, we have $a_5 = b_5 = 0$. Taking the inner product with x, y, and w then yields, respectively $b_2 = 0$, $c_2 = 0$, and $a_2 = 0$, which completes the proof of Assertion (2).

To prove Assertion (3), we use the curvature symmetries to compute:

$$\begin{aligned} e_5 + f_5 + g_5 &= \mathcal{J}_{xy}w + \mathcal{J}_{yw}x + \mathcal{J}_{wx}y \\ &= \frac{1}{2}\{\mathcal{A}_{xw}y + \mathcal{A}_{yw}x + \mathcal{A}_{yx}w + \mathcal{A}_{wx}y + \mathcal{A}_{wy}x + \mathcal{A}_{xy}w\} \\ &= 0\,. \end{aligned}$$

Assertion (4) is immediate from Assertion (2). □

10.2.1 Classification theorem

If $V = V_1 \oplus V_2$ is a non-trivial orthogonal direct sum decomposition of V with respect to $\langle\cdot,\cdot\rangle$, which moreover induces a decomposition $A = A_1 \oplus A_2$, then $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is said to be decomposable and we write $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_i := (V_i, \langle\cdot,\cdot\rangle|_{V_i}, A_i)$; $\mathcal{V}$ is said to be indecomposable otherwise. A pseudo-Riemannian manifold $\mathcal{M} = (M, g)$ is said to be reducible at a point $p \in M$ if there is a neighborhood $\mathcal{O}$ of p in M and a Cartesian product $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$ which induces an orthogonal decomposition $g_{\mathcal{O}} = g_{\mathcal{O}_1} \oplus g_{\mathcal{O}_2}$; $\mathcal{M}$ is locally irreducible at p if $\mathcal{M}$ is not reducible at p. Also recall that we say an algebraic model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is skew Tsankov if

$$\mathcal{A}(x,y)\mathcal{A}(z,w) = \mathcal{A}(z,w)\mathcal{A}(x,y) \quad \text{for all} \quad x, y, z, w.$$

The following result is a useful classification result which relates a condition on the Jacobi operator with the skew Tsankov one, that we will study later in Chapter 11.

Theorem 10.2.3 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ be a model. The following statements are equivalent:*

1. *$\mathcal{V}$ is indecomposable and $\mathcal{J}_x\mathcal{J}_y = 0$ for all $x, y \in V$.*
2. *$\mathcal{V}$ is indecomposable and $\mathcal{A}_{x_1x_2}\mathcal{A}_{x_3x_4} = 0$ for all $x_i \in V$.*
3. *We can decompose $V = W \oplus \bar{W}$ and $A = A_W \oplus 0$ where (W, A_W) is irreducible and where W and $\bar{W}$ are totally isotropic subspaces of V.*

We first give a series of technical lemmas we will use to complete the proof of Theorem 10.2.3.

Lemma 10.2.4 *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A)$. If $\mathcal{A}_{x_1x_2}\mathcal{A}_{x_3x_4} = 0$ for all $x_i \in V$, then $\mathcal{J}_x\mathcal{J}_y = 0$ for all $x, y \in V$.*

Proof. Suppose $\mathcal{A}_{x_1x_2}\mathcal{A}_{x_3x_4} = 0$ for all $x_i \in V$. Then compute

$$\begin{aligned} 0 &= -\langle \mathcal{A}_{x_1x_2}\mathcal{A}_{x_3x_4}x_3, x_1\rangle \\ &= \langle \mathcal{A}_{x_3x_4}x_3, \mathcal{A}_{x_1x_2}x_1\rangle \\ &= \langle \mathcal{J}_{x_3}x_4, \mathcal{J}_{x_1}x_2\rangle \\ &= \langle \mathcal{J}_{x_1}\mathcal{J}_{x_3}x_4, x_2\rangle. \end{aligned}$$

This shows $\mathcal{J}_{x_1}\mathcal{J}_{x_3} = 0$ for all $x_1, x_3 \in V$. □

Lemma 10.2.5 *Let $\mathcal{V}$ be as in Theorem 10.2.3 (3). Then $\mathcal{V}$ is indecomposable and the curvature operator satisfies $\mathcal{A}_{x_1x_2}\mathcal{A}_{x_3x_4} = 0$ for all $x_i \in V$.*

Proof. Let $\mathcal{V} = (V, \langle\cdot,\cdot,\rangle, A)$ be as in Theorem 10.2.3 (3). This means that $V = W \oplus \bar{W}$, that $A = A_W \oplus 0$, that W and $\bar{W}$ are totally isotropic, and that (W, A_W) is irreducible. Suppose there is a non-trivial decomposition

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \quad \text{where} \quad \mathcal{V}_i = (V_i, \langle\cdot,\cdot\rangle_i, A_i\rangle\,.$$

This would then induce a non-trivial decomposition of (W, A_W). Since (W, A_W) is assumed indecomposable, either $W \subset V_1$ or $W \subset V_2$; we suppose without loss of generality that $W \subset V_1$. Since W and $\bar{W}$ are totally isotropic and since $V = W \oplus \bar{W}$, $W^\perp = W$. Since $V_2 \perp W$, we have $V_2 \subset W$. Since $V_2 \cap V_1 = \{0\}$, this implies $V_2 = \{0\}$ which is false. Consequently $\mathcal{V}$ is indecomposable.

Choose a basis $\{e_i\}$ for W and choose a basis $\{\bar{e}_i\}$ for $\bar{W}$ so that the only non-zero components of the inner product are $\langle e_i, \bar{e}_i\rangle = 1$. The only non-zero components of $\mathcal{A}$ are

$$\mathcal{A}(e_i, e_j)e_k = \sum_l A_W(e_i, e_j, e_k, e_l)\bar{e}_l\,.$$

This shows $\mathcal{A}_{x_1x_2}\mathcal{A}_{x_3x_4} = 0$. □

Lemma 10.2.6 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ and suppose that $\mathcal{J}_x y = 0$ for all $x \in V$. Then $A(x_1, x_2, x_3, y) = 0$ for all $x_i \in V$.*

Proof. We compute:

$$\begin{aligned} A(x_1, x_2, x_3, y) + A(x_3, x_2, x_1, y) &= 2\langle \mathcal{J}_{x_1x_3}x_2, y\rangle \\ &= 2\langle x_2, \mathcal{J}_{x_1x_3}y\rangle \\ &= 0\,. \end{aligned}$$

Consequently $A(x_1,x_2,x_3,y)=-A(x_3,x_2,x_1,y)$ for all $x_i\in V$. Thus

$$\begin{aligned}0 &= A(x_1,x_2,x_3,y)+A(x_2,x_3,x_1,y)+A(x_3,x_1,x_2,y)\\ &= A(x_1,x_2,x_3,y)-A(x_3,x_2,x_1,y)-A(x_2,x_1,x_3,y)\\ &= A(x_1,x_2,x_3,y)+A(x_1,x_2,x_3,y)+A(x_1,x_2,x_3,y)\\ &= 3A(x_1,x_2,x_3,y) \qquad \text{for all } x_i\in V.\end{aligned}$$

□

Proof of Theorem 10.2.3. Lemma 10.2.4 shows Assertion (2) implies Assertion (1), Lemma 10.2.5 shows Assertion (3) implies Assertion (2). Therefore, we complete the proof by showing that Assertion (1) implies Assertion (3).

Suppose that $\mathcal{V}$ is indecomposable and that $\mathcal{J}_x\mathcal{J}_y=0$ for all $x,y\in V$. Set

$$\begin{aligned}&W^*:=\operatorname{Span}_{v_1,v_2\in V}\{\mathcal{J}_{v_1}v_2\},\\ &U:=\{v\in V:\mathcal{J}_{v_1}v=0\ \forall v_1\in V\}\,.\end{aligned}$$

Then by assumption, $W^*\subset U$. Furthermore, by Lemma 10.2.6, $A(v_1,v_2,v_3,v_4)=0$ if any of the $v_i\in U$. Choose a complementary subspace W_1 so that $V=U\oplus W_1$.

If $w^*\in W^*$, then $w^*=\sum_j\mathcal{J}_{x_j}y_j$. Thus if $u\in U$,

$$\langle w^*,u\rangle=\langle\sum_j\mathcal{J}_{x_j}y_j,u\rangle=\sum_j\langle y_j,\mathcal{J}_{x_j}u\rangle=0\,. \tag{10.2}$$

As $\langle\cdot,\cdot\rangle$ is non-degenerate, there must exist $\tilde w\in W_1$ so $\langle\tilde w,w^*\rangle\neq0$. Fix a basis $\{w_1^*,...,w_k^*\}$ for W^*. The argument given above shows we can find corresponding elements $\{\tilde w_1,...,\tilde w_k\}$ in W_1 so

$$\langle\tilde w_i,w_j^*\rangle=\delta_{ij}\,.$$

If $\{\tilde w_1,...,\tilde w_k\}$ does not span W_1, choose $0\neq\tilde w\in W_1$ so that $\tilde w\perp W^*$. Since $\tilde w\notin U$, there exists y so that $\mathcal{J}_y\tilde w\neq0$. Choose $z\in V$ so

$$0\neq\langle\mathcal{J}_y\tilde w,z\rangle=\langle\tilde w,\mathcal{J}_yz\rangle\,.$$

This contradicts the fact that $\tilde w\perp W^*$. Thus $\{\tilde w_1,...,\tilde w_k\}$ is a basis for W_1. We set $w_i:=\tilde w_i-\frac12\sum_j\langle\tilde w_i,\tilde w_j\rangle w_j^*$. We then have

$$\begin{aligned}&W:=\operatorname{Span}\{w_i\}, &&\langle w_i,w_j\rangle=0,\\ &W^*=\operatorname{Span}\{w_i^*\}, &&\langle w_i^*,w_j^*\rangle=0,\\ &V=W\oplus U, &&\langle w_i,w_j^*\rangle=\delta_{ij}\,.\end{aligned}$$

Let $\{w_1^*,...,w_k^*,\tilde u_1,...,\tilde u_l\}$ be a basis for U. Set

$$u_i:=\tilde u_i-\sum_j\langle w_j,\tilde u_i\rangle w_j^*\,.$$

By Equation (10.2), $\langle w_i^*, \tilde{u}_j\rangle = 0$. Consequently $\langle u_i, w_j\rangle = \langle u_i, w_j^*\rangle = 0$ for $1 \leq i \leq l$ and $1 \leq j \leq k$. Let $T := \text{Span}\{u_i\}$. Then

$$(V, \langle\cdot,\cdot\rangle, A) = (W \oplus W^*, \langle\cdot,\cdot\rangle|_{W\oplus W^*}, A|_W \oplus 0) \ \oplus\ (T, \langle\cdot,\cdot\rangle|_T, 0)\,.$$

Since $(V, \langle\cdot,\cdot\rangle, A)$ is indecomposable, $T = \{0\}$. Since W and W^* are totally isotropic, the implication follows and completes the proof of the Theorem. □

10.3 A Jacobi Tsankov model with $\mathcal{J}_x\mathcal{J}_y \neq 0$

Theorem 10.2.1 shows that the Jacobi Tsankov condition is equivalent to $\mathcal{J}_x\mathcal{J}_y = 0$ for all x, y for $n \leq 13$. In this section we show that $n \leq 13$ is a sharp assumption by giving a 14-dimensional example at the algebraic level. We study some properties of such model, which is Jacobi Tsankov, indecomposable and does not have the form given in Theorem 10.2.3. Furthermore, we show that this model is geometrically realizable in manifolds with different interesting features.

10.3.1 The algebraic model $\mathcal{V}_{14}$

First of all, we define the algebraic model as follows.

Definition 10.3.1 Let $\{\alpha_i, \alpha_i^*, \beta_{i,1}, \beta_{i,2}, \beta_{4,1}, \beta_{4,2}\}$ be a basis for $\mathbb{R}^{14}$ where we shall let the index i range from 1 through 3. Let $\mathcal{V}_{14} := (\mathbb{R}^{14}, \langle\cdot,\cdot\rangle, A)$ be the model where the non-zero components of $\langle\cdot,\cdot\rangle$ and of A are given, up to the usual symmetries, by:

$$\begin{aligned}
&\langle\alpha_i, \alpha_i^*\rangle = \langle\beta_{i,1}, \beta_{i,2}\rangle = 1,\\
&\langle\beta_{4,1}, \beta_{4,1}\rangle = \langle\beta_{4,2}, \beta_{4,2}\rangle = -\tfrac{1}{2}, \quad \langle\beta_{4,1}, \beta_{4,2}\rangle = \tfrac{1}{4},\\
&A(\alpha_1, \alpha_2, \alpha_1, \beta_{2,1}) = A(\alpha_1, \alpha_3, \alpha_1, \beta_{3,1}) = 1,\\
&A(\alpha_2, \alpha_3, \alpha_2, \beta_{3,2}) = A(\alpha_2, \alpha_1, \alpha_2, \beta_{1,2}) = 1,\\
&A(\alpha_3, \alpha_1, \alpha_3, \beta_{1,1}) = A(\alpha_3, \alpha_2, \alpha_3, \beta_{2,2}) = 1,\\
&A(\alpha_1, \alpha_2, \alpha_3, \beta_{4,1}) = A(\alpha_1, \alpha_3, \alpha_2, \beta_{4,1}) = \tfrac{1}{2},\\
&A(\alpha_2, \alpha_3, \alpha_1, \beta_{4,2}) = A(\alpha_2, \alpha_1, \alpha_3, \beta_{4,2}) = \tfrac{1}{2}\,.
\end{aligned} \tag{10.3}$$

Note that the model is well defined and the signature of the metric is $(8, 6)$.

Let $Sl_\pm(3)$ be the group of all 3×3 matrices of determinant ± 1 and let $\mathcal{G}(\mathcal{V}_{14})$ be the group of isomorphisms of $\mathcal{V}_{14}$. The following result describes the basic properties of the model $\mathcal{V}_{14}$.

Theorem 10.3.1 *Let $\mathcal{V}_{14}$ be the model of Definition 10.3.1.*

1. *$\mathcal{V}_{14}$ is Jacobi Tsankov.*

2. *There exist* $x_i \in V$ *so* $\mathcal{A}_{x_1x_2}\mathcal{A}_{x_3x_4} \neq \mathcal{A}_{x_3x_4}\mathcal{A}_{x_1x_2}$. *Thus* $\mathcal{V}_{14}$ *is not skew Tsankov. Furthermore, there exist* $x, y \in V$ *so* $\mathcal{J}_x\mathcal{J}_y \neq 0$.
3. *There is a short exact sequence* $1 \to \mathbb{R}^{21} \to \mathcal{G}(\mathcal{V}_{14}) \to Sl_{\pm}(3) \to 1$.
4. *One has* $\mathcal{A}_{x_1x_2}\mathcal{J}_{x_3} = \mathcal{J}_{x_3}\mathcal{A}_{x_1x_2}$ *for all* $x_i \in V$.

Proof. Let us begin by establishing the following notational conventions. The following spaces are invariantly defined:

$$\begin{aligned} V_{\beta,\alpha^*} &:= \operatorname{Span}_{\xi_i\in\mathbb{R}^{14}}\{\mathcal{J}_{\xi_1}\xi_2\} = \operatorname{Span}_{1\le i\le 3, 1\le j\le 2}\{\beta_{i,j}, \beta_{4,j}, \alpha_i^*\}, \\ V_{\alpha^*} &:= \operatorname{Span}_{\xi_i\in\mathbb{R}^{14}}\{\mathcal{J}_{\xi_1}\mathcal{J}_{\xi_2}\xi_3\} = \operatorname{Span}_{1\le i\le 3}\{\alpha_i^*\}\,. \end{aligned} \tag{10.4}$$

Define

$$\beta^*_{4,1} := -\tfrac{8}{3}\beta_{4,1} - \tfrac{4}{3}\beta_{4,2}, \quad \beta^*_{4,2} := -\tfrac{4}{3}\beta_{4,1} - \tfrac{8}{3}\beta_{4,2}\,.$$

One then has that

$$\langle \beta^*_{4,i}, \beta_{4,j}\rangle = \delta_{ij}\,.$$

Since the proof of each assertion is a little bit long, we label each of them inside the complete proof of the theorem.

Proof of Assertion (1). If $\xi \in \mathbb{R}^{14}$, then

$$\mathcal{J}_\xi\alpha_i \subset V_{\beta,\alpha^*}, \quad \mathcal{J}_\xi\beta_{ij} \subset V_{\alpha^*}, \quad \text{and} \quad \mathcal{J}_\xi\alpha_i^* = 0\,.$$

Thus to show $\mathcal{J}_x\mathcal{J}_y = \mathcal{J}_y\mathcal{J}_x$ for all x, y, it suffices to show

$$\mathcal{J}_x\mathcal{J}_y\alpha_i = \mathcal{J}_y\mathcal{J}_x\alpha_i$$

for all x, y, i. Since $\mathcal{J}_x\mathcal{J}_y\alpha_i \in V_{\alpha^*}$, this can be done by establishing:

$$\langle \mathcal{J}_x\alpha_i, \mathcal{J}_y\alpha_j\rangle = \langle \mathcal{J}_y\alpha_i, \mathcal{J}_x\alpha_j\rangle$$

for all x, y, i, j. Since $\mathcal{J}_{x_1x_2}\alpha_i \in V_{\alpha^*}$ if either x_1 or $x_2 \in V_{\beta,\alpha^*}$, we may take $x_1 = \alpha_i$ and $x_2 = \alpha_j$. Let $\mathcal{J}_{ijk} := \mathcal{J}_{\alpha_i\alpha_j}\alpha_k$. We must show:

$$\langle \mathcal{J}_{i_1i_2i_3}, \mathcal{J}_{j_1j_2j_3}\rangle = \langle \mathcal{J}_{i_1i_2j_3}, \mathcal{J}_{j_1j_2i_3}\rangle \quad \forall i_1i_2i_3j_1j_2j_3\,.$$

The non-zero components of $\mathcal{J}_{ijk} = \mathcal{J}_{jik}$ are:

$$\begin{array}{llllll} \mathcal{J}_{112} = \beta_{2,2}, & \mathcal{J}_{113} = \beta_{3,2}, & \mathcal{J}_{221} = \beta_{1,1}, \\ \mathcal{J}_{223} = \beta_{3,1}, & \mathcal{J}_{331} = \beta_{1,2}, & \mathcal{J}_{332} = \beta_{2,1}, \\ \mathcal{J}_{121} = -\tfrac{1}{2}\beta_{2,2}, & \mathcal{J}_{122} = -\tfrac{1}{2}\beta_{1,1}, & \mathcal{J}_{131} = -\tfrac{1}{2}\beta_{3,2}, \\ \mathcal{J}_{133} = -\tfrac{1}{2}\beta_{1,2}, & \mathcal{J}_{232} = -\tfrac{1}{2}\beta_{3,1}, & \mathcal{J}_{233} = -\tfrac{1}{2}\beta_{2,1}, \\ \mathcal{J}_{132} = \tfrac{1}{4}\beta^*_{4,1} - \tfrac{1}{2}\beta^*_{4,2} = \beta_{4,2}, \\ \mathcal{J}_{231} = -\tfrac{1}{2}\beta^*_{4,1} + \tfrac{1}{4}\beta^*_{4,2} = \beta_{4,1}, \\ \mathcal{J}_{123} = \tfrac{1}{4}\beta^*_{4,1} + \tfrac{1}{4}\beta^*_{4,2} = -\beta_{4,1} - \beta_{4,2}. \end{array}$$

The non-zero inner products are given by:

$$\begin{array}{llllll}
\langle\mathcal{J}_{112},\mathcal{J}_{332}\rangle &= 1, & \langle\mathcal{J}_{112},\mathcal{J}_{233}\rangle &= -\frac{1}{2}, & \langle\mathcal{J}_{121},\mathcal{J}_{332}\rangle &= -\frac{1}{2},\\
\langle\mathcal{J}_{121},\mathcal{J}_{233}\rangle &= \frac{1}{4}, & \langle\mathcal{J}_{113},\mathcal{J}_{223}\rangle &= 1, & \langle\mathcal{J}_{113},\mathcal{J}_{232}\rangle &= 6-\frac{1}{2},\\
\langle\mathcal{J}_{131},\mathcal{J}_{223}\rangle &= -\frac{1}{2}, & \langle\mathcal{J}_{232},\mathcal{J}_{131}\rangle &= \frac{1}{4}, & \langle\mathcal{J}_{221},\mathcal{J}_{331}\rangle &= 1,\\
\langle\mathcal{J}_{221},\mathcal{J}_{133}\rangle &= -\frac{1}{2}, & \langle\mathcal{J}_{122},\mathcal{J}_{331}\rangle &= -\frac{1}{2}, & \langle\mathcal{J}_{122},\mathcal{J}_{133}\rangle &= \frac{1}{4},\\
\langle\mathcal{J}_{123},\mathcal{J}_{123}\rangle &= \star, & \langle\mathcal{J}_{123},\mathcal{J}_{132}\rangle &= \frac{1}{4}, & \langle\mathcal{J}_{123},\mathcal{J}_{231}\rangle &= \frac{1}{4},\\
\langle\mathcal{J}_{132},\mathcal{J}_{132}\rangle &= \star, & \langle\mathcal{J}_{132},\mathcal{J}_{231}\rangle &= \frac{1}{4}, & \langle\mathcal{J}_{231},\mathcal{J}_{231}\rangle &= \star.
\end{array}$$

The desired symmetries are now immediate:

$$\begin{array}{llll}
\langle\mathcal{J}_{112},\mathcal{J}_{233}\rangle = -\frac{1}{2} &= \langle\mathcal{J}_{113},\mathcal{J}_{232}\rangle, & \langle\mathcal{J}_{123},\mathcal{J}_{132}\rangle = \frac{1}{4} &= \langle\mathcal{J}_{122},\mathcal{J}_{133}\rangle,\\
\langle\mathcal{J}_{121},\mathcal{J}_{332}\rangle = -\frac{1}{2} &= \langle\mathcal{J}_{122},\mathcal{J}_{331}\rangle, & \langle\mathcal{J}_{123},\mathcal{J}_{231}\rangle = \frac{1}{4} &= \langle\mathcal{J}_{121},\mathcal{J}_{233}\rangle,\\
\langle\mathcal{J}_{131},\mathcal{J}_{223}\rangle = -\frac{1}{2} &= \langle\mathcal{J}_{133},\mathcal{J}_{221}\rangle, & \langle\mathcal{J}_{132},\mathcal{J}_{231}\rangle = \frac{1}{4} &= \langle\mathcal{J}_{131},\mathcal{J}_{232}\rangle.
\end{array}$$

This completes the proof of Assertion (1).

Proof of Assertion (2). It is immediate from the definition that

$$\mathcal{J}_{\alpha_3}\mathcal{J}_{\alpha_2}\alpha_1 = \mathcal{J}_{\alpha_3}\beta_{1,1} = \alpha_1^*,$$

so there exist $x, y \in V$ so $\mathcal{J}_x\mathcal{J}_y \neq 0$. Let $\mathcal{A}_{ij} := \mathcal{A}(\alpha_i,\alpha_j)$. One shows $\mathcal{V}_{14}$ is not skew Tsankov by computing:

$$\begin{aligned}
&\mathcal{A}_{12}\mathcal{A}_{13}\alpha_3 = -\mathcal{A}_{12}\beta_{1,2} = -\alpha_2^*,\\
&\mathcal{A}_{13}\mathcal{A}_{12}\alpha_3 = \tfrac{1}{2}\mathcal{A}_{13}\{\beta_{4,1}^* - \beta_{4,2}^*\} = \mathcal{A}_{13}\{-\tfrac{2}{3}\beta_{4,1} + \tfrac{2}{3}\beta_{4,2}\} = \tfrac{1}{3}\alpha_2^*.
\end{aligned}$$

This establishes Assertion (2).

Proof of Assertion (3). Let $\mathcal{G} = \mathcal{G}(\mathcal{V}_{14})$ be the group of symmetries of the model $\mathcal{V}_{14}$. Note that the spaces V_{β,α^*} and V_{α^*} defined in Equation (10.4) are preserved by $\mathcal{G}$. Consequently one has that

$$TV_{\alpha^*} \subset V_{\alpha^*} \quad\text{and}\quad TV_{\beta,\alpha^*} \subset V_{\beta,\alpha^*} \quad\text{if}\quad T \in \mathcal{G}. \tag{10.5}$$

Let $\tau : \mathcal{G} \to Gl(3)$ be the restriction of T to $V_{\alpha^*} = \mathbb{R}^3$. We will prove Assertion (3) by showing:

$$Sl_{\pm}(3) = \tau(\mathcal{G}) \quad\text{and}\quad \ker(\tau) = \mathbb{R}^{21}.$$

We argue as follows to show $Sl_{\pm}(3) \subset \tau(\mathcal{G})$. Let $\beta_{4,3} := -\beta_{4,1} - \beta_{4,2}$. Define a linear map T of $\mathbb{R}^{14}$ which interchanges the first 2 coordinates of $\mathbb{R}^3$ by setting

$$\begin{array}{llll}
T:\alpha_1 \leftrightarrow \alpha_2, & T:\alpha_3 \leftrightarrow \alpha_3, & T:\alpha_1^* \leftrightarrow \alpha_2^*, & T:\alpha_3^* \leftrightarrow \alpha_3^*,\\
T:\beta_{1,1} \leftrightarrow \beta_{2,2}, & T:\beta_{1,2} \leftrightarrow \beta_{2,1}, & T:\beta_{3,1} \leftrightarrow \beta_{3,2}, & T:\beta_{4,1} \leftrightarrow \beta_{4,2}.
\end{array}$$

It is then immediate by inspection that T preserves the relations of Definition 10.3.1 and hence $T \in \mathcal{G}$. One may define a map $T \in \mathcal{G}$ which interchanges the first and third coordinates by setting:

$$\begin{array}{llll} T:\alpha_1 \leftrightarrow \alpha_3, & T:\alpha_2 \leftrightarrow \alpha_2, & T:\alpha_1^* \leftrightarrow \alpha_3^*, & T:\alpha_2^* \leftrightarrow \alpha_2^*, \\ T:\beta_{1,1} \leftrightarrow \beta_{3,1}, & T:\beta_{1,2} \leftrightarrow \beta_{3,2}, & T:\beta_{2,1} \leftrightarrow \beta_{2,2}, & T:\beta_{4,1} \leftrightarrow \beta_{4,3}, \\ T:\beta_{4,2} \leftrightarrow \beta_{4,2}\,. & & & \end{array}$$

To define a map $T \in \mathcal{G}$ which induces a rotation in the first two coordinates, we set

$$\begin{array}{ll} T_\theta : \alpha_1 \to \cos\theta\alpha_1 + \sin\theta\alpha_2, & T_\theta : \alpha_2 \to -\sin\theta\alpha_1 + \cos\theta\alpha_2, \\ T_\theta : \alpha_1^* \to \cos\theta\alpha_1^* + \sin\theta\alpha_2^*, & T_\theta : \alpha_2^* \to -\sin\theta\alpha_1^* + \cos\theta\alpha_2^*, \\ T_\theta : \alpha_3 \to \alpha_3, & T_\theta : \alpha_3^* \to \alpha_3^*, \\ T_\theta : \beta_{1,1} \to \cos\theta\beta_{1,1} + \sin\theta\beta_{2,2}, & T_\theta : \beta_{1,2} \to \cos\theta\beta_{1,2} + \sin\theta\beta_{2,1}, \\ T_\theta : \beta_{2,1} \to -\sin\theta\beta_{1,2} + \cos\theta\beta_{2,1}, & T_\theta : \beta_{2,2} \to -\sin\theta\beta_{1,1} + \cos\theta\beta_{2,2}, \\ \multicolumn{2}{l}{T_\theta : \beta_{3,1} \to \sin^2\theta\beta_{3,2} - 2\sin\theta\cos\theta\beta_{4,3} + \cos^2\theta\beta_{3,1},} \\ \multicolumn{2}{l}{T_\theta : \beta_{3,2} \to \cos^2\theta\beta_{3,2} + 2\cos\theta\sin\theta\beta_{4,3} + \sin^2\theta\beta_{3,1},} \\ \multicolumn{2}{l}{T_\theta : \beta_{4,1} \to \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,2} - \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,1} - \sin^2\theta\beta_{4,2} + \cos^2\theta\beta_{4,1},} \\ \multicolumn{2}{l}{T_\theta : \beta_{4,2} \to \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,2} - \tfrac{1}{2}\sin\theta\cos\theta\beta_{3,1} + \cos^2\theta\beta_{4,2} - \sin^2\theta\beta_{4,1}.} \end{array}$$

Finally, we show that the dilatations of determinant 1 belong to Range $\{\tau\}$. Suppose $a_1a_2a_3 = 1$. We may define $T \in \mathcal{G}$ by setting:

$$\begin{array}{rclrclrclrcl} T\alpha_1 &=& a_1\alpha_1, & T\alpha_2 &=& a_2\alpha_2, & T\alpha_3 &=& a_3\alpha_3, & T\alpha_1^* &=& \frac{1}{a_1}\alpha_1^*, \\ T\alpha_2^* &=& \frac{1}{a_2}\alpha_2^*, & T\alpha_3^* &=& \frac{1}{a_3}\alpha_3^*, & T\beta_{1,1} &=& \frac{a_2}{a_3}\beta_{1,1}, & T\beta_{1,2} &=& \frac{a_3}{a_2}\beta_{1,2}, \\ T\beta_{2,1} &=& \frac{a_3}{a_1}\beta_{2,1}, & T\beta_{2,2} &=& \frac{a_1}{a_3}\beta_{2,2}, & T\beta_{3,1} &=& \frac{a_2}{a_1}\beta_{3,1}, & T\beta_{3,2} &=& \frac{a_1}{a_2}\beta_{3,2}, \\ T\beta_{4,1} &=& \beta_{4,1}, & T\beta_{4,2} &=& \beta_{4,2}\,. & & & & & & \end{array}$$

Since these elements acting on V_{α^*} generate $Sl_\pm(3)$,

$$Sl_\pm(3) \subset \tau(\mathcal{G})\,.$$

Conversely, let $T \in \mathcal{G}$. We must show $\tau(T) \in Sl_\pm(3)$. Since $Sl_\pm(3) \subset \text{Range}\,(\tau)$, there exists $S \in \mathcal{G}$ so that $\tau(TS)$ is diagonal. Thus without loss of generality, we may assume $\tau(T)$ is diagonal and hence:

$$\begin{aligned} T\alpha_i &= a_i\alpha_i + \sum_\nu b_i^\nu\beta_\nu + \sum_j c_i^j\alpha_j^*, \\ T\beta_\nu &= b_\nu\beta_\nu + \sum_i d_\nu^i\alpha_i^*, \\ T\alpha_i^* &= a_i^{-1}\alpha_i^*\,. \end{aligned}$$

We have the relations

$$\begin{aligned}\tfrac{1}{2} &= A(T\alpha_1, T\alpha_2, T\alpha_3, T\beta_{4,1}) = \tfrac{1}{2}a_1a_2a_3b_{4,1},\\ -\tfrac{1}{2} &= \langle T\beta_{4,1}, T\beta_{4,1}\rangle = -\tfrac{1}{2}b_{4,1}b_{4,1}\,.\end{aligned}$$

These relations show that $b_{4,1}^2 = 1$ and thus $a_1a_2a_3 = \pm 1$. Consequently, one has that Range $(\tau) = Sl_\pm(3)$.

We complete the proof of Assertion (3) by studying $T \in \ker(\tau)$. One has

$$\begin{aligned}T\alpha_i &= \alpha_i + \sum_\nu b_i^\nu \beta_\nu + \sum_j c_i^j \alpha_j^*,\\ T\beta_\nu &= \beta_\nu + \sum_i d_\nu^i \alpha_i^*,\\ T\alpha_i^* &= \alpha_i^*\,.\end{aligned}$$

Using the relations $A(\alpha_i, \alpha_j, \alpha_k, \alpha_l) = 0$ then leads to the following 6 linear equations the coefficients b_i^ν must satisfy:

$$\begin{aligned}0 &= A(T\alpha_1, T\alpha_2, T\alpha_1, T\alpha_2) = 2A(\alpha_1, \alpha_2, \alpha_1, b_2^{2,1}\beta_{2,1}) + 2A(\alpha_2, \alpha_1, \alpha_2, b_1^{1,2}\beta_{1,2})\\ &= 2b_2^{2,1} + 2b_1^{1,2},\end{aligned}$$

$$\begin{aligned}0 &= A(T\alpha_1, T\alpha_3, T\alpha_1, T\alpha_3) = 2A(\alpha_1, \alpha_3, \alpha_1, b_3^{3,1}\beta_{3,1}) + 2A(\alpha_3, \alpha_1, \alpha_3, b_1^{1,1}\beta_{1,1})\\ &= 2b_3^{3,1} + 2b_1^{1,1},\end{aligned}$$

$$\begin{aligned}0 &= A(T\alpha_2, T\alpha_3, T\alpha_2, T\alpha_3) = 2A(\alpha_2, \alpha_3, \alpha_2, b_3^{3,2}\beta_{3,2}) + 2A(\alpha_3, \alpha_2, \alpha_3, b_2^{2,2}\beta_{2,2})\\ &= 2b_3^{3,2} + 2b_2^{2,2},\end{aligned}$$

$$\begin{aligned}0 &= A(T\alpha_1, T\alpha_2, T\alpha_1, T\alpha_3)\\ &= A(\alpha_1, b_2^{3,1}\beta_{3,1}, \alpha_1, \alpha_3) + A(\alpha_1, \alpha_2, \alpha_1, b_3^{2,1}\beta_{2,1})\\ &\quad + A(b_1^{4,1}\beta_{4,1} + b_1^{4,2}\beta_{4,2}, \alpha_2, \alpha_1, \alpha_3) + A(\alpha_1, \alpha_2, b_1^{4,1}\beta_{4,1} + b_1^{4,2}\beta_{4,2}, \alpha_3)\\ &= b_2^{3,1} + b_3^{2,1} - \frac{1}{2}b_1^{4,1} - \frac{1}{2}b_1^{4,1} + \frac{1}{2}b_1^{4,2},\end{aligned}$$

$$\begin{aligned}0 &= A(T\alpha_2, T\alpha_1, T\alpha_2, T\alpha_3)\\ &= A(\alpha_2, b_1^{3,2}\beta_{3,2}, \alpha_2, \alpha_3) + A(\alpha_2, \alpha_1, \alpha_2, b_3^{1,2}\beta_{1,2})\\ &\quad + A(b_2^{4,1}\beta_{4,1} + b_2^{4,2}\beta_{4,2}, \alpha_1, \alpha_2, \alpha_3) + A(\alpha_2, \alpha_1, b_2^{4,1}\beta_{4,1} + b_2^{4,2}\beta_{4,2}, \alpha_3)\\ &= b_1^{3,2} + b_3^{1,2} - \frac{1}{2}b_2^{4,2} + \frac{1}{2}b_2^{4,1} - \frac{1}{2}b_2^{4,2},\end{aligned}$$

$$\begin{aligned} 0 &= A(T\alpha_3, T\alpha_1, T\alpha_3, T\alpha_2)\\ &= A(\alpha_3, b_1^{2,2}\beta_{2,2}, \alpha_3, \alpha_2) + A(\alpha_3, \alpha_1, \alpha_3, b_2^{1,1}\beta_{1,1})\\ &\quad + A(b_3^{4,1}\beta_{4,1} + b_3^{4,2}\beta_{4,2}, \alpha_1, \alpha_3, \alpha_2) + A(\alpha_3, \alpha_1, b_3^{4,1}\beta_{4,1} + b_3^{4,2}\beta_{4,2}, \alpha_2)\\ &= b_1^{2,2} + b_2^{1,1} + \frac{1}{2}b_3^{4,2} + \frac{1}{2}b_3^{4,1}. \end{aligned}$$

These equations are linearly independent so there are 18 degrees of freedom in choosing the b's. Once the b's are known, the coefficients d_ν^i are determined;

$$0 = \langle T\alpha_i, T\beta_\nu\rangle = d_\nu^i + \sum_\mu \langle\beta_\nu, \beta_\mu\rangle b_i^\mu\,.$$

The relation $\langle T\alpha_i, T\alpha_j\rangle = \delta_{ij}$ implies $c_i^j + c_j^i = 0$; this creates an additional 3 degrees of freedom. Thus $\ker(\tau)$ is isomorphic to the additive group $\mathbb{R}^{21}$. This completes the proof of Assertion (3).

Proof of Assertion (4) Let $\xi_i \in V$. We wish to show

$$\mathcal{A}_{\xi_1\xi_2}\mathcal{J}_{\xi_3} = \mathcal{J}_{\xi_3}\mathcal{A}_{\xi_1\xi_2} \quad \text{for all} \quad \xi_i \in V\,.$$

Since $\mathcal{A}_{\xi_1\xi_2}\mathcal{J}_{\xi_3} = \mathcal{J}_{\xi_3}\mathcal{A}_{\xi_1\xi_2} = 0$ if any of the $\xi_i \in V_{\beta,\alpha^*}$, we may work modulo V_{β,α^*} and suppose that $\xi_i \in \operatorname{Span}\{\alpha_i\}$. Since $\mathcal{A}_{\xi_1\xi_2} = 0$ if the ξ_i are linearly dependent, we suppose ξ_1 and ξ_2 are linearly independent.

There are 2 cases to be considered. We first suppose $\xi_3 \in \operatorname{Span}\{\xi_1, \xi_2\}$. The argument given above shows that a subgroup of $\mathcal{G}$ isomorphic to $Sl_\pm(3)$ acts on $\operatorname{Span}\{\alpha_i\}$. After reparametrizing by this action, we may suppose

$$\operatorname{Span}\{\xi_1, \xi_2\} = \operatorname{Span}\{\alpha_1, \alpha_2\} \quad \text{and} \quad \xi_3 = \alpha_1\,.$$

Furthermore, because $\mathcal{A}_{\xi_1\xi_2} = c\mathcal{A}_{\alpha_1\alpha_2}$, we may also assume $\xi_1 = \alpha_1$ and $\xi_2 = \alpha_2$. We set $\mathcal{A}_{ij} := \mathcal{A}_{\alpha_i\alpha_j}$ and $\mathcal{J}_k := \mathcal{J}_{\alpha_k}$. The desired result is obtained by computing:

$$\begin{aligned} \mathcal{A}_{12}\mathcal{J}_1\alpha_1 &= 0, & \mathcal{J}_1\mathcal{A}_{12}\alpha_1 &= \mathcal{J}_1\beta_{2,2} = 0,\\ \mathcal{A}_{12}\mathcal{J}_1\alpha_2 &= \mathcal{A}_{12}\beta_{2,2} = 0, & \mathcal{J}_1\mathcal{A}_{12}\alpha_2 &= -\mathcal{J}_1\beta_{1,1} = 0,\\ \mathcal{A}_{12}\mathcal{J}_1\alpha_3 &= \mathcal{A}_{12}\beta_{3,2} = 0, & \mathcal{J}_1\mathcal{A}_{12}\alpha_3 &= \tfrac{1}{2}\mathcal{J}_1(\beta_{4,1}^* - \beta_{4,2}^*) = 0\,. \end{aligned}$$

On the other hand, if $\{\xi_1, \xi_2, \xi_3\}$ are linearly independent, we can apply a symmetry in $\mathcal{G}$ and rescale to assume $\xi_i = \alpha_i$. We compute:

$$\begin{aligned} \mathcal{A}_{12}\mathcal{J}_3\alpha_1 &= \mathcal{A}_{12}\beta_{1,2} = \alpha_2^*, & \mathcal{J}_3\mathcal{A}_{12}\alpha_1 &= \mathcal{J}_3\beta_{2,2} = \alpha_2^*,\\ \mathcal{A}_{12}\mathcal{J}_3\alpha_2 &= \mathcal{A}_{12}\beta_{2,1} = -\alpha_1^*, & \mathcal{J}_3\mathcal{A}_{12}\alpha_2 &= -\mathcal{J}_3\beta_{1,1} = -\alpha_1^*,\\ \mathcal{A}_{12}\mathcal{J}_3\alpha_3 &= 0, & \mathcal{J}_3\mathcal{A}_{12}\alpha_3 &= \tfrac{1}{2}\mathcal{J}_3(\beta_{4,1}^* - \beta_{4,2}^*) = 0\,. \end{aligned}$$

Assertion (4) follows and therefore, the proof is complete. □

10.3.2 Some geometric realizations of $\mathcal{V}_{14}$

In this section we show $\mathcal{V}_{14}$ is geometrically realizable by considering the following family of examples:

Definition 10.3.2 Let $\{x_i, x_i^*, y_{i,1}, y_{i,2}, y_{4,1}, y_{4,2}\}$ be coordinates on $\mathbb{R}^{14}$ where the index i ranges from 1 through 3. Suppose given a collection of functions $\Phi := \{\phi_{i,1}, \phi_{i,2}\} \in C^\infty(\mathbb{R})$ with $\phi'_{i,1}\phi'_{i,2} = 1$. Let $\mathcal{M}_\Phi := (\mathbb{R}^{14}, g_\Phi)$ where the non-zero components of g_Φ are, up to the usual $\mathbb{Z}_2$ symmetry, given by:

$$\begin{aligned}
&g_\Phi(\partial_{x_i}, \partial_{x_i^*}) = g_\Phi(\partial_{y_{i,1}}, \partial_{y_{i,2}}) = 1,\\
&g_\Phi(\partial_{y_{4,1}}, \partial_{y_{4,1}}) = g_\Phi(\partial_{y_{4,2}}, \partial_{y_{4,2}}) = -\tfrac{1}{2}, \quad g_\Phi(\partial_{y_{4,1}}, \partial_{y_{4,2}}) = \tfrac{1}{4},\\
&g_\Phi(\partial_{x_1}, \partial_{x_1}) = -2\phi_{2,1}(x_2)y_{2,1} - 2\phi_{3,1}(x_3)y_{3,1},\\
&g_\Phi(\partial_{x_2}, \partial_{x_2}) = -2\phi_{3,2}(x_3)y_{3,2} - 2\phi_{1,2}(x_1)y_{1,2},\\
&g_\Phi(\partial_{x_3}, \partial_{x_3}) = -2\phi_{1,1}(x_1)y_{1,1} - 2\phi_{2,2}(x_2)y_{2,2},\\
&g_\Phi(\partial_{x_1}, \partial_{x_3}) = x_2 y_{4,2}, \quad g_\Phi(\partial_{x_2}, \partial_{x_3}) = x_1 y_{4,1}.
\end{aligned}$$

Theorem 10.3.2 *Let $\mathcal{M}_\Phi := (\mathbb{R}^{14}, g_\Phi)$ be as in Definition 10.3.2.*

1. *$\mathcal{M}_\Phi$ is geodesically complete.*
2. *For all $P \in \mathbb{R}^{14}$, $\exp_P$ is a diffeomorphism from $T_P(\mathbb{R}^{14})$ to $\mathbb{R}^{14}$.*
3. *$\mathcal{M}_\Phi$ realizes the model $\mathcal{V}_{14}$.*

In order to proof Assertions (1) and (2) of Theorem 10.3.2, we introduce the following more general construction:

Definition 10.3.3 Let $\{x_i, x_i^*, y_\mu\}$ be coordinates on $\mathbb{R}^{2a+b}$ where $1 \le i \le a$ and $1 \le \mu \le b$. Given a non-degenerate symmetric matrix $C_{\mu\nu}$ and smooth functions $\psi_{ij\mu} = \psi_{ij\mu}(\vec{x})$ with $\psi_{ij\mu} = \psi_{ji\mu}$, define the following pseudo-Riemannian manifold

$$\mathcal{M}_{C,\psi} := (\mathbb{R}^{2a+b}, g_{C,\psi}),$$

where

$$g_{C,\psi}(\partial_{x_i}, \partial_{x_j}) = 2\textstyle\sum_k y_\mu \psi_{ij\mu}, \quad g_{C,\psi}(\partial_{x_i}, \partial_{x_i^*}) = 1, \quad g_{C,\psi}(\partial_{y_\mu}, \partial_{y_\nu}) = C_{\mu\nu}\,.$$

Next lemma describes some aspects of the geometry of manifolds in Definition 10.3.3.

