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Abstract

The notion of symmetry underlies a large number of new ideas and major advances in Science, Engineering and Art. From the mathematical viewpoint, the intuitive idea of symmetry as the balanced correspondence of shape along space translates to the existence of a transformation group acting on such space. The first natural field for the study of symmetry is then geometry. Conversely, in his influential Erlanger Programm, Felix Klein described geometry as the study of those properties of a space that are invariant under a transformation group. Hence, symmetry lies in the very core of geometry.

In Riemannian geometry, the natural group to consider is the isometry group, that is, the group of those transformations of the space that preserve distances. The action of a subgroup of the isometry group of a given manifold is called an *isometric action*. Its cohomogeneity is the lowest codimension of its orbits. Each one of the orbits of such an isometric action is called an *(extrinsically) homogeneous submanifold*, and the collection of all the orbits is the orbit foliation of the action.

The main objects of study in this thesis are certain kinds of submanifolds with a particularly high degree of symmetry. Our ultimate goal is to decide whether the intuitive notion of symmetry is reflected in the mathematical notion of symmetry, namely if the correspondence of shape at different parts of the submanifold implies that the submanifold is homogeneous.

On the one hand, the study of isometric actions in general turns out to be a very difficult problem. This motivated the introduction of special kinds of actions whose investigation could be more manageable. This is the case of *polar actions*, that is, isometric actions that admit totally geodesic submanifolds intersecting all orbits orthogonally.

On the other hand, several geometric notions have been proposed to try to characterize homogeneous submanifolds. Thus, in this thesis we will consider *hypersurfaces with constant principal curvatures* and *isoparametric submanifolds*. However, in some cases, these notions admit *inhomogeneous* examples. This phenomenon was thought to be rare, but our work will show that it is much more common than it was believed.

Orbit foliations of isometric actions are the standard examples of the more general notion of *singular Riemannian foliation*. A singular Riemannian foliation is a kind of decomposition of a manifold into equidistant submanifolds (called leaves) of possibly different dimensions. This concept was introduced by Molino [103] and has become very attractive in the last years, see [5]. The reason for this is that it provides the appropriate unifying framework for the study of different classes of geometric objects, apart from orbit folia-

tions. For instance, isoparametric submanifolds define locally, and sometimes globally, a singular Riemannian foliation that we will call *isoparametric foliation*.

The investigation of homogeneous submanifolds and their generalizations has produced an influential and fruitful area of research along the last decades. Historically, the case of codimension one was the first one to be addressed. The classification problem of homogeneous hypersurfaces in spaces of constant curvature traces back to the works of Somigliana [128], Levi-Civita [91], Segre [125] and Cartan [28], [30] in the first decades of the 20th century. In fact, they studied the so-called *isoparametric hypersurfaces*, that is, hypersurfaces whose sufficiently close parallel hypersurfaces have constant mean curvature. The isoparametric foliation of codimension one determined by an isoparametric hypersurface is usually called isoparametric family of hypersurfaces. These objects arose naturally in certain problem of geometric optics. Cartan proved that, in space forms, a hypersurface is isoparametric if and only if it has constant principal curvatures. This condition is quite strong, and actually characterizes homogeneous hypersurfaces in Euclidean and real hyperbolic spaces.

The problem in spheres is much more involved. The classification of homogeneous hypersurfaces (or, equivalently, of cohomogeneity one isometric actions) in spheres had to wait until the work of Hsiang and Lawson [77]. It turns out that each such hypersurface has $g \in \{1, 2, 3, 4, 6\}$ constant principal curvatures. Münzner [108] showed that this restriction also holds more generally for isoparametric hypersurfaces. But, in this case, not every isoparametric hypersurface is homogeneous. All known counterexamples were constructed by Ferus, Karcher and Münzner [63] using representations of Clifford algebras. These examples considerably raised the interest in this topic, to the extent that the classification problem was included in Yau's list of open problems in geometry [160]. Many mathematicians have contributed to this task, but in the last five years notable progress has been made. The works of Cecil, Chi and Jensen [34], Immervoll [78], Chi [36], [37] and Miyaoka [101] complete the classification of isoparametric hypersurfaces in spheres, except for only one open case corresponding to hypersurfaces with four principal curvatures with multiplicities $(7, 8)$.

When the ambient space has nonconstant curvature, the complexity of the problem increases, and the results could only be obtained more recently and, in most of the cases, just when the ambient manifold is a Riemannian symmetric space. The research in this thesis mostly focuses on *rank one symmetric spaces*, where still many problems remain open. We pay especial attention to the case of *complex space forms*. We now present the main contributions of this work.

Real hypersurfaces with constant principal curvatures in complex space forms

The first original result in this thesis concerns some partial classification of real hypersurfaces with constant principal curvatures in nonflat complex space forms. The constancy of the principal curvatures of a hypersurface seems to be quite a restrictive condition, even more than the isoparametric condition if the space has nonconstant curvature. Apart from the examples in spheres by Ferus, Karcher and Münzner, and an example in the Cayley hyperbolic plane that we will construct in Chapter 5 (see below), it seems that all complete hypersurfaces with constant principal curvatures in symmetric spaces known so far are homogeneous. Thus, the study of hypersurfaces with constant principal curvatures has its own interest.

The main result of Chapter 3 falls within the line initiated by Kimura [83] and Berndt [7], who classified Hopf real hypersurfaces with constant principal curvatures in complex space forms. Given a real hypersurface M with unit normal vector field ξ, its Hopf vector field is the tangent vector field $J\xi$, where J is the complex structure of the ambient space. Let us define h as the integer valued function on M given by the number of nontrivial projections of $J\xi$ onto the eigenspaces of the shape operator of M. Then M is said to be Hopf if $h = 1$ along the hypersurface.

Thus, we will *classify real hypersurfaces with constant principal curvatures and* $h = 2$ *in complex projective and hyperbolic spaces*. This is, hence, the natural step to be taken after Kimura and Berndt's classification. It will turn out that complex projective spaces $\mathbb{C}P^n$ do not admit such hypersurfaces, whereas the examples appearing in complex hyperbolic spaces $\mathbb{C}H^n$ are all homogeneous and, more specifically, open parts of some homogeneous hypersurfaces constructed by Lohnherr [92], Berndt and Brück [10]. Homogeneous hypersurfaces in $\mathbb{C}H^n$ have been classified by Berndt and Tamaru [21].

Isoparametric hypersurfaces in complex hyperbolic spaces

Motivated by our objective of characterizing homogeneous hypersurfaces and, in particular, the Lohnherr-Berndt-Brück hypersurfaces, in Chapter 4 we address the study of isoparametric hypersurfaces in complex hyperbolic spaces.

We first show that isoparametric hypersurfaces have a nice behaviour with respect to the Hopf fibration associated with $\mathbb{C}H^n$, that is, a hypersurface in $\mathbb{C}H^n$ is isoparametric if and only if its pullback to the anti-De Sitter space H_1^{2n+1} under the Hopf map is a Lorentzian isoparametric hypersurface. As the sectional curvature of the anti-De Sitter space is constant, the study of isoparametric hypersurfaces in H_1^{2n+1} is then equivalent to the study of hypersurfaces with constant principal curvatures in H_1^{2n+1}. This condition turns out to be easier to handle.

Then, by going through the four possible types that the shape operator of the lifted hypersurface in H_1^{2n+1} can adopt, we are able to find a *pointwise restriction on the number* g *of principal curvatures of an isoparametric hypersurface in* $\mathbb{C}H^n$, $g \in \{2, 3, 4, 5\}$. We also show that *the number of nontrivial projections of the Hopf vector field onto the principal curvature spaces must satisfy* $h \in \{1, 2, 3\}$ *pointwise*.

Moreover, it occurs that the principal curvatures of isoparametric hypersurfaces in $\mathbb{C}H^n$ are pointwise the same as those of the homogeneous examples, with only a couple of exceptions. In fact, if $h \leq 2$ everywhere, then a hypersurface is isoparametric if and only if it has constant principal curvatures, and thus it is homogeneous (the same is true for complex projective spaces $\mathbb{C}P^n$). This would support the idea that a homogeneity result could be proven. However, one of the consequences of Chapter 5 is that such a homogeneity result cannot exist.

New isoparametric hypersurfaces in Damek-Ricci spaces

In Chapter 5 we develop a method of construction of isoparametric hypersurfaces in the so-called Damek-Ricci spaces. These spaces are harmonic manifolds among which we find the noncompact rank one symmetric spaces; the other nonsymmetric spaces are counterexamples to the Lichnerowicz conjecture. Our method allows to construct many *examples of isoparametric families of hypersurfaces as the collection of tubes around certain minimal focal submanifolds.* Our idea, which generalizes a method developed by Berndt and Brück [10], makes use of the algebraic structure of Damek-Ricci spaces, of Jacobi field theory and, more crucially, of the introduction of the concept of *generalized Kähler angle* of a subspace of a Clifford module. When we restrict our attention to noncompact symmetric spaces of rank one, that is, the hyperbolic spaces, some important consequences can be derived.

Firstly, we show the existence of *uncountably many inhomogeneous isoparametric families of hypersurfaces with nonconstant principal curvatures in complex and quaternionic hyperbolic spaces* $\mathbb{C}H^n$ *and* $\mathbb{H}H^n$. The notion of Kähler angle allows us to distinguish between the homogeneous and the inhomogeneous examples in $\mathbb{C}H^n$.

Secondly, we construct *uncountably many new examples of cohomogeneity one actions on quaternionic hyperbolic spaces* $\mathbb{H}H^n$. It is important to say that these manifolds are the only rank one symmetric spaces for which a classification of cohomogeneity one actions is still open. Such classification in complex hyperbolic spaces $\mathbb{C}H^n$ and the Cayley hyperbolic plane $\mathbb{O}H^2$ has been obtained by Berndt and Tamaru [21].

Finally, we come across an intriguing example. We find an *inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane* $\mathbb{O}H^2$. This is the only known example of inhomogeneous family of hypersurfaces with constant principal curvatures in a symmetric space different from a sphere.

Classification of polar actions on complex hyperbolic spaces

In Chapter 6 (as well as in Chapter 7) our attention moves from codimension one to the arbitrary codimension case, where the complexity of the problems is usually bigger. The objects of our research in Chapter 6 are polar actions, which naturally generalize the notion of cohomogeneity one action to arbitrary cohomogeneity.

Polar actions on Euclidean spaces and on spheres were classified by Dadok [42]. Such actions are orbit equivalent to the isotropy representations of symmetric spaces. The problem was then tackled in symmetric spaces of compact type. Podestà and Thorbergsson [123] classified polar actions on compact symmetric spaces of rank one and showed that there are polar, non-hyperpolar actions. A hyperpolar action is a polar action whose sections are flat. A complete classification of hyperpolar actions was achieved by Kollross [85] in symmetric spaces of higher rank. Kollross subsequently analyzed the classification problem of polar actions on irreducible symmetric spaces of compact type [86], [87]. This problem was recently solved by Kollross and Lytchak [89] by showing that such polar actions are always hyperpolar.

As for cohomogeneity one actions, results in the noncompact setting are scarcer. Wu classified polar actions on real hyperbolic spaces [155]. Very recently, Berndt and Díaz-Ramos classified polar actions on the complex hyperbolic plane $\mathbb{C}H^2$ in [16], while in [17] they determined those polar actions inducing regular foliations on $\mathbb{C}H^n$.

Chapter 6 completes the *classification of polar actions on complex hyperbolic spaces* $\mathbb{C}H^n$ *up to orbit equivalence*. This result, which includes the description of many new examples of polar actions on $\mathbb{C}H^n$, constitutes the only known classification of polar actions on a whole family of symmetric spaces of noncompact type and nonconstant curvature.

The proof has two main parts depending on whether the group acting upon leaves a totally geodesic subspace invariant or is contained in a maximal parabolic subgroup of $\mathrm{SU}(1, n)$, the isometry group of $\mathbb{C}H^n$. The properties of the Iwasawa decomposition associated with the symmetric space $\mathbb{C}H^n$ and the usage of certain criterion of polarity are two aspects that play an important role in our arguments.

Isoparametric foliations on complex projective spaces

The introduction of isoparametric submanifolds of arbitrary codimension in space forms by Harle [72], Carter and West [32] and, more crucially, by Terng [140], was motivated by the eventual characterization of the orbits of polar actions by means of a more geometric notion. Every isoparametric submanifold in a space form extends to an isoparametric foliation that fills the whole space. Contrary to the situation in codimension one, where inhomogeneous examples exist, every irreducible isoparametric foliation of codimension at least two on a sphere is homogeneous and, more specifically, it is the orbit foliation of the isotropy representation of a symmetric space. This important result was obtained by Thorbergsson [143]. More recently, Heintze, Liu and Olmos [74] proposed a definition of isoparametric submanifold of an arbitrary Riemannian ambient space which generalizes the notion of isoparametric hypersurface of a Riemannian manifold and of isoparametric submanifold of a space form.

Our aim in Chapter 7 is to investigate isoparametric submanifolds of complex projective spaces. It turns out that, as for space forms, every isoparametric submanifold determines a globally defined isoparametric foliation that fills the whole space. Chapter 7 contains a thorough investigation of the behaviour of isoparametric foliations with respect to the Hopf fibration of $\mathbb{C}P^n$. Surprisingly, as detected by Xiao [158], an isoparametric hypersurface M in a sphere S^{2n+1} can be projected to $\mathbb{C}P^n$ producing *noncongruent* isoparametric hypersurfaces, which might be inhomogeneous with independence of the homogeneity of M. We show that this phenomenon also takes place in higher codimension. Thus, we obtain the *classification of irreducible isoparametric foliations of codimension greater than one on complex projective spaces* and we show that most of the examples are *inhomogeneous foliations*. As far as we know, these provide the first examples of inhomogeneous irreducible isoparametric foliations of codimension greater than one on symmetric spaces.

We also study the codimension one case, which turns out to be related to the open problem of the classification of isoparametric hypersurfaces in spheres. Thus, by investigating the possible projections via the Hopf map of the isoparametric examples constructed by

Ferus, Karcher and Münzner, we are able to *classify isoparametric hypersurfaces in complex projective spaces* $\mathbb{C}P^n$*, for any* $n \neq 15$. Hence, this classification generalizes results by Takagi [131], Wang [150], Xiao [158] and Ge, Tang and Yan [66]. Again, we find many inhomogeneous examples.

The study of the homogeneity of isoparametric hypersurfaces gives us as a by-product the following result: *every irreducible isoparametric foliation on* $\mathbb{C}P^n$ *is homogeneous if and only if* $n+1$ *is prime.*

The main tool developed in Chapter 7 is a method to study singular Riemannian foliations with closed leaves on complex projective spaces. For this study, we introduce certain kind of graph that we call *lowest weight diagram* and which generalizes extended Vogan diagrams of inner symmetric spaces.

Structure of the thesis

This memoir is organized as follows.

Chapter 1 introduces the basic definitions, terminology and conventions needed for this thesis with respect to the following topics: semi-Riemannian manifolds (§1.1), theory of submanifolds (§1.2), Jacobi field theory (§1.3), singular Riemannian foliations (§1.4), isometric actions (§1.5), symmetric spaces (§1.6) and complex space forms (§1.7).

Chapter 2 contains an exposition on some known results on isoparametric hypersurfaces (§2.1) and cohomogeneity one actions (§2.2). We explain with certain detail the current state of the classification problem of isoparametric hypersurfaces in real space forms (§2.3). After presenting the fundamental language employed in the investigation of real hypersurfaces of complex space forms in §2.4, then we focus on describing cohomogeneity one actions and the corresponding homogeneous hypersurfaces in complex projective spaces (§2.5) and in complex hyperbolic spaces (§2.6).

The original contributions of this thesis are presented in Chapters 3 to 7.

In Chapter 3 we classify real hypersurfaces with constant principal curvatures in nonflat complex space forms satisfying that their Hopf vector field has nontrivial projections onto two principal curvature spaces (i.e. $h = 2$).

We carry out an investigation of the possible principal curvatures of isoparametric hypersurfaces in complex hyperbolic spaces in Chapter 4, where some bounds on g and on h are obtained. We also derive a complete classification of isoparametric hypersurfaces in $\mathbb{C}H^n$ and $\mathbb{C}P^n$ satisfying $h \leq 2$ everywhere (§4.4).

In Chapter 5 we propose a method of construction of new isoparametric families of hypersurfaces in Damek-Ricci harmonic spaces. We pay especial attention to the new examples in noncompact symmetric spaces of rank one (§5.4).

Chapter 6 contains the complete classification of polar actions on complex hyperbolic spaces $\mathbb{C}H^n$.

Finally, in Chapter 7 we study isoparametric foliations of arbitrary codimension q on complex projective spaces $\mathbb{C}P^n$ and derive their classification for every (q, n) different from $(1, 15)$.

Contents

Chapter 1

Preliminaries and conventions

In this chapter we introduce the basic notions and terminology needed for this thesis. The notations and conventions described will be used throughout this work unless otherwise stated.

In Section 1.1, the concept of semi-Riemannian manifold and our sign convention for the curvature tensor are introduced. In Section 1.2 we explain the basics of submanifold geometry. Section 1.3 describes how one can apply Jacobi field theory to the study of submanifolds of Riemannian manifolds. In Section 1.4 we review the notion of singular Riemannian foliation, which embraces most of the geometric objects in this work. In Section 1.5 we present some terminology regarding isometric actions. In Section 1.6 we recall the notion and basic facts on Riemannian symmetric spaces. Finally, in Section 1.7 we provide the construction of nonflat complex space forms as base spaces of certain Hopf fibrations.

1.1 Semi-Riemannian manifolds

Although this thesis will mostly deal with Riemannian manifolds, at some point it will become important the use of arguments involving semi-Riemannian manifolds. That is why we will focus our attention here and in Section 1.2 not only on Riemannian geometry, but on the more general setting of semi-Riemannian geometry.

Let M be a differentiable manifold of dimension n and class C^∞. We will always assume that manifolds are second countable and, hence, paracompact. For each $p \in M$, we denote by T_pM the tangent space of M at p. The tangent bundle is denoted by TM. If $\mathcal{D}$ is a distribution along M, $\Gamma(\mathcal{D})$ will denote the module of sections of such distribution, that is, those vector fields X on M such that $X_p \in \mathcal{D}_p$ for all $p \in M$.

A symmetric bilinear tensor T in a vector space is said to be nondegenerate if $T(x, y) = 0$ for all y implies $x = 0$. Any nondegenerate symmetric bilinear tensor in a vector space is linearly congruent to a diagonal matrix $\operatorname{diag}(1, \overset{r}{\cdots}, 1, -1, \overset{s}{\cdots}, -1)$. The signature of the tensor is then the pair (r, s).

A *semi-Riemannian manifold* is a pair $(M, \langle\cdot,\cdot\rangle)$, where M is a manifold and $\langle\cdot,\cdot\rangle$

is a nondegenerate symmetric bilinear tensor field of type $(0,2)$ and constant signature. Thus, each tangent space T_pM is endowed with a nondegenerate symmetric bilinear tensor $\langle\cdot,\cdot\rangle_p$. If the signature of these bilinear tensors is (r,s), then the manifold is said to have signature (r,s). *Riemannian manifolds* are precisely those semi-Riemannian manifolds with signature $(n,0)$, while *Lorentzian manifolds* are those with signature $(n-1,1)$. If M is Riemannian and X is a vector or a vector field on M, then $\|X\|$ will denote its norm, $\|X\| = \sqrt{\langle X,X\rangle}$. The Riemannian exponential map of M will be denoted by exp.

The central concept in semi-Riemannian geometry is curvature. The curvature information of a semi-Riemannian manifold M is codified in its *curvature tensor* R, which is a tensor of type $(1,3)$ that we define with the following sign convention:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X,Y,Z \in \Gamma(TM),$$

where ∇ is the *Levi-Civita connection* of M, that is, the unique torsion-free metric connection on M. When the curvature of a manifold vanishes identically, we say that the manifold is flat.

A semi-Riemannian manifold is said to have *constant curvature* c if its curvature tensor can be written as $R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$ for any vector fields X, Y and Z on M. In Riemannian geometry, spaces of constant curvature are the simplest ones from the point of view of their curvature tensor, and the value c is precisely the sectional curvature of any tangent 2-plane to M. It is known that the only connected, simply connected Riemannian manifolds of constant curvature are Euclidean spaces $\mathbb{R}^n$ $(c=0)$, spheres S^n $(c>0)$ and real hyperbolic spaces $\mathbb{R}H^n$ $(c<0)$. These are the so-called *(real) space forms.*

1.2 Geometry of submanifolds

This section introduces the basic terminology and fundamental formulas for the study of submanifolds of semi-Riemannian manifolds. For a comprehensive introduction to this topic, we refer to [11, Chapters 2 and 8] for the Riemannian case and to [116, Chapter 4] for the general case of arbitrary signature.

Let $(\bar{M},\langle\cdot,\cdot\rangle)$ be a semi-Riemannian manifold and M an embedded submanifold of $\bar{M}$. The restriction of $\langle\cdot,\cdot\rangle$ to M provides a symmetric bilinear tensor field on M. However, this tensor field can be degenerate. When it is not, that is, when M is itself a semi-Riemannian manifold, M is called a *semi-Riemannian submanifold* or a *nondegenerate submanifold* of $\bar{M}$. If $\bar{M}$ is Riemannian, every submanifold of $\bar{M}$ is a Riemannian submanifold. Throughout this work, unless otherwise stated, we assume that submanifolds are embedded and come equipped with the induced semi-Riemannian metric (whenever the restriction of the ambient metric is nondegenerate). The terminology that we will explain below also applies to the setting of immersed submanifolds, since immersed submanifolds are locally embedded and the concepts below only involve local geometry.

From now on, assume that M is a semi-Riemannian submanifold of $\bar{M}$. The *normal bundle* of M, that is, the bundle of vectors orthogonal to the tangent space of M, is denoted by νM. By $\Gamma(\nu M)$ we denote the module of all normal vector fields to M. A canonical isomorphism holds at each point $p \in M$, namely, $T_p\bar{M} = T_pM \oplus \nu_pM$. In this work, the symbol $\oplus$ will always denote direct sum (not necessarily orthogonal direct sum). Given a vector field X of $\bar{M}$ along M we denote by $X^\top$ the orthogonal projection of X onto TM and by $X^\perp$ the orthogonal projection onto νM.

If V is a vector space with symmetric bilinear form $\langle \cdot, \cdot \rangle$ and $W \subset V$ is a vector subspace, we will denote by $V \ominus W$ the vector subspace $\{v \in V : \langle v, w \rangle = 0, \forall w \in W\}$. If $\langle \cdot, \cdot \rangle$ is positive definite, this notation stands for the orthogonal complement of W in V. For example, we have $\nu_pM = T_p\bar{M} \ominus T_pM$.

The curvature tensor of any semi-Riemannian manifold is said to be an intrinsic geometric invariant. One may study the intrinsic geometry of both $\bar{M}$ and M. Nonetheless, one can also investigate the geometry of M in relation to the geometry of $\bar{M}$. This is the extrinsic geometry of M, which is encoded in its second fundamental form.

Let us denote by $\bar{\nabla}$ and $\bar{R}$ the Levi-Civita connection and the curvature tensor of $\bar{M}$, respectively, and by ∇ and R the corresponding objects for M. The *second fundamental form* of M is defined by the *Gauss formula*

$$\bar{\nabla}_X Y = \nabla_X Y + I\!I(X, Y)$$

for any X, $Y \in \Gamma(TM)$. Hence, $I\!I(X, Y) = (\bar{\nabla}_X Y)^\perp$. Let $\xi \in \Gamma(\nu M)$ be a unit normal vector field. The *shape operator* of M associated with ξ is the self-adjoint operator on M defined by $\langle \mathcal{S}_\xi X, Y \rangle = \langle I\!I(X, Y), \xi \rangle$, where X, $Y \in \Gamma(TM)$. Moreover, denote by $\nabla^\perp$ the normal connection of M, that is, $\nabla^\perp_X \xi = (\bar{\nabla}_X \xi)^\perp$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Then we have the *Weingarten formula*

$$\bar{\nabla}_X \xi = -\mathcal{S}_\xi X + \nabla^\perp_X \xi.$$

The relation between the curvature tensors of $\bar{M}$ and M is given by means of the second fundamental form and is known as the *Gauss equation*:

$$\langle \bar{R}(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle - \langle I\!I(Y, Z), I\!I(X, W) \rangle + \langle I\!I(X, Z), I\!I(Y, W) \rangle.$$

The *Codazzi equation* is also important in our work

$$(\bar{R}(X, Y)Z)^\perp = (\nabla^\perp_X I\!I)(Y, Z) - (\nabla^\perp_Y I\!I)(X, Z),$$

where the covariant derivative of the second fundamental form is given by

$$(\nabla^\perp_X I\!I)(Y, Z) = \nabla^\perp_X I\!I(Y, Z) - I\!I(\nabla_X Y, Z) - I\!I(Y, \nabla_X Z).$$

The last of the three fundamental equations of second order in submanifold theory is the *Ricci equation*

$$\langle R^\perp(X, Y)\xi, \eta \rangle = \langle \bar{R}(X, Y)\xi, \eta \rangle + \langle [\mathcal{S}_\xi, \mathcal{S}_\eta]X, Y \rangle,$$

where X, $Y \in \Gamma(TM)$, ξ, $\eta \in \Gamma(\nu M)$ and $R^\perp$ is the curvature tensor of the normal vector bundle of M, which is defined by $R^\perp(X, Y)\xi = [\nabla^\perp_X, \nabla^\perp_Y]\xi - \nabla^\perp_{[X,Y]}\xi$.

We say that a submanifold is *totally geodesic* if its second fundamental form vanishes identically, $I\!I = 0$. This is equivalent to saying that every geodesic in M is also a geodesic in $\bar{M}$. If M is complete and totally geodesic we have that $M = \exp_p(T_pM)$ for any $p \in M$.

A submanifold is said to be *totally umbilical* if there exists a constant λ such that $I\!I = \lambda\langle\cdot,\cdot\rangle$. Clearly, if $\lambda = 0$, then M is totally geodesic.

The *mean curvature vector* H of a semi-Riemannian submanifold M is defined as the trace of the second fundamental form. Hence, with respect to a local orthonormal basis $\{E_i\}$ of TM we may write $H = \sum_i \langle E_i, E_i\rangle I\!I(E_i, E_i)$. If $\xi \in \Gamma(\nu M)$, then the *mean curvature* of M with respect to ξ is the trace of the shape operator $\mathcal{S}_\xi$. A submanifold is said to be *minimal* if and only if its mean curvature vector vanishes. Minimal submanifolds appear in a natural way as the critical points of the volume functional and they are a topic of current interest in differential geometry.

We say that a submanifold M has *globally flat* normal bundle if every normal vector can be extended to a parallel normal vector field along M. The fact that $\xi \in \Gamma(\nu M)$ is a parallel normal vector field on M means that ξ is parallel with respect to the normal connection of M, i.e. $\nabla^\perp \xi = 0$. When every point in M admits a neighbourhood which is globally flat, then M is said to have *flat* normal bundle.

Two semi-Riemannian submanifolds M_1 and M_2 of $\bar{M}$ are said to be *congruent* if there exists an isometry of $\bar{M}$ that takes M_1 into M_2.

Assume now that M is a hypersurface of $\bar{M}$, that is, an embedded submanifold of codimension one. Then, locally and up to sign, there exists a unique unit normal vector field $\xi \in \Gamma(\nu M)$. We write $\epsilon = \langle \xi, \xi\rangle \in \{-1, 1\}$. Hence the second fundamental form $I\!I$ is a multiple of ξ.

We will denote by $\mathcal{S} = \mathcal{S}_\xi$ the shape operator with respect to ξ. The Gauss formula and the Weingarten equation can now be written as

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + \epsilon\,\langle \mathcal{S}X, Y\rangle \xi,\\ \bar{\nabla}_X \xi &= -\mathcal{S}X.\end{aligned}$$

Then, the Gauss and Codazzi equations reduce to

$$\begin{aligned}\langle \bar{R}(X,Y)Z, W\rangle &= \langle R(X,Y)Z, W\rangle - \epsilon\,\langle \mathcal{S}Y, Z\rangle\langle \mathcal{S}X, W\rangle + \epsilon\,\langle \mathcal{S}X, Z\rangle\langle \mathcal{S}Y, W\rangle,\\ \langle \bar{R}(X,Y)Z, \xi\rangle &= \langle (\nabla_X \mathcal{S})Y - (\nabla_Y \mathcal{S})X, Z\rangle,\end{aligned}$$

whereas the Ricci equation does not give further information for hypersurfaces.

The mean curvature vector H is proportional to the vector ξ. Thus, when dealing with hypersurfaces, one usually talks about the *mean curvature* of the hypersurface, which is defined as the trace of its shape operator $\mathcal{S}$.

Let ξ be a unit normal vector field defined on an open subset U of the hypersurface M. We say that $\lambda\colon U \subset M \to \mathbb{R}$ is a *principal curvature* of M (associated with ξ) if there exists a vector field $X \in \Gamma(TU)$ such that $\mathcal{S}X = \lambda X$. If $\bar{M}$ is a Riemannian manifold, the shape operator $\mathcal{S}$ is diagonalizable at every point because it is a self-adjoint map and the metric is positive definite.

If λ is a principal curvature we denote by $T_\lambda(p)$ the eigenspace of $\lambda(p)$ and call it the *principal curvature space* associated with $\lambda(p)$. If $X \in T_\lambda(p)$, $X \neq 0$, we say that X is a *principal curvature vector* of λ at p. Let us point out that, in general, the dimension of the principal curvature spaces associated with a principal curvature λ may vary from point to point. This dimension, that is $\dim\ker(\mathcal{S} - \lambda\,\mathrm{Id})$, is called the *geometric multiplicity* of the principal curvature λ, while the multiplicity of λ as a zero of the characteristic polynomial of $\mathcal{S}$ is called the *algebraic multiplicity* of λ. In the Riemannian setting both values coincide; then we simply speak about the *multiplicity* of λ.

A connected hypersurface is said to have *constant principal curvatures* if the eigenvalues of the shape operator are the same at every point. In this case the principal curvature spaces associated with an eigenvalue λ have the same dimension at any point, whenever the ambient manifold $\bar{M}$ is Riemannian. In this situation, we denote by T_λ the distribution on M formed by the principal curvature spaces of λ and by $\Gamma(T_\lambda)$ we denote the set of all sections of T_λ, that is, the vector fields $X \in \Gamma(TM)$ such that $\mathcal{S}X = \lambda X$.

1.3 Jacobi field theory

A rather useful method in submanifold theory is based on employing Jacobi vector fields for the study of the geometric behaviour of a submanifold when this is moved along normal directions. In this section we will briefly present the main features of this technique in the Riemannian setting, as it will be needed for our work. A more thorough discussion of this method can be found in [11, Chapter 8].

Let $\bar{M}$ be a Riemannian manifold of dimension n and $M \subset \bar{M}$ a Riemannian submanifold of $\bar{M}$. For fixed $r > 0$, we define the set

$$M^r = \{\exp(r\xi) : \xi \in \nu M,\ \|\xi\| = 1\}.$$

In general M^r is not a submanifold of $\bar{M}$. But if M^r is a hypersurface then we say that M^r is the *tube* of radius r around M. Locally, if r is sufficiently small, such a set is always a tube. If M^r is a submanifold of $\bar{M}$ but with codimension greater than one, we call it *focal submanifold* of M. In particular, if M^r is a tube around M and M has codimension greater than one, then M is a focal submanifold of M^r.

Let p be any point of M and $\gamma\colon [0,1] \to \bar{M}$ a unit speed geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) \in \nu M$. Here and henceforth, $\dot{\gamma}$ denotes the tangent vector field to the curve γ. Let $F(s,t) = \gamma_s(t)$ be a geodesic variation of $\gamma = \gamma_0$ such that $c(s) = F(s,0) = \gamma_s(0) \in M$ and $\xi(s) = \dot{\gamma}_s(0) \in \nu M$ for all s. Let ζ be the variational vector field of F. Then ζ is a solution to the initial value problem

$$\zeta'' + \bar{R}(\zeta,\dot{\gamma})\dot{\gamma} = 0, \qquad \zeta(0) = \dot{c}(0) \in T_pM, \qquad \zeta'(0) = -\mathcal{S}_{\xi(0)}\zeta(0) + \nabla^{\perp}_{\zeta(0)}\xi,$$

where $\mathcal{S}$ is the shape operator of M and the prime $'$ denotes covariant derivative of a vector field along a curve. A Jacobi vector field ζ along γ satisfying $\zeta(0) \in T_pM$ and $\zeta'(0) + \mathcal{S}_{\gamma'(0)}\zeta(0) \in \nu_pM$ is called an *M-Jacobi vector field*.

We say that $\gamma(r)$ is a *focal point* of M along γ if there exists an M-Jacobi vector field ζ along γ such that $\zeta(r) = 0$. A focal point arising from a Jacobi vector field ζ such that $\zeta(0) = 0$, $\zeta'(0) \in \nu M$ and $\zeta(r) = 0$ is a conjugate point of p in $\bar{M}$ along γ.

Assume now that M^r is a submanifold of $\bar{M}$. Let ξ be a smooth curve in νM with $\xi(0) = \dot{\gamma}(0)$ such that $\|\xi(s)\| = 1$ for all s. Then $F(s,t) = \exp\big(t\,\xi(s)\big)$ is a smooth geodesic variation of γ consisting of geodesics intersecting M perpendicularly. Let ζ be the corresponding M-Jacobi vector field which is the variational vector field of F. Then ζ is determined by the initial values $\zeta(0) = \dot{c}(0)$ and $\zeta'(0) = \xi'(0)$, where $c(s) = F(s,0)$. For any r, the curve $c_r(s) = F(r,s) = \exp(r\,\xi(s))$ is a smooth curve in M^r. Then,

$$T_{\gamma(r)}M^r = \{\zeta(r) : \zeta \text{ is an } M\text{-Jacobi vector field along } \gamma\}.$$

Let us denote by $\mathcal{S}^r$ the shape operator of M^r. Then it follows that

$$\mathcal{S}^r_{\dot{\gamma}(r)}\zeta(r) = -\zeta'(r)^\top.$$

If M^r is a tube, that is, if M^r is a hypersurface, its shape operator can be described in an efficient way, as we now explain. Let $X \in T_p\bar{M} \ominus \mathbb{R}\dot{\gamma}(0)$, where $\ominus$ denotes the orthogonal complement. We introduce the following notation. By B_X we denote the parallel translation of X along the geodesic γ. Let ζ_X be the M-Jacobi vector field along γ given by the following initial conditions

$$\zeta_X(0) = X, \qquad \zeta'_X(0) = -\mathcal{S}_{\dot{\gamma}(0)}X, \qquad \text{if } X \in T_pM,$$

$$\zeta_X(0) = 0, \qquad \zeta'_X(0) = X, \qquad \text{if } X \in \nu_pM \ominus \mathbb{R}\dot{\gamma}.$$

We define $D(r)$ by $D(r)B_X(r) = \zeta_X(r)$ for all $X \in T_p\bar{M} \ominus \mathbb{R}\dot{\gamma}(0)$. In other words, D is the $\operatorname{End}(\dot{\gamma}^\perp)$-valued tensor field along γ determined by the following initial value problem

$$D'' + \bar{R}_{\dot{\gamma}} \circ D = 0, \qquad D(0) = \begin{pmatrix} \mathrm{Id}_{T_pM} & 0 \\ 0 & 0 \end{pmatrix}, \qquad D'(0) = \begin{pmatrix} -\mathcal{S}_{\dot{\gamma}(0)} & 0 \\ 0 & \mathrm{Id}_{\nu_pM\ominus\mathbb{R}\dot{\gamma}(0)} \end{pmatrix},$$

where $\bar{R}_{\dot{\gamma}}(v) = \bar{R}(v,\dot{\gamma})\dot{\gamma}$ for $v \in \dot{\gamma}^\perp$. The endomorphism $D(r)$ is singular if and only if $\gamma(r)$ is a focal point of M along γ. If this is not the case, M^r is a tube and its shape operator in the direction of $\dot{\gamma}(r)$ is given by

$$\mathcal{S}^r_{\dot{\gamma}(r)} = -D'(r)\,D(r)^{-1}.$$

Of special interest is the case when M is a hypersurface. Let us now have a closer look at this case.

Let $M \subset \bar{M}$ be a hypersurface and ξ a unit normal vector field on an open set of M. Our objective is the study of local geometric properties of the displacement of M in the direction given by ξ at a certain distance r. We can hence assume that ξ is globally defined on M. For $r > 0$ we define the map

$$\begin{aligned} \Phi^r\colon M &\longrightarrow \bar{M} \\ p &\mapsto \Phi^r(p) = \exp(r\xi_p). \end{aligned}$$

We denote by η the vector field along Φ^r such that $\eta^r(p) = \dot{\gamma}_p(r)$ for each $p \in M$, where γ_p is the geodesic of $\bar{M}$ determined by the initial conditions $\gamma_p(0) = p$ and $\dot{\gamma}_p(0) = \xi_p$. The map Φ^r is smooth and parametrizes the tube M^r of radius r around M. Clearly, M^r is an immersed submanifold of $\bar{M}$ if and only if Φ^r is an immersion. It may happen, however, that M^r is a focal submanifold. The fact that M^r has higher codimension depends on the rank of Φ^r.

Let ζ_X be an M-Jacobi vector field. We have $X = \zeta_X(0) \in TM$ and $\zeta_X'(0) = -\mathcal{S}X$ because ξ has unit length and the normal bundle of M has rank one. Then it follows that

$$\Phi^r_* X = \zeta_X(r), \qquad \bar{\nabla}_X \eta^r = \zeta_X'(r).$$

Thus, Φ^r is not an immersion at $p \in M$ if and only if $\Phi^r(p)$ is a focal point of M along the geodesic γ_p. In this case, the dimension of the kernel of Φ^r_{*p} is called the *multiplicity* of the focal point. If there exists a positive integer k such that $\Phi^r(q)$ is a focal point of M along γ_q with multiplicity k for all q in some open neighbourhood U of p, then, if U is sufficiently small, $\Phi^r|_U$ parametrizes and embedded $(n-1-k)$–dimensional submanifold of $\bar{M}$, which is a focal submanifold of M. If $\Phi^r(q)$ is not a focal point of M along γ_q for any q in a sufficiently small neighbourhood U of p, then $\Phi^r|_U$ parametrizes an embedded hypersurface of $\bar{M}$, which is called an *equidistant hypersurface* to M in $\bar{M}$.

If M^r is a hypersurface, its shape operator can be calculated using the endomorphism-valued tensor field D defined above. In this case the initial conditions simplify slightly and D is determined by the initial value problem

$$D'' + \bar{R}_{\dot{\gamma}} \circ D = 0, \qquad D(0) = \mathrm{Id}_{T_pM}, \qquad D'(0) = -\mathcal{S}_{\xi_p}.$$

Finally, let us mention that the notion of equidistant hypersurface can be generalized to arbitrary codimension in the following way. Let M be a submanifold of $\bar{M}$. Assume that M has globally flat normal bundle. For each parallel normal vector field ξ and each sufficiently small $r > 0$, we can consider the set $M^{r,\xi} = \{\exp(r\xi_p) : p \in M\}$. If such a set is a submanifold, then we call it a *parallel submanifold* of M determined by the vector field ξ. Locally and for r sufficiently small, $M^{r,\xi}$ is always a parallel submanifold. Note as well that if M^r is a tube, then it is foliated by parallel submanifolds $M^{r,\xi}$ of M.

1.4 Singular Riemannian foliations

This work investigates certain geometric objects that can be seen as particular instances of the notion of singular Riemannian foliation, which we briefly recall in this section. Singular Riemannian foliations were introduced by Molino [103] in his study of Riemannian foliations and constitute nowadays an active field of research. See the articles [5], [96] and [146] for more information.

Let $\mathcal{F}$ be a decomposition of a Riemannian manifold $\bar{M}$ into connected injectively immersed submanifolds, called *leaves*, which may have different dimensions. We say that $\mathcal{F}$ is a *singular Riemannian foliation* if the following conditions are satisfied:

(i) $\mathcal{F}$ is a *transnormal system*, that is, every geodesic orthogonal to one leaf remains orthogonal to all the leaves that it intersects, and

(ii) $\mathcal{F}$ is a *singular foliation*, that is, $T_pL = \{X_p : X \in \mathcal{X}_\mathcal{F}\}$ for every leaf L in $\mathcal{F}$ and every $p \in L$, where $\mathcal{X}_\mathcal{F}$ is the module of smooth vector fields on the ambient manifold that are everywhere tangent to the leaves of $\mathcal{F}$.

If $\bar{M}$ is complete, the transnormality condition implies that the leaves are equidistant to each other.

The leaves of maximal dimension are called *regular* and the other ones are *singular*. The points of $\bar{M}$ are said to be regular or singular according to the leaves through them. A singular Riemannian foliation is called *regular* if all leaves are regular, that is, if it is a Riemannian foliation. The *dimension* of $\mathcal{F}$ is the maximal dimension of the leaves and its *codimension* is $\dim \bar{M} - \dim \mathcal{F}$.

In this work, for the sake of brevity, we will refer to singular Riemannian foliations simply as *foliations* and we will use the term *regular foliation* to mean (regular) Riemannian foliation.

For each $s \leq \dim \mathcal{F}$, let us denote by Π_s the subset of all points $p \in \bar{M}$ such that the leaf of $\mathcal{F}$ through p has dimension s. Then Π_s is an embedded submanifold of $\bar{M}$ and the restriction of $\mathcal{F}$ to Π_s is a regular foliation. The subset $\Pi_{\dim \mathcal{F}}$ is open, dense and connected in $\bar{M}$. It is the *regular stratum* of the foliation. The connected components of the different Π_s, $s < \dim \mathcal{F}$, are called *singular strata*.

Now we will comment on some examples of singular Riemannian foliations that will be particularly important for our work.

The first set of examples is given by isometric actions on Riemannian manifolds (in Section 1.5 we will deepen into the notation and main concepts concerning isometric actions). Let G be a Lie group that acts on a Riemannian manifold $\bar{M}$ by isometries. Then, the set $\mathcal{F}$ of orbits is called the *orbit foliation* of the action; $\mathcal{F}$ is then a *homogeneous foliation* and its orbits are called *(extrinsically) homogeneous submanifolds*. It is clear that $\mathcal{F}$ is a singular foliation since the set of values of the Killing fields induced by the action at a point $p \in \bar{M}$ coincides with the tangent space $T_p(G \cdot p)$ at p of the orbit $G \cdot p$. The transnormality of $\mathcal{F}$ follows from the fact that $\bar{\nabla} X$ is a skew-symmetric tensor field on $\bar{M}$ for every Killing field X. Hence $\mathcal{F}$ is a singular Riemannian foliation.

Another important example is that of *polar foliations*, also called *singular Riemannian foliations with sections* in the terminology of Alexandrino [3]. Let $\mathcal{F}$ be a foliation on $\bar{M}$. Then $\mathcal{F}$ is said to be polar if, for each point $p \in \bar{M}$, there is an immersed submanifold Σ_p, called *section*, that passes through p and that meets all the leaves and always perpendicularly. It follows that Σ_p is totally geodesic and that the dimension of Σ_p is equal to the codimension of $\mathcal{F}$. When the sections of a polar foliation are flat submanifolds, the foliation is called *hyperpolar*.

If the ambient manifold $\bar{M}$ is complete, the condition of polarity turns out to be equivalent to saying that the distribution made up of the normal spaces to the regular leaves is integrable. In this case, the sections are complete. Moreover, the leaves of a polar foliation on a complete, simply connected Riemannian manifold are always closed submanifolds with globally flat normal bundle (see [97, Theorem 1.2]). Note that, in a complete ambient manifold, codimension one foliations are always polar.

One important question in the study of polar foliations is to decide when polar foliations are orbit foliations of isometric actions. In this case, such homogeneous polar foliations are precisely the orbit foliations of the so-called polar actions, which we will define in Section 1.5.

Another kind of foliations that is fundamental in this work is that of *isoparametric foliations*. These can be defined as those foliations whose regular leaves are isoparametric submanifolds. There have been several different approaches to define isoparametric submanifolds, particularly in space forms, as we will review in Chapters 2 and 7. The definition that we will consider in this work is the one due to Heintze, Liu and Olmos [74]. Thus, we will say that a submanifold M of a Riemannian manifold is an *isoparametric submanifold* if the following properties are satisfied:

(i) The normal bundle νM is flat.

(ii) Locally, the parallel submanifolds of M have constant mean curvature in radial directions (see below for explanation).

(iii) M admits sections, i.e. for each $p \in M$ there exists a totally geodesic submanifold Σ_p that meets M at p orthogonally and whose dimension is the codimension of M.

Let us explain the meaning of condition (ii). Since νM is flat, every point $p \in M$ admits an open neighbourhood U where every normal vector can be extended to a parallel normal field. By restricting U further if necessary, we can assume that there is an $s > 0$ such that for all $r < s$ and for every parallel normal field ξ on U, the set $U^{r,\xi} = \{\exp(r\xi_p) : p \in U\}$ is an embedded parallel submanifold of $U \subset M$. The radial vector field $\partial/\partial r = \operatorname{grad} r$ is normal to every such $U^{r,\xi}$. Then we say that locally the parallel submanifolds of M have constant mean curvature in radial directions if the mean curvature of each $U^{r,\xi}$ is constant with respect to the normal field $\partial/\partial r$.

It was proved in [74, Theorem 2.4] that condition (ii) above may be replaced by the following condition (ii') without changing the notion of isoparametric submanifold:

(ii') Locally, the parallel submanifolds $M^{r,\xi}$ of M have constant mean curvature with respect to any parallel normal field of $M^{r,\xi}$.

This implies that the locally defined parallel submanifolds of an isoparametric submanifold are isoparametric as well [74, Corollary 2.5], and thus define locally a regular foliation where all leaves are isoparametric. Of course, globally, isoparametric foliations need not be regular foliations.

In complete ambient manifolds, isoparametric foliations are always polar, since the distribution of the normal spaces to the regular leaves is integrable, because of condition (iii) in the definition of isoparametric submanifold. The converse is not true in general; see [152,

p. 89, Remark 1] for counterexamples in the real hyperbolic space. However, homogeneous polar foliations, i.e. orbit foliations of polar actions, are isoparametric (see [74, p. 2], [11, Corollary 3.2.5]).

Note that, for the codimension one case, the definition of isoparametric submanifold above simplifies. Thus, a hypersurface in a Riemannian manifold $\bar{M}$ is isoparametric if, locally, it and its sufficiently close equidistant hypersurfaces have constant mean curvature. Chapter 2 contains further information on isoparametric hypersurfaces, whereas a review of isoparametric submanifolds of arbitrary codimension is postponed to Chapter 7.

1.5 Isometric actions

Our purpose here is to review the basic terminology and concepts that arise in the study of isometric actions on Riemannian manifolds. A more detailed reference is [11, Chapter 3].

Let $\bar{M}$ be a Riemannian manifold and G a Lie group acting smoothly on $\bar{M}$ by isometries. This means that we have an *isometric action*, that is, a smooth map

$$\varphi\colon G \times \bar{M} \to \bar{M}, \qquad (g, p) \mapsto gp$$

satisfying $(gg')p = g(g'p)$ for all g, $g' \in G$ and $p \in \bar{M}$, and such that the map

$$\varphi_g\colon \bar{M} \to \bar{M}, \qquad p \mapsto gp$$

is an isometry of $\bar{M}$ for every $g \in G$. If we denote by $I(\bar{M})$ the isometry group of $\bar{M}$, which is known to be a Lie group [111], then we have a Lie group homomorphism $\rho\colon G \to I(\bar{M})$ given by $\rho(g) = \varphi_g$.

For each point $p \in \bar{M}$, the *orbit* of the action of G through p is

$$G \cdot p = \{gp : g \in G\}$$

and the *isotropy group* or *stabilizer* at p is

$$G_p = \{g \in G : gp = p\}.$$

If $G \cdot p = \bar{M}$ for some $p \in \bar{M}$, and hence for each $p \in \bar{M}$, the G-action is said to be *transitive* and $\bar{M}$ is a *homogeneous G-space*. If all leaves are points, the action is said to be *trivial*. An action is called *effective* if the associated map ρ above is injective, which means that G is isomorphic to a subgroup of $I(\bar{M})$. When for every $p \in \bar{M}$ and every g, $h \in G$, the equality $gp = hp$ implies $g = h$, then the action is *free*. If a G-action on $\bar{M}$ is free and transitive we say that G acts *simply transitively* on $\bar{M}$.

Consider two isometric actions $G \times \bar{M} \to \bar{M}$ and $G \times \bar{M}' \to \bar{M}'$. They are said to be *conjugate* or *equivalent* if there is a Lie group isomorphism $\psi\colon G \to G'$ and an isometry $f\colon \bar{M} \to \bar{M}'$ such that $f(gp) = \psi(g)f(p)$ for all $p \in \bar{M}$ and $g \in G$. We say that both isometric actions are *orbit equivalent* if there is an isometry $f\colon \bar{M} \to \bar{M}'$ that maps the orbits of the G-action on $\bar{M}$ to the orbits of the G'-action on $\bar{M}'$. Clearly, two conjugate actions are orbit equivalent.

We will be mostly interested in studying the extrinsic geometry of the orbits of isometric actions. An *(extrinsically) homogeneous submanifold* of $\bar{M}$ is an orbit of an isometric action on $\bar{M}$. In general, these orbits will only be immersed submanifolds of $\bar{M}$. With respect to the induced metric, each orbit $G \cdot p$ is a Riemannian homogeneous space $G \cdot p = G/G_p$, on which G acts transitively by isometries.

Each isometric action induces certain orthogonal representations in a natural way. Recall that a representation of a Lie group G on a vector space V is a Lie group homomorphism $\rho\colon G \to \mathrm{GL}(V)$ or, equivalently, an action $G \times V \to V$ given by automorphisms of V; when V is a Euclidean space and the automorphisms $\rho(g)$, $g \in G$, are orthogonal transformations of V, we have an orthogonal representation $\rho\colon G \to \mathrm{O}(V)$. Let $\varphi\colon G \times \bar{M} \to \bar{M}$ be an isometric action on a Riemannian manifold $\bar{M}$, and let $p \in \bar{M}$. Since the isotropy group G_p fixes p and G_p leaves the orbit $G \cdot p$ invariant, the differential of each isometry $\varphi_g\colon \bar{M} \to \bar{M}$, $p \mapsto gp$, for $g \in G_p$, leaves the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$ invariant. Thus, the action

$$G_p \times T_p(G \cdot p) \to T_p(G \cdot p), \qquad (g, X) \mapsto (\varphi_g)_{*p} X,$$

is called the *isotropy representation* of the action φ at p, while

$$G_p \times \nu_p(G \cdot p) \to \nu_p(G \cdot p), \qquad (g, \xi) \mapsto (\varphi_g)_{*p} \xi,$$

is called the *slice representation* of the action φ at p.

Let $\bar{M}/G$ be the set of orbits of the action of G on $\bar{M}$, and equip $\bar{M}/G$ with the quotient topology relative to the canonical projection $\bar{M} \to \bar{M}/G$, $p \mapsto G \cdot p$. In general, $\bar{M}/G$ is not a Hausdorff space. In order to avoid this behaviour, the particular type of proper isometric actions was introduced. Thus, the action of G on $\bar{M}$ is *proper* if, for any two points p, $q \in \bar{M}$, there exist open neighbourhoods U_p and U_q of p and q in $\bar{M}$, respectively, such that $\{g \in G : gU_p \cap U_q \neq \emptyset\}$ is relatively compact in G. Equivalently, the map

$$G \times \bar{M} \to \bar{M} \times \bar{M}, \qquad (g, p) \mapsto (p, gp)$$

is a proper map, i.e. the inverse image of each compact set in $\bar{M} \times \bar{M}$ is also compact in $G \times \bar{M}$. Every compact Lie group action is proper. If G is a subgroup of $I(\bar{M})$, then the G-action is proper if and only if G is closed in $I(\bar{M})$. Moreover, if G acts properly on $\bar{M}$, then $\bar{M}/G$ is a Hausdorff space, each isotropy group G_p is compact, and each orbit $G \cdot p$ is closed in $\bar{M}$ and hence an embedded submanifold. In fact, the orbits of an isometric action are closed if and only if the action is orbit equivalent to a proper isometric action, see [45].

We can distinguish three different kinds of orbits of a proper isometric action: principal, exceptional and singular orbits. An orbit $G \cdot p$ is called a *principal orbit* if for each $q \in \bar{M}$ the isotropy group G_p at p is conjugate in G to some subgroup of G_q. The union of all principal orbits is a dense and open subset of $\bar{M}$ and any orbit $G \cdot p$ of a proper action is principal if and only if the slice representation at p is trivial. Each principal orbit is an orbit of maximal dimension. The codimension of any principal orbit is the *cohomogeneity* of the action. A non-principal orbit of maximal dimension is called an *exceptional orbit.* Finally, a *singular orbit* is an orbit whose dimension is less than the dimension of a principal orbit or, equivalently, an orbit whose codimension is greater than the cohomogeneity.

Another important kind of isometric actions are polar actions. An isometric action of a group G on a Riemannian manifold $\bar{M}$ is called *polar* if its orbit foliation is polar, i.e. if there exists an immersed submanifold Σ of M that intersects all the orbits of the G-action, and for each $p \in \Sigma$, the tangent space of Σ at p, $T_p\Sigma$, and the tangent space of the orbit through p at p, $T_p(G \cdot p)$, are orthogonal. In such a case, the submanifold Σ is totally geodesic and is called a *section* of the G-action. If, in addition, the section Σ is flat in its induced Riemannian metric, the action is called *hyperpolar*. Any polar action admits sections through any given point.

Polar actions are much more rigid than arbitrary isometric actions. For complete, simply connected ambient manifolds $\bar{M}$, the orbits of polar actions are always closed submanifolds, none of them is exceptional, and the image of the group G on the isometry group $I(\bar{M})$ is closed (see [97, Corollary 1.3]). This, in particular, implies that polar actions on complete, simply connected manifolds are orbit equivalent to proper actions. Furthermore, if φ is a polar action of a connected group G on $\bar{M}$, $p \in \bar{M}$ and Σ is a section through p, then the slice representation of such action at p is polar with section $T_p\Sigma$.

1.6 Symmetric spaces

Symmetric spaces constitute a particularly nice class of homogeneous spaces. They share, moreover, many connections with the theory of polar actions and, thus, will be important in this work. Here we provide a quick review on some basic facts about these spaces. Standard references for this topic are [76], [94, 95] and [161].

Firstly, let us fix some notation concerning Lie groups and Lie algebras. As customary, the Lie algebra of a Lie group G will be written with the corresponding gothic letter, in this case, $\mathfrak{g}$. The Lie exponential map will be denoted by Exp. Given $g \in G$, we have the conjugation map $I_g \colon G \to G$, $h \mapsto ghg^{-1}$. Its differential at the identity element $e \in G$ allows to define the Lie group adjoint map $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$, $g \to (I_g)_*$, where $\mathrm{Aut}(\mathfrak{g})$ is the group of automorphisms of the Lie algebra $\mathfrak{g}$, i.e. those linear transformations $\varphi \colon \mathfrak{g} \to \mathfrak{g}$ such that $\varphi[X,Y] = [\varphi X, \varphi Y]$ for all X, $Y \in \mathfrak{g}$. The differential of Ad at e yields the Lie algebra adjoint map $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $X \mapsto \mathrm{ad}(X) = [X, \cdot\,]$. The Killing form of a real Lie algebra $\mathfrak{g}$ is the bilinear form $\mathcal{B} = \mathcal{B}_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X,Y) \mapsto \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$.

Let now M be a Riemannian manifold. Let $o \in M$. Take $r > 0$ sufficiently small so that normal coordinates are defined on the open ball $B_r(o)$. We define the local geodesic symmetry at o as the map $s_o \colon B_r(o) \to B_r(o)$ given by $s_o(\exp_o(tv)) = \exp_o(-tv)$ for $t \in \mathbb{R}$ and $v \in T_oM$. In general, this map is defined only locally. A Riemannian manifold M is said to be *locally symmetric* if at each point there is a ball such that the corresponding local geodesic symmetry is a local isometry. A locally symmetric space is characterized by the fact that $\nabla R = 0$. A connected Riemannian manifold M is called a *(Riemannian) symmetric space* if each local geodesic symmetry s_o can be extended to a global isometry $s_o \colon M \to M$. Since isometries are characterized by their differential at a point, this is equivalent to saying that for each point $o \in M$ there is an involutive isometry of M such that o is an isolated fixed point of that isometry; this involutive isometry turns out to be s_o.

If M is a connected, complete, locally symmetric Riemannian manifold, then its universal covering is a symmetric space. In particular, every locally symmetric space is locally isometric to a symmetric space. Moreover, every symmetric space is complete and homogeneous.

Now we give a more algebraic description of symmetric spaces. Denote by $G = I(M)^0$ the connected component of the identity of the isometry group $I(M)$ and by $\mathfrak{g}$ the Lie algebra of G. Let $o \in M$ and s_o the geodesic symmetry at o. Define K as the isotropy group of G at o, that is, $K = G_o$, which is compact. The coset space G/K is diffeomorphic to M by means of the map $\Phi\colon G/K \to M$, $gK \mapsto g(o)$. If $\langle\cdot,\cdot\rangle$ denotes the metric obtained by pulling back the metric of M, then Φ becomes an isometry and the metric $\langle\cdot,\cdot\rangle$ is G-invariant, that is, the map $gK \to hgK$ is an isometry for each $h \in G$. The *isotropy representation* of the symmetric space $M \cong G/K$ at o is the orthogonal representation defined by $K \times T_oM \to T_oM$, $(k, v) \mapsto k_*v$.

The map $\sigma\colon G \to G$, $g \mapsto s_o g s_o$, is an involutive automorphism of G, and $G_\sigma^0 \subset K \subset G_\sigma$, where $G_\sigma = \{g \in G : \sigma(g) = g\}$, and G_σ^0 is the connected component of the identity of G_σ . Let θ be the differential of σ at the identity. The Lie algebra of K is given by $\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}$, and we define $\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}$. The space $\mathfrak{p}$ may be identified with T_oM by using the map Φ and taking into account that $\mathfrak{p}$ is a complementary subspace to $\mathfrak{k}$ in $\mathfrak{g}$. Thus, $\mathfrak{p}$ inherits an inner product from T_oM which turns out to be $\mathrm{Ad}(K)$-invariant. In fact, the isotropy representation of G/K is equivalent to the adjoint representation of K on $\mathfrak{p}$, $K \times \mathfrak{p} \to \mathfrak{p}$, $(k, X) \mapsto \mathrm{Ad}(k)X$. Moreover, we have the Lie bracket relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to the involution θ (or the point $o \in M$), and θ is called the *Cartan involution*.

The pair (G, K) defined above is an effective (Riemannian) symmetric pair. In general, if G is a connected Lie group and K a compact subgroup, the pair (G, K) is called a *(Riemannian) symmetric pair* if there exists an involutive automorphism σ of G such that $G_\sigma^0 \subset K \subset G_\sigma$, and (G, K) is *effective* if the action of G on $M \cong G/K$ is effective. The isotropy representation of an effective symmetric pair is effective, since isometries are determined by their derivatives. The infinitesimal counterpart of a symmetric pair is the notion of orthogonal symmetric pair. Given a real Lie algebra $\mathfrak{g}$ and a compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, we will say that $(\mathfrak{g}, \mathfrak{k})$ is an *orthogonal symmetric pair* if $\mathfrak{k}$ is the fixed point set of an involutive automorphism θ of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{k})$ is said to be effective if $\mathfrak{k} \cap Z(\mathfrak{g}) = 0$, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. Any effective symmetric pair (G, K) determines an effective orthogonal symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

Let $M = G/K$ be a symmetric space. The long homotopy sequence of $K \to G \to G/K$ implies that K is connected if M is simply connected and G is connected. Conversely, if G is simply connected and K connected, then M is simply connected.

Let M be a symmetric space and $\widetilde{M}$ its universal covering. Then the De Rham theorem guarantees that $\widetilde{M}$ can be decomposed as $\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k$. Here $\widetilde{M}_0$ is the

Euclidean factor, that is, $\widetilde{M}_0$ is locally isometric to a Euclidean space, and each $\widetilde{M}_i$, $i = 1, \ldots, k$, is a simply connected, irreducible symmetric space. A symmetric space $M = G/K$ is *irreducible* if its isotropy representation restricted to the identity connected component K^0 of K is an irreducible representation, and is *reducible* otherwise. M is irreducible if and only if its universal covering $\widetilde{M}$ is irreducible.

A *semisimple* symmetric space (or symmetric pair) is one for which the Euclidean factor of its universal covering space has dimension zero. In this case, the Lie algebra of the isometry group of $\widetilde{M}$ is semisimple. A semisimple symmetric space (or symmetric pair) is said to be of *compact type* if all the De Rham factors of its universal covering are compact. It is said to be of *noncompact type* if all the De Rham factors of its universal covering are non-Euclidean, irreducible and noncompact. Again, the Lie algebra $\mathfrak{g}$ of the isometry group of a symmetric space of compact (resp. noncompact) type is compact (resp. noncompact). By definition, an irreducible symmetric space must be one of these three: of Euclidean type (i.e. flat), of compact type, or of noncompact type. If $\mathcal{B}$ is the Killing form of $\mathfrak{g}$, then G/K is of compact type if and only if $\mathcal{B}|_{\mathfrak{p}}$ is negative definite, is of noncompact type if and only if $\mathcal{B}|_{\mathfrak{p}}$ is positive definite, and is of Euclidean type if and only if $\mathcal{B}|_{\mathfrak{p}} = 0$. Moreover, if (G, K) is an effective irreducible symmetric pair of non-Euclidean type, then either G is a simple Lie group, or $(G, K) = (K \times K, \Delta K)$ and G/K is isometric to a compact simple Lie group with bi-invariant metric; here ΔK stands for the diagonal of $K \times K$. If (G, K) is an effective symmetric pair with no Euclidean factor, then $G = I(M)^0$.

There is a duality between symmetric spaces of compact and noncompact type which we explain now. Assume (G, K) is an effective symmetric pair with no Euclidean factor and such that $M = G/K$ is simply connected. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as defined above. We consider the real Lie subalgebra $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$ of the complexification $\mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$, where i is the imaginary unit. Let G^* be the simply connected real Lie group with Lie algebra $\mathfrak{g}^*$. Then we have that G^*/K is a simply connected symmetric space, which we call the *dual* symmetric space of G/K. If G/K is of compact type, then G^*/K is of noncompact type, and if G/K is of noncompact type, then G^*/K is of compact type. Dual symmetric spaces have the same isotropy representation. Duality establishes a one-to-one correspondence between simply connected symmetric spaces of compact and noncompact type, which respects the irreducibility.

Riemannian symmetric spaces have been classified by Cartan. One can find a list of irreducible simply connected symmetric spaces in [76, p. 515–520].

An important subclass of symmetric spaces is that of the Hermitian ones, which we review below. But, first, let us recall some definitions concerning complex, Hermitian and Kähler manifolds. See [159] for more details and proofs.

To start with, let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. In this work, by *complex structure* on the vector space V we will always understand an orthogonal transformation J of V such that $J^2 = -\operatorname{Id}$. Thus, any endomorphism J of V is a complex structure if and only if any two of the following properties are satisfied: (i) $\langle Jv, Jw \rangle = \langle v, w \rangle$ for all $v, w \in V$ (that is, $J \in \mathrm{O}(V)$); (ii) $J^2 = -\operatorname{Id}$; and (iii) $\langle Jv, w \rangle = -\langle v, Jw \rangle$ for all $v, w \in V$ (that is, $J \in \mathfrak{so}(V)$).

A *complex manifold* is a manifold that admits charts with image onto open subsets of $\mathbb{C}^n$ such that the coordinate transitions are holomorphic. This induces an almost complex structure J on M, i.e. an endomorphism of the tangent bundle of M such that $J^2 = -\operatorname{Id}$. If M is Riemannian and complex, and the complex structure J is orthogonal (equivalently, J restricts to a complex structure of each tangent space T_pM, $p \in M$), then M is called a *Hermitian manifold.* A *Kähler manifold* is a Hermitian manifold M satisfying $\nabla J = 0$, where ∇ is the Levi-Civita connection of M. The endomorphism J is known as the *Kähler structure* or the *complex structure* of M.

Thus, a symmetric space M is *Hermitian* if it is a Hermitian manifold and the geodesic symmetries s_p, $p \in M$, are holomorphic transformations. It occurs that every Hermitian symmetric space is Kähler. A symmetric space M is Hermitian if and only if its dual is Hermitian, and every Hermitian symmetric space is simply connected. If, in addition, M is irreducible, then the complex structure J is unique up to sign. An irreducible symmetric pair (G, K) is Hermitian if and only if K is not semisimple. Given an effective irreducible Hermitian symmetric pair (G, K) of non-Euclidean type, then the center of K is isomorphic to $\mathrm{U}(1)$ and the induced complex structure J on $\mathfrak{p} \equiv T_o(G/K)$ is given by the $\operatorname{Ad}(K)$-invariant transformation $J = \operatorname{Ad}(i)$, where i stands for the imaginary unit in $\mathrm{U}(1)$. Furthermore, every isometry in $I(M)^0$ is holomorphic, and $M = G/K$ is an *inner* or *equal-rank symmetric space*, which means that $\operatorname{rank} G = \operatorname{rank} K$.

It is also possible to define the *rank* of a symmetric space M. This is by definition the dimension of a maximal flat, totally geodesic submanifold of M, or equivalently, the dimension of a maximal abelian subspace of $\mathfrak{p}$. The isotropy representation of a semisimple symmetric space is said to be an *s-representation.* It turns out that the isotropy representation of a semisimple symmetric space M is a polar action on the Euclidean space $T_oM \cong \mathfrak{p}$, and its cohomogeneity is precisely the rank of M. In fact, any maximal abelian subspace of $\mathfrak{p}$ is a section of this representation. This action also induces a polar action on the unit sphere of $T_oM \cong \mathfrak{p}$, which in this case has cohomogeneity equal to the rank of M minus one. A remarkably result by Dadok [42] says that the only homogeneous polar foliations on spheres are the orbit foliations of s-representations.

Of particular interest for our work are the *rank one symmetric spaces.* Together with Euclidean spaces $\mathbb{R}^n$ (which have rank n), rank one symmetric spaces are precisely those manifolds M which are *homogeneous and isotropic.* This means that for any two points p, $q \in M$ and any two tangent vectors $v \in T_pM$, $w \in T_qM$, there is an isometry f of M such that $f(p) = q$ and $f_*(v) = w$. Equivalently, these are the so-called *two-point homogeneous spaces*, i.e. for any four points p_1, p_2, q_1, $q_2 \in M$, there is an isometry f of M such that $f(p_i) = q_i$ for $i = 1, 2$.

Simply connected rank one symmetric spaces of non-Euclidean type are shown in Table 1.1. Duality of symmetric spaces allows to classify these manifolds into two groups. Those spaces of compact type are spheres and the projective spaces over the algebras of the complex numbers $\mathbb{C}$, of the quaternions $\mathbb{H}$ and of the octonions $\mathbb{O}$. Those of noncompact type are the hyperbolic spaces over the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

The spaces in the first row of Table 1.1 (i.e. spheres S^n and real hyperbolic spaces

Compact	Noncompact	G^{compact}	$G^{\text{noncomp.}}$	K
Spheres S^n	Real hyperbolic spaces $\mathbb{R}H^n$	$\mathrm{SO}(n+1)$	$\mathrm{SO}(1,n)$	$\mathrm{SO}(n)$
Complex projective spaces $\mathbb{C}P^n$	Complex hyperbolic spaces $\mathbb{C}H^n$	$\mathrm{SU}(n+1)$	$\mathrm{SU}(1,n)$	$\mathrm{S}(\mathrm{U}(1)\mathrm{U}(n))$
Quaternionic projective spaces $\mathbb{H}P^n$	Quaternionic hyperbolic spaces $\mathbb{H}H^n$	$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(1,n)$	$\mathrm{Sp}(1)\mathrm{Sp}(n)$
Cayley projective plane $\mathbb{O}P^2$	Cayley hyperbolic plane $\mathbb{O}H^2$	F_4	F_4^{-20}	$\mathrm{Spin}(9)$

Table 1.1: Duality in rank one symmetric spaces

$\mathbb{R}H^n$) together with Euclidean spaces, are precisely the real space forms, which are the Riemannian manifolds with the simplest curvature tensor.

The second row of Table 1.1 contains the nonflat *complex space forms*: complex projective spaces $\mathbb{C}P^n$ and complex hyperbolic spaces $\mathbb{C}H^n$. These are Kähler manifolds, so they become Hermitian symmetric spaces. Section 1.7 will be devoted to describe these manifolds with certain detail.

Quite analogously to the complex case, the third row of Table 1.1 shows the so-called *quaternionic space forms* $\mathbb{H}P^n$ and $\mathbb{H}H^n$, which can be seen as the most basic examples of quaternionic-Kähler manifolds.

The last row of Table 1.1 is constituted by two somehow exceptional 16-dimensional manifolds: the Cayley projective and hyperbolic planes, $\mathbb{O}P^2$ and $\mathbb{O}H^2$.

1.7 Complex space forms

Most of this memoir deals with certain geometric objects on two families of symmetric spaces of rank one: complex projective and hyperbolic spaces. The purpose of this section is to give a description of these two spaces. Other references are [53] and [114].

In Kähler geometry, spaces of constant curvature are not very relevant because Kähler manifolds of constant curvature and dimension greater than two are necessarily flat. If $\bar{M}$ is a Kähler manifold with complex structure J and curvature tensor $\bar{R}$, the *holomorphic sectional curvature* $\bar{K}_{hol}$ of $\bar{M}$ is defined as the restriction of the sectional curvature $\bar{K}$ to J-invariant 2-dimensional subspaces of the tangent space. Since these subspaces are generated by pairs of the form $\{v, Jv\}$, with $v \in T_p\bar{M}$, $p \in \bar{M}$, $\bar{K}_{hol}$ can be regarded as a function that maps each unit tangent vector $v \in T\bar{M}$ to the real number $\bar{K}_{hol}(v) = \bar{K}(v, Jv) = \langle \bar{R}(v, Jv)Jv, v\rangle$.

A Kähler manifold is said to have *constant holomorphic curvature* if $\bar{K}_{hol}$ is constant for any unit tangent vector of $\bar{M}$. If $\bar{M}$ has constant holomorphic curvature c then its curvature tensor can be written as

$$\bar{R}(X,Y)Z = \frac{c}{4}\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ\right).$$

A complete, simply-connected Kähler manifold of constant holomorphic curvature c is isometric to one of the following spaces: a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, a complex projective space $\mathbb{C}P^n$ if $c > 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. These are the so-called *complex space forms*. The first manifold of this list is nothing but an even dimensional Euclidean space $\mathbb{R}^{2n}$ equipped with a flat Kähler metric. Thus, below we will focus on complex projective and hyperbolic spaces and describe their construction.

1.7.1 The complex projective space $\mathbb{C}P^n$

The complex projective space of complex dimension n is defined as the space of complex lines of $\mathbb{C}^{n+1}$ through the origin, or equivalently, as the quotient manifold $S^{2n+1}(r)/\sim$, with respect to the equivalence relation given by $z \sim \lambda z$, $z \in \mathbb{C}^{n+1}$, $\lambda \in S^1 \subset \mathbb{C}$. We denote by π the canonical projection of the sphere of radius r, $S^{2n+1}(r)$, onto the complex projective space, $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$. Then π is a smooth surjective submersion called the *Hopf map*. The Riemannian metric of $\mathbb{C}P^n$ is by definition the metric induced by $S^{2n+1}(r)$ via the Hopf map. We now give a more precise description.

Consider a complex structure J on $\mathbb{R}^{2n+2}$, which allows us to identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, where the multiplication by the imaginary unit i is induced by J. Then the real part of the standard Hermitian inner product of $\mathbb{C}^{n+1}$ yields the standard Euclidean metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n+2}$. The $(2n+1)$-dimensional sphere of radius r is $S^{2n+1}(r) = \{z \in \mathbb{C}^{n+1} : \langle z, z \rangle = r^2\}$. A unit normal vector field ξ along $S^{2n+1}(r)$ is given by $\xi_z = \frac{1}{r} z$.

We consider the equivalence relation on $S^{2n+1}(r)$ generated by $z \sim \lambda z$ with $\lambda \in S^1 \subset \mathbb{C}$. This defines a principal fiber bundle over $\mathbb{C}P^n$ with total space S^{2n+1}, fiber S^1 and projection map $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$. Define $V = J\xi$. Obviously, V is a unit tangent vector field to $S^{2n+1}(r)$ and we can write

$$TS^{2n+1}(r) = \mathbb{R}V \oplus V^\perp,$$

where $V^\perp$ is the orthogonal complement of V. Actually, if $z \in S^{2n+1}(r)$, then $\mathbb{R}V_z$ is the kernel of π_{*z}, where π_* denotes the differential of π. Hence, π_{*z} maps $V_z^\perp$ isomorphically onto $T_{\pi(z)}\mathbb{C}P^n$, and for each $X \in T_{\pi(z)}\mathbb{C}P^n$ we can define the horizontal lift X_z^L of X to z, as the unique tangent vector in $V_z^\perp$ such that $\pi_* X_z^L = X$. The map $t \mapsto \varphi_t(z) = e^{it} z$ is exactly the geodesic on $S^{2n+1}(r)$ starting at z with initial speed $Jz = iz = rV_z$. We have $\pi \circ \varphi_t = \pi$, and thus $X^L_{\varphi_t(z)} = (\varphi_t)_{*z} X_z^L$.

The complex structure J on $\mathbb{C}P^n$ is then defined by

$$JX = \pi_*(JX^L)$$

for each $X \in T\mathbb{C}P^n$, whereas the metric on $\mathbb{C}P^n$ is given by

$$\langle X, Y \rangle = \left\langle X^L, Y^L \right\rangle$$

for all X, $Y \in T\mathbb{C}P^n$. This metric, called the *Fubini-Study metric* of $\mathbb{C}P^n$, makes $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$ a Riemannian submersion. It also satisfies $\langle JX, JY \rangle = \langle X, Y \rangle$ for any tangent vectors X and Y. By virtue of the formulas for Riemannian submersions [115], the Levi-Civita connection of $\mathbb{C}P^n$ is given by

$$\bar{\nabla}_X Y = \pi_* \left(\widetilde{\nabla}_{X^L} Y^L \right),$$

for tangent vector fields X, Y on $\mathbb{C}P^n$. Using this formula one can show that J is Kähler.

The theory of semi-Riemannian submersions [115] also allows to calculate the holomorphic sectional curvature of $\mathbb{C}P^n$, which turns out to be $\bar{K}_{hol}(X) = 4/r^2$ for every $X \in T\mathbb{C}P^n$. Therefore, $\mathbb{C}P^n$ is a space of constant holomorphic curvature $c = 4/r^2$.

The unitary group $\mathrm{U}(n+1) = \{A \in \mathrm{GL}(n, \mathbb{C}) : AA^* = \mathrm{Id}\}$, where A^* denotes conjugate transpose, respects the standard metric of $\mathbb{R}^{2n+2} \equiv \mathbb{C}^{n+2}$. Since it preserves complex lines through the origin of $\mathbb{C}^{n+1}$ and acts transitively on them, $\mathrm{U}(n+1)$ acts transitively by isometries on $\mathbb{C}P^n$ by $A(p) = \pi(Az)$ where $p = \pi(z) \in \mathbb{C}P^n$, and $A \in \mathrm{U}(n+1)$. However, the action is not effective, as all transformations of the form $z\,\mathrm{Id}$ with $|z| = 1$, act trivially on $\mathbb{C}P^n$. The subgroup $\mathrm{SU}(n+1)$ of those matrices in $\mathrm{U}(n+1)$ with determinant one keeps acting transitively on $\mathbb{C}P^n$ but with finite kernel constituted by the matrices $z\,\mathrm{Id}$ with z an $(n+1)$-th root of the unit.

Therefore $\mathbb{C}P^n$ is a homogeneous space. The isotropy group at, for example, the point $p = \pi(r, 0, \dots, 0) \in \mathbb{C}P^n$ is $\mathrm{S}(\mathrm{U}(1)\mathrm{U}(n))$, which is isomorphic to $\mathrm{U}(n)$. Thus, the complex projective space turns out to be the Hermitian symmetric space of rank one given by

$$\mathbb{C}P^n = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1)\mathrm{U}(n)).$$

The fact that $\mathbb{C}P^n$ has rank one follows, for instance, from the following classification of totally geodesic submanifolds, which implies that any totally geodesic, flat submanifold of maximal dimension in $\mathbb{C}P^n$ is a geodesic.

Theorem 1.1. [154] *Let M be a totally geodesic submanifold of $\mathbb{C}P^n$. Then M is holomorphically congruent to an open part of a real projective space $\mathbb{R}P^k$ for some $k \in \{1, \dots, n\}$ or to a complex projective space $\mathbb{C}P^k$ for some $k \in \{0, \dots, n\}$. Any two totally geodesic submanifolds of $\mathbb{C}P^n$ are locally holomorphically congruent to each other if and only if they are locally isometric.*

1.7.2 The complex hyperbolic space $\mathbb{C}H^n$

The construction of the complex hyperbolic space is formally very similar to the construction of the complex projective space. However, their geometries turn out to be very different. In this subsection we summarize the basic facts of this construction, trying to be as close as possible to the description of §1.7.1.

As above, take a complex structure J on $\mathbb{R}^{2n+2}$, and identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$. Now we consider the scalar product on $\mathbb{C}^{n+1}$ given by

$$\langle z, w \rangle = \mathrm{Re} \left(-z_0 \bar{w}_0 + \sum_{k=1}^{n} z_k \bar{w}_k \right),$$

for each $z = (z_0, z_1, \ldots, z_n)$, $w = (w_0, w_1, \ldots, w_n) \in \mathbb{C}^{n+1}$. This scalar product does no longer induce the usual inner product of $\mathbb{R}^{2n+2}$, but a standard semi-Riemannian metric of signature $(2n, 2)$.

The *anti-De Sitter space* (of radius r) is the Lorentzian analogue of the real hyperbolic space, and it is defined as

$$H_1^{2n+1}(r) = \left\{ z \in \mathbb{C}^{n+1} : \langle z, z \rangle = -r^2 \right\}.$$

Its tangent space at $z \in H_1^{2n+1}(r)$ is $T_z H_1^{2n+1}(r) = \{w \in \mathbb{C}^{n+1} : \langle z, w \rangle = 0\}$. The restriction of the above inner product yields a Lorentzian metric of constant sectional curvature $-1/r^2$ on $H_1^{2n+1}(r)$. A unit normal vector field ξ along $H_1^{2n+1}(r)$ is given by $\xi_z = \frac{1}{r} z$, but in this case it satisfies $\langle \xi, \xi \rangle = -1$.

