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THE GEOMETRY

OF

LORENTZIAN RICCI SOLITONS

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Memoria realizada no Departamento de Xeometría e Topoloxía da Facultade de Matemáticas, baixo a dirección dos profesores Eduardo García Río e Miguel Brozos Vázquez, para obter o grao de Doutora en Matemáticas pola Universidade de Santiago de Compostela.

Levouse a cabo a súa defensa o día 2 de Xullo de 2012 na Facultade de Matemáticas de dita Universidade, obtendo a calificación de Apto (Cum Laude).

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Resumo

As variedades de Einstein constituen un aspecto central no estudo da xeometría pseudo-Riemanniana. O seu interese non só reside na súa xeometría particular, senón que trascende ó campo da Física e en particular ó da cosmoloxía. A ecuación de Einstein, que describe tales variedades, pode ser xeneralizada dende diversos puntos de vista. Aquí adoptaremos un deles que se vincula a distintos campos da Xeometría Diferencial e das Matemáticas.

A investigación dos fenómenos de rixidez é un tema central en Xeometría pseudo-Riemanniana. Os resultados de rixidez poden aparecer a nivel métrico, como nos teoremas de descomposición, ou a nivel topolóxico, como sucede cos teoremas de compacidade e os resultados que involucran o primeiro grupo fundamental. Ademais, se a variedade está equipada cunha estructura adicional, esta pode impoñer novas restriccións sobre a variedade a calquera dos dous niveis.

Nesta tese consideránse variedades Lorentzianas dotadas dunha estrutura adicional dada por certas ecuacións diferenciais: as ecuacións de solitón de Ricci e de métrica quasi-Einstein. Tradicionalmente, no campo da Análise, discútese a existencia dunha solución non trivial dunha ecuación diferencial nun dominio determinado. Porén, en Xeometría tamén se adoita discutir a existencia dunha variedade que sexa un dominio axeitado para que a ecuación diferencial teña unha solución non trivial. Isto habitualmente proporciona un resultado de rixidez para a estrutura correspondente. En xeral, neste traballo adoptaremos un punto de vista local para o estudo destas ecuacións, pero tamén en certos casos faremos consideracións globais relacionadas principalmente coa completitude da variedade.

Os solitóns de Ricci e as métricas quasi-Einstein conxugan os dous aspectos que vimos de mencionar. Por un lado poden considerarse xeneralizacións das variedades de Einstein, pero ademais dan lugar a resultados de rixidez. A pesar de amosar certas similitudes, a motivación das condicións de solitón de Ricci e de métrica quasi-Einsten proveñen de problemas diferentes. O obxectivo final das distintas ecuacións de evolución xeométrica é producir (ou deducir a existencia de) variedades cun comportamento óptimo con respecto ós invariantes propostos: o fluxo de Ricci fai posible a construción de métricas de Einstein baixo certas condicións, mentres que o fluxo de curvatura media permite deformar certas superficies noutras nas que a curvatura media é constante. Porén, hai condicións baixo as cales a estrutura inicial non evoluciona baixo o fluxo senón que permanece como un punto fixo da mesma. Os solitóns de Ricci son os puntos fixos xeométricos (módulo homotecias e difeomorfismos) do fluxo de Ricci. Por outra banda, xa que aparecen como singularidades

do fluxo, a análise da súa xeometría é un paso importante de cara á comprensión do fluxo de Ricci. A ecuación das métricas quasi-Einstein preséntase como a ecuación natural asociada ó tensor de Ricci Lichnerowicz-Bakry-Emery, que é unha xeneralización da curvatura de Ricci a variedades con densidade. Ademais, a ecuación das métricas quasi-Einstein codifica información necesaria e suficiente para a construción de métricas de Einstein que son produtos warped.

O obxectivo deste traballo foi investigar os solitóns de Ricci e as métricas quasi-Einstein baixo certas condicións naturais sobre a curvatura. Centrámonos fundamentalmete na familia de solitóns chamados gradiente, isto é, aqueles nos que o campo de vectores é un campo de vectores gradiente. A análise desenvolvida descubriu algúns fenómenos de rixidez para este tipo de estruturas. En esencia, ámbalas dúas condicións proporcionan información sobre os conxuntos de nivel das correspondentes funcións potenciais e sobre a curvatura de Ricci da variedade subxacente. Polo tanto, centrámonos de forma natural nos espazos localmente conformemente chans, xa que neste caso o tensor de Ricci determina a curvatura, amosando unha rixidez local para as estruturas que se estudian. Cómpre destacar aquí que traballar no ámbito Lorentziano é menos ríxido que no campo Riemanniano, xa que a xeometría de Lorentz permite a existencia de hipersuperficies dexeneradas que poden aparecer como conxuntos de nivel das solucións das ecuacións diferenciais que se consideran.

Tamén prestamos especial atención á existencia de solucións para a ecuación do solitón de Ricci en variedades cun certo grao de homoxeneidade. Os nosos resultados son especialmente concluíntes no caso tridimensional, onde se dá unha descrición completa dos solitóns de Ricci gradiente homoxéneos, así coma no caso de variedades con grupo de isometrías grande, demostrando que todas elas admiten solitóns de Ricci expansivos, estables e contractivos.

O esquema xeral desta memoria é o seguinte. Preséntase un capítulo inicial de preliminares co propósito de establecer as principais definicións e algúns resultados básicos que se precisarán posteriormente. Estúdianse as métricas de Walker, prestando especial atención ás *pp-waves* e ás *plane waves* que aparecerán en diversas ocasións nos capítulos seguintes. Os solitóns de Ricci introdúcense como triples da forma (M, g, X), onde (M, g) é unha variedade pseudo-Riemanniana e X é un campo de vectores, de xeito que se satisfai a ecuación seguinte

$$\mathcal{L}_X g + \rho = \lambda\, g,$$

onde $\mathcal{L}$ denota a derivada de Lie, ρ é o tensor de Ricci e $\lambda \in \mathbb{R}$. Os solitóns de Ricci gradiente son triples (M, g, f) satisfacendo

$$\mathrm{Hes}_f + \rho = \lambda\, g,$$

pois corresponden ó caso especial en que o campo de vectores X é un campo gradiente $X = \frac{1}{2}\nabla f$ para unha certa función potencial f, e o solitón dise que é expansivo, estable e contractivo segundo sexa $\lambda < 0$, $\lambda = 0$ e $\lambda > 0$, respectivamente. Recórdanse algúns exemplos e resultados coñecidos sobre este tipo de estruturas no marco Riemanniano, que tamén motivan o noso traballo. Lévase a cabo un estudio detallado dos solitóns de Ricci gradiente en dimensión dous, dando lugar á construción do solitón cigarro Lorentziano na Sección 1.4.4. A continuación, na Sección 1.4.5, considérase a existencia de solitóns de Ricci gradiente que son variedades Einstein, amosando a existencia de exemplos non triviais sen homólogo Riemanniano. Todos estes exemplos están relacionados coa existencia de campos de vectores nulos paralelos.

Concluídos os preliminares, o corpo principal da tese divídese en dúas partes ben diferenciadas. A Parte I ocúpase da xeometría dos solitóns de Ricci gradiente. A continuación, a Parte II está dedicada ó estudio dos solitóns de Ricci xerais e das métricas quasi-Einstein, consideradas como xeneralizacións naturais da ecuación do solitón de Ricci gradiente.

A Parte I divídese en tres capítulos. O Capítulo 2 está destinado ó estudio de solitóns de Ricci gradiente localmente conformemente chans. A súa estrutura local vén dada polo seguinte:

Teorema 2.1 *Sexa (M, g, f) un solitón de Ricci gradiente localmente conformemente chan.*

(i) Nun entorno de calquera punto onde $\|\nabla f\| \neq 0$, M é localmente isométrica a un produto warped $I \times_\varphi N$ con métrica $\varepsilon\, dt^2 + \varphi^2 g_N$, onde I é un intervalo real e (N, g_N) é un espazo de curvatura seccional constante c.

(ii) Se $\|\nabla f\| = 0$ nun conxunto aberto non baleiro, entón (M, g) é localmente isométrica a unha plane wave, i.e., M é localmente difeomorfa a $\mathbb{R}^2 \times \mathbb{R}^n$ con métrica

$$g = 2\, dudv + H(u, x_1, \ldots, x_n) du^2 + \sum_{i=1}^{n} dx_i^2,$$

onde $H(u, x_1, \ldots, x_n) = a(u) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i(u) x_i + c(u)$ para funcións a, b_i, c arbitrarias e a función potencial vén dada por $f(u, v, x_1, \ldots, x_n) = f_0(u)$, verificando a condición $f_0''(u) = -\rho\left(\frac{\partial}{\partial_u}, \frac{\partial}{\partial_u}\right) = n\, a(u)$.

Unha das consecuencias de que unha variedade sexa localmente conformemente chá é que o tensor de Schouten é Codazzi, o que, como veremos, implica que ∇f é un autovector do operador de Ricci. Esta propiedade permítenos considerar bases locais adaptadas e investigar a xeometría dos conxuntos de nivel da función potencial f por medio do seu Hessiano, dando lugar a unha descomposición de produto deformado (warped) no caso non dexenerado e á estrutura dunha *pp-wave* no caso dexenerado.

Motivados polos resultados do Capítulo 2 e dado que as *pp-waves* xorden dun xeito natural neste contexto, no Capítulo 3 realizamos un estudo pormenorizado dos solitóns de Ricci, tanto gradientes como xerais, en variedades deste tipo. Estes resultados, ademais de complementar a clasificación obtida no capítulo anterior, teñen interese por si mesmos. Dentro da familia de *pp-waves*, as *plane waves* desempeñan un importante e destacado papel, polo que incidimos particularmente nesta subfamilia. Próbase que todas as *plane waves* son solitóns de Ricci gradiente estables ($\lambda = 0$).

Teorema 3.1 *Toda plane wave (M, g_{ppw}) é un solitón de Ricci gradiente estable non trivial con función potencial f dada por $f(u, v, x_1, \ldots, x_n) = f_0(u)$, onde*

$$f_0''(u) = -\,\rho\left(\frac{\partial}{\partial_u}, \frac{\partial}{\partial_u}\right) = \sum_{i=1}^{n} a_{ii}(u).$$

Ademais, $\|\nabla f\| \geq 0$ e ∇f é un campo de vectores xeodésico. Adicionalmente, o solitón de Ricci gradiente (M, g_{ppw}, f) é completo se $H(u, x_1, \ldots, x_n)$ está definida en todo $\mathbb{R}^{n+2}$.

A pesar da súa simplicidade, as *plane waves* son a estrutura subxacente a moitas situacións xeométricas Lorentzianas sen análogo Riemanniano. Por exemplo, as variedades Lorentzianas localmente conformemente chás con curvatura recurrente [55, 98], as variedades Lorentzianas dos-simétricas [1, 10], as variedades de Lorentz conformemente simétricas [47], as variedades de Lorentz que admiten unha estrutura homoxénea dexenerada de tipo lineal [3, 72], e as variedades de Lorentz con grupo de isometrías grande clasificadas por Patrangenaru [82] son algúns exemplos. De especial importancia son os espazos simétricos Cahen-Wallach:

Teorema 3.5 *Os espazos Lorentzianos simétricos indescompoñibles pero non irreducibles son solitóns de Ricci gradiente isotrópicos e estables.*

Os solitóns de Ricci gradiente homoxéneos estúdianse no Capítulo 4. Como feito técnico, amósase que a cada campo de vectores de Killing sobre a variedade pódeselle asociar un campo de vectores paralelo que induce unha descomposición local baixo certas circunstancias (cf. Lema 4.1). Este feito resulta ser moi restritivo en signatura Riemanniana, pero non o é tanto en signatura indefinida. Para realizar a análise deste tipo de variedades, obteremos en primeiro lugar algúns resultados xerais sobre solitóns de Ricci gradiente con curvatura escalar constante, para logo abordar o estudio das variedades homoxéneas. Este realizarase distinguindo os solitóns en función da constante λ. Así, primeiro analizaremos o caso $\lambda \neq 0$ e posteriormente o $\lambda = 0$.

Para solitóns de Ricci non estables obtemos unha descrición detallada da xeometría dos solitóns, obtendo a seguinte descomposición local

Teorema 4.7 *Sexa (M, g) unha variedade de Lorentz homoxénea. Entón se (M, g, f) é un solitón de Ricci gradiente non estable, a variedade (M, g) descomponse como un produto $M = N \times \mathbb{R}^k$ para algún $k \geq 0$, onde ou ben*

(a) (N, g_N) é unha variedade Lorentziana Einstein e o solitón é ríxido, ou

(b) (N, g_N) *é unha variedade Lorentziana de Walker admitindo un campo de vectores nulo paralelo.*

A xeometría dos solitóns de Ricci estables é en moitos casos máis flexible. Dado que a función potencial de calquera solitón de Ricci gradiente homoxéneo estable é unha solución da ecuación Eikonal, centrámonos nas distintas posibilidades segundo sexa ∇f temporal, nulo ou espacial. O caso en que é temporal é moi ríxido, e amósase no Teorema 4.10 que iso conleva que a variedade é chá. No caso nulo obtemos algúns resultados parciais que amosan que o operador de Ricci é ou ben nilpotente en tres pasos ou nilpotente en dous pasos; de darse esta última posibilidade a variedade admite un campo de vectores nulo paralelo (Lemas 4.11 e 4.13), polo que é unha variedade de Walker estrita. Por último, o problema de estudar o caso espacial permanece aberto e supón un reto para futuras investigacións. Como aplicación dos resultados anteriores, damos no Teorema 4.14 unha descrición de todos os solitóns de Ricci gradientes simétricos Lorentzianos.

En dimensión tres o problema é máis manexable dado que o tensor curvatura está completamente determinado polo tensor de Ricci. Na Sección 4.5 describimos completamente todos os solitóns de Ricci gradientes homoxéneos de dimensión tres. Esencialmente amósase que calquera destes solitóns é ou ben trivial, ou ben ríxido, ou ben a variedade subxacente ten curvatura recurrente (cf. Teorema 4.15). Existen ademais outras características propias desta dimensión que fan dela un caso especial. Así, por exemplo, as variedades de Walker estrictas en dimensión tres coinciden coas *pp-waves*, feito que non se dá en dimensións superiores.

Na Parte II da tese abórdase o caso xeral dos solitóns de Ricci, é dicir, estúdianse os solitóns de Ricci que veñen dados por un campo de vectores arbitrario (non necesariamente gradiente), e tamén o das variedades de Lorentz quasi-Einstein.

Dado que calquera variedade Lorentziana homoxénea é ou ben simétrica ou ben un grupo de Lie [27], analízase no Capítulo 5 a existencia de solitóns de Ricci invariantes á esquerda en grupos de Lie tridimensionais. Considéranse por separado os casos unimodular e non unimodular. Os grupos de Lie Lorentzianos unimodulares descríbense en termos do produto vectorial inducido polos para-cuaternios e certo operador autoadxunto L. Polo tanto, como diferencia coa signatura Riemanniana, un debe considerar non só a estrutura de autovalores, senón tamén a forma normal de Jordan de L, dando lugar a catro familias de álxebras de Lie unimodulares [89]. Ó considerar os grupos non unimodulares, analizamos o núcleo unimodular e as distintas posibilidades para as restricións da métrica: definida positiva, Lorentziana e dexenerada. Estas determinan tres familias de álxebras de Lie non unimodulares [42]. Acadamos o noso obxectivo probando a existencia de solitóns de Ricci en álxebras de Lie unimodulares e non unimodulares, ningún deles con análogo Riemanniano.

Teorema 5.2 *Sexa G un grupo de Lie Lorentziano de dimensión tres equipado con unha métrica invariante á esquerda. $(G, \langle\,\cdot\,,\,\cdot\,\rangle, X)$ é un solitón de Ricci invariante á esquerda se e só se se corresponde cun dos seguintes casos:*

(i) G é un grupo de Lie unimodular cunha das seguintes álxebras de Lie:

$$(i.1)\quad \begin{aligned}&[e_1,e_2]=\tfrac{1}{2}e_2-(\beta-\tfrac{1}{2})e_3,\\ &[e_1,e_3]=-(\beta+\tfrac{1}{2})e_2-\tfrac{1}{2}e_3,\\ &[e_2,e_3]=\alpha e_1,\end{aligned}$$

con $\alpha=0$ ou $\alpha=\beta\neq 0$. Se $\alpha=0$ entón $G=E(1,1)$, mentres que se $\alpha=\beta\neq 0$ entón $G=O(1,2)$ ou $G=SL(2,\mathbb{R})$.

$$(i.2)\quad \begin{aligned}&[e_1,e_2]=-\tfrac{1}{\sqrt{2}}e_1-\alpha e_3,\\ &[e_1,e_3]=-\tfrac{1}{\sqrt{2}}e_1-\alpha e_2,\\ &[e_2,e_3]=\alpha e_1+\tfrac{1}{\sqrt{2}}e_2-\tfrac{1}{\sqrt{2}}e_3.\end{aligned}$$

Se $\alpha=0$ entón $G=E(1,1)$, mentres que se $\alpha\neq 0$ entón ou $G=O(1,2)$ ou $G=SL(2,\mathbb{R})$.

(ii) G é un grupo de Lie non unimodular con álxebra de Lie dada por

$$\begin{aligned}[e_1,e_2]&=-\tfrac{1}{\sqrt{2}}\left(\alpha e_1+\tfrac{1}{\sqrt{2}}\beta(e_2+e_3)\right),\\ [e_1,e_3]&=\tfrac{1}{\sqrt{2}}\left(\alpha e_1+\tfrac{1}{\sqrt{2}}\beta(e_2+e_3)\right),\qquad \alpha+\delta\neq 0,\ \delta\neq 0,\\ [e_2,e_3]&=\tfrac{1}{\sqrt{2}}\delta(e_2+e_3).\end{aligned}$$

En todos os casos anteriores $\{e_1,e_2,e_3\}$ é unha base ortonormal de signatura $(++-)$ da correspondente álxebra de Lie.

Desde un punto de vista xeométrico, a existencia de solitóns de Ricci non triviais (i.e., asociados a métricas que non son Einstein) invariantes en grupos de Lie caracterízase polo seguinte:

Teorema 5.5 *Un grupo de Lie Lorentziano non simétrico de dimensión tres é un solitón de Ricci non trivial se e só se o operador de Ricci ten exactamente tres autovalores iguais.*

Obsérvase que a existencia dun único autovalor para o operador de Ricci non implica que a variedade sexa Einstein debido a que, en xeral, os operadores autoadxuntos no ámbito Lorentziano non son diagonalizables. Facemos fincapé en que os grupos de Lie correspondentes ó Teorema 5.2–(*ii*) permiten todas as clases (expansivos, estables e contractivos) de solitóns de Ricci. Esta situación, que non se dá para métricas definidas positivas, está estreitamente relacionada coa existencia de solitóns de Yamabe [31].

Finalmente, o Capítulo 6 dedícase ó estudio de variedades quasi-Einstein. Unha variedade de Lorentz (M,g) dise que é quasi-Einstein se existe unha función diferenciable f en M e unha constante $\mu\in\mathbb{R}$ tal que

$$\mathrm{Hes}_f+\rho-\mu\,df\otimes df=\lambda\,g$$

para unha constante $\lambda \in \mathbb{R}$. Claramente as variedades quasi-Einstein xeneralizan ás variedades de Einstein (obtidas para f constante) e ós solitóns de Ricci gradiente (obtidos para $\mu = 0$). As métricas quasi-Einstein aparecen como unha deformación conforme das métricas de Einstein e, reciprocamente, para calquera métrica quasi-Einstein con $\mu = -\frac{1}{n}$ a métrica conforme $\tilde{g} = e^{-\frac{2}{n}f} g$ é unha métrica de Einstein, sendo $\dim(M) = n + 2$. Ademais, a ecuación quasi-Einstein xorde dun xeito moi natural ó considerar o tensor de Ricci Lichnerowicz-Bakry-Emery, que vén dado pola expresión $\rho_f^m = \rho + \mathrm{Hes}_f - \frac{1}{m} df \otimes df$, onde $0 < m \leq \infty$. Este tensor defínese en variedades con densidade $(M, g, e^{-f} dvol_g)$ e a ecuación xorde ó impoñer a condición análoga á de Einstein, é dicir, que ρ_f^m sexa un múltiplo escalar da métrica g. Ademais, esta mesma condición está estreitamente relacionada coa busca de métricas de Einstein en produtos warped. En efecto, se $M \times_\varphi F$ é un produto warped Einstein, entón (M, g) é quasi-Einstein para $f = -(\dim(F)) \log(\varphi)$ e $\mu = \frac{1}{\dim(F)}$. Reciprocamente, se (M, g) é quasi-Einstein, entón para certas f e μ existen fibras axeitadas (F, g_F) tal que o produto warped $M \times_\varphi F$ é Einstein [66].

Coa motivación que outorgan os resultados do Capítulo 2, determínase a estrutura subxacente das métricas Lorentzianas quasi-Einstein localmente conformemente chás.

Teorema 6.2 *Sexa (M, g, f, μ) unha variedade quasi-Einstein localmente conformemente chá.*

(i) *Se $\mu = -\frac{1}{n}$, entón (M, g) é globalmente conformemente equivalente a un space form.*

(ii) *Se $\mu \neq -\frac{1}{n}$, entón*

  (a) *nunha veciñanza dun punto onde $\|\nabla f\| \neq 0$, M é localmente isométrica a un produto warped $I \times_\varphi F$, onde I é un intervalo real e F é unha fibra $(n+1)$-dimensional de curvatura seccional constante.*

  (b) *se $\|\nabla f\| = 0$, entón (M, g) é localmente isométrica a unha plane wave, i.e., (M, g) é localmente isométrica a $\mathbb{R}^2 \times \mathbb{R}^n$ con métrica*

$$g = 2\, du dv + H(u, x_1, \ldots, x_n) du^2 + \sum_{i=1}^{n} dx_i^2,$$

onde $H(u, x_1, \ldots, x_n) = a(u) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i(u) x_i + c(u)$, para funcións a, b_i, c arbitrarias e unha función $f(u, v, x_1, \ldots, x_n) = f_0(u)$ verificando a condición $f_0''(u) - \mu(f_0'(u))^2 - n\, a(u) = 0$.

O Teorema 6.2 amosa que as métricas quasi-Einstein localmente conformemente chás son ou ben localmente equivalentes a un *space form* (se $\mu = -\frac{1}{n}$), ou ben localmente un produto warped de tipo Robertson-Walker (sempre que os conxuntos de nivel da función potencial f sexan hipersuperficies non dexeneradas), ou ben localmente isométricas a certas *pp-waves* (cando os conxuntos de nivel da función potencial son dexenerados).

Aínda que este resultado se parece ó do Teorema 2.1, aparecen diferenzas importantes cando se considera a estrutura global. Mentres a existencia de solitóns de Ricci gradiente isotrópicos en *plane waves* está controlada por unha ecuación linear de segunda orde, o potencial quasi-Einstein está definido por unha ecuación diferencial de tipo Ricatti. Polo tanto hai que buscar solucións positivas de certa ecuación de segunda orde non linear (cf. Teorema 6.10). Aplicando técnicas estándar de Sturm-Liouville amósase que calquera *plane wave* homoxénea é tamén quasi-Einstein, aínda que existen *plane waves* que son solitóns de Ricci gradientes e que non admiten ningunha estrutura quasi-Einstein con $\mu \neq 0$.

Ó remate de cada unha das partes, Parte I e Parte II, plantexamos algúns problemas abertos que xorden como consecuencia dos resultados obtidos ou estreitamente relacionados con eles e que son merecentes dunha investigación futura.

Introduction

The investigation of rigidity phenomena is a central and broad topic in pseudo-Riemannian geometry. Rigidity results may appear at the metric level, like splitting theorems, or at the topological level, being compactness theorems or results involving the first fundamental group classical examples. Moreover, if the manifold is equipped with some additional structure, one analyzes its behavior as it often gives rise to restrictions at both levels.

In this thesis we consider Lorentzian manifolds equipped with an additional structure given by certain differential equations: the Ricci soliton and the quasi-Einstein equations. Traditionally, in Analysis, one is mostly interested in the existence of a nontrivial solution to a differential equation on a certain domain. However, from a more geometric point of view, one can also argue the existence of a domain manifold or structure for a differential equation to provide a nontrivial solution, and this leads to rigidity results for the corresponding structure.

Ricci solitons and quasi-Einstein metrics can be viewed as generalizations of Einstein manifolds, although the motivation for their study comes from different problems. The ultimate aim of the different geometric evolution equations is to produce (or deduce the existence of) manifolds with an optimal behavior with respect to given invariants: the Ricci flow makes it possible to construct Einstein metrics under certain conditions, whereas the mean curvature flow makes it possible to deform certain submanifolds into other ones whose mean curvature is constant. However, there are conditions under which the initial structure does not evolve under the flow but remains as a fixed point of it. Ricci solitons are the geometric fixed points (modulo homotheties and diffeomorphisms) of the Ricci flow. Moreover, since they appear as singular models for the flow, analyzing their geometry is an important step towards an understanding of the Ricci flow itself. The quasi-Einstein equation appears as a natural equation associated to the Lichnerowicz-Bakry-Emery Ricci tensor, which is a modified Ricci curvature on weighted manifolds. Furthermore, the quasi-Einstein equation encodes necessary and sufficient information to construct Einstein warped product metrics.

It has been our purpose in this work to investigate Ricci solitons and quasi-Einstein metrics under some natural curvature conditions and to show some rigidity phenomena for both structures. Within the family of Ricci solitons, we have paid special attention to gradient Ricci solitons. Essentially, both conditions, namely gradient Ricci solitons and quasi-Einstein, provide information about the level sets of the corresponding potential function and the Ricci curvature of the underlying manifold. Hence, one naturally focusses on locally conformally flat spaces, as in this case the Ricci tensor determines the curvature and one can show a local rigidity for the structures in question. It is worth emphasizing here that, in general, the Lorentzian setting is less rigid than its Riemannian analog. This is due to the fact that Lorentzian geometry allows degenerate hypersurfaces which may occur as level sets of the solutions of the differential equations under consideration.

Special attention is also paid to the existence of solutions for the Ricci soliton equation in manifolds with a certain degree of homogeneity. Our results are the most complete in the three-dimensional context, where we provide a complete description of homogeneous gradient Ricci solitons. Also, for manifolds with large isometry group, we show that they all support expanding, steady and shrinking Ricci solitons.

The general scheme of this memoir goes as follows.

A preliminary chapter, Chapter 1, is presented with the purpose of establishing the main definitions and some basic results on the subject that will be needed later on. Walker metrics are discussed, with special attention to *pp*-waves and plane waves. Ricci solitons are introduced as triples (M, g, X), where (M, g) is a pseudo-Riemannian manifold and X is a vector field so that the following equation is satisfied:

$$\mathcal{L}_X g + \rho = \lambda\, g,$$

where $\mathcal{L}$ denotes the Lie derivative, ρ is the Ricci tensor and $\lambda \in \mathbb{R}$. Gradient Ricci solitons are triples (M, g, f) obeying the equation

$$\mathrm{Hes}_f + \rho = \lambda\, g,$$

which corresponds to the special case of a gradient vector field $X = \frac{1}{2}\nabla f$ for some potential function f. The soliton is said to be expanding, steady or shrinking according to $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively. We recall some examples and known results from the Riemannian setting, which also motivate our study. A detailed study of two-dimensional gradient Ricci solitons is carried out, leading to the construction of the Lorentzian cigar soliton in Section 1.4.4. Then the existence of Einstein gradient Ricci solitons is considered in Section 1.4.5, showing the existence of non-trivial examples without Riemannian counterpart. All these examples are related to the existence of null parallel vector fields. The main body of the thesis is divided into two different parts. Part I deals with the geometry of gradient Ricci solitons, while Part II is devoted to the study of general Ricci solitons and quasi-Einstein metrics, as they are natural generalizations of gradient Ricci solitons.

Part I begins with Chapter 2, which is devoted to the study of locally conformally flat gradient Ricci solitons. Their local structure is given by the following theorem:

Theorem 2.1 *Let (M, g, f) be a locally conformally flat Lorentzian gradient Ricci soliton.*

(i) In a neighborhood of any point where $\|\nabla f\| \neq 0$, M is locally isometric to a Robertson-Walker warped product $I \times_\varphi N$ with metric $\varepsilon\, dt^2 + \varphi^2 g_N$, where I is a real interval and (N, g_N) is a space of constant sectional curvature c.

(ii) If $\|\nabla f\| = 0$ on a non-empty open set, then (M, g) is locally isometric to a plane wave, i.e., M is locally diffeomorphic to $\mathbb{R}^2 \times \mathbb{R}^n$ with metric

$$g = 2\, dudv + H(u, x_1, \ldots, x_n) du^2 + \sum_{i=1}^{n} dx_i^2,$$

where $H(u, x_1, \ldots, x_n) = a(u) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i(u) x_i + c(u)$ for some functions a, b_i, c and the potential function is given by $f(u, v, x_1, \ldots, x_n) = f_0(u)$, satisfying the condition $f_0''(u) = -\rho\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = n\, a(u)$.

One important consequence of the locally conformally flat condition is given by the fact that the Schouten tensor is Codazzi, which implies that ∇f is an eigenvector of the Ricci operator. This property allows us to specialize local frames and to investigate the geometry of the level sets of the potential function f by means of its Hessian, leading to a warped product decomposition in the non-degenerate case and to the structure of a *pp*-wave in the degenerate case.

Motivated by the special properties of *pp*-waves, we investigate the existence of Ricci solitons in *pp*-waves (with emphasis on plane waves) in Chapter 3. We show that all plane waves are steady gradient Ricci solitons.

Theorem 3.1 *Let (M, g_{ppw}) be a plane wave. Then it results in a nontrivial steady gradient Ricci soliton with potential function f given by $f(u, v, x_1, \ldots, x_n) = f_0(u)$, where*

$$f_0''(u) = -\rho\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \sum_{i=1}^{n} a_{ii}(u).$$

Moreover, $\|\nabla f\| \geq 0$ and ∇f is a geodesic vector field. The gradient Ricci soliton (M, g_{ppw}, f) is complete if $H(u, x_1, \ldots, x_n)$ is defined on the whole $\mathbb{R}^{n+2}$.

Besides their simplicity, plane waves are the underlying structure of many Lorentzian geometric situations without Riemannian analog. For instance, locally conformally flat Lorentzian manifolds with recurrent curvature [55, 98], two-symmetric Lorentzian manifolds [1, 10], conformally symmetric Lorentzian manifolds [47], Lorentzian manifolds admitting a degenerate homogeneous structure of linear type [3, 72], and the Lorentzian manifolds with large isometry group classified by Patrangenaru [82] are representative examples. Of special significance are Cahen-Wallach symmetric spaces:

Theorem 3.5 *Indecomposable but not irreducible Lorentzian symmetric spaces are isotropic steady gradient Ricci solitons.*

Homogeneous gradient Ricci solitons are studied in Chapter 4. As a technical fact, we show that given a Killing vector field on the manifold, one can associate a parallel vector field which induces a local splitting under some circumstances (cf. Lemma 4.1). In the non-steady case, one has a local decomposition.

Theorem 4.7 *Let (M, g) be a homogeneous Lorentzian manifold. If it is a non-steady gradient Ricci soliton, then it splits as a product $M = N \times \mathbb{R}^k$ for some $k \geq 0$, where either*

(a) (N, g_N) is a Lorentzian Einstein manifold and the soliton is rigid, or

(b) (N, g_N) is a Lorentzian Walker manifold admitting a parallel null vector field.

The steady case is somehow more flexible. Observing that the potential function of any steady homogeneous gradient Ricci soliton is a solution of the Eikonal equation (i.e., $\|\nabla f\| = \mu$ for some constant μ), we focus on the different possibilities of timelike, null and spacelike ∇f. The timelike case is very rigid, and we show in Theorem 4.10 that in such a case the manifold is flat. In the null case we obtain some partial results showing that the Ricci operator is either three-step nilpotent or two-step nilpotent, moreover in the later case the manifold admits a null parallel vector field (Lemmas 4.11 and 4.13). As an application of previous results, we give a description of all symmetric Lorentzian gradient Ricci solitons in Theorem 4.14.

The three-dimensional context is more tractable and we give in Section 4.5 a complete description of all three-dimensional homogeneous gradient Ricci solitons. Essentially we show that any such soliton is either trivial (i.e., Einstein), rigid (i.e., the product of an Einstein manifold and a Euclidean or Minkowskian factor), or the underlying manifold has recurrent curvature (cf. Theorem 4.15).

In the second part of this thesis, Part II, we consider general (not necessarily gradient) Ricci solitons and quasi-Einstein Lorentzian manifolds.

Since any complete and simply connected three-dimensional homogeneous Lorentzian manifold is either symmetric or a Lie group [27], we analyze the existence of invariant Ricci solitons on three-dimensional Lie groups in Chapter 5. We consider the unimodular and non-unimodular cases separately. Unimodular Lorentzian Lie groups are described in terms of the vector product induced by the unit para-quaternions and a self-adjoint operator L. Hence, as a difference with the Riemannian setting, one must consider not only the eigenvalue structure but also the Jordan normal form of L, giving rise to four families of unimodular Lie algebras [89]. When considering the non-unimodular case, one studies the unimodular kernel and the possibilities for the restriction of the metric (positive-definite, Lorentzian and degenerate) which determine three families of non-unimodular Lie algebras [42]. We show the existence of Ricci solitons on unimodular and non-unimodular Lie algebras, none of them with Riemannian analog.

Theorem 5.2 *Let G be a three-dimensional Lorentzian Lie group equipped with a left-invariant metric. $(G, \langle \cdot , \cdot \rangle, X)$ is a left-invariant Ricci soliton if and only if it corresponds to one of the following:*

(i) G is a unimodular Lie group with one of the following Lie algebras:

$$(i.1) \quad \begin{aligned} &[e_1, e_2] = \tfrac{1}{2} e_2 - (\beta - \tfrac{1}{2}) e_3, \\ &[e_1, e_3] = -(\beta + \tfrac{1}{2}) e_2 - \tfrac{1}{2} e_3, \\ &[e_2, e_3] = \alpha e_1, \end{aligned}$$

with either $\alpha = 0$ or $\alpha = \beta \neq 0$. If $\alpha = 0$ then $G = E(1,1)$, while if $\alpha = \beta \neq 0$ then $G = O(1,2)$ or $G = SL(2,\mathbb{R})$.

$$(i.2) \quad \begin{aligned} &[e_1, e_2] = -\tfrac{1}{\sqrt{2}} e_1 - \alpha e_3, \\ &[e_1, e_3] = -\tfrac{1}{\sqrt{2}} e_1 - \alpha e_2, \\ &[e_2, e_3] = \alpha e_1 + \tfrac{1}{\sqrt{2}} e_2 - \tfrac{1}{\sqrt{2}} e_3. \end{aligned}$$

If $\alpha = 0$ then $G = E(1,1)$, while if $\alpha \neq 0$ then either $G = O(1,2)$ or $G = SL(2,\mathbb{R})$.

(ii) G is a non-unimodular Lie group with Lie algebra given by

$$\begin{aligned} [e_1, e_2] &= -\tfrac{1}{\sqrt{2}} \left(\alpha e_1 + \tfrac{1}{\sqrt{2}} \beta (e_2 + e_3) \right), \\ [e_1, e_3] &= \tfrac{1}{\sqrt{2}} \left(\alpha e_1 + \tfrac{1}{\sqrt{2}} \beta (e_2 + e_3) \right), \qquad \alpha + \delta \neq 0, \ \delta \neq 0 \\ [e_2, e_3] &= \tfrac{1}{\sqrt{2}} \delta (e_2 + e_3). \end{aligned}$$

In all the cases above, $\{e_1, e_2, e_3\}$ is an orthonormal basis of signature $(++-)$ of the corresponding Lie algebra.

From a geometric point of view, the existence of invariant Ricci solitons on Lie groups is characterized by the following.

Theorem 5.5 *A non-symmetric three-dimensional Lorentzian Lie group is a non-trivial Ricci soliton if and only if the Ricci operator has exactly three equal eigenvalues.*

Observe that the existence of a single eigenvalue for the Ricci operator does not imply that the manifold is Einstein. We emphasize that Lie groups corresponding to Theorem 5.2–(*ii*) support all classes of (expanding, steady and shrinking) Ricci solitons. This situation, which is not possible in the Riemannian case, is closely related to the existence of Yamabe solitons [31].