Lemma 10.3.3 *Let $\mathcal{M}_{C,\psi} = (\mathbb{R}^{2a+b}, g_{C,\psi})$ be as in Definition 10.3.3. Then*

1. *$\mathcal{M}_{C,\psi}$ is geodesically complete.*
2. *For all $P \in \mathbb{R}^{2a+b}$, $\exp_P$ is a diffeomorphism from $T_P(\mathbb{R}^{2a+b})$ to $\mathbb{R}^{2a+b}$.*

3. *The possibly non-zero components of the curvature tensor are, up to the usual* $\mathbb{Z}_2$ *symmetries given by:*

$$\begin{aligned} R(\partial_{x_i},\partial_{x_j},\partial_{x_k},\partial_{y_\nu}) &= \partial_{x_i}\psi_{jk\nu}-\partial_{x_j}\psi_{ik\nu},\\ R(\partial_{x_i},\partial_{x_j},\partial_{x_k},\partial_{x_l}) &= \sum_{\nu\mu} C^{\nu\mu}\left\{\psi_{il\mu}\psi_{jk\nu}-\psi_{ik\mu}\psi_{jl\nu}\right\}\\ &\quad +\sum_{\nu} y_\nu\left\{\partial_{x_i}\partial_{x_l}\psi_{jk\nu}+\partial_{x_j}\partial_{x_k}\psi_{il\nu}\right.\\ &\qquad\left.-\partial_{x_i}\partial_{x_k}\psi_{jl\nu}-\partial_{x_j}\partial_{x_l}\psi_{ik\nu}\right\}. \end{aligned}$$

Proof. The non-zero Christoffel symbols of the first kind are given by:

$$\begin{aligned} g(\nabla_{\partial_{x_i}}\partial_{x_j},\partial_{x_k}) &= \sum_\mu\{\partial_{x_i}\psi_{jk\mu}+\partial_{x_j}\psi_{ik\mu}-\partial_{x_k}\psi_{ij\mu}\}y_\mu,\\ g(\nabla_{\partial_{x_i}}\partial_{x_j},\partial_{y_\nu}) &= -\psi_{ij\nu},\\ g(\nabla_{\partial_{x_i}}\partial_{y_\nu},\partial_{x_k}) &= g(\nabla_{\partial_{y_\nu}}\partial_{x_i},\partial_{x_k})=\psi_{ik\nu}, \end{aligned}$$

and the non-zero Christoffel symbols of the second kind are given by:

$$\begin{aligned} \nabla_{\partial_{x_i}}\partial_{x_j} &= \sum_\mu y_\mu\{\partial_{x_i}\psi_{jk\mu}+\partial_{x_j}\psi_{ik\mu}-\partial_{x_k}\psi_{ij\mu}\}\partial_{x_k^*}-\sum_{\mu\nu}C^{\nu\mu}\psi_{ij\nu}\partial_{y_\mu},\\ \nabla_{\partial_{x_i}}\partial_{y_\nu} &= \nabla_{\partial_{y_\nu}}\partial_{x_i}=\sum_k\psi_{ik\nu}\partial_{x_k^*}. \end{aligned}$$

This shows that $\mathcal{M}$ is a generalized plane wave manifold; Assertions (1) and (2) then follow from results in [86]. Assertion (3) now follows by a direct calculation. □

Now, we are ready to give the proof of the Theorem.

Proof of Theorem 10.3.2. Assertions (1) and (2) of Theorem 10.3.2 follow by specializing the corresponding results of Lemma 10.3.3. We use Assertion (3) of Lemma 10.3.3 to see that the possibly non-zero components of the curvature tensor defined by the metric of Definition 10.3.2 are:

$$\begin{aligned} R(\partial_{x_{i_1}},\partial_{x_{i_2}},\partial_{x_{i_3}},\partial_{x_{i_4}}) &= \star,\\ R(\partial_{x_1},\partial_{x_2},\partial_{x_1},\partial_{y_{2,1}}) &= \partial_{x_2}\phi_{2,1}, & R(\partial_{x_1},\partial_{x_3},\partial_{x_1},\partial_{y_{3,1}}) &= \partial_{x_3}\phi_{3,1},\\ R(\partial_{x_2},\partial_{x_3},\partial_{x_2},\partial_{y_{3,2}}) &= \partial_{x_3}\phi_{3,2}, & R(\partial_{x_2},\partial_{x_1},\partial_{x_2},\partial_{y_{1,2}}) &= \partial_{x_1}\phi_{1,2},\\ R(\partial_{x_3},\partial_{x_1},\partial_{x_3},\partial_{y_{1,1}}) &= \partial_{x_1}\phi_{1,1}, & R(\partial_{x_3},\partial_{x_2},\partial_{x_3},\partial_{y_{2,2}}) &= \partial_{x_2}\phi_{2,2},\\ R(\partial_{x_2},\partial_{x_1},\partial_{x_3},\partial_{y_{4,1}}) &= R(\partial_{x_3},\partial_{x_1},\partial_{x_2},\partial_{y_{4,1}})=-\tfrac12,\\ R(\partial_{x_1},\partial_{x_2},\partial_{x_3},\partial_{y_{4,2}}) &= R(\partial_{x_3},\partial_{x_2},\partial_{x_1},\partial_{y_{4,2}})=-\tfrac12. \end{aligned}$$

We introduce the following basis as a first step in the proof of Assertion (3). Let the index i range from 1 to 3 and the index j run from 1 to 2. Set:

$$\bar\alpha_i := \partial_{x_i}, \quad \alpha_i^* := \partial_{x_i^*}, \quad \bar\beta_{4,j} := \partial_{y_{4,j}}, \quad \bar\beta_{i,j} := \{\phi_{i,j}'\}^{-1}\partial_{y_{i,j}}\,. \tag{10.6}$$

Since $\phi_{i,1}'\cdot\phi_{i,2}' = 1$, the relations of (10.3) are satisfied. However, we still have the following potentially non-zero terms to deal with:

$$g(\bar\alpha_i,\bar\alpha_j) = \star \quad\text{and}\quad R(\bar\alpha_i,\bar\alpha_j,\bar\alpha_k,\bar\alpha_l) = \star\,.$$

To deal with the extra curvature terms, we introduce a modified basis setting:

$$\begin{array}{rcl}
\tilde\alpha_1 &:=& \bar\alpha_1 + R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_1,\bar\alpha_3)\bar\beta_{4,1} - \frac12 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_1,\bar\alpha_2)\bar\beta_{1,2},\\
\tilde\alpha_2 &:=& \bar\alpha_2 + R(\bar\alpha_2,\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)\bar\beta_{4,2} - \frac12 R(\bar\alpha_2,\bar\alpha_3,\bar\alpha_2,\bar\alpha_3)\bar\beta_{2,2},\\
\tilde\alpha_3 &:=& \bar\alpha_3 - 2R(\bar\alpha_3,\bar\alpha_1,\bar\alpha_3,\bar\alpha_2)\bar\beta_{4,1} - \frac12 R(\bar\alpha_1,\bar\alpha_3,\bar\alpha_1,\bar\alpha_3)\bar\beta_{3,1},\\
\beta_{1,1} &:=& \bar\beta_{1,1} + \frac12 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_1,\bar\alpha_2)\alpha_1^*, \qquad \beta_{1,2} := \bar\beta_{1,2},\\
\beta_{2,1} &:=& \bar\beta_{2,1} + \frac12 R(\bar\alpha_2,\bar\alpha_3,\bar\alpha_2,\bar\alpha_3)\alpha_2^*, \qquad \beta_{2,2} := \bar\beta_{2,2},\\
\beta_{3,2} &:=& \bar\beta_{3,2} + \frac12 R(\bar\alpha_1,\bar\alpha_3,\bar\alpha_1,\bar\alpha_3)\alpha_3^*, \qquad \beta_{3,1} := \bar\beta_{3,1},\\
\beta_{4,1} &:=& \bar\beta_{4,1} + \frac12 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_1,\bar\alpha_3)\alpha_1^* - \frac14 R(\bar\alpha_2,\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)\alpha_2^*\\
&& -R(\bar\alpha_3,\bar\alpha_1,\bar\alpha_3,\bar\alpha_2)\alpha_3^*,\\
\beta_{4,2} &:=& \bar\beta_{4,2} - \frac14 R(\bar\alpha_1,\bar\alpha_2,\bar\alpha_1,\bar\alpha_3)\alpha_1^* + \frac12 R(\bar\alpha_2,\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)\alpha_2^*\\
&& +\frac12 R(\bar\alpha_3,\bar\alpha_1,\bar\alpha_3,\bar\alpha_2)\alpha_3^*\,.
\end{array} \tag{10.7}$$

All the normalizations of Equation (10.3) are satisfied except for the unwanted metric terms $g(\tilde\alpha_i,\tilde\alpha_j)$. To eliminate these terms and to exhibit a basis with the required normalizations, we set:

$$\alpha_i := \tilde\alpha_i - \tfrac12 \textstyle\sum_j g(\tilde\alpha_i,\tilde\alpha_j)\alpha_j^*\,. \tag{10.8}$$

This completes the proof of the Theorem. □

Isometry invariants

An important task, when one studies geometric realizations of algebraic models is to find isometry invariants as a useful tool to understand the geometry of the built manifold. That is our purpose here and the goal of the following theorem, which specializes a bit examples of Definition 10.3.2 to show that those manifolds are not locally homogeneous.

Theorem 10.3.4 *Adopt notation of Definition 10.3.2. Let $\phi_{2,1}(x_2) = \phi_{2,2}(x_2) = x_2$ and $\phi_{3,1}(x_3) = \phi_{3,2}(x_3) = x_3$. Let $\{\phi_{1,1},\phi_{1,2}\}$ be real analytic with $\phi_{1,1}'\phi_{1,2}' = 1$ and with $\phi_{1,j}'' \neq 0$. Then*

1. $\Xi := \{1 - \phi'_{1,1}\phi'''_{1,1}(\phi''_{1,1})^{-2}\}^2$ *is a local isometry invariant of* $\mathcal{M}_\Phi$.
2. *If* $\phi'_{1,1}(x_1) \neq be^{cx_1}$, *then* Ξ *is not locally constant and hence* $\mathcal{M}_\Phi$ *is not locally homogeneous.*

Before we proceed with the proof of the theorem, we need some analysis on the curvature of the manifolds under consideration.

Lemma 10.3.5 *Adopt the assumptions of Theorem 10.3.4. Let* $\{\alpha_i, \beta_\nu, \alpha_i^*\}$ *be defined by Equations (10.6), (10.7) and (10.8). Set* $\phi_1 := \phi'_{1,1}$ *and* $\phi_2 := \phi'_{1,2}$. *Then:*

1. $\nabla R(v_1, v_2, v_3, v_4; v_5) = 0$ *if at least one of the* $v_i \in V_{\alpha^*}$.
2. $\nabla R(v_1, v_2, v_3, v_4; v_5) = 0$ *if at least two of the* $v_i \in V_{\beta,\alpha^*}$.
3. $\nabla^k R(\alpha_2, \alpha_1, \alpha_2, \beta_{1,2}; \alpha_1, ..., \alpha_1) = \phi_2^{-1}\phi_2^{(k)}$.
4. $\nabla^k R(\alpha_3, \alpha_1, \alpha_3, \beta_{1,1}; \alpha_1, ..., \alpha_1) = \phi_1^{-1}\phi_1^{(k)}$.
5. $\nabla R(\alpha_i, \alpha_j, \alpha_k, \beta_\nu; \alpha_{l_1}, ..., \alpha_{l_k}) = 0$ *in cases other than those given in (3) and (4) up to the usual* $\mathbb{Z}_2$ *symmetry in the first 2 entries.*

Proof. Let v_i be coordinate vector fields. To prove Assertion (1), we suppose some $v_i \in V_{\alpha^*}$. We use the second Bianchi identity and the other curvature symmetries to assume, without loss of generality, that $v_1 \in V_{\alpha^*}$. Assertion (1) follows since $\nabla_{v_5} v_1 = 0$ and $R(v_1, \cdot, \cdot, \cdot) = 0$. The proof of Assertion (2) is similar and uses the fact that $R(\cdot, \cdot, \cdot, \cdot) = 0$ if 2-entries belong to V_{β,α^*}. The proof of the remaining assertions is similar and uses the particular form of the warping functions $\phi_{i,j}$; the factor of $\phi_{1,j}^{-1}$ arising from the normalization in Equation (10.6). □

Definition 10.3.4 *We say that a basis* $\tilde{\mathcal{B}} := \{\tilde\alpha_i, \tilde\beta_\nu, \tilde\alpha_i^*\}$ *is* 0*-normalized if the normalizations of Equation (10.3) are satisfied and* 1*-normalized if it is* 0*-normalized and if additionally*

$$\begin{aligned} \nabla R(\tilde\alpha_3, \tilde\alpha_1, \tilde\alpha_3, \tilde\beta_{1,1}; \tilde\alpha_1) &\neq 0, \\ \nabla R(\tilde\alpha_2, \tilde\alpha_1, \tilde\alpha_2, \tilde\beta_{1,2}; \tilde\alpha_1) &\neq 0, \\ \nabla R(\tilde\alpha_i, \tilde\alpha_j, \tilde\alpha_k, \tilde\beta_\nu; \tilde\alpha_l) &= 0 \quad \textit{otherwise}. \end{aligned}$$

Lemma 10.3.6 *Adopt the assumptions of Theorem 10.3.4. Then:*

1. *There exists a* 1*-normalized basis.*
2. *If* $\tilde{\mathcal{B}}$ *is a* 1*-normalized basis, then there exist constants* a_i *so* $a_1a_2a_3 = \varepsilon$ *for* $\varepsilon = \pm1$ *and so that exactly one of the following conditions holds:*
   (a) $\tilde\alpha_1 = a_1\alpha_1$, $\tilde\alpha_2 = a_2\alpha_2$, $\tilde\alpha_3 = a_3\alpha_3$, $\tilde\beta_{1,1} = \varepsilon\frac{a_2}{a_3}\beta_{1,1}$, $\tilde\beta_{1,2} = \varepsilon\frac{a_3}{a_2}\beta_{1,2}$.

(b) $\tilde\alpha_1 = a_1\alpha_1$, $\tilde\alpha_2 = a_3\alpha_3$, $\tilde\alpha_3 = a_2\alpha_2$, $\tilde\beta_{1,1} = \varepsilon\frac{a_3}{a_2}\beta_{1,2}$, $\tilde\beta_{1,2} = \varepsilon\frac{a_2}{a_3}\beta_{1,1}$.

Proof. We use Equation (10.6), Equation (10.7), and Equation (10.8) to construct a 0-normalized basis and then apply Lemma 10.3.5 to see this basis is 1-normalized. On the other hand, if $\tilde{\mathcal{B}}$ is a 1-normalized basis, we may expand:

$$\begin{array}{rclrcl}
\tilde\alpha_1 &=& a_{11}\alpha_1 + a_{12}\alpha_2 + a_{13}\alpha_3 + ..., && & \\
\tilde\alpha_2 &=& a_{21}\alpha_1 + a_{22}\alpha_2 + a_{23}\alpha_3 + ..., & \tilde\beta_{1,2} &=& b_{21}\beta_{1,1} + b_{22}\beta_{1,2} + ..., \\
\tilde\alpha_3 &=& a_{31}\alpha_1 + a_{32}\alpha_2 + a_{33}\alpha_3 + ..., & \tilde\beta_{1,1} &=& b_{11}\beta_{1,1} + b_{12}\beta_{1,2} + ...
\end{array}$$

Because

$$\begin{aligned}
0 \neq \nabla R(\tilde\alpha_2, \tilde\alpha_1, \tilde\alpha_2, \tilde\beta_{1,2}; \tilde\alpha_1) &= a_{11}\left\{(a_{11}a_{22} - a_{12}a_{21})a_{22}b_{22})\phi_2^{-1}\phi_2' \right.\\
&\quad \left. + (a_{11}a_{33} - a_{13}a_{31})a_{33}b_{21})\phi_1^{-1}\phi_1'\right\},
\end{aligned}$$

we have $a_{11} \neq 0$. Because

$$0 = \nabla R(\tilde\alpha_2, \tilde\alpha_1, \tilde\alpha_2, \tilde\beta_{1,2}; \tilde\alpha_2) = \tfrac{a_{21}}{a_{11}} \nabla R(\tilde\alpha_2, \tilde\alpha_1, \tilde\alpha_2, \tilde\beta_{1,2}; \tilde\alpha_1),$$

we have $a_{21} = 0$; similarly $a_{31} = 0$. Since $\text{Span}\{\alpha_i\} = \text{Span}\{\tilde\alpha_i\} \bmod V_{\beta,\alpha*}$,

$$a_{22}\, a_{33} - a_{23}\, a_{32} \neq 0\,.$$

By hypothesis we have that $R(\tilde\alpha_1, \tilde\alpha_2, \tilde\alpha_3, \beta; \tilde\alpha_1) = 0$ for any β which belongs to the subspace $\text{Span}\{\tilde\beta_\nu, \tilde\alpha_i^*\} = V_{\beta,\alpha^*}$, so

$$\begin{aligned}
0 &= R(\tilde\alpha_1, \tilde\alpha_2, \tilde\alpha_3, \beta_{1,2}; \tilde\alpha_1) = -a_{11}^2\, a_{22}\, a_{32}\phi_2^{-1}\phi_2', \\
0 &= R(\tilde\alpha_1, \tilde\alpha_2, \tilde\alpha_3, \beta_{1,1}; \tilde\alpha_1) = -a_{11}^2\, a_{23}\, a_{33}\phi_1^{-1}\phi_1'\,.
\end{aligned}$$

Suppose that $a_{22} \neq 0$. Since $a_{11}^2\, a_{22}\, a_{32} = 0$ and $a_{11} \neq 0$, $a_{32} = 0$. Since $a_{22}\, a_{33} - a_{23}\, a_{32} \neq 0$, $a_{33} \neq 0$. Since $a_{11}^2 a_{23}\, a_{33} = 0$, we also have $a_{23} = 0$. Since the basis is also 0-normalized,

$$diag(a_{11}^{-1}, a_{22}^{-1}, a_{33}^{-1}) \in Sl_\pm(3)$$

from the discussion in Subsection 10.3.1. Thus

$$\varepsilon := a_{11}a_{22}a_{33} = \pm 1, \quad b_{11} = \varepsilon\tfrac{a_{33}}{a_{22}}, \quad b_{22} = \varepsilon\tfrac{a_{22}}{a_{33}}\,.$$

These are the relations of Assertion (2a). The argument is similar if $a_{32} \neq 0$; we simply reverse the roles of $\tilde\alpha_2$ and $\tilde\alpha_3$ to establish the relations of Assertion (2b). □

We are now ready to give the proof of the main theorem in this subsection.

Proof of Theorem 10.3.4. Let

$$\Xi(\mathcal{B}) := \frac{1}{4}\left\{\frac{\nabla^2 R(\tilde\alpha_2,\tilde\alpha_1,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1,\tilde\alpha_1)}{\{\nabla R(\tilde\alpha_2,\tilde\alpha_1,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1)\}^2} - \frac{\nabla^2 R(\tilde\alpha_3,\tilde\alpha_1,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1,\tilde\alpha_1)}{\{\nabla R(\tilde\alpha_3,\tilde\alpha_1,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1)\}^2}\right\}^2 .$$

We apply Lemma 10.3.6. Suppose the conditions of Assertion (2a) hold. Then:

$$\begin{aligned}
\nabla R(\tilde\alpha_2,\tilde\alpha_1,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1) &= a_1\phi_2^{-1}\phi_2',\\
\nabla^2 R(\tilde\alpha_2,\tilde\alpha_1,\tilde\alpha_2,\tilde\beta_{1,2};\tilde\alpha_1,\tilde\alpha_1) &= a_1^2\phi_2^{-1}\phi_2'',\\
\nabla R(\tilde\alpha_3,\tilde\alpha_1,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1) &= a_1\phi_1^{-1}\phi_1',\\
\nabla^2 R(\tilde\alpha_3,\tilde\alpha_1,\tilde\alpha_3,\tilde\beta_{1,1};\tilde\alpha_1,\tilde\alpha_1) &= a_1^2\phi_1^{-1}\phi_1'' .
\end{aligned}$$

Consequently one has that

$$\Xi(\mathcal{B}) = \frac{1}{4}\left\{\frac{\phi_2\phi_2''}{\phi_2'\phi_2'} - \frac{\phi_1\phi_1''}{\phi_1'\phi_1'}\right\}^2 .$$

The roles of ϕ_1 and ϕ_2 are reversed if Assertion (2b) holds. It now follows that Ξ is a local isometry invariant. Since

$$\phi_2 = \phi_1^{-1},\quad \phi_2' = -\phi_1^{-2}\phi_1',\quad \phi_2'' = 2\phi_1^{-3}\phi_1'\phi_1' - \phi_1^{-2}\phi_1'',$$

we may establish Assertion (1) of Theorem 10.3.4 by computing

$$\frac{\phi_2\phi_2''}{\phi_2'\phi_2'} = \frac{\phi_1^{-1}(2\phi_1^{-3}\phi_1'\phi_1' - \phi_1^{-2}\phi_1'')}{\phi_1^{-4}\phi_1'\phi_1'} = 2 - \frac{\phi_1\phi_1''}{\phi_1'\phi_1'} .$$

Hence

$$\Xi = \frac{1}{4}\left\{2 - 2\frac{\phi_1\phi_1''}{\phi_1'\phi_1'}\right\}^2 .$$

If $\mathcal{M}_\Phi$ is locally homogeneous, then Ξ must be constant. Conversely, if Ξ is constant, then $\phi_1\phi_1'' = k\phi_1'\phi_1'$ for some $k \in \mathbb{R}$. The solutions to this ordinary differential equation take the form $\phi_1(t) = a(t+b)^c$ if $k \neq 1$ and $\phi_1(t) = ae^{bt}$ if $k = 1$ for suitably chosen constants a and b and for $c = c(k)$. The first family is ruled out as ϕ_1 and ϕ_1' must be invertible for all t. Thus $\phi_1(t)$ is a pure exponential; Assertion (2) of Theorem 10.3.4 follows. □

10.3.3 A symmetric space with model $\mathcal{V}_{14}$

In the following result we show that there are symmetric spaces which have model $\mathcal{V}_{14}$. Thus, this shows that model $\mathcal{V}_{14}$, which was found as an algebraic counterexample, is realized geometrically by a great variety of manifolds with interesting properties such as being symmetric.

Definition 10.3.5 Let $\{x_i, x_i^*, y_{i,1}, y_{i,2}, y_{4,1}, y_{4,2}\}$ for $1 \le i \le 3$ be coordinates on $\mathbb{R}^{14}$. Let $A := \{a_{i,j}\}$ be a collection of real constants. Let $\mathcal{M}_A := (\mathbb{R}^{14}, g_A)$ where the non-zero components of g_A are given, up to the usual $\mathbb{Z}_2$ symmetry, by:

$$\begin{aligned}
g_A(\partial_{x_i}, \partial_{x_i^*}) &= g_A(\partial_{y_{i,1}}, \partial_{y_{i,2}}) = 1, \\
g_A(\partial_{y_{4,1}}, \partial_{y_{4,1}}) &= g_A(\partial_{y_{4,2}}, \partial_{y_{4,2}}) = -\tfrac{1}{2}, \qquad g_A(\partial_{y_{4,1}}, \partial_{y_{4,2}}) = \tfrac{1}{4}, \\
g_A(\partial_{x_1}, \partial_{x_1}) &= -2a_{2,1}x_2y_{2,1} - 2a_{3,1}x_3y_{3,1}, \\
g_A(\partial_{x_2}, \partial_{x_2}) &= -2a_{3,2}x_3y_{3,2} - 2a_{1,2}x_1y_{1,2}, \\
g_A(\partial_{x_3}, \partial_{x_3}) &= -2a_{1,1}x_1y_{1,1} - 2a_{2,2}x_2y_{2,2}, \\
g_A(\partial_{x_1}, \partial_{x_2}) &= 2(1-a_{2,1})x_1y_{2,1} + 2(1-a_{1,2})x_2y_{1,2}, \\
g_A(\partial_{x_2}, \partial_{x_3}) &= x_1y_{4,1} + 2(1-a_{3,2})x_2y_{3,2} + 2(1-a_{2,2})x_3y_{2,2}, \\
g_A(\partial_{x_1}, \partial_{x_3}) &= x_2y_{4,2} + 2(1-a_{3,1})x_1y_{3,1} + 2(1-a_{1,1})x_3y_{1,1}.
\end{aligned}$$

The following theorem describes the geometry of these examples.

Theorem 10.3.7 *Let $\mathcal{M}_A$ be described by Definition 10.3.5. Then $\mathcal{M}_A$ is a generalized plane wave manifold with model $\mathcal{V}_{14}$. Furthermore $\mathcal{M}_A$ is locally symmetric if and only if the following equations are satisfied:*

$$\begin{aligned}
a_{1,1} + a_{2,2} + a_{3,1}a_{3,2} &= 2, \\
3a_{2,1} + 3a_{3,1} + 3a_{1,2}a_{1,1} &= 4, \\
3a_{1,2} + 3a_{3,2} + 3a_{2,1}a_{2,2} &= 4.
\end{aligned}$$

Proof. Let $\mathcal{M}_A$ be as described in Definition 10.3.5. By Lemma 10.3.3 one has that

$$\begin{aligned}
R(\partial_{x_1}, \partial_{x_2}, \partial_{x_1}, \partial_{y_{2,1}}) &= R(\partial_{x_1}, \partial_{x_3}, \partial_{x_1}, \partial_{y_{3,1}}) = 1, \\
R(\partial_{x_2}, \partial_{x_3}, \partial_{x_2}, \partial_{y_{3,2}}) &= R(\partial_{x_2}, \partial_{x_1}, \partial_{x_2}, \partial_{y_{1,2}}) = 1, \\
R(\partial_{x_3}, \partial_{x_1}, \partial_{x_3}, \partial_{y_{1,1}}) &= R(\partial_{x_3}, \partial_{x_2}, \partial_{x_3}, \partial_{y_{2,2}}) = 1, \\
R(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{y_{4,1}}) &= R(\partial_{x_1}, \partial_{x_3}, \partial_{x_2}, \partial_{y_{4,1}}) = \tfrac{1}{2}, \\
R(\partial_{x_2}, \partial_{x_3}, \partial_{x_1}, \partial_{y_{4,2}}) &= R(\partial_{x_2}, \partial_{x_1}, \partial_{x_3}, \partial_{y_{4,2}}) = \tfrac{1}{2}.
\end{aligned}$$

The same argument constructing a 0-normalized basis which was given in the proof of Theorem 10.3.1 can then be used to construct a 0-normalized basis in this setting and establish that $\mathcal{M}_A$ has algebraic model $\mathcal{V}_{14}$.

We can also apply Lemma 10.3.3 to see:

$$
\begin{aligned}
&R(\partial_{x_2},\partial_{x_1},\partial_{x_2},\partial_{x_1}) = -a_{3,1}a_{3,2}x_3^2,\\
&R(\partial_{x_3},\partial_{x_1},\partial_{x_3},\partial_{x_1}) = -\tfrac{1}{3}\,(2+3a_{2,1}a_{2,2})x_2^2,\\
&R(\partial_{x_2},\partial_{x_3},\partial_{x_2},\partial_{x_3}) = -\tfrac{1}{3}\,(2+3a_{1,1}a_{1,2})x_1^2,\\
&R(\partial_{x_1},\partial_{x_2},\partial_{x_1},\partial_{x_3}) = (1-a_{1,1}-a_{1,2}+a_{1,1}a_{1,2}+a_{2,1}-a_{2,1}a_{2,2}+a_{3,1}-a_{3,1}a_{3,2})\,x_2x_3,\\
&R(\partial_{x_2},\partial_{x_1},\partial_{x_2},\partial_{x_3}) = (1+a_{1,2}-a_{2,1}-a_{1,1}a_{1,2}-a_{2,2}+a_{2,1}a_{2,2}+a_{3,2}-a_{3,1}a_{3,2})\,x_1x_3,\\
&R(\partial_{x_3},\partial_{x_1},\partial_{x_3},\partial_{x_2}) = \left(\tfrac{2}{3}+a_{1,1}-a_{1,1}a_{1,2}+a_{2,2}-a_{2,1}a_{2,2}-a_{3,1}-a_{3,2}+a_{3,1}a_{3,2}\right)x_1x_2.
\end{aligned}
$$

The Christoffel symbols describing $\nabla_{\partial_{x_i}}\partial_{x_j}$ are given by:

$$
\begin{aligned}
\nabla_{\partial_{x_1}}\partial_{x_1} &= (2-a_{2,1})y_{2,1}\partial_{x_2^*}+(2-a_{3,1})y_{3,1}\partial_{x_3^*}+a_{2,1}x_2\partial_{y_{2,2}}+a_{3,1}x_3\partial_{y_{3,2}},\\
\nabla_{\partial_{x_2}}\partial_{x_2} &= (2-a_{1,2})y_{1,2}\partial_{x_1^*}+(2-a_{3,2})y_{3,2}\partial_{x_3^*}+a_{1,2}x_1\partial_{y_{1,1}}+a_{3,2}x_3\partial_{y_{3,1}},\\
\nabla_{\partial_{x_3}}\partial_{x_3} &= (2-a_{1,1})y_{1,1}\partial_{x_1^*}+(2-a_{2,2})y_{2,2}\partial_{x_2^*}+a_{2,2}x_2\partial_{y_{2,1}}+a_{1,1}x_1\partial_{y_{1,2}},\\
\nabla_{\partial_{x_1}}\partial_{x_2} &= -a_{2,1}y_{2,1}\partial_{x_1^*}-a_{1,2}y_{1,2}\partial_{x_2^*}+\frac{y_{4,1}+y_{4,2}}{2}\partial_{x_3^*}\\
&\quad+(a_{1,2}-1)x_2\partial_{y_{1,1}}+(a_{2,1}-1)x_1\partial_{y_{2,2}},\\
\nabla_{\partial_{x_1}}\partial_{x_3} &= -a_{3,1}y_{3,1}\partial_{x_1^*}+\frac{y_{4,1}-y_{4,2}}{2}\partial_{x_2^*}-a_{1,1}y_{1,1}\partial_{x_3^*}\\
&\quad+(a_{1,1}-1)x_3\partial_{y_{1,2}}+(a_{3,1}-1)x_1\partial_{y_{3,2}}+\frac{2x_2}{3}\partial_{y_{4,1}}+\frac{4x_2}{3}\partial_{y_{4,2}},\\
\nabla_{\partial_{x_2}}\partial_{x_3} &= \frac{-y_{4,1}+y_{4,2}}{2}\partial_{x_1^*}-a_{3,2}y_{3,2}\partial_{x_2^*}-a_{2,2}y_{2,2}\partial_{x_3^*}\\
&\quad+(a_{2,2}-1)x_3\partial_{y_{2,1}}+(a_{3,2}-1)x_2\partial_{y_{3,1}}+\frac{4x_1}{3}\partial_{y_{4,1}}+\frac{2x_1}{3}\partial_{y_{4,2}}.
\end{aligned}
$$

It is now easy to show that the non-zero components of ∇R are:

$$
\begin{aligned}
\nabla R(\partial_{x_1},\partial_{x_2},\partial_{x_1},\partial_{x_2};\partial_{x_3}) &= -2(-2+a_{1,1}+a_{2,2}+a_{3,1}a_{3,2})x_3,\\
\nabla R(\partial_{x_1},\partial_{x_3},\partial_{x_1},\partial_{x_3};\partial_{x_2}) &= -\frac{2}{3}(-4+3a_{1,2}+3a_{3,2}+3a_{2,1}a_{2,2})x_2,\\
\nabla R(\partial_{x_2},\partial_{x_3},\partial_{x_2},\partial_{x_3};\partial_{x_1}) &= -\frac{2}{3}(-4+3a_{2,1}+3a_{3,1}+3a_{1,1}a_{1,2})x_1,\\
\nabla R(\partial_{x_1},\partial_{x_2},\partial_{x_1},\partial_{x_3};\partial_{x_2}) &= (2-a_{1,1}-a_{1,2}+a_{2,1}-a_{2,2}+a_{3,1}-a_{3,2}\\
&\qquad+a_{1,1}a_{1,2}-a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_3,
\end{aligned}
$$

$$\begin{aligned}
\nabla R(\partial_{x_1},\partial_{x_2},\partial_{x_1},\partial_{x_3};\partial_{x_3}) &= (2-a_{1,1}-a_{1,2}+a_{2,1}-a_{2,2}+a_{3,1}-a_{3,2}\\
&\qquad +a_{1,1}a_{1,2}-a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_2,\\
\nabla R(\partial_{x_2},\partial_{x_1},\partial_{x_2},\partial_{x_3};\partial_{x_1}) &= (2-a_{1,1}+a_{1,2}-a_{2,1}-a_{2,2}-a_{3,1}+a_{3,2}\\
&\qquad -a_{1,1}a_{1,2}+a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_3,\\
\nabla R(\partial_{x_2},\partial_{x_1},\partial_{x_2},\partial_{x_3};\partial_{x_3}) &= (2-a_{1,1}+a_{1,2}-a_{2,1}-a_{2,2}-a_{3,1}+a_{3,2}\\
&\qquad -a_{1,1}a_{1,2}+a_{2,1}a_{2,2}-a_{3,1}a_{3,2})x_1,\\
\nabla R(\partial_{x_3},\partial_{x_1},\partial_{x_3},\partial_{x_2};\partial_{x_1}) &= (\frac{2}{3}+a_{1,1}-a_{1,2}-a_{2,1}+a_{2,2}-a_{3,1}-a_{3,2}\\
&\qquad -a_{1,1}a_{1,2}-a_{2,1}a_{2,2}+a_{3,1}a_{3,2})x_2,\\
\nabla R(\partial_{x_3},\partial_{x_1},\partial_{x_3},\partial_{x_2};\partial_{x_2}) &= (\frac{2}{3}+a_{1,1}-a_{1,2}-a_{2,1}+a_{2,2}-a_{3,1}-a_{3,2}\\
&\qquad -a_{1,1}a_{1,2}-a_{2,1}a_{2,2}+a_{3,1}a_{3,2})x_1.
\end{aligned}$$

We set $\nabla R = 0$ to obtain the desired equations of Theorem 10.3.7; note that the first 3 equations generate the last 6. □

Chapter 11

Skew Tsankov manifolds

In this chapter we study manifolds whose Riemann curvature operators commute, i.e. which satisfy $\mathcal{R}(x_1,x_2)\mathcal{R}(x_3,x_4)=\mathcal{R}(x_3,x_4)\mathcal{R}(x_1,x_2)$ for all tangent vectors x_i. As usual, we begin by analyzing the skew Tsankov condition at the algebraic level for metrics of positive definite signature before giving examples at the differentiable level.

There are many examples of skew Tsankov manifolds in the higher signature context; these examples indicate that, even in the algebraic setting, the classification is likely to be far more complicated and that this is a fruitful subject for further inquiry.

The results of this chapter are collected in [37].

11.1 Classification of Riemannian skew Tsankov models

We concentrate on the Riemannian case and use the fact that the metric is positive definite to obtain the following classification result.

Theorem 11.1.1 *Let $\mathcal{V}:=(V,\langle\cdot,\cdot\rangle,A)$ be a Riemannian algebraic model.*

1. *$\mathcal{V}$ is skew Tsankov if and only if there exists an orthogonal direct sum decomposition $V=V_1\oplus...\oplus V_k\oplus W$ decomposing $A=A_1\oplus...\oplus A_k\oplus 0$ where $dim\,V_i=2$ for all i.*

2. *$\mathcal{V}$ is skew Tsankov and indecomposable if and only if $dim\,V=2$ and $A\neq 0$.*

Proof. Suppose given an orthogonal direct sum decomposition $V=V_1\oplus...\oplus V_k\oplus W$ so $A=A_1\oplus...\oplus A_k\oplus 0$ where $dim\,V_i=2$ for $1\leq i\leq k$. Let $\{e_i^1,e_i^2\}$ be an orthonormal basis for V_i. Given $x,y\in V$, there exist coefficients $\varepsilon_i(x,y)\in\mathbb{R}$ with

$$\mathcal{A}(x,y)\xi=\begin{cases}-\varepsilon_i(x,y)e_i^2 & \text{if}\quad \xi=e_i^1,\\ \varepsilon_i(x,y)e_i^1 & \text{if}\quad \xi=e_i^2,\\ 0 & \text{if}\quad \xi\perp\operatorname{Span}\{e_i^1,e_i^2\}\,.\end{cases}\tag{11.1}$$

We may then show $\mathcal{V}$ is skew Tsankov by computing

$$\mathcal{A}(x,y)\mathcal{A}(\bar{x},\bar{y})\xi = \begin{cases} -\varepsilon_i(x,y)\varepsilon_i(\bar{x},\bar{y})e_i^1 & \text{if} \quad \xi = e_i^1, \\ -\varepsilon_i(x,y)\varepsilon_i(\bar{x},\bar{y})e_i^2 & \text{if} \quad \xi = e_i^2, \\ 0 & \text{if} \quad \xi \perp \operatorname{Span}_{1\le i\le k}\{e_i^1, e_i^2\}. \end{cases}$$

Conversely, suppose that $\mathcal{V}$ is skew Tsankov. One may simultaneously skew-diagonalize the collection $\{\mathcal{A}(x,y)\}_{x,y\in V}$ of commuting skew-adjoint linear operators to find an orthonormal set $\{e_i^1, e_i^2\}$ and functions $\varepsilon_i(x,y)$ where $1 \le i \le k$ so that Equation (11.1) holds. Extend this to a full orthonormal basis $\mathcal{B} := \{e_1^1, e_1^2, ..., e_k^1, e_k^2, f_1, ..., f_l\}$ for V. Then the only non-zero entries in the curvature tensor relative to this base are $A(\cdot,\cdot,e_i^1,e_i^2)$ modulo the usual $\mathbb{Z}_2$ symmetry. Interchanging the first 2 entries with the last 2 entries shows the only non-zero curvatures are

$$A(e_i^1, e_i^2, e_j^1, e_j^2)\,.$$

On the other hand, if $i \ne j$, we can use the Bianchi identity to express

$$A(e_i^1, e_i^2, e_j^1, e_j^2) = -A(e_i^1, e_j^1, e_j^2, e_i^2) - A(e_i^1, e_j^2, e_i^2, e_j^1) = 0$$

and hence the only non-zero curvatures are $a_i := A(e_i^1, e_i^2, e_i^1, e_i^2)$. Thus, the desired decomposition $A = A_1 \oplus ... \oplus A_k \oplus 0$ yields just setting $V_i := \operatorname{Span}\{e_i^1, e_i^2\}$. Assertions (1) and (2) now follow. □

11.2 Examples of Riemannian manifolds

In this section we illustrate, by means of geometric examples, that the skew Tsankov condition is much more complicated when we pass from the algebraic to the differentiable setting. As we did in Part II, we take advantage of the tractable geometry of warped products to build examples of skew Tsankov manifolds in low dimensions.

11.2.1 Irreducible skew Tsankov manifolds of dimension 3

We begin by constructing irreducible 3-dimensional examples by taking a product of the interval $(0,\infty)$ with a Riemann surface.