We define the equivalence relation on $H_1^{2n+1}(r)$ generated by $z \sim \lambda z$ with $\lambda \in S^1 \subset \mathbb{C}$. By definition, the complex hyperbolic space, as a smooth manifold, is the quotient manifold $\mathbb{C}H^n = H_1^{2n+1}(r)/\sim$ or, equivalently, the space of timelike complex lines through the origin of $\mathbb{C}^{n+1}$. The canonical projection is denoted by $\pi \colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ and is called the *Hopf map* of $\mathbb{C}H^n$. As a Riemannian manifold, the metric on $\mathbb{C}H^n$ will be induced by the metric on the anti-De Sitter space through the map π.

Define $V = J\xi$. Then V is a unit tangent vector field to $H_1^{2n+1}(r)$, where now "unit" means, similarly as for ξ, that $\langle V, V \rangle = -1$. Hence, both ξ and V are timelike vector fields. We can write

$$T H_1^{2n+1}(r) = \mathbb{R}V \oplus V^\perp,$$

where $V^\perp$ is the orthogonal complement of V with respect to the Lorentzian metric on $H_1^{2n+1}(r)$. Actually, if $z \in H^{2n+1}(r)$, then $\mathbb{R}V_z$ is the kernel of π_{*z}. Hence, π_{*z} maps $V_z^\perp$ isomorphically onto $T_{\pi(z)}\mathbb{C}H^n$, and for each $X \in T_{\pi(z)}\mathbb{C}H^n$ we can define the horizontal lift X_z^L of X to z as the unique tangent vector in $V_z^\perp$ such that $\pi_* X_z^L = X$. The map $t \mapsto \varphi_t(z) = e^{it} z$ is exactly the geodesic on $H^{2n+1}(r)$ starting at z with initial speed $Jz = iz = rV_z$. We have $\pi \circ \varphi_t = \pi$, and thus $X^L_{\varphi_t(z)} = (\varphi_t)_{*z} X_z^L$.

The complex structure J on $\mathbb{C}H^n$ is then defined by

$$JX = \pi_*(JX^L)$$

for each $X \in T\mathbb{C}H^n$, whereas the metric on $\mathbb{C}H^n$ is given by

$$\langle X, Y \rangle = \left\langle X^L, Y^L \right\rangle$$

for all X, $Y \in T\mathbb{C}H^n$. An important point here is the fact that the metric of $H_1^{2n+1}(r)$ is positive definite on $V_z^\perp$ and, hence, the metric on $\mathbb{C}H^n$ is positive definite and thus $\mathbb{C}H^n$ becomes a Riemannian manifold. This metric, called the *Bergman metric* of $\mathbb{C}H^n$, makes $\pi \colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ a semi-Riemannian submersion. Moreover, the Bergman metric is Hermitian, i.e. it satisfies $\langle JX, JY \rangle = \langle X, Y \rangle$ for any tangent vectors X and Y. By virtue of the formulas for semi-Riemannian submersions (see [115] or [116, p. 213]), the Levi-Civita connection of $\mathbb{C}H^n$ is given by

$$\bar{\nabla}_X Y = \pi_* \left(\widetilde{\nabla}_{X^L} Y^L \right),$$

for tangent vector fields X, Y on $\mathbb{C}H^n$. Using this formula one can show that J is Kähler.

Again, the theory of semi-Riemannian submersions allows to calculate the holomorphic sectional curvature of $\mathbb{C}H^n$, which turns out to be $\bar{K}_{hol}(X) = -4/r^2$ for every $X \in T\mathbb{C}H^n$. Therefore, $\mathbb{C}H^n$ is a space of constant holomorphic curvature $c = -4/r^2$.

The indefinite unitary group $\mathrm{U}(1,n) = \{A \in \mathrm{GL}(n,\mathbb{C}) : AI_{1,n}A^* = I_{1,n}\}$, where $I_{1,n}$ is the diagonal matrix $\mathrm{diag}(-1,1,\dots,1)$, respects the metric of $\mathbb{R}^{2n+2} \equiv \mathbb{C}^{n+2}$ with signature $(2,2n)$ considered above. It also preserves timelike complex lines through the origin of $\mathbb{C}^{n+1}$ and acts transitively on them. Then it follows that $\mathrm{U}(1,n)$ acts transitively by isometries on $\mathbb{C}H^n$ and, like with $\mathbb{C}P^n$, we can restrict to $\mathrm{SU}(1,n)$, the group of the matrices of $\mathrm{U}(1,n)$ with determinant one, which still acts transitively on $\mathbb{C}H^n$. This shows that $\mathbb{C}H^n$ is a homogeneous space. Even more, the complex hyperbolic space is a Hermitian symmetric space that has the following expression as coset space:

$$\mathbb{C}H^n = \mathrm{SU}(1,n)/\mathrm{S}(\mathrm{U}(1)\mathrm{U}(n)).$$

The following result completely explains both the intrinsic and extrinsic geometry of totally geodesic submanifolds of $\mathbb{C}H^n$, and implies that complex hyperbolic spaces have rank one as symmetric spaces. Note the analogy with Theorem 1.1, from where it can be obtained using duality of symmetric spaces (cf. [11, §9.1]).

Theorem 1.2. *Let M be a totally geodesic submanifold of $\mathbb{C}H^n$. Then M is holomorphically congruent to an open part of a real hyperbolic space $\mathbb{R}H^k$ for some $k \in \{1,\dots,n\}$ or to a complex hyperbolic space $\mathbb{C}H^k$ for some $k \in \{0,\dots,n\}$. Any two totally geodesic submanifolds of $\mathbb{C}H^n$ are locally holomorphically congruent to each other if and only if they are locally isometric.*

1.7.3 The complex hyperbolic space as a symmetric space and a solvable Lie group

In this subsection we deepen into the structure of the complex hyperbolic space as a symmetric space. The notation that we introduce below will be fundamental for the rest of the work. Our main aim will be to provide a model of the complex hyperbolic space $\mathbb{C}H^n$ as a solvable Lie group AN equipped with a left-invariant metric. This model is not exclusive to $\mathbb{C}H^n$: every symmetric space of noncompact type is a solvable Lie group and its metric is left-invariant with respect to the Lie group structure. The proof of this general fact, which is based on the Iwasawa decomposition of the noncompact symmetric space, follows along the same lines as for $\mathbb{C}H^n$. We will content ourselves with presenting the construction without giving the proofs. The reader is referred to [53, Chapter 2] for a more detailed description and to [84, §6.4] for general information on the Iwasawa decomposition of semisimple Lie groups.

The complex hyperbolic space $\mathbb{C}H^n$ is a rank one Hermitian symmetric space of noncompact type and, as we have seen, admits the representation as a coset space G/K, where $G = \mathrm{SU}(1,n)$ is the identity connected component of the isometry group of $\mathbb{C}H^n$, and

$K = \mathrm{S}(\mathrm{U}(1)\mathrm{U}(n))$ is the isotropy group at some point $o \in \mathbb{C}H^n$. Denote by $\mathfrak{g} = \mathfrak{su}(1,n)$ and $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$ the Lie algebras of G and K, respectively. Let ad and Ad be the adjoint maps of $\mathfrak{g}$ and G, respectively. Let $\mathcal{B}$ be the Killing form of $\mathfrak{g}$, that is, $\mathcal{B}\colon (X,Y) \in \mathfrak{g} \times \mathfrak{g} \mapsto \mathcal{B}(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)) \in \mathbb{R}$, which is a nondegenerate bilinear form by virtue of Cartan's criterion for semisimple Lie algebras ($\mathfrak{g}$ is actually a simple Lie algebra). Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the *Cartan decomposition* of $\mathfrak{g}$ with respect to $o \in \mathbb{C}H^n$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\mathcal{B}$. This means that we have the bracket relations $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, and $\mathcal{B}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

The *Cartan involution* θ corresponding to the Cartan decomposition above is the automorphism of the Lie algebra $\mathfrak{g}$ defined by $\theta(X) = X$ for all $X \in \mathfrak{k}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$. Hence, the orthogonal projection maps onto $\mathfrak{k}$ and $\mathfrak{p}$ are $\frac{1}{2}(1+\theta)$ and $\frac{1}{2}(1-\theta)$, respectively. Moreover, it turns out that $\mathcal{B}_\theta(X,Y) = -\mathcal{B}(\theta X, Y)$ defines a positive definite inner product on $\mathfrak{g}$ satisfying the relation $\mathcal{B}_\theta(\mathrm{ad}(X)Y, Z) = -\mathcal{B}_\theta(Y, \mathrm{ad}(\theta X)Y)$ for all X, Y, $Z \in \mathfrak{g}$.

We take now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. It can be easily proved that the dimension of $\mathfrak{a}$ is one, which is equivalent to the fact that the rank of the symmetric space $G/K = \mathbb{C}H^n$ is precisely one. The set $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ is a family of commuting self-adjoint (with respect to $\mathcal{B}_\theta$) endomorphisms of $\mathfrak{g}$, and hence simultaneously diagonalizable. By definition, their common eigenspaces are the *(restricted) root spaces* of the simple Lie algebra $\mathfrak{g}$, and their nonzero eigenvalues (which do depend on $H \in \mathfrak{a}$) are the *(restricted) roots* of $\mathfrak{g}$. Denoting by $\mathfrak{a}^*$ the dual vector space of $\mathfrak{a}$, if we define for each $\lambda \in \mathfrak{a}^*$

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H,X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\},$$

then the *(restricted) root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{a}$ has the form

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$$

for a certain covector $\alpha \in \mathfrak{a}^*$. These five mutually $\mathcal{B}_\theta$-orthogonal subspaces are precisely the root spaces, while -2α, $-\alpha$, α and 2α are the roots of $\mathfrak{g}$. Moreover, $\mathfrak{a} \subset \mathfrak{g}_0$, and for every $\lambda, \mu \in \mathfrak{a}^*$, we have that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. If one writes down the matrices of $\mathfrak{g} = \mathfrak{su}(1,n)$ that belong to each root space, one can show that $\dim \mathfrak{g}_{2\alpha} = \dim \mathfrak{g}_{-2\alpha} = \dim \mathfrak{a} = 1$ and $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 2n-2$. Furthermore, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k} \cong \mathfrak{u}(n-1)$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$. The root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ are both normalized by $\mathfrak{k}_0$.

Now we fix a criterion of positivity in the set of roots; in our case, let us say that α is a positive root. Define $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ as the sum of the root spaces corresponding to all positive roots. Due to the properties of the root space decomposition, $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ with center $\mathfrak{g}_{2\alpha}$; in fact $\mathfrak{n}$ is isomorphic to the $(2n-1)$-dimensional Heisenberg algebra (see [23, Chapter 3] for a description of generalized Heisenberg algebras). Then $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, since $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$ is nilpotent.

The direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is called the *Iwasawa decomposition* of the semisimple Lie algebra $\mathfrak{g}$. It is important to mention that, even though $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$, the previous decomposition of $\mathfrak{g}$ is just a decomposition in a direct sum of vector subspaces, but neither an orthogonal decomposition, nor a direct sum of Lie algebras.

Let A, N and AN be the connected subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$, respectively. The Iwasawa decomposition theorem at the Lie group level ensures that the product map $(k, a, m) \in K \times A \times N \mapsto kam \in G$ is a diffeomorphism. Again, we just mean that G and $K \times A \times N$ are diffeomorphic as manifolds, but not that G is isomorphic to the direct product of the groups K, A and N. It follows from the Iwasawa decomposition that the solvable group AN acts simply transitively on $\mathbb{C}H^n$.

Consider now the differentiable map

$$\phi\colon h \in G \mapsto h(o) \in \mathbb{C}H^n.$$

Since AN acts simply transitively on $\mathbb{C}H^n$, the map $\phi|_{AN}\colon AN \to \mathbb{C}H^n$ is a diffeomorphism, and one can identify $\mathfrak{a} \oplus \mathfrak{n}$ with the tangent space $T_o\mathbb{C}H^n$. The Bergman metric g of the complex hyperbolic space $\mathbb{C}H^n$ induces a metric $\phi^* g$ on AN. The Riemannian manifolds $(AN, \phi^* g)$ and $(\mathbb{C}H^n, g)$ are then trivially isometric. Let us denote by L_h the left translation in G by the element $h \in G$. As the metric g on $\mathbb{C}H^n$ is invariant under isometries (and then under elements of G), it follows that

$$L_h^*(\phi^* g) = L_h^* \phi^* (h^{-1})^* g = (h^{-1} \circ \phi \circ L_h)^* g = \phi^* g, \quad \text{for all } h \in G,$$

because $(h^{-1} \circ \phi \circ L_h)(h') = h^{-1}(hh'(o)) = h'(o) = \phi(h')$ for all $h' \in G$. Therefore the metric $\phi^* g$ on AN is left-invariant. From now on, we will denote this metric by $\langle \cdot, \cdot \rangle_{AN}$. Thus, we have obtained that $\mathbb{C}H^n$ can be seen as a solvable Lie group AN endowed with a left-invariant metric. Moreover, if one takes an appropriate multiple of the metric $\mathcal{B}_\theta$ on $\mathfrak{g}$ and denote this new inner product by $\langle \cdot, \cdot \rangle$, one can show that

$$\langle X, Y \rangle_{AN} = \langle X_{\mathfrak{a}}, Y_{\mathfrak{a}} \rangle + \frac{1}{2} \langle X_{\mathfrak{n}}, Y_{\mathfrak{n}} \rangle,$$

for X, $Y \in \mathfrak{a} \oplus \mathfrak{n}$, and where subscripts mean the $\mathfrak{a}$ and $\mathfrak{n}$ components respectively. Of course, this new metric also satisfies that $\langle \operatorname{ad}(X)Y, Z \rangle = -\langle Y, \operatorname{ad}(\theta X)Y \rangle$ for all X, Y, $Z \in \mathfrak{g}$.

By means of $\phi|_{AN}$ we can also equip AN with the Kähler structure induced by the one in $\mathbb{C}H^n$, and we obtain the corresponding complex structure J on AN, and also on $\mathfrak{a} \oplus \mathfrak{n}$. Some calculations with matrices would show that the complex structure J on $\mathfrak{a} \oplus \mathfrak{n}$ leaves $\mathfrak{g}_\alpha$ invariant and $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$.

Thus, we have obtained a model for the complex hyperbolic space $\mathbb{C}H^n$ as a solvable Lie group AN with left-invariant Riemannian metric whose Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ can be identified with the tangent space $T_o\mathbb{C}H^n$, and such that $\mathfrak{g}_\alpha$ can be seen as a complex vector space $\mathbb{C}^{n-1}$.

Let $B \in \mathfrak{a}$ be a vector such that $\langle B, B \rangle = \langle B, B \rangle_{AN} = 1$ and define $Z = JB \in \mathfrak{g}_{2\alpha}$. Then $\langle Z, Z \rangle = 2\langle Z, Z \rangle_{AN} = 2$. Let now a, b, x, y be real numbers and U, $V \in \mathfrak{g}_\alpha$. One can show that the Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$ is given by

$$\frac{1}{\sqrt{-c}}[aB + U + xZ, bB + V + yZ] = -\frac{b}{2}U + \frac{a}{2}V + \left(-bx + ay + \frac{1}{2}\langle JU, V \rangle\right) Z,$$

where c is the constant holomorphic sectional curvature of $\mathbb{C}H^n$. Furthermore, the Levi-Civita connection $\bar{\nabla}$ of $(AN, \langle\cdot,\cdot\rangle_{AN})$ can be calculated by the expression (cf. [15, §2]):

$$\frac{1}{\sqrt{-c}}\bar{\nabla}_{aB+U+xZ}(bB+V+yZ) = \left(xy+\frac{1}{2}\langle U,V\rangle_{AN}\right)B - \frac{1}{2}\left(bU+yJU+xJV\right) + \left(-bx+\frac{1}{2}\langle JU,V\rangle_{AN}\right)Z.$$

Let us also define $\mathfrak{p}_\lambda = (1-\theta)\mathfrak{g}_\lambda$, the projection onto $\mathfrak{p}$ of the restricted root spaces. Then $\mathfrak{p} = \mathfrak{a}\oplus\mathfrak{p}_\alpha\oplus\mathfrak{p}_{2\alpha}$. If the complex structure on $\mathfrak{p}$ is denoted by i, then we have that $2iB = (1-\theta)Z$, and $i(1-\theta)U = (1-\theta)JU$ for every $U\in\mathfrak{g}_\alpha$.

We state now two lemmas that provide some extra information on the structure of the solvable model of $\mathbb{C}H^n$ and that will be used frequently along Chapter 6.

Lemma 1.3. [17, Lemma 2.1] *We have:*

(a) $\frac{1}{\sqrt{-c}}[\theta X, Z] = -JX$ *for each* $X\in\mathfrak{g}_\alpha$.

(b) $\langle T, (1+\theta)[\theta X, Y]\rangle = 2\langle [T,X], Y\rangle$, *for any* X, $Y\in\mathfrak{g}_\alpha$ *and* $T\in\mathfrak{k}_0$.

Lemma 1.4. *The orthogonal projection map* $\frac{1}{2}(1-\theta)\colon \mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}\to\mathfrak{a}\oplus\mathfrak{p}_\alpha\oplus\mathfrak{p}_{2\alpha}$ *defines an equivalence between the adjoint* K_0*-representation on* $\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ *and the adjoint* K_0*-representation on* $\mathfrak{p} = \mathfrak{a}\oplus\mathfrak{p}_\alpha\oplus\mathfrak{p}_{2\alpha}$. *Moreover, this equivalence is an isometry between* $(\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}, \langle\cdot,\cdot\rangle_{AN})$ *and* $(\mathfrak{p}, \langle\cdot,\cdot\rangle)$, *and* $\frac{1}{2}(1-\theta)\colon\mathfrak{g}_\alpha\to\mathfrak{p}_\alpha$ *is a complex linear map.*

Proof. The first part follows from the fact that θ is a K-equivariant, hence K_0-equivariant, map on $\mathfrak{g}$. The other claims follow from the facts stated above in this subsection. □

To conclude this chapter, we will give the fundamental ideas of the geometric interpretation of the groups A and N arising in the Iwasawa decomposition of $G = \mathrm{SU}(1,n)$. Details can be found in [53, §2.2]; see also [58, Chapter 1].

Two unit speed curves γ and σ in a nonpositively curved, complete, simply connected Riemannian manifold $\bar{M}$ are called asymptotic if there is a positive constant C such that $\bar{d}(\gamma(t),\sigma(t)) \leq C$ for all $t\geq 0$, where $\bar{d}$ denotes the Riemannian distance in $\bar{M}$. This definition establishes an equivalence relation in the collection of complete geodesics of $\bar{M}$. Each equivalence class is called *point at infinity* of $\bar{M}$. The set of the points at infinity of $\bar{M}$ is the *ideal boundary* of $\bar{M}$ and is denoted by $\bar{M}(\infty)$.

In the case $\bar{M} = \mathbb{C}H^n$, it is possible to endow $\mathbb{C}H^n\cup\mathbb{C}H^n(\infty)$ with a topology (the so-called cone topology) that makes $\mathbb{C}H^n\cup\mathbb{C}H^n(\infty)$ homeomorphic to the closed unit ball of $\mathbb{R}^{2n}$ in such a way that $\mathbb{C}H^n(\infty)$ corresponds to the unit sphere of $\mathbb{R}^{2n}$. In this model, two geodesics in $\mathbb{C}H^n$ are asymptotic if they converge to the same point of the unit sphere. Moreover, for each $p\in\mathbb{C}H^n$ and $x\in\mathbb{C}H^n(\infty)$ there is a unique geodesic $\gamma_{px}\colon\mathbb{R}\to\mathbb{C}H^n$ such that $\|\dot{\gamma}_{px}\| = 1$, $\gamma_{px}(0) = p$ and $\lim_{t\to\infty}\gamma_{px}(t) = x$.

The Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is a 1-dimensional abelian subspace of $\mathfrak{p}$. In $\mathfrak{p}\equiv T_o\mathbb{C}H^n$, the Riemannian exponential map and the Lie group exponential map coincide, that is,

$\mathrm{Exp}(tX) \cdot o = \exp_o(tX)$ for all $X \in \mathfrak{p}$ and $t \in \mathbb{R}$. It follows that the orbit of the group A through o is the geodesic through o with tangent space at o given by $\mathfrak{a} \subset \mathfrak{p} \equiv T_o\mathbb{C}H^n$. This geodesic determines two points at infinity; let x be one of them. Thus, the submanifold A of AN corresponds to $\gamma_{ox}(\mathbb{R})$ under the isometry $\phi|_{AN} \colon AN \to \mathbb{C}H^n$. In other words, $\gamma_{ox}(\mathbb{R})$ is the orbit $A \cdot o$ of the action of A on $\mathbb{C}H^n$, while the rest of the orbits are equidistant curves to $A \cdot o$.

Now, let us comment on the action of the nilpotent part N of the Iwasawa decomposition. First notice that N has dimension $2n-1$. This, together with the fact that AN acts simply transitively on $\mathbb{C}H^n$, implies that N acts isometrically with cohomogeneity one on $\mathbb{C}H^n$. It turns out that the orbits of this action are hypersurfaces in $\mathbb{C}H^n$ which are orthogonal at every point to the integral curves of the left-invariant vector field $B \in \mathfrak{a}$. These integral curves are all geodesics with a common point at infinity (x, according to the notation above).

More specifically, the orbits of the N-action are the *horospheres* of $\mathbb{C}H^n$ determined by the point at infinity x. In order to recall this concept, consider a unit speed geodesic γ in $\mathbb{C}H^n$. The real function $f \colon \mathbb{C}H^n \to \mathbb{R}$ given by $f_\gamma(p) = \lim_{t\to\infty} \big(\bar{d}(\gamma(t), p) - t\big)$ is said to be the *Busemann function* with respect to γ. Then, horospheres are defined as the level sets of a Busemann function, and these are parallel real hypersurfaces of $\mathbb{C}H^n$ defining a regular Riemannian foliation, each of whose leaves has a unique adherent point at infinity. Thus, it turns out that the orbits of the N-action on $\mathbb{C}H^n$ are the horospheres determined by the geodesic γ_{ox} or, in other words, the horospheres adherent to x.

As we have said above, once the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is chosen, the orbit $A \cdot o$ is a geodesic that detemines two points at infinity, say x and y. The fact that the horospheres given by the N-action have x as point at infinity and not y is equivalent to the choice of the positivity criterion in the set of restricted roots $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$. Thus, there are two equivalent ways of defining a concrete Iwasawa decomposition of the Lie algebra $\mathfrak{g}$. On the one hand, we have the algebraic way described above in this section and which depends on the choice of a particular Cartan decomposition, a maximal abelian subspace of $\mathfrak{p}$ and a positivity criterion in the set of roots. On the other hand, we have a more geometric way, by means of which a concrete Iwasawa decomposition is determined by a point of the space (o in our notation) and by a point at infinity (x in our notation).

Chapter 2

Homogeneous and isoparametric hypersurfaces

This chapter contains an exposition on some known results concerning homogeneous hypersurfaces and, more generally, isoparametric hypersurfaces. Particular emphasis is put on the description and characterization of homogeneous hypersurfaces in complex space forms.

We start in Section 2.1 with the notion of isoparametric hypersurface, which comes motivated by a problem in geometric optics. Since homogeneous hypersurfaces (i.e. principal orbits of cohomogeneity one orbits) constitute the first natural examples of isoparametric hypersurfaces, in Section 2.2 we recall some important facts on cohomogeneity one actions on general ambient manifolds and give an overview of the classification results known so far for such actions. The classification problem of isoparametric hypersurfaces in space forms is presented in Section 2.3. The last three sections of this chapter focus on the description and on certain properties of homogeneous real hypersurfaces in nonflat complex space forms. Section 2.4 contains some important concepts for the study of such real hypersurfaces, whereas Sections 2.5 and 2.6 deal with the concrete cases of complex projective and hyperbolic spaces, respectively. Of particular relevance for some of the original contributions of this thesis is the description of homogeneous hypersurfaces in complex hyperbolic spaces which is provided in Section 2.6.

2.1 Isoparametric hypersurfaces

The history of isoparametric hypersurfaces traces back (at least) to the work [128] of Somigliana in 1919, where the following problem of geometric optics was studied. Consider a solution φ to the wave equation

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial t^2},$$

where Δ is the Laplace operator of $\mathbb{R}^3$ (that is, with respect to the space variables), and t is the time variable. The wavefronts of φ are the set of points that have a common phase (same oscillating state) at a given instant $t = t_0$. Mathematically, they are the level surfaces of $\varphi(\cdot, t_0)$. Let us assume that the wavefronts of φ are parallel, that is, equidistant to each other. Somigliana refers to this condition as Huygens principle. What are then the possible wavefronts? He then showed that these level surfaces must have constant mean curvature, and from this, he deduced that only very particular wavefronts satisfy this kind of Huygens principle, namely: concentric spheres, coaxial cylinders and parallel planes.

The term *isoparametric hypersurface* was probably introduced by Levi-Civita [91] in the year 1937, and it is motivated by a classical terminology that we explain now. Let $f\colon \bar{M} \to \mathbb{R}$ be a smooth function, where $\bar{M}$ is a Riemannian manifold (in [91], $\bar{M} = \mathbb{R}^3$). The *first* and the *second differential parameters* of f are, respectively,

$$\Delta_1 f = \|\operatorname{grad} f\|^2 \quad \text{and} \quad \Delta_2 f = \Delta f,$$

where Δ is the Laplace-Beltrami operator of $\bar{M}$ and $\operatorname{grad} f$ denotes the gradient of f. When the first and the second differential parameters of a nonconstant function f are constant along the level sets of f, we say that f is an *isoparametric function.* Its regular level sets are then called *isoparametric hypersurfaces*, and the collection of all the level sets of f is called an *isoparametric family of hypersurfaces.* Note that f is isoparametric if and only if there exist real functions F_1 and F_2 of real variable such that

$$\Delta_1 f = F_1(f) \quad \text{and} \quad \Delta_2 f = F_2(f).$$

In order to avoid pathological examples, it is usual to require that the function F_1 is smooth and the function F_2 is continuous. See [153] for more details.

In an arbitrary ambient space, the concept of isoparametric hypersurface not only has perfect mathematical meaning, but it is also motivated by certain physical problem that subsumes in a natural way the problem of geometric optics studied by Somigliana. Indeed, given a wave in any Riemannian manifold, if this wave is stationary and its wavefronts are parallel hypersurfaces, then the family of all such wavefronts constitutes an isoparametric family of hypersurfaces.

From the geometric point of view, the constancy of the first differential parameter along the level sets means that the level sets are parallel, while for the second differential parameter the condition means that these level sets have *constant mean curvature.* In fact, Cartan showed that a hypersurface is isoparametric if and only if it and its *sufficiently close parallel hypersurfaces have constant mean curvature* [30]. Since this characterization holds in every Riemannian manifold, we will take it as definition of isoparametric hypersurface along this work.

The study of isoparametric hypersurfaces has today a long history which has revealed many connections with different areas of mathematics, such as Riemannian geometry, but also Lie group theory, algebraic geometry, algebraic topology, differential equations and Hilbert spaces. Even some applications in physics have been found. For instance, see [122] and [126] for the appearance of isoparametric hypersurfaces in some problems of fluid mechanics, or [67] for certain relation between isoparametric families and Dirac operators.

Although in Section 2.3 we will continue with some of the most important results on isoparametric hypersurfaces in space forms, our exposition here does not attempt to be complete. For a more detailed introduction to this topic and its relation to other subjects (such as isoparametric submanifolds of higher codimension, equifocal submanifolds, Dupin hypersurfaces and polar actions), we refer the reader to the excellent surveys [144], [35], [146] and [38], and to the books [121] and [11].

2.2 Cohomogeneity one actions

An important collection of examples of isoparametric hypersurfaces is given by those isometric actions of cohomogeneity one. A *cohomogeneity one action* of a Lie group G on a Riemannian manifold $\bar{M}$ is an isometric action of G on $\bar{M}$ such that the codimension of every principal orbit is one. In such a case, $\bar{M}$ is called a *cohomogeneity one manifold* and the principal orbits of the action are called *(extrinsically) homogeneous hypersurfaces.* It is clear that homogeneous hypersurfaces have constant principal curvatures. Moreover, since nearby principal orbits of a cohomogeneity one action are parallel hypersurfaces, we obtain that homogeneous hypersurfaces are always isoparametric with constant principal curvatures.

The classification of cohomogeneity one actions up to orbit equivalence (which is equivalent to the classification of homogeneous hypersurfaces up to isometric congruence) is an important problem in differential geometry. The main reason is that, if $\bar{M}$ is a cohomogeneity one manifold, certain partial differential equations that can arise on $\bar{M}$ can be reduced to ordinary differential equations, which can make its resolution easier. This procedure has proved to be successful, for example, for the construction of Einstein, Einstein-Kähler and Einstein-Weyl structures as in [6] and [25], in order to investigate Yang-Mills equations [148], and to construct hyper-Kähler Calabi metrics [41] and special Lagrangian submanifolds [81].

The orbit space $\bar{M}/G$ of a cohomogeneity one action is homeomorphic to $\mathbb{R}$, S^1, $[0,1]$ or $[0,\infty)$, as shown by Mostert [105] and Bérard-Bergery [6]. This result implies that a cohomogeneity one action has at most two singular or exceptional orbits, which correspond to the boundary points of $\bar{M}/G$. If there is one singular orbit, every principal orbit is a tube around the singular orbit. If all orbits are principal, and hence $\bar{M}/G$ is homeomorphic to $\mathbb{R}$ or S^1, the orbits of the action of G on $\bar{M}$ constitute a (homogeneous) regular Riemannian foliation on $\bar{M}$.

The first spaces where a classification of cohomogeneity one actions was achieved were space forms. In nonpositive curvature, this traces back to the works of Somigliana [128], Levi-Civita [91], Segre [125] and Cartan [28]. The corresponding classification in spheres had to wait until the the work [77] of Hsiang and Lawson in the seventies. We will come back to these classifications in Section 2.3.

In spaces of nonconstant curvature, the complexity of the problem increases and, in most of the cases, results could only be obtained quite recently and taking a symmetric space as ambient manifold.

Symmetric spaces of compact type were the next candidates for a study of cohomogeneity one actions. The classification on projective spaces was obtained by Takagi [131] for the complex case, and by Iwata for the quaternionic [79] and the Cayley [80] cases. Cohomogeneity one actions on irreducible compact symmetric spaces of rank higher than one were classified much later by Kollross [85].

The methods used by these authors do not extend to the noncompact duals, where the problem is still open. However, there is a classification for complex hyperbolic spaces and the quaternionic and Cayley hyperbolic planes. This classification, due to Berndt and Tamaru [21], leaves open the problem for $\mathbb{H}H^n$, $n \geq 3$. In fact, in §5.4.2 we construct some new examples of such actions. Also Berndt and Tamaru have made some progress concerning higher rank in [19], [20] and, especially, in [22], where a conceptual approach to the classification of cohomogeneity one actions on noncompact symmetric spaces is proposed. But, apart from some particular symmetric spaces, the problem is still open in the noncompact case.

2.3 Isoparametric hypersurfaces in space forms

In this section, we will give an idea of the main aspects of the history of isoparametric hypersurfaces in the ambient manifolds where their study was first developed: in real space forms. We also say a word about some basic facts on isoparametric hypersurfaces in space forms of arbitrary signature.

2.3.1 The Euclidean and real hyperbolic cases

In the paper [91] published in 1937, Levi-Civita classified isoparametric hypersurfaces in $\mathbb{R}^3$. He was probably not aware that a similar result had been obtained almost two decades ago by Somigliana [128]. In 1938, Segre [125] explains that one can extend the results of [128] and [91] to Euclidean spaces $\mathbb{R}^n$ of arbitrary dimension. Segre shows that isoparametric hypersurfaces in $\mathbb{R}^n$ have constant principal curvatures (i.e. the eigenvalues of the shape operator are independent of the point in the hypersurface), he proves that there are at most two principal curvatures and from this he derives a complete classification, in which there are again three types of examples: concentric spheres, generalized coaxial cylinders (i.e. tubes around an affine subspace of dimension at least one) or parallel hyperplanes. This examples are all homogeneous, so this result implies the following classification of cohomogeneity one actions on Euclidean spaces.

Theorem 2.1. [125] *Let G be a Lie subgroup of the isometry group of $\mathbb{R}^n$, $\mathbb{R}^n \rtimes \mathrm{O}(n)$, acting on $\mathbb{R}^n$ with cohomogeneity one. Then the action of G is orbit equivalent to one of the following actions:*

(i) *The action of $\mathrm{SO}(n) \subset \mathbb{R}^n \rtimes \mathrm{O}(n)$. The singular orbit is a point and the principal orbits are spheres centered at that point.*

(ii) *The action of* $\mathbb{R}^k \rtimes \mathrm{SO}(n-k) \subset \mathbb{R}^n \rtimes \mathrm{O}(n)$ *for some* $k \in \{1, \dots, n-2\}$. *There is one singular orbit which is a totally geodesic* $\mathbb{R}^k \subset \mathbb{R}^n$ *and the principal orbits are tubes around it.*

(iii) *The action of* $\mathbb{R}^{n-1} \subset \mathbb{R}^n \rtimes \mathrm{O}(n)$. *All orbits are principal and totally geodesic hyperplanes.*

In the late thirties, Cartan also addressed the study of isoparametric hypersurfaces. In [28] he characterized isoparametric hypersurfaces in real space forms by the property of having *constant principal curvatures*. This equivalence turns out to be very helpful in the investigation of isoparametric hypersurfaces, and sometimes even the constancy of the principal curvatures is taken as the definition of isoparametric hypersurface. However, this should only be done in real space forms, since in spaces of nonconstant curvature both notions are different, as we will see in Section 2.5.

From now on in this section, we will denote by g the number of distinct constant principal curvatures of an isoparametric hypersurface, $\lambda_1, \dots, \lambda_g$ will be the values of the principal curvatures, and $m_1, \dots, m_g$ their corresponding multiplicities. Cartan derived the following *fundamental formula* of an isoparametric hypersurface in a real space form of constant sectional curvature κ:

$$\sum_{j=1,\, \lambda_j \neq \lambda_i}^{g} m_j \frac{\kappa + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0, \quad \text{for each } i = 1, \dots, g.$$

From this relation, it is easy to show that if $\kappa \leq 0$, then $g \in \{1, 2\}$. Using this fact, Cartan was able to classify isoparametric hypersurfaces in real hyperbolic spaces $\mathbb{R}H^n$. The examples that appear in this classification are: geodesic spheres, totally geodesic real hyperbolic hyperspaces $\mathbb{R}H^{n-1}$ and their equidistant hypersurfaces, tubes around totally geodesic real hyperbolic subspaces $\mathbb{R}H^k$ ($1 \leq k \leq n-2$) and horospheres. Again, all these examples are homogeneous hypersurfaces, from where one can obtain the following classification of cohomogeneity one actions on real hyperbolic spaces.

Theorem 2.2. [28] *Every cohomogeneity one action on the real hyperbolic space* $\mathbb{R}H^n = \mathrm{SO}^0(1, n)/\mathrm{SO}(n)$ *is orbit equivalent to one of the following cohomogeneity one actions:*

(i) *The action of* $\mathrm{SO}(n) \subset \mathrm{SO}^0(1, n)$. *The singular orbit is a point and the principal orbits are geodesic spheres centered at that point.*

(ii) *The action of* $\mathrm{SO}^0(1, k) \times \mathrm{SO}(n-k) \subset \mathrm{SO}^0(1, n)$ *for some* $k \in \{1, \dots, n-2\}$. *This action has one singular orbit, which is a totally geodesic* $\mathbb{R}H^k \subset \mathbb{R}H^n$, *and the principal orbits are tubes around this* $\mathbb{R}H^k$.

(iii) *The action of* $\mathrm{SO}^0(1, n-1) \subset \mathrm{SO}^0(1, n)$. *All the orbits are principal, one orbit is a totally geodesic* $\mathbb{R}H^{n-1} \subset \mathbb{R}H^n$ *and the others are equidistant hypersurfaces to it.*

(iv) *The action of the nilpotent subgroup in an Iwasawa decomposition of* $\mathrm{SO}^0(1, n)$. *All the orbits are principal and the resulting foliation is the horosphere foliation on* $\mathbb{R}H^n$.

2.3.2 The problem in spheres

Cartan also investigated isoparametric hypersurfaces in spheres in the articles [29], [30], [31]. In this setting, since $\kappa > 0$, the fundamental formula does not provide much information. In fact, the problem in spheres is much more involved and rich. Cartan was able to classify isoparametric hypersurfaces in spheres S^n with $g \in \{1, 2, 3\}$. The examples with $g = 1$ are just geodesic spheres, while those with $g = 2$ are tubes around totally geodesic submanifolds S^k of S^n with $1 \leq k \leq n-2$. For $g = 3$, Cartan showed that all three multiplicities m_i are equal, and one has $m = m_1 = m_2 = m_3 \in \{1, 2, 4, 8\}$. He also proved that the corresponding isoparametric hypersurfaces are tubes around certain embedding of the projective plane $\mathbb{F}P^2$ in S^{3m+1}, where $\mathbb{F}$ is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, for $m = 1, 2, 4, 8$, respectively. Moreover, Cartan found two examples of isoparametric hypersurfaces with four principal curvatures in S^5 and in S^9, but he could get neither a classification for $g \geq 4$, nor an upper bound on g (as for $\mathbb{R}^n$ and $\mathbb{R}H^n$).

However, Cartan noticed that all the isoparametric hypersurfaces known to him (those in spheres, but also those in $\mathbb{R}^n$ and $\mathbb{R}H^n$) were homogeneous. This observation led him to ask the following question: is every isoparametric hypersurface extrinsically homogeneous? A surprising negative answer would only come several decades later.

The study of isoparametric hypersurfaces was taken up again in the early seventies. Nomizu [112] shows that the focal manifolds of an isoparametric family of hypersurfaces in a sphere are minimal; the *focal manifolds* of an isoparametric family are those elements of the family with codimension greater than one. About that time Hsiang and Lawson [77] derived the classification of cohomogeneity one actions on spheres:

Theorem 2.3. [77] *Each cohomogeneity one action on a sphere S^n is orbit equivalent to the isotropy representation of a Riemannian symmetric space of rank* 2*. Every such action has exactly two singular orbits, while the other orbits are principal and tubes around each one of the singular ones.*

Based on the work of Hsiang and Lawson, Takagi and Takahashi [135] determined the principal curvatures of homogeneous (isoparametric) hypersurfaces in spheres. According to these results, every homogeneous hypersurface in a sphere is a principal orbit of the isotropy representation of a Riemannian symmetric space of rank two. In Table 2.1 all symmetric spaces of rank 2 are shown, together with their dimensions, the number g of principal curvatures and the multiplicities of the corresponding homogeneous hypersurfaces.

A consequence of Takagi and Takahashi's work is that the number of principal curvatures g of a homogeneous hypersurface in a sphere satisfies $g \in \{1, 2, 3, 4, 6\}$. In two remarkable articles [108], [109] (that were written around 1973, but published in 1980-1981), Münzner was able to prove that the same restriction on g holds for every (not necessarily homogeneous) isoparametric hypersurface in a sphere. Münzner's papers contain a deep analysis of the structure of isoparametric families of hypersurfaces in spheres, using both geometric and topological methods. Apart from the restriction on g, we emphasize other two consequences of Münzner's work. The first one is that, if $\lambda_1 < \cdots < \lambda_g$ are

g	Multiplicities	Symmetric space G/K	$\dim G/K$
1	$l-2$	$S^1 \times S^{l-1}$	l
2	$(k, l-k-2)$	$S^{k+1} \times S^{l-k-1}$	l
3	1	SU(3)/SO(3)	5
3	2	SU(3)	8
3	4	SU(6)/Sp(3)	14
3	8	E_6/F_4	26
4	$(2, 2)$	Sp(2)	10
4	$(4, 5)$	SO(10)/U(5)	20
4	$(1, k-2)$	$\mathrm{SO}(k+2)/\mathrm{SO}(2) \times SO(k)$	$2k$
4	$(2, 2k-3)$	$\mathrm{SU}(k+2)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(k))$	$4k$
4	$(4, 4k-5)$	$\mathrm{Sp}(k+2)/\mathrm{Sp}(2) \times \mathrm{Sp}(k)$	$8k$
4	$(9, 6)$	$E_6/\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	32
6	$(1, 1)$	$G_2/\mathrm{SO}(4)$	8
6	$(2, 2)$	G_2	14

Table 2.1: Compact symmetric spaces of rank 2 corresponding to the homogeneous isoparametric families in spheres

the principal curvatures of an isoparametric hypersurface in a sphere, and $m_1, \dots, m_g$ their corresponding multiplicities, then $m_i = m_{i+2}$ (indices modulo g); in particular, if g is odd, all the multiplicities coincide, and if g is even, there are at most two different multiplicities. The second result is the algebraic character of isoparametric hypersurfaces in spheres. More precisely, a hypersurface M in S^n is isoparametric if and only if $M \subset F^{-1}(c) \cap S^n$, where F is a homogeneous polynomial of degree g on $\mathbb{R}^{n+1}$ satisfying the differential equations

$$\begin{aligned} &\| \operatorname{grad} F(x)\|^2 = g^2 \|x\|^{2g-2}, \\ &\Delta F(x) = \frac{1}{2}(m_2 - m_1) g^2 \|x\|^{g-2}, \qquad x \in \mathbb{R}^{n+1}. \end{aligned}$$

The intersection of S^n with the level sets of such an F form an isoparametric family of hypersurfaces in S^n. From this result, it also follows that every isoparametric hypersurface in S^n is an open part of a complete isoparametric hypersurface in S^n, which is in turn a leaf of an isoparametric foliation that fills the whole S^n (this happened also for $\mathbb{R}^n$ and $\mathbb{R}H^n$). Moreover, each such isoparametric family has exactly two focal submanifolds of codimensions $m_1 + 1$ and $m_2 + 1$. A polynomial F like the one above is called a *Cartan-Münzner polynomial*. Notice that, according to this result, the classification problem of isoparametric hypersurfaces in spheres is reduced to a problem of algebraic geometry, but a very difficult one.

Since the restriction on g obtained by Münzner coincides with the one for homogeneous hypersurfaces, Cartan's question on the homogeneity of isoparametric hypersurfaces became even more attractive. However, in 1975 Ozeki and Takeuchi gave a negative answer to this question [118]. They constructed some Cartan-Münzner polynomials that give rise to isoparametric hypersurfaces with $g = 4$ that are not homogeneous, because their multiplicities do not coincide with the possible multiplicities of the homogeneous examples.

Some years later, Ferus, Karcher and Münzner [63] found a much larger family of inhomogeneous examples that included the ones given by Ozeki and Takeuchi. For each representation of a Clifford algebra they constructed a Cartan-Münzner polynomial that yields an isoparametric family of hypersurfaces with $g = 4$. We call these examples *of FKM-type* or *of Clifford type*. In §7.4.2 we will briefly describe their construction. Most of these examples are inhomogeneous, and this inhomogeneity was proved in [63] in a direct way, without using the classification of homogeneous hypersurfaces. As a consequence of this result, one gets the existence of an infinite countable collection of noncongruent inhomogeneous isoparametric families in spheres. This made the study of isoparametric hypersurfaces in spheres a much more appealing and interesting topic of research.

Even today, all known isoparametric hypersurfaces in spheres are either homogeneous or of FKM-type; and all those hypersurfaces with $g = 4$ are of FKM-type, with the exception of two homogeneous families of hypersurfaces with multiplicities $(2, 2)$ and $(4, 5)$. A first step towards a classification would be to determine the possible triples (g, m_1, m_2) that an isoparametric hypersurface with $g = 4$ or $g = 6$ can take. Several authors have contributed to this question (we just mention some of them, and refer to the surveys [144] and [35] for further references). In [108] and [109], Münzner already found some restrictions, which were improved by Abresch [1]. In particular, Abresch showed that the only possible triples with $g = 6$ are $(6, 1, 1)$ and $(6, 2, 2)$; moreover, there exist homogeneous examples in both cases. The determination of all possible triples with $g = 4$ was established by Stolz in 1999 [129]. He proved that every isoparametric hypersurface with $g = 4$ constant principal curvatures in a sphere has the multiplicities of one of the known homogeneous or inhomogeneous examples; in other words, the possible triples $(4, m_1, m_2)$ are $(4, 2, 2)$, $(4, 4, 5)$ and the ones of FKM-type hypersurfaces (see Table 7.1 in §7.4.2).

As we mentioned before, isoparametric hypersurfaces in spheres with $g \in \{1, 2, 3\}$ had been classified by Cartan. In 1976, Takagi [134] showed that if $g = 4$ and one of the multiplicities is one, then the hypersurface is homogeneous and of FKM-type. Ozeki and Takeuchi [119] proved that those isoparametric hypersurfaces with $g = 4$ and one multiplicity equal to 2 are homogeneous and, except for the case of multiplicities $(2, 2)$ (which corresponds to the homogeneous example in S^9 obtained by Cartan), also of FKM-type. In 1985, Dorfmeister and Neher [57] proved the uniqueness of the hypersurface with triple $(6, 1, 1)$, which is hence homogeneous. Quite recently, in 2007-2008, Cecil, Chi and Jensen [34], and independently Immervoll [78], proved that, with a few possible exceptions, every isoparametric hypersurface with $g = 4$ is one of the known examples. More precisely, if the multiplicities (m_1, m_2) of an isoparametric hypersurface with $g = 4$ in a sphere satisfy $m_2 \geq 2m_1 - 1$, then such hypersurface must be of FKM-type. Together with other known results, this one gives a classification of the case $g = 4$ with the exception of the pairs of multiplicities $(3, 4)$, $(4, 5)$, $(6, 9)$ and $(7, 8)$. The methods used in both articles are different: while Cecil, Chi and Jensen make use of the theory of moving frames and commutative algebra, Immervoll uses the tool of isoparametric triple systems developed by Dorfmeister and Neher [57]. In the last years, on the one hand, Chi went on studying the exceptional cases with $g = 4$ in [36] and [37], leaving only open the case of multiplicities $(7, 8)$. On the other hand, the investigation of the case $(g, m_1, m_2) = (6, 2, 2)$ by Miyaoka has just culminated in the article [102], where she shows the uniqueness and homogeneity of such isoparametric family.

Therefore, the solution to the classification problem of isoparametric hypersurfaces in spheres (and hence the solution to Problem 34 in Yau's list of important problems in geometry [160]) seems to be close. Only the case $(g, m_1, m_2) = (4, 7, 8)$ remains open.

Beyond their classification, some new directions in the study of isoparametric hypersurfaces have come out in the last years. As an application of the classication results above, Tang and Yan [137] have recently verified, for the case of minimal isoparametric hypersurfaces, a conjecture of Yau that states that the first eigenvalue of every compact minimal hypersurface in S^{n+1} is n. Another application is the construction, by Tang, Xie and Yang [136], of new manifolds with positive scalar curvature. Ge and Tang [65] studied isoparametric functions on exotic spheres, whereas Ma and Ohnita investigated the Hamiltonian stability of the Gauss images of isoparametric hypersurfaces as Lagrangian submanifolds of complex hyperquadrics [99]. Finally, we mention the study of the so-called n-Sasakian manifolds by Dearricott [44].

2.3.3 Isoparametric hypersurfaces in indefinite space forms

Although we will mostly deal with the notion of isoparametric hypersurface in Riemannian manifolds, in Chapter 4 we will make use of the notion and properties of isoparametric hypersurfaces in certain semi-Riemannian manifold (the anti-De Sitter space). That is why we include here a quick review of the results that we will need. This review is based on the work [71] of Hahn, who first studied the notion of isoparametric hypersurface in a general indefinite setting.

The definition of isoparametric hypersurface in an arbitrary semi-Riemannian manifold is the same as for Riemannian manifolds, provided that one assumes that the hypersurface is nondegenerate, i.e. the induced metric on the hypersurface is nondegenerate. In this case, the normal bundle is either spacelike or timelike, so there exists (locally) a normal vector field ξ such that $\epsilon = \langle \xi, \xi \rangle \in \{\pm 1\}$, and this allows to define sufficiently close equidistant hypersurfaces via normal exponentiation. When these close-by hypersurfaces have constant mean curvature, we say that they are isoparametric.

The first fundamental result involving isoparametric hypersurfaces in semi-Riemannian space forms is the following:

Theorem 2.4. [71] *For a nondegenerate connected hypersurface M in a semi-Riemannian space form, the following conditions are equivalent:*

(i) *M is isoparametric,*

(ii) *the principal curvatures of M and their algebraic multiplicities are constant along M,*

(iii) *the shape operator of M has the same characteristic polynomial at all points of M.*

The proof of this result, which can be found in [27, Section 2.1], is just an adaptation of the proof of the analogous Riemannian theorem (see [62] or [113]).

Remark 2.5. Given an endomorphism $\mathcal{S}$ of a vector space, if λ is an eigenvalue of $\mathcal{S}$, then the multiplicity of λ as a zero of the characteristic polynomial of $\mathcal{S}$ is called the algebraic multiplicity of λ. The dimension of $\ker(\mathcal{S} - \lambda I)$ is called the geometric multiplicity of λ. When $\mathcal{S}$ is the shape operator of a hypersurface, we obtain the definitions of algebraic and geometric multiplicities of the principal curvatures.

For an isoparametric hypersurface in an indefinite space form, the geometric multiplicity of a principal curvature does not need to coincide with its algebraic multiplicity, and it can be non constant (see examples in [71]).

In [100] and [156], Magid and Xiao define an isoparametric hypersurface (in the Lorentz spacetime and in the anti-De Sitter spacetime, respectively) as a hypersurface whose shape operator has a constant minimal polynomial or, equivalently, as a hypersurface with constant principal curvatures with constant geometric multiplicities. But this does not coincide with the classical definition of isoparametric hypersurface as element of a family of parallel hypersurfaces with constant mean curvature (this follows from the examples in [71] and the theorem above).

Nevertheless, it is reasonable to assume the constancy of the minimal polynomial at a first step and, later, analyze the possibility of gluing together different pieces of hypersurfaces with distinct minimal polynomials but common characteristic polynomial.

Also following the lines of the proof for the Riemannian case (cf. [62]), one can prove the following indefinite analogue of Cartan's fundamental formula (see [71] and [27, Section 2.3]):

Theorem 2.6. [71] *Let M be an isoparametric hypersurface in a semi-Riemannian space form of curvature κ, and set $\epsilon = \langle \xi, \xi \rangle$ and $\delta = \epsilon\kappa$. If its (possibly complex) principal curvatures are $\lambda_1, \ldots, \lambda_g$ with algebraic multiplicities $m_1, \ldots, m_g$, respectively, and if for some $i \in \{1, \ldots, g\}$ the principal curvature λ_i is real and its algebraic and geometric multiplicities coincide, then:*

$$\sum_{j=1,\, j\neq i}^{g} m_j \frac{\delta + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0.$$

2.4 Real hypersurfaces in complex space forms

This section contains some basic definitions and notation for the study of real hypersurfaces in complex space forms. A thorough introduction to this topic can be found in [114].

First of all, recall that a complex space form is a simply connected complete Kähler manifold with constant holomorphic sectional curvature. These manifolds are classified in three families according to the value of their constant holomorphic sectional curvature c:

complex projective spaces $\mathbb{C}P^n$ if $c > 0$, complex Euclidean spaces $\mathbb{C}^n$ if $c = 0$ and complex hyperbolic spaces $\mathbb{C}H^n$ if $c < 0$. We will denote by J the almost complex structure of a complex space form. In what follows we disregard the flat case. Note as well that $\mathbb{C}P^1$ is isometric to a 2-sphere and $\mathbb{C}H^1$ to a real hyperbolic plane $\mathbb{R}H^2$.

As for any other Kähler manifold, one would have in principle two natural notions of hypersurface in a complex space form: either a complex hypersurface or a real hypersurface. We are interested in the latter, which refers to a submanifold with real codimension one (and not complex codimension one).

Let M be a real hypersurface in a complex space form $\bar{M}$, and let ξ be a unit normal vector field on (an open part of) M. Then the vector field $J\xi$ is tangent to M, and is called the *Hopf vector field* of the hypersurface M (also the *Reeb vector field* or the *structure vector field* of M). We will denote by $g(p)$ the number of principal curvatures of the hypersurface M at a point $p \in M$. Since we are not assuming that M has constant principal curvatures, g may vary from point to point, whence the notation $g(p)$.

Another notation that will be relevant later is the following. For each point $p \in M$ we will write $h(p)$ for the number of nontrivial projections of the Hopf vector field $J\xi$ onto the distinct principal curvature spaces at p (that is, onto the distinct eigenspaces of the shape operator of M at p). Again, as well as g, h is an integer-valued function on M. Note that, obviously, $h(p) \leq g(p)$ for each $p \in M$. When $h = 1$ along M, that is, when $J\xi$ is an eigenvector of the shape operator at every point, we say that M is a *Hopf hypersurface*.

A related notion is that of *curvature-adapted* hypersurface, which refers to a hypersurface whose shape operator and normal Jacobi operator commute. Recall that the normal Jacobi operator of a hypersurface M in an ambient manifold $\bar{M}$ is the self-adjoint (local) $(1,1)$-tensor on M defined by $\bar{R}(\cdot,\xi)\xi$, where $\bar{R}$ is the $(1,3)$-curvature tensor field of $\bar{M}$ and ξ is a unit normal vector field on M. In real space forms, every hypersurface is curvature-adapted, but in spaces of nonconstant curvature, the curvature-adaptedness imposes restrictions on the geometry of a hypersurface. For instance, in nonflat complex space forms, a hypersurface is curvature-adapted if and only if it is Hopf.

Curvature-adaptedness is quite a common condition in the investigation of hypersurfaces in spaces of nonconstant curvature, because it simplifies the Gauss and Codazzi equations of the hypersurface and, however, it still allows to obtain some interesting examples. Apart from the classifications of Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ that we will review in the next sections, some partial results concerning curvature-adapted hypersurfaces in certain symmetric spaces have been derived. For instance, Berndt [8] classified curvature-adapted hypersurfaces in quaternionic projective spaces $\mathbb{H}P^n$, and also in quaternionic hyperbolic spaces $\mathbb{H}H^n$ under the extra assumption of having constant principal curvatures. More recently, Murphy [110] classified curvature-adapted hypersurfaces with constant principal curvatures in the Cayley projective plane $\mathbb{O}P^2$.

2.5 Homogeneous hypersurfaces in complex projective spaces

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^n$ was obtained by Takagi [131] in 1973. The idea of Takagi's classification is the following. Every homogeneous hypersurface in $\mathbb{C}P^n$ is the projection of a homogeneous hypersurface in S^{2n+1} via the Hopf map $\pi\colon S^{2n+1} \to \mathbb{C}P^n$. However, not every homogeneous hypersurface in S^{2n+1} is invariant under the action of the fiber S^1; if it is not, then it cannot be projected to a homogeneous hypersurface in $\mathbb{C}P^n$. Takagi showed, using the classification of homogeneous hypersurfaces in spheres, that the ones that project to homogeneous hypersurfaces of $\mathbb{C}P^n$ are precisely the ones that arise from isotropy representations of Hermitian symmetric spaces of rank two. The classification theorem, stated in terms of homogeneous hypersurfaces, is:

Theorem 2.7. [131] *A real hypersurface in $\mathbb{C}P^n$, $n \geq 2$, is homogeneous if and only if it is holomorphically congruent to an open part of one of the following hypersurfaces:*

(i) *a tube around a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$, for some $k \in \{0, \ldots, n-1\}$,*

(ii) *a tube around the complex quadric $Q^{n-1} = \{[z] \in \mathbb{C}P^n : z_0^2 + \ldots + z_n^2 = 0\}$ in $\mathbb{C}P^n$ or, equivalently, a tube around a totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$,*

(iii) *a tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ in $\mathbb{C}P^{2k+1}$ for some $k \geq 2$,*

(iv) *a tube around the Plücker embedding of the complex 2-plane Grassmanian $\mathrm{Gr}_2(\mathbb{C}^5)$ in $\mathbb{C}P^9$,*

(v) *a tube around the half-spin embedding of the Hermitian symmetric space $\mathrm{SO}(10)/\mathrm{U}(5)$ in $\mathbb{C}P^{15}$.*

The corresponding Hermitian symmetric spaces of rank two whose isotropy representations give rise to the corresponding cohomogeneity one actions are: (i) $\mathbb{C}P^{k+1} \times \mathbb{C}P^{n-k}$, (ii) $\mathrm{Gr}_2^+(\mathbb{R}^{n+3})$, (iii) $\mathrm{Gr}_2(\mathbb{C}^{k+3})$, (iv) $\mathrm{SO}(10)/\mathrm{U}(5)$, (v) $\mathrm{E}_6/(\mathrm{U}(1) \cdot \mathrm{Spin}(10))$.

For the sake of completeness, let us mention that the Segre embedding of $\mathbb{C}P^n \times \mathbb{C}P^m$ into $\mathbb{C}P^{(n+1)(m+1)-1}$ is given by

$$([z_0, \ldots, z_n], [w_0, \ldots, w_m]) \longmapsto [z_0w_0, z_0w_1, \ldots, z_0w_m, z_1w_0, \ldots, z_iw_j, \ldots, z_nw_m],$$

where all products of the form z_iw_j, $0 \leq i \leq n$, $0 \leq j \leq m$, were taken in lexicographic order.

The Plücker embedding of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^m)$ into $\mathbb{C}P^r$, with $r = \binom{m}{k} - 1$, is given by

$$\mathrm{span}(v_1, \ldots, v_k) \longmapsto [\det A_0, \ldots, \det A_r],$$

where $A_0, A_1, \ldots, A_r$ are all minors of order k of the matrix of order $k \times m$ whose rows are the components of the vectors $v_1, \ldots, v_k$.

Finally, in order to define the half-spin embedding, let Δ^+ and Δ^- be the real half-spin representations corresponding to the action of Spin(10) on $\mathbb{R}^{32} \equiv \mathbb{C}^{16}$ (see [90, Chapter 1] for a reference on spin representations), and let ϑ be the canonical representation of U(1) on $\mathbb{C}^{16}$ given by the multiplication by unit complex numbers. Then, the representation $\vartheta^3 \otimes \Delta^+ + \vartheta^{-3} \otimes \Delta^-$ of $\mathrm{U}(1) \times \mathrm{Spin}(10)$ on $\mathbb{R}^{32} \equiv \mathbb{C}^{16}$ induces, via the Hopf map $\pi\colon S^{31} \to \mathbb{C}P^{15}$, an embedding of SO(10)/U(5) in $\mathbb{C}P^{15}$. See [11, p. 88 and §9.3d] for more details.

An important consequence of Takagi's result is the fact that every homogeneous hypersurface in $\mathbb{C}P^n$ is Hopf. Another consequence of Takagi's theorem is that the number g of constant principal curvatures of a homogeneous hypersurface in $\mathbb{C}P^n$ satisfies $g \in \{2, 3, 5\}$.

The study of hypersurfaces with constant principal curvatures, particularly in complex space forms, has been a fruitful area of research in the last decades. Here, and in next section, we will explain the main results in this area. We refer the reader to the surveys [13] and [54] for further information on this topic.

The first partial classification of real hypersurfaces with constant principal curvatures is due to Takagi, who classified real hypersurfaces in $\mathbb{C}P^n$ with 2 and 3 constant principal curvatures in the papers [132] ($g = 2$) and [133] ($g = 3, n \geq 3$). The case $g = 3$, $n = 2$ was solved by Wang [151].

Theorem 2.8. [132] *Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 2$, with two distinct constant principal curvatures. Then M is an open part of a geodesic sphere in $\mathbb{C}P^n$.*

Theorem 2.9. [133], [151] *Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 2$, with three distinct constant principal curvatures. Then M is an open part of one of the following hypersurfaces:*

(i) *a tube around a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$, for some $k \in \{1, \dots, n-2\}$,*

(ii) *a tube around a complex quadric Q^{n-1} in $\mathbb{C}P^n$ or, equivalently, a tube around a totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$.*

From these results, it follows that real hypersurfaces with $g \leq 3$ constant principal curvatures in $\mathbb{C}P^n$ are open parts of homogenous hypersurfaces. Without the assumption of constant principal curvatures, Tashiro and Tachibana [139] showed that there are no totally umbilical real hypersurfaces in $\mathbb{C}P^n$ or $\mathbb{C}H^n$, $n \geq 2$, and Cecil and Ryan [33] proved that every real hypersurface in $\mathbb{C}P^n$, $n \geq 3$, with at most two distinct principal curvatures at each point has constant principal curvatures (the problem for $n = 2$ is still open nowadays).

Kimura [83] classified Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$. The examples in this classification are exactly the same as in the one by Takagi of homogeneous hypersurfaces (Theorem 2.7). Kimura's theorem rests on the use of the results of Takagi [131] and Münzner [108]. In particular, it makes use of the fact that the number of constant principal curvatures of a Hopf real hypersurface is restricted to the set $\{2, 3, 5\}$ due to Münzner's results.

So far, the existence of non-Hopf (equivalently, inhomogeneous) hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ is an open question. One does not even know any restriction on the possible values of g for such hypersurfaces.

As shown by Cartan, in a space of constant curvature, an isoparametric hypersurface is the same as a hypersurface with constant principal curvatures. However, this equivalence does not hold in general, so one might in principle use both conditions individually to try to characterize homogeneous hypersurfaces in spaces of nonconstant curvature.

One of the most simple ambient manifolds of nonconstant curvature where a study of isoparametric hypersurfaces could be addressed is the complex projective space. It quickly turned out that there are examples of inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in $\mathbb{C}P^n$, for certain values of n. The first such examples were constructed by Wang [151], by means of projecting some of the inhomogeneous isoparametric hypersurfaces of FKM-type in odd-dimensional spheres S^{2n+1} to $\mathbb{C}P^n$ via the Hopf map. Wang also showed the following result:

Theorem 2.10. [151] *Let M an isoparametric hypersurface in $\mathbb{C}P^n$. Then the following conditions are equivalent:*

(i) *One focal submanifold of M is complex,*

(ii) *M is a Hopf hypersurface,*

(iii) *M has constant principal curvatures.*

Combining this with Kimura's classification of Hopf hypersurfaces with constant principal curvatures, one derives the following

Corollary 2.11. [83], [151] *Let M be a real hypersurface in $\mathbb{C}P^n$. Then, any two of the following conditions implies the third one:*

(i) *M is Hopf,*

(ii) *M is isoparametric,*

(iii) *M has constant principal curvatures.*

In such a case, M is an open part of a homogeneous hypersurface in $\mathbb{C}P^n$ (see Theorem 2.7).

More examples of inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in complex projective spaces were constructed by Xiao [158] and Ge, Tang and Yan [66]. Again, these examples are related to isoparametric hypersurfaces in spheres. However, now these hypersurfaces in spheres do not need to be inhomogeneous: it is possible to project a homogeneous hypersurface in a sphere to an inhomogeneous hypersurface in a complex projective space. Chapter 7 will be devoted to a thorough investigation of this phenomenon, which will allow us to obtain an almost complete classification of isoparametric hypersurfaces in complex projective spaces.

2.6 Homogeneous hypersurfaces in complex hyperbolic spaces

Unlike the projective case, the first investigations involving homogeneous hypersurfaces in complex hyperbolic spaces did not directly focus on their classification, which, indeed, had to wait until 2007, as we will soon see.

Montiel [104] found in 1985 one of the first such results, which consisted in the hyperbolic analogue of the classification of real hypersurfaces with two principal curvatures in $\mathbb{C}P^n$, $n \geq 3$, obtained by Cecil and Ryan [33].

Theorem 2.12. [104] *Let M be a real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$, of holomorphic sectional curvature c. Assume that M has at most two principal curvatures at each point. Then M is an open part of one of the following hypersurfaces:*

(i) *a horosphere,*

(ii) *a geodesic sphere,*

(iii) *a tube around a totally geodesic complex hyperbolic subspace $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$,*

(iv) *a tube of radius $r = \frac{1}{\sqrt{-c}} \ln(2 + \sqrt{3})$ around a totally geodesic real hyperbolic subspace $\mathbb{R}H^n$ in $\mathbb{C}H^n$.*

As in the projective case, all examples in Theorem 2.12 are homogeneous Hopf hypersurfaces.

Hopf hypersurfaces with constant principal curvatures were classified by Berndt [7] in 1989:

Theorem 2.13. *Let M be a Hopf hypersurface in $\mathbb{C}H^n$, $n \geq 2$, with constant principal curvatures. Then M is an open part of one of the following hypersurfaces:*

(i) *a horosphere,*

(ii) *a tube around a totally geodesic complex hyperbolic subspace $\mathbb{C}H^k$ in $\mathbb{C}H^n$, for some $k \in \{0, \dots, n-1\}$,*

(iii) *a tube around a totally geodesic real hyperbolic subspace $\mathbb{R}H^n$ in $\mathbb{C}H^n$.*

Berndt's proof makes use of a modified version of Cartan's fundamental formula for isoparametric hypersurfaces in real space forms. In order to derive such formula, the assumption that M is Hopf is central. In the general case, the Gauss and Codazzi equations seem to be too complicated in order to obtain a manageable formula.

All Hopf hypersurfaces above are open parts of homogeneous ones. However, the classification problem of homogeneous hypersurfaces in $\mathbb{C}H^n$ remained open. In 1998, Lohnherr [92] (cf. [93]) found a homogeneous, ruled, minimal, non-Hopf hypersurface in $\mathbb{C}H^n$, $n \geq 2$; we will denote this hypersurface by W^{2n-1}. This meant that the classification of homogeneous hypersurfaces could be more difficult than expected. Later, Berndt and Brück

[10] generalized this construction to obtain many other non-Hopf homogeneous examples, which are related to certain submanifolds W_{φ}^{2n-k}. Thanks to these discoverings, Berndt and Tamaru [21] obtained the classification of cohomogeneity one actions on $\mathbb{C}H^n$ up to orbit equivalence. The extrinsic geometry of the corresponding homogeneous hypersurfaces was studied in [15]. In Table 2.2 at the end of this chapter we present a description of the principal curvatures and corresponding multiplicities of each one of these homogeneous hypersurfaces.

Theorem 2.14. [21] *A real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, is homogeneous if and only if it is holomorphically congruent to one of the following hypersurfaces:*

(i) *a tube around a totally geodesic $\mathbb{C}H^k$, for some $k \in \{0, \ldots, n-1\}$,*

(ii) *a tube around a totally geodesic $\mathbb{R}H^n$,*

(iii) *a horosphere,*

(iv) *the Lohnherr hypersurface W^{2n-1} or one of its equidistant hypersurfaces,*

(v) *a tube around a Berndt-Brück submanifold W_{φ}^{2n-k} for some $\varphi \in (0, \pi/2]$ and some $k \in \{2, \ldots, n-1\}$, where k is even if $\varphi \neq \pi/2$.*

Examples (i) and (ii) correspond to cohomogeneity one actions with one totally geodesic singular orbit. The families (iii) and (iv) provide homogeneous regular foliations on $\mathbb{C}H^n$ and, hence, the corresponding cohomogeneity one actions do not have singular orbits. The families in (v) correspond to cohomogeneity one actions with one non-totally geodesic singular orbit. The connected closed subgroups of $\mathrm{SU}(1,n)$ that give rise to each one of the cohomogeneity one actions (up to orbit equivalence) are: (i) $\mathrm{S}(\mathrm{U}(1,k) \times \mathrm{U}(n-k))$, (ii) $\mathrm{SO}^0(1,n)$, (iii) N, i.e. the nilpotent part of the Iwasawa decomposition of $\mathrm{SU}(1,n)$, (iv) the connected Lie subgroup H of AN whose Lie algebra is $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is a linear hyperplane of $\mathfrak{g}_\alpha$, (v) the connected Lie subgroup H of AN whose Lie algebra is $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is a (real) subspace of $\mathfrak{g}_\alpha$ such that $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ has dimension k and constant Kähler angle φ. In the next subsection we will explain in detail the construction of the cohomogeneity one actions corresponding to cases (iv) and (v) above.

2.6.1 The Lohnherr-Berndt-Brück submanifolds and the geometry of the non-Hopf homogeneous hypersurfaces

For some time it was believed that, as in the case of the complex projective space, every homogeneous hypersurface in the complex hyperbolic space was Hopf. However, as we have already said, Lohnherr constructed a counterexample: the minimal ruled hypersurface W^{2n-1} in $\mathbb{C}H^n$. Later, Berndt and Brück generalized this construction to the minimal ruled submanifolds W_{φ}^{2n-k}. As a consequence of the classification of homogeneous hypersurfaces in $\mathbb{C}H^n$ stated in Theorem 2.14, tubes around these submanifolds constitute the only nonclassical (and non-Hopf) examples of homogeneous hypersurfaces in the complex hyperbolic space. The aim of this section is to construct the submanifolds W^{2n-k}_{φ}, which we are going to call *Berndt-Brück submanifolds* (*Lohnherr hypersurface* for the particular case of W^{2n-1}), and explain some of the properties of the non-Hopf real hypersurfaces that they give rise to. Our exposition here is based on the article [15] by Berndt and Díaz-Ramos; see also the paper [10] by Berndt and Brück.

First of all, we need to introduce here some terminology concerning real subspaces of complex vector spaces. This will be important for the construction below, but also for the rest of this thesis.

Let us denote by J the complex structure of the complex vector space $\mathbb{C}^n$. We view $\mathbb{C}^n$ as a Euclidean vector space with the scalar product given by the real part of the standard Hermitian scalar product. We define a *real subspace* of $\mathbb{C}^n$ to be an $\mathbb{R}$-linear subspace of the real vector space obtained from $\mathbb{C}^n$ by restricting the scalars to the real numbers. Sometimes we may remove the word "real" if there is no danger of confusion.

Let V be a real subspace of a complex vector space $\mathbb{C}^n$. The *Kähler angle* of a nonzero vector $v \in V$ with respect to V is defined to be the angle between Jv and V or, equivalently, the value $\varphi \in [0, \pi/2]$ such that $\langle \pi_V Jv, \pi_V Jv \rangle = \cos^2(\varphi)\langle v, v\rangle$, where π_V denotes the orthogonal projection map onto V. We say that V has *constant Kähler angle* φ if the Kähler angle of every nonzero vector $v \in V$ with respect to V is φ. In particular, V is a *complex subspace* if and only if it has constant Kähler angle 0, and it is a *totally real subspace* if and only if it has constant Kähler angle $\pi/2$. Subspaces of constant Kähler angle are completely described by the following results.

Proposition 2.15. [10, Proposition 7] *Let V be some real subspace of $\mathbb{C}^n$ with constant Kähler angle $\varphi \in (0, \pi/2)$. Then the real dimension of V is even, say $\dim V = 2k$, and there exist $2k$ $\mathbb{C}$-orthonormal vectors $e_1, \dots, e_{2k}$ in $\mathbb{C}^n$ such that*

$$e_1, \cos(\varphi)Je_1 + \sin(\varphi)Je_2, \dots, e_{2k-1}, \cos(\varphi)Je_{2k-1} + \sin(\varphi)Je_{2k}$$

is an orthonormal basis of V. Conversely, if $\varphi \in (0, \pi/2)$ and $e_1, \dots, e_{2k}$ are $\mathbb{C}$-orthonormal vectors in $\mathbb{C}^n$, then the vectors

$$e_1, \cos(\varphi)Je_1 + \sin(\varphi)Je_2, \dots, e_{2k-1}, \cos(\varphi)Je_{2k-1} + \sin(\varphi)Je_{2k}$$

span a $2k$-dimensional real subspace of $\mathbb{C}^n$ with constant Kähler angle φ.

Remark 2.16. Based on the previous proposition it is possible to show the following alternative description of subspaces with constant Kähler angle. If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are $\mathbb{C}$-orthonormal bases of $\mathbb{C}^n$, then the real subspace of $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ generated by

$$\cos(\tfrac{\varphi}{2})e_1 + \sin(\tfrac{\varphi}{2})Jf_1, \cos(\tfrac{\varphi}{2})Je_1 + \sin(\tfrac{\varphi}{2})f_1, \dots, \cos(\tfrac{\varphi}{2})e_n + \sin(\tfrac{\varphi}{2})Jf_n, \cos(\tfrac{\varphi}{2})Je_n + \sin(\tfrac{\varphi}{2})f_n$$

has constant Kähler angle $\varphi \in [0, \pi/2)$. Conversely, any subspace of constant Kähler angle $\varphi \in [0, \pi/2)$ and dimension $2n$ of $\mathbb{C}^{2n}$ can be constructed in this way.

We can now proceed with the definition of the Berndt-Brück submanifolds. Their construction relies on the root space and Iwasawa decompositions of the isometry group of $\mathbb{C}H^n$ and on the model of the complex hyperbolic space as a solvable Lie group AN with left-invariant metric. It is hence important to recall at this point the notation and results stated in §1.7.3.

Fix a point $o \in \mathbb{C}H^n$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g} = \mathfrak{su}(1,n)$ with respect to o. Let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the root space decomposition of $\mathfrak{g}$ determined by a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the corresponding Iwasawa decomposition determined by the choice $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. Recall that $\mathfrak{g}_\alpha$ is a complex vector space $\mathbb{C}^{n-1}$ with complex structure J and induced inner product $\langle \cdot, \cdot \rangle$, whereas $\mathfrak{a}$ and $\mathfrak{g}_{2\alpha}$ are 1-dimensional.

Let $\mathfrak{w}$ be a (real) subspace of the root space $\mathfrak{g}_\alpha \equiv \mathbb{C}^{n-1}$ such that its orthogonal complement $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ in $\mathfrak{g}_\alpha$ has constant Kähler angle $\varphi \in [0, \pi/2]$. As we have said above, this means that, for all nonzero $v \in \mathfrak{w}^\perp$, the angle between Jv and $\mathfrak{w}^\perp$ is φ or, equivalently, the projection of Jv onto $\mathfrak{w}^\perp$ has length $\cos(\varphi)\,\|v\|$.

Now we define $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. According to the properties of the root space decomposition, $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Let us denote by S the connected subgroup of AN with Lie algebra $\mathfrak{s}$, and set $k = \dim \mathfrak{w}^\perp$. The group S is a simply connected closed subgroup of AN of dimension $2n - k$, since $\mathrm{Exp}|_{\mathfrak{a}\oplus\mathfrak{n}} \colon \mathfrak{a} \oplus \mathfrak{n} \to AN$ is a diffeomorphism. The Berndt-Brück submanifolds are then defined as the orbits through the point o of the isometric action of S on $\mathbb{C}H^n$:

$$W_\varphi^{2n-k} = S \cdot o \qquad \text{and} \qquad W^{2n-k} = W_{\pi/2}^{2n-k}.$$

If $\varphi = 0$, then $\mathfrak{w}^\perp$ is a complex subspace of $\mathfrak{g}_\alpha$ of complex dimension $k/2$, and W_0^{2n-k} turns out to be a totally geodesic complex hyperbolic subspace $\mathbb{C}H^{n-\frac{k}{2}}$. This is a degenerate case.

If $\varphi = \pi/2$, then $\mathfrak{w}^\perp$ is a k-dimensional totally real subspace of $\mathfrak{g}_\alpha$. If $k = 1$, the corresponding hypersurface $W_{\pi/2}^{2n-1}$ will be denoted simply by W^{2n-1}; this is precisely the Lohnherr hypersurface constructed in [92] and [93]. If $k > 1$, then $W_{\pi/2}^{2n-k}$ is a $(2n-k)$-dimensional submanifold with totally real normal bundle of rank k. We will also put W^{2n-k} instead of $W_{\pi/2}^{2n-k}$.

If $0 < \varphi < \pi/2$, then k is even (see Proposition 2.15) and W_φ^{2n-k} is a $(2n-k)$-dimensional submanifold with normal bundle of constant Kähler angle φ and rank k.

Since $\mathbb{C}H^n$ is a two-point homogeneous space, the construction of the Lohnherr-Berndt-Brück submanifolds does not depend on the choice of the Iwasawa decomposition of $\mathfrak{g}$. In other words, all possible choices give rise to submanifolds which are holomorphically congruent to each other.

The Lohnherr-Berndt-Brück submanifolds arise as orbits of cohomogeneity one actions on $\mathbb{C}H^n$, as stated in Theorem 2.14 and shown in [10]. It is instructive to sketch here an idea of this fact. Let $N_K^0(S)$ be the connected component of the identity element of K of the normalizer of S in K,

$$N_K(S) = \left\{ k \in K : kSk^{-1} \subset S \right\},$$

which consists of all the elements of K that leave $S \cdot o$ invariant. Therefore, $S \cdot o$ is an orbit of the action of $N^0_K(S)S$ on $\mathbb{C}H^n$. It follows that $N^0_K(S)$ leaves invariant the unit sphere of the normal bundle of $S \cdot o$, because the tangent bundle is also invariant under $N^0_K(S)$. To conclude that the action of $N^0_K(S)S$ on $\mathbb{C}H^n$ is of cohomogeneity one, it remains to see that $N^0_K(S)$ acts transitively on the unit sphere of the normal bundle of $S \cdot o$. This can be consulted in [10], although it also follows from the investigation of polar actions on $\mathbb{C}H^n$ that we will carry out in Chapter 6. We conclude that $W^{2n-k}_\varphi = N^0_K(S)S \cdot o = S \cdot o$ is the orbit through o of the cohomogeneity one action of $N^0_K(S)S$ on $\mathbb{C}H^n$. In particular, when $k = 1$, then $\varphi = \pi/2$, and the orbits of this action generate a codimension one homogeneous regular foliation.

The maximal holomorphic subbundle of the submanifold W^{2n-k}_φ is autoparallel, and hence the leaves of the induced foliation on W^{2n-k}_φ are totally geodesic $\mathbb{C}H^{n-k}$. This implies that the Lohnherr-Berndt-Brück submanifolds are ruled. Furthermore, apart from the algebraic construction that we have described above, these submanifolds admit a more geometric description:

Proposition 2.17. [15] *Let $k \in \{1, \dots, n-1\}$, and fix a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$ and points $o \in \mathbb{C}H^{n-k}$ and $x \in \mathbb{C}H^{n-k}(\infty)$. Let KAN be the Iwasawa decomposition of $SU(1,n)$ with respect to o and x, and let H' be the subgroup of AN which acts simply transitively on $\mathbb{C}H^{n-k}$. Next, let W be a subspace of $\nu_o\mathbb{C}H^{n-k}$ with constant Kähler angle $\varphi \in (0, \pi/2]$ such that $\mathbb{C}W = \nu_o\mathbb{C}H^{n-k}$. Left translation of W by H' to all points in $\mathbb{C}H^{n-k}$ determines a subbundle $\mathfrak{V}$ of the normal bundle $\nu\mathbb{C}H^{n-k}$. At each point $p \in \mathbb{C}H^{n-k}$ attach the horocycles determined by x and the linear lines in $\mathfrak{V}_p$. The resulting subset M of $\mathbb{C}H^n$ is holomorphically congruent to the ruled submanifold W^{2n-k}.*

For the case $\varphi = \pi/2$, the geometric construction can be simplified as follows. Fix a horosphere $\mathcal{H}$ in a totally geodesic real hyperbolic subspace $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$. Attach at each point $p \in \mathcal{H}$ the totally geodesic $\mathbb{C}H^{n-k}$ which is tangent to the orthogonal complement of the complex span of the tangent space of $\mathcal{H}$ at p. The resulting submanifold is congruent to W^{2n-k}.