Finally, Chapter 6 is devoted to the study of quasi-Einstein manifolds. A Lorentzian manifold (M, g) is said to be quasi-Einstein if there exists a smooth function f on M and a constant $\mu \in \mathbb{R}$ such that

$$\mathrm{Hes}_f + \rho - \mu \, df \otimes df = \lambda \, g$$

for a constant $\lambda \in \mathbb{R}$. Clearly, quasi-Einstein manifolds generalize Einstein manifolds (obtained whenever the function f is constant) and gradient Ricci solitons (which correspond to $\mu = 0$). Quasi-Einstein metrics appear as conformal deformations of Einstein metrics and, conversely: for any quasi-Einstein metric with $\mu = -\frac{1}{n}$ the conformal metric $\tilde{g} = e^{-\frac{2}{n}f}g$ is an Einstein metric where $\dim(M) = n + 2$. Moreover, their own interest comes from the consideration of the Lichnerowicz-Bakry-Emery Ricci tensor $\rho_f^m = \rho + \mathrm{Hes}_f - \frac{1}{m} df \otimes df$ in manifolds with density $(M, g, e^{-f} dvol_g)$, where $0 < m \leq \infty$. Furthermore, a search for warped product Einstein metrics leads to the quasi-Einstein condition which appears naturally in this context too. Indeed, if $M \times_\varphi F$ is an Einstein warped product, then (M, g) is quasi-Einstein for $f = -(\dim(F)) \log(\varphi)$ and $\mu = \frac{1}{\dim(F)}$ and, conversely, if (M, g) is quasi-Einstein, then there exist suitable fibers (F, g_F) such that the warped product manifold $M \times_\varphi F$ is Einstein [66].

Motivated by the results in Chapter 2, we determine the underlying structure of locally conformally flat Lorentzian quasi-Einstein metrics. Theorem 6.2 shows that locally conformally flat quasi-Einstein metrics are either locally equivalent to a space form (if $\mu = -\frac{1}{n}$), locally a warped product of Robertson-Walker type (whenever the level sets of the potential function f are non-degenerate hypersurfaces), or locally isometric to certain *pp*-waves (when the level sets of the potential function become degenerate). Although this result resembles that of Theorem 2.1, important differences appear when considering the global structure. While the existence of isotropic gradient Ricci solitons on plane waves is controlled by a linear second order equation, the quasi-Einstein potential is defined by a Ricatti-type differential equation. Hence one must search for positive solutions of a certain second order non-linear equation (cf. Theorem 6.10). Applying standard Sturm-Liouville techniques we show that any homogeneous plane-wave is also quasi-Einstein but there exist plane waves resulting in gradient Ricci solitons that do not admit a quasi-Einstein structure with $\mu \neq 0$.

Contents

Chapter 1

Preliminaries

In this chapter we introduce the main concepts we are going to work with in this memoir. In first section we recall some basic notions and objects from pseudo-Riemannian geometry and establish the corresponding notation. Among the families of metrics that we will find in this work, warped products play an important role, so we treat them in Section 1.2. Indecomposable but not irreducible Lorentzian manifolds are discussed in Section 1.3, with special attention to *pp*-waves. In Section 1.4 the notion of a Ricci soliton, which constitutes the core of this work, is introduced. Also, we recall some known examples and results from the Riemannian setting which motivate several questions discussed later on.

1.1 Basic notions

The context of this work is pseudo-Riemannian geometry. For this reason we start the first chapter by recalling the principal geometric objects of a pseudo-Riemannian manifold and fixing the notation that we will use in this memoir.

A *pseudo-Riemannian manifold* is a differentiable manifold of dimension $n+2$ equipped with a metric tensor g (i.e., a symmetric and non-degenerate $(0,2)$-tensor) of signature $(\nu, n+2-\nu)$. The pair (M,g) will denote a pseudo-Riemannian manifold of signature $(\nu, n+2-\nu)$. As a matter of terminology, we will say that a pseudo-Riemannian manifold is *Riemannian* if the metric is positive definite and *Lorentzian* if the signature is $(1, n+1)$. We will denote by T_pM the tangent space at the point $p \in M$ and by TM the tangent bundle of the manifold M.

We will consider $\mathfrak{X}(M)$ the space of all tangent vector fields of M. As a general rule, vector fields will be denoted by capital letters X, Y, Z, V, $W, \dots$ and the tangent vectors at a given point of the manifold by small letters x, y, z, v, $w, \dots$. If $(x_1, x_2, \dots)$ are local coordinates on M, then we denote partial derivatives by $\{\partial_{x_1} := \frac{\partial}{\partial x_1}, \partial_{x_2} := \frac{\partial}{\partial x_2}, \dots\}$. Following the usual notation in pseudo-Riemannian geometry, we will say that a non-zero vector $z \in T_pM$ is *timelike* if $g_p(z,z) < 0$, *spacelike* if $g_p(z,z) > 0$, and *null* if $g_p(z,z) = 0$ (in Lorentzian geometry null vectors are also called *lightlike*).

For a pseudo-Riemannian manifold (M, g), the *Levi-Civita connection* ∇ is the unique torsion-free connection which makes the metric g parallel, i.e., for X, $Y \in \mathfrak{X}(M)$ the connection ∇ satisfies

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0, \text{ and} \tag{1.1}$$

$$\nabla g = 0. \tag{1.2}$$

The *Koszul formula* gives the expression of such connection:

$$\begin{aligned} 2\, g(\nabla_X Y, Z) &= X g(Y, Z) + Y g(X, Z) - Z g(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

where X, Y, $Z \in \mathfrak{X}(M)$ and $[\cdot, \cdot]$ denotes the Lie bracket.

We also denote by ∇ the *gradient operator* in M. Note that the gradient of a smooth function $f : M \longrightarrow \mathbb{R}$ is the vector field ∇f determined by

$$g(\nabla f, X) = X(f), \ X \in \mathfrak{X}(M).$$

Let $(x_1, \ldots, x_{n+2})$ be a system of local coordinates in M. Then the gradient of a function is given in local coordinates as

$$\nabla f = g^{ik} \frac{\partial f}{\partial x_k} \frac{\partial}{\partial x_i}, \tag{1.3}$$

where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$, which expresses the metric tensor in local coordinates. We adopt the Einstein convention and sum over repeated indices.

Now we are going to define some differential operators that will play an important role along this memoir. Let $f : M \longrightarrow \mathbb{R}$ be a smooth function, we define the *Hessian operator*, hes_f, of the function f as

$$\mathrm{hes}_f(X) = \nabla_X \nabla f,$$

where $X \in \mathfrak{X}(M)$. Additionally the $(0, 2)$ symmetric tensor, Hes_f, defined as

$$\mathrm{Hes}_f(X, Y) = g\left(\mathrm{hes}_f(X), Y\right),$$

is called the *Hessian tensor* of f, where again X, $Y \in \mathfrak{X}(M)$. In local coordinates it expresses as

$$\mathrm{Hes}_f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{1}{2} g^{kl} \left(\frac{\partial g_{ij}}{\partial x_l} - \frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{li}}{\partial x_j}\right) \frac{\partial f}{\partial x_k}.$$

We define the *Laplacian* of f and the *divergence* of a vector field X as follows

$$\Delta f = \mathrm{tr}\,(\mathrm{hes}_f), \qquad \text{and} \qquad \mathrm{div}\,(X) = \mathrm{tr}\,(\nabla X),$$

respectively, where ∇X is the operator $Z \mapsto \nabla_Z X$, for all vector fields Z on M.

The *Lie derivative* of a function is the directional derivative of the function, so if f is a real function on M we have that $\mathcal{L}_X f = X(f) = \nabla_X f$. The *Lie derivative* of a vector field is the Lie bracket, therefore, if Y is a vector field $\mathcal{L}_X Y = [X, Y]$.

We extend the definition of Lie derivative to a tensor field T as follows

$$(\mathcal{L}_X T)_p = \frac{d}{dt}\,|_{t=0}\,((\psi(t))^* T)_p,$$

where $\psi : I \times M \subset \mathbb{R} \times M \to M$ is the local flow induced by X and $(\psi(t))^*$ is the pullback along the diffeomorphism $\psi(t)$ for all $t \in I$.

As a particular case, the Lie derivative of the metric is defined as a usual derivation on a $(0, 2)$-tensor by

$$(\mathcal{L}_X g)(Y, Z) = Xg(Y, Z) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z),$$

which, by (1.1) and (1.2), can also be written as

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y).$$

A vector field X in M is said to be *Killing* if the Lie derivative of the metric g with respect to X vanishes identically $\mathcal{L}_X g = 0$ or equivalently if the local flow of X is performed by isometries. A vector field X is said to be *conformal* if it satisfies

$$\mathcal{L}_X g = \phi\, g,$$

for some function ϕ. Equivalently, its local flow consists of conformal transformations. Having in mind that $\operatorname{tr}(\mathcal{L}_X g) = 2\operatorname{div}(X)$, we get that a vector field X is conformal if and only if $\mathcal{L}_X g = \frac{2}{n+2}\operatorname{div}(X)\, g$. Whenever $\operatorname{div}(X)$ is constant, a conformal vector field is called *homothetic*, i.e., its local flow is performed by homotheties.

Using the Levi-Civita connection we define the $(1, 3)$-curvature tensor by

$$R(X, Y)Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y]Z,$$

and the $(0, 4)$-curvature tensor is given by $R(X, Y, Z, V) = g(R(X, Y)Z, V)$.

The curvature tensor satisfies the following algebraic symmetries:

$$\begin{aligned} &(a) \quad R(X,Y,Z,V) &=& -R(Y,X,Z,V) = -R(X,Y,V,Z), \\ &(b) \quad R(X,Y,Z,V) &+& R(Y,Z,X,V) + R(Z,X,Y,V) = 0, \\ &(c) \quad R(X,Y,Z,V) &=& R(Z,V,X,Y), \end{aligned} \tag{1.4}$$

and the differential identity:

$$(d) \quad (\nabla_X R)(Y, Z, U, V) + (\nabla_Y R)(Z, X, U, V) + (\nabla_Z R)(X, Y, U, V) = 0. \tag{1.5}$$

We will refer to the identities (b) and (d) as the *first* and the *second Bianchi identity*, respectively.

The *sectional curvature* of a Riemannian manifold (M, g) is the real function K defined on the Grassmannian of two-planes as

$$K(\pi) := \frac{R(x,y,x,y)}{g(x,x)g(y,y) - g(x,y)^2},$$

for every two-plane $\pi = \mathrm{span}\{x, y\}$ in T_pM. In the pseudo-Riemannian setting, the previous definition should be restricted to the Grassmannian of non-degenerate two-planes (i.e., those planes $\pi = \mathrm{span}\{x, y\}$ satisfying $g(x,x)g(y,y) - g(x,y)^2 \neq 0$). The possibility of continuously extending K to the whole Grassmannian is indeed equivalent to the constancy of K [43].

In a purely algebraic context, a $(0,4)$-tensor satisfying the identities in equation (1.4) is called an *algebraic curvature tensor*. A standard procedure to build an algebraic curvature tensor from two symmetric bilinear forms D and B is provided by the *Kulkarni-Nomizu product*, which is defined by

$$\begin{aligned}(D \odot B)(X,Y,Z,V) \;&=\; D(X,Z)B(Y,V) + D(Y,V)B(X,Z)\\ &\quad -D(X,V)B(Y,Z) - D(Y,Z)B(X,V).\end{aligned}$$

Now, it is well-known that a pseudo-Riemannian manifold (M, g) has constant sectional curvature c if and only if the curvature tensor is determined by the metric tensor as $R = \frac{c}{2}\, g \odot g$.

The *Ricci tensor* ρ is defined as the trace of the curvature tensor as follows

$$\rho(x,y) = \mathrm{tr}\{z \mapsto R(x,z)y\}.$$

Sometimes we will work with the Ricci operator Ric which is defined from the Ricci tensor by $g(\mathrm{Ric}(X), Y) = \rho(X, Y)$. Now, the *scalar curvature* τ is defined as follows

$$\tau = \mathrm{tr}\,(\mathrm{Ric})\,.$$

In a system of local coordinates $(x_1, \ldots, x_{n+2})$, the Ricci tensor and the scalar curvature are given by

$$\rho(x,y) = \sum_{i,j=1}^{n+2} g^{ij} R(x, \partial_{x_i}, y, \partial_{x_j}), \text{ and } \qquad \tau = \sum_{i,j=1}^{n+2} g^{ij} \rho(\partial_{x_i}, \partial_{x_j}).$$

Remark 1.1 Since the Ricci tensor is symmetric, the Ricci operator is self-adjoint and hence diagonalizable in Riemannian signature. Therefore the curvature tensor of any three-dimensional Riemannian manifold is determined by the eigenvalues of the Ricci operator. The Lorentzian setting, however, requires a more detailed analysis since a self-adjoint operator may have non-trivial Jordan normal form.

For the purpose of subsequent chapters, recall that a self-adjoint operator L on a three-dimensional Lorentzian vector space corresponds to one of the following

(I_a) The operator L is diagonalizable, i.e.,

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

(I_b) The operator L has complex eigenvalues, i.e.,

$$L = \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

where $\beta \neq 0$.

(II) There is a double root of the minimal polinomial of L, i.e.,

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 1 & \beta \end{pmatrix}.$$

(III) There is a triple root of the minimal polinomial of L, i.e.,

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 1 & \alpha \end{pmatrix}.$$

Tracing (1.5) appropriately, one gets the so-called *contracted second Bianchi identity*

$$\nabla_X \tau = 2\,\mathrm{div}(\mathrm{Ric}(X)). \tag{1.6}$$

Also, if f is a function on M, the *Bochner formula* relates differential operators of the function with the geometry of the manifold as follows

$$\mathrm{div}(\mathrm{Hes}_f)(X) = \rho(\nabla f, X) + g(\nabla \Delta f, X). \tag{1.7}$$

A pseudo-Riemannian manifold (M, g) is said to be *Einstein* if its Ricci tensor is a scalar multiple of the metric. In such a case one has that $\rho = \frac{\tau}{n+2}\, g$.

The *Schouten tensor* of a pseudo-Riemannian manifold of dimension $n+2$ is defined by

$$C = \frac{1}{n}\left(\rho - \frac{\tau}{2(n+1)}\, g\right). \tag{1.8}$$

Note that the curvature tensor of any two-dimensional manifold is determined by the metric as $R = \frac{\tau}{2}\, g \odot g$, where $\frac{\tau}{2}$ corresponds to the Gaussian curvature of the surface. Also, it is well-known that the curvature tensor of any three-dimensional pseudo-Riemannian manifold is completely determined by its Ricci tensor, so it has the form $R = C \odot g$.

The geometric meaning of the Schouten tensor appears in the study of conformal geometry. Define the *Weyl tensor* of a pseudo-Riemannian manifold as $W = R - C \odot g$, hence

$$
\begin{aligned}
W(X,Y,Z,V) = R(X,Y,Z,V) &+ \frac{\tau}{n(n+1)}\left\{g(X,Z)g(Y,V) - g(Y,Z)g(X,V)\right\} \\
&- \frac{1}{n}\left\{\rho(X,Z)g(Y,V) - \rho(Y,Z)g(X,V)\right. \\
&\qquad \left. +\rho(Y,V)g(X,Z) - \rho(X,V)g(Y,Z)\right\},
\end{aligned}
\tag{1.9}
$$

for X, Y, Z, $V \in \mathfrak{X}(M)$.

A pseudo-Riemannian manifold (M, g) is said to be *locally conformally flat* if for each point $p \in M$ there exists an open neighborhood U and a positive function $e^\sigma : U \to \mathbb{R}$ such that $g = e^\sigma g_0$, where g_0 is a pseudo-Euclidean metric on $\mathbb{E}^{n+2}$.

Locally conformally flat manifolds of dimension three are characterized by the fact that the Schouten tensor is *Codazzi*, that is $(\nabla_X C)(Y,Z) = (\nabla_Y C)(X,Z)$, while higher dimensional locally conformally flat manifolds are characterized by the vanishing of the corresponding Weyl tensor. Further more, observe that the Schouten tensor of any locally conformally flat manifold is a Codazzi tensor, a fact that will be used in subsequent chapters.

Remark 1.2 The action of the orthogonal group decomposes the space of algebraic curvature tensors into three irreducible modules, except in dimension four that there are four irreducible modules. Thus, the curvature tensor R of any pseudo-Riemannian manifold can be written as

$$R = \frac{\tau}{2(n+2)(n+1)}\, g \odot g + \frac{1}{n}\, \rho_0 \odot g + W,$$

where ρ_0 is the traceless Ricci tensor: $\rho_0 = \rho - \dfrac{\tau}{n+2}\, g$.

1.2 Warped products

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and $\varphi : B \to \mathbb{R}^+$ a positive function on B. The *warped product* $M = B \times_\varphi F$ is the product manifold $M = B \times F$ equipped with the metric

$$g = \pi^*(g_B) + (\varphi \circ \pi)^2 \sigma^*(g_F),$$

where π denotes the canonical projection of M in B and σ is the canonical projection of M on F. The function φ is called the *warping function* of the warped product, B is called the *base* and F the *fiber*.

The relation of a warped product to the base B is almost as simple as in the special case of a pseudo-Riemannian product; however, the relation to the fiber F often involves the warping function φ. To approach the geometry of warped products, we begin by seeing the expression of its Levi-Civita connection with respect to vectors tangent to B and F.

Lemma 1.3 [81] *Let $M = B \times_\varphi F$ be a warped product manifold. Let X, $Y \in \mathfrak{X}(B)$ and U, $V \in \mathfrak{X}(F)$. The Levi-Civita connection of M is given by:*

1. $\nabla_X Y$ *is the lift to* $B \times F$ *of* $\nabla^B_X Y$,
2. $\nabla_X U = \nabla_U X = \frac{X(\varphi)}{\varphi} U$,
3. $\mathrm{nor}(\nabla_U V) = \mathrm{II}(U,V) = -\frac{g(U,V)}{\varphi} \nabla\varphi$,
4. $\tan(\nabla_U V)$ *is the lift to* $B \times F$ *of* $\nabla^F_U V$,

where ∇^B and ∇^F denote the Levi-Civita connections of (B, g_B) and (F, g_F), respectively, and II *represents the second fundamental form of the fibers.*

Lemma 1.4 [81] *Let $M = B \times_\varphi F$ be a warped product. Let X, Y, $Z \in \mathfrak{X}(B)$ and U, V, $W \in \mathfrak{X}(F)$. The curvature tensor R of M is determined by:*

1. $R(X,Y)Z$ *is the lift to* $B \times F$ *of* $R^B(X,Y)Z$ *on* B,
2. $R(U,X)Y = \frac{\mathrm{Hes}_\varphi(X,Y)}{\varphi}$,
3. $R(X,Y)U = R(U,V)X = 0$,
4. $R(X,U)V = \frac{g(U,V)}{\varphi} \nabla_X(\nabla\varphi)$,
5. $R(U,V)W = R^F(U,V)W - \frac{g(\nabla\varphi,\nabla\varphi)}{\varphi^2} (g(U,W)V - g(V,W)U)$,

where R^B and R^F are the curvature tensors of the base B and the fiber F, respectively.

Lemma 1.5 [81] *Let $M = B \times_\varphi F$ be a warped product with $d = \dim(F) > 1$. Let X, $Y \in \mathfrak{X}(B)$ and U, $V \in \mathfrak{X}(F)$. Then the Ricci tensor of M is given by:*

1. $\rho(X,Y) = \rho^B(X,Y) - \frac{d}{\varphi} \mathrm{Hes}_\varphi(X,Y)$,
2. $\rho(X,U) = 0$,
3. $\rho(U,V) = \rho^F(U,V) - g(U,V) \left(\frac{\Delta\varphi}{\varphi} + (d-1) \frac{g(\nabla\varphi,\nabla\varphi)}{\varphi^2} \right)$,

where ρ^B and ρ^F are the Ricci tensors of B and F, respectively.

Remark 1.6 Let $I \times_\varphi F$ be a warped product with metric $\varepsilon\, dt^2 \oplus \varphi^2 g_F$, where $\varepsilon = \pm 1$ and $d = \dim(F)$. We specialize the expressions of Lemmas 1.3 and 1.5 for a warped product with one-dimensional base as follows. Let $U, V \in \mathfrak{X}(F)$, then the Levi-Civita connection is given by

1. $\nabla_{\partial t}\partial t = 0$,
2. $\nabla_{\partial t} U = \nabla_U \partial t = \frac{\varphi'}{\varphi}\, U$,
3. $\mathrm{nor}(\nabla_U V) = \mathrm{II}(U,V) = -\varepsilon\, \frac{\varphi'}{\varphi}\, g(U,V)\partial t$,
4. $\tan(\nabla_U V)$ is the lift of $\nabla^F_U V$,

whereas the Ricci tensor is given by:

$$\begin{aligned} \rho(\partial t, \partial t) &= -\varepsilon\, d\, \frac{\varphi''}{\varphi}, \qquad \rho(\partial t, U) = 0, \\ \rho(U,V) &= \rho^F(U,V) - \varepsilon \left(\frac{\varphi''}{\varphi} + (d-1)\left(\frac{\varphi'}{\varphi}\right)^2 \right) g(U,V). \end{aligned} \tag{1.10}$$

An important result related with completeness of warped products with one-dimensional base, that we will need in our subsequent study is as follows:

Theorem 1.7 [33] *Let $M = I \times_\varphi F$ be a warped product where $I = (\alpha, \beta)$ is a (possibly unbounded) real interval and (F, g_F) is a geodesically complete manifold. Then M is timelike, spacelike and null geodesically complete if and only if for some $\gamma \in (\alpha, \beta)$ one has that*

$$\int_\alpha^\gamma \frac{\varphi}{\sqrt{1+\varphi^2}} dt = \int_\gamma^\beta \frac{\varphi}{\sqrt{1+\varphi^2}} dt = +\infty.$$

We turn our attention back to general warped products. Using the fact that any warped product metric is in the conformal class of a direct product metric, a characterization of locally conformally flat warped products was given in [22] as follows

Theorem 1.8 [22] *Let $M = B \times_\varphi F$ be a pseudo-Riemannian warped product. Then the following holds:*

(i) *If $\dim(B) = 1$, then $M = B \times_\varphi F$ is locally conformally flat if and only if (F, g_F) has constant sectional curvature.*

(ii) *If $\dim(B) > 1$ and $\dim(F) > 1$, then $M = B \times_\varphi F$ is locally conformally flat if and only if*

  (ii.a) *(F, g_F) has constant sectional curvature c_F.*

  (ii.b) *The function $\varphi : B \to \mathbb{R}^+$ defines a global conformal deformation on B such that $\left(B, \frac{1}{\varphi^2}\, g_B\right)$ has constant sectional curvature $\tilde{c}_B = -c_F$.*

(iii) *If* $\dim(F) = 1$, *then* $M = B \times_\varphi F$ *is locally conformally flat if and only if the function* $\varphi : B \to \mathbb{R}^+$ *defines a conformal deformation on* B *such that* $\left(B, \frac{1}{\varphi^2} g_B\right)$ *has constant sectional curvature.*

A generalization of the warped product metrics previously considered is given by the *twisted products*. This is defined formally in the same way than the warped structure $g = \pi^*(g_B) + (\varphi \circ \pi)^2 \sigma^*(g_F)$ but the twisting function $\varphi : M \to \mathbb{R}$, i.e. the function φ is defined on M instead of B.

The different possible metric structures we have defined on a direct product manifold address several differences on the geometry of the canonical foliations of the product and vice versa. Thus, for $M = B \times F$, let $\mathcal{L}_B$ denote the canonical horizontal foliation and $\mathcal{L}_F$ the canonical vertical foliation. One has the following

Theorem 1.9 [88] *Let* $M = B \times F$ *be a pseudo-Riemannian manifold such that the canonical foliations* $\mathcal{L}_B$ *and* $\mathcal{L}_F$ *intersect orthogonally with respect to a metric* g. *Then*

1. (M, g) *is a direct product if and only if* $\mathcal{L}_B$ *and* $\mathcal{L}_F$ *are totally geodesic.*

2. (M, g) *is a warped product if and only if* $\mathcal{L}_B$ *is totally geodesic and* $\mathcal{L}_F$ *is spherical.*

3. (M, g) *is a twisted product if and only if* $\mathcal{L}_B$ *is totally geodesic and* $\mathcal{L}_F$ *is totally umbilic.*

Although more general a priori, under certain curvature conditions the twisted structure reduces to the warped one.

Theorem 1.10 [53] *Let* $B \times_\varphi F$ *be a twisted product with* $\dim(F) > 1$. *If* $\rho(X, U) = 0$ *for all* X, U *with* X *tangent to* B *and* U *tangent to* F, *then* $B \times_\varphi F$ *can be expressed as a warped product* $B \times_{\bar{\varphi}} F$ *of* (B, g_B) *and* $(F, \bar{g}_F)$, *where* $\bar{g}_F$ *is a metric conformally equivalent to* g_F.

1.3 Holonomy and Walker metrics

Let $\gamma : [a, b] \to M$ be a smooth curve and $v \in T_{\gamma(a)}M$, then the parallel transport of v along γ is given by solving the following equation

$$\nabla_{\gamma(t)} v(t) = 0.$$

Let γ be a closed curve, i.e., $\gamma(a) = \gamma(b) = p \in M$. The map $v(a) \to v(b)$ given by parallel transport around γ defines an isomorphism $L_\gamma : T_pM \to T_pM$. The set of all such linear maps forms a group called the *holonomy group* of the connection. Holonomy groups corresponding to different points in a connected manifold are all isomorphic and thus the role of the basepoint p is usually suppressed. The holonomy group is a closed subgroup of $GL(T_pM)$ and therefore is a Lie group. When one uses the Levi-Civita connection of a pseudo-Riemannian metric, the holonomy group is a subgroup of the orthogonal group, since parallel transport is realized by isometries, and we refer to it as the holonomy group of the pseudo-Riemannian manifold.

For a Riemannian metric the holonomy group acts completely reducibly, i.e., the tangent space decomposes into subspaces which are invariant under its action and where it acts trivially or irreducibly, but for indefinite metrics the situation is more subtle. One says that the holonomy group acts indecomposably if the metric is degenerate on any invariant proper subspace. If this happens we also say that the manifold is *indecomposable*. Of course, for Riemannian manifolds, indecomposability is equivalent to irreducibility.

A remarkable property is that the holonomy group of a product of Riemannian manifolds (i.e., equipped with the product metric) is the product of the holonomy groups of these manifolds (with the corresponding representation on the direct sum). Furthermore, a converse of this statement is true in the following sense: let M be a connected pseudo-Riemannian manifold whose tangent space at a single point (and hence at every point) admits an orthogonal direct sum decomposition into non-degenerate subspaces which are invariant under the holonomy representation, then M is locally isometric to a product of pseudo-Riemannian manifolds corresponding to the invariant subspaces. Moreover, the holonomy group is the product of the groups acting on the corresponding invariant subspaces. A global version of this statement under the assumption that the manifold is simply connected and complete was proven by G. de Rham [90] for Riemannian manifolds and by H. Wu [99] in arbitrary signature.

Theorem 1.11 [99] *Any simply connected complete pseudo-Riemannian manifold M is isometric to a product of simply connected complete pseudo-Riemannian manifolds one of which can be flat and the others have an indecomposably acting holonomy group. Moreover, the holonomy group of (M, g) is the product of these indecomposably acting holonomy groups.*

For indefinite metrics there exists the possibility that one of the factors in the previous theorem is indecomposable, but not irreducible. This means that the holonomy representation admits an invariant subspace on which the metric is degenerate, but no proper non-degenerate invariant subspaces. An attempt to classify holonomy groups for indefinite metrics has to provide a classification of these indecomposable, not irreducible, holonomy groups. If a holonomy group acts indecomposably, but not irreducibly, with a degenerate invariant subspace $V \subset T_pM$, it admits a totally isotropic invariant subspace $S := V \cap V^{\perp}$.

1.3.1 Walker coordinates

Walker [97] studied pseudo-Riemannian manifolds (M, g) with a parallel field of null planes $\mathcal{D}$ and derived a canonical form for their metric in adapted coordinates. An r-dimensional distribution $\mathcal{D}$ on a manifold (i.e., a section of the Grassmann bundle $Gr_k(TM)$) is said to be *parallel* if $\nabla_X \mathcal{D} \subset \mathcal{D}$, i.e., if $\nabla_X Y \in \mathcal{D}$ for all $Y \in \mathcal{D}$ and any $X \in \mathfrak{X}(M)$. Motivated by this seminal work of A. G. Walker, a pseudo-Riemannian manifold (M, g) which admits a parallel null distribution $\mathcal{D}$ is said to be a *Walker manifold*. Walker manifolds constitute the underlying structure of many strictly pseudo-Riemannian situations with no Riemannian counterpart: indecomposable but not irreducible manifolds, Einstein hypersurfaces with nilpotent shape operators, some classes of non-symmetric Osserman metrics and para-Kähler manifolds are typical examples. We refer to [21] for more information and examples on Walker manifolds.

Canonical forms were known previously for parallel non-degenerate distributions. In this case, the metric tensor, in matrix notation, expresses in canonical form as

$$(g_{ij}) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A is a symmetric $r \times r$ matrix whose coefficients are functions of $(x_1, \dots, x_r)$ and B is a symmetric $(n+2-r) \times (n+2-r)$ matrix whose coefficients are functions of $(x_{r+1}, \dots, x_{n+2})$. In the case of a parallel null distribution one has

Theorem 1.12 [97] *A canonical form for an $(n+2)$-dimensional pseudo-Riemannian manifold M admitting a parallel field of null r-dimensional planes $\mathcal{D}$ is given by the metric tensor in matrix form as*

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & Id_r \\ 0 & A & H \\ Id_r & H^t & B \end{pmatrix},$$

where Id_r is the $r \times r$ identity matrix and A, B, H are matrices whose coefficients are functions of the coordinates satisfying the following:

(a) A and B are symmetric matrices of order $(n+2-2r) \times (n+2-2r)$ and $r \times r$ respectively. H is a matrix of order $(n+2-2r) \times r$ and H^t stands for the transpose of H.

(b) A and H are independent of the coordinates $(x_1, \dots, x_r)$.

Furthermore, the null parallel r-plane $\mathcal{D}$ is locally generated by the coordinate vector fields $\{\partial_{x_1}, \dots, \partial_{x_r}\}$.

Following the terminology of [97], a field of r-planes $\mathcal{D}$ is said to be *strictly parallel* if it is locally generated by orthogonal parallel null vectors.

Theorem 1.13 [97] *A canonical form for an $(n+2)$-dimensional pseudo-Riemannian manifold M admitting a strictly parallel field of null r-dimensional planes $\mathcal{D}$ is given by the metric tensor in Theorem 1.12, where B is independent of the coordinates $(x_1, \dots, x_r)$.*

Remark 1.14 We emphasize that the local coordinates in Theorems 1.12 and 1.13 are not unique.

The special case of Lorentzian manifolds was previously investigated by Brinkmann [13], who showed the following

Theorem 1.15 [13] *Let (M, g) be a Lorentzian manifold of dimension $n+2$ with a null recurrent vector field. Then there are local coordinates $(u, v, x_1, \ldots, x_n)$ in which the metric g has the form*

$$g = 2\,dudv + f\,du^2 + \sum_{i=1}^{n} a_i\,dudx_i + \sum_{i,j=1}^{n} g_{ij}\,dx_i dx_j \tag{1.11}$$

where $\frac{\partial}{\partial v} g_{ij} = \frac{\partial}{\partial v} a_i = 0$. Moreover $\frac{\partial}{\partial v} f = 0$ if and only if the recurrent vector field can be re-scaled to a parallel vector field, in which case the coordinates can be chosen so that $a_i = 0$ and even that $f = 0$.

1.3.2 *pp*-waves

Lorentzian manifolds admitting a parallel null vector V are of interest both in Physics and Mathematics. A special class of such manifolds [21] are *pp-waves*, which occurs whenever the Ricci tensor is completely determined by the parallel null vector V (i.e., $\rho = \omega V^\flat \otimes V^\flat$ for a function ω and where $V^\flat(\,\cdot\,) = g(V, \cdot\,)$) and the metric is transversally flat.

The general form of an $(n+2)$-dimensional *pp*-wave is the following: the ambient space is $\mathbb{R}^{n+2}$ $(n \geq 0)$ with coordinates $(u, v, x_1, \ldots, x_n)$, and the Lorentzian metric g is given by

$$g_{ppw} = 2\,dudv + H(u, x_1, \ldots, x_n)du^2 + \sum_{i=1}^{n} dx_i^2, \tag{1.12}$$

where $H(u, x_1, \ldots, x_n)$ is an arbitrary smooth function usually called the potential function of the *pp*-wave.

The possibly non-zero components of the Levi-Civita connection of a *pp*-wave in the basis of coordinate vector fields $\{\partial_u = \frac{\partial}{\partial u}, \partial_v = \frac{\partial}{\partial v}, \partial_{x_i} = \frac{\partial}{\partial x_i}\}$ are

$$\nabla_{\partial_u}\partial_u = \frac{1}{2}\,\partial_u H\,\partial_v - \frac{1}{2}\sum_{i=1}^{n}\partial_{x_i} H\,\partial_{x_i}, \qquad \nabla_{\partial_u}\partial_{x_i} = \frac{1}{2}\,\partial_{x_i} H\,\partial_v, \quad i = 1, \ldots, n. \tag{1.13}$$

This shows that the null vector field ∂_v is parallel. Adopt notation $\partial^2_{x_i x_j} := \partial_{x_i}\partial_{x_j}$ henceforth. The possibly non-vanishing components of the curvature tensor are given (up to the usual symmetries) by

$$R(\partial_u, \partial_{x_i}, \partial_u, \partial_{x_j}) = -\frac{1}{2}\,\partial^2_{x_i x_j} H, \quad i, j = 1, \ldots, n. \tag{1.14}$$

Next, we obtain that the only possibly non-vanishing component of the Ricci tensor is

$$\rho(\partial_u, \partial_u) = -\frac{1}{2}\sum_{i=1}^{n}\partial^2_{x_i x_i} H, \tag{1.15}$$

and, hence, the Ricci operator is described in local coordinates as follows:

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2}\sum\limits_{i=1}^{n}\partial^2_{x_i x_i} H & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1.16}$$

The scalar curvature τ is zero, since the Ricci tensor is determined by (1.15). Therefore, a *pp*-wave is Einstein (and hence Ricci flat) if and only if the space-Laplacian of the defining function H vanishes identically, i.e., $\rho(\partial_u, \partial_u) = -\frac{1}{2}\,\Delta_x H = 0$, where $\Delta_x H = \sum\limits_{i=1}^{n}\partial^2_{x_i x_i} H$ denotes the Laplacian of H with respect to $x = (x_1, \dots, x_n)$.

The existence of a parallel null plane $\mathcal{D}$ on a pseudo-Riemannian manifold influences the curvature as follows (see, for example, [48])

$$R(\mathcal{D}, \mathcal{D}^\perp, \cdot, \cdot) = 0, \quad R(\mathcal{D}, \mathcal{D}, \cdot, \cdot) = 0, \quad \text{and } R(\mathcal{D}^\perp, \mathcal{D}^\perp, \mathcal{D}, \cdot) = 0. \tag{1.17}$$

The following curvature characterization of *pp*-waves given by Leistner [71] will be useful for our purposes in subsequent chapters.

Theorem 1.16 [71] *A Lorentzian manifold* (M, g) *admitting a parallel null distribution* $\mathcal{D}$ *is locally isometric to a pp-wave if and only if the curvature tensor satisfies the condition* $R(\mathcal{D}^\perp, \mathcal{D}^\perp, \cdot, \cdot) = 0$ *and the image of the Ricci operator is totally isotropic (i.e.,* $g(\mathrm{Ric}(X), \mathrm{Ric}(X)) = 0$ *for all* $X \in \mathfrak{X}(M)$*).*

Plane waves

A special class of *pp*-waves is constituted by the so-called plane waves. In spite of their simplicity, plane waves occur as the underlying structure of many interesting geometrical situations, as we shall see along this section.

A *pp*-wave whose potential function $H(u, \cdot)$ defines a quadratic form on $\mathbb{R}^n$ is called a *plane wave*. Thus, for any plane wave, we can write

$$H(u, x_1, \dots, x_n) = \sum_{i,j=1}^{n} a_{ij}(u) x_i x_j, \tag{1.18}$$

where a_{ij} are arbitrary functions that are the components of a $n \times n$ symmetric matrix.

Theorem 1.17 [32] *Let* $(\mathbb{R}^{n+2}, g_{ppw})$ *with metric given by* (1.12) *and* (1.18) *be a plane wave. Then it is geodesically complete.*

Remark 1.18 Locally conformally flat *pp*-waves are given by (1.12) with

$$H(u, x_1, \ldots, x_n) = a(u) \sum_{i,j=1}^{n} x_i x_j + \sum_{i=1}^{n} b_i(u) x_i + c(u). \tag{1.19}$$

Moreover, by a suitable change of coordinates $H(u, x_1, \ldots, x_n)$ reduces to (1.18), and thus they are plane waves.