Theorem 11.2.1 *Let $\mathcal{N} := (N, g_N)$ be a Riemann surface which does not have constant sectional curvature $+1$. Endow $\mathcal{M} := (M = (0,\infty)\times N, g_M)$ with the warped product metric $g_M := dt^2 + t^2 g_N$ for $t \in (0,\infty)$. Then $\mathcal{M}$ is an irreducible skew Tsankov manifold with scalar curvature $\tau_{\mathcal{M}} = t^{-2}\{\tau_{\mathcal{N}} - 2\}$.*

Proof. Choose isothermal coordinates to locally express $ds_N^2 = e^{2\alpha}(dx_1^2 + dx_2^2)$. Let $\partial_1 := \partial_{x_1}$, $\partial_2 := \partial_{x_2}$ and $\partial_3 := \partial_t$. Let $\alpha_i := \partial_i(\alpha)$ and $\alpha_{ij} := \partial_i\partial_j(\alpha)$. We have

$$g_M(\partial_1,\partial_1) = g_M(\partial_2,\partial_2) = t^2 e^{2\alpha} \quad \text{and} \quad g_M(\partial_3,\partial_3) = 1\,.$$

The non-zero Christoffel symbols of the first kind must have at least one repeated index different from 3:

$$\begin{array}{rrrrrrrrrrr} & & \Gamma_{111} &=& \alpha_1 t^2 e^{2\alpha}, & & & \Gamma_{112} &=& -\alpha_2 t^2 e^{2\alpha}, & \Gamma_{113} = -te^{2\alpha},\\ \Gamma_{121} &=& \Gamma_{211} &=& \alpha_2 t^2 e^{2\alpha}, & \Gamma_{131} &=& \Gamma_{311} &=& te^{2\alpha}, & \\ & & \Gamma_{221} &=& -\alpha_1 t^2 e^{2\alpha}, & & & \Gamma_{222} &=& \alpha_2 t^2 e^{2\alpha}, & \Gamma_{223} = -te^{2\alpha},\\ \Gamma_{122} &=& \Gamma_{212} &=& \alpha_1 t^2 e^{2\alpha}, & \Gamma_{322} &=& \Gamma_{232} &=& te^{2\alpha}\,. & \end{array}$$

Since the metric is diagonal, we can raise indices to see that

$$\begin{array}{rrrrl} & & \nabla_{\partial_1}\partial_1 &=& \alpha_1\partial_1 - \alpha_2\partial_2 - te^{2\alpha}\partial_3,\\ \nabla_{\partial_1}\partial_2 &=& \nabla_{\partial_2}\partial_1 &=& \alpha_2\partial_1 + \alpha_1\partial_2,\\ \nabla_{\partial_1}\partial_3 &=& \nabla_{\partial_3}\partial_1 &=& t^{-1}\partial_1,\\ & & \nabla_{\partial_2}\partial_2 &=& -\alpha_1\partial_1 + \alpha_2\partial_2 - te^{2\alpha}\partial_3,\\ \nabla_{\partial_2}\partial_3 &=& \nabla_{\partial_3}\partial_2 &=& t^{-1}\partial_2\,. \end{array}$$

One determines the curvature operator to obtain

$$\begin{array}{rcl} \mathcal{R}_{\mathcal{M}}(\partial_1,\partial_2)\partial_1 &=& -(\alpha_{11}+\alpha_{22}+e^{2\alpha})\partial_2,\\ \mathcal{R}_{\mathcal{M}}(\partial_2,\partial_1)\partial_2 &=& -(\alpha_{11}+\alpha_{22}+e^{2\alpha})\partial_1,\\ \mathcal{R}_{\mathcal{M}}(\partial_3,\partial_1)\partial_3 &=& \mathcal{R}_{\mathcal{M}}(\partial_3,\partial_2)\partial_3 = 0\,. \end{array}$$

As the only non-zero curvature is $R_{\mathcal{M}}(\partial_1,\partial_2,\partial_1,\partial_2) = -t^2e^{2\alpha}(\alpha_{11}+\alpha_{22}+e^{2\alpha})$, Theorem 11.1.1 implies that $\mathcal{M}$ is skew Tsankov. This calculation also yields

$$\tau_{\mathcal{M}} = t^{-2}\{-2e^{-2\alpha}(\alpha_{11}+\alpha_{22})-2\}\,.$$

An analogous computation on $\mathcal{N}$ yields

$$\begin{array}{rclrclrcrcl} \Gamma_{111} &=& \alpha_1 t^2e^{2\alpha}, & \Gamma_{112} &=& -\alpha_2 t^2 e^{2\alpha}, & \Gamma_{121} &=& \Gamma_{211} &=& \alpha_2 t^2 e^{2\alpha},\\ \Gamma_{221} &=& -\alpha_1 t^2 e^{2\alpha}, & \Gamma_{222} &=& \alpha_2 t^2 e^{2\alpha}, & \Gamma_{122} &=& \Gamma_{212} &=& \alpha_1 t^2 e^{2\alpha}, \end{array}$$

so the Christoffel symbols of the second kind and the curvature are given by

$$\begin{array}{ll} \nabla_{\partial_1}\partial_1 = \alpha_1\partial_1 - \alpha_2\partial_2, & \nabla_{\partial_2}\partial_2 = -\alpha_1\partial_1 + \alpha_2\partial_2,\\ \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \alpha_2\partial_1 + \alpha_1\partial_2, & \mathcal{R}_{\mathcal{N}}(\partial_1,\partial_2)\partial_1 = -(\alpha_{11}+\alpha_{22})\partial_2\,. \end{array}$$

$\mathcal{M}$ is indecomposable because Range $\{\mathcal{R}\}$ = Span $\{\partial_1,\partial_2\}$ and $\tau_{\mathcal{M}} = t^{-2}(\tau_{\mathcal{N}}-2)$ exhibits non-trivial dependence on t; hence the result follows. □

Remark 11.2.2 Let $f(x_1,x_2)$ be an isometric embedding of a Riemann surface N in $S^3 \subset \mathbb{R}^4$. Define an embedding of $(0,\infty)\times N$ in $\mathbb{R}^4$ by setting $F(t,x) := tf(x)$. Theorem 11.2.1 may then be used to see the resulting hypersurface in $\mathbb{R}^4$ is skew Tsankov; such hypersurfaces appear in Tsankov [167].

Remark 11.2.3 Choose a point $x \in N$ where $\tau_{\mathcal{N}}(x) \neq 2$ and let $\gamma_x(t) := t \times x$. Then γ_x is a unit speed geodesic and $\lim_{t\to 0} |\tau_{\mathcal{M}}(\gamma_x(t))| = \infty$. Thus $\mathcal{M}$ exhibits scalar curvature blowup at finite time. This shows $\mathcal{M}$ is geodesically incomplete and can not be embedded isometrically in a geodesically complete manifold.

11.2.2 Irreducible skew Tsankov manifolds of dimension 4

We take a warped product metric with a flat base and a flat fiber. Denote the usual coordinates on $\mathbb{R}^4$ by (x_1, x_2, x_3, x_4). Let $\partial_i := \partial_{x_i}$ and let

$$\mathcal{O} := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_3 > 0, x_4 > 0\}\,.$$

Theorem 11.2.4 *For $\beta > 0$, let $\mathcal{M}_\beta := (\mathcal{O}, g_\beta)$ where*

$$\begin{array}{rclrcl} g_\beta(\partial_1,\partial_1) &=& x_3^2, & g_\beta(\partial_2,\partial_2) &=& (x_3+\beta x_4)^2, \\ g_\beta(\partial_3,\partial_3) &=& 1, & g_\beta(\partial_4,\partial_4) &=& 1\,. \end{array}$$

1. *$\mathcal{M}_\beta$ is an indecomposable skew Tsankov manifold.*
2. *The scalar curvature $\tau_{\mathcal{M}_\beta} = -2x_3^{-1}(x_3+\beta x_4)^{-1}$.*
3. *$\mathcal{M}_{\beta_1}$ is not isometric to $\mathcal{M}_{\beta_2}$ for $\beta_1 \neq \beta_2$.*

Proof. The non-zero Christoffel symbols are given by:

$$\begin{array}{rclrclcl} \Gamma_{113} &=& -x_3, & \Gamma_{131} &=& \Gamma_{311} &=& x_3, \\ \Gamma_{223} &=& -(x_3+\beta x_4), & \Gamma_{232} &=& \Gamma_{322} &=& x_3+\beta x_4, \\ \Gamma_{224} &=& -\beta(x_3+\beta x_4), & \Gamma_{242} &=& \Gamma_{422} &=& \beta(x_3+\beta x_4)\,. \end{array}$$

Since the metric is diagonal, we may raise indices to compute:

$$\begin{array}{rclcl} & & \nabla_{\partial_1}\partial_1 &=& -x_3\partial_3, \\ \nabla_{\partial_1}\partial_3 &=& \nabla_{\partial_3}x_1 &=& x_3^{-1}\partial_1, \\ & & \nabla_{\partial_2}\partial_2 &=& -(x_3+\beta x_4)\partial_3 - \beta(x_3+\beta x_4)\partial_4, \\ \nabla_{\partial_2}\partial_3 &=& \nabla_{\partial_3}\partial_2 &=& (x_3+\beta x_4)^{-1}\partial_2, \\ \nabla_{\partial_2}\partial_4 &=& \nabla_{\partial_4}\partial_2 &=& \beta(x_3+\beta x_4)^{-1}\partial_2\,. \end{array}$$

The curvature operator can now be determined after a straightforward computation:

$$\begin{array}{rcl} \mathcal{R}(\partial_1,\partial_2)\partial_1 &=& -x_3(x_3+\beta x_4)^{-1}\partial_2, \\ \mathcal{R}(\partial_2,\partial_1)\partial_2 &=& -x_3^{-1}(x_3+\beta x_4)\partial_1\,. \end{array}$$

The remaining curvatures vanish so the only non-zero curvature is

$$\mathcal{R}(\partial_1,\partial_2,\partial_1,\partial_2) = -x_3(x_3+\beta x_4),$$

and hence $\mathcal{M}$ is skew Tsankov by Theorem 11.1.1. This establishes Assertion (1); Assertion (2) follows from the computations performed above.

Let $\mathcal{E} := \text{Range}\{\mathcal{R}\} = \text{Span}\{\partial_1, \partial_2\}$ and let $\mathcal{F} := \mathcal{E}^\perp = \text{Span}\{\partial_3, \partial_4\}$. These spaces are invariantly defined. We have

$$\begin{aligned}
\ln|\tau| &= \ln(2) - \ln(x_3) - \ln(x_3 + \beta x_4), \\
\nabla^2\{\ln|\tau|\}|_{\mathcal{F}} &= \begin{pmatrix} x_3^{-2} + (x_3 + \beta x_4)^{-2} & \beta(x_3 + \beta x_4)^{-2}, \\ \beta(x_3 + \beta x_4)^{-2} & \beta^2(x_3 + \beta x_4)^{-2} \end{pmatrix}, \\
\det(\nabla^2\{\ln|\tau|\}|_{\mathcal{F}}) &= \beta^2 x_3^{-2}(x_3 + \beta x_4)^{-2} = \tfrac{1}{4}\beta\, \tau_{\mathcal{M}_\beta}^2 .
\end{aligned}$$

This shows that β is an isometry invariant of $\mathcal{M}_\beta$. Furthermore since $H|_{\mathcal{F}}$ has rank 2, $\mathcal{M}$ is irreducible. □

Remark 11.2.5 As in the example described in Section 11.2.1, the scalar curvature blows up at finite time along the geodesic $\gamma(t) = (1, 1, t, 1)$; thus $\mathcal{M}_\beta$ can not be isometrically embedded as an open subset of a complete manifold.

11.3 Higher signature examples

Although we have been able to give an algebraic classification of the skew Tsankov condition for Riemannian models, we have seen in the previous section that it looks far more complicated at the differentiable level. The situation presents itself as a difficult task in the higher signature setting too. In order to justify such a claim we devote the remaining of this chapter to exhibit several geometric examples.

11.3.1 Strict Walker manifolds

It turns out that many manifolds which appeared in other different contexts are also skew Tsankov. In Section 3.2.4 we introduced strict Walker manifolds in dimension 4. These are similarly defined in dimension $2p$ as manifolds with a strictly parallel null p-plane and in appropriate coordinates $\{x_1, \ldots, x_p, y_1, \ldots, y_p\}$ they have the form

$$\left(\begin{array}{c|c} 0 & I \\ \hline I & B \end{array}\right)$$

where B depends only on $y_1, \ldots, y_p$ [169].

The next theorem shows that strict Walker manifolds are also skew Tsankov with nilpotent skew-symmetric curvature operators of order 2, this is, $\mathcal{R}(x_1, x_2)\mathcal{R}(x_3, x_4) = 0$ for all x_1, x_2, x_3, x_4.

Theorem 11.3.1 *Let $(x_1, ..., x_p, y_1, ..., y_p)$ be coordinates on $\mathbb{R}^{2p}$. Let ψ_{ij} be given functions such that $\psi_{ij}(x) = \psi_{ji}(x)$. Let $\mathcal{M} := (\mathbb{R}^{2p}, g)$ be the manifold of neutral signature (p, p) where*

$$g(\partial_{x_i}, \partial_{x_j}) = \psi_{ij}(x) \quad \text{and} \quad g(\partial_{x_i}, \partial_{y_i}) = 1\,.$$

Then $\mathcal{M}$ is skew Tsankov and $\mathcal{R}$ is nilpotent of order 2.

Proof. We use notation $\psi_{ij/k} := \partial_{x_k}\psi_{ij}$ to express the non-zero Christoffel symbols as follows:

$$\begin{aligned} g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{x_k}) &= \frac{1}{2}(\psi_{ik/j} + \psi_{jk/i} - \psi_{ij/k}), \\ \nabla_{\partial_{x_i}}\partial_{x_j} &= \frac{1}{2}\sum_k(\psi_{ik/j} + \psi_{jk/i} - \psi_{ij/k})\partial_{y_k}\,. \end{aligned}$$

From this it follows that the possibly non-zero entries in curvature tensor R are:

$$R_{ijkl} = \frac{1}{2}\sum_l(\psi_{il/jk} + \psi_{jk/il} - \psi_{ik/jl} - \psi_{jl/ik})\,.$$

Consequently $\mathcal{R}(\partial_{x_i}, \partial_{x_j})\partial_{x_k} = \sum_\ell R_{ijkl}\partial_{y_\ell}$. This shows that

$$\text{Range}\,(\mathcal{R}) \subset \text{Span}\,\{\partial_{y_i}\} \quad \text{and} \quad \text{Span}\,\{\partial_{y_i}\} \subset \ker(\mathcal{R})\,.$$

Thus $\mathcal{R}(\xi_1, \xi_2)\mathcal{R}(\xi_3, \xi_4) = 0$ for all $\xi_1, \xi_2, \xi_3, \xi_4$ so $\mathcal{M}$ is skew Tsankov. □

Remark 11.3.2 These manifolds have been studied extensively (see [63]). They are generalized plane wave manifolds so, in particular, they are all geodesically complete and the exponential map is a global diffeomorphism. If $\psi_{ij} = \partial_{x_i}f\partial_{x_j}f$ for some function f, then $\mathcal{M}$ is realizable as a hypersurface in $\mathbb{R}^{(p,p+1)}$. Certain manifolds in this family are curvature homogeneous but not homogeneous. We shall emphasize that, in contrast to the examples we showed in the Riemannian setting, these are geodesically complete.

11.3.2 Fiedler manifolds

Certain Fiedler manifolds have been shown to be nilpotent Osserman of arbitrarily high order [71]. The next result shows that Fiedler manifolds are also skew Tsankov and that their skew-symmetric curvature operator is nilpotent of order 3.

Theorem 11.3.3 *Let $(x, u_1, ..., u_\nu, y)$ be coordinates on $\mathbb{R}^{\nu+2}$. Let $f \in C^\infty(\mathbb{R}^\nu)$ and let $\Xi = \Xi_{ab}$ be an invertible symmetric $\nu \times \nu$ matrix of signature (r, s). Define a metric g of signature $(r + 1, s + 1)$ on $\mathbb{R}^{\nu+2}$ by setting:*

$$g(\partial_x, \partial_x) = -2f(\vec{u}), \qquad g(\partial_x, \partial_y) = 1, \qquad \text{and} \quad g(\partial_{u_a}, \partial_{u_b}) = \Xi_{ab}.$$

Then $\mathcal{M}$ is skew Tsankov and $\mathcal{R}$ is nilpotent of order 3.

Proof. Since $d\Xi = 0$, the potentially non-zero Christoffel symbols are:

$$\begin{aligned} g(\nabla_{\partial_x}\partial_x, \partial_{u_a}) &= \partial_{u_a}(f), \\ g(\nabla_{\partial_{u_a}}\partial_x, \partial_x) &= g(\nabla_{\partial_x}\partial_{u_a}, \partial_x) = -\partial_{u_a}(f) . \end{aligned}$$

Let Ξ^{ab} be the inverse matrix. Then

$$\begin{aligned} \nabla_{\partial_x}\partial_x &= \sum_{ab} \Xi^{ab}\partial_{u_a}(f)\partial_{u_b}, \\ \nabla_{\partial_x}\partial_{u_a} &= \nabla_{\partial_{u_a}}\partial_x = -\partial_{u_a}(f)\partial_y . \end{aligned}$$

The quadratic terms in the Christoffel symbols play no role in the calculation of R. Let $f_{ab} := \partial_{u_a}\partial_{u_b} f$. The possibly non-zero components of R and of $\mathcal{R}$ are given by

$$R(\partial_x, \partial_{u_a}, \partial_x, \partial_{u_b}) = f_{ab},$$

and

$$\begin{aligned} \mathcal{R}(\partial_x, \partial_{u_a})\partial_{u_b} &= -f_{ab}\partial_y, \\ \mathcal{R}(\partial_x, \partial_{u_a})\partial_x &= \Xi^{bc} f_{ac}\partial_{u_b} . \end{aligned}$$

Thus the only potentially non-zero quadratic terms in the curvature are

$$\mathcal{R}(\partial_x, \partial_{u_d})\mathcal{R}(\partial_x, \partial_{u_a})\partial_x = -\Xi^{bc} f_{ac} f_{db}\partial_y .$$

It now follows that $\mathcal{R}(\partial_x, \partial_{u_d})\mathcal{R}(\partial_x, \partial_{u_a})\partial_x = \mathcal{R}(\partial_x, \partial_{u_a})\mathcal{R}(\partial_x, \partial_{u_d})\partial_x$. This shows that $\mathcal{M}$ is skew Tsankov and that $\mathcal{R}$ is nilpotent of order 3. $\square$

Remark 11.3.4 These manifolds are irreducible for generic f. They are complete for certain choices of the warping function but they are not geodesically complete in general.

Open problems

It has been mentioned in the introduction of this part that Tsankov theory is a rather new field. This is one of the reasons why one can set many different open problems. Since there are a lot of operators associated to the curvature tensor, one may study the implications of commutativity properties of such operators. As we have said, the references in the literature about this subject are growing fast. Therefore, we present here some open problems intimately related to Jacobi Tsankov and skew Tsankov manifolds, avoiding question related to commutativity properties of other different operators.

- Although we have characterized Jacobi Tsankov models in Riemannian and Lorentzian signature, a complete understanding of the higher signature case is still in search.
- We have seen that the Jacobi Tsankov condition is a strong one. Thus, by weakening this condition to the orthogonally Jacobi Tsankov, we may get more interesting classifications, as that given for the Riemannian setting in Chapter 9. Therefore, it would be interesting to understand orthogonally Jacobi Tsankov manifolds in indefinite signature; especially in Lorentzian signature, where Jacobi Tsankov manifolds are classified.
- Skew Tsankov irreducible models are well-understood, however this situation does not correspond with our knowledge about skew Tsankov manifolds, even for definite signature.
- Although we have mentioned orthogonally skew Tsankov manifolds, we have not included any result dealing with this concept. Also, this appears in [167] in the context of hypersurfaces. A classification of these manifolds is open even in the Riemannian setting and it looks like an interesting task.

Part IV

Complex Osserman manifolds

Recall from Section 1.4.4 that the higher order Jacobi operator of a k-plane Π is defined in the Riemannian setting as

$$\mathcal{J}(\Pi) = \sum_{i=1}^{k} \mathcal{J}(e_i)\,;$$

where $\{e_1, ..., e_k\}$ is an orthonormal basis of Π. Thus, an algebraic model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A)$ is said to be *k-Osserman* if the eigenvalues of $\mathcal{J}(\Pi)$ are constant on the Grassmannian $Gr_k(V)$ of k-planes.

The geometry of k-Osserman manifolds is very rigid, since only spaces of constant curvature may occur. Therefore, as a part of a general program to understand the rank-one symmetric spaces, we consider a different condition aimed to characterized complex space forms.

This part is devoted to the study of the complex Jacobi operator in Riemannian signature. Let J denote a Hermitian almost complex structure on $(V, \langle\cdot,\cdot\rangle)$ with $\langle\cdot,\cdot\rangle$ positive definite; this means that J is an isometry of $(V, \langle\cdot,\cdot\rangle)$ with $J^2 = -Id$. A 2-plane is said to be holomorphic if it is J-invariant and a real linear transformation T of V is said to be complex linear if $TJ = JT$. We let $\mathbb{CP}(V, J)$ be the set of all holomorphic 2 planes. If $x \in S(V, \langle\cdot,\cdot\rangle)$ is a unit vector, let $\pi_x := \operatorname{Span}\{x, Jx\}$. The natural map $x \to \pi_x$ defines the Hopf fibration from $S(V, \langle\cdot,\cdot\rangle)$ to $\mathbb{CP}(V, J)$. Let

$$\mathcal{J}(\pi_x) := \mathcal{J}(x) + \mathcal{J}(Jx)$$

be the complex Jacobi operator; this is the restriction of the higher order Jacobi operator to the set of complex 2-planes.

The complex Osserman condition is defined by the complex Jacobi operator; thus, we say a model $\mathcal{V}$ is *complex Osserman* if the eigenvalues of $\mathcal{J}(\pi)$ are constant on $S(\pi)$. Similarly, an almost Hermitian manifold (M, g, J) is said to be complex Osserman if $\mathcal{J}(\pi)$ is constant on $S_p(\pi)$ for all $p \in M$.

In order to study complex Osserman manifolds, we first examine if the complex Jacobi operator completely determines the curvature tensor. In other words, to face the complex Osserman problem, first we shall establish which is the appropriate framework to study this condition. This is the purpose of Chapter 12, where we show that the complex Jacobi operator completely determines the curvature tensor if either (M, g, J) is Hermitian or nearly Kähler. We shall point out that this result fails in general for almost Hermitian manifolds as we will see in Theorem 12.1.2.

Afterwards, we move on to study some preliminary aspects of the complex Osserman problem in Chapter 13. Thus, we concentrate in studying some general properties of complex Osserman algebraic models and characterize complex Osserman algebraic curvature tensors which are given by a Clifford family. Here we act motivated by the historic way which led to the final proof of Osserman conjecture in Riemannian signature. Firstly we obtain all possible eigenvalue structures for the complex Jacobi operators (Theorem 13.1.7) and classify essentially all complex models which are given by a Clifford family (Theorems 13.3.1 and 13.4.1).

In Chapter 14 we restrict to a specific natural framework, namely Kähler geometry, to study the complex Osserman condition. We prove that a four-dimensional Kähler manifold is complex Osserman if and only if it is a complex space form.

Chapter 12

The complex Jacobi operator

In this chapter we recall some basic facts on complex geometry and we derive some interesting properties of the complex Jacobi operator. We will develop our analysis in two steps; as usual we begin our study in a purely algebraic context, using Gray's decomposition of the space of curvature tensors in the almost Hermitian setting. Afterwards, we translate those results to the geometric context, thus obtaining results which concern some of the most important families of almost Hermitian manifolds: *Hermitian*, *nearly Kähler* or *almost Kähler*. Results in this chapter are collected in [41].

12.1 Introduction

As a matter of motivation, note that the usual Jacobi operator determines the full curvature tensor. The following theorem is well known; Assertion (2) in the geometric setting is an immediate consequence of the corresponding Assertion (1) in the algebraic setting.

Theorem 12.1.1

1. *Let $\mathcal{V}_i = (V_i, \langle\cdot,\cdot\rangle_i, A_i)$ be algebraic models for $i = 1, 2$. Suppose there exists an isometry $\theta : (V_1, \langle\cdot,\cdot\rangle_1) \to (V_2, \langle\cdot,\cdot\rangle_2)$ so that $\mathcal{J}_{\mathcal{V}_2}(\theta x)\theta = \theta\mathcal{J}_{\mathcal{V}_1}(x)$ for all $x \in V_1$. Then $\theta^* A_2 = A_1$.*

2. *Let $\mathcal{M}_i = (M_i, g_i)$ be Riemannian manifolds for $i = 1, 2$. Suppose there is an isometry $\theta : (T_pM_1, g_1) \to (T_qM_2, g_2)$ so that $\mathcal{J}_{\mathcal{M}_2,q}(\theta x)\theta = \theta\mathcal{J}_{\mathcal{M}_1,p}(x)$ for all $x \in T_pM_1$. Then $\theta^* R_2(q) = R_1(p)$.*

Similarly, the higher order Jacobi operator also determines the full curvature tensor. To see that this is true, one may argue as follows. Let $2 \leq p \leq n-1$. Assume that $\mathcal{J}_R(\sigma) = 0$ for every p-plane σ. Let x and y be unit vectors in V. Choose additional unit vectors $\{e_2, ..., e_p\}$ so that $\{x, e_2, ..., e_p\}$ is an orthonormal basis for a p-plane σ_x and so that $\{y, e_2, ..., e_p\}$ is an orthonormal basis for a p-plane σ_y. Then

$$\mathcal{J}_R(x)y = (\mathcal{J}_R(x) - \mathcal{J}_R(y))y = (\mathcal{J}_R(\sigma_x) - \mathcal{J}_R(\sigma_y))y = 0\,.$$

This shows that $\mathcal{J}_R = 0$ and hence $R = 0$ by Theorem 12.1.1.

In view of these considerations, it arises as a natural question under which conditions the complex Jacobi operator determines the curvature tensor in the complex setting. The main purpose of this chapter is to answer this question. We will see that in a Hermitian or a nearly Kähler manifold the complex Jacobi operator completely determines the curvature tensor. However, this is not true in general for an almost Hermitian manifold, as we show in the following example.

Let $\mathcal{C} = (M, g, J)$ be an almost Hermitian manifold. We let $\mathcal{U}_\mathcal{C}$ be the bundle of complex isometries of TM; the fibers of this bundle are the associated unitary group of the fibers. If $\Theta \in C^\infty\{\mathcal{U}_\mathcal{C}\}$ and if $p \in M$, then $\theta_p := \Theta(p)$ is a complex isometry of (T_pM, g_p, J_p) for any $p \in M$.

Theorem 12.1.2 *Let $n \equiv 0$ mod 4. There exists an almost Hermitian manifold $\mathcal{C}$ and there exists $\Theta \in C^\infty\{\mathcal{U}_\mathcal{C}\}$ so that for any point p in M we have:*

1. $\theta_p \mathcal{J}_\mathcal{C}(\pi) = \mathcal{J}_\mathcal{C}(\theta_p \pi)\theta_p$ *for all* $\pi \in \mathbb{CP}(T_pM, J)$.
2. $\theta_p \mathcal{R}_\mathcal{C}(\pi) = \mathcal{R}_\mathcal{C}(\theta_p \pi)\theta_p$ *for all* $\pi \in \mathbb{CP}(T_pM, J)$.
3. $\theta_p^* R_p \neq R_p$.

12.1.1 Basic notation and terminology

$\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ is said to be a *complex model* if J is a unitary complex structure and if A is an algebraic curvature tensor. Any point p of an almost Hermitian manifold $\mathcal{C}$ determines a corresponding complex model $\mathfrak{C}(\mathcal{C}, p) := (T_pM, g_p, R_p, J_p)$ in a natural fashion. We say a 2-dimensional subspace π of V is a *complex line* if $J\pi = \pi$. We denote by $\mathbb{CP}(V, J)$ the complex projective space of complex lines in V.

Let $\mathcal{V}$ be a complex model. We recall the definition of the Ricci tensor ρ and introduce the $\star$-Ricci tensor $\rho^\star$; both are defined by contracting indices. If $\{e_1, ..., e_n\}$ is an orthonormal basis for V, then

$$\rho(x, y) = \sum_{i=1}^n R(x, e_i, y, e_i) \quad \text{and} \quad \rho^\star(x, y) := \sum_{i=1}^n R(x, e_i, Jy, Je_i)\,.$$

Note that $\rho^\star$ is not in general a symmetric 2-tensor; however one does have that $\rho^\star(x, y) = \rho^\star(y, x)$ if the compatibility condition given in Lemma 12.2.1 is satisfied. The scalar curvature τ and the $\star$-scalar curvature $\tau^\star$ are defined by a final contraction:

$$\tau = \sum_{i=1}^n \rho(e_i, e_i) \quad \text{and} \quad \tau^\star = \sum_{i=1}^n \rho^\star(e_i, e_i)\,.$$

Let x be a unit vector and $\pi_x := \text{Span}\,\{x, Jx\}$ be the corresponding complex line. We recall the definitions for the holomorphic sectional curvature $Q(\pi_x)$, the complex Jacobi operator $\mathcal{J}(\pi_x)$, and the complex skew-symmetric curvature operator $\mathcal{R}(\pi_x)$ for $\pi_x \in \mathbb{CP}(V, J)$:

$$Q(\pi_x) := A(x, Jx, x, Jx), \quad \mathcal{J}(\pi_x) := \mathcal{J}(x) + \mathcal{J}(Jx), \qquad (12.1)$$
$$\mathcal{R}(\pi_x) := \mathcal{R}(x, Jx) \qquad \text{for any } x \in S(V)\,.$$

There are several important families of almost Hermitian manifolds; a complete classification can be found in [92]. The following are some of the most important and will be studied along this chapter.

- $\mathcal{C}$ is called *Kähler* if the complex structure is parallel, i.e. $\nabla J = 0$.
- $\mathcal{C}$ is said to be *Hermitian* if the Nijenhuis tensor vanishes, i.e. if

  $$[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0 \quad \text{for all} \quad X, Y\,.$$

  Equivalently, see [133], this means that we can find local holomorphic coordinates $z_\nu = x_\nu + \sqrt{-1}y_\nu$ for $1 \le \nu \le \frac{1}{2}n$ so that $J\partial_{x_\nu} = \partial_{y_\nu}$ and $J\partial_{y_\nu} = -\partial_{x_\nu}$; the transition functions relating two such coordinate systems are then complex analytic.
- One says that $\mathcal{C}$ is *nearly Kähler* if $(\nabla_x J)x = 0$ for all tangent vectors x; we refer to [132] for further information concerning this class of manifolds.
- We say that $\mathcal{C}$ is *almost Kähler* if the two form $\Omega(x, y) := \langle Jx, y\rangle$ is closed; we refer to [5] for a survey and to [4], [110] and [141] for some recent results concerning this class of manifolds.

12.1.2 Main result

The following result, which generalizes Theorem 12.1.1 to the complex setting, is central in this chapter and motivates subsequent work. It shows that the full curvature tensor is determined either by the complex Jacobi operator or by the complex curvature operator in certain natural geometric contexts.

Theorem 12.1.3 *Let $\mathcal{M}_1 = (M_1, g_1, J_1)$ and $\mathcal{M}_2 = (M_2, g_2, J_2)$ be either Hermitian or nearly Kähler manifolds (not necessarily both of the same class). Let $\theta : (T_pM_1, g_1, J_1) \longrightarrow (T_qM_2, g_2, J_2)$ be a complex isometry. The following assertions are equivalent:*

1. $\theta\mathcal{J}_{\mathcal{M}_1,p}(\pi) = \mathcal{J}_{\mathcal{M}_2,q}(\theta\pi)\theta$ *for all* $\pi \in \mathbb{CP}(T_pM_1, J_{1,p})$.
2. $\theta R_{\mathcal{M}_1,p}(\pi) = R_{\mathcal{M}_2,q}(\theta\pi)\theta$ *for all* $\pi \in \mathbb{CP}(T_pM_1, J_{1,p})$.
3. $\theta^* R_{\mathcal{M}_2,q} = R_{\mathcal{M}_1,p}\,.$

Part of the interest of this result relies on the fact that geometric conditions, such as being Hermitian or nearly Kähler, imply desirable properties at the algebraic level.

We do not have an analogous of Theorem 12.1.3 for almost Kähler manifolds, but a bit weaker result:

Theorem 12.1.4 *Let $\mathcal{C} = (M, g, J)$ be an almost Kähler manifold. Assume either that $\mathcal{J}(\pi) = 0$ for all $\pi \in \mathbb{CP}(TM, J)$ or that $\mathcal{R}(\pi) = 0$ for all $\pi \in \mathbb{CP}(TM, J)$. Then M is flat.*

12.2 Algebraic curvature tensors in Hermitian vector spaces

We devote this section to recall some known facts about curvature tensors in complex geometry and introduce the appropriate framework we will work in.

Let $\mathfrak{A}(V)$ be the vector space of all algebraic curvature tensors on V. We use $\langle \cdot, \cdot \rangle$ to define a natural inner product on $\mathfrak{A}(V)$ by setting:

$$\langle A_1, A_2 \rangle := \sum_{i,j,k,l} A_1(e_i, e_j, e_k, e_l) A_2(e_i, e_j, e_k, e_l)\,;$$

this is independent of the particular orthonormal basis $\{e_i\}$ chosen. The following subspaces are invariant under the action of the unitary group $\mathcal{U}(n)$ [91]:

$$\begin{aligned}
\mathfrak{A}_1(V, J) &= \{A \in \mathfrak{A}(V) : A(x, y, z, w) = A(Jx, Jy, z, w)\}\,,\\
\mathfrak{A}_2(V, J) &= \{A \in \mathfrak{A}(V) : A(x, y, z, w) = A(Jx, Jy, z, w)\\
&\qquad + A(Jx, y, Jz, w) + A(Jx, y, z, Jw)\},\\
\mathfrak{A}_3(V, J) &= \{A \in \mathfrak{A}(V) : A(x, y, z, w) = A(Jx, Jy, Jz, Jw)\}\,.
\end{aligned}$$

Moreover, we have

$$\mathfrak{A}_1(V, J) \subset \mathfrak{A}_2(V, J) \subset \mathfrak{A}_3(V, J) \subset \mathfrak{A}(V, J)\,.$$

Note that $\mathfrak{A}_1(V, J)$ is the space of algebraic curvature tensors which verify Kähler identity. Denote

$\mathfrak{A}_1^\perp(V, J) \equiv$ orthogonal complement of $\mathfrak{A}_1(V, J)$ in $\mathfrak{A}_2(V, J)$,

$\mathfrak{A}_2^\perp(V, J) \equiv$ orthogonal complement of $\mathfrak{A}_2(V, J)$ in $\mathfrak{A}_3(V, J)$,

$\mathfrak{A}_3^\perp(V, J) \equiv$ orthogonal complement of $\mathfrak{A}_3(V, J)$ in $\mathfrak{A}(V, J)$.

The next lemma characterizes curvature tensors in $\mathfrak{A}_3(V, J)$ by means of the commutativity of the complex structure with the complex Jacobi operator or the complex skew-symmetric curvature operator. For a complex model $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A, J)$, we say $\mathcal{V}$ is *compatible* if $A \in \mathfrak{A}_3(V, J)$ or, equivalently, any of the conditions below holds. Generally, a manifold satisfying this condition at every point is known in the literature as a *RK*-manifold. The equivalence between 1. and 3. in the following Lemma can be found in [84] as well as other equivalent conditions.

Lemma 12.2.1 *Let $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A, J)$ be a complex model. The following conditions are equivalent:*

1. *$J^*A = A$, i.e. $A(x, y, z, t) = A(Jx, Jy, Jz, Jt)$ for all $x, y, z, t \in V$.*

2. $\mathcal{J}(\pi)J = J\mathcal{J}(\pi)$ *for all* $\pi \in \mathbb{CP}(V,J)$.

3. $\mathcal{A}(\pi)J = J\mathcal{A}(\pi)$ *for all* $\pi \in \mathbb{CP}(V,J)$.

Proof. Suppose first that Assertion (1) holds, i.e. $A(x,y,z,t) = A(Jx,Jy,Jz,Jt)$ for all x, y, z, t. Then

$$A(x,y,x,Jz) + A(Jx,y,Jx,Jz) = -A(x,Jy,x,z) - A(Jx,Jy,Jx,z)\,, \tag{12.2}$$

which implies $\langle\mathcal{J}(\pi_x)y,Jz\rangle = -\langle\mathcal{J}(\pi_x)Jy,z\rangle$ and, hence, $\mathcal{J}(\pi_x)J = J\mathcal{J}(\pi_x)$. Assume conversely that $\mathcal{J}(\pi_x)J = J\mathcal{J}(\pi_x)$ or, equivalently, that Equation (12.2) holds for all x. Polarizing this identity and replacing z by $-Jz$ yields

$$\begin{aligned} &A(x,y,w,z) + A(w,y,x,z) + A(Jx,y,Jw,z) + A(Jw,y,Jx,z)\\ &\quad= A(x,Jy,w,Jz) + A(w,Jy,x,Jz) + A(Jx,Jy,Jw,Jz) + A(Jw,Jy,Jx,Jz)\,. \end{aligned} \tag{12.3}$$

Interchanging arguments $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ in the curvature tensors then yields:

$$\begin{aligned} &A(y,x,z,w) + A(y,w,z,x) + A(y,Jx,z,Jw) + A(y,Jw,z,Jx)\\ &\quad= A(Jy,x,Jz,w) + A(Jy,w,Jz,x) + A(Jy,Jx,Jz,Jw) + A(Jy,Jw,Jz,Jx)\,. \end{aligned}$$

If we interchange x and y and we interchange z and w in this identity, we get

$$\begin{aligned} &A(x,y,w,z) + A(x,z,w,y) + A(x,Jy,w,Jz) + A(x,Jz,w,Jy)\\ &\quad= A(Jx,y,Jw,z) + A(Jx,z,Jw,y) + A(Jx,Jy,Jw,Jz) + A(Jx,Jz,Jw,Jy)\,. \end{aligned} \tag{12.4}$$

Adding (12.3) and (12.4) and simplifying yields:

$$A(x,y,w,z) + A(w,y,x,z) = A(Jx,Jy,Jw,Jz) + A(Jw,Jy,Jx,Jz)\,. \tag{12.5}$$

We permute the indices in Equation (12.5) to change $y \to x \to w \to y$. This yields:

$$A(w,x,y,z) + A(y,x,w,z) = A(Jw,Jx,Jy,Jz) + A(Jy,Jx,Jw,Jz)\,. \tag{12.6}$$

We add 2(12.5) and (12.6) and use the Bianchi identity to see

$$\begin{aligned} 3A(w,y,x,z) &= A(x,y,w,z) + 2A(w,y,x,z) + A(w,x,y,z)\\ &= A(Jx,Jy,Jw,Jz) + 2A(Jw,Jy,Jx,Jz) + A(Jw,Jx,Jy,Jz)\\ &= 3A(Jw,Jy,Jx,Jz)\,. \end{aligned}$$

The desired identity now follows.

We now prove that Assertion (1) implies Assertion (3). We have:

$$\begin{aligned} &\langle J\mathcal{A}(\pi_x)y,z\rangle = -\langle\mathcal{A}(\pi_x)y,Jz\rangle = -A(x,Jx,y,Jz)\\ &\quad= -A(Jx,JJx,Jy,JJz) = A(x,Jx,Jy,z) = \langle\mathcal{A}(\pi_x)Jy,z\rangle\,. \end{aligned}$$

Thus $J\mathcal{A}(\pi_x) = \mathcal{A}(\pi_x)J$ as desired.

We finally show that Assertion (3) implies Assertion (1). We have

$$\begin{aligned} &J\mathcal{A}(x, Jx) = \mathcal{A}(x, Jx)J,\\ &\quad\Rightarrow \langle JA(x, Jx)z, w\rangle - \langle A(x, Jx)Jz, w\rangle = 0,\\ &\quad\Rightarrow A(x, Jx, z, Jw) + A(x, Jx, Jz, w) = 0\,. \end{aligned}$$

Polarizing yields an identity for all x, y, z, w:

$$0 = A(y, Jx, z, Jw) + A(x, Jy, z, Jw) + A(y, Jx, Jz, w) + A(x, Jy, Jz, w)\,.$$

Interchange the first two arguments in the first and third term to see:

$$0 = -A(Jx, y, z, Jw) + A(x, Jy, z, Jw) - A(Jx, y, Jz, w) + A(x, Jy, Jz, w)\,.$$

Replace (x, w) by (Jx, Jw) to show:

$$0 = -A(x, y, z, w) - A(Jx, Jy, z, w) + A(x, y, Jz, Jw) + A(Jx, Jy, Jz, Jw)\,. \tag{12.7}$$

Interchange the first two arguments with the final two arguments:

$$0 = -A(z, w, x, y) - A(z, w, Jx, Jy) + A(Jz, Jw, x, y) + A(Jz, Jw, Jx, Jy)\,.$$

Change notation to interchange x and z, and y and w, to see:

$$0 = -A(x, y, z, w) - A(x, y, Jz, Jw) + A(Jx, Jy, z, w) + A(Jx, Jy, Jz, Jw)\,. \tag{12.8}$$

We add Equations (12.7) and (12.8) to conclude

$$-A(x, y, z, w) + A(Jx, Jy, Jz, Jw) = 0$$

and complete the proof that Assertion (3) implies Assertion (1). □

Remark 12.2.2 We note (see Lemma 12.4.1) that if $\mathcal{C} = (M, g, J)$ is a nearly Kähler manifold, then $\mathcal{V}(\mathcal{C}, p) = (T_pM, g_p, R_p, J_p)$ is a compatible complex model for any point $p \in M$. Thus, this is a rather natural condition.

The following result, which is due to Vanhecke [168], was originally stated in a purely geometrical setting. It expresses $A(x, y, x, y)$ for a compatible complex model in terms of the holomorphic sectional curvature Q and in terms of an additional tensor λ defined as follows

$$\lambda(x, y) = R(x, y, x, y) - R(x, y, Jx, Jy)\,. \tag{12.9}$$

Lemma 12.2.3 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a compatible complex model. Then*

$$\begin{aligned}32A(x,y,x,y) &= 3Q(x+Jy) + 3Q(x-Jy) - Q(x+y) - Q(x-y)\\ &\quad -4Q(x) - 4Q(y) + 4\{5\lambda(x,y) + \lambda(x,Jy)\}\,.\end{aligned}$$

Proof. First note that

$$\begin{aligned}Q(x+y) &= A(x+y, Jx+Jy, x+y, Jx+Jy)\\ &= A(x,Jx,x,Jx) + A(x,Jy,x,Jy)\\ &\quad +A(y,Jx,y,Jx) + A(y,Jy,y,Jy)\\ &\quad +2A(x,Jx,y,Jx) + 2A(x,Jy,y,Jy) + 2A(x,Jx,x,Jy)\\ &\quad +2A(y,Jx,y,Jy) + 2A(x,Jx,y,Jy) + 2A(x,Jy,y,Jx).\end{aligned}$$

Hence

$$\begin{aligned}&3Q(x+Jy) + 3Q(x-Jy) - Q(x+y) - Q(x-y) - 4Q(x) - 4Q(y)\\ &\quad = 3\{A(x,Jx,Jx,x) + A(x,y,y,x) + A(Jy,Jx,Jx,Jy) + A(Jy,y,y,Jy)\\ &\qquad +2A(x,Jx,Jx,Jy) + 2A(x,y,y,Jy) - 2A(x,Jx,y,x)\\ &\qquad -2A(Jy,Jx,y,Jy) - 2A(x,Jx,y,Jy) - 2A(x,y,Jx,Jy)\}\\ &\qquad +3\{A(x,Jx,Jx,x) + A(x,y,y,x) + A(Jy,Jx,Jx,Jy) + A(Jy,y,y,Jy)\\ &\qquad -2A(x,Jx,Jx,Jy) - 2A(x,y,y,Jy) + 2A(x,Jx,y,x)\\ &\qquad +2A(Jy,Jx,y,Jy) - 2A(x,Jx,y,Jy) - 2A(x,y,Jx,Jy)\}\\ &\qquad -\{A(x,Jx,Jx,x) + A(x,Jy,Jy,x) + A(y,Jx,Jx,y) + A(y,Jy,Jy,y)\\ &\qquad +2A(x,Jx,Jx,y) + 2A(x,Jy,Jy,y) + 2A(x,Jx,Jy,x)\\ &\qquad +2A(y,Jx,Jy,y) + 2A(x,Jx,Jy,y) + 2A(x,Jy,Jx,y)\}\\ &\qquad -\{A(x,Jx,Jx,x) + A(x,Jy,Jy,x) + A(y,Jx,Jx,y) + A(y,Jy,Jy,y)\\ &\qquad -2A(x,Jx,Jx,y) - 2A(x,Jy,Jy,y) - 2A(x,Jx,Jy,x)\\ &\qquad -2A(y,Jx,Jy,y) + 2A(x,Jx,Jy,y) + 2A(x,Jy,Jx,y)\}\\ &\qquad -4\{A(x,Jx,Jx,x) + A(y,Jy,Jy,y)\}\\ &\quad = 6A(x,y,y,x) + 6A(Jx,Jy,Jy,Jx) - 2A(x,Jy,Jy,x) - 2A(y,Jx,Jx,y)\\ &\qquad -12A(x,y,Jx,Jy) - 12A(x,Jx,y,Jy) - 4A(x,Jx,Jy,y) - 4A(x,Jy,Jx,y)\\ &\quad = 12A(x,y,y,x) - 4A(x,Jy,Jy,x)\\ &\qquad -12A(x,y,Jx,Jy) - 8A(x,Jx,y,Jy) - 4A(x,Jy,Jx,y).\end{aligned}$$

We may now compute:

$$\begin{aligned}
&3Q(x+Jy)+3Q(x-Jy)-Q(x+y)-Q(x-y)\\
&\quad -4Q(x)-4Q(y)+4\{5\lambda(x,y)+\lambda(x,Jy)\}\\
&\quad = 12A(x,y,x,y)-4A(x,Jy,x,Jy)\\
&\qquad -12A(x,y,Jy,Jx)-8A(x,Jx,Jy,y)-4A(x,Jy,y,Jx)\\
&\qquad +20(A(x,y,x,y)-A(x,y,Jx,Jy))\\
&\qquad +4(A(x,Jy,x,Jy)+A(x,Jy,Jx,y))\\
&\quad = 32A(x,y,x,y)-8A(x,y,Jx,Jy)-8A(x,Jx,Jy,y)-8A(x,Jy,y,Jx).
\end{aligned}$$

The desired identity now follows from the First Bianchi identity. □

The following result is due to Sato [154]; again, it was stated in a geometrical context but may be translated to a purely algebraic one.