Let $\mathfrak{c}$ be the maximal complex subspace of the Lie algebra $\mathfrak{s}$ defining a Lohnherr-Berndt-Brück submanifold W^{2n-k}_φ. The following result codifies the extrinsic geometry of the submanifolds W^{2n-k}_φ:

Proposition 2.18. [15] *Let $c < 0$ be the constant holomorphic sectional curvature of $\mathbb{C}H^n$. Then, the second fundamental form $I\!I$ of W^{2n-k} is given by*

$$I\!I\big(aB + U + P\xi + xZ, bB + V + P\eta + yZ\big) = \sqrt{-c}\,\frac{\sin^2(\varphi)}{2}\big(y\xi + x\eta\big)$$

for all $\xi, \eta \in \mathfrak{w}^\perp$, $U, V \in \mathfrak{c} \ominus (\mathfrak{a} + \mathfrak{g}_{2\alpha})$ and $a, b, x, y \in \mathbb{R}$.

In other words, $I\!I$ is given by the trivial symmetric bilinear extension of $2I\!I(Z, P\xi) = \sqrt{-c}\sin^2(\varphi)\xi$ for all $\xi \in \mathfrak{w}^\perp$.

An important consequence of this result is the minimality of these submanifolds:

Corollary 2.19. [15] W_φ^{2n-k} *is a minimal ruled submanifold of* $\mathbb{C}H^n$.

The above equation for the second fundamental form in fact characterizes the submanifolds W_φ^{2n-k}: this is a rigidity result proved in [12] for the case $\varphi = \pi/2$ and in [15] for the general case.

Theorem 2.20. [15] *Let* M *be a* $(2n-k)$*-dimensional connected submanifold in* $\mathbb{C}H^n$, $n \geq 2$, *with normal bundle* $\nu M \subset T\mathbb{C}H^n$ *of constant Kähler angle* $\varphi \in (0, \pi/2]$. *Assume that there exists a unit vector field* Z *tangent to the maximal complex distribution on* M *such that the second fundamental form* $I\!I$ *of* M *is given by the trivial symmetric bilinear extension of*

$$2I\!I(Z, P\xi) = \sqrt{-c}\sin^2(\varphi)\,\xi \tag{2.1}$$

for all $\xi \in \nu M$, *where* $P\xi$ *is the tangential component of* $J\xi$.

Then M *is holomorphically congruent to an open part of the ruled minimal submanifold* W_φ^{2n-k}.

Finally, we end this section with Table 2.2, which summarizes some important information about the homogeneous hypersurfaces of $\mathbb{C}H^n$: the corresponding subgroup of $\mathrm{SU}(1,n)$ acting with cohomogeneity one, the principal curvatures and corresponding multiplicities of the resulting hypersurfaces, the value h of nontrivial projections of the Hopf vector field onto the distinct principal curvature spaces, and some relevant comments. All this information is basically extracted from the article [15], although there, $\mathbb{C}H^n$ is assumed to have holomorphic sectional curvature -1, whereas here this value is an arbitrary $c < 0$.

Table 2.2: Principal curvatures of homogeneous hypersurfaces in $\mathbb{C}H^n$

Group	Principal curvatures	Multiplicities	h	Geometric realization of M and comments
$S(U(1,k)U(n-k))$	$\lambda_1 = \sqrt{-c}\coth(r\sqrt{-c})$	1	1	Tube of radius $r>0$ around totally geodesic $\mathbb{C}H^k$,
	$\lambda_2 = \frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$	$2k$		$0 \leq k \leq n-1$.
	$\lambda_3 = \frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$	$2(n-k-1)$		$g=2$ if $k \in \{0, n-1\}$.
$SO^0(1,n)$	$\lambda_1 = \sqrt{-c}\tanh(r\sqrt{-c})$	1	1	Tube of radius $r>0$ around totally geodesic $\mathbb{R}H^n$.
	$\lambda_2 = \frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$	$n-1$		$\lambda_1 = \lambda_3$ if $r = \frac{1}{\sqrt{-c}}\ln(2+\sqrt{3})$.
	$\lambda_3 = \frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$	$n-1$		
N (nilpotent part of	$\lambda_1 = \sqrt{-c}$	1	1	Horosphere
Iwasawa decomposition)	$\lambda_2 = \frac{\sqrt{-c}}{2}$	$2(n-1)$		
S (Lie algebra of S:	$\lambda_1 = \sqrt{-c}\left(\frac{3}{4}\tanh(\frac{\sqrt{-c}}{2}r) + \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2(\frac{\sqrt{-c}}{2}r)}\right)$	1	2	$r=0$: Ruled minimal real hypersurface W^{2n-1}
$\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ with	$\lambda_2 = \sqrt{-c}\left(\frac{3}{4}\tanh(\frac{\sqrt{-c}}{2}r) - \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2(\frac{\sqrt{-c}}{2}r)}\right)$	1		$r>0$: Equidistant hypersurface to W^{2n-1}
$\mathfrak{w}$ hyperplane in $\mathfrak{g}_\alpha$)	$\lambda_3 = \frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$	$2n-3$		at distance r.
$N_K^0(S)S$ (Lie algebra of S:	$\lambda_1 = \sqrt{-c}\left(\frac{3}{4}\tanh(\frac{\sqrt{-c}}{2}r) + \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2(\frac{\sqrt{-c}}{2}r)}\right)$	1	2	Tube of radius $r>0$ around $W^{2n-k}_{\pi/2}$, $2 \leq k \leq n-1$.
$\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ with	$\lambda_2 = \sqrt{-c}\left(\frac{3}{4}\tanh(\frac{\sqrt{-c}}{2}r) - \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2(\frac{\sqrt{-c}}{2}r)}\right)$	1		$\lambda_2 = \lambda_4$ if $r = \frac{1}{\sqrt{-c}}\ln(2+\sqrt{3})$.
$\mathfrak{w}$ such that $\mathfrak{g}_\alpha \ominus \mathfrak{w}$ is	$\lambda_3 = \frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$	$2n-k-2$		
totally real)	$\lambda_4 = \frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$	$k-1$		
$N_K^0(S)S$ (Lie algebra of S:	Each one of the three zeroes $(\lambda_1, \lambda_2, \lambda_3)$ of the polynomial	1	3	Tube of radius $r>0$ around W^{2n-k}, $2 \leq k \leq n-1$,
$\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ with	$p(x) = -x^3 + \left(-\frac{c}{4\lambda} + 3\lambda\right)x^2 + \frac{1}{2}\left(c - 6\lambda^2\right)x$	1		k even.
$\mathfrak{w}$ such that $\mathfrak{g}_\alpha \ominus \mathfrak{w}$ has	$+ \frac{1}{32\lambda}\left(-c^2 - 16c\lambda^2 + 16\lambda^4 + (c+4\lambda^2)^2\cos(2\varphi)\right)$	1		
constant Kähler angle	$\lambda_4 = \lambda = \frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$	$2n-k-2$		
$0 < \varphi < \pi/2$)	$\lambda_5 = \frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$	$k-2$		$g=4$ if $k=2$.

For each one of the cohomogeneity one actions on $\mathbb{C}H^n$, this table shows the principal curvatures, their multiplicities and the value of h of the corresponding homogeneous hypersurfaces. The complex hyperbolic space $\mathbb{C}H^n$ is assumed to have holomorphic sectional curvature c. The Hopf vector field of each of the hypersurfaces has nontrivial projections onto the principal curvature spaces associated to the first h principal curvatures $\lambda_1, \ldots, \lambda_h$.

Chapter 3

Non-Hopf real hypersurfaces with constant principal curvatures in complex space forms

The purpose of this chapter is to obtain the following classification of real hypersurfaces in nonflat complex space forms with constant principal curvatures and whose Hopf vector field has two nontrivial projections onto the principal curvature spaces:

Theorem 3.1. *We have:*

(a) *There are no real hypersurfaces with constant principal curvatures in* $\mathbb{C}P^n$*,* $n \geq 2$*, whose Hopf vector field has* $h = 2$ *nontrivial projections onto the principal curvature spaces.*

(b) *Let* M *be a connected real hypersurface in* $\mathbb{C}H^n$*,* $n \geq 2$*, with constant principal curvatures and whose Hopf vector field has* $h = 2$ *nontrivial projections onto the principal curvature spaces of* M*. Then,* M *has* $g \in \{3, 4\}$ *principal curvatures and is holomorphically congruent to an open part of:*

   (i) *a ruled minimal real hypersurface* W^{2n-1} *or one of the equidistant hypersurfaces to* W^{2n-1}*, or*

   (ii) *a tube around a ruled minimal Berndt-Brück submanifold with totally real normal bundle* W^{2n-k}*, for some* $k \in \{2, \dots, n-1\}$*.*

In particular, M *is an open part of a homogeneous real hypersurface of* $\mathbb{C}H^n$*.*

This result constitutes the next natural step after Kimura and Berndt's classifications of Hopf (i.e. $h = 1$) real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$, see [83] and [7]. Moreover, it provides a characterization of all Lohnherr-Berndt-Brück submanifolds with totally real normal bundle. We refer the reader to Tables 3.1 and 3.2 at the end of this chapter for a visual summary of all known classification results of real hypersurfaces with constant principal curvatures in nonflat complex space forms depending on the values of g and h. The content of this chapter has been published in the papers [47] and [54].

The proof of Theorem 3.1 has several parts. First, we use the Gauss and Codazzi equations to derive some algebraic properties of the eigenvalue structure of the shape operator. The methods used for this are similar to those of [12], although a bit more general. Whenever we use a method similar to one in [12] we explicitly point it out and skip the details as much as possible. On the other hand, we focus on the new techniques and results, especially on Section 3.5. The most crucial step of the proof is to show that the number g of constant principal curvatures satisfies $g \leq 4$. For this we use a novel approach based on the study of some inequalities satisfied by the principal curvatures. Using standard Jacobi field theory one can deduce the geometry of the focal submanifolds of these hypersurfaces and then the result follows from the rigidity result in Theorem 2.20.

This chapter is organized as follows. In Section 3.1 we introduce the equations of submanifold geometry that we will use in the rest of the chapter. The proof of Theorem 3.1 is divided in several steps. Some vector fields and functions arise naturally in our proof (Sections 3.2 and 3.3). We get some of their properties in Section 3.4. In Section 3.5 we show that the number g of principal curvatures satisfies $g \in \{3, 4\}$. We summarize all the eigenvalue structure in Section 3.6. In Section 3.7 we use standard Jacobi field theory to finish the proof of the Theorem 3.1.

3.1 The equations of a real hypersurface with constant principal curvatures in a complex space form

In this section we write down the Gauss and Codazzi equations of a real hypersurface with constant principal curvatures in a complex space form.

Let $\bar{M}(c)$ be a complex space form of constant holomorphic sectional curvature $c \neq 0$ and complex dimension n. If $c > 0$ then $\bar{M}(c)$ is a complex projective space $\mathbb{C}P^n$, and if $c < 0$ then $\bar{M}(c)$ is a complex hyperbolic space $\mathbb{C}H^n$. As usual, we denote by $\langle \cdot, \cdot \rangle$ its inner product, by J its Kähler structure, by $\bar{\nabla}$ its Levi-Civita connection and by $\bar{R}$ its curvature tensor.

Let M be a connected real hypersurface of $\bar{M}(c)$. We denote by ∇ and R its Levi-Civita connection and its curvature tensor respectively. Fix $\xi \in \Gamma(\nu M)$ a (local) unit normal vector field. As usual, we denote by $\mathcal{S}$ the shape operator of M with respect to ξ.

We assume from now on that M has constant principal curvatures, that is, the eigenvalues of the shape operator $\mathcal{S}$ are constant. For each principal curvature λ of M we denote by T_λ the distribution on M formed by the principal curvature spaces of λ along M.

The Codazzi equation implies (see [12, Section 2] for a proof)

Lemma 3.2.

(i) *Let $p \in M$. If the orthogonal projection of $J\xi_p$ onto $T_\alpha(p)$ is nonzero, then $T_\alpha(p)$ is a totally real subspace of $T_p\bar{M}(c)$, that is, $JT_\alpha(p)$ is orthogonal to $T_\alpha(p)$.*

(ii) *Let $X, Y \in \Gamma(T_\alpha)$ and $Z \in \Gamma(T_\beta)$ with $\alpha \neq \beta$. Then*

$$\langle \nabla_X Y, Z\rangle = \frac{c}{4(\alpha-\beta)}\left(\langle JY, Z\rangle\langle X, J\xi\rangle + \langle JX, Y\rangle\langle Z, J\xi\rangle + 2\langle JX, Z\rangle\langle Y, J\xi\rangle\right).$$

(iii) *Let $X \in \Gamma(T_\alpha)$, $Y \in \Gamma(T_\beta)$ and $Z \in \Gamma(T_\gamma)$. Then*

$$\langle \bar{R}(X,Y)Z, \xi\rangle = (\beta-\gamma)\langle \nabla_X Y, Z\rangle - (\alpha-\gamma)\langle \nabla_Y X, Z\rangle.$$

The Gauss equation implies (again, see [12, Lemma 4] for a proof)

Lemma 3.3. *Let $X \in \Gamma(T_\alpha)$ and $Y \in \Gamma(T_\beta)$, with $\alpha \neq \beta$, be unit vector fields. Then*

$$\begin{aligned} 0 = {} & (\beta-\alpha)(-c - 4\alpha\beta - 2c\langle JX, Y\rangle^2 + 8\langle \nabla_X Y, \nabla_Y X\rangle - 4\langle \nabla_X X, \nabla_Y Y\rangle) \\ & - 4c\langle JX, Y\rangle(X\langle Y, J\xi\rangle + Y\langle X, J\xi\rangle) \\ & - c\langle X, J\xi\rangle(3Y\langle JX, Y\rangle + \langle \nabla_Y X, JY\rangle - 2\langle \nabla_X Y, JY\rangle) \\ & - c\langle Y, J\xi\rangle(3X\langle JX, Y\rangle - \langle \nabla_X Y, JX\rangle + 2\langle \nabla_Y X, JX\rangle). \end{aligned}$$

3.2 Notation and setup

Let M be a connected real hypersurface with $g > 1$ distinct constant principal curvatures in a complex space form $\bar{M}(c)$, $c \neq 0$. Since the calculations that follow are local we may assume that we have a globally defined unit normal vector field ξ. We denote by $\lambda_1, \ldots, \lambda_g$ the principal curvatures of M.

If M satisfies the conditions of Theorem 3.1, then the number of nontrivial projections of $J\xi$ onto the principal curvature distributions T_{λ_i}, $i \in \{1, \ldots, g\}$, is $h = 2$. By relabelling the indices we may also assume that $J\xi$ has nontrivial projections onto T_{λ_1} and T_{λ_2}. Hence, there exist unit vector fields $U_i \in \Gamma(T_{\lambda_i})$, $i \in \{1, 2\}$, and positive smooth functions $b_i \colon M \to \mathbb{R}$, $i \in \{1, 2\}$, such that

$$J\xi = b_1 U_1 + b_2 U_2.$$

Obviously, $b_1^2 + b_2^2 = 1$. Moreover,

Lemma 3.4. *We have $g \geq 3$, $\langle JU_1, U_2\rangle = 0$ and there exists a unit vector field $A \in \Gamma(\oplus_{k=3}^g T_{\lambda_k})$ such that*

$$\begin{aligned} JU_i &= (-1)^i b_j A - b_i \xi, \quad (i, j \in \{1, 2\}, i \neq j), \\ JA &= b_2 U_1 - b_1 U_2. \end{aligned}$$

Proof. The proof is similar to that of [12, Lemma 7], so we just sketch it. We will assume in what follows $i, j \in \{1, 2\}$, $i \neq j$, and $k \in \{3, \ldots, g\}$.

Since T_{λ_i}, $i \in \{1, 2\}$, is totally real by Lemma 3.2 (i), we can write $JU_i = \langle JU_i, U_j\rangle U_j + W_{ij} + \sum_{k=3}^g W_{ik} - b_i \xi$, where $W_{ij} \in \Gamma(T_{\lambda_j} \ominus \mathbb{R}U_j)$ and $W_{ik} \in \Gamma(T_{\lambda_k})$. (Recall that the symbol $\ominus$ is used to denote orthogonal complement.) From $J\xi = b_1 U_1 + b_2 U_2$ we get

$$-\xi = J^2\xi = b_2(\langle JU_2, U_1\rangle U_1 + W_{21}) + b_1(\langle JU_1, U_2\rangle U_2 + W_{12}) + \sum_{k=3}^g (b_1 W_{1k} + b_2 W_{2k}) - \xi.$$

Thus, $g \geq 3$, $\langle JU_1, U_2\rangle = 0$, $W_{12} = W_{21} = 0$, and $b_1 W_{1k} + b_2 W_{2k} = 0$ for all k. If we define $A \in \Gamma(\oplus_{k=3}^{g} T_{\lambda_k})$ by $\sum_{k=3}^{g} W_{ik} = (-1)^i b_j A$, then the last equality implies $\sum_{k=3}^{g} W_{jk} = (-1)^j b_i A$ (recall $i, j \in \{1, 2\}$, $i \neq j$). This gives the desired expression for JU_i, $i \in \{1, 2\}$. Finally, from $b_1^2 + b_2^2 = 1$ and $-U_1 = J(JU_1) = -b_2 JA - b_1 J\xi = -b_2 JA - U_1 + b_2^2 U_1 - b_1 b_2 U_2$ we obtain $JA = b_2 U_1 - b_1 U_2$. □

3.3 The vector field A

In view of Lemma 3.4 we may write

$$A = \sum_{k=3}^{g} A_k, \text{ with } A_k \in \Gamma(T_{\lambda_k}), k \in \{3, \ldots g\}.$$

The aim of this section is to show that all but one A_k are zero and hence we can assume for example that $A \in \Gamma(T_{\lambda_3})$ (Proposition 3.6). The main difficulty here is the fact that g is not known. We start with the following

Lemma 3.5. *Let $i, j \in \{1, 2\}$ with $i \neq j$. Then we have*

$$\nabla_{U_i} U_i = \sum_{k=3}^{g} (-1)^j \frac{3cb_1 b_2}{4(\lambda_k - \lambda_i)} A_k, \qquad \nabla_{U_i} U_j = \sum_{k=3}^{g} (-1)^j \left(\lambda_i - \frac{3cb_i^2}{4(\lambda_k - \lambda_i)}\right) A_k.$$

Proof. Again, this is quite similar to [12, Lemma 8]. We assume $i, j \in \{1, 2\}$, $i \neq j$, and $k \in \{3, \ldots, g\}$. Let $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$ and $W_k \in \Gamma(T_{\lambda_k} \ominus \mathbb{R}A_k)$.

Since U_i has unit length, $\langle \nabla_{U_i} U_i, U_i\rangle = 0$. Lemma 3.2 (ii) yields $\langle \nabla_{U_i} U_i, U_j\rangle = \langle \nabla_{U_i} U_i, W_j\rangle = \langle \nabla_{U_i} U_i, W_k\rangle = 0$ and $\langle \nabla_{U_i} U_i, A_k\rangle = 3(-1)^j cb_1 b_2/(4(\lambda_k - \lambda_i))$. From $\bar{\nabla} J = 0$, the Weingarten formula and Lemma 3.4, we obtain $\langle W_i, \bar{\nabla}_{U_i} J\xi\rangle = -\lambda_i \langle W_i, JU_i\rangle = 0$. Hence, using $J\xi = b_1 U_1 + b_2 U_2$, and Lemma 3.2 (ii), we get

$$0 = U_i \langle W_i, J\xi\rangle = \langle \nabla_{U_i} W_i, J\xi\rangle + \langle W_i, \bar{\nabla}_{U_i} J\xi\rangle = -b_i \langle \nabla_{U_i} U_i, W_i\rangle.$$

Since $b_i \neq 0$ the expression for $\nabla_{U_i} U_i$ follows.

As U_j has unit length, $\langle \nabla_{U_i} U_j, U_j\rangle = 0$. From Lemma 3.2 (ii) we obtain $\langle \nabla_{U_i} U_j, U_i\rangle = \langle \nabla_{U_i} U_j, W_i\rangle = 0$. Now, the Weingarten formula and Lemma 3.4 imply $\langle W_j, \bar{\nabla}_{U_i} J\xi\rangle = -\lambda_i \langle W_j, JU_i\rangle = 0$, and thus, Lemma 3.2 (ii), yields

$$0 = U_i \langle W_j, J\xi\rangle = \langle \nabla_{U_i} W_j, J\xi\rangle + \langle W_j, \bar{\nabla}_{U_i} J\xi\rangle = b_j \langle \nabla_{U_i} W_j, U_j\rangle.$$

This implies $\langle \nabla_{U_i} W_j, U_j\rangle = 0$. A similar calculation gives $\langle \nabla_{U_i} W_k, U_j\rangle = 0$. Finally, by Lemma 3.2 (ii) and Lemma 3.4 we have

$$\begin{aligned} 0 = U_i \langle A_k, J\xi\rangle &= \langle \nabla_{U_i} A_k, J\xi\rangle + \langle A_k, \bar{\nabla}_{U_i} J\xi\rangle \\ &= (-1)^i \frac{3cb_i^2 b_j}{4(\lambda_k - \lambda_i)} - b_j \langle \nabla_{U_i} U_j, A_k\rangle - (-1)^i \lambda_i b_j, \end{aligned}$$

from where we get $\langle \nabla_{U_i} U_j, A_k\rangle$. Altogether this yields the formula for $\nabla_{U_i} U_j$. □

Now we can prove the main result of this section.

Proposition 3.6. *$A \in \Gamma(T_{\lambda_k})$ for some $k \in \{3, \dots, g\}$.*

Proof. On the contrary, assume that there exists a point $p \in M$ and two distinct integers $r, s \in \{3, \dots, g\}$ such that $(A_r)_p, (A_s)_p \neq 0$. Hence, in a neighbourhood of p we have $A_r, A_s \neq 0$ as well. We will work in that neighbourhood from now on.

Applying Lemma 3.2 (iii) to the vector fields U_1, U_2 and A_k, $k \in \{r, s\}$, and using Lemma 3.5 we easily get

$$\frac{3c(\lambda_2 - \lambda_k)}{4(\lambda_1 - \lambda_k)} b_1^2 + \frac{3c(\lambda_1 - \lambda_k)}{4(\lambda_2 - \lambda_k)} b_2^2 = -\frac{c}{4} - \lambda_1(\lambda_2 - \lambda_k) - \lambda_2(\lambda_1 - \lambda_k), \quad k \in \{r, s\}. \tag{3.1}$$

Together with $b_1^2 + b_2^2 = 1$, this yields a linear system of three equations with unknowns b_1^2 and b_2^2. This system must be compatible. We show it is determined (that is, it has a unique solution). If it were not, the rank of the system would, at most, be one. In particular,

$$\begin{vmatrix} \frac{3c(\lambda_2-\lambda_k)}{4(\lambda_1-\lambda_k)} & \frac{3c(\lambda_1-\lambda_k)}{4(\lambda_2-\lambda_k)} \\ 1 & 1 \end{vmatrix} = 3c\frac{(\lambda_2 - \lambda_1)(\lambda_1 + \lambda_2 - 2\lambda_k)}{4(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} = 0, \quad k \in \{r, s\},$$

which implies $\lambda_1 + \lambda_2 - 2\lambda_k = 0$, $k \in \{r, s\}$, and hence $\lambda_r = \lambda_s$, contradiction. We conclude that the above system is determined. Therefore, we can find an expression for b_1^2 and b_2^2 in terms of the principal curvatures and c. Since these are constant, it follows that b_1 and b_2 are constant.

We take $i, j \in \{1, 2\}$, $i \neq j$, and $k \in \{r, s\}$. Since b_i is constant and U_i has unit length, using $J\xi = b_1 U_1 + b_2 U_2$, the Weingarten formula and Lemma 3.4 we get

$$0 = A_k(b_i) = A_k\langle U_i, J\xi\rangle = \langle \nabla_{A_k} U_i, J\xi\rangle + \langle U_i, \bar{\nabla}_{A_k} J\xi\rangle = b_j\langle \nabla_{A_k} U_i, U_j\rangle - (-1)^j b_j \lambda_k,$$

and thus, $\langle \nabla_{A_k} U_i, U_j\rangle = (-1)^j \lambda_k$. Taking this, Lemma 3.4 and Lemma 3.5 into account, Lemma 3.2 (iii) for A_k, U_1 and U_2 yields

$$\frac{c}{4}(2b_2^2 - b_1^2) = \langle \bar{R}(A_k, U_1)U_2, \xi\rangle = (\lambda_1 - \lambda_2)\lambda_k + (\lambda_k - \lambda_2)\left(\lambda_1 - \frac{3cb_1^2}{4(\lambda_k - \lambda_1)}\right),$$

for $k \in \{r, s\}$. We can rearrange this as:

$$\left(\frac{c}{4} - \frac{3c(\lambda_k - \lambda_2)}{4(\lambda_k - \lambda_1)}\right) b_1^2 - \frac{c}{2} b_2^2 = (\lambda_2 - \lambda_1)\lambda_k + \lambda_1(\lambda_2 - \lambda_k), \quad k \in \{r, s\}. \tag{3.2}$$

Hence, (3.1), (3.2) and $b_1^2 + b_2^2 = 1$ give a linear system of five equations with unknowns b_1^2 and b_2^2. This system is compatible by assumption, so it has rank two. Then, all minors of order three of the augmented matrix of the system vanish. This implies (take (3.1), (3.2) and $b_1^2 + b_2^2 = 1$, with $k \in \{r, s\}$, and then both equations in (3.2) and $b_1^2 + b_2^2 = 1$):

$$\frac{3c(\lambda_1 - \lambda_2)^2(-12\lambda_k^2 + 8\lambda_1\lambda_k + 8\lambda_2\lambda_k + c - 4\lambda_1\lambda_2)}{16(\lambda_1 - \lambda_k)(\lambda_k - \lambda_2)} = 0, \quad k \in \{r, s\}, \tag{3.3}$$

$$\frac{3c(\lambda_2 - \lambda_1)(\lambda_r - \lambda_s)(4\lambda_1^2 - 4\lambda_r\lambda_1 - 4\lambda_s\lambda_1 + c + 2\lambda_2\lambda_r + 2\lambda_2\lambda_s)}{8(\lambda_1 - \lambda_r)(\lambda_1 - \lambda_s)} = 0. \tag{3.4}$$

In particular, (3.3) implies $-12\lambda_k^2 + 8\lambda_1\lambda_k + 8\lambda_2\lambda_k + c - 4\lambda_1\lambda_2 = 0$. Putting $k = r$ and $k = s$, and subtracting, we get $4(2\lambda_1 + 2\lambda_2 - 3\lambda_r - 3\lambda_s)(\lambda_r - \lambda_s) = 0$, from where we obtain $\lambda_r + \lambda_s = 2(\lambda_1 + \lambda_2)/3$. Taking this into account, (3.4) gives $(4\lambda_1^2 - 4\lambda_1\lambda_2 + 4\lambda_2^2 + 3c)/3 = 0$. The discriminant of $-12\lambda_k^2 + 8\lambda_1\lambda_k + 8\lambda_2\lambda_k + c - 4\lambda_1\lambda_2 = 0$ as a quadratic equation in λ_k is precisely $16(4\lambda_1^2 - 4\lambda_1\lambda_2 + 4\lambda_2^2 + 3c)$, so this discriminant vanishes. As a consequence, this quadratic equation has a unique solution and hence $\lambda_r = \lambda_s$. This is a contradiction. Therefore, all but one A_k, $k \in \{3, \ldots, g\}$, are zero for each p. The result follows by continuity. □

3.4 Some properties of the principal curvature spaces

In view of Proposition 3.6, we may assume from now on that $A \in \Gamma(T_{\lambda_3})$. Moreover, we can choose an orientation on M and a relabelling of the indices so that

$$\lambda_1 < \lambda_2 \quad \text{and} \quad \lambda_3 \geq 0.$$

We will follow this convention from now on.

First we calculate some covariant derivatives.

Lemma 3.7. *Let $i, j \in \{1, 2\}$ with $i \neq j$. Then we have*

$$\nabla_{U_i} U_i = (-1)^j \frac{3cb_1b_2}{4(\lambda_3 - \lambda_i)} A, \tag{3.5}$$

$$\nabla_{U_i} U_j = (-1)^j \left(\lambda_i - \frac{3cb_i^2}{4(\lambda_3 - \lambda_i)} \right) A, \tag{3.6}$$

$$\nabla_{U_i} A = (-1)^i \frac{3cb_1b_2}{4(\lambda_3 - \lambda_i)} U_i + (-1)^i \left(\lambda_i - \frac{3cb_i^2}{4(\lambda_3 - \lambda_i)} \right) U_j, \tag{3.7}$$

$$\nabla_A U_i = \frac{(-1)^j}{\lambda_i - \lambda_j} \left(\frac{c(2b_j^2 - b_i^2)}{4} + (\lambda_j - \lambda_3) \left(\lambda_i - \frac{3cb_i^2}{4(\lambda_3 - \lambda_i)} \right) \right) U_j, \tag{3.8}$$

$$\nabla_A A = 0. \tag{3.9}$$

Proof. The proof is similar to that of [12, Lemma 8]. Equations (3.5) and (3.6) are a direct consequence of Lemma 3.5 and Proposition 3.6. Assume $i, j \in \{1, 2\}$, $i \neq j$, and $k \in \{4, \ldots, g\}$. Let $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$, $W_3 \in \Gamma(T_{\lambda_3} \ominus \mathbb{R}A)$ and $W_k \in \Gamma(T_{\lambda_k})$.

According to (3.5) and (3.6), in order to prove (3.7) we have to show $\langle \nabla_{U_i} A, A \rangle = 0$ (obvious because A is a unit vector field) and $\langle \nabla_{U_i} A, W_l \rangle = 0$ for all $l \in \{1, \ldots, g\}$. The latter follows from $\bar{\nabla} J = 0$, the Weingarten formula, Lemma 3.4 and (3.5), with

$$\begin{aligned} 0 = U_i \langle JU_i, W_l \rangle &= \langle \bar{\nabla}_{U_i} JU_i, W_l \rangle + \langle JU_i, \bar{\nabla}_{U_i} W_l \rangle \\ &= -\langle \nabla_{U_i} U_i, JW_l \rangle + (-1)^i b_j \langle A, \nabla_{U_i} W_l \rangle - b_i \langle \xi, \bar{\nabla}_{U_i} W_l \rangle = (-1)^j b_j \langle \nabla_{U_i} A, W_l \rangle. \end{aligned}$$

We now prove (3.8). Obviously, $\langle\nabla_A U_i, U_i\rangle = 0$, and by Lemma 3.2 (ii) we get $\langle\nabla_A U_i, A\rangle = 0$. Applying Lemma 3.2 (iii) to A, U_i and U_j, using Lemma 3.4 and (3.6), gives

$$\frac{c}{4}(-1)^i(b_i^2 - 2b_j^2) = (\lambda_i - \lambda_j)\langle\nabla_A U_i, U_j\rangle - (\lambda_3 - \lambda_j)(-1)^i\left(\lambda_i - \frac{3cb_i^2}{4(\lambda_3 - \lambda_i)}\right),$$

from where we get $\langle\nabla_A U_i, U_j\rangle$. For $l \in \{j, 3, \ldots, g\}$, a similar argument with Lemma 3.2 (iii) applied to A, U_i, and W_l, taking Lemma 3.4 and (3.7) into account, yields $\langle\nabla_A U_i, W_l\rangle = 0$. Finally, the previous equality (interchanging i and j and putting $l = i$) gives

$$\begin{aligned} 0 &= A\langle W_i, J\xi\rangle = \langle\nabla_A W_i, J\xi\rangle + \langle W_i, \bar{\nabla}_A J\xi\rangle \\ &= b_i\langle\nabla_A W_i, U_i\rangle + b_j\langle\nabla_A W_i, U_j\rangle - \lambda_3\langle W_i, JA\rangle = -b_i\langle\nabla_A U_i, W_i\rangle. \end{aligned}$$

Altogether this proves (3.8).

We have $\langle\nabla_A A, A\rangle = 0$, and $\langle\nabla_A A, U_i\rangle = \langle\nabla_A A, W_l\rangle = 0$ for $l \in \{1, 2, 4, \ldots, g\}$ by Lemma 3.2 (ii). From $\bar{\nabla} J = 0$, (3.8), Lemma 3.4 and the Weingarten formula we get

$$\begin{aligned} 0 &= A\langle JU_i, W_3\rangle = \langle\bar{\nabla}_A JU_i, W_3\rangle + \langle JU_i, \bar{\nabla}_A W_3\rangle \\ &= -\langle\nabla_A U_i, JW_3\rangle + (-1)^i b_j\langle A, \nabla_A W_3\rangle - b_i\langle\xi, \bar{\nabla}_A W_3\rangle = (-1)^j b_j\langle\nabla_A A, W_3\rangle. \end{aligned}$$

from where (3.9) follows. □

Our main difficulty from now on is the fact that the number g of principal curvatures is not known. In fact, the aim of Section 3.5 is to obtain a bound on g. An important step in the proof is the following

Proposition 3.8. *The functions b_1 and b_2 are constant. In fact*

$$b_i^2 = \frac{4(\lambda_j - 2\lambda_3)(\lambda_i - \lambda_3)^2}{c(\lambda_i - \lambda_j)}, \quad (i, j \in \{1, 2\}, i \neq j).$$

Moreover, $c - 4\lambda_1\lambda_2 + 8(\lambda_1 + \lambda_2)\lambda_3 - 12\lambda_3^2 = 0$.

Proof. First we show that the functions b_1 and b_2 are constant.

We apply Lemma 3.3 to U_1 and U_2, using Lemma 3.4 and Lemma 3.7,

$$\begin{aligned} 0 &= (\lambda_2 - \lambda_1)(-c - 4\lambda_1\lambda_2 + 8\langle\nabla_{U_1}U_2, \nabla_{U_2}U_1\rangle - 4\langle\nabla_{U_1}U_1, \nabla_{U_2}U_2\rangle) \\ &\quad - cb_1(\langle\nabla_{U_2}U_1, JU_2\rangle - 2\langle\nabla_{U_1}U_2, JU_2\rangle) + cb_2(\langle\nabla_{U_1}U_2, JU_1\rangle - 2\langle\nabla_{U_2}U_1, JU_1\rangle) \\ &= -\frac{3c^2}{2(\lambda_3 - \lambda_1)}b_1^4 + \frac{3c^2}{2(\lambda_3 - \lambda_2)}b_2^4 + \frac{3c^2(\lambda_1 - \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}b_1^2 b_2^2 \\ &\quad + \frac{c(6\lambda_2^2 - 7\lambda_1\lambda_2 - 2\lambda_1^2 + 2\lambda_1\lambda_3 + \lambda_2\lambda_3)}{\lambda_3 - \lambda_1}b_1^2 \\ &\quad - \frac{c(6\lambda_1^2 - 7\lambda_1\lambda_2 - 2\lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_1\lambda_3)}{\lambda_3 - \lambda_2}b_2^2 - (\lambda_2 - \lambda_1)(c + 12\lambda_1\lambda_2). \end{aligned}$$

Now we substitute b_2^2 by $1 - b_1^2$ to get

$$0 = \frac{9c^2(\lambda_2 - \lambda_1)}{2(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} b_1^4 + \Lambda_1 b_1^2 + \Lambda_0,$$

where Λ_1 and Λ_0 are constants depending on c, λ_1, λ_2 and λ_3. This equation is a quadratic equation in b_1^2 and the coefficient of b_1^4 does not vanish. Hence, it has at most two real solutions depending on the constants c, λ_1, λ_2 and λ_3. Since M is connected it follows that b_1 and b_2 are constant.

From the argument above one might derive an explicit expression for b_i, $i \in \{1,2\}$. However, that expression would involve square roots that would make later calculations difficult. Instead, we use the constancy of these functions to give an alternative formula which is easier to handle. For $i \in \{1,2\}$, using lemmas 3.4 and 3.7, and the Weingarten formula, we get

$$\begin{aligned} 0 = A(b_i) &= A\langle U_i, J\xi\rangle = \langle \nabla_A U_i, J\xi\rangle + \langle U_i, \bar{\nabla}_A J\xi\rangle = b_j\langle \nabla_A U_i, U_j\rangle - \lambda_3\langle U_i, JA\rangle \\ &= (-1)^i b_j \left(c\frac{-\lambda_i + 3\lambda_j - 2\lambda_3}{4(\lambda_i - \lambda_j)(\lambda_3 - \lambda_i)} b_i^2 - \frac{c}{2(\lambda_i - \lambda_j)} b_j^2 + \frac{2\lambda_i\lambda_3 - \lambda_j\lambda_3 - \lambda_i\lambda_j}{\lambda_i - \lambda_j} \right). \end{aligned}$$

Together with $b_1^2 + b_2^2 = 1$, this gives a linear system of three equations with unknowns b_1^2 and b_2^2. Since this system is compatible by hypothesis, its rank is two and hence the determinant of its augmented matrix is zero. This implies

$$\frac{3c}{16(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} (c - 4\lambda_1\lambda_2 + 8\lambda_3(\lambda_1 + \lambda_2) - 12\lambda_3^2) = 0.$$

Solving the above system is only a matter of linear algebra. After some calculations we get $b_i^2 = 4(\lambda_j - 2\lambda_3)(\lambda_i - \lambda_3)^2/(c(\lambda_i - \lambda_j))$ from where the result follows. ☐

We are now able to derive an important relation among λ_1, λ_2 and λ_3.

Proposition 3.9. *We have $c < 0$. In this case, we get*

$$\lambda_i = \frac{1}{2}\left(3\lambda_3 + (-1)^i \sqrt{-c - 3\lambda_3^2}\right), \quad (i,j \in \{1,2\}, i \neq j).$$

In particular, $\lambda_1 < \lambda_3 < \lambda_2$. Moreover, $c + 4\lambda_3^2 < 0$, or equivalently, $0 \leq \lambda_3 < \sqrt{-c}/2$.

Proof. Let $i,j \in \{1,2\}$ with $i \neq j$. Using Lemma 3.4, the constancy of b_i and then Lemma 3.7, we get by Lemma 3.3 applied to U_i and A

$$\begin{aligned} 0 &= (\lambda_3 - \lambda_i)(-c - 4\lambda_i\lambda_3 - 2cb_j^2 + 8\langle \nabla_{U_i} A, \nabla_A U_i\rangle) \\ &\quad - cb_i((-1)^i b_i\langle \nabla_A U_i, U_j\rangle - 2(-1)^j b_j\langle \nabla_{U_i} A, U_i\rangle - 2(-1)^i b_i\langle \nabla_{U_i} A, U_j\rangle) \\ &= \frac{c^2(\lambda_i - 15\lambda_j + 14\lambda_3)}{4(\lambda_3 - \lambda_i)(\lambda_i - \lambda_j)} b_i^4 + \frac{c^2(-10\lambda_i + 3\lambda_j + 7\lambda_3)}{2(\lambda_3 - \lambda_i)(\lambda_i - \lambda_j)} b_i^2 b_j^2 \\ &\quad - \frac{11c\lambda_i(\lambda_3 - \lambda_j)}{\lambda_i - \lambda_j} b_i^2 - \frac{2c(\lambda_3 - \lambda_i)(3\lambda_i - \lambda_j)}{\lambda_i - \lambda_j} b_j^2 \\ &\quad - \frac{(\lambda_3 - \lambda_i)(c\lambda_i - c\lambda_j + 8\lambda_i^2\lambda_j - 4\lambda_3\lambda_i\lambda_j - 4\lambda_3\lambda_i^2)}{\lambda_i - \lambda_j}. \end{aligned}$$

Now substituting b_i^2, $i \in \{1,2\}$, by the expressions given in Proposition 3.8, after multiplying by $(\lambda_j - \lambda_i)/(\lambda_i - \lambda_3)$ and some long calculations we get

$$72\lambda_3^3 - 48\lambda_i\lambda_3^2 - 108\lambda_j\lambda_3^2 + 4\lambda_i^2\lambda_3 + 32\lambda_j^2\lambda_3 + 72\lambda_i\lambda_j\lambda_3 - 16\lambda_i\lambda_j^2 - c\lambda_i - 8\lambda_i^2\lambda_j + c\lambda_j = 0.$$

Subtracting the above equation for $i = 2$ from the one with $i = 1$ we get $2(\lambda_1 - \lambda_2)(c - 4\lambda_1\lambda_2 + 14\lambda_3(\lambda_1 + \lambda_2) - 30\lambda_3^2) = 0$. Combining this with $c - 4\lambda_1\lambda_2 + 8(\lambda_1 + \lambda_2)\lambda_3 - 12\lambda_3^2 = 0$ (Proposition 3.8), we get $6\lambda_3(\lambda_1 + \lambda_2 - 3\lambda_3) = 0$. If $\lambda_3 = 0$, then the equation above gives $c(\lambda_i - \lambda_j) + 8\lambda_i^2\lambda_j + 16\lambda_i\lambda_j^2 = 0$, which combined with Proposition 3.8 yields $3c(\lambda_1 + \lambda_2) = 0$. This implies $\lambda_1 + \lambda_2 - 3\lambda_3 = \lambda_1 + \lambda_2 = 0$, so it suffices to deal with the case $\lambda_1 + \lambda_2 - 3\lambda_3 = 0$. In this situation we substitute λ_1 by $-\lambda_2 + 3\lambda_3$ in the equation in Proposition 3.8, thus obtaining $c + 4\lambda_2^2 - 12\lambda_2\lambda_3 + 12\lambda_3^2 = 0$. This is a quadratic equation with unknown λ_2 and discriminant $-c - 3\lambda_3^2$. So that this discriminant is nonnegative we already need $c < 0$, proving the first claim of this proposition. The solution to this equation is one of

$$\frac{1}{2}\left(3\lambda_3 \pm \sqrt{-c - 3\lambda_3^2}\right).$$

On the other hand, λ_1 is also one of the two values above. Since $\lambda_1 < \lambda_2$ by hypothesis, we get $c + 3\lambda_3^2 < 0$ and $\lambda_i = \frac{1}{2}\left(3\lambda_3 + (-1)^i\sqrt{-c - 3\lambda_3^2}\right)$.

Finally, we show that $0 \leq \lambda_3 < \sqrt{-c}/2$. We already know that $0 \leq \lambda_3 < \sqrt{-c/3}$. Substituting the above expression for λ_i, $i \in \{1,2\}$, in Proposition 3.8 we get

$$b_i^2 = -\frac{\left((-1)^i\lambda_3 + \sqrt{-c - 3\lambda_3^2}\right)^3}{2c\sqrt{-c - 3\lambda_3^2}}, \quad i \in \{1,2\}.$$

If $\sqrt{-c}/2 \leq \lambda_3 < \sqrt{-c/3}$, then $-c - 4\lambda_3^2 \leq 0$, and hence $-\lambda_3 + \sqrt{-c - 3\lambda_3^2} \leq 0$. This implies $b_1^2 \leq 0$, a contradiction. Therefore $0 \leq \lambda_3 < \sqrt{-c}/2$ and the result follows. ☐

Proposition 3.9 already implies that there are no hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$, $n \geq 2$, whose Hopf vector field has $h = 2$ nontrivial projections onto the principal curvature spaces. From now on we can assume $c < 0$.

Corollary 3.10. *The distribution T_{λ_k} is totally real for all $k \in \{4, \dots, g\}$.*

Proof. Let $k \in \{4, \dots, g\}$ and take unit vector fields $V_k, W_k \in \Gamma(T_{\lambda_k})$. Using the Weingarten formula, Lemma 3.2 (ii), Proposition 3.8, and $\lambda_1 + \lambda_2 - 3\lambda_3 = 0$ (by Proposition 3.9) we get

$$\begin{aligned} 0 = V_k\langle W_k, J\xi\rangle &= \langle \nabla_{V_k} W_k, b_1U_1 + b_2U_2\rangle + \langle W_k, \bar{\nabla}_{V_k} J\xi\rangle \\ &= \left(\frac{cb_1^2}{4(\lambda_k - \lambda_1)} + \frac{cb_2^2}{4(\lambda_k - \lambda_2)} - \lambda_k\right)\langle JV_k, W_k\rangle = \frac{(\lambda_3 - \lambda_k)^3}{(\lambda_k - \lambda_1)(\lambda_k - \lambda_2)}\langle JV_k, W_k\rangle. \end{aligned}$$

Since $\lambda_k \neq \lambda_3$, we get $\langle JV_k, W_k\rangle = 0$. As V_k and W_k are arbitrary, the result follows. ☐

3.5 A bound on the number of principal curvatures

In this section we show, using the Gauss equation and some inequalities involving the principal curvatures, that the number g of distinct principal curvatures satisfies $g \in \{3, 4\}$. This allows us to obtain further properties of the principal curvature spaces (see Proposition 3.14). We start with the Gauss equation.

Lemma 3.11. *Let us denote by $(\cdot)_i$, $i \in \{1, 2\}$, the orthogonal projection onto the distribution $T_{\lambda_i} \ominus \mathbb{R}U_i$, and by $(\cdot)_k$, $k \in \{4, \dots, g\}$, the orthogonal projection onto T_{λ_k}. Then we have:*

(i) *Let $i \in \{1, 2\}$ and $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$ be a unit vector field. If $j \in \{1, 2\}$ and $j \neq i$ then*

$$0 = -(c + 4\lambda_3\lambda_i) + 8\frac{\lambda_i - \lambda_j}{\lambda_3 - \lambda_j}\|(\nabla_A W_i)_j\|^2 + 8\sum_{k=4}^{g}\frac{\lambda_i - \lambda_k}{\lambda_3 - \lambda_k}\|(\nabla_A W_i)_k\|^2.$$

(ii) *Let $k \in \{4, \dots, g\}$ and $W_k \in \Gamma(T_{\lambda_k})$ be a unit vector field. Then*

$$\begin{aligned} 0 = & -(c + 4\lambda_3\lambda_k) + 8\frac{\lambda_k - \lambda_1}{\lambda_3 - \lambda_1}\|(\nabla_A W_k)_1\|^2 + 8\frac{\lambda_k - \lambda_2}{\lambda_3 - \lambda_2}\|(\nabla_A W_k)_2\|^2 \\ & + 8\sum_{l=4,\, l\neq k}^{g}\frac{\lambda_k - \lambda_l}{\lambda_3 - \lambda_l}\|(\nabla_A W_k)_l\|^2. \end{aligned}$$

Proof. As usual, let $i, j \in \{1, 2\}$ with $i \neq j$ and $k \in \{4, \dots, g\}$.

Let $W_i \in \Gamma(T_{\lambda_i}) \ominus \mathbb{R}U_i$ be a unit vector field. Applying Lemma 3.3 to W_i and A we get

$$-c - 4\lambda_3\lambda_i + 8\langle\nabla_{W_i}A, \nabla_A W_i\rangle = 0. \tag{3.10}$$

If $W_3 \in \Gamma(T_{\lambda_3})$, we get from Lemma 3.2 (ii) that $\langle\nabla_A W_i, W_3\rangle = 0$. This and Lemma 3.7 yield $\nabla_A W_i \in \Gamma((T_{\lambda_1} \ominus \mathbb{R}U_1) \oplus (T_{\lambda_2} \ominus \mathbb{R}U_2) \oplus T_{\lambda_4} \oplus \dots \oplus T_{\lambda_g})$. Similarly, Lemma 3.2 (ii) implies $\nabla_{W_i}A \in \Gamma(T_{\lambda_j} \oplus (T_{\lambda_3} \ominus \mathbb{R}A) \oplus T_{\lambda_4} \oplus \dots \oplus T_{\lambda_g})$. Hence $\langle\nabla_{W_i}A, \nabla_A W_i\rangle = \langle\nabla_{W_i}A, (\nabla_A W_i)_j\rangle + \sum_{k=4}^{g}\langle\nabla_{W_i}A, (\nabla_A W_i)_k\rangle$. For each addend of this sum we apply Lemma 3.2 (iii). Since $\langle\bar{R}(W_i, A)(\nabla_A W_i)_l, \xi\rangle = 0$ for all $l \in \{j, 4, \dots, g\}$ we get

$$\langle\nabla_{W_i}A, \nabla_A W_i\rangle = \frac{\lambda_i - \lambda_j}{\lambda_3 - \lambda_j}\langle\nabla_A W_i, (\nabla_A W_i)_j\rangle + \sum_{k=4}^{g}\frac{\lambda_i - \lambda_k}{\lambda_3 - \lambda_k}\langle\nabla_A W_i, (\nabla_A W_i)_k\rangle.$$

Now, part (i) follows by substituting the previous expression in (3.10).

Part (ii) follows in a similar way by applying Lemma 3.3 to W_k and A. □

We will use the following technical lemma several times in what follows.

Lemma 3.12. *Assume $g \geq 4$ and let $k \in \{4, \dots, g\}$. Assume that one of the following statements is true:*

(i) $\dim T_{\lambda_1} = \dim T_{\lambda_2} = 1$, *or*

(ii) $\dim T_{\lambda_1} = 1$ *and* $\lambda_k < \lambda_2$, *or*

(iii) $\lambda_1 < \lambda_k < \lambda_2$.

Then, $c + 4\lambda_3\lambda_k \geq 0$.

Proof. On the contrary, assume $c + 4\lambda_3\lambda_k < 0$. Let $W_k \in \Gamma(T_{\lambda_k})$ be a (local) unit vector field. When we apply Lemma 3.11 (ii) to W_k, any of the assumptions ensures that the first three addends of the equation given in Lemma 3.11 (ii) are nonnegative with the first one strictly positive. This already implies $g > 4$. In this case, it follows that there exists $r \in \{4, \dots, g\}$, $r \neq k$, such that $(\lambda_k - \lambda_r)/(\lambda_3 - \lambda_r) < 0$. We may choose λ_r to be the principal curvature that minimizes $|\lambda_3 - \lambda_l|$ among all λ_l, $l \in \{4, \dots, g\}$, $l \neq k$, with $(\lambda_k - \lambda_l)/(\lambda_3 - \lambda_l) < 0$. In particular we have

$$\text{either} \quad \lambda_k < \lambda_r < \lambda_3 \quad \text{or} \quad \lambda_3 < \lambda_r < \lambda_k. \tag{3.11}$$

It follows that λ_r satisfies the same assumption as λ_k: this is obvious for (i) and a consequence of (3.11) and $\lambda_1 < \lambda_3 < \lambda_2$ for (ii) and (iii). Using (3.11), $\lambda_3 \geq 0$, $c + 4\lambda_3^2 < 0$ (Proposition 3.9) and $c + 4\lambda_3\lambda_k < 0$, we also get $c + 4\lambda_3\lambda_r \leq c + 4\lambda_3 \max\{\lambda_3, \lambda_k\} < 0$. Thus we may apply Lemma 3.11 (ii) to a unit vector field $W_r \in \Gamma(T_{\lambda_r})$, from where it follows, as before, that there exists $s \in \{4, \dots, g\}$, $s \neq r$, such that $(\lambda_r - \lambda_s)/(\lambda_3 - \lambda_s) < 0$. This implies either $\lambda_r < \lambda_s < \lambda_3$ or $\lambda_3 < \lambda_s < \lambda_r$, and taking (3.11) into account we easily obtain

$$\text{either} \quad \lambda_k < \lambda_r < \lambda_s < \lambda_3 \quad \text{or} \quad \lambda_3 < \lambda_s < \lambda_r < \lambda_k. \tag{3.12}$$

In both cases (3.12) yields $s \neq k$, $(\lambda_k - \lambda_s)/(\lambda_3 - \lambda_s) < 0$ and $|\lambda_3 - \lambda_s| < |\lambda_3 - \lambda_r|$. This contradicts the definition of λ_r. Therefore, $c + 4\lambda_3\lambda_k \geq 0$. ☐

From the previous lemma we easily derive the first important consequence.

Proposition 3.13. *We have* $\dim T_{\lambda_1} = 1$.

Proof. On the contrary, assume $\dim T_{\lambda_1} > 1$ and let $W_1 \in \Gamma(T_{\lambda_1} \ominus \mathbb{R}U_1)$ be a (local) unit vector field. Since $c + 4\lambda_1\lambda_3 \leq c + 4\lambda_3^2 < 0$ by Proposition 3.9, from Lemma 3.11 (i) we deduce the existence of $k \in \{4, \dots, g\}$ such that $(\lambda_1 - \lambda_k)/(\lambda_3 - \lambda_k) < 0$. Since $\lambda_1 < \lambda_3$ we get $\lambda_1 < \lambda_k < \lambda_3 < \lambda_2$ and hence Lemma 3.12 (iii) yields $c + 4\lambda_3\lambda_k \geq 0$. This contradicts $c + 4\lambda_3\lambda_k \leq c + 4\lambda_3^2 < 0$. Therefore we have $\dim T_{\lambda_1} = 1$. ☐

This is the most crucial step of the proof.

Proposition 3.14. *We have*

(i) $g \in \{3, 4\}$.

(ii) *If* $g = 3$ *and* $\dim T_{\lambda_2} > 1$ *then* $\lambda_1 = 0$, $\lambda_2 = \frac{\sqrt{-3c}}{2}$ *and* $\lambda_3 = \frac{\sqrt{-c}}{2\sqrt{3}}$.

(iii) *If* $g = 4$ *then* $\dim T_{\lambda_2} = 1$, $0 \neq \lambda_3 \neq \frac{\sqrt{-c}}{2\sqrt{3}}$ *and* $\lambda_4 = -\frac{c}{4\lambda_3}$.

Proof. If $g = 3$ and $\dim T_{\lambda_2} > 1$, take a (local) unit vector field $W_2 \in \Gamma(T_{\lambda_2} \ominus \mathbb{R}U_2)$ and apply Lemma 3.11 (i). Note that the last two addends vanish since $\dim T_{\lambda_1} = 1$ and $g = 3$. Then, $c + 4\lambda_2\lambda_3 = 0$, and from Proposition 3.9 we get $\lambda_1 = 0$, $\lambda_2 = \sqrt{-3c}/2$ and $\lambda_3 = \sqrt{-c}/(2\sqrt{3})$. This implies (ii).

Assume $g \geq 4$. We first have $\lambda_3 < \lambda_k$ for all $k \in \{4, \dots, g\}$; otherwise, if $\lambda_k < \lambda_3 < \lambda_2$ we would get $c + 4\lambda_3\lambda_k \leq c + 4\lambda_3^2 < 0$ contradicting Lemma 3.12 (ii) (by Proposition 3.13).

We show that $\dim T_{\lambda_2} = 1$. On the contrary, assume $\dim T_{\lambda_2} > 1$ and let $W_2 \in \Gamma(T_{\lambda_2} \ominus \mathbb{R}U_2)$ be a (local) unit vector field. If $c + 4\lambda_2\lambda_3 < 0$, then Lemma 3.11 (i) applied to W_2 (and taking Proposition 3.13 into account) implies that there exists $k \in \{4, \dots, g\}$ such that $(\lambda_2 - \lambda_k)/(\lambda_3 - \lambda_k) < 0$. Then, $\lambda_3 < \lambda_k < \lambda_2$, and thus $c + 4\lambda_3\lambda_k \leq c + 4\lambda_3\lambda_2 < 0$, which contradicts Lemma 3.12 (ii). Hence we can assume from now on that $c + 4\lambda_2\lambda_3 \geq 0$. This inequality does not hold if $\lambda_3 = 0$ so we already get $\lambda_3 > 0$.

We claim that there exists $r \in \{4, \dots, g\}$ such that $\lambda_2 < \lambda_r$. If $c + 4\lambda_2\lambda_3 = 0$, then the assertion is true for all $k \geq 4$; otherwise, if $\lambda_k < \lambda_2$, we would get $c + 4\lambda_3\lambda_k < c + 4\lambda_3\lambda_2 = 0$, contradicting Lemma 3.12 (ii). Hence, we have to prove our claim for the case $c + 4\lambda_2\lambda_3 > 0$. In this case we apply Lemma 3.11 (i) to W_2. Then, there exists $r \in \{4, \dots, g\}$ such that $(\lambda_2 - \lambda_r)/(\lambda_3 - \lambda_r) > 0$. Since $\lambda_3 < \lambda_r$ this implies $\lambda_2 < \lambda_r$ as claimed.

In any case, there exists $r \in \{4, \dots, g\}$ such that $\lambda_2 < \lambda_r$. In fact, we may assume that λ_r is the largest principal curvature. Now, we have $c + 4\lambda_3\lambda_r > c + 4\lambda_3\lambda_2 \geq 0$, and hence Lemma 3.11 (ii) applied to a (local) unit vector field $W_r \in \Gamma(T_{\lambda_r})$ implies the existence of $l \in \{4, \dots, g\}$, $l \neq r$, such that $(\lambda_r - \lambda_l)/(\lambda_3 - \lambda_l) > 0$. Since $\lambda_3 < \lambda_l$, we get $\lambda_r < \lambda_l$ which contradicts the fact that λ_r is the largest principal curvature. Altogether this implies $\dim T_{\lambda_2} = 1$.

From Lemma 3.12 (i) we obtain $c + 4\lambda_3\lambda_k \geq 0$ for all $k \geq 4$. In particular this implies $\lambda_3 > 0$. Assume that for some $r \in \{4, \dots, g\}$ we have strict inequality $c + 4\lambda_3\lambda_r > 0$ and let λ_r, $r \in \{4, \dots, g\}$, be the largest principal curvature satisfying this condition. Applying Lemma 3.11 (ii) once more to a (local) unit vector field $W_r \in \Gamma(T_{\lambda_r})$ (note that the second addend now vanishes) yields the existence of $l \in \{4, \dots, g\}$, $l \neq r$, such that $(\lambda_r - \lambda_l)/(\lambda_3 - \lambda_l) > 0$. Since $\lambda_3 < \lambda_l$ we get $\lambda_r < \lambda_l$. Obviously, $c + 4\lambda_3\lambda_l > c + 4\lambda_3\lambda_r > 0$, which contradicts the fact that λ_r is the largest principal curvature satisfying this condition.

As a consequence, $c + 4\lambda_3\lambda_k = 0$ for all $k \geq 4$. Since $\lambda_3 \neq 0$ and the principal curvatures are different, this immediately implies $g = 4$ and $\lambda_4 = -c/(4\lambda_3)$. Eventually, this also yields $c + 4\lambda_3\lambda_2 \neq 0$ and thus, by Proposition 3.9, $\lambda_3 \neq \sqrt{-c}/(2\sqrt{3})$ (otherwise the principal curvatures would not be different). This concludes the proof of (i) and (iii). ☐

Part (ii) of Proposition 3.14 had already been obtained in [12] by different methods. We have included a proof here as it is almost effortless to do so.

3.6 The eigenvalue structure of the shape operator

We summarize the results obtained so far:

Theorem 3.15. *We have:*

(a) *There are no real hypersurfaces with constant principal curvatures in* $\mathbb{C}P^n$, $n \geq 2$, *whose Hopf vector field has* $h = 2$ *nontrivial projections onto the principal curvature spaces.*

(b) *Let* M *be a connected real hypersurface with* g *distinct constant principal curvatures* $\lambda_1, \dots, \lambda_g$ *in* $\mathbb{C}H^n$, $n \geq 2$, *such that the number of nontrivial projections of its Hopf vector field* $J\xi$ *onto the principal curvature spaces of* M *is* $h = 2$. *Then,* $g \in \{3, 4\}$ *and, with a suitable labeling of the principal curvatures and a suitable choice of the normal vector field* ξ, *we have:*

   (i) *The Hopf vector field can be written as* $J\xi = b_1 U_1 + b_2 U_2$, *where* $U_i \in \Gamma(T_{\lambda_i})$, $i \in \{1, 2\}$, *are unit vector fields, and* b_1 *and* b_2 *are positive constants satisfying*

$$b_i^2 = \frac{4(\lambda_j - 2\lambda_3)(\lambda_i - \lambda_3)^2}{c(\lambda_i - \lambda_j)}, \quad (i, j \in \{1, 2\}, i \neq j).$$

   (ii) *There exists a unit vector field* $A \in \Gamma(T_{\lambda_3})$ *such that*

$$JU_i = (-1)^i b_j A - b_i \xi, \quad (i, j \in \{1, 2\}, i \neq j), \quad \text{and} \quad JA = b_2 U_1 - b_1 U_2.$$

   (iii) *We have* $0 \leq \lambda_3 < \frac{1}{2}\sqrt{-c}$ *and*

$$\lambda_i = \frac{1}{2}\left(3\lambda_3 + (-1)^i \sqrt{-c - 3\lambda_3^2}\right), \quad (i, j \in \{1, 2\}, i \neq j).$$

   (iv) $\dim T_{\lambda_1} = 1$.

   (v) *If* $g = 4$ *then* $\dim T_{\lambda_2} = 1$. *We define* $k = \dim T_{\lambda_4} + 1$, *and thus,* $k \in \{2, \dots, n-1\}$. *The distribution* T_{λ_4} *is totally real with* $JT_{\lambda_4} \subset T_{\lambda_3} \ominus \mathbb{R}A$,

$$0 \neq \lambda_3 \neq \frac{\sqrt{-c}}{2\sqrt{3}}, \quad \text{and} \quad \lambda_4 = -\frac{c}{4\lambda_3}.$$

   (vi) *If* $g = 3$ *there are two possibilities:*

   (A) $\dim T_{\lambda_2} = 1$; *in this case we define* $k = 1$.

   (B) $\dim T_{\lambda_2} > 1$; *in this case we define* $k = \dim T_{\lambda_2} \in \{2, \dots, n-1\}$ *and we have that* $T_{\lambda_2} \ominus \mathbb{R}U_2$ *is a totally real distribution with* $J(T_{\lambda_2} \ominus \mathbb{R}U_2) \subset T_{\lambda_3} \ominus \mathbb{R}A$ *and*

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\sqrt{-3c}}{2}, \quad \lambda_3 = \frac{\sqrt{-c}}{2\sqrt{3}}.$$

Remark 3.16. Part (a) of Theorem 3.15 already provides a proof for part (a) of the Theorem 3.1.

We know that $\mathbb{R}U_1 \oplus \mathbb{R}U_2 \oplus \mathbb{R}A \oplus \mathbb{R}\xi$ is a complex subbundle on M by Lemma 3.4. Thus, in part (bv) of Theorem 3.15, the fact that T_{λ_4} is totally real (Corollary 3.10) implies $JT_{\lambda_4} \subset T_{\lambda_3} \ominus \mathbb{R}A$ as claimed. Similarly, in Theorem 3.15 b(vi)B, the assertion $J(T_{\lambda_2} \ominus \mathbb{R}U_2) \subset T_{\lambda_3} \ominus \mathbb{R}A$ follows from the fact that T_{λ_2} is totally real by Lemma 3.2 (i).

The definition of k above might seem a bit artificial at the moment, but it will be useful in the next section where we conclude the proof of Theorem 3.1 ($k-1$ will be the dimension of the kernel of the differential of the map $\Phi^r\colon M \to \mathbb{C}H^n$, $p \mapsto \exp_p(r\xi_p)$).

If we examine the proof of our theorem, so far we have actually shown that for any point $p \in M$ there exists a neighbourhood of p where the conclusion of Theorem 3.15 is satisfied. However, by the connectedness of M and a continuity argument, it can be easily shown that M is orientable and that the conclusion of Theorem 3.15 is satisfied globally.

3.7 Jacobi field theory and rigidity of focal submanifolds

In this last section we finish the proof of part (b) of the Theorem 3.1. Since we use standard Jacobi field theory, we provide the reader just with the fundamental details and skip the long calculations. According to [12] we just have to take care of the case $g = 4$. However, it is not much overload to deal with the two cases simultaneously, so for the sake of completeness we will do so in what follows.

Let M be a real hypersurface of $\mathbb{C}H^n$ in the conditions of Theorem 3.15 (b). For $r \in \mathbb{R}$ we define the map $\Phi^r\colon M \to \mathbb{C}H^n$, $p \mapsto \exp_p(r\xi_p)$, where $\exp_p$ is the Riemannian exponential map of $\mathbb{C}H^n$ at p. Then, $\Phi^r(M)$ is obtained by moving M a distance r along its normal direction. The singularities of Φ^r are the focal points of M. We will find a particular distance r for which Φ^r_* has constant rank, where Φ^r_* denotes the differential of Φ^r. Then we will apply Theorem 2.20 to $\Phi^r(M)$ for this choice of r. This way, $\Phi^r(M)$ will be an open part of the ruled minimal Berndt-Brück submanifold W^{2n-k}, $k \in \{1, \dots, n-1\}$, and hence M will be an open part of a tube around this ruled minimal submanifold W^{2n-k}. (If $k = 1$ then M will be an equidistant hypersurface to the ruled minimal hypersurface W^{2n-1} at distance r.)

Let $p \in M$ and denote by γ_p the geodesic determined by the initial conditions $\gamma_p(0) = p$ and $\dot\gamma_p(0) = \xi_p$. For any $v \in T_pM$ let B_v be the parallel vector field along the geodesic γ_p such that $B_v(0) = v$, and let ζ_v be the Jacobi field along γ_p with initial conditions $\zeta_v(0) = v$ and $\zeta'(0) = -\mathcal{S}_p v$. Here $'$ denotes covariant derivative along γ_p. Since ζ_v is a solution to the differential equation $4\zeta_v'' + c\zeta_v + 3c\langle\zeta_v, J\dot\gamma_p\rangle J\dot\gamma_p = 0$, if $v \in T_{\lambda_i}(p)$ then

$$\zeta_v(t) = f_i(t)B_v(t) + \langle v, J\xi\rangle g_i(t) J\dot\gamma_p(t),$$

where

$$f_i(t) = \cosh\left(\frac{t\sqrt{-c}}{2}\right) - \frac{2\lambda_i}{\sqrt{-c}}\sinh\left(\frac{t\sqrt{-c}}{2}\right),$$
$$g_i(t) = \left(\cosh\left(\frac{t\sqrt{-c}}{2}\right) - 1\right)\left(1 + 2\cosh\left(\frac{t\sqrt{-c}}{2}\right) - \frac{2\lambda_i}{\sqrt{-c}}\sinh\left(\frac{t\sqrt{-c}}{2}\right)\right).$$

We also define the smooth vector field η^r along Φ^r by $\eta^r_p = \dot{\gamma}_p(r)$. It is known that $\zeta_v(r) = \Phi^r_* v$ and $\zeta'_v(r) = \bar{\nabla}_{\Phi^r_* v}\eta^r$.

We now determine the value of r. Since $0 \leq \lambda_3 < \sqrt{-c}/2$ we can find a real number $r \geq 0$ such that

$$\lambda_3 = \frac{\sqrt{-c}}{2}\tanh\left(\frac{r\sqrt{-c}}{2}\right).$$

Let $p \in M$. We define $u_i = (U_i)_p$, $i \in \{1, 2\}$. Let $v_2 \in T_{\lambda_2}(p) \ominus \mathbb{R}u_2$ and $v_k \in T_{\lambda_k}(p)$ for $3 \leq k \leq g$ (whenever these spaces are nontrivial). The explicit solution to the Jacobi equation above implies

$$(\Phi^r_* u_1, \Phi^r_* u_2) = (B_{u_1}(r), B_{u_2}(r))D(r),$$

$$\Phi^r_* v_2 = 0, \qquad \Phi^r_* v_3 = \operatorname{sech}\left(\frac{r\sqrt{-c}}{2}\right) B_{v_3}(r), \qquad \Phi^r_* v_4 = 0,$$

where

$$D(t) = \begin{pmatrix} f_1(t) + b_1^2 g_1(t) & b_1 b_2 g_2(t) \\ b_1 b_2 g_1(t) & f_2(t) + b_2^2 g_2(t) \end{pmatrix}.$$

Since $\det(D(r)) = \operatorname{sech}^3\left(r\sqrt{-c}/2\right)$ we conclude that Φ^r_* has constant rank $2n - k$ (see Theorem 3.15 (bv)-(bvi) for the definition of k). Then, for each point $p \in M$ there exists an open neighbourhood $\mathcal{V}$ of p such that $\mathcal{W} = \Phi^r(\mathcal{V})$ is an embedded submanifold of $\mathbb{C}H^n$ and $\Phi^r \colon \mathcal{V} \to \mathcal{W}$ is a submersion. (If $k = 1$, then Φ^r is actually a local diffeomorphism.)

Let $q = \Phi^r(p) \in \mathcal{W}$. The expression above for Φ^r_* shows that the tangent space $T_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $\mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus T_{\lambda_3}(p)$ along the geodesic γ_p from $p = \gamma_p(0)$ to $q = \gamma_p(r)$. Therefore, the normal space $\nu_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $(\ker \Phi^r_{*p}) \oplus \mathbb{R}\xi_p$ along γ_p from $p = \gamma_p(0)$ to $q = \gamma_p(r)$. The latter is $(T_{\lambda_2} \ominus \mathbb{R}u_2) \oplus \mathbb{R}\xi_p$ if $g = 3$ (see Theorem 3.15 (bvi)), or $T_{\lambda_4}(p) \oplus \mathbb{R}\xi_p$ if $g = 4$ (see Theorem 3.15 (bv)). In any case, by Theorem 3.15 (bv)-(bvi) it follows that $\mathcal{W}$ has totally real normal bundle of rank k.

We have that $\eta^r_p = B_{\xi_p}(r)$ is a unit normal vector of $\mathcal{W}$ at q. If $\mathcal{S}^r$ denotes the shape operator of $\mathcal{W}$, then it is known that $\mathcal{S}^r_{\eta^r_p}\Phi^r_* v = -(\zeta'_v(r))^\top$, where $(\cdot)^\top$ denotes orthogonal projection onto the tangent space of $\mathcal{W}$. Using the explicit expression for ζ_v above, we get

$$(\mathcal{S}^r_{\eta^r_p} B_{u_1}(r), \mathcal{S}^r_{\eta^r_p} B_{u_2}(r)) = (B_{u_1}(r), B_{u_2}(r))C(r), \quad \text{and}$$
$$\mathcal{S}^r_{\eta^r_p} B_{v_3}(r) = 0 \text{ for all } v_3 \in T_{\lambda_3}(p),$$

where $C(r) = -D'(r)D(r)^{-1}$. A lengthy and tedious calculation shows that

$$C(r) = \frac{\sqrt{-c}}{2}\begin{pmatrix} -2b_1b_2 & b_1^2 - b_2^2 \\ b_1^2 - b_2^2 & 2b_1b_2 \end{pmatrix}.$$

Since $J\eta_p^r = B_{J\xi_p}(r) = b_1B_{u_1}(r) + b_2B_{u_2}(r)$, and $B_{JA_p}(r) = b_2B_{u_1}(r) - b_1B_{u_2}(r)$, the above expression for $C(r)$ implies

$$\mathcal{S}^r_{\eta_p^r}B_{JA_p}(r) = -\frac{\sqrt{-c}}{2}J\eta_p^r, \qquad \mathcal{S}^r_{\eta_p^r}J\eta_p^r = -\frac{\sqrt{-c}}{2}B_{JA_p}(r),$$

and $\mathcal{S}^r_{\eta_p^r}$ vanishes on the orthogonal complement of $\mathbb{R}J\eta_p^r \oplus \mathbb{R}B_{JA_p}(r)$ in $T_q\mathcal{W}$.

We have that $J(\nu_q\mathcal{W} \ominus \mathbb{R}\eta_p^r)$ is contained in the parallel translation along γ_p of $T_{\lambda_3}(p)$. This follows from Theorem 3.15 (bv)-(bvi) and the fact that $\nu_q\mathcal{W} \ominus \mathbb{R}\eta_p^r$ is the parallel translation along γ_p from $\gamma_p(0) = p$ to $\gamma_p(r) = q$ of $T_{\lambda_2}(p) \ominus \mathbb{R}u_2$ if $g = 3$, and of $T_{\lambda_4}(p)$ if $g = 4$. The linearity of $\mathcal{S}^r_{\eta_r^p}$ implies

$$\mathcal{S}^r_{\eta_p^r}J\tilde{\eta} = -\frac{\sqrt{-c}}{2}\langle \eta_p^r, \tilde{\eta}\rangle B_{JA_p}(r), \text{ for all } \tilde{\eta} \in \nu_q\mathcal{W}. \tag{3.13}$$

From the Gauss formula and $\bar{\nabla}J = 0$ we have that for each $X \in T_q\mathcal{W}$,

$$\langle \mathcal{S}^r_{\tilde{\eta}}J\eta_p^r, X\rangle = \langle II(X, J\eta_p^r), \tilde{\eta}\rangle = \langle \bar{\nabla}_X J\eta_p^r, \tilde{\eta}\rangle = \langle \eta_p^r, \bar{\nabla}_X J\tilde{\eta}\rangle = \langle \eta_p^r, II(X, J\tilde{\eta})\rangle = \langle \mathcal{S}^r_{\eta_p^r}J\tilde{\eta}, X\rangle,$$

from where it follows that $\mathcal{S}^r_{\tilde{\eta}}J\eta_p^r = \mathcal{S}^r_{\eta_p^r}J\tilde{\eta}$, and hence, $\mathcal{S}^r_{\tilde{\eta}}J\eta_p^r = 0$ for all $\tilde{\eta} \in \nu_q\mathcal{W} \ominus \mathbb{R}\eta_p^r$. Let α be a curve in $(\Phi^r)^{-1}(\{q\}) \cap \mathcal{V}$ with $\alpha(0) = p$. Since η_p^r and $\eta^r_{\alpha(t)} - \langle \eta^r_{\alpha(t)}, \eta_p^r\rangle\eta_p^r$ are perpendicular, $\mathcal{S}^r_{\tilde{\eta}}J\eta_p^r = 0$ and the linearity of $\eta \mapsto \mathcal{S}^r_\eta$ imply

$$0 = \mathcal{S}^r_{\eta^r_{\alpha(t)} - \langle \eta^r_{\alpha(t)}, \eta_p^r\rangle\eta_p^r}J\eta_p^r = \mathcal{S}^r_{\eta^r_{\alpha(t)}}J\eta_p^r + \frac{\sqrt{-c}}{2}\langle \eta^r_{\alpha(t)}, \eta_p^r\rangle B_{JA_p}(r),$$

which together with (3.13) (with $\alpha(t)$ instead of p) yields

$$-\frac{\sqrt{-c}}{2}\langle \eta^r_{\alpha(t)}, \eta_p^r\rangle B_{JA_p}(r) = \mathcal{S}^r_{\eta^r_{\alpha(t)}}J\eta_p^r = -\frac{\sqrt{-c}}{2}\langle \eta^r_{\alpha(t)}, \eta_p^r\rangle B_{JA_{\alpha(t)}}(r).$$

Since α is arbitrary we get that the map $\tilde{p} \mapsto B_{JA_{\tilde{p}}}(r)$ is constant in the connected component $\mathcal{V}_0$ of $(\Phi^r)^{-1}(\{q\}) \cap \mathcal{V}$ containing p. Thus it makes sense to define the unit vector $z = -B_{JA_{\tilde{p}}}(r) \in T_q\mathcal{W}$ for any $\tilde{p} \in \mathcal{V}_0$.

We may consider η^r as a map from $\mathcal{V}_0$ to the unit sphere of $\nu_q\mathcal{W}$. The tangent space of $\mathcal{V}_0$ at p is given by the kernel of Φ^r_{*p}. If $v \in \ker \Phi^r_{*p}$, then $\eta^r_{*p}v = \zeta_v'(r)$. If $g = 3$, then $v \in \ker \Phi^r_{*p} = T_{\lambda_2}(p) \ominus \mathbb{R}u_2$ and $\eta^r_{*p}v = -\sqrt{-c/2}\, B_v(r)$. If $g = 4$, then $v \in \ker \Phi^r_{*p} = T_{\lambda_4}(p)$ and $\eta^r_{*p}v = -\operatorname{csch}(r\sqrt{-c}/2)B_v(r)$. In any case, we get that η^r is a local diffeomorphism from $\mathcal{V}_0$ into the unit sphere of $\nu_q\mathcal{W}$ (note that this is trivial if $g = 3$ and $k = 1$). Hence,

$\eta^r(\mathcal{V}_0)$ is an open subset of the unit sphere of $\nu_q\mathcal{W}$. But since $\eta \mapsto \mathcal{S}^r_\eta$ depends analytically on η we conclude

$$\mathcal{S}^r_\eta J\eta = \frac{\sqrt{-c}}{2} z, \qquad \mathcal{S}^r_\eta z = \frac{\sqrt{-c}}{2} J\eta, \qquad \mathcal{S}^r_\eta v = 0,$$

for all unit $\eta \in \nu_q\mathcal{W}$ and $v \in T_q\mathcal{W} \ominus (\mathbb{R}J\eta \oplus \mathbb{R}z)$. Therefore, the second fundamental form $I\!I^r$ of $\mathcal{W}$ at q is given by the trivial symmetric bilinear extension of $I\!I^r(z, J\eta) = (\sqrt{-c}/2)\eta$ for all $\eta \in \nu_q\mathcal{W}$. By construction, z depends smoothly on the point $q \in \mathcal{W}$ and hence gives rise to a vector field Z which is tangent to the maximal holomorphic distribution of $\mathcal{W}$. The relation $\mathcal{S}^r_\eta J\eta = (\sqrt{-c}/2)Z$ ensures that Z can actually be defined on $\Phi^r(M)$, and hence, the second fundamental form of $\Phi^r(M)$ is given by the trivial symmetric bilinear extension of $I\!I^r(Z, J\eta) = (\sqrt{-c}/2)\eta$ for all $\eta \in \Gamma(\nu\,\Phi^r(M))$. Since $\Phi^r(M)$ has totally real normal bundle of rank k we conclude from Theorem 2.20 and the remark that follows, that $\Phi^r(M)$ is holomorphically congruent to an open part of the ruled minimal Berndt-Brück submanifold W^{2n-k}. This readily implies that M is an open part of a tube (an equidistant hypersurface if $g = 3$ and $k = 1$) of radius r around the ruled minimal Berndt-Brück submanifold W^{2n-k}.

Finally, let us point out that if $g = 3$ and $\lambda_3 = 0$, then $r = 0$ and M is an open part of the ruled minimal hypersurface W^{2n-1}. Also, if $g = 3$ and $k > 1$ then $\lambda_3 = \sqrt{-c}/(2\sqrt{3})$ according to Theorem 3.15 b(vi)B, and hence $r = (1/\sqrt{-c})\log(2+\sqrt{3})$. The tube around the ruled minimal submanifold W^{2n-k}, $k > 1$, of radius $r = (1/\sqrt{-c})\log(2+\sqrt{3})$ has $g = 3$ principal curvatures whereas if $r \neq (1/\sqrt{-c})\log(2+\sqrt{3})$ the tube of radius r around the ruled minimal submanifold W^{2n-k}, $k > 1$, has $g = 4$ principal curvatures. This finishes the proof of the Theorem 3.1.