Cahen-Wallach symmetric spaces

Recall that the notion of irreducibility is very strong in the pseudo-Riemannian setting. Indeed, irreducible Lorentzian symmetric spaces are necessarily of constant sectional curvature [24]. Indecomposable Lorentzian symmetric spaces are either irreducible or the so-called *Cahen-Wallach symmetric spaces* which are given as follows [24, 25]. Take $M = \mathbb{R}^{n+2}$ with coordinates as before and define a metric tensor by

$$g_{cw} = 2\, du dv + \left(\sum_{i=1}^{n} a_i\, x_i^2 \right) du^2 + \sum_{i=1}^{n} dx_i^2 \tag{1.20}$$

for some non-zero constants a_i, $i = 1, \ldots, n$.

The Levi-Civita connection is determined by the non-zero Christoffel symbols as follows

$$\nabla_{\partial_u} \partial_u = -\sum_{i=1}^{n} a_i\, x_i\, \partial_{x_i}, \qquad \nabla_{\partial_u} \partial_{x_i} = \nabla_{\partial_{x_i}} \partial_u = a_i\, x_i\, \partial_v\, .$$

The above shows that all constants a_j must be non-zero if the manifold is indecomposable but not irreducible. The only non-zero components of the $(0, 4)$-curvature tensor are given by

$$R(\partial_u, \partial_{x_i}, \partial_u, \partial_{x_i}) = -a_i, \quad i = 1, \ldots, n.$$

The Ricci tensor satisfies $\rho(\partial_u, \partial_u) = -\sum_{i=1}^{n} a_i$, the other terms being zero.

Two-symmetric Lorentzian manifolds

A Lorentzian manifold (M, g) is said to be *two-symmetric* if the second covariant derivative of the curvature tensor vanishes, i.e., $\nabla^2 R = 0$. While this condition implies local symmetry for Riemannian manifolds, there exist non-symmetric two-symmetric Lorentzian manifolds. The local structure of such manifolds was given independently by Alekseevsky and Galaev [1] and Blanco, Sánchez and Senovilla [11] showing that any non-symmetric two-symmetric Lorentzian manifold is a plane wave.

Theorem 1.19 [11] *A non-symmetric two-symmetric Lorentzian manifold (M,g) is locally isometric to $\mathbb{R}^{n+2}$ with metric*

$$g = 2\,dudv + \left(\sum_{i,j=1}^{n} a_{ij}(u)x_i x_j\right) du^2 + \sum_{i=1}^{n} dx_i^2$$

where $a_{ij}(u) = \alpha_{ij}\,u + \beta_{ij}$ for some constants α_{ij}, β_{ij}, where at least one of the α_{ij} is non-zero.

In this case the Ricci tensor satisfies $\rho(\partial_u, \partial_u) = -\sum_{i=1}^{n} (\alpha_{ii}\,u + \beta_{ii})$, the other terms being zero.

Moreover, the previous local characterization is global if (M,g) is assumed to be geodesically complete and simply connected.

Conformally symmetric Lorentzian manifolds

A pseudo-Riemannian manifold is said to be *conformally symmetric* if the Weyl tensor is parallel (i.e., $\nabla W = 0$). In the Riemannian setting any conformally symmetric manifold is either locally conformally flat or locally symmetric, but the Lorentzian signature admits non-trivial examples. Derdzinski and Roter [47] showed that any non-trivial conformally symmetric Lorentzian manifold is a plane wave:

Theorem 1.20 [47] *Let (M,g) be a non-trivial conformally symmetric Lorentzian manifold. Then it is locally isometric to $\mathbb{R}^{n+2}$ with metric*

$$g = 2\,dudv + \left(\sum_{i,j=1}^{n} a_{ij}(u)x_i x_j\right) du^2 + \sum_{i=1}^{n} dx_i^2$$

where $a_{ij}(u) = a(u)\alpha_{ij} + \beta_{ij}$ for some non-constant function $a(u)$.

The Ricci tensor satisfies $\rho(\partial_u, \partial_u) = -\sum_{i=1}^{n} (a(u)\alpha_{ii} + \beta_{ii})$, the other terms being zero.

Homogeneous Lorentzian structures of linear type

The characterization of Cartan of locally symmetric spaces (i.e., $\nabla R = 0$) was extended to homogeneous spaces by Ambrose and Singer [2] in the Riemannian case and by Gadea and Oubiña [54] in the reductive homogeneous pseudo-Riemannian setting. Hence, a pseudo-Riemannian manifold is reductive homogeneous if and only if there exists a $(1,2)$-tensor field S such that the connection $\tilde{\nabla} = \nabla - S$ satisfies the Ambrose-Singer equations

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0.$$

The tensor field S is called a homogeneous structure.

Considering the action of the orthogonal group on the space of algebraic tensors with the symmetries of a homogeneous structure on a vector space V,

$$\mathcal{T}(V) = \{S \in \otimes^3 V^*; S_{XYZ} = -S_{XZY},\ X,Y,Z \in V\},$$

Tricerri and Vanhecke [96] obtained a decomposition of such space into irreducible components in Riemannian signature. Later Gadea and Oubiña [54] considered indefinite metrics, showing that $\mathcal{T}(V)$ decomposes into a direct sum of three invariant and irreducible subspaces, $\mathcal{T}(V) = \mathcal{T}_1(V) \oplus \mathcal{T}_2(V) \oplus \mathcal{T}_3(V)$, where

$$\begin{aligned}
\mathcal{T}_1(V) &= \{S \in \mathcal{T}(V);\ S_{XYZ} = \langle X,Y\rangle\theta(Z) - \langle X,Z\rangle\theta(Y),\ \theta \in V^*\},\\
\mathcal{T}_2(V) &= \{S \in \mathcal{T}(V);\ \mathfrak{S}_{XYZ} S_{XYZ} = 0,\ c_{12}(S) = 0\},\ \text{and}\\
\mathcal{T}_3(V) &= \{S \in \mathcal{T}(V);\ S_{XYZ} + S_{YXZ} = 0\},
\end{aligned}$$

here $\mathfrak{S}_{XYZ}$ denotes the cyclic sum and $c_{12}(S)(Z) = \sum g^{ij} S_{e_i e_j Z}$ for any basis $\{e_i\}$ of V.

A homogeneous pseudo-Riemannian structure S on (M,g) is said to be of class $\mathcal{T}_i$ if S_p belongs to $\mathcal{T}_i(T_pM)$ for each $p \in M$. Moreover, a homogeneous pseudo-Riemannian structure of type $\mathcal{T}_1$ is said to be of *linear type*, in which case there exists a vector field ξ on M such that $S_X Y = g(X,Y)\xi - g(\xi,Y)X$. Now the condition $\tilde{\nabla} S = 0$ is equivalent to $\nabla_X \xi = g(X,\xi)\xi - g(\xi,\xi)X$ and it follows that ξ spans a parallel one-dimensional distribution whenever it is a null vector field.

A Riemannian manifold admits a linear homogeneous structure if and only if it is of constant negative curvature [96]. In the Lorentzian setting, a linear homogeneous structure occurs if and only if the sectional curvature is constant whenever ξ is non-null, and the sign of the sectional curvature depends on the causality of ξ. According to Montesinos [3] the case of linear structures associated to null vector fields occur only if the manifold is a plane wave. This result was later extended by Meessen [72] to homogeneous structures of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ showing that, if the vector field ξ is null, the underlying geometry of any such manifold is that of a plane wave.

Lorentzian manifolds with recurrent curvature

A pseudo-Riemannian manifold (M,g) is said to be *recurrent* (or with *recurrent curvature*) if $\nabla R = \sigma \otimes R$ for some one-form σ. While Riemannian manifolds with recurrent curvature are locally symmetric, non-symmetric examples exist in the Lorentzian setting.

Recurrent Lorentzian manifolds have been classified by Walker [98] (see also Galaev [55] for a modern exposition). Non-symmetric Lorentzian recurrent manifolds are *pp*-waves, and thus the metric tensor is given by (1.12). Moreover, they correspond to one of the following two families

Type I. The defining function satisfies $H(u, x_1, \dots, x_n) = H(u, x_1)$ where $\partial^2_{x_1 x_1} H$ is not constant and the only possibly non-vanishing component of the Ricci tensor is $\rho(\partial_u, \partial_u) = -\frac{1}{2}\,\partial^2_{x_1 x_1} H(u, x_1)$.

Type II. The defining function is given by $H(u, x_1, \ldots, x_n) = a(u)\left(\sum_{i=1}^{n} b_i\, x_i^2\right)$ for constants $b_1,\ldots,b_n$ satisfying $|b_1| \geq \cdots \geq |b_n|$, $b_2 \neq 0$, and a function a such that $a'(u) \neq 0$. In this case the only possibly non-vanishing component of the Ricci tensor is $\rho(\partial_u, \partial_u) = -a(u)\sum_{i=1}^{n} b_i$.

Observe that any Lorentzian manifold with recurrent curvature of Type II is a plane wave and, moreover, that it is locally conformally flat. On the other hand, recurrent Lorentz manifolds of Type I are not locally conformally flat if the dimension is greater than three.

Lorentzian manifolds with large isometry group

It is well-known that the isometry group of an $(n+2)$-dimensional pseudo-Riemannian manifold (M, g) has dimension at most $\frac{1}{2}(n+2)(n+3)$ and that it is exactly $\frac{1}{2}(n+2)(n+3)$ if and only if (M, g) has constant sectional curvature. Lower bounds on the dimension of the isometry group still guarantee the constancy of the sectional curvature in some particular contexts. Thus, Riemannian and Lorentzian manifolds whose isometry group is larger than $\frac{1}{2}(n+2)(n+1)+2$ are still of constant curvature, but there are examples of manifolds with non-constant curvature whose isometry group has dimension $\frac{1}{2}(n+2)(n+1)+1$.

Riemannian manifolds whose isometry group is larger than $\frac{1}{2}(n+2)(n+1)+1$ are either of constant sectional curvature or products of a space of constant sectional curvature and the real line (and thus they are locally conformally flat). In the Lorentzian setting there are other two examples which are still locally conformally flat: Egorov and ε-spaces (assuming the dimension is greater than five and different from seven) [82].

An *Egorov space* is a Lorentzian manifold $(\mathbb{R}^{n+2}, g_E)$, $n \geq 1$, where E is a positive function of one variable and the metric is given by the warped product

$$g_E = 2\, du dv + E(u)\sum_{i=1}^{n} dx_i^2.$$

Egorov spaces are not homogeneous in general. However the Ricci tensor is recurrent and so is the curvature tensor since they are locally conformally flat (see [7, 29]). Hence they are *pp*-waves corresponding to Type II above, and thus plane waves.

An *ε-space* is the Lorentzian manifold $(\mathbb{R}^{n+2}, g)$ with metric given by

$$g = 2\, du dv + \varepsilon\sum_{i=1}^{n} x_i^2 du^2 + \sum_{i=1}^{n} dx_i^2.$$

Cahen-Wallach symmetric spaces are locally conformally flat if and only if $a_1 = \cdots = a_n$, in which case the resulting manifold is an ε-space (we refer to [7, 29] for more information on the geometry of ε-spaces).

1.4 Ricci solitons

A number of geometric evolution equations, corresponding to different problems, have been recently studied in Riemannian geometry. The curve shortening flow and the mean curvature flow are just two important examples which attracted considerable attention. Our work is related to another important evolution equation: the Ricci flow.

A one-parameter family of metrics $g(t)$ on a manifold M defined on some time interval $I \subset \mathbb{R}$ is a solution of the *Ricci flow equation* if

$$\frac{\partial}{\partial t} g(t) = -2\rho(g(t)). \tag{1.21}$$

It was shown by Hamilton [58] that for any C^∞ metric g_0 on a closed manifold M, there exists a unique solution of the Ricci flow equation $g(t)$, $t \in [0, \epsilon)$, for some $\epsilon > 0$, with $g(0) = g_0$.

The genuine fixed points of the Ricci flow are given by Ricci flat metrics. However, if (M, g_0) is an Einstein metric with constant $\lambda \neq 0$, then

$$g(t) = (1 - 2\lambda t)g_0 \tag{1.22}$$

is a solution of the Ricci flow. Observe that $g(t)$ differs from g_0 by a homothety. Hence if one looks for geometric fixed points of the flow, i.e., considering the Ricci flow in the space of metrics modulo diffeomorphisms and homotheties, Einstein metrics also arise as fixed points of the flow. Moreover, observe that if $\lambda < 0$, then the solution $g(t)$ is defined for all $t > \frac{1}{2\lambda}$ and it expands with t, while if $\lambda > 0$, then $g(t)$ is defined for $t < \frac{1}{2\lambda}$ and it shrinks.

Generalizing the behavior of Einstein metrics, and allowing the initial metric to change not only by homotheties but also by diffeomorphisms, a solution $g(t)$ of the Ricci flow is said to be *self-similar* if there exists a positive function $\sigma(t)$ and a one-parameter group of diffeomorphisms $\psi(t) : M \to M$ such that

$$g(t) = \sigma(t)\psi(t)^* g(0). \tag{1.23}$$

Remark 1.21 If (1.23) defines a solution of the Ricci flow (1.21), then differentiating (1.23) yields

$$-2\,\rho(g(t)) = \sigma'(t)\psi(t)^* g_0 + \sigma(t)\psi(t)^*(\mathcal{L}_X g_0), \tag{1.24}$$

where $g_0 = g(0)$, X is the time-dependent vector field such that $X(\psi(t)(p)) = \frac{d}{dt}(\psi(t)(p))$ for any $p \in M$, and $\sigma' = \frac{d\sigma}{dt}$.

Since $\rho(g(t)) = \psi(t)^*\rho(g_0)$, one can drop the pullbacks in (1.24) and get:

$$-2\,\rho(g_0) = \sigma'(t)g_0 + \mathcal{L}_{\tilde{X}(t)} g_0, \tag{1.25}$$

where $\tilde{X}(t) = \sigma(t)X(t)$. Put $\lambda = -\frac{1}{2}\dot{\sigma}(0)$ and $X_0 = \frac{1}{2}\tilde{X}(0)$, so that equation (1.25) becomes

$$-2\,\rho(g_0) = -2\lambda\, g_0 + 2\,\mathcal{L}_{X_0} g_0, \text{ at } t = 0.$$

This shows that for any self-similar solution of the Ricci flow there exists a vector field on M satisfying

$$\mathcal{L}_X g + \rho = \lambda\, g.$$

Conversely, let X be a complete vector field on a pseudo-Riemannian manifold (M, g) and denote by $\psi(t) : M \to M$ with $\psi(0) = id_M$ the family of diffeomorphisms generated by X according to

$$\frac{\partial}{\partial t}\psi(t)(p) = \frac{1}{1-2\lambda t} X(\psi(t)(p)),$$

which is defined for all $t < \frac{1}{2\lambda}$ if $\lambda > 0$ and for all $t > \frac{1}{2\lambda}$ if $\lambda < 0$. Considering now the one-parameter family of metrics

$$g(t) = (1 - 2\lambda t)\psi(t)^* g,$$

one has

$$\begin{aligned} \tfrac{\partial}{\partial t}\, g(t) &= -2\lambda\, \psi(t)^* g + (1-2\lambda)\psi(t)^* \left(\mathcal{L}_{\frac{1}{1-2\lambda t}X}\, g\right) \\ &= \psi(t)^* \left(-2\lambda\, g + \mathcal{L}_{X(\psi(t)(p))} g\right). \end{aligned}$$

Now, if the vector field X satisfies (1.26), then

$$\frac{\partial}{\partial t}\, g(t) = \psi(t)^*(-2\,\rho) = -2\,\psi(t)^*\rho = -2\,\rho(\psi(t)^* g) = -2\,\rho(g(t)),$$

which shows that $g(t)$ is a solution of the Ricci flow given by (1.21).

The above remark motivates the following definition.

Definition 1.22 *A triple (M, g, X) where (M, g) is a pseudo-Riemannian manifold and X is a vector field on M satisfying*

$$\mathcal{L}_X g + \rho = \lambda\, g \tag{1.26}$$

is called a Ricci soliton. *A Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.*

A Ricci soliton is a gradient Ricci soliton *if there exists a smooth function f on M such that $X = \frac{1}{2}\nabla f$, i.e., if*

$$\mathrm{Hes}_f + \rho = \lambda\, g. \tag{1.27}$$

In this case we say that the triple (M, g, f) is a gradient Ricci soliton and we refer to f as the potential function of the soliton.

Remark 1.23 If (M, g, f) is a gradient Ricci soliton with g positive definite, then the completeness of (M, g) guaranties the completeness of the vector field ∇f [101].

Remark 1.24 Since the Ricci tensor is invariant by homotheties, one can re-scale any Ricci soliton (M, g, X) as follows. Set $\tilde{g} = \mu g$ and $\tilde{X} = \frac{1}{\mu} X$ for any positive constant μ, then

$$\rho = \lambda\, g - \mathcal{L}_X g = \lambda \frac{1}{\mu}\, \tilde{g} - \frac{1}{\mu}\, \mu\, \mathcal{L}_{\tilde{X}} \tilde{g} = \frac{\lambda}{\mu}\, \tilde{g} - \mathcal{L}_{\tilde{X}} \tilde{g}.$$

This shows that $(M, \tilde{g}, \tilde{X})$ is a Ricci soliton with constant $\frac{\lambda}{\mu}$ and that only the sign, but not the absolute value of the constant, matters. Hence, one may assume that a expanding (resp., shrinking) Ricci soliton is given by a vector field X on (M, g) satisfying $\mathcal{L}_X g + \rho = -g$ (resp., $\mathcal{L}_X g + \rho = g$).

1.4.1 Some examples of Ricci solitons in Riemannian geometry

Next we recall some examples of Riemannian Ricci solitons. As already discussed, the first trivial examples correspond to Einstein metrics. If g_0 is an Einstein metric with Einstein constant λ (i.e, $\rho = \lambda\, g_0$), then

$$g(t) = (1 - 2\lambda\, t) g_0$$

is a solution of the Ricci flow and thus a trivial Ricci soliton.

The Gaussian soliton

Although $\mathbb{R}^{n+2}$ endowed with the Euclidean metric g_0 is clearly a steady Ricci soliton since it is Ricci flat, $(\mathbb{R}^{n+2}, g_0)$ also admits a structure of non-steady gradient Ricci soliton. Consider the potential function $f(x) = \frac{\lambda}{2} \|x\|^2$; it is straightforward to check that $(\mathbb{R}^{n+2}, g_0, f)$ defines a gradient Ricci soliton which is expanding or shrinking depending on the sign of $\lambda \neq 0$. This soliton is known in the literature as the *Gaussian soliton* and it depends on the existence of homothetic vector fields on the Euclidean space which are not Killing.

The Gaussian soliton shows that a possible extension of Myers' theorem for shrinking Ricci solitons is not valid unless one assumes some additional conditions on the norm of the Ricci soliton [51]. Moreover, while any compact Riemannian expanding or steady Ricci soliton is trivial, there exist non-trivial expanding Ricci solitons in the complete non-compact case.

Rigid solitons

As a generalization of the Gaussian soliton, a gradient Ricci soliton (M, g, f) is said to be *rigid* if (M, g) is isometric to a quotient of $N \times \mathbb{R}^k$, where N is an Einstein manifold with Einstein constant λ and the potential function f is defined on the Euclidean factor as $f = \frac{\lambda}{2} \|x\|^2$ [86].

Although rigidity is a rather restrictive condition, rigid Ricci solitons are the only examples in many important situations. A characterization of gradient shrinking Ricci solitons was given in [52, 74], where it is shown that a complete gradient shrinking Ricci soliton is rigid if and only if the Weyl curvature is harmonic (or, equivalently, the Schouten tensor is Codazzi).

The cigar soliton

Hamilton's *cigar soliton*, also known in the Physics literature as *Witten's black hole*, is the steady gradient Ricci soliton given by the complete Riemannian surface $\mathbb{R}^2$ equipped with the metric $g = \frac{1}{1+x^2+y^2}(dx^2+dy^2)$ and the potential function $f(x,y) = -\log(x^2+y^2+1)$.

Some characterizations of the cigar soliton are the following (see, for example, [41])

- Positively curved two-dimensional gradient steady soliton:
  It is a straightforward calculation to check that the metric g above has positive Gauss curvature. Moreover, one has that if (M,g) is a complete two-dimensional steady gradient Ricci soliton with positive curvature, then (M,g) is isometric to the cigar soliton.

- Uniqueness of the cigar:
  If $(\mathbb{R}^2, g, f)$ is a steady gradient Ricci soliton with g conformal to the Euclidean metric g_0 on $\mathbb{R}^2$, then it is either the cigar soliton or the flat metric.

The Bryant soliton

As a generalization of the cigar soliton, Bryant [23] extended the construction of Hamilton's cigar soliton to arbitrary dimensions by considering the Euclidean space in radial coordinates. Let g_{can} denote the standard metric on the unit $(n+1)$-dimensional sphere and let $(0,\infty)\times_\varphi \mathbb{S}^{n+1}$ be $\mathbb{R}^{n+2}\setminus\{0\}$ viewed as a warped product. A steady gradient Ricci soliton whose potential function only depends on the radial coordinate is given by the equations

$$f'' = (n+1)\frac{\varphi''}{\varphi}; \qquad \varphi\varphi' f' = -n(1-(\varphi')^2) + \varphi\,\varphi''.$$

In order to define a complete steady gradient Ricci soliton, one must show the existence of a solution to the above equations which can be smoothly extended through the origin. This requires an analysis of the phase portrait corresponding to the system above. (A detailed exposition is available at [41]).

For our purposes of pointing out the existence of many complete non-compact gradient Ricci solitons, we emphasize that the Bryant construction was generalized by Ivey [63] and Dancer and Wang [44, 45] who constructed expanding, steady and shrinking gradient Ricci solitons on manifolds whose underlying structure is that of a multiply warped product.

As well as for the cigar, Bryant soliton is also characterized by its curvature properties; we refer to [49] for specific results.

Algebraic Ricci solitons

The examples discussed so far are gradient Ricci solitons. There is however, no scarcity of non-gradient Ricci solitons. All the examples below are expanding or steady Ricci solitons which are constructed on Lie groups by considering the more general class of *algebraic Ricci solitons* introduced by Lauret [70].

A Lie group G equipped with a left-invariant pseudo-Riemannian metric g is said to be an algebraic Ricci soliton if $\operatorname{Ric} = \lambda \operatorname{Id} + D$, where D is a derivation of the Lie algebra $\mathfrak{g}$ of G, i.e., $D[X,Y] = [DX,Y] + [X,DY]$ for all X, $Y \in \mathfrak{g}$. It was shown in [70, 79] that any algebraic Ricci soliton on a solvable Lie group corresponds to a Ricci soliton. Moreover, all these examples are either expanding or steady.

1.4.2 Some results in the Riemannian setting

In this section we recall some results which are specific to positive definite signature. The theory of Lorentzian Ricci solitons is somehow richer than its Riemannian analog, and the purpose of this section is to point out some aspects where the Riemannian signature forces some rigidity which does not occur in the Lorentzian case, as we shall see presently.

Gradient Ricci solitons

Perelman [84] showed that if (M, g, X) is a compact Ricci soliton then X is the sum of a gradient and a Killing vector field (see also [46] for a direct Riemannian proof of Perelman's theorem, not invoking the Ricci flow). Moreover, this result was extended by Naber [75] to the complete non-compact case under a boundedness condition on the curvature.

Compactness is a very restrictive condition when dealing with Ricci solitons. Indeed it is a consequence of the expression

$$\Delta\tau = 2\langle \nabla\tau, X\rangle - 2\left(\|\rho\|^2 - \frac{\tau^2}{n+2}\right) - \frac{\tau}{n+2}\left(\tau - \frac{1}{\operatorname{vol}(M,g)}\int_M \tau\right)$$

and the Hopf maximum principle that non-trivial compact Ricci solitons are necessarily shrinking [59].

Homogeneous Ricci solitons

Petersen and Wylie proved that homogeneity is a very rigid condition when considering gradient Ricci solitons, as next theorem shows.

Theorem 1.25 [85] *Any Riemannian homogeneous gradient Ricci soliton is rigid.*

Note that the above result fails when passing from the Riemannian to the Lorentzian setting. Indeed, indecomposable but not irreducible Lorentzian symmetric spaces provide examples of non-trivial steady gradient Ricci solitons which are not rigid. We will discuss such examples in Chapter 3.

Left-invariant Ricci solitons on solvable Lie groups were studied by di Cerbo [39], who showed that any such soliton is necessarily expanding and moreover that left-invariant unimodular Lie groups cannot result in Ricci solitons in the Riemannian setting. These results depend on an analysis of the evolution of certain curvature quantities along the Ricci flow. Once again, the Lorentzian situation is richer allowing some non-trivial invariant three-dimensional Ricci solitons on Lie groups (Chapter 5).

On the other hand, there are a number of expanding Ricci solitons constructed from algebraic Ricci solitons on nilpotent Lie algebras [5, 70]. It is still an open question whether all Riemannian homogeneous Ricci solitons are algebraic, although counterexamples are known in the Lorentzian setting [8].

Locally conformally flat gradient Ricci solitons

Hamilton [59] and Ivey [62] showed that any compact Ricci soliton of dimension two or three has constant sectional curvature. Since the Weyl tensor vanishes in both dimensions, the curvature tensor is determined by the Ricci tensor. As a generalization of this situation to higher dimensions, there has been an special interest in understanding the geometry of locally conformally flat gradient Ricci solitons.

From a more philosophical point of view, note that the gradient Ricci soliton equation provides information about the Hessian of the potential function (and thus the geometry of its level sets) and the curvature of the manifold through its Ricci tensor. Since the Ricci tensor determines the curvature whenever the Weyl tensor vanishes, one expects to be able to determine the structure of the manifold from such information.

The classification of complete locally conformally flat gradient shrinking Ricci solitons has been finally achieved as a result of several previous works. The compact case was settled by Derdzinski [46] and Eminenti, La Nave and Mantegazza [50] showed that the only possibilities are the standard sphere or one of its quotients. Ni and Wallach [76] classified complete locally conformally flat gradient shrinking Ricci solitons under the assumptions of non-negative Ricci curvature and that the norm of the curvature tensor has at most exponencial growth. In such a case the soliton must be $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{R}\times\mathbb{S}^{n-1}$ or one of their quotients. Cao, Wang and Zhang [35] improved previous result by relaxing the assumption on the Ricci curvature and assuming only that the Ricci curvature is bounded from below. Petersen and Wylie [87] got the same result by using a different assumption, namely that

$$\int_M \|R\|^2 e^{-f} < \infty, \tag{1.28}$$

where f is any potential function of the gradient shrinking Ricci soliton. Zhang [100] showed that the classification is true for all locally conformally flat gradient shrinking Ricci solitons, proving that they have non-negative curvature operator and that the growth of its norm is at most exponential. Munteanu and Sesum [74] gave a different proof of this result, showing that the integral inequality (1.28) holds for shrinking gradient Ricci solitons with vanishing Weyl tensor.

In what refers to the classification of complete steady gradient Ricci solitons, Bryant [23] proved that there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci soliton on $\mathbb{R}^{n+2}$, together with the trivial steady Gaussian soliton. Cao and Chen [34] proved that these are the only possibilities under the assumptions of being steady and locally conformally flat.

Theorem 1.26 [34] *Let (M, g, f) be a complete locally conformally flat gradient Ricci soliton.*

(a) If (M, g, f) is shrinking, then it is Einstein or a product $\mathbb{R} \times N$, where (N, g_N) is a space of constant curvature.

(b) If (M, g, f) is steady, then it is flat or the Bryant soliton.

Working at the local level, Riemannian locally conformally flat gradient Ricci solitons are locally warped products of the form $I \times_\varphi N$, where $I \subset \mathbb{R}$ is a real interval and (N, g_N) is of constant curvature [22]. Properties of the Ricci flow then show that (N, g_N) is indeed of positive sectional curvature in the steady or shrinking case, from where the result is obtained as a consequence of the classification of rotationally symmetric gradient Ricci solitons [69]. The fact that no classification results are available for expanding gradient Ricci solitons is not surprising since a classification of locally conformally flat manifolds with negative Ricci curvature is not available either.

One of the purposes of this work is to determine the local structure of locally conformally flat Lorentzian gradient Ricci solitons. We show in Theorem 2.1 that even their local structure is richer than the corresponding Riemannian one.

Manifolds admitting different kinds of Ricci solitons

Let (M, g, X), (M, g, Y) be two Ricci solitons with the same underlying Riemannian manifold (M, g) for some constants λ_X and λ_Y, i.e. X, Y are vector fields on M satisfying $\mathcal{L}_X g + \rho = \lambda_X\, g$ and $\mathcal{L}_Y g + \rho = \lambda_Y\, g$. Set $\xi = X - Y$, then

$$\mathcal{L}_\xi g = \mathcal{L}_X g - \mathcal{L}_Y g = (\lambda_X - \lambda_Y) g,$$

which shows that ξ is a homothetic vector field on (M, g). Conversely, if (M, g, X) is a Ricci soliton and ξ is a homothetic vector field on (M, g) (i.e., $\mathcal{L}_\xi g = \kappa\, g$ for some constant κ), then $Y = \xi + X$ satisfies

$$\mathcal{L}_Y g = \mathcal{L}_\xi g + \mathcal{L}_X g = (\kappa + \lambda) g - \rho$$

which shows that (M, g, Y) is a Ricci soliton.

The above shows that any two Ricci solitons with the same underlying Riemannian manifold differ by a homothetic vector field. Hence, a Riemannian manifold (M, g) admits two distinct non-trivial Ricci solitons (M, g, X) and (M, g, Y) with $\lambda_X \neq \lambda_Y$ if and only if there exists a non-Killing homothetic vector field on (M, g). This has strong consequences if the manifold is assumed to be Riemannian and complete. Indeed, a complete Riemannian manifold admits a non-Killing homothetic vector field if and only if it is locally Euclidean [95]. Hence, a complete Riemannian manifold results in two distinct Ricci solitons if and only if it is the Euclidean space and the Ricci solitons are Gaussian solitons.

In this memoir we construct geodesically complete Lorentzian manifolds that yield all classes of Ricci solitons without being the Gaussian soliton, even if the manifold is assumed to be symmetric (see Section 3.5).

1.4.3 General formulae

Due to the indefiniteness of the metric, when investigating the geometry of Lorentzian (gradient) Ricci solitons we will work with the defining equation (1.27) rather than with the Ricci flow itself. In this section we derive some formulae which are direct consequences of the defining equation (1.27) and that will be used through this work. Since the analysis we present in the following chapters is developed within the context of Lorentzian geometry, we concentrate on that particular signature henceforth.

Lemma 1.27 *A Lorentzian gradient Ricci soliton with potential function f satisfies*

$$\nabla\tau = 2\,\mathrm{Ric}(\nabla f), \tag{1.29}$$

$$\tau + \|\nabla f\|^2 - 2\lambda f = \text{const}\,. \tag{1.30}$$

Proof.
Let (M, g, f) be a gradient Ricci soliton, i.e., $\mathrm{Hes}_f + \mathrm{Ric} = \lambda\, g$. Tracing this equation one obtains $\Delta f + \tau = (n+2)\lambda$, hence $\nabla\tau = -\nabla\Delta f$. Moreover, since $\nabla_Z\tau = 2\,\mathrm{div}(\mathrm{Ric}(Z))$ by (1.6) and $\mathrm{div}(\mathrm{Hes}_f)(X) = \rho(\nabla f, X) + g(\nabla\Delta f, X)$ by (1.7), one has

$$\begin{aligned} 0 &= \mathrm{div}(\lambda\, g)(X) \\ &= \mathrm{div}(\rho + \mathrm{Hes}_f)(X) \\ &= \tfrac{1}{2}\, g(X, \nabla\tau) + \rho(\nabla f, X) - g(\nabla\tau, X) \\ &= \rho(\nabla f, X) - \tfrac{1}{2}\, g(\nabla\tau, X) \end{aligned}$$

and hence $\nabla\tau = 2\,\mathrm{Ric}(\nabla f)$, which proves (1.29).

Moreover

$$\begin{aligned} 0 &= \rho(\nabla f, X) + \tfrac{1}{2}\, g(\nabla\Delta f, X) \\ &= \lambda\, g(X, \nabla f) - \mathrm{Hes}_f(X, \nabla f) + \tfrac{1}{2}\, g(\nabla\Delta f, X) \\ &= \lambda\, g(X, \nabla f) - g(X, \nabla_{\nabla f}\nabla f) + \tfrac{1}{2}\, g(\nabla\Delta f, X) \\ &= g(X, \lambda\nabla f - \nabla_{\nabla f}\nabla f + \tfrac{1}{2}\nabla\Delta f) \\ &= g(X, \lambda\nabla f - \tfrac{1}{2}\nabla\|\nabla f\|^2 - \tfrac{1}{2}\nabla\tau), \end{aligned}$$

where we have used that $\nabla_{\nabla f}\nabla f = \frac{1}{2}\nabla\|\nabla f\|^2$. This shows that $2\lambda f - \tau - \|\nabla f\|^2$ is constant, thus proving (1.30). □

Remark 1.28 As a consequence of Lemma 1.27, there are several particular situations in which ∇f is an eigenvector of the Ricci operator, a technical fact which will be extensively used in our analysis of (1.27). Thus, if τ is constant, from (1.29) it follows that ∇f is an eigenvector for the Ricci operator associated to the eigenvalue zero. Also, if ∇f is null, then from (1.30) one has $\tau = \text{const} + 2\lambda f$; now substitute in (1.29) to see that $\operatorname{Ric}(\nabla f) = \lambda \nabla f$.

1.4.4 Two-dimensional case

If (M, g, X) is a two-dimensional Lorentzian Ricci soliton, then X is a conformal vector field since $\rho = \frac{\tau}{2}\, g$ and the soliton equation (1.26) reduces to $\mathcal{L}_X g = (\lambda - \frac{\tau}{2})g$. Consider the canonical para-Kähler structure $\mathfrak{J}$ (i.e., $\mathfrak{J}^2 = Id$, $g(\mathfrak{J}\,\cdot\,, \mathfrak{J}\,\cdot\,) = -g(\,\cdot\,,\,\cdot\,)$, $\nabla\mathfrak{J} = 0$) on (M, g) defined by any local orientation. Then for any (M, g, f) two-dimensional gradient Ricci soliton it follows from the following argument that $\mathfrak{J}(\nabla f)$ is a Killing vector field:

$$\begin{aligned}
\left(\mathcal{L}_{\mathfrak{J}(\nabla f)} g\right)(X, Y) &= g(\nabla_X \mathfrak{J}\nabla f, Y) + g(\nabla_Y \mathfrak{J}\nabla f, X) \\
&= g(\mathfrak{J}\nabla_X \nabla f, Y) + g(\mathfrak{J}\nabla_Y \nabla f, X) \\
&= -g(\nabla_X \nabla f, \mathfrak{J}Y) - g(\nabla_Y \nabla f, \mathfrak{J}X) \\
&= -\operatorname{Hes}_f(X, \mathfrak{J}Y) - \operatorname{Hes}_f(Y, \mathfrak{J}X) \\
&= \tfrac{1}{2}\left(\tfrac{\tau}{2} - \lambda\right)\{g(X, \mathfrak{J}Y) + g(Y, \mathfrak{J}X)\} = 0,
\end{aligned}$$

where we have used that ∇f is a conformal vector field.

Now, proceeding as in [41], we have the following

Lemma 1.29 *A non-flat Lorentzian surface with a non-zero Killing vector field K is locally a warped product. In particular, any non-trivial gradient Ricci soliton on a surface is locally a warped product.*

Proof.
Let (M, g) be a surface with a non-null Killing vector field K. Proceeding in a analogous way as in [41], one has that M is locally a warped product of the form $g = \varepsilon\, dr^2 - \varepsilon\, f(r)^2 d\theta^2$, where $\varepsilon = \pm 1$ depending on the causal character of the Killing vector field K.