Lemma 12.2.4 *Let $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,A,J)$ be a compatible complex model.*

1. *If $\mathcal{V}$ has constant holomorphic sectional curvature c, then*

$$\begin{aligned}
A(x,y,z,w) &= \tfrac{c}{4}\{A_0(x,y,z,w)+A_J(x,y,z,w)\}\\
&\quad +\tfrac{1}{8}\{5A(x,y,z,w)-3A(x,y,Jz,Jw)+A(x,z,Jw,Jy)\\
&\quad -A(x,w,Jz,Jy)-A(x,Jz,w,Jy)+A(x,Jw,z,Jy)\}.
\end{aligned}$$

2. *If $\mathcal{V}$ has constant zero holomorphic sectional curvature, then*

$$\begin{aligned}
3A(x,y,z,w)+3A(x,y,Jz,Jw) &= A(x,w,Jy,Jz)-A(x,z,Jy,Jw)\\
&\quad +A(x,Jz,Jy,w)-A(x,Jw,Jy,z)\,.
\end{aligned}$$

Proof. As the holomorphic sectional curvature is constant, $Q(x)=c\langle x,x\rangle^2$. We use the identity of Lemma 12.2.3 to see:

$$\begin{aligned}
32A(x,y,x,y) &= 3c\{\langle x,x\rangle+\langle y,y\rangle+2\langle x,Jy\rangle\}^2+3c\{\langle x,x\rangle+\langle y,y\rangle-2\langle x,Jy\rangle\}^2\\
&\quad -c\{\langle x,x\rangle+\langle y,y\rangle+2\langle x,y\rangle\}^2-c\{\langle x,x\rangle+\langle y,y\rangle-2\langle x,y\rangle\}^2\\
&\quad -4c\langle x,x\rangle^2-4c\langle y,y\rangle^2+4\{5\lambda(x,y)+\lambda(x,Jy)\}\\
&= 8c\{\langle x,x\rangle\langle y,y\rangle-\langle x,y\rangle^2+3\langle x,Jy\rangle^2\}+4\{5\lambda(x,y)+\lambda(x,Jy)\}.
\end{aligned}$$

We now polarize this identity to see:

$$\begin{aligned}
&8(A(x,y,z,w)+A(x,z,y,w))\\
&\quad = 2c\{2\langle x,w\rangle\langle y,z\rangle-\langle x,y\rangle\langle z,w\rangle-\langle x,z\rangle\langle w,y\rangle\\
&\qquad +3\langle x,Jy\rangle\langle w,Jz\rangle+3\langle x,Jz\rangle\langle w,Jy\rangle\}\\
&\qquad +5\{A(x,y,z,w)+A(x,z,y,w)-A(x,y,Jz,Jw)-A(x,z,Jy,Jw)\}\\
&\qquad +A(x,Jy,z,Jw)+A(x,Jz,y,Jw)+A(x,Jy,Jz,w)+A(x,Jz,Jy,w)\,.
\end{aligned}\tag{12.10}$$

Interchanging x and y in Equation (12.10) yields:

$$\begin{aligned}
&8(A(y,x,z,w)+A(y,z,x,w))\\
&\quad= 2c\{2\langle y,w\rangle\langle x,z\rangle-\langle y,x\rangle\langle z,w\rangle-\langle y,z\rangle\langle w,x\rangle\\
&\qquad+3\langle y,Jx\rangle\langle w,Jz\rangle+3\langle y,Jz\rangle\langle w,Jx\rangle\}\\
&\qquad+5\{A(y,x,z,w)+A(y,z,x,w)-A(y,x,Jz,Jw)-A(y,z,Jx,Jw)\}\\
&\qquad+A(y,Jx,z,Jw)+A(y,Jz,x,Jw)+A(y,Jx,Jz,w)+A(y,Jz,Jx,w)\,.
\end{aligned} \tag{12.11}$$

We now subtract Equation (12.11) from Equation (12.10) and simplify to obtain:

$$\begin{aligned}
&8(A(x,y,z,w)+A(x,z,y,w)-A(y,x,z,w)-A(y,z,x,w))\\
&\quad= 24A(x,y,z,w)\\
&\quad= 2c\{3\langle x,w\rangle\langle y,z\rangle-3\langle x,z\rangle\langle w,y\rangle\\
&\qquad+3\langle x,Jw\rangle\langle y,Jz\rangle-3\langle x,Jz\rangle\langle y,Jw\rangle-6\langle x,Jy\rangle\langle z,Jw\rangle\}\\
&\qquad+10A(x,y,z,w)+5A(x,w,z,y)-5A(x,z,w,y)\\
&\qquad-10A(x,y,Jz,Jw)+A(x,Jw,Jz,y)-A(x,Jz,Jw,y)\\
&\qquad-5A(x,w,Jz,Jy)+A(x,Jy,Jz,w)-A(x,Jz,w,Jy)+A(x,Jy,Jz,w)\\
&\qquad+5A(x,z,Jw,Jy)+A(x,Jy,z,Jw)+A(x,Jw,z,Jy)+A(x,Jy,z,Jw)\\
&\quad= 2c\{3A_0(x,y,z,w)+3A_J(x,y,z,w)\}+15A(x,y,z,w)-9A(x,y,Jz,Jw)\\
&\qquad-3A(x,w,Jz,Jy)-3A(x,Jz,w,Jy)+3A(x,Jw,z,Jy)+3A(x,z,Jw,Jy).
\end{aligned}$$

Now Assertion (1) follows. It may be specialized to derive Assertion (2). □

12.3 Algebraic results

Since our goal is to analyze whether the complex Jacobi operator determines the curvature tensor, as a first approach we concentrate in examining the condition $\mathcal{J}(\cdot)=0$.

Lemma 12.3.1 *Let $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,A,J)$ be a complex model. The following conditions are equivalent:*

1. $\mathcal{J}(\pi)=0$ *for all* $\pi\in\mathbb{CP}(V,J)$.
2. $\mathcal{A}(x,y)=-\mathcal{A}(Jx,Jy)$ *for all* x,y.
3. $\mathcal{A}(\pi)=0$ *for all* $\pi\in\mathbb{CP}(V,J)$.
4. $\mathcal{A}(Jx,y)z=\mathcal{A}(x,Jy)z=\mathcal{A}(x,y)Jz$ *for all* x,y,z.

Proof. Suppose Condition (1) holds. Then $\mathcal{V}$ is compatible. Furthermore,

$$Q(x) = A(x, Jx, x, Jx) = A(x, Jx, x, Jx) + A(x, x, x, x) = \langle \mathcal{J}(\pi_x)x, x\rangle = 0\,,$$

so $\mathcal{V}$ has constant holomorphic sectional curvature 0. Thus Lemma 12.2.4 (2) applies and we show that Condition (2) holds by computing:

$$\begin{aligned} & 3A(x, y, z, w) + 3A(x, y, Jz, Jw) \\ = \; & A(x, w, Jy, Jz) - A(x, z, Jy, Jw) + A(x, Jz, Jy, w) - A(x, Jw, Jy, z) \\ = \; & \langle\{\mathcal{J}(\pi_z) + \mathcal{J}(\pi_{Jw}) - \mathcal{J}(\pi_{z+Jw})\}x, Jy\rangle = 0\,. \end{aligned}$$

Suppose Condition (2) holds. We establish Condition (3) by computing:

$$\mathcal{A}(\pi_x) = \mathcal{A}(x, Jx) = -\mathcal{A}(Jx, JJx) = \mathcal{A}(Jx, x) = -\mathcal{A}(x, Jx) = -\mathcal{A}(\pi_x)\,.$$

Suppose Condition (3) holds. Then $\mathcal{V}$ is compatible. Again Lemma 12.2.4 is applicable. We set $w = Jz$ in Lemma 12.2.4 (2) to show Condition (1) holds by computing:

$$\begin{aligned} 0 &= 6\langle\mathcal{A}(\pi_z)x, y\rangle = 6A(x, y, z, Jz) \\ &= 2A(x, z, Jy, z) + 2A(x, Jz, Jy, Jz) = 2\langle\mathcal{J}(\pi_z)x, Jy\rangle\,. \end{aligned}$$

We have shown that Conditions (1), (2) and (3) are equivalent. Suppose that Condition (4) holds. We show that Condition (3) holds by computing:

$$\mathcal{A}(\pi_x) = \mathcal{A}(x, Jx) = \mathcal{A}(Jx, x) = -\mathcal{A}(x, Jx) = -\mathcal{A}(\pi_x)\,.$$

Finally suppose that Condition (1) holds. We must show Condition (4) holds. Since Condition (1) implies Condition (2), we have $\mathcal{A}(x, y) = -\mathcal{A}(Jx, Jy)$. Thus

$$A(Jx, y)z = -A(JJx, Jy)z = \mathcal{A}(x, Jy)z\,.$$

Since $\mathcal{J}(\pi_y) = 0$ and since $\mathcal{V}$ is compatible,

$$0 = A(y, x, y, w) + A(Jy, x, Jy, w) = A(y, x, y, w) + A(y, Jx, y, Jw)\,.$$

Polarize this identity to see

$$0 = A(y, x, z, w) + A(z, x, y, w) + A(y, Jx, z, Jw) + A(z, Jx, y, Jw)\,. \tag{12.12}$$

Since $\mathcal{A}(x, w) = -\mathcal{A}(Jx, Jw)$, $A(Jx, Jw, y, z) = -A(x, w, y, z)$. We may therefore use the First Bianchi identity to see:

$$\begin{aligned} 0 &= A(Jx, y, z, Jw) + A(Jx, Jw, y, z) + A(Jx, z, Jw, y) \\ &\quad + A(x, y, z, w) + A(x, w, y, z) + A(x, z, w, y) \\ &= A(Jx, y, z, Jw) - A(Jx, z, y, Jw) + A(x, y, z, w) - A(x, z, y, w)\,. \end{aligned} \tag{12.13}$$

Subtract (12.12) to (12.13), then we get

$$0 = 2A(x, y, z, w) + 2A(Jx, y, z, Jw)\,.$$

Replacing w by Jw and changing the order of the arguments then shows Condition (1) implies Condition (4) since:

$$A(x, y, Jw, z) = A(Jx, y, w, z)\,.$$

This completes the proof of the Lemma. □

A result in [166] shows that an algebraic curvature tensor satisfies Assertion (4) in Lemma 12.3.1 if and only if it belongs to the invariant subspace $\mathfrak{A}_2^\perp(V, J)$. Although it is equivalent to the result in [166], we use projection $\mathcal{P}$ to show in the next lemma that $\mathfrak{A}_2^\perp(V, J)$ is exactly the space of curvature tensors with vanishing complex Jacobi operator.

Lemma 12.3.2 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex model. The following assertions are equivalent:*

1. *$\mathcal{J}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V, J)$.*
2. *$A \in \mathfrak{A}_2^\perp(V, J)$.*

Proof. Let $\mathcal{P} : \mathfrak{A}(V, J) \to \mathfrak{A}(V, J)$ be defined by

$$\begin{aligned}\mathcal{P}(A)(x, y, z, w) \;=\; & \tfrac{1}{8}\{A(x, y, z, w) + A(Jx, Jy, Jx, Jw) \\ & -A(Jx, Jy, z, w) - A(Jx, y, Jz, w) - A(Jx, y, z, Jw) \\ & -A(x, y, Jz, Jw) - A(x, Jy, z, Jw) - A(x, Jy, Jz, w)\}\,.\end{aligned}$$

Note that $\mathcal{P}(A) = 0$ for all $A \in \mathfrak{A}_2(V, J)$. Also note that $\mathcal{P}(A) \in \mathfrak{A}_3(V, J)$ since

$$\mathcal{P}(A)(x, y, z, w) = \mathcal{P}(A)(Jx, Jy, Jz, Jw)\,.$$

Moreover, $\mathcal{P}^2 = \mathcal{P}$, so $\mathcal{P}$ is the projection on $\mathfrak{A}_2^\perp(V, J)$.

Since $\mathcal{P}$ is the projection on $\mathfrak{A}_2^\perp(V, J)$, we can express

$$\mathfrak{A}_2^\perp(V, J) = \{A \in \mathfrak{A}(V, J) : \mathcal{P}(A) = A\}.$$

Thus, if $\mathcal{J}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V, J)$, we may use equivalences of Lemma 12.3.1 to see $\mathcal{P}(A) = A$ and hence $A \in \mathfrak{A}_2^\perp(V, J)$.

Now, if $A \in \mathfrak{A}_2^\perp(V, J)$, for any vectors x, y, z, w we have

$$A(x, y, z, w) + A(Jx, Jy, z, w) = \mathcal{P}(A)(x, y, z, w) + \mathcal{P}(A)(Jx, Jy, z, w) = 0\,.$$

Therefore $\mathcal{A}(x, y) = -\mathcal{A}(Jx, Jy)$ and, by Lemma 12.3.1, we obtain that $\mathcal{J}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V, J)$. □

Remark 12.3.3 Algebraic curvature tensors in $\mathfrak{A}_2^\perp(V,J)$ have several useful properties. Let $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,A,J)$ be a complex model with $A\in\mathfrak{A}_2^\perp(V,J)$, then:

- $\mathcal{V}$ is compatible. This is a direct consequence of the definition of $\mathfrak{A}_2^\perp(V,J)$.
- We have that
$$2\rho(x,y)=\sum_{i=1}^n\{A(x,e_i,y,e_i)+A(x,Je_i,y,Je_i)\}=\sum_{i=1}^n\langle\mathcal{J}(\pi_{e_i})x,y\rangle)=0,$$
and
$$\rho^\star(x,y)=\sum_{i=1}^n A(x,e_i,Jy,Je_i)=-\sum_{i=1}^n A(x,e_i,y,e_i)=-\rho(x,y)=0,$$
where $\{e_1,\dots,e_n\}$ forms an orthonormal basis. This shows $\mathcal{V}$ is Ricci flat and $\star$-Ricci flat.
- $\mathfrak{A}_2^\perp(V,J)$ is one of the 10 invariant subspaces of $\mathfrak{A}(V,J)$ (see [166] for the complete decomposition and other interesting properties). Moreover, $\mathfrak{A}_2^\perp(V,J)$ is conformally invariant.

The next example shows that Theorem 12.1.3 does not have a purely algebraic analogue even if we impose the compatibility condition of Lemma 12.2.1. Note that the domain of the usual Jacobi operator is V which is n-dimensional. The domain of the higher order Jacobi operator is the dimension of the p-dimensional Grassmannian which has dimension greater than n for $2\le p\le n-2$. However, the domain of the complex Jacobi operator is $\mathbb{CP}(V,\langle\cdot,\cdot\rangle,J)$ which is $n-2$ dimensional. This permits constructing a model with vanishing complex Jacobi operator, but with curvature tensor different from zero.

Theorem 12.3.4 *If $n\equiv 0$ mod 4, there exists a complex model $\mathcal{V}=(V,\langle\cdot,\cdot\rangle,A,J)$ with $A\ne 0$ so that $\mathcal{J}(\pi)=0$ and so that $\mathcal{A}(\pi)=0$ for every $\pi\in\mathbb{CP}(V,J)$.*

Proof. Since the dimension of V is a multiple of 4, we may choose almost complex structures J,K such that $JK+KJ=0$. Now consider the algebraic curvature tensor $A:=A_K-A_{JK}$ where A_K and A_{JK} are given by expression (9.1). Note that the Jacobi operator is given by
$$\mathcal{J}(x)y=3\langle Kx,y\rangle Kx-3\langle JKx,y\rangle JKx\,,$$
and, since $\mathcal{J}(x)Kx=Kx$ for any unit vector x, $A\ne 0$. However, the complex Jacobi operator vanishes identically:
$$\begin{aligned}\mathcal{J}(\pi_x)&=3\langle Kx,y\rangle Kx-3\langle JKx,y\rangle JKx\\&\quad+3\langle KJx,y\rangle KJx-3\langle JKJx,y\rangle JKJx\\&=0\,.\end{aligned}$$
We may now apply Lemma 12.3.1 to see that $\mathcal{A}(\pi)$ vanishes identically as well. □

This shows that the complex Jacobi operator has a different behavior at the algebraic level than the Jacobi operator.

12.4 Geometrical results

We begin our study of the geometrical context by recalling several well known results. We refer to Gray [91] for the proof of Assertions (1) and (2) (see also [90]) in the following Lemma:

Lemma 12.4.1 *Let $\mathcal{C} = (M, g, J)$ be a Hermitian manifold and let $\mathcal{V} = \mathcal{V}(\mathcal{C}, p)$ be the almost complex model determined by $\mathcal{C}$ at a point $p \in M$. Then:*

1. *If $\mathcal{C}$ is Hermitian, then $\mathcal{P}(R) = 0$.*
2. *If $\mathcal{C}$ is nearly Kähler, then $R \in \mathfrak{A}_2(M, J)$. Thus, in particular, $\mathcal{P}(R) = 0$.*
3. *If $\mathcal{C} = (M, g, J)$ is an almost Kähler manifold, then the following identity from [91] holds*

$$\begin{aligned} &-2\langle(\nabla_x J)y - (\nabla_y J)x, (\nabla_z J)w - (\nabla_w J)z\rangle = \\ &R(x,y,z,w) + R(Jx,Jy,Jz,Jw) - R(Jx,Jy,x,w) - R(x,y,Jz,Jw) \\ &+R(Jx,y,Jz,w) + R(x,Jy,z,Jw) + R(Jx,y,z,Jw) + R(x,Jy,Jz,w). \end{aligned} \tag{12.14}$$

Also, from Yano [174] we have

$$\|\nabla J\|^2 = 2(\tau^\star - \tau). \tag{12.15}$$

Next we proceed to prove Theorem 12.1.3. It will follow from the following Lemma:

Lemma 12.4.2 *Let $\mathcal{C}_i$ be almost Hermitian manifolds. Suppose given a complex isometry*

$$\theta : (T_pM_1, g_1, J_1) \to (T_qM_2, g_2, J_2).$$

Let $V = T_pM_1$, $\langle\cdot,\cdot\rangle = g_1$, $J = J_1$, and $A := R_1 - \theta^* R_2$. Let

$$\mathcal{V} = \mathcal{V}(\mathcal{C}_1, p, \mathcal{C}_2, q, \theta) := (V, \langle\cdot,\cdot\rangle, A, J)\,.$$

Assume that $\mathcal{C}_i$ are Hermitian or nearly Kähler manifolds. Then the following assertions are equivalent:

1. *$\mathcal{J}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V, J)$.*
2. *$\mathcal{R}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V, J)$.*
3. *$R = 0$.*

Proof. Assume that either Condition (1) or Condition (2) holds; these are equivalent by Lemma 12.3.1. Since the curvature tensors $R_{\mathcal{C}_i}$ satisfy $\mathcal{P}(R_{\mathcal{C}_i}) = 0$ by Lemma 12.4.1, so does their difference at any point. We use the relations provided by Lemma 12.3.1 to show that $R = 0$ by computing:

$$\begin{aligned} 0 = \mathcal{P}(R) &= \frac{1}{8}\{R(x,y,z,w) + R(Jx,Jy,Jz,Jw) - R(Jx,Jy,z,w) - R(x,y,Jz,Jw) \\ &\quad -R(Jx,y,Jz,w) - R(x,Jy,z,Jw) - R(Jx,y,z,Jw) - R(x,Jy,Jz,w)\} \\ &= \frac{1}{8}\{R(x,y,z,w) + R(x,y,z,w) - R(JJx,y,z,w) - R(JJx,y,z,w) \\ &\quad -R(JJx,y,z,w) - R(JJx,y,z,w) - R(JJx,y,z,w) - R(JJx,y,z,w)\} \\ &= R(x,y,z,w)\,. \end{aligned}$$

Conversely, of course, if $R = 0$, then $\mathcal{J}(\pi) = \mathcal{R}(\pi) = 0$ for all $\pi \in \mathbb{CP}(V,J)$. □

Proof of Theorem 12.1.4. We use Lemma 12.3.1 to see that $\mathcal{M}$ is both Ricci flat and $\star$-Ricci flat. Hence $\tau = \tau^\star = 0$. Therefore, by Lemma 12.4.1 (3), $\|\nabla J\| = 0$, hence $\nabla J = 0$ and the manifold is Kähler. This implies the almost complex structure is in fact integrable so $\mathcal{C}$ is Hermitian. The desired conclusion now follows from Theorem 12.1.3. □

Remark 12.4.3 These previous result holds in the Riemannian setting and fails in higher signature. The proof relies on the fact that the metric is positive definite to obtain that the complex structure is parallel.

Remark 12.4.4 As we have previously mentioned, Theorem 12.1.4 is not as strong as Theorem 12.1.3. One may impose $\mathcal{P}(R) = 0$ in expression (12.14) to obtain necessary and sufficient conditions for an almost Kähler manifold to satisfy the thesis of Theorem 12.1.3. Thus, one gets that the complex Jacobi operator (or the complex skew-symmetric curvature operator) determines the curvature tensor in an almost Kähler manifold if and only if the following identity is satisfied:

$$\begin{aligned} &R(Jx,y,Jz,w) + R(Jx,y,z,Jw) + R(x,Jy,z,Jw) + R(x,Jy,Jz,w) \\ &\quad = -\langle (\nabla_x J)y - (\nabla_y J)x, (\nabla_z J)w - (\nabla_w J)z \rangle. \end{aligned}$$

This identity establishes a non-trivial relation between the curvature tensor and the covariant derivative of the complex structure, which has no meaning on a purely algebraic level.

We show that Theorem 12.1.3 fails in the almost Hermitian context with the following proof.

Proof of Theorem 12.1.2. Our construction is motivated by the construction of Theorem 12.3.4 and is based on work of Sato [155]. Let $n = 4m$. Let $(\mathbb{CP}^{2m}, g, J)$ be complex projective space with the Fubini-Study metric g and usual complex structure J; this is a Kähler manifold. The canonical embedding of $\mathbb{C}^{2m} \subset \mathbb{C}^{2m+1}$ defines an isometric embedding of $\mathbb{CP}^{2m-1}$ in $\mathbb{CP}^{2m}$. Let $M := \mathbb{CP}^{2m} - \mathbb{CP}^{2m-1}$. Let $\mathcal{H}$ be the fiber bundle of all unitary quaternion structures $\{J_1, J_2, J_3\}$ on the tangent bundle of M which satisfy $J_1 = J$. Since M is contractable, $\mathcal{H}$ is a trivial fiber bundle so we can define a global quaternion structure $\{J_1, J_2, J_3\}$ on TM so that $J = J_1$. This is, of course, just the usual twistor construction.

Let $\mathcal{C} := (M, g, J_2)$. Let

$$\Theta : x \to (1 + J_2)/\sqrt{2}\,.$$

This defines an isometry of T_pM with $\Theta J_2 = J_2\Theta$. Furthermore

$$\Theta J_1 = -J_3\Theta \quad \text{and} \quad \Theta J_3 = J_1\Theta\,.$$

The curvature tensor of the Fubini-Study metric is given by $R_0 + R_{J_1}$. Let x be a unit tangent vector. Then

$$\mathcal{J}_R(x)y = \begin{cases} 0 & \text{if} \quad y \in \operatorname{Span}\{x\}, \\ 4y & \text{if} \quad y \in \operatorname{Span}\{J_1x\}, \\ y & \text{if} \quad y \perp \operatorname{Span}\{x, J_1x\}\,. \end{cases}$$

As $\Theta^*R = R_0 + R_{J_3}$ and as $J_1x \perp J_3x$, $\Theta^*R \neq R$. Since $\mathcal{J}_R(\pi_x) = \mathcal{J}_R(x) + \mathcal{J}_R(J_2x)$,

$$\mathcal{J}_R(\pi_x)y = \begin{cases} y & \text{if} \quad y \in \operatorname{Span}\{x, J_2x\}, \\ 5y & \text{if} \quad y \in \operatorname{Span}\{J_1x, J_1J_2x = J_3x\}, \\ 2y & \text{if} \quad y \perp \operatorname{Span}\{x, J_1x, J_2x, J_3x\}\,. \end{cases}$$

Since J_1 and J_3 play symmetric roles in this identity, $\mathcal{J}_{\Theta^*R}(\pi_x) = \mathcal{J}_R(\pi_x)$ as desired. Lemma 12.3.1 now shows $\mathcal{R}_{\Theta^*R}(\pi_x) = \mathcal{R}_R(\pi_x)$ as well. □

12.4.1 Conformal and almost Kähler geometry

There are many examples in the literature of almost Kähler manifolds which are not Kähler [1, 54, 170]. Also, a great interest has been shown in finding conditions for an almost Kähler manifold to be Kähler. For example, the Goldberg conjecture states: *A compact Einstein almost Kähler manifold is Kähler.* This conjecture has generated extensive literature, see for example [158, 159]. Many other conditions have been studied which might imply that an almost Kähler manifold is Kähler, see [4, 8] for example. The following result follows that line.

Theorem 12.4.5 *Let $\mathcal{C} = (M, g, J)$ be an almost Kähler manifold such that there exists a Kähler metric $\tilde{g}$ on (M, J) with $\mathcal{R}(\pi) = \tilde{\mathcal{R}}(\pi)$ for all $\pi \in \mathbb{CP}(TM, J)$ (equivalently, $\mathcal{J}_R(\pi) = \mathcal{J}_{\tilde{R}}(\pi)$ for all $\pi \in \mathbb{CP}(TM, J)$). Then $\mathcal{C}$ is Kähler and $R = \tilde{R}$.*

Proof. Suppose $\mathcal{R}(x, Jx) = \tilde{\mathcal{R}}(x, Jx)$. Set $\bar{R} := R - \tilde{R}$. Then $\bar{\mathcal{R}}(\pi) = 0$ for all π. Since $\tau^\star = \tau$ for any Kähler manifold and $\bar{R}$ is Ricci flat and $\star$-Ricci flat by Lemma 12.3.1 (2), one has $\|\nabla J\|^2 = 0$ by Lemma 12.4.1 (3); hence $\nabla J = 0$ and the original manifold $\mathcal{C}$ is indeed Kähler. That the curvature tensors are equal follows from Theorem 12.1.3. $\square$

We conclude this chapter showing that for almost Hermitian manifolds in the same conformal class, to have equal complex Jacobi operators means to have the same curvature tensor.

Theorem 12.4.6 *Let $\mathcal{C} := (M, g, J)$ and $\mathcal{C}^\alpha := (M, e^\alpha g, J)$ be conformally equivalent almost Hermitian manifolds. If $\mathcal{J}_{\mathcal{C}}(\pi_x) = \mathcal{J}_{\mathcal{C}^\alpha}(\pi)$ for all $\pi \in \mathbb{CP}(TM, J)$, then $R = R^\alpha$.*

Proof. A priori projection $\mathcal{P}$ on $\mathfrak{A}_2^\perp(g, J)$ depends on the choice of g. It is, however conformally invariant, i.e. $\mathfrak{A}_2^\perp(g, J) = \mathfrak{A}_2^\perp(e^\alpha g, J)$ and $\mathcal{P}(g, J) = \mathcal{P}(e^\alpha g, J)$. For notational simplicity set $\sigma := \mathcal{P}(g, J)$. Note that, see [166], $\sigma(R) = \sigma(R^\alpha)$; thus $\sigma(R - R^\alpha) = 0$. Since by hypothesis $(\mathcal{J}_{\mathcal{C}} - \mathcal{J}_{\mathcal{C}^\alpha})(\pi_x) = 0$ for all x, we use the argument given in Lemma 12.4.2 to conclude as desired that $R = R^\alpha$. $\square$

Chapter 13

Clifford families and Complex Osserman models

Recall that an almost Hermitian manifold is said to be complex Osserman if the complex Jacobi operator $\mathcal{J}(\pi)$ has constant eigenvalues and if $\mathcal{J}(\pi)$ is complex linear which forces compatibility. In this chapter we work at the purely algebraic level to determine the eigenvalue structure of complex Osserman models. In Section 13.1 we show some basic important properties, we give necessary and sufficient conditions so that a complex model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ is complex Osserman and we show that $\mathcal{V}$ is necessarily Einstein if it is complex Osserman. In Theorem 13.1.7 we make use of algebraic topology techniques to control the eigenvalue structure of $\mathcal{J}(\pi)$ if $\mathcal{V}$ is complex Osserman.

Work of Nikolayevsky shows that any Osserman model $(V, \langle\cdot,\cdot\rangle, A)$ is given by a Clifford family except in dimension 16. This motivates our study of complex Osserman algebraic curvature tensors given by a Clifford family. We divide our study into two cases depending on the rank κ of the structure in question. In Section 13.2, we recall results of Adams on the existence of Clifford families and discuss some reparametrization results. We essentially classify complex Osserman algebraic curvature tensors given by a Clifford family, thus we take advantage of those results to show Theorem 13.1.7 is sharp.

Finally we point out two basic differences between the Osserman and the complex Osserman conditions. While the eigenvalue structure of an Osserman algebraic curvature tensor is very rich, it is very restrictive for the complex Osserman case (Theorem 13.1.7). Moreover, any Osserman algebraic curvature tensor is given by a Clifford family (except for dimension 16), but this is not true in the complex Osserman setting (Theorem 13.2.7).

This chapter is based on results in [39]. We refer to [85] for a broader exposition.

13.1 Algebraic preliminaries

In this section we present some foundational results. Let $Spec\{\mathcal{J}(\pi_x)\}$ be the spectrum of $\mathcal{J}(\pi_x)$ and let $E_\lambda(\pi_x)$ be the eigenspace associated to the eigenvalue λ of $\mathcal{J}(\pi_x)$. Since

$\mathcal{J}(\pi_x)$ is self-adjoint,

$$\begin{aligned}\langle \mathcal{J}(\pi_x)y, z\rangle &= A(x,y,x,z) + A(Jx,y,Jx,z)\\ &= A(x,z,x,y) + A(Jx,z,Jx,y)\\ &= \langle \mathcal{J}(\pi_x)z, y\rangle,\end{aligned}$$

$\mathcal{J}(\pi_x)$ is diagonalizable with real eigenvalues. Thus we have an orthogonal direct sum decomposition

$$V = \oplus_\lambda E_\lambda(\pi_x)$$

for any $x \in S(V, \langle\cdot,\cdot\rangle)$. The following lemma provides a criterion for complex Osserman curvature tensors:

Lemma 13.1.1 $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ *is complex Osserman if and only if*

1. $JE_\lambda(\pi_x) = E_\lambda(\pi_x)$ *for all* $\pi_x \in \mathbb{CP}(V, J)$ *and* $\lambda \in Spec\{\mathcal{J}(\pi_x)\}$.
2. $Spec\{\mathcal{J}(\pi_x)\} = Spec\{\mathcal{J}(\pi_y)\}$ *for all* $\pi_x, \pi_y \in \mathbb{CP}(V, J)$.

Proof. If $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ is complex Osserman then J and A are compatible, so J preserves the eigenspaces of $\mathcal{J}(\pi_x)$ and Condition 1. holds. Moreover, the eigenvalue structure does not depend on π_x, so Condition 2. holds. On the other hand, if Condition 1. holds then J and $\mathcal{J}(\pi_x)$ commute and, by Lemma 12.2.1, J and R are compatible. This, together with Condition 2. implies $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ is complex Osserman. □

A model $(V, \langle\cdot,\cdot\rangle, A)$ is said to be Einstein if the Ricci tensor is a multiple of the metric tensor, i.e. $\rho(\cdot,\cdot) = c\,\langle\cdot,\cdot\rangle$ for a constant c. In general, p-Osserman models are Einstein. This result generalizes to become:

Lemma 13.1.2 *Let* $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ *be complex Osserman. Then* $\mathcal{V}$ *is Einstein.*

Proof. Assume that $\mathcal{V}$ is complex Osserman. Let $x \in S(V, \langle\cdot,\cdot\rangle)$. As A is compatible, by Lemma 12.2.1 we have that $A(Jx,y,Jx,z) = A(x,Jy,x,Jz)$ and thus $\mathcal{J}(Jx) = -J\mathcal{J}(x)J$. Consequently

$$\rho(x,x) = \operatorname{tr}\{\mathcal{J}(x)\} = \operatorname{tr}\{\mathcal{J}(Jx)\} = \tfrac{1}{2}\operatorname{tr}\{\mathcal{J}(\pi_x)\} = \tfrac{1}{2}\textstyle\sum_\lambda \lambda \dim\{E_\lambda(\pi_x)\}$$

is independent of $x \in S(V, \langle\cdot,\cdot\rangle)$. Thus $\rho(\cdot,\cdot) = c\,\langle\cdot,\cdot\rangle$ so $\mathcal{V}$ is Einstein. □

Recall from [150] the Rakić duality principle. It states that in an Osserman Riemannian model $\mathcal{J}(x)y = \lambda y$ if and only if $\mathcal{J}(y)x = \lambda x$ for any $x, y \in S(V, \langle\cdot,\cdot\rangle)$. We begin the proper study of complex Osserman models by giving a complex version of Rakić duality (see [84] and [150] for more information on Rakić duality).

In the following Lemma we prove this duality for extremal eigenvalues.

Lemma 13.1.3 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex Osserman model. Suppose λ is the minimum or the maximum eigenvalue and $\mathcal{J}(\pi_x)y = \lambda\, y$ for $x, y \in S(V, \langle\cdot,\cdot\rangle)$. Then $\mathcal{J}(\pi_y)x = \lambda\, x$.*

Proof. Assume $\mathcal{V}$ is complex Osserman and $\mathcal{J}(\pi_x)y = \lambda\, y$. Assume λ is the maximum eigenvalue. Then

$$\lambda = max_{z\in S(V,\langle\cdot,\cdot\rangle)}\langle \mathcal{J}(\pi_x)z, z\rangle,$$

and if z realizes the maximum then z is an eigenvector. Hence the result follows from the following sequence of equalities:

$$\begin{aligned}\lambda = \langle \mathcal{J}(\pi_x)y, y\rangle &= A(x, y, x, y) + A(Jx, y, Jx, y)\\ &= A(x, y, x, y) + A(x, Jy, x, Jy)\\ &= \langle \mathcal{J}(\pi_y)x, x\rangle.\end{aligned}$$

If λ is the minimum eigenvalue, the same argument applies. □

Let ρ_τ denote orthogonal projection on τ. In order to give the general complex Rakić duality principle we need the following technical results.

Lemma 13.1.4 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex algebraic model. Let S be a complex self-adjoint linear map with eigenvalues $\lambda_1 = \lambda_2 \le \lambda_3 = \lambda_4 \le \ldots \le \lambda_{n-1} = \lambda_n$. Set $\Lambda_k := \lambda_1 + \ldots + \lambda_{2k}$. Then $\Lambda_k = min_{\tau\in Gr_k(\mathbb{CP}V)}Tr\{\rho_\tau S\}$. Moreover, $\Lambda_k = Tr\{\rho_\tau S\}$ if and only if S preserves τ and if the eigenvalues of S restricted to τ are $\lambda_1 \le \ldots \le \lambda_{2k}$.*

Proof. Recall from [84] that $\Lambda_k = min_{\sigma\in Gr_{2k}(V)}Tr\{\rho_\sigma S\}$ and $\Lambda_k = Tr\{\rho_\sigma S\}$ if and only if S preserves σ and the eigenvalues of S restricted to σ are $\lambda_1 \le \ldots \le \lambda_{2k}$. Hence we shall show that there exists $\tau \in Gr_k(\mathbb{CP}V)$ which minimizes $Tr\{\sigma_\sigma S\}$. Since S is complex, each eigenvalue has even multiplicity and $Sv = \lambda_j v \Leftrightarrow SJv = \lambda_j Jv$. Therefore we can construct an adapted basis $\{v_1, Jv_1, \ldots, v_k, Jv_k\}$ such that $\tau = \text{Span}\,\{v_1, Jv_1, \ldots, v_k, Jv_k\}$ minimizes $Tr\{\rho_\sigma S\}$ and then the result follows. □

The following is a particular case of Lemma 3.4.1 in [84].

Lemma 13.1.5 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex model. Let τ and σ be complex subspaces of V (i.e. $J(\tau) \subset \tau$ and $J(\sigma) \subset \sigma$). Then $Tr\{\rho_\tau \mathcal{J}_R(\sigma)\} = Tr\{\rho_\sigma \mathcal{J}_R(\tau)\}$.*

Proof. Let $\{v_1, v_2, \ldots, v_r\}$ and $\{w_1, w_2, \ldots, w_s\}$ be orthonormal bases of τ and σ. Then we have:

$$\begin{aligned}Tr\{\rho_\tau \mathcal{J}(\sigma)\} &= \textstyle\sum_{i=1}^r \langle \mathcal{J}_R(\sigma)v_i, v_i\rangle = \sum_{i=1}^r \sum_{j=1}^s R(w_j, v_i, w_j, v_i)\\ &= \textstyle\sum_{i=1}^r \sum_{j=1}^s R(v_i, w_j, v_i, w_j) = \sum_{j=1}^r \langle \mathcal{J}_R(\tau)w_j, w_j\rangle = Tr\{\rho_\sigma \mathcal{J}(\tau)\}.\end{aligned}$$ □

The following Theorem establishes the complex Rakić duality. The proof we give here is essentially adapted from the proof given in [84] for the Rakić duality principle.

Theorem 13.1.6 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex Osserman model. Let λ be an eigenvalue of $\mathcal{J}(\pi)$. Then $\mathcal{J}(\pi_x)y = \lambda y$ if and only if $\mathcal{J}(\pi_y)x = \lambda x$.*

Proof. Assume $\mathcal{J}(\pi_x)y = \lambda y$. Let $\lambda_1(x) = \lambda_2(x) \leq \lambda_3(x) = \lambda_4(x) \leq \ldots \leq \lambda_n(x)$ be the eigenvalues of $\mathcal{J}(\pi_x)$. If λ is a minimal or a maximal eigenvalue then the result follows from Lemma 13.1.3. Let $\lambda = \lambda_{2i-1} = \lambda_{2i}$.

We define $\Lambda_i(\sigma)$ for any subspace $\sigma \subset V$ as

$$\Lambda_i(\sigma) := \mu_1(\sigma) + \ldots + \mu_{2i}(\sigma),$$

where the μ_i's are the ordered eigenvalues of $\mathcal{J}(\sigma)$. Thus, note that $\Lambda_i(\pi_x)$ is the sum of the first $2i$ eigenvalues of $\mathcal{J}(\pi_x)$ and it is constant since $\mathcal{V}$ is complex Osserman. Now, let the set of vectors $\{v_1, v_2 = Jv_1, \ldots, y = v_{2i-1}, Jy = v_{2i}, \ldots, v_{n-1}, v_n = Jv_{n-1}\}$ be an adapted basis of orthonormal eigenvectors such that

$$\mathcal{J}(\pi_x)v_{2j-1} = \lambda_j v_{2j-1} \quad \text{and} \quad \mathcal{J}(\pi_x)v_{2j} = \lambda_j v_{2j}.$$

Let $\tau_i := \operatorname{Span}\{v_1, \ldots, v_{2i}\}$. We apply Lemma 13.1.5 to get that for any complex line π

$$\Lambda_i(\pi_x) = \Lambda_i(\pi) \leq Tr\{\rho_{\tau_i}\mathcal{J}(\pi)\} = Tr\{\rho_\pi \mathcal{J}(\tau_i)\}.$$

Since π is arbitrary we use Lemma 13.1.4 to obtain:

$$\Lambda_i(\pi_x) \leq min_{\pi\in\mathbb{CP}V} Tr\{\rho_\pi \mathcal{J}_R(\tau_i)\} = \Lambda_2(\tau_i) = 2\mu_1(\tau_i)$$

where μ_1 is the minimal eigenvalue of $\mathcal{J}_R(\tau_i)$. On the other hand we have

$$2\mu_1(\tau_i) = min_{\pi\in\mathbb{CP}V} Tr\{\rho_\pi \mathcal{J}_R(\tau_i)\} \leq Tr\{\rho_{\pi_x}\mathcal{J}_R(\tau_i)\} = Tr\{\rho_{\tau_i}\mathcal{J}_R(\pi_x)\} = \Lambda_i(\pi_x).$$

Hence we conclude the equality

$$2\mu_1(\tau_i) = Tr\{\rho_{\pi_x}\mathcal{J}_R(\tau_i)\}$$

and, by Lemma 13.1.4, $\mathcal{J}_R(\tau_i)x = \mu_1 x$. Therefore, x is an eigenvector of $\mathcal{J}(\pi_y)$, since

$$\mathcal{J}(\pi_y)x = \mathcal{J}(\tau_i)x - \mathcal{J}(\tau_{i-1})x = \nu x$$

for a certain ν. Now note that

$$\nu = \langle \mathcal{J}(\pi_y)x, x\rangle = \langle \mathcal{J}(\pi_x)y, y\rangle = \lambda\,,$$

so $\mathcal{J}(\pi_y)x = \lambda x$, which completes the proof. □

Methods of algebraic topology can be used to control the eigenvalue structure of a complex Osserman model. In particular, no more than 3 distinct eigenvalues may occur.