Table 3.1: Real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$

	h = 1 Kimura [83]	Hermitian symmetric space of rank 2	**h = 2** Theorem 3.1	**h ≥ 3**
g = 1 Tachibana, Tashiro [139]	Impossible			
g = 2 Takagi [132]	Geodesic hypersphere	$\mathbb{C}P^1 \times \mathbb{C}P^n$	Impossible	
g = 3 Takagi [133] Wang [151]	Tube around a totally geodesic $\mathbb{C}P^k$, $1 \leq k \leq n-2$ Tube around a totally geodesic $\mathbb{R}P^n$	$\mathbb{C}P^{k+1} \times \mathbb{C}P^{n-k}$ $\mathrm{Gr}_2^+(\mathbb{R}^{n+3})$	Impossible	?
g = 4	Impossible		Impossible	?
g = 5	Tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ in $\mathbb{C}P^{2k+1}$, $k \geq 2$ Tube around the Plücker embedding of the complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^5)$ in $\mathbb{C}P^9$ Tube around the half spin embedding of SO(10)/U(5) in $\mathbb{C}P^{15}$	$\mathrm{Gr}_2(\mathbb{C}^{k+3})$ SO(10)/U(5) $E_6/(U(1) \cdot \mathrm{Spin}(10))$	Impossible	?
g ≥ 6	Impossible		Impossible	?

1. In Tables 3.1 and 3.2 all known classification results and examples of real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ are shown, up to holomorphic congruence.
2. A shaded cell just means that $h > g$ is impossible.
3. When for a particular case of g and h there are some known examples, but a classification is missing, we write *Not yet classified.* If neither any example nor a classification is known, we write a question mark ?.
4. For each homogeneous hypersurface in $\mathbb{C}P^n$, we indicate the associated Hermitian symmetric space of rank 2 whose isotropy representation give rise to that homogeneous hypersurface, via the projection of a principal orbit by the Hopf fibration.

Table 3.2: Real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$

	h = 1 Berndt [7]	**h = 2** Theorem 3.1	**h = 3**	**h ≥ 4**
g = 1 Tachibana, Tashiro [139]	Impossible			
g = 2 Montiel [104] Berndt, Díaz-Ramos [14]	Horosphere Geodesic hypersphere Tube around a totally geodesic $\mathbb{C}H^{n-1}$ Tube of radius $\frac{1}{\sqrt{-c}} \ln\left(2+\sqrt{3}\right)$ around a totally geodesic $\mathbb{R}H^n$	Impossible		
g = 3 Berndt, Díaz-Ramos [12], [14]	Tube around a totally geodesic $\mathbb{C}H^k$, $1 \leq k \leq n-2$ Tube of radius $r \neq \frac{1}{\sqrt{-c}} \ln\left(2+\sqrt{3}\right)$ around a totally geodesic $\mathbb{R}H^n$	Hypersurface W^{2n-1} and its equidistant hypersurfaces Tube of radius $\frac{1}{\sqrt{-c}} \ln\left(2+\sqrt{3}\right)$ around a submanifold W^{2n-k}, $2 \leq k \leq n-1$	Impossible	
g = 4	Impossible	Tube of radius different from $\frac{1}{\sqrt{-c}} \ln\left(2+\sqrt{3}\right)$ around a submanifold W^{2n-k}, $2 \leq k \leq n-1$	Tube around a submanifold W_φ^{2n-2}, $0 < \varphi < \frac{\pi}{2}$ *Not yet classified*	?
g = 5	Impossible	Impossible	Tube around a submanifold W_φ^{2n-k}, $0 < \varphi < \frac{\pi}{2}$, k even, $4 \leq k \leq n-1$ *Not yet classified*	?
g ≥ 6	Impossible	Impossible	?	?

Chapter 4

Isoparametric hypersurfaces in complex hyperbolic spaces

Two natural conditions to try to characterize homogeneous hypersurfaces are the constancy of the principal curvatures and the isoparametric condition. In the previous chapter we studied the first condition for real hypersurfaces of complex space forms, under certain extra assumption. This chapter deals with the second condition, for the case of complex hyperbolic spaces. The projective case requires different techniques and new ideas and will be thoroughly investigated in Chapter 7.

The main result of this chapter is a study of the possible principal curvatures of isoparametric hypersurfaces in complex hyperbolic spaces, showing that they are pointwise the same as those of homogeneous real hypersurfaces, with a few possible exceptions. In particular, we prove that the number of principal curvatures of such hypersurfaces is $g \in \{2, 3, 4, 5\}$ and the number of nontrivial projections of the Hopf vector field onto the principal curvature spaces is $h \in \{1, 2, 3\}$. Most of the contents of this chapter can also be found in the preprint [51].

Recall that, for isoparametric hypersurfaces in real Euclidean and hyperbolic spaces, a bound on g can be obtained by using Cartan's fundamental formula. For spheres, the known restriction $g \in \{1, 2, 3, 4, 6\}$ was proved by Münzner [108] using algebraic topological methods. Xiao [158] showed that, for every isoparametric hypersurface in a complex projective space, the restriction $g \in \{2, 3, 4, 5, 7\}$ holds, and, moreover, the constancy of g characterizes the homogeneous examples. Xiao's method is based on the behaviour of isoparametric hypersurfaces with respect to the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ and on well-known results on isoparametric hypersurfaces in spheres. Our method here will also make use of the Hopf map $H_1^{2n+1} \to \mathbb{C}H^n$, but unlike the projective case, we will have to deal with the Lorentzian character of H_1^{2n+1} and, more importantly, with the absence of correct structure results for isoparametric hypersurfaces in the anti-De Sitter space.

This chapter is organized in the following way. First, in Section 4.1 we will see the relation between the extrinsic geometry of a hypersurface in the complex hyperbolic space and the extrinsic geometry of the lift of such hypersurface under the Hopf map. Every such lift is a Lorentzian hypersurface in the anti-De Sitter space. Moreover, it will also follow that a hypersurface in $\mathbb{C}H^n$ is isoparametric if and only if its lift is isoparametric in H_1^{2n+1}. This motivates the investigation of the eigenvalue structure of the shape operator of Lorentzian isoparametric hypersurfaces in the anti-De Sitter space; this is carried out in Section 4.2. It will turn out that there are four basic types of such shape operators. In Section 4.3 we will go through these four types and obtain the possible principal curvatures of the associated isoparametric hypersurfaces in $\mathbb{C}H^n$, following a case-by-case investigation. Finally, in Section 4.4 we apply our results to classify isoparametric hypersurfaces with $h \leq 2$ in nonflat complex space forms.

4.1 The behaviour of hypersurfaces with respect to the Hopf map

In this section, we will explain certain well-known aspects of the behaviour of hypersurfaces with respect to the Hopf map. For more details, we refer the reader to [149].

First, let us briefly recall that the anti-De Sitter space is a Lorentzian space form of constant negative curvature. We will consider odd dimensional anti-De Sitter spaces $H_1^{2n+1} \subset \mathbb{C}^{n+1}$ of sectional curvature $c/4 < 0$. An S^1-action can be defined on H_1^{2n+1} by means of $z \mapsto \lambda z$, with $\lambda \in \mathbb{C}$, $|\lambda| = 1$. One can define a vector field V on H_1^{2n+1} by means of $V_z = i\sqrt{-c}z/2$ for each $z \in H_1^{2n+1}$. This vector field is tangent to the S^1-flow and has length -1.

The Hopf map $\pi\colon H_1^{2n+1} \to \mathbb{C}H^n$ is a semi-Riemannian submersion with timelike totally geodesic fibers, whose tangent spaces are generated by the vertical vector field V, and where $\mathbb{C}H^n$ has constant holomorphic sectional curvature c. We have the isomorphism

$$T_z H_1^{2n+1} \cong T_{\pi(z)}\mathbb{C}H^n \oplus \mathbb{R}V$$

and the following relations between the Levi-Civita connections $\widetilde{\nabla}$ and $\bar{\nabla}$ of H_1^{2n+1} and $\mathbb{C}H^n$, respectively:

$$\widetilde{\nabla}_{X^L} Y^L = (\bar{\nabla}_X Y)^L + \frac{\sqrt{-c}}{2}\langle JX^L, Y^L\rangle V, \tag{4.1}$$

$$\widetilde{\nabla}_V X^L = \widetilde{\nabla}_{X^L} V = \frac{\sqrt{-c}}{2}(JX)^L = \frac{\sqrt{-c}}{2} JX^L, \tag{4.2}$$

for all X, $Y \in \Gamma(T\mathbb{C}H^n)$, and where X^L denotes the horizontal lift of X and J is the complex structure on $\mathbb{C}^{n+1}$. These formulas follow from the fundamental equations of semi-Riemannian submersions (see [115] and [116, Chapter 7]).

Let now M be a real hypersurface in $\mathbb{C}H^n$. Then $\widetilde{M} = \pi^{-1}(M)$ is a hypersurface in H_1^{2n+1} which is invariant under the S^1-action. Thus $\pi|_{\widetilde{M}} : \widetilde{M} \to M$ is a semi-Riemannian submersion with timelike totally geodesic S^1-fibers. Conversely, if $\widetilde{M}$ is a Lorentzian hypersurface in H_1^{2n+1} which is invariant under the S^1-action, then $M = \pi(\widetilde{M})$ is a real hypersurface in $\mathbb{C}H^n$, and $\pi|_{\widetilde{M}}\colon \widetilde{M} \to M$ is a semi-Riemannian submersion with timelike totally geodesic fibers. If ξ is a (local) unit normal vector field to M, then ξ^L is a (local) spacelike unit normal vector field to $\widetilde{M}$. In order to simplify the notation, we will denote by ∇ the Levi-Civita connections of M and of $\widetilde{M}$. Denote by $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ the shape operators of M and $\widetilde{M}$, respectively.

The Gauss and Weingarten formulas for the hypersurface $\widetilde{M}$ in H_1^{2n+1} are:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle \widetilde{\mathcal{S}} X, Y \rangle \xi^L, \qquad \widetilde{\nabla}_X \xi^L = -\widetilde{\mathcal{S}} X.$$

Using (4.1) and (4.2), for any $X \in \Gamma(TM)$, we have

$$\widetilde{\mathcal{S}} X^L = (\mathcal{S} X)^L + \frac{\sqrt{-c}}{2} \langle J\xi^L, X^L \rangle V, \tag{4.3}$$

$$\widetilde{\mathcal{S}} V = -\frac{\sqrt{-c}}{2} J\xi^L. \tag{4.4}$$

Let $X_1, \ldots, X_{2n-1}$ be a local frame on M consisting of principal directions with corresponding principal curvatures $\lambda_1, \ldots, \lambda_{2n-1}$ (obviously, some can be repeated). Then $X_1^L, \ldots, X_{2n-1}^L, V$ is a local frame on $\widetilde{M}$ with respect to which $\widetilde{\mathcal{S}}$ is represented by the matrix

$$\begin{pmatrix} \lambda_1 & & 0 & -\frac{b_1\sqrt{-c}}{2} \\ & \ddots & & \vdots \\ 0 & & \lambda_{2n-1} & -\frac{b_{2n-1}\sqrt{-c}}{2} \\ \frac{b_1\sqrt{-c}}{2} & \cdots & \frac{b_{2n-1}\sqrt{-c}}{2} & 0 \end{pmatrix}, \tag{4.5}$$

where $b_i = \langle J\xi, X_i \rangle$, $i = 1, \ldots, 2n-1$, are S^1-invariant functions on (an open set of) $\widetilde{M}$.

As a consequence of (4.3) and (4.4), M and $\widetilde{M}$ have the same mean curvatures. Therefore, M is isoparametric if and only if $\widetilde{M}$ is isoparametric. This allows us to study isoparametric hypersurfaces in $\mathbb{C}H^n$ by analyzing which Lorentzian isoparametric hypersurfaces in H_1^{2n+1} can result by lifting isoparametric hypersurfaces in $\mathbb{C}H^n$ to the anti-De Sitter space. This is the approach that we will follow in this chapter. It is instructive to note that, whereas the isoparametric condition behaves well with respect to the Hopf map, this is not so for the constancy of the principal curvatures of a hypersurface, since the functions b_i might be nonconstant.

4.2 Lorentzian isoparametric hypersurfaces in anti-De Sitter space

In this section we will obtain the possible eigenvalue structures of the shape operator of a Lorentzian isoparametric hypersurface in H_1^{2n+1}.

Let $\widetilde{M}$ be a Lorentzian isoparametric hypersurface in H_1^{2n+1}. Then we know by Theorem 2.4 that it has constant principal curvatures with constant algebraic multiplicities.

The shape operator $\widetilde{\mathcal{S}}$ of $\widetilde{M}$ is a self-adjoint $(1,1)$-tensor field along $\widetilde{M}$. At any point $p \in \widetilde{M}$, $\widetilde{\mathcal{S}}_p$ is a self-adjoint endomorphism of $T_p\widetilde{M}$. It is known (see [116, Chapter 9]) that there exists a basis of $T_p\widetilde{M}$ where $\widetilde{\mathcal{S}}$ adopts one of the following Jordan canonical forms:

I.

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2n} \end{pmatrix}$$

II.

$$\begin{pmatrix} \lambda_1 & 0 & & & \\ \varepsilon & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{2n-1} \end{pmatrix}, \qquad \varepsilon = \pm 1$$

III.

$$\begin{pmatrix} \lambda_1 & 0 & 1 & & & \\ 0 & \lambda_1 & 0 & & & \\ 0 & 1 & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{2n-2} \end{pmatrix}$$

IV.

$$\begin{pmatrix} a & b & & & \\ -b & a & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_{2n} \end{pmatrix}$$

where the $\lambda_i \in \mathbb{R}$ can be repeated and, in case IV, $\lambda_1 = a + ib, \lambda_2 = a - ib$ $(b \neq 0)$ are the complex eigenvalues of $\widetilde{\mathcal{S}}_p$. In cases I and IV the basis with respect to which $\widetilde{\mathcal{S}}$ is represented is orthonormal (with the first vector being timelike) while in cases II and III the basis is semi-null. A semi-null basis is a basis $\{u, v, e_1, \ldots, e_{m-2}\}$ for which all inner products are zero excepting $\langle u, v \rangle = \langle e_i, e_i \rangle = 1$, for all $i = 1, \ldots, m-2$. We will say that a point $p \in \widetilde{M}$ is of type I, II, III or IV if the canonical form of $\widetilde{\mathcal{S}}_p$ is of type I, II, III or IV, respectively.

Remark 4.1. For the study of Lorentzian isoparametric hypersurfaces in Lorentzian space forms, one cannot assume (as Magid [100] and Xiao [156] do) that $\widetilde{\mathcal{S}}$ has one of the previous canonical forms for a certain local frame. In principle, the canonical form may change from one point to another, as follows from Remark 2.5. However, if there is a point of type IV, then all points are of type IV, and if there is a point of type I, II or III, then all points are of one of these types (due to the constancy of the principal curvatures and the existence or inexistence of nonreal complex principal curvatures).

Applying Theorem 2.6 to our case, we have

Lemma 4.2. *Let $\widetilde{M}$ be a Lorentzian isoparametric hypersurface in the anti-De Sitter space H_1^{2n+1} of curvature $c/4$. If its (possibly complex) principal curvatures are $\lambda_1, \dots, \lambda_{\widetilde{g}}$ with algebraic multiplicities $m_1, \dots, m_{\widetilde{g}}$, respectively, and if for some $i \in \{1, \dots, \widetilde{g}\}$ the principal curvature λ_i is real and its algebraic and geometric multiplicities coincide, then:*

$$\sum_{j=1,\, j\neq i}^{\widetilde{g}} m_j \frac{c + 4\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0$$

This lemma will allow us to show the following results. They have also been proved in [156] (the second one in a slightly different way), but we include the proofs here for the sake of completeness.

Lemma 4.3. *Let $q \in \widetilde{M}$ be a point of type I, II or III. Then the number $\widetilde{g}$ of constant principal curvatures at q satisfies $\widetilde{g} \in \{1, 2\}$. Moreover, if $\widetilde{g} = 2$ and the principal curvatures are λ and μ then $c + 4\lambda\mu = 0$.*

Proof. Let $\lambda_1, \dots, \lambda_{\widetilde{g}}$ be the principal curvatures of $\widetilde{M}$ at q, with corresponding multiplicities $m_1, \dots, m_{\widetilde{g}}$.

If q is of type II or III, then for only one principal curvature of $\widetilde{M}$ at q its algebraic and geometric multiplicities do not coincide. Assume this one is λ_1. By Lemma 4.2, we have

$$\sum_{j=1,j\neq i}^{\widetilde{g}} m_j \frac{c + 4\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0,$$

for any $i \in \{2, \dots, \widetilde{g}\}$. This implies

$$\begin{aligned} m_1 \sum_{j=2}^{\widetilde{g}} m_j \frac{c + 4\lambda_1\lambda_j}{\lambda_1 - \lambda_j} &= \sum_{i=1}^{\widetilde{g}} m_i \left(\sum_{j=1,j\neq i}^{\widetilde{g}} m_j \frac{c + 4\lambda_i\lambda_j}{\lambda_i - \lambda_j} \right) \\ &= \sum_{i<j} m_i m_j (c + 4\lambda_i\lambda_j) \left(\frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_j - \lambda_i} \right) = 0. \end{aligned}$$

As $m_1 \neq 0$, we have that the fundamental formula is also satisfied for $i = 1$.

Now let q be a point of type I, II or III. Then we have

$$\sum_{j=1, j\neq i}^{\widetilde{g}} m_j \frac{c + 4\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0, \tag{4.6}$$

for all $i \in \{1, \ldots, \widetilde{g}\}$. By a suitable choice of the normal vector field, we can assume that there are positive principal curvatures (otherwise, there would be only one principal curvature $\lambda_1 = 0$, and hence $\widetilde{g} = 1$). Define, if it is possible, $\lambda = \max\{\lambda_i : 0 < \lambda_i \le \frac{\sqrt{-c}}{2}\}$ and $\lambda' = \min\{\lambda_i : \frac{\sqrt{-c}}{2} < \lambda_i\}$. We distinguish three cases:

1. λ exists and there is no principal curvature in $\left(\lambda, -\frac{c}{4\lambda}\right)$. Then

$$m_i \frac{c + 4\lambda\lambda_i}{\lambda - \lambda_i} = 4m_i\lambda \frac{\lambda_i + \frac{c}{4\lambda}}{\lambda - \lambda_i} < 0$$

for all $\lambda_i \neq \lambda$, except maybe, $\lambda_i = -\frac{c}{4\lambda}$. Hence, by (4.6) we have $\widetilde{g} \in \{1, 2\}$ and, if $\widetilde{g} = 2$, then the principal curvatures λ and μ satisfy $c + 4\lambda\mu = 0$.

2. λ exists and there is a principal curvature η in $\left(\lambda, -\frac{c}{4\lambda}\right)$. Then also $\lambda' \in \left(\lambda, -\frac{c}{4\lambda}\right)$ (otherwise, as $\lambda' > \lambda$, we would have $\lambda < \eta < -\frac{c}{4\lambda} \le \lambda'$, which contradicts the definition of λ'). Furthermore, there is no principal curvature in $\left(-\frac{c}{4\lambda'}, \lambda'\right)$, so we can apply the same argument as above to λ' instead of λ.

3. λ does not exist. Then λ' exists, and again $\left(-\frac{c}{4\lambda'}, \lambda'\right)$ does not contain any principal curvature. ☐

Lemma 4.4. *Let $q \in \widetilde{M}$ be a point of type IV and let $a \pm ib$ ($b \neq 0$) be the nonreal complex conjugate principal curvatures at q. Then $\widetilde{g} \in \{3, 4\}$ and*

$$a(-c + 4\lambda_i^2) - \lambda_i(4a^2 + 4b^2 - c) = 0, \qquad i \ge 3.$$

If $a = 0$, then $\widetilde{g} = 3$ and $\lambda_3 = 0$. If $\widetilde{g} = 4$, the real principal curvatures λ_3 and λ_4 satisfy $c + 4\lambda_3\lambda_4 = 0$.

Proof. Let $\lambda_1 = a + ib$, $\lambda_2 = a - ib$ ($b \neq 0$) be the two complex nonreal principal curvatures (both with multiplicity one) and $\lambda_3, \ldots, \lambda_{\widetilde{g}}$ the real principal curvatures with multiplicities $m_3, \ldots, m_{\widetilde{g}}$. By Lemma 4.2, for each $i \in \{3, \ldots, \widetilde{g}\}$ we have

$$2\frac{a(4\lambda_i^2 - c) - \lambda_i(4a^2 + 4b^2 - c)}{(\lambda_i - a)^2 + b^2} + \sum_{j=3, j\neq i}^{\widetilde{g}} m_j \frac{c + 4\lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0. \tag{4.7}$$

Take the normal vector field in such a way that $a \le 0$ and assume that there are positive principal curvatures. Then, one can argue as in the final part of the proof of Lemma 4.3

to obtain a principal curvature $\lambda_i > 0$ such that $m_j \frac{c+4\lambda_i\lambda_j}{\lambda_i-\lambda_j} \leq 0$ for all $j \in \{3, \dots, \widetilde{g}\}$, $j \neq i$. Since for such a positive λ_i we also have

$$2\frac{a(4\lambda_i^2 - c) - \lambda_i(4a^2 + 4b^2 - c)}{(\lambda_i - a)^2 + b^2} < 0,$$

we derive a contradiction with (4.7). Therefore, all real principal curvatures are nonpositive if $a \leq 0$. Similarly we get that all real principal curvatures are nonnegative if $a \geq 0$. In particular, if $a = 0$ then $\lambda_i = 0$ for all $i \in \{3, \dots, \widetilde{g}\}$ and hence, $\widetilde{g} = 3$ (notice that $\widetilde{g} = 2$ implies $n = 1$, and we are considering an anti-De Sitter space of dimension $2n + 1$ with $n \geq 2$).

From now on we will assume that $a > 0$. Then all real principal curvatures are nonnegative. But from (4.7) one sees that in fact $\lambda_i > 0$ for all $i \in \{3, \dots, \widetilde{g}\}$. Define the following map $f\colon x \in \mathbb{R} \mapsto f(x) = a(4x^2 - c) - x(4a^2 + 4b^2 - c) \in \mathbb{R}$. It is a quadratic function with discriminant $(c + 4a^2 - 4b^2)^2 + 64a^2b^2 > 0$, so f has exactly two zeroes, say x_1 and x_2, and they are positive (their product is $-\frac{c}{4} > 0$ and their sum is $\frac{a^2+b^2-c/4}{a} > 0$), so we will assume that $0 < x_1 < x_2$. Moreover, f is positive in $(-\infty, x_1)$ and $(x_2, +\infty)$ and negative in (x_1, x_2). If $f(x) = 0$ then $0 = -\frac{c}{4x^2} f(x) = f\left(-\frac{c}{4x}\right)$ which means that $\{x_1, x_2\} = \{x, -\frac{c}{4x}\}$ ($x = -\frac{c}{4x}$ is impossible because, in that case, x would be the abscissa of the vertex of the parabola defined by $f = 0$, and so $f(x)$ would not be zero). Therefore $c + 4x_1x_2 = 0$ and $0 < x_1 < \frac{\sqrt{-c}}{2} < x_2$.

Now assume that there is a principal curvature λ_i in (x_1, x_2). As $\lambda_i > 0$ one can use the argument of the final part of Lemma 4.3 to prove that $m_j \frac{c+4\lambda_k\lambda_j}{\lambda_k-\lambda_j} \leq 0$ for all $j \in \{3, \dots, \widetilde{g}\}$ and for a certain $\lambda_k \in (x_1, x_2)$. But then $f(\lambda_k) < 0$, and we get a contradiction with (4.7).

Suppose now that there is a principal curvature λ_i in $(0, x_1)$ or in $(x_2, +\infty)$. Define $\eta_1 = \min\{\lambda_i : i = 3, \dots, \widetilde{g}\}$ and $\eta_2 = \max\{\lambda_i : i = 3, \dots, \widetilde{g}\}$. If $-\frac{c}{4\eta_1} > \eta_2$ define $\lambda_k = \eta_1$; otherwise, set $\lambda_k = \eta_2$. This definition of λ_k means that all real principal curvatures lie between λ_k and $-\frac{c}{4\lambda_k}$. Then $m_j \frac{c+4\lambda_k\lambda_j}{\lambda_k-\lambda_j} \geq 0$ for all $j = 3, \dots, \widetilde{g}$. But $f(\lambda_k) > 0$, which gives a contradiction with (4.7).

Therefore, if $a > 0$, then $\{\lambda_3, \dots, \lambda_{\widetilde{g}}\} \subset \{x_1, x_2\}$, and hence $\widetilde{g} \in \{3, 4\}$. If $\widetilde{g} = 4$ then $c + \lambda_3\lambda_4 = 0$ and $f(\lambda_3) = f(\lambda_4) = 0$. □

4.3 Principal curvatures of an isoparametric hypersurface in $\mathbb{C}H^n$

As above, let M be an isoparametric real hypersurface in $\mathbb{C}H^n$ and $\widetilde{M}$ its lift to H_1^{2n+1}. We know that $\widetilde{M}$ is a Lorentzian isoparametric hypersurface in the anti-De Sitter space. We denote by g the number of principal curvatures of M and by h the number of nontrivial projections of the Hopf vector field $J\xi$ of M onto the principal curvature spaces.

The objective of this section is to analyze the eigenvalue structure of the shape operator of an isoparametric hypersurface in $\mathbb{C}H^n$. As a corollary, we can derive bounds on h and on g for isoparametric hypersurfaces in complex hyperbolic spaces. More specifically, we have:

Theorem 4.5. *Let M be an isoparametric hypersurface in $\mathbb{C}H^n$ and $p \in M$. Then $h(p) \in \{1, 2, 3\}$ and $g(p) \in \{2, 3, 4, 5\}$.*

It is important to note that this is a *pointwise* result, and not a local one. In other words, the functions g and h might be nonconstant along the hypersurface. In fact, we will show the existence of examples with variable g and h in Chapter 5. The results in this section provide much more information on the possible principal curvatures and multiplicities of isoparametric hypersurfaces in $\mathbb{C}H^n$. For instance, by comparing these results with Table 2.2 one can observe that principal curvatures of isoparametric hypersurfaces in $\mathbb{C}H^n$ must coincide (pointwise) with those of homogeneous hypersurfaces in $\mathbb{C}H^n$, except, maybe, for some particular cases. These exceptional cases correspond to Proposition 4.8(ii)(a) for $g(p) = h(p) = 2$, Proposition 4.8(ii)(b) if λ is not a principal curvature of M at p, and Proposition 4.8(iii) for $g(p) = h(p) = 3$ or if λ is not a principal curvature of M at p. So far, there are no known isoparametric hypersurfaces in $\mathbb{C}H^n$ under the conditions of these exceptional cases.

Our approach here will be mostly based on elementary algebraic arguments. The proof will be carried out by analyzing each one of the four possible canonical forms of the shape operator of $\widetilde{M}$. Thus, we will study each case separately. It will follow that, for each one of the four possible types, there are homogeneous hypersurfaces in $\mathbb{C}H^n$ whose pullback under the Hopf map has shape operator of that type. More specifically, Table 4.1 shows the classification of homogeneous hypersurfaces in $\mathbb{C}H^n$ with respect to the canonical form of the shape operator of the pullback of such hypersurface under the Hopf map. This distribution follows from the information that we will provide for each one of the four possible types and from its comparison with the principal curvatures of homogeneous hypersurfaces in $\mathbb{C}H^n$ (Table 2.2).

Type	Homogeneous hypersurfaces
I	Tubes around a totally geodesic $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$
II	Horospheres
III	A Lohnherr hypersurface W^{2n-1} and its equidistant hypersurfaces
	Tubes around a Berndt-Brück submanifold W^{2n-k}_{φ}, $\varphi \in (0, \pi/2]$
IV	Tubes around a totally geodesic $\mathbb{R}H^n$

Table 4.1: Types of the shape operator of the lifts of homogeneous hypersurfaces in $\mathbb{C}H^n$

In what follows we will consider an isoparametric hypersurface M of $\mathbb{C}H^n$, with (local) unit normal vector field ξ, and which lifts to an isoparametric hypersurface $\widetilde{M}$ of the anti-De Sitter space under the Hopf map π. Fixed a point $q \in \widetilde{M}$, the shape operator $\widetilde{\mathcal{S}}_q$ of $\widetilde{M}$ at q with respect to ξ^L can adopt one of the four possible types described in Section 4.2.

We will analyze the possible principal curvatures of M at the point $p = \pi(q)$ going through the four cases. We will make extensive use of the relations (see (4.4) and (4.5))

$$\widetilde{\mathcal{S}}V = -\frac{\sqrt{-c}}{2}J\xi^L \qquad \text{and} \qquad \langle\widetilde{\mathcal{S}}V, V\rangle = 0,$$

where V is a timelike unit vector field on H_1^{2n+1} tangent to the fibers of the Hopf map π. In order to simplify the notation, we will put $v = V_q$ and remove the base point of a vector field from the notation whenever it does not lead to confusion.

4.3.1 Type I points

Proposition 4.6. *If $q \in \widetilde{M}$ is of type I and $p = \pi(q)$, then $h(p) = 1$ and $g(p) \in \{2, 3\}$. The principal curvatures of M at p are: a nonzero real number $\lambda \in \left(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}\right)$, $\mu = \frac{-c}{4\lambda}$ and $\lambda + \mu$. The first two principal curvatures coincide with those of $\widetilde{M}$ (one of them might not exist as a principal curvature of M at p) and the last one is of multiplicity one and corresponds to the Hopf vector.*

Proof. Let λ and $\mu = -c/4\lambda$ be the eigenvalues of $\widetilde{\mathcal{S}}_q$ (μ might not exist). Assume that $\widetilde{\mathcal{S}}_q$ has a type I matrix expression with respect to an orthonormal basis $\{e_1, e_2, \dots, e_{2n}\}$, where the first k vectors belong to T_λ and the others belong to T_μ.

First, suppose that there exist two principal curvatures $\lambda \neq 0$ and $\mu \neq 0$. As v is timelike, one can assume that $v = su + tw$, where $u \in T_\lambda$, $\langle u, u\rangle = -1$, $w \in T_\mu$, $\langle w, w\rangle = 1$ and $s, t \in \mathbb{R}$. Since $-1 = \langle v, v\rangle = -s^2 + t^2$ we have

$$0 = \langle\widetilde{\mathcal{S}}_q v, v\rangle = -\lambda s^2 + \mu t^2 = (\mu - \lambda)t^2 - \lambda,$$

whence $t^2 = \frac{\lambda}{\mu-\lambda}$ and $s^2 = \frac{\mu}{\mu-\lambda}$. In addition:

$$J\xi^L = -\frac{2}{\sqrt{-c}}\widetilde{\mathcal{S}}_q v = -\frac{2}{\sqrt{-c}}(\lambda s u + \mu t w).$$

Therefore, there are $k - 1$ linearly independent orthogonal vectors in T_λ and $2n - k - 1$ linearly independent orthogonal vectors in T_μ such that they are also orthogonal to $J\xi^L$ and v. By (4.3) the projections of these $2n - 2$ vectors onto $\mathbb{C}H^n$ are eigenvectors of $\mathcal{S}_p$ (with eigenvalues λ and μ) which are orthogonal to $J\xi \in T_pM$. Then $J\xi$ belongs to one eigenspace of $\mathcal{S}_p$; in other words, $h(p) = 1$. Again, by (4.3),

$$\mathcal{S}_p J\xi = \frac{-2}{\sqrt{-c}}(\lambda^2 s\,\pi_* u + \mu^2 t\,\pi_* w) - \frac{\sqrt{-c}}{2}(s\,\pi_* u + t\,\pi_* w) = s\frac{c - 4\lambda^2}{2\sqrt{-c}}\,\pi_* u + t\frac{c - 4\mu^2}{2\sqrt{-c}}\,\pi_* w.$$

As $c + 4\lambda\mu = 0$, one can show that $\mathcal{S}_p J\xi = \frac{4\lambda^2 - c}{4\lambda}J\xi = \frac{4\mu^2 - c}{4\mu}J\xi$. Therefore M has $g(p) \in \{2, 3\}$ principal curvatures at p: λ, μ and $\frac{4\lambda^2 - c}{4\lambda} = \lambda + \mu$, where one of the first two might not exist and where the last one is of multiplicity one and corresponds to the Hopf vector.

Interchanging λ and μ if necessary, one also gets $\lambda \in \left(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}\right)$; notice that $\lambda = \pm\frac{\sqrt{-c}}{2}$ is not possible, because it would imply $\lambda = \mu$.

Now assume that there is just one principal curvature λ. Then $\widetilde{\mathcal{S}}_q v = \lambda v$ and $0 = \langle \widetilde{\mathcal{S}}_q v, v\rangle = -\lambda$, but then $J\xi^L = -\frac{2}{\sqrt{-c}}\widetilde{\mathcal{S}}_q v = 0$, which makes no sense. So this case is impossible. □

Note that for a certain $r \in \mathbb{R}$, one can write $\lambda = \frac{\sqrt{-c}}{2}\tanh\left(\frac{r\sqrt{-c}}{2}\right)$, $\mu = \frac{\sqrt{-c}}{2}\coth\left(\frac{r\sqrt{-c}}{2}\right)$ and $\lambda+\mu = \sqrt{-c}\coth(r\sqrt{-c})$. Therefore, if M is an isoparametric hypersurface that lifts to a type I hypersurface, then M is a Hopf real hypersurface with constant principal curvatures and, according to the classification of Hopf real hypersurfaces with constant principal curvatures in the complex hyperbolic space (Theorem 2.13) and to the principal curvatures of M, it is an open part of a tube around a totally geodesic $\mathbb{C}H^k$, $k = 0, 1, \dots, n-1$ (cf. Table 2.2). It is important to remark that, to make this assertion, we are imposing to M the condition that its lift is of type I at every point, not only at one point.

4.3.2 Type II points

Proposition 4.7. *If $q \in \widetilde{M}$ is of type II and $p = \pi(q)$, then $h(p) = 1$ and $g(p) = 2$. $\widetilde{M}$ has just one principal curvature $\lambda = \pm\sqrt{-c}$, and the principal curvatures of M at p are λ and 2λ. The second one has multiplicity one and corresponds to the Hopf vector.*

Proof. Let λ and $\mu = -c/4\lambda$ be the eigenvalues of $\widetilde{\mathcal{S}}_q$ (μ might not exist). Assume that $\widetilde{\mathcal{S}}_q$ has a type II matrix expression with respect to a semi-null basis $\{e_1, e_2, \dots, e_{2n}\}$, where $\widetilde{\mathcal{S}}_q e_1 = \lambda e_1 + \varepsilon e_2$, $\varepsilon \in \{-1, 1\}$, $T_\lambda = \operatorname{span}\{e_2, \dots, e_k\}$ and $T_\mu = \operatorname{span}\{e_{k+1}, \dots, e_{2n}\}$.

First, suppose that there exist two distinct principal curvatures $\lambda \neq 0$ and $\mu \neq 0$. We can assume that $v = r_1 e_1 + r_2 e_2 + u + w$, where $u \in T_\lambda$, $\langle e_1, u\rangle = \langle e_2, u\rangle = 0$, $w \in T_\mu$ and $r_1, r_2 \in \mathbb{R}$. We have $-1 = \langle v, v\rangle = 2r_1r_2 + \langle u, u\rangle + \langle w, w\rangle$ and $\widetilde{\mathcal{S}}_q v = r_1\lambda e_1 + r_1\varepsilon e_2 + r_2\lambda e_2 + \lambda u + \mu w$, and hence $J\xi^L = -\frac{2}{\sqrt{-c}}(r_1\lambda e_1 + (r_1\varepsilon + r_2\lambda)e_2 + \lambda u + \mu w)$. Taking into account that $\langle u, u\rangle = -1 - 2r_1r_2 - \langle w, w\rangle$ we have

$$\begin{aligned} 1 = \langle J\xi^L, J\xi^L\rangle &= -\frac{4}{c}\left(2r_1^2\lambda\varepsilon + 2r_1r_2\lambda^2 + \langle u, u\rangle\lambda^2 + \langle w, w\rangle\mu^2\right) \\ &= -\frac{4}{c}\left(2r_1^2\lambda\varepsilon - \lambda^2 + \langle w, w\rangle(\mu^2 - \lambda^2)\right), \\ 0 = \langle \widetilde{\mathcal{S}}_q v, v\rangle &= 2r_1r_2\lambda + r_1^2\varepsilon + \langle u, u\rangle\lambda + \langle w, w\rangle\mu = r_1^2\varepsilon - \lambda + \langle w, w\rangle(\mu - \lambda). \end{aligned}$$

Therefore we have the following linear system in the unknowns r_1^2 and $\langle w, w\rangle$:

$$\left.\begin{array}{rcl} 2\lambda\varepsilon r_1^2 + (\mu^2 - \lambda^2)\langle w, w\rangle &=& -\frac{c}{4} + \lambda^2 \\ \varepsilon r_1^2 + (\mu - \lambda)\langle w, w\rangle &=& \lambda \end{array}\right\}$$

As $\lambda \neq \mu$ and $c + 4\lambda\mu = 0$, it is immediate to prove that this system is compatible and determined and the solution is $r_1^2 = 0$, $\langle w, w\rangle = -\frac{c+4\lambda^2}{4(\lambda-\mu)^2}$. But then $\langle u, u\rangle = -1 - \langle w, w\rangle < 0$, which is impossible. Therefore, the case $\lambda \neq \mu$ is not possible.

If $\widetilde{\mathcal{S}}_q$ has just one eigenvalue λ, similar calculations as above (or just setting $w = 0$ everywhere) gives us that $2\lambda\varepsilon r_1^2 = -\frac{c}{4} + \lambda^2$ and $\varepsilon r_1^2 = \lambda$, which is only possible if $\lambda = \pm\frac{\sqrt{-c}}{2}$ and $r_1^2 = \frac{\sqrt{-c}}{2}$.

There are $2n-2$ linearly independent orthogonal vectors in T_λ which are also orthogonal to v and $J\xi^L$: $2n-3$ of them are trivial to find (take an orthogonal basis of $T_\lambda \ominus (\mathbb{R}e_1 \oplus \mathbb{R}u)$), and another one is $\langle u, u\rangle e_2 - r_1 u$. By (4.3) these vectors project onto $2n-2$ linearly independent orthogonal vectors of $T_p\mathbb{C}H^n$ which are eigenvectors of $\mathcal{S}_p$ (with eigenvalue λ) and which are orthogonal to $J\xi$. Then $h(p) = 1$. Furthermore, again by (4.3) and using $c + 4\lambda^2 = 0$ we get

$$\begin{aligned}\mathcal{S}_p J\xi &= -\frac{2}{\sqrt{-c}}(r_1\lambda^2\pi_* e_1 + (2r_1\varepsilon\lambda + r_2\lambda^2)\pi_* e_2 + \lambda^2\pi_* u) - \frac{\sqrt{-c}}{2}(r_1\pi_* e_1 + r_2\pi_* e_2 + \pi_* u)\\ &= \frac{1}{2\sqrt{-c}}\left(r_1(c-4\lambda^2)\pi_* e_1 + (-8r_1\lambda\varepsilon + r_2(c-4\lambda^2))\pi_* e_2 + (c-4\lambda^2)\pi_* u\right) = 2\lambda J\xi.\end{aligned}$$

In conclusion, M has $g(p) = 2$ principal curvatures at p. One is $\lambda = \pm\frac{\sqrt{-c}}{2}$, which coincides with the unique principal curvature of $\widetilde{M}$, and the other one is $2\lambda = \pm\sqrt{-c}$, which has multiplicity one and corresponds to the Hopf vector. □

Notice that if M is an isoparametric hypersurface of $\mathbb{C}H^n$ that lifts to a type II hypersurface, then M is a Hopf real hypersurface with constant principal curvatures $\frac{\sqrt{-c}}{2}$ and $\sqrt{-c}$ (for certain choice of the normal vector field) and, according to the classification of Hopf real hypersurfaces with constant principal curvatures in the complex hyperbolic space (Theorem 2.13) and to the principal curvatures of M, this is an open part of a horosphere (cf. Table 2.2). Again, note that we are imposing the condition that every point of the lift of M is of type II, not just one point.

4.3.3 Type III points

Proposition 4.8. *Let $q \in \widetilde{M}$ be a point of type III, set $p = \pi(q)$ and let λ be the principal curvature of $\widetilde{M}$ at q whose algebraic and geometric multiplicities do not coincide. Then $h(p) \in \{2, 3\}$ and $\lambda \in \left(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}\right)$.*

The type III point $q \in \widetilde{M}$ defines a real number $\varphi \in \left(0, \frac{\pi}{2}\right]$ such that $h(p) = 2$ if and only if $\varphi = \frac{\pi}{2}$ (in this case $g(p) \in \{2, 3, 4\}$), and $h(p) = 3$ if and only if $\varphi \in \left(0, \frac{\pi}{2}\right)$ (in this case $g(p) \in \{3, 4, 5\}$).

Moreover, we have the following:

(i) *If $\varphi = \frac{\pi}{2}$ and $g = 4$, then $0 \neq \lambda \neq \pm\frac{\sqrt{-c}}{2\sqrt{3}}$, and the principal curvatures of M at p are: $\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right)$ (both principal curvature spaces are one dimensional and the Hopf vector has nontrivial projections onto both of them), λ and $\mu = -\frac{c}{4\lambda}$.*

(ii) *If $\varphi = \frac{\pi}{2}$ and $g \in \{2, 3\}$, then we have two cases:*

(a) *If* $\lambda = \pm\frac{\sqrt{-c}}{2\sqrt{3}}$, *then the principal curvatures of* M *at* p *are* 0 *(its corresponding principal curvature space is one dimensional),* $\mu = -\frac{c}{4\lambda} = \pm\frac{\sqrt{-3c}}{2}$ *and maybe* $\lambda = \pm\frac{\sqrt{-c}}{2\sqrt{3}}$, *and the Hopf vector has nontrivial projections onto the principal curvature spaces corresponding to the two first principal curvatures.*

(b) *If* $\lambda \neq \pm\frac{\sqrt{-c}}{2\sqrt{3}}$, *then the principal curvatures of* M *at* p *are* $\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right)$ *(both principal curvature spaces are one dimensional and the Hopf vector has nontrivial projections onto both of them) and* λ *or* $\mu = -\frac{c}{4\lambda}$.

(iii) *If* $\varphi \in \left(0, \frac{\pi}{2}\right)$, *then* $\lambda \neq 0$ *and the zeroes of the following polynomial in* x

$$f(x) = -x^3 + \left(-\frac{c}{4\lambda} + 3\lambda\right) x^2 + \frac{1}{2}\left(c - 6\lambda^2\right) x + \frac{-c^2 - 16c\lambda^2 + 16\lambda^4 + (c+4\lambda^2)^2\cos(2\varphi)}{32\lambda}$$

are three different principal curvatures of M *at* p, *which are also different from* λ *and* $-\frac{c}{4\lambda}$. *Therefore,* M *has* $g(p) \in \{3,4,5\}$ *principal curvatures at* p*: the zeroes of* p *(the corresponding principal curvature spaces are one dimensional and the Hopf vector belongs to the sum of these spaces), and maybe,* λ *and/or* $\mu = -\frac{c}{4\lambda}$.

Proof. Let λ and $\mu = -c/4\lambda$ be the eigenvalues of $\widetilde{\mathcal{S}}_q$ (μ might not exist). Assume that $\widetilde{\mathcal{S}}_q$ has a type III matrix expression with respect to a semi-null basis $\{e_1, e_2, \ldots, e_{2n}\}$, where $\widetilde{\mathcal{S}}_q e_2 = \lambda e_2 + e_3$, $\widetilde{\mathcal{S}}_q e_3 = e_1 + \lambda e_3$, $T_\lambda = \operatorname{span}\{e_1, e_4, \ldots, e_k\}$ and $T_\mu = \operatorname{span}\{e_{k+1}, \ldots, e_{2n}\}$.

First, assume that there exist two distinct principal curvatures $\lambda \neq 0$ and $\mu \neq 0$. We can suppose that $v = r_1e_1 + r_2e_2 + r_3e_3 + u + w$, where $u \in T_\lambda$, $\langle e_2, u\rangle = 0$, $w \in T_\mu$. Taking an appropriate orientation of e_1, e_2, e_3 we can further assume $r_2 \geq 0$. We have $-1 = \langle v, v\rangle = 2r_1r_2 + r_3^2 + \langle u, u\rangle + \langle w, w\rangle$ and $\widetilde{\mathcal{S}}_q v = (r_1\lambda + r_3)e_1 + r_2\lambda e_2 + (r_2 + r_3\lambda)e_3 + \lambda u + \mu w$, and hence $J\xi^L = -\frac{2}{\sqrt{-c}}\left((r_1\lambda + r_3)e_1 + r_2\lambda e_2 + (r_2 + r_3\lambda)e_3 + \lambda u + \mu w\right)$. Taking into account that $\langle u, u\rangle = -1 - 2r_1r_2 - r_3^2 - \langle w, w\rangle$ we have

$$\begin{aligned} 1 = \langle J\xi^L, J\xi^L\rangle &= -\frac{4}{c}\left(2r_1r_2\lambda^2 + 4r_2r_3\lambda + r_2^2 + r_3^2\lambda^2 + \langle u, u\rangle\lambda^2 + \langle w, w\rangle\mu^2\right) \\ &= -\frac{4}{c}\left(4r_2r_3\lambda + r_2^2 - \lambda^2 + \langle w, w\rangle(\mu^2 - \lambda^2)\right), \end{aligned}$$

$$0 = \langle \widetilde{\mathcal{S}}_q v, v\rangle = 2r_1r_2\lambda + 2r_2r_3 + r_3^2\lambda + \langle u, u\rangle\lambda + \langle w, w\rangle\mu = 2r_2r_3 - \lambda + \langle w, w\rangle(\mu - \lambda).$$

Hence we have:

$$\left.\begin{aligned} 4r_2r_3\lambda + r_2^2 + (\mu^2 - \lambda^2)\langle w, w\rangle &= -\tfrac{c}{4} + \lambda^2 \\ 2r_2r_3 + (\mu - \lambda)\langle w, w\rangle &= \lambda \end{aligned}\right\} \tag{4.8}$$

Multiplying by 2λ the second equation, subtracting from the first one and using $c + 4\lambda\mu = 0$, we get

$$r_2^2 + \frac{\left(\frac{c}{4} + \lambda^2\right)^2}{\lambda^2}\langle w, w\rangle = -\frac{c}{4} - \lambda^2.$$

Here we deduce that $\lambda \in \left(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}\right)$, $\lambda \neq 0$ ($|\lambda| = \frac{\sqrt{-c}}{2}$ is impossible because that would imply $r_2 = 0$ and, then, $\langle u, u\rangle = -1 - r_3^2 - \langle w, w\rangle < 0$).

Since we assumed r_2 to be nonnegative, we can then set $r_2 = \sqrt{-\frac{c}{4} - \lambda^2}\sin(\varphi)$ and $\sqrt{\langle w, w\rangle} = \frac{2\lambda}{\sqrt{-c-4\lambda^2}}\cos(\varphi)$ for a suitable $\varphi \in \left(0, \frac{\pi}{2}\right]$; $\varphi = 0$ is impossible because it would imply $\langle u, u\rangle = -1 - r_3^2 - \langle w, w\rangle < 0$. From (4.8) we also get $r_3 = \frac{\lambda}{\sqrt{-c-4\lambda^2}}\sin(\varphi)$.

There are $k-3$ linearly independent orthogonal vectors in T_λ which are also orthogonal to v and $J\xi^L$: in the case $u \neq 0$, we easily find $k-4$ of them (take an orthogonal basis of $T_\lambda \ominus (\mathbb{R}e_2 \oplus \mathbb{R}u)$), and another one is $u_1 = \langle u, u\rangle e_1 - r_2 u$; if $u = 0$, we just take an orthogonal basis of $T_\lambda \ominus \mathbb{R}e_2$. There are $2n-k-1$ linearly independent orthogonal vectors in T_μ which are also orthogonal to v and $J\xi$. Define L as the subspace of $T_q\widetilde{M}$ which is orthogonal to v and to these $2n-4$ eigenvectors. By (4.3) these $2n-4$ vectors project onto $2n-4$ linearly independent orthogonal vectors of $T_p\mathbb{C}H^n$ which are eigenvectors of $\mathcal{S}_p$ (the first $k-3$ with eigenvalue λ, and the last $2n-k-1$ with eigenvalue μ) and which are orthogonal to $J\xi$. Then $h(p) \leq 3$. Furthermore, by (4.5) we see that $h(p) \neq 1$, because otherwise $\widetilde{\mathcal{S}}_q$ would contain at most a 2×2 nondiagonal block, and never a 3×3 nondiagonal block, so q would not be of type III.

Assume that $w \neq 0$ (equivalently, $\varphi \in \left(0, \frac{\pi}{2}\right)$) and consider the following basis of L: $l_1 = (r_1 r_2 + \langle u, u\rangle)e_1 - r_2^2 e_2 - r_2 u$, $l_2 = r_2 r_3 e_1 - r_2^2 e_3$ and $l_3 = 2\lambda\langle w, w\rangle r_2 e_1 - 2\lambda r_2^2 w$. We have that $\operatorname{span}\{e_1, e_2, e_3, u, w\} = L \oplus \mathbb{R}v \oplus \mathbb{R}u_1$, where $u_1 = \langle u, u\rangle e_1 - r_2 u$. After some long calculations we get that the matrix expression of the shape operator of M at p restricted to $\pi_* L$, with respect to the basis $\{\pi_* l_1, \pi_* l_2, \pi_* l_3\}$ is the following:

$$\begin{pmatrix} \lambda + r_2 r_3 & r_2^2 & -\frac{1}{2}\langle w, w\rangle(c+4\lambda^2)r_2 \\ 1 + r_3^2 & \lambda + r_2 r_3 & -\frac{1}{2}\langle w, w\rangle(c+4\lambda^2)r_3 \\ \frac{r_3}{2\lambda} & \frac{r_2}{2\lambda} & -\frac{c+\langle w, w\rangle(c+4\lambda^2)}{4\lambda} \end{pmatrix}.$$

Using the expressions we got for r_2, r_3 and $\langle w, w\rangle$, we can calculate the characteristic polynomial of the previous matrix, which is

$$f(x) = -x^3 + \left(3\lambda - \frac{c}{4\lambda}\right)x^2 + \frac{1}{2}\left(c - 6\lambda^2\right)x + \frac{-c^2 - 16c\lambda^2 + 16\lambda^4 + (c+4\lambda^2)^2\cos(2\varphi)}{32\lambda}.$$

This is the same characteristic polynomial as that of the nontrivial part of the shape operator of a tube around a Berndt-Brück submanifold W_φ^{2n-k}, see [15] or Table 2.2. We have that $f(\lambda) = -\frac{(c+4\lambda^2)^2\sin^2(\varphi)}{16\lambda}$ and $f(\mu) = \frac{(c+4\lambda^2)^2\cos^2(\varphi)}{16\lambda}$. Therefore, as we are in the case $\varphi \in \left(0, \frac{\pi}{2}\right)$, neither λ nor μ are eigenvalues of the matrix above. Moreover, the same argument as in [15, p. 146] proves that the three zeroes of f are different. Hence, if $\varphi \in \left(0, \frac{\pi}{2}\right)$, M has $g(p) \in \{3, 4, 5\}$ principal curvatures at p: the zeroes of f (all of them with multiplicity one), maybe λ (depending on whether or not e_1 generates the whole λ-eigenspace of $\widetilde{\mathcal{S}}_q$) and maybe μ (depending on whether or not w generates the whole μ-eigenspace of $\widetilde{\mathcal{S}}_q$).

Let us prove that, in the case $\varphi \neq \frac{\pi}{2}$, we have $h(p) = 3$. Define $l_1' = r_1 e_1 + u$. Then $\{l_1', e_2, e_3, w\}$ is a basis of $L \oplus \mathbb{R}v$. With respect to this basis, the shape operator $\widetilde{\mathcal{S}}_q$ of $\widetilde{M}$

at q restricted to $L \oplus \mathbb{R}v$ adopts the form

$$\begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}.$$

The characteristic polynomial of this matrix (in the variable x) is

$$x^4 + (-3\lambda - \mu)x^3 + 3\lambda(\lambda + \mu)x^2 - \lambda^2(\lambda + 3\mu)x + \lambda^3\mu.$$

Define x_1, x_2, x_3 to be unit eigenvectors of $\mathcal{S}_p$ whose corresponding eigenvalues are the three different zeroes λ_1, λ_2, λ_3 of the polynomial f, respectively. Set $b_i = \langle J\xi, x_i\rangle$, for $i = 1, 2, 3$. Then the shape operator $\widetilde{\mathcal{S}}_q$ of $\widetilde{M}$ at q restricted to $L \oplus \mathbb{R}v = \operatorname{span}\{x_1^L, x_2^L, x_3^L, v\}$ with respect to the basis $\{x_1^L, x_2^L, x_3^L, v\}$ is given by (4.5)

$$\begin{pmatrix} \lambda_1 & 0 & 0 & -b_1\frac{\sqrt{-c}}{2} \\ 0 & \lambda_2 & 0 & -b_2\frac{\sqrt{-c}}{2} \\ 0 & 0 & \lambda_3 & -b_3\frac{\sqrt{-c}}{2} \\ b_1\frac{\sqrt{-c}}{2} & b_2\frac{\sqrt{-c}}{2} & b_3\frac{\sqrt{-c}}{2} & 0 \end{pmatrix}.$$

Using that $b_1^2 + b_2^2 + b_3^2 = 1$, we get the characteristic polynomial of this matrix (in the variable x), which is

$$\begin{aligned} & x^4 + (-\lambda_1 - \lambda_2 - \lambda_3)\, x^3 \\ & \quad + \frac{1}{4}(-c + 4\lambda_1\lambda_2 + 4\lambda_1\lambda_3 + 4\lambda_2\lambda_3)\, x^2 \\ & \quad + \frac{1}{4}\left(b_1^2 c\lambda_2 + b_1^2 c\lambda_3 + b_2^2 c\lambda_1 + b_3^2 c\lambda_1 + b_3^2 c\lambda_2 + b_2^2 c\lambda_3 - 4\lambda_1\lambda_2\lambda_3\right) x \\ & \quad - \frac{1}{4}c\left(b_1^2\lambda_2\lambda_3 + b_3^2\lambda_1\lambda_2 + b_2^2\lambda_1\lambda_3\right). \end{aligned}$$

Both characteristic polynomials of the two matrices above must coincide, because both matrices represent the same endomorphism of $L \oplus \mathbb{R}v$, but in different bases. Thus, we obtain the following linear system in the variables b_1^2, b_2^2, b_3^2:

$$\left.\begin{aligned} \tfrac{1}{4}(c\lambda_2 + c\lambda_3)\, b_1^2 + \tfrac{1}{4}(c\lambda_1 + c\lambda_3)\, b_2^2 + \tfrac{1}{4}(c\lambda_1 + c\lambda_2)\, b_3^2 &= \lambda_1\lambda_2\lambda_3 - \lambda^2(\lambda + 3\mu) \\ -\tfrac{1}{4}c\lambda_2\lambda_3 b_1^2 - \tfrac{1}{4}c\lambda_1\lambda_3 b_2^2 - \tfrac{1}{4}c\lambda_1\lambda_2 b_3^2 &= \lambda^3\mu \\ b_1^2 + b_2^2 + b_3^2 &= 1 \end{aligned}\right\}$$

The determinant of the matrix of this linear system is $\frac{c^2}{16}(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3) \neq 0$, so the system is compatible and determined. Using the relations among λ, μ, λ_1, λ_2 and

λ_3 that the equality of the characteristic polynomials impose, namely

$$\lambda_1 + \lambda_2 + \lambda_3 = 3\lambda + \mu,$$
$$-\frac{c}{4} + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 3\lambda(\lambda + \mu),$$

one can check that the solution to the linear system above is given by:

$$b_i^2 = -\frac{4(\lambda - \lambda_i)^3(\lambda_i - \mu)}{c(\lambda_{i+1} - \lambda_i)(\lambda_i - \lambda_{i+2})}, \qquad i = 1,\, 2,\, 3, \qquad \text{(indices modulo 3)}.$$

Since μ and λ are both different from any λ_i, $i \in \{1,2,3\}$, we conclude that $b_i \neq 0$ for all $i \in \{1,2,3\}$, whence $h(p) = 3$.

Now let us consider the case $w = 0$, or, equivalently, $\varphi = \frac{\pi}{2}$. In this case $r_2 = \sqrt{-\frac{c}{4} - \lambda^2}$, $r_3 = \frac{\lambda}{\sqrt{-c-4\lambda^2}}$, $2r_2r_3 = \lambda$. Then there are $2n-3$ linearly independent orthogonal eigenvectors of $\widetilde{\mathcal{S}}_q$, which are also orthogonal to v and $J\xi^L$ ($k-3$ belong to T_λ, and $2n-k$ belong to T_μ). By (4.3), their projections onto $T_p\mathbb{C}H^n$ are $2n-3$ linearly independent orthogonal eigenvectors of $\mathcal{S}_p$ ($2n-k$ of them with eigenvalue μ) which are orthogonal to $J\xi$. So, in this case, we have that $h(p) = 2$. Defining l_1, l_2 and l_3 as above, we have that $l_3 = 0$. Now the shape operator of M at p restricted to $\operatorname{span}\{\pi_* l_1, \pi_* l_2\}$, with respect to the basis $\{\pi_* l_1, \pi_* l_2\}$, turns out to be

$$\begin{pmatrix} \lambda + r_2r_3 & r_2^2 \\ 1 + r_3^2 & \lambda + r_2r_3 \end{pmatrix} = \begin{pmatrix} \frac{3\lambda}{2} & -\left(\frac{c}{4} + \lambda^2\right) \\ \frac{c+3\lambda^2}{c+4\lambda^2} & \frac{3\lambda}{2} \end{pmatrix}$$

Thus, the eigenvalues of the shape operator of M at p restricted to $\operatorname{span}\{\pi_* l_1, \pi_* l_2\}$ are $\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right)$. These eigenvalues are different and also different from λ.

For $\lambda = \frac{\sqrt{-c}}{2\sqrt{3}}$ (resp. $\lambda = -\frac{\sqrt{-c}}{2\sqrt{3}}$) we have $\mu = \frac{1}{2}\left(3\lambda + \sqrt{-c-3\lambda^2}\right)$ (resp. $\mu = \frac{1}{2}\left(3\lambda - \sqrt{-c-3\lambda^2}\right)$), hence $g(p) \in \{2,3\}$ ($g(p) = 3$ if e_1 does not span the λ-eigenspace of $\widetilde{\mathcal{S}}_q$, i.e. if λ is a principal curvature of M at p). Moreover, $\frac{1}{2}\left(3\lambda - \sqrt{-c-3\lambda^2}\right) = 0$ (resp. $\frac{1}{2}\left(3\lambda + \sqrt{-c-3\lambda^2}\right) = 0$) has multiplicity one, and the Hopf vector has nontrivial projections onto the principal curvature spaces corresponding to 0 and μ.

But for $\lambda \neq \pm\frac{\sqrt{-c}}{2\sqrt{3}}$ we have $g(p) \in \{3,4\}$ ($g(p) = 4$ if e_1 does not generate the whole λ-eigenspace of $\widetilde{\mathcal{S}}_q$, that is, if λ is a principal curvature of M at p), the principal curvatures $\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right)$ have both multiplicity one, and the Hopf vector has nontrivial projections onto their corresponding principal curvature spaces.

Finally, we have to consider the case where $\widetilde{M}$ has just one principal curvature λ at q. Here, calculations are very similar to what we did up to now, just putting $w = 0$. We get $\lambda \in \left(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}\right)$, $r_2 = \sqrt{-\frac{c}{4} - \lambda^2}$, $r_3 = \frac{\lambda}{\sqrt{-c-4\lambda^2}}$ and $2r_2r_3 = \lambda$, so doing the same as before in the case $\varphi = \frac{\pi}{2}$, we obtain $h(p) = 2$ and $g(p) = 3$, where the principal curvatures of M at p are $\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right)$ (with multiplicity one, and with the Hopf vector projecting nontrivially onto their corresponding principal curvature spaces) and λ. Also notice that now we can have $\lambda = 0$ and then, the other principal curvatures are $\pm\frac{\sqrt{-c}}{2}$. □

4.3.4 Type IV points

Proposition 4.9. *If $q \in \widetilde{M}$ is of type IV and $p = \pi(q)$, then $h(p) = 1$ and $g(p) \in \{2, 3\}$. Let λ and μ (this one could not exist) be the real principal curvatures of $\widetilde{M}$ at q.*

The principal curvatures of M at p are λ, $\mu = -\frac{c}{4\lambda}$ (one of them might not exist) and $2a = -\frac{4c\lambda}{-c+4\lambda^2}$. This last one has multiplicity one, unless it coincides with μ, and corresponds to the Hopf vector. In any case, the Hopf principal curvature satisfies $2a \in \left(-\sqrt{-c}, \sqrt{-c}\right)$, and the non-Hopf principal curvature(s) is (are) different from $\pm\frac{\sqrt{-c}}{2}$.

Proof. Let $a \pm ib$ be the nonreal complex eigenvalues of $\widetilde{\mathcal{S}}_q$ ($b \neq 0$). Let λ and $\mu = -c/4\lambda$ be the real eigenvalues of $\widetilde{\mathcal{S}}_q$ (μ might not exist). Assume that $\widetilde{\mathcal{S}}_q$ has a type IV matrix expression with respect to an orthonormal basis $\{e_1, e_2, \ldots, e_{2n}\}$, where $\widetilde{\mathcal{S}}_q e_1 = ae_1 - be_2$, $\widetilde{\mathcal{S}}_q e_2 = be_1 + ae_2$, $T_\lambda = \operatorname{span}\{e_3, \ldots, e_k\}$ and $T_\mu = \operatorname{span}\{e_{k+1}, \ldots, e_{2n}\}$.

First of all, let us prove that $4a^2 + 4b^2 \geq -c$. As $\widetilde{M}$ is of type IV at q, due to the constancy of the principal curvatures, $\widetilde{M}$ is of type IV in a neighbourhood of q, with the same principal curvatures. Take an orthonormal frame $\{E_1, E_2, \ldots, E_{2n}\}$ defined locally around q (as usual, the first vector field is unit timelike) such that $\widetilde{\mathcal{S}}E_1 = aE_1 - bE_2$, $\widetilde{\mathcal{S}}E_2 = bE_1 + aE_2$, $T_\lambda = \operatorname{span}\{E_3, \ldots, E_k\}$ and $T_\mu = \operatorname{span}\{E_{k+1}, \ldots, E_{2n}\}$.

If we apply the Codazzi equation to (E_1, E_2, E_1), and then to (E_2, E_1, E_2), we get $\langle \nabla_{E_1} E_1, E_2\rangle = \langle \nabla_{E_2} E_2, E_1\rangle = 0$. Again, by applying the Codazzi equation to (E_1, E_i, E_1), (E_1, E_i, E_2), (E_2, E_i, E_1) and (E_2, E_i, E_2), we obtain

$$
\begin{aligned}
(a-\lambda_i)\langle \nabla_{E_1} E_1, E_i\rangle - b\langle \nabla_{E_1} E_2, E_i\rangle =& 2b\langle \nabla_{E_i} E_1, E_2\rangle,\\
(a-\lambda_i)\langle \nabla_{E_1} E_2, E_i\rangle + b\langle \nabla_{E_1} E_1, E_i\rangle =& 0,\\
(a-\lambda_i)\langle \nabla_{E_2} E_1, E_i\rangle - b\langle \nabla_{E_2} E_2, E_i\rangle =& 0,\\
(a-\lambda_i)\langle \nabla_{E_2} E_2, E_i\rangle + b\langle \nabla_{E_2} E_1, E_i\rangle =& 2b\langle \nabla_{E_i} E_1, E_2\rangle,
\end{aligned}
$$

where $\lambda_i = \lambda$ if $i \in \{3, \ldots, k\}$ and $\lambda_i = \mu$ if $i \in \{k+1, \ldots, 2n\}$. From this we get the following relations:

$$
\begin{aligned}
\langle \nabla_{E_1} E_1, E_i\rangle = \langle \nabla_{E_2} E_2, E_i\rangle =& -\frac{2b(\lambda_i - a)}{(\lambda_i - a)^2 + b^2}\langle \nabla_{E_i} E_1, E_2\rangle,\\
\langle \nabla_{E_1} E_2, E_i\rangle = -\langle \nabla_{E_2} E_1, E_i\rangle =& -\frac{2b^2}{(\lambda_i - a)^2 + b^2}\langle \nabla_{E_i} E_1, E_2\rangle.
\end{aligned}
$$

The Gauss equation applied to the vector fields E_1, E_2, E_2, E_1 gives the following:

$$
\begin{aligned}
-\frac{c}{4} =& \langle \widetilde{R}(E_1, E_2)E_2, E_1\rangle = \langle R(E_1, E_2)E_2, E_1\rangle - \langle \widetilde{\mathcal{S}}E_2, E_2\rangle\langle \widetilde{\mathcal{S}}E_1, E_1\rangle + \langle \widetilde{\mathcal{S}}E_2, E_1\rangle\langle \widetilde{\mathcal{S}}E_2, E_1\rangle\\
=& -\langle \nabla_{E_1} E_1, \nabla_{E_2} E_2\rangle + \langle \nabla_{E_1} E_2, \nabla_{E_2} E_1\rangle - \langle \nabla_{\sum_{i=3}^{2n}\langle \nabla_{E_1} E_2, E_i\rangle E_i} E_2, E_1\rangle\\
&+ \langle \nabla_{\sum_{i=3}^{2n}\langle \nabla_{E_2} E_1, E_i\rangle E_i} E_2, E_1\rangle + a^2 + b^2
\end{aligned}
$$

$$= - \sum_{i=3}^{2n} \langle \nabla_{E_1} E_1, E_i \rangle \langle \nabla_{E_2} E_2, E_i \rangle + \sum_{i=3}^{2n} \langle \nabla_{E_1} E_2, E_i \rangle \langle \nabla_{E_2} E_1, E_i \rangle$$
$$- \sum_{i=3}^{2n} \langle \nabla_{E_1} E_2, E_i \rangle \langle \nabla_{E_i} E_2, E_1 \rangle + \sum_{i=3}^{2n} \langle \nabla_{E_2} E_1, E_i \rangle \langle \nabla_{E_i} E_2, E_1 \rangle + a^2 + b^2$$
$$= - \sum_{i=3}^{2n} \frac{4b^2(\lambda_i - a)^2}{(b^2 + (a - \lambda_i)^2)^2} \langle \nabla_{E_i} E_1, E_2 \rangle^2 - \sum_{i=3}^{2n} \frac{4b^4}{(b^2 + (a - \lambda_i)^2)^2} \langle \nabla_{E_i} E_1, E_2 \rangle^2$$
$$- \sum_{i=3}^{2n} \frac{4b^2}{b^2 + (a - \lambda_i)^2} \langle \nabla_{E_i} E_1, E_2 \rangle^2 + a^2 + b^2 \leq a^2 + b^2.$$

From this we deduce that $a^2 + b^2 \geq -\frac{c}{4}$, as desired.

Now, let us restrict to the case that there exist two distinct real principal curvatures $\lambda \neq 0$ and $\mu \neq 0$. We can assume that $v = r_1 e_1 + r_2 e_2 + u + w$, where $u \in T_\lambda$, $w \in T_\mu$ and r_1, $r_2 \in \mathbb{R}$. We have $-1 = \langle v, v \rangle = -r_1^2 + r_2^2 + \langle u, u \rangle + \langle w, w \rangle$ and $\widetilde{\mathcal{S}}_q v = (r_1 a + r_2 b) e_1 + (r_2 a - r_1 b) e_2 + \lambda u + \mu w$, and hence $J\xi^L = -\frac{2}{\sqrt{-c}} \left((r_1 a + r_2 b) e_1 + (r_2 a - r_1 b) e_2 + \lambda u + \mu w \right)$. Taking into account that $\langle u, u \rangle = -1 + r_1^2 - r_2^2 - \langle w, w \rangle$ we have

$$(4.9) \qquad 1 = \langle J\xi^L, J\xi^L \rangle = -\frac{4}{c} \left((-a^2 + b^2 + \lambda^2) r_1^2 - (-a^2 + b^2 + \lambda^2) r_2^2 - 4ab r_1 r_2 + (\mu^2 - \lambda^2) \langle w, w \rangle - \lambda^2 \right),$$
$$(4.10) \qquad 0 = \langle \widetilde{\mathcal{S}}_q v, v \rangle = (\lambda - a) r_1^2 - (\lambda - a) r_2^2 - 2b r_1 r_2 + (\mu - \lambda) \langle w, w \rangle - \lambda.$$

Multiplying by $\lambda - a$ the first equation, by $-a^2 + b^2 + \lambda^2$ the second one, and subtracting, after some calculations and using $c + 4\lambda\mu = 0$ we get

$$2 r_1 r_2 = \frac{-\frac{c\lambda}{4} + \frac{ca}{4} - a\lambda^2 + a^2\lambda - b^2\lambda}{b\left((a - \lambda)^2 + b^2\right)} \left(1 + \left(\frac{c}{4\lambda^2} + 1 \right) \langle w, w \rangle \right).$$

Suppose $\lambda \neq a$. Using (4.10) and the previous expression for $2r_1r_2$, we have

$$r_1^2 - r_2^2 = \frac{1}{\lambda - a} \left(2b r_1 r_2 + (\lambda - \mu) \langle w, w \rangle + \lambda \right) = \frac{-c - 8a\lambda + 4\lambda^2}{16\lambda^2((a - \lambda)^2 + b^2)} (4\lambda^2 + (c + 4\lambda^2) \langle w, w \rangle).$$

Now, if $\lambda = a$, using the equation (4.9) instead of (4.10) we obtain that $r_1^2 - r_2^2 = -\frac{c + 4\lambda^2}{16 b^2 \lambda^2} (4\lambda^2 + (c + 4\lambda^2) \langle w, w \rangle)$. So, the above formula for $r_1^2 - r_2^2$ is valid even for the case $\lambda = a$. Then we have

$$0 \leq \langle u, u \rangle = -1 + r_1^2 - r_2^2 - \langle w, w \rangle = -\frac{(c + 4a\lambda)^2 + 16 b^2 \lambda^2}{16\lambda^2((a - \lambda)^2 + b^2)} \langle w, w \rangle - \frac{4a^2 + 4b^2 + c}{4((a - \lambda)^2 + b^2)}.$$

Hence, as we knew that $4a^2 + 4b^2 + c \geq 0$, we must have that $4a^2 + 4b^2 + c = 0$ and $u = w = 0$.

As a consequence, there are $2n-2$ linearly independent orthogonal vectors in $T_q\widetilde{M}$, which are orthogonal to v and $J\xi^L$ and are eigenvectors of $\widetilde{\mathcal{S}}_q$ with eigenvalues λ and μ (take a basis of T_λ and a basis of T_μ). Therefore, they project onto $2n-2$ linearly independent orthogonal vectors of $T_p\mathbb{C}H^n$, which are orthogonal to $J\xi$ and eigenvectors of $\mathcal{S}_p$, with eigenvalues λ and μ. Then $h(p)=1$. In fact, using that $4a^2+4b^2+c=0$, we have

$$\begin{aligned}\mathcal{S}_pJ\xi &= -\frac{2}{\sqrt{-c}}((r_1a^2+2r_2ab-r_1b^2)e_1+(-2r_1ab-r_2b^2+r_2a^2)e_2)-\frac{\sqrt{-c}}{2}(r_1e_1+r_2e_2)\\ &=\frac{1}{2\sqrt{-c}}8a\left((r_1a+r_2b)e_1+(r_2a-r_1b)e_2\right)=2aJ\xi.\end{aligned}$$

From Lemma 4.4 and $4a^2+4b^2+c=0$ we know that $a=-\frac{2c\lambda}{-c+4\lambda^2}$. Note that $\lambda\neq\pm\frac{\sqrt{-c}}{2}$ and $|2a|<\sqrt{-c}$. In fact, suppose the first assertion is not true. Then $2a$ would be $\pm\sqrt{-c}$, so the second assertion would not hold either. If $|2a|\geq\sqrt{-c}$, then $0=4a^2+4b^2+c\geq 4b^2$, which is impossible because $b\neq 0$. Therefore only one of λ or μ belongs to $\left(-\frac{\sqrt{-c}}{2},\frac{\sqrt{-c}}{2}\right)$; there is no restriction of generality if we assume it is λ. It is then easy to see that $\lambda=2a$ is impossible (because this would imply $|\lambda|>\frac{\sqrt{-c}}{2}$), and $\mu=2a$ if and only if $\lambda=\pm\frac{\sqrt{-c}}{2\sqrt{3}}$. Therefore, if $\lambda\neq\pm\frac{\sqrt{-c}}{2\sqrt{3}}$, M has three principal curvatures at p: $2a=-\frac{4c\lambda}{-c+4\lambda^2}$ (with multiplicity one and corresponding to the Hopf vector), λ and $\mu=-\frac{c}{4\lambda}$. If $\lambda=\pm\frac{\sqrt{-c}}{2\sqrt{3}}$, M has two principal curvatures at p: λ and μ (this one corresponding to the Hopf vector).

Finally, if $\widetilde{M}$ only has one real principal curvature λ at q, calculations are fairly similar to the ones above, just putting $w=0$. One also gets $4a^2+4b^2+c=0$ and $h(p)=1$. The principal curvatures now would be $2a=-\frac{4c\lambda}{-c+4\lambda^2}$ (with multiplicity one and corresponding to the Hopf vector) and λ. Notice that $2a\neq\lambda$ because there are no totally umbilical hypersurfaces in $\mathbb{C}H^n$. The relations $\lambda\neq\pm\frac{\sqrt{-c}}{2}$ and $|2a|<\sqrt{-c}$ also hold in this case. □

Corollary 4.10. *Let M be a connected isoparametric hypersurface in $\mathbb{C}H^n$ which lifts to a type IV hypersurface in H_1^{2n+1} at some point. Then M is a Hopf real hypersurface with $g\in\{2,3\}$ constant principal curvatures. There exists an $r\in\mathbb{R}$ such that the principal curvatures of M are $\frac{\sqrt{-c}}{2}\tanh\left(\frac{r\sqrt{-c}}{2}\right)$, $\frac{\sqrt{-c}}{2}\coth\left(\frac{r\sqrt{-c}}{2}\right)$ and $\sqrt{-c}\tanh\left(r\sqrt{-c}\right)$, where the second and the third coincide if $r=\pm\frac{1}{\sqrt{-c}}\ln(2+\sqrt{3})$, and where the Hopf vector belongs to the principal space corresponding to the last principal curvature.*

Moreover, M is an open part of a tube of radius r around a totally geodesic $\mathbb{R}H^n$.

Proof. Let M be a connected isoparametric hypersurface in $\mathbb{C}H^n$ which lifts to a hypersurface $\widetilde{M}$ that contains a point $q\in\widetilde{M}$ of type IV. Due to the constancy of the principal curvatures of $\widetilde{M}$, every point of $\widetilde{M}$ is of type IV. From the previous result, we obtain that M is Hopf and has $g\in\{2,3\}$ constant principal curvatures. From Table 2.2 and the classification of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ stated in Theorem 2.13, it follows that the unique such hypersurface whose Hopf principal curvature is less than $\sqrt{-c}$ in absolute value is a tube around a totally geodesic $\mathbb{R}H^n$. The rest of the claims of the corollary follow from the eigenvalue structure of the shape operator of a tube around a totally geodesic $\mathbb{R}H^n$ (see Table 2.2). □

4.4 Isoparametric hypersurfaces versus hypersurfaces with constant principal curvatures

Cartan proved that a hypersurface in a real space form is isoparametric if and only if it has constant principal curvatures. We have commented in Section 2.5 that this equivalence does not necessarily hold in spaces of nonconstant curvature. However, it might still be true under certain additional hypotheses. For example, it follows from results by Kimura [83] and Wang [150] that, for a Hopf real hypersurface in a complex projective space $\mathbb{C}P^n$, both conditions turn out to be equivalent (see Corollary 2.11).

We can now prove an even stronger result, which is also valid for complex hyperbolic spaces $\mathbb{C}H^n$. In particular, by means of the classifications of homogeneous hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ (Theorems 2.7 and 2.14), this result provides a complete classification of isoparametric hypersurfaces with $h \leq 2$ in nonflat complex space forms.

Theorem 4.11. *Let M be a connected real hypersurface in $\mathbb{C}P^n$ or in $\mathbb{C}H^n$ satisfying $h \leq 2$ at every point. Then M is isoparametric if and only if it has constant principal curvatures. In this case, h is constant and M is an open part of a homogeneous hypersurface.*

Proof. Assume first that M has constant principal curvatures. If $h(p) = 2$ at some point $p \in M$, then h takes the value 2 in an open neighbourhood around p. This is impossible if $M \subset \mathbb{C}P^n$, in view of Theorem 3.1. If $M \subset \mathbb{C}H^n$, then the functions b_i defined in Section 4.1 are constant in an open neighbourhood around p, according to Proposition 3.8. Since M is connected and the b_i are continuous functions, then the b_i are constant and h takes the value 2 along all M. It follows from Theorem 3.1 that M is an open part of a homogeneous hypersurface in $\mathbb{C}H^n$ and hence, M is isoparametric. If $h(p) \neq 2$ for every $p \in M$, then M is a Hopf hypersurface with constant principal curvatures in $\mathbb{C}P^n$ or $\mathbb{C}H^n$. According to Kimura's and Berndt's classifications (see Sections 2.5 and 2.6), it follows that M is an open part of a homogeneous (and hence isoparametric) hypersurface.

Now, let us prove the converse implication. Assume first that M is isoparametric in $\mathbb{C}H^n$. According to (4.5), we can assume that the shape operator of the lift of M to the anti-De Sitter space adopts the form

$$\begin{pmatrix} \lambda_{2n-1} & & & & & 0 \\ & \ddots & & \text{\Large 0} & & \vdots \\ & & \lambda_3 & & & 0 \\ & \text{\Large 0} & & \lambda_2 & & -\frac{\sqrt{-c}}{2}b_2 \\ & & & & \lambda_1 & -\frac{\sqrt{-c}}{2}b_1 \\ 0 & \cdots & 0 & \frac{\sqrt{-c}}{2}b_2 & \frac{\sqrt{-c}}{2}b_1 & 0 \end{pmatrix},$$

with respect to a basis $\{X^L_{2n-1}, \dots, X^L_2, X^L_1, V\}$, and where $\lambda_1 \neq \lambda_2$. Taking into account that $b_1^2 + b_2^2 = 1$, the characteristic polynomial of this matrix takes the form

$$q(x) = \Big(-x^3 + (\lambda_1 + \lambda_2)x^2 + (\frac{c}{4} - \lambda_1\lambda_2)x - \frac{c}{4}(b_1^2\lambda_2 + b_2^2\lambda_1)\Big) \prod_{i=3}^{2n-1} (\lambda_i - x).$$

Since M is isoparametric, its lift $\widetilde{M}$ is also isoparametric in H_1^{2n+1} and, in view of Theorem 2.4, the shape operator of $\widetilde{M}$ has constant characteristic polynomial. In particular, λ_i for $i = 3, \ldots, 2n-1$, $\lambda_1 + \lambda_2$ and $\frac{c}{4} - \lambda_1\lambda_2$ are constant functions on (an open subset of) $\widetilde{M}$, which implies that the principal curvatures $\lambda_1, \ldots, \lambda_{2n-1}$ of M are constant.