On the other hand, assuming that K is a null vector field, choose coordinates (x_1, x_2) so that the null Killing vector field is $K = \partial_{x_2}$. The metric tensor takes following form $g = a(x_1, x_2)dx_1^2 + b(x_1, x_2)dx_1 dx_2$ for some functions a and b. Now, the fact that K is Killing implies that $\frac{\partial}{\partial x_2} a = 0$ and $\frac{\partial}{\partial x_2} b = 0$. This shows that g is flat and K is indeed parallel. □

Two-dimensional steady gradient Ricci solitons

Recall from Subsection 1.4.1 that Hamilton's cigar soliton is a two-dimensional complete steady gradient Ricci soliton. Here, we are going to extend the Hamilton's cigar soliton to the Lorentzian setting. Let (M, g, f) be a two-dimensional steady gradient Ricci soliton with ∇f a timelike vector field (the spacelike case is similar). Set $M = I \times N$ with metric $g = -dt^2 + \varphi(t)^2 ds^2$ and assume that f only depends on t. Then a straightforward calculation from (1.27) gives that (M, g, f) is a steady gradient Ricci soliton if and only if

$$f''(t) - \frac{\varphi''(t)}{\varphi(t)} = 0 \quad \text{and} \quad -f'(t)\varphi'(t) + \varphi''(t) = 0.$$

Hence $f''\varphi - f'\varphi' = 0$ and we integrate it to see that $f'(t) = \kappa\,\varphi(t)$ for some constant κ. Equations above then reduce to $\kappa\,\varphi\,\varphi' - \varphi'' = 0$. Hence the possible solutions, depending on the sign of κ, are the following:

(i) If $\kappa = 0$; then $\varphi(t) = a\,t + b$ for some constants a and b. In this case M is flat and f is constant.

(ii) If $\kappa = r^2$; then $\varphi(t) = \frac{a\sqrt{2}}{r} \tan r\sqrt{2}(a\,t + b)$, where a and b are constants. The potential function is $f(t) = d - 2\log\left(\cos\left(\frac{r(a\,t+b)}{\sqrt{2}}\right)\right)$ for a constant d, and the scalar curvature is $\tau = 2a\,r^2 \sec^2\left(r\frac{(a\,t+b)}{\sqrt{2}}\right)$.

(iii) If $\kappa = -r^2$; then $\varphi(t) = \frac{a\sqrt{2}}{r} \tanh r\sqrt{2}(a\,t + b)$ for some constants a and b. The potential function is $f(t) = d + 2\log\left(\cosh\left(\frac{r(a\,t+b)}{\sqrt{2}}\right)\right)$ for a constant d, and the scalar curvature is $\tau = -2a\,r^2 \operatorname{sech}^2\left(r\frac{(a\,t+b)}{\sqrt{2}}\right)$.

Analyzing geodesic completeness in the Lorentzian case is a subtle task. Indeed Lorentzian warped products of geodesically complete manifolds need not be complete, as occurs in positive definite signature. Necessary and sufficient conditions for geodesic completeness of Lorentzian warped products were summarized in Lemma 1.7 (see [33] for details). As a consequence, the Lorentzian warped products given by (ii) above are not geodesically complete, while those given by (iii) are. Thus, (iii) generalizes Hamilton's cigar soliton (see [59]) to the Lorentzian setting.

We summarize the above as follows

Theorem 1.30 *Let $M = I \times_\varphi N$ be a warped product with metric $g = -dt^2 + \varphi(t)^2 ds^2$, where $\varphi(t) = \pm\frac{a\sqrt{2}}{r} \tanh r\sqrt{2}(at + b)$. Then (M, g) is a two-dimensional complete steady gradient Ricci soliton.*

Remark 1.31 Note that the metric described in the previous theorem is conformally equivalent to the flat metric $g = -\frac{1}{\varphi(t)^2}\,dt^2 + ds^2$, as occurs with Hamilton's cigar soliton, which is conformally equivalent to the Euclidean space. As a difference with the Riemannian case, the Lorentzian cigar described in Theorem 1.30 has negative Gauss curvature and ∇f is timelike. By reversing the metric, one has a Lorentzian cigar with positive Gauss curvature and spacelike ∇f.

1.4.5 Einstein Ricci solitons

Ricci solitons are generalizations of Einstein metrics. If (M, g) is a complete Riemannian Einstein manifold, then either (M, g, f) is a gradient Ricci soliton if and only if $\mathrm{Hes}_f = 0$, or otherwise it is the Gaussian soliton [86]. The next result describes the local structure of Einstein gradient Ricci solitons in the Lorentzian setting.

Theorem 1.32 *Let (M, g) be a Lorentzian Einstein manifold. If (M, g, f) is a gradient Ricci soliton with non-constant f, then (M, g) is Ricci flat. Moreover:*

(i) If $\|\nabla f\| \neq 0$, then (M, g) is locally a warped product of the form $I \times_{f'} N$ and the potential function is given by $f(t) = \frac{\lambda}{2}\, t^2 + a\, t + b$.

(ii) If $\|\nabla f\| = 0$, then there exist coordinates $(u, v, x_1, \dots, x_n)$ in which the metric has the form $g = 2dudv + \tilde{g}$, where the n-dimensional metric $\tilde{g}$ does not depend on v. Moreover, the potential function f is given by any function $f(u)$ with $f''(u) = 0$ and the soliton is steady.

Proof.
Let (M, g, f) be an Einstein gradient Ricci soliton. Since the scalar curvature is constant, it follows from (1.29) that either the potential function f is constant or otherwise (M, g) is Ricci flat.

Assume (M, g) is Ricci flat. The soliton equation (1.27) reduces to $\mathrm{Hes}_f = \lambda g = \dfrac{\Delta f}{n+2} g$. This equation was previously investigated by Brinkmann [13] (see [65] for a modern exposition) showing that in a neighborhood of any point where $\|\nabla f\| \neq 0$ the manifold (M, g) decomposes locally as a warped product of a real interval $I \subset \mathbb{R}$ and an Einstein manifold (N, g_N) so that $g = \varepsilon\, dt^2 + (f')^2 g_N$, where f is a real function defined on I with $f' \neq 0$. Now, since (M, g) is Ricci flat, a direct computation of the Ricci tensor for the metric $\varepsilon\, dt^2 + (f')^2 g_N$ shows that g_N is Einstein and f must satisfy $f''' = 0$ and $f' f''' + n\, \varepsilon (f'')^2 = \frac{\tau_N}{n+1}$. Hence $f(t) = \frac{\lambda}{2}\, t^2 + a\, t + b$ and $\tau_N = n(n+1)\varepsilon\, \lambda^2$.

Now assume $\|\nabla f\| = 0$ identically. Then (1.30) shows that either f is constant or the gradient Ricci soliton is steady. If $\lambda = 0$ the Ricci soliton equation (1.27) reduces to $\mathrm{Hes}_f = 0$. Then ∇f is a parallel isotropic vector field and the metric tensor can be written in suitable Rosen coordinates $(u, v, x_1, \dots, x_n)$ as $g = 2\, dudv + \tilde{g}$, where the n-dimensional metric $\tilde{g}(u)$ is Ricci flat for any fixed u and does not depend on v [13, 65]. Moreover, in this coordinates $\nabla f = \frac{\partial}{\partial v}$ and the potential function depends only on the variable u. Now the result follows by computing the Hessian of f. □

Part I

Gradient Ricci solitons

Chapter 2

Locally conformally flat gradient Ricci solitons

Let (M, g, f) be a gradient Ricci soliton. The Ricci soliton equation (1.27) relates the geometry of the level sets of the potential function to the geometry of (M, g). More concretely the curvature of the manifold is involved in the Ricci soliton equation by means of the Ricci tensor, which imposes some restrictions on the geometry of (M, g). Considering the decomposition of the curvature tensor as in Remark 1.2, there is a lack of information on the conformally invariant component represented by the Weyl tensor. This is one of the reasons why a complete classification of gradient Ricci solitons is hard to attain.

In this chapter we simplify the curvature of the manifold by concentrating on the family of metrics which are locally conformally flat. We will see that within this setting the Ricci soliton equation provides enough information to obtain a local classification.

The purpose of this chapter is to investigate locally conformally flat gradient Ricci solitons in the Lorentzian setting by focusing on their local structure in order to prove the following result.

Theorem 2.1 *Let (M, g, f) be a locally conformally flat Lorentzian gradient Ricci soliton.*

(i) In a neighborhood of any point where $\|\nabla f\| \neq 0$, M is locally isometric to a warped product $I \times_\varphi N$ with metric $\varepsilon\, dt^2 + \varphi^2 g_N$, where I is a real interval and (N, g_N) is a space of constant sectional curvature c.

(ii) If $\|\nabla f\| = 0$ on a non-empty open set, then (M, g) is locally isometric to a plane wave, i.e., M is locally diffeomorphic to $\mathbb{R}^2 \times \mathbb{R}^n$ with metric

$$g_{ppw} = 2dudv + H(u, x_1, \ldots, x_n)du^2 + \sum_{i=1}^{n} dx_i^2,$$

where $H(u, x_1, \ldots, x_n) = a(u)\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i(u)x_i + c(u)$ for some functions a, b_i, c

and the potential function is given by $f(u, v, x_1, \dots, x_n) = f_0(u)$*, satisfying condition* $f_0''(u) = -\rho(\partial_u, \partial_u) = n\, a(u)$.

While Riemannian locally conformally flat gradient Ricci solitons correspond to those manifolds discussed in Theorem 2.1–(i), due to holonomy action, there exist other possibilities in Lorentzian signature, as Theorem 2.1–(ii) shows.

Remark 2.2 The causal character of ∇f may vary from one point to another as the Lorentzian analog of the Gaussian soliton shows. Let $(\mathbb{L}^{n+2}, g)$ be the flat Minkowski space and let $f(x_1, \dots, x_{n+2}) = \frac{\lambda}{2}\left(-x_1^2 + x_2^2 + \cdots + x_{n+2}^2\right)$ be defined on $\mathbb{L}^{n+2}$. The gradient of f is given by $\nabla f = \lambda\,(x_1 + x_2 + \cdots + x_{n+2})$ and the Hessian is $\mathrm{Hes}_f = \lambda\, g$. Hence the soliton equation (1.27) is satisfied for any given λ. Note that $\|\nabla f\|^2 = \lambda^2\left(-x_1^2 + x_2^2 + \cdots + x_{n+2}^2\right)$ is positive, zero or negative depending on $(x_1, \dots, x_{n+2})$, so the causality of ∇f varies with the point.

As a matter of notation, a gradient Ricci soliton is said to be *non-isotropic* if $\|\nabla f\| \neq 0$ and *isotropic* if $\|\nabla f\| = 0$ but $\nabla f \neq 0$. In the following sections we will study both cases separately.

The chapter is organized as follows. In Section 2.1 we give some sufficient conditions to guarantee that ∇f is an eigenvector of the Ricci operator; this will be crucial in the proof of Theorem 2.1. We devote Section 2.2 to analyze locally conformally flat non-isotropic gradient Ricci solitons and Section 2.3 to study the isotropic case, showing that the underlying structure of such a soliton is a *pp*-wave. In Section 2.4 we discuss which *pp*-waves are gradient Ricci solitons in general, without any assumption on isotropicity or local conformal flatness. Thus the restriction of this discussion to locally conformally flat isotropic *pp*-waves completes the proof of Theorem 2.1. The main results of this chapter are summarized in [16].

2.1 General remarks on locally conformally flat gradient Ricci solitons

Although locally conformally flat gradient Ricci solitons will be more deeply analyzed in Sections 2.2 and 2.3, we begin here by giving the expression of the curvature tensor of a locally conformally flat manifold and by establishing a technical lemma. If the dimension of a manifold is greater than or equal to four, local conformal flatness is characterized by the fact that the Weyl conformal tensor vanishes. Hence, by (1.9), the curvature tensor is given by the Ricci tensor as follows

$$
\begin{aligned}
R(X,Y,Z,T) \;=\; & \frac{\tau}{n(n+1)}\{g(X,T)g(Y,Z) - g(X,Z)g(Y,T)\} \\
& + \tfrac{1}{n}\{\rho(X,Z)g(Y,T) + \rho(Y,T)g(X,Z) \\
& \quad -\rho(X,T)g(Y,Z) - \rho(Y,Z)g(X,T)\}\,.
\end{aligned}
\tag{2.1}
$$

Proceeding in a similar way to that developed in [52], one has the following:

Lemma 2.3 *Let (M, g, f) be a locally conformally flat gradient Ricci soliton. Then ∇f is an eigenvector of the Ricci operator.*

Proof.
Since (M, g) is locally conformally flat, the Schouten tensor is Codazzi. This means that, $(\nabla_X C)(Y, Z) = (\nabla_Y C)(X, Z)$ for all vector fields X, Y, Z. Hence by (1.8)

$$(\nabla_X \rho)(Y, Z) - \frac{X(\tau)}{2(n+1)}\, g(Y, Z) = (\nabla_Y \rho)(X, Z) - \frac{Y(\tau)}{2(n+1)}\, g(X, Z). \tag{2.2}$$

From (1.27) and using that $\mathrm{Hes}_f(X, Y) = g(\nabla_X \nabla f, Y)$ one has

$$\begin{aligned}(\nabla_X \rho)(Y, Z) &= -(\nabla_X \mathrm{Hes}_f)(Y, Z) \\ &= -Xg(\nabla_Y \nabla f, Z) + g(\nabla_{\nabla_X Y} \nabla f, Z) + g(\nabla_Y \nabla f, \nabla_X Z) \\ &= -g(\nabla_X \nabla_Y \nabla f, Z) + g(\nabla_{\nabla_X Y} \nabla f, Z).\end{aligned}$$

Substituting this expression into (2.2) we get

$$\begin{aligned}g(\nabla_X \nabla_Y \nabla f, Z) - g(\nabla_{\nabla_X Y} \nabla f, Z) &+ \frac{X(\tau)}{2(n+1)}\, g(Y, Z) \\ &= g(\nabla_Y \nabla_X \nabla f, Z) - g(\nabla_{\nabla_Y X} \nabla f, Z) + \frac{Y(\tau)}{2(n+1)}\, g(X, Z).\end{aligned}$$

Thus

$$g(\nabla_X \nabla_Y \nabla f - \nabla_Y \nabla_X \nabla f - \nabla_{[X,Y]} \nabla f, Z) = -\frac{X(\tau)}{2(n+1)}\, g(Y, Z) + \frac{Y(\tau)}{2(n+1)}\, g(X, Z),$$

that is,

$$R(X, Y, Z, \nabla f) = -\frac{X(\tau)}{2(n+1)}\, g(Y, Z) + \frac{Y(\tau)}{2(n+1)}\, g(X, Z),$$

or equivalently, using (1.29),

$$R(X, Y, Z, \nabla f) = -\frac{1}{n+1}\, \rho(X, \nabla f) g(Y, Z) + \frac{1}{n+1}\, \rho(Y, \nabla f) g(X, Z). \tag{2.3}$$

Let $Z = \nabla f$ in (2.3) to obtain

$$\rho(Y, \nabla f) g(X, \nabla f) = \rho(X, \nabla f) g(Y, \nabla f).$$

Now choose X so that $g(X, \nabla f) = 1$ to see that for all $Y \perp \nabla f$ one has

$$0 = \rho(Y, \nabla f) = -\mathrm{Hes}_f(Y, \nabla f)$$

and conclude that ∇f is an eigenvector of the Ricci operator Ric. Note that ∇f is also an eigenvector of the Hessian operator hes_f. □

Remark 2.4 We emphasize here that if (M,g,f) is an isotropic gradient Ricci soliton, then ∇f is an eigenvector of the Ricci operator (see Remark 1.28), without any assumption on the curvature of (M,g).

2.2 Locally conformally flat non-isotropic gradient Ricci solitons

Next we show that in a neighborhood of any point where $\|\nabla f\| \neq 0$ the underlying manifold has the local structure of a warped product, thus proving Theorem 2.1–(i).

Lemma 2.5 *Let (M,g,f) be a locally conformally flat Lorentzian gradient Ricci soliton with $\|\nabla f\|_p \neq 0$ for some point $p \in M$. Then, on a neighborhood of p, (M,g) is a warped product of a real interval and a space of constant sectional curvature c.*

Proof.
If the Weyl tensor of (M,g) vanishes, then the curvature tensor is given by (2.1).

Consider the unit vector $V = \frac{\nabla f}{\|\nabla f\|}$ on the tangent space T_pM, which can be timelike or spacelike (we set $g(V,V) = \varepsilon$). Extend this definition to a neighborhood of p where $\|\nabla f\| \neq 0$ and complete it to a local orthonormal frame $\{V, E_1, \dots, E_{n+1}\}$ where $g(E_i,E_i) = \varepsilon_i$. Then from (2.3) one has

$$R(V,E_i,E_i,V) = -\frac{1}{n+1}\,\rho(V,V)\varepsilon_i\,,$$

while from (2.1) one gets

$$R(V,E_i,E_i,V) = \frac{\tau}{n(n+1)}\,\varepsilon\,\varepsilon_i - \frac{1}{n}\,\rho(V,V)\varepsilon_i - \frac{1}{n}\,\rho(E_i,E_i)\varepsilon\,.$$

Hence for all $i = 1, \dots, n+1$:

$$-\frac{1}{n+1}\,\rho(V,V)\varepsilon_i = -\frac{1}{n}\,\rho(V,V)\varepsilon_i - \frac{1}{n}\,\rho(E_i,E_i)\varepsilon + \frac{\tau}{n(n+1)}\,\varepsilon\,\varepsilon_i,$$

from where $\rho(E_i,E_i)\varepsilon = \frac{1}{n+1}(\tau\,\varepsilon - \rho(V,V))\varepsilon_i$. Using (1.27) we have

$$\mathrm{Hes}_f(E_i,E_i) = \lambda\,\varepsilon_i + \frac{1}{n+1}\,(\rho(V,V)\varepsilon - \tau)\,\varepsilon_i,$$

which shows that the level sets of f are totally umbilical hypersurfaces. Hence (M,g) decomposes locally as a twisted product of the form $I \times_\varphi N$ by Theorem 1.9 (see [88] for details). Now, since ∇f is an eigenvector of the Ricci operator by Lemma 2.3, it follows that $\rho(V,E_i) = 0$ for all $i = 1, \dots, n+1$, and therefore the twisted product reduces to a warped product by Theorem 1.10. Hence (M,g) is locally a warped product $(I \times N, \varepsilon\, dt^2 + \varphi(t)^2 g_N)$ where (N, g_N) is a Riemannian or a Lorentzian manifold of constant sectional curvature c by applying Theorem 1.8. □

Remark 2.6 The potential function f in Lemma 2.5 is a radial function $f(t)$, and hence a direct computation from the soliton equation (1.27) shows that it is given as a solution to the equations:

$$f'' = \varepsilon\,\lambda + (n+1)\frac{\varphi''}{\varphi}, \qquad \varepsilon\,\varphi\,\varphi' f' = \lambda\,\varphi^2 - n\,c + \varepsilon(\varphi\,\varphi'' + n(\varphi')^2).$$

Note that these equations impose restrictions on the warping function φ, thus the warped product is not arbitrary.

2.3 Locally conformally flat isotropic gradient Ricci solitons

In this section we study gradient Ricci solitons (M, g, f) with $\|\nabla f\| = 0$.

Lemma 2.7 *Any isotropic locally conformally flat Lorentzian gradient Ricci soliton is steady and the underlying manifold is locally a pp-wave.*

Proof.
Let (M, g, f) be a gradient Ricci soliton with $\|\nabla f\| = 0$. In what follows we will show that ∇f spans a parallel null line field and use Theorem 1.16 to prove that (M, g) is a *pp*-wave. Set $V = \nabla f$. Since V is a null vector, there exist orthogonal vectors S, T satisfying $g(S, S) = -g(T, T) = \frac{1}{2}$ such that $V = S + T$. Define $U = S - T$, which is a null vector such that $g(U, V) = g(S, S) - g(T, T) = 1$, and consider a pseudo-orthonormal basis $\{U, V, E_1, \ldots, E_n\}$. For any vector field Z, from equations (2.3) and (2.1) we get

$$\begin{aligned} R(Z, E_i, E_j, V) &= -\tfrac{1}{n+1}\,\rho(Z, V)\delta_{ij} + \tfrac{1}{n+1}\,\rho(E_i, V)g(Z, E_j) \\ &= \tfrac{\tau}{n(n+1)}\,g(Z, V)\delta_{ij} - \tfrac{\tau}{n(n+1)}\,g(E_i, V)g(Z, E_j) \\ &\quad -\tfrac{1}{n}\,\rho(Z, V)\delta_{ij} - \tfrac{1}{n}\,\rho(E_i, E_j)g(Z, V) \\ &\quad +\tfrac{1}{n}\,\rho(Z, E_j)g(E_i, V) + \tfrac{1}{n}\,\rho(E_i, V)g(Z, E_j). \end{aligned} \tag{2.4}$$

We use the fact that $\mathrm{Ric}(V) = \lambda\,V$ (see Remark 1.28) to check that

$$\rho(V, V) = 0,\ \rho(U, V) = \lambda,\ \rho(V, E_i) = 0 \text{ for all } i = 1, \ldots, n.$$

On the other hand compute $R(U, E_i, E_j, V)$ in expression (2.4) to get that

$$\begin{aligned} R(U, E_i, E_j, V) &= -\frac{1}{n+1}\,\lambda\,\delta_{ij} \\ &= \frac{\tau}{n(n+1)}\,\delta_{ij} - \frac{1}{n}\,\lambda\,\delta_{ij} - \frac{1}{n}\,\rho(E_i, E_j). \end{aligned}$$

Hence $\rho(E_i, E_j) = 0$ if $i \neq j$ and $\rho(E_i, E_i) = \frac{\tau-\lambda}{n+1}$ for all $i = 1, \ldots, n$. Now, compute

$$\tau = 2\rho(U, V) + n\,\rho(E_i, E_i) = \frac{(n+2)\,\lambda + n\,\tau}{n+1}$$

to see that $\tau = (n+2)\lambda$. Hence the scalar curvature τ is constant and from (1.29) we have $0 = \nabla\tau = 2\,\mathrm{Ric}(V) = 2\lambda\,V$. Therefore we conclude that $\lambda = 0 = \tau$ and the only possibly non-zero Ricci component is $\rho(U, U)$, so (M, g, f) is a steady gradient Ricci soliton with nilpotent Ricci operator.

Since the soliton is steady, from (1.27) we have $\mathrm{hes}_f = -\,\mathrm{Ric}$. Now since $\mathrm{Ric}(V) = 0$, it follows that $\nabla_V V = 0$, which shows that V is a geodesic vector field.

The gradient of the potential function is a recurrent vector field (i.e., the null line field $\mathcal{D} = \mathrm{span}\{\nabla f\}$ is parallel) if and only if $\nabla_X \nabla f = \mathrm{hes}_f(X) = \sigma(X)\nabla f$ for some one-form σ and for all X. Since (M, g, f) is a steady gradient Ricci soliton, it follows from the expressions above for the Ricci operator that

$$\begin{aligned}
\mathrm{hes}_f(U) &= -\,\mathrm{Ric}(U) = -\rho(U, U)V,\\
\mathrm{hes}_f(V) &= -\,\mathrm{Ric}(V) = 0,\\
\mathrm{hes}_f(E_i) &= -\,\mathrm{Ric}(E_i) = 0.
\end{aligned}$$

This shows that V is a recurrent vector field with one-form σ given by $\sigma(U) = -\rho(U, U)$, $\sigma(V) = 0$ and $\sigma(E_i) = 0$ for all $i = 1, \ldots, n$.

It follows now from (2.1), the expressions of the Ricci tensor above and the vanishing of the scalar curvature that

$$R(\mathcal{D}^\perp, \mathcal{D}^\perp, \cdot, \cdot) = 0.$$

Moreover note that the Ricci tensor is isotropic and thus that (M, g) is indeed a *pp*-wave by Theorem 1.16. □

Remark 2.8 Note that although (M, g) is a *pp*-wave, and hence it admits a null parallel vector field, ∇f is not in general parallel.

2.4 Gradient Ricci solitons on *pp*-waves

In this section we analyze the existence of gradient Ricci solitons on *pp*-waves. Due to their simplicity, one can explicitly integrate the gradient Ricci soliton equations when the underlying Lorentzian structure corresponds to a *pp*-wave. Theorem 2.1–(ii) will follow as a consequence of Lemma 2.7 and the following analysis.

Set $M = \mathbb{R}^{n+2}$ with coordinates $(u, v, x_1, \ldots, x_n)$, and let g_{ppw} be given by (1.12) for some arbitrary function $H(u, x_1, \ldots, x_n)$. Then, one has

Theorem 2.9 *(M, g_{ppw}, f) is a non-trivial gradient Ricci soliton if and only if it is steady and the potential function f is given by $f(u,v,x_1,\dots,x_n) = f_0(u) + \sum_{i=1}^{n} \kappa_i\, x_i$, where*

$$f_0''(u) = -\rho(\partial_u,\partial_u) - \frac{1}{2}\sum_{i=1}^{n}\kappa_i\,\partial_{x_i}H(u,x_1,\dots,x_n) \tag{2.5}$$

for some constants $\kappa_1,\dots,\kappa_n$.

Proof.
Let f be a function on $\mathbb{R}^{n+2}$. Then the gradient of the previous function is given by $\nabla f = (\partial_v f, \partial_u f - H\partial_v f, \partial_{x_1} f, \dots, \partial_{x_n} f)$ and thus (1.27) becomes

$$\begin{cases} \frac{1}{2}\sum_{i=1}^{n}\partial_{x_i}H\,\partial_{x_i}f + \partial^2_{uu}f - \frac{1}{2}\,\partial_u H\,\partial_v f + \rho(\partial_u,\partial_u) = \lambda\, H, & \\ \partial^2_{ux_i}f - \frac{1}{2}\,\partial_{x_i}H\,\partial_v f = 0, & 1\le i\le n, \\ \partial^2_{x_ix_i}f = \lambda, & 1\le i\le n, \\ \partial^2_{uv}f = \lambda, & \\ \partial^2_{x_ix_j}f = \partial^2_{vx_i}f = \partial^2_{vv}f = 0, & 1\le i\ne j\le n. \end{cases} \tag{2.6}$$

Integrating equations $\partial^2_{vx_i}f = \partial^2_{vv}f = 0$ in (2.6) we obtain that the potential function splits as $f(u,v,x_1,\dots,x_n) = f_0(u,x_1,\dots,x_n) + v\, f_1(u)$ for some functions f_0, f_1. Moreover equations $\partial^2_{uv}f = \lambda$ and $\partial^2_{x_ix_j}f = 0$ now show that $f(u,v,x_1,\dots,x_n) = \sum_{i=1}^{n} f_i(u,x_i) + v(\lambda\, u + \kappa)$ for some constant κ and functions f_i, $i = 1,\dots,n$. Hence (2.6) reduces to

$$\begin{cases} \frac{1}{2}\sum_{i=1}^{n}\partial_{x_i}H\,\partial_{x_i}f_i + \sum_{i=1}^{n}\partial^2_{uu}f_i - \frac{1}{2}(\lambda\, u+\kappa)\partial_u H + \rho(\partial_u,\partial_u) = \lambda\, H, & \\ \partial^2_{ux_i}f_i - \frac{1}{2}(\lambda\, u+\kappa)\partial_{x_i}H = 0, & 1\le i\le n, \\ \partial^2_{x_ix_i}f_i = \lambda, & 1\le i\le n. \end{cases} \tag{2.7}$$

Integrating the last equations in (2.7) we have

$$f_i(u,x_i) = f_{0i}(u) + x_i\,\kappa_i(u) + \frac{\lambda}{2}\,x_i^2,$$

for some functions f_{0i} and κ_i. Substituting the above into (2.7) and differentiating the second set of equations we get

$$0 = \partial^3_{ux_ix_i}f_i = (\lambda\, u+\kappa)\partial^2_{x_ix_i}H,$$

which shows that either, $\partial^2_{x_i x_i} H = 0$ for all i (and hence the *pp*-wave is Ricci flat) or otherwise that $\lambda = \kappa = 0$.

The first case, when (M, g_{ppw}) is Ricci flat, was already analyzed in Theorem 1.32. The second case, $\lambda = \kappa = 0$ shows that non Einstein gradient Ricci solitons are steady and f becomes $f(u, v, x_1, \dots, x_n) = f_0(u) + \sum_{i=1}^{n} \kappa_i(u) x_i$. Now the second set of equations in (2.7) reduces to $\kappa_i'(u) = 0$ for all $i = 1, \dots, n$ and hence $\kappa_i(u) = \kappa_i$ for real constants κ_i, which gives

$$f(u, v, x_1, \dots, x_n) = f_0(u) + \sum_{i=1}^{n} \kappa_i \, x_i. \tag{2.8}$$

Finally, it follows from the first equations in (2.7) that the function $f_0(u)$ is given by the differential equation

$$f_0'' = \frac{1}{2} \left(\sum_{i=1}^{n} \partial^2_{x_i x_i} H \right) - \frac{1}{2} \sum_{i=1}^{n} \kappa_i \, \partial_{x_i} H = -\rho(\partial_u, \partial_u) - \frac{1}{2} \sum_{i=1}^{n} \kappa_i \, \partial_{x_i} H,$$

which completes the proof. □

Remark 2.10 In general, equation (2.5) does not have a solution, since the derivatives $\partial_{x_i} H(u, x_1, \dots, x_n)$ and $\partial^2_{x_i x_i} H$ may be functions of the x_i's. Further note that ∇f is not isotropic in general since $\|\nabla f\| = \sum_{i=1}^{n} \kappa_i^2$, although it is a geodesic vector field since $\nabla_{\nabla f} \nabla f = -\operatorname{Ric}(\nabla f) = 0$.

Remark 2.11 It is easy to show the existence of non-isotropic steady gradient Ricci solitons on *pp*-waves. For instance, let $\mathbb{R}^{n+2}$ be the *pp*-wave with metric g given by (1.12) for a function $H(u, x_1, \dots, x_n) = a(u) e^{x_1 + \cdots + x_n}$. Since $f(u, v, x_1, \dots, x_n) = \sum_{i=1}^{n} x_i$ satisfies (2.5), it is immediate from Theorem 2.9 that $(\mathbb{R}^{n+2}, g, f)$ is a steady gradient Ricci soliton. Moreover, $\nabla f = \partial_{x_1} + \cdots + \partial_{x_n}$ is a spacelike vector field on $\mathbb{R}^{n+2}$.

2.4.1 Gradient Ricci solitons on locally conformally flat *pp*-waves

It follows from the expressions (1.14) and (1.15), that a *pp*-wave is locally conformally flat if and only if the defining function H takes the form

$$H(u, x_1, \dots, x_n) = a(u) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i(u) x_i + c(u), \tag{2.9}$$

where a, b_1, $\dots$, b_n, c are arbitrary smooth functions of the variable u.

In this case condition (2.5) reduces to

$$f_0'' = -\rho(\partial_u, \partial_u) - \frac{1}{2}\sum_{i=1}^{n}\kappa_i\, b_i(u) - a(u)\sum_{i=1}^{n}\kappa_i\, x_i, \tag{2.10}$$

where $\rho(\partial_u, \partial_u) = -n\, a(u)$. So, if we differentiate (2.10) with respect to x_i we get that $a(u)\kappa_i = 0$ for all $i = 1, \ldots, n$. Then, unless the manifold is flat, it follows that necessarily $\kappa_i = 0$ for all i, and the potential function is given by

$$f(u, v, x_1, \ldots, x_n) = f_0(u), \quad \text{where} \quad f_0''(u) = -\rho(\partial_u, \partial_u) = n\, a(u).$$

This completes the proof of Theorem 2.1.

2.4.2 Ricci solitons on locally conformally flat *pp*-waves

Gradient Ricci solitons are a special class of Ricci solitons. So far, we have shown that *pp*-waves support steady gradient Ricci solitons in many cases. The existence of (non-gradient) Ricci solitons is a weaker condition, and in what follows we show that any locally conformally flat *pp*-wave admits many non-gradient expanding and shrinking Ricci solitons.

Let $X = X_u\partial_u + X_v\partial_v + \sum_{i=1}^{n} X_i\partial_{x_i}$ be an arbitrary vector field on $(\mathbb{R}^{n+2}, g_{ppw})$, where the metric g_{ppw} is given by (1.19). Then (1.26) becomes

$$\begin{cases} \frac{1}{2}\sum_{i=1}^{n}\partial_{x_i}H X_i + \frac{1}{2}\,\partial_u H X_u + H\partial_u X_u + \partial_u X_v + \rho(\partial_u, \partial_u) = \lambda\, H, & \\ H\partial_v X_u + \partial_v X_v + \partial_u X_u = 2\lambda, & \\ H\partial_{x_i} X_u + \partial_{x_i} X_v + \partial_u X_i = 0, & 1 \le i \le n, \\ \partial_{x_i} X_j + \partial_{x_j} X_i = 0, & 1 \le i \ne j \le n, \\ \partial_{x_i} X_u + \partial_v X_i = 0, & 1 \le i \le n, \\ \partial_v X_u = 0; \quad \partial_{x_i} X_i = \lambda, & 1 \le i \le n. \end{cases} \tag{2.11}$$

While we are not explicitly integrating (2.11), in what follows we point out the existence of non-gradient solutions.

Consider the vector field

$$X = \left(p(u) - \sum_{i=1}^{n} q_i'(u)x_i + 2\lambda\, v\right)\partial_v + \sum_{i=1}^{n}\left(q_i(u) + \lambda\, x_i\right)\partial_{x_i}, \tag{2.12}$$

where functions p and q_i satisfy the following conditions

$$\text{(2.13)}\qquad \begin{cases} a(u)q_i(u) - q_i''(u) = \frac{\lambda}{2}\, b_i(u), & 1 \leq i \leq n, \\ \\ \frac{1}{2}\sum_{i=1}^{n} b_i(u)q_i(u) + \rho(\partial_u, \partial_u) + p'(u) = \lambda\, c(u). \end{cases}$$

with functions a and b_i given by (1.19). Note that one can always find p, q_i being solutions of (3.8). A straightforward calculation from (2.11) shows that (M, g, X) is a Ricci soliton. Also observe that λ is the constant of equation (1.26) and can be chosen arbitrarily.

Opposite to the gradient case, we obtain that *any locally conformally flat pp-wave* (M, g) *admits appropriate vector fields resulting in expanding, steady and shrinking Ricci solitons.*

Remark 2.12 As discussed in Section 1.4.2, two vector fields X and Y satisfying (1.26) differ by a homothetic vector field. Since complete non-flat Riemannian manifolds do not admit proper homothetic vector fields (see [67, 95] for this classical result), a complete Riemannian Ricci soliton (M, g, X) which is shrinking cannot be expanding or steady for another vector field Y unless it is flat. Nevertheless the situation is different in Lorentzian signature: locally conformally flat *pp*-waves are geodesically complete Lorentzian manifolds which admit vector fields so that the corresponding triples are expanding, steady and shrinking Ricci solitons. This is due to the existence of non-Killing homothetic vector fields on *pp*-waves [64, 94].

Chapter 3

Special families of gradient Ricci solitons

In this chapter we point out the existence of gradient Ricci solitons on special classes of Lorentzian manifolds related to the family of *pp*-waves. Since plane waves are the underlying structure of many Lorentzian situations without Riemannian counterpart, they are a natural family to look for new examples of complete gradient Ricci solitons exhibiting nice geometrical properties.

Henceforth we specialize a *pp*-wave (1.12) to be a plane wave, i.e.,

$$H(u, x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}(u) x_i x_j$$

for some functions a_{ij}. Let A be the symmetric matrix $A = (a_{ij}(u))$ and consider the vector $\overrightarrow{\kappa} = (\kappa_1, \ldots, \kappa_n)$. Now, a direct application of Theorem 2.9 shows

Theorem 3.1 *Let (M, g_{ppw}) be a plane wave. Then it is a non-trivial steady gradient Ricci soliton with potential function f given by $f(u, v, x_1, \ldots, x_n) = f_0(u) + \sum_{i=1}^{n} \kappa_i x_i$, where $A \cdot \overrightarrow{\kappa} = 0$ and*

$$f_0''(u) = -\rho(\partial_u, \partial_u) = \sum_{i=1}^{n} a_{ii}(u). \tag{3.1}$$

Moreover, $\|\nabla f\| = \sum_{i=1}^{n} \kappa_i^2 \geq 0$ and ∇f is a geodesic vector field.

Proof.