Theorem 13.1.7 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be complex Osserman. If $\mathcal{J}(\pi)$ is not a multiple of the identity (i.e. if $\mathcal{J}(\pi)$ has at least 2 distinct eigenvalues), then:*

1. *If $n \equiv 2$ (mod 4), there are 2 eigenvalues with multiplicities $(n-2, 2)$.*
2. *If $n \equiv 0$ (mod 4), then one of the following holds:*
   (a) *There are 2 eigenvalues with multiplicities $(n-2, 2)$.*
   (b) *There are 2 eigenvalues with multiplicities $(n-4, 4)$.*
   (c) *There are 3 eigenvalues with multiplicities $(n-4, 2, 2)$.*

Proof. Let $\mathbb{V} := \mathbb{CP}(V, \langle\cdot,\cdot\rangle, J) \times V$ be the trivial bundle over projective space. Lemma 13.1.1 shows that the eigenspaces

$$E_{\lambda_i}(\pi) := \{v \in V : \mathcal{J}(\pi)v = \lambda_i v\}$$

have constant rank and patch together to define smooth vector bundles $E_{\lambda_i}(\pi)$ over $\mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$ where $\{\lambda_0, ..., \lambda_k\}$ denote the distinct eigenvalues of $\mathcal{J}(\pi)$ for any, and hence for all, $\pi \in \mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$. This gives the following direct sum decomposition

$$\mathbb{V} = E_{\lambda_0} \oplus \ldots \oplus E_{\lambda_k}.$$

This decomposition is in the category of complex vector bundles since the eigenbundles are invariant under J.

A sub-bundle E of $\mathbb{V}$ is said to be a *geometrically symmetric vector bundle* if for all complex lines σ, τ in $\mathbb{CP}(V, \langle\cdot,\cdot\rangle, J)$, $\tau \subset E(\sigma)$ implies $\sigma \subset E(\tau)$. Note that Theorem 13.1.6 implies that the bundle E_λ is geometrically symmetric for any eigenvalue λ. Hence, following results in [83], the theorem is obtained. □

We shall show that this result is sharp in Remark 13.2.6 by giving an example for each possibility.

13.2 Clifford families and associated curvature tensors

We say that a set $\mathcal{F} = \{J_1, \ldots, J_\kappa\}$ of Hermitian almost complex structures on $(V, \langle\cdot,\cdot\rangle)$ is a *Clifford family of rank* κ if they are subject to the commutation rules

$$J_i J_j + J_j J_i = -2\delta_{ij} Id.$$

Recall the definition of the following algebraic curvature tensors, where J is an arbitrary almost complex structure:

$$\begin{aligned} A_0(x,y,z,w) &= \langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle, \\ A_J(x,y,z,w) &= \langle Jx,z\rangle\langle Jy,w\rangle - \langle Jy,z\rangle\langle Jx,w\rangle + 2\langle Jx,y\rangle\langle Jz,w\rangle. \end{aligned}$$

We say that a model $(V, \langle\cdot,\cdot\rangle, A)$ is *given by a Clifford family* $\mathcal{F}$ *of rank* κ if there exist constants c_i with $c_i \neq 0$ for $1 \leq i \leq \kappa$ so that

$$A = c_0 A_0 + c_1 A_{J_1} + ... + c_\kappa A_{J_\kappa} \,. \tag{13.1}$$

We shall also sometimes simply say that A is given by a Clifford family in this setting. Note that

$$\mathcal{J}_{A_0}(x)y = y - \langle y, x\rangle x \quad \text{and} \quad \mathcal{J}_{A_J}(x)y = 3\langle y, Jx\rangle Jx \,. \tag{13.2}$$

This relations yield:

$$\mathcal{J}(x)y = c_0\{y - \langle y, x\rangle x\} + 3c_1\langle y, J_1x\rangle J_1x + ... + 3c_\kappa\langle y, J_\kappa x\rangle J_\kappa x \,. \tag{13.3}$$

From this it follows immediately that

$$\begin{aligned} \mathcal{J}(\pi_x)y &= c_0\{2y - \langle y, x\rangle x - \langle y, Jx\rangle Jx\} \\ &\quad + \sum_{i=1}^{\kappa} 3c_i\{\langle y, J_ix\rangle J_ix + \langle y, J_iJx\rangle J_iJx\} \,. \end{aligned} \tag{13.4}$$

A Clifford family $\mathcal{F} = \{J_1, J_2, J_3\}$ of rank 3 is called a *quaternion structure* if $J_1J_2 = J_3$. Note that V admits a quaternion structure if and only if $dim\, V$ is divisible by 4. One defines the *Adams number* $\nu(n)$ by setting $\nu(1) = 0$, $\nu(2) = 1$, $\nu(4) = 3$, $\nu(8) = 7$, $\nu(16r) = \nu(r) + 8$ and $\nu(n2^s) = \nu(2^s)$ for n odd. One then has the following well known result of Atiyah, Bott and Shapiro [7] which is closely related to work of Adams [2] concerning vector fields on spheres:

Lemma 13.2.1 *There exists a Clifford family of rank* κ *on* V *if and only if* $\kappa \leq \nu(n)$.

Remark 13.2.2 Notice that Theorem 13.1.7 points out a big difference between the Osserman and the complex Osserman problems. While the possible eigenvalue structure for Riemannian complex Osserman models is restricted to 4 different possibilities, one may use Clifford families to construct Osserman models with a wide range of possible eigenvalues and multiplicities. Indeed, from expression (13.3) one gets that the Jacobi operator of

$$A = c_0 A_0 + \frac{c_1 - c_0}{3} A_{J_1} + ... + \frac{c_\kappa - c_0}{3} A_{J_\kappa}$$

has eigenvalues $(0, c_1, \ldots, c_\kappa, c_0, \overset{n-\kappa-1}{\ldots\ldots}, c_0)$.

We now present a useful technical result:

Lemma 13.2.3 *Let* V *and* W *be vector spaces and let* $\mathcal{T} = \{T_1, \ldots, T_\kappa\}$ *be a family of linear maps* $T_i : V \to W$. *Assume there is an integer* μ *so that for any set of constants* a_i, *not all of which are zero, one has* $\operatorname{Rank}\{a_1T_1 + \ldots + a_\kappa T_\kappa\} \geq \mu$. *Then the following assertions hold:*

1. *If $\mu \geq \kappa$, there exists $x \in V$ so that $\{T_1 x, \ldots, T_\kappa x\}$ is a set of linearly independent vectors.*

2. *If $\mu \geq 2\kappa$, there exists $x, y \in V$ so that $\{T_1 x, \ldots, T_\kappa x, T_1 y, \ldots, T_\kappa y\}$ is a set of linearly independent vectors.*

3. *Let $T : V \longrightarrow W$ be a linear map so that $Tx \in \operatorname{Span}\{T_1 x, \ldots, T_\kappa x\}$ for all $x \in V$. If $\mu \geq 2\kappa$, then $T \in \operatorname{Span}\{T_1, \ldots, T_\kappa\}$.*

Proof. In order to prove Assertion (1), suppose $\mu \geq \kappa$. For a given $x \in V$, choose $r(x)$ maximal so that $\{T_1 x, \ldots, T_r x\}$ is a linearly independent set of r vectors. Take $x \in V$ so that $r(x)$ is maximal. If $r(x) = \kappa$, then clearly Assertion (1) holds. Suppose $r(x) < \kappa$. We argue for a contradiction. Choose $(a_1, \ldots, a_r)$ so that $a_1 T_1 x + \ldots + a_r T_r x + T_{r+1} x = 0$ and let

$$S := a_1 T_1 + \ldots + a_r T_r + T_{r+1} .$$

As $\operatorname{Rank}\{S\} \geq \mu \geq \kappa$, there is $y \in V$ so that $\{T_1 x, \ldots, T_r x, Sy\}$ is a set of $r+1$ linearly independent vectors. Hence, by continuity, there exists $\epsilon > 0$ such that $\{T_1(x + \epsilon y), \ldots, T_r(x + \epsilon y), Sy\}$ is a set of $r+1$ linearly independent vectors. Consequently $\{T_1(x + \epsilon y), \ldots, T_r(x + \epsilon y), T_{r+1}(x + \epsilon y)\}$ also is a set of $r+1$ linearly independent vectors. Therefore $r(x + \epsilon y) \geq r + 1$ which contradicts the choice of x. This contradiction establishes Assertion (1).

Now suppose $\mu \geq 2\kappa$. Using Assertion (1) we may choose $x \in V$ in such a way that $\{T_1 x, \ldots, T_\kappa x\}$ is a linearly independent set of κ vectors. Consider the vector space $W_0 := \operatorname{Span}\{T_1 x, \ldots, T_\kappa x\}$ and let $\pi : W \longrightarrow W/W_0$ be the natural projection. We apply Assertion (1) to the linear maps $\bar{T}_i := \pi T_i : V \longrightarrow W/W_0$ with $\bar{\mu} = \mu - \kappa \geq \kappa$ to complete the proof of Assertion (2).

We complete the proof by establishing Assertion (3). By assumption, for every $z \in V$, there exist coefficients $a_i(z)$ so that $Tz = a_1(z) T_1 z + ... + a_\kappa(z) T_\kappa z$. In order to show that $T \in \operatorname{Span}\{T_1, \ldots, T_\kappa\}$, we must show that the coefficients can be chosen to be independent of z.

There are vectors $x, y \in S(V, \langle \cdot, \cdot \rangle)$ so $\{T_1 x, ..., T_\kappa x, T_1 y, ..., T_\kappa y\}$ is a collection of 2κ linearly independent vectors by Assertion (2). Then, by continuity, this remains true on some open neighborhoods $\mathcal{O}_x$ and $\mathcal{O}_y$ of x and y, respectively. Let $z \in \mathcal{O}_x$ and let $t \in \mathcal{O}_y$. We may then express:

$$\begin{aligned} T(z+t) &= \sum_{i=1}^{\kappa} a_i(z+t) T_i(z+t) = \sum_{i=1}^{\kappa} a_i(z+t)(T_i z + T_i t) \\ &= Tz + Tt = \sum_{i=1}^{\kappa} \{a_i(z) T_i z + a_i(t) T_i t\} . \end{aligned}$$

Since the vectors $\{T_1 z, ..., T_\kappa z, T_1 t, ..., T_\kappa t\}$ are linearly independent, this implies that

$a_i(z) = a_i(z+t) = a_i(t)$ for $z \in \mathcal{O}_x$ and $t \in \mathcal{O}_y$. Thus, for $a_i := a_i(t)$,

$$Tz = \sum_{i=1}^{\kappa} a_i T_i z \quad \text{for all} \quad z \in \mathcal{O}_x \,.$$

This polynomial identity holds on a non-empty open set and thus holds on all V. This establishes Assertion (3). □

We specialize this result for Clifford families.

Corollary 13.2.4 Let $\mathcal{F} := \{J_1, \ldots, J_\kappa\}$ be a Clifford family of rank κ on a vector space of dimension n.

1. Suppose that $n \geq \kappa$. Then there exists x in V so that the set $\{J_i x\}_{1\leq i\leq\kappa}$ consists of κ linearly independent vectors.
2. Suppose that $n \geq 2\kappa$. Then there exist x and y in V such that the set $\{J_i x, J_i y\}_{1\leq i\leq\kappa}$ consists of 2κ linearly independent vectors. Furthermore, if $Tx \in \operatorname{Span}_{1\leq i\leq\kappa}\{J_i x\}$ for all x in V, then $T \in \operatorname{Span}_{1\leq i\leq\kappa}\{J_i\}$.
3. Suppose that $n \geq \kappa(\kappa-1)$. Then there exists x in V such that the set $\{J_j J_k x\}_{1\leq j<k\leq\kappa}$ consists of $\frac{1}{2}\kappa(\kappa-1)$ linearly independent vectors.
4. Suppose that $n \geq 2\kappa(\kappa-1)$. Then there exist x and y in V so that the set $\{J_j J_k x, J_j J_k y\}_{1\leq j<k\leq\kappa}$ consists of $\kappa(\kappa-1)$ linearly independent vectors. Moreover, if $Tx \in \operatorname{Span}_{1\leq j<k\leq\kappa}\{J_j J_k x\}$ for all x in V, then $T \in \operatorname{Span}_{1\leq j<k\leq\kappa}\{J_j J_k\}$.

Proof. One verifies that $(a_1 J_1 + \ldots + a_\kappa J_\kappa)^2 = -(a_1^2 + \ldots + a_\kappa^2) Id$ and thus one has that $\operatorname{Rank}(a_1 J_1 + \ldots + a_\kappa J_\kappa) = n$ if any coefficient is non-zero. Assertions (1) and (2) now follow from Lemma 13.2.3. If not all the coefficients vanish, one shows similarly that:

$$\operatorname{Rank}\left(\sum_{j=1}^{\kappa-1}\sum_{k=j+1}^{\kappa} a_{jk} J_j J_k\right) \geq \frac{n}{2} \,.$$

The remaining assertions of the Lemma now follow. □

Let $A = (A_{ij}) \in O(\kappa)$ be an orthogonal matrix. Set

$$\tilde{\mathcal{F}} := \{\tilde{J}_i = A_{i1} J_1 + \ldots + A_{i\kappa} J_\kappa\} \,.$$

This new Clifford family is said to be a *reparametrization* of $\mathcal{F}$; this defines an equivalence relation on the collection of Clifford families.

We now describe some general properties of models given by Clifford families.

Lemma 13.2.5

1. *Suppose that J is a Hermitian almost complex structure on $(V, \langle\cdot,\cdot\rangle)$ (i.e. for all $x, y \in V$ we have $\langle Jx, Jy\rangle = \langle x, y\rangle$). Then $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, c_0A_0 + c_1A_J, J)$ is complex Osserman.*

2. *Suppose that $\{J_1, J_2, J_3\}$ is a Hermitian quaternion structure on $(V, \langle\cdot,\cdot\rangle)$. Then $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, c_0A_0 + c_1A_{J_1} + c_2A_{J_2} + c_3A_{J_3}, J_1)$ is complex Osserman.*

3. *Let $\mathcal{F} := \{J_1, \ldots, J_\kappa\}$ be a Clifford family and let $\tilde{\mathcal{F}} := \{\tilde{J}_1, \ldots, \tilde{J}_\kappa\}$ be a reparametrization of $\mathcal{F}$. Then $A_{J_1} + \ldots + A_{J_\kappa} = A_{\tilde{J}_1} + \ldots + A_{\tilde{J}_\kappa}$.*

Proof. Let $\mathcal{V}$ be as in Assertion (1). We use Equation (13.4) to see that:

$$\mathcal{J}(\pi_x)y = \begin{cases} (c_0 + 3c_1)y & \text{if } y \in \operatorname{Span}\{x, Jx\}, \\ 2c_0 & \text{if } y \perp \operatorname{Span}\{x, Jx\}. \end{cases}$$

Hence J and $\mathcal{J}(\pi)$ commute and the eigenvalues are constant. Thus $\mathcal{V}$ is complex Osserman by Lemma 13.1.1; the proof of Assertion (2) is similar and follows from a calculation in this instance that:

$$\mathcal{J}_{A_J}(\pi_x)y = \begin{cases} (c_0 + 3c_1)y & \text{if } y \in \operatorname{Span}\{x, J_1x\}, \\ (2c_0 + 3c_2 + 3c_3)y & \text{if } y \in \operatorname{Span}\{J_2x, J_3x\}, \\ 2c_0 & \text{if } y \perp \operatorname{Span}\{x, J_1x, J_2x, J_3x\}. \end{cases}$$

We complete the proof by verifying that Assertion (3) holds. If $x \in S(V)$, then the vectors $\{J_1x, ..., J_\kappa x\}$ form an orthonormal set. Let $\rho_{\mathcal{F}}x$ be orthogonal projection on the subspace

$$S_1^{\mathcal{F}}(x) := \operatorname{Span}\{J_1x, ..., J_\kappa x\}.$$

We then have $\sum_i \langle x, J_ix\rangle J_ix = \rho_{\mathcal{F}}x$. Let $A = A_{J_1} + ... + A_{J_\kappa}$. By Equation (13.3), $\mathcal{J}(x) = 3\rho_{\mathcal{F}}(x)$. If $\tilde{\mathcal{F}}$ is a reparametrization of $\mathcal{F}$, then $S_1^{\mathcal{F}}(x) = S_1^{\tilde{\mathcal{F}}}(x)$. Consequently $\mathcal{J}(x) = \mathcal{J}_{\tilde{A}}(x)$ so by Theorem 12.1.1, $A = \tilde{A}$. □

Remark 13.2.6 Theorem 13.1.7 places restrictions on the possible eigenvalue multiplicities of the complex Jacobi operator defined by a complex Osserman model. We may use Lemma 13.2.5 to show that in fact all these possibilities occur, hence Theorem 13.1.7 is sharp. Suppose first that the dimension n of V is even. Let J be a Hermitian almost complex structure on $(V, \langle\cdot,\cdot\rangle)$.

1. If $A = 3A_0 + A_J$, then $\mathcal{J}_A(\pi) = 6Id$.

2. If $A = A_0 + A_J$, then the eigenvalues of $\mathcal{J}_A(\pi)$ are $(2, 4)$ and the eigenvalue multiplicities are $(n - 2, 2)$.

If n is divisible by 4, there are additional eigenvalue multiplicities which can be realized. Let $\{J_1, J_2, J_3\}$ be a quaternion structure on $(V, \langle\cdot,\cdot\rangle)$ and let $J = J_1$.

1. If $A = 3A_0 + 3A_{J_1} + A_{J_2} + A_{J_3}$, then the eigenvalues of $\mathcal{J}_A(\pi)$ are $(6, 12)$ and the eigenvalue multiplicities are $(n-4, 4)$.

2. If $A = A_0 + A_{J_1} + A_{J_2} + A_{J_3}$, then the eigenvalues of $\mathcal{J}_A(\pi)$ are $(2, 4, 8)$ and the eigenvalue multiplicities are $(n-4, 2, 2)$.

Recall here that any Osserman algebraic curvature tensor in dimension different from 16 is given by a Clifford family [135, 136]. The following result shows this does not hold for the complex Osserman condition and also extends Theorem 12.3.4.

Theorem 13.2.7 *Let V be a vector space of dimension n. Assume n is divisible by 4 and that n is at least 8. Then there exists a model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ which is complex Osserman, which is not Osserman, which is not given by a Clifford family, and which has $\mathcal{J}(\pi_x) = 0$ for all x.*

Proof. Since the dimension of V is divisible by 4, we can find a quaternion structure $\{K_1, K_2, K_3\}$ on V. Since $n \geq 8$, we may take a non-trivial decomposition of V as a quaternion module in the form $V = V_+ \oplus V_-$. Define a new Clifford family on V which is not a quaternion structure by setting $J_1 := K_1$, $J_2 := K_2$, and $J_3 := \mp J_1 J_2$ on $V_\pm$. We then have $J_1 J_2 J_3 x = \pm x$ for $x \in V_\pm$. Define

$$A := A_{J_2} - A_{J_1 J_2} - A_{J_3} + A_{J_1 J_3}\,.$$

Let $x_\pm \in S(V_\pm)$. Equation (13.2) yields that:

$$\mathcal{J}(x_+)y = \begin{cases} 6y & \text{if} \quad y \in \operatorname{Span}\{J_2 x_+\} = \operatorname{Span}\{J_1 J_3 x_+\}, \\ -6y & \text{if} \quad y \in \operatorname{Span}\{J_3 x_+\} = \operatorname{Span}\{J_1 J_2 x_+\}\,. \end{cases}$$

On the other hand, if we take $x_0 = (x_+ + x_-)/\sqrt{2}$, then

$$\mathcal{J}(x_0)y = \begin{cases} 3y & \text{if} \quad y \in \operatorname{Span}\{J_2 x_0, J_1 J_3 x_0\} = \operatorname{Span}\{J_2 x_+, J_2 x_-\}, \\ -3y & \text{if} \quad y \in \operatorname{Span}\{J_1 J_2 x_0, J_3 x_0\} = \operatorname{Span}\{J_3 x_+, J_3 x_-\}\,. \end{cases}$$

This shows that $\mathcal{V}$ is not Osserman. As any model given by a Clifford family is necessarily Osserman, $\mathcal{V}$ is not given by a Clifford family. On the other hand, the complex Jacobi operator with respect to $J = J_1$ is given by

$$\begin{aligned} \mathcal{J}(\pi_x)y &= 3\langle y, J_2 x\rangle J_2 x + 3\langle y, J_2 J_1 x\rangle J_2 J_1 x - 3\langle y, J_1 J_2 x\rangle J_1 J_2 x \\ &\quad -3\langle y, J_1 J_2 J_1 x\rangle J_1 J_2 J_1 x - 3\langle y, J_3 x\rangle J_3 x - 3\langle y, J_3 J_1 x\rangle J_3 J_1 x \\ &\quad +3\langle y, J_1 J_3 x\rangle J_1 J_3 x + 3\langle y, J_1 J_3 J_1 x\rangle J_1 J_3 J_1 x \\ &= 0\,. \end{aligned}$$

This shows $\mathcal{J}(\pi_x) = 0$ for all x as desired. Thus $\mathcal{V}$ is complex Osserman. □

13.3 Curvature and higher order Clifford families

In this section, we study models with

$$A = c_0 A_0 + c_1 A_{J_1} + ... + c_\kappa A_{J_\kappa}$$

where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank $\kappa \geq 4$ on $(V, \langle \cdot, \cdot \rangle)$. We remark that the work of [51, 135, 136] shows that tensors of this kind do not arise in the geometric context. In Section 13.3.1 we study the case $c_0 = 0$ and in Section 13.3.2 we study the case $c_0 \neq 0$. We shall always assume that the constants c_1, ..., c_κ are non-zero.

The remaining of this section is devoted to give a proof of the following theorem, which summarizes the study of higher order Clifford families.

Theorem 13.3.1 *Let $\mathcal{V} = (V, \langle \cdot, \cdot \rangle, A, J)$ where $A = c_0 A_0 + c_1 A_{J_1} + ... + c_\kappa A_{J_\kappa}$ is given by a Clifford family of rank $\kappa \geq 4$ on a vector space V of dimension n. The following assertions hold:*

1. *Let $c_0 = 0$. If $\kappa = 4, 5$, assume $n \geq 2^\kappa$ and, if $\kappa \geq 6$, assume $n \geq \kappa(\kappa - 1)$. Then $\mathcal{V}$ is not complex Osserman.*

2. *Let $c_0 \neq 0$. If $\kappa = 4$ assume $n \geq 32$, if $\kappa = 5, 6, 7$ assume $n \geq 2^\kappa$, if $\kappa \geq 8$ assume $n \geq \kappa(\kappa - 1)$. Then $\mathcal{V}$ is not complex Osserman.*

Note that, as a consequence of Lemma 13.2.1 in the previous section, the hypothesis $n \geq \kappa(\kappa - 1)$ in Theorem 13.3.1 is not a restriction when $\kappa \geq 16$. Consequently, there are only a finite number of possibly exceptional dimensions and ranks when $\kappa \geq 4$.

13.3.1 Curvature given by a Clifford family with $c_0 = 0$.

Throughout this section we assume that

$$A = c_1 A_{J_1} + ... + c_\kappa A_{J_\kappa}$$

where c_1, ..., c_k are non-zero constants and where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank κ on a vector space V of dimension n. Let $\mathcal{V} := (V, \langle \cdot, \cdot \rangle, A, J)$. We suppose that $\mathcal{V}$ is complex Osserman. We first show that this implies that J has the form $J = \sum_{i<j} c_{ij} J_i J_j$. We then derive a contradiction by studying the eigenvalue structure and by studying the coefficients c_{ij}. The eigenvalue multiplicity estimated in Theorem 13.1.7 will play a crucial role in our analysis.

We shall have to impose certain conditions on n; these conditions are automatic for κ large. We begin with a technical result:

Lemma 13.3.2 *Let $\mathcal{V} := (V, \langle \cdot, \cdot \rangle, A = c_1 A_{J_1} + \ldots + c_\kappa A_{J_\kappa}, J)$ be a complex Osserman model where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank κ on a vector space of dimension n. Assume that $\kappa \geq 4$ and that $n \geq 2\kappa + 5$. If $x \in S(V, \langle \cdot, \cdot \rangle)$, then*

1. Rank $\{\mathcal{J}(\pi)\} \leq 4$.

2. $Jx \in \operatorname{Span}_{i\leq 4, i\neq j}\{J_iJ_jx\}$.

Proof. Equation (13.4) shows Rank $\{\mathcal{J}(\pi)\} \leq 2\kappa$. Consequently 0 is an eigenvalue of multiplicity at least $n - 2\kappa \geq 5$. Theorem 13.1.7 then shows that 0 is an eigenvalue of multiplicity at least $n-4$. Consequently, as desired, Rank $\{\mathcal{J}(\pi)\} \leq 4$.

The vectors $\{J_1x, ..., J_\kappa x\}$ form an orthonormal set for any $x \in S(V, \langle\cdot,\cdot\rangle)$, so let $\alpha_i(x) := \langle J_ix, J_1Jx\rangle$ be the Fourier coefficients of J_1Jx. Let

$$\begin{aligned} U(x) &:= \operatorname{Span}\{J_1x, \dots, J_\kappa x, J_1Jx\}, \\ V(x) &:= \operatorname{Span}\{J_2Jx, \dots, J_\kappa Jx\}, \\ W(x) &:= U(x) + V(x). \end{aligned}$$

Note that Range $\{\mathcal{J}(\pi_x)\} \subset W(x)$. If $\dim\{U(x)\} \leq \kappa$, then $J_1Jx \in \operatorname{Span}_i\{J_ix\}$. Since $J_1Jx \perp J_1x$, we have that $Jx \in \operatorname{Span}_{i>1}\{J_1J_ix\}$ and Assertion (2) follows.

Suppose on the other hand that $\dim\{U(x)\} = \kappa + 1$ or equivalently that

$$\alpha_1^2 + \ldots + \alpha_\kappa^2 < 1\,. \tag{13.5}$$

Let ρ be the projection on $W(x)/V(x)$. Then

$$\begin{aligned} \rho\mathcal{J}(\pi_x)J_ix &= \rho\{3c_iJ_ix + 3c_1\alpha_iJ_1Jx\}, \\ \rho\mathcal{J}(\pi_x)J_1Jx &= \rho\{3c_1JJ_1x + 3c_1\alpha_1J_1x + ... + 3c_\kappa\alpha_\kappa J_\kappa x\}. \end{aligned}$$

Hence $\rho\mathcal{J}(\pi_x) = \rho M$ on $U(x)$, where

$$M := 3\begin{pmatrix} c_1 & 0 & \dots & 0 & c_1\alpha_1 \\ 0 & c_2 & \dots & 0 & c_2\alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_\kappa & c_\kappa\alpha_\kappa \\ c_1\alpha_1 & c_1\alpha_2 & \dots & c_1\alpha_\kappa & c_1 \end{pmatrix}.$$

We compute $\det(M) = 3^{\kappa+1}c_1^2c_2\dots c_\kappa(1-\alpha_1^2-\dots-\alpha_\kappa^2)$. Thus by Equation (13.5), $\det(M) \neq 0$ so M is invertible. Consequently,

$$\dim\{\rho U(x)\} = \dim\{\rho MU(x)\} = \dim\{\rho\mathcal{J}(\pi_x)U(x)\} \leq \operatorname{Rank}\{\mathcal{J}(\pi_x)\} \leq 4.$$

The short exact sequence

$$0 \to V(x) \to W(x) \to W(x)/V(x) = \rho U(x) \to 0$$

shows that $\dim\{W(x)\} = \dim\{V(x)\} + \dim\{\rho U(x)\} \leq (\kappa - 1) + 4$. Therefore

$$\begin{aligned} \dim\{\operatorname{Span}_{i\leq 4}\{J_ix\} \cap \operatorname{Span}_i\{J_iJx\}\} &= 4 - (\dim\{W(x)\} - \kappa) \\ &\geq 4 + \kappa - (\kappa+3) > 0\,. \end{aligned}$$

Hence, there exist non-zero constants a_i and b_j so that

$$a_1 J_1 x + a_2 J_2 x + a_3 J_3 x + a_4 J_4 x = b_1 J_1 J x + \ldots + b_\kappa J_\kappa J x.$$

We now multiply by $b_1 J_1 + \ldots + b_\kappa J_\kappa$ to invert this relation and conclude thereby that $Jx \in \operatorname{Span}\{x, \{J_i J_j x\}_{i \le 4, i \ne j}\}$. Since $Jx \perp x$, we conclude $Jx \in \operatorname{Span}_{i \le 4, i \ne j}\{J_i J_j x\}$ as desired. □

We continue our study by reducing to the cases $\kappa = 4$ and $\kappa = 5$:

Lemma 13.3.3 *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_1 A_{J_1} + \ldots + c_\kappa A_{J_\kappa}, J)$ be a complex Osserman model where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank κ on a vector space of dimension n. Assume that $\mathcal{V}$ is complex Osserman, that $n \ge \kappa(\kappa - 1)$, and that $\kappa \ge 4$. Then $\kappa \le 5$.*

Proof. Suppose $\kappa \ge 6$. By Corollary 13.2.4 we know that there exists $x \in V$ so $\{J_i J_j x\}_{i<j}$ is a linearly independent set of $\frac{1}{2}\kappa(\kappa - 1)$ vectors. By Lemma 13.3.2,

$$Jx = \sum_{1 \le i \le 6, i<j} a_{ij}(x) J_i J_j x.$$

Moreover, the sum may be restricted to $i \le 4$ and, since the coefficients a_{ij} are uniquely determined, we get $a_{56}(x) = 0$. By permuting the role of the indices we may conclude that all the coefficients vanish. As this is not possible, $\mathcal{V}$ can not be a complex Osserman model. □

The analysis of the cases $\kappa = 4$ and $\kappa = 5$ to complete the proof of Theorem 13.3.1 (1) is a bit technical. We shall outline the proof but omit details in the interests of brevity (we refer to [85] for a complete proof). We assume $\dim(V) \ge 16$ throughout.

Lemma 13.3.4 *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_1 A_{J_1} + \ldots + c_\kappa A_{J_\kappa}, J)$ where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank κ on a vector space of dimension n. Assume that $n \ge 2^\kappa$ and that $\kappa = 4, 5$. Then:*

1. *Suppose that $\mathcal{V}$ is complex Osserman. Then there exists a reparametrization $\tilde{\mathcal{F}} = \{\tilde{J}_1, ..., \tilde{J}_\kappa\}$ of the family $\mathcal{F} = \{J_1, ..., J_\kappa\}$ so that $J = \tilde{J}_1 \tilde{J}_2$ and so that $A = \tilde{c}_1 A_{\tilde{J}_1} + \ldots + \tilde{c}_\kappa A_{\tilde{J}_\kappa}$.*
2. *If $\kappa = 5$, then $\mathcal{V}$ is not complex Osserman.*
3. *If $\kappa = 4$, then $\mathcal{V}$ is not complex Osserman.*

Proof. Since $\kappa = 4$ or $\kappa = 5$ we have $2\kappa + 5 < 16 \le n$. Thus Lemma 13.3.2 implies $Jx \in \operatorname{Span}_{i \ne j}\{J_i J_j x\}$ for all $x \in S(V, \langle\cdot,\cdot\rangle)$. One can show there exist $x, y \in V$ so

$\{J_jJ_kx, J_jJ_ky\}_{j<k}$ is an orthonormal set of $\kappa(\kappa-1)$ linearly independent vectors. Thus the argument used to establish Lemma 13.2.3 proves that

$$J = \sum_{i=1}^{\kappa-1} \sum_{j=i+1}^{\kappa} a_{ij} J_i J_j \,.$$

One can now show that there exists a suitable reparametrization; as the argument is straightforward, if a bit lengthy, we shall omit the details.

Suppose that $\kappa = 5$. By Assertion (1), we may suppose that $J = J_1J_2$. As noted above, there exists $x \in S(V, \langle\cdot,\cdot\rangle)$ such that $\{J_iJ_jx\}_{i<j}$ is an orthonormal set and, thus, $\{J_1x, J_2x, J_3x, J_4x, J_5x, J_1J_2J_3x, J_1J_2J_4x, J_1J_2J_5x\}$ is also an orthonormal set. Therefore

$$\mathcal{J}(\pi_x)y = \begin{cases} 3(c_1+c_2)y & \text{if } y \in \operatorname{Span}\{J_1x, J_2x\}, \\ 3c_3y & \text{if } y \in \operatorname{Span}\{J_3x, J_1J_2J_3x\}, \\ 3c_4y & \text{if } y \in \operatorname{Span}\{J_4x, J_1J_2J_4x\}, \\ 3c_5y & \text{if } y \in \operatorname{Span}\{J_5x, J_1J_2J_5x\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\operatorname{Rank}\{\mathcal{J}(\pi_x)y\} > 4$. Hence, by Lemma 13.1.7, A is not complex Osserman. Assertion (2) now follows.

Finally suppose $\kappa = 4$. Again, we may suppose $J = J_1J_2$. As $(J_1J_2J_3)^2 = Id$, there exists $x \in S(V, \langle\cdot,\cdot\rangle)$ such that $J_1J_2J_3x = \pm x$, and hence

$$\{x, J_1x, J_2x, J_3x, J_4x, J_1J_2J_4x\}$$

is an orthonormal set. Note that

$$\mathcal{J}(\pi_x)y = \begin{cases} 3c_3y & \text{if } y \in \operatorname{Span}\{x, J_3x\}, \\ 3(c_1+c_2)y & \text{if } y \in \operatorname{Span}\{J_1x, J_2x\}, \\ 3c_4y & \text{if } y \in \operatorname{Span}\{J_4x, J_1J_2J_4x\}, \\ 0 & \text{if } y \perp \operatorname{Span}\{x, J_3x, J_1x, J_2x, J_4x, J_1J_2J_4x\}. \end{cases}$$

Because $(J_1J_2J_3J_4)^2 = Id$, there exists $y \in S(V, \langle\cdot,\cdot\rangle)$ such that $J_1J_2J_3J_4y = \pm y$ and

$$\mathcal{J}(\pi_x)y = \begin{cases} 3(c_1+c_2)y & \text{if } y \in \operatorname{Span}\{J_1x, J_2x\}, \\ 3(c_3+c_4)y & \text{if } y \in \operatorname{Span}\{J_3x, J_4x\}, \\ 0 & \text{if } y \perp \operatorname{Span}\{J_1x, J_2x, J_3x, J_4x\}. \end{cases}$$

Since the eigenvalues are different, A is not complex Osserman. □

13.3.2 Curvature given by a Clifford family with $c_0 \neq 0$.

This section is devoted to the proof of Assertion (2) of Theorem 13.3.1. Although there is some parallelism between cases $c_0 = 0$ and $c_0 \neq 0$, the approach we follow now is slightly different. However, in the interests of brevity, we will refer to arguments in Section 13.3.1 whenever possible. We begin by studying a reduced complex Jacobi operator where the effect of c_0 has been normalized.

Lemma 13.3.5 *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_0A_0 + c_1A_{J_1} + \ldots + c_\kappa A_{J_\kappa}, J)$ be a complex Osserman model where $\{J_1, ..., J_\kappa\}$ is a Clifford family of rank κ on a vector space of dimension n. Assume that $\kappa \geq 4$. If $4 \leq \kappa \leq 7$, assume that $n \geq 2^\kappa$. If $\kappa \geq 8$, assume that $n \geq \kappa(\kappa - 1)$. Let $\tilde{\mathcal{J}}(\pi) = \mathcal{J}(\pi) - 2c_0 Id$. Then:*

1. $\operatorname{Rank}\{\tilde{\mathcal{J}}(\pi)\} \leq 4$.
2. $Jx \in \operatorname{Span}\{J_ix, J_jJ_kx\}_{i,j<k}$ *for all* $x \in V$.
3. *If* $\kappa \geq 6$, *then* $Jx \in \operatorname{Span}\{J_iJ_jx\}_{i\leq 6}$ *for all* $x \in V$.
4. $\kappa \leq 5$.

Proof. We use Equation (13.4) to see that:

$$\tilde{\mathcal{J}}(\pi_x)y = -c_0\langle y, x\rangle x - c_0\langle y, Jx\rangle Jx + 3\sum c_i(\langle y, J_ix\rangle J_ix + \langle y, J_iJx\rangle J_iJx)\,.$$

Consequently $\operatorname{Rank}\{\tilde{\mathcal{J}}(\pi_x)\} \leq 2\kappa + 2$ and 0 is an eigenvalue with multiplicity at least $n-2\kappa-2$. Since $n-2\kappa-2 > 4$ and as we have simply shifted the spectrum, Theorem 13.1.7 may be used to derive Assertion (1).

To prove Assertion (2), we compute that:

$$\begin{aligned}
&\tilde{\mathcal{J}}(\pi_x)x = -c_0x + \textstyle\sum_i 3c_i\langle x, J_iJx\rangle J_iJx,\\
&\tilde{\mathcal{J}}(\pi_x)Jx = -c_0Jx + \textstyle\sum_i 3c_i\langle Jx, J_ix\rangle J_ix,\\
&\tilde{\mathcal{J}}(\pi_x)J_ix = -c_0\langle J_ix, Jx\rangle Jx + 3c_iJ_ix + \textstyle\sum_j 3c_j\langle J_ix, J_jJx\rangle J_jJx\,.
\end{aligned} \tag{13.6}$$

Define:

$$\begin{aligned}
&M := diag(-c_0, 3c_1, ..., 3c_\kappa),\\
&U(x) := \operatorname{Span}\{x, J_1x, ..., J_\kappa x\},\\
&V(x) := \operatorname{Span}\{Jx, J_1Jx, ..., J_\kappa Jx\},\\
&W(x) := U(x) + V(x)\,.
\end{aligned}$$

Let ρ denote projection on $W(x)/V(x)$. We then have that $\rho\tilde{\mathcal{J}}(\pi_x) = \rho M$ on $U(x)$. As M is invertible, the following inequalities hold:

$$\begin{aligned}
&\dim\{\rho U(x)\} = \dim\{\rho\tilde{\mathcal{J}}(\pi_x)U(x)\} \leq 4,\\
&\dim\{W(x)\} \leq 4 + \kappa + 1,\\
&\dim\{U(x) \cap V(x)\} \geq \kappa + 1 + \kappa + 1 - \kappa - 5 = \kappa - 3 > 0\,.
\end{aligned}$$

Therefore, there exists a non-trivial relationship

$$(a_0 + a_1 J_1 + ... + a_\kappa J_\kappa) Jx = (b_0 + b_1 J_1 + ... + b_\kappa J_\kappa) x \,.$$

We invert this relationship by multiplying by $(a_0 - a_1 J_1 - ... - a_\kappa J_\kappa)$. Since $Jx \perp x$, one has that $Jx \in \operatorname{Span}\{J_i x, J_j J_k x\}$ which establishes Assertion (2).

If $\kappa \geq 6$, then we can derive a stronger result. We estimate that:

$$\begin{aligned} &\dim\{\{\operatorname{Span}\{J_1 x, ..., J_6 x\} \cap \operatorname{Span}\{J_1 Jx, ..., J_\kappa Jx\}\} \\ &\geq 6 + \kappa - \dim(W) \geq 6 + \kappa - \kappa - 5 > 0 \,. \end{aligned}$$

Assertion (3) now follows using a similar argument to that used to establish Assertion (2).

To establish Assertion (4), we assume to the contrary that $\kappa \geq 6$ and argue for a contradiction. By Assertion (3), we have that $Jx \in \operatorname{Span}\{J_i J_j x\}_{i \leq 6, j \neq i}$. The argument used to establish Lemma 13.3.3 shows that $\kappa \leq 7$. Thus we have that $\kappa = 6$ or $\kappa = 7$. Since $n \geq 2\kappa(\kappa - 1)$, Corollary 13.2.4 and Assertion (3) show that $J \in \operatorname{Span}\{J_i J_j\}$. One may show there exists $x \in V$ such that $x \perp J_i J_j J_k x$ for any i, j, k and such that $J_1 J_2 x \perp \operatorname{Span}\{J_i J_j x\}_{(i,j) \neq (1,2)}$. Thus, since $Jx \perp J_i x$ for this specific x, Equation (13.6) yields:

$$\begin{aligned} &\tilde{\mathcal{J}}(\pi_x) x = -c_0 x, \qquad \tilde{\mathcal{J}}(\pi_x) Jx = -c_0 Jx, \\ &\tilde{\mathcal{J}}(\pi_x) J_i x = 3 c_i J_i x + \sum_{j=1}^{\kappa} 3 c_j \langle J_i x, J_j Jx \rangle J_j Jx \,. \end{aligned}$$

Hence the subspace $\operatorname{Span}\{x, Jx\}$ is invariant under $\tilde{\mathcal{J}}(\pi_x)$. We clear the previous notation. By applying the argument used to prove Assertion (2) to the sets

$$\begin{aligned} U(x) &:= \operatorname{Span}\{J_1 x, ..., J_\kappa x\}, \\ V(x) &:= \operatorname{Span}\{J_1 Jx, ..., J_\kappa Jx\}, \\ W(x) &:= U(x) + V(x), \end{aligned}$$

we obtain $Jx = \sum_{i \leq 3, i < j} a_{ij} J_i J_j x$. Thus in particular $a_{45} = 0$. Since the coefficients a_{ij} were universal and independent of x, we can permute the indices to see that $a_{ij} = 0$ for all $i < j$, which is impossible. □

It remains to show that a Clifford family of rank $\kappa = 4$ or $\kappa = 5$ can not give a complex Osserman model. As in the case $c_0 = 0$ these ranks are treated independently. However, the present situation is a bit more difficult. We present sketch of proofs describing the main ideas involved; full details are available in [85] but are omitted here in the interests of brevity.

Lemma 13.3.6 *Let* $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_0 A_0 + c_1 A_{J_1} + \ldots + c_\kappa A_{J_\kappa}, J)$ *be a complex Osserman model where* $\{J_1, ..., J_\kappa\}$ *is a Clifford family of rank* $\kappa = 4$ *or* $\kappa = 5$ *on a vector space of dimension* $n \geq 32$.