If $M \subset \mathbb{C}P^n$, a completely analogous argument works with only some sign changes in the equations of Section 4.1. □

Let us mention that, although an arbitrary isoparametric hypersurface in $\mathbb{C}H^n$ satisfies $h \leq 3$, the classification of isoparametric hypersurfaces in $\mathbb{C}H^n$ would not follow from Theorem 4.11 and an eventual classification of the case $h = 3$. The reason is that the function h may be nonconstant along the hypersurface, which means that there can be (and in fact, there are) isoparametric hypersurfaces whose function h takes the value 3 at some points, but not at every point.

Chapter 5

Isoparametric hypersurfaces in Damek-Ricci spaces

In the previous chapter we presented an investigation of the extrinsic geometry of isoparametric hypersurfaces in complex hyperbolic spaces. This is probably the first such study in a symmetric space of noncompact type and nonconstant curvature. The present chapter aims at going further in this line of studying isoparametric hypersurfaces in noncompact symmetric spaces, but, in this case, our goal will be to obtain new examples. The contents of this chapter can also be found in the articles [48], [49] and [55].

More specifically, we will provide a construction method of isoparametric hypersurfaces in Damek-Ricci harmonic spaces. These spaces constitute a family of homogeneous manifolds that contains the rank one noncompact symmetric spaces as particular cases. They were constructed by Damek and Ricci in [43] and they provide counterexamples to the so-called Lichnerowicz conjecture, stating that every Riemannian harmonic manifold is locally isometric to a two-point homogeneous space. The hypersurfaces that we introduce arise as tubes around certain homogeneous minimal submanifolds whose construction extends the one proposed by Berndt and Brück [10]. On the one hand, our construction provides uncountably many isoparametric families of hypersurfaces in most Damek-Ricci spaces. This should be compared with the case of spheres, where the known set of isoparametric families up to congruence is countable. On the other hand, most of these new hypersurfaces are inhomogeneous and have nonconstant principal curvatures.

The particular case of noncompact rank one symmetric spaces deserves some extra attention. To our knowledge, our construction yields the first examples of inhomogeneous isoparametric hypersurfaces in symmetric spaces whose construction does not depend on isoparametric hypersurfaces in spheres. But when we look at the implications of our results on each concrete rank one symmetric space, we can derive other important consequences.

In first place, for each complex hyperbolic space $\mathbb{C}H^n$, with $n \geq 3$, we obtain inhomogeneous isoparametric families of hypersurfaces with nonconstant principal curvatures. Although the results in the previous chapter might point in the opposite direction, the new examples show that not every isoparametric hypersurface in a complex hyperbolic space is homogeneous.

Secondly, apart from new inhomogeneous hypersurfaces, we will also construct new cohomogeneity one actions on quaternionic hyperbolic spaces that were unknown up to now. We should mention here that Berndt and Tamaru [21] classified cohomogeneity one actions on all noncompact rank one symmetric spaces, except on quaternionic hyperbolic spaces $\mathbb{H}H^n$, $n \geq 3$. Although we do not address this classification here, the generality of our construction suggests that no other examples exist.

Finally, regarding the Cayley hyperbolic plane $\mathbb{O}H^2$, our method only provides one new family of hypersurfaces, but an interesting one. This is an inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures. The only such families known so far were the FKM examples in spheres constructed by Ferus, Karcher and Münzner [63] (cf. [66, p. 7]).

The main concept introduced in this chapter is that of *generalized Kähler angle*, which generalizes previous notions of Kähler angle and quaternionic Kähler angle [10]. Among the isoparametric hypersurfaces we construct, the ones with constant principal curvatures are precisely those whose focal submanifolds have normal spaces of constant generalized Kähler angle (Theorem 5.8).

This chapter is organized as follows. In Section 5.1 we set up the fundamental definitions and results on Damek-Ricci spaces. The definition of generalized Kähler angle is presented in Section 5.2. In Section 5.3 the new examples of isoparametric hypersurfaces in Damek-Ricci spaces are introduced. We start by defining the focal set of the new examples in §5.3.1, and then in §5.3.2 we investigate the properties of the tubes around these submanifolds using Jacobi field theory. The main result of this chapter is stated in Theorem 5.8. Finally, in Section 5.4 we consider some particular cases in the rank one symmetric spaces of noncompact type. In §5.4.1 we study the case of the complex hyperbolic space, in §5.4.2 we construct new examples of cohomogeneity one actions on quaternionic hyperbolic spaces (Theorem 5.11), and in §5.4.3 we give an example of an inhomogeneous isoparametric hypersurface with constant principal curvatures in the Cayley hyperbolic plane (Theorem 5.13).

5.1 Generalized Heisenberg groups and Damek-Ricci spaces

In this section we recall the construction of Damek-Ricci spaces, presenting some of the properties that we will use later. Since the description of such spaces depends on the so-called generalized Heisenberg algebras, we begin by defining these structures. The main reference for all these notions is [23], where one can find the proofs of the results presented below, as well as further information on Damek-Ricci spaces.

5.1.1 Generalized Heisenberg algebras and groups

Let $\mathfrak{v}$ and $\mathfrak{z}$ be real vector spaces and $\beta : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ a skew-symmetric bilinear map. Define the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and endow it with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular. Define a linear map $J\colon Z \in \mathfrak{z} \mapsto J_Z \in \operatorname{End}(\mathfrak{v})$ by

$$\langle J_Z U, V\rangle = \langle \beta(U,V), Z\rangle, \qquad \text{for all } U, V \in \mathfrak{v},\ Z \in \mathfrak{z},$$

and a Lie algebra structure on $\mathfrak{n}$ by

$$[U + X, V + Y] = \beta(U,V), \qquad \text{for all } U, V \in \mathfrak{v},\ X, Y \in \mathfrak{z},$$

or equivalently, by

$$\langle [U,V], X\rangle = \langle J_X U, V\rangle, \quad [X,V] = [U,Y] = [X,Y] = 0, \qquad \text{for all } U, V \in \mathfrak{v},\ X, Y \in \mathfrak{z}.$$

Then, $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center $\mathfrak{z}$, and if $J_Z^2 = -\langle Z, Z\rangle \operatorname{Id}_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$, $\mathfrak{n}$ is said to be a *generalized Heisenberg algebra* or an *H-type algebra*. The associated simply connected nilpotent Lie group N, endowed with the induced left-invariant Riemannian metric, is called a *generalized Heisenberg group* or an *H-type group*.

Let U, $V \in \mathfrak{v}$ and X, $Y \in \mathfrak{z}$. In this chapter, we will make use of the following properties of generalized Heisenberg algebras without explicitly referring to them:

$$J_X J_Y + J_Y J_X = -2\langle X, Y\rangle \operatorname{Id}_{\mathfrak{v}}, \qquad [J_X U, V] - [U, J_X V] = -2\langle U, V\rangle X,$$
$$\langle J_X U, J_X V\rangle = \langle X, X\rangle\langle U, V\rangle, \qquad \langle J_X U, J_Y U\rangle = \langle X, Y\rangle\langle U, U\rangle.$$

In particular, for any unit $Z \in \mathfrak{z}$, J_Z is a complex structure on $\mathfrak{v}$.

The map $J\colon \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ can be extended to the Clifford algebra $\operatorname{Cl}(\mathfrak{z}, q)$, where q is the quadratic form given by $q(Z) = -\langle Z, Z\rangle$, in such a way that $\mathfrak{v}$ becomes now a Clifford module over $\operatorname{Cl}(\mathfrak{z}, q)$ (see [23, Chapter 3]). The classification of generalized Heisenberg algebras is known (it follows from the classification of representations of Clifford algebras of vector spaces with negative definite quadratic forms). In particular, for each $m \in \mathbb{N}$ there exist an infinite number of non-isomorphic generalized Heisenberg algebras with $\dim \mathfrak{z} = m$.

5.1.2 Damek-Ricci spaces

We proceed now with the definition of Damek-Ricci spaces. The construction tries to imitate the model of a noncompact rank one symmetric space as the solvable part AN of its Iwasawa decomposition, where AN is endowed with a left-invariant metric.

Let $\mathfrak{a}$ be a one-dimensional real vector space, B a nonzero vector in $\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ a generalized Heisenberg algebra, where $\mathfrak{z}$ is the center of $\mathfrak{n}$. We denote the inner product and the Lie bracket on $\mathfrak{n}$ by $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$ and $[\cdot,\cdot]_{\mathfrak{n}}$, respectively, and consider a new vector space $\mathfrak{a} \oplus \mathfrak{n}$ as the vector space direct sum of $\mathfrak{a}$ and $\mathfrak{n}$.

From now on in this section, let s, $r \in \mathbb{R}$, U, $V \in \mathfrak{v}$ and X, $Y \in \mathfrak{z}$. We now define an inner product $\langle\cdot,\cdot\rangle$ and a Lie bracket $[\cdot,\cdot]$ on $\mathfrak{a} \oplus \mathfrak{n}$ by

$$\langle rB + U + X, sB + V + Y\rangle = rs + \langle U + X, V + Y\rangle_{\mathfrak{n}}, \qquad \text{and}$$
$$[rB + U + X, sB + V + Y] = [U, V]_{\mathfrak{n}} + \frac{1}{2}rV - \frac{1}{2}sU + rY - sX.$$

Thus, $\mathfrak{a} \oplus \mathfrak{n}$ becomes a solvable Lie algebra with an inner product. The corresponding simply connected Lie group AN, equipped with the induced left-invariant Riemannian metric, is a solvable extension of the H-type group N, and is called a *Damek-Ricci space.*

The Levi-Civita connection $\bar{\nabla}$ of a Damek-Ricci space is given by

$$\bar{\nabla}_{sB+V+Y}(rB+U+X) = -\frac{1}{2}J_X V - \frac{1}{2}J_Y U - \frac{1}{2}rV - \frac{1}{2}[U,V] - rY + \frac{1}{2}\langle U,V\rangle B + \langle X,Y\rangle B.$$

From this expression, one can obtain the curvature tensor $\bar{R}$ of AN, where as usual we agree to take the convention $\bar{R}(W_1, W_2) = [\bar{\nabla}_{W_1}, \bar{\nabla}_{W_2}] - \bar{\nabla}_{[W_1,W_2]}$.

A Damek-Ricci space AN is a symmetric space if and only if AN is isometric to a rank one symmetric space. In this case, AN is either isometric to a complex hyperbolic space $\mathbb{C}H^n$ with constant holomorphic sectional curvature -1 (in this case, $\dim \mathfrak{z} = 1$), or to a quaternionic hyperbolic space $\mathbb{H}H^n$ with constant quaternionic sectional curvature -1 (here $\dim \mathfrak{z} = 3$), or to the Cayley hyperbolic plane $\mathbb{O}H^2$ with minimal sectional curvature -1 ($\dim \mathfrak{z} = 7$). As a limit case, which we will disregard in what follows, one would obtain the real hyperbolic space $\mathbb{R}H^n$ if one puts $\mathfrak{z} = 0$.

The non-symmetric Damek-Ricci spaces are counterexemples to the so-called Lichnerowicz conjecture, stating that every Riemannian harmonic manifold is locally isometric to a two-point homogeneous space. There are several equivalent conditions for a manifold to be harmonic; see [23, §2.6]. One of them is the following: a manifold is harmonic if and only if its sufficiently small geodesic spheres are isoparametric. However, while geodesic spheres in symmetric Damek-Ricci spaces are homogeneous isoparametric hypersurfaces with constant principal curvatures, geodesic spheres in non-symmetric Damek-Ricci spaces are inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures (see [43] and [23, §4.4 and §4.5]).

5.2 Generalized Kähler angle

In this section we introduce the new notion of generalized Kähler angle of a vector of a subspace of a Clifford module with respect to that subspace. This notion will be crucial for the rest of the chapter.

Let $\mathfrak{v}$ be a Clifford module over $\mathrm{Cl}(\mathfrak{z}, q)$ and denote by $J\colon \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ the restriction to $\mathfrak{z}$ of the Clifford algebra representation. We equip $\mathfrak{z}$ with the inner product induced by polarization of $-q$, and extend it to an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, so that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular, and J_Z is an orthogonal map for each unit $Z \in \mathfrak{z}$. Then, $\mathfrak{n}$ has the structure of a generalized Heisenberg algebra as defined above.

Let $\mathfrak{w}$ be a subspace of $\mathfrak{v}$. We denote by $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{v}$. For each $Z \in \mathfrak{z}$ and $\xi \in \mathfrak{w}^\perp$, we write $J_Z\xi = P_Z\xi + F_Z\xi$, where $P_Z\xi$ is the orthogonal projection of $J_Z\xi$ onto $\mathfrak{w}$, and $F_Z\xi$ is the orthogonal projection of $J_Z\xi$ onto $\mathfrak{w}^\perp$. We define the Kähler angle of $\xi \in \mathfrak{w}^\perp$ with respect to the element $Z \in \mathfrak{z}$ (or, equivalently, with respect to J_Z) and the subspace $\mathfrak{w}^\perp \subset \mathfrak{v}$ as the angle $\varphi \in [0, \pi/2]$ between $J_Z\xi$ and $\mathfrak{w}^\perp$; thus φ satisfies $\langle F_Z\xi, F_Z\xi\rangle = \cos^2(\varphi)\langle Z, Z\rangle\langle \xi, \xi\rangle$. It readily follows from $J_Z^2 = -\langle Z, Z\rangle \operatorname{Id}_{\mathfrak{v}}$

that $\langle P_Z\xi, P_Z\xi\rangle = \sin^2(\varphi)\langle Z, Z\rangle\langle \xi, \xi\rangle$. Hence, if Z and ξ have unit length, φ is determined by the fact that $\cos(\varphi)$ is the length of the orthogonal projection of $J_Z\xi$ onto $\mathfrak{w}^\perp$.

The following theorem is a generalization of [10, Lemma 3] (which concerned only the case of the quaternionic hyperbolic space $\mathbb{H}H^n$). The proof is new and simpler than in [10]. This result will be fundamental for the calculations we will carry out later.

Theorem 5.1. *Let $\mathfrak{w}^\perp$ be some vector subspace of $\mathfrak{v}$ and let $\xi \in \mathfrak{w}^\perp$ be a nonzero vector. Then there exists an orthonormal basis $\{Z_1, \dots, Z_m\}$ of $\mathfrak{z}$ and a uniquely defined m-tuple $(\varphi_1, \dots, \varphi_m)$ such that:*

(a) *φ_i is the Kähler angle of ξ with respect to J_{Z_i}, for each $i = 1, \dots, m$.*

(b) $\langle P_{Z_i}\xi, P_{Z_j}\xi\rangle = \langle F_{Z_i}\xi, F_{Z_j}\xi\rangle = 0$ *whenever $i \neq j$.*

(c) $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_m \leq \pi/2$.

(d) *φ_1 is minimal and φ_m is maximal among the Kähler angles of ξ with respect to all the elements of $\mathfrak{z}$.*

Proof. Since the map $Z \in \mathfrak{z} \mapsto F_Z\xi \in \mathfrak{w}^\perp$ is linear, we can define the quadratic form

$$Q_\xi \colon Z \in \mathfrak{z} \mapsto \langle F_Z\xi, F_Z\xi\rangle \in \mathbb{R}.$$

Observe that φ is the Kähler angle of ξ with respect to $Z \in \mathfrak{z}$ ($Z \neq 0$) and the subspace $\mathfrak{w}^\perp \subset \mathfrak{v}$ if and only if $Q_\xi(Z) = \cos^2(\varphi)\langle Z, Z\rangle\langle \xi, \xi\rangle$.

Let $\{Z_1, \dots, Z_m\}$ be an orthonormal basis of $\mathfrak{z}$ for which the quadratic form Q_ξ assumes a diagonal form. Define the real numbers $\varphi_1, \dots, \varphi_m \in [0, \pi/2]$ by the expression $Q_\xi(Z_i) = \cos^2(\varphi_i)\langle \xi, \xi\rangle$, for every $i = 1, \dots, m$. We can further assume that $\varphi_1 \leq \cdots \leq \varphi_m$, by reordering the elements of the basis in a suitable way.

If L_ξ is the symmetric bilinear form associated with Q_ξ, then $L_\xi(X, Y) = \langle F_X\xi, F_Y\xi\rangle$, for each X, $Y \in \mathfrak{z}$. But then the fact that $\{Z_1, \dots, Z_m\}$ is an orthonormal basis for which Q_ξ assumes a diagonal form is equivalent to $0 = L_\xi(Z_i, Z_j) = \langle F_{Z_i}\xi, F_{Z_j}\xi\rangle$ for all $i \neq j$. This, together with the ordering of $(\varphi_1, \dots, \varphi_m)$ and the fact that $\{Z_1, \dots, Z_m\}$ is an orthonormal basis, implies that the m-tuple $(\varphi_1, \dots, \varphi_m)$ is uniquely defined for a fixed $\mathfrak{w}^\perp$ and a fixed $\xi \in \mathfrak{w}^\perp$. Moreover, due to the bilinearity of L_ξ, it is clear that φ_1 is minimal and φ_m is maximal among the Kähler angles of ξ with respect to all the elements of $\mathfrak{z}$. Finally, we also have that $\langle P_{Z_i}\xi, P_{Z_j}\xi\rangle = \langle J_{Z_i}\xi, J_{Z_j}\xi\rangle - \langle F_{Z_i}\xi, F_{Z_j}\xi\rangle = 0$, whenever $i \neq j$. □

Motivated by Theorem 5.1, we define the *generalized Kähler angle* of ξ with respect to $\mathfrak{w}^\perp$ as the m-tuple $(\varphi_1, \dots, \varphi_m)$ satisfying properties (a)-(d) of Theorem 5.1.

Remark 5.2. Observe that the Kähler angles $\varphi_1, \dots, \varphi_m$ depend, not only on the subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$, but also on the vector $\xi \in \mathfrak{w}^\perp$.

Assuming the notation of the previous theorem, we will say that the subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$ has *constant generalized Kähler angle* $(\varphi_1, \dots, \varphi_m)$ if the m-tuple $(\varphi_1, \dots, \varphi_m)$ is independent of the unit vector $\xi \in \mathfrak{w}^\perp$.

If $\mathfrak{v} = \mathbb{C}^n$ and $\mathfrak{z} = \mathbb{R}$, then the complex structure of $\mathbb{C}^n$ is $J = J_1$. For a given subspace $\mathfrak{w}$ of $\mathbb{C}^n$, we denote $F = F_1$ and $P = P_1$, and we define $\bar{F}\xi = F\xi/\|F\xi\|$ if $F\xi \neq 0$. We will need the following result from [10, Lemma 2]:

Lemma 5.3. *Let $\mathfrak{w}^\perp$ be some linear subspace of $\mathbb{C}^n$, and $\xi \in \mathfrak{w}^\perp$ a unit vector with Kähler angle $\varphi \in (0, \pi/2)$. Then, there exists a unique vector $\eta \in \mathbb{C}^n \ominus \mathbb{C}\xi$ such that $\bar{F}\xi = \cos(\varphi)J\xi + \sin(\varphi)J\eta$.*

5.3 The new examples

The new isoparametric hypersurfaces will be tubes around certain homogeneous submanifolds of a Damek-Ricci space. Thus, in this section, we proceed first with the construction of these submanifolds and then determine their extrinsic geometry. This is done in §5.3.1. The geometry of the tubes around these focal submanifolds is studied in §5.3.2, where their main properties are given.

5.3.1 The focal manifold of the new examples

As we explained above, the new examples are constructed as tubes around certain homogeneous submanifolds. Each isoparametric family will have at most one submanifold that is not a hypersurface. This is the focal submanifold of the family, and we define it in this subsection.

Let AN be a Damek-Ricci space with Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$, where $\dim \mathfrak{z} = m$. Let $\mathfrak{w}$ be a proper subspace of $\mathfrak{v}$ and define $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$, the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{v}$. Then,

$$\mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$$

is a solvable Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, as one can easily check from the bracket relations in §5.1.2. Let $S_{\mathfrak{w}}$ be the corresponding connected subgroup of AN whose Lie algebra is $\mathfrak{s}_{\mathfrak{w}}$. Since AN acts by isometries on itself and $S_{\mathfrak{w}}$ is a subgroup of AN, $S_{\mathfrak{w}}$ is also a homogeneous submanifold of AN.

Let $\xi \in \mathfrak{w}^\perp$ be a unit normal vector field along the submanifold $S_{\mathfrak{w}}$. Let $\{Z_1, \dots, Z_m\}$ be an orthonormal basis of $\mathfrak{z}$ satisfying the properties of Theorem 5.1. In order to simplify the notation, for each $i \in \{1, \dots, m\}$, we set J_i, P_i and F_i instead of J_{Z_i}, P_{Z_i} and F_{Z_i}, respectively. It is convenient to define

$$m_0 = \max\{i : \varphi_i = 0\} + 1 \quad \text{and} \quad m_{\pi/2} = \min\{i : \varphi_i = \pi/2\} - 1,$$

where φ_i is the Kähler angle of ξ with respect to $Z_i \in \mathfrak{z}$ (set $m_0 = 1$ if $\varphi_i > 0$ for all i, and $m_{\pi/2} = m$ if $\varphi_i < \pi/2$ for all i). Thus, m_0 is the first index i for which $\varphi_i > 0$, and $m_{\pi/2}$ is the last index i for which $\varphi_i < \pi/2$. It might of course happen that $m_0 > m$ if $\varphi_i = 0$ for all i, or $m_{\pi/2} < 1$ if $\varphi_i = \pi/2$ for all i, in which case some of the equations that follow are just disregarded.

With this notation we can now define

$$\bar{P}_i\xi = \frac{1}{\sin(\varphi_i)} P_i\xi, \text{ for } i = m_0, \dots, m, \quad \text{and} \quad \bar{F}_i\xi = \frac{1}{\cos(\varphi_i)} F_i\xi, \text{ for } i = 1, \dots, m_{\pi/2}.$$

Since ξ is of unit length, so are $\bar{P}_i\xi$ and $\bar{F}_i\xi$ whenever they exist. Moreover, by Theorem 5.1, the set $\{\bar{P}_{m_0}\xi, \dots, \bar{P}_m\xi, \bar{F}_1\xi, \dots, \bar{F}_{m_{\pi/2}}\xi\}$ constitutes an orthonormal system of vector fields along $S_{\mathfrak{w}}$, the first $m - m_0 + 1$ of which being tangent, and the rest normal to $S_{\mathfrak{w}}$.

We are now interested in calculating the shape operator $\mathcal{S}$ of $S_{\mathfrak{w}}$. Recall that the shape operator $\mathcal{S}_\xi$ of $S_{\mathfrak{w}}$ with respect to a unit normal $\xi \in \nu S_{\mathfrak{w}}$ is defined by $\mathcal{S}_\xi X = -(\bar{\nabla}_X \xi)^\top$, for any $X \in TS_{\mathfrak{w}}$, and where $(\cdot)^\top$ denotes orthogonal projection onto the tangent space. The expression for the Levi-Civita connection of the Damek-Ricci space AN allows us to calculate the shape operator of $S_{\mathfrak{w}}$ for left-invariant vector fields:

$$\begin{aligned}
\mathcal{S}_\xi B &= 0, \\
\mathcal{S}_\xi Z_i &= \frac{1}{2} P_i\xi = 0, \qquad \text{if } i = 1, \dots, m_0 - 1, \\
\mathcal{S}_\xi Z_i &= \frac{1}{2} P_i\xi = \frac{1}{2}\sin(\varphi_i)\bar{P}_i\xi, \qquad \text{if } i = m_0, \dots, m, \\
\mathcal{S}_\xi \bar{P}_i\xi &= \frac{1}{2}[\xi, \bar{P}_i\xi]^\top = \frac{1}{2}\sum_{j=1}^{m} \langle J_j\xi, \bar{P}_i\xi\rangle Z_j = \frac{1}{2}\sin(\varphi_i) Z_i, \qquad \text{if } i = m_0, \dots, m, \\
\mathcal{S}_\xi U &= \frac{1}{2}[\xi, U]^\top = \frac{1}{2}\sum_{j=1}^{m} \langle J_j\xi, U\rangle Z_j = 0, \qquad \text{if } U \in \mathfrak{w} \ominus \Big(\bigoplus_{i=m_0}^{m} \mathbb{R}\bar{P}_i\xi\Big).
\end{aligned}$$

From the expressions above, we obtain that the principal curvatures of $S_{\mathfrak{w}}$ with respect to the unit normal vector ξ are

$$0, \quad \frac{1}{2}\sin\varphi_i, \quad \text{and} \quad -\frac{1}{2}\sin\varphi_i,$$

and their corresponding principal spaces are, respectively,

$$\mathfrak{a} \oplus \Big(\mathfrak{w} \ominus \Big(\bigoplus_{j=m_0}^{m} \mathbb{R}\bar{P}_j\xi\Big)\Big) \oplus \Big(\bigoplus_{j=1}^{m_0-1} Z_j\Big), \quad \mathbb{R}(Z_i + \bar{P}_i\xi), \quad \text{and} \quad \mathbb{R}(Z_i - \bar{P}_i\xi),$$

where $i = m_0, \dots, m$. In any case, the submanifold $S_{\mathfrak{w}}$ is minimal (even austere) and, if $\dim \mathfrak{w}^\perp = 1$, then $S_{\mathfrak{w}}$ is a minimal hypersurface of AN.

Remark 5.4. We emphasize that, although the dependance on ξ is not made explicit in the notation, $(\varphi_1, \dots, \varphi_m)$, $\{Z_1, \dots, Z_m\}$, m_0, and $m_{\pi/2}$ do depend on ξ.

5.3.2 Solving the Jacobi equation

Denote by M^r the tube of radius r around the submanifold $S_{\mathfrak{w}}$ that was described in the previous subsection. We claim that, for every $r > 0$, M^r is an isoparametric hypersurface which has, in general, nonconstant principal curvatures.

In order to show that M^r has the properties mentioned above, we will make use of Jacobi field theory. The main step of our approach is to write down the Jacobi equation along a geodesic normal to $S_{\mathfrak{w}}$ and to solve some initial value problems for this equation. Unlike the standard method used in this kind of arguments, we will express the Jacobi fields in terms of left-invariant vector fields and not in terms of parallel translated vector fields. The relevance of Theorem 5.1 will become clear with our approach.

Given a unit speed geodesic γ in the Damek-Ricci space AN, a vector field ζ along γ is called a Jacobi vector field if it satisfies the Jacobi equation in AN along γ, namely

$$\zeta'' + \bar{R}(\zeta, \dot{\gamma})\dot{\gamma} = 0,$$

where $\dot{\gamma}$ is the tangent vector of γ, and $'$ stands for covariant differentiation along the geodesic γ.

Let $p \in S_{\mathfrak{w}}$ be an element of the submanifold, and $\xi \in \nu_p S_{\mathfrak{w}}$ a unit normal vector at p. Let γ be the geodesic of AN such that $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. Denote by $\dot{\gamma}(t)^\perp$ the orthogonal complement of $\dot{\gamma}(t)$ in $T_{\gamma(t)}AN$ and by $\mathcal{S}$ the shape operator of the submanifold $S_{\mathfrak{w}}$. We denote by ζ_v the Jacobi vector field with initial conditions

(5.1) $\zeta_v(0) = v^\top$, $\zeta_v'(0) = -\mathcal{S}_\xi v^\top + v^\perp$, where $v = v^\top + v^\perp$, $v^\top \in \mathfrak{s}_{\mathfrak{w}}$, and $v^\perp \in \mathfrak{w}^\perp \ominus \mathbb{R}\xi$.

We define the $\operatorname{Hom}\big((\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi, \dot{\gamma}^\perp\big)$-valued tensor fields C and E along γ satisfying $C(r)v = \zeta_v(r)$ and $E(r)v = (\zeta_v'(r))^\top$, for every $r \in \mathbb{R}$ and every left-invariant vector field $v \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi$, where now $(\cdot)^\top$ denotes the projection onto $\dot{\gamma}^\perp$. Standard Jacobi field theory ensures that if $C(r)$ is nonsingular for every unit $\xi \in \nu S_{\mathfrak{w}}$, then the tube M^r of radius r around $S_{\mathfrak{w}}$ is a hypersurface of AN. Moreover, in this case, the shape operator $\mathcal{S}^r$ of M^r at the point $\gamma(r)$ with respect to the unit vector $-\dot{\gamma}(r)$ is given by $\mathcal{S}^r\zeta_v(r) = (\zeta_v'(r))^\top$, for every $v \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi$, that is, $\mathcal{S}^r = E(r)C(r)^{-1}$.

Therefore, our objective in what follows is to determine an explicit expression for the Jacobi fields whose initial conditions are given by (5.1). In order to achieve this goal, we fix here and henceforth an orthonormal basis $\{Z_1, \dots, Z_m\}$ of $\mathfrak{z}$ satisfying the properties of Theorem 5.1, and let $(\varphi_1, \dots, \varphi_m)$ be the corresponding generalized Kähler angle of ξ with respect to $\mathfrak{w}^\perp$. Recall that $(\varphi_1, \dots, \varphi_m)$ and $\{Z_1, \dots, Z_m\}$ depend on ξ, but we remove this dependence from the notation for the sake of simplicity. Let $\{U_1, \dots, U_l\}$ be an orthonormal basis of $\mathfrak{w} \ominus \big(\oplus_{j=m_0}^m \mathbb{R}\bar{P}_j\xi\big)$, and let $\{\eta_1, \dots, \eta_h\}$ be an orthonormal basis of $\mathfrak{w}^\perp \ominus \big(\mathbb{R}\xi \oplus \big(\oplus_{j=1}^{m_{\pi/2}} \bar{F}_j\xi\big)\big)$. Then the set

$$\left\{ \xi, B, U_1, \dots, U_l, \eta_1, \dots, \eta_h, \bar{P}_{m_0}\xi, \dots, \bar{P}_m\xi, \bar{F}_1\xi, \dots, \bar{F}_{m_{\pi/2}}\xi, Z_1, \dots, Z_m \right\} \tag{5.2}$$

constitutes an orthonormal basis of left-invariant vector fields of $\mathfrak{a} \oplus \mathfrak{n}$.

The main step of the proof of Theorem 5.8 is the following

Proposition 5.5. *With the notation as above we have*

$$\begin{aligned}
\zeta_B(t) &= B + \sinh\Big(\frac{t}{2}\Big)\xi,\\
\zeta_{U_i}(t) &= \cosh\Big(\frac{t}{2}\Big)U_i, \quad i = 1,\dots,l,\\
\zeta_{\eta_i}(t) &= 2\sinh\Big(\frac{t}{2}\Big)\eta_i, \quad i = 1,\dots,h,\\
\zeta_{\bar{P}_i\xi}(t) &= \cosh\Big(\frac{t}{2}\Big)\bar{P}_i\xi - \sin(\varphi_i)\sinh(t)Z_i, \quad i = m_0,\dots,m,\\
\zeta_{\bar{F}_i\xi}(t) &= 2\sinh\Big(\frac{t}{2}\Big)\bar{F}_i\xi - 2\cos(\varphi_i)\sinh^2\Big(\frac{t}{2}\Big)Z_i, \quad i = 1,\dots,m_{\pi/2},\\
\zeta_{Z_i}(t) &= \sinh\Big(\frac{t}{2}\Big)F_i\xi + \Big(1 + \sin^2(\varphi_i)\sinh^2\Big(\frac{t}{2}\Big)\Big)Z_i, \quad i = 1,\dots,m.
\end{aligned}$$

Proof. In order to prove this result it suffices to take the expressions above and show that they satisfy the Jacobi equation and the initial conditions (5.1). The calculations are long so we will first show an example of how they are performed for ζ_{Z_i} and, then, we will write down some intermediate calculations for the general case.

First of all, recall that $p \in S_{\mathfrak{w}}$, and $\xi \in \nu_p S_{\mathfrak{w}}$ is a unit normal vector at p. The geodesic γ satisfies $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. By [23, §4.1.11, Theorem 2] we know that $\dot{\gamma}(t) = \operatorname{sech}\big(\frac{t}{2}\big)\xi - \tanh\big(\frac{t}{2}\big)B$, for every $t \in \mathbb{R}$, where ξ and B are considered as left-invariant vector fields on AN. Actually, in [23] this result is stated only for the case when $p = e$ is the identity element of AN. However, since $\gamma_p = L_p \circ \gamma_e$, the homogeneity of $S_{\mathfrak{w}}$ implies that

$$\dot{\gamma}_p(t) = L_{p*}\dot{\gamma}_e(t) = \operatorname{sech}\Big(\frac{t}{2}\Big)\xi - \tanh\Big(\frac{t}{2}\Big)B, \tag{5.3}$$

for every $t \in \mathbb{R}$, where L_p denotes the left multiplication by p in the group AN, γ_e is the normal geodesic through the identity element e with initial velocity ξ, and γ_p is the normal geodesic through the point $p \in AN$ with initial velocity $L_{p*}\xi = \xi$.

It is easy to check that $\zeta_{Z_i}(0) = Z_i$, which is a tangent vector to $\mathfrak{s}_{\mathfrak{w}}$. Now we have to calculate ζ'_{Z_i}. By the Leibniz rule we get

$$\begin{aligned}
\zeta'_{Z_i}(t) = \bar{\nabla}_{\dot{\gamma}(t)}\zeta_{Z_i} = \frac{1}{2}\cosh\Big(\frac{t}{2}\Big)F_i\xi + \sinh\Big(\frac{t}{2}\Big)\bar{\nabla}_{\dot{\gamma}(t)}F_i\xi + \sin^2(\varphi_i)\sinh\Big(\frac{t}{2}\Big)\cosh\Big(\frac{t}{2}\Big)Z_i\\
+ \Big(1 + \sin^2(\varphi_i)\sinh^2\Big(\frac{t}{2}\Big)\Big)\bar{\nabla}_{\dot{\gamma}(t)}Z_i.
\end{aligned} \tag{5.4}$$

Using (5.3), and the formula for the Levi-Civita connection in §5.1.2, we obtain

$$\begin{aligned}
\bar{\nabla}_{\dot{\gamma}(t)}F_i\xi &= \operatorname{sech}\Big(\frac{t}{2}\Big)\bar{\nabla}_\xi F_i\xi - \tanh\Big(\frac{t}{2}\Big)\bar{\nabla}_B F_i\xi = \frac{1}{2}\operatorname{sech}\Big(\frac{t}{2}\Big)[\xi, F_i\xi], \quad \text{and}\\
\bar{\nabla}_{\dot{\gamma}(t)}Z_i &= \operatorname{sech}\Big(\frac{t}{2}\Big)\bar{\nabla}_\xi Z_i - \tanh\Big(\frac{t}{2}\Big)\bar{\nabla}_B Z_i = -\frac{1}{2}\operatorname{sech}\Big(\frac{t}{2}\Big)J_{Z_i}\xi.
\end{aligned}$$

Now, for $j \in \{1, \dots, m\}$, the bracket relations from §5.1.1 yield $\langle [\xi, F_i\xi], Z_j\rangle = \langle J_{Z_j}\xi, F_i\xi\rangle = \langle F_j\xi, F_i\xi\rangle = \cos^2(\varphi_i)\delta_{ij}$, where δ is the Kronecker delta. Thus, $[\xi, F_i\xi] = \cos^2(\varphi_i)Z_i$. Using $J_{Z_i}\xi = P_i\xi + F_i\xi$, and inserting these expressions into (5.4) we obtain

$$\begin{aligned}\zeta'_{Z_i}(t) = &-\frac{1}{2}\operatorname{sech}\Big(\frac{t}{2}\Big)\Big(1+\sin^2(\varphi_i)\sinh^2\Big(\frac{t}{2}\Big)\Big)P_i\xi + \frac{1}{2}\cos^2(\varphi_i)\sinh\Big(\frac{t}{2}\Big)\tanh\Big(\frac{t}{2}\Big)F_i\xi\\ &+\frac{1}{2}\Big(\sin^2(\varphi_i)\sinh(t)+\cos^2(\varphi_i)\tanh\Big(\frac{t}{2}\Big)\Big)Z_i, \quad i=1,\dots,m.\end{aligned}$$

From this expression, and the shape operator of $S_{\mathfrak{w}}$ obtained in §5.3.1 we easily get $\zeta'_{Z_i}(0) = -\frac{1}{2}P_i\xi = -\frac{1}{2}\sin\varphi_i\bar{P}_i\xi = -\mathcal{S}_\xi Z_i$, so the initial conditions (5.1) are satisfied.

The very same approach can be used to calculate ζ''_{Z_i}. We omit the explicit calculations here, which are very similar to those shown above, and give the result:

$$\begin{aligned}\zeta''_{Z_i}(t) = &-\frac{3}{4}\sin^2(\varphi_i)\sinh\Big(\frac{t}{2}\Big)P_i\xi + \frac{1}{16}(6\cos(2\varphi_i)-2)\sinh\Big(\frac{t}{2}\Big)F_i\xi\\ &+\frac{1}{4}\Big(\cosh(t)-2\cos(2\varphi_i)\sinh^2\Big(\frac{t}{2}\Big)\Big)Z_i.\end{aligned}$$

Finally, we need to calculate $\bar{R}(\zeta_{Z_i}(t), \dot\gamma(t))\dot\gamma(t))$. We have the following identities for the curvature tensor, where U, $V \in \mathfrak{v}$, and $Z \in \mathfrak{z}$ are of unit length (for the complete formula, see [23, §4.1.7]):

$$\begin{aligned}&\bar{R}(U,Z)B = \frac{1}{4}J_Z U, && \bar{R}(B,Z)B = Z, && \bar{R}(U,V)V = \frac{1}{4}\big(\langle U,V\rangle V - U + 3J_{[U,V]}V\big),\\ &\bar{R}(U,V)B = \frac{1}{2}[U,V], && \bar{R}(B,U)B = \frac{1}{4}U, && \bar{R}(U,B)V = \frac{1}{4}\langle U,V\rangle B + \frac{1}{4}[U,V],\\ &\bar{R}(B,Z)U = \frac{1}{2}J_Z U, && \bar{R}(U,Z)U = \frac{1}{4}Z, && \bar{R}(U,B)U = \frac{1}{4}B.\end{aligned}$$

Using the properties of the curvature tensor and the formulas above, we get after some calculations $\zeta''_{Z_i} + \bar{R}(\zeta_{Z_i}, \dot\gamma)\dot\gamma = 0$ as we wanted to show.

For the sake of completeness, we provide now some intermediate calculations of the initial value problem (5.1) for the general case. Thus, let ζ be a Jacobi field along the geodesic γ and express it in terms of the basis (5.2):

$$\zeta(t) = x(t)\xi + b(t)B + \sum_{i=1}^{l} u_i(t)U_i + \sum_{i=1}^{h} s_i(t)\eta_i + \sum_{i=m_0}^{m} p_i(t)\bar{P}_i\xi + \sum_{i=1}^{m_{\pi/2}} f_i(t)\bar{F}_i\xi + \sum_{i=1}^{m} z_i(t)Z_i.$$

In order to allow a unified approach, we set $p_i = 0$ if $i \in \{1, \dots, m_0 - 1\}$ and $f_i = 0$ if $i \in \{m_{\pi/2}+1, \dots, m\}$.

Now we write down separately the two terms of the Jacobi equation. Taking into account the formula of the Levi-Civita connection of AN and (5.3) one gets

$$\begin{aligned}
\zeta''(t) = & \left(-\operatorname{sech}(t/2)b'(t) + b(t)\sinh^3(t/2)\operatorname{csch}^2(t) + x''(t) - \frac{1}{4}x(t)\operatorname{sech}^2(t/2)\right)\xi \\
& + \left(b''(t) - \frac{1}{4}b(t)\operatorname{sech}^2(t/2) + \operatorname{sech}(t/2)x'(t) + x(t)\sinh^3(t/2)\left(-\operatorname{csch}^2(t)\right)\right)B \\
& + \sum_{i=1}^{h} s_i''(t)\eta_i + \sum_{i=1}^{l} u_i''(t)U_i \\
& + \frac{1}{4}\sum_{i=m_0}^{m}\big(4p_i''(t) - 4\operatorname{sech}(t/2)\sin(\varphi_i)z_i'(t) - \operatorname{sech}^2(t/2)f_i(t)\sin(\varphi_i)\cos(\varphi_i) \\
& - \operatorname{sech}^2(t/2)p_i(t)\sin^2(\varphi_i) + \tanh(t/2)\operatorname{sech}(t/2)z_i(t)\sin(\varphi_i)\big)\bar{P}_i\xi \\
& + \frac{1}{4}\sum_{i=1}^{m_{\pi/2}}\big(4f_i''(t) - 4\operatorname{sech}(t/2)\cos(\varphi_i)z_i'(t) - \operatorname{sech}^2(t/2)f_i(t)\cos^2(\varphi_i) \\
& - \operatorname{sech}^2(t/2)p_i(t)\sin(\varphi_i)\cos(\varphi_i) + \tanh(t/2)\operatorname{sech}(t/2)z_i(t)\cos(\varphi_i)\big)\bar{F}_i\xi \\
& + \sum_{i=1}^{m}\Big(\operatorname{sech}(t/2)\cos(\varphi_i)f_i'(t) + \operatorname{sech}(t/2)\sin(\varphi_i)p_i'(t) + z_i''(t) - \frac{1}{4}\operatorname{sech}^2(t/2)z_i(t) \\
& -\frac{1}{4}\tanh(t/2)\operatorname{sech}(t/2)f_i(t)\cos(\varphi_i) - \sinh^3(t/2)\operatorname{csch}^2(t)p_i(t)\sin(\varphi_i)\Big)Z_i.
\end{aligned}$$

Using the tensoriality of $\bar{R}$ and the relations for the curvature tensor $\bar{R}$ of a Damek-Ricci space, after some computations we obtain

$$\begin{aligned}
\bar{R}(\zeta,\dot{\gamma})\dot{\gamma} = & -\operatorname{csch}^2(t)\sinh^3(t/2)\left(b(t) + \sinh(t/2)x(t)\right)\xi \\
& - \frac{1}{4}\operatorname{sech}^2(t/2)\left(b(t) + \sinh(t/2)x(t)\right)B - \frac{1}{4}\sum_{i=1}^{l}u_i(t)U_i - \frac{1}{4}\sum_{i=1}^{h}s_i(t)\eta_i \\
& - \frac{1}{8}\operatorname{sech}^2(t/2)\sum_{i=m_0}^{m}\left(3\sin(2\varphi_i)f_i(t) + (4 - 3\cos(2\varphi_i) + \cosh(t))p_i(t)\right. \\
& \left. -6\sin(\varphi_i)\sinh(t/2)z_i(t)\right)\bar{P}_i\xi \\
& - \frac{1}{8}\operatorname{sech}^2(t/2)\sum_{i=1}^{m_{\pi/2}}\left((4 + 3\cos(2\varphi_i) + \cosh(t))f_i(t)\right. \\
& \left. +6\cos(\varphi_i)\left(\sin(\varphi_i)p_i(t) - \sinh(t/2)z_i(t)\right)\right)\bar{F}_i\xi \\
& + \frac{1}{4}\operatorname{sech}(t/2)\sum_{i=1}^{m}\left(3\cos(\varphi_i)\tanh(t/2)f_i(t) + 3\sin(\varphi_i)\tanh(t/2)p_i(t)\right. \\
& \left. +(1 - 2\cosh(t))\operatorname{sech}(t/2)z_i(t)\right)Z_i.
\end{aligned}$$

Hence, the Jacobi equation for the Jacobi vector field ζ is equivalent to the following system of second order differential equations:

$$\begin{aligned}
0 &= x''(t) - \operatorname{sech}(t/2)b'(t) - \frac{1}{4}x(t),\\
0 &= b''(t) + \operatorname{sech}(t/2)x'(t) - \frac{1}{2}b(t)\operatorname{sech}^2(t/2) - 2\operatorname{csch}^2(t)\sinh^3(t/2)x(t),\\
0 &= 4u_i''(t) - u_i(t), \quad i = 1,\dots,l,\\
0 &= 4s_i''(t) - s_i(t), \quad i = 1,\dots,h,\\
0 &= p_i''(t) - \operatorname{sech}(t/2)\sin(\varphi_i)\left(z_i'(t) + \cos(\varphi_i)\operatorname{sech}(t/2)f_i(t) - \tanh(t/2)z_i(t)\right)\\
&\quad - \frac{1}{8}(5 - 4\cos(2\varphi_i) + \cosh(t))\operatorname{sech}^2(t/2)p_i(t), \qquad i = m_0,\dots,m,\\
0 &= f_i''(t) - \operatorname{sech}(t/2)\cos(\varphi_i)\left(z_i'(t) + \operatorname{sech}(t/2)\sin(\varphi_i)p_i(t) - \tanh(t/2)z_i(t)\right)\\
&\quad - \frac{1}{8}(5 + 4\cos(2\varphi_i) + \cosh(t))\operatorname{sech}^2(t/2)f_i(t), \qquad i = 1,\dots,m_{\pi/2},\\
0 &= 2z_i''(t) + 2\operatorname{sech}(t/2)\left(\sin(\varphi_i)p_i'(t) + \cos(\varphi_i)f_i'(t)\right) - (1 + \tanh^2(t/2))z_i(t)\\
&\quad + \operatorname{sech}(t/2)\tanh(t/2)\left(\sin(\varphi_i)p_i(t) + \cos(\varphi_i)f_i(t)\right), \qquad i = 1,\dots,m.
\end{aligned}$$

Now, the initial conditions (5.1) can be rewritten in the following way (the coefficient functions that are not given vanish identically):

- If $v = B$, then $b(0) = 1$, $x(0) = 0$, $b'(0) = 0$ and $x'(0) = \frac{1}{2}$.
- If $v = U_i$, $i \in \{1,\dots,l\}$, then $u_i(0) = 1$ and $u_i'(0) = 0$.
- If $v = \eta_i$, $i \in \{1,\dots,h\}$, then $s_i(0) = 0$ and $s_i'(0) = 1$.
- If $v = \bar{P}_i\xi$, $i \in \{m_0,\dots,m\}$, then $p_i(0) = 1$, $f_i(0) = 0$, $z_i(0) = 0$, $p_i'(0) = 0$, $f_i'(0) = 0$ and $z_i'(0) = -\sin(\varphi_i)$.
- If $v = \bar{F}_i\xi$, $i \in \{1,\dots,m_{\pi/2}\}$, then $p_i(0) = 0$, $f_i(0) = 0$, $z_i(0) = 0$, $p_i'(0) = 0$, $f_i'(0) = 1$ and $z_i'(0) = 0$.
- If $v = Z_i$, $i \in \{1,\dots,m\}$, then $p_i(0) = 0$, $f_i(0) = 0$, $z_i(0) = 1$, $p_i'(0) = 0$, $f_i'(0) = \frac{1}{2}\cos(\varphi_i)$ and $z_i'(0) = 0$.

Just some elementary calculations are needed to check that the vector fields in the statement of this proposition are indeed the solutions to the initial value problems described above. □

Our aim in what follows is to finish the calculation of the shape operator $\mathcal{S}^r$ of M^r at $\gamma(r)$. Recall that we first need to calculate $C(r)\colon (\mathfrak{a}\oplus\mathfrak{n})\ominus\mathbb{R}\xi \to \dot{\gamma}^\perp = T_{\gamma(r)}M^r$, $v \mapsto \zeta_v(r)$. In order to describe this operator we consider the following distributions on $\mathfrak{a}\oplus\mathfrak{n}$:

$$\begin{aligned}
\mathfrak{U} &= \oplus_{j=1}^{l}\mathbb{R}U_j, & \mathfrak{F}_i &= \mathbb{R}\bar{F}_i\xi \oplus \mathbb{R}Z_i, & i &= 1,\dots,m_0-1,\\
\mathfrak{H} &= \oplus_{j=1}^{h}\mathbb{R}\eta_j, & \mathfrak{M}_i &= \mathbb{R}\bar{P}_i\xi \oplus \mathbb{R}\bar{F}_i\xi \oplus \mathbb{R}Z_i, & i &= m_0,\dots,m_{\pi/2},\\
& & \mathfrak{P}_i &= \mathbb{R}\bar{P}_i\xi \oplus \mathbb{R}Z_i, & i &= m_{\pi/2}+1,\dots,m.
\end{aligned}$$

Then, we can decompose

$$(\mathfrak{a}\oplus\mathfrak{n})\ominus\mathbb{R}\xi=\mathbb{R}B\oplus\mathfrak{U}\oplus\mathfrak{H}\oplus\Big(\bigoplus_{i=1}^{m_0-1}\mathfrak{F}_i\Big)\oplus\Big(\bigoplus_{i=m_0}^{m_{\pi/2}}\mathfrak{M}_i\Big)\oplus\Big(\bigoplus_{i=m_{\pi/2}+1}^{m}\mathfrak{P}_i\Big),\text{ and}$$

$$T_{\gamma(r)}M^r=\mathbb{R}\Big(\operatorname{sech}\Big(\frac{r}{2}\Big)B+\tanh\Big(\frac{r}{2}\Big)\xi\Big)\oplus\mathfrak{U}\oplus\mathfrak{H}\oplus\Big(\bigoplus_{i=1}^{m_0-1}\mathfrak{F}_i\Big)\oplus\Big(\bigoplus_{i=m_0}^{m_{\pi/2}}\mathfrak{M}_i\Big)\oplus\Big(\bigoplus_{i=m_{\pi/2}+1}^{m}\mathfrak{P}_i\Big).$$

With respect to these decompositions, a direct application of Proposition 5.5 shows that the operator $C(r)$ can be written as

$$\begin{aligned}C(r)=\;&\cosh\Big(\frac{r}{2}\Big)\operatorname{Id}_{l+1}\oplus\Big(2\sinh\Big(\frac{r}{2}\Big)\operatorname{Id}_h\Big)\oplus\left(\bigoplus_{i=1}^{m_0-1}\begin{pmatrix}2\sinh(\frac{r}{2}) & \sinh(\frac{r}{2})\\ -2\sinh^2(\frac{r}{2}) & 1\end{pmatrix}\right)\\ &\oplus\left(\bigoplus_{i=m_0}^{m_{\pi/2}}\begin{pmatrix}\cosh(\frac{r}{2}) & 0 & 0\\ 0 & 2\sinh(\frac{r}{2}) & \cos(\varphi_i)\sinh(\frac{r}{2})\\ -\sin(\varphi_i)\sinh(r) & -2\cos(\varphi_i)\sinh^2(\frac{r}{2}) & 1+\sin^2(\varphi_i)\sinh^2(\frac{r}{2})\end{pmatrix}\right)\\ &\oplus\left(\bigoplus_{i=m_{\pi/2}+1}^{m}\begin{pmatrix}\cosh(\frac{r}{2}) & 0\\ -\sinh(r) & \cosh^2(\frac{r}{2})\end{pmatrix}\right).\end{aligned}$$

In particular, the determinant of $C(r)$ is

$$\det(C(r))=2^{h+m_{\pi/2}}\Big(\cosh\Big(\frac{r}{2}\Big)\Big)^{2+l+3m-m_0}\Big(\sinh\Big(\frac{r}{2}\Big)\Big)^{h+m_{\pi/2}},$$

which is nonzero for every $r>0$, and hence, the tubes M^r around $S_{\mathfrak{w}}$ are hypersurfaces for every $r>0$.

The next step is to consider the operator $E(r)\colon(\mathfrak{a}\oplus\mathfrak{n})\ominus\mathbb{R}\xi\to T_{\gamma(r)}M^r$, $v\mapsto(\zeta_v'(r))^\top$. To that end, we need to calculate the covariant derivative along γ of the Jacobi vector fields given in Proposition 5.5. We omit the explicit calculations, which follow the procedure described in the proof of Proposition 5.5, and give $E(r)$ with respect to the same decomposition as above:

$$\begin{aligned}E(r)=\;&\frac{1}{2}\sinh\Big(\frac{r}{2}\Big)\operatorname{Id}_{l+1}\oplus\Big(\cosh\Big(\frac{r}{2}\Big)\operatorname{Id}_h\Big)\oplus\left(\bigoplus_{i=1}^{m_0-1}\begin{pmatrix}\cosh(r)\operatorname{sech}(\frac{r}{2}) & \frac{1}{2}\sinh(\frac{r}{2})\tanh(\frac{r}{2})\\ \tanh(\frac{r}{2})-\sinh(r) & \frac{1}{2}\tanh(\frac{r}{2})\end{pmatrix}\right)\\ &\oplus\left(\bigoplus_{i=m_0}^{m_{\pi/2}}A_i(r)\right)\oplus\left(\bigoplus_{i=m_{\pi/2}+1}^{m}\begin{pmatrix}\frac{3}{2}\sinh(\frac{r}{2}) & -\frac{1}{2}\cosh(\frac{r}{2})\\ \frac{1}{2}-\cosh(r) & \frac{1}{2}\sinh(r)\end{pmatrix}\right),\end{aligned}$$

where $A_i(r)$ is

$$\frac{1}{2}\begin{pmatrix}(2-\cos(2\varphi_i))\sinh(\frac{r}{2}) & 2\sin(2\varphi_i)\operatorname{csch}(r)\sinh^3(\frac{r}{2}) & -\sin(\varphi_i)\operatorname{sech}(\frac{r}{2})(1+\sin^2(\varphi_i)\sinh^2(\frac{r}{2}))\\ \sin(2\varphi_i)\sinh(\frac{r}{2}) & 2\cosh(\frac{r}{2})(1+\cos^2(\varphi_i)\tanh^2(\frac{r}{2})) & \cos^3(\varphi_i)\sinh(\frac{r}{2})\tanh(\frac{r}{2})\\ \sin(\varphi_i)(1-2\cosh(r)) & 2\cos(\varphi_i)(\tanh(\frac{r}{2})-\sinh(r)) & \sin^2(\varphi_i)\sinh(r)+\cos^2(\varphi_i)\tanh(\frac{r}{2})\end{pmatrix}.$$

Using the expression $\mathcal{S}^r = E(r)C(r)^{-1}$ and some tedious but elementary calculations we get to the main results of this section.

Proposition 5.6. *The shape operator $\mathcal{S}^r$ of the tube M^r around the homogeneous submanifold $S_{\mathfrak{w}}$ of AN with respect to the decomposition $T_{\gamma(r)}M^r = \mathbb{R}(\operatorname{sech}(\frac{r}{2})B + \tanh(\frac{r}{2})\xi) \oplus \mathfrak{U} \oplus \mathfrak{H} \oplus (\oplus_{i=1}^{m_0-1}\mathfrak{F}_i) \oplus (\oplus_{i=m_0}^{m_{\pi/2}}\mathfrak{M}_i) \oplus (\oplus_{i=m_{\pi/2}+1}^{m}\mathfrak{P}_i)$ is given by*

$$\begin{aligned}\mathcal{S}^r = {} & \left(\frac{1}{2}\tanh\Big(\frac{r}{2}\Big)\operatorname{Id}_{l+1}\right) \oplus \left(\frac{1}{2}\coth\Big(\frac{r}{2}\Big)\operatorname{Id}_h\right) \oplus \left(\bigoplus_{i=1}^{m_0-1}\begin{pmatrix}\frac{1}{2}\coth(\frac{r}{2}) & -\frac{1}{2}\operatorname{sech}(\frac{r}{2})\\ -\frac{1}{2}\operatorname{sech}(\frac{r}{2}) & \tanh(\frac{r}{2})\end{pmatrix}\right)\\ & \oplus \left(\bigoplus_{i=m_0}^{m_{\pi/2}}\begin{pmatrix}\frac{1}{2}\tanh(\frac{r}{2}) & 0 & -\frac{1}{2}\sin(\varphi_i)\operatorname{sech}(\frac{r}{2})\\ 0 & \frac{1}{2}\coth(\frac{r}{2}) & -\frac{1}{2}\cos(\varphi_i)\operatorname{sech}(\frac{r}{2})\\ -\frac{1}{2}\sin(\varphi_i)\operatorname{sech}(\frac{r}{2}) & -\frac{1}{2}\cos(\varphi_i)\operatorname{sech}(\frac{r}{2}) & \tanh(\frac{r}{2})\end{pmatrix}\right)\\ & \oplus \left(\bigoplus_{i=m_{\pi/2}+1}^{m}\begin{pmatrix}\frac{1}{2}\tanh(\frac{r}{2}) & -\frac{1}{2}\operatorname{sech}(\frac{r}{2})\\ -\frac{1}{2}\operatorname{sech}(\frac{r}{2}) & \tanh(\frac{r}{2})\end{pmatrix}\right).\end{aligned}$$

As a consequence, we immediately get

Corollary 5.7. *The mean curvature $\mathcal{H}^r$ of the tube M^r at the point $\gamma(r)$ is*

$$\begin{aligned}\mathcal{H}^r(\gamma(r)) &= \frac{1}{2}\left((h + m_{\pi/2})\coth\left(\frac{r}{2}\right) + (2 + l + 3m - m_0)\tanh\left(\frac{r}{2}\right)\right)\\ &= \frac{1}{2}\left((\operatorname{codim} S_{\mathfrak{w}} - 1)\coth\left(\frac{r}{2}\right) + (\dim S_{\mathfrak{w}} + \dim\mathfrak{z})\tanh\left(\frac{r}{2}\right)\right).\end{aligned}$$

Therefore, for every $r > 0$, the tube M^r around $S_{\mathfrak{w}}$ is a hypersurface with constant mean curvature, and hence, the collection of tubes around the submanifold $S_{\mathfrak{w}}$ constitute an isoparametric family of hypersurfaces in AN, that is, every tube M^r is an isoparametric hypersurface.

We can also give the characteristic polynomial of $\mathcal{S}^r$, which can be written as

$$p_{r,\xi}(x) = (\lambda - x)^{l+1}\left(\frac{1}{4\lambda} - x\right)^h \prod_{i=1}^{m} q^i_{r,\xi}(x),$$

where $\lambda = \frac{1}{2}\tanh\left(\frac{r}{2}\right)$, and

$$\begin{aligned}q^i_{r,\xi}(x) &= x^2 - \left(2\lambda + \frac{1}{4\lambda}\right)x + \frac{1}{4} + \lambda^2, \qquad \text{if } i = 1, \dots, m_0 - 1,\\ q^i_{r,\xi}(x) &= -x^3 + \left(3\lambda + \frac{1}{4\lambda}\right)x^2 - \frac{1}{2}\left(6\lambda^2 + 1\right)x + \frac{16\lambda^4 + 16\lambda^2 - 1 + (4\lambda^2 - 1)^2\cos 2\varphi_i}{32\lambda},\\ &\qquad \text{if } i = m_0, \dots, m_{\pi/2},\\ q^i_{r,\xi}(x) &= x^2 - 3\lambda x - \frac{1}{4} + 3\lambda^2, \qquad \text{if } i = m_{\pi/2} + 1, \dots, m.\end{aligned}$$

The zeroes of $p_{r,\xi}$ are the principal curvatures of the tube M^r at the point $\gamma(r)$. Notice that the zeroes of $q^i_{r,\xi}$ for $i \in \{1, \dots, m_0 - 1\}$ are $\lambda = \frac{1}{2}\tanh\left(\frac{r}{2}\right)$ and $\lambda + \frac{1}{4\lambda} = \coth(r)$, while for $i \in \{m_{\pi/2} + 1, \dots, m\}$ they are $\frac{1}{2}\left(3\lambda \pm \sqrt{1 - 3\lambda^2}\right)$. If $i \in \{m_0, \dots, m_{\pi/2}\}$ the zeroes of $q^i_{r,\xi}$ are given by complicated expressions, because they are solutions to a cubic polynomial. This polynomial coincides with the one in [15, p. 146], where an analysis of its zeroes is carried out.

From these results we deduce that, in general, the principal curvatures of M^r, and even the number of principal curvatures of M^r, may vary from point to point, which implies that, in general, M^r is an inhomogeneous hypersurface. Actually, *the principal curvatures of M^r are constant if and only if $\mathfrak{w}^\perp$ has constant generalized Kähler angle*, that is, if the m-tuple $(\varphi_1, \dots, \varphi_m)$ does not depend on ξ.

We summarize the main results obtained so far.

Theorem 5.8. *Let AN be a Damek-Ricci space with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ is one-dimensional and $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is a generalized Heisenberg algebra with center $\mathfrak{z}$. Let $S_\mathfrak{w}$ be the connected subgroup of AN whose Lie algebra is $\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$, where $\mathfrak{w}$ is any proper subspace of $\mathfrak{v}$.*

Then, the tubes around the submanifold $S_\mathfrak{w}$ are isoparametric hypersurfaces of AN, and have constant principal curvatures if and only if $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ has constant generalized Kähler angle.

5.4 Rank-one symmetric spaces of noncompact type

In this section we present some particular examples of isoparametric families of hypersurfaces in the noncompact rank one symmetric spaces of nonconstant curvature. Note that, in the case of real hyperbolic spaces, our method only gives rise to tubes around totally geodesic real hyperbolic subspaces, which are well-known examples.

5.4.1 Complex hyperbolic spaces $\mathbb{C}H^n$

When the Damek-Ricci space under consideration is a complex hyperbolic space, it is possible to completely characterize, not only the examples with constant principal curvatures arisen from our construction, but also the homogeneous examples.

First notice that the notion of generalized Kähler angle reduces to the concept of Kähler angle if $m = \dim \mathfrak{z} = 1$, i.e. if $AN = \mathbb{C}H^n$. It was proved in [10] that the tubes around the submanifold $S_\mathfrak{w}$ are homogeneous precisely when $\mathfrak{w}^\perp$ has constant Kähler angle, that is, when $\varphi = \varphi_1$ is independent of the vector $\xi \in \mathfrak{w}^\perp$ (this will also follow from the considerations in Section 6.6). Indeed, the Berndt-Brück submanifolds W^{2n-k}_φ that we have defined in §2.6.1 are precisely those $S_\mathfrak{w}$ for which $\mathfrak{w}^\perp$ has *constant* Kähler angle φ and $k = \dim \mathfrak{w}^\perp$. Then, we have the following improvement of Theorem 5.8 for the case $AN = \mathbb{C}H^n$.

Theorem 5.9. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of the isometry group $G = \mathrm{SU}(1,n)$ of $\mathbb{C}H^n$ with respect to a point $o \in \mathbb{C}H^n$. Assume $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace and let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the root space decomposition with respect to $\mathfrak{a}$. Let $S_\mathfrak{w}$ be the connected subgroup of G whose Lie algebra is $\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is any proper linear subspace of $\mathfrak{g}_\alpha$.*

Then, the tubes around the minimal submanifold $S_\mathfrak{w}$ are isoparametric hypersurfaces of $\mathbb{C}H^n$. Moreover, the following conditions are equivalent:

(i) *each tube around $S_\mathfrak{w}$ is a homogeneous hypersurface of $\mathbb{C}H^n$,*

(ii) *each tube around $S_\mathfrak{w}$ has constant principal curvatures,*

(iii) *$\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ has constant Kähler angle φ,*

(iv) *$S_\mathfrak{w}$ is a Lohnherr-Berndt-Brück submanifold W^{2n-k}_φ.*

In view of the information provided in the previous section, it follows that the expression for the characteristic polynomial of the shape operator $\mathcal{S}^r$ of the tube M^r of radius r around the submanifold $S_\mathfrak{w}$ adopts the following form

$$p_{r,\xi}(x) = (\lambda - x)^{2n-k-2}\left(\frac{1}{4\lambda} - x\right)^{k-2} q_{r,\xi}(x),$$

where $k = \dim \mathfrak{w}^\perp$, $\lambda = \frac{1}{2}\tanh\frac{r}{2}$ and

$$q_{r,\xi}(x) = -x^3 + \left(3\lambda + \frac{1}{4\lambda}\right)x^2 - \frac{1}{2}\left(6\lambda^2 + 1\right)x + \frac{16\lambda^4 + 16\lambda^2 - 1 + (4\lambda^2-1)^2\cos 2\varphi}{32\lambda}.$$

It is important to remark that, *pointwise*, M^r has the same principal curvatures as the tubes around the Berndt-Brück submanifolds W^{2n-k}_φ, $\varphi \in [0, \pi/2]$, $k \in \{1, \ldots, n-1\}$ (see Table 2.2 and recall that here $c = -1$); notice that for $\varphi = 0$ these are tubes around a totally geodesic $\mathbb{C}H^{k'}$, $k' \in \{1, \ldots, n-1\}$, in $\mathbb{C}H^n$. In other words, at each point, the tubes around $S_\mathfrak{w}$ have the same principal curvatures as the homogeneous hypersurfaces that arise as tubes around the W^{2n-k}_φ. However, in general, the principal curvatures, and even the number of principal curvatures, vary from point to point in M^r, due to the fact that φ may depend on ξ. Again, when φ is independent of ξ, then $S_\mathfrak{w}$ is a Lohnherr-Berndt-Brück submanifold.

If $n = 2$, then either $\mathfrak{w}^\perp$ is 1-dimensional, in which case $S_\mathfrak{w}$ is a Lohnherr hypersurface W^3, whose equidistant hypersurfaces are homogeneous, or $\mathfrak{w}^\perp = \mathfrak{g}_\alpha$, which gives a totally geodesic $\mathbb{C}H^1$ and thus the tubes around it are also homogeneous. In any case, for $n = 2$ we do not get inhomogeneous examples.

If $n = 3$, then $\mathfrak{g}_\alpha \equiv \mathbb{C}^2$. If $\dim \mathfrak{w}^\perp \in \{1, 4\}$, the situation is similar as above, and $S_\mathfrak{w}$ is a Lohnherr hypersurface W^5 if $\dim \mathfrak{w}^\perp = 1$, or a totally geodesic $\mathbb{C}H^1$ if $\dim \mathfrak{w}^\perp = 4$. If $\dim \mathfrak{w}^\perp = 2$, then $\mathfrak{w}^\perp$ must have constant Kähler angle $\varphi \in [0, \pi/2]$, so $S_\mathfrak{w}$ is a Berndt-Brück submanifold W^4_φ. Again, all tubes around $S_\mathfrak{w}$ are homogeneous. But if

$\dim \mathfrak{w}^\perp = 3$ one obtains inhomogeneous examples. In fact, any 3-dimensional subspace $\mathfrak{w}^\perp$ of $\mathfrak{g}_\alpha$ has the form $\text{span}\{e_1, Je_1, e_2\}$ for some orthonormal basis $\{e_1, Je_1, e_2, Je_2\}$ of $\mathfrak{g}_\alpha$, from where it follows that the vectors of $\mathfrak{w}^\perp$ have Kähler angles that take all values in the interval $[0, \pi/2]$. However, note that if $n = 3$, our construction provides only one family of inhomogeneous isoparametric hypersurfaces up to isometric congruence, since any two 3-dimensional subspaces of $\mathfrak{g}_\alpha \equiv \mathbb{C}^2$ are mapped into each other by a unitary transformation of $\mathrm{U}(2) \cong \mathrm{Ad}(K_0)$.

If $n \geq 4$, then we always obtain uncountably many inhomogeneous isoparametric families. To see this, let us consider an orthonormal basis of $\mathfrak{g}_\alpha \equiv \mathbb{C}^{n-1}$ of the form $\{e_1, Je_1, \dots, e_{n-1}, Je_{n-1}\}$, and for each $\psi \in [0, \pi/2]$ define the subspaces

$$\mathfrak{w}_\psi^\perp = \text{span}\{e_1, Je_1, e_2, \cos(\psi)Je_2 + \sin(\psi)e_3\}.$$

Then it is easy to show that the Kähler angles of the vectors in $\mathfrak{w}_\psi^\perp$ take all values in the interval $[0, \psi]$. It follows that, for each $\psi \in (0, \pi/2]$, our construction gives an inhomogeneous isoparametric family of hypersurfaces, and different values of ψ produce noncongruent examples since the corresponding hypersurfaces have different principal curvatures in view of the characteristic polinomial $p_{r,\xi}$ of $\mathcal{S}^r$.

5.4.2 Quaternionic hyperbolic spaces $\mathbb{H}H^n$

Our definition of generalized Kähler angle includes as a particular case the notion of quaternionic Kähler angle introduced in [10]. The construction of several examples of subspaces of $\mathbb{H}^{n-1}$ with constant quaternionic Kähler angle led Berndt and Brück to some examples of cohomogeneity one actions on $\mathbb{H}H^n$. In [21], Berndt and Tamaru proved that these examples exhaust all cohomogeneity one actions on $\mathbb{H}H^n$ with a non-totally geodesic singular orbit whenever $n = 2$ or the codimension of the singular orbit is two.

Moreover, they reduced the problem of classifying cohomogeneity one actions on $\mathbb{H}H^n$ to the following one: *find all subspaces $\mathfrak{w}^\perp$ of $\mathfrak{v} = \mathbb{H}^{n-1}$ with constant quaternionic Kähler angle and determine for which of them there exists a subgroup of* $\mathrm{Sp}(1)\mathrm{Sp}(n-1)$ *that acts transitively on the unit sphere of $\mathfrak{w}^\perp$ (via the standard representation on $\mathbb{H}^{n-1}$).* However, a complete classification of cohomogeneity one actions on quaternionic hyperbolic spaces is not yet known, and neither is a classification of the subspaces of $\mathbb{H}^{n-1}$ with constant quaternionic Kähler angle, which seems to be a difficult linear algebra problem. Furthermore, it is not clear whether an answer to this latter problem would directly lead to the answer of the former. In fact, in view of Theorem 5.8 a subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$ with constant quaternionic Kähler angle gives rise to an isoparametric hypersurface in $\mathbb{H}H^n$ with constant principal curvatures, but then one would have to decide whether this hypersurface is homogeneous or not. Nonetheless, what Theorem 5.8 guarantees, as well as in the case of complex hyperbolic spaces, is the existence of inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in $\mathbb{H}H^n$, for every $n \geq 3$.

The subspaces of $\mathbb{H}^{n-1}$ with constant quaternionic Kähler angle known up to now can take the following values of quaternionic Kähler angles [21]: $(0, 0, 0)$, $(0, \pi/2, \pi/2)$,

$(\pi/2, \pi/2, \pi/2)$, $(0, 0, \pi/2)$, $(\varphi, \pi/2, \pi/2)$ and $(0, \varphi, \varphi)$. In this subsection, we will give new examples of subspaces of $\mathbb{H}^{n-1}$, $n \geq 5$, with constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$, with $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$, and $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$. This includes, for example, the cases $(\varphi, \varphi, \varphi)$, with $0 < \varphi < \pi/2$, and $(\varphi_1, \varphi_2, \pi/2)$, with $\cos(\varphi_1)+\cos(\varphi_2) < 1$. Theorem 5.8 ensures that these new subspaces yield new examples of isoparametric hypersurfaces with constant principal curvatures in $\mathbb{H}H^n$. In fact, these hypersurfaces are homogeneous, as shown in Theorem 5.11. This provides a large new family of cohomogeneity one actions on quaternionic hyperbolic spaces.

From now on in this subsection, $(i, i+1, i+2)$ will always be a cyclic permutation of $(1, 2, 3)$. Fix a canonical basis $\{J_1, J_2, J_3\}$ of the quaternionic structure of $\mathbb{H}^{n-1}$, that is, $J_i^2 = -\operatorname{Id}$ and $J_i J_{i+1} = J_{i+2} = -J_{i+1} J_i$, with $i \in \{1, 2, 3\}$.

Let $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$ with $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$, and consider a four dimensional totally real subspace of $\mathbb{H}^{n-1}$ and a basis of unit vectors $\{e_0, e_1, e_2, e_3\}$ of it, where $\langle e_0, e_i\rangle = 0$, for $i = 1, 2, 3$, and

$$\langle e_i, e_{i+1}\rangle = \frac{\cos(\varphi_{i+2}) - \cos(\varphi_i)\cos(\varphi_{i+1})}{\sin(\varphi_i)\sin(\varphi_{i+1})}, \quad i = 1, 2, 3. \tag{5.5}$$

The existence of these vectors is ensured by the following

Lemma 5.10. *We have:*

(a) *Let* α_1, α_2, $\alpha_3 \in \mathbb{R}$. *Then, there exists a basis of unit vectors* $\{e_1, e_2, e_3\}$ *of* $\mathbb{R}^3$ *such that* $\langle e_i, e_{i+1}\rangle = \alpha_{i+2}$ *if and only if* $|\alpha_i| < 1$ *for all* i *and* $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 < 1 + 2\alpha_1\alpha_2\alpha_3$.

(b) *Assume* $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$. *Then, there exists a basis of unit vectors* $\{e_1, e_2, e_3\}$ *of* $\mathbb{R}^3$ *with the inner products as in* (5.5) *if and only if* $\cos(\varphi_1)+\cos(\varphi_2) < 1+\cos(\varphi_3)$.

Proof. The proof of (a) is elementary so we omit it.

For the proof of (b), we define $x_i = \cos(\varphi_i)$. With this notation, the conditions $|\alpha_i| < 1$ in part (a) are equivalent to $x_1^2+x_2^2+x_3^2 < 1+2x_1x_2x_3$, whereas the condition $\alpha_1^2+\alpha_2^2+\alpha_3^2 < 1 + 2\alpha_1\alpha_2\alpha_3$ turns out to be equivalent to

$$\frac{(x_1 - x_2 - x_3 + 1)(x_1 + x_2 - x_3 - 1)(x_1 - x_2 + x_3 - 1)(x_1 + x_2 + x_3 + 1)}{(x_1^2 - 1)(x_2^2 - 1)(x_3^2 - 1)} < 0.$$

Since $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$, we have $0 \leq x_3 \leq x_2 \leq x_1 < 1$, and thus the equation above is equivalent to $x_1+x_2-x_3-1 < 0$. Finally, it is not hard to show that $0 \leq x_3 \leq x_2 \leq x_1 < 1$ and $x_1 + x_2 - x_3 - 1 < 0$ imply $x_1^2 + x_2^2 + x_3^2 < 1 + 2x_1x_2x_3$. □

For the sake of simplicity let us define $\varphi_0 = 0$ and $J_0 = \operatorname{Id}$. Notice that $\langle J_j e_k, e_l\rangle = 0$ for $j \in \{1, 2, 3\}$ and k, $l \in \{0, 1, 2, 3\}$, because $\operatorname{span}\{e_0, e_1, e_2, e_3\}$ is a totally real subspace of $\mathbb{H}^{n-1}$. Then we can define

$$\xi_k = \cos(\varphi_k)J_k e_0 + \sin(\varphi_k)J_k e_k, \quad k \in \{0, 1, 2, 3\}.$$

(Note that $\xi_0 = e_0$.) We consider the subspace $\mathfrak{w}^\perp$ generated by these four vectors, for which $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ is an orthonormal basis. Now, taking a generic unit vector $\xi = a_0\xi_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_3 \in \mathfrak{w}^\perp$, some straightforward calculations show that the matrix of the quadratic form Q_ξ defined in Theorem 5.1 with respect to the basis $\{J_1, J_2, J_3\}$ (i.e. the matrix whose entries are $\langle F_j\xi, F_k\xi\rangle = \sum_{l=0}^{3}\langle J_j\xi, \xi_l\rangle\langle J_k\xi, \xi_l\rangle$, for j, $k \in \{1, 2, 3\}$) is the diagonal matrix with entries $\cos^2(\varphi_1)$, $\cos^2(\varphi_2)$, $\cos^2(\varphi_3)$. Therefore, $\mathfrak{w}^\perp$ has constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$.

Next, we show that the submanifold $S_\mathfrak{w}$ is the singular orbit of a cohomogeneity one action on $\mathbb{H}H^n$, and hence, the tubes around $S_\mathfrak{w}$ are homogeneous isoparametric hypersurfaces. Let $G = \mathrm{Sp}(1, n)$ be the connected component of the identity of the isometry group of $\mathbb{H}H^n$, and let $K = \mathrm{Sp}(1)\mathrm{Sp}(n)$ be the isotropy group of G at the identity element of $AN = \mathbb{H}H^n$. Denote by $N_K(S_\mathfrak{w}) = \{k \in K : kS_\mathfrak{w}k^{-1} \subset S_\mathfrak{w}\}$ the normalizer of $S_\mathfrak{w}$ in K, and by $N_K^0(S_\mathfrak{w})$ the connected component of the identity. Notice that $S_\mathfrak{w}$ can be seen as a submanifold of $AN = \mathbb{H}H^n$, and also as a subgroup of $AN \subset G$. We have:

Theorem 5.11. *Let $\mathfrak{w}^\perp$ be the subspace of $\mathfrak{v} = \mathbb{H}^{n-1}$ of constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$, with $0 < \varphi_1 \le \varphi_2 \le \varphi_3 \le \pi/2$, and $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$, as defined above, and consider $\mathfrak{w} = \mathfrak{v} \ominus \mathfrak{w}^\perp$. Then:*

(a) *The tubes around the submanifold $S_\mathfrak{w}$ are isoparametric hypersurfaces with constant principal curvatures.*

(b) *There is a subgroup of $\mathrm{Sp}(1)\mathrm{Sp}(n-1)$ that acts transitively on the unit sphere of $\mathfrak{w}^\perp$ (via the standard representation on $\mathfrak{v} = \mathbb{H}^{n-1}$).*

(c) *The subgroup $N_K^0(S_\mathfrak{w})S_\mathfrak{w}$ of G acts isometrically with cohomogeneity one on $\mathbb{H}H^n$, $S_\mathfrak{w}$ is a singular orbit of this action, and the other orbits are tubes around $S_\mathfrak{w}$.*

Proof. Assertion (a) follows from previous calculations in this section. Part (c) follows from part (b) using [10, p. 220] (cf. [21, Theorem 4.1(i)]). Let us then prove (b). For the case $(\varphi, \pi/2, \pi/2)$, with $0 < \varphi \le \pi/2$, assertion (b) is already known to be true [10]. If $0 < \varphi_1 \le \varphi_2 < \varphi_3 = \pi/2$, then the proof given below needs to be adapted: we would get $F_3 = 0$, and hence $\bar{F}_3$ would not be defined; in that case we would explicitly define $\bar{F}_3 = \bar{F}_1\bar{F}_2$. Thus, we will assume $\varphi_3 < \pi/2$ in what follows.