Specialize Theorem 2.9 by taking $H(u, x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}(u) x_i x_j$. Then the potential function of any gradient Ricci soliton, which is of the form (2.8), satisfies (2.5). Differentiating (2.5) with respect to x_j one gets

$$\begin{aligned} 0 &= -\partial_{x_j}\rho(\partial_u,\partial_u) - \frac{1}{2}\sum_{i=1}^{n}\kappa_i\,\partial^2_{x_ix_j}H(u,x_1,\dots,x_n) \\ &= -\partial_{x_j}\sum_{i=1}^{n}a_{ii}(u) - \frac{1}{2}\sum_{i=1}^{n}\kappa_i\,\partial^2_{x_ix_j}\sum_{r,s=1}^{n}a_{rs}(u)x_rx_s \\ &= -\sum_{i=1}^{n}\kappa_i\,a_{ij}(u) = -A\cdot\overrightarrow{\kappa}, \end{aligned}$$

and hence equation (2.5) becomes

$$\begin{aligned} f_0''(u) &= -\rho(\partial_u,\partial_u) - \tfrac{1}{2}\sum_{i=1}^{n}\kappa_i\,\partial_{x_i}H(u,x_1,\dots,x_n) \\ &= -\rho(\partial_u,\partial_u) - \tfrac{1}{2}\sum_{i=1}^{n}\kappa_i\,\partial_{x_i}\sum_{r,s=1}^{n}a_{rs}(u)x_rx_s \\ &= -\rho(\partial_u,\partial_u) - \sum_{i=1}^{n}\kappa_i\sum_{r=1}^{n}a_{ri}(u)x_r \\ &= -\rho(\partial_u,\partial_u) - \sum_{r=1}^{n}x_r\sum_{i=1}^{n}\kappa_i a_{ri}(u) \\ &= -\rho(\partial_u,\partial_u) = \sum_{i=1}^{n}a_{ii}(u) \end{aligned}$$

Finally, note that if $\kappa_i = 0$ for all $i = 1,\dots,n$, then the potential function is given by $f(u,v,x_1,\dots,x_n) = f_0(u)$, and in this case, $\|\nabla f\| = 0$. On the other hand, if there exist some constant $\kappa_j \neq 0$, then $\|\nabla f\| = \kappa_j^2 > 0$, thus resulting in a non-isotropic Lorentzian gradient steady Ricci soliton. □

Remark 3.2 Let $(\mathbb{R}^{n+2}, g_{ppw}, f)$ be a complete plane wave. Then it is a complete steady gradient Ricci soliton. Moreover, the causal structure of any gradient Ricci soliton depends on the symmetric matrix $A = (a_{ij}(u))$, since the first equation at (3.1) is $A\cdot\overrightarrow{\kappa} = 0$. Hence any plane wave gradient soliton is isotropic if A is non-degenerate and spacelike otherwise.

In next sections we point out some applications of the previous theorem to special classes of Lorentzian manifolds (Cahen-Wallach symmetric spaces, two-symmetric Lorentzian manifolds, conformally symmetric Lorentzian manifolds, etc.) showing that all of them are naturally equipped with a non-trivial gradient Ricci soliton structure. Moreover, these results will also show some of the differences between the gradient Ricci solitons and the quasi-Einstein metrics, which will be discussed in Chapter 6. In the final section of this chapter, the existence of non-steady Ricci solitons on plane waves is considered. The results in this chapter are summarized in [6, 16].

3.1 Lorentzian manifolds with recurrent curvature

Recall that a pseudo-Riemannian manifold (M, g) is said to be *recurrent* (or with *recurrent curvature*) if $\nabla R = \sigma \otimes R$ for some one-form σ.

Recurrent Lorentz manifolds have been classified by Walker [98] (see also Galaev [55] for a modern exposition based on holonomy theory). As already pointed out in the introduction, non-symmetric Lorentzian recurrent manifolds are *pp*-waves which correspond to one of the following two families

Type I The defining function satisfies $H(u, x_1, \ldots, x_n) = H(u, x_1)$ where $\partial^2_{x_1 x_1} H$ is not constant.

Type II The defining function is given by $H(u, x_1, \ldots, x_n) = a(u) \left(\sum_{i=1}^{n} b_i x_i^2 \right)$ for constants $b_1, \ldots, b_n$ with $|b_1| \geq \cdots \geq |b_n|$, $b_2 \neq 0$, and a function a such that $a'(u) \neq 0$.

Observe here that manifolds of Type II are locally conformally flat plane waves, while manifolds of Type I are not plane waves. We analyze both cases separately.

For a recurrent manifold of Type I, by Theorem 2.9 we have that the existence of a gradient Ricci soliton is equivalent to the possibility of solving the equation

$$
\begin{aligned}
f_0''(u) &= -\rho(\partial_u, \partial_u) - \frac{1}{2} \kappa_1 \partial_{x_1} H(u, x_1) \\
&= \frac{1}{2} \partial^2_{x_1 x_1} H(u, x_1) - \frac{1}{2} \kappa_1 \partial_{x_1} H(u, x_1),
\end{aligned}
\tag{3.2}
$$

for some constant κ_1.

Differentiating in (3.2) with respect to x_1 we get

$$
\frac{\kappa_1}{2} \partial^2_{x_1 x_1} H(u, x_1) - \frac{1}{2} \partial^3_{x_1 x_1 x_1} H(u, x_1) = 0,
$$

and hence the defining function $H(u, x_1)$ becomes

$$
H(u, x_1) = \frac{1}{\kappa_1^2} e^{\kappa_1 x_1} h_0(u) + h_1(u) + x_1 h_2(u).
$$

Then the soliton is given by the potential function $f(u, v, x_1, \ldots, x_n) = f_0(u) + \sum_{i=1}^{n} \kappa_i x_i$ (see (2.8)-(3.1)), where

$$
f_0''(u) = -\frac{\kappa_1}{2} h_2(u).
$$

Note that in this case ∇f is always spacelike (since $\kappa_1 \neq 0$) and the underlying manifold is not locally conformally flat (unless it is flat which occurs if $h_0(u) = 0$).

For a recurrent manifold of Type II, it follows by Theorem 3.1 that the potential function of any gradient Ricci soliton must be of the form $f(u,v,x_1,\dots,x_n) = f_0(u) + \sum_{i=1}^{n} \kappa_i\, x_i$, where

$$f_0''(u) = -\rho(\partial_u, \partial_u) - a(u)\sum_{i=1}^{n} \kappa_i\, b_i\, x_i. \tag{3.3}$$

Now, since $\rho(\partial_u, \partial_u) = -a(u)\sum_{i=1}^{n} b_i$ in this case, taking derivatives with respect to x_i in (3.3) we get that $\kappa_i\, b_i a(u) = 0$ for all i and therefore, as $b_1 \neq 0 \neq b_2$ there are two different possibilities:

(i) If $b_i \neq 0$ for all i, unless the manifold is flat, it follows that $\kappa_i = 0$ for all i and the potential function is $f(u,v,x_1,\dots,x_n) = f_0(u)$ with $f_0''(u) = -\rho(\partial_u,\partial_u) = a(u)\sum_{i=1}^{n} b_i$. In this case the Ricci soliton is isotropic ($\|\nabla f\| = 0$).

(ii) If $b_j = 0$ for some $j \in \{3,\dots,n\}$, then $\kappa_i = 0$ for $i < j$ and the potential function is given by

$$f(u,v,x_1,\dots,x_n) = f_0(u) + \sum_{i=j}^{n} \kappa_i\, x_i\,, \text{ where } f_0''(u) = -\rho(\partial_u,\partial_u) = a(u)\sum_{i=1}^{j} b_i.$$

Further observe that in this case ∇f is spacelike.

Summarizing the above, we have that

Theorem 3.3 *Let (M,g) be a Lorentzian manifold with recurrent curvature.*

I. *If (M,g) is of Type I, then it is a gradient Ricci soliton if and only if the defining function is $H(u,x_1) = \frac{1}{\kappa_1^2}\, e^{\kappa_1 x_1}\, h_0(u) + h_1(u) + x_1 h_2(u)$. In this case the potential function is of the form $f(u,v,x_1,\dots,x_n) = f_0(u) + \sum_{i=1}^{n} \kappa_i\, x_i$, where*

$$f_0''(u) = -\frac{\kappa_1}{2}\, h_2(u).$$

The Ricci soliton is steady and $\|\nabla f\|$ is spacelike.

II. *If (M,g) is of Type II, then it is a steady gradient Ricci soliton. Moreover*

II-(i). *If $b_i \neq 0$ for all $i = 1,\dots,n$, then the potential function is given by $f(u,v,x_1,\dots,x_n) = f_0(u)$, satisfying $f_0''(u) = -\rho(\partial_u,\partial_u) = a(u)\sum_{i=1}^{n} b_i$ and $\|\nabla f\| = 0$.*

II-(ii). *If $b_j = 0$ for some $j \in \{3, \dots, n\}$, then the potential function is given by $f(u, v, x_1, \dots, x_n) = f_0(u) + \sum_{i=j}^{n} \kappa_i x_i$, where $f_0''(u) = -\rho(\partial_u, \partial_u) = a(u) \sum_{i=1}^{j} b_i$ and ∇f is spacelike.*

Remark 3.4 The existence of Ricci solitons in Lorentzian manifolds with large isometry groups was investigated in [6]. It follows from the work of Patrangenaru [82] that $(n+2)$-dimensional Lorentzian manifolds whose isometry group is of dimension at least $\frac{1}{2}(n+2)(n+1)+1$ are one of the following: manifolds of constant curvature, the product of a line with a space of constant curvature, an ε-space or an Egorov space.

The class of ε-spaces can be viewed as a degenerate class of the Type II family above just considering $a(u) = \varepsilon$ and $b_i = 1$ for all $i = 1, \dots, n$. Moreover, Egorov spaces are locally conformally flat, non-symmetric and with recurrent curvature (see [29]). Hence they are *pp*-waves of Type II–(ii), as discussed in the previous theorem.

Hence, $(n+2)$*-dimensional Lorentzian manifolds whose isometry group is of dimension at least* $\frac{1}{2}(n+2)(n+1)+1$ *are isotropic steady gradient Ricci solitons.*

Another special case of Theorem 3.3 are the Cahen-Wallach symmetric spaces, which can be viewed as a limit case of Type II recurrent manifolds when $a(u)$ is a constant function.

Theorem 3.5 *Indecomposable but not irreducible Lorentzian symmetric spaces are isotropic steady gradient Ricci solitons.*

3.2 Two-symmetric Lorentzian manifolds

Recall that a Lorentzian manifold is said to be *two-symmetric* if $\nabla^2 R = 0$ but $\nabla R \neq 0$. As already discussed in the introduction, two-symmetric Lorentzian manifolds are a special class of plane waves given by (see [1, 11])

$$H(u, x_1, \dots, x_n) = \sum_{i,j=1}^{n} (a_{ij}\, u + b_{ij}) x_i x_j \,, \tag{3.4}$$

where (a_{ij}) is a diagonal matrix with the diagonal elements $a_{11} \leq \dots \leq a_{nn}$ non-null real numbers and (b_{ij}) an arbitrary symmetric matrix of real numbers.

Now, an immediate application of Theorem 3.1 shows that *two-symmetric Lorentzian manifolds are steady gradient Ricci solitons.* If we differentiate (3.1) with respect to u and x_i, then one gets $a_{ii}\, \kappa_i = 0$, but as all the terms a_{ii} are non-null we conclude that $\kappa_i = 0$ for all $i = 1, \dots, n$ and the potential function is given by $f(u, v, x_1, \dots, x_n) = f_0(u)$ where

$$f_0''(u) = -\rho(\partial_u, \partial_u) = \sum_{i=1}^{n} (a_{ii}\, u + b_{ii}).$$

Finally observe that ∇f is a geodesic vector field and (M, g) is geodesically complete. Moreover ∇f is isotropic.

3.3 Conformally symmetric Lorentzian manifolds

A Lorentzian manifold is said to be *conformally symmetric* if the covariant derivative of the Weyl tensor vanishes identically ($\nabla W = 0$). Conformally symmetric Lorentzian manifolds which are neither locally conformally flat nor locally symmetric have recurrent Ricci tensor and have been described locally by Derdzinski and Roter [47]. It turns out that all of them are plane waves given by

$$H(u, x_1, \dots, x_n) = (a(u)\alpha_{ij} + \beta_{ij})x_i x_j$$

where (β_{ij}) is a non-zero symmetric matrix with $\sum_{i=1}^{n} b_{ii} = 0$.

Now Theorem 3.1 shows that *any conformally symmetric Lorentzian manifold admits a function f resulting in a steady gradient Ricci soliton with isotropic ∇f.*

3.4 Homogeneous structures of linear type

As already mentioned in Section 1.3.2, Montesinos characterized Lorentzian manifolds admitting a degenerate homogeneous structures of type $\mathcal{T}_1$.

Theorem 3.6 [3] *Let (M, g) be an $(n+2)$-dimensional connected Lorentzian manifold with a degenerate homogeneous structure of type $\mathcal{T}_1$. Then (M, g) is locally isometric to $\mathbb{R}^{n+2}$ with the pseudo-Riemannian metric*

$$g = 2dudv + (b(x, x) + 2u)dv^2 + h, \tag{3.5}$$

where b and h are symmetric bilinear forms in $\mathbb{R}^n$, h is non-degenerate, x is the position vector in $\mathbb{R}^n$ and u, v are the coordinates in $\mathbb{R}^2$.

This result was later extended by Meessen to the $\mathcal{T}_1 \oplus \mathcal{T}_3$ case, showing that

Theorem 3.7 [72] *A connected homogeneous Lorentzian space admitting a non-degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure is a locally symmetric space.*

The underlying geometry of a connected homogeneous Lorentzian space that admits a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure is that of a singular homogeneous plane wave.

Homogeneous Lorentzian manifolds admitting a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure are special cases of plane waves and therefore again by Theorem 3.1 *any Lorentzian manifold admitting a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ homogeneous structure is a steady gradient Ricci soliton.*

3.5 Ricci solitons on plane waves

Recall that gradient Ricci solitons are a special class of Ricci solitons. So far, we have shown that the family of plane waves support steady gradient Ricci solitons in many cases. The existence of (non-gradient) Ricci solitons is a weaker condition, and in what follows we show that any plane wave admits many non-gradient expanding and shrinking Ricci solitons.

Let $X = X_u\partial_u + X_v\partial_v + \sum_{i=1}^{n} X_i\partial_{x_i}$ be an arbitrary vector field on $(\mathbb{R}^{n+2}, g_{ppw})$, where the metric g_{ppw} is given by the function $H(u, x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}(u)x_i x_j$. Then (1.26) becomes

$$(3.6)\quad \begin{cases} \frac{1}{2}\sum_{i=1}^{n}\partial_{x_i}HX_i + \frac{1}{2}\,\partial_u HX_u + H\partial_u X_u + \partial_u X_v + \rho(\partial_u,\partial_u) = \lambda\, H, & \\ H\partial_v X_u + \partial_v X_v + \partial_u X_u = 2\lambda, & \\ H\partial_{x_i}X_u + \partial_{x_i}X_v + \partial_u X_i = 0, & 1 \le i \le n, \\ \partial_{x_i}X_j + \partial_{x_j}X_i = 0, & 1 \le i \ne j \le n, \\ \partial_{x_i}X_u + \partial_v X_i = 0, & 1 \le i \le n, \\ \partial_v X_u = 0; \quad \partial_{x_i}X_i = \lambda, & 1 \le i \le n, \end{cases}$$

as in (2.11) on previous chapter, but now specialized to the previous function H.

While we are not explicitly integrating previous system of differential equations, in what follows we point out the existence of non-gradient solutions.

Consider the vector field

$$(3.7)\qquad X = \left(p(u) - \sum_{i=1}^{n} q_i'(u)x_i + 2\lambda\, v\right)\partial_v + \sum_{i=1}^{n}\left(q_i(u) + \lambda\, x_i\right)\partial_{x_i},$$

where functions p and q_i satisfy the following conditions

$$(3.8)\qquad \begin{cases} \sum_{j=1}^{n} a_{ij}(u)q_i(u) - q_i''(u) = 0, & 1 \le i \le n, \\ \rho(\partial_u,\partial_u) + p'(u) = 0, & \end{cases}$$

where a_{ij} are the functions that appear in the definition function of a plane wave as in (1.18). Note that one can always find p, q_i being solutions of (3.8). A straightforward calculation from (2.11) shows that (M, g, X) is a Ricci soliton. Also observe that λ is the constant of equation (1.26) and can be chosen arbitrarily.

In contrast to the gradient case, we obtain that

Theorem 3.8 *Any plane wave (M, g) admits vector fields resulting in expanding, steady and shrinking Ricci solitons.*

Remark 3.9 We refer to [6] for an explicit integration of (3.6) in the special case of the Cahen-Wallach symmetric spaces.

Chapter 4

Homogeneous gradient Ricci solitons

The geometry of homogeneous gradient Ricci solitons has been recently investigated in the Riemannian setting [85], showing that any homogeneous Riemannian gradient Ricci soliton is *rigid* (i.e., M is a product $M = N \times \mathbb{R}^k$, where (N, g_N) is an Einstein manifold with Ricci tensor $\rho_N = \lambda\, g_N$ and the potential function of the soliton is given by $f(x) = \frac{\lambda}{2}\|\pi(x)\|^2$, where π is the orthogonal projection on the Euclidean factor). The Lorentzian situation allows other possibilities closely related to the existence of degenerate parallel line fields. For instance it was shown in the previous chapter that indecomposable but not irreducible Lorentzian symmetric spaces are non-rigid gradient Ricci solitons.

In this chapter we analyze the geometry of gradient Ricci solitons which are homogeneous, showing that three-dimensional homogeneous expanding or shrinking gradient Ricci solitons are rigid. On the other hand, we show the existence of non-rigid homogeneous steady gradient Ricci solitons when the background metric corresponds to that of a Walker manifold. We describe the structure of the chapter in more detail as follows. First we give some general properties of gradient Ricci solitons with constant scalar curvature in Section 4.1. Afterwards we study the non-steady and the steady homogeneous cases separately in Sections 4.2 and 4.3. As a particular family of homogeneous manifolds, symmetric spaces are considered in Section 4.4. Finally we classify three-dimensional homogeneous gradient Ricci solitons in Section 4.5. The main results in this chapter are summarized in references [15, 18].

4.1 Gradient Ricci solitons with constant scalar curvature

Since homogeneous spaces have constant scalar curvature, we begin by analyzing some properties of gradient Ricci solitons with this characteristic. Note that by Remark 1.28, the gradient of the potential function is an eigenvector of the Ricci operator associated to the eigenvalue zero. This fact will be crucial to describe the structure of an homogeneous soliton. Moreover, the study of homogeneous spaces is linked with the geometry of Killing vector fields. Recall from Section 1.1 that a vector field X on (M,g) is Killing if and only if $\mathcal{L}_X g = 0$. Now, choose local adapted coordinates $(x_1,\dots,x_{n+2})$ so that $X=\partial_{x_1}$. Denote $g_{ij} := g(\partial_{x_i},\partial_{x_j})$ and observe that

$$\begin{aligned}\partial_{x_1} g_{ij} &= g(\nabla_{\partial_{x_1}}\partial_{x_i},\partial_{x_j}) + g(\partial_{x_i},\nabla_{\partial_{x_1}}\partial_{x_j})\\ &= g(\nabla_{\partial_{x_i}}\partial_{x_1},\partial_{x_j}) + g(\partial_{x_i},\nabla_{\partial_{x_j}}\partial_{x_1}) = (\mathcal{L}_{\partial_{x_1}} g)(\partial_{x_i},\partial_{x_j}).\end{aligned}$$

Hence, if $X=\partial_{x_1}$ is a Killing vector field then $\partial_{x_1} g_{ij}=0$, and thus $\partial_{x_1}\Gamma_{ij}^k = 0$ as well.

Let f be a smooth function on M. Then

$$\begin{aligned}(\mathcal{L}_{\partial_{x_1}}\operatorname{Hes}_f)(\partial_{x_i},\partial_{x_j}) &= \mathcal{L}_{\partial_{x_1}}\operatorname{Hes}_f(\partial_{x_i},\partial_{x_j})\\ &= \mathcal{L}_{\partial_{x_1}}\left(\partial^2_{x_ix_j}(f) - \Gamma_{ij}^k\partial_{x_k}(f)\right)\\ &= \partial^3_{x_1x_ix_j}(f) - \partial_{x_1}(\Gamma_{ij}^k)\partial_{x_k}(f) - \Gamma_{ij}^k\partial^2_{x_1x_k}(f),\end{aligned}$$

and

$$\begin{aligned}\operatorname{Hes}_{\partial_{x_1}(f)}(\partial_{x_i},\partial_{x_j}) &= \partial^2_{x_ix_j}\partial_{x_1}(f) - \Gamma_{ij}^k\partial_{x_k}\partial_{x_1}(f)\\ &= \partial^3_{x_1x_ix_j}(f) - \Gamma_{ij}^k\partial^2_{x_1x_k}(f),\end{aligned}$$

which shows that $(\mathcal{L}_X\operatorname{Hes}_f) = \operatorname{Hes}_{X(f)}$ for all Killing vector fields X on (M,g).

Although we usually denote the gradient of a function by the symbol ∇, for notational clarity we will use $\operatorname{grad} f$ instead along this chapter.

Lemma 4.1 *Let (M,g,f) be a gradient Ricci soliton with constant scalar curvature. If X is a Killing vector field, then $\operatorname{grad} X(f)$ is a parallel vector field. Moreover, if $\lambda\neq 0$, then $\operatorname{grad} X(f)=0$ if and only if $X(f)=0$.*

Proof.
Assume X is a Killing vector field, i.e. $\mathcal{L}_X g=0$, hence $\mathcal{L}_X\rho=0$. Also it follows from (1.27) that $\mathcal{L}_X\operatorname{Hes}_f=0$, and therefore $\operatorname{Hes}_{X(f)}=0$. This shows that $\operatorname{grad} X(f)$ is parallel.

It is clear that $\operatorname{grad} X(f)=0$ if $X(f)=0$. Assume now that $\operatorname{grad} X(f)=0$. Then $X(f)=\kappa$ for some constant κ. Now, since the scalar curvature is constant, from (1.29) we have that $\operatorname{Ric}(\nabla f)=0$ and since X is Killing the following sequence of equalities holds

$$\begin{aligned}0=\nabla f(\kappa)=\nabla f(X(f)) &= \nabla_{\nabla f} g(\nabla f, X)\\ &= g(\nabla_{\nabla f}\nabla f, X) + g(\nabla f, \nabla_{\nabla f} X)\\ &= \operatorname{Hes}_f(\nabla f, X) + \tfrac{1}{2}(\mathcal{L}_X g)(\nabla f,\nabla f)\\ &= -\rho(\nabla f, X) + \lambda\, g(\nabla f, X)\\ &= \lambda\,\kappa,\end{aligned}$$

from where it follows that $\kappa=0$, and thus that $\operatorname{grad} X(f)=0$ if and only if $X(f)=0$. □

Lemma 4.2 *Let (M, g, f) be a gradient Ricci soliton with constant scalar curvature. Then the following relation holds*

$$\lambda((n+2)\lambda - \tau) = \| \operatorname{hes}_f \|^2. \tag{4.1}$$

Proof.
Recall from Lemma 1.27 that $\operatorname{Ric}(\nabla f) = 0$. Also recall the following Bochner identity

$$\frac{1}{2} \Delta g(\nabla f, \nabla f) = \| \operatorname{hes}_f \|^2 + \rho(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f).$$

From (1.30) we have that $\tau + \|\nabla f\|^2 - 2\lambda f = \text{const}$, so the left-hand side in the Bochner formula becomes $\frac{1}{2} \Delta g(\nabla f, \nabla f) = \lambda \Delta f - \frac{1}{2} \Delta \tau$. Taking the trace of the Ricci soliton equation (1.27) show us that $\Delta f = (n+2)\lambda - \tau$ and hence $\frac{1}{2} \Delta g(\nabla f, \nabla f) = \lambda((n+2)\lambda - \tau)$. On the other hand, since $\operatorname{Ric}(\nabla f) = 0$ and $\nabla \Delta f = -\nabla \tau = 0$, the right-hand side in Bochner formula reduces to $\| \operatorname{hes}_f \|^2$, thus getting (4.1). □

Remark 4.3 An immediate application of Lemma 4.2 shows that any Riemannian steady gradient Ricci soliton with constant scalar curvature is Ricci flat. In the Lorentzian case, steady solitons with the same property have isotropic hes_f, and thus, isotropic Ricci operator.

4.2 Non-steady homogeneous gradient Ricci solitons

Theorem 4.4 *Let (M, g, f) be an isotropic non-steady gradient Ricci soliton with constant scalar curvature. Then (M, g) is Einstein.*

Proof.
From Lemma 1.27 we have that $\tau + \|\nabla f\|^2 - 2\lambda f = \text{const}$. Modify the potential function $\tilde{f} = f + \text{const}$ so that $\tau + \|\nabla f\|^2 - 2\lambda f = 0$. Now, if (M, g, f) is isotropic, one has $2\lambda \tilde{f} = -\tau$, which shows that $\tilde{f}$ is constant (and thus (M, g) is Einstein) if and only if the scalar curvature is constant. □

Recall that a Lorentzian manifold is said to be irreducible if the holonomy representation does not admit any nontrivial invariant subspace. Moreover, (M, g) is said to be indecomposable, but not irreducible if the holonomy representation admits a nontrivial invariant subspace on which the metric is degenerate. Hence, we have

Theorem 4.5 *Let (M, g) be a homogeneous irreducible Lorentzian manifold admitting a non-steady gradient Ricci soliton. Then (M, g) is Einstein.*

Proof.
The fact that the holonomy representation does not admit any nontrivial invariant subspace is equivalent to the non-existence of any parallel k-dimensional distribution on (M, g). This implies that all parallel vector fields $\operatorname{grad} X(f)$ must vanish, and hence, since the soliton is non-steady, $X(f) = 0$ for all Killing vector field X on M. This shows that f is constant and the metric is Einstein. □

The following observation by Petersen and Wylie was made in the Riemannian setting, but it extends immediately to arbitrary signature.

Lemma 4.6 [85] *If a gradient Ricci soliton (M, g, f) splits as a pseudo-Riemannian product $(M, g) = (M_1 \times M_2, g_1 \oplus g_2)$, then the potential function $f(x_1 + x_2) = f_1(x_1) + f_2(x_2)$ also splits in such a way that each factor (M_i, g_i, f_i), $i = 1, 2$, is a gradient Ricci soliton.*

We now use the result of Lemma 4.1 to obtain a splitting result generalizing Theorem 4.5.

Theorem 4.7 *Let (M, g) be a homogeneous Lorentzian manifold. If (M, g, f) is a non-steady gradient Ricci soliton, then it splits as a product $M = N \times \mathbb{R}^k$ for some $k \geq 0$, where either*

(a) (N, g_N) is a Lorentzian Einstein manifold and the soliton is rigid, or

(b) (N, g_N) is a Lorentzian Walker manifold admitting a parallel null vector field.

Proof.
Recall from Lemma 4.1 that if X is a Killing vector field then $\operatorname{grad} X(f)$ is parallel. If $\operatorname{grad} X(f)$ is spacelike or timelike then splits a one-dimensional factor from (M, g). Hence, after considering all such possible Killing vector fields, (M, g) splits as $M = N \times \mathbb{R}^k$ for some $k \geq 0$ such that no Killing vector field Y exists on (N, g_N) with $\operatorname{grad} Y(f)$ spacelike or timelike.

If $Y(f) = 0$ for all Killing vector fields on (N, g_N), then it is an Einstein manifold and the rigidity is obtained as in [85]. Otherwise, assume there is a Killing vector field Z such that $\operatorname{grad} Z(f)$ is null. This shows that (N, g_N) is a strict Walker manifold. □

Remark 4.8 In the proof of Theorem 4.7, there is a unique parallel null vector field of the form $\operatorname{grad} Z(f)$, Z being a Killing vector field. Indeed, if Z_1, Z_2 are Killing vector fields such that $\operatorname{grad} Z_1(f)$ and $\operatorname{grad} Z_2(f)$ are null vectors, then they span a Lorentzian vector space $\pi = \operatorname{span}\{\operatorname{grad} Z_1(f), \operatorname{grad} Z_2(f)\}$. Since no orthogonal null vectors may exist in Lorentzian signature, it follows that $\operatorname{grad} Z_1(f) \pm \operatorname{grad} Z_2(f)$ are non-null parallel vector fields on (N, g_N), but this contradicts the assumption that no such vector fields exist.

4.3 Steady homogeneous gradient Ricci solitons

First of all, observe from Lemma 4.2 that $\| \operatorname{hes}_f \|^2 = 0$ for all homogeneous steady gradient Ricci solitons (and thus also $\| \operatorname{Ric} \|^2 = 0$). Moreover, since $\operatorname{hes}_f = -\operatorname{Ric}$ one has from Lemma 1.27 that $\operatorname{hes}_f(\nabla f) = 0$, which shows that ∇f is a geodesic vector field. Next, using the identity $\tau + \|\nabla f\|^2 - 2\lambda f = \text{const}$, one has that $\|\nabla f\|^2$ is constant and therefore f is a solution of the Eikonal equation $\|\nabla f\|^2 = \mu$.

Remark 4.9 Observe that all the conclusions above still remain true under the more general hypothesis that (M, g) has constant scalar curvature.

In what follows we will consider separately the different situations corresponding to the possible values of μ. We begin by analyzing the case $\mu < 0$.

Theorem 4.10 *Let (M, g, f) be a homogeneous steady gradient Ricci soliton such that $\|\nabla f\|^2 = \mu < 0$. Then (M, g) splits isometrically as a product $(\mathbb{R} \times N, -dt^2 + g_N)$, where (N, g_N) is a flat Riemannian manifold and f is the projection on $\mathbb{R}$.*

Proof.
Under these assumptions f is a solution of the timelike Eikonal equation. The special significance of this case comes from the fact that $\operatorname{hes}_f(\nabla f) = 0$, and thus one can consider the restriction of the Hessian tensor to $\nabla f^\perp$, which is now a positive definite vector space. Hence the Hessian operator diagonalizes and thus the condition $\| \operatorname{hes}_f \|^2 = 0$ in Lemma 4.2 shows that indeed $\operatorname{hes}_f = 0$.

Now, $\operatorname{hes}_f = 0$ shows that ∇f is a parallel vector field, and thus (M, g) is locally a product $(\mathbb{R} \times N, -dt^2 + g_N)$, where (N, g_N) is a homogeneous Riemannian manifold (see, for example, [57]). Moreover, it follows from Lemma 4.6 that (N, g_N) is a homogeneous steady gradient Ricci soliton, and therefore Einstein. Since the soliton is steady, one has that (N, g_N) is Ricci flat and, finally, given that all Ricci flat homogenous manifolds are flat, that (N, g_N) is indeed flat. □

The cases when $\mu \geq 0$ are less rigid than the previous one and they allow the existence of non-trivial homogeneous steady gradient Ricci solitons. In the isotropic case one has some restrictions on the Ricci operator, which must be nilpotent according to the following

Lemma 4.11 *Let (M, g, f) be an isotropic homogeneous steady gradient Ricci soliton. There exists a local pseudo-orthonormal basis $\{U, V, E_1, \ldots, E_n\}$ such that*

$$\operatorname{hes}_f = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ a & 0 & b_1 & b_2 & \ldots & b_n \\ b_1 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{4.2}$$

Moreover, the Ricci operator is two-step nilpotent if $b_1 = b_2 = b_n = 0$, or three-step nilpotent otherwise.

Proof.
Since ∇f is a null vector, choose orthogonal vectors X and Y so that $V = \nabla f = X + Y$, $U = X - Y$ and $\langle U, V \rangle = 1$. Let $\overline{\nabla f}^{\perp}$ denote the non-degenerate normal space giving by $\overline{\nabla f}^{\perp} = \nabla f^{\perp} / \operatorname{span}\{\nabla f\}$ and denote by $\pi : \nabla f^{\perp} \to \overline{\nabla f}^{\perp}$ the projection. Now the fact that $\operatorname{hes}_f(\nabla f) = 0$ allow us to consider the induced Hessian operator on $\overline{\nabla f}^{\perp}$ defined by $\overline{\operatorname{hes}_f}(\overline{x}) = \pi(\operatorname{hes}_f(x))$ for any x such that $\pi(x) = \overline{x}$, which remains self-adjoint.

Since the induced metric on the n-dimensional non-degenerate normal space $\overline{\nabla f}^{\perp}$ is positive definite, $\overline{\operatorname{hes}_f}$ diagonalizes in an orthonormal basis $\{\bar{E}_1, \dots, \bar{E}_n\}$ with eigenvalues $\overline{\lambda}_1, \dots, \overline{\lambda}_n$. Then, it follows that $\|\overline{\operatorname{hes}_f}\|^2 = \| \operatorname{hes}_f \|^2 = 0$, and thus $\overline{\lambda}_1 = \cdots = \overline{\lambda}_n = 0$. Now the result follows by using that hes_f is self-adjoint and $\operatorname{hes}_f(\nabla f) = 0$. □

Remark 4.12 There exists homogeneous Lorentzian manifolds with two and three-step nilpotent Ricci operators. However, in dimension three, no homogeneous Lorentzian manifold may have three-step nilpotent Ricci operator [26].

Observe that if (M, g, f) is an isotropic homogeneous steady gradient Ricci soliton with two-step nilpotent Ricci operator, then hes_f is given by (4.2) with $b_1 = \cdots = b_n = 0$ and hence ∇f is a recurrent vector field. This implies that there exists a parallel one-dimensional null distribution on (M, g). Moreover, one has the following

Lemma 4.13 *Let (M, g, f) be a non-trivial isotropic homogeneous steady gradient Ricci soliton with two-step nilpotent Ricci operator. Then there exists a parallel null vector field on (M, g).*

Proof.
Let (M, g, f) be a homogeneous steady gradient Ricci soliton with two-step nilpotent Ricci operator. Adopt the notation of Lemma 4.11 and consider a basis $\{U, V, E_1, \dots, E_n\}$ so that hes_f has associated matrix as in (4.2) at given point $p \in M$. Extend E_i, $i = 1, \dots, n$ being Killing vector fields and U so that $g(U, U) = 0$ and $g(U, V) = 1$. Now, since E_i is Killing, by Lemma 4.1, $\operatorname{grad}(E_i(f))$ is a parallel vector field. Several possibilities may occur now, we analyze them separately.

First we show that $\operatorname{grad}(E_i(f))$ cannot be non-null for all $i = 1, \dots, n$. Assume on the contrary that $\operatorname{grad}(E_i(f))$ is non-null for all $i = 1, \dots, n$, then the manifold splits as $M = N \times \mathbb{R}^n$ where N is a two-dimensional gradient Ricci solitons by Lemma 4.6. Note that (N, g_N) can be Riemannian or Lorentzian, but an argument as in Section 1.4.4 shows that the only possibility for a steady homogeneous manifold is that (N, g_N) is flat and the potential function is constant. So M reduces to a product $M = \mathbb{R}^2 \times \mathbb{R}^n$ with $\operatorname{hes}_f = 0$, which contradicts the non-triviality assumption.

Secondly, if $\| \operatorname{grad}(E_i(f))\|^2 = 0$ for some $i \in \{1, \dots, n\}$, then this is a parallel null vector field.

Finally, if $\operatorname{grad}(E_i(f)) = 0$ for $i = 1, \dots, k$ and $\| \operatorname{grad}(E_i(f))\|^2 \neq 0$ for $i = k+1, \dots, n$, where it may be $k = n$. Then the manifold splits as $N \times \mathbb{R}^{n-k}$ and N is a gradient Ricci soliton by Lemma 4.6. If N has positive signature then the soliton is trivial, so we assume N is Lorentzian. Then $E_i(f) = \kappa_i$ where κ_i is a constant for $i = 1, \dots, k$, but since $E_i(f) = 0$ at p, we have that $g(E_i, V) = 0$ everywhere. This shows that hes_f has associated matrix as in (4.2) everywhere with $b_1 = \cdots = b_k = 0$ since it is two-step nilpotent.

Note that

$$0 = \rho(U, V) = R(U, V, V, U) + \sum_{i,j=1}^{n} R(U, E_i, V, E_j) = R(U, V, V, U),$$

so $R(U, V, U, V) = 0$. Also, in general, if X is a Killing vector field, then we have that $R(X, Y)Z = -\nabla_Y \nabla_Z X + \nabla_{\nabla_Y Z} X$ for all Y, $Z \in \mathfrak{X}(M)$ (see, for example, [68, 77]). Since E_i are Killing vector fields, we compute

$$\begin{aligned} R(E_i, U, U, V) &= -g(\nabla_U \nabla_U E_i, V) + g(\nabla_{\nabla_U U} E_i, V) \\ &= -U\, g(\nabla_U E_i, V) + g(\nabla_U E_i, \nabla_U V) + (\nabla_U U) g(E_i, V) - g(E_i, \nabla_{\nabla_U U} V) \\ &= -U(U\, g(E_i, V) - g(E_i, \nabla_U V)) + g(\nabla_U E_i, aV) \\ &= -U\, g(E_i, a\, V) + a\, g(\nabla_U E_i, V) = 0 \end{aligned}$$

since $g(\nabla_U E_i, V) = U\, g(E_i, V) - g(E_i, \nabla_U V) = -g(E_i, a\, V) = 0$.