1. *If* $\kappa = 5$*, then* $\mathcal{V}$ *is not complex Osserman.*

2. *If* $\kappa = 4$*, then* $\mathcal{V}$ *is not complex Osserman.*

Proof. Suppose that $\kappa = 5$, that $n \geq 32$, and that $\mathcal{V}$ is complex Osserman. We argue for a contradiction. Using similar techniques to those which were used to prove Lemma 13.3.5, one shows that $J \notin \text{Span}\,\{J_i J_j\}_{i \neq j}$. Consider the set

$$C := \{x \in V : Jx \in \text{Span}\,\{J_i x\}\}\,.$$

One shows that C is a closed nowhere dense set. So, working in the complementary set C^c and using similar arguments to those which were used to prove Lemma 13.3.2 applied to the sets

$$\begin{aligned} U(x) &:= \text{Span}\,\{J_1 x, ..., J_5 x, Jx\}, \\ V(x) &:= \text{Span}\,\{J_1 J x, ..., J_5 J x\}, \\ W(x) &:= U(x) + V(x), \end{aligned}$$

one shows that $Jx \in \text{Span}\,\{J_i J_j x\}_{i \neq j}$ and, therefore, $J \in \text{Span}\,\{J_i J_j\}_{i \neq j}$, which is false. This proves Assertion (1).

Suppose that $\kappa = 4$. By Lemma 13.3.5 we know that $Jx \in \text{Span}\,\{J_i x, J_j J_k x\}_{j<k}$ for all $x \in V$. Since $n \geq 32$, one can show that there exist $x, y \in S(V, \langle\cdot,\cdot\rangle)$ so that $\{J_i x, J_{jk} x, J_i y, J_{jk} y\}_{j<k}$ is an orthonormal set. The argument given to establish Lemma 13.2.3 (3) then shows there exist constants a_i and a_{jk} so that

$$J = \sum_{i=1}^{4} a_i J_i + \sum_{j<k} a_{jk} J_j J_k\,.$$

The compatibility between J and A shows that the constants a_i vanish so

$$J = \sum_{i<j} a_{ij} J_i J_j\,.$$

In this situation one may reparametrize the Clifford family so $J = \tilde{J}_1 \tilde{J}_2$. A straightforward calculation now shows $\text{Rank}\{\tilde{\mathcal{J}}\} \geq 6$, which contradicts Theorem 13.1.7. □

13.4 Classification for Clifford families of lower rank

In this section we study complex Osserman models which are given by Clifford families of rank κ for $0 \leq \kappa \leq 3$. The following theorem gives a characterization depending on such a rank.

Theorem 13.4.1 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$. Let $\mathcal{F} = \{J_i\}$ be a Clifford family on V, $n = dim V$. Let $c_i \neq 0$ be given for $1 \leq i \leq \kappa$ where $\kappa \leq 3$.*

1. Rank $\kappa = 0$. *Let $A = c_0 A_0$. Then $\mathcal{V}$ is complex Osserman.*
2. Rank $\kappa = 1$. *Let $A = c_0 A_0 + c_1 A_{J_1}$.*
   (a) *If $c_0 = 0$, then $\mathcal{V}$ is complex Osserman if and only if $JJ_1 = \pm J_1 J$.*
   (b) *If $c_0 \neq 0$, then $\mathcal{V}$ is complex Osserman if and only if either $J = \pm J_1$ or $JJ_1 = -J_1 J$.*
3. Rank $\kappa = 2$. *Let $A = c_0 A_0 + c_1 A_{J_1} + c_2 A_{J_2}$. Then $\mathcal{V}$ is complex Osserman if and only if there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2\}$ of $\mathcal{F}$ so that one has $A = c_0 A_0 + \tilde{c}_1 A_{\tilde{J}_1} + \tilde{c}_2 A_{\tilde{J}_2}$ and so that one of the following holds:*
   (a) *$c_0 = 0$, $J\tilde{J}_1 = \tilde{J}_1 J$ and $J\tilde{J}_2 = -\tilde{J}_2 J$.*
   (b) *Either $J = \tilde{J}_1$ or $J = \tilde{J}_1\tilde{J}_2$.*
4. Rank $\kappa = 3$. *Let $A = c_0 A_0 + c_1 A_{J_1} + c_2 A_{J_2} + c_3 A_{J_3}$.*
   (a) *Assume $n \geq 12$. If $c_0 = 0$, then $\mathcal{V}$ is complex Osserman if and only if there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ of $\mathcal{F}$ so that one has that $A = \tilde{c}_1 A_{\tilde{J}_1} + \tilde{c}_2 A_{\tilde{J}_2} + \tilde{c}_3 A_{\tilde{J}_3}$ and that $J = \tilde{J}_1$ or $J = \tilde{J}_2\tilde{J}_3$.*
   (b) *Assume $n \geq 16$. If $c_0 \neq 0$, then $\mathcal{V}$ is complex Osserman if and only if there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ of $\mathcal{F}$ so that one has that $A = c_0 A_0 + \tilde{c}_1 A_{\tilde{J}_1} + \tilde{c}_2 A_{\tilde{J}_2} + \tilde{c}_3 A_{\tilde{J}_3}$, that $J = \tilde{J}_1$, and that $\tilde{J}_1\tilde{J}_2\tilde{J}_3 = Id$.*

Section 13.4.1 deals with the case $\kappa = 0$, Section 13.4.2 deals with $\kappa = 1$, and Section 13.4.3 deals with $\kappa = 2$. We shall omit much of the analysis when discussing the case $\kappa = 3$ in Section 13.4.4 in the interests of brevity as it is similar to the other cases, details can be seen in [85]. Throughout Section 13.4, we suppose that $A = c_0 A_0 + c_1 A_{J_1} + ... + c_\kappa A_{J_\kappa}$.

13.4.1 Clifford families of rank 0

Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A = c_0 A_0, J)$. Then we have

$$\mathcal{J}(\pi_x)y = c_0(2y - \langle y, x\rangle x - \langle y, Jx\rangle Jx).$$

Hence we have that $J\mathcal{J}(\pi_x) = \mathcal{J}(\pi_x)J$ and that the eigenvalues are $(c_0, 2c_0)$ with multiplicities $(2, n-2)$ for any $x \in S(V, \langle\cdot,\cdot\rangle)$. Consequently, A is complex Osserman.

13.4.2 Clifford families of rank 1

We summarize the complete analysis in the following lemma:

Lemma 13.4.2 *Let J and J_1 be Hermitian almost complex structures on $(V, \langle\cdot,\cdot\rangle)$.*

1. *Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_1 A_{J_1}, J)$ where $c_1 \neq 0$. The following assertions are equivalent:*
   - *(a) A and J are compatible.*
   - *(b) $JJ_1 = J_1 J$ or $JJ_1 = -J_1 J$.*
   - *(c) $\mathcal{V}$ is complex Osserman.*
2. *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A = c_0 A_0 + c_1 A_{J_1}, J)$ where $c_0 c_1 \neq 0$. Then $\mathcal{V}$ is complex Osserman if and only if $J = \pm J_1$ or $JJ_1 = -J_1 J$.*

Proof. Suppose that $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_1 A_{J_1}, J)$ and that J and A are compatible. By Equation (13.4),

$$\mathcal{J}(\pi_x)y = 3\langle y, J_1 x\rangle J_1 x + 3\langle y, J_1 J x\rangle J_1 J x\,.$$

Hence $\operatorname{Range}\{\mathcal{J}(\pi_x)\} = \operatorname{Span}\{J_1 x, J_1 J x\}$ and, since J and A are compatible, we have $J(\operatorname{Span}\{J_1 x, J_1 J x\}) \subset \operatorname{Span}\{J_1 x, J_1 J x\}$. Since $JJ_1 x \perp J_1 x$, we have $JJ_1 x = \epsilon_x J_1 J x$, where $\epsilon_x = \pm 1$. By continuity, since $S(V, \langle\cdot,\cdot\rangle)$ is connected, ϵ_x is constant. Hence $JJ_1 = J_1 J$ or $JJ_1 = -J_1 J$. If this condition holds, then it is easily verified that $\mathcal{V}$ is complex Osserman. Finally, if $\mathcal{V}$ is complex Osserman, then A and J are compatible by definition. Assertion (1) now follows.

Next suppose that $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_0 A_0 + c_1 A_{J_1}, J)$ is a complex Osserman model where $c_0 \neq 0$ and $c_1 \neq 0$. Since A and J are compatible and since A_0 and J are compatible, A_{J_1} and J are compatible as well. Thus by Assertion (1), $JJ_1 = J_1 J$ or $JJ_1 = -J_1 J$. We now show that $JJ_1 = J_1 J$ implies $J = \pm J_1$. We suppose to the contrary that $J \neq \pm J_1$ and argue for a contradiction. Because $(JJ_1)^2 = Id$, JJ_1 may be used to define a $\mathbb{Z}_2$ grading on V by decomposing $V = V_+ \oplus V_-$ where $J = \pm J_1$ on $V_\pm$.

Let $x_\pm \in S(V_\pm)$ and let $x_0 = (x_+ + x_-)/\sqrt{2}$. Then one has that:

$$\begin{aligned}
\mathcal{J}(\pi_{x_+})y &= \begin{cases} (c_0 + 3c_1)y & \text{if } y \in \operatorname{Span}\{x_+, Jx_+\}, \\ 2c_0 y & \text{if } y \perp \operatorname{Span}\{x_+, Jx_+\}, \end{cases} \\
\mathcal{J}(\pi_{x_0})y &= \begin{cases} c_0 y & \text{if } y \in \operatorname{Span}\{x_+, Jx_0\}, \\ (2c_0 + 3c_1)y & \text{if } y \in \operatorname{Span}\{J_1 x_0, J_1 J x_0\}, \\ 2c_0 y & \text{if } y \perp \operatorname{Span}\{x_0, Jx_0, J_1 x_0, JJ_1 x_0\}. \end{cases}
\end{aligned}$$

Thus the eigenvalues of $\mathcal{J}(\pi_{x_+})$ are $(c_0 + 3c_1, 2c_0)$ with multiplicities $(2, n-2)$ (except if $3c_1 = c_0$ that $2c_0$ has multiplicity n). Furthermore, the eigenvalues of $\mathcal{J}(\pi_{x_0})$ are

$(c_0, 2c_0 + 3c_1, 2c_0)$ with multiplicities $(2, 2, n-4)$. So the eigenvalues are different in both cases. This contradiction shows that if $JJ_1 = J_1J$, then $J = \pm J_1$.

Conversely, if $JJ_1 = -J_1J$ or if $J = \pm J_1$, then a straightforward calculation shows $\mathcal{V}$ is complex Osserman. □

13.4.3 Clifford families of rank 2

As we have done for higher rank Clifford families, we divide our present analysis in two cases; namely, when $c_0 = 0$ and when $c_0 \neq 0$. We first assume $c_0 = 0$.

Lemma 13.4.3 *Let J be a Hermitian almost complex structure and let $\{J_1, J_2\}$ be a Clifford family on $(V, \langle\cdot,\cdot\rangle)$. Let $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, A = c_1A_{J_1} + c_2A_{J_2}, J)$ be complex Osserman. If x is a unit vector, set $\alpha(x) := \langle J_1J_2x, Jx\rangle$. Then:*

1. *$\alpha(x)$ is constant on $S(V, \langle\cdot,\cdot\rangle)$.*
2. *Either $\alpha = 0$, or $\alpha = 1$, or $\alpha = -1$.*
3. *Suppose that $\alpha = \pm 1$. Then $J = \pm J_1J_2$ and* $\operatorname{Rank}\{\mathcal{J}(\pi_x)\} = 2$.
4. *Suppose that $\alpha = 0$. Then* $\operatorname{Rank}\{\mathcal{J}(\pi_x)\} = 4$. *Furthermore:*
   (a) *if $c_1 \neq c_2$ then $JJ_1 = J_1J$ and $JJ_2 = -J_2J$ or otherwise $JJ_1 = -J_1J$ and $JJ_2 = J_2J$.*
   (b) *if $c_1 = c_2$ then there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2\}$ of $\{J_1, J_2\}$ so that $A = c_1A_{\tilde{J}_1} + c_2A_{\tilde{J}_2}$, $J\tilde{J}_1 = \tilde{J}_1J$ and $J\tilde{J}_2 = -\tilde{J}_2J$.*

Proof. Since $\mathcal{V}$ is complex Osserman, Equation (13.4) shows that

$$\operatorname{Range}\{\mathcal{J}(\pi_x)\} \subset \operatorname{Span}\{J_1x, J_1Jx, J_2x, J_2Jx\}.$$

Consequently,

$$\begin{aligned}
\mathcal{J}(\pi_x)J_1x &= 3c_1J_1x + 3\alpha(x)c_2J_2Jx,\\
\mathcal{J}(\pi_x)J_2Jx &= 3\alpha(x)c_1J_1x + 3c_2J_2Jx,\\
\mathcal{J}(\pi_x)J_1Jx &= 3c_1J_1Jx - 3\alpha(x)c_2J_2x,\\
\mathcal{J}(\pi_x)J_2x &= -3\alpha(x)c_1J_1Jx + 3c_2J_2x\,.
\end{aligned}$$

Thus $V_1(x) := \operatorname{Span}\{J_1x, J_2Jx\}$ and $V_2(x) := \operatorname{Span}\{J_2x, J_1Jx\}$ are $\mathcal{J}(\pi_x)$ invariant subspaces. Note that $J(V_1(x)) = V_2(x)$, that $V_1(x) \perp V_2(x)$, and that

$$\operatorname{Range}(\mathcal{J}(\pi_x)) = V_1(x) \oplus V_2(x)\,.$$

If $\alpha(\bar{x}) = \pm 1$ for some $\bar{x} \in S(V, \langle\cdot,\cdot\rangle)$, then $\text{Rank}\{\mathcal{J}(\pi_{\bar{x}})\} = 2$. Since A is complex Osserman, $\mathcal{J}(\pi_x)$ has constant rank. In such a case we get $\alpha(x) = \pm 1$ for all $x \in S(V, \langle\cdot,\cdot\rangle)$. On the other hand if $\alpha(x) \neq \pm 1$, then

$$\mathcal{J}(\pi)|_{V_1(x)} = \begin{pmatrix} 3c_1 & 3\alpha(x)c_1 \\ 3\alpha(x)c_2 & 3c_2 \end{pmatrix}, \quad \text{and}$$
$$\mathcal{J}(\pi)|_{V_2(x)} = \begin{pmatrix} 3c_1 & -3\alpha(x)c_1 \\ -3\alpha(x)c_2 & 3c_2 \end{pmatrix}.$$

Consequently, $\det\{\mathcal{J}(\pi_x)|_{V_1(x)+V_2(x)}\} = (9c_1c_2(1-\alpha(x)^2))^2$. Since the eigenvalues of $\mathcal{J}(\cdot)$ are constant, the determinant of $\mathcal{J}(\cdot)$ is constant and consequently $\alpha(x)$ does not depend on x. This establishes Assertion (1). The proof of Assertion (2) is a bit technical and is omitted in the interests of brevity. It relies on the fact that J preserves the eigenspaces of $\mathcal{J}(\pi)$.

The possible values of $\text{Rank}\{\mathcal{J}(\pi)\}$ are 2 and 4, which correspond to $\alpha = \pm 1$ or $\alpha \neq \pm 1$, respectively. If $\alpha = \pm 1$, then $J = \pm J_1J_2$ since Jx and J_1J_2x are unit vectors. Assertion (3) now follows.

On the other hand, if $\alpha = 0$ then polarizing the identity $\langle J_1J_2x, Jx\rangle = 0$ yields

$$\begin{aligned} 0 &= \langle J_1J_2x, Jy\rangle + \langle J_1J_2y, Jx\rangle \\ &= \langle x, (J_2J_1J - JJ_1J_2)y\rangle \\ &= -\langle x, (J_1J_2J + JJ_1J_2)y\rangle, \end{aligned}$$

so $J_1J_2J + JJ_1J_2 = 0$. Furthermore, $\{J_1x, J_1Jx, J_2x, J_2Jx\}$ is an orthonormal set for any $x \in S(V, \langle\cdot,\cdot\rangle)$.

Suppose that $c_1 \neq c_2$ and that $\mathcal{J}$ has three different eigenvalues $(0, 3c_1, 3c_2)$. Since J preserves the eigenspaces of $\mathcal{J}(\pi_x)$, we conclude that J preserves $\text{Span}\{J_1x, J_1Jx\}$ and $\text{Span}\{J_2x, J_2Jx\}$. Consequently, $JJ_1 = \pm J_1J$ and $JJ_2 = \pm J_2J$. Since one has that $JJ_1J_2 + J_1J_2J = 0$, the only possibilities are $JJ_1 = J_1J$ and $JJ_2 = -J_2J$ or $JJ_1 = -J_1J$ and $JJ_2 = J_2J$.

Suppose that $c_1 = c_2$. In such a case there are only two distinct eigenvalues for $\mathcal{J}(\pi_x)$ and $\text{Range}\{\mathcal{J}(\pi_x)\} = \text{Span}\{J_1x, J_2x, J_1Jx, J_2Jx\}$ is a 4-dimensional eigenspace. Since J preserves this eigenspace and $J_1Jx \perp J_1x, J_2x$, we have

$$JJ_1x = \langle JJ_1x, J_1Jx\rangle J_1Jx + \langle JJ_1x, J_2Jx\rangle J_2Jx\,.$$

Set $\Theta_1 = JJ_1$ and $\Theta_2 = JJ_2$, then $\langle\Theta_1^2x, x\rangle^2 + \langle\Theta_2\Theta_1x, x\rangle^2 = 1$. Also note that

$$\begin{aligned} &\Theta_1\Theta_1^* = JJ_1J_1J = Id, \Theta_2\Theta_2^* = JJ_2J_2J = Id, \\ &\Theta_1\Theta_2^* + \Theta_2\Theta_1^* = JJ_1J_2J + JJ_2J_1J = 0, \\ &\Theta_1\Theta_2 = JJ_1JJ_2 = JJ_1JJ_1J_2J_1 = -JJ_1J_1J_2JJ_1 = \Theta_2\Theta_1\,. \end{aligned}$$

Consequently, Θ_1 and Θ_2 are commuting orthogonal maps. Let

$$V = V_+ \oplus V_- \oplus V_1 \oplus \ldots \oplus V_k$$

be a skew-diagonalization of Θ_1, such that $\Theta_1 = \pm Id$ on $V_\pm$ and Θ_1 is a rotation through an angle θ_i, $0 < \theta_i < \pi$, on V_i. After some technical fuss, one may show that there is a reparametrization $\{\tilde{J}_1, \tilde{J}_2\}$ such that the previous decomposition is reduced to $V = V_+ \oplus V_-$ and hence $J\tilde{J}_1 = \tilde{J}_1 J$. Also, since $JJ_1J_2 = -J_1J_2J$ as noted above, $J\tilde{J}_2 = -\tilde{J}_2 J$. □

We complete the proof of Theorem 13.4.1 (3) by studying models with $c_0 \neq 0$. First we establish the following consequence of the compatibility between J and A for a Clifford family of rank at most 3.

Lemma 13.4.4 *Let $A = c_1 A_{J_1} + c_2 A_{J_2} + c_3 A_{J_3}$ be an algebraic curvature tensor given by a Clifford family of rank 3. Suppose A is compatible with a Hermitian almost complex structure J. If $Jx = (a_1 J_1 + a_2 J_2 + a_3 J_3)x$ for all $x \in V$, then $(c_i - c_j)a_i a_j = 0$ for $i \neq j$.*

Proof. Compute

$$\begin{aligned} JA(x, Jx)x &= c_0 x - 3c_1 a_1 JJ_1 x - 3c_2 a_2 JJ_2 x - 3c_3 a_3 JJ_3 x, \\ A(x, Jx)Jx &= c_0 x - 3c_1 a_1 J_1 Jx - 3c_2 a_2 J_2 Jx - 3c_3 a_3 J_3 Jx. \end{aligned}$$

Now, since A and J are compatible, $JA(x, Jx)x = A(x, Jx)Jx$ so

$$(c_1 - c_2)a_1 a_2 J_1 J_2 x + (c_1 - c_3)a_1 a_3 J_1 J_3 x + (c_2 - c_3)a_2 a_3 J_2 J_3 x = 0.$$

Since $\{J_1J_2x, J_1J_3x, J_2J_3x\}$ is an orthogonal set, the desired equalities follow. □

Now, we show that the complex structure is closely related to the complex structures of the Clifford family.

Lemma 13.4.5 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A = c_0 A_0 + c_1 A_{J_1} + c_2 A_{J_2}, J)$ be complex Osserman. If $\dim\{V\} \geq 12$, then there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2\}$ of $\{J_1, J_2\}$ such that $A = c_0 A_0 + \tilde{c}_1 A_{\tilde{J}_1} + \tilde{c}_2 A_{\tilde{J}_2}$ and either $J = \tilde{J}_1$ or $J = \tilde{J}_1 \tilde{J}_2$.*

Proof. Let $\tilde{\mathcal{J}}(\pi) = \mathcal{J}(\pi) - 2c_0 Id$ be the reduced complex Jacobi operator. Because $\mathcal{J}(\pi)$ is complex Osserman, the rank of $\tilde{\mathcal{J}}(\pi_x)$ is at most 4. Let $\alpha(x) := \langle J_1J_2x, Jx\rangle$, $\alpha_1(x) := \langle J_1x, Jx\rangle$ and $\alpha_2(x) := \langle J_2x, Jx\rangle$. Then

$$\begin{aligned} \tilde{\mathcal{J}}(\pi_x)x &= -c_0 x - 3c_1\alpha_1(x)J_1Jx - 3c_2\alpha_2(x)J_2Jx, \\ \tilde{\mathcal{J}}(\pi_x)Jx &= -c_0 Jx + 3c_1\alpha_1(x)J_1x + 3c_2\alpha_2(x)J_2x, \\ \tilde{\mathcal{J}}(\pi_x)J_1x &= -c_0\alpha_1(x)Jx + 3c_1 J_1x + 3c_2\alpha(x)J_2Jx, \\ \tilde{\mathcal{J}}(\pi_x)J_2x &= -c_0\alpha_2(x)Jx - 3c_1\alpha(x)J_1Jx + 3c_2 J_2x, \\ \tilde{\mathcal{J}}(\pi_x)J_1Jx &= c_0\alpha_1(x)x + 3c_1 J_1Jx - 3c_2\alpha(x)J_2x, \\ \tilde{\mathcal{J}}(\pi_x)J_2Jx &= c_0\alpha_2(x)x + 3c_1\alpha(x)J_1x + 3c_2 J_2Jx. \end{aligned}$$

Consider the subspace $W(x) := \text{Span}\{x, J_1x, J_2x, Jx, J_1Jx, J_2Jx\}$. Then notice that $\text{Range}\{\tilde{\mathcal{J}}(\pi_x)\} \subset W(x)$. We wish to show that $\dim\ W(x) < 6$. On the contrary, suppose $\dim\{W(x)\} = 6$. From the previous calculations we get the matrix associated to $\tilde{\mathcal{J}}(\pi_x)_{|W(x)}$ and compute:

$$\det(\tilde{\mathcal{J}}(\pi_x)_{|W(x)}) = 3^4c_0^2c_1^2c_2^2(-1+\alpha(x)^2+\alpha_1(x)^2+\alpha_2(x)^2)^2\,.$$

Since $\dim\{V\} \geq 12$ we apply Theorem 13.1.7 to get $\det(\tilde{\mathcal{J}}(\pi_x)_{|W(x)}) = 0$ and hence $\alpha^2+\alpha_1^2+\alpha_2^2 = 1$. Since $\alpha(x)$, $\alpha_1(x)$ and $\alpha_2(x)$ are the Fourier coefficients of Jx with respect to $\{J_1J_2x, J_1x, J_2x\}$, we get $Jx = \alpha(x)J_1J_2x + \alpha_1(x)J_1x + \alpha_2(x)J_2x$ which contradicts the assumption that $\dim\{W(x)\} = 6$.

Hence $\dim\{W(x)\} \leq 5$ and $\text{Span}\{x, J_1x, J_2x\} \cap \text{Span}\{Jx, J_1Jx, J_2Jx\}$ is non-trivial. Moreover, there exists a unit vector $(\rho_0, \rho_1, \rho_2) \in \mathbb{A}^3$ such that

$$(\rho_0 + \rho_1J_1 + \rho_2J_2)Jx \in \text{Span}\{x, J_1x, J_2x\}\,.$$

Let $\{J_1, J_2, J_1J_2\}$ give V a quaternion structure $\mathbb{H}$. As $Jx \in \mathbb{H}x$,

$$Jx = a_1(x)J_1x + a_2(x)J_2x + a_3(x)J_3x.$$

The following argument shows that $a_i(\cdot)$ are constant functions in $S(V, \langle\cdot,\cdot\rangle)$. Let $x, y \in S(V, \langle\cdot,\cdot\rangle)$. Since $\dim\{\mathbb{H}x + \mathbb{H}y\} \leq 8$, there exists $z \in S(V, \langle\cdot,\cdot\rangle)$ such that $z \perp \mathbb{H}x, \mathbb{H}y$. Then $\mathbb{H}x \perp \mathbb{H}z$ and for $w := \frac{1}{\sqrt{2}}(x+z)$ we have:

$$J(w) = \frac{1}{\sqrt{2}}\sum_i a_i(w)J_i(x+z) = \frac{1}{\sqrt{2}}\sum_i (a_i(x)J_i(x) + a_i(z)J_i(z))\,,$$

which implies that $a_i(x) = a_i(w) = a_i(z)$. Similarly, $a_i(y) = a_i(z)$.

Therefore $J = a_1J_1 + a_2J_2 + a_3J_1J_2$. By Lemma 13.4.4 with $c_3 = 0$, we have:

$$(c_1 - c_2)a_1a_2 = c_2a_2a_3 = c_1a_1a_3 = 0\,.$$

Then either $J = \pm J_3$ or $J = a_1J_1 + a_2J_2$ and we may reparametrize $\{J_1, J_2\}$ by $\{\tilde{J}_1, \tilde{J}_2\}$ so that $J = \tilde{J}_1$. □

13.4.4 Clifford families of rank 3

Let $\{J_1, J_2, J_3\}$ be a Clifford family on V. The dual structure, which is always a quaternion structure, is given by

$$\{J_1^* := J_2J_3, J_2^* := J_3J_1, J_3^* := J_1J_2\}\,.$$

We use this structure to establish Assertion (4) of Theorem 13.4.1; in the interest of brevity we shall simply outline the proof rather than giving full details. Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A = c_0A_0 + c_1A_{J_1} + c_2A_{J_2} + c_3A_{J_3}, J)$ be complex Osserman, where c_0 may be 0. Then one has the following:

1. If $J = a_1 J_1 + a_2 J_2 + a_3 J_3$, then there exists a reparametrization $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ so that $A = c_0 A_0 + \tilde{c}_1 A_{\tilde{J}_1} + \tilde{c}_2 A_{\tilde{J}_2} + \tilde{c}_3 A_{\tilde{J}_3}$ and $J = \tilde{J}_1$.
2. Suppose $J_1 J_2 \neq J_3$.
   (a) Then $J \neq J_1$. Furthermore, if $c_0 \neq 0$, then $J \neq J_2 J_3$.
   (b) Suppose that $Jx \in \operatorname{Span}\{J_1 x, J_2 x, J_3 x, J_1^* x, J_2^* x, J_3^* x\}$ for some element $x \in V$ with $x = (x_+ + x_-)/\sqrt{2}$ where $J_1 J_2 x_\pm = J_3 x_\pm$. Then $c_0 = 0$, and there is a reparametrization $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ so that $A = \tilde{c}_1 A_{\tilde{J}_1} + \tilde{c}_2 A_{\tilde{J}_2} + \tilde{c}_3 A_{\tilde{J}_3}$ and $J = \tilde{J}_2 \tilde{J}_3$.

The classification in Theorem 13.4.1 (4) follows from these observations and from a careful analysis of the rank of the matrix associated to $\mathcal{J}(\pi)$. The technique is similar to that developed in Lemma 13.4.5.

13.4.5 Conclusions

We finish this chapter with the following geometric conclusions we obtain from Theorem 13.4.1:

1. Let (M, g) be a manifold of constant sectional curvature. Then (M, g) is complex Osserman with respect to any Hermitian almost complex structure J.
2. Let (M, g, J) be a Kähler manifold which has constant holomorphic sectional curvature. Then (M, g, J) is complex Osserman with respect to J.
3. Let $(M, g, \{J_1, J_2, J_3\})$ be a quaternionic Kähler manifold which has constant quaternionic sectional curvature, where $\{J_1, J_2, J_3\}$ forms a locally defined quaternionic structure. Then, for any $J \in \operatorname{Span}\{J_1, J_2, J_3\}$, (M, g, J) is complex Osserman.

Note that if (M, g) is Osserman of dimension different from 16, then it is isometric to one of the three examples above [51, 135, 136]. Thus, except for dimension 16, we have that this is the complete classification of manifolds which are both Osserman and complex Osserman.

Chapter 14

Complex Osserman Kähler manifolds

In the previous chapter we have studied the complex Osserman condition on an algebraic model $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ with $A \in \mathfrak{A}_3(V)$. In order to give some partial results on the classification of complex Osserman models and manifolds, which is the purpose of this chapter, we impose some stronger assumptions: we assume that the algebraic curvature tensor verifies the Kähler identity, i.e., $A \in \mathfrak{A}_1(V)$. Thus, in particular, $R \in \mathfrak{A}_3$ and the algebraic results in the previous chapter apply. Moreover, the Kähler identity allows us to relate the complex and the Ivanov-Petrova conditions. A complete description of complex Osserman Kähler algebraic models is obtained in dimension 4 in Theorem 14.2.8 and a complete geometric classification is then derived in Theorem 14.2.9. Results in higher dimensions are not so conclusive (see Section 14.3).

Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$. Henceforth we assume $A \in \mathfrak{A}_1(V)$. This is,

$$\mathcal{A}(Jx, Jy) = \mathcal{A}(x, y). \tag{14.1}$$

Under this assumption we refer to $\mathcal{V}$ as a *complex Kähler model.*

Remark 14.0.1 We use Kähler identity (14.1) and First Bianchi Identity to express the complex Jacobi operator in the following way:

$$\begin{aligned} \mathcal{J}(\pi_x)y &= \mathcal{J}(x)y + \mathcal{J}(Jx)y \\ &= A(x,y)x + A(Jx,y)Jx \\ &= A(Jx,Jy)x + A(Jy,x)Jx \\ &= A(Jx,x)Jy \end{aligned}$$

so we obtain $\mathcal{J}(\pi_x) = A(Jx, x)J$. This shows that the complex Osserman problem is closely related to the complex Ivanov-Petrova one.

14.1 General lemmas

In this section we give a couple of general technical lemmas which will be useful afterwards. First we recall a well known result of Nomizu [140], originally stated in the geometrical setting, that we translate to our context:

Lemma 14.1.1 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex Kähler model. Then $\mathcal{V}$ has constant holomorphic sectional curvature c if and only if*

$$A(x, Jx)x = c\, Jx \qquad \text{for all } x \in S(V).$$

Remark 14.1.2 Note from Remark 14.0.1 and Lemma 13.1.3 that if a model $\mathcal{V}$ is complex Osserman then

$$A(Jx, x)Jy = \lambda\, y \Leftrightarrow A(Jy, y)Jx = \lambda\, x.$$

We have seen previously (Theorem 13.1.7) that there are only four cases as concerns the eigenvalue structure of the complex Jacobi operator in a complex Osserman model. Here we assume the complex model has two distinct eigenvalues (μ, λ) with multiplicities $(2, n-2)$, which is necessarily the case if $dim\, V \equiv 2 (mod\, 4)$.

Lemma 14.1.3 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex Osserman model such that the complex Jacobi operator has two distinct eigenvalues (μ, λ) with multiplicities $(2, n-2)$. Assume there exist orthonormal vectors x, y satisfying*

$$A(Jx, x, Jx, x) = \mu\,, \quad A(Jy, y, Jy, y) = \mu\,, \quad A(Jx, x, Jy, y) = \lambda\,.$$

Then the following assertions hold:

1. $\mathcal{J}(y)(\mathrm{Span}\,\{x, Jx\}) \subset \mathrm{Span}\,\{x, Jx\}$,
2. $\mathcal{J}(x)(\mathrm{Span}\,\{y, Jy\}) \subset \mathrm{Span}\,\{y, Jy\}$,
3. $\|\mathcal{J}(y)z\| = \left|\frac{\lambda-\mu}{2}\right|$ *for any* $z \in \mathrm{Span}\,\{x, Jx\}$.

Proof. Let $D := \mathrm{Span}\,\{x, y, Jx, Jy\}$. Let $z \in D$ and $w \in D^\perp$ be unit vectors, then w is an eigenvector associated to λ for the complex Jacobi operators $\mathcal{J}(\pi_x)$ and $\mathcal{J}(\pi_y)$, or equivalently, for $A(Jx, x)J$ and $A(Jy, y)J$. Now, Remark 14.1.2 implies $A(Jz, z)Jw = \lambda\, w$. Hence, since w was chosen arbitrarily, there must exist $v \in D$ with $\|v\| = 1$ such that $A(Jz, z)Jv = \mu\, v$. Without loss of generality we may write $z = ax + by$ and $v = v_1 x + v_2 y + v_3 Jx + v_4 Jy$.

A preliminary calculation shows

$$\begin{aligned} A(Jz, z)J &= A(aJx + bJy, ax + by)J \\ &= a^2 A(Jx, x)J + 2ab A(Jx, y)J + b^2 A(Jy, y)J\,. \end{aligned}$$

For $w \in D^\perp$, we have

$$\begin{aligned}\lambda w &= A(aJx + bJy, ax + by)Jw \\ &= a^2 A(Jx, x)Jw + 2abA(Jx, y)Jw + b^2 A(Jy, y)Jw \\ &= \lambda w + 2abA(Jx, y)Jw\,.\end{aligned}$$

Hence $A(Jx, y)Jw = 0$ for all $w \in D^\perp$; therefore, since $D^\perp$ is a complex subspace, $A(Jx, y, y, w) = 0$ for all $w \in D^\perp$. Then $A(Jx, y)y \in D$. Since $A(Jx, y, y, Jy) = 0$ and $A(Jx, y, y, y) = 0$, we conclude $A(Jx, y)y \in \operatorname{Span}\{x, Jx\}$. This proves Assertion (1). Assertion (2) is similar.

In order to prove Assertion (3), consider the vector v and compute

$$\begin{aligned}\mu v &= A(z, Jz)Jv \\ &= A(aJx + bJy, ax + by)J(v_1x + v_2y + v_3Jx + v_4Jy) \\ &= a^2v_1A(Jx, x)Jx + 2v_1abA(Jx, y)Jx + b^2v_1A(Jy, y)Jx \\ &\quad +a^2v_2A(Jx, x)Jy + 2v_2abA(Jx, y)Jy + b^2v_2A(Jy, y)Jy \\ &\quad +a^2v_3A(Jx, x)JJx + 2v_3abA(Jx, y)JJx + b^2v_3A(Jy, y)JJx \\ &\quad +a^2v_4A(Jx, x)JJy + 2v_4abA(Jx, y)JJy + b^2v_4A(Jy, y)JJy \\ &= a^2v_1\mu x + 2v_1abA(Jx, y)Jx + b^2v_1\lambda x \\ &\quad +a^2v_2\lambda y + 2v_2abA(Jx, y)Jy + b^2v_2\mu y \\ &\quad +a^2v_3\mu Jx + 2v_3abA(Jx, y)JJx + b^2v_3\lambda Jx \\ &\quad +a^2v_4\lambda Jy + 2v_4abA(Jx, y)JJy + b^2v_4\mu Jy\,.\end{aligned}$$

Taking the inner product with x, y, Jx and Jy we obtain the following system of linear equations in v_1, v_2, v_3 and v_4:

$$\begin{aligned}v_1\,\mu &= (a^2\mu + b^2\lambda)v_1 + 2abA(x, Jy, x, Jy)v_2 + 2abA(Jx, y, x, y)v_4\,, \\ v_2\,\mu &= 2abA(Jx, y, Jx, y)v_1 + (a^2\lambda + b^2\mu)v_2 - 2abA(Jx, y, x, y)v_3\,, \\ v_3\,\mu &= -2abA(Jx, y, x, y)v_2 + (a^2\mu + b^2\lambda)v_3 + 2abA(x, Jy, x, Jy)v_4\,, \\ v_4\,\mu &= 2abA(Jx, y, x, y)v_1 + 2abA(x, Jy, x, Jy)v_3 + (a^2\lambda + b^2\mu)v_4\,.\end{aligned}$$

Let $S := A(Jx, y, Jx, y)$ and let $T := A(Jx, y, x, y)$. Then

$$\begin{aligned}0 &= b^2(\lambda - \mu)v_1 + 2abSv_2 + 2abTv_4\,, \\ 0 &= 2abSv_1 + a^2(\lambda - \mu)v_2 - 2abTv_3\,, \\ 0 &= -2abTv_2 + b^2(\lambda - \mu)v_3 + 2abSv_4\,, \\ 0 &= 2abTv_1 + 2abSv_3 + a^2(\lambda - \mu)v_4\,.\end{aligned} \tag{14.2}$$

The matrix L of this homogeneous linear system is given by

$$L = \begin{pmatrix} b^2(\lambda-\mu) & 2abS & 0 & 2abT \\ 2abS & a^2(\lambda-\mu) & -2abT & 0 \\ 0 & -2abT & b^2(\lambda-\mu) & 2abS \\ 2abT & 0 & 2abS & a^2(\lambda-\mu) \end{pmatrix}.$$

Now, compute $det(L) = a^4b^4(-4S^2-4T^2+(\lambda-\mu)^2)^2$. Since we know the system has non trivial solution, necessarily $det(L) = 0$, from where

$$S^2 + T^2 = \left(\frac{\lambda-\mu}{2}\right)^2.$$

Note that previous arguments can be done for any $t \in \operatorname{Span}\{x, Jx\}$ just taking $z = at+by$. Hence last equation holds for any $t \in \operatorname{Span}\{x, Jx\}$ changing S by $S(t) = A(t, Jy, Jy, t)$ and T by $T(t) = A(Jt, y, y, t)$.

Now, from Assertion (1), we have $A(Jt, y)y = A(Jt, y, y, t)t + A(Jt, y, y, Jt)Jt$ for all $t \in \operatorname{Span}\{x, Jx\}$, hence

$$\|A(Jt, y)y\| = \sqrt{A(Jt, y, y, t)^2 + A(Jt, y, y, Jt)^2} = \left|\frac{\lambda-\mu}{2}\right|.$$

Assertion (3) now follows. □

14.2 Four-dimensional complex Osserman manifolds

We begin by studying complex Osserman models and manifolds in dimension 4. As a preliminary approach to the problem, we build new algebraic examples which are complex Osserman. First note that the algebraic curvature tensors A_0 and A_J are not Kähler with respect to J. However, the sum of them, $A_0 + A_J$, is complex Osserman with respect to J, as we have seen in Theorem 13.4.1.

14.2.1 Exotic example in dimension 4

Consider an algebraic curvature tensor of the following form:

$$A = c_0A_0 + c_1A_J + c_2A_L,$$

where J and L are arbitrary complex structures on V satisfying $JL = LJ$. We look for a model satisfying our conditions, this is, it has to be Kähler and complex Osserman. We do a two step analysis, first we fix coefficients c_0, c_1 and c_2 so that A verifies the Kähler identity and then we find conditions on the same coefficients so that A is complex Osserman.

Lemma 14.2.1 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex model with $\dim V = 4$. Then*

1. *A_J is not a Kähler algebraic curvature tensor with respect to J.*
2. *$A_0 + A_J$ is a Kähler algebraic curvature tensor with respect to J.*
3. *$A_J + A_L$ is a Kähler algebraic curvature tensor with respect to J.*
4. *$c_0A_0 + c_1A_J + c_2A_L$ is a Kähler algebraic curvature tensor with respect to J if and only if $c_1 = c_0 + c_2$.*

Proof. Assertions (1) and (2) are well known. We study Assertion (3). We decompose $V = V_+ \oplus V_-$ where

$$V_\pm := \{v \in V : Jv = \pm Lv\}\,.$$

Since $n = 4$, we may write $V = \operatorname{Span}\{x_+, Jx_+, x_-, Jx_-\}$. We study condition (14.1), i.e., $\mathcal{A}(x, y) = \mathcal{A}(Jx, Jy)$; we suppose without loss of generality that $x = x_+$; the cases $x = Jx_+$, $x = x_-$, and $x = Jx_-$ being similar. Since $\mathcal{A}(x, x) = 0$ and since $\mathcal{A}(Jx, JJx) = \mathcal{A}(Jx, -x) = \mathcal{A}(x, Jx)$, we suppose without loss of generality that $y = x_-$; the case $y = Jx_-$ being similar. Let $A = A_J + A_L$. We use the fact that $Jx_\pm = \pm Lx_\pm$ to compute:

$$\begin{aligned} A(x_+, x_-, z, w) &= \langle Jx_+, z\rangle\langle Jx_-, w\rangle + \langle Lx_+, z\rangle\langle Lx_-, w\rangle \\ &\quad -\langle Jx_+, w\rangle\langle Jx_-, z\rangle - \langle Lx_+, w\rangle\langle Lx_-, z\rangle \\ &\quad +2\langle Jx_+, x_-\rangle\langle Jz, w\rangle + 2\langle Lx_+, x_-\rangle\langle Lz, w\rangle \\ &= 2\langle Jx_+, x_-\rangle\langle (J + L)z, w\rangle\,. \end{aligned}$$

Since $\langle JJx_+, Jx_-\rangle = \langle Jx_+, x_-\rangle$, Assertion (3) follows. Now, decompose

$$A = c_0(A_0 + A_J) + c_2(A_J + A_L) + (c_1 - c_0 - c_2)A_J,$$

and use Assertions (1), (2) and (3) to derive Assertion (4). □

Now we study when these models are complex Osserman. Although we will only use the following result in dimension 4, we give it in general.

Lemma 14.2.2 *Let J and L be two commuting complex structures on V. Let A be an algebraic curvature tensor as follows:*

$$A = c_0A_0 + c_1A_J + c_2A_L.$$

Then $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ is complex Osserman if and only if either $c_2 = 0$ or $c_0 = 3c_1$.