Let $i \in \{1, 2, 3\}$. As above, henceforth $(i, i+1, i+2)$ will be a cyclic permutation of $(1, 2, 3)$. First note that $F_i\xi_0 = \langle J_i\xi_0, \xi_i\rangle\xi_i = \cos(\varphi_i)\xi_i$, and thus $\bar{F}_i\xi_0 = \xi_i$. This implies $\langle F_i\xi_{i+1}, \xi_0\rangle = -\langle \xi_{i+1}, F_i\xi_0\rangle = -\cos(\varphi_i)\langle \xi_{i+1}, \xi_i\rangle = 0$. The skew-symmetry of J_i yields $\langle F_i\xi_{i+1}, \xi_{i+1}\rangle = 0$, and from $J_i\xi_{i+1} = \cos(\varphi_{i+1})J_{i+2}e_0 + \sin(\varphi_{i+1})J_{i+2}e_{i+1}$ we get $\langle F_i\xi_{i+1}, \xi_i\rangle = 0$ using $\langle J_je_k, e_l\rangle = 0$. Altogether this implies that $F_i\xi_{i+1}$ must be a multiple of ξ_{i+2}, and hence one readily gets $\bar{F}_i\xi_{i+1} = \xi_{i+2} = -\bar{F}_{i+1}\xi_i$. Applying these results twice, we get $\bar{F}_i\bar{F}_{i+1} = \bar{F}_{i+2} = -\bar{F}_{i+1}\bar{F}_i$, and $\bar{F}_i^2 = -\mathrm{Id}$, so $\{\bar{F}_1, \bar{F}_2, \bar{F}_3\}$ is a quaternionic structure on $\mathfrak{w}^\perp$.

Let $\eta_0 \in \mathfrak{w}^\perp$ be an arbitrary unit vector. We define $f_0 = \eta_0$ and apply Lemma 5.3 to find unit vectors f_1, f_2, $f_3 \in \mathbb{H}^{n-1}$ orthogonal to f_0, such that $\eta_i = \bar{F}_if_0 = \cos(\varphi_i)J_if_0 +$

$\sin(\varphi_i)J_i f_i$. Then $f_i = -(J_i\bar{F}_i f_0 + \cos(\varphi_i)f_0)/\sin(\varphi_i)$. We easily obtain $\langle J_i\bar{F}_i f_0, f_0\rangle = -\langle \bar{F}_i f_0, F_i f_0\rangle = -\cos(\varphi_i)$, and using $\bar{F}_{i+2}\bar{F}_{i+1} = -\bar{F}_i$ we get

$$\begin{aligned}\langle J_i\bar{F}_i f_0, J_{i+1}\bar{F}_{i+1}f_0\rangle &= -\langle \bar{F}_i f_0, J_{i+2}\bar{F}_{i+1}f_0\rangle = -\cos(\varphi_{i+2})\langle \bar{F}_i f_0, \bar{F}_{i+2}\bar{F}_{i+1}f_0\rangle \\ &= \cos(\varphi_{i+2})\langle \bar{F}_i f_0, \bar{F}_i f_0\rangle = \cos(\varphi_{i+2}).\end{aligned}$$

Altogether this implies

$$\langle f_i, f_{i+1}\rangle = \frac{\cos(\varphi_{i+2}) - \cos(\varphi_i)\cos(\varphi_{i+1})}{\sin(\varphi_i)\sin(\varphi_{i+1})} = \langle e_i, e_{i+1}\rangle.$$

We show that $\langle J_j f_k, f_l\rangle = 0$ for all $j \in \{1,2,3\}$, and $k,\ l \in \{0,1,2,3\}$. For example,

$$\langle J_i f_{i+1}, f_{i+2}\rangle = \frac{1}{\sin(\varphi_{i+1})\sin(\varphi_{i+2})}\langle J_{i+2}\bar{F}_{i+1}f_0 + \cos(\varphi_{i+1})J_i f_0, J_{i+2}\bar{F}_{i+2}f_0 + \cos(\varphi_{i+2})f_0\rangle.$$

Using the properties of the generalized Kähler angle (see Theorem 5.1), and the definition of $\bar{F}_i$, we obtain $\langle J_{i+2}\bar{F}_{i+1}f_0, J_{i+2}\bar{F}_{i+2}f_0\rangle = \langle \bar{F}_{i+1}f_0, \bar{F}_{i+2}f_0\rangle = 0$. Similarly, one gets $\langle J_{i+2}\bar{F}_{i+1}f_0, f_0\rangle = \langle J_i f_0, J_{i+2}\bar{F}_{i+2}f_0\rangle = \langle J_i f_0, f_0\rangle = 0$, and hence $\langle J_i f_{i+1}, f_{i+2}\rangle = 0$. Other combinations of indices can be handled analogously to obtain $\langle J_j f_k, f_l\rangle = 0$.

Now, one can apply the Gram-Schmidt process to $\{e_0, e_1, e_2, e_3\}$ to obtain an $\mathbb{H}$-orthonormal set $\{e'_0, e'_1, e'_2, e'_3\}$, and similarly with $\{f_0, f_1, f_2, f_3\}$, to obtain an $\mathbb{H}$-orthonormal set $\{f'_0, f'_1, f'_2, f'_3\}$. Then there exists an element $T \in \mathrm{Sp}(4) \subset \mathrm{Sp}(n-1) \subset \mathrm{Sp}(1)\mathrm{Sp}(n-1)$ such that $Te'_i = f'_i$ for $i = 0,1,2,3$, and $TJ_l = J_l T$ for $l = 1,2,3$. Since the transition matrices from $\{e'_i\}$ to $\{e_i\}$, and from $\{f'_i\}$ to $\{f_i\}$ coincide, we get $Te_i = f_i$, and hence $T\xi_i = \eta_i$ for $i = 0,1,2,3$. Therefore, $T\xi_0 = \eta_0$ and $T\mathfrak{w}^\perp = \mathfrak{w}^\perp$. Since $\eta_0 \in \mathfrak{w}^\perp$ is arbitrary, (b) follows. ☐

Remark 5.12. This construction can be extended to subspaces of $\mathbb{H}^{n-1}$ (for n sufficiently high) with real dimension multiple of four and with constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$ as before, just by considering orthogonal sums of subspaces $\mathfrak{w}^\perp$ like the one constructed above. Theorem 5.11 can easily be extended to show the homogeneity of the corresponding isoparametric hypersurfaces.

5.4.3 The Cayley hyperbolic plane $\mathbb{O}H^2$

Based on some results of [10], Berndt and Tamaru achieved the classification of homogeneous hypersurfaces in the Cayley hyperbolic plane $\mathbb{O}H^2$ [21]. Some of these homogeneous examples appear as particular cases of the construction we have developed. If we put $k = \dim \mathfrak{w}^\perp$, then for $k \in \{1,2,3,4,6,7,8\}$ the tubes M^r around $S_\mathfrak{w}$ are homogeneous hypersurfaces for every $r > 0$ and, together with $S_\mathfrak{w}$, constitute the orbits of a cohomogeneity one action; if $k = 5$, none of the tubes around $S_\mathfrak{w}$ is homogeneous [10, p. 233].

Therefore, our method yields a family of inhomogeneous isoparametric hypersurfaces if and only if the codimension of $S_\mathfrak{w}$ is $k = 5$. It can be shown that all isoparametric families that arise in this way with $k = 5$ are congruent, see [21, p. 3436]. But for this case, something surprising happens: these hypersurfaces have constant principal curvatures.

Let ξ be a unit vector in $\mathfrak{w}^\perp$. Taking into account that $\mathfrak{v} = \mathbb{R}^8$ is an irreducible Clifford module of $\mathfrak{z} = \mathbb{R}^7$, the properties of generalized Heisenberg algebras imply that the linear map $Z \in \mathfrak{z} \mapsto J_Z\xi \in \mathfrak{v} \ominus \mathbb{R}\xi$ is an isometry. Hence, we can find an orthonormal basis $\{Z_1, \ldots, Z_7\}$ of the vector space $\mathfrak{z}$ such that $\mathfrak{w} = \operatorname{span}\{J_{Z_5}\xi, J_{Z_6}\xi, J_{Z_7}\xi\}$ and $\mathfrak{w}^\perp = \operatorname{span}\{\xi, J_{Z_1}\xi, J_{Z_2}\xi, J_{Z_3}\xi, J_{Z_4}\xi\}$. It is then clear that $J_{Z_5}, J_{Z_6}, J_{Z_7}$ map ξ into $\mathfrak{w}$ and $J_{Z_1}, J_{Z_2}, J_{Z_3}, J_{Z_4}$ map ξ into $\mathfrak{w}^\perp$. By definition, the generalized Kähler angle of ξ with respect to $\mathfrak{w}^\perp$ is $(0, 0, 0, 0, \pi/2, \pi/2, \pi/2)$.

As the above argument is valid for every unit $\xi \in \mathfrak{w}^\perp$, we conclude that $\mathfrak{w}^\perp$ has constant Kähler angle and so, by Theorem 5.8 and the inhomogeneity result in [10], we obtain:

Theorem 5.13. *The tubes around the homogeneous submanifolds $S_\mathfrak{w}$ with* $\dim \mathfrak{w}^\perp = 5$ *are inhomogeneous isoparametric hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane.*

Let us consider now the case $k = 4$. A similar argument as above can be used to show that the generalized Kähler angle of $\mathfrak{w}^\perp$ is $(0, 0, 0, \pi/2, \pi/2, \pi/2, \pi/2)$. Then, the calculations before Theorem 5.8 show that for any choice of $\mathfrak{w}^\perp$ the principal curvatures, and their corresponding multiplicities, of the tube of radius r around $S_\mathfrak{w}$, depend only on r. By [21, Theorem 4.7], it follows that there are uncountably many orbit equivalence classes of cohomogeneity one actions on $\mathbb{O}H^2$ arising from this method with $k = 4$. Therefore, we obtain an *uncountable set of noncongruent homogeneous isoparametric families with the same constant principal curvatures, counted with multiplicities.* This phenomenon was known in the inhomogeneous case for spheres [63], and in the homogeneous case for noncompact symmetric spaces of rank higher than two [19].

Chapter 6

Polar actions on complex hyperbolic spaces

In the investigation of isoparametric hypersurfaces carried out in the previous chapters, an important role has been played by the classification of homogeneous hypersurfaces in complex hyperbolic spaces. Homogeneous hypersurfaces are principal orbits of cohomogeneity one actions. In this chapter we study the so-called polar actions, which are an important kind of isometric actions which subsumes cohomogeneity one actions as a particular case. Our goal here will be to obtain the classification of polar actions on complex hyperbolic spaces up to orbit equivalence (Theorem 6.4). The results in this chapter can also be found in the paper [50].

In order to contextualize this problem, in Section 6.1 we give the definition of polar action and review some important milestones in the study of such actions on symmetric spaces. In Section 6.2 we recall the (already known) classification of polar actions on complex projective spaces. From the differences between both classification results (that is, on complex projective and complex hyperbolic spaces), it will be apparent that the classification on $\mathbb{C}H^n$ cannot be obtained from the known classification on $\mathbb{C}P^n$ using duality of symmetric spaces. In Section 6.3, we exhibit what was known about polar actions on $\mathbb{C}H^n$ before we had obtained the complete classification, and we state our classification result in Theorem 6.4.

Sections 6.4 to 6.9 develop the ingredients for the proof of our result. In Section 6.4 we state a useful criterion to decide when an action on a noncompact symmetric space is polar, and we also show that sections of polar actions on complex hyperbolic spaces must be totally real. In Section 6.5 we present some important facts on real subspaces of complex vector spaces. These facts are crucial to understand the new examples of polar actions. These new examples are constructed in Section 6.6, where we also present an outline of the proof of the classification theorem. This proof has two main parts depending on whether the group acting upon leaves a totally geodesic subspace invariant (Section 6.7) or is contained in a maximal parabolic subgroup of $\mathrm{SU}(1,n)$ (Section 6.8). Finally, the proof of Theorem 6.4 is concluded in Section 6.9.

6.1 Polar actions

Let M be a Riemannian manifold and $I(M)$ its isometry group. It is known that $I(M)$ is a Lie group [111]. Let H be a connected closed subgroup of $I(M)$. The action of H on M is called *polar* if there exists an immersed submanifold Σ of M such that:

1. Σ intersects all the orbits of the H-action, and
2. for each $p \in \Sigma$, the tangent space of Σ at p, $T_p\Sigma$, and the tangent space of the orbit through p at p, $T_p(H \cdot p)$, are orthogonal.

In such a case, the submanifold Σ is called a *section* of the H-action. The action of H is called *hyperpolar* if the section Σ is flat in its induced Riemannian metric.

Recall that two isometric Lie group actions on two Riemannian manifolds M and N are said to be *orbit equivalent* if there is an isometry $M \to N$ which maps connected components of orbits onto connected components of orbits. They are said to be *conjugate* if there exists an equivariant isometry $M \to N$.

A first important objective in the study of polar actions is to classify them on certain Riemannian manifolds of particular interest, as is the case of Riemannian symmetric spaces. From the geometric viewpoint, that is, if one is mainly interested in the geometry of the orbits, it is enough to obtain such classifications up to orbit equivalence. The aim of the last sections of this chapter is, precisely, to classify polar actions on complex hyperbolic spaces up to orbit equivalence.

We now give a quick overview of the historical evolution of the study of polar actions. The reader is referred to the survey articles [145], [146] and [46] for more detailed information and references on polar actions.

The notion of polar action was pioneered by Szenthe [130] and by Palais and Terng [120], who investigated the fundamental properties of polar actions on Riemannian manifolds. The first classification on a concrete manifold was given by Dadok [42] who classified polar representations on Euclidean spaces up to orbit equivalence. It follows from Dadok's work that polar actions on spheres are orbit equivalent to isotropy representations of Riemannian symmetric spaces.

Several years later, the interest of classifying polar and hyperpolar actions on symmetric spaces of compact type is stated in [75]. The classification of polar actions up to orbit equivalence on compact symmetric spaces of rank one was obtained by Podestà and Thorbergsson [123]. This classification shows that there are examples of polar actions on symmetric spaces of rank one that are not hyperpolar.

Hyperpolar actions on irreducible symmetric spaces of compact type were classified by Kollross in [85]. The lack of examples of polar actions that are not hyperpolar on irreducible symmetric spaces of compact type and higher rank, led Biliotti [24] to formulate the following conjecture: a polar action on an irreducible symmetric space of compact type and higher rank is hyperpolar. Kollross answered this question in the affirmative for symmetric spaces with simple isometry group [86], and for the exceptional simple Lie groups [87]. The final step was given by Kollross and Lytchak [89] who showed that

Biliotti's conjecture is true: a polar action on an irreducible symmetric space of compact type and rank higher than one is hyperpolar and, hence, the classification follows from [85]. It is worthwhile to mention that the classification of polar actions on reducible symmetric spaces cannot be obtained from the corresponding classification in irreducible ones.

Beyond symmetric spaces, Fang, Grove and Thorbergsson [61] have recently studied polar actions on simply connected, compact, positively curved manifolds of cohomogeneity greater than one. Their main result shows that these actions are equivariantly diffeomorphic to polar actions on compact rank one symmetric spaces.

While there has been certain progress in the study of polar actions on compact symmetric spaces, the situation in the noncompact case remains largely open. Wu [155] classified polar actions on real hyperbolic spaces and showed that, up to orbit equivalence, the groups acting upon are products of a noncompact factor, which is either the isometry group of a lower dimensional real hyperbolic space or the nilpotent part of its Iwasawa decomposition, and a compact factor, which comes from the isotropy representation of a symmetric space. In particular, there are finitely many examples of polar actions on a real hyperbolic space up to orbit equivalence. Berndt and Díaz-Ramos obtained in [16] the classification of polar actions on the complex hyperbolic plane $\mathbb{C}H^2$. No other classification of polar actions was known on a symmetric space of noncompact type.

An important fact to bear in mind here is that, in general, duality cannot be applied to derive classifications of polar actions on noncompact symmetric spaces from the corresponding classifications in the compact setting. For example, a horosphere foliation on a real hyperbolic space is polar but cannot be obtained from duality. Nevertheless, there are certain situations where duality can be used to obtain partial classifications. Díaz-Ramos and Kollross derived in [52] the classification of polar actions with a fixed point on symmetric spaces using this method. Remarkably, it can be shown that a polar action with a fixed point on a reducible symmetric space splits as a product of polar actions on each factor. Kollross explored this idea a bit further and obtained a classification of polar actions by algebraic reductive subgroups using duality in [88].

Berndt and Tamaru [21] classified cohomogeneity one actions on complex hyperbolic spaces, the quaternionic hyperbolic plane, and the Cayley hyperbolic plane. Note that in rank one an isometric action is hyperpolar if and only if it is of cohomogeneity one. The classification remains open in quaternionic hyperbolic spaces $\mathbb{H}H^n$, $n \geq 3$, and in symmetric spaces of higher rank. See [22] for more information on cohomogeneity one actions on symmetric spaces of noncompact type. As we mentioned earlier, a polar action on a symmetric space of compact type always has singular orbits. Motivated by this fact Berndt, Tamaru and Díaz-Ramos studied hyperpolar actions on symmetric spaces that have no singular orbits [18] and obtained a complete classification. It was also shown in this paper that there are polar actions on symmetric spaces of noncompact type and rank higher than one that are not hyperpolar unlike in the compact setting. This classification can be improved in complex hyperbolic spaces, where Berndt and Díaz-Ramos classified polar homogeneous regular foliations [17]. The main result of this chapter contains [17], [16] and the part of [21] corresponding to $\mathbb{C}H^n$ as particular cases.

6.2 Polar actions on complex projective spaces

As we commented in Section 6.1, polar actions on irreducible symmetric spaces of compact type are nowadays relatively well understood. The situation is completely different in the noncompact setting. Indeed, the classification of polar actions on complex hyperbolic spaces provided in this chapter is the first one in a whole family of noncompact symmetric spaces of nonconstant curvature. In this section, we recall the classification of polar actions on complex projective spaces, with a view to appreciate the similarities and differences between both dual families of spaces.

Since the complex projective space has rank one, it is clear that a hyperpolar action on $\mathbb{C}P^n$ must be of cohomogeneity one, that is, the minimum codimension of an orbit is one, and this codimension is precisely the dimension of the section. Hence, the classification of hyperpolar actions on $\mathbb{C}P^n$ follows from Takagi's classification of cohomogeneity one actions on $\mathbb{C}P^n$, see Theorem 2.7.

Takagi's result can be considered the first step towards the classification of polar actions on complex projective spaces. The complete classification of polar actions on $\mathbb{C}P^n$ was obtained by Podestà and Thorbergsson in [123]. On the one hand it includes Takagi's classification and, on the other hand, the existence of polar actions with cohomogeneity greater than one yields examples of polar actions that are not hyperpolar.

Theorem 6.1. [123] *If H acts polarly on a complex projective space $\mathbb{C}P^n$, then the action of H is, up to orbit equivalence, induced by the isotropy representation of a Hermitian symmetric space.*

Let us explain the statement of Theorem 6.1.

A Hermitian symmetric space $M = G/K$ is a Riemannian symmetric space endowed with a complex structure J invariant under the geodesic symmetries. Assume M has complex dimension $n+1$. Then, the tangent space at o has a complex structure J_o, which commutes with the isometries of K, and which turns T_oM into a complex vector space $\mathbb{C}^{n+1}$. As we have already said, the isotropy representation $K \times T_oM \to T_oM$ is polar; moreover, the section $\mathfrak{a}$ is totally real (that is, $J\mathfrak{a}$ is orthogonal to $\mathfrak{a}$). The action of K induces a polar action on the unit sphere S^{2n+1} of $\mathbb{C}^{n+1}$. A complex projective space can be defined as $\mathbb{C}P^n = S^{2n+1}/S^1$, and since J_o is invariant by the isometries of K, the action of K on $T_oM \cong \mathbb{C}^{n+1}$ descends to an isometric action on $\mathbb{C}P^n$. Using the fact that $\mathfrak{a}$ is totally real, it is not difficult to see that $(\mathfrak{a} \cap S^{2n+1})/S^1$ is a section of the induced action on $\mathbb{C}P^n$. Theorem 6.1 implies that any polar action on $\mathbb{C}P^n$ can be obtained, up to orbit equivalence, in this way. Note as well that the cohomogeneity of a polar action on $\mathbb{C}P^n$ coincides with the rank of the Hermitian symmetric space $M = G/K$ minus one.

6.3 Previous results and main theorem

In this section we review some partial classifications of polar actions on $\mathbb{C}H^n$ due to Berndt and Díaz-Ramos [16], [17], and we also state the main result of this chapter, which completes such classification.

Let us briefly bring to mind the notation introduced in §1.7.3, which will be important to understand the statements of the results below. Let $\mathbb{C}H^n = G/K$ be the complex hyperbolic n-space, where $G = \mathrm{SU}(1,n)$, and $K = \mathrm{S}(\mathrm{U}(1)\mathrm{U}(n))$ is the isotropy group of G at some point o. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to o, with associated Cartan involution θ. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the root space decomposition with respect to $\mathfrak{a}$. Set $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0 \cong \mathfrak{u}(n-1)$. Since $\mathfrak{k}_0$ acts on the root space $\mathfrak{g}_\alpha$, the center of $\mathfrak{k}_0$ induces a natural complex structure J on $\mathfrak{g}_\alpha$ which makes it isomorphic to $\mathbb{C}^{n-1}$. Recall also that we say that a subset of $\mathfrak{g}_\alpha$ is a real subspace of $\mathfrak{g}_\alpha$ if it is a linear subspace of $\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is viewed as a real vector space. The solvable Lie algebra $\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is endowed with certain inner product $\langle \cdot, \cdot \rangle_{AN}$ which is induced naturally from the metric on $\mathbb{C}H^n$. A real subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ is said to be totally real if $\langle \mathfrak{w}, J(\mathfrak{w}) \rangle_{AN} = 0$.

Apart from the classification of polar actions of cohomogeneity one due to Berndt and Tamaru [21] (cf. Theorem 2.14), the first step towards a classification of polar actions on complex hyperbolic spaces was given by Berndt and Díaz-Ramos in [17], where the authors provide the classification of polar actions giving rise to regular Riemannian foliations, or equivalently, the classification of polar actions without singular orbits. It turns out that there are, up to isometric congruence, $2n-1$ homogeneous polar regular foliations on $\mathbb{C}H^n$, apart from the trivial foliations (the foliation whose leaves are points, and the foliation with only one leaf).

Theorem 6.2. [17] *Every nontrivial homogeneous polar regular foliation on $\mathbb{C}H^n$, $n \geq 2$, is up to isometric congruence one of the following:*

(i) *The homogeneous polar foliation induced by the connected subgroup of* $\mathrm{SU}(1,n)$ *with Lie algebra $\mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ is a totally real subspace of $\mathfrak{g}_\alpha$. In this case the codimension of the foliation is equal to* $\dim \mathfrak{w}^\perp + 1$ *and all leaves are contained in horospheres of $\mathbb{C}H^n$.*

(ii) *The homogeneous polar foliation induced by the connected subgroup of* $\mathrm{SU}(1,n)$ *with Lie algebra $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ is a nonzero totally real subspace of $\mathfrak{g}_\alpha$. In this case the codimension of the foliation is equal to* $\dim \mathfrak{w}^\perp$ *and no leaf is contained in a horosphere of $\mathbb{C}H^n$.*

Berndt and Díaz-Ramos [16] have also recently derived the classification of polar actions on the complex hyperbolic plane $\mathbb{C}H^2$. This was the first such classification in a noncompact symmetric space of nonconstant curvature. Apart from the trivial and transitive actions, there are exactly nine orbit equivalence classes of polar actions on $\mathbb{C}H^2$. The result is the following (in the statement, $\mathfrak{g}_\alpha^{\mathbb{R}}$ is any one-dimensional real subspace of $\mathfrak{g}_\alpha$).

Theorem 6.3. [16] *For each of the subalgebras $\mathfrak{h}$ of $\mathfrak{su}(1,2)$ listed below the connected closed subgroup H of* $\mathrm{SU}(1,2)$ *with Lie algebra $\mathfrak{h}$ acts polarly on $\mathbb{C}H^2$:*

(i) *Actions of cohomogeneity one - the section Σ is a totally geodesic real hyperbolic line $\mathbb{R}H^1 \subset \mathbb{C}H^2$:*

(a) $\mathfrak{h} = \mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \cong \mathfrak{u}(2)$; *the orbits are* $\{o\}$ *and the geodesic spheres centered at* o.

(b) $\mathfrak{h} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha} = \mathfrak{s}(\mathfrak{u}(1,1) \oplus \mathfrak{u}(1)) \cong \mathfrak{u}(1,1)$; *the orbits are a totally geodesic complex hyperbolic line* $\mathbb{C}H^1 \subset \mathbb{C}H^2$ *and the tubes around* $\mathbb{C}H^1$.

(c) $\mathfrak{h} = \theta(\mathfrak{g}_\alpha^{\mathbb{R}}) \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}} \cong \mathfrak{so}(1,2)$; *the orbits are a totally geodesic real hyperbolic plane* $\mathbb{R}H^2 \subset \mathbb{C}H^2$ *and the tubes around* $\mathbb{R}H^2$.

(d) $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ *or* $\mathfrak{h} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$; *the orbits form a foliation of* $\mathbb{C}H^2$ *by horospheres.*

(e) $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$; *the orbits form a foliation of* $\mathbb{C}H^2$; *one of its leaves is the Lohnherr hypersurface* W^3 *of* $\mathbb{C}H^2$, *and the other leaves are its equidistant hypersurfaces.*

(ii) *Actions of cohomogeneity two - the section* Σ *is a totally geodesic real hyperbolic plane* $\mathbb{R}H^2 \subset \mathbb{C}H^2$:

(a) $\mathfrak{h} = \mathfrak{k} \cap (\mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha}) = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)) \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1)$; *the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (a) *and* (b) *in* (i): *the action has one fixed point* o, *and on each distance sphere centered at* o *the orbits are two circles as singular orbits and* 2*-dimensional tori as principal orbits.*

(b) $\mathfrak{h} = \mathfrak{g}_0$; *the action leaves a totally geodesic* $\mathbb{C}H^1 \subset \mathbb{C}H^2$ *invariant. On this* $\mathbb{C}H^1$ *the action induces a foliation by a totally geodesic real hyperbolic line* $\mathbb{R}H^1 \subset \mathbb{C}H^1$ *and its equidistant curves in* $\mathbb{C}H^1$. *The other orbits are* 2*-dimensional cylinders whose axis is one of the curves in that* $\mathbb{C}H^1$.

(c) $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$; *the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (b) *and* (d) *in* (i): *the action leaves a horosphere foliation invariant, and on each horosphere the orbits consist of a complex horocycle and the tubes around it.*

(d) $\mathfrak{h} = \mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha}$; *the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (d) *and* (e) *in* (i): *the action leaves a horosphere foliation invariant, and on each horosphere the action induces a foliation for which the minimally embedded Euclidean plane and its equidistant surfaces are the leaves.*

Every polar action on $\mathbb{C}H^2$ *is either trivial, transitive, or orbit equivalent to one of the polar actions described above.*

Finally, we state the classification result that we will prove in this chapter. Of course, it includes Theorems 6.2 and 6.3 as particular cases.

Theorem 6.4. *For each of the Lie algebras* $\mathfrak{h}$ *below, the corresponding connected subgroup of* $U(1,n)$ *acts polarly on* $\mathbb{C}H^n$:

(i) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{so}(1,k) \subset \mathfrak{u}(n-k) \oplus \mathfrak{su}(1,k)$, $k \in \{0,\ldots,n\}$, *where* $\mathfrak{q}$ *is a subalgebra of* $\mathfrak{u}(n-k)$ *such that the corresponding subgroup* Q *of* $\mathrm{U}(n-k)$ *acts polarly with a totally real section on* $\mathbb{C}^{n-k}$.

(ii) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{su}(1,n)$, *where* $\mathfrak{b}$ *is a linear subspace of* $\mathfrak{a}$, $\mathfrak{w}$ *is a real subspace of* $\mathfrak{g}_\alpha$, *and* $\mathfrak{q}$ *is a subalgebra of* $\mathfrak{k}_0$ *which normalizes* $\mathfrak{w}$ *and such that the connected subgroup of* $\mathrm{SU}(1,n)$ *with Lie algebra* $\mathfrak{q}$ *acts polarly with a totally real section on the orthogonal complement of* $\mathfrak{w}$ *in* $\mathfrak{g}_\alpha$.

Conversely, every nontrivial polar action on $\mathbb{C}H^n$ *is orbit equivalent to one of the actions above.*

In case (i) of Theorem 6.4, one orbit of the H-action is a totally geodesic $\mathbb{R}H^k$ and the other orbits are contained in the distance tubes around it. In case (ii), if $\mathfrak{b} = \mathfrak{a}$, one H-orbit of minimal orbit type contains a geodesic line, while if $\mathfrak{b} = 0$, any H-orbit of minimal orbit type is contained in a horosphere.

It is important to remark here that Theorem 6.4 actually provides many examples of polar actions on $\mathbb{C}H^n$. Indeed, for every choice of a real subspace $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, there is at least one polar action as described in part (ii) of Theorem 6.4, as will see in Section 6.6.

6.4 Criterion of polarity. Sections are totally real

This short section includes two results that will be essential for our approach in this chapter. The first one is a criterion that will allow us to decide whether an action on a symmetric space of noncompact type (in our case, on $\mathbb{C}H^n$) is polar or not. The second result shows that the sections of nontrivial polar actions on $\mathbb{C}H^n$ are totally real submanifolds. This implies that they are totally geodesic real hyperbolic subspaces $\mathbb{R}H^k$ inside $\mathbb{C}H^n$, $k \in \{0, \dots, n\}$.

The study of polar actions on specific manifolds usually requires developing specific criteria in order to determine when an isometric action is polar. Examples of polarity criteria for compact symmetric spaces are the ones proposed by Gorodski [68, Proposition] and Kollross [86, Proposition 4.1]. A generalization of these results to general Riemannian manifolds was provided by Díaz-Ramos and Kollross [52, Theorem 7]. As a corollary of this, Berndt and Díaz-Ramos derived a criterion for symmetric spaces of noncompact type in [16, Corollary 3.2], which is basically the criterion that we will use here.

Proposition 6.5. *Let* $M = G/K$ *be a Riemannian symmetric space of noncompact type, and let* Σ *be a connected totally geodesic submanifold of* M *with* $o \in \Sigma$. *Let* H *be a closed subgroup of* $I(M)$. *Then* H *acts polarly on* M *with section* Σ *if and only if* $T_o\Sigma$ *is a section of the slice representation of* H_o *on* $\nu_o(H \cdot o)$, *and* $\langle \mathfrak{h}, T_o\Sigma \oplus [T_o\Sigma, T_o\Sigma] \rangle = 0$.

In this case, the following conditions are satisfied:

(a) $T_o\Sigma \oplus [\mathfrak{h}_o, \xi] = \nu_o(H \cdot o)$ *for each regular normal vector* $\xi \in \nu_o(H \cdot o)$.

(b) $T_o\Sigma \oplus [\mathfrak{h}_o, T_o\Sigma] = \nu_o(H \cdot o)$.

(c) $\mathrm{Ad}(H_o)T_o\Sigma = \nu_o(H \cdot o)$.

Proof. The first claim follows directly from [16, Corollary 3.2]. Claims (a) to (c) follow from well-known facts on polar representations of compact groups [42], taking into account that the slice representation of H_o on $\nu_o(H \cdot o)$ is polar with section $T_o\Sigma$. □

Now we will show an important result concerning the sections of a polar action on $\mathbb{C}H^n$. For that, let us recall that, if N is a submanifold of $\mathbb{C}H^n$, then N is said to be totally real if for each $p \in N$ the tangent space T_pN is a totally real subspace of $T_p\mathbb{C}H^n$, that is, JT_pN is orthogonal to T_pN. The next theorem shows that sections are necessarily totally real.

Proposition 6.6. *Let H act nontrivially, nontransitively, and polarly on the complex hyperbolic space $\mathbb{C}H^n$, and let Σ be a section of this action. Then, Σ is a totally real submanifold of $\mathbb{C}H^n$ and, hence, it is a totally geodesic $\mathbb{R}H^k$ in $\mathbb{C}H^n$, for some $k \in \{1, \dots, n\}$.*

Proof. Since the action of H is polar, the section Σ is a totally geodesic submanifold of $\mathbb{C}H^n$, hence Σ is either totally real or complex. Assume that Σ is complex.

Since all sections are of the form $h(\Sigma)$, with $h \in H$, and the isometries of H are holomorphic, it follows that any principal orbit is almost complex. It is a well-known fact that an almost complex submanifold in a Kähler manifold is Kähler. Every H-equivariant normal vector field on a principal orbit is parallel with respect to the normal connection [11, Corollary 3.2.5]. This implies that the normal bundle of such a principal orbit is flat. But it is known that there are no proper Kähler submanifolds of $\mathbb{C}H^n$ with flat normal bundle (see for example [2, Theorem 19]), which gives us a contradiction. Therefore Σ is totally real. Since, moreover, sections are totally geodesic, it follows that Σ is an $\mathbb{R}H^k$ embedded in $\mathbb{C}H^n$ in a totally geodesic way. □

6.5 The structure of a real subspace of a complex vector space

This section will be devoted to investigating real subspaces of a complex vector space. It will turn out that every real subspace is the orthogonal sum of real subspaces with constant Kähler angles.

The reader will need to bring to mind the notions of real subspace and Kähler angle introduced in §2.6.1. Given a real subspace V of $\mathbb{C}^n$, we will denote by π_V the orthogonal projection map onto V. The following structure result provides a useful description of arbitrary real subspaces of a complex vector space.

Theorem 6.7. *Let V be any real subspace of $\mathbb{C}^n$. Then V can be decomposed in a unique way as an orthogonal sum of subspaces V_i, $i = 1, \dots, r$, such that:*

(a) *Each real subspace V_i of $\mathbb{C}^n$ has constant Kähler angle φ_i.*

(b) $\mathbb{C}V_i \perp \mathbb{C}V_j$*, for every* $i \neq j$, $i, j \in \{1, \dots, r\}$.

(c) $\varphi_1 < \varphi_2 < \dots < \varphi_r$.

Proof. The endomorphism $P = \pi_V \circ J$ of V is clearly skew-symmetric, i.e. $\langle Pv, w\rangle = -\langle v, Pw\rangle$ for every v, $w \in V$. Then, there exists an orthonormal basis of V for which P takes a block diagonal form with 2×2 skew-symmetric matrix blocks, and maybe one zero matrix block. Since P is skew-symmetric, its nonzero eigenvalues are imaginary. Assume then that the distinct eigenvalues of P are $\pm i\lambda_1, \dots, \pm i\lambda_r$ (maybe one of them is zero). We can and will further assume that $|\lambda_1| > \cdots > |\lambda_r|$.

Now consider the quadratic form $\Psi\colon V \to \mathbb{R}$ defined by $\Psi(v) = \langle Pv, Pv\rangle = -\langle P^2 v, v\rangle$ for $v \in V$. The matrix of this quadratic form Ψ (or of the endomorphism $-P^2$) with respect to the basis fixed above is diagonal with entries $\lambda_1^2, \dots, \lambda_r^2$. For each $i = 1, \dots, r$, let V_i be the eigenspace of $-P^2$ corresponding to the eigenvalue λ_i^2. Let $v \in V_i$ be a unit vector. Then

$$\langle \pi_{V_i} Jv, \pi_{V_i} Jv\rangle = \langle Pv, \pi_{V_i} Jv\rangle = \langle Pv, Pv\rangle = \Psi(v) = \lambda_i^2,$$

where in the second and last equalities we have used that $Pv \in V_i$. This means that each subspace V_i has constant Kähler angle φ_i, where φ_i is the unique value in $[0, \frac{\pi}{2}]$ such that $\lambda_i^2 = \cos^2(\varphi_i)$.

By construction, it is clear that $V_i \perp V_j$ and $JV_i \perp JV_j$ for $i \neq j$. Since for every $v \in V_i$ and $w \in V_j$, $i \neq j$, we have that $\langle Jv, w\rangle = \langle Pv, w\rangle = 0$, we also get that $JV_i \perp V_j$ if $i \neq j$. Hence $\mathbb{C}V_i \perp \mathbb{C}V_j$ if $i \neq j$.

Property (c) follows from the assumption that $|\lambda_1| > \cdots > |\lambda_r|$, and this also implies the uniqueness of the decomposition. □

It is convenient to change the notation of Theorem 6.7 slightly. Let V be any real subspace of $\mathbb{C}^n$, and let $V = \bigoplus_{\varphi \in \Phi} V_\varphi$ be the decomposition stated in Theorem 6.7, where V_φ has constant Kähler angle $\varphi \in [0, \pi/2]$, and Φ is the set of all Kähler angles arising in this decomposition. Note that according to Theorem 6.7, this decomposition is unique up to the order of the factors. We agree to write $V_\varphi = 0$ if $\varphi \notin \Phi$. The subspaces V_0 and $V_{\pi/2}$ (which can be zero) play a somewhat distinguished role in the calculations that follow, so we will denote $\Phi^* = \{\varphi \in \Phi : \varphi \neq 0, \pi/2\}$. Then, the above decomposition is written as

$$V = V_0 \oplus \left(\bigoplus_{\varphi \in \Phi^*} V_\varphi \right) \oplus V_{\pi/2}.$$

For each $\varphi \in \Phi^* \cup \{0\}$, we define $J_\varphi\colon V_\varphi \to V_\varphi$ by $J_\varphi = \frac{1}{\cos(\varphi)}(\pi_{V_\varphi} \circ J)$. This is clearly a skew-symmetric and orthogonal endomorphism of V_φ (see the proof of Theorem 6.7). Therefore (V_φ, J_φ) is a complex vector space for every $\varphi \in \Phi^* \cup \{0\}$. Note that $J_0 = J|_{V_0}$. Let $\mathrm{U}(V_\varphi)$ be the group of all unitary transformations of the complex vector space (V_φ, J_φ).

Lemma 6.8. *Let V be a real subspace of constant Kähler angle $\varphi \neq 0$ in $\mathbb{C}^n$. Then the real subspace $\mathbb{C}V \ominus V$ of $\mathbb{C}^n$ has the same dimension as V and constant Kähler angle φ.*

Proof. See for example [15, p. 135]. □

Let $V^\perp = \mathbb{C}^n \ominus V$, where as usual $\ominus$ denotes the orthogonal complement. Then, Lemma 6.8 implies that the decomposition stated in Theorem 6.7 can be written as

$$V^\perp = V_0^\perp \oplus \left(\bigoplus_{\varphi \in \Phi^*} V_\varphi^\perp \right) \oplus V_{\pi/2}^\perp, \quad \text{where } \mathbb{C}V_\varphi = V_\varphi \oplus V_\varphi^\perp \text{ for each } \varphi \in \Phi^* \cup \{\pi/2\}.$$

We define $m_\varphi = \dim V_\varphi$ and $m_\varphi^\perp = \dim V_\varphi^\perp$. For every $\varphi \neq 0$ we have $m_\varphi = m_\varphi^\perp$ by Lemma 6.8, but V_0 and $V_0^\perp$ are both complex subspaces of $\mathbb{C}^n$, possibly of different dimension.

Lemma 6.9. *Let V be a real subspace of $\mathbb{C}^n$. Let $\mathrm{U}(n)_V$ be the subgroup of $\mathrm{U}(n)$ consisting of all the elements $A \in \mathrm{U}(n)$ such that $AV = V$. Then, we have the canonical isomorphism*

$$\mathrm{U}(n)_V \cong \left[\prod_{\varphi \in \Phi^* \cup \{0\}} \mathrm{U}(V_\varphi) \right] \times \mathrm{O}(V_{\pi/2}) \times \mathrm{U}(V_0^\perp).$$

where we assume that V_φ, $\varphi \in \Phi^* \cup \{0\}$, is endowed with the complex structure given by $J_\varphi = \frac{1}{\cos(\varphi)}(\pi_{V_\varphi} \circ J)$, and that $V_0^\perp$ is endowed with the complex structure given by the restriction of J.

Proof. Let $A \in \mathrm{U}(n)$ be such that $AV = V$. Then A commutes with J and π_V and hence leaves the eigenspaces of $-P^2$ invariant (see the proof of Theorem 6.7). Thus $AV_\varphi = V_\varphi$. Since we also have $AV^\perp = V^\perp$, it follows that $AV_\varphi^\perp = V_\varphi^\perp$.

Let $\varphi \in \Phi \cup \{0\}$. Since $AV_\varphi = V_\varphi$ and $AV_\varphi^\perp = V_\varphi^\perp$ we have $A\mathbb{C}V_\varphi = \mathbb{C}V_\varphi$. Clearly, $A \circ \pi_{V_\varphi}|_{V_\varphi} = \pi_{V_\varphi} \circ A|_{V_\varphi}$, and $A \circ \pi_{V_\varphi}|_{\mathbb{C}^n \ominus V_\varphi} = 0 = \pi_{V_\varphi} \circ A|_{\mathbb{C}^n \ominus V_\varphi}$. Hence, $A \circ \pi_{V_\varphi} = \pi_{V_\varphi} \circ A$. Since $AJ = JA$ as well, we have that $A \circ J_\varphi|_{V_\varphi} = J_\varphi \circ A|_{V_\varphi}$ on V_φ, and thus, $A|_{V_\varphi} \in \mathrm{U}(V_\varphi)$. If $\varphi = \pi/2$ then we have $AV_{\pi/2} = V_{\pi/2}$, and clearly, $A|_{V_{\pi/2}}$ is an orthogonal transformation of $V_{\pi/2}$. Moreover, we have $A|_{V_0^\perp} \in \mathrm{U}(V_0^\perp)$. We define a map

$$F \colon \mathrm{U}(n)_V \to \left[\prod_{\varphi \in \Phi^* \cup \{0\}} \mathrm{U}(V_\varphi) \right] \times \mathrm{O}(V_{\pi/2}) \times \mathrm{U}(V_0^\perp)$$

by requiring that the projection onto each factor is given by the corresponding restriction, that is, the $\mathrm{U}(V_\varphi)$-projection of $F(A)$ is given by $A|_{V_\varphi}$, the $\mathrm{O}(V_{\pi/2})$-projection of $F(A)$ is $A|_{V_{\pi/2}}$, and the $\mathrm{U}(V_0^\perp)$-projection of $F(A)$ is $A|_{V_0^\perp}$.

Since every element in $\mathrm{U}(n)_V$ leaves the subspaces V_φ, $\varphi \in \Phi$, and $V_0^\perp$ invariant, the map thus defined is a homomorphism. Let us show injectivity and surjectivity. Let $A_\varphi \in \mathrm{U}(V_\varphi)$ for each $\varphi \in \Phi^* \cup \{0\}$, let $A_{\pi/2} \in \mathrm{O}(V_{\pi/2})$, and let $A_0^\perp \in \mathrm{U}(V_0^\perp)$. If $A \in \mathrm{U}(n)_V$ and $v \in JV_\varphi$ for $\varphi \in \Phi$, then Av is determined by A_φ and v, since $Av = -AJ^2v = -JAJv = -JA_\varphi(Jv)$. Since we have the direct sum decomposition

$$\mathbb{C}^n = \left[\bigoplus_{\varphi \in \Phi} \mathbb{C}V_\varphi \right] \oplus V_0^\perp,$$

it follows that the unitary map A on $\mathbb{C}^n$ is uniquely determined by the maps A_φ, $\varphi \in \Phi$, and $A_0^\perp$. This shows injectivity.

Conversely, let $A \in \left[\prod_{\varphi\in\Phi^*\cup\{0\}} \mathrm{U}(V_\varphi)\right] \times \mathrm{O}(V_{\pi/2}) \times \mathrm{U}(V_0^\perp)$, and denote by A_φ the $\mathrm{U}(V_\varphi)$-projection, by $A_{\pi/2}$ the $\mathrm{O}(V_{\pi/2})$-projection, and by $A_0^\perp$ the $\mathrm{U}(V_0^\perp)$-projection. Then, we may construct a map $A \in \mathrm{U}(n)_V$ be defining $A(v + Jw) = A_\varphi v + JA_\varphi w$ for all $v, w \in V_\varphi$, $\varphi \in \Phi$, $Av = A_0^\perp v$ for $v \in V_0^\perp$, and extending linearly. For the map A thus defined we have $A|_{V_\varphi} = A_\varphi$ for $\varphi \in \Phi$, and $A|_{V_0^\perp} = A_0^\perp$. This proves surjectivity. ☐

6.6 New examples of polar actions

We will now construct new examples of polar actions on complex hyperbolic spaces. Here and henceforce, we will use the notation, conventions and results from §1.7.3, sometimes without actually referring to them. We will further assume that $\mathbb{C}H^n$ has constant holomorphic sectional curvature $c = -1$.

Recall that the root space $\mathfrak{g}_\alpha$ is a complex vector space, which we will identify with $\mathbb{C}^{n-1}$. Let $\mathfrak{w}$ be a real subspace of $\mathfrak{g}_\alpha$ and

$$\mathfrak{w} = \bigoplus_{\varphi\in\Phi} \mathfrak{w}_\varphi = \mathfrak{w}_0 \oplus \left(\bigoplus_{\varphi\in\Phi^*} \mathfrak{w}_\varphi\right) \oplus \mathfrak{w}_{\pi/2}$$

its decomposition as in Theorem 6.7, where Φ is the set of all possible Kähler angles of vectors in $\mathfrak{w}$, $\Phi^* = \{\varphi \in \Phi : \varphi \neq 0, \pi/2\}$, and $\mathfrak{w}_\varphi$ has constant Kähler angle $\varphi \in [0, \pi/2]$. Similarly, define $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ and let

$$\mathfrak{w}^\perp = \mathfrak{w}_0^\perp \oplus \left(\bigoplus_{\varphi\in\Phi^*} \mathfrak{w}_\varphi^\perp\right) \oplus \mathfrak{w}_{\pi/2}^\perp$$

be the corresponding decomposition as in Theorem 6.7. We define $m_\varphi = \dim \mathfrak{w}_\varphi$ and $m_\varphi^\perp = \dim \mathfrak{w}_\varphi^\perp$, and recall that $m_\varphi = m_\varphi^\perp$ if $\varphi \in (0, \pi/2]$. Recall also that K_0, the connected subgroup of $G = \mathrm{SU}(1, n)$ with Lie algebra $\mathfrak{k}_0$, is isomorphic to $\mathrm{U}(n - 1)$ and acts on $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ in the standard way. We know from Lemma 6.9 that the normalizer $N_{K_0}(\mathfrak{w})$ of $\mathfrak{w}$ in K_0 has the form

$$N_{K_0}(\mathfrak{w}) \cong \left[\prod_{\varphi\in\Phi^*\cup\{0\}} \mathrm{U}(\mathfrak{w}_\varphi)\right] \times \mathrm{O}(\mathfrak{w}_{\pi/2}) \times \mathrm{U}(\mathfrak{w}_0^\perp). \tag{6.1}$$

This group leaves invariant each $\mathfrak{w}_\varphi$ and each $\mathfrak{w}_\varphi^\perp$, and acts transitively on the unit sphere of these subspaces of constant Kähler angle. Moreover, it acts polarly on $\mathfrak{w}^\perp$, see Remark 6.11 below. We will denote by $\mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$ the normalizer of $\mathfrak{w}$ in $\mathfrak{k}_0$, which coincides with the Lie algebra of $N_{K_0}(\mathfrak{w})$.

The following result provides a large family of new examples of polar actions on $\mathbb{C}H^n$.

Theorem 6.10. *Let* $\mathfrak{w}$ *be a real subspace of* $\mathfrak{g}_\alpha$ *and* $\mathfrak{b}$ *a subspace of* $\mathfrak{a}$*. Let* $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{su}(1,n)$*, where* $\mathfrak{q}$ *is any Lie subalgebra of* $\mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$ *such that the corresponding connected subgroup* Q *of* K *acts polarly on* $\mathfrak{w}^\perp$ *with section* $\mathfrak{s}$*. Assume* $\mathfrak{s}$ *is a totally real subspace of* $\mathfrak{g}_\alpha$*. Then the connected subgroup* H *of* G *with Lie algebra* $\mathfrak{h}$ *acts polarly on* $\mathbb{C}H^n$ *with section* $\Sigma = \exp_o((\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s})$*.*

Proof. We have that $T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s}$ and $\nu_o(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{w}^\perp$. Since $\mathfrak{s} \subset \mathfrak{w}^\perp$, it follows that $T_o\Sigma \subset \nu_o(H \cdot o)$. The slice representation of H_o on $\nu_o(H \cdot o)$ leaves the subspaces $\mathfrak{a} \ominus \mathfrak{b}$ and $(1-\theta)\mathfrak{w}^\perp$ invariant. For the first one the action is trivial, while for the second one the action is equivalent to the representation of Q on $\mathfrak{w}^\perp$ (see Lemma 1.4), which is polar with section $\mathfrak{s}$. Hence, the slice representation of H_o on $\nu_o(H \cdot o)$ is polar and $T_o\Sigma$ is a section of it. Let v, $w \in \mathfrak{s} \subset \mathfrak{w}^\perp$. We have:

$$[(1-\theta)v, (1-\theta)w] = (1+\theta)[v,w] - (1+\theta)[\theta v, w] = -(1+\theta)[\theta v, w].$$

The last equality holds because v and w lie in $\mathfrak{s}$, which is a totally real subspace of $\mathfrak{g}_\alpha$, and then $[v,w] = \frac{1}{2}\langle Jv, w\rangle Z = 0$. Since v, $w \in \mathfrak{g}_\alpha$, then $\theta v \in \mathfrak{g}_{-\alpha}$ and $[\theta v, w] \in \mathfrak{g}_0$. Hence $-(1+\theta)[\theta v, w] \in \mathfrak{k}_0$. Let $X = T + aB + U + xZ \in \mathfrak{h}$, where $T \in \mathfrak{q}$, $U \in \mathfrak{w}$ and a, $x \in \mathbb{R}$. Since $\mathfrak{k}_0$ is orthogonal to $\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, we have:

$$\langle [(1-\theta)v, (1-\theta)w], X\rangle = -\langle (1+\theta)[\theta v, w], T\rangle = -2\langle [T,v], w\rangle = -4\langle [T,v], w\rangle_{AN} = 0,$$

where in the last equality we have used that the action of Q on $\mathfrak{w}^\perp$ is a polar representation with section $\mathfrak{s}$. If $\mathfrak{b} = \mathfrak{a}$, the result then follows using the criterion in Proposition 6.5.

If $\mathfrak{b} \neq \mathfrak{a}$ then $\mathfrak{b} = 0$. In this case, let $v \in \mathfrak{s}$ and $X = T + U + xZ \in \mathfrak{h}$, where $T \in \mathfrak{q}$, $U \in \mathfrak{w}$, $x \in \mathbb{R}$. Then:

$$\langle [B, (1-\theta)v], X\rangle = \langle (1+\theta)[B,v], X\rangle = \frac{1}{2}\langle (1+\theta)v, U\rangle = 0.$$

Since $[B,B] = 0$, by linearity and the skew-symmetry of the Lie bracket, it follows that $\langle [T_o\Sigma, T_o\Sigma], \mathfrak{h}\rangle = 0$. Again by Proposition 6.5, the result follows also in case $\mathfrak{b} \neq \mathfrak{a}$. □

Remark 6.11. In the special case $Q = N_{K_0}(\mathfrak{w})$, we obtain a polar action on $\mathbb{C}H^n$, since the whole normalizer $N_{K_0}(\mathfrak{w})$ acts polarly on $\mathfrak{w}^\perp$. Indeed, let $\mathfrak{s}_\varphi$ be any one-dimensional subspace of $\mathfrak{w}^\perp_\varphi$ if $\mathfrak{w}^\perp_\varphi \neq 0$, and define $\mathfrak{s} = \bigoplus_{\varphi \in \Phi \cup \{0\}} \mathfrak{s}_\varphi$. Then $\mathfrak{s}$ is a section of the action of $N_{K_0}(\mathfrak{w})$ on $\mathfrak{w}^\perp$. The cohomogeneity one examples introduced by Berndt and Brück in [10] correspond to the case where $\mathfrak{w}^\perp$ has constant Kähler angle, $\mathfrak{b} = \mathfrak{a}$ and $Q = N_{K_0}(\mathfrak{w})$.

Remark 6.12. It is straightforward to describe all polar actions of closed subgroups Q in Theorem 6.10 up to orbit equivalence. In fact, the action of the group $N_{K_0}(\mathfrak{w})$ is given by the products of the natural representations of the direct factors in (6.1) on the spaces $\mathfrak{w}^\perp_\varphi$. By the main result of Dadok [42], a representation is polar if and only if it is orbit equivalent to the isotropy representation of some Riemannian symmetric space. Therefore, we obtain a representative for each orbit equivalence class of polar actions on $\mathfrak{w}^\perp$ given by closed subgroups of $N_{K_0}(\mathfrak{w})$ in the following manner. Given $\mathfrak{w}$, for each $\varphi \in \Phi \cup \{0\}$ choose a Riemannian symmetric space M_φ such that $\dim M_\varphi = \dim \mathfrak{w}_\varphi^\perp$. In case $\pi/2 \in \Phi$, choose the symmetric spaces such that all of them except possibly $M_{\pi/2}$ are Hermitian symmetric; in case $\pi/2 \notin \Phi$, choose all these symmetric spaces to be Hermitian without exception. Then the isotropy representation of $\prod_{\varphi\in\Phi\cup\{0\}} M_\varphi$ defines a closed subgroup of $N_{K_0}(\mathfrak{w})$, which acts polarly on $\mathfrak{w}^\perp$ with a section $\mathfrak{s}$, which is a totally real subspace of $\mathfrak{g}_\alpha$, see [123]. This construction exhausts all orbit equivalent classes of closed subgroups in K_0 leaving $\mathfrak{w}$ invariant and acting polarly on $\mathfrak{w}^\perp$ with totally real section.

Remark 6.13. There is a curious relation between some of the new examples of polar actions in Theorem 6.10 and the isoparametric hypersurfaces constructed in Chapter 5 (§5.4.1). The orbit $H \cdot o$ of any of the polar actions described in Theorem 6.10 with $\mathfrak{b} = \mathfrak{a}$ is always a minimal (even austere) submanifold of $\mathbb{C}H^n$ that satisfies the following property: the distance tubes around it are isoparametric hypersurfaces which are hence foliated by orbits of the H-action. Moreover, these hypersurfaces have constant principal curvatures if and only if they are homogeneous (i.e. they are the principal orbits of the cohomogeneity one action resulting from choosing $\mathfrak{q} = \mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$ in Theorem 6.10); this happens precisely when the real subspace $\mathfrak{w}^\perp$ of $\mathfrak{g}_\alpha$ has constant Kähler angle.

The remaining of this chapter will be devoted to the proof of the classification result stated in Theorem 6.4. In order to justify the content of the following sections, we will provide here a sketch of the proof of Theorem 6.4, and leave the details for the following sections.

Assume that H is a closed subgroup of $\mathrm{SU}(1, n)$ that acts polarly on $\mathbb{C}H^n$. Any subgroup of $\mathrm{SU}(1, n)$ is contained in a maximal proper subgroup L of $\mathrm{SU}(1, n)$. We will see that each maximal subgroup of $\mathrm{SU}(1, n)$ either leaves a totally geodesic proper subspace of $\mathbb{C}H^n$ invariant or it is a parabolic subgroup. In the first case, L leaves invariant a lower dimensional totally geodesic complex hyperbolic space $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$, or a totally geodesic real hyperbolic space $\mathbb{R}H^n$. The first possibility is tackled in §6.7.1, and it will follow that, roughly speaking, the action of H splits, up to orbit equivalence, as the product of a polar action on the totally geodesic $\mathbb{C}H^k$, and a polar action with a fixed point on its normal space. Hence, the problem is reduced to the classification of polar actions on lower dimensional complex hyperbolic spaces, which will allow us to use an induction argument. The second possibility is addressed in §6.7.2 where we show that the action of H is orbit equivalent to the action of $\mathrm{SO}(1, n)$, which is a cohomogeneity one action whose orbits are tubes around a totally geodesic $\mathbb{R}H^n$. If the group L is parabolic, this means that its Lie algebra is of the form $\mathfrak{l} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, for certain root space decomposition of $\mathfrak{su}(1, n)$. We will show in Section 6.8 that the Lie algebra of the group H (up to orbit equivalence) must be of the form $\mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, with $\mathfrak{q} \subset \mathfrak{k}_0$, $\mathfrak{b} \subset \mathfrak{a}$, and $\mathfrak{w} \subset \mathfrak{g}_\alpha$, or of the form $\mathfrak{q} \oplus \mathfrak{a}$, with $\mathfrak{q} \subset \mathfrak{k}_0$. A bit more work leads us to the examples described in Theorem 6.10. Combining the different cases, we will conclude in Section 6.9 the proof of Theorem 6.4.

6.7 Actions leaving a totally geodesic subspace invariant

The results in this section show that in order to classify polar actions leaving a totally geodesic complex hyperbolic subspace invariant it suffices to study polar actions on the complex hyperbolic spaces of lower dimensions. We will also show that actions leaving a totally geodesic $\mathbb{R}H^n$ invariant are orbit equivalent to the cohomogeneity one action of $\mathrm{SO}(1,n)$. Note that if an isometric action leaves a totally geodesic $\mathbb{R}H^k$ invariant, it also leaves a totally geodesic $\mathbb{C}H^k$ invariant.

The following is well known. Let H be closed connected subgroup of $\mathrm{SU}(1,n)$. If the natural action of H on $\mathbb{C}H^n$ leaves a totally geodesic proper submanifold of $\mathbb{C}H^n$ invariant, then there is an element $g \in \mathrm{SU}(1,n)$ such that gHg^{-1} is contained in one of the subgroups $\mathrm{S}(\mathrm{U}(1,k)\mathrm{U}(n-k))$ or $\mathrm{SO}(1,n)$ of $\mathrm{SU}(1,n)$.

6.7.1 Actions leaving a totally geodesic complex hyperbolic space invariant

Let $L = \mathrm{S}(\mathrm{U}(1,k)\mathrm{U}(n-k)) \subset G = \mathrm{SU}(1,n)$. Let M_1 be the totally geodesic $\mathbb{C}H^k$ given by the orbit $L \cdot o$. Let M_2 be the totally geodesic $\mathbb{C}H^{n-k}$ which is the image of the normal space $\nu_o M_1$ under the Riemannian exponential map $\exp_o$. Let H be a closed connected subgroup of L. Then the H-action on $\mathbb{C}H^n$ leaves M_1 invariant and the H-action on $\mathbb{C}H^n$ restricted to the isotropy subgroup H_o leaves M_2 invariant. Let $\pi_1 \colon L \to \mathrm{U}(1,k)$ and $\pi_2 \colon L \to \mathrm{U}(n-k)$ be the natural projections.

Theorem 6.14. *Assume the H-action on $\mathbb{C}H^n$ is nontrivial. Then it is polar if and only if the following hold:*

(i) *The action of H on M_1 is polar and nontrivial.*

(ii) *The action of H_o on M_2 is polar and nontrivial.*

(iii) *The action of $\pi_1(H) \times \pi_2(H_o)$ on $\mathbb{C}H^n$ is orbit equivalent to the H-action.*

Proof. Assume first that the H-action on $\mathbb{C}H^n$ is polar and Σ is a section. Let Σ_i be the connected component of $\Sigma \cap M_i$ containing o for $i = 1, 2$. Obviously, the H-orbits on M_1 intersect Σ_1 orthogonally. Let p be an arbitrary point in M_1. Then the intersection of the orbit $H \cdot p$ with Σ is non-empty. Let $q \in (H \cdot p) \cap \Sigma$. Since H leaves M_1 invariant, we have that $q \in M_1$. Both the Riemannian exponential maps of M_1 and of Σ at the point o are diffeomorphisms by the Cartan-Hadamard theorem. Hence there is a unique shortest geodesic segment β in Σ connecting o with q and there is also a unique shortest geodesic segment γ in M_1 connecting o with q. Since both Σ and M_1 are totally geodesic submanifolds of $\mathbb{C}H^n$ it follows that β and γ are both also totally geodesic segments of $\mathbb{C}H^n$ connecting the points o and q and must coincide by the Cartan-Hadamard theorem. Hence

$\beta = \gamma$ both lie in Σ_1. This shows that Σ_1 meets the H-orbit through p (namely, at the point q) and completes the proof that (i) holds.

Obviously, the H_o-orbits on M_2 intersect Σ_2 orthogonally. Since T_oM_2 is a submodule of the slice representation of H_o on $\nu_o(H \cdot o)$, the linear H_o-action on T_oM_2 is polar with section $T_o\Sigma_2$. The map $\exp_o \colon T_oM_2 \to M_2$ is an H_o-equivariant diffeomorphism by the Cartan-Hadamard theorem. In particular, it follows that Σ_2 meets all H_o-orbits in M_2, since $T_o\Sigma_2$ meets all H_o-orbits in T_oM_2. Thus (ii) holds.

Consider the polar slice representation of H_o at $T_o\mathbb{C}H^n$ with section $T_o\Sigma$. By [42, Theorem 4], it follows that $T_o\Sigma = T_o\Sigma_1 \oplus T_o\Sigma_2$. Since $H \subset \pi_1(H) \times \pi_2(H)$, it follows that the actions of the two groups on $\mathbb{C}H^n$ are orbit equivalent.

Now let us prove the other direction of the equivalence. Assume $H \subset L$ is a closed subgroup such that (i), (ii) and (iii) hold. Because of (iii) we may replace H by $\pi_1(H) \times \pi_2(H)$. Let Σ_1 be the section of the H-action on M_1 and let Σ_2 be the section of H-action on M_2. Then by Proposition 6.6, the tangent spaces $T_o\Sigma_1$ and $T_o\Sigma_2$ are totally real subspaces of $T_o\mathbb{C}H^n$; moreover, $\mathbb{C}T_o\Sigma_1 \perp \mathbb{C}T_o\Sigma_2$. Thus the sum $T_o\Sigma_1 \oplus T_o\Sigma_2$ is a totally real Lie triple system in $T_o\mathbb{C}H^n$. Let Σ be the corresponding totally geodesic submanifold.

Using Proposition 6.5, we will show that the H-action on $\mathbb{C}H^n$ is polar and Σ is a section. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $o \in \mathbb{C}H^n$. We have $\mathfrak{p} = T_oM_1 \oplus T_oM_2$. Furthermore, the direct sum decomposition

$$\nu_o(H \cdot o) = (\nu_o(H \cdot o) \cap T_oM_1) \oplus T_oM_2 \tag{6.2}$$

holds. The slice representation of the H-action on M_1 at the point o is orbit equivalent to the submodule $\nu_o(H \cdot o) \cap T_oM_1$ of the slice representation of the H-action on $\mathbb{C}H^n$ at o. The slice representation of the H_o-action on M_2 at the point o is orbit equivalent to the submodule T_oM_2 of the slice representation of the H-action on $\mathbb{C}H^n$ at o. By [42, Theorem 4], we conclude that the slice representation of H_o on $\nu_o(H \cdot o)$ is polar and a section is $T_o\Sigma = T_o\Sigma_1 \oplus T_o\Sigma_2$. We have to show $\langle [v, w], X\rangle = 0$ for all v, $w \in T_o\Sigma \subset \mathfrak{p}$ and all $X \in \mathfrak{h}$. We may identify the tangent space $T_o\mathbb{C}H^n = \mathfrak{p}$ with the space of complex $(n+1) \times (n+1)$-matrices of the form

$$\left(\begin{array}{c|ccc} 0 & \bar{z}_1 & \dots & \bar{z}_n \\ \hline z_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ z_n & 0 & \dots & 0 \end{array}\right). \tag{6.3}$$

The subspace T_oM_1 is given by the matrices where $z_{k+1} = \ldots = z_n = 0$. On the other hand, T_oM_2 consists of those matrices where $z_1 = \ldots = z_k = 0$. Let $v, w \in T_o\Sigma_1$. Then $[v, w]$ is a matrix all of whose nonzero entries are located in the $(k+1) \times (k+1)$-submatrix in the upper left-hand corner, and it follows from (i) and Proposition 6.5 that all vectors in $\mathfrak{h}$ are orthogonal to $[v, w]$. Now assume $v, w \in T_o\Sigma_2$. Then $[v, w]$ is a matrix all of whose nonzero entries are located in the $(n-k) \times (n-k)$-submatrix in the bottom right-hand corner. It follows from (ii) and Proposition 6.5 that all vectors in $\mathfrak{h}$ are orthogonal to

$[v, w]$. Finally assume $v \in T_o\Sigma_1$ and $w \in T_o\Sigma_2$. In this case, the bracket $[v, w]$ is contained in the orthogonal complement of the Lie algebra of L in $\mathfrak{su}(1, n)$; in particular, $[v, w]$ is orthogonal to $\mathfrak{h}$. We conclude that the H-action on $\mathbb{C}H^n$ is polar by Proposition 6.5. □

6.7.2 Actions leaving a totally geodesic real hyperbolic space invariant

Now we assume that the polar action leaves a totally geodesic $\mathbb{R}H^n$ invariant. We have:

Theorem 6.15. *Assume that H is a closed subgroup of $\mathrm{SO}(1, n) \subset \mathrm{SU}(1, n)$. If the H-action on $\mathbb{C}H^n$ is polar and nontrivial, then it is orbit equivalent to the $\mathrm{SO}(1, n)$-action on $\mathbb{C}H^n$; in particular, it is of cohomogeneity one.*

Proof. This proof is divided in three steps.

Claim 1. The group H induces a homogeneous polar regular foliation on the totally geodesic submanifold $\mathbb{R}H^n$ given by the $\mathrm{SO}(1, n)$-orbit through o.

Let M_1 be the totally geodesic $\mathbb{R}H^n$ given by the $\mathrm{SO}(1, n)$-orbit through o. Obviously, the H-action leaves M_1 invariant. Assume the H-action on M_1 has a singular orbit $H \cdot p$, where $p = g(o) \in M_1$. Consider the action of H' on $\mathbb{C}H^n$, where H' is the conjugate subgroup $H' = gHg^{-1}$ of $\mathrm{SU}(1, n)$. The action of H' is conjugate to the H-action on $\mathbb{C}H^n$, hence polar. We have the splitting (6.2) for the normal space of the H'-orbit through o as in the proof of Lemma 6.14, where in this case M_2 is the totally geodesic $\mathbb{R}H^n$ such that $T_oM_2 = i(T_oM_1)$. Since o is a singular orbit of the H'-action on M_1, the slice representation of H'_o on $V = \nu_o(H' \cdot o) \cap T_oM_1$ is nontrivial. The space T_oM_1 consists of all matrices in (6.3) where the entries $z_1, \ldots, z_n$ are real. Consequently, the space iV is contained in the normal space $\nu_o(H' \cdot o)$ and it follows that the slice representation of H'_o with respect to the H'-action on $\mathbb{C}H^n$ contains the submodule $V \oplus iV$ with two equivalent nontrivial H'_o-representations and is hence non-polar by [85, Lemma 2.9], a contradiction. Hence the H-action on M_1 does not have singular orbits, i.e. H induces a homogeneous regular foliation on M_1.

Claim 2. The homogeneous polar regular foliation induced on the invariant totally geodesic real hyperbolic space consists of only one leaf or the H-action on $\mathbb{C}H^n$ is trivial.

Consider the point $o \in M_1$ as in the proof of Claim 1. The tangent space of M_1 at o splits as

$$T_oM_1 = T_o(H \cdot o) \oplus (\nu_o(H \cdot o) \cap T_oM_1).$$

The action of the isotropy group H_o on T_oM_1 respects this splitting. Moreover, the action is trivial on $V = \nu_o(H \cdot o) \cap T_oM_1$, as this is a submodule of the slice representation at o, which lies in a principal orbit of the H-action on M_1. It follows that the action of H_o on iV is trivial as well and the only possibly nontrivial submodule of the slice representation at o is iW, where we define $W = T_o(H \cdot o)$. It follows that the action of the isotropy group

H_o on iW is polar by Proposition 6.5. Let Σ' be a section of this action. Let Σ be a section of the H-action on $\mathbb{C}H^n$. Then we have

$$T_o\Sigma = V \oplus iV \oplus \Sigma'.$$

By Proposition 6.6, Σ is either totally real or $\Sigma = \mathbb{C}H^n$. In the first case, V must be 0, so the action of H on M_1 is transitive. In the second case, the action of H on $\mathbb{C}H^n$ is trivial.

Claim 3. The H-action on $\mathbb{C}H^n$ is orbit equivalent to the $\mathrm{SO}(1,n)$ action.

Assume the H-action is nontrivial and polar with section Σ. We will use the notation of §1.7.3. By Claim 2, H acts transitively on $M_1 = \mathbb{R}H^n$. By Lemma 1.4, the tangent space $T_o(H \cdot o) = T_oM_1$ coincides with $\mathfrak{a} \oplus (1-\theta)\mathfrak{g}_\alpha^{\mathbb{R}}$, where $\mathfrak{g}_\alpha^{\mathbb{R}}$ is a totally real subspace of the root space $\mathfrak{g}_\alpha$ satisfying $\mathbb{C}\mathfrak{g}_\alpha^{\mathbb{R}} = \mathfrak{g}_\alpha$. Moreover $\nu_oM_1 = i(T_oM_1)$. The action of the isotropy subgroup $H_o = H \cap K$ on ν_oM_1 by the slice representation is polar with section $T_o\Sigma$. Since $iB \in \nu_oM_1$, by conjugating the section with a suitable element in H_o we can then assume that $iB \in T_o\Sigma$.

According to [18, Proposition 2.2], the group H contains a solvable subgroup S which acts transitively on $M_1 = \mathbb{R}H^n$. Since S is solvable, it is contained in a Borel subgroup of $\mathrm{SO}(1,n)$. As shown in the proof of [17, Proposition 4.2], we may assume that the Lie algebra of such a Borel subgroup is maximally noncompact, i.e. its Lie algebra is $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}}$, where $\mathfrak{t}$ is an abelian subalgebra of $\mathfrak{k} \cap \mathfrak{so}(n)$ such that $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{so}(1,n)$, see [107]. Note that the Cartan decomposition of $\mathfrak{so}(1,n)$ with respect to the point $o \in M_1 = \mathbb{R}H^n$ is $\mathfrak{so}(1,n) = (\mathfrak{k} \cap \mathfrak{so}(1,n)) \oplus \mathfrak{p}^{\mathbb{R}}$, where $\mathfrak{p}^{\mathbb{R}} = \mathfrak{a} \oplus (1-\theta)\mathfrak{g}_\alpha^{\mathbb{R}} \cong T_oM_1$, and $\mathfrak{g}_\alpha^{\mathbb{R}}$ is the only positive root space of $\mathfrak{so}(1,n)$ with respect to the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}^{\mathbb{R}}$, for a fixed order in the roots.

Now assume the H-action on $\mathbb{C}H^n$ is not of cohomogeneity one. Then $T_o\Sigma \subset \nu_oM_1$ is a Lie triple system containing iB and a nonzero vector iw such that iB, $iw \in \mathfrak{p}$ are orthogonal. By Lemma 1.4, there is a vector $W \in \mathfrak{g}_\alpha^{\mathbb{R}}$ such that $w = (1-\theta)W$. Then, using Lemma 1.3(a), we have

$$[iB, iw] = \frac{1}{2}[(1-\theta)Z, (1-\theta)JW] = \frac{1}{2}(1+\theta)[\theta JW, Z] = \frac{1}{2}(1+\theta)W.$$

Since $T_o(S \cdot o) = T_oM_1$, it follows that the orthogonal projection of the Lie algebra of S onto $\mathfrak{p}$ is $\mathfrak{p}^{\mathbb{R}} = \mathfrak{a} \oplus (1-\theta)\mathfrak{g}_\alpha^{\mathbb{R}}$. This implies that $\mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}}$ is contained in the Lie algebra of S, and hence, also in $\mathfrak{h}$. But then $W \in \mathfrak{h}$ and

$$\langle [iB, iw], W\rangle = \frac{1}{2}\langle (1+\theta)W, W\rangle = \frac{1}{2}\langle W, W\rangle \neq 0,$$

so we have arrived at a contradiction with the criterion for polarity in Proposition 6.5. □

6.8 The parabolic case

As above, let $G = \mathrm{SU}(1,n)$ be the identity connected component of the isometry group of $\mathbb{C}H^n$, and $K = \mathrm{S}(\mathrm{U}(1)\mathrm{U}(n))$ the isotropy group at some point o. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G with respect to o, and choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. As usual we consider $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, where α is a simple positive restricted root. The normalizer of $\mathfrak{n}$ in $\mathfrak{k}$ is denoted by $\mathfrak{k}_0$. Then $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra of $\mathfrak{g}$, and the semi-direct product K_0AN is a maximal parabolic subgroup of G. All parabolic subalgebras (resp. subgroups) of $\mathfrak{g} = \mathfrak{su}(1,n)$ (resp. of $G = \mathrm{SU}(1,n)$) can be obtained in this way [117, Chapter 6, §1.5].

The aim of this section is to prove the following decomposition theorem.

Theorem 6.16. *Let H be a connected closed subgroup of K_0AN acting polarly and non-trivially on $\mathbb{C}H^n$. Then the action of H is orbit equivalent to the action of a subgroup of K_0AN whose Lie algebra can be written as one of the following:*

(a) $\mathfrak{q} \oplus \mathfrak{a}$, *where* $\mathfrak{q}$ *is a subalgebra of* $\mathfrak{k}_0$.

(b) $\mathfrak{q} \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, *where* $\mathfrak{w}$ *is a subspace of* $\mathfrak{g}_\alpha$, *and* $\mathfrak{q}$ *is a subalgebra of* $\mathfrak{k}_0$.

(c) $\mathfrak{q} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, *where* $\mathfrak{w}$ *is a subspace of* $\mathfrak{g}_\alpha$, *and* $\mathfrak{q}$ *is a subalgebra of* $\mathfrak{k}_0$.

Let Q be a maximal compact subgroup of H. Any two maximal compact subgroups of a connected Lie group H are connected and conjugate by an element of H [117, p. 148–149]. By Cartan's fixed point theorem, Q fixes a point $p \in \mathbb{C}H^n$, and hence $Q = H_p$, the isotropy group of H at p. Since AN acts simply transitively on $\mathbb{C}H^n$, we can take the unique element g in AN such that $g(o) = p$, and consider the group $H' = I_{g^{-1}}(H) = g^{-1}Hg$, whose action on $\mathbb{C}H^n$ is conjugate to the one of H. Moreover, $Q' = I_{g^{-1}}(Q) = g^{-1}Qg$ fixes the point o. Since $\mathfrak{a} \oplus \mathfrak{n}$ normalizes $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, we get that AN normalizes $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. In particular, $\mathrm{Ad}(g^{-1})\mathfrak{h} \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ and therefore $H' \subset K_0AN$. Since we are interested in the study of polar actions up to orbit equivalence, it is not restrictive to assume that the group $H \subset K_0AN$ acting polarly on $\mathbb{C}H^n$ admits a maximal connected compact subgroup Q that fixes the point o, and hence $Q \subset K_0$. We will assume this from now on in this section.

As a matter of notation, given two subspaces $\mathfrak{m}$, $\mathfrak{l}$, and a vector v of $\mathfrak{g}$, by $\mathfrak{m}_\mathfrak{l}$ (resp. by $v_\mathfrak{l}$) we will denote the orthogonal projection of $\mathfrak{m}$ (resp. of v) onto $\mathfrak{l}$.

The crucial part of the proof of Theorem 6.16 is contained in the following assertion:

Proposition 6.17. *Let H be a connected closed subgroup of K_0AN acting polarly on $\mathbb{C}H^n$. Let Q be a maximal subgroup of H that fixes the point $o \in \mathbb{C}H^n$. Let $\mathfrak{b}$ be a subspace of $\mathfrak{a}$, $\mathfrak{w}$ a subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{r}$ a subspace of $\mathfrak{g}_{2\alpha}$. Assume that $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, and let $\hat{H}$ be the connected subgroup of K_0AN whose Lie algebra is $\hat{\mathfrak{h}}$. If $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$, then the actions of H and $\hat{H}$ are orbit equivalent.*

The proof of Proposition 6.17 is carried out in several steps. We start with a few basic remarks.

Since $\mathfrak{a}$ and $\mathfrak{g}_{2\alpha}$ are one dimensional, $\mathfrak{b}$ is either 0 or $\mathfrak{a}$, and $\mathfrak{r}$ is either 0 or $\mathfrak{g}_{2\alpha}$. Moreover, if $\mathfrak{r} = 0$ then $\mathfrak{w}$ has to be a totally real subspace of the complex vector space $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, so that $\hat{\mathfrak{h}}$ is a Lie subalgebra. Using the properties of the root space decomposition, it is then easy to check that $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ if and only if $[\mathfrak{q}, \mathfrak{w}] \subset \mathfrak{w}$.

Let Σ be a section of the action of H on $\mathbb{C}H^n$ through $o \in \mathbb{C}H^n$, and let $T_o\Sigma$ be its tangent space at o. The normal space of the orbit through the origin is $\nu_o(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)\mathfrak{w}) \oplus (\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r})$. Since $[\mathfrak{k}_0, \mathfrak{a}] = [\mathfrak{k}_0, \mathfrak{g}_{2\alpha}] = 0$, $[\mathfrak{k}_0, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$, and $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum of vector subspaces) by Proposition 6.5, it follows that $\mathfrak{a} \ominus \mathfrak{b} \subset T_o\Sigma$ and $\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r} \subset T_o\Sigma$. Moreover, since sections are totally real by Proposition 6.6, we can write the tangent space at o of any section as $T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s} \oplus (\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r})$, where $\mathfrak{s}$ is a totally real subspace of $\mathfrak{g}_\alpha$, with $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Furthermore, the fact that $T_o\Sigma$ is totally real, and $i\mathfrak{a} = \mathfrak{p}_{2\alpha}$ (where i is the complex structure on $\mathfrak{p}$), implies that $\mathfrak{a} \ominus \mathfrak{b} = 0$ or $\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r} = 0$, or equivalently, $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{r} = \mathfrak{g}_{2\alpha}$ (that is, $\mathfrak{a} \subset \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$ or $\mathfrak{g}_{2\alpha} \subset \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$).

Let $T + aB + U + xZ$ be an arbitrary element of $\mathfrak{h}$, with $T \in \mathfrak{h}_{\mathfrak{k}_0}$, $U \in \mathfrak{w}$, and $a, x \in \mathbb{R}$. Let ξ, η be arbitrary vectors of $\mathfrak{s}$. By Proposition 6.5, and since $\mathfrak{s}$ is totally real, we have, using Lemma 1.3(b):

$$0 = \langle T + aB + U + xZ, [(1-\theta)\xi, (1-\theta)\eta]\rangle = -\langle T, (1+\theta)[\theta\xi, \eta]\rangle = -2\langle [T, \xi], \eta\rangle,$$

from where it follows that $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$.

Moreover, if $T \in \mathfrak{q}$ and $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$, $U \in \mathfrak{w}$ are such that $S_U + U \in \mathfrak{h}$, then $[T, S_U] + [T, U] = [T, S_U + U] \in \mathfrak{h}$, so $[T, U] \in \mathfrak{w}$. In particular, if $\xi \in \mathfrak{s}$, then $0 = \langle [T, U], \xi\rangle = -\langle [T, \xi], U\rangle$, which proves $[\mathfrak{q}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$.