Now, as we saw in the proof of Lemma 2.3, the identity $R(X, Y, Z, \nabla f) = (\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z)$ holds for any gradient Ricci soliton for any X, Y, $Z \in \mathfrak{X}(M)$, so we have that

$$\begin{aligned} 0 &= R(U, V, U, V) = (\nabla_U \rho)(V, U) - (\nabla_V \rho)(U, U) \\ 0 &= R(E_i, U, U, V) = (\nabla_{E_i} \rho)(U, U) - (\nabla_U \rho)(E_i, U). \end{aligned}$$

Taking into account that $\rho = -\operatorname{Hes}_f$, this gives rise to

$$\begin{aligned} -V(a) &= V \rho(U, U) \\ &= (\nabla_V \rho)(U, U) + 2\, \rho(\nabla_V U, U) \\ &= (\nabla_U \rho)(V, U) - 2\, g(\nabla_V U, a\, V) \\ &= U\, \rho(V, U) - \rho(\nabla_U V, U) - \rho(V, \nabla_U U) - 2a(V g(U, V) - g(U, \nabla_V V)) = 0 \end{aligned}$$

and for all $i = 1, \dots, n$ we have

$$
\begin{aligned}
-E_i(a) &= E_i\rho(U,U) \\
&= (\nabla_{E_i}\rho)(U,U) + 2\,\rho(\nabla_{E_i}U,U) \\
&= (\nabla_U\rho)(E_i,U) - 2\,g(\nabla_{E_i}U,a\,V) \\
&= U\,\rho(E_i,U) - \rho(\nabla_U E_i,U) - \rho(E_i,\nabla_U U) - 2a(E_i g(U,V) - g(U,\nabla_{E_i}V)) \\
&= g(\nabla_U E_i, a\,V) = \\
&= a\,U g(E_i,V) - a\,g(E_i,a\,V) = 0.
\end{aligned}
$$

Next, let $h : M \to \mathbb{R}$ be a function on M which is constant in every direction except in the direction of U, thus $V(h) = E_i(h) = 0$ and such that $U(h) + h\,a = 0$. Such an h exists precisely because $V(a)$ and $E_i(a) = 0$ for all $i = 1, \ldots, k$. Consider the vector field $h\,V$ and compute:

$$
\begin{aligned}
\nabla_V(h\,V) &= V(h)V + h\nabla_V V = 0\,, \\
\nabla_U(h\,V) &= U(h)V + h\nabla_U V = (U(h) + h\,a)V = 0\,, \\
\nabla_{E_i}(h\,V) &= E_i(h)V + h\nabla_{E_i}V = 0\,.
\end{aligned}
$$

Hence $h\,V$ is a parallel null vector field. □

4.4 Symmetric gradient Ricci solitons

Let (M,g) be a symmetric Lorentzian manifold. If (M,g) admits a non-steady gradient Ricci soliton, then Theorem 4.7 shows that it splits as a product $M = N \times \mathbb{R}^k$, where either (N,g_N) is an irreducible Lorentzian symmetric manifold (and thus of constant sectional curvature) and the soliton is rigid or, otherwise, (N,g_N) is an indecomposable but not irreducible Lorentzian symmetric space (hence a Cahen-Wallach symmetric space). Now, it follows from the results in previous chapter that any gradient Ricci soliton on (N,g_N) is steady, and thus any symmetric non-steady Lorentzian gradient Ricci soliton is rigid.

Next, assume (M,g,f) to be a Lorentzian steady symmetric gradient Ricci soliton. Using the de Rham-Wu decomposition of the manifold, (M,g) splits as a product $M = \mathbb{R}^k_\nu \times N \times M_1 \times \cdots \times M_l$, where $\mathbb{R}^k_\nu$ is either the Euclidean or the Minkowskian space, (N,g_N) is a Cahen-Wallach symmetric space and M_i are irreducible symmetric spaces. Since irreducible symmetric spaces are Einstein, the induced soliton is either trivial or otherwise the scalar curvature vanishes, which implies that M_i is Ricci flat. Now, observe that if M_i is Riemannian, then it is flat since Ricci flat homogeneous spaces are flat in the Riemannian setting [9]. Moreover, if M_i is Lorentzian, then it is flat since irreducible Lorentzian symmetric spaces are of constant sectional curvature [24]. Hence, if the gradient Ricci soliton is steady, then the decomposition above reduces to $M = N \times \mathbb{R}^k$, where (N,g_N) is a Cahen-Wallach symmetric space. Hence,

Theorem 4.14 *Let (M, g, f) be a symmetric Lorentzian gradient Ricci soliton.*

(i) If (M, g, f) is not steady, then (M, g) splits as a product $M = N \times \mathbb{R}^k$, where (N, g_N) is Einstein and the soliton is rigid.

(ii) If (M, g, f) is steady, then (M, g) splits as a product $M = N \times \mathbb{R}^k$, where (N, g_N) is a Cahen-Wallach symmetric space and the potential function of the soliton is given by Theorem 3.1.

4.5 Three-dimensional homogeneous gradient Ricci solitons

The special properties of dimension three allow us to obtain some more conclusive results about homogeneous gradient Ricci solitons. Due to the fact that the Weyl tensor vanishes, the Ricci operator completely determines the curvature. The purpose of this section is to prove the following classification result for three-dimensional homogeneous gradient Ricci solitons.

Theorem 4.15 *Let (M, g, f) be a three-dimensional homogeneous Lorentzian gradient Ricci soliton. Then (M, g) is of constant sectional curvature and the soliton is trivial, or otherwise*

(i) (M, g) splits as $\mathbb{R} \times N$, where (N, g_N) is a surface of constant curvature, and the soliton is rigid.

(ii) (M, g) admits a parallel null vector field and either

(ii.1) there exists local coordinates (t, x, y) so that the metric g takes the form

$$g = 2dtdy + dx^2 + \phi(x, y)dy^2$$

for some function

$$\phi(x, y) = \frac{1}{\alpha^2}\, a(y)\, e^{\alpha x} + x\, b(y) + c(y)$$

and the potential function of the soliton is given by $f(x, y) = x\,\alpha + \gamma(y)$ with $\gamma''(y) = -\frac{1}{2}\,\alpha\, b(y)$.

(ii.2) or (M, g) is locally conformally flat and hence a plane wave.

Moreover, in both cases $(ii.1)$ and $(ii.2)$ the Ricci soliton is steady.

Remark 4.16 Observe that metrics in case (ii) above have two-step nilpotent Ricci operator and hence they have recurrent curvature [30]. Indeed, cases $(ii.1)$ and $(ii.2)$ correspond to types I and II of Lorentzian manifolds with recurrent curvature as discussed in Section 3.1.

Remark 4.17 Homogeneous plane waves have been characterized in [12], showing that in the three-dimensional case the metric (1.12) reduces to one of the following two possibilities

(i) A symmetric Cahen-Wallach space, i.e., the Euclidean space $\mathbb{R}^3$ with coordinates (u, v, x) and metric tensor

$$g = 2dudv + \kappa\, x^2 du^2 + dx^2,$$

where κ is an arbitrary non-zero constant, or

(ii) a non-symmetric homogeneous plane wave, i.e., an open set $\mathcal{U} \subset \mathbb{R}^3$ with coordinates (u, v, x) and metric tensor

$$g = 2dudv + \frac{\kappa}{u^2}\, x^2 du^2 + dx^2,$$

where κ is an arbitrary non-zero constant.

In both cases they admit steady gradient Ricci solitons, which are constructed as discussed in Chapter 3.

Remark 4.18 Proceeding as in Section 1.4.2, one sees that a Lorentzian manifold admits two-distinct gradient Ricci solitons if and only if it admits a gradient homothetic vector field. Hence, it follows from results in [31] that a three-dimensional homogeneous Lorentzian manifold admits two distinct gradient Ricci solitons if and only if it is flat, which shows the uniqueness of steady gradient Ricci solitons considered in Theorem 4.15 – (*ii*).

The aim of this section is to prove Theorem 4.15.

First of all, since Walker metrics play a distinguished role in dimension three, we begin by characterizing which manifolds in this family result in a gradient Ricci soliton.

Lemma 4.19 *Let (M, g) be a non-flat three-dimensional Lorentzian manifold admitting a parallel null vector field. Then (M, g, f) is a gradient Ricci soliton if and only if one of the following occurs:*

(i) There exist coordinates (t, x, y) so that the metric g takes the form

$$g = 2dtdy + dx^2 + \phi(x, y)dy^2$$

where

$$\phi(x, y) = \frac{1}{\alpha^2}\, a(y)\, e^{\alpha x} + x\, b(y) + c(y)$$

for some arbitrary functions a, b and c. Moreover, in this case the potential function of the soliton is given by $f(t, x, y) = x\,\alpha + \gamma(y)$, where $\gamma''(y) = -\frac{1}{2}\,\alpha b(y)$.

(ii) There exist coordinates (t, x, y) *so that the metric* g *takes the form*

$$g = 2dtdy + dx^2 + \phi(x, y)dy^2$$

where

$$\phi(x, y) = x^2\, a(y) + x\, b(y) + c(y)$$

for some arbitrary functions a, b *and* c. *Moreover the potential function of the soliton is given by* $f(t, x, y) = \gamma(y)$, *where* $\gamma''(y) = \frac{1}{4}\, a(y)$.

Moreover, in both cases the Ricci soliton is steady.

Proof.
Choose adapted Walker coordinates (t, x, y) so that the metric expresses as follows (we refer to [21] for a broad exposition on Walker manifolds)

$$g = 2dtdy + dx^2 + \phi(x, y)dy^2. \tag{4.3}$$

where ϕ is an arbitrary function. Let $f(t, x, y)$ be an arbitrary function on M. In order to simplify the notation we denote with sub-indices the corresponding partial derivatives, thus, for example, $f_t = \frac{\partial f}{\partial t}$, $f_{tx} = \frac{\partial^2 f}{\partial t \partial x}, \dots$. Then a straightforward calculation shows that (1.27) is equivalent to the following

$$f_{tt} = f_{tx} = 0, \tag{4.4}$$

$$f_{xx} - \lambda = f_{ty} - \lambda = 0, \tag{4.5}$$

$$2f_{xy} - \phi_x f_t = 0, \tag{4.6}$$

$$2\lambda\, \phi + \phi_{xx} - 2f_{yy} - \phi_x f_x + \phi_y f_t = 0. \tag{4.7}$$

Now, it follows from (4.4)–(4.5) that the potential function of any gradient Ricci soliton satisfies

$$f(t, x, y) = t(\lambda\, y + \kappa) + \frac{1}{2}\, \lambda\, x^2 + \alpha(y)\, x + \gamma(y)$$

for some functions α, γ and a constant κ. Hence, the previous system of PDEs reduces to

$$2\, \alpha'(y) - (\lambda\, y + \kappa)\ \phi_x = 0, \tag{4.8}$$

$$2\, \lambda\, \phi - 2\, \gamma''(y) - 2\, x\, \alpha''(y) + (\lambda\, y + \kappa)\, \phi_y - (\lambda\, x + \alpha(y))\, \phi_x + \phi_{xx} = 0. \tag{4.9}$$

Next differentiate (4.8) with respect to x to obtain

$$0 = (\lambda\, y + \kappa)\, \phi_{xx}. \tag{4.10}$$

Note here that a simple calculation gives that the Ricci operator of any three-dimensional strict Walker metric satisfies

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\,\phi_{xx} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which shows that (4.3) is flat if and only if $\phi_{xx} = 0$. Hence, assuming the Walker metric is non-flat, one has that $\lambda = \kappa = 0$, so the gradient Ricci soliton is steady.

Now, from equations (4.8) and (4.10) we get that the potential function of any gradient Ricci soliton on a non-flat Walker manifold (4.3) is given by $f(t, x, y) = \alpha\, x + \gamma(y)$. Moreover, in this case (4.9) reduces to

$$2\,\gamma''(y) + \alpha\,\phi_x - \phi_{xx} = 0. \tag{4.11}$$

Hence, differentiating with respect to x, one has that $\alpha\,\phi_{xx} = \phi_{xxx}$. Two possibilities may occur:

- $\alpha \neq 0$. In this case the metric (4.3) is given by a function

  $$\phi(x, y) = \frac{1}{\alpha^2}\, a(y)\, e^{\alpha x} + x\, b(y) + c(y)$$

  for some arbitrary functions $a(y) \neq 0$, $b(y)$ and $c(y)$. Moreover the potential function of the soliton is given by $f(t, x, y) = \alpha\, x + \gamma(y)$, where $\gamma''(y) = -\frac{1}{2}\,\alpha\, b(y)$.

  In this case $\nabla f = \gamma'(y)\,\partial_t + \alpha\,\partial_x$ is spacelike.

- $\alpha = 0$. In this case the metric (4.3) is given by a function

  $$\phi(x, y) = x^2\, a(y) + x\, b(y) + c(y)$$

  for some arbitrary functions $a(y) \neq 0$, $b(y)$ and $c(y)$. Moreover the potential function of the soliton is given by $f(t, x, y) = \gamma(y)$, where $\gamma''(y) = \frac{1}{4}\, a(y)$.

  In this case $\nabla f = \gamma'(y)\,\partial_t$ is a null and recurrent vector field.

This completes the proof. □

Now, as an application of Lemmas 4.1 and 4.19, one has

Lemma 4.20 *Any three-dimensional non-steady homogeneous Lorentzian gradient Ricci soliton is either trivial or rigid.*

Proof.
For any Killing vector field X on (M, g), proceeding as in Section 4.1, either $X(f) = 0$ for all Killing vector fields or there exist some Killing vector fields such that $X(f) \neq 0$. If

$X(f) = 0$ for all Killing vector fields then f is constant and (M, g) is of constant sectional curvature. Assume, on the contrary, that there exists a Killing vector field X such that $X(f) \neq 0$. If $\operatorname{grad} X(f)$ is non-null, then it induces a local splitting of the form $I \times N$, where I is a real interval and (N, g_N) is a two-dimensional manifold of constant curvature, from where it follows that the soliton is rigid. If $\operatorname{grad} X(f)$ is null, then the manifold has a null parallel vector field and by Lemma 4.19 it cannot be non-steady unless it is flat. □

Now we we are going to analyze the steady case.

Lemma 4.21 *Let (M, g, f) be a three-dimensional homogeneous steady gradient Ricci soliton. Then either it is rigid or it admits a parallel null vector field.*

Proof.
Let (M, g, f) be a three-dimensional homogeneous steady gradient Ricci soliton. Hence, as discussed in previous section, the potential function is a solution of the Eikonal equation $\|\nabla f\|^2 = \mu$. In the case $\mu < 0$, the manifold splits as a product and the soliton is rigid (Theorem 4.10). In the isotropic case, the Ricci operator must be two- or three-step nilpotent (Lemma 4.11). As it is shown in [26] there are no homogeneous Walker manifolds with three-step nilpotent Ricci operator. If the Ricci operator is two-step nilpotent, then the manifold admits a parallel null vector field (Lemma 4.13) and the existence of gradient Ricci solitons has been already considered in Lemma 4.19. Finally, consider the spacelike case $\|\nabla f\|^2 = \mu > 0$. Since the scalar curvature is constant, the Ricci operator satisfies $\operatorname{Ric}(\nabla f) = 0$. This shows that either f is constant, or else the Ricci operator has a zero eigenvalue. We now consider the different possibilities for the kernel of Ric.

Assume $\dim(\ker(\operatorname{Ric})) = 1$. It follows from [27] that (M, g) is either a symmetric space or a Lie group. If (M, g) is symmetric, then it is one of the following: a manifold of constant sectional curvature, a product $\mathbb{R} \times N$ where (N, g_N) is of constant curvature, or a three-dimensional Cahen-Wallach symmetric space. Hence, in all the cases, any gradient Ricci soliton is trivial, rigid or the underlying manifold admits a null parallel vector field. Now we concentrate on Lie groups. Since the eigenspaces of the Ricci operator are left-invariant and ∇f has constant norm $\mu > 0$, it follows from the fact that $\dim(\ker(\operatorname{Ric})) = 1$ that ∇f is a left-invariant vector field. By Theorem 5.5 in next chapter, the existence of left-invariant Ricci solitons is equivalent to the fact that the Ricci operator has exactly one-single eigenvalue, which must be zero since $\operatorname{Ric}(\nabla f) = 0$. This shows that the Ricci operator is two-step nilpotent and the manifold admits a null parallel vector field [21, 30]. Hence, the existence of gradient Ricci solitons is covered by Lemma 4.19.

Next assume $\dim(\ker(\operatorname{Ric})) = 2$. In this case the Ricci operator is either diagonalizable or two-step nilpotent. The later implies that the manifold admits a null parallel vector field, and the result is covered by Lemma 4.19, while in the first case the result follows from the splitting theorem in [57] (note that since $\operatorname{Hes}_f = -\rho$, the Ricci operator is diagonalizable if and only if so is the Hessian operator). □

Open problems I

There are many open questions on the geometry of Lorentzian gradient Ricci solitons. Some interesting problems which are closely related to our work are the following:

- Compact Riemannian Ricci solitons are necessarily gradient and, moreover, compact expanding or steady Ricci solitons are necessarily Einstein [41]. These results depend on the validity of maximum principles for the Laplace operator. It is an open question whether they remain valid in the Lorentzian setting.

- Riemannian homogeneous gradient Ricci solitons are rigid [85]. We showed that this is not true in the Lorentzian setting and gave some partial results on the characterization of Lorentzian homogeneous gradient Ricci solitons. One might wonder if the underlying structure of isotropic homogeneous gradient Ricci solitons is a *pp*-wave in view of the results in Section 4.3.

  However, it may happen that some additional hypothesis on the curvature of such homogeneous spaces is needed. One such condition could be the harmonicity of the Weyl tensor as it provided some useful information in the Riemannian setting [52].

- Riemannian Ricci solitons whose curvature operator $\mathcal{R} : \Lambda^2(M) \to \Lambda^2(M)$ is non-negative have been extensively investigated. Although bounds on the sectional curvature of Lorentzian manifolds are very rigid, one may expect to obtain some information by considering the sign of the curvature tensor as in [4].

Part II

Generalizations of gradient Ricci solitons

Chapter 5

Homogeneous Ricci solitons

The purpose of this chapter is to discuss homogeneous Lorentzian Ricci solitons in dimension three. Three-dimensional locally homogeneous Lorentzian manifolds are either locally symmetric or locally isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric. Moreover, three-dimensional locally symmetric Lorentzian manifolds which are not of constant sectional curvature either are locally isometric to a Lorentzian product of a real line and a surface of constant Gauss curvature, or they are Walker manifolds with two-step nilpotent Ricci operator [27].

As proved in [39] (see also [60, 83]), three-dimensional Lie groups do not admit left-invariant Riemannian Ricci solitons. In this chapter we study the corresponding existence problem in Lorentzian signature. This context was also investigated in [80], showing the existence of expanding, steady and shrinking left-invariant Ricci solitons. Besides their geometric interest, homogeneous Ricci solitons are also important in Physics, as they have been used in the construction of black holes [60, 61].

We organize this chapter as follows. First of all we consider the existence of Ricci solitons given by left-invariant vector fields on Lie groups; we provide a complete description and study some geometric properties in Section 5.1. Secondly, in Section 5.2, we focus on the study of Ricci solitons on three-dimensional Walker manifolds with nilpotent Ricci operator, proving the existence of expanding, steady and shrinking locally symmetric Ricci solitons. The results of this chapter are collected in [14].

5.1 Invariant Ricci solitons on three-dimensional Lie groups

Let $(G, \langle \cdot, \cdot \rangle)$ be a three-dimensional Lie group equipped with a left-invariant metric.

Definition 5.1 $(G, \langle \cdot, \cdot \rangle, X)$ *is an left-invariant Ricci soliton if* (1.26) *holds and* X *is a left-invariant vector field.*

A complete description of left-invariant Ricci solitons on three-dimensional Lie groups, is given in [14]. The result is summarized as follows

Theorem 5.2 *Let G be a three-dimensional Lorentzian Lie group equipped with a left-invariant metric. $(G, \langle\cdot\,,\cdot\rangle, X)$ is a left-invariant Ricci soliton if and only if it corresponds to one of the following:*

(i) G is a unimodular Lie group with one of the following Lie algebras:

$$(i.1)\quad \begin{aligned} &[e_1, e_2] = \tfrac{1}{2}\, e_2 - (\beta - \tfrac{1}{2})\, e_3,\\ &[e_1, e_3] = -(\beta + \tfrac{1}{2})\, e_2 - \tfrac{1}{2}\, e_3,\\ &[e_2, e_3] = \alpha\, e_1, \end{aligned}$$

with either $\alpha = 0$ or $\alpha = \beta \neq 0$. If $\alpha = 0$ then $G = E(1,1)$, while if $\alpha = \beta \neq 0$ then $G = O(1,2)$ or $G = SL(2,\mathbb{R})$.

$$(i.2)\quad \begin{aligned} &[e_1, e_2] = -\tfrac{1}{\sqrt{2}}\, e_1 - \alpha\, e_3,\\ &[e_1, e_3] = -\tfrac{1}{\sqrt{2}}\, e_1 - \alpha\, e_2,\\ &[e_2, e_3] = \alpha\, e_1 + \tfrac{1}{\sqrt{2}}\, e_2 - \tfrac{1}{\sqrt{2}}\, e_3. \end{aligned}$$

If $\alpha = 0$ then $G = E(1,1)$, while if $\alpha \neq 0$ then either $G = O(1,2)$ or $G = SL(2,\mathbb{R})$.

(ii) G is a non-unimodular Lie group with Lie algebra given by

$$\begin{aligned} [e_1, e_2] &= -\tfrac{1}{\sqrt{2}}\left(\alpha\, e_1 + \tfrac{1}{\sqrt{2}}\,\beta\,(e_2 + e_3)\right),\\ [e_1, e_3] &= \tfrac{1}{\sqrt{2}}\left(\alpha\, e_1 + \tfrac{1}{\sqrt{2}}\,\beta\,(e_2 + e_3)\right), \qquad \alpha + \delta \neq 0,\ \delta \neq 0\\ [e_2, e_3] &= \tfrac{1}{\sqrt{2}}\,\delta\,(e_2 + e_3). \end{aligned}$$

In all the cases above, $\{e_1, e_2, e_3\}$ is an orthonormal basis of signature $(++-)$ of the corresponding Lie algebra.

Remark 5.3 The invariant Ricci soliton structures in previous theorem satisfy the following:

(i.1) Ricci solitons are steady ($\lambda = 0$) if $\alpha = 0$ and the left-invariant vector field is given by $X = -\beta\, e_1$.

Ricci solitons are expanding $\left(\lambda = -\frac{1}{2}\,\beta^2\right)$ if $\alpha = \beta \neq 0$, and there exists a one-parameter family of left-invariant Ricci solitons given by $X = -\frac{1}{2}\,\beta\, e_1 + t\, e_2 + t\, e_3$, for any $t \in \mathbb{R}$.

(i.2) Ricci solitons are expanding $\left(\lambda = -\frac{1}{2}\,\alpha^2\right)$ and the left-invariant vector field is given by $X = \alpha\, e_1 - \frac{1}{\sqrt{2}}\, e_2 + \frac{1}{\sqrt{2}}\, e_3$.

(ii) In the general case, Ricci solitons are steady and the left-invariant vector field is given by $X = \frac{\alpha^2 - \alpha\delta}{2\delta\sqrt{2}}\,(e_2 + e_3)$. Moreover, in the special case when $\alpha = \frac{1}{2}\,\delta$, the left-invariant Ricci solitons are given by

$$X = -\frac{2\beta\lambda}{\delta^2}\, e_1 - \frac{\delta^4 + 8(\delta^2 - 2\beta^2)\lambda}{8\delta^3\sqrt{2}}\, e_2 - \frac{\delta^4 - 8(\delta^2 + 2\beta^2)\lambda}{8\delta^3\sqrt{2}}\, e_3$$

and they can be expanding, steady or shrinking, depending on the value of λ.

Remark 5.4 Note that none of the Lie algebras in Theorem 5.2 have a Riemannian counterpart. This agrees with the results in [39] on non-existence of non-trivial left-invariant Ricci solitons on Riemannian Lie groups. On the other hand, some three-dimensional Lie groups admit non-trivial Ricci solitons which are not left-invariant [41].

Taking into account the classification given in Theorem 5.2, we have the following

Theorem 5.5 *A non-symmetric three-dimensional Lorentzian Lie group with a left-invariant metric results in a non-trivial Ricci soliton if and only if the Ricci operator* Ric *has exactly three equal eigenvalues.*

Let $\times$ denote the Lorentzian vector product on $\mathbb{R}^3_1$ induced by the product of the para-quaternions (i.e., $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of signature $(+ + -)$). The Lie bracket $[\,\cdot\,,\cdot\,]$ defines the corresponding Lie algebra $\mathfrak{g}$, which is unimodular if and only if the endomorphism L defined by $[Z, Y] = L(Z \times Y)$ is self-adjoint [89] and non-unimodular if L is not self-adjoint. We recall that the Ricci operator, Ric, being self-adjoint, is always diagonalizable in the Riemannian case, while four different possibilities can occur at each point of a Lorentzian manifold (cf. Remark 1.1).

For the sake of completeness we include a brief description of three-dimensional unimodular and non-unimodular Lie groups. Theorem 5.2 will be a consequence of the analysis carried out in Subsections 5.1.1 and 5.1.2. Theorem 5.5 will follow from the subsequent analysis and the proof will be completed in Subsection 5.1.3.

5.1.1 Unimodular Lie groups

Lie groups having unimodular Lie algebras compatible with the Ricci soliton equation (1.26), and listed in Theorem 5.2, can be deduced from [89] (see also [27]). Considering the different types of L, we have the following four classes of unimodular three-dimensional Lie algebras (we follow notation in [56]):

Type (I_a).

If L is diagonalizable with eigenvalues $\{\alpha, \beta, \gamma\}$ with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+ + -)$, the corresponding Lie algebra is given by

$$\text{(5.1)} \qquad \mathfrak{g}_{I_a}: \qquad [e_1, e_2] = -\gamma\, e_3, \quad [e_1, e_3] = -\beta\, e_2, \quad [e_2, e_3] = \alpha\, e_1.$$

Up to symmetries, the only non-vanishing components of the curvature tensor are given by

$$\begin{aligned} R_{1221} &= \tfrac{1}{4}\left(\alpha^2+\beta^2-3\gamma^2-2\alpha\beta+2\alpha\gamma+2\beta\gamma\right), \\ R_{1313} &= \tfrac{1}{4}\left(\alpha^2-3\beta^2+\gamma^2+2\alpha\beta-2\alpha\gamma+2\beta\gamma\right), \\ R_{2332} &= \tfrac{1}{4}\left(3\alpha^2-\beta^2-\gamma^2-2\alpha\beta-2\alpha\gamma+2\beta\gamma\right), \end{aligned}$$

and the Ricci operator is diagonalizable (that is, of type (I_a)) with respect to the basis $\{e_1, e_2, e_3\}$ with eigenvalues

$$\lambda_1 = \tfrac{1}{2}\left((\beta-\gamma)^2-\alpha^2\right), \quad \lambda_2 = \tfrac{1}{2}\left((\alpha-\gamma)^2-\beta^2\right), \quad \lambda_3 = \tfrac{1}{2}\left((\alpha-\beta)^2-\gamma^2\right). \tag{5.2}$$

For an arbitrary vector $X = \sum_{i=1}^{3} X_i e_i$, from equation (5.1) we get

$$\mathcal{L}_X g = \begin{pmatrix} 0 & X_3(\alpha-\beta) & X_2(\gamma-\alpha) \\ X_3(\alpha-\beta) & 0 & X_1(\beta-\gamma) \\ X_2(\gamma-\alpha) & X_1(\beta-\gamma) & 0 \end{pmatrix}.$$

Hence, by (1.26), there exist a Ricci soliton of this type if and only if the following system of equations is satisfied:

$$\begin{cases} (\beta-\gamma)^2-\alpha^2 = 2\lambda, \\ (\alpha-\gamma)^2-\beta^2 = 2\lambda, \\ (\alpha-\beta)^2-\gamma^2 = 2\lambda, \\ X_1(\beta-\gamma) = 0, \\ X_2(\alpha-\gamma) = 0, \\ X_3(\alpha-\beta) = 0. \end{cases} \tag{5.3}$$

Now, from (5.2), it is clear that any solution of (5.3) gives rise to an Einstein metric. Therefore *there are no homogeneous non-trivial Ricci solitons of type* (I_a).

Type (I_b)**.**

Assume L has a complex eigenvalue. Then

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \gamma & -\beta \\ 0 & \beta & \gamma \end{pmatrix}, \qquad \beta \neq 0,$$

with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$. The corresponding Lie algebra is given by

$$\mathfrak{g}_{I_b}: \quad [e_1,e_2] = \beta\, e_2 - \gamma\, e_3, \quad [e_1,e_3] = -\gamma\, e_2 - \beta\, e_3, \quad [e_2,e_3] = \alpha\, e_1.$$

The non-zero components of the curvature tensor (up to symmetries) are

$$R_{1221} = R_{1313} = \tfrac{1}{4}\left(\alpha^2 + 4\beta^2\right), \quad R_{2332} = \tfrac{3}{4}\,\alpha^2 + \beta^2 - \alpha\gamma, \quad R_{1231} = \beta(\alpha - 2\gamma).$$

The Ricci operator, with respect to the basis $\{e_1, e_2, e_3\}$, is described as follows:

$$\mathrm{Ric} = \begin{pmatrix} -\frac{1}{2}(\alpha^2 + 4\beta^2) & 0 & 0 \\ 0 & \frac{1}{2}\,\alpha(\alpha - 2\gamma) & -\beta(\alpha - 2\gamma) \\ 0 & \beta(\alpha - 2\gamma) & \frac{1}{2}\,\alpha(\alpha - 2\gamma) \end{pmatrix}, \qquad \beta \neq 0.$$

Hence, Ric is of type (I_b) if $\alpha \neq 2\gamma$ and (I_a) if $\alpha = 2\gamma$. For $X = \sum_{i=1}^{3} X_i e_i$, one has

$$\mathcal{L}_X g = \begin{pmatrix} 0 & X_2\beta + X_3(\alpha - \gamma) & X_3\beta + X_2(\gamma - \alpha) \\ X_2\beta + X_3(\alpha - \gamma) & -2X_1\beta & 0 \\ X_3\beta + X_2(\gamma - \alpha) & 0 & -2X_1\beta \end{pmatrix},$$

and thus, we have a homogeneous Ricci soliton of type (I_b) if and only if

$$(5.4) \qquad \begin{cases} \alpha^2 + 4\beta^2 = -2\lambda, \\ \alpha^2 - 2\alpha\,\gamma - 4X_1\beta = 2\lambda, \\ \alpha^2 - 2\alpha\,\gamma + 4X_1\beta = 2\lambda, \\ X_3(\alpha - \gamma) + X_2\beta = 0, \\ X_2(\alpha - \gamma) - X_3\beta = 0, \\ \beta(\alpha - 2\gamma) = 0. \end{cases}$$

Since $\beta \neq 0$, the last equation in (5.4) gives $\alpha - 2\gamma = 0$. Hence, the second and third equations simplify to $-4X_1\beta = 2\lambda$ and $4X_1\beta = 2\lambda$, respectively, which imply $X_1 = \lambda = 0$. Finally, from the first equation one gets that there are no solutions of (5.4) with $\beta \neq 0$. Therefore *there are no homogeneous Ricci solitons of type* (I_b).

Type **(II).**

Assume L has a double root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, one has

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{2} + \beta & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} + \beta \end{pmatrix}$$

and the corresponding Lie algebra is given by

$$\mathfrak{g}_{II}: \quad [e_1, e_2] = \tfrac{1}{2}\,e_2 - \left(\beta - \tfrac{1}{2}\right) e_3, \quad [e_1, e_3] = -\left(\beta + \tfrac{1}{2}\right) e_2 - \tfrac{1}{2}\,e_3, \quad [e_2, e_3] = \alpha\, e_1.$$

The non-zero components of the curvature tensor are (up to symmetries) given by

$$\begin{aligned} R_{1221} &= \tfrac{1}{4}\left(\alpha^2 - 2\alpha + 4\beta\right), & R_{1313} &= \tfrac{1}{4}\left(\alpha^2 + 2\alpha - 4\beta\right), \\ R_{2332} &= \tfrac{1}{4}\,\alpha(3\alpha - 4\beta), & R_{1231} &= \tfrac{1}{2}\,\alpha - \beta. \end{aligned}$$

Hence the Ricci operator takes the form

$$\text{Ric} = \begin{pmatrix} -\frac{1}{2}\,\alpha^2 & 0 & 0 \\ 0 & \frac{1}{2}\,(\alpha+1)(\alpha-2\beta) & -\frac{1}{2}\,\alpha+\beta \\ 0 & \frac{1}{2}\,\alpha-\beta & \frac{1}{2}\,(\alpha-1)(\alpha-2\beta) \end{pmatrix}, \tag{5.5}$$

with eigenvalues $\lambda_1 = -\frac{1}{2}\,\alpha^2$ and $\lambda_2 = \lambda_3 = \frac{1}{2}\,\alpha(\alpha - 2\beta)$. Thus, Ric is of type (II) if and only if $\alpha = 0$ or $\alpha = \beta$.

For a vector field $X = \sum_{i=1}^{3} X_i e_i$, we get

$$\mathcal{L}_X g = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{12} & -X_1 & X_1 \\ a_{13} & X_1 & -X_1 \end{pmatrix},$$

where $a_{12} = \dfrac{1}{2}\,(X_2 + X_3(2\alpha - 2\beta - 1))$ and $a_{13} = \dfrac{1}{2}\,(X_3 + X_2(2\beta - 2\alpha - 1))$. Necessary and sufficient conditions for the existence of a homogeneous Ricci soliton of type (II) are then given by

$$\begin{cases} \alpha^2 = -2\lambda, \\ \alpha^2 - 2\alpha\,\beta + \alpha - 2\beta - 2X_1 = 2\lambda, \\ \alpha^2 - 2\alpha\,\beta - \alpha + 2\beta + 2X_1 = 2\lambda, \\ \alpha - 2\beta - 2X_1 = 0, \\ (2\alpha - 2\beta)X_3 + X_2 - X_3 = 0, \\ (2\alpha - 2\beta)X_2 + X_2 - X_3 = 0. \end{cases} \tag{5.6}$$

From the second and forth equation in (5.6) one gets $\alpha^2 - 2\alpha\,\beta - 2\lambda = 0$. Substituting this relation into the first equation, we then obtain $\alpha(\alpha - \beta) = 0$. Hence, either $\alpha = 0 \neq \beta$ or $\alpha = \beta \neq 0$. (We excluded the case $\alpha = \beta = 0$, since by (5.5) this corresponds to a flat manifold.)

First case: $\alpha = 0 \neq \beta$. From the first equation in (5.6) one gets $\lambda = 0$, the last two equations give $X_2 = X_3$ and the fourth equation yields $X_1 = -\beta$. Therefore, the (space-like) vector field

$$X = -\beta\, e_1 \tag{5.7}$$

defines a homogeneous (steady) Ricci soliton. By (5.5), the Ricci operator is two-step nilpotent but non-vanishing (since $\beta \neq 0$), that is, of type (II) with a single eigenvalue equal to zero.

Second case: $\alpha = \beta \neq 0$. In this case, one easily gets from (5.6) that $\lambda = -\frac{1}{2}\beta^2$, that $X_1 = -\frac{1}{2}\beta$ and that $X_2 = X_3$; thus, *there exists a one-parameter family of homogeneous expanding Ricci solitons*, given by

$$X = -\frac{1}{2}\beta\, e_1 + \delta\, e_2 + \delta\, e_3, \quad \delta \in \mathbb{R}. \tag{5.8}$$

Note that the causality of X is again fixed and one can only find examples of solitons for X spacelike but not null or timelike. Since $\alpha = \beta \neq 0$, (5.5) yields that the Ricci operator is of type (II), with one non-zero eigenvalue equal to $-\frac{1}{2}\alpha^2$.

Remark 5.6 A Lie group of type (II) with $\alpha = 0$ or $\alpha = \beta$ is locally symmetric if and only if $\beta = 0$ (see [26]). This shows that previous examples are not locally symmetric.