Proof. We decompose $V = V_+ \oplus V_-$ where

$$V_\pm := \{v \in V : Jv = \pm Lv\}\,;$$

by hypothesis $V_\pm \neq \emptyset$. Let $x \in S(V - V_+ - V_-)$ and let $W := \operatorname{Span}\{x, Jx, Lx, LJx\}$; this is 4-dimensional and $\mathcal{J}(x) = 2c_0$ on $W^\perp$. Let $\varepsilon := \langle Lx, Jx\rangle \in (-1, 1)$. We compute

$$\begin{aligned}
\mathcal{J}_{c_0A_0}(\pi_x)x &= c_0x,\\
\mathcal{J}_{c_0A_0}(\pi_x)Jx &= c_0Jx,\\
\mathcal{J}_{c_0A_0}(\pi_x)Lx &= \mathcal{J}_{c_0A_0}\{\varepsilon Jx + (Lx - \varepsilon Jx)\} = \varepsilon c_0Jx + 2c_0(Lx - \varepsilon Jx)\\
&= -c_0\varepsilon Jx + 2c_0Lx,\\
\mathcal{J}_{c_0A_0}(\pi_x)JLx &= c_0\varepsilon x + 2c_0JLx\,.
\end{aligned}$$

It is now easy to compute:

$$\begin{aligned}
\mathcal{J}(\pi_x)x &= (c_0 + 3c_1)x - 3c_2\varepsilon JLx,\\
\mathcal{J}(\pi_x)Jx &= (c_0 + 3c_1)Jx + 3c_2\varepsilon Lx,\\
\mathcal{J}(\pi_x)Lx &= \varepsilon(-c_0 + 3c_1)Jx + 3(c_0 + c_2)Lx,\\
\mathcal{J}(\pi_x)JLx &= \varepsilon(c_0 - 3c_1)x + 3(c_0 + c_2)JLx\,.
\end{aligned}$$

This shows that $W_1 := \operatorname{Span}\{x, JLx\}$ and $W_2 := \operatorname{Span}\{Jx, Lx\}$ are invariant under the action of $\mathcal{J}(\pi_x)$ and that the matrices of the action of $\mathcal{J}(\pi_x)$ are given by:

$$\begin{aligned}
M_1 &:= \begin{pmatrix} c_0 + 3c_1 & \varepsilon(c_0 - 3c_1)\\ -3\varepsilon c_2 & 3c_2 \end{pmatrix} \qquad \text{in } W_1,\\
M_2 &:= \begin{pmatrix} c_0 + 3c_1 & \varepsilon(-c_0 + 3c_1)\\ 3\varepsilon c_2 & 3c_2 \end{pmatrix} \qquad \text{in } W_2\,.
\end{aligned}$$

The matrices M_1 and M_2 are similar and have the same eigenvalues. Their trace is independent of ε. We have

$$\det(A_1) = (c_0 + 3c_1)(3c_2) + 3\varepsilon^2c_2(c_0 - 3c_1)\,.$$

Thus, $\mathcal{V}$ is complex Osserman on $S(V - V_+ - V_-)$ if and only if $\det(M_1)$ is independent of ε or, equivalently, if and only if $c_2 = 0$ or $c_0 - 3c_1 = 0$. Under these conditions, the eigenvalue structure is constant on $S(V - V_+ - V_-)$ and hence, by continuity, on $S(V)$.□

As a direct consequence of Lemmas 14.2.1 and 14.2.2 we obtain the following characterization.

Theorem 14.2.3 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex model with $\dim V = 4$ and*

$$A = c_0A_0 + c_1A_J + c_2A_L,$$

where J and L are commuting complex structures on V. Then $\mathcal{V}$ is a Kähler complex Osserman model if and only if one of the following holds:

1. there exists μ *such that*

$$A = \frac{\mu}{2} A_0 + \frac{\mu}{6} A_J - \frac{\mu}{3} A_L \,, \text{ or} \tag{14.3}$$

2. there exists λ *such that*

$$A = \lambda(A_0 + A_J).$$

Definition 14.2.1 *Let* $\mathfrak{C}\mathfrak{C}^4$ *be the model*

$$\mathfrak{C}\mathfrak{C}^4 = (V, \langle\cdot,\cdot\rangle, A, J),$$

where $\dim V = 4$ *and* $A = \frac{\mu}{2}A_0 + \frac{\mu}{6}A_J - \frac{\mu}{3}A_L$ *for* J *and* L *satisfying* $J \neq L$ *and* $JL = LJ$.

$\mathfrak{C}\mathfrak{C}^4$ *is complex Osserman with eigenvalues* $(\mu, 0)$ *of multiplicities* $(2, 2)$.

Remark 14.2.4 Let $\mathfrak{C}\mathfrak{C}^4$ be the model defined in Definition 14.2.1. Let $x_+, x_- \in V$ be unit vectors such that $JLx_+ = x_+$ and $JLx_- = -x_-$. Define $x = (x_+ + x_-)/\sqrt{2}$. Hence $\{x, Jx, y = Lx, Jy = JLx\}$ is an orthonormal basis and the components of A are given by:

$$\begin{array}{rclrclr}
A(x,Jx,x,Jx) &=& \mu\,, & A(y,Jy,y,Jy) &=& \mu\,, & A(x,Jx,y,Jy)=0\,,\\
A(x,y,x,y) &=& \frac{-\mu}{2}\,, & A(x,Jy,x,Jy) &=& \frac{\mu}{2}\,, & \\
A(Jx,y,Jx,y) &=& \frac{\mu}{2}\,, & A(Jx,Jy,Jx,Jy) &=& \frac{-\mu}{2}\,, & \\
A(x,y,Jx,Jy) &=& \frac{-\mu}{2}\,, & A(x,Jy,Jx,y) &=& \frac{-\mu}{2}\,. &
\end{array}$$

Let v be a unit vector and write $v = (v_1, v_2, v_3, v_4)$ with respect to the basis $\{x, Jx, y, Jy\}$. Then the eigenvalues of the Jacobi operator $\mathcal{J}(v)$ are

$$\left(0\,,\ \frac{\mu}{2}\,,\ \frac{1}{4}\left(1-\sqrt{\xi}\right)\mu\,,\ \frac{1}{4}\left(1+\sqrt{\xi}\right)\mu\right),$$

where $\xi = 9((v_1^2+v_2^2)^2 + (v_3^2+v_4^2)^2) + 64v_1v_2v_3v_4 - 2v_2^2(7v_3^2 - 9v_4^2) + 2v_1^2(9v_3^2 - 7v_4^2)$.

Next, we compute the Ricci tensor, the scalar curvature and the self-dual and anti-self-dual Weyl operators for $\mathfrak{C}\mathfrak{C}^4$:

$$\rho_{\mathfrak{C}\mathfrak{C}^4} = \begin{pmatrix} \mu & 0 & 0 & 0\\ 0 & \mu & 0 & 0\\ 0 & 0 & \mu & 0\\ 0 & 0 & 0 & \mu \end{pmatrix}, \quad \tau_{\mathfrak{C}\mathfrak{C}^4} = 4\mu\,,$$

$$W^+_{\mathfrak{C}\mathfrak{C}^4} = \begin{pmatrix} -\frac{2\mu}{3} & 0 & 0\\ 0 & \frac{\mu}{3} & 0\\ 0 & 0 & \frac{\mu}{3} \end{pmatrix}, \quad W^-_{\mathfrak{C}\mathfrak{C}^4} = \begin{pmatrix} -\frac{2\mu}{3} & 0 & 0\\ 0 & \frac{4\mu}{3} & 0\\ 0 & 0 & -\frac{2\mu}{3} \end{pmatrix}.$$

Note that for A_0 we have

$$\rho_{A_0} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tau_{A_0} = 12\,,$$

$$W^+_{A_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W^-_{A_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For A_J we have:

$$\rho_{A_J} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tau_{A_J} = 12\,,$$

$$W^+_{A_J} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad W^-_{A_J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And for A_L we have:

$$\rho_{A_L} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tau_{A_L} = 12\,,$$

$$W^+_{A_L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W^-_{A_L} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

These calculations lead again to expression (14.3).

14.2.2 Classification of 4-dimensional complex Osserman models

In this subsection we give a complete classification of 4-dimensional complex Osserman models which is of inherent interest; it is also the key for the subsequent geometric classification. First of all note that in 4-dimensional complex Osserman models, there are only two possibilities for the eigenvalues of the complex Jacobi operators, namely there is only one eigenvalue with multiplicity 4 or there are two distinct eigenvalues of multiplicities $(2, 2)$. We begin by analyzing the case of two eigenvalues in the following lemma.

Lemma 14.2.5 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex Osserman algebraic model with $\dim V = 4$ and (μ, λ) eigenvalues for the complex Jacobi operator ($\mu \neq \lambda$). Suppose there exist orthonormal vectors x, y satisfying*

$$A(Jx, x, Jx, x) = \mu\,, \quad A(Jy, y, Jy, y) = \mu\,, \quad A(Jx, x, Jy, y) = \lambda\,.$$

Then one of the following holds:

1. $\mu = 2\lambda$ *and* $A = \frac{\mu}{4}(A_0 + A_J)$.

2. $\mu \neq 0$ *and* $\lambda = 0$, *or* $\mu = 0$ *and* $\lambda \neq 0$, *and* $\mathcal{V}$ *is isomorphic to* $\mathfrak{CC}^4$.

Proof. Lemma 14.1.3 shows $\mathcal{J}(y)(\mathrm{Span}\,\{x, Jx\}) \subset \mathrm{Span}\,\{x, Jx\}$. Since the Jacobi operator is self-adjoint it is diagonalizable and, by Assertion (3) of Lemma 14.1.3 the possible eigenvalues are: $\{-\frac{\lambda-\mu}{2}, -\frac{\lambda-\mu}{2}\}$, $\{\frac{\lambda-\mu}{2}, -\frac{\lambda-\mu}{2}\}$ and $\{\frac{\lambda-\mu}{2}, \frac{\lambda-\mu}{2}\}$.

Change notation if necessary, so that x and Jx are the corresponding eigenvectors. Hence $T = 0$ and the linear system in (14.2) reduces to:

$$\begin{array}{rcl} 0 &=& b^2(\lambda-\mu)v_1 + 2abSv_2\,, \\ 0 &=& 2abSv_1 + a^2(\lambda-\mu)v_2\,, \\ 0 &=& b^2(\lambda-\mu)v_3 + 2abSv_4\,, \\ 0 &=& 2abSv_3 + a^2(\lambda-\mu)v_4\,. \end{array} \tag{14.4}$$

We analyze the three cases separately:

<u>*Case 1:*</u> *eigenvalues* $\{-\frac{\lambda-\mu}{2}, -\frac{\lambda-\mu}{2}\}$.
Note that

$$\begin{array}{rcl} \lambda &=& \mathcal{J}(\pi_y)x \\ &=& A(x,y,x,y) + A(x,Jy,x,Jy) \\ &=& A(x,y,x,y) + A(Jx,y,Jx,y) \\ &=& -\frac{\lambda-\mu}{2} - \frac{\lambda-\mu}{2} \\ &=& \mu - \lambda\,, \end{array}$$

hence $\mu = 2\lambda$. The linear system (14.4) simplifies to the following two equations

$$\begin{array}{rcl} 0 &=& bv_1 - av_2\,, \\ 0 &=& bv_3 - av_4\,. \end{array}$$

Hence the set of solutions of the linear system is given by $\mathrm{Span}\,\{(a,b,0,0),(0,0,a,b)\}$. We conclude that $A(Jz,z,Jz,z) = \mu$ for all unit $z \in \mathrm{Span}\,\{x,y\}$, i.e., all $z \in \mathrm{Span}\,\{x,y\}$ are eigenvectors for $A(Jz,z)J$. After a bit of technical fuss this can be extended to the whole unitary sphere, this is,

$$A(Jz,z,Jz,z) = \mu \qquad \text{for all } z \in S(V).$$

Lemma 14.1.1 shows this is equivalent to the holomorphic sectional curvature to be constant. This completes case 1.

<u>*Case 2:*</u> *eigenvalues* $\{\frac{\lambda-\mu}{2}, -\frac{\lambda-\mu}{2}\}$.

Suppose that $\mathcal{J}(y)x = \frac{\lambda-\mu}{2}$ and $\mathcal{J}(y)Jx = -\frac{\lambda-\mu}{2}$, otherwise one simply changes notation x by Jx. Then compute:

$$\begin{aligned}\lambda &= A(x,y,y,x)+A(x,Jy,Jy,x)\\ &= A(x,y,y,x)+A(Jx,y,y,Jx)\\ &= \tfrac{\lambda-\mu}{2}-\tfrac{\lambda-\mu}{2}\\ &= 0.\end{aligned}$$

The linear system (14.4) simplifies to the following pair of equations:

$$\begin{aligned}0 &= bv_1+av_2\,,\\ 0 &= bv_3+av_4\,.\end{aligned}$$

Hence solutions are given by Span$\{(-b,a,0,0),(0,0,-b,a)\}$. For the basis $\{x,y,Jx,Jy\}$, the curvature tensor is given, up to the usual symmetries, by

$$\begin{aligned}&A(Jx,x,Jx,x)=A(Jy,y,Jy,y)=\mu\,, A(Jx,x,Jy,y)=\lambda\,,\\ &A(x,y,x,y)=\tfrac{-\mu}{2}\,, A(x,Jy,x,Jy)=\tfrac{\mu}{2}\,,\\ &A(Jx,y,Jx,y)=\tfrac{\mu}{2}\,, A(Jx,Jy,Jx,Jy)=\tfrac{-\mu}{2}\,,\\ &A(x,y,Jx,Jy)=\tfrac{-\mu}{2}\,, A(x,Jy,Jx,y)=\tfrac{-\mu}{2}\,.\end{aligned}$$

From Remark 14.2.4, $\mathcal{V}$ is isomorphic to $\mathfrak{C}\mathfrak{C}^4$.

<u>Case 3:</u> *eigenvalues* $\{\frac{\lambda-\mu}{2},\frac{\lambda-\mu}{2}\}$.
First observe that

$$\begin{aligned}\lambda &= \mathcal{J}(\pi_y)x\\ &= A(y,x,y,x)+A(Jy,x,Jy,x)\\ &= A(y,x,y,x)+A(y,Jx,y,Jx)\\ &= \tfrac{\lambda-\mu}{2}+\tfrac{\lambda-\mu}{2}\\ &= \lambda-\mu\,,\end{aligned}$$

hence $\mu=0$. Then the linear system (14.4) simplifies to

$$\begin{aligned}0 &= bv_1+av_2\,,\\ 0 &= bv_3+av_4\,.\end{aligned}$$

Hence the solutions are given by Span$\{(-a,b,0,0),(0,0,-a,b)\}$. Now, for the particular choice $a=1/\sqrt{2}, b=1/\sqrt{2}$, we have

$$A\left(\frac{Jx+Jy}{\sqrt{2}},\frac{x+y}{\sqrt{2}},\frac{-Jx+Jy}{\sqrt{2}},\frac{-x+y}{\sqrt{2}}\right)=0\,.$$

Hence, since $\langle\frac{x+y}{\sqrt{2}},\frac{-x+y}{\sqrt{2}}\rangle=0$, we have

$$A\left(\frac{Jx+Jy}{\sqrt{2}},\frac{x+y}{\sqrt{2}},\frac{Jx+Jy}{\sqrt{2}},\frac{x+y}{\sqrt{2}}\right)=\lambda\,,$$

$$A\left(\frac{-Jx+Jy}{\sqrt{2}},\frac{-x+y}{\sqrt{2}},\frac{-Jx+Jy}{\sqrt{2}},\frac{-x+y}{\sqrt{2}}\right)=\lambda\,.$$

On the other hand compute

$$A\left(\frac{x+y}{\sqrt{2}},\frac{-x+y}{\sqrt{2}},\frac{x+y}{\sqrt{2}},\frac{-x+y}{\sqrt{2}}\right)=A(x,y,x,y)=\frac{\lambda}{2}\,.$$

Set notation $z=\frac{x+y}{\sqrt{2}}$ and $v=\frac{-x+y}{\sqrt{2}}$ to see

$$\begin{aligned}&A(z,Jz,Jz,z)=A(v,Jv,Jv,v)=\lambda\,,A(z,Jz,Jv,v)=\lambda\,,\\&A(z,v,v,z)=\tfrac{-\lambda}{2}\,,A(z,Jv,Jv,z)=\tfrac{\lambda}{2}\,,\\&A(Jz,v,v,Jz)=\tfrac{\lambda}{2}\,,A(Jz,Jv,Jv,Jz)=\tfrac{-\lambda}{2}\,,\\&A(z,v,Jv,Jz)=\tfrac{-\lambda}{2}\,,A(z,Jv,Jz,v)=\tfrac{\lambda}{2}\,.\end{aligned}$$

By Remark 14.2.4, these relations show that our model is in this case again isomorphic to $\mathfrak{CC}^4$. □

In previous lemmas we have classified complex Osserman models with x,y vectors satisfying a suitably chosen condition with respect to the complex Jacobi operator. Our next objective is to show that we can always find x,y which satisfy those conditions.

As we have done before, consider an orthonormal basis $\{x,y,Jx,Jy\}$. Henceforth let $A(\pi_z,\pi_v):=A(z,Jz,Jv,v)$, for any $z,v\in V$, and let $\mathbb{P}(V):=\{\pi_z:z\in V\}$ be the set of complex lines. We say $\pi_z\perp\pi_v$ if and only if $\langle z,v\rangle=\langle z,Jv\rangle=0$. Thus, $\pi_z^\perp$ is the only $\pi_v\in\mathbb{P}(V)$ such that $\pi_z\perp\pi_v$.

In a complex Osserman model with eigenvalues $\{\mu,\lambda\}$, for any $z\in V$ there exist $v,w\in V$ such that $A(\pi_z,\pi_v)=\mu$ and $A(\pi_z,\pi_w)=\lambda$. Therefore, we have two maps

$$\Gamma,\Lambda:\mathbb{P}(V)\to\mathbb{P}(V)$$

defined by

$$A(\pi_z,\Gamma(\pi_z))=\mu\,,\quad A(\pi_z,\Lambda(\pi_z))=\lambda\,.$$

The following Lemma follows directly from Lemma 13.1.3.

Lemma 14.2.6 *Adopt the notation established above. We have:*

1. $\Gamma^2=\Lambda^2=Id$,
2. $\Lambda\Gamma(\pi_z)=\pi_z^\perp=\Gamma\Lambda(\pi_z)$.

We use the Hopf fibration to identify $\mathbb{P}(V) \cong S^2$. With this identification, we have $\pi_z^\perp = -\pi_z$.

Lemma 14.2.7 *The following assertions hold in S^2:*

1. $\Lambda\Gamma(\pi_z) = \Gamma\Lambda(\pi_z) = -\pi_z$,
2. $\Lambda(-\pi_z) = -\Lambda(\pi_z)$, $\Gamma(-\pi_z) = -\Gamma(\pi_z)$,
3. $\pi_z \perp \pi_w \Rightarrow \Lambda(\pi_z) \perp \Lambda(\pi_w)$, $\Gamma(\pi_z) \perp \Gamma(\pi_w)$.

Proof. Assertion (1) is Assertion (2) in Lemma 14.2.6. To proof Assertion (2), note that

$$\Gamma(-\pi_z) = \Gamma(\Lambda\Gamma)(\pi_z) = (\Gamma\Lambda)\Gamma(\pi_z) = -\Gamma(\pi_z)\,,$$

and analogously for Λ.

For Assertion (3), $\pi_z \perp \pi_w$ implies $\pi_w = -\pi_z$. Hence $\Gamma(\pi_z) \perp \Gamma(-\pi_z)$ is equivalent to $\Gamma(\pi_z)^\perp = -\Gamma(\pi_z)$ which is clearly true. □

Now we are ready to give the complete classification of 4-dimensional complex Osserman models.

Theorem 14.2.8 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a 4-dimensional complex Kähler model. If $\mathcal{V}$ is complex Osserman, then one of the following holds:*

1. *$\mathcal{V}$ has constant holomorphic sectional curvature c (i.e., $A = \frac{c}{4}(A_0 + A_J)$).*
2. *$\mathcal{V}$ is isomorphic to $\mathfrak{CC}^4$.*

Proof. Since the dimension of the model is 4, the complex Jacobi operator may have one eigenvalue of multiplicity 4 or two distinct eigenvalues of multiplicities $(2,2)$. We study the two cases separately.

Suppose the complex Jacobi operator has one eigenvalue λ. Then

$$\lambda Jx = \lambda \mathcal{J}(\pi_x) Jx = A(x, Jx)x.$$

Lemma 14.1.1 shows $\mathcal{V}$ has constant holomorphic sectional curvature. This is a contradiction, since the complex Jacobi operator associated to these manifolds have two distinct eigenvalues.

Now, assume there are two eigenvalues (μ, λ). By Lemma 14.2.7, $\Gamma\Lambda = -Id$. Therefore, either Γ preserves the orientation on the sphere and Λ reverses it, or Λ preserves the orientation on the sphere and Γ reverses it. Since the roles of Γ and Λ are symmetric, we suppose without loss of generality that Γ preserves orientation. Since $\Gamma^2 = Id$, as a consequence of the Lefschetz fixed point formula, Γ has at least two fixed points, say x and y (see, for instance, [53]). This shows

$$A(Jx, x, Jx, x) = A(Jy, y, Jy, y) = \mu \quad \text{and} \quad A(Jx, x, Jy, y) = \lambda\,.$$

Now Lemma 14.2.5 applies and gives the desired classification. □

14.2.3 4-dimensional complex Osserman manifolds

Once we have completed the classification of complex Osserman models, the next step is to investigate if they are geometrically realizable. In other words, our aim here is to classify complex Osserman manifolds. Since we know that one of the possible algebraic models corresponds to constant holomorphic sectional curvature, we know that manifolds of constant holomorphic sectional curvature are taking part in such classification. Next result shows even more, they are the only possibility.

Theorem 14.2.9 *Let $\mathcal{M} = (M, g, J)$ be a 4-dimensional Kähler manifold. Then $\mathcal{M}$ is pointwise complex Osserman if and only if it is of constant holomorphic sectional curvature.*

Proof. From Theorem 14.2.8 we have two possibilities at each point. Adopt notation in previous section. Since $\mu = 2\lambda \neq 0$ or $\lambda = 0$ at each point, we have that at every point $\mu = 2\lambda \neq 0$ and $\mathcal{M}$ has constant holomorphic sectional curvature, or at every point $\mathcal{M}$ realizes the model $\mathfrak{C}\mathfrak{C}^4$. We are going to show that last option can not happen.

Suppose $\mathcal{M}$ realizes the model $\mathfrak{C}\mathfrak{C}^4$ at every point. Since $\mathcal{M}$ is Einstein, the scalar curvature ($\tau = 4\mu$) is constant, so μ is also constant. Recall that the self-dual and the anti-self-dual Weyl components are given by

$$W^+ = \begin{pmatrix} -\frac{2\mu}{3} & 0 & 0 \\ 0 & \frac{\mu}{3} & 0 \\ 0 & 0 & \frac{\mu}{3} \end{pmatrix}, \quad W^- = \begin{pmatrix} -\frac{2\mu}{3} & 0 & 0 \\ 0 & \frac{4\mu}{3} & 0 \\ 0 & 0 & -\frac{2\mu}{3} \end{pmatrix}.$$

This shows that both the self-dual and the anti-self-dual components have exactly two eigenvalues. Therefore, one applies results in [56], to obtain that (M, g, J) is globally conformally equivalent to a Kähler metric $\bar{g}$, $g = e^\sigma \bar{g}$, where e^σ is a multiple of $g(W^+, W^+)^{1/3}$. Since $g(W^+, W^+)$ is constant in our context, we conclude (M, g) is indeed Kähler, so $\nabla J = 0$.

Consider now the almost complex structure L given by the 2-form associated to the eigenvalue $\frac{4\mu}{3}$ in W^-. We apply the previous argument again to show $\nabla L = 0$. The fact that there are two complex structures which induce opposite orientations shows the manifold is a product [22], but this is obviously not true, so we conclude $\mathfrak{C}\mathfrak{C}^4$ is not geometrically realizable. □

14.3 Complex Osserman models in higher dimension

In dimension four there are only two possible eigenvalue structures for the complex Jacobi operator, namely only one eigenvalue or two eigenvalues of multiplicity $(2, 2)$. However, there are in principle four different possibilities in higher dimension, as we have seen in Theorem 13.1.7, and all of them are realized by complex models, as we have seen in

Remark 13.2.6. In this section we concentrate on those complex Osserman models with two eigenvalues (μ, λ) of multiplicities $(2, \mu - 2)$.

Suppose $A(x, Jx, Jx, x) = \lambda$. If $n = dim\ V \geq 6$, then there exists $y \in \pi_x^\perp$ such that $A(x, Jx, Jy, y) = \lambda$. Under these conditions next lemma implies $\lambda = 0$.

Lemma 14.3.1 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex model with $n \geq 6$. Suppose there exists $x \in V$ such that $A(x, Jx)Jx = \lambda x$, then $\lambda = 0$.*

Proof. Let $E_\lambda(x) = \{y \in V : \mathcal{J}(\pi_x)y = \lambda y\}$. Since $dim\, E_\lambda(x) \geq 4$ we may choose $y \in E_\lambda(x)$ such that $y \perp \mathrm{Span}\,\{x, Jx\}$. We use Lemma 13.1.3 to obtain

$$\begin{aligned} \lambda x &= A(\cos\theta Jx + \sin\theta Jy, \cos\theta x + \sin\theta y)Jx \\ &= \cos^2\theta A(Jx, x)Jx + 2\cos\theta\sin\theta A(Jx, y)Jx + \sin^2\theta A(Jy, y)Jx \\ &= \lambda x + 2\cos\theta\sin\theta A(Jx, y)Jx \end{aligned}$$

from where $A(Jx, Jy)Jx = 0$. We repeat the argument for Jx instead of x to obtain $A(x, y)x = 0$. Hence $\lambda y = A(Jx, x)Jy = \mathcal{J}(\pi_x)y = 0$. □

The following theorem describes the possibilities for a complex Osserman model of the kind under consideration.

Theorem 14.3.2 *Let $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ be a complex Kähler model. Assume $\mathcal{V}$ is complex Osserman with two distinct eigenvalues (μ, λ) of multiplicities $(2, n-2)$. Then one of the following holds:*

1. *$\mu = 2\lambda$ and $\mathcal{V}$ has constant holomorphic sectional curvature μ (i.e. A is given by $A = \frac{\mu}{4}(A_0 + A_J)$).*
2. *$\lambda = 0$ and μ is arbitrary.*

Proof. If V has dimension 2, then the holomorphic sectional curvature is constant. If V has dimension 4, Theorem 14.2.8 shows $\mathcal{V}$ has constant holomorphic sectional curvature or is isomorphic to $\mathfrak{CC}^4$. So, in both cases Assertion (1) or (2) holds.

Henceforth we assume $n \geq 6$. If $A(Jx, x, Jx, x) = \mu$ for all $x \in V$, then the model has constant holomorphic sectional curvature by Lemma 14.1.1. Suppose on the contrary that there exist x, y such that $A(x, Jx, Jy, y) = \mu$. Consider the subspace $D := \{x, y, Jx, Jy\}$. Due to Lemma 13.1.3 the complex Jacobi operator preserves the subspaces D and $D^\perp$. Hence, we restrict the curvature tensor to D and apply Theorem 14.2.8 to see that the restriction of $\mathcal{V}$ to D is isomorphic to $\mathfrak{CC}^4$. Thus, in particular, one has $\mu = 0$ or $\lambda = 0$. If $\mu = 0$, since in $\mathfrak{CC}^4$ there exist z such that $A(z, Jz, Jz, z) = \lambda$, we apply Lemma 14.3.1 to show $\lambda = 0$ and hence $A = 0$. Therefore, we conclude $\lambda = 0$. □

The following example shows that there exist algebraic models satisfying Assertion (2) in Theorem 14.3.2.

Example 14.3.3 *Let* $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ *be a* $2n$*-dimensional complex model and let* $\{x_i, Jx_i\}$ *be an orthonormal basis. Define* A*, up to the usual symmetries, by:*

$$\begin{aligned}
&A(Jx_i, x_i, Jx_i, x_i) = \mu\,,\\
&A(x_i, x_j, x_i, x_j) = \tfrac{-\mu}{2}\,, A(x_i, Jx_j, x_i, Jx_j) = \tfrac{\mu}{2}\,,\\
&A(Jx_i, x_j, Jx_i, x_j) = \tfrac{\mu}{2}\,, A(Jx_i, Jx_j, Jx_i, Jx_j) = \tfrac{-\mu}{2}\,,\\
&A(x_i, x_j, Jx_i, Jx_j) = \tfrac{-\mu}{2}\,, A(x_i, Jx_j, Jx_i, x_j) = \tfrac{-\mu}{2}\,.
\end{aligned}$$

Then it is straightforward to compute the complex Jacobi operator and to see that the model $\mathfrak{E}\mathfrak{E}^n = (V, \langle\cdot,\cdot\rangle, J, A)$ *is complex Osserman with two eigenvalues* $(\mu, 0)$ *of multiplicities* $(2, n-2)$*. We omit the details in the interest of brevity.*

Since the possible eigenvalue structure depends on the dimension of the model, we obtain the following consequence of Theorem 14.3.2.

Corollary 14.3.4 *Let* $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, A, J)$ *be a complex Osserman Kähler model with* $n \equiv 2(mod\,4)$*. Then* $\mathcal{V}$ *has two distinct eigenvalues* (μ, λ) *with multiplicities* $(2, n-2)$ *and, moreover,* $\mu = 2\lambda$ *and* $\mathcal{V}$ *has constant holomorphic sectional curvature* μ *(in this case* $A = \frac{\mu}{4}(A_0 + A_J)$*) or, otherwise,* $\lambda = 0$*.*

Proof. Since $n \equiv 2(mod\,4)$ we have, from Theorem 13.1.7 that $\mathcal{V}$ has one eigenvalue or two distinct eigenvalues (μ, λ) with multiplicities $(2, n-2)$. The result follows from Theorem 14.3.2 as soon as we show $\mathcal{V}$ can not have only one eigenvalue. Assume on the contrary $\mathcal{V}$ has one eigenvalue λ. Then $A(x, Jx)x = \lambda Jx$ and by Lemma 14.1.1 the model has constant holomorphic sectional curvature. This is a contradiction and completes the proof. $\square$

Open problems

In Chapter 12 we characterized the space of algebraic curvature tensors where the complex Jacobi operator determines the curvature tensor. Hence it looks natural to study the complex Osserman problem from an algebraic point of view in that space. At the geometric level, Kähler, Nearly Kähler or Hermitian manifolds are some of the possible settings where that condition is guaranteed.

Also, the compatibility condition we imposed in Chapter 13 appears to be very natural and many classes of manifolds, for example Nearly Kähler manifolds, satisfy that condition. However, in Chapter 14 we have restricted to Kähler manifolds. Thus, one may aboard the complex Osserman problem in a more general class such as Hermitian or Nearly Kähler manifolds. Here we consider the following more specific open problems:

- We have shown in Chapter 12 that almost Kähler Hermitian manifolds with vanishing complex Jacobi operator are necessarily flat. However, we still do not know if the complex Jacobi operator fully determines the curvature tensor in an arbitrary almost Kähler manifold.
- In Chapter 14 we classify complex Osserman Kähler manifolds in dimension 4. Moreover, it looks natural to generalize this result to higher dimension but restricted to the case of 2 eigenvalues with multiplicities $(2, n-2)$. We have obtain some partial results in this line; more specifically, we have shown that the Kähler manifold has constant holomorphic sectional curvature or, otherwise, the eigenvalue of multiplicity $n-2$ is zero. Furthermore, we have given exotic algebraic models realizing this last possibility. We still do not know if those models are unique (up to isomorphism) as it happens in dimension 4. Even more, we also do not know if they are geometrically realizable.
- Of course, a complete classification of complex Osserman Kähler manifolds is desirable and it is the final goal.
- It would be interesting to know to what extent is the Kähler condition essential in the 4-dimensional classification of complex Osserman manifolds. We may rephrase the question in the following way: is the same classification of Theorem 14.2.9 true if we remove the Kähler condition? A natural option would be to substitute Kähler by Hermitian.

Resumo en galego

Consecuencias xeométricas de condicións alxébricas en operadores asociados á curvatura

O papel que desempeña a curvatura en Xeometría riemanniana aínda non se alcanza a entender na súa totalidade. Como invariante riemanniano, a curvatura e as súas derivadas son os invariantes alxébricos máis naturais de entre os que se obteñen a partir da conexión. Este feito suxire que o tensor curvatura encerra gran información sobre a xeometría dunha variedade riemanniana. Por estes motivos podemos dicir que a curvatura é un concepto central en Xeometría diferencial.

O principal obxectivo desta tese é obter consecuencias xeométricas de condicións alxébricas impostas ó tensor de curvatura. Por mor da complexidade que encerra o manexo deste, normalmente impoñemos estas condicións en operadores asociados e non no tensor curvatura directamente. Como norma xeral traballamos no amplo campo das variedades semi-riemannianas; porén, en algúns capítulos ou seccións restrinximos a nosa análise a métricas definidas positivas. Se ben tratamos de desenvolver o noso estudio na xeneralidade que proporcionan as métricas non dexeneradas, o campo da xeometría riemanniana está moito máis desenvolvido que o da estrictamente semi-riemanniana, o cal posibilitará que en certos casos sexamos quen de afondar máis se a métrica é definida positiva. A pesar disto, tamén cobra unha salientable importancia o caso lorentziano, que ademais de atopar unha grande inspiración na Física, presenta en moitas ocasións propiedades que fan a súa xeometría máis tanxible que o caso semi-riemanniano de signatura superior.

O tratamento que facemos á hora de encarar os problemas plantexados nesta tese estará, en moitos casos, motivado polo papel que desempeñou nos últimos anos a conxetura de Osserman e os traballos que dela se derivan. Así, parte deste traballo consiste no estudio do problema de Osserman en signatura superior ou na análise de problemas tipo Osserman, é dicir, problemas que xorden de xeneralizar o problema de Osserman a un contexto diferente. En ambos casos, achamos unha parte importante da nosa inspiración no problema de Osserman riemanniano, empregando en ocasións técnicas similares ás xa coñecidas por medio de adaptacións específicas ó contexto baixo consideración. Noutros casos, o tratamento que facemos do problema é novedoso e vén perfilado polas súas características concretas.

Sen restrinxírmonos exclusivamente a problemas de tipo Osserman, cómpre salientar que parte deste traballo (fundamentalmente a Parte II) está motivado tamén por elementos que aparecen na Física e, máis concretamente, na Cosmoloxía. Se ben é certo que o tratamento que facemos aquí ten un carácter meramente teórico, tamén debemos salientar que o estudio levado a cabo se aplica, como se fai patente nesta memoria, a modelos cosmolóxicos explícitos, contribuindo así a clarificar a súa xeometría.

O esquema que segue esta memoria é o seguinte. Un capítulo preliminar establece as principais definicións e algúns resultados básicos no campo, que serán de utilidade no desenvolvemento posterior; estamos a falar do Capítulo 1. A continuación, o que podemos denominar corpo da tese divídese en catro partes diferenciadas. A pesar de estar relacionadas entre si, pois todas elas perseguen describir a xeometría de variedades sobre as que se impón unha condición alxébrica sobre a curvatura, estas catro partes son esencialmente independentes. Se ben esta independencia permite unha lectura individual, a orde na que se presentan aquí é natural e atoparemos como algunha parte motiva á que precede, sendo este o principal motivo da nosa escolla.

I Variedades conformemente Osserman

A primeira parte desta memoria trata certos aspectos do problema de Osserman. Dende que a conxetura de Osserman foi formulada en [145] para variedades riemannianas, tanto a propia conxetura (excepto en dimensión 16) como moitos problemas relacionados foron resoltos [51, 52, 135, 136]. Un claro exemplo deste feito é a clasificación das variedades Osserman en signatura lorentziana, a cal mostra que son necesariamente de curvatura seccional constante [16, 75]. Motivados por esta conxetura xurdiron interesantes problemas que aínda hoxe permanecen sen resposta. As variedades de dimensión catro e signatura neutra preséntanse como o caso máis sinxelo sen resolver. É por isto que nos Capítulos 2, 3 e 4 nos centramos no estudio de variedades Osserman en signatura $(2, 2)$. En primeiro lugar, relacionamos a estructura de autovalores do operador de Jacobi dun tensor de curvatura alxébrico Osserman coa estructura de autovalores do operador de Weyl autodual (anti-autodual). Así, móstrase que as catro posibilidades (dependendo dos autovalores e a forma de Jordan) están en correspondencia unívoca entre os dous operadores. A continuación empregamos métricas de Walker para proporcionar exemplos que realizan estes catro tipos para o operador de Jacobi conforme; é dicir, móstrase a existencia de variedades conformemente Osserman que realizan tódalas posibilidades alxébricas do espectro. Escollendo unha ampla familia de métricas de Walker, obtense unha caracterización de cando esas métricas son Osserman. O seguinte resultado, xa coñecido no caso riemanniano, cúmprese tamén en signatura neutra:

Teorema 3.1.2. *Sexa (M, g) unha variedade semi-riemanniana de dimensión catro. Entón (M, g) é conformemente Osserman se e só se é autodual ou anti-autodual.*

A continuación, empregando esta equivalencia entre autodualidade (anti-autodualidade) e conformalidade Osserman, analizamos unha gran variedade de exemplos conformemente Osserman desa familia, mostrando que as catro posibilidades descritas anteriormente son realizadas por variedades da dita familia. Ademais, entre os exemplos que presentamos, cobran especial importancia aqueles que non están na clase conforme dunha variedade Osserman.

A pesar de que a propiedade de Osserman conforme é local, tamén interesan as propiedades globais como a completitude ou a homoxeneidade. Por este motivo, estudiamos a completitude xeodésica en diversos exemplos de variedades Osserman e conformemente Osserman, sendo todos eles variedades de Walker. Ademais veremos que os máis dos exemplos dados non son curvatura homoxéneos e, polo tanto, non son homoxéneos.

Outra condición topolóxica global como a compacidade permítenos esclarecer a condición Jordan Osserman en signatura $(2, 2)$. Isto queda reflexado no seguinte resultado:

Teorema 4.1.3. *Sexa* (M, g) *unha variedade Jordan Osserman compacta con métrica de signatura* $(2, 2)$*. Entón* (M, g) *ten curvatura seccional constante ou operadores de Jacobi nilpotentes.*

No último capítulo desta parte estudiamos a condición de Osserman conforme e a propia condición de Osserman en variedades cunha estructura prefixada, máis concretamente coa estructura de produto deformado (*warped*). Dun xeito máis pormenorizado, vemos que a estructura de produto deformado é suficientemente ríxida para que a condición de Osserman conforme sexa equivalente á conformalidade chá local para produtos deformados riemannianos. Estas consideracións, xunto co feito de que a condición de Osserman é equivalente á de Osserman conforme e o carácter einstein, facilitan a comprensión das variedades Osserman que teñen como estructura subxacente a dun produto deformado. En particular conclúese o seguinte:

Corolario 5.3.3. *Nin* $\mathbb{C}P^n$ *nin o seu dual de curvatura negativa poden ser descompostos como produto twisted.*

O feito de existiren exemplos de produtos deformados conformemente Osserman que non son localmente conformemente chans, pon de manifesto unha vez máis o distinto comportamento que presenta o problema de Osserman segundo sexa a signatura da variedade. Ademais, e a pesar de que o estudio realizado supón un novo chanzo na comprensión da xeometría dos produtos deformados, existen estructuras, como a dos produtos twisted, cunha maior xeneralidade que fai difícil o seu tratemento en signatura superior. Por estes motivos, atopamos multitude de problemas abertos, que xorden do estudio aquí realizado.

II Variedades localmente conformemente chás con estructura de produto deformado

Motivados polos resultados do Capítulo 5, desenvolvemos na Parte II unha exhaustiva análise das variedades localmente conformemente chás con estructura de produto deformado. Facendo uso do feito de que todo produto deformado está na clase conforme dun produto directo, no Capítulo 6 caracterizamos a conformalidade chá local como segue:

Teorema 6.1.2. *Sexa* $(M, g) = B \times_f F$ *un produto deformado semi-riemanniano. Entón tense:*

(i) *Se* $\dim B = 1$*, entón* (M, g) *é localmente conformemente chá se e só se* (F, g_F) *é un espacio de curvatura constante.*

(ii) *Se* $\dim B > 1$ *e* $\dim F > 1$*, entón* (M, g) *é localmente conformemente chá se e só se*

   (ii.a) (F, g_F) *é un espacio de curvatura constante* K_F*.*

   (ii.b) *A función* $f : B \to \mathbb{R}^+$ *define unha deformación conforme global en* B *tal que* $(B, \frac{1}{f^2} g_B)$ *é un espacio de curvatura constante* $\tilde{K}_B = -K_F$*.*

(iii) *Se* $\dim F = 1$*, entón* (M, g) *é localmente conformemente chá se e só se a función* $f : B \to \mathbb{R}^+$ *define unha deformación conforme en* B *tal que* $(B, \frac{1}{f^2} g_B)$ *é un espacio de curvatura constante.*

Debido a que o dominio dunha función de deformación f é o factor base B, as súas propiedades xeométricas inflúen fortemente na xeometría de toda a variedade $B \times_f F$. Así, préstase especial atención ós produtos deformados con base un espacio modelo (euclídeo, hiperbólico e esférico). Ademais, móstrase que propiedades globais na base, como a compacidade ou a completitude xeodésica, teñen interesantes consecuencias na variedade produto. Un exemplo disto é o seguinte resultado:

Teorema 6.2.3. *Sexa* $B \times_f F$ *un produto deformado localmente conformemente chan con* $(B, \frac{1}{f^2} g_B)$ *unha variedade riemanniana completa. Se existe unha función de deformación* $\hat{f} \neq cf$ *en* B *tal que* $B \times_{\hat{f}} F$ *é tamén localmente conformemente chá, entón tense unha das posibilidades seguintes:*

1. $(B, \frac{1}{f^2} g_B)$ *é un espacio completo e simplemente conexo de curvatura seccional constante e a función de deformación* $\hat{f}/f$ *queda determinada polo Teorema 6.1.7, ou*

2. $(B, \frac{1}{f^2} g_B)$ *é un produto deformado* $\mathbb{R} \times_{\alpha \exp(\alpha t + \beta)} N$*, onde* (N, g_N) *é unha variedade riemanniana completa e chá; ademais as funcións de deformación satisfán*

$$\hat{f}/f = \exp(\alpha t + \beta) + \kappa$$

para certas constantes reais $\alpha > 0$*,* β*,* $\kappa \geq 0$*, onde* $\alpha^2 = -\frac{\tau^B}{n(n-1)}$ *e* τ^B *denotan a curvatura escalar de* $(B, \frac{1}{f^2} g_B)$*.*

Existen diversos xeitos de xeneralizar a estructura de deformación dunha variedade produto deformado. Unha consiste en aumentar o dominio da función de deformación; obtéñense así os chamados produtos twisted. En [68] probouse que un produto twisted einstein é en realidade un produto deformado (excepto se a fibra ten dimensión un). De modo similar mostramos que un produto twisted localmente conformemente chan é tamén un produto deformado se a dimensión dos seus factores é estrictamente maior que un. Polo tanto, pódese aplicar o estudio feito previamente a esta clase de variedades.