Summarizing what we have obtained about sections we can state:

Lemma 6.18. *If Σ is a section of the action of H on $\mathbb{C}H^n$ through o, then*

$$T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s} \oplus (\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r}),$$

where $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ is a totally real subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{r} = \mathfrak{g}_{2\alpha}$. Moreover, $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$, and $[\mathfrak{q}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$.

We will need to calculate the isotropy group at certain points.

Lemma 6.19. *Let $\xi \in \mathfrak{g}_\alpha$ and write $g = \operatorname{Exp}(\lambda\xi)$, with $\lambda \in \mathbb{R}$. Then, the Lie algebra of the isotropy group H_p of H at $p = g(o)$ is $\mathfrak{h}_p = \mathfrak{h} \cap \operatorname{Ad}(g)\mathfrak{k} = \mathfrak{q} \cap \ker \operatorname{ad}(\xi)$.*

Proof. First notice that $\mathfrak{h} \cap \operatorname{Ad}(g)\mathfrak{k}$ is the Lie algebra of $H_p = H \cap I_g(K)$. Let v be the unique element in $\mathfrak{p} = T_o\mathbb{C}H^n$ such that $\exp_o(v) = p$. We show that the isotropy group H_p coincides with the isotropy group of the slice representation of Q at v, Q_v. By [147, §2] we know that the normal exponential map $\exp\colon \nu(H \cdot o) \to \mathbb{C}H^n$ is an H-equivariant diffeomorphism. Let $h \in H_p$. Since $\exp_o(v) = p = h(p) = h \exp_o(v) = \exp_{h(o)}(h_{*o}v)$, we get that $h(o) = o$ and $h_{*o}v = v$, and hence, $h \in Q_v$. The H-equivariance of exp also shows the converse inclusion. Therefore $H_p = Q_v$.

We can write $v = aB + b(1-\theta)\xi$ for certain a, $b \in \mathbb{R}$. In fact, $\operatorname{Exp}(\lambda\xi)(o)$ belongs to the totally geodesic $\mathbb{R}H^2$ given by $\exp_o(\mathfrak{a} \oplus \mathbb{R}(1-\theta)\xi)$, and $b \neq 0$ if $\lambda\xi \neq 0$. Then, the Lie algebra of $H_p = Q_v$ is $\{T \in \mathfrak{q} : [T, aB + b(1-\theta)\xi] = 0\} = \{T \in \mathfrak{q} : [T, \xi] = 0\}$, which is $\mathfrak{q} \cap \ker \operatorname{ad}(\xi)$. ∎

By definition, we say that a vector $\xi \in \mathfrak{s}$ is *regular* if $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$. We have

Lemma 6.20. *The set $\{\xi \in \mathfrak{s} : \xi$ is regular$\}$ is an open dense subset of $\mathfrak{s}$.*

Proof. An element of $T_o\Sigma$ can be written, according to Lemma 6.18, as $v = aB + (1-\theta)\xi + x(1-\theta)Z$ where a, $x \in \mathbb{R}$, and $\xi \in \mathfrak{s}$. We have $[\mathfrak{q}, v] = (1-\theta)[\mathfrak{q}, \xi]$ and $\nu_o(H \cdot o) \ominus T_o\Sigma = (1-\theta)(\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}))$. An element of $T_o\Sigma$ is regular (that is, belongs to a principal orbit of the slice representation $Q \times \nu_o(H \cdot o) \to \nu_o(H \cdot o)$) if and only if $[\mathfrak{q}, v] = \nu_o(H \cdot o) \ominus T_o\Sigma$. The previous equalities, and the fact that $(1-\theta)\colon \mathfrak{g}_\alpha \to \mathfrak{p}_\alpha$ is an isomorphism implies that v is regular if and only if $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$. Since the set of regular points of a section is open and dense, the result follows. ☐

Lemma 6.21. *For each regular vector $\xi \in \mathfrak{s}$ we have $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$.*

Proof. Let $\xi \in \mathfrak{s}$ be a regular vector, that is, $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$. In order to prove the lemma, it is enough to show that $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$, since $\mathfrak{q} \subset \mathfrak{h}_{\mathfrak{k}_0}$ and, by Lemma 6.18, $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$.

First, consider the case $\mathfrak{r} = 0$. By Lemma 6.18, $T_o\Sigma = (1-\theta)\mathfrak{s} \oplus \mathbb{R}(1-\theta)Z$ for each section Σ through o, where $\mathfrak{s}$ is some totally real subspace of $\mathfrak{g}_\alpha$. By Proposition 6.5 we have $\nu_o(H \cdot o) = \mathrm{Ad}(Q)(T_o\Sigma)$ and, thus, for any $\eta \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$ we can find a section Σ through o such that $\eta \in \mathfrak{s}$ by conjugating by a suitable element in Q. Then using Lemma 1.3, we have that $(1+\theta)J\eta = [(1-\theta)\eta, (1-\theta)Z] \in [T_o\Sigma, T_o\Sigma]$. Let $W \in \mathfrak{w}$ and $T_W \in \mathfrak{h}_{\mathfrak{k}_0}$ be such that $T_W + W \in \mathfrak{h}$. Since by Proposition 6.5 we have $\langle \mathfrak{h}, [T_o\Sigma, T_o\Sigma] \rangle = 0$, then $0 = \langle T_W + W, (1+\theta)J\eta \rangle = \langle W, J\eta \rangle$. We have then shown that $J(\mathfrak{g}_\alpha \ominus \mathfrak{w})$ is orthogonal to $\mathfrak{w}$, that is, $\mathfrak{g}_\alpha \ominus \mathfrak{w}$ is a complex subspace of $\mathfrak{g}_\alpha$. Since $\mathfrak{w}$ is totally real, we deduce $\mathfrak{w} = 0$. But then $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ holds trivially.

For the rest of the proof, we assume that $\mathfrak{r} = \mathfrak{g}_{2\alpha}$.

Let $T_B \in \mathfrak{h}_{\mathfrak{k}_0}$ and $a \in \mathbb{R}$ such that $T_B + aB \in \mathfrak{h}$. Note that, if $\mathfrak{b} = 0$, then $a = 0$, $T_B \in \mathfrak{q}$ and there is nothing to prove. For each $U \in \mathfrak{w}$ take an $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ with $S_U + U \in \mathfrak{h}$. Then $[T_B, S_U] + [T_B, U] + \frac{a}{2}U = [T_B + aB, S_U + U] \in \mathfrak{h}$, so $[T_B, U] + \frac{a}{2}U \in \mathfrak{w}$, from where $[T_B, U] \in \mathfrak{w}$. Hence, $\langle [T_B, \xi], U \rangle = -\langle \xi, [T_B, U] \rangle = 0$, so we get $[T_B, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$.

Now let $T_Z \in \mathfrak{h}_{\mathfrak{k}_0}$ and $x \in Z$ with $T_Z + xZ \in \mathfrak{h}$. For each $U \in \mathfrak{w}$ take an $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ with $S_U + U \in \mathfrak{h}$. Then $[T_Z, S_U] + [T_Z, U] = [T_Z + Z, S_U + U] \in \mathfrak{h}$, so $[T_Z, U] \in \mathfrak{w}$. As above, we conclude $[T_Z, \xi] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$.

Finally, we have to prove that for each $U \in \mathfrak{w}$, if $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$ is such that $T_U + U \in \mathfrak{h}$, then $[T_U, \xi] \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. This will require some effort.

Let $U \in \mathfrak{w}$ and $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$ with $T_U + U \in \mathfrak{h}$. By Lemma 6.18, $[T_U, \xi] \in \mathfrak{g}_\alpha \ominus \mathfrak{s} = \mathfrak{w} \oplus (\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}))$. Since $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$, we can find an $S \in \mathfrak{q}$ so that $[T_U + S, \xi] \in \mathfrak{w}$. Therefore we can define the map

$$F_\xi \colon \mathfrak{w} \to \mathfrak{w}, \quad U \mapsto [T_U, \xi], \quad \text{where } T_U \in \mathfrak{h}_{\mathfrak{k}_0},\, T_U + U \in \mathfrak{h}, \text{ and } [T_U, \xi] \in \mathfrak{w}.$$

The map F_ξ is well-defined. Indeed, if T_U, $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$, $U \in \mathfrak{w}$, $T_U + U$, $S_U + U \in \mathfrak{h}$, and $[T_U, \xi]$, $[S_U, \xi] \in \mathfrak{w}$, then $T_U - S_U \in \mathfrak{q}$, so $[T_U, \xi] - [S_U, \xi] = [T_U - S_U, \xi] \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$, and $[T_U, \xi] - [S_U, \xi] \in \mathfrak{w}$. Hence $[T_U, \xi] = [S_U, \xi]$. It is also easy to check that F_ξ is linear.

Furthermore, F_ξ is self-adjoint. To see this, let T_U, $S_V \in \mathfrak{h}_{\mathfrak{k}_0}$, U, $V \in \mathfrak{w}$, with $T_U + U$, $S_V + V \in \mathfrak{h}$, and $[T_U, \xi]$, $[S_V, \xi] \in \mathfrak{w}$. Then we have

$$\begin{aligned}0 &= \langle [T_U + U, S_V + V], \xi\rangle = \langle [T_U, V], \xi\rangle - \langle [S_V, U], \xi\rangle = -\langle V, [T_U, \xi]\rangle + \langle U, [S_V, \xi]\rangle \\ &= -\langle F_\xi(U), V\rangle + \langle F_\xi(V), U\rangle.\end{aligned}$$

Assume now that $F_\xi \neq 0$. Then F_ξ admits an eigenvector $U \in \mathfrak{w}$ with nonzero eigenvalue $\lambda \in \mathbb{R}$: $F_\xi(U) = \lambda U \neq 0$. We will get a contradiction with this.

Let $g = \operatorname{Exp}(-\frac{1}{\lambda}\xi)$, and consider $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T_U + U \in \mathfrak{h}$ and $F_\xi(U) = [T_U, \xi] = \lambda U$. We also consider an element $S \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S + Z \in \mathfrak{h}$ and $[S, \xi] = 0$; this is possible because $[S, \xi] \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}) = [\mathfrak{q}, \xi]$ and $\mathfrak{q} \subset \mathfrak{h}$. If we define $R = T_U - \frac{1}{4\lambda}\langle J\xi, U\rangle S \in \mathfrak{h}_{\mathfrak{k}_0}$, then we have

$$\begin{aligned}\operatorname{Ad}(g)R = e^{-\frac{1}{\lambda}\operatorname{ad}(\xi)}R &= T_U - \frac{1}{\lambda}[\xi, T_U] + \frac{1}{2\lambda^2}[\xi, [\xi, T_U]] - \frac{1}{4\lambda}\langle J\xi, U\rangle S \\ &= (T_U + U) - \frac{1}{4\lambda}\langle J\xi, U\rangle (S + Z) \in \mathfrak{h} \cap \operatorname{Ad}(g)(\mathfrak{k}).\end{aligned}$$

However, $\operatorname{Ad}(g)R \notin \mathfrak{q} \cap \ker \operatorname{ad}(\xi)$. By virtue of Lemma 6.19, this gives a contradiction. Thus we must have $F_\xi = 0$, from where the result follows. □

Lemma 6.22. *The subspace $\mathfrak{h}_{\mathfrak{k}_0}$ is a subalgebra of $\mathfrak{k}_0$, and $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{w}] \subset \mathfrak{w}$.*

Proof. If $T + aB + U + xZ$, $S + bB + V + yZ \in \mathfrak{h}$, with T, $S \in \mathfrak{h}_{\mathfrak{k}_0}$, U, $V \in \mathfrak{w}$, and a, b, x, $y \in \mathbb{R}$, then the bracket $[T + aB + U + xZ, S + bB + V + yZ] = [T, S] + [T, V] - [S, U] + \frac{a}{2}V - \frac{b}{2}U + \left(\frac{1}{2}\langle JU, V\rangle + ay - bx\right) Z$ belongs to $\mathfrak{h}$. In particular $[T, S] \in \mathfrak{h}_{\mathfrak{k}_0}$, so $\mathfrak{h}_{\mathfrak{k}_0}$ is a Lie subalgebra of $\mathfrak{k}_0$. Taking $U = 0$, $a = b = x = y = 0$ we obtain that $[\mathfrak{q}, \mathfrak{w}] \subset \mathfrak{w}$ and hence $[\mathfrak{q}, \mathfrak{g}_\alpha \ominus \mathfrak{w}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$.

Now let $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. For any section through o we have $\operatorname{Ad}(Q)(T_o\Sigma) = \nu_0(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w}) \oplus (1 - \theta)(\mathfrak{g}_{2\alpha} \ominus \mathfrak{r})$, and $(\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{r}) \subset T_o\Sigma$ by Lemma 6.18. Hence, for $(1 - \theta)X \in (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w})$ we can find a section Σ such that $(1 - \theta)X \in T_o\Sigma$ (after conjugation by an element of Q if necessary). Then, if X is regular, Lemma 6.21 implies $[\mathfrak{h}_{\mathfrak{k}_0}, X] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Since the set of regular vectors is dense, X can always be approximated by a sequence of regular vectors, and hence, by continuity we also obtain $[\mathfrak{h}_{\mathfrak{k}_0}, X] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ for non-regular vectors. Therefore, $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{g}_\alpha \ominus \mathfrak{w}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Finally, the skew-symmetry of the elements of $\operatorname{ad}(\mathfrak{k}_0)$ implies $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{w}] \subset \mathfrak{w}$. □

We can now finish the proof of Proposition 6.17.

Proof of Proposition 6.17. The fact that $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, and Lemma 6.22, imply that $\tilde{\mathfrak{h}} = \mathfrak{h}_{\mathfrak{k}_0} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a Lie subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$ and $\hat{\mathfrak{h}}$. Let $\tilde{H}$ be the connected subgroup of G whose Lie algebra is $\tilde{\mathfrak{h}}$. Since $T_o(H \cdot o) = T_o(\tilde{H} \cdot o) = T_o(\hat{H} \cdot o) = \mathfrak{b} \oplus (1 - \theta)\mathfrak{w} \oplus (1 - \theta)\mathfrak{r}$ and $H \subset \tilde{H}$, $\hat{H} \subset \tilde{H}$, the orbits through o of the groups H, $\tilde{H}$, and $\hat{H}$ coincide. The slice representations at o of H and $\tilde{H}$ have the same principal orbits. Indeed, for a section Σ through o and $v = aB + (1 - \theta)\xi + x(1 - \theta)Z \in T_o\Sigma$ with $\xi \in \mathfrak{s}$ regular, Lemma 6.21 implies $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}) = [\mathfrak{q}, \xi]$. Thus, the tangent spaces at v of the orbits of the slice representations of H and $\tilde{H}$ through v coincide, and since $H \subset \tilde{H}$, both orbits coincide. Then, the slice representations at o of H and $\tilde{H}$ are orbit equivalent. Since the codimension of an orbit of H (resp. of $\tilde{H}$) through $\exp_o(v)$ coincides with the codimension of the orbit of the slice representation of H (resp. of $\tilde{H}$) through $v \in \nu_o(H \cdot o) = \nu_o(\tilde{H} \cdot o)$, and since the orbits of H are contained in the orbits of $\tilde{H}$, we conclude that the actions of H and $\tilde{H}$ on $\mathbb{C}H^n$ have the same orbits. Similarly, an analogous argument with $\hat{H}$ instead of H allows to show that the actions of $\hat{H}$ and $\tilde{H}$ on $\mathbb{C}H^n$ are orbit equivalent, and this completes the proof. □

We now proceed with the proof of Theorem 6.16.

Let H be a closed subgroup of the isometry group of $\mathbb{C}H^n$ acting polarly on $\mathbb{C}H^n$, and assume that the Lie algebra of H is contained in a maximal parabolic subalgebra $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. As we argued at the beginning of this section, there is a maximal compact subgroup Q of H, and we can assume that $o \in \mathbb{C}H^n$ is a fixed point of Q, that is, the isotropy group of H at o is Q. We are now interested in $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$, the orthogonal projection of $\mathfrak{h}$ on $\mathfrak{a} \oplus \mathfrak{n}$. It is clear that $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$ can be written in one of the following forms: $\mathfrak{w}$, $\mathbb{R}(B + X) \oplus \mathfrak{w}$, $\mathbb{R}(B + X + xZ) \oplus \mathfrak{w}$ (with $x \neq 0$), $\mathfrak{w} \oplus \mathbb{R}(Y + Z)$, or $\mathbb{R}(B + X) \oplus \mathfrak{w} \oplus \mathbb{R}(Y + Z)$, where $\mathfrak{w} \subset \mathfrak{g}_\alpha$, and X, $Y \in \mathfrak{g}_\alpha$.

In order to conclude the proof of Theorem 6.16 we deal with these five possibilities separately.

Case 1: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{w}$, with $\mathfrak{w}$ a subspace of $\mathfrak{g}_\alpha$

Here $\mathfrak{h}$ is in the hypotheses of Proposition 6.17, and it readily follows from Lemma 6.18 that this case is not possible.

Case 2: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B + X) \oplus \mathfrak{w}$, with $\mathfrak{w}$ a subspace of $\mathfrak{g}_\alpha$, and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$

Assume first that $X \neq 0$. Then, $\nu_o(H \cdot o) = \mathbb{R}(-\|X\|^2 B + (1-\theta)X) \oplus (1-\theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w}) \oplus \mathfrak{p}_{2\alpha}$. Let Σ be a section through o. Since $T_o\Sigma \subset \nu_o(H \cdot o)$, $[\mathfrak{q}, -\|X\|^2 B + (1-\theta)X] \subset \mathfrak{p}_\alpha$, $[\mathfrak{q}, \mathfrak{p}_{2\alpha}] = 0$, and $[\mathfrak{q}, \mathfrak{p}_\alpha] \subset \mathfrak{p}_\alpha$, we get that $[\mathfrak{q}, T_o\Sigma]$ is orthogonal to $\mathfrak{a}$ and $\mathfrak{p}_{2\alpha}$. As $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum) by Proposition 6.5, we readily get that $\mathfrak{p}_{2\alpha} \subset T_o\Sigma$. Moreover, let $T \in \mathfrak{h}_{\mathfrak{k}_0}$ be such that $T + B + X \in \mathfrak{h}$; then $T + B + X$ is orthogonal to $[\mathfrak{q}, T_o\Sigma]$, and since $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$ we obtain that X is orthogonal to $[\mathfrak{q}, T_o\Sigma]$. The fact that the direct sum $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ is orthogonal implies that $-\|X\|^2 B + (1-\theta)X \in T_o\Sigma$. However, since $T_o\Sigma$ is totally real we have

$$0 = \langle i(-\|X\|^2 B + (1-\theta)X), (1-\theta)Z\rangle = \langle -\frac{1}{2}\|X\|^2(1-\theta)Z + (1-\theta)JX, (1-\theta)Z\rangle = -2\|X\|^2,$$

which is not possible because $X \neq 0$.

Therefore we must have $X = 0$, and thus $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w}$. Note that the fact that $\mathfrak{h}$ is a subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ implies that $\mathfrak{w}$ is a totally real subspace of $\mathfrak{g}_\alpha$. We are now in the hypotheses of Proposition 6.17 and, as shown in the proof of Lemma 6.21, $\mathfrak{w} = 0$. We conclude that the action of H is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a}$. This corresponds to Theorem 6.16(a).

Case 3: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B + X + xZ) \oplus \mathfrak{w}$**, with** $\mathfrak{w}$ **a subspace of** $\mathfrak{g}_\alpha$**,** $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$**, and** $x \in \mathbb{R}$**,** $x \neq 0$

Let $g = \operatorname{Exp}(xZ) \in G$, and let $T + r(B + X + xZ) + V$ be a generic element of $\mathfrak{h}$, with $V \in \mathfrak{w}$, $r \in \mathbb{R}$. Clearly, since $g \in AN$ we have $\operatorname{Ad}(g)(\mathfrak{h}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then, it is easy to obtain

$$\operatorname{Ad}(g)(T + r(B + X + xZ) + V) = T + r(B + X + xZ) + V - rxZ = T + r(B + X) + V.$$

Hence $(\operatorname{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B + X) \oplus \mathfrak{w}$, and $\operatorname{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$. Since Q is a maximal compact subgroup of $I_g(H) = gHg^{-1}$, and the orthogonal projection of the Lie algebra of $I_g(H)$ onto $\mathfrak{a} \oplus \mathfrak{n}$ is $\mathbb{R}(B + X) \oplus \mathfrak{w}$, the new group $I_g(H)$ satisfies the conditions of Case 2. Therefore, the action of H is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a}$. This also corresponds to Theorem 6.16(a).

Case 4: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{w} \oplus \mathbb{R}(Y + Z)$**, with** $\mathfrak{w}$ **a subspace of** $\mathfrak{g}_\alpha$**, and** $Y \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$

Assume that $Y \neq 0$. Then, $\nu_o(H \cdot o) = \mathfrak{a} \oplus (1-\theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w}) \oplus \mathbb{R}(2(1-\theta)Y - \|Y\|^2(1-\theta)Z)$. Let Σ be a section through o. Then, by Proposition 6.5 we have $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum). Since $[\mathfrak{q}, 2(1-\theta)Y - \|Y\|^2(1-\theta)Z] \subset \mathfrak{p}_\alpha$, $[\mathfrak{q}, \mathfrak{a}] = 0$, and $[\mathfrak{q}, \mathfrak{p}_\alpha] \subset \mathfrak{p}_\alpha$, we get that $[\mathfrak{q}, T_o\Sigma]$ is orthogonal to $\mathfrak{a}$ and $\mathfrak{p}_{2\alpha}$. Then, $\mathfrak{a} \subset T_o\Sigma$. On the other hand, if $T \in \mathfrak{h}_{\mathfrak{k}_0}$ is such that $T + Y + Z \in \mathfrak{h}$, then $T + Y + Z$ is orthogonal to $[\mathfrak{q}, T_o\Sigma] \subset \nu_o(H \cdot o)$, and since $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$ we also obtain that Y is orthogonal to $[\mathfrak{q}, T_o\Sigma]$. Thus, $2(1-\theta)Y - \|Y\|^2(1-\theta)Z \in T_o\Sigma$. But, since $T_o\Sigma$ is totally real, we get

$$0 = \langle B, i(2(1-\theta)Y - \|Y\|^2(1-\theta)Z)\rangle = \langle B, 2(1-\theta)JY + 2\|Y\|^2 B\rangle = 2\|Y\|^2,$$

which contradicts $Y \neq 0$.

Therefore we have $Y = 0$, and thus, $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. We are now in the hypotheses of Proposition 6.17, and we conclude that the action of H is orbit equivalent to the action of the connected subgroup $\hat{H}$ of the isometry group of $\mathbb{C}H^n$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, with $\mathfrak{w}$ a subspace of $\mathfrak{g}_\alpha$. This corresponds to Theorem 6.16(c).

Case 5: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B + X) \oplus \mathfrak{w} \oplus \mathbb{R}(Y + Z)$**, with** $\mathfrak{w} \subset \mathfrak{g}_\alpha$**, and** X**,** $Y \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$

This final possibility is more involved.

Our first aim is to show that $Y = 0$. So, assume for the moment that $Y \neq 0$.

Lemma 6.23. *We have* $X = \gamma Y + \frac{2}{\|Y\|^2} JY$*, with* $\gamma \in \mathbb{R}$*.*

Proof. Assume that X and Y are linearly dependent, that is, $X = \lambda Y$, with $\lambda \in \mathbb{R}$. Then, $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B + \lambda Y) \oplus \mathfrak{w} \oplus \mathbb{R}(Y + Z)$, and there exist T, $S \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T + B + \lambda Y$, $S + Y + Z \in \mathfrak{h}$. Then,

$$[T, S] + [T, Y] - \lambda[S, Y] + \frac{1}{2}Y + Z = [T + B + \lambda Y, S + Y + Z] \in \mathfrak{h}.$$

Since $[T, Y] - \lambda[S, Y] \in \mathfrak{g}_\alpha \ominus \mathbb{R}Y$ by the skew-symmetry of the elements of $\mathrm{ad}(\mathfrak{k}_0)$, we get $\frac{1}{2}Y + Z \in \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$, which is not possible.

Therefore, we can assume that X and Y are linearly independent vectors of $\mathfrak{g}_\alpha$. In particular, $X \neq 0$. Take and fix for the rest of the calculations T, $S \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T + B + X$, $S + Y + Z \in \mathfrak{h}$.

In this case, the normal space to the orbit through the origin o can be written as

$$\begin{aligned}\nu_o(H \cdot o) = \mathbb{R}(-\|X\|^2 B + (1-\theta)X - \frac{1}{2}\langle X, Y\rangle(1-\theta)Z) \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)(\mathfrak{w} \oplus \mathbb{R}X \oplus \mathbb{R}Y)) \\ \oplus \mathbb{R}(-\langle X, Y\rangle B + (1-\theta)Y - \frac{1}{2}\|Y\|^2(1-\theta)Z).\end{aligned}$$

Let Σ be a section of the action of H on $\mathbb{C}H^n$ through the point $o \in \mathbb{C}H^n$. By Proposition 6.5 we have $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum). In particular the vectors $T + B + X$ and $S + Y + Z$ are orthogonal to $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$ (because $[\mathfrak{k}_0, \mathfrak{a}] = [\mathfrak{k}_0, \mathfrak{g}_{2\alpha}] = 0$). This implies that X and Y are already orthogonal to $[\mathfrak{q}, T_o\Sigma]$, and thus, so are $-\|X\|^2 B + (1-\theta)X - \frac{1}{2}\langle X, Y\rangle(1-\theta)Z$ and $-\langle X, Y\rangle B + (1-\theta)Y - \frac{1}{2}\|Y\|^2(1-\theta)Z$. Hence, they are in $T_o\Sigma$ and we can write

$$\begin{aligned}T_o\Sigma = \mathbb{R}(-\|X\|^2 B + (1-\theta)X - \frac{1}{2}\langle X, Y\rangle(1-\theta)Z) \\ \oplus (1-\theta)\mathfrak{s} \oplus \mathbb{R}(-\langle X, Y\rangle B + (1-\theta)Y - \frac{1}{2}\|Y\|^2(1-\theta)Z),\end{aligned}$$

where $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ is totally real, and $\mathbb{C}X \oplus \mathbb{C}Y$ is orthogonal to $\mathfrak{s}$ (because sections are totally real). The fact that $T_o\Sigma$ is totally real also implies

$$\begin{aligned}0 &= \langle i(-\|X\|^2 B + (1-\theta)(X - \frac{1}{2}\langle X, Y\rangle Z)), -\langle X, Y\rangle B + (1-\theta)(Y - \frac{1}{2}\|Y\|^2 Z)\rangle \\ &= \langle (1-\theta)(-\frac{1}{2}\|X\|^2 Z + JX) + \langle X, Y\rangle B), -\langle X, Y\rangle B + (1-\theta)(Y - \frac{1}{2}\|Y\|^2 Z)\rangle \\ &= \|X\|^2\|Y\|^2 - \langle X, Y\rangle^2 + 2\langle JX, Y\rangle.\end{aligned} \tag{6.4}$$

Now, using Lemma 1.3(a), and (6.4), we compute

$$\begin{aligned}&[-\|X\|^2 B + (1-\theta)(X - \frac{1}{2}\langle X, Y\rangle Z), -\langle X, Y\rangle B + (1-\theta)(Y - \frac{1}{2}\|Y\|^2 Z)] \\ &\quad = \frac{1}{2}(1+\theta)\big(-2[\theta X, Y] + \langle X, Y\rangle X - \|X\|^2 Y - \|Y\|^2 JX + \langle X, Y\rangle JY - \langle JX, Y\rangle Z\big).\end{aligned}$$

This vector is in $[T_o\Sigma, T_o\Sigma]$, which is orthogonal to $\mathfrak{h}$ by Proposition 6.5, so taking inner product with $S+Y+Z$, and using Lemma 1.3(b) and (6.4), we get $0 = -2\langle [S,X], Y\rangle - \frac{1}{2}\|Y\|^2\langle JX, Y\rangle$, which implies

$$\langle [S,X], Y\rangle = -\frac{1}{4}\|Y\|^2\langle JX, Y\rangle. \tag{6.5}$$

We also have

$$[T+B+X, S+Y+Z] = [T,S] + [T,Y] - [S,X] + \frac{1}{2}Y + \big(1+\frac{1}{2}\langle JX,Y\rangle\big)Z,$$

which is in $\mathfrak{h}$, so taking inner product with $-\langle X,Y\rangle B + (1-\theta)(Y - \frac{1}{2}\|Y\|^2 Z)$, and using (6.5), we obtain

$$0 = -\langle [S,X],Y\rangle + \frac{1}{2}\|Y\|^2 - \|Y\|^2\big(1+\frac{1}{2}\langle JX,Y\rangle\big) = -\frac{1}{2}\|Y\|^2\big(1+\frac{1}{2}\langle JX,Y\rangle\big).$$

Since $Y \neq 0$, we get $\langle JX, Y\rangle = -2$ and thus (6.4) can be written as

$$\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2 = 4 = \langle JX,Y\rangle^2.$$

Now put $X = \gamma Y + \delta JY + E$ with E orthogonal to $\mathbb{C}Y$, and $\gamma, \delta \in \mathbb{R}$. Then, the previous equation reads $\|E\|^2\|Y\|^2 = 0$, which yields $E = 0$. This implies the result. □

Therefore the situation now is $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B + \gamma Y + \frac{2}{\|Y\|^2}JY) \oplus \mathfrak{w} \oplus \mathbb{R}(Y+Z)$, with $\mathbb{C}Y \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$. The normal space can be rewritten as

$$\begin{aligned}\nu_o(H\cdot o) = {} & \mathbb{R}(-2B + (1-\theta)JY) \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)(\mathfrak{w}\oplus \mathbb{C}Y)) \\ & \oplus \mathbb{R}(-\gamma\|Y\|^2 B + (1-\theta)Y - \frac{1}{2}\|Y\|^2(1-\theta)Z),\end{aligned}$$

and arguing as above, if Σ is a section through o, then

$$T_o\Sigma = \mathbb{R}(-2B + (1-\theta)JY) \oplus (1-\theta)\mathfrak{s} \oplus \mathbb{R}(-\gamma\|Y\|^2 B + (1-\theta)Y - \frac{1}{2}\|Y\|^2(1-\theta)Z), \tag{6.6}$$

where $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{C}Y)$ is a totally real subspace of $\mathfrak{g}_\alpha$.

Lemma 6.24. *If $S \in \mathfrak{h}_{\mathfrak{k}_0}$ is such that $S+Y+Z \in \mathfrak{h}$ then $[S, JY] = \frac{1}{4}\|Y\|^2 Y$.*

Proof. First of all, by the properties of root systems and the skew-symmetry of the elements of $\operatorname{ad}(\mathfrak{k}_0)$, we have $[S, JY] \in \mathfrak{g}_\alpha \ominus \mathbb{R}JY$.

Lemma 1.3(a) yields

$$\begin{aligned}&[-2B + (1-\theta)JY, -\gamma\|Y\|^2 B + (1-\theta)(Y - \frac{1}{2}\|Y\|^2 Z)] \\ &\quad = (1+\theta)\Big(-[\theta JY, Y] + \Big(\frac{1}{2}\|Y\|^2 - 1\Big)Y + \frac{\gamma}{2}\|Y\|^2 JY + \frac{1}{2}\|Y\|^2 Z\Big),\end{aligned} \tag{6.7}$$

which is a vector in $[T_o\Sigma, T_o\Sigma]$.

Take $U \in \mathfrak{w}$, and let $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$ be such that $T_U + U \in \mathfrak{h}$. Taking inner product with (6.7) and using Lemma 1.3(b) we get $0 = 2\langle [T_U, JY], Y\rangle$. Using this equality and since $\mathfrak{h}$ is a Lie subalgebra, we now have

$$0 = \langle [S+Y+Z, T_U+U], -2B+(1-\theta)JY\rangle = \langle [S,T_U]+[S,U]-[T_U,Y], JY\rangle = \langle [S,U], JY\rangle,$$

and since $U \in \mathfrak{w}$ is arbitrary, $[S, JY] \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}JY)$.

Let $\xi \in \mathfrak{s}$. Proposition 6.5 implies

$$0 = \langle S + Y + Z, [-2B + (1-\theta)JY, (1-\theta)\xi]\rangle = -\langle S, (1+\theta)[\theta JY, \xi]\rangle = -2\langle [S, JY], \xi\rangle.$$

Let $\eta \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{C}Y)$ be an arbitrary vector. Since $\mathrm{Ad}(Q)(T_o\Sigma) = \nu_o(H \cdot o)$ by Proposition 6.5, we can conjugate the section Σ in such a way that $\eta \in \mathfrak{s}$. (Note that $-2B + (1-\theta)JY$ and $-\gamma\|Y\|^2 B + (1-\theta)Y - \frac{1}{2}\|Y\|^2(1-\theta)Z$ always belong to $T_o\Sigma$ by (6.6).) Hence, the equation above shows that $[S, JY]$ is orthogonal to $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{C}Y)$. Altogether this implies $[S, JY] \in \mathbb{R}Y$.

Finally, taking inner product of (6.7) with $S + Y + Z \in \mathfrak{h}$ we get, using Lemma 1.3(a), $0 = 2\langle [S,Y], JY\rangle + \frac{1}{2}\|Y\|^4$, and hence $[S, JY] = \frac{1}{4}\|Y\|^2 Y$ as we wanted. □

We define $g = \mathrm{Exp}(-4JY/\|Y\|^2)$. Recall that the Lie algebra of the isotropy group of H at $g(o)$ is $\mathfrak{h}_{g(o)} = \mathrm{Ad}(g)(\mathfrak{k}) \cap \mathfrak{h} = \mathfrak{q} \cap \ker \mathrm{ad}(JY)$, according to Lemma 6.19. Let $S \in \mathfrak{h}_{\mathfrak{k}_0}$ be such that $S + Y + Z \in \mathfrak{h}$. Then, Lemma 6.24 yields

$$\mathrm{Ad}(g)(S) = S - \frac{4}{\|Y\|^2}[JY, S] + \frac{8}{\|Y\|^4}[JY, [JY, S]] = S + Y + Z \in \mathrm{Ad}(g)(\mathfrak{k}) \cap \mathfrak{h}.$$

However, it is clear that $S + Y + Z \notin \mathfrak{q} \cap \ker \mathrm{ad}(JY)$, which gives a contradiction.

Therefore we have proved that $Y = 0$. Thus $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B+X) \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. If $X = 0$ then $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, and we are under the hypotheses of Proposition 6.17, which implies that the action of H is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. This corresponds to Theorem 6.16(b).

For the rest of this case we assume $X \neq 0$. Note that the normal space to the orbit through o is $\nu_o(H \cdot o) = \mathbb{R}(-\|X\|^2 B + (1-\theta)X) \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)(\mathfrak{w} \oplus \mathbb{R}X))$. If Σ is a section through o, since $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum), and $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$, it is easy to deduce, as in previous cases, that

$$T_o\Sigma = \mathbb{R}(-\|X\|^2 B + (1-\theta)X) \oplus (1-\theta)\mathfrak{s},$$

where $\mathbb{R}X \oplus \mathfrak{s}$ is a real subspace of $\mathfrak{g}_\alpha$.

We define $g = \mathrm{Exp}(2X)$. We will show $(\mathrm{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ and $\mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$, which will allow us to apply Proposition 6.17. From now on we take $T \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T + B + X \in \mathfrak{h}$.

Let $S \in \mathfrak{q}$. Then $[S,T] + [S,X] = [S, T + B + X] \in \mathfrak{h}$, and thus $[S, X] \in \mathfrak{w}$. Now let $U \in \mathfrak{w}$ be an arbitrary vector, and let $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_U + U \in \mathfrak{h}$. We have

$0 = \langle [S, S_U + U], -\|X\|^2 B + (1-\theta)X \rangle = -\langle [S, X], U \rangle$, which together with the previous assertion implies $[S, X] = 0$. Then $\mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$. In particular this implies that Q is a maximal compact subgroup of $I_g(H) = gHg^{-1}$.

Now we calculate $[T, X]$. Let $U \in \mathfrak{w}$ and $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_U + U \in \mathfrak{h}$. Then, by the skew-symmetry of the elements of $\mathrm{ad}(\mathfrak{k}_0)$ we have $0 = \langle [T+B+X, S_U+U], -\|X\|^2 B + (1-\theta)X \rangle = -\langle [T, X], U \rangle$, so $[T, X] \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Let now $\xi \in \mathfrak{s}$. By Proposition 6.5 we get, using Lemma 1.3(b), $0 = \langle T + B + X, [-\|X\|^2 B + (1-\theta)X, (1-\theta)\xi] \rangle = -\langle T, (1+\theta)[\theta X, \xi] \rangle = -2\langle [T, X], \xi \rangle$. Using again Proposition 6.5 we have $\nu_o(H \cdot o) = \mathrm{Ad}(Q)(T_o\Sigma)$, and thus, for any $\eta \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$ we can find a section through o such that $(1-\theta)\eta \in T_o\Sigma$ (note that $-\|X\|^2 B + (1-\theta)X \in T_o\Sigma$ for any section). Hence the previous argument shows $\langle [T, X], \eta \rangle = 0$, and altogether this means $[T, X] = 0$. Therefore, $\mathrm{Ad}(g)(T+B+X) = T+B$, so the projection of this vector onto $\mathfrak{a} \oplus \mathfrak{n}$ is in $\mathfrak{a} \subset \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$.

Fix $U \in \mathfrak{w}$ and $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_U + U \in \mathfrak{h}$. We calculate $[S_U, X]$. For any $\xi \in \mathfrak{s}$, by Proposition 6.5 and Lemma 1.3(b), we get $0 = \langle S_U + U, [-\|X\|^2 B + (1-\theta)X, (1-\theta)\xi] \rangle = -2\langle [S_U, X], \xi \rangle$. As in the previous paragraph, one can argue that ξ can be taken arbitrarily in $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$ by changing the tangent space to the section, if necessary, by an element of $\mathrm{Ad}(Q)$. Hence $[S_U, X] \in \mathfrak{w}$, which yields $\mathrm{Ad}(g)(S_U+U) = S_U + U - 2[S_U, X] + \frac{1}{2}(\langle JX, U \rangle - 2\langle JX, [S_U, X] \rangle)Z$, and thus, its projection onto $\mathfrak{a} \oplus \mathfrak{n}$ belongs to $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$.

Finally, let $S_Z \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_Z + Z \in \mathfrak{h}$. For each $\xi \in \mathfrak{s}$ we obtain $0 = \langle S_Z + Z, [-\|X\|^2 B + (1-\theta)X, (1-\theta)\xi] \rangle = -2\langle [S_Z, X], \xi \rangle$, and since ξ can be taken to be in $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$ by a suitable conjugation of the section by an element in $\mathrm{Ad}(Q)$, we deduce $[S_Z, X] \in \mathfrak{w}$. Hence, $\mathrm{Ad}(g)(S_Z + Z) = S_Z - 2[S_Z, X] + (1 - \langle JX, [S_Z, X] \rangle)Z$, and the orthogonal projection of this vector onto $\mathfrak{a} \oplus \mathfrak{n}$ belongs to $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$.

These last calculations show that $(\mathrm{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}} \subset \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. Since $g \in AN$ normalizes $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, we have that $\mathrm{Ad}(g)(\mathfrak{h}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then the kernel of the projection of $\mathrm{Ad}(g)(\mathfrak{h})$ onto $\mathfrak{a} \oplus \mathfrak{n}$ is precisely $\mathrm{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0$, which is a compact subalgebra of $\mathrm{Ad}(g)(\mathfrak{h})$ containing $\mathfrak{q} = \mathrm{Ad}(g)(\mathfrak{q})$. By the maximality of $\mathfrak{q}$ we get that $\mathrm{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0 = \mathfrak{q}$. But then by elementary linear algebra

$$\begin{aligned}\dim(\mathrm{Ad}(g)\mathfrak{h})_{\mathfrak{a}\oplus\mathfrak{n}} &= \dim \mathrm{Ad}(g)(\mathfrak{h}) - \dim(\mathrm{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0)\\ &= \dim \mathfrak{h} - \dim \mathfrak{q} = \dim \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \dim(\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}).\end{aligned}$$

All in all we have shown that the Lie algebra $\mathrm{Ad}(g)(\mathfrak{h})$ of $I_g(H) = gHg^{-1}$ satisfies $(\mathrm{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, and that Q is a maximal compact subgroup of $I_g(H)$. Therefore, we can apply Proposition 6.17 to $I_g(H)$. This implies that the action of H on $\mathbb{C}H^n$ is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. This corresponds to Theorem 6.16(b).

Altogether, we have concluded the proof of Theorem 6.16.

6.9 Proof of the main result

In this section we conclude the proof of Theorem 6.4 using the results of Sections 6.7 and 6.8.

Proof of Theorem 6.4. The actions described in part (i) are polar by virtue of Lemma 6.14 and Theorem 6.15, whereas the polarity of the actions in part (ii) follows from Theorem 6.10.

Since $\mathfrak{a} \subset \mathfrak{p}$, the actions in (ii) with $\mathfrak{b} = \mathfrak{a}$ contain the geodesic line $\exp_o \mathfrak{a}$. On the other hand, a horospherical foliation of $\mathbb{C}H^n$ is given by the action on $\mathbb{C}H^n$ of the connected subgroup N of G with Lie algebra $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. This shows that an orbit of minimal type for the actions with $\mathfrak{b} = 0$ is contained in a horosphere.

An action of a subgroup H of the isometry group $I(M)$ of a Riemannian manifold M is proper if and only if H is a closed subgroup of $I(M)$. Hence we may assume $H \subset \mathrm{SU}(1,n)$ is closed. Since the polarity of the action depends only on the Lie algebra of H by Proposition 6.5, we may assume that H is connected.

Thus, let H be a connected closed subgroup of $\mathrm{SU}(1,n)$ acting polarly on $\mathbb{C}H^n$. The Lie algebra $\mathfrak{h}$ of H is contained in a maximal subalgebra of $\mathfrak{su}(1,n)$. By [117, Theorem 1.9, Chapter 6], the maximal nonsemisimple subalgebras of a semisimple real Lie algebra are parabolic or coincide with the centralizer of a pseudotoric subalgebra. (A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is called *pseudotoric* if $\mathrm{Exp}\,\mathrm{ad}\,\mathfrak{t} \subset \mathrm{Int}\,\mathfrak{g}$ is a torus.) The maximal subalgebras of simple real Lie algebras which are centralizers of pseudotoric subalgebras have been classified in [138]. However, it is easy to determine them in the case of $\mathfrak{su}(1,n)$. Indeed, it follows from [117, Theorem 3.3, Chapter 4] that for all pseudotoric subalgebras $\mathfrak{t}$ of $\mathfrak{su}(1,n)$ there is an element $g \in \mathrm{SU}(1,n)$ such that $\mathrm{Ad}(g)\mathfrak{t}$ is contained in the subalgebra comprised of all diagonal matrices in $\mathfrak{su}(1,n)$. Since we are interested in maximal subalgebras which are centralizers of pseudotoric subalgebras $\mathfrak{t}$ we may restrict ourselves to one-dimensional pseudotoric subalgebras $\mathfrak{t}$. For such a subalgebra we have $\mathfrak{t} = \mathbb{R}\,\mathrm{diag}(it_0, \ldots, it_n)$, for $t_0, \ldots, t_n \in \mathbb{R}$ such that $t_0 + \cdots + t_n = 0$. The centralizers of such $\mathfrak{t}$ are the subalgebras of the form $\mathfrak{s}(\mathfrak{u}(1,n_1) \oplus \mathfrak{u}(n_2) \oplus \cdots \oplus \mathfrak{u}(n_\ell))$ where $n_1 + \cdots + n_\ell = n$. In particular, any maximal connected subgroup of $\mathrm{SU}(1,n)$ whose Lie algebra is the centralizer of a pseudotoric subalgebra is conjugate to one of the maximal subgroups $\mathrm{S}(\mathrm{U}(1,k)\mathrm{U}(n-k))$, $k = 0, \ldots, n-1$.

First, let us assume H is contained (after conjugation) in a maximal subgroup of the form $\mathrm{S}(\mathrm{U}(1,k)\mathrm{U}(n-k))$ or in a semisimple maximal subgroup of $\mathrm{SU}(1,n)$. In both cases, the action of H on $\mathbb{C}H^n$ leaves a totally geodesic submanifold invariant; this follows from the Karpelevich-Mostow Theorem [82], [106] (it is obvious in the first case). This situation has been studied in Section 6.7.

If the action of H leaves a totally geodesic $\mathbb{R}H^n$ invariant, then Theorem 6.15 applies and the H-action is orbit equivalent to the cohomogeneity one action of $\mathrm{SO}(1,n)$. This corresponds to case (i) with $k = n$ in Theorem 6.4. If the action of H leaves a totally geodesic $\mathbb{R}H^k$ invariant, with $k < n$, then it also leaves a totally geodesic $\mathbb{C}H^k$ invariant.

Let then k be the smallest complex dimension of a totally geodesic complex hyperbolic subspace left invariant by the H-action. If $k = 0$, then the H-action has a fixed point. In this case, it follows from [52] that H is a subgroup of $\mathrm{S}(\mathrm{U}(1)\mathrm{U}(n)) \cong \mathrm{U}(n)$ that corresponds to a polar action on $\mathbb{C}P^{n-1}$, and therefore is induced by the isotropy representation of a Hermitian symmetric space. This corresponds to case (i) with $k = 0$ in Theorem 6.4.

Let us assume from now on that $k \geq 1$. Lemma 6.14 guarantees that the H-action is orbit equivalent to the product action of a closed subgroup H_1 of $\mathrm{SU}(1,k)$ acting polarly on $\mathbb{C}H^k$ times a closed subgroup H_2 of $\mathrm{U}(n-k)$ acting polarly (and with a fixed point) on $\mathbb{C}H^{n-k}$. By assumption, the H_1-action on $\mathbb{C}H^k$ does not leave any totally geodesic $\mathbb{C}H^l$ or $\mathbb{R}H^l$ with $l < k$ invariant. Hence, either the H_1-action on $\mathbb{C}H^k$ is orbit equivalent to the $\mathrm{SO}(1,k)$-action on $\mathbb{C}H^k$, or H_1 is contained in a maximal parabolic subgroup of $\mathrm{SU}(1,k)$. The first case corresponds to part (i) with $k \in \{1,\dots,n\}$. Note that for $Q = H_2$, the Q-action on $\mathbb{C}H^{n-k}$ is determined by its slice representation at the fixed point, so Q acts polarly with a totally real section on $T_o\mathbb{C}H^{n-k} \cong \mathbb{C}^{n-k}$.

Let us consider the second case, that is, H_1 is contained in a maximal parabolic subgroup of $\mathrm{SU}(1,k)$, $k \in \{1,\dots,n\}$. As explained at the beginning of Section 6.8, we may assume $\mathfrak{h}_1 \subset \mathfrak{k}_0^1 \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^1 \oplus \mathfrak{g}_{2\alpha}$, where now $\mathfrak{g}_\alpha^1$ is a complex subspace of $\mathfrak{g}_\alpha$ with complex dimension $k-1$, and $\mathfrak{k}_0^1 \cong \mathfrak{u}(k-1)$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k} \cap \mathfrak{su}(1,k)$. It follows that the H_1-action is orbit equivalent to the action of a closed subgroup of $\mathrm{SU}(1,k)$ with one of the Lie algebras described in Theorem 6.16: (a) $\mathfrak{q}^1 \oplus \mathfrak{a}$, (b) $\mathfrak{q}^1 \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, or (c) $\mathfrak{q}^1 \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is a real subspace of $\mathfrak{g}_\alpha^1$, and $\mathfrak{q}^1 \subset \mathfrak{k}_0^1$ normalizes $\mathfrak{w}$. Since $H_2 \subset \mathrm{U}(n-k)$ acts on $\mathbb{C}H^{n-k}$, we can define $\mathfrak{q} = \mathfrak{q}^1 \oplus \mathfrak{h}_2$, which is a subalgebra of $\mathfrak{k}_0$. Part (a) of Theorem 6.16 is then a particular case of Theorem 6.4(i) for $k = 1$, while parts (b) and (c) of Theorem 6.16 correspond to Theorem 6.4(ii), where $\mathfrak{b} = \mathfrak{a}$ and $\mathfrak{b} = 0$, respectively. Lemma 1.4, Proposition 6.6 and the fact that the slice representation of a polar action is also polar, guarantee that the action of $\mathfrak{q}$ on the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$ is polar with a totally real section. $\square$

Chapter 7

Isoparametric foliations on complex projective spaces

In Chapters 4 and 5 the objects of our study were isoparametric families of hypersurfaces or, in other words, isoparametric (singular Riemannian) foliations of codimension one. In this chapter we relax the condition on the codimension and focus on the case of complex projective spaces as ambient manifolds. Thus, we will classify irreducible isoparametric foliations of arbitrary codimension q on complex projective spaces $\mathbb{C}P^n$, for $(q, n) \neq (1, 15)$. Most of the contents of this chapter have been included in the paper [56].

There is also a connection with the objects of study of the previous chapter, where we investigated polar actions up to orbit equivalence or, in other words, homogeneous polar (singular Riemannian) foliations. In the case of complex projective spaces $\mathbb{C}P^n$, the classification of such objects was achieved by Podestà and Thorbergsson [123]. However, little was known about polar foliations on $\mathbb{C}P^n$ without the assumption of homogeneity. The work in this chapter sheds light on this point, because polar and isoparametric foliations are equivalent concepts on complex projective spaces.

The most remarkable consequence of our investigation here is the existence of many inhomogeneous irreducible isoparametric foliations, even of codimension higher than one. In fact, as a by-product of our analysis we will show that every irreducible isoparametric foliation on $\mathbb{C}P^n$ is homogeneous if and only if $n + 1$ is a prime number. It seems that the existence of inhomogeneous examples of codimension greater than one was something unknown so far. This is surprising in view of the situation in spheres, where there are no such examples, as shown by Thorbergsson [143].

The main tool we develop in this chapter is a method to study singular Riemannian foliations with closed leaves on complex projective spaces. This method is based on certain graph that generalizes extended Vogan diagrams of inner symmetric spaces.

This chapter has been organized in the following manner. In Section 7.1 we review some important results involving isoparametric foliations of arbitrary codimension and related notions. In Section 7.2 we give the idea that motivates our approach and we state our main results. Section 7.3 is devoted to the study the behaviour of isoparametric submanifolds with respect to the Hopf map. In Section 7.4 we find the group of automorphisms of homogeneous polar foliations on Euclidean spaces (§7.4.1) and of most FKM-foliations (§7.4.2). In Section 7.5 we study general foliations with closed leaves on $\mathbb{C}P^n$, first characterizing the complex structures that preserve a given foliation on a sphere (§7.5.1) and then studying the congruence of the projected foliations (§7.5.2). We particularize this theory and obtain the corresponding classifications for homogeneous polar foliations in Section 7.6 and for FKM-foliations in Section 7.7. Finally, in Section 7.8 we study the homogeneity of the resulting isoparametric foliations on $\mathbb{C}P^n$.

7.1 Isoparametric foliations of arbitrary codimension

In Chapter 2 we presented an introduction to the theory of isoparametric hypersurfaces and other related topics, such as homogeneous hypersurfaces and hypersurfaces with constant principal curvatures. This section can be seen as an extension of that exposition to the more general field of isoparametric submanifolds of arbitrary codimension. Thus, here we will review the most relevant milestones in the study of these objects, in order to proper contextualize our research. There are several excellent references to deepen into this and other related topics. Let us mention the articles [144], [145], [146], [74] and [5], as well as the books [121] and [11].

Isoparametric submanifolds of arbitrary codimension on real space forms were first studied by Harle [72], Carter and West [32] and, more crucially, by Terng [140]. A submanifold of a space form is called *isoparametric* if its normal bundle is flat and if it has constant principal curvatures in the direction of any parallel normal field. Any isoparametric submanifold in the Euclidean space is the product of an isoparametric submanifold in a sphere times an affine subspace. Thus, it is equivalent to study these objects in Euclidean spaces or in spheres. The case of real hyperbolic spaces was tackled by Wu [155], who reduced the classification problem of isoparametric submanifolds in $\mathbb{R}H^n$ to the problem in spheres.

In real space forms, every isoparametric submanifold extends to a global isoparametric foliation, as follows from the works by Terng and Wu. Indeed, these foliations can be seen as the level sets of the so-called isoparametric maps (see [140]). Moreover, Terng developed a structure theory for isoparametric foliations that associates certain Coxeter group to each such foliation.

A foliation on an arbitrary Riemannian manifold is said to be *polar* if there is a totally geodesic immersed submanifold Σ_p through every point p intersecting all leaves orthogonally (Σ_p is then called a *section*). This notion of polar foliation was proposed by Alexandrino [3] under the terminology of *singular Riemannian foliation with sections*. Previously, Boualem [26] had already studied foliations such that the distribution of the normal spaces of the regular leaves is integrable, which is equivalent to the notion of polar foliation [4]. It turns out that a foliation on a space form of nonnegative curvature is isoparametric if and only if it is polar. The fact that isoparametric foliations on space forms are polar follows from the general theory of isoparametric submanifolds developed by Terng, whereas the converse is a consequence of the constant curvature of the ambient space (see [142, p. 669] and [3, Theorem 2.7]). However, polar foliations on $\mathbb{R}H^n$ are not so rigid and are not isoparametric in general (see [152, p. 89, Remark 1]).

We have explained in Section 2.3 that, for isoparametric foliations of codimension one on spheres, there are many inhomogeneous examples and the classification problem is still open. However, the situation for higher codimension is very different. Using theory of Tits buildings, Thorbergsson [143] showed that all such examples are homogeneous. More precisely:

Theorem 7.1. [143] *Every irreducible isoparametric foliation of codimension higher than one on a sphere is the orbit foliation of an s-representation.*

The attempts to generalize isoparametric foliations to ambient spaces of nonconstant curvature have led to several different but related concepts. We have already mentioned the notion of polar foliation introduced by Alexandrino [3]. Another important concept, which was introduced by Terng and Thorbergsson [142], is that of *equifocal submanifold* of a compact symmetric space. A closed submanifold of a compact symmetric space is equifocal if it has globally flat and abelian normal bundle and its focal directions and distances are invariant under parallel translation in the normal bundle. This notion of equifocality has been modified by eliminating the requirement of having abelian normal bundle (or, equivalently, having flat sections). Thus, Alexandrino [3] defines an immersed submanifold of a complete Riemannian manifold to be equifocal if it has globally flat normal bundle, its focal directions and distances are invariant under parallel translation in the normal bundle, and it admits sections. Here, admitting sections means that, for every point p in the submanifold M, there exists a complete, immersed, totally geodesic submanifold Σ_p such that $\nu_p M = T_p\Sigma_p$. It turns out that, with this definition of equifocal submanifold, the regular leaves of polar foliations are equifocal. The converse, the fact that the partition of a Riemannian manifold into the parallel submanifolds determined by an equifocal submanifold is a polar foliation, is also true under some mild assumption; see [5, §4] for more details. Thus, the original notion of equifocality with flat sections turns out to be equivalent to that of *hyperpolar foliation*, i.e. polar foliation with flat sections.

Terng and Thorbergsson [142] developed a powerful method to study equifocal submanifolds, based on the use of a Riemannian submersion $\mathcal{H} \to G/K$ from a Hilbert space $\mathcal{H}$ of paths to a compact symmetric space G/K, which allows to lift equifocal submanifolds (equivalently, hyperpolar foliations) from G/K to $\mathcal{H}$. This technique was employed by Christ [40] to show the homogeneity of every irreducible hyperpolar foliation of codimension at least two on a simply connected compact symmetric space. Christ's theorem makes use of a homogeneity result for isoparametric submanifolds of a Hilbert space $\mathcal{H}$ (with codimension greater than one if $\mathcal{H}$ is infinite dimensional, or with codimension greater than two if $\mathcal{H}$ is finite dimensional). This result is due to Heintze and Liu [73] and provides a different proof of Thorbergsson's theorem when applied to a finite dimensional Hilbert space.

In this chapter, the definition of isoparametric submanifold that we will consider is the one due to Heintze, Liu and Olmos [74], which we presented in Section 1.4. This definition extends the notions of isoparametric hypersurface in any Riemannian manifold and of isoparametric submanifold of a real space form. Moreover, the locally defined parallel submanifolds of an isoparametric submanifold are isoparametric as well, and thus define locally an *isoparametric foliation*.

Practically at the same time that the results of this chapter were published in the preprint [56], Lytchak proved in [98] that every polar foliation of codimension at least three on an irreducible compact symmetric space of rank higher than one is hyperpolar. Combining this result with Christ's homogeneity theorem [40] and with the classification of hyperpolar actions on irreducible simply connected compact symmetric spaces due to Kollross [85], the work by Lytchak allows to get a classification of irreducible polar foliations of codimension greater than two on irreducible simply connected compact symmetric spaces of rank greater than one. Note that, in this case, this classification is then equivalent to that of isoparametric foliations on these spaces, and all examples are homogeneous.

To our knowledge, the classification presented in this chapter and the one that follows from Lytchak's, Christ's and Kollross' papers [98], [40], [85], constitute the first classifications of isoparametric or polar foliations of arbitrary codimension on symmetric spaces of nonconstant curvature. However, an important difference between both contexts is the question of the homogeneity: while for symmetric spaces of rank greater than one the classified examples are homogeneous, rank one symmetric spaces of nonconstant curvature seem to allow a greater diversity of examples, including inhomogeneous foliations of large codimension, as we will see below in this chapter.

7.2 Motivation and main results

The aim of this chapter is to address the classification problem of isoparametric submanifolds of arbitrary codimension on complex projective spaces $\mathbb{C}P^n$. A first important observation is that, as for real space forms, every isoparametric submanifold in $\mathbb{C}P^n$ extends to a globally defined isoparametric foliation, as we will see in Remark 7.6. Thus, we will mostly talk about isoparametric foliations instead of isoparametric submanifolds.

In order to give some of the fundamental ideas and main results of our work, it is convenient to settle some terminology.

Let $\mathcal{F}$ be a foliation of the unit sphere S^{2n+1} of $\mathbb{R}^{2n+2}$ and let J be a complex structure on $\mathbb{R}^{2n+2}$. Recall that, in this work, by complex structure we mean an orthogonal, skew-symmetric transformation, and by foliation we mean singular Riemannian foliation. We will say that J preserves $\mathcal{F}$ if $\mathcal{F}$ is the pullback of a foliation on $\mathbb{C}P^n$ under the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ determined by J or, equivalently, if the leaves of $\mathcal{F}$ are foliated by the Hopf circles determined by J. We will say that a foliation $\mathcal{G}$ of a complex projective space $\mathbb{C}P^n$ is irreducible if there is no totally geodesic $\mathbb{C}P^k$, $k \in \{0, \dots, n-1\}$, which is foliated by leaves of $\mathcal{G}$.

An important set of examples of isoparametric foliations on spheres come from s-representations of symmetric spaces. Thus, if (G, K) is a semisimple symmetric pair, we will denote by $\mathcal{F}_{G/K}$ the orbit foliation of the isotropy representation of G/K restricted to the unit sphere of the tangent space $T_{eK}(G/K)$. Here e denotes the identity element of G.

Our investigation was initially motivated by the following observation of Xiao [158]. He noticed that, if G/K is the real Grassmann manifold $\mathrm{Gr}_2(\mathbb{R}^{n+3}) = \mathrm{SO}(n+3)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(n+1))$ with odd n, one can find two complex structures J_1 and J_2 on $T_{eK}(G/K)$ that preserve $\mathcal{F}_{G/K}$ and with the following property: the projections of any fixed regular leaf of $\mathcal{F}_{G/K}$ via the corresponding Hopf maps $\pi_1, \pi_2 \colon S^{2n+1} \subset T_{eK}(G/K) \to \mathbb{C}P^n$ yield two noncongruent isoparametric hypersurfaces of $\mathbb{C}P^n$, one of which is homogeneous while the other one is not.

Therefore, it seems natural to address the following problem: given an isoparametric foliation $\mathcal{F}$ on the sphere S^{2n+1}, find the set $\mathcal{J}_\mathcal{F}$ of complex structures on $\mathbb{R}^{2n+2}$ that preserve $\mathcal{F}$, determine the quotient set $\mathcal{J}_\mathcal{F}/\sim$, where $\sim$ stands for the equivalence relation "give rise to congruent foliations on $\mathbb{C}P^n$", and finally decide which elements of $\mathcal{J}_\mathcal{F}/\sim$ provide homogeneous foliations on the corresponding $\mathbb{C}P^n$. Note that determining $\mathcal{J}_\mathcal{F}/\sim$ is then equivalent to classifying (up to congruence in $\mathbb{C}P^n$) those foliations on $\mathbb{C}P^n$ that pull back under the Hopf map to a foliation congruent to $\mathcal{F}$. We will write $N(\mathcal{F})$ for the cardinality of $\mathcal{J}_\mathcal{F}/\sim$.

Firstly, we will carry out this investigation for all isoparametric foliations $\mathcal{F}_{G/K}$ arisen from s-representations, thus obtaining the following result. Recall that G/K is called inner if $\operatorname{rank} G = \operatorname{rank} K$. In order to keep the statement as short as possible, we have made use of Cartan's notation for symmetric spaces (see [76, p. 518] or Section 7.6.3 below in this chapter).

Theorem 7.2. *Let G/K be an irreducible inner compact symmetric space of rank greater than one and set $n = \frac{1}{2}\dim G/K - 1$. Then, up to congruence in $\mathbb{C}P^n$, there are exactly $N(\mathcal{F}_{G/K}) \geq 1$ isoparametric foliations on $\mathbb{C}P^n$ whose pullback under the Hopf map gives a foliation congruent to $\mathcal{F}_{G/K}$, where:*

- $N(\mathcal{F}_{G/K}) = 1 + \left[\frac{\nu}{2}\right] + \left[\frac{p-\nu+1}{2}\right]$, *if* $G/K = \mathrm{Gr}_\nu(\mathbb{C}^{p+1})$ *with* $2\nu \neq p+1$,
- $N(\mathcal{F}_{G/K}) = 1 + \left[\frac{\nu}{2}\right]$, *if* $G/K = \mathrm{Gr}_\nu(\mathbb{C}^{p+1})$ *with* $2\nu = p+1$,
- $N(\mathcal{F}_{G/K}) = 2$, *if* $G/K = \mathrm{Gr}_\nu(\mathbb{H}^p)$ *with* $2\nu \neq p$, *or if* $G/K = \mathrm{Gr}_{2\nu}(\mathbb{R}^{2p})$ *with* $2\nu \neq p$, *or if* $G/K \in \{\mathbf{D\,III}, \mathbf{E\,II}, \mathbf{E\,III}, \mathbf{E\,VI}\}$,
- $N(\mathcal{F}_{G/K}) = 1$, *otherwise.*

Moreover, if G/K is Hermitian, exactly one of those $N(\mathcal{F}_{G/K})$ foliations is homogeneous. If G/K is not Hermitian, all $N(\mathcal{F}_{G/K})$ foliations are inhomogeneous.

Conversely, if $\mathcal{G}$ is an irreducible isoparametric foliation of codimension greater than one on $\mathbb{C}P^n$, then there exist an irreducible inner compact symmetric space G/K of dimension $2n+2$ and a complex structure J on $T_{eK}(G/K)$ preserving $\mathcal{F}_{G/K}$ such that $\mathcal{G}$ is the projection of $\mathcal{F}_{G/K}$ by the Hopf map associated with J.

In this chapter we will investigate the possible projections not only of orbit foliations arisen from s-representations, but also of the other isoparametric foliations on spheres which are not of this type, i.e. the inhomogeneous FKM-foliations. We will restrict this study to FKM-foliations satisfying $m_1 \leq m_2$. To understand this condition and Theorem 7.3 below, let us briefly recall some known facts; see §7.4.2 for more details. Given a symmetric Clifford system $(P_0, \dots, P_m)$ on $\mathbb{R}^{2n+2}$, the corresponding FKM-foliation $\mathcal{F}_\mathcal{P}$ depends only on the $(m+1)$-dimensional vector space of symmetric matrices $\mathcal{P} = \operatorname{span}\{P_0, \dots, P_m\}$. The hypersurfaces of $\mathcal{F}_\mathcal{P}$ have $g = 4$ principal curvatures with multiplicities $(m_1, m_2) = (m, n-m)$. Let Cl^*_{m+1} be the Clifford algebra of $\mathbb{R}^{m+1}$ with positive definite quadratic form. There is one equivalence class $\mathfrak{d}$ of irreducible Cl^*_{m+1}-modules if $m \not\equiv 0 \pmod 4$, and two equivalence classes $\mathfrak{d}_+$, $\mathfrak{d}_-$ if $m \equiv 0 \pmod 4$. Then $\mathcal{P}$ determines a representation of Cl^*_{m+1} on $\mathbb{R}^{2n+2}$, which is then equivalent to $\oplus_{i=1}^{k}\mathfrak{d}$ for some k if $m \not\equiv 0 \pmod 4$, or to $(\oplus_{i=1}^{k_+}\mathfrak{d}_+) \oplus (\oplus_{i=1}^{k_-}\mathfrak{d}_-)$ for some k_+, k_- if $m \equiv 0 \pmod 4$.

The condition $m_1 \leq m_2$ always holds, except for 8 FKM-examples. However, some of these exceptions are homogeneous or congruent to other FKM-foliations, so only two examples remain unsettled, namely: both FKM-foliations with $(m_1, m_2) = (8, 7)$. Intriguingly, such examples belong to the only open case in the classification of isoparametric hypersurfaces in spheres. Now, combining these results with Theorem 7.1, our work classifies all irreducible isoparametric foliations of arbitrary codimension q on $\mathbb{C}P^n$, except if $n = 15$ and $q = 1$. More explicitly, we have:

Theorem 7.3. *Let $\mathcal{F}_\mathcal{P}$ be an FKM-foliation on S^{2n+1} with $\dim \mathcal{P} = m+1$. Assume that $m_1 \leq m_2$. Then, up to congruence in $\mathbb{C}P^n$, there are exactly $N(\mathcal{F}_\mathcal{P}) \geq 1$ isoparametric foliations on $\mathbb{C}P^n$ that pull back under the Hopf map to a foliation congruent to $\mathcal{F}_\mathcal{P}$, where:*

- *$N(\mathcal{F}_\mathcal{P}) = 2$, if $m \equiv 0 \pmod 8$ with k_+ and k_- even, or if $m \equiv 1, 7 \pmod 8$ with k even, or if $m \equiv 3, 4, 5 \pmod 8$,*
- *$N(\mathcal{F}_\mathcal{P}) = 2 + \left[\frac{k}{2}\right]$, if $m \equiv 2, 6 \pmod 8$,*
- *$N(\mathcal{F}_\mathcal{P}) = 1$, otherwise.*

Conversely, if $\mathcal{G}$ is an isoparametric foliation of codimension one on $\mathbb{C}P^n$, then there is a foliation $\mathcal{F}$ on S^{2n+1} and a complex structure J on $\mathbb{R}^{2n+2}$ preserving $\mathcal{F}$ such that $\mathcal{G}$ is the projection of $\mathcal{F}$ by the Hopf map associated to J, where

- *$\mathcal{F} = \mathcal{F}_\mathcal{P}$ is an FKM-foliation satisfying $m_1 \leq m_2$, or*
- *$\mathcal{F} = \mathcal{F}_{G/K}$ for some inner compact symmetric space G/K of rank 2, or*
- *$\mathcal{F}$ is an inhomogeneous isoparametric foliation of codimension one on S^{31} whose hypersurfaces have $g = 4$ principal curvatures with multiplicities $(7, 8)$.*

Our results could have been stated in terms of polar foliations instead of isoparametric foliations. In fact, a foliation on a complex projective space is isoparametric if and only if it is polar, as follows from the analogous result for spheres in combination with the good behaviour of isoparametric and polar foliations with respect to the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ (see Section 7.3 and [98, Proposition 9.1]). Thus, our work gives an almost complete classification of polar foliations on complex projective spaces.

An important consequence of Theorem 7.2 is the existence of *irreducible inhomogeneous isoparametric foliations of higher codimension* on complex projective spaces. This shows the impossibility of extending Thorbergsson's homogeneity theorem from spheres to complex projective spaces. Moreover, it gives the first examples of irreducible inhomogeneous polar foliations of codimension greater than one on a compact symmetric space. We also generalize known results due to Wang [150], Xiao [158] and Ge, Tang and Yan [66] on the existence of inhomogeneous examples of codimension one on $\mathbb{C}P^n$. As shown by Ge, Tang and Yan, these examples exist if and only if n is odd and $n \geq 3$ (cf. Theorem 7.36(i)).

Several ingredients are fundamental in the proof of Theorems 7.2 and 7.3. The classification results of isoparametric foliations on spheres and the nice behaviour of isoparametric submanifolds with respect to the Hopf map constitute the starting point of our arguments. However, the main tool we develop is certain general theory for the study of foliations with closed leaves on complex projective spaces. This is based, on the one hand, on the consideration of the automorphism group of foliations $\mathcal{F} \subset S^{2n+1}$, i.e. the group of orthogonal transformations of $\mathbb{R}^{2n+2}$ that map leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. This motivates the calculation of this group for homogeneous polar foliations on Euclidean spaces and for FKM-foliations satisfying $m_1 \leq m_2$. On the other hand, our method requires the study of the symmetries of certain graph (the *lowest weight diagram*) that we associate with $\mathcal{F}$. If G/K is inner and $\mathcal{F} = \mathcal{F}_{G/K}$, such a diagram amounts to the *extended Vogan diagram* of G/K. Finally, a subtle improvement of a result of Podestà and Thorbergsson [123] gives us a criterion to decide when an isoparametric foliation on $\mathbb{C}P^n$ is homogeneous, from where we obtain some nice consequences, for example:

Theorem 7.4. *Every irreducible isoparametric foliation on $\mathbb{C}P^n$ is homogeneous if and only if $n+1$ is a prime number.*

At some point of the investigation that we collect in this chapter, we came across another article of Xiao [157], where he claims to obtain the classification of isoparametric submanifolds in $\mathbb{C}P^n$. However, the arguments and classification in [157] seem to have several crucial gaps. Firstly, the author uses the maximality property for s-representations (see §7.4.1) without actually referring to it. Secondly, the study of the inner symmetric spaces **E V** and **E VIII** is missing there. But more importantly, although he mentions that there are pairs of noncongruent isoparametric submanifolds in $\mathbb{C}P^n$ with congruent inverse images, surprisingly this is not reflected in his classification, since for each inner symmetric space G/K considered, only one complex structure is specified. Therefore, the congruence problem (which is the main difficulty in our work) is completely disregarded, as well as the study of the homogeneity.

7.3 Isoparametric submanifolds and the Hopf map

In this section we study the behaviour of isoparametric submanifolds with respect to the Hopf map and comment on some related questions.

Let us first recall that the construction of the complex projective space $\mathbb{C}P^n$ depends on the choice of a complex structure J on the Euclidean space $\mathbb{R}^{2n+2}$. This J induces a principal fiber bundle with total space the unit sphere S^{2n+1}, with base space the complex projective space $\mathbb{C}P^n$ and with structural group S^1. The corresponding projection $\pi\colon S^{2n+1} \to \mathbb{C}P^n$ is called the Hopf map and the Fubini-Study metric of constant holomorphic sectional curvature 4 on $\mathbb{C}P^n$ turns this map into a Riemannian submersion. See Section 1.7 for more details.

Let us denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of $\mathbb{C}P^n$ and S^{2n+1}, respectively. Then, for all tangent vector fields X, Y on $\mathbb{C}P^n$ we have that $\widetilde{\nabla}_{X^L} Y^L = (\nabla_X Y)^L + O'N(X^L, Y^L)$. Here $(\cdot)^L$ denotes the horizontal lift of a vector field, whereas $O'N$ is one of the tensors of O'Neill (denoted by A in [115]). This tensor satisfies $O'N(X^L, Y^L) = (\widetilde{\nabla}_{X^L} Y^L)^{\mathcal{V}} = \frac{1}{2}[X^L, Y^L]^{\mathcal{V}}$, where $(\cdot)^{\mathcal{V}}$ denotes orthogonal projection onto the vertical space.

Heintze, Liu and Olmos showed in [74, Theorem 3.4] that, if $\pi\colon E \to B$ is a Riemannian submersion with minimal fibers and $M \subset B$ an embedded submanifold, then $\widetilde{M} = \pi^{-1}M$ is isoparametric with horizontal sections if and only if M is isoparametric and $O'N = 0$ on all horizontal lifts of tangent vectors to sections of M; moreover, in this situation, π maps sections of $\widetilde{M}$ to sections of M. Using this result, we can show the following.

Proposition 7.5. *Let M be an embedded submanifold of $\mathbb{C}P^n$ of positive dimension and $\widetilde{M} = \pi^{-1}M$ its lift to S^{2n+1}. Then M is isoparametric if and only if $\widetilde{M}$ is isoparametric.*

In this situation, π maps sections of $\widetilde{M}$ (which are horizontal) to sections of M (which are totally real).

Proof. First notice that the fibers of the Hopf map are minimal (in fact, totally geodesic) and that, if $\widetilde{M}$ is isoparametric, it necessarily has horizontal sections (since $\widetilde{M}$ is union of S^1-fibers). Therefore, by the result in [74], if $\widetilde{M}$ is isoparametric, then M is isoparametric.

Assume now that M is isoparametric. Let X, Y be arbitrary tangent vector fields to the sections of M. Denote by ξ the outer unit normal vector field to S^{2n+1}, so $J\xi$ is a vertical vector field on S^{2n+1}. Let D be the Levi-Civita connection of $\mathbb{R}^{2n+2}$. We have:

$$\begin{aligned}\langle O'N(X^L, Y^L), J\xi\rangle &= \langle \widetilde{\nabla}_{X^L} Y^L, J\xi\rangle = \langle D_{X^L} Y^L, J\xi\rangle \\ &= -\langle D_{X^L} JY^L, \xi\rangle = \langle JY^L, X^L\rangle = \langle JY, X\rangle,\end{aligned}$$

since S^{2n+1} is a totally umbilical hypersurface in $\mathbb{R}^{2n+2}$ and J preserves the horizontal distribution. Hence, the proposition will follow from the result in [74] once we show that sections of M are totally real.

It is known that any totally geodesic submanifold of $\mathbb{C}P^n$ must be either a totally real or a complex submanifold. By continuity, if one section of M is complex, then all sections of M are complex. However, if the sections of M were complex, then M would be an almost complex submanifold of $\mathbb{C}P^n$, and hence Kähler, but this is impossible because there are no Kähler submanifolds of positive dimension with flat normal bundle in $\mathbb{C}P^n$ (see, for example, [2, Theorem 19]). Hence, all sections of M are totally real and the result follows. □

Proposition 7.5 guarantees that every isoparametric submanifold in a complex projective space can be obtained by projecting some isoparametric submanifold in a sphere under the Hopf map.

Remark 7.6. As well as for space forms, every isoparametric submanifold in $\mathbb{C}P^n$ can be extended to a global isoparametric foliation on $\mathbb{C}P^n$. Let us show this. Every isoparametric submanifold extends locally to an isoparametric foliation. By Proposition 7.5 the lift of this local foliation to an open set U of S^{2n+1} is again isoparametric. By [141, Theorem 3.4] and [140, Theorem D], a local isoparametric foliation of S^{2n+1} can be extended to an isoparametric foliation $\mathcal{F}$ of the whole sphere in a unique way; moreover, this foliation is defined by the level sets of the restriction $F|_{S^{2n+1}}$ of a polynomial function $F = (F_1, \dots, F_k)\colon \mathbb{R}^{2n+2} \to \mathbb{R}^k$, where k is the lowest codimension of the leaves and the gradients $\operatorname{grad} F_1, \dots, \operatorname{grad} F_k$ define k global normal vector fields on every leaf; on each regular leaf these fields conform a basis of the normal space. Consider the analytic function $f\colon S^{2n+1} \to \mathbb{R}^k$, defined by $x \mapsto (\langle Jx, (\operatorname{grad} F_1)_x\rangle, \dots, \langle Jx, (\operatorname{grad} F_k)_x\rangle)$. Since f is constantly equal to zero in U (the leaves of this local foliation are foliated by Hopf fibers), by analiticity we get that $f = 0$ identically on S^{2n+1}, and therefore, $\mathcal{F}$ can be projected to a global isoparametric foliation on $\mathbb{C}P^n$.

Every isoparametric foliation on a sphere determines an isoparametric foliation on the whole Euclidean space via homotheties. Conversely, if the leaves of an isoparametric foliation on a Euclidean space are compact, then they are contained in concentric spheres. Moreover, an isoparametric foliation of codimension $k-1$ on a sphere is said to be irreducible if its associated Coxeter system of rank k (in the sense of Terng [140]) is irreducible, or equivalently, if there is no proper totally geodesic submanifold of the sphere being a union of leaves of the foliation. Similarly, we will say that an isoparametric foliation on a complex projective space $\mathbb{C}P^n$ is *irreducible* if there is no proper totally geodesic complex projective subspace $\mathbb{C}P^k$, $k < n$, that is a union of leaves of the foliation. Hence, an isoparametric foliation on a complex projective space is irreducible if and only if its lift to the sphere S^{2n+1} is an irreducible isoparametric foliation. This follows from the fact that the only totally geodesic submanifolds of S^{2n+1} which are foliated by Hopf circles are intersections of S^{2n+1} with complex subspaces of $\mathbb{C}^{n+1}$.

According to Proposition 7.5, the problem of classifying irreducible isoparametric foliations on $\mathbb{C}P^n$ amounts to determining which irreducible isoparametric foliations on S^{2n+1} are such that their leaves contain the S^1-fibers of the Hopf map. Our approach lies, therefore, on the classification of isoparametric foliations on spheres. For irreducible isoparametric foliations of codimension greater than one, Thorbergsson's result (Theorem 7.1) guarantees that they are exactly orbit foliations $\mathcal{F}_{G/K}$ of isotropy representations of irreducible semisimple symmetric spaces G/K. In codimension one, Münzner's result [108] ensures that the number of principal curvatures of an isoparametric hypersurface in a sphere is $g \in \{1, 2, 3, 4, 6\}$ and the corresponding multiplicities $m_1, \dots, m_g$ satisfy $m_i = m_{i+2}$ (indices modulo g). As already commented, the classification of isoparametric hypersurfaces in spheres has been completed, except if $(g, m_1, m_2) = (4, 7, 8)$. We refer to Section 2.3 for more information on this problem.

These results imply that every irreducible isoparametric foliation on a sphere is an FKM-foliation or a homogeneous polar foliation, excluding the exceptional case of codimension one. We finish this section with a result that will be needed later.