Type **(III).**

Assume L has a triple root of its minimal polynomial. Then

$$L = \begin{pmatrix} \alpha & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \alpha & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \alpha \end{pmatrix}$$

with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, and the corresponding Lie algebra is given by

$$\mathfrak{g}_{III} : \begin{cases} [e_1, e_2] = -\frac{1}{\sqrt{2}}\, e_1 - \alpha\, e_3, & [e_1, e_3] = -\frac{1}{\sqrt{2}}\, e_1 - \alpha\, e_2, \\ [e_2, e_3] = \alpha\, e_1 + \frac{1}{\sqrt{2}}\, e_2 - \frac{1}{\sqrt{2}}\, e_3. & \end{cases}$$

Hence the non-zero components of the curvature tensor (up to symmetries) are

$$\begin{array}{lll} R_{1221} = \frac{1}{4}\left(\alpha^2 + 4\right), & R_{1331} = 1 - \frac{1}{4}\alpha^2, & R_{2323} = \frac{1}{4}\alpha^2, \\ R_{1231} = 1, & R_{1223} = R_{1323} = \frac{1}{\sqrt{2}}\alpha. & \end{array}$$

The Ricci operator, expressed in terms of the basis $\{e_1, e_2, e_3\}$, becomes

$$\mathrm{Ric} = \begin{pmatrix} -\frac{1}{2}\alpha^2 & -\frac{1}{\sqrt{2}}\alpha & -\frac{1}{\sqrt{2}}\alpha \\ -\frac{1}{\sqrt{2}}\alpha & -\frac{1}{2}(\alpha^2 + 2) & -1 \\ \frac{1}{\sqrt{2}}\alpha & 1 & 1 - \frac{1}{2}\alpha^2 \end{pmatrix},$$

with a single eigenvalue $-\frac{1}{2}\alpha^2$. If $\alpha \neq 0$, then Ric is of type (III), while Ric is two-step nilpotent if $\alpha = 0$.

For a vector $X = \sum_{i=1}^{3} X_i e_i$, the Lie derivative has the following expression

$$\mathcal{L}_X g = \frac{1}{\sqrt{2}} \begin{pmatrix} -2(X_2 + X_3) & X_1 & X_1 \\ X_1 & 2X_3 & X_3 - X_2 \\ X_1 & X_3 - X_2 & -2X_2 \end{pmatrix}.$$

Thus the Ricci soliton condition (1.26) on $\mathfrak{g}_{III}$ gives rise to the following system of equations:

$$\text{(5.9)} \qquad \begin{cases} \frac{\alpha^2}{2} + \sqrt{2}X_2 + \sqrt{2}X_3 = -\lambda, \\ \frac{\alpha^2}{2} - \sqrt{2}X_3 + 1 = -\lambda, \\ \frac{\alpha^2}{2} - \sqrt{2}X_2 - 1 = -\lambda, \\ \frac{1}{\sqrt{2}}(X_1 - \alpha) = 0, \\ X_2 - X_3 + \sqrt{2} = 0. \end{cases}$$

If we subtract half of the second and third equations from the first equation in (5.9), we see that $X_2 = -X_3$ and therefore $\lambda = -\frac{1}{2}\alpha^2$. Moreover from the fourth equation $X_1 = \alpha$. Hence *any type* (III) *unimodular Lie group is a homogeneous Ricci soliton* for

$$\text{(5.10)} \qquad X = \alpha\, e_1 - \frac{1}{\sqrt{2}}\, e_2 + \frac{1}{\sqrt{2}}\, e_3.$$

Remark 5.7 A vector field X defining a homogeneous Ricci soliton on $\mathfrak{g}_{III}$ satisfies $\langle X, X \rangle = \alpha^2$ and thus it is either spacelike or null. Correspondingly, the homogeneous Ricci soliton is either expanding or steady. Note also that type (III) unimodular Lie groups are not symmetric (see also [26]).

Our discussion above proves (i) of Theorem 5.2. The results we proved are summarized in the following

Theorem 5.8 *The following are all non-trivial homogeneous Lorentzian Ricci solitons realized as unimodular Lorentzian Lie groups* G*:*

a) $G = E(1,1)$*, with Lie algebra as in Theorem 5.2–*$(i.1)$*,* $\alpha = 0 \neq \beta$*. The homogeneous Ricci soliton is steady and defined by a spacelike vector field* (5.7)*.*

b) $G = O(1,2)$ *or* $SL(2,\mathbb{R})$*, with Lie algebra as in Theorem 5.2–*$(i.1)$*,* $\alpha = \beta \neq 0$*. The homogeneous Ricci soliton is expanding and defined by a spacelike vector field* (5.8)*.*

c) $G = O(1,2)$ *or* $SL(2,\mathbb{R})$*, with Lie algebra as in Theorem 5.2–*$(i.2)$*,* $\alpha \neq 0$*. The homogeneous Ricci soliton is expanding and defined by a spacelike vector field* (5.10)*.*

d) $G = E(1,1)$*, with Lie algebra as in Theorem 5.2–*$(i.2)$*,* $\alpha = 0$*. The homogeneous Ricci soliton is steady and defined by a null vector field* (5.10)*.*

Remark 5.9 Ricci solitons listed in Theorem 5.8 are locally conformally flat if and only if they correspond to $G = E(1,1)$, which is not locally symmetric.

5.1.2 Non-unimodular Lie groups

Following [42], now we consider the rest of the three-dimensional Lie algebras $\mathfrak{g}$. We note in passing that these are in fact all solvable. We shall exclude from our study the class $\mathfrak{G}$ corresponding to those Lie algebras where $[x, y] = l(x)y - l(y)x$ for all x, y for some linear map l on the Lie algebra. Briefly, all left-invariant Lorentzian metrics on groups of this class have constant curvature, and this constant can be any real number [73, 78].

Given a non-unimodular $\mathfrak{g}$ not in $\mathfrak{G}$, let $\mathfrak{u}$ denote its unimodular kernel,

$$\mathfrak{u} = \ker(\operatorname{tr} ad : \mathfrak{g} \to \mathbb{R}),$$

and let $\langle \cdot, \cdot \rangle$ be a Lorentzian metric tensor on $\mathfrak{g}$. There are two cases that we consider separately.

If $\mathfrak{u}$ is not a null plane (i.e., $\mathfrak{u}$ is not tangent to the lightcone of $\langle \cdot, \cdot \rangle$ in $\mathfrak{g}$), then we can choose a $\langle \cdot, \cdot \rangle$-orthonormal basis $\{e_1, e_2, e_3\}$ with $e_3 \perp \mathfrak{u}$ and $[e_1, e_3] \perp [e_2, e_3]$ in $\mathfrak{u}$, where $\perp$ denotes $\langle \cdot, \cdot \rangle$-orthogonal. If $\mathfrak{u}$ is a spacelike plane, then e_3 is a timelike vector and we have signature $(- - +)$. If $\mathfrak{u}$ is a spacetime plane (also called a timelike plane in relativity), then e_3 is spacelike and we have signature $(+ - -)$ or $(- + -)$. Taking into account the semidirect product structure of our $\mathfrak{g}$, it easily follows that these last two signatures produce equivalent geometries. We shall consider only $(+ - -)$ explicitly.

If $\mathfrak{u}$ is a null plane, then we can choose a null basis $\{e_1, e_2, e_3\}$ with e_3 a null vector, e_1, $e_2 \in \mathfrak{u}$, and with

$$\begin{aligned} \langle e_3, e_3 \rangle = \langle e_2, e_2 \rangle = \langle e_1, e_3 \rangle = \langle e_1, e_2 \rangle &= 0, \\ \langle e_3, e_2 \rangle = -\langle e_1, e_1 \rangle &= 1, \\ [e_1, e_3] &\perp [e_2, e_3]. \end{aligned}$$

Then, non-unimodular Lorentzian Lie algebras of non-constant sectional curvature are given, with respect to a suitable basis $\{e_1, e_2, e_3\}$, by

$$\mathfrak{g}_{IV}: \quad [e_1, e_2] = 0, \quad [e_1, e_3] = \alpha\, e_1 + \beta\, e_2, \quad [e_2, e_3] = \gamma\, e_1 + \delta\, e_2, \tag{5.11}$$

where $\alpha + \delta \neq 0$ and one of the following holds:

IV.1 $\{e_1, e_2, e_3\}$ is orthonormal with $\langle e_1, e_1 \rangle = -\langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = -1$ and the structure constants satisfy $\alpha\,\gamma - \beta\,\delta = 0$.

IV.2 $\{e_1, e_2, e_3\}$ is orthonormal with $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = 1$ and the structure constants satisfy $\alpha\,\gamma + \beta\,\delta = 0$.

IV.3 $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis with

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

and the structure constants satisfy $\alpha\,\gamma = 0$.

We analyze the three cases separately.

Type $(IV.1)$.

The non-zero components of the curvature tensor are given by

$$\begin{aligned} R_{1212} &= \tfrac{1}{4}\left(\beta^2+\gamma^2+4\alpha\,\delta-2\beta\,\gamma\right),\\ R_{1313} &= \tfrac{1}{4}\left(4\alpha^2-3\beta^2+\gamma^2+2\beta\,\gamma\right),\\ R_{2332} &= \tfrac{1}{4}\left(\beta^2-3\gamma^2+4\delta^2+2\beta\,\gamma\right). \end{aligned}$$

Hence the Ricci operator is diagonalizable with eigenvalues

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(\beta^2-\gamma^2-2\alpha(\alpha+\delta)),\\ \lambda_2 &= \tfrac{1}{2}(\gamma^2-\beta^2-2\delta(\alpha+\delta)),\\ \lambda_3 &= \tfrac{1}{2}((\beta-\gamma)^2-2(\alpha^2+\delta^2)). \end{aligned}$$

The Lie derivative of the metric for an arbitrary vector $X=\sum_{i=1}^{3}X_i e_i$ is given by

$$\mathcal{L}_X g = \begin{pmatrix} -2\alpha X_3 & X_3(\beta-\gamma) & X_1\alpha+X_2\gamma \\ X_3(\beta-\gamma) & 2X_3\delta & -X_1\beta-X_2\delta \\ X_1\alpha+X_2\gamma & -X_1\beta-X_2\delta & 0 \end{pmatrix},$$

and thus necessary and sufficient conditions for the existence of an invariant homogeneous Ricci soliton (1.26) on $\mathfrak{g}_{IV.1}$ are given by

$$\text{(5.12)}\qquad \begin{cases} \beta^2-\gamma^2-2\alpha(\alpha+\delta)+4X_3\,\alpha=2\lambda,\\ \gamma^2-\beta^2-2\delta(\alpha+\delta)+4X_3\,\delta=2\lambda,\\ (\beta-\gamma)^2-2(\alpha^2+\delta^2)=2\lambda,\\ X_1\,\alpha+X_2\,\gamma=0,\\ X_1\,\beta+X_2\,\delta=0,\\ X_3(\beta-\gamma)=0. \end{cases}$$

If $X_3=0$, then the first three equations in (5.12) imply $\lambda_1=\lambda_2=\lambda_3$. On the other hand, if $X_3\neq 0$, then the last equation in (5.12) gives $\beta=\gamma$. Since $\alpha\,\gamma-\beta\,\delta=0$, from $\alpha+\delta\neq 0$ and (5.12) we obtain $\alpha=\delta$ and hence $\lambda_1=\lambda_2=\lambda_3$. Thus, all solutions of (5.12) are Einstein, and hence of constant sectional curvature.

Type $(IV.2)$.

Assume the non-unimodular Lie algebra $\mathfrak{g}_{IV}$ has a basis as in IV.2. Then, a straightforward calculation shows that the non-zero components of the curvature tensor are given by

$$\begin{aligned} R_{1212} &= \alpha\,\delta-\tfrac{1}{4}(\beta+\gamma)^2,\\ R_{1331} &= \tfrac{1}{4}\left(4\alpha^2+3\beta^2-\gamma^2+2\beta\gamma\right),\\ R_{2323} &= \tfrac{1}{4}\left(\beta^2-3\gamma^2-4\delta^2-2\beta\gamma\right). \end{aligned}$$

Therefore, the Ricci operator is diagonalizable with eigenvalues

$$\begin{array}{rcl} \lambda_1 & = & \frac{1}{2}(\beta^2-\gamma^2+2\alpha(\alpha+\delta)), \\ \lambda_2 & = & \frac{1}{2}(\gamma^2-\beta^2+2\delta(\alpha+\delta)), \\ \lambda_3 & = & \frac{1}{2}((\beta+\gamma)^2+2(\alpha^2+\delta^2)). \end{array}$$

A straightforward calculation from (5.11), using the fact that the structure constants satisfy $\alpha\,\gamma+\beta\,\delta=0$ and $\alpha+\delta\neq 0$, shows that the Lie derivative of the metric with respect to a vector $X=\sum_{i=1}^{3} X_i e_i$ is given by

$$\mathcal{L}_X g=\begin{pmatrix} 2X_3\alpha & X_3(\beta+\gamma) & -X_1\alpha-X_2\gamma \\ X_3(\beta+\gamma) & 2X_3\delta & -X_1\beta-X_2\delta \\ -X_1\alpha-X_2\gamma & -X_1\beta-X_2\delta & 0 \end{pmatrix}.$$

Then an invariant Ricci soliton must satisfy

$$(5.13)\qquad \begin{cases} \beta^2-\gamma^2+2\alpha(\alpha+\delta)+4X_3\alpha=2\lambda, \\ \gamma^2-\beta^2+2\delta(\alpha+\delta)+4X_3\delta=2\lambda, \\ (\beta+\gamma)^2+2(\alpha^2+\delta^2)=2\lambda, \\ X_1\alpha+X_2\gamma=0, \\ X_1\beta+X_2\delta=0, \\ X_3(\beta+\gamma)=0. \end{cases}$$

A similar analysis to that developed for type $(IV.1)$ shows that an invariant homogeneous Ricci solitons of type $(IV.2)$ are necessarily of constant sectional curvature.

Type $(IV.3)$**.**

Let now $\mathfrak{g}_{IV}$ admit a pseudo-orthonormal basis as in IV.3. We then consider the orthonormal basis

$$\tilde{e}_1 := e_1, \quad \tilde{e}_2 := \tfrac{1}{\sqrt{2}}\,(e_2-e_3), \quad \tilde{e}_3 := \tfrac{1}{\sqrt{2}}\,(e_2+e_3),$$

with signature $(++-)$. Then the non-zero components of the curvature tensor are given by

$$R_{1212}=\tfrac{1}{4}\,(2\alpha\,\delta-2\alpha^2-\gamma(2\beta+\gamma)), \quad R_{1213}=\tfrac{1}{2}\,(\alpha^2+\beta\,\gamma-\alpha\,\delta),$$
$$R_{1313}=\tfrac{1}{4}\,(2\alpha\,\delta-2\alpha^2+\gamma(\gamma-2\beta)), \quad R_{2323}=-\tfrac{3}{4}\,\gamma^2.$$

The Ricci operator in the new basis $\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3\}$ becomes

$$\mathrm{Ric}=\begin{pmatrix} -\frac{1}{2}\,\gamma^2 & 0 & 0 \\ 0 & \frac{1}{2}\,(\alpha(\delta-\alpha)+\gamma(\gamma-\beta)) & \frac{1}{2}\,(\alpha(\alpha-\delta)+\beta\gamma) \\ 0 & -\frac{1}{2}\,(\alpha(\alpha-\delta)+\beta\gamma) & \frac{1}{2}\,(\alpha(\alpha-\delta)+\gamma(\beta+\gamma)) \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -\frac{1}{2}\gamma^2$ and $\lambda_2 = \lambda_3 = \frac{1}{2}\gamma^2$. Thus, Ric is of type (II) (see Remark 1.1).

For an arbitrary vector $X = \sum_{i=1}^{3} X_i \tilde{e}_i$, the Lie derivative of the metric becomes

$$\mathcal{L}_X g = \begin{pmatrix} \sqrt{2}\alpha(X_3 - X_2) & a_{12} & a_{13} \\ a_{12} & X_1\beta + \sqrt{2}X_3\delta & -X_1\beta - \frac{1}{\sqrt{2}}\delta(X_2 + X_3) \\ a_{13} & -X_1\beta - \frac{1}{\sqrt{2}}\delta(X_2 + X_3) & X_1\beta + \sqrt{2}X_2\delta \end{pmatrix},$$

where

$$\begin{aligned} a_{12} &= \tfrac{1}{2}\left(X_3(\beta + 2\gamma) + \sqrt{2}X_1\alpha - X_2\beta\right), \\ a_{13} &= \tfrac{1}{2}\left(X_2(\beta - 2\gamma) - \sqrt{2}X_1\alpha - X_3\beta\right). \end{aligned}$$

Now, since $\alpha\gamma = 0$, we consider the possibilities $\alpha = 0$ and $\gamma = 0$ separately. Note that if $\alpha = \gamma = 0$, then the metric $\mathfrak{g}_{IV}$ is flat.

Assume first that $\alpha = 0 \neq \gamma$. Then, (1.26) holds if and only if

$$(5.14) \qquad \begin{cases} \gamma^2 = -2\lambda, \\ \gamma^2 - \beta\gamma + 2X_1\beta + 2\sqrt{2}X_3\delta = 2\lambda, \\ \gamma^2 + \beta\gamma - 2X_1\beta - 2\sqrt{2}X_2\delta = 2\lambda, \\ -2\gamma X_3 + X_2\beta - X_3\beta = 0, \\ 2\gamma X_2 - X_2\beta + X_3\beta = 0, \\ \beta\gamma - 2X_1\beta - \sqrt{2}X_2\delta - \sqrt{2}X_3\delta = 0, \end{cases}$$

for a vector $X = \sum_{i=1}^{3} X_i \tilde{e}_i$. From the fourth and fifth equation in (5.14) we get $X_2 = X_3$, which implies, using the second and third equations, that (5.14) admits no solutions.

Assume now $\alpha \neq 0 = \gamma$. Then, (1.26) reduces to the following system of equations

$$(5.15) \qquad \begin{cases} 2\sqrt{2}\alpha(X_2 - X_3) = -2\lambda, \\ \alpha\delta - \alpha^2 + 2X_1\beta + 2\sqrt{2}X_3\delta = 2\lambda, \\ \alpha^2 - \alpha\delta - 2X_1\beta - 2\sqrt{2}X_2\delta = 2\lambda, \\ \sqrt{2}X_1\alpha - X_2\beta + X_3\beta = 0, \\ \alpha^2 - \alpha\delta - 2X_1\beta = \sqrt{2}\delta(X_2 + X_3), \end{cases}$$

for a vector $X = \sum_{i=1}^{3} X_i \tilde{e}_i$. We subtract the third equation to the second one and conclude, using the first equation, that either $2\alpha = \delta$ or $X_2 = X_3$. We analyze both cases separately.

Set first $\alpha = \frac{1}{2}\delta \neq 0$. Then there exists homogeneous Ricci solitons for

$$(5.16) \qquad X = -\frac{2\beta\lambda}{\delta^2}\,\tilde{e}_1 - \frac{\delta^4 + 8(\delta^2 - 2\beta^2)\lambda}{8\sqrt{2}\delta^3}\,\tilde{e}_2 - \frac{\delta^4 - 8(\delta^2 + 2\beta^2)\lambda}{8\sqrt{2}\delta^3}\,\tilde{e}_3.$$

Note that the corresponding solitons may be expanding, steady or shrinking depending on the value of λ, which can be chosen arbitrarily.

Set now $X_2 = X_3$. Then necessarily $\lambda = 0$ and $X_1 = 0$. The remaining equation $\alpha^2 - \alpha\,\delta - 2\sqrt{2}X_2\delta = 0$ in (5.15) gives rise to homogeneous steady Ricci solitons for

$$X = \frac{\alpha^2 - \alpha\,\delta}{2\sqrt{2}\delta}\,(\tilde{e}_2 + \tilde{e}_3)\,. \tag{5.17}$$

In all cases above the Ricci operator is two-step nilpotent and the metric is non-symmetric whenever $\alpha\,\delta(\alpha - \delta) \neq 0$. Furthermore, for the particular choice $\delta = 0 \neq \alpha$, the resulting metric is symmetric but not of constant curvature [26].

The results of this subsection prove case (*ii*) of Theorem 5.2 and are summarized in the following

Theorem 5.10 *A non-unimodular Lie group G equipped with a left-invariant Lorentzian metric results in a non-trivial invariant homogeneous Ricci soliton if and only if its non-unimodular Lie algebra $\mathfrak{g}_{IV}$ satisfies $\alpha \neq 0 = \gamma$.*

Steady Ricci solitons, defined by null vector fields (5.17)*, exist for any choice of $\alpha \neq 0$, β and δ.*

In the special case $\delta = 2\alpha$, there exist expanding, steady and shrinking Ricci solitons, defined by vector fields (5.16)*, whose causal character depends on λ.*

Remark 5.11 Non-trivial Ricci solitons in Theorem 5.10 are locally conformally flat if and only if $\gamma = \beta = 0$, in which case they are not locally symmetric. Therefore, Theorem 5.10 provides examples of complete locally conformally flat expanding, steady and shrinking Ricci solitons.

5.1.3 Some remarks on left-invariant Ricci solitons

Remark 5.12 Since the Weyl tensor of any three-dimensional Lorentzian manifold is identically zero, the whole curvature is completely determined by the Ricci operator. Further more observe that although the Ricci operator is self-adjoint, it may have non-trivial Jordan form due to the indefiniteness of the metric. The Ricci operators corresponding to the Ricci solitons in Theorem 5.2 are described below.

- The Ricci operator corresponding to a unimodular Lie group with Lie algebra as in (i.1) is given by

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta & \beta \\ 0 & -\beta & \beta \end{pmatrix}$$

  if $\alpha = 0$, and by

$$\mathrm{Ric} = -\frac{1}{2}\begin{pmatrix} \beta^2 & 0 & 0 \\ 0 & \beta(1+\beta) & -\beta \\ 0 & \beta & \beta(\beta-1) \end{pmatrix}$$

if $\alpha = \beta \neq 0$. Hence the Ricci operator is two-step nilpotent (if $\alpha = 0$) or otherwise it has single eigenvalue $-\frac{1}{2}\beta^2$, which is a double root of the minimal polynomial (if $\alpha = \beta \neq 0$).

- The Ricci operator corresponding to a unimodular Lie group with Lie algebra as in $(i.2)$ is given by

$$\mathrm{Ric} = \begin{pmatrix} -\frac{\alpha^2}{2} & -\frac{\alpha}{\sqrt{2}} & -\frac{\alpha}{\sqrt{2}} \\ -\frac{\alpha}{\sqrt{2}} & -\frac{1}{2}(\alpha^2+2) & -1 \\ \frac{\alpha}{\sqrt{2}} & 1 & 1-\frac{\alpha^2}{2} \end{pmatrix}.$$

  Hence the Ricci operator has a single eigenvalue $-\frac{1}{2}\alpha^2$, which is a triple root of the minimal polynomial (if $\alpha \neq 0$) or a double root of the minimal polynomial (if $\alpha = 0$).

- The Ricci operator corresponding to a non-unimodular Lie group with Lie algebra as in (ii) is given by

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\alpha(\delta-\alpha) & -\frac{1}{2}\alpha(\delta-\alpha) \\ 0 & \frac{1}{2}\alpha(\delta-\alpha) & -\frac{1}{2}\alpha(\delta-\alpha) \end{pmatrix}.$$

  Hence the Ricci operator is two-step nilpotent.

This proves Theorem 5.5.

Remark 5.13 Two natural generalizations of locally symmetric spaces are the classes of $\mathfrak{P}$-spaces and $\mathfrak{C}$-spaces defined as follows. A Lorentzian manifold (M, g) is said to be a $\mathfrak{P}$-space if the eigenspaces of the Jacobi operators are parallel along timelike geodesics. A Lorentzian manifold (M, g) is said to be a $\mathfrak{C}$-space if the eigenvalues of the Jacobi operators are constant along timelike geodesics. Moreover, a Lorentzian manifold is simultaneously a $\mathfrak{P}$-space and a $\mathfrak{C}$-space if and only if it is locally symmetric [19].

It was shown in [30] that a homogeneous Lorentzian three-dimensional manifold is a $\mathfrak{P}$-space if and only if the Ricci operator is two-step nilpotent. Hence it follows from Remark 5.12 that

- *Left-invariant Ricci solitons corresponding to* $(i.1)$ *and* $(i.2)$ *with* $\alpha = 0$*, as well as any left-invariant Ricci soliton corresponding to* (ii) *are* $\mathfrak{P}$*-spaces.*

- *Any non-symmetric homogeneous* $\mathfrak{P}$*-space admits a left-invariant Ricci soliton.*

On the other hand, three-dimensional Lorentzian $\mathfrak{C}$-spaces are characterized by the fact that their Ricci tensor is cyclic parallel (i.e., $\nabla_X \rho(X, X) = 0$). Now, as an application of the results in [26, 30] one has

- *Left-invariant Ricci solitons corresponding to* $(i.1)$ *with* $\alpha = \beta \neq 0$ *are* $\mathfrak{C}$*-spaces.*

- *Any non-symmetric homogeneous* $\mathfrak{C}$*-space with non-diagonalizable Ricci operator is a left-invariant Ricci soliton.*

Remark 5.14 A straightforward calculation shows that Lie groups given by $(i.1)$ and $(i.2)$ in Theorem 5.2 are not locally symmetric unless they are flat. On the other hand, Lie groups given by (ii) are non-flat locally symmetric if and only if $\delta = 0$. In this case, they are Cahen-Wallach symmetric spaces and they admit expanding, steady and shrinking Ricci solitons as an immediate application of Theorem 3.8. However note that none of them are left-invariant, since no left-invariant Ricci soliton exists on a Lie algebra of type $(IV.3)$ with $\gamma = \delta = 0$, unless the metric is flat (see Remark 5.3).

It follows from the discussion in Section 1.4.2 that a Lorentzian manifold admits two-distinct Ricci solitons if and only if it admits a homothetic vector field. The existence of homothetic vector fields on three-dimensional Lie groups was discussed in [31], from where it follows that

Theorem 5.15 [31] *A non-flat three-dimensional homogeneous Lorentzian manifold* (M, g) *admits two distinct Ricci solitons if and only if it is locally conformally flat and its Ricci operator is two-step nilpotent.*

Remark 5.16 Three-dimensional locally conformally flat Lorentzian Lie groups correspond to cases $(i.2)$ for $\alpha = 0$ and (ii) in Theorem 5.2. Note that left-invariant homothetic vector fields exist in non-unimodular Lie groups as in Theorem 5.2–(ii) with $\alpha = \frac{1}{2}\delta$ but not in unimodular Lie groups corresponding to Theorem 5.2–$(i.2)$ [31]. This agrees with the different possibilities of Ricci solitons described in Remark 5.3. Hence, although locally conformally flat Lie groups with two-step nilpotent Ricci operators admit two distinct Ricci solitons, these are not left-invariant in general.

5.2 Ricci solitons on three-dimensional Walker manifolds

We now consider three-dimensional Lorentzian manifolds (M, g) admitting a parallel null vector field $\mathcal{U}$. We refer to [21, 40] for more information on the geometry of three-dimensional Walker metrics. Although we are going to use the classical notation on Walker metrics, these manifolds correspond to three-dimensional *pp*-waves (see Section 1.3.2). As in Chapter 4, in order to simplify the notation we denote with sub-indices the corresponding partial derivatives, thus, for example, $f_t = \frac{\partial f}{\partial t}$, $f_{tx} = \frac{\partial^2 f}{\partial t \partial x}, \dots$.

Following [97] choose adapted coordinates (t, x, y) where the Lorentzian metric tensor expresses as

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \phi(x,y) \end{pmatrix}, \tag{5.18}$$

for some function $\phi(x,y)$, where the parallel null vector field is $\mathcal{U} = \frac{\partial}{\partial t}$. Then, the associated Levi-Civita connection is described by

$$\nabla_{\partial_x}\partial_y = \tfrac{1}{2}\,\partial_x\phi\,\partial_t, \qquad \nabla_{\partial_y}\partial_y = \tfrac{1}{2}\,\partial_y\phi\,\partial_t - \tfrac{1}{2}\,\partial_x\phi\,\partial_x. \tag{5.19}$$

As shown in [40], the Ricci tensor ρ and the Ricci operator Ric of a metric g as in (5.18), expressed in the coordinate basis, take the form

$$\rho = -\frac{1}{2}\,\phi_{xx}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{Ric} = -\frac{1}{2}\,\phi_{xx}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.20}$$

Hence, if $\phi_{xx} = 0$, then (M,g) is flat, while for $\phi_{xx} \neq 0$ the Ricci operator Ric is two-step nilpotent.

Next, let $X = (\mathcal{A}(t,x,y), \mathcal{B}(t,x,y), \mathcal{C}(t,x,y))$ be an arbitrary vector field on M. A straightforward calculation from (5.19) shows that the Lie derivative of the metric $\mathcal{L}_X g$ expresses in the coordinate basis as follows:

$$\mathcal{L}_X g = \begin{pmatrix} 2\mathcal{C}_t & \varepsilon\,\mathcal{B}_t + \mathcal{C}_x & \mathcal{A}_t + \mathcal{C}_y + \phi\,\mathcal{C}_t \\ \varepsilon\,\mathcal{B}_t + \mathcal{C}_x & 2\varepsilon\,\mathcal{B}_x & \mathcal{A}_x + \varepsilon\,\mathcal{B}_y + \phi\,\mathcal{C}_x \\ \mathcal{A}_t + \mathcal{C}_y + \phi\,\mathcal{C}_t & \mathcal{A}_x + \varepsilon\,\mathcal{B}_y + \phi\,\mathcal{C}_x & \mathcal{B}\,\phi_x + \mathcal{C}\,\phi_y + 2(\mathcal{A}_y + \phi\,\mathcal{C}_y) \end{pmatrix}. \tag{5.21}$$

Now, from (5.20) and (5.21) we obtain the following necessary and sufficient conditions for a strict Walker metric (5.18) to be a Ricci soliton:

$$\begin{cases} 2\mathcal{C}_t = 0, \\ \mathcal{C}_x + \varepsilon\,\mathcal{B}_t = 0, \\ \mathcal{C}_y + \mathcal{A}_t + \phi\,\mathcal{C}_t = \lambda, \\ 2\mathcal{B}_x = \lambda, \\ \mathcal{A}_x + \varepsilon\,\mathcal{B}_y + \phi\,\mathcal{C}_x = 0, \\ 2\mathcal{A}_y + \phi_x\mathcal{B} + \phi_y\mathcal{C} - \frac{1}{2\varepsilon}\,\phi_{xx} = \phi(\lambda - 2\mathcal{C}_y). \end{cases} \tag{5.22}$$

The first equation in (5.22) gives $\mathcal{C} = \mathcal{C}(x,y)$ and simplifies the third one. Since $\mathcal{C}$ does not depend on t, we can easily integrate the second and third equations in (5.22) to get

$$\mathcal{A} = (\lambda - \mathcal{C}_y)t + G(x,y), \qquad \mathcal{B} = -\frac{\mathcal{C}_x t}{\varepsilon} + H(x,y).$$

Therefore, the fourth equation in (5.22) now gives

$$-2t\,\mathcal{C}_{xx} + 2\varepsilon\,H_x = \varepsilon\,\lambda. \tag{5.23}$$

Since (5.23) must hold for any value of t, it implies at once $\mathcal{C}_{xx} = 0$ and $2H_x = \lambda$. By integration we then have $\mathcal{C} = u(y)x + v(y)$ and $H = \frac{1}{2}\,\lambda\,x + w(y)$. Then the fifth equation in (5.22) becomes

$$\phi\,u(y) - 2t\,u'(y) + w'(y) + G_x = 0$$

from where it follows that $u(y)$ is constant: $u(y) = \alpha$. Then, the system (5.22) now reduces to

$$(5.24)\qquad \begin{cases} \alpha\,\phi + \varepsilon w'(y) + G_x = 0, \\ 2\phi\, v'(y) - \lambda\,\phi - 2t\, v''(y) + \phi_y(\alpha\, x + v(y)) + 2G_y \\ \qquad\qquad + \phi_x\left(w(y) + \frac{\lambda}{2}\, x - \varepsilon\,\alpha\, t\right) = \frac{1}{2\varepsilon}\,\phi_{xx}. \end{cases}$$

The second equation in (5.24) holds for any value of t. Hence $\frac{\alpha\,\phi_x}{\varepsilon} = -2v''(y)$ and thus $\alpha\phi_{xx} = 0$. Since $\phi_{xx} = 0$ if and only if the Walker metric is flat, we assume $\alpha = 0$ and (5.24) reduces to

$$(5.25)\qquad \begin{cases} \varepsilon\, w'(y) + G_x = 0, \\ 2\phi\, v'(y) - \lambda\,\phi - 2t\, v''(y) + \phi_y\, v(y) + 2G_y + \phi_x\left(w(y) + \frac{\lambda}{2}\, x\right) = \frac{1}{2\varepsilon}\,\phi_{xx}. \end{cases}$$

The first equation in (5.25) gives $G(x, y) = -xw'(y) + \mu(y)$. Then, since the second equation in (5.25) must be independent of t, one gets $v''(y) = 0$ and hence $v(t) = \beta\, y + \gamma$.

Finally we conclude that there exist non-trivial Ricci solitons given by strict Walker metrics (5.18) if and only if the vector field X takes the form

$$X(t, x, y) = \left(t\,(\lambda - \beta) - \varepsilon\, x\, w'(y) + \mu(y), \frac{1}{2}\,\lambda\, x + w(y), \beta\, y + \gamma\right),$$

for some real constants β, γ and smooth functions w and μ, satisfying the partial differential equation

$$(5.26)\qquad 2\beta\,\phi - \lambda\,\phi + 2\mu'(y) - 2\varepsilon\, x\, w''(y) + \phi_y(\beta\, y + \gamma) + \phi_x\left(\frac{\lambda}{2}\, x + w(y)\right) = \frac{1}{2\varepsilon}\,\phi_{xx}.$$

One can not expect the partial differential equation (5.26) to admit solutions in general. We now turn our attention to the special case when the Walker metric is locally symmetric. Locally symmetric Walker metrics (5.18) are characterized by the fact that their defining function $\phi(x, y)$ is given by (see [21, 40])

$$(5.27)\qquad \phi(x, y) = x^2\kappa + x\, P(y) + Q(y),$$

for arbitrary functions P and Q, and constant κ which vanishes if and only if the metric is flat. When ϕ satisfies (5.27), a straightforward calculation shows that equation (5.26) becomes

$$(5.28)\qquad \begin{aligned} 0 = 2x^2\beta\,\kappa + x\left(2\beta\, P(y) - \tfrac{1}{2}\,\lambda\, P(y) + 2\kappa\, w(y) + (\beta\, y + \gamma)P'(y) - 2w''(y)\right) \\ - \kappa + 2\beta\, Q(y) - \lambda\, Q(y) + P(y)w(y) + (\beta\, y + \gamma)Q'(y) + 2\mu'(y). \end{aligned}$$

But (5.28) must hold for all values of x. Therefore, it gives $\beta = 0$ (excluding the flat case $\kappa = 0$) and reduces to the system

$$(5.29)\qquad \begin{cases} 2w''(y) - 2\kappa\, w(y) = \gamma\, P'(y) - \frac{1}{2}\,\lambda P(y), \\ 2\mu'(y) = \kappa - P(y)w(y) + \lambda\, Q(y) - \gamma Q'(y). \end{cases}$$

The second equation in (5.29), by direct integration, permits to express μ in terms of w and P, Q. The first equation in (5.29) is a second order linear ordinary differential equation for w determined by the smooth function $\gamma\, P'(y) - \frac{\lambda}{2}\, P(y)$, which has globally defined solutions. Therefore, we have proved the following

Theorem 5.17 *Any three-dimensional symmetric Walker metric* (5.18) *is a Ricci soliton, which can be expanding, steady or shrinking and is defined by vector fields*

$$X(t,x,y) = \left(\lambda\, t - x\, w'(y) + \mu(y), \frac{1}{2}\,\lambda\, x + w(y), \gamma\right), \tag{5.30}$$

where λ *is the constant in* (1.26), γ *is an arbitrary constant and the functions* w *and* μ *are arbitrary solutions of* (5.29). *In general, the causal character of* X *may vary with the point.*

Chapter 6

Quasi-Einstein manifolds

In this chapter we focus on another generalization of gradient Ricci solitons which is related to the geometry of smooth metric measure spaces. A smooth metric measure space is a Riemannian manifold with a measure which is conformal to the Riemannian one. Formally, it is a triple $(M, g, e^{-f} dvol_g)$, where M is a complete n-dimensional smooth manifold with Riemannian metric g, f is a smooth real valued function on M, and $dvol_g$ is the Riemannian volume density on (M, g). This is also sometimes called a manifold with density.