Outro xeito de xeneralizar a estructura de produto deformado é engadindo novas fibras coas correspondentes funcións de deformación; obtendo deste modo os denominados produtos deformados múltiples. No Capítulo 7 estudiamos a conformalidade chá local en variedades con esta estructura. Distinguimos dous casos dependendo de se a dimensión da base é un ou superior. Dos Teoremas 7.2.6 e 7.3.1 e da Observación 7.3.4 obtemos as seguintes conclusións:

> *Sexa* $B \times_{f_1} F_1 \times \ldots \times_{f_k} F_k$ *un produto deformado múltiple localmente conformemente chan. Entón:*
>
> - *o número de fibras é menor ou igual que* $\dim B + 2$*,*
> - *tódalas fibras teñen curvatura seccional constante,*
> - *o signo da curvatura das fibras depende da signatura da base; por exemplo, para signatura riemanniana, hai como moito unha fibra de curvatura negativa.*

Analogamente, acádanse algúns resultados interesantes en variedades con esta estructura de produto deformado múltiple que teñen curvatura seccional constante.

As variedades localmente conformemente chás están lonxe de ser completamente clasificadas. Non obstante, coñécense algúns resultados se a curvatura de Ricci é positiva. Por exemplo, unha variedade completa, simplemente conexa e localmente conformemente chá con curvatura de Ricci non negativa está na clase conforme de $\mathbb{S}^n$, $\mathbb{R}^n$ ou $\mathbb{R} \times \mathbb{S}^{n-1}$. No Capítulo 8 aplicamos os resultados dos Capítulos 6 e 7 para construír novos exemplos de variedades completas localmente conformemente chás con curvatura (de Ricci) negativa. A diversidade de exemplos que se mostra e a sinxeleza do método de construcción suxiren que unha clasificación similar para este conxunto de variedades non é posible.

Posto que moitos dos modelos cosmolóxicos empregados para describir a xeometría do Universo, tanto isótropos como anisótropos, son produtos deformados múltiples, empréganse as caracterizacións dadas previamente para acadar unha mellor comprensión da xeometría de tales modelos. Estas aplicacións expóñense no Capítulo 8. En vista das características mostradas polos modelos cosmolóxicos coñecidos como Robertson-Walker, parece natural buscar unha xeneralización destes en espacios deformados múltiples que presentan a característica de conformalidade chá local. Posto que a estructura subxacente dos espacios Robertson-Walker é a dun produto deformado con base de dimensión 1, é unha consecuencia inmediata do Teorema 6.1.2 que son localmente conformemente chás. Os resultados obtidos no Capítulo 7 suxiren como xeneralizar os espacios Robertson-Walker a espacios deformados múltiples conservando, entre outras, a propiedade de conformalidade chá local. Dende un punto de vista cosmolóxico, un posible modo de describir a xeometría do Universo é aumentando a dimensión do modelo, que posúe un espacio observable (*espacio externo*) de signatura lorentziana e diversos *espacios internos* compactos e de tamaño moi reducido. O estudio realizado no Capítulo 7 e os exemplos dados na Sección 8.3.2 mostran que é posible construír modelos multidimensionais con fibras (espacios internos) compactas e que sexan localmente conformemente chans.

III Conmutatividade de operadores asociados ó tensor curvatura: variedades de Tsankov

Recordemos que o principal obxectivo ó estudiarmos problemas de tipo Osserman é obter consecuencias xeométricas da constancia dos autovalores de certos operadores que están intimamente ligados ó tensor de curvatura. Na Parte III tomamos como obxectivo o clasificar variedades con operadores curvatura que conmutan. Así, a base do noso estudio aquí non é os autovalores dun operador curvatura, senón os correspondentes autoespacios, que analizamos por medio de propiedades de conmutación. Por exemplo, o obxectivo do Capítulo 9 é probar o Teorema 9.3.1, que se traduce no seguinte:

> *Sexa (M, g) unha variedade riemanniana de dimensión n. Entón os operadores de Jacobi conmutan para direccións arbitrarias, i.e.*
>
> $$\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x) \qquad \text{para calesquera} \quad x, y,$$
>
> *se e só se (M, g) é chá; e se $n \geq 3$, os operadores de Jacobi conmutan para direccións ortogonais, i.e.*
>
> $$\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x) \qquad \text{para} \quad x \perp y,$$
>
> *se e só se (M, g) ten curvatura seccional constante.*

No Capítulo 10 obtemos algúns resultados parciais nesta dirección para variedades de signatura superior, onde este resultado non é certo. Tales resultados resúmense no Teorema 10.1.1:

> *Sexa (M, g) unha variedade semi-riemanniana tal que os operadores de Jacobi commutan, i.e. $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ para calesquera x, y. Entón*
>
> - *$\mathcal{J}(x)^2 = 0$ para todo x, e*
> - *se a signatura é lorentziana entón a variedade é chá.*

Por exemplo, móstrase que para dimensión estrictamente menor que 14, a condición $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ é equivalente a $\mathcal{J}(x)\mathcal{J}(y) = 0$ para calesquera x, y. A menor dimensión onde non se verifica esta equivalencia é en dimensión 14, o cal se pon en evidencia coa construcción dun exemplo a nivel alxébrico e co posterior estudio das propiedades xeométricas de diversas realizacións a nivel diferenciable.

A maioría das condicións que se poden estudiar sobre o operador de Jacobi pódense trasladar ó operador curvatura antisimétrico. De feito, unha vez que o problema de Osserman chamou a atención á comunidade matemática, xurdiron as variedades Ivanov-Petrova como un problema paralelo motivado polo de Osserman e o comportamento do tensor curvatura ó longo de círculos unitarios. Polo tanto, no Capítulo 11 estúdianse modelos alxébricos con operadores curvatura que conmutan e dase a seguinte clasificación alxébrica de modelos indescompoñibles para signatura definida positiva.

Teorema 11.1.1. *Sexa $(V, \langle\cdot,\cdot\rangle, A)$ un modelo alxébrico riemanniano.*

1. *Cúmprese que $A(x_1, x_2)A(x_3, x_4) = A(x_3, x_4)A(x_1, x_2)$ se e só se existe unha suma directa ortogonal $V = V_1 \oplus ... \oplus V_k \oplus W$ descompoñendo $A = A_1 \oplus ... \oplus A_k \oplus 0$ onde $\dim V_i = 2$ para todo i.*

2. *Ademais o modelo é indescompoñible se e só se $\dim V = 2$ e $A \neq 0$.*

Neste mesmo capítulo danse exemplos xeométricos que suxiren que unha clasificación a nivel diferenciable é moito máis complicada. Este comentario presenta o problema de conmutación de operadores curvatura como unha liña de investigación por explorar na que a clasificación a nivel alxébrico non é máis ca un primeiro paso.

Cómpre mencionar que, se ben a conmutación de operadores asociados á curvatura foi historicamente estudiada ligada ó campo de subvariedades, actualmente existe unha bibliografía crecente que trata o problema de clasificación de variedades semi-riemannianas por medio de propiedades de commutación de operadores como o operador de Ricci, Jacobi, Szabó,. . . .

IV Variedades complexas Osserman

Sexa π un k-plano no espacio tanxente ó punto p. Para unha base $\{e_1, \dots, e_k\}$ de π, defínese o operador de Jacobi de orde k como

$$\mathcal{J}(\pi) := \sum_{i=1}^{k} \mathcal{J}(e_i)\,.$$

Como problema de tipo Osserman, Gilkey [82] caracterizou as variedades riemannianas tales que o seu operador de Jacobi de orde k ten autovalores constantes, mostrando que, para $2 \leq k \leq n-2$, estas son as variedades de curvatura seccional constante. Sexa (M, g, J) unha variedade case hermítica. Unha xeneralización natural do operador de Jacobi é o operador de Jacobi complexo, que vén dado por

$$\mathcal{J}(\pi_x) = \mathcal{J}(x) + \mathcal{J}(Jx),$$

onde $\pi_x = \mathrm{Span}\,\{x, Jx\}$. Unha das propiedades que fai do operador de Jacobi usual un obxecto tan especial é o feito de que determina o tensor curvatura. Así, parece natural preguntarse se isto mesmo é certo para o operador de Jacobi complexo que vimos de definir. A primeira tarefa a afrontar na Parte IV é o dar resposta a esta pregunta e o entendermos cando o operador de Jacobi complexo determina o tensor de curvatura para unha variedade case hermítica. Isto lévase a cabo no Capítulo 12, onde amosaremos no Teorema 12.1.3 que:

> *Para (M, g, J) unha variedade hermítica ou nearly Kähler, o operador de Jacobi complexo determina completamente o tensor curvatura.*

Este feito non se pode extender ó contexto máis xeral de variedades case hermíticas (véxase o Teorema 12.1.2 para un exemplo explícito). Tamén se obteñen resultados referentes a variedades almost Kähler ou variedades conformemente equivalentes.

Posteriormente comézase o estudio das variedades complexas Osserman, definidas de maneira natural como as variedades case hermíticas con operadores de Jacobi complexos que teñen autovalores constantes. No Capítulo 13 obtemos algúns resultados xerais, mostrando que este tipo de variedades son einstein e, ademais, que a estructura de autovalores do operador de Jacobi complexo vén controlada do seguinte xeito:

> *Sexa (M, g, J) unha variedade case hermítica de dimensión n que verifica a condición de compatibilidade $J^*R = R$ e tal que o operador de Jacobi determina o tensor curvatura (i.e. $R \in \mathfrak{R}_2$). Entón, se os autovalores do operador de Jacobi complexo son constantes, verifícase:*
>
> 1. *Se $n \equiv 2$ (mod 4), hai 2 autovalores con multiplicidades $(n-2, 2)$.*
> 2. *Se $n \equiv 0$ (mod 4), entón cúmprese unha das seguintes posibilidades:*
>   (a) *Hai 2 autovalores con multiplicidades $(n-2, 2)$.*
>   (b) *Hai 2 autovalores con multiplicidades $(n-4, 4)$.*
>   (c) *Hai 3 autovalores con multiplicidades $(n-4, 2, 2)$.*

Ademais empréganse as estructuras de familia de Clifford para construír exemplos, principalmente a nivel alxébrico, e mostrar que todas estas posibilidades realmente ocorren. Aínda máis, profundízase na estructura dos tensores curvatura alxébricos que veñen dados por familias de Clifford para analizar cales son Osserman complexos.

No derradeiro capítulo concentrámonos nas variedades Kähler. Así, o feito de que a estructura complexa é paralela ten consecuencias xeométricas que nos permiten obter algúns resultados parciais cando o operador de Jacobi complexo ten 2 autovalores de multiplicidades $(2, n-2)$. Máis concretamente, neste caso se a variedade non é de curvatura seccional holomorfa constante entón o autovalor de maior multiplicidade é 0. O principal resultado do capítulo é a completa clasificación das variedades Kähler complexas Osserman en dimensión 4:

Teorema 14.2.9. *Sexa (M, g, J) unha variedade Kähler de dimensión 4. Entón (M, g, J) é Osserman complexa se e só se é de curvatura seccional holomorfa constante.*

A pesar de que estes resultados supoñen os primeiros avances no problema de caracterizar as variedades complexas Osserman, unha caracterización completa segue aberta incluso no caso en que a variedade sexa Kähler. Os anteriores resultados suxiren un achegamento ó problema consistente en analizar, un a un, os distintos casos de posibles autovalores e multiplicidades. Posto que estas variedades non poden ter un único autovalor para o operador de Jacobi complexo, habería que analizar os casos de dous e tres autovalores. Ademais, parece natural que un amplíe o dominio de variedades Kähler a variedades, cando menos, Hermíticas. Neste contexto as variedades complexas Osserman preséntanse coma un reto natural de gran interese.

Bibliography

[1] E. Abbena; An example of an almost Kähler manifold which is not Kählerian, *Boll. Un. Mat. Ital.* **3A** (1984), 383–392.

[2] J. Adams; Vector fields on spheres, *Ann. of Math.* **75** (1962), 603–632.

[3] D. M. Alekseevski, N. Blažić, N. Bokan, and R. Rakić; Self-duality and pointwise Osserman spaces, *Arch. Math. (Brno)* **35** (1999), 193–201.

[4] V. Apostolov, J. Armstrong, and T. Drăghici; Local models and integrability of certain almost Kähler 4-manifolds, *Math. Ann.* **323** (2002), 633–666.

[5] V. Apostolov, and T. Drăghici; The curvature and the integrability of almost-Kähler manifolds: a survey, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), 25–53, Fields Inst. Commun. **35** Amer. Math. Soc., Providence, RI, 2003.

[6] V. Apostolov, J. Davidov, and O. Muškarov; Compact self-dual Hermitian surfaces, *Trans. Amer. Math. Soc.* **348**, n 8, (1996).

[7] M. F. Atiyah, R. Bott, and A. Shapiro; Clifford Modules, *Topology* **3** suppl. 1 (1964), 3–38.

[8] A. Balas, and P. Gauduchon; Any Hermitian Metric of Constant Non-Positive (Hermitian) Holomorphic Sectional Curvature on a Compact Complex Surface is Kähler, *Math. Z.* **190** (1985), 39–43.

[9] W. Barth, C. Peters, and A. Van de Ven; *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, (1984).

[10] M. Berger, P. Gauduchon, and E. Mazet; *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math. **194**, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[11] J. Berndt; Real hypersurfaces in quaternionic space forms, *J. Reine Angew. Math.* **419** (1991), 9–26.

[12] M. Bertola, and D. Gouthier; Warped products with special Riemannian curvature, *Bol. Soc. Brasil. Mat. (N.S.)* **32** (2001), 45–62.

[13] R. L. Bishop, and B. O'Neill; Manifolds of negative curvature, *Trans. Amer. Math. Soc.* **145** (1969), 1–49.

[14] I. Bivens, J. P. Bourguignon, A. Derdzinski, D. Ferus, O. Kowalski, T. Klotz Milnor, V. Oliker, U. Simon, W. Struebing, and K. Voss; Discussion on Codazzi-tensors, *Global differential geometry and global analysis (Berlin, 1979)*, Lecture Notes in Math., **838**, pp. 243–299, Springer, Berlin-New York, 1981.

[15] N. Blažić; Natural curvature operators of bounded spectrum, *Differential Geom. Appl.* **24** (2006), no. 6, 563–566.

[16] N. Blažić, N. Bokan, and P. Gilkey; A Note on Osserman Lorentzian manifolds, *Bull. London Math. Soc.* **29** (1997), 227–230.

[17] N. Blažić, N. Bokan, and Z. Rakić; Osserman pseudo-Riemannian manifolds of signature $(2,2)$, *J. Aust. Math. Soc.* **71** (2001), 367–395.

[18] N. Blažić, and P. Gilkey; Conformally Osserman manifolds and conformally complex space forms, *Int. J. Geom. Methods Mod. Phys.* **1** (2004), no. 1-2, 97–106.

[19] N. Blažić, and P. Gilkey; Conformally Osserman manifolds and self-duality in Riemannian geometry, *Proceedings DGA Conf.*, Czech Republic 2004 (2005), 15–18.

[20] N. Blažić, P. Gilkey, S. Nikčević, and U. Simon; The spectral geometry of the Weyl conformal tensor, *PDEs, submanifolds and affine differential geometry*, Banach Center Publ., **69**, Polish Acad. Sci., Warsaw, 2005, 195-203.

[21] N. Bokan; On the complete decomposition of curvature tensors of Riemannian manifolds with symmetric connection, *Rend. Circ. Mat. Palermo* **XXIX** (1990), 331–380.

[22] A. Bonome, R. Castro, E. García-Río, L. Hervella, and Y. Matsushita; Almost complex manifolds with holomorphic distributions, *Rend. Mat. Appl. (7)* **14** (1994), 567–589.

[23] A. Bonome, R. Castro, E. García-Río, L. Hervella, and Y. Matsushita; Pseudo-Chern classes and opposite Chern classes of indefinite almost Hermitian manifolds, *Acta Math. Hungar.* **75** (1997), 299–316.

[24] A. Bonome, R. Castro, E. García-Río, L. Hervella, and R. Vázquez-Lorenzo; Pseudo-Riemannian manifolds with simple Jacobi operators. *J. Math. Soc. Japan* **54** (2002), 847–875.

[25] A. Borowiec, M. Francaviglia, and I. Volovich; Anti-Kählerian manifolds, *Differential Geom. Appl.* **12** (2000), 281–289.

[26] M. Brozos-Vázquez; Variedades Semi-Riemannianas con Tensor de Curvatura Especial, Publicaciones del Departamento de Geometría y Topología **100**, Universidade de Santiago de Compostela, 2003.

[27] M. Brozos-Vázquez, E. García-Río, and R. Vázquez-Lorenzo; Pointwise Osserman four-dimensional manifolds with local structure of twisted product, *Proceedings of the II International Meeting on Lorentzian Geometry*, Publ. RSME, Vol. XX (2004), 2–7.

[28] M. Brozos-Vázquez; Propiedades conformes de productos deformados, Publicaciones del Departamento de Geometría y Topología **105**, Universidade de Santiago de Compostela, 2004.

[29] M. Brozos-Vázquez, E. García-Río, and R. Vázquez-Lorenzo; Locally conformally flat multidimensional cosmological models and generalized Friedmann-Robertson-Walker spacetimes, *J. Cosmol. Astropart. Phys.* **12** (008) 1 (2004), 13pp.

[30] M. Brozos-Vázquez, E. García-Río, and R. Vázquez-Lorenzo; Some remarks on locally conformally flat static spacetimes, *J. Math. Phys.* **46** (2005), 11pp.

[31] M. Brozos-Vázquez, E. García-Río, and R. Vázquez-Lorenzo; Warped product metrics and locally conformally flat structures, *Mat. Contemp.*, **28** (2005), 91-110, Sociedade Brasileira de Matemática.

[32] M. Brozos-Vázquez, E. García-Río, and R. Vázquez-Lorenzo; Locally conformally flat multidimensional cosmological models with a higher-dimensional external spacetime, *Class. Quantum Grav.* **22** (2005) 3119-3133.

[33] M. Brozos-Vázquez, E. García–Río, and R. Vázquez-Lorenzo; Conformal Osserman four-dimensional manifolds whose conformal Jacobi operators have complex eigenvalues, *Proc. Royal Society A* **462** (2006), 1425–1441.

[34] M. Brozos-Vázquez, E. García-Río, and R. Vázquez-Lorenzo; Complete locally conformally flat manifolds of negative curvature, *Pacific J. Math.* **226** (2006), no. 2, 201–219.

[35] M. Brozos-Vázquez, and P. Gilkey; Manifolds with commuting Jacobi operators, *J. Geom.* **86** (2006), 21–30.

[36] M. Brozos-Vázquez, and P. Gilkey; Pseudo-Riemannian manifolds with commuting Jacobi operators, *Rend. Circ. Mat. Palermo* (2) **55** (2006), no. 2, 163–174.

[37] M. Brozos-Vázquez, and P. Gilkey; The global geometry of Riemannian manifolds with commuting curvature operators, *J. Fixed Point Theory and Appl.* **1** (2007), no. 1, 87–96.

[38] M. Brozos-Vázquez, P. Gilkey, and S. Nikčević; Jacobi-Tsankov manifolds which are not 2-step nilpotent, *Proceedings of the Conference Contemporary Geometry and Related Topics*, 63–79.

[39] M. Brozos-Vázquez, and P. Gilkey; Complex Osserman algebraic curvature tensors and Clifford families, *Houston J. Math.*, to appear.

[40] M. Brozos-Vázquez, E. García-Río, P. Gilkey, and R. Vázquez-Lorenzo; Completeness, Ricci blowup, the Osserman and the conformal Osserman condition for Walker signature (2,2) manifolds, *RSME*, *Procedings of XV International Workshop on Geometry and Physics*, to appear.

[41] M. Brozos-Vázquez, E. García-Río, and P. Gilkey; Relating the curvature tensor and the complex Jacobi operator of an almost Hermitian manifold, *Adv. Geom.*, to appear.

[42] M. Brozos-Vázquez, E. García-Río, P. Gilkey, and R. Vázquez-Lorenzo; Examples of signature $(2,2)$ manifolds with commuting curvature operators, *J. Phys. A: Math. Theor.* **40** (2007) 13149–13159.

[43] M. Brozos-Vázquez, E. García-Río, P. Gilkey, and R. Vázquez-Lorenzo; Compact Osserman manifolds with neutral signature, preprint.

[44] E. Calabi, and L. Markus; Relativistic space forms, *Ann. Math.* **75** (1962), 63–76.

[45] A. M. Candela, J. L. Flores, and M. Sánchez; On general plane fronted waves. Geodesics, *Gen. Rel. Grav.* **35** (2003), 631–649.

[46] G. Carron, and M. Herzlich, Conformally flat manifolds with nonnegative Ricci curvature, *Compos. Math.* **142** (2006), no. 3, 798–810.

[47] M. Chaichi, E. García-Río, and Y. Matsushita; Curvature properties of four-dimensional Walker metrics, *Classical Quantum Gravity* **22** (2005), 559–577.

[48] S. Chatterjee; A class of multidimensional cosmological models and its possible astrophysical consequences, *Ann. Phys.* **218** (1992), 121–138.

[49] B.-Y. Chen; Geometry of Submanifolds and its Applications, *Science University of Tokyo*, 1981.

[50] Q. M. Cheng; Compact locally conformally flat Riemannian manifolds, *Bull. London Math. Soc.* **33** (2001), 459–465.

[51] Q.-S. Chi; A curvature characterization of certain locally rank-one symmetric spaces, *J. Differential Geom.* **28** (1988), 187–202.

[52] Q.-S. Chi; Curvature characterization and classification of rank-one symmetric spaces, *Pacific J. Math.* **150** (1991), 31–42.

[53] A. Constantin, and B. Kolev; The theorem of Kerékjártó on periodic homeomorphisms of the disc and the sphere, *Enseign. Math.* (2) **40** (1994), no. 3-4, 193–204.

[54] L. A. Cordero, M. Fernández, and M. de León; Examples of compact non-Kähler almost Kähler manifolds, *Proc. Amer. Math. Soc.* **95** (1985), 280–286.

[55] M. Dajczer, and K. Nomizu; On the boundedness of Ricci curvature of an indefinite metric, *Bol. Soc. Brasil. Mat.* **11** (1980), no. 1, 25–30.

[56] A. Derdziński; Self-dual Kähler manifolds and Einstein manifolds of dimension four, *Compositio Mathematica* **49** (1983), 405–433.

[57] A. Derdziński; Einstein metrics in dimension four, *Handbook of Differential Geometry. Vol. I*, North-Holland, Amsterdam, (2000), 419–707.

[58] A. Derdziński; Curvature-homogeneous indefinite Einstein metrics in dimension four: the diagonalizable case, *Contemp. Math.* **337**, Amer. Math. Soc., Providence, RI (2003), 21–38.

[59] J. C. Díaz-Ramos, and E. García-Río; A note on the structure of algebraic curvature tensors, *Linear Alg. Appl.* **382** (2004), 271–277.

[60] J. C. Díaz-Ramos, E. García-Río, and R. Vázquez-Lorenzo; New examples of Osserman metrics with nondiagonalizable Jacobi operators, *Differential Geom. Appl.* **24** (2006), 433–442.

[61] J. C. Díaz-Ramos, E. García-Río, and R. Vázquez-Lorenzo; Four-dimensional Osserman metrics with nondiagonalizable Jacobi operators, *J. Geom. Anal.* **16** (2006), 39–52.

[62] M. Dunajski; Anti-self-dual four-manifolds with a parallel real spinor, *Proc. R. Soc. London A* **458** (2002), 1205–1222.

[63] C. Dunn, and P. Gilkey; Curvature homogeneous pseudo-Riemannian manifolds which are not locally homogeneous, *Complex, Contact and Symmetric Manifolds, Progr. Math. 234*, Birkhäuser Boston, Boston, MA, 2005, 145–152.

[64] C. Dunn, P. Gilkey, and S. Nikčević; Curvature homogeneous signature $(2,2)$ manifolds, *Differential Geom. Appl.*, Proc. Conf. Prague, August 30-September 3, 2004 Charles University, Prague (Czech Republic), editors Bures, Kowalski, Krupka, Slovak, MATFYZPRESS (2005), 29–44.

[65] A. Fabbri, D. Langlois, D. A. Steer, and R. Zegers; Brane cosmology with an anisotropic bulk, *J. High Energy Phys.* **9** (2004) 025, 20 p.

[66] J. C. Fabris, and J. Tossa; Qualitative study of multidimensional cosmological models, *Phys. Lett. B* **296** (1992), 307–310.

[67] M. Fernández López; *Resultados de descomposición asociados á ecuación de Möbius*, Publicaciones del Departamento de Geometría y Topología **96**, Universidade de Santiago de Compostela, 2002.

[68] M. Fernández López, E. García Río, D. N. Kupeli, and B. Ünal; A curvature condition for a twisted product to be a warped product, *Manuscripta Math.* **106** (2001), 213–217.

[69] A. V. Frolov; Kasner-AdS spacetime and anisotropic brane-world cosmology, *Phys. Lett. B* **514** (2001), 213–216.

[70] B. Fiedler; Determination of the structure of algebraic curvature tensors by means of Young symmetrizers, *Seminaire Lotharingien de Combinatorie* B48d (2003).

[71] B. Fiedler, and P. Gilkey; Nilpotent Szabó, Osserman and Ivanov–Petrova pseudo-Riemannian manifolds, *Contemp. Math.* **337**, Amer. Math. Soc. (2003), 53–64.

[72] T. Fukui, S. S. Seahra, and P. S. Wesson; Cosmological implications of a nonseparable 5D solution of the vacuum Einstein field equations, *J. Math. Phys.* **42** (2001), no. 11, 5195–5201.

[73] E. García-Río, and D. N. Kupeli; Null and infinitesimal isotropy in semi-Riemannian geometry', *J. Geom. Phys.* **13** (1994), 207–222.

[74] E. García-Río, and D. N. Kupeli; *Semi-Riemannian Maps and Their Applications*, Mahtematics and Its Applications. **475**, Kluwer Academic Publishers, Dordrecht-Boston-London, 1999.

[75] E. García-Río, D. N. Kupeli, and M. E. Vázquez-Abal; On a problem of Osserman in Lorentzian geometry, *Differential Geom. Appl.* **7** (1997), 85–100.

[76] E. García–Río, D. N. Kupeli, M. E. Vázquez-Abal, and R. Vázquez-Lorenzo, Osserman affine connections and their Riemannian extensions, *Differential Geom. Appl.* **11** (1999), 145–153.

[77] E. García-Río, D. N. Kupeli, and R. Vázquez-Lorenzo; *Osserman Manifolds in Semi-Riemannian Geometry*, Lectures Notes in Math., **1777**, Springer-Verlag, New York (2002).

[78] E. García-Río, Z. Rakić, and M. E. Vázquez-Abal; 4-dimensional indefinite Kähler Osserman manifolds, *J. Math. Phys.* **46** (2005), 11pp.

[79] A. Gebarowski; On nearly conformally symmetric warped product spacetimes, *Soochow J. Math.* **20** (1994), 61–75.

[80] A. Gebarowski; On conformally flat doubly warped products, *Soochow J. Math.* **21** (1995), 125–129.

[81] P. Gilkey; Manifolds whose curvature operator has constant eigenvalues at the basepoint, *J. Geom. Anal.* **4** (1994), 155–158.

[82] P. Gilkey; Algebraic curvature tensors which are p-Osserman, *Differential Geom. Appl.* **14** (2001), 297–311.

[83] P. Gilkey; Bundles over projective spaces and algebraic curvature tensors, *J. Geom* **71** (2001), 54–67.

[84] P. Gilkey; *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific (2001).

[85] P. Gilkey; *The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds*, Imperial College Press, ICP Advanced Texts in Mathematics, Vol. 2.

[86] P. Gilkey, and S. Nikčević; Complete k-curvature homogeneous pseudo-Riemannian manifolds, *Ann. Global Anal. Geom.* **27** (2005), 87–100.

[87] P. Gilkey, and S. Nikčević; Pseudo-Riemannian Jacobi-Videv manifolds, *Int. J. Geom. Meth. Modern Phys.* **4** (2007), 727–738.

[88] P. Gilkey, E. Puffini, and V. Videv; Puffini-Videv Models and Manifolds, preprint (math.DG/0605464).

[89] P. Gilkey, A. Swann, and L. Vanhecke; Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator, *Quart. J. Math. Oxford* **46** (1995), 299–320.

[90] A. Gray; Vector cross products on manifolds, *Trans. Amer. Math. Soc.* **141** (1969), 465–504.

[91] A. Gray; Curvature identities for Hermitian and almost Hermitian manifolds, *Tôhoku Math. J.* **28** (1976), 601–612.

[92] A. Gray, and L. M. Hervella; The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.* (4) **123** (1980), 35–58.

[93] U. Günther, P. Moniz, and A. Zhuk; Asymptotical AdS space from nonlinear gravitational models with stabilized extra dimensions, *Phys. Rev. D (3)* **66** (2002), no. 4, 044014, 9 pp.

[94] U. Günther, P. Moniz, and A. Zhuk; Nonlinear multidimensional cosmological models with form fields: stabilization of extra dimensions and the cosmological constant problem, *Phys. Rev. D (3)* **68** (2003), no. 4, 044010, 16 pp.

[95] U. Günther, A. Starobinsky, and A. Zhuk; Multidimensional cosmological models: Cosmological and astrophysical implications and constraints, *Phys. Rev. D (3)* **69** (2004), no. 4, 044003, 12 pp.

[96] U. Günther, and A. Zhuk; Gravitational excitons from extra dimensions, *Phys. Rev. D (3)* **56** (1997), no. 10, 6391–6402.

[97] U. Günther, and A. Zhuk; Observable effects from extra dimensions, *Gravit. Cosmol.* **5** (1999), 167–169.

[98] U. Günther, and A. Zhuk; Stabilization of internal spaces in multidimensional cosmology, *Phys. Rev. D (3)* **61** (2000), no. 12, 124001, 10 pp.

[99] S. Harris; A characterization of Robertson-Walker spaces by null sectional curvature, *Gen. Rel. Grav.* **17** (1985), 493–498.

[100] A. Harvey; Geodesics in Kasner universes, *Phys. Rev. D* **39** (1989), 673–676.

[101] S. T. Hong, J. Choi, and Y. J. Park; $(2+1)$ BTZ Black hole and multiply warped product spacetimes, *Gen. Relativity Gravitation* **35** (2003), 2105–2116.

[102] S. Ivanov, and S. Zamkovoy; Parahermitian and paraquaternionic manifolds, *Differential Geom. Appl.* **23** (2005), 205–234.

[103] M. Ivanova, and V. Videv; Four-dimensional Riemannian manifolds with commuting Stanilov curvature operators., *Mathematics and education in mathematics (Bulgarian)*, Union of Bulgarian Mathematicians, Sofia, 2004, 184–188.

[104] V. D. Ivashchuk, and V. N. Melnikov; Perfect-fluid type solution in multidimensional cosmology, *Phys. Lett. A* **136** (1989), 465–467.

[105] V. D. Ivashchuk, and V. N. Melnikov; Billiard representation for multidimensional cosmology with intersecting p-branes near the singularity, *J. Math. Phys.* **41** (2000), 6341–6363.

[106] V. D. Ivashchuk, V. N. Melnikov, and A. B. Selivanov; Cosmological solutions in multidimensional model with multiple exponential potential, *J. High Energy Phys.* **9** (2003) 059, 20 p.

[107] M. Kanai; On a differential equation characterizing a Riemannian structure of a manifold, *Tokyo J. Math.* **6** (1983), 143–151.

[108] U. Kasper, M. Rainer, and A. Zhuk; Integrable multicomponent perfect fluid multidimensional cosmology II: scalar fields, *Gen. Relativity Gravitation* **29** (1997), 1123–1162.

[109] I. M. Khalatnikov, and A. Yu. Kamenshchik; A generalisation of the Heckmann-Schucking cosmological solution, *Phys. Lett. B* **553** (2003), 119–125.

[110] K.-D. Kirchberg; Some integrability conditions for almost Kähler manifolds, *J. Geom. Phys.* **49** (2004), 101–115; see also Integrability Conditions For Almost Hermitian And Almost Kaehler 4-Manifolds; math.DG/0605611.

[111] L. Koch-Sen; Infinitesimal Null Isotropy and Robertson-Walker Metrics, *J. Math. Phys.* **26** (1985), 407–410.

[112] C. Kozameh, E. T. Newman, and P. Nurowski; Conformal Einstein equations and Cartan conformal connection, *Class. Quantum Grav.* **20**, 3029–3035 (2003).

[113] W. Kühnel; *Differential Geometry, Curves-Surfaces-Manifolds*, Student Mathematical Library **16**, American Mathematical Society, Providence, RI, 2002.

[114] W. Kühnel; Conformal transformations between Einstein spaces, Conformal geometry (Bonn, 1985/1986), 105–146, Aspects Math., **E12**, Vieweg, Braunschweig, 1988.

[115] W. Kühnel, and H.-B. Rademacher; Conformal diffeomorphisms preserving the Ricci tensor, *Proc. Amer. Math. Soc.* **123** (1995), 2841–2848.

[116] N. H. Kuiper; On conformally flat spaces in the large, *Ann. Math.* **50** (1949), 916–924.

[117] R. S. Kulkarni; Curvature structures and conformal transformations, *J. Differential Geom.* **54** (1969), 425–451.

[118] R. S. Kulkarni; The values of sectional curvature in indefinite metrics, *Comment. Math. Helv.* **54** (1979), no. 1, 173–176.

[119] J. Lafontaine; *Conformal geometry from the Riemannian viewpoint*, Conformal geometry (Bonn, 1985/1986), 65–92, Aspects Math., **E12**, Vieweg, Braunschweig, 1988.

[120] P. R. Law; Neutral Einstein metrics in four dimensions, *J. Math. Phys.* **32** (1991), 3039–3042.

[121] P. R. Law, and Y. Matsushita; Hitchin-Thorpe-type inequalities for pseudo-Riemannian 4-manifolds of metric signature $(++--)$, *Geom. Dedicata* **87** (2001), 65–89.

[122] J. M. Lee; *Riemannian manifolds. An introduction to curvature.* Graduate Texts in Mathematics, **176**. Springer-Verlag, New York, 1997.

[123] J. E. Lidsey, D. Wands, and E. J. Copeland; Superstring cosmology, *Phys. Rep.* **337** (2000), 343–492.

[124] J. Lohkamp; Metrics of negative Ricci curvature, *Ann. of Math. (2)* **140** (1994), no. 3, 655–683.

[125] D. Lorenz-Petzold; Higher-dimensional perfect fluid cosmologies, *Phys. Lett. B* **153** (1985), 134–136.

[126] N. Loureiro Santos, O. J. C. Dias, and J. P. S. Lemos; Global embedding of D-dimensional black holes with a cosmological constant in Minkowskian spacetimes: Matching between Hawking temperature and Unruh temperature, *Phys. Rev. D* (3) **70** (2004), no. 12, 124033, 7 pp.

[127] Y. Matsushita; Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds, *Math. Z.* **207** (1991), 281–291.

[128] Y. Matsushita; Walker 4-manifolds with proper almost complex structures, *J. Geom. Phys.* **52** (2004), 89–99.

[129] R. Mazzeo, and F. Pacard; Poincaré–Einstein metrics and the Schouten tensor, *Pacific J. Math.* **212**, 169–185 (2003).

[130] M. Meumertzheim, and H. Reckziegel; Decomposition of twisted and warped product nets, *Results Math.* **36** (1999), 297–312.

[131] S. Mignemi, and H.-J. Schmidt; Classification of multidimensional inflationary models, *J. Math. Phys.* **39** (1998), 998–1010.

[132] P. Nagy; On nearly-Kähler geometry, *Ann. Global Anal. Geom.* **22** (2002), 167–178.

[133] A. Newlander, and L. Nirenberg; Complex analytic coordinates in almost complex manifolds, *Ann. of Math.* **65** (1957), 391–404.

[134] Y. Nikolayevsky; Two theorems on Osserman manifolds, *Differential Geom. Appl.* **18** (2003), 239–253.

[135] Y. Nikolayevsky; Osserman manifolds of dimension 8, *Manuscr. Math.* **115** (2004), 31–53.

[136] Y. Nikolayevsky; Osserman Conjecture in dimension $n \neq 8, 16$, *Math. Ann.* **331** (2005), 505–522.

[137] M. Obata; Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Japan* **14**, 333–340 (1962).

[138] Y. Ogawa; On conformally flat spaces with warped product Riemannian metric, *Nat. Sci. Rep. Ochanomizu Univ.* **29**, 117–127 (1978).

[139] B. O'Neill; *Semi-Riemannian Geometry*, (New York: Academic Press) (1983).

[140] K. Nomizu; Conditions for constancy of the holomorphic sectional curvature, *J. Differential Geom.* **8** (1973), 335–339.

[141] T. Oguro, and K. Sekigawa; Notes on strictly almost Kähler Einstein manifolds of dimension four, *Yokohama Math. J.* **51** (2004), 19–27.

[142] Z. Olszak; On the existence of generalized complex space forms, *Israel J. Math.* **65** (1989), no. 2, 214–218.

[143] M. Ortega, J. de D. Pérez, and Y. J. Suh; Real hypersurfaces in quaternionic projective spaces with commuting tangent Jacobi operators, *Glasg. Math. J.* **45** (2003), no. 1, 79–89.

[144] B. Osgood, and D. Stowe; The Schwarzian Derivative and Conformal Mapping of Riemannian manifolds, *Duke Math. J.* **67** (1992), 57–99.

[145] R. Osserman; Curvature in the eighties, *Amer. Math. Monthly*, **97** (1990), 731–756.

[146] J. Petean; Indefinite Kähler-Einstein metrics on compact complex surfaces, *Comm. Math. Phys.* **189** (1997), 227–235.

[147] R. Ponge, and H. Reckziegel; Twisted products in pseudo-Riemannian geometry, *Geom. Dedicata* **48** (1993), 15–25.

[148] M. Rainer, and A. Zhuk; Tensor-multiscalar theories from multidimensional cosmology, *Phys. Rev. D (3)* **54** (1996), no. 10, 6186–6192.

[149] M. Rainer, and A. Zhuk; Einstein and Brans-Dicke frames in multidimensional cosmology, *Gen. Rel. Grav.* **32** (1) 79 (2000).

[150] Z. Rakić; On duality principle in Osserman manifolds, *Linear Algebra Appl.* **296** (1999), 183–189.

[151] J. G. Ratcliffe; *Foundations of Hyperbolic Manifolds* Graduate Texts in Mathematics **149**, Springer-Verlag, New York (1994).

[152] A. Romero, and M. Sánchez; New properties and examples of incomplete Lorentzian tori, *J. Math. Phys.* **35** (1994), 1992–1997.

[153] T. Sakai, *Riemannian Geommetry*, Providence, RI, 1996.

[154] T. Sato; On some almost Hermitian manifolds with constant holomorphic sectional curvature, *Kyungpook Math. J.* **29** (1989), 11–25.

[155] T. Sato; Almost Hermitian structures induced from a Kähler structure which has constant holomorphic sectional curvature, *Proc. Amer. Math. Soc.* **131** (2003), 2903–2909.

[156] R. Schoen, and S.-T. Yau; Conformally flat manifolds, Kleinian groups and scalar curvature, *Invent. Math.* **92** (1988), 47–71.

[157] S. S. Seahra, and P. S. Wesson; Universes encircling five-dimensional black holes, *J. Math. Phys.* **44** (12) 5664 (2003).

[158] K. Sekigawa; On some compact Einstein almost Kähler manifolds, *J. Math. Soc. Japan* **39** (1987), 677–684.

[159] K. Sekigawa, and L. Vanhecke; Four-Dimensional Almost Kähler Einstein Manifolds, *Ann. Mat. Pura Appl.* **CLVII** (IV) (1990), 149–160.

[160] G. Stanilov, and V. Videv; On a generalization of the Jacobi operator in the Riemannian geometry, *Annuaire Univ. Sofia Fac. Math. Inform.* **86** (1992), 27–34.

[161] G. Stanilov, and V. Videv; On the commuting of curvature operators., *Mathematics and education in mathematics (Bulgarian)*, 176–179, Union of Bulgarian Mathematicians, Sofia, 2004.

[162] M. Tani; *On a conformally flat Riemannian manifold*, Tohoku Math. J. **19** (1967), 205–214.

[163] Y. Tashiro; *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275.

[164] R. Tojeiro; *Conformal de Rham decomposition of Riemannian manifolds*, *Houston J. Math.* **32** (2006), 725–743.

[165] R. Tojeiro, *Conformal immersions of warped products into Euclidean space*, preprint.

[166] F. Tricerri, and L. Vanhecke; Curvature tensors on almost hermitian manifolds, *Trans. Amer. Math. Soc.* **267** (1981), 365–398.

[167] Y. Tsankov; A characterization of n-dimensional hypersurface in $\mathbb{R}^{n+1}$ with commuting curvature operators, PDEs, submanifolds and affine differential geometry, *Banach Center Publ.* **69**, Polish Acad. Sci., Warsaw, 2005, 205–209.

[168] L. Vanhecke; Some almost Hermitian manifolds with constant holomorphic sectional curvature, *J. Differential Geom.* **12** (1977), 461–471.

[169] A. G. Walker; Canonical form for a Riemannian space with a parallel field of null planes, *Quart. J. Math. Oxford* **(2) 1** (1950), 69–79.

[170] B. Watson; New examples of strictly almost Kähler manifolds, *Proc. Amer. Math. Soc.* **88** (1983), 541–544.

[171] J. A. Wolf; *Spaces of Constant Curvature* (Third Edition) (Boston, Mass.: Publish or Perish, Inc) 1974.

[172] H. Wu; On the de Rham decomposition theorem, *Illinois J. Math.* **8** (1964), 291-311.

[173] X. Xu; On the existence and uniqueness of solutions of Möbius equations, *Trans. Amer. Math. Soc.* **337** (1993), 927–945.

[174] K. Yano; *Differential geometry on complex and almost complex spaces*, Pergamon Press, New York (1965).

[175] S. T. Yau; Remarks on conformal transformations, *J. Differential Geom.* **8** (1973), 369–381.

[176] S. Zhu; The classification of complete locally conformally flat manifolds of nonnegative Ricci curvature, *Pacific J. Math.* **163** (1994), 189–199.