A natural extension of the Ricci tensor to smooth metric measure spaces is the m-Bakry-Emery Ricci tensor

$$\rho_f^m = \rho + \mathrm{Hes}_f - \frac{1}{m}\, df \otimes df, \qquad \text{for} \qquad 0 < m \leq \infty.$$

When f is constant, this is the usual Ricci tensor. We call a quadruple $(M, g, f, -\frac{1}{m})$ quasi-Einstein if it satisfies the equation $\rho_f^m = \lambda\, g$, for some $\lambda \in \mathbb{R}$. Quasi-Einstein manifolds are a generalization of Einstein metrics and they contain gradient Ricci solitons as a limit case when $m = \infty$. Moreover they are also closely related to the construction of warped product Einstein metrics. Recall here that a complete classification of warped product Einstein metrics is still an open problem in spite of their interest in describing standard stationary metrics. Quasi-Einstein metrics and, more generally, the m-Bakry-Emery Ricci tensor allows some generalizations of the celebrated Hawking-Penrose singularity theorems and the Lorentzian splitting theorem [36].

We are going to work in a slightly more general setting.

Definition 6.1 *Let (M, g) be a Lorentzian manifold of dimension $n + 2$, with $n \geq 1$. Let f be a smooth function on M and let $\mu \in \mathbb{R}$ be an arbitrary constant. We say that (M, g, f, μ) is* quasi-Einstein *if there exists $\lambda \in \mathbb{R}$ so that*

$$\mathrm{Hes}_f + \rho - \mu\, df \otimes df = \lambda g. \tag{6.1}$$

If we specialize the elements of this quadruple, then we get some important families of manifolds. Thus, for example, if $\mu = 0$ then quasi-Einstein manifolds correspond to gradient Ricci solitons and if f is constant (6.1) reduces to the Einstein equation. There are other interesting relations between quasi-Einstein manifolds and some well-known structures:

(a) If $\left(M, g, f, -\frac{1}{n}\right)$ is quasi-Einstein, then the conformal metric $\widetilde{g} = e^{-\frac{2}{n}f} g$ is Einstein since, from the expression of the Ricci tensor of a conformal metric, we get [91]

$$\begin{aligned}\rho_{\widetilde{g}} &= \rho_g + \mathrm{Hes}_f + \frac{1}{n}\, df \otimes df + \frac{1}{n}\,(\Delta f - \|\nabla f\|^2) g \\ &= \frac{1}{n}\,(\Delta f - \|\nabla f\|^2 + n\,\lambda) e^{\frac{2}{n}f}\widetilde{g}.\end{aligned} \tag{6.2}$$

In particular, if g is also locally conformally flat, then $\widetilde{g}$ has constant curvature.

(b) Let $M \times_\varphi F$ be an Einstein warped product. Then it follows from the expressions of the Ricci tensor of the warped product (see, for example, [53]) that $\left(M, g, -(\dim(F)) \log(\varphi), \frac{1}{\dim(F)}\right)$ is quasi-Einstein.

Moreover, it follows from [66] that for any quasi-Einstein manifold (M, g, f, μ) with $\mu = \frac{1}{m} > 0$, there exist suitable fibers (F, g_F) so that the warped product $M \times_\varphi F$ is Einstein, where $\varphi = e^{-\mu f}$.

Locally conformally flat complete quasi-Einstein Riemannian manifolds were recently classified in [38]. The purpose of this chapter is to describe the local structure of locally conformally flat quasi-Einstein manifolds in the Lorentzian setting. The main result is summarized as follows

Theorem 6.2 *Let (M, g, f, μ) be a locally conformally flat Lorentzian quasi-Einstein manifold.*

(i) If $\mu = -\frac{1}{n}$, then (M, g) is globally conformally equivalent to a space form.

(ii) If $\mu \neq -\frac{1}{n}$, then

(a) In a neighborhood of any point where $\|\nabla f\| \neq 0$, M is locally isometric to a warped product $I \times_\varphi F$, where I is a real interval and F is a $(n+1)$-dimensional fiber of constant sectional curvature.

(b) If $\|\nabla f\| = 0$, then (M, g) is locally isometric to a plane wave, i.e., (M, g) is locally isometric to $\mathbb{R}^2 \times \mathbb{R}^n$ with metric

$$g = 2dudv + H(u, x_1, \dots, x_n)du^2 + \sum_{i=1}^{n} dx_i^2,$$

where $H(u, x_1, \dots, x_n) = a(u)\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i(u)x_i + c(u)$, for some functions a, b_i, c, and function $f(u, v, x_1, \dots, x_n) = f_0(u)$ with $f_0''(u) - \mu\,(f_0'(u))^2 - n\,a(u) = 0$.

Depending on the character of ∇f we say that a quasi-Einstein manifold is *non-isotropic* if $\|\nabla f\| \neq 0$ or *isotropic* if $\|\nabla f\| = 0$. In the following sections we will study both cases separately.

The chapter is organized as follows. In Section 6.1 we study some formulae and properties involving geometric objects of quasi-Einstein manifolds, specially the Ricci tensor. The analysis of case (ii)–(a) in Theorem 6.2 is carried out in Section 6.2. In Section 6.3 we obtain some results on the isotropic case and, finally, we study locally conformally flat quasi-Einstein *pp*-waves in Section 6.4 to complete the proof of Theorem 6.2. The main results of this chapter are collected in [17, 20].

6.1 General remarks on locally conformally flat quasi-Einstein manifolds

Let (M, g, f, μ) be a Lorentzian quasi-Einstein manifold. As observed above, if $\mu = -\frac{1}{n}$, the manifold is globally conformally equivalent to a space form. In what follows, we are going to introduce some results and definitions that we will use in subsequent sections to prove Theorem 6.2–(ii).

6.1.1 General formulae

Although the next result was given in [38], we include a sketch of a proof following a different strategy in order to make this memoir as self-contained as possible.

Lemma 6.3 *A Lorentzian quasi-Einstein manifold* (M, g, f, μ) *satisfies*

$$\tau + \Delta f - \mu \|\nabla f\|^2 = (n+2)\,\lambda, \tag{6.3}$$

$$\nabla \tau = 2(\lambda - (n+2)\lambda\,\mu + \mu(1-\mu)\|\nabla f\|^2 + \mu\,\tau)\nabla f + (\mu - 1)\nabla\|\nabla f\|^2. \tag{6.4}$$

Proof.
Equation (6.3) is obtained by simply contracting equation (6.1). Taking into account the contracted Second Bianchi identity, $\nabla\tau = 2\,\mathrm{div}(\rho)$, and the following Bochner formula, $\mathrm{div}(\nabla\nabla f) = \rho(\nabla f) + \nabla\Delta f$, we compute the divergence of Equation (6.1):

$$\begin{aligned} 0 &= \mathrm{div}(\lambda\, g)(X) \\ &= \mathrm{div}(\rho + \mathrm{Hes}_f - \mu\, df \otimes df)(X) \\ &= \frac{1}{2}\, g(\nabla\tau, X) + \rho(\nabla f, X) + g(\nabla\Delta f, X) \\ &\quad - \mu\, g((\mathrm{div}\,\nabla f)\nabla f, X) - \mu\, g(\nabla_{\nabla f}\nabla f, X). \end{aligned}$$

Covariantly differentiating Equation (6.3) we get $\nabla\tau = -\nabla\Delta f + \mu\nabla\|\nabla f\|^2$. Using this equation and that $\nabla_{\nabla f}\nabla f = \frac{1}{2}\nabla\|\nabla f\|^2$ we see that

$$
\begin{aligned}
0 =& \frac{1}{2}\, g(\nabla\tau, X) + \lambda\, g(\nabla f, X) + \mu\,\|\nabla f\|^2 g(\nabla f, X) - g(\nabla_{\nabla f}\nabla f, X) \\
&+ g(\mu\,\nabla\|\nabla f\|^2, X) - g(\nabla\tau, X) - \mu\, g(\Delta f\nabla f, X) - \mu\, g(\nabla_{\nabla f}\nabla f, X) \\
=& g\left(-\frac{1}{2}\,\nabla\tau + (\lambda + \mu\,\|\nabla f\|^2 - \mu\,\Delta f)\nabla f + \left(\frac{\mu - 1}{2}\right)\nabla\|\nabla f\|^2, X\right).
\end{aligned}
$$

Now we replace Δf by $(n+2)\,\lambda + \mu\,\|\nabla f\|^2 - \tau$ to obtain Equation (6.4). □

Remark 6.4 If the quasi-Einstein manifold (M, g, f, μ) is isotropic, i.e., $\|\nabla f\| = 0$, then Equation (6.4) reduces to

$$\nabla\tau = 2(\lambda - \mu((n+2)\lambda - \tau))\nabla f.$$

Also note that, from Equation (6.1), one can write the Ricci operator in the direction of ∇f as

$$2\,\mathrm{Ric}(\nabla f) = 2\lambda\nabla f + 2\mu\,\|\nabla f\|^2\nabla f - \nabla\|\nabla f\|^2,$$

so, if $\|\nabla f\| = 0$, then $\mathrm{Ric}(\nabla f) = \lambda\nabla f$ and ∇f is an eigenvector of the Ricci operator associated to the eigenvalue λ.

6.1.2 Some curvature properties of locally conformally flat quasi-Einstein manifolds.

We proceed as in [52] to continue the study of the spectrum of the Ricci operator.

Lemma 6.5 *Let (M, g, f, μ) be a locally conformally flat quasi-Einstein manifold of dimension $n+2$. Then, if $\mu \neq -\frac{1}{n}$, ∇f is an eigenvector of the Ricci operator.*

Proof.
Since (M, g) is locally conformally flat the Schouten tensor is Codazzi, so

$$(\nabla_X\rho)(Y, Z) - \frac{X(\tau)}{2(n+1)}\, g(Y, Z) = (\nabla_Y\rho)(X, Z) - \frac{Y(\tau)}{2(n+1)}\, g(X, Z), \tag{6.5}$$

for all vector fields X, Y, Z.

Using Equation (6.1) we can write

$$
\begin{aligned}
(\nabla_X\rho)(Y, Z) =& -(\nabla_X \mathrm{Hes}_f)(Y, Z) + \mu\,(\nabla_X df \otimes df)(Y, Z) \\
=& -X(\mathrm{Hes}_f)(Y, Z) + g(\nabla_{\nabla_X Y}\nabla f, Z) + g(\nabla_Y\nabla f, \nabla_X Z) \\
&+ \mu\,\left(X(df \otimes df(Y, Z)) - g(\nabla f, \nabla_X Y)Z(f) - g(\nabla f, \nabla_X Z)Y(f)\right) \\
=& -g(\nabla_X\nabla_Y\nabla f, Z) + g(\nabla_{\nabla_X Y}\nabla f, Z) \\
&+ \mu\,(\mathrm{Hes}_f(X, Y)g(\nabla f, Z) + \mathrm{Hes}_f(X, Z)g(\nabla f, Y)).
\end{aligned}
$$

We substitute this expression in (6.5) to get

$$\begin{aligned} &-g(\nabla_X\nabla_Y\nabla f, Z) + g(\nabla_{\nabla_X Y}\nabla f, Z) - \frac{X(\tau)}{2(n+1)}\, g(Y,Z) + \mu\,\mathrm{Hes}_f(X,Z) g(\nabla f, Y) \\ =&-g(\nabla_Y\nabla_X\nabla f, Z) + g(\nabla_{\nabla_Y X}\nabla f, Z) - \frac{Y(\tau)}{2(n+1)}\, g(X,Z) + \mu\,\mathrm{Hes}_f(Y,Z) g(\nabla f, X). \end{aligned}$$

Reorganizing the terms of this expression and using that the curvature tensor is given by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we obtain

$$\begin{aligned} R(X,Y,Z,\nabla f) =& -\frac{X(\tau)}{2(n+1)}\, g(Y,Z) + \frac{Y(\tau)}{2(n+1)}\, g(X,Z) \\ &+ \mu\,(\mathrm{Hes}_f(X,Z) g(\nabla f, Y) - \mathrm{Hes}_f(Y,Z) g(\nabla f, X)). \end{aligned} \tag{6.6}$$

We choose vector fields $Z = \nabla f$ and X such that $g(X, \nabla f) = 1$ to get that

$$0 = \frac{Y(\tau)}{2(n+1)} - \mu\,\mathrm{Hes}_f(Y, \nabla f)$$

for all $Y \perp \nabla f$. Now, from Equation (6.4) we have that $Y(\tau) = 2(\mu - 1)\,\mathrm{Hes}_f(Y, \nabla f)$ so

$$0 = -\frac{n\,\mu + 1}{n+1}\,\mathrm{Hes}_f(Y, \nabla f),$$

if $Y \perp \nabla f$. Hence, either $\mu = -\frac{1}{n}$ or ∇f is an eigenvector of the Hessian operator $\mathrm{hes}_f(X) = \nabla_X \nabla f$. Assume the latter, from (6.1) we have that $\rho(Y, \nabla f) = -\,\mathrm{Hes}_f(Y, \nabla f)$ and therefore

$$0 = \frac{n\,\mu + 1}{n+1}\rho(Y, \nabla f),$$

for $Y \perp \nabla f$, showing that ∇f is also an eigenvector of the Ricci operator, unless the parameter μ takes the value $\mu = -\frac{1}{n}$.

□

6.2 Non-isotropic locally conformally flat quasi-Einstein manifolds

In this section we show that in a neighborhood of any point where $\|\nabla f\| \neq 0$ the underlying manifold has the local structure of a warped product, thus proving Theorem 6.2–(ii)–(a).

Lemma 6.6 *Let (M, g, f, μ) be a locally conformally flat Lorentzian quasi-Einstein manifold with $\|\nabla f\|_p \neq 0$ in some point $p \in M$. Then, if $\mu \neq -\frac{1}{n}$, (M, g) is a warped product of a real interval and a space of constant sectional curvature on a neighborhood of p.*

Proof.
Since $\|\nabla f\|_p \neq 0$, $\|\nabla f\| \neq 0$ on a neighborhood $\mathcal{U}$ of p. Consider the unit vector $V = \frac{\nabla f}{\|\nabla f\|}$ on $\mathcal{U}$, which can be timelike or spacelike (we set $\varepsilon = g(V,V) = \pm 1$). Consider a local orthonormal frame $\{V = E_0, E_1, \dots, E_{n+1}\}$ and set $\varepsilon_i = g(E_i, E_i)$.

From Lemma 6.5 we have that $\rho(E_i, V) = \mathrm{Hes}_f(E_i, V) = 0$ for all $i = 1, \dots, n+1$. Hence from Equation (6.4) we obtain

$$E_i(\tau) = 2\|\nabla f\|(1-\mu)\rho(V, E_i) = 0.$$

We compute $R(V, E_i, E_i, V)$ in Equations (6.6) and (2.1) to see that

$$-\frac{V(\tau)}{2(n+1)\|\nabla f\|}\,\varepsilon_i - \mu\,\mathrm{Hes}_f(E_i,E_i)\varepsilon = \frac{\tau}{n(n+1)}\,\varepsilon_i\varepsilon - \frac{1}{n}\,\rho(V,V)\varepsilon_i - \frac{1}{n}\,\rho(E_i,E_i)\varepsilon,$$

which shows that

$$\left(\mu + \frac{1}{n}\right)\mathrm{Hes}_f(E_i,E_i)\varepsilon = \left(-\frac{\tau}{n(n+1)}\,\varepsilon + \frac{1}{n}\,\rho(V,V) + \frac{1}{n}\,\lambda\varepsilon - \frac{V(\tau)}{2(n+1)\|\nabla f\|}\right)\varepsilon_i.$$

This shows that, when $\mu \neq -\frac{1}{n}$, the level sets of f are totally umbilical hypersurfaces. Hence, as the normal foliations are totally geodesic ($g(\nabla_{\nabla f}\nabla f, E_i) = \mathrm{Hes}_f(\nabla f, E_i) = 0$, $\forall i = 1, \dots, n+1$), (M,g) decomposes locally as a twisted product (see [88]). Now, since $\rho(V, E_i) = 0$ for all $i = 1, \dots, n+1$, the twisted product reduces to a warped product [53]. In conclusion (M,g) is locally a warped product $(I \times N, \varepsilon\, dt^2 + \psi(t)^2 g_N)$, which is locally conformally flat by hypothesis and hence (N, g_N) is a Riemannian or a Lorentzian manifold of constant sectional curvature [22]. □

Remark 6.7 Note that in Riemannian signature a quasi-Einstein manifold satisfies the hypothesis of Lemma 6.6 and analogous arguments to those given here apply. Thus one has that locally conformally flat quasi-Einstein manifolds are either conformally equivalent to a space form or locally isometric to a warped product of a real interval and a space of constant sectional curvature. We refer to [38] for a different proof of this result.

6.3 Isotropic locally conformally flat quasi-Einstein manifolds

In order to prove Theorem 6.2–(ii)–(b), in this section we are going to study isotropic locally conformally flat Lorentzian quasi-Einstein manifolds.

Lemma 6.8 *Let (M, g, f, μ) be an isotropic locally conformally flat Lorentzian quasi-Einstein manifold. Then, if $\mu \neq -\frac{1}{n}$, around any regular point of f, the manifold (M, g) is locally a pp-wave.*

Proof.
Let (M, g, f, μ) be quasi-Einstein with $\|\nabla f\| = 0$. We will see that ∇f spans a parallel null line field, but we start by analyzing the Ricci tensor. First we choose an appropriate basis to work with. Set $V = \nabla f$. Since V is a null vector, there exist orthogonal vectors S, T satisfying $g(S, S) = -g(T, T) = \frac{1}{2}$ such that $V = S + T$. Define $U = S - T$, which is a null vector such that $g(U, V) = g(S, S) - g(T, T) = 1$, and consider a basis $\{U, V, E_1, \dots, E_n\}$, with $g(E_i, E_i) = 1$ for all $i = 1, \dots, n$. We begin the study of the Ricci tensor by noting that, since $\mathrm{Ric}(V) = \lambda V$ by Remark 6.4, we have

$$\rho(V, V) = 0,\ \rho(U, V) = \lambda \text{ and } \rho(V, E_i) = 0 \quad \forall i = 1, \dots, n.$$

We compute $R(U, E_i, E_j, V)$ both in Equation (6.6) and in Equation (2.1) to see that

$$\frac{-U(\tau)}{2(n+1)}\, \delta_{ij} - \mu\, \mathrm{Hes}_f(E_i, E_j) = \frac{\tau}{n(n+1)}\, \delta_{ij} - \frac{1}{n}\, \lambda\, \delta_{ij} - \frac{1}{n}\, \rho(E_i, E_j). \tag{6.7}$$

From Equation (6.1) we get that $\mathrm{Hes}_f(E_i, E_j) = \lambda\, \delta_{ij} - \rho(E_i, E_j)$ and from Remark 6.4 that $U(\tau) = 2(\lambda - \mu((n+2)\lambda - \tau))$. Hence, since $\mu \neq -\frac{1}{n}$, from Equation (6.7) we conclude that, if $i \neq j$, then $\rho(E_i, E_j) = 0$, whereas $\rho(E_i, E_i) = \frac{\tau - \lambda}{n+1}$ for all $i = 1, \dots, n$. Now, compute the scalar curvature

$$\tau = 2\rho(U, V) + n\, \rho(E_i, E_i) = 2\lambda + n\, \frac{\tau - \lambda}{n+1},$$

to see that

$$\tau = (n+2)\lambda,$$

so τ is constant. Moreover, since $\nabla \tau = 0$, by Remark 6.4 we have $\lambda - \mu((n+2)\lambda - \tau) = 0$ and consequently $\lambda = 0 = \tau$. From (2.1) we compute

$$\rho(U, E_i) = R(U, V, E_i, V) + \sum_{j \neq i} R(U, E_j, E_i, E_j) = \frac{n-1}{n}\, \rho(U, E_i).$$

Thus $\rho(U, E_i) = 0$ for $i = 1, \dots, n$ and the only possibly nonzero term of the Ricci tensor is $\rho(U, U)$. Therefore the Ricci operator is two-step nilpotent and its image is totally isotropic.

Now we are going to show that the line field $\mathcal{D} = \mathrm{span}\{V\}$ is parallel by checking that V is a recurrent vector field. Thus, from Equation (6.1) we compute

$$\begin{aligned}
\mathrm{hes}_f(U) &= -\mathrm{Ric}(U) + \mu\, df \otimes df(U) = -\rho(U, U)U + \mu\, U, \\
\mathrm{hes}_f(V) &= -\mathrm{Ric}(V) + \mu\, df \otimes df(V) = 0, \\
\mathrm{hes}_f(E_i) &= -\mathrm{Ric}(E_i) + \mu\, df \otimes df(E_i) = 0.
\end{aligned}$$

This shows that $\nabla_X \nabla f = \sigma(X) \nabla f$ for the one-form σ given by $\sigma(U) = \mu - \rho(U, U)$, $\sigma(V) = 0$, $\sigma(E_i) = 0$ for all $i = 1, \dots, n$.

It is now easy to see from (2.1) and the information on the Ricci tensor that

$$R(\mathcal{D}^\perp, \mathcal{D}^\perp, \cdot, \cdot) = 0.$$

Hence, by Theorem 1.16, it follows that (M, g) is a *pp*-wave. □

Remark 6.9 Note that although (M, g) is a *pp*-wave, and hence it admits a null parallel vector field, ∇f is not in general parallel.

6.4 Locally conformally flat quasi-Einstein *pp*-waves

Let (M, g_{ppw}) be a *pp*-wave given in local coordinates as in (1.12).

As a consequence of expressions (1.14) and (1.15), a *pp*-wave as in (1.12) is locally conformally flat if

$$H(u, x_1, \dots, x_n) = a(u)\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} b_i(u) x_i + c(u), \tag{6.8}$$

for arbitrary smooth functions a, b_1, …, b_n, c. Furthermore, note that in this case the Ricci tensor is only a function of u, $\rho(\partial_u, \partial_u) = -n\, a(u)$. Recall that a *pp*-wave whose defining function H is a quadratic form on the variables $x_1, \dots, x_n$ is called a plane wave. Thus, one can change variables in Equation (6.8) to see that locally conformally flat *pp*-waves are a particular family of plane waves.

The following result characterizes locally conformally flat quasi-Einstein *pp*-waves.

Theorem 6.10 *Let (M, g_{ppw}, f, μ) be a non-trivial locally conformally flat quasi-Einstein pp-wave with g_{ppw} given as in (1.12). Then one of the following holds:*

(i) $\mu = 0$ and (M, g_{ppw}, f) is a steady gradient Ricci soliton with f a function of u satisfying $f''(u) = -\rho(\partial_u, \partial_u) = n\, a(u)$.

(ii) $\mu = -\frac{1}{n}$ and (M, g_{ppw}) is conformally equivalent to a manifold of constant sectional curvature $c \leq 0$. Moreover, if M is three-dimensional or $\|\nabla f\| = 0$, it is conformally equivalent to a flat manifold.

(iii) $\mu \neq 0$ and $\mu \neq -\frac{1}{n}$, then the function f is given by

$$f(u, v, x_1, \dots, x_n) = -\frac{1}{\mu} \log(f_0(u)) \tag{6.9}$$

with

$$f_0''(u) = \mu\, \rho(\partial_u, \partial_u) f_0(u) = -\mu\, n\, a(u) f_0(u). \tag{6.10}$$

Proof.
If $\mu = 0$ then (M, g_{ppw}, f) is a gradient Ricci soliton and the result follows from Section 2.4.

Now assume $\mu \neq 0$. Let $f(u, v, x_1, \ldots, x_n)$ be an arbitrary function. The gradient of f with respect to (1.12) is given by $\nabla f = (\partial_v f, \partial_u f - H\partial_v f, \partial_{x_1} f, \ldots, \partial_{x_n} f)$ and thus (6.1) reduces to the following equations:

$$\frac{1}{2}\sum_{i=1}^{n} \partial_{x_i} H \, \partial_{x_i} f + \partial^2_{uu} f - \frac{1}{2}\partial_u H \, \partial_v f + \rho(\partial_u, \partial_u) - \mu(\partial_u f)^2 = \lambda H, \tag{6.11}$$

$$\partial^2_{ux_i} f - \frac{1}{2}\partial_{x_i} H \, \partial_v f - \mu \, \partial_{x_i} f \partial_u f = 0, \qquad 1 \le i \le n, \tag{6.12}$$

$$\partial^2_{x_i x_i} f - \mu(\partial_{x_i} f)^2 = \lambda, \qquad 1 \le i \le n, \tag{6.13}$$

$$\partial^2_{uv} f - \mu \, \partial_u f \, \partial_v f = \lambda, \tag{6.14}$$

$$\partial^2_{vv} f - \mu(\partial_v f)^2 = 0, \tag{6.15}$$

$$\partial^2_{vx_i} f - \mu \, \partial_{x_i} f \, \partial_v f = 0, \qquad 1 \le i \le n, \tag{6.16}$$

$$\partial^2_{x_i x_j} f - \mu \, \partial_{x_i} f \, \partial_{x_j} f = 0, \qquad 1 \le i \neq j \le n. \tag{6.17}$$

First we are going to show that $\partial_v f = 0$ and that $\lambda = 0$. We argue as follows. Differentiate (6.14) with respect to v to see that $\partial^3_{uvv} f - \mu \, \partial^2_{uv} f \, \partial_v f - \mu \, \partial_u f \partial^2_{vv} f = 0$. Differentiate (6.15) with respect to u to see that $\partial^3_{uvv} f = 2\mu \, \partial_v f \partial^2_{uv} f$ and substitute the third order term to get that $\mu \, \partial_v f \partial^2_{uv} f - \mu \, \partial_u f \partial^2_{vv} f = 0$. Now use (6.14) and (6.15) to reduce higher order terms to first order terms and get $\lambda \, \mu \, \partial_v f = 0$. Since $\mu \neq 0$, either $\lambda = 0$ or $\partial_v f = 0$. If $\partial_v f = 0$ then, as a consequence of (6.14), we also have that $\lambda = 0$.

Now if $\mu = -\frac{1}{n}$, then it follows that (M, g_{ppw}) is conformally equivalent to an Einstein manifold and, since (M, g_{ppw}) is locally conformally flat, it is conformally equivalent to a manifold of constant sectional curvature. Furthermore since $\tau = \lambda = 0$, we obtain from Equation (6.3) that $\Delta f = \frac{-1}{n}\|\nabla f\|^2$. Hence the conformal factor reduces to $\frac{1}{n}\left(\frac{1}{n} - 1\right)\|\nabla f\|^2$, which is zero whenever $n = 1$ or $\|\nabla f\| = 0$, or otherwise positive. This proves (ii).

Next assume $\mu \neq -\frac{1}{n}$ and $\lambda = 0$. Differentiate (6.12) with respect to x_i to get

$$\partial^3_{ux_i x_i} f - \frac{1}{2}\left(\partial^2_{x_i x_i} H \, \partial_v f + \partial_{x_i} H \, \partial^2_{vx_i} f\right) - \mu\left(\partial^2_{x_i x_i} f \partial_u f + \partial_{x_i} f \, \partial^2_{ux_i} f\right) = 0. \tag{6.18}$$

Differentiate (6.13) with respect to u to see that $\partial^3_{ux_i x_i} f = 2\mu \, \partial_{x_i} f \, \partial^2_{ux_i} f$. Simplify (6.18) using (6.12), (6.13) and (6.16) to obtain

$$\frac{1}{2}\partial^2_{x_i x_i} H \, \partial_v f = -\lambda \, \mu \, \partial_u f = 0.$$

Since $\partial^2_{x_i x_i} H \neq 0$ for some i (otherwise the manifold is flat), we conclude that $\partial_v f = 0$.

From equations $\partial^2_{x_ix_i} f - \mu(\partial_{x_i} f)^2 = 0$ we get that

$$f(u,v,x_1,\dots,x_n) = -\frac{1}{\mu}\log\left(f_0(u) + \sum_{i=1}^{n}\kappa_i\, x_i\right), \tag{6.19}$$

for some constants κ_i and some function $f_0(u)$. Replace H by the expression in (6.8) and simplify to see that equations (6.11)–(6.17) reduce to the condition

$$\mu\,\rho(\partial_u,\partial_u)\left(f_0(u) + \sum_{i=1}^{n}\kappa_i\, x_i\right) = f_0''(u) + a(u)\sum_{i=1}^{n}\kappa_i\, x_i + \frac{1}{2}\sum_{i=1}^{n}\kappa_i\, b_i(u).$$

Since $\rho(\partial_u,\partial_u) = -n\,a(u)$, we may differentiate with respect to x_i to see that

$$-n\,\mu\,\kappa_i\,a(u) = \kappa_i\,a(u).$$

Since $a(u) \neq 0$ (otherwise the manifold is flat) and $\mu \neq -\frac{1}{n}$, we conclude that $\kappa_i = 0$ for all $i = 1,\dots,n$ and the result follows. □

Proof of Theorem 6.2. The result follows from Lemmas 6.6, 6.8 and 6.10.

Remark 6.11 Let (M,g,f,μ) be a quasi-Einstein *pp*-wave with f given by (6.19). Then

$$\nabla f = \left(0, \frac{-f_0'(u)}{\mu\left(f_0(u) + \sum_{i=1}^{n}\kappa_i\, x_i\right)}, \frac{-\kappa_1}{\mu\left(f_0(u) + \sum_{i=1}^{n}\kappa_i\, x_i\right)}, \dots, \frac{-\kappa_n}{\mu\left(f_0(u) + \sum_{i=1}^{n}\kappa_i\, x_i\right)}\right)$$

and hence $\|\nabla f\|^2 = \sum_{i=1}^{n}\frac{\kappa_i^2}{\mu^2\left(f_0(u) + \sum_{i=1}^{n}\kappa_i\, x_i\right)^2}$. Now if $\mu \neq 0$ and $\mu \neq -\frac{1}{n}$ then $\kappa_i = 0$ for all $i = 1,\dots,n$ and (M,g,f,μ) is isotropic.

Remark 6.12 It was shown in Section 2.4 that a *pp*-wave (1.12) is a gradient Ricci soliton (i.e., a quasi-Einstein metric with $\mu = 0$) if and only if there exists a function $f(u,v,x_1,\dots,x_n) = f_0(u) + \sum_{i=1}^{n}\kappa_i\, x_i$ satisfying $f_0''(u) = -\rho(\partial_u,\partial_u) - \frac{1}{2}\sum_{i=1}^{n}\kappa_i\,\partial_{x_i} H$.

6.4.1 Quasi-Einstein plane waves

As an immediate application of Lemma 6.10 we have the following result:

Theorem 6.13 *Every plane wave is locally an isotropic quasi-Einstein manifold.*

Proof.
For the function given in (6.9) we have that $\|\nabla f\| = 0$ (see Remark 6.11). Moreover, the necessary and sufficient condition given in (6.10) in Theorem 6.10 becomes

$$f_0''(u) = \mu\,\rho(\partial_u, \partial_u) f_0(u). \tag{6.20}$$

Now, since the Ricci tensor of a plane wave (1.18) satisfies $\rho(\partial_u, \partial_u) = -\sum_{i=1}^{n} a_{ii}(u)$, one can always guarantee the existence of positive local solutions $f_0(u)$ of (6.20). □

Remark 6.14 Note that no function $f(u, v, x_1, \dots, x_n) = -\frac{1}{\mu}\log\left(f_0(u) + \sum_{i=1}^{n} \kappa_i x_i\right)$ can be globally defined on $\mathbb{R}^{n+2}$ unless $\kappa_i = 0$ for all $i = 1, \dots, n$. Hence, in searching for global solutions of (6.1) within the class of plane waves one must assume that $f(u, v, x_1, \dots, x_n) = -\dfrac{1}{\mu}\log(f_0(u))$ and look for positive solutions of (6.20), which can be approached by standard Sturm-Liouville arguments.

Theorem 6.15 *Any plane wave whose Ricci tensor is nowhere zero results in an isotropic quasi-Einstein manifold for appropriate f and μ.*

Proof.
Let f_0 be a solution of (6.20) satisfying the initial conditions $f_0(0) = 1$, $f_0'(0) = 0$ and choose μ such that $\mu\,\rho(\partial_u, \partial_u) > 0$. Then f_0 is positive in a neighborhood of zero and, since $\mu\,\rho(\partial_u, \partial_u) > 0$, f_0'' remains positive in such interval. Hence f_0 has a minimum at zero and the assumption $\mu\,\rho(\partial_u, \partial_u) > 0$ guarantees that no other extremum of f_0 may occur. Therefore f_0 remains positive for all time. □

Remark 6.16 Any plane wave (1.18) results in a gradient Ricci soliton (i.e., quasi-Einstein with $\mu = 0$) just considering a function $f(u, v, x_1, \dots, x_n) = f_0(u)$ satisfying $f_0''(u) = -\rho(\partial_u, \partial_u)$ as shown in Theorem 3.1.

Remark 6.17 For the special choice of $\mu = -\frac{1}{n}$, Theorem 6.13 shows that any plane wave is a locally conformally Einstein manifold. Furthermore, since $\tau = \|\nabla f\| = 0$ and $\lambda = 0$, we get from (6.3) that $\Delta f = 0$ as well, so the conformal factor in (6.2) vanishes and we get that a plane wave is indeed locally conformally equivalent to a Ricci flat manifold.

6.4.2 Special classes of quasi-Einstein plane waves

Recall that indecomposable Lorentzian symmetric spaces are either irreducible or the so-called Cahen-Wallach symmetric spaces which are given by a plane wave metric (1.18) with $H(u, x_1, \dots, x_n) = \sum_{i=1}^{n} a_i x_i^2$ where a_i are non-zero real numbers for all $i = 1, \dots, n$. It was shown in [6] that Cahen-Wallach symmetric spaces are steady gradient Ricci solitons. Moreover, since the Ricci tensor satisfies $\rho(\partial_u, \partial_u) = -\sum_{i=1}^{n} a_i$, one has that

Theorem 6.18 *Every Cahen-Wallach symmetric space is an isotropic quasi-Einstein manifold.*

Remark 6.19 Classical Sturm-Liouville theory investigates the existence of zeroes for second order ordinary differential equations of this form. It is well known that any nontrivial (i.e., not identically zero) solution of a differential equation of the form $y'' + q(x)y = 0$ has at most one zero if $q(x) < 0$.

On the other hand any nontrivial solution of $y'' + q(x)y = 0$ has infinitely many zeroes if $q(x) > 0$ for all $x > \alpha$ and

$$\int_{\alpha}^{\infty} q(x)dx = +\infty.$$

See, for example the discussion in [93].

Remark 6.20 Obstructions to the existence of positive solutions of the differential equation $f_0''(u) = -\rho(\partial_u, \partial_u)\, f_0(u)$ easily follow by Sturm-Liouville arguments [92]. In fact, if $\int_1^{\infty} \rho(\partial_u, \partial_u) = \infty$, then any solution of $f_0''(u) = -\rho(\partial_u, \partial_u)\, f_0(u)$ has an infinite number of zeros.

Recall that the only non-zero component of the Ricci tensor of a two-symmetric Lorentzian manifold is given by $\rho(\partial_u, \partial_u) = -\sum_{i=1}^{n} (\alpha_{ii}\, u + \beta_{ii})$, which shows that the Ricci tensor changes sign. Hence any two-symmetric Lorentzian manifold is locally but not globally quasi-Einstein spaces.

Also, as an application of the results in the previous section we get that

Theorem 6.21 *Every Lorentzian space admitting a degenerate homogeneous structure of linear type is a quasi-Einstein manifold.*

Remark 6.22 Meessen [72] proved that the underlying geometry of a singular homogeneous plane wave is that of a connected homogeneous Lorentzian space that admits a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$-structure and, conversely, any Lorentzian manifold admitting a degenerate homogeneous $\mathcal{T}_1 \oplus \mathcal{T}_3$-structure is a singular homogeneous plane wave. Therefore, Theorem 6.21 extends to the broader class $\mathcal{T}_1 \oplus \mathcal{T}_3$.

Open problems II

The geometry of general Ricci solitons and quasi-Einstein metrics is still far from being completely understood. There are many open problems, but we mention just a few of them which are closely related to our work:

- Invariant Ricci solitons on four-dimensional homogeneous spaces have been recently investigated in [28]. Although some progress has been made, the geometric structure of Ricci solitons in [28] needs to be elucidated.

- Algebraic Ricci solitons, which involve the underlying Lie group structure, have a strong connection with Ricci solitons and Einstein metrics, as shown in [70]. Lorentzian algebraic Ricci solitons have been recently investigated in [79], showing the existence of steady algebraic Ricci solitons. Moreover Lorentzian Ricci solitons need not be algebraic Ricci solitons [8], a question that still remains open in the Riemannian setting.

- The existence of Lorentzian manifolds admitting different kinds of Ricci solitons with a background *pp*-wave metric suggests that this could be the underlying structure of Lorentzian manifolds admitting homothetic vector fields. This is indeed the case in many situations [94] but the general problem remains open.

- A rigidity result for quasi-Einstein Riemannian metrics was proven in [37] under the conditions of harmonic Weyl tensor and zero radial Weyl curvature, showing that they are locally warped products. In view of the results in Chapter 6, one could expect *pp*-waves to play an important role in Lorentzian signature whenever the level sets of the potential function f become degenerate.

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