Proposition 7.7. *Let M be an isoparametric hypersurface in a sphere with $(g, m_1, m_2) \in \{(4,2,2),(6,2,2)\}$. Then M is not the pullback of a hypersurface in $\mathbb{C}P^n$ ($n = 4, 6$) under some Hopf map.*

Proof. Münzner [109] determined the cohomology rings $H^*(M, \mathbb{Z}_2)$ of isoparametric hypersurfaces M in spheres. It follows from this result that $H^q(M, \mathbb{Z}_2) = 0$ for all odd integers $q \in \{1, \ldots, \dim M\}$ and that $2g = \dim_{\mathbb{Z}_2} H^*(M, \mathbb{Z}_2)$. Therefore the Euler characteristic of M is $\chi(M) = 2g \neq 0$. This implies that M is not foliated by Hopf circles: otherwise, the complex structure J would determine a globally defined non-vanishing tangent vector field on M, which would imply $\chi(M) = 0$ because of the Hopf index theorem. □

7.4 The group of automorphisms of an isoparametric foliation

Our aim in this section is to determine the whole (not necessarily connected) group of orthogonal transformations leaving invariant a given isoparametric foliation on a sphere. We will call these transformations the automorphisms of the foliation, and denote by $\mathrm{Aut}(\mathcal{F})$ the group of automorphisms of the foliation $\mathcal{F}$. We carry out this study for the case of orbit foliations of s-representations (or, equivalently, for homogeneous polar foliations) in §7.4.1 and for the case of FKM-foliations satisfying $m_1 \leq m_2$ in §7.4.2.

7.4.1 The group of automorphisms of a homogeneous polar foliation

Dadok [42] classified homogeneous polar foliations (or equivalently, polar actions up to orbit equivalence) on Euclidean spaces. He proved that these foliations are orbit foliations of isotropy representations of Riemannian symmetric spaces.

Since any homogeneous polar foliation on a Euclidean space is the product of a homogeneous polar foliation with compact leaves times an affine subspace, we will just consider homogeneous polar foliations with compact leaves. This means that the symmetric space G/K whose isotropy representation defines the foliation is semisimple. Moreover, since the duality between symmetric spaces of compact and noncompact type preserves their isotropy representations, we will assume that G/K is of compact type.

Given a compact symmetric pair (G, K), we will write the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of the isotropy group K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $\mathcal{B}_{\mathfrak{g}}$ of $\mathfrak{g}$, which is negative definite. Moreover, $\mathfrak{p}$ is endowed with the metric $\langle \cdot, \cdot \rangle = -\mathcal{B}_{\mathfrak{g}}|_{\mathfrak{p}\times\mathfrak{p}}$. The s-representation of (G, K) can be seen as the adjoint representation $K \to \mathrm{O}(\mathfrak{p})$, $k \mapsto \mathrm{Ad}(k)|_{\mathfrak{p}}$.

We will say that a symmetric pair (G, K) satisfies the *maximality property* if it is effective, K is connected and $\mathrm{Ad}(K)|_{\mathfrak{p}}$ is the maximal connected subgroup of $\mathrm{O}(\mathfrak{p})$ acting on $\mathfrak{p}$ with the same orbits as the s-representation of (G, K).

Let (G, K) be an effective compact symmetric pair, with K connected. Eschenburg and Heintze proved that, if G/K is irreducible and of rank greater than two, then (G, K) satisfies the maximality property [60]; the same holds if G/K is irreducible and of rank two (this follows from [42]; cf. [59, p. 392, Remark 1]). If G/K is reducible, then G/K satisfies the maximality property whenever all irreducible factors of rank equal to one and dimension n are assumed to be spheres represented by the symmetric pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ (see [59, p. 391]). Therefore, for the study of geometric properties of the foliations induced by s-representations, it is not a restriction of generality to assume that the corresponding symmetric pairs satisfy the maximality property.

This property allows us to determine the whole group $\mathrm{Aut}(\mathcal{F}_{G/K})$ of automorphisms of the orbit foliation $\mathcal{F}_{G/K}$ of the isotropy representation of G/K. If (G, K) satisfies the maximality property, then the identity connected component of $\mathrm{Aut}(\mathcal{F}_{G/K})$ is $\mathrm{Ad}(K)|_{\mathfrak{p}}$, but $\mathrm{Aut}(\mathcal{F}_{G/K})$ might have several connected components.

Theorem 7.8. *Let (G, K) be a compact symmetric pair that satisfies the maximality property and with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

Then there is a Lie group isomorphism between the group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ of automorphisms of $\mathfrak{g}$ that restrict to automorphisms of $\mathfrak{k}$ and the group $\mathrm{Aut}(\mathcal{F}_{G/K})$ of orthogonal transformations of $\mathfrak{p}$ that map leaves of $\mathcal{F}_{G/K}$ to leaves of $\mathcal{F}_{G/K}$.

Proof. We will show that the restriction map $\Psi\colon \mathrm{Aut}(\mathfrak{g}, \mathfrak{k}) \to \mathrm{Aut}(\mathcal{F}_{G/K})$, $\varphi \mapsto \varphi|_{\mathfrak{p}}$ yields the desired isomorphism. Let $\varphi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$. Clearly $\varphi|_{\mathfrak{p}} \in \mathrm{O}(\mathfrak{p})$. Since φ preserves $\mathfrak{k}$, if we fix $P \in \mathfrak{p}$, we easily get that $\varphi(\mathrm{Ad}(K)P) = \mathrm{Ad}(K)\varphi(P)$. Hence $\varphi|_{\mathfrak{p}}$ sends the orbit through P to the orbit through $\varphi(P)$, from where it follows that Ψ is a well-defined Lie group homomorphism. In order to show that Ψ is one-to-one, let $\varphi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ with $\varphi|_{\mathfrak{p}} = \mathrm{Id}_{\mathfrak{p}}$ and take arbitrary elements $X \in \mathfrak{k}$ and $P \in \mathfrak{p}$. Then $[X, P] = \varphi[X, P] = [\varphi X, P]$, so $\mathrm{ad}(\varphi X - X)|_{\mathfrak{p}} = 0$. By the effectiveness of (G, K), we have $\varphi X = X$, and hence $\varphi = \mathrm{Id}$.

It remains to prove that Ψ is onto. Let $A \in \mathrm{Aut}(\mathcal{F}_{G/K})$. The maximality property implies that $A\,\mathrm{Ad}(K)|_{\mathfrak{p}}A^{-1} = \mathrm{Ad}(K)|_{\mathfrak{p}}$. Then the effectiveness of (G, K) entails the existence of an automorphism ϕ_A of $\mathfrak{k}$ defined by $\mathrm{ad}(\phi_A(X))|_{\mathfrak{p}} = A\,\mathrm{ad}(X)|_{\mathfrak{p}}A^{-1}$, for all $X \in \mathfrak{k}$.

Now for each $A \in \mathrm{Aut}(\mathcal{F}_{G/K})$ we construct an automorphism $\varphi_A \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ whose restriction to $\mathfrak{p}$ is A. Define φ_A as the linear endomorphism of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ given by $\varphi_A|_{\mathfrak{k}} = \phi_A$ and $\varphi_A|_{\mathfrak{p}} = A$. Clearly, it is a linear isomorphism preserving the Cartan decomposition and with the desired restriction to $\mathfrak{p}$. We just have to see that it respects the Lie bracket.

Let $P_1, P_2 \in \mathfrak{p}$ and $X \in \mathfrak{k}$ be arbitrary elements. Denote by $\mathcal{B}_{\mathfrak{k}}$ and $\mathcal{B}_{\mathfrak{g}}$ the Killing forms of $\mathfrak{k}$ and $\mathfrak{g}$, respectively. Then, using that $\varphi_A|_{\mathfrak{k}} = \phi_A \in \mathrm{Aut}(\mathfrak{k})$, the definition of φ_A and the invariance of the trace operator under conjugation, we get

$$\begin{aligned}\mathcal{B}_{\mathfrak{g}}(\varphi_A X, \varphi_A[P_1, P_2]) &= \mathcal{B}_{\mathfrak{k}}(\varphi_A X, \varphi_A[P_1, P_2]) + \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}(\varphi_A X)\,\mathrm{ad}(\varphi_A[P_1, P_2])) \\ &= \mathcal{B}_{\mathfrak{k}}(X, [P_1, P_2]) + \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}(X)\,\mathrm{ad}([P_1, P_2])) = \mathcal{B}_{\mathfrak{g}}(X, [P_1, P_2]).\end{aligned}$$

Using this property, the definition of φ_A and the fact that $A \in \mathrm{O}(\mathfrak{p})$, now we have:

$$\begin{aligned}\mathcal{B}_{\mathfrak{g}}(X, \varphi_A[P_1, P_2]) &= \mathcal{B}_{\mathfrak{g}}(\varphi_A\varphi_A^{-1}X, \varphi_A[P_1, P_2]) = \mathcal{B}_{\mathfrak{g}}(\varphi_A^{-1}X, [P_1, P_2]) = \mathcal{B}_{\mathfrak{g}}(\mathrm{ad}(\varphi_A^{-1}X)P_1, P_2) \\ &= \mathcal{B}_{\mathfrak{g}}(A^{-1}\,\mathrm{ad}(X)AP_1, P_2) = \mathcal{B}_{\mathfrak{g}}(\mathrm{ad}(X)AP_1, AP_2) = \mathcal{B}_{\mathfrak{g}}(X, [\varphi_A P_1, \varphi_A P_2]).\end{aligned}$$

Since $X \in \mathfrak{k}$ is arbitrary, $\varphi_A[P_1, P_2] \in \mathfrak{k}$ and $\mathcal{B}_{\mathfrak{g}}$ is nondegenerate, we get that $\varphi_A[P_1, P_2] = [\varphi_A P_1, \varphi_A P_2]$, for every $P_1, P_2 \in \mathfrak{p}$. Furthermore, using the previous properties, we obtain:

$$\begin{aligned}\mathcal{B}_{\mathfrak{g}}(\varphi_A[P_1, X], P_2) &= \mathcal{B}_{\mathfrak{g}}([P_1, X], \varphi_A^{-1}P_2) = \mathcal{B}_{\mathfrak{g}}(X, [\varphi_A^{-1}P_2, P_1]) = \mathcal{B}_{\mathfrak{g}}(\varphi_A X, \varphi_A[\varphi_A^{-1}P_2, P_1]) \\ &= \mathcal{B}_{\mathfrak{g}}(\varphi_A X, [P_2, \varphi_A P_1]) = \mathcal{B}_{\mathfrak{g}}([\varphi_A P_1, \varphi_A X], P_2),\end{aligned}$$

from where $\varphi_A[P, X] = [\varphi_A P, \varphi_A X]$ for all $P \in \mathfrak{p}$ and all $X \in \mathfrak{k}$. We conclude that $\varphi_A \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$. Therefore, Ψ is onto and the proof is finished. □

By [95, Chapter VII, Proposition 4.1] the group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ is isomorphic to the isotropy group of the base point of G/K in the whole isometry group of G/K, and also to the group $\mathrm{Aut}(\mathfrak{p})$ of automorphisms of $\mathfrak{p}$, that is, the group of linear isomorphisms A of $\mathfrak{p}$ such that $A[P_1, [P_2, P_3]] = [AP_1, [AP_2, AP_3]]$, for all $P_1, P_2, P_3 \in \mathfrak{p}$.

Let us conclude this subsection with the following observation.

Remark 7.9. A homogeneous polar foliation defined by the s-representation of a compact symmetric pair satisfying the maximality property determines the corresponding orthogonal symmetric pair (up to a permutation of its irreducible factors). Let us give a quick argument for this claim. It is enough to check it for irreducible symmetric pairs. Let (G, K) and (G', K') be compact irreducible symmetric pairs satisfying the maximality property and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ be their Cartan decompositions. If the foliations $\mathcal{F}_{G/K}$ and $\mathcal{F}_{G'/K'}$ are congruent, there is an orthogonal map A between $\mathfrak{p}$ and $\mathfrak{p}'$ that maps leaves of $\mathcal{F}_{G/K}$ to leaves of $\mathcal{F}_{G'/K'}$. Hence $A\,\mathrm{Ad}(K)|_{\mathfrak{p}}A^{-1}$ acts on $\mathfrak{p}'$ with the same orbits as $\mathrm{Ad}(K')|_{\mathfrak{p}'}$. The maximality property implies $A\,\mathrm{Ad}(K)|_{\mathfrak{p}}A^{-1} = \mathrm{Ad}(K')|_{\mathfrak{p}'}$, and hence $\mathfrak{k}$ and $\mathfrak{k}'$ must be isomorphic. Moreover, the ranks of G/K and G'/K' must be equal (so that the codimensions of both foliations agree), as well as the dimensions of both symmetric spaces (so that $\dim \mathfrak{p} = \dim \mathfrak{p}'$). But one can check (by direct inspection, see [76, p. 516–519]) that these invariants determine the compact irreducible orthogonal symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

7.4.2 The group of automorphisms of an FKM-foliation

Our goal now is to determine the group of automorphisms of the isoparametric foliations on spheres constructed by Ferus, Karcher and Münzner in [63]. We will do this for almost all such examples, with only some exceptions mentioned below. However, these exceptions can be reduced to only two, namely the inhomogeneous FKM-examples whose multiplicities are $(m_1, m_2) = (8, 7)$.

Let us begin by reminding the reader about the construction of the FKM-foliations. For details missing here we refer to the original paper [63]. For more information on Clifford algebras and their representations, see [90, Chapter I].

Let $\mathrm{Cl}(\mathcal{E}) = \mathrm{Cl}^*_{m+1}$ be the Clifford algebra associated with $\mathcal{E} = \mathbb{R}^{m+1}$ endowed with the standard positive definite quadratic form. Thus $\mathrm{Cl}(\mathcal{E})$ can be regarded as the algebra generated by an orthonormal basis $\{E_0, \dots, E_m\}$ of $\mathcal{E}$ (and the unit 1) subject to the relations $E_iE_j + E_jE_i = 2\delta_{ij}1$ for every $i, j \in \{0, \dots, m\}$, where δ_{ij} is the Kronecker delta. Set $V = \mathbb{R}^{2n+2}$ and let $\chi\colon \mathrm{Cl}(\mathcal{E}) \to \mathrm{End}(V)$ be a representation of the Clifford algebra $\mathrm{Cl}(\mathcal{E})$. Endow V with a positive definite $\mathrm{Pin}(\mathcal{E})$-invariant inner product $\langle\cdot,\cdot\rangle$. Let us put $P_i = \chi(E_i)$ for $i = 0, \dots, m$. Then $(P_0, \dots, P_m)$ is what in [63] is called a *(symmetric) Clifford system*, i.e. an $(m+1)$-tuple of symmetric matrices on V which satisfy $P_iP_j + P_jP_i = 2\delta_{ij}\,\mathrm{Id}$ for all $i, j \in \{0, \dots, m\}$. We also define $\mathcal{P} = \mathrm{span}\{P_0, \dots, P_m\}$ and endow this vector space with the inner product induced by χ, which turns out to be given by $\langle P, P'\rangle = (1/\dim V)\,\mathrm{tr}(PP')$, for $P, P' \in \mathcal{P}$.

Assume that $m_2 = n - m > 0$. Then the FKM-foliation $\mathcal{F}_{\mathcal{P}}$ associated with the Clifford system $(P_0, \dots, P_m)$ is defined by the level sets of $F|_{S(V)}$, where $S(V)$ is the unit sphere of V and $F\colon V \to \mathbb{R}$ is the Cartan-Münzner polynomial:

$$F(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^{m}\langle P_i x, x\rangle^2.$$

The corresponding isoparametric hypersurfaces have $g = 4$ principal curvatures with multiplicities $(m_1, m_2) = (m, n - m)$. This construction does not depend on the particular matrices $P_0, \dots, P_m$, but only on the unit sphere $S(\mathcal{P})$ of $\mathcal{P}$. $S(\mathcal{P})$ is called the Clifford sphere of the foliation. Moreover, two FKM-foliations are congruent if and only if their Clifford spheres are conjugate under an orthogonal transformation of V.

For each integer $m \geq 1$, we define $\delta(m)$ as the smallest natural number such that there exists a Clifford system $(P_0, \dots, P_m)$ on $V = \mathbb{R}^{2\delta(m)}$. Equivalently, $2\delta(m)$ is the dimension of any irreducible Clifford Cl^*_{m+1}-module. In addition, if $(P_0, \dots, P_m)$ is a Clifford system on $V = \mathbb{R}^{2n+2}$, then there is a natural number k such that $n + 1 = k\delta(m)$. Conversely, for a fixed $m \geq 1$, if k is a natural number such that $m_2 = n - m \geq 1$, with $n = k\delta(m) - 1$, then there exists a Clifford system $(P_0, \dots, P_m)$ on $V = \mathbb{R}^{2n+2} = \mathbb{R}^{2k\delta(m)}$ that gives rise to an FKM-foliation on S^{2n+1}.

The classification result of FKM-foliations given in [63] ensures that for $m \not\equiv 0 \pmod 4$, there exists only one isoparametric FKM-foliation for each natural $k \geq (m+2)/\delta(m)$, up to congruence. However, if $m \equiv 0 \pmod 4$, for each natural $k \geq (m+2)/\delta(m)$ there are exactly $[k/2] + 1$ FKM-foliations up to congruence. Here $[\cdot]$ denotes the integer part of a real number. This different behaviour, depending on whether m is multiple of 4 or not, is due to the fact that, if $m \equiv 0 \pmod 4$, there exist exactly two irreducible representations $\mathfrak{d}_+$, $\mathfrak{d}_-$ of the Clifford algebra Cl^*_{m+1} up to equivalence, whereas there is only one, say $\mathfrak{d}$, if $m \not\equiv 0 \pmod 4$. Thus, every representation of Cl^*_{m+1} on $V = \mathbb{R}^{2n+2}$ has the form $\oplus_{i=1}^{k}\mathfrak{d}$ if $m \equiv 0 \pmod 4$, or the form $(\oplus_{i=1}^{k_+}\mathfrak{d}_+) \oplus (\oplus_{i=1}^{k_-}\mathfrak{d}_-)$ if $m \not\equiv 0 \pmod 4$, for certain integers k_+, k_- such that $k = k_+ + k_-$. We will come back to this point later.

In Table 7.1 we show the pairs of multiplicities $(m_1, m_2) = (m, n - m)$ of the principal curvatures of the hypersurfaces of FKM type, for low values of m and k. When a pair (m_1, m_2) is not underlined, we will understand that there is only one FKM-foliation with those multiplicities, up to congruence; the underlinings $\underline{(m_1, m_2)}$, $\underline{\underline{(m_1, m_2)}}$... point at the existence of two, three... FKM-foliations with multiplicities (m_1, m_2), respectively.

m	1	2	3	4	5	6	7	8	9	10	...
$\delta(m)$	1	2	4	4	8	8	8	8	16	32	...
$k=1$	–	–	–	–	$(5,2)$	$(6,1)$	–	–	$(9,6)$	$(10,21)$	...
$k=2$	–	$(2,1)$	$(3,4)$	$\underline{(4,3)}$	$(5,10)$	$(6,9)$	$(7,8)$	$\underline{(8,7)}$	$(9,22)$	$(10,53)$	...
$k=3$	$(1,1)$	$(2,3)$	$(3,8)$	$\underline{(4,7)}$	$(5,18)$	$(6,17)$	$(7,16)$	$\underline{(8,15)}$	$(9,38)$	$(10,85)$	...
$k=4$	$(1,2)$	$(2,5)$	$(3,12)$	$\underline{\underline{(4,11)}}$	$(5,26)$	$(6,25)$	$(7,24)$	$\underline{\underline{(8,23)}}$	$(9,54)$	$(10,117)$	...
$k=5$	$(1,3)$	$(2,7)$	$(3,16)$	$\underline{\underline{(4,15)}}$	$(5,34)$	$(6,33)$	$(7,32)$	$\underline{\underline{(8,31)}}$	$(9,70)$	$(10,149)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 7.1: Small multiplicities (m_1, m_2) of the FKM-hypersurfaces

For a fixed m, the examples in the corresponding column of Table 7.1 are mutually noncongruent. Nevertheless, it can happen that examples in two different columns (i.e. with different m) are congruent to each other. In fact, this is the case for several pairs of examples: the families with pairs of multiplicities $(2, 1)$, $(6, 1)$, $(5, 2)$ and one of the two families with pair $(4, 3)$ are congruent, respectively, to those families with pairs $(1, 2)$, $(1, 6)$, $(2, 5)$ e $(3, 4)$. These exhaust all coincidences up to congruence. Moreover, the only FKM-foliations that are homogeneous correspond to those families with multiplicities $(1, k-2)$ for $k \geq 3$, $(2, 2k-3)$ for $k \geq 2$, one of the examples with multiplicities $(4, 4k-5)$ for each $k \geq 2$, and the foliations with multiplicities $(5, 2)$, $(6, 1)$ and $(9, 6)$.

We address now the problem of calculating the group $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ of automorphisms of an FKM-foliation $\mathcal{F}_{\mathcal{P}}$ arising from a Clifford system with span $\mathcal{P}$.

The two focal submanifolds of an FKM-foliation are never congruent (they have codimensions $m_1+1 \neq m_2+1$ except for the pair $(1, 1)$, but for this see [108, p. 59]). Since a focal submanifold determines the whole isoparametric foliation, it follows that every automorphism of $\mathcal{F}_{\mathcal{P}}$ maps each leaf onto itself. Hence $A \in \mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ if and only if $F(Ax) = F(x)$ for all $x \in V$, where F is the Cartan-Münzner polynomial of $\mathcal{F}_{\mathcal{P}}$. This means that the Clifford systems $(P_0, \dots, P_m)$ and $(A^{-1}P_0A, \dots, A^{-1}P_mA)$ define the same foliation. Therefore $A \in \mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ if and only if $\mathcal{F}_{\mathcal{P}} = \mathcal{F}_{A^{-1}\mathcal{P}A}$.

Let $\mathrm{SO}(\mathcal{P}) \cup \mathrm{O}^-(\mathcal{P})$ and $\mathrm{Spin}(\mathcal{P}) \cup \mathrm{Pin}^-(\mathcal{P})$ be the decompositions of the orthogonal group $\mathrm{O}(\mathcal{P})$ and of the pin group $\mathrm{Pin}(\mathcal{P})$ in connected components, respectively. We define the following subsets of the orthogonal group of V:

$$\mathrm{U}^+(\mathcal{P}) = \{U \in \mathrm{O}(V) : PU = UP \text{ for all } P \in \mathcal{P}\},$$
$$\mathrm{U}^-(\mathcal{P}) = \{U \in \mathrm{O}(V) : PU = -UP \text{ for all } P \in \mathcal{P}\},$$

and $\mathrm{U}^{\pm}(\mathcal{P}) = \mathrm{U}^{+}(\mathcal{P}) \cup \mathrm{U}^{-}(\mathcal{P})$. The set $\mathrm{U}^{-}(\mathcal{P})$ might be empty. Elements in $\mathrm{U}^{+}(\mathcal{P})$ commute with those in $\mathrm{Pin}(\mathcal{P})$, while elements in $\mathrm{U}^{-}(\mathcal{P})$ commute with the ones in $\mathrm{Spin}(\mathcal{P})$ and anticommute those in $\mathrm{Pin}^{-}(\mathcal{P})$. Moreover, $\mathrm{Pin}(\mathcal{P})$ and $\mathrm{U}^{\pm}(\mathcal{P})$ are subgroups of $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$.

An important remark for our work is that if $m_1 \leq m_2$, then the FKM-foliation determines the Clifford sphere $S(\mathcal{P})$, or equivalently, the space $\mathcal{P}$ (see [63, §4.6]). This observation allows us to show the following structure result for $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$.

Theorem 7.10. *Let $\mathcal{F}_{\mathcal{P}}$ be an FKM-foliation satisfying $m_1 \leq m_2$. We have:*

(i) *If m is odd, then* $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}}) \cong \mathrm{Pin}(\mathcal{P}) \cdot \mathrm{U}^{+}(\mathcal{P})$.

(ii) *If m is even, then* $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}}) \cong \mathrm{Spin}(\mathcal{P}) \cdot \mathrm{U}^{\pm}(\mathcal{P})$.

In both cases $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ *is isomorphic to a direct product modulo the center* $Z(\mathrm{Spin}(\mathcal{P}))$ *of* $\mathrm{Spin}(\mathcal{P})$, *which is* $\{\pm \mathrm{Id}, \pm P_0 \cdots P_m\} \cong \mathbb{Z}_4$ *if* $m \equiv 1 \pmod 4$; $\{\pm \mathrm{Id}, \pm P_0 \cdots P_m\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *if* $m \equiv 3 \pmod 4$; *and* $\{\pm \mathrm{Id}\} \cong \mathbb{Z}_2$ *if m is even.*

Proof. We saw that $A \in \mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ if and only if $\mathcal{F}_{\mathcal{P}} = \mathcal{F}_{A^{-1}\mathcal{P}A}$. If $m_1 \leq m_2$, the previous condition is equivalent to $\mathcal{P} = A\mathcal{P}A^{-1}$.

Assume that m is odd. Then the adjoint representation $\mathrm{Ad}\colon \mathrm{Pin}(\mathcal{P}) \to \mathrm{O}(\mathcal{P})$ is onto and $\mathrm{Ad}(\mathrm{Pin}^{-}(\mathcal{P})) = \mathrm{O}^{-}(\mathcal{P})$. Consider the group homomorphism $\Psi\colon \mathrm{Pin}(\mathcal{P}) \times \mathrm{U}^{+}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$, $(Q, U) \mapsto QU$. Its kernel is isomorphic to $\mathrm{Pin}(\mathcal{P}) \cap \mathrm{U}^{+}(\mathcal{P})$, which is exactly $Z(\mathrm{Spin}(\mathcal{P}))$ (note that $\mathrm{Pin}^{-}(\mathcal{P}) \cap \mathrm{U}^{+}(\mathcal{P}) = \emptyset$). We show that Ψ is onto. Let $A \in \mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$. Then $\varphi_A\colon \mathcal{P} \to \mathcal{P}$, $P \mapsto APA^{-1}$, is an orthogonal transformation of $\mathcal{P}$. Since the adjoint representation is onto, we can find a $Q \in \mathrm{Pin}(\mathcal{P})$ such that $\mathrm{Ad}(Q) = \varphi_A$, that is, $QP_iQ^{-1} = AP_iA^{-1}$ for all $i = 0, \ldots, m$. Hence $U = Q^{-1}A \in \mathrm{U}^{+}(\mathcal{P})$ and then $\Psi(Q, U) = A$. Thus we have proved (i).

Let m be even. Consider the group homomorphism $\Psi\colon \mathrm{Spin}(\mathcal{P}) \times \mathrm{U}^{\pm}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$, $(Q, U) \mapsto QU$. Its kernel is isomorphic to $\mathrm{Spin}(\mathcal{P}) \cap \mathrm{U}^{\pm}(\mathcal{P})$. Since $m+1$ is odd, then $-\mathrm{Id} \in \mathrm{O}^{-}(\mathcal{P})$, so $\mathrm{Spin}(\mathcal{P}) \cap \mathrm{U}^{-}(\mathcal{P}) = \emptyset$ and $\mathrm{Spin}(\mathcal{P}) \cap \mathrm{U}^{\pm}(\mathcal{P}) = Z(\mathrm{Spin}(\mathcal{P}))$. Now if $A \in \mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$, then $\varphi_A\colon \mathcal{P} \to \mathcal{P}$, $P \mapsto APA^{-1}$, is an orthogonal transformation of $\mathcal{P}$. On the one hand, if $\varphi_A \in \mathrm{SO}(\mathcal{P})$, there exists a $Q \in \mathrm{Spin}(\mathcal{P})$ such that $\mathrm{Ad}(Q) = \varphi_A$ and $\Psi(Q, U) = A$, where $U = Q^{-1}A \in \mathrm{U}^{+}(\mathcal{P})$. On the other hand, if $\varphi_A \in \mathrm{O}^{-}(\mathcal{P})$, since $m+1$ is odd, then $-\varphi_A \in \mathrm{SO}(\mathcal{P})$, so there is a $Q \in \mathrm{Spin}(\mathcal{P})$ so that $\mathrm{Ad}(Q) = -\varphi_A$, and hence $\Psi(Q, U) = A$, where $U = Q^{-1}A \in \mathrm{U}^{-}(\mathcal{P})$. This proves (ii).

Finally, the claims involving $Z(\mathrm{Spin}(\mathcal{P}))$ are well known (see [127, Theorem VII.7.5]). □

Up to congruence, there are only 8 FKM-foliations for which $m_1 > m_2$. These are the ones with multiplicities (m_1, m_2) equal to $(2,1)$, $(4,3)$ (two noncongruent examples), $(5,2)$, $(6,1)$, $(8,7)$ (two noncongruent examples) and $(9,6)$. However, on the one hand, the FKM-foliations with pairs $(2,1)$, $(6,1)$ and $(5,2)$ are congruent to those FKM-foliations with pairs $(1,2)$, $(1,6)$ and $(2,5)$, respectively; and one of the examples with pair $(4,3)$ is congruent to the FKM-foliation with pair $(3,4)$. On the other hand, the other example with multiplicities $(4,3)$ is homogeneous, as well as the FKM-foliation with pair $(9,6)$. Therefore, in our investigation of FKM-foliations we are putting aside only the two inhomogeneous FKM-foliations with pair $(m_1, m_2) = (8, 7)$.

Our purpose now is to calculate $\mathrm{U}^+(\mathcal{P})$. To do this, we need first to recall some facts about representations of the Clifford algebras Cl^*_{m+1}. See [90, Chapter I] for details.

Each Clifford algebra Cl^*_{m+1} is a matrix algebra over some associative division algebra: $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We state the classification of low-dimensional Clifford algebras in Table 7.2, where $\mathbb{K}(r)$ denotes the algebra of $(r \times r)$-matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The higher-dimensional Clifford algebras Cl^*_{m+1} can be obtained recursively by means of $\mathrm{Cl}^*_{m+8} = \mathrm{Cl}^*_m \otimes \mathbb{R}(16)$.

m	0	1	2	3	4	5	6	7	...
Cl^*_{m+1}	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	...

Table 7.2: Classification of the Clifford algebras Cl^*_{m+1}

The classification of Cl^*_{m+1}-modules is obtained directly from the classification of the corresponding Clifford algebras. The algebra $\mathbb{K}(r)$ has only one equivalence class $\mathfrak{d}$ of irreducible representations (on $\mathbb{K}^r$), whereas the algebra $\mathbb{K}(r) \oplus \mathbb{K}(r)$ has exactly two equivalence classes of irreducible representations $\mathfrak{d}_+$, $\mathfrak{d}_-$ (both on $\mathbb{K}^r$), given by the projections onto each one of the factors. If $\mathrm{Cl}^*_{m+1} = \mathbb{K}(r)$, we will let $\chi\colon \mathrm{Cl}^*_{m+1} \to \mathrm{End}(\mathfrak{d})$ be the corresponding irreducible representation, and if $\mathrm{Cl}^*_{m+1} = \mathbb{K}(r) \oplus \mathbb{K}(r)$, the irreducible representations will be denoted by $\chi_+\colon \mathrm{Cl}^*_{m+1} \to \mathrm{End}(\mathfrak{d}_+)$ and $\chi_-\colon \mathrm{Cl}^*_{m+1} \to \mathrm{End}(\mathfrak{d}_-)$. Therefore, if $m \not\equiv 0 \,(\mathrm{mod}\, 4)$, each Cl^*_{m+1}-module V is isomorphic to $\oplus_{i=1}^k \mathfrak{d}$ for certain positive integer k, while if $m \equiv 0 \,(\mathrm{mod}\, 4)$ each Cl^*_{m+1}-module V is isomorphic to $(\oplus_{i=1}^{k_+} \mathfrak{d}_+) \oplus (\oplus_{i=1}^{k_-} \mathfrak{d}_-)$ for nonnegative integers k_-, k_+; we will write $k = k_+ + k_- > 0$. The corresponding Clifford algebra representations will be denoted by χ_k and χ_{k_+,k_-}, respectively. Furthermore, in the case $m \equiv 0 \,(\mathrm{mod}\, 4)$ we can and will assume that $\chi_- = \chi_+ \circ \alpha$, where α is the canonical involution of $\mathrm{Cl}^*_{m+1} = \mathrm{Cl}(\mathcal{E})$ that extends the map $-\mathrm{Id}$ on $\mathcal{E}$.

Theorem 7.11. *Let $\mathcal{P}$ be a symmetric Clifford system. The group $\mathrm{U}^+(\mathcal{P})$ is isomorphic to*

$$\begin{aligned} &\mathrm{O}(k), && \text{if } m \equiv 1,7 \,(\mathrm{mod}\, 8), && && \\ &\mathrm{U}(k), && \text{if } m \equiv 2,6 \,(\mathrm{mod}\, 8), && \mathrm{O}(k_+) \times \mathrm{O}(k_-), && \text{if } m \equiv 0 \,(\mathrm{mod}\, 8), \\ &\mathrm{Sp}(k), && \text{if } m \equiv 3,5 \,(\mathrm{mod}\, 8), && \mathrm{Sp}(k_+) \times \mathrm{Sp}(k_-), && \text{if } m \equiv 4 \,(\mathrm{mod}\, 8). \end{aligned}$$

Proof. First, assume that $m \not\equiv 0 \,(\mathrm{mod}\, 4)$. The real endomorphisms U of $V = \oplus_{i=1}^k \mathfrak{d}$ can be identified with matrices (U_{ij}) with $U_{ij} \in \mathrm{End}(\mathfrak{d})$ for $i, j = 1, \dots, k$. The endomorphisms $U = (U_{ij})$ that commute with the elements in $\mathrm{Cl}(\mathcal{P}) = \chi_k(\mathrm{Cl}^*_{m+1})$ are exactly those whose U_{ij} commute with the elements in $\chi(\mathrm{Cl}^*_{m+1}) = \mathbb{K}(r)$. Equivalently, the U_{ij} belong to the commuting subalgebra of $\mathbb{K}(r)$, which is isomorphic to $\mathbb{K}$. Hence the algebra of endomorphisms U that commute with $\mathrm{Cl}(\mathcal{P})$ is isomorphic to $\mathbb{K}(k)$. Now $\mathrm{U}^+(\mathcal{P})$ is the set of those endomorphisms U commuting with $\mathrm{Cl}(\mathcal{P})$ that are orthogonal transformations of V.

Since $\mathbb{R}(k) \cap \mathrm{O}(k) = \mathrm{O}(k)$, $\mathbb{C}(k) \cap \mathrm{O}(2k) = \mathrm{U}(k)$ and $\mathbb{H}(k) \cap \mathrm{O}(4k) = \mathrm{Sp}(k)$, we get that $\mathrm{U}^+(\mathcal{P})$ is isomorphic to $\mathrm{O}(k)$ if $m \equiv 1, 7 \,(\mathrm{mod}\, 8)$, $\mathrm{U}(k)$ if $m \equiv 2, 6 \,(\mathrm{mod}\, 8)$, or $\mathrm{Sp}(k)$ if $m \equiv 3, 5 \,(\mathrm{mod}\, 8)$.

Let $m \equiv 0 \,(\mathrm{mod}\, 4)$ and put $V = (\oplus_{i=1}^{k_+} \mathfrak{d}_+) \oplus (\oplus_{i=1}^{k_-} \mathfrak{d}_-)$. Arguing as above, one can show that the algebra of endomorphisms $U = (U_{ij})$ that commute with the elements of $\chi_{k_+,k_-}(\mathrm{Cl}^*_{m+1})$ is isomorphic to $\mathbb{K}(k_+) \oplus \mathbb{K}(k_-)$; note that if, for example, $i \in \{1, \dots, k_+\}$ and $j \in \{k_+ + 1, \dots, k_+ + k_-\}$, then $U_{ij}\chi_-(f) = \chi_+(f)U_{ij}$ for all $f \in \mathrm{Cl}^*_{m+1}$ if and only if $U_{ij} = 0$, since χ_+ and χ_- are inequivalent representations. Restricting to orthogonal transformations of V, one readily finishes the proof. □

Let $\{e_1, \dots, e_k\}$ be the canonical $\mathbb{K}$-basis of $\mathbb{K}^k$, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let us regard $\mathfrak{d}$, $\mathfrak{d}^\pm$ and $\mathbb{K}^k$ as right vector spaces, in order to deal also with the quaternionic case. Assume, for example, that m is odd and let $\widetilde{\mathrm{U}}^+(\mathcal{P})$ be the corresponding classical group in Theorem 7.11. Therefore, Theorems 7.10 and 7.11 establish the following isomorphism of groups

$$\mathrm{Pin}(\mathcal{P}) \cdot \mathrm{U}^+(\mathcal{P}) \to \mathrm{Pin}(m+1) \cdot \widetilde{\mathrm{U}}^+(\mathcal{P}), \quad QU \mapsto \widetilde{Q} \otimes \widetilde{U},$$

where $\widetilde{Q}$ and $\widetilde{U}$ are defined as follows: for each $Q \in \mathrm{Pin}(\mathcal{P})$, let $f \in \mathrm{Pin}(\mathcal{E})$ so that $Q = \chi_k(f)$ and define $\widetilde{Q} = \chi(f)$; given $U \in \mathrm{U}^+(\mathcal{P})$, put $U = (U_{ij})$ with $U_{ij} \in \mathrm{End}_{\mathbb{R}}(\mathfrak{d})$ and define $\widetilde{U} = (\widetilde{u}_{ij})$, where $\widetilde{u}_{ij} \in \mathbb{K}$ and $U_{ij}v = v\widetilde{u}_{ij}$ for all $v \in \mathfrak{d}$ and each $i, j = 1, \dots, k$. Moreover, it is straightforward to check that the map $\bigoplus_{i=1}^k \mathfrak{d} \to \mathfrak{d} \otimes_{\mathbb{K}} \mathbb{K}^k$, $(v_1, \dots, v_k) \mapsto \sum_{i=1}^k v_i \otimes e_i$, gives an equivalence between the representation of $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}}) \cong \mathrm{Pin}(\mathcal{P}) \cdot \mathrm{U}^+(\mathcal{P})$ on $V = \bigoplus_{i=1}^k \mathfrak{d}$ and the tensor product representation of $\mathrm{Pin}(m+1) \cdot \widetilde{\mathrm{U}}^+(\mathcal{P})$ on $\mathfrak{d} \otimes_{\mathbb{K}} \mathbb{K}^k$. If m is even, the previous argument applies (with small changes) to the subgroup $\mathrm{Spin}(\mathcal{P}) \cdot \mathrm{U}^+(\mathcal{P})$ of $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$. Thus, we obtain:

Theorem 7.12. *Let $\mathcal{F}_{\mathcal{P}}$ be an FKM-foliation satisfying $m_1 \le m_2$.*

If m is odd, the representation of the group $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ on V is equivalent to the action of

$$\begin{aligned} &\mathrm{Pin}(m+1) \cdot \mathrm{O}(k), && \textit{if } m \equiv 1, 7 \,(\mathrm{mod}\, 8), \quad \textit{or} \\ &\mathrm{Pin}(m+1) \cdot \mathrm{Sp}(k), && \textit{if } m \equiv 3, 5 \,(\mathrm{mod}\, 8), \end{aligned}$$

on $\mathfrak{d} \otimes_{\mathbb{K}} \mathbb{K}^k$, given by the pin representation $\mathfrak{d}$ on the left factor and by the standard representation of $\mathrm{O}(k)$ or $\mathrm{Sp}(k)$ on the right factor $\mathbb{R}^k$ or $\mathbb{H}^k$, respectively.

If $m \equiv 2 \,(\mathrm{mod}\, 4)$, the representation of the subgroup $\mathrm{Spin}(\mathcal{P}) \cdot \mathrm{U}^+(\mathcal{P})$ of $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ on V is equivalent to the action of $\mathrm{Spin}(m+1) \cdot \mathrm{U}(k)$ on $\mathfrak{d} \otimes_{\mathbb{C}} \mathbb{C}^k$, given by the spin representation $\mathfrak{d}$ on the left factor and by the standard representation of $\mathrm{U}(k)$ on the right factor $\mathbb{C}^k$.

In case $m \equiv 0 \,(\mathrm{mod}\, 4)$, the representation of the subgroup $\mathrm{Spin}(\mathcal{P}) \cdot \mathrm{U}^+(\mathcal{P})$ of $\mathrm{Aut}(\mathcal{F}_{\mathcal{P}})$ on V is equivalent to the action of

$$\begin{aligned} &\mathrm{Spin}(m+1) \cdot (\mathrm{O}(k_+) \times \mathrm{O}(k_-)), && \textit{if } m \equiv 0 \,(\mathrm{mod}\, 8), \quad \textit{or} \\ &\mathrm{Spin}(m+1) \cdot (\mathrm{Sp}(k_+) \times \mathrm{Sp}(k_-)), && \textit{if } m \equiv 4 \,(\mathrm{mod}\, 8), \end{aligned}$$

on $(\mathfrak{d}_+ \otimes_{\mathbb{K}} \mathbb{K}^{k_+}) \oplus (\mathfrak{d}_- \otimes_{\mathbb{K}} \mathbb{K}^{k_-})$, given by the spin representations $\mathfrak{d}_+$, $\mathfrak{d}_-$ on the left factors and by the standard representations of $\mathrm{O}(k_\pm)$ or $\mathrm{Sp}(k_\pm)$ on the right factors $\mathbb{R}^{k_\pm}$ or $\mathbb{H}^{k_\pm}$.

If m is even, the description of the group of automorphisms of $\mathcal{F}_{\mathcal{P}}$ depends on $\mathrm{U}^-(\mathcal{P})$. We will only need a description of this set for the case $m \equiv 0 \,(\mathrm{mod}\, 4)$.

Proposition 7.13. *Let $\mathcal{P}$ be a symmetric Clifford system with $m \equiv 0 \,(\mathrm{mod}\, 4)$. We have that $\mathrm{U}^-(\mathcal{P}) = \emptyset$ if $k_+ \neq k_-$, or $\mathrm{U}^-(\mathcal{P}) = \tau U^+(\mathcal{P})$ if $k_+ = k_-$, where τ is the orthogonal transformation of $V = (\oplus_{i=1}^{k_+} \mathfrak{d}_+) \oplus (\oplus_{i=1}^{k_-} \mathfrak{d}_-)$ defined by*

$$\tau(v_1, \dots, v_{k_+}, v_{k_++1}, \dots, v_k) = (v_{k_++1}, \dots, v_k, v_1, \dots, v_{k_+}),$$

where $v_i \in \mathfrak{d}_+$ for $i = 1, \dots, k_+$ and $v_i \in \mathfrak{d}_-$ for $i = k_+ + 1, \dots, k$.

Proof. First note that if σ is an element in $\mathrm{U}^-(\mathcal{P})$, then $\mathrm{U}^-(\mathcal{P}) = \sigma \mathrm{U}^+(\mathcal{P})$. With the notation as above, let $\sigma = (\sigma_{ij})$ anticommute with the endomorphisms in $\chi_{k_+,k_-}(\mathcal{E})$. Equivalently, for all $f \in \mathcal{E}$ we have that $\sigma_{ij}\chi_+(f) = -\chi_+(f)\sigma_{ij}$ if $i, j \in \{1, \dots, k_+\}$, $\sigma_{ij}\chi_-(f) = -\chi_-(f)\sigma_{ij}$ if $i, j \in \{k_++1, \dots, k\}$, $\sigma_{ij}\chi_+(f) = -\chi_-(f)\sigma_{ij}$ if $i \in \{k_++1, \dots, k\}$ and $j \in \{1, \dots, k_+\}$, and $\sigma_{ij}\chi_-(f) = -\chi_+(f)\sigma_{ij}$ if $j \in \{k_+ + 1, \dots, k\}$ and $i \in \{1, \dots, k_+\}$. Since $\chi_- = \chi_+ \circ \alpha$ and χ_+, χ_- are not equivalent, these conditions imply that $\sigma_{ij} = 0$ if $i, j \in \{1, \dots, k_+\}$ or $i, j \in \{k_+ + 1, \dots, k\}$.

If $k_+ \neq k_-$ then σ is not invertible, so $\mathrm{U}^-(\mathcal{P}) = \emptyset$. If $k_+ = k_-$, the orthogonal transformation $\tau = (\tau_{ij})$ given above satisfies $\tau_{ij} = \mathrm{Id}$ if $i = k_+ + j$ or $j = k_+ + i$, and $\tau_{ij} = 0$ otherwise. Then τ anticommutes with the elements of $\chi_{k_+,k_-}(\mathcal{E})$. ☐

7.5 Singular Riemannian foliations on complex projective spaces

In this section we present some general theory for the study of singular Riemannian foliations with closed leaves on $\mathbb{C}P^n$. In §7.5.1 we obtain a criterion to determine all complex structures preserving a given foliation. The congruence of foliations on $\mathbb{C}P^n$ projected using different complex structures is analyzed in §7.5.2.

7.5.1 Complex structures preserving foliations

Let $V = \mathbb{R}^{2n+2}$ and let $\mathcal{F}$ be a foliation on $S(V) = S^{2n+1}$. We will say that a complex structure J in V *preserves the foliation* $\mathcal{F}$ if $\mathcal{F}$ is the lift of some foliation of the complex projective space $\mathbb{C}P^n$ under the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ determined by J; or equivalently, if the leaves of $\mathcal{F}$ are foliated by the Hopf circles determined by J. Since Hopf fibrations are Riemannian submersions, each foliation on $\mathbb{C}P^n$ is obtained by projecting some foliation $\mathcal{F}$ on S^{2n+1} by some Hopf map whose J preserves $\mathcal{F}$. Therefore, the study of foliations on complex projective spaces is reduced to the study of the complex structures that preserve foliations on odd-dimensional spheres.

It is equivalent to give a foliation of a sphere $S(V)$ and to give a foliation of the Euclidean space V whose leaves are contained in concentric spheres with center at the origin: simply extend the given foliation on $S(V)$ by homotheties to V, or inversely, restrict the foliation on V to $S(V)$. Sometimes along this chapter we will implicitly take this fact into account.

Fix a foliation $\mathcal{F}$ of the sphere $S(V) \subset V$. Consider an effective representation $\rho\colon K \to \mathrm{O}(V)$ of a Lie group K such that $\rho(K)$ is the maximal connected group of orthogonal transformations of V that send each leaf of $\mathcal{F}$ onto itself. Let $\rho_*\colon \mathfrak{k} \to \mathfrak{so}(V)$ be the Lie algebra homomorphism defined by ρ.

Proposition 7.14. *With $\mathcal{F}$, K and ρ as above, we have:*

(i) *A complex structure J on V preserves $\mathcal{F}$ if and only if $J = \rho_*(X)$ for some $X \in \mathfrak{k}$.*

(ii) *Assume that K is compact and fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. If $H \in \mathfrak{t}$ and $k \in K$, then $\rho_*(H)$ is a complex structure on V if and only if $\rho_*(\mathrm{Ad}(k)H)$ is a complex structure on V. Moreover, a complex structure J on V preserves $\mathcal{F}$ if and only if $J = \rho_*(\mathrm{Ad}(k)H)$ for some $k \in K$ and $H \in \mathfrak{t}$.*

(iii) *Let $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_r$ be a product foliation on $V = \bigoplus_{i=1}^r V_i$, where each $\mathcal{F}_i$ is the extension of a foliation on $S(V_i)$ to V_i. Then $K = \prod_{i=1}^r K_i$ for certain subgroups K_i of K, where $\rho(K_i)$ is the maximal connected group of orthogonal transformations of V that act trivially on the orthogonal complement of V_i in V and map the leaves of $\mathcal{F}_i$ onto themselves. If $X = \sum_{i=1}^r X_i \in \mathfrak{k} = \bigoplus_{i=1}^r \mathfrak{k}_i$, then $\rho_*(X)$ is a complex structure on V if and only if $\rho_*(X_i)|_{V_i}$ is a complex structure on V_i, for every i.*

Proof. If $J = \rho_*(X)$ is a complex structure on V, its Hopf circles are integral curves of the Hopf vector field $J\xi$, where $\xi_v = v$ for $v \in S(V)$, and each Hopf circle is contained in one leaf of $\mathcal{F}$, since for every $v \in V$, $Jv = \rho_*(X)v$ is tangent to the leaf through v.

Conversely, assume that J is a complex structure on V that preserves $\mathcal{F}$. Then $\mathcal{T}^1 = \{\cos(t)\,\mathrm{Id} + \sin(t)J : t \in \mathbb{R}\}$ is a 1-dimensional group which preserves $\mathcal{F}$. Let K' be the subgroup of $\mathrm{O}(V)$ generated by $\rho(K)$ and $\mathcal{T}^1$, which is connected and leaves every leaf of $\mathcal{F}$ invariant. By the maximality of $\rho(K)$, we have that $K' \subset \rho(K)$ and then $\mathcal{T}^1$ is a subgroup of $\rho(K)$. If we differentiate, we get that $J \in \rho_*(\mathfrak{k})$, which shows (i).

Every transformation of V of the form $\rho_*(X)$, with $X \in \mathfrak{k}$, is a complex structure if and only if $\rho_*(X)^2 = -\mathrm{Id}$, since $\rho_*(X) \in \mathfrak{so}(V)$ is skew-symmetric. Then, with the notation of (ii), we have that $\rho_*(\mathrm{Ad}(k)H) = \mathrm{Ad}(\rho(k))\rho_*(H) = \rho(k)\rho_*(H)\rho(k)^{-1}$ and, hence, $\rho_*(\mathrm{Ad}(k)H)^2 = \rho(k)\rho_*(H)^2\rho(k)^{-1}$. Since K is a connected compact Lie group, then $\mathfrak{k} = \bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{t}$, from where we get (ii).

Finally, (iii) follows from the effectiveness of ρ and from the facts that the leaves of $\mathcal{F}$ are products of leaves of the foliations $\mathcal{F}_i$ and the V_i are invariant subspaces for ρ. □

From now on, $\mathcal{F}$ will be a foliation with closed leaves on $S(V)$. Then K is compact. We also fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Let $(\cdot)^{\mathbb{C}}$ denote complexification. We will use some known facts on compact Lie groups that can be consulted in [84, Chapter IV].

Let $\Delta_{\mathfrak{k}} = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ be the root system of $\mathfrak{k}$ with respect to $\mathfrak{t}$, that is, the set of nonzero elements $\alpha \in (\mathfrak{t}^{\mathbb{C}})^*$ such that the corresponding eigenspace $\mathfrak{k}_\alpha = \{X \in \mathfrak{k}^{\mathbb{C}} : \mathrm{ad}(H)X = i\alpha(H)X,$ for all $H \in \mathfrak{t}\}$ is nonzero. Let $\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta_{\mathfrak{k}}} \mathfrak{k}_\alpha$ be the root space decomposition of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Recall that $\mathfrak{t} = Z(\mathfrak{k}) \oplus \mathfrak{t}'$, where $Z(\mathfrak{k})$ is the center of $\mathfrak{k}$ and $\mathfrak{t}'$ is a maximal abelian subalgebra of the semisimple Lie algebra $[\mathfrak{k}, \mathfrak{k}]$. The roots in $\Delta_{\mathfrak{k}}$ vanish on $Z(\mathfrak{k}^{\mathbb{C}})$ and $\Delta_{\mathfrak{k}}$ is an abstract reduced root system in the subspace $((\mathfrak{t}')^{\mathbb{C}})^*$ of $(\mathfrak{t}^{\mathbb{C}})^*$.

Let $\Delta_V = \Delta(V^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ be the set of weights of the representation $\rho_*^{\mathbb{C}} \colon \mathfrak{k}^{\mathbb{C}} \to \mathfrak{gl}(V^{\mathbb{C}})$, that is, those elements $\lambda \in (\mathfrak{t}^{\mathbb{C}})^*$ such that the subspace $V_\lambda = \{v \in V^{\mathbb{C}} : \rho_*^{\mathbb{C}}(H)v = i\lambda(H)v, \text{ for all } H \in \mathfrak{t}\}$ is nonzero. Then we have the weight space decomposition $V^{\mathbb{C}} = \bigoplus_{\lambda \in \Delta_V} V_\lambda$. Notice that, according to our notation, all roots and weights are real on $\mathfrak{t}$.

Proposition 7.15. *Let $H \in \mathfrak{t}$. Then $\rho_*(H)$ is a complex structure on V if and only if $\lambda(H) \in \{\pm 1\}$ for every weight λ of the representation $\rho_*^{\mathbb{C}}$.*

Proof. The skew-symmetric transformation $\rho_*(H)$ is a complex structure on V if and only if $\rho_*(H)^2 = -\operatorname{Id}$, or equivalently, if $\rho_*^{\mathbb{C}}(H)^2 = -\operatorname{Id}$. For an arbitrary $\lambda \in \Delta_V$, let $v_\lambda \in V_\lambda$. Then $\rho_*^{\mathbb{C}}(H)^2 v_\lambda = -\lambda(H)^2 v_\lambda$. Hence $\rho_*^{\mathbb{C}}(H)^2 = -\operatorname{Id}$ if and only if $\lambda(H)^2 = 1$ for all $\lambda \in \Delta_V$. □

In view of Propositions 7.14 and 7.15, if 0 is a weight of $\rho_*^{\mathbb{C}}$, then $\mathcal{F}$ cannot be projected to the complex projective space. Moreover, once one knows the maximal connected group of orthogonal transformations preserving the leaves of the foliation $\mathcal{F}$, these propositions allow to determine all possible complex structures preserving $\mathcal{F}$ in a computational way.

7.5.2 Congruence of projected foliations

Now we focus on the study of the congruence of foliations on complex projective spaces. We start with the following basic result.

Proposition 7.16. *Let $V = \mathbb{R}^{2n+2}$. Let J_1, J_2 be two complex structures on V, $\mathbb{C}P_1^n$, $\mathbb{C}P_2^n$ the corresponding complex projective spaces, and π_1, π_2 the corresponding Hopf maps.*

Two foliations $\mathcal{G}_1 \subset \mathbb{C}P_1^n$ and $\mathcal{G}_2 \subset \mathbb{C}P_2^n$ are congruent if and only if there exists an orthogonal transformation $A \in \mathrm{O}(V)$ satisfying $AJ_1A^{-1} = \pm J_2$ and mapping leaves of $\pi_1^{-1}\mathcal{G}_1$ to leaves of $\pi_2^{-1}\mathcal{G}_2$.

Proof. $\mathcal{G}_1$ and $\mathcal{G}_2$ are congruent if and only if there exists a unitary or anti-unitary transformation A between (V, J_1) and (V, J_2) (i.e. $A \in \mathrm{O}(V)$ and $AJ_1A^{-1} = \pm J_2$) whose induced isometry $[A] : \mathbb{C}P_1^n \to \mathbb{C}P_2^n$ takes the leaves of $\mathcal{G}_1$ to the leaves of $\mathcal{G}_2$. But this condition is equivalent to the one in the statement. □

In particular, a necessary condition for two foliations on a complex projective space to be congruent is that their lifts to the sphere are congruent. In view of this, in order to study the congruence of foliations on complex projective spaces it suffices to decide when two complex structures preserving some fixed foliation give rise to congruent foliations on the corresponding complex projective spaces.

Let $\mathcal{F}$, K, ρ and $\mathfrak{t}$ be as in §7.5.1. Consider two complex structures $J_i = \rho_*(X_i)$, $i = 1, 2$, on V preserving the foliation $\mathcal{F}$. Let us say that J_1 and J_2 are equivalent, and write $J_1 \sim J_2$, if J_1 and J_2 give rise to congruent foliations on the complex projective space.

We also denote by $\sim$ the corresponding equivalence relation on the subset $\mathcal{J}$ of those X in $\mathfrak{k}$ such that $\rho_*(X)$ is a complex structure preserving $\mathcal{F}$. The problem of the congruence of foliations on $\mathbb{C}P^n$ is then reduced to the determination of the $\sim$-equivalence classes of $\mathcal{J}$.

Let $\operatorname{Aut}(\mathcal{F})$ be the group of automorphisms of the foliation $\mathcal{F}$, i.e. the group of those orthogonal transformations of V that map leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. Clearly, $\rho(K)$ is a subgroup of $\operatorname{Aut}(\mathcal{F})$. Due to the effectiveness of ρ, each $A \in \operatorname{Aut}(\mathcal{F})$ defines an automorphism $\phi_A \in \operatorname{Aut}(\mathfrak{k})$ of the Lie algebra $\mathfrak{k}$, by means of the relation $A\rho_*(X)A^{-1} = \rho_*(\phi_A(X))$. Consider the group $\operatorname{Aut}(\mathfrak{k}, \mathcal{F})$ of those linear isomorphisms $\varphi_A \colon \mathfrak{k} \oplus V \to \mathfrak{k} \oplus V$ defined by $\varphi_A|_{\mathfrak{k}} = \phi_A$ and $\varphi_A|_V = A$, where A runs over $\operatorname{Aut}(\mathcal{F})$. Note that $(\operatorname{Ad} \oplus \rho)(K) = \{\varphi_{\rho(k)} : k \in K\}$ is a subgroup of $\operatorname{Aut}(\mathfrak{k}, \mathcal{F})$.

In view of this notation, Proposition 7.16 asserts that two complex structures $J_i = \rho_*(X_i)$, $i = 1, 2$, are equivalent (i.e. $X_1 \sim X_2$) if and only if there exists $A \in \operatorname{Aut}(\mathcal{F})$ with $A\rho_*(X_1)A^{-1} = \pm\rho_*(X_2)$, or equivalently, if there exists $\varphi \in \operatorname{Aut}(\mathfrak{k}, \mathcal{F})$ such that $\varphi X_1 = \pm X_2$.

Every $\sim$-equivalence class intersects the maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, since $\mathfrak{k} = \bigcup_{k \in K} \operatorname{Ad}(k)\mathfrak{t}$ and $\operatorname{Ad}(K) \subset \operatorname{Aut}(\mathfrak{k}, \mathcal{F})|_{\mathfrak{k}}$. We can therefore restrict $\sim$ to $\mathfrak{t}$ and analyze the set $\mathcal{J} \cap \mathfrak{t}$ and its partition in $\sim$-equivalence classes.

Proposition 7.17. *Let $T_1, T_2 \in \mathcal{J} \cap \mathfrak{t}$. Then $T_1 \sim T_2$ if and only if there exists an automorphism $\varphi \in \operatorname{Aut}(\mathfrak{k}, \mathcal{F})$ preserving $\mathfrak{t}$ such that $\varphi T_1 = \pm T_2$.*

Proof. The sufficiency is clear according to the previous remarks. For the necessity we will use a well-known argument in the study of compact groups (cf. [84, Prop 4.53]).

Let $\phi \in \operatorname{Aut}(\mathfrak{k}, \mathcal{F})$ be such that $\phi T_1 = \pm T_2$. The centralizer $Z_K(T_2)$ of T_2 in K is a compact group, and $\mathfrak{t}$, $\phi(\mathfrak{t})$ are maximal abelian subalgebras of $Z_{\mathfrak{k}}(T_2)$, which is the Lie algebra of $Z_K(T_2)$. Hence there exists $k \in Z_K(T_2)$ such that $\operatorname{Ad}(k)\mathfrak{t} = \phi(\mathfrak{t})$. Define $\varphi = (\operatorname{Ad} \oplus \rho)(k^{-1}) \circ \phi \in \operatorname{Aut}(\mathfrak{k}, \mathcal{F})$. Then $\varphi(\mathfrak{t}) = \mathfrak{t}$ and $\varphi(T_1) = \pm \operatorname{Ad}(k^{-1})T_2 = \pm T_2$. □

Since the leaves of $\mathcal{F}$ are closed and equidistant, it follows that the group $\operatorname{Aut}(\mathcal{F})$ is compact, so $\operatorname{Aut}(\mathfrak{k}, \mathcal{F})$ and $\operatorname{Aut}(\mathfrak{k}, \mathcal{F})|_{\mathfrak{k}}$ are also compact. Hence, there exists a positive definite $\operatorname{Aut}(\mathfrak{k}, \mathcal{F})|_{\mathfrak{k}}$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$. Then $\langle \mathfrak{t}', Z(\mathfrak{k}) \rangle = 0$. Moreover, $\langle \cdot, \cdot \rangle$ restricted to each simple ideal of $\mathfrak{k}$ is a negative multiple of the Killing form of such ideal.

For each $\lambda \in \mathfrak{t}^*$, we define $H_\lambda \in \mathfrak{t}$ by $\langle H_\lambda, H \rangle = \lambda(H)$, for all $H \in \mathfrak{t}$. Then $\langle \cdot, \cdot \rangle$ induces an inner product on $\mathfrak{t}^*$ in a natural way, by means of $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$, for $\lambda, \mu \in \mathfrak{t}^*$. If $\alpha \in \Delta_{\mathfrak{k}}$ is a root of $\mathfrak{k}$, the corresponding H_α will be called a coroot, whereas if $\lambda \in \Delta_V$ is a weight of $\rho_*^{\mathbb{C}}$, we will say that H_λ is a coweight. Note that the coroots belong to $\mathfrak{t}'$, since $\mathfrak{t} = Z(\mathfrak{k}) \oplus \mathfrak{t}'$ and the roots vanish on $Z(\mathfrak{k})$.

We will say that an orthogonal transformation of $\mathfrak{t}$ is an automorphism of $\Delta_{\mathfrak{k}}$ if it maps the set of coroots $\{H_\alpha : \alpha \in \Delta_{\mathfrak{k}}\}$ onto itself. The group of automorphisms of $\Delta_{\mathfrak{k}}$ is denoted by $\operatorname{Aut}(\Delta_{\mathfrak{k}})$. The subgroup of those automorphisms of $\Delta_{\mathfrak{k}}$ that map the set of coweights $\{H_\lambda : \lambda \in \Delta_V\}$ onto itself will be denoted by $\operatorname{Aut}(\Delta_{\mathfrak{k}}, \Delta_V)$. The action of $\operatorname{Aut}(\Delta_{\mathfrak{k}})$ on $\mathfrak{t}$ induces an action of $\operatorname{Aut}(\Delta_{\mathfrak{k}})$ on $\mathfrak{t}^*$ by means of $\varphi(\alpha) = \alpha \circ \varphi^{-1}$ for $\alpha \in \mathfrak{t}^*$ and $\varphi \in \operatorname{Aut}(\Delta_{\mathfrak{k}})$.

Proposition 7.18. *The restriction to $\mathfrak{t}$ of each element of $\operatorname{Aut}(\mathfrak{k}, \mathcal{F})$ preserving $\mathfrak{t}$ gives an element of $\operatorname{Aut}(\Delta_{\mathfrak{k}}, \Delta_V)$.*

Proof. Consider an element $\varphi \in \mathrm{Aut}(\mathfrak{k}, \mathcal{F})$ with $\varphi(\mathfrak{t}) = \mathfrak{t}$, and let $\phi = \varphi^{\mathbb{C}}$, which is a linear automorphism of $\mathfrak{k}^{\mathbb{C}} \oplus V^{\mathbb{C}}$ such that $\phi|_{\mathfrak{k}^{\mathbb{C}}}$ is a Lie algebra automorphism of $\mathfrak{k}^{\mathbb{C}}$. If $\alpha \in \Delta_{\mathfrak{k}}$ and $X \in \mathfrak{k}_\alpha$, then $[\phi H, \phi X] = \alpha(H)\phi X$ for all $H \in \mathfrak{t}^{\mathbb{C}}$, so $\phi(\mathfrak{k}_\alpha) = \mathfrak{k}_\beta$, where $\beta = \alpha \circ \phi^{-1}|_{\mathfrak{t}^{\mathbb{C}}} \in \Delta_{\mathfrak{k}}$. Moreover, $\beta(H) = \alpha(\phi^{-1}H) = \langle \phi^{-1}H, H_\alpha \rangle = \langle H, \phi H_\alpha \rangle$ for all $H \in \mathfrak{t}$, and thus $\phi H_\alpha = H_\beta$. If $\lambda \in \Delta_V$ and $X \in V_\lambda$, then $\rho_*^{\mathbb{C}}(\phi H)\phi X = \lambda(H)\phi X$ for all $H \in \mathfrak{t}^{\mathbb{C}}$, so $\rho_*^{\mathbb{C}}(H)\phi X = \mu(H)\phi X$, where $\mu = \lambda \circ \phi^{-1}|_{\mathfrak{t}^{\mathbb{C}}} \in \Delta_V$. Hence $\phi V_\lambda = V_\mu$ and, similarly as for the coroots, we get that $\phi H_\lambda = H_\mu$. We have thus shown that $\varphi|_{\mathfrak{t}} = \phi|_{\mathfrak{t}} \in \mathrm{Aut}(\Delta_{\mathfrak{k}}, \Delta_V)$. □

We will denote by $\mathrm{Aut}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ the subgroup of those automorphisms in $\mathrm{Aut}(\Delta_{\mathfrak{k}}, \Delta_V)$ that are restriction of automorphisms in $\mathrm{Aut}(\mathfrak{k}, \mathcal{F})$ preserving $\mathfrak{t}$, and by $\mathrm{Aut}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ the group generated by $\mathrm{Aut}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ and $-\mathrm{Id}_{\mathfrak{t}}$. The elements of $\mathrm{Aut}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ leave Δ_V invariant; indeed $-\mathrm{Id}_{\mathfrak{t}} \in \mathrm{Aut}(\Delta_{\mathfrak{k}}, \Delta_V)$ since $\rho_*^{\mathbb{C}}$ is a complex representation of real type, hence self dual. A straightforward consequence of Propositions 7.17 and 7.18 is the following

Corollary 7.19. *If $T_1, T_2 \in \mathcal{J} \cap \mathfrak{t}$, then $T_1 \sim T_2$ if and only if there exists $\varphi \in \mathrm{Aut}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ such that $\varphi T_1 = T_2$.*

Let us fix a set of simple roots $\Pi_{\mathfrak{k}} = \{\alpha_1, \ldots, \alpha_r\}$ for the root system of $\mathfrak{k}$. Let $W(\Delta_{\mathfrak{k}}) = N_K(\mathfrak{t})/Z_K(\mathfrak{t})$ be the Weyl group of the root system $\Delta_{\mathfrak{k}}$; we will regard $W(\Delta_{\mathfrak{k}})$ as a group of transformations of $\mathfrak{t}$ that leave $\mathfrak{t}'$ invariant. Since $W(\Delta_{\mathfrak{k}}) \subset \mathrm{Ad}(K) \subset \mathrm{Aut}(\mathfrak{k}, \mathcal{F})|_{\mathfrak{t}}$, Proposition 7.18 implies that $W(\Delta_{\mathfrak{k}}) \subset \mathrm{Aut}(\Delta_{\mathfrak{k}}, \Delta_V)$, i.e. the Weyl group action on $\mathfrak{t}$ leaves the set of coroots and the set of coweights invariant. It is known that $W(\Delta_{\mathfrak{k}})$ is generated by the reflections in $\mathfrak{t}$ around the hyperplanes of equation $\alpha = 0$, for all $\alpha \in \Delta_{\mathfrak{k}}$, or even for all $\alpha \in \Pi_{\mathfrak{k}}$. The coroot H_α is a normal vector to the hyperplane $\alpha = 0$.

The subset $\bar{C}$ of $\mathfrak{t}$ defined by the inequalities $\alpha \geq 0$, for every $\alpha \in \Pi_{\mathfrak{k}}$, constitutes a fundamental domain for the action of $W(\Delta_{\mathfrak{k}})$ on $\mathfrak{t}$, that is, every $W(\Delta_{\mathfrak{k}})$-orbit intersects $\bar{C}$ in exactly one point. The set $\bar{C}$ is the Cartesian product of the closure of a Weyl chamber of $\Delta_{\mathfrak{k}} = \Delta_{[\mathfrak{k},\mathfrak{k}]}$ in $\mathfrak{t}'$ times the center of $\mathfrak{k}$. We will denote by $\mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ (resp. $\mathrm{Out}(\Delta_{\mathfrak{k}})$, $\mathrm{Out}(\Delta_{\mathfrak{k}}, \Delta_V)$) the subgroup of $\mathrm{Aut}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ (resp. $\mathrm{Aut}(\Delta_{\mathfrak{k}})$, $\mathrm{Aut}(\Delta_{\mathfrak{k}}, \Delta_V)$) of all automorphisms leaving $\bar{C}$ invariant, or equivalently, leaving invariant the simple coroots.

Since $W(\Delta_{\mathfrak{k}}) \subset \mathrm{Aut}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ and $\bar{C}$ is a fundamental domain for the action of $W(\Delta_{\mathfrak{k}})$, the problem of understanding the partition of $\mathcal{J} \cap \mathfrak{t}$ in its $\sim$-equivalence classes is then reduced to determining the set $\mathcal{J} \cap \bar{C}$ and deciding which of its elements are $\sim$-related.

Proposition 7.20. *Let $T_1, T_2 \in \mathcal{J} \cap \bar{C}$. Then $T_1 \sim T_2$ if and only if there is $\varphi \in \mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ such that $\varphi(T_1) = T_2$.*

Proof. The sufficiency of the claim is clear. Let us assume $T_1 \sim T_2$. Then there is $\phi \in \mathrm{Aut}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ such that $\phi(T_1) = T_2$. In particular $\phi \in \mathrm{Aut}(\Delta_{\mathfrak{k}}) \cong W(\Delta_{\mathfrak{k}}) \ltimes \mathrm{Out}(\Delta_{\mathfrak{k}})$, so we can put $\phi = \phi' \circ \varphi$, with $\phi' \in W(\Delta_{\mathfrak{k}})$ and $\varphi \in \mathrm{Out}(\Delta_{\mathfrak{k}})$. Then $\varphi(T_1) \in \bar{C}$ and $\phi'(\varphi(T_1)) = \phi(T_1) = T_2 \in \bar{C}$. But $\bar{C}$ is a fundamental domain for the action of $W(\Delta_{\mathfrak{k}})$, hence $\varphi(T_1) = T_2$. Finally notice that $\varphi \in \mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$, because $\phi, \phi' \in \mathrm{Aut}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$. □

We introduce now one of the key ideas of this chapter, namely: the usage of certain generalization of the so-called extended Vogan diagrams of inner symmetric spaces. This particular case will be discussed in Section 7.6.

Given a complex finite dimensional representation η of a compact Lie algebra $\mathfrak{k}$, the *lowest weight diagram* of η is constructed as follows. Consider the Dynkin diagram of $\mathfrak{k}$, where each simple root of $\Delta_{\mathfrak{k}}$ is represented by a white node, and draw as many black nodes as lowest weights of η, counted with multiplicity. Join each black node corresponding to a lowest weight λ to those white nodes corresponding to roots α with $\langle \alpha, \lambda \rangle \neq 0$ by means of a simple line. Finally, attach to each one of these new edges the integer value $2\langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle$ as a label; if no label is attached, we understand that the associated value is -1.

An automorphism (or symmetry) of a lowest weight diagram is a permutation of its nodes preserving the graph, the black nodes and the labels of the edges between black nodes and white nodes. Having in mind the notation of this section, we will talk about the lowest weight diagram of $\rho_*^{\mathbb{C}}$ or, directly, of the foliation $\mathcal{F}$. The study of the symmetries of the lowest weight diagrams of certain foliations will be a useful tool in this chapter. The following result gives a first idea of the interest of these diagrams.

Proposition 7.21. *Each automorphism in* $\operatorname{Out}(\Delta_{\mathfrak{k}}, \Delta_V)$ *induces an automorphism of the lowest weight diagram of* $\rho_*^{\mathbb{C}}$ *in a natural way. This correspondence is injective if the set of simple roots and lowest weights generates* $\mathfrak{t}^*$.

Proof. Let $\varphi \in \operatorname{Out}(\Delta_{\mathfrak{k}}, \Delta_V)$ and $\lambda \in \Delta_V$. Then λ is a lowest weight of $\rho_*^{\mathbb{C}}$ if and only if $\lambda - \sum_{\alpha_i \in \Pi_{\mathfrak{k}}} n_i \alpha_i$ (with all $n_i \in \mathbb{N} \cup \{0\}$) is not a weight unless all n_i vanish. Since $\varphi(H_\lambda - \sum_{\alpha_i \in \Pi_{\mathfrak{k}}} n_i H_{\alpha_i}) = \varphi(H_\lambda) - \sum_{\alpha_i \in \Pi_{\mathfrak{k}}} n_i \varphi(H_{\alpha_i})$ and φ preserves the set of simple coroots and the set of coweights, we get that φ preserves the set of lowest coweights of $\rho_*^{\mathbb{C}}$. As moreover φ is an orthogonal transformation of $\mathfrak{t}$, we conclude that φ induces a symmetry of the lowest weight diagram of $\rho_*^{\mathbb{C}}$. The last assertion of the statement is immediate. $\square$

7.6 Projecting homogeneous polar foliations

Our goal in this section is to classify isoparametric foliations on complex projective spaces obtained by projection of homogeneous polar foliations on spheres. In §7.6.1 we characterize the homogeneous polar foliations that can be projected to the complex projective space and determine the complex structures that can be used with that end. In §7.6.2 we investigate the congruence of the corresponding projected foliations. In §7.6.3 we derive the classification.

First of all, we recall the notion of inner symmetric space, we introduce some known facts about Vogan diagrams and the Borel-de Siebenthal theory and we fix some notation that will be used throughout this section.

Let (G, K) be a symmetric pair, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition and θ the corresponding Cartan involution. Then G/K (or (G, K)) is said to be *inner* or *equal-rank* if its involution θ is inner. This happens exactly when $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank [76, Chapter IX, Theorem 5.6], or when the Euler characteristic of G/K is nonzero [70, Chapter XI, Theorem VII].

For the study of Vogan diagrams and Borel-de Siebenthal theory, we refer to [84, §VI.8–10 and Appendix C.3–4] (where the pictures of Vogan diagrams can be found) and [69, Chapter 8]. Here we give a quick overview for the particular case of inner symmetric spaces.

Assume that (G, K) is an inner compact symmetric pair. Then a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is also a maximal abelian subalgebra of $\mathfrak{g}$. Let $\Delta_{\mathfrak{g}}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta_{\mathfrak{g}}} \mathfrak{g}_\alpha$ be the root space decomposition. For every $\alpha \in \Delta_{\mathfrak{g}}$, either $\mathfrak{g}_\alpha \subset \mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}^{\mathbb{C}}$ holds. In the first case we say that α is *compact*; in the second case, α is *noncompact*. This terminology is motivated by the consideration of the real semisimple Lie algebra $\mathfrak{k} \oplus i\mathfrak{p}$ (see [84, p. 390]).

Choose a set $\Pi_{\mathfrak{g}}$ of simple roots for $\Delta_{\mathfrak{g}}$. The *Vogan diagram* of the inner orthogonal symmetric pair $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{t}$ and $\Pi_{\mathfrak{g}}$ is the Dynkin diagram of $\Pi_{\mathfrak{g}}$ with its nodes painted or not, depending on whether the corresponding simple root is noncompact or compact.

There can be several Vogan diagrams corresponding to the same orthogonal symmetric pair. This redundancy is eliminated by the Borel-de Siebenthal theorem (see [84, Theorem 6.96]). For our purposes, what this result asserts is the following: given an irreducible inner compact orthogonal symmetric pair $(\mathfrak{g}, \mathfrak{k})$ and given $\mathfrak{t}$ as above, there is a set of simple roots for $\Delta_{\mathfrak{g}}$ whose corresponding Vogan diagram has exactly one painted node. That is, we can assume that there is a set $\Pi_{\mathfrak{g}}$ of simple roots for $\Delta_{\mathfrak{g}}$ with precisely one noncompact root. Furthermore, the set $\Delta_{\mathfrak{k}}$ of compact roots corresponds to the root system of $\mathfrak{k}$ and is a root subsystem of $\Delta_{\mathfrak{g}}$, whereas the noncompact roots are exactly the weights of the adjoint $\mathfrak{k}^{\mathbb{C}}$-representation on $\mathfrak{p}^{\mathbb{C}}$.

Applying the Borel-de Siebenthal theorem, we fix a set of simple roots $\Pi_{\mathfrak{g}} = \{\alpha_1, \dots, \alpha_p\}$ for $\Delta_{\mathfrak{g}}$, where α_ν is noncompact, for certain $\nu \in \{1, \dots, p\}$, and the other simple roots are compact. Let $\{h_1, \dots, h_p\} \subset \mathfrak{t}$ be the dual basis of $\Pi_{\mathfrak{g}}$.

Let $\mu = \sum_{i=1}^{p} y_i\alpha_i$ ($y_i \in \mathbb{N}$) be the highest root of $\Delta_{\mathfrak{g}}$ and put $\alpha_0 = -\mu$. Then G/K is Hermitian if and only if $y_\nu = 1$; otherwise, $y_\nu = 2$ (see Table 7.3). If G/K is Hermitian, we will consider $\Pi_{\mathfrak{k}} = \{\alpha_1, \dots, \alpha_p\} \setminus \{\alpha_\nu\}$ as a set of simple roots for $\Delta_{\mathfrak{k}}$. In this case, if $\mathfrak{t}'$ is the hyperplane of $\mathfrak{t}$ generated by the compact coroots, its normal space with respect to the Killing form $\mathcal{B}_{\mathfrak{g}}$ is $\mathbb{R}h_\nu$. If G/K is not Hermitian, we will take $\Pi_{\mathfrak{k}} = \{\alpha_0, \alpha_1, \dots, \alpha_p\} \setminus \{\alpha_\nu\}$ as a set of simple roots for $\Delta_{\mathfrak{k}}$. For a justification of these choices, see [69, Chapter 8].

An enumeration of the roots in $\Delta_{\mathfrak{g}}$ shows that there is a unique highest noncompact root λ, in the following sense: λ is the unique noncompact root that is greater than any other noncompact root, according to the lexicographic ordering defined by the set of simple roots $\Pi_{\mathfrak{g}}$. Notice that, if $\lambda = \sum_{i=1}^{p} z_i\alpha_i$, one has that $z_\nu = 1$ and $z_i > 0$ for every i (see Table 7.3, cf. [84, Appendix C.1,2,4]).

7.6.1 Complex structures preserving homogeneous polar foliations

Given a compact symmetric pair (G, K) satisfying the maximality property, we denote by $\mathcal{F}_{G/K}$ the orbit foliation of the s-representation of G/K restricted to the unit sphere of $\mathfrak{p}$, and we refer to it as the foliation determined by G/K (or by (G, K)). The theory developed in Section 7.5 applies to these foliations $\mathcal{F}_{G/K}$, where $V = \mathfrak{p}$ and $\rho = \mathrm{Ad}\colon K \to \mathrm{O}(\mathfrak{p})$. The following result completely characterizes those s-representations whose orbit foliations can be projected to the corresponding complex projective space.

Theorem 7.22. *Let (G, K) be a compact symmetric pair satisfying the maximality property, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition and $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{k}$. Then there exists a complex structure in $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$ if and only if G/K is inner.*

In this situation, let $T \in \mathfrak{t}$. Then $\mathrm{ad}(T)|_{\mathfrak{p}}$ is a complex structure in $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$ if and only if $\alpha(T) \in \{\pm 1\}$ for all (positive) noncompact roots α of $\mathfrak{g}$.

Proof. If G/K is not inner, the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ of the form $\mathfrak{t} \oplus \mathfrak{a}$ with $0 \neq \mathfrak{a} \subset \mathfrak{p}$. Then 0 is a weight of the adjoint $\mathfrak{k}^{\mathbb{C}}$-representation $\rho_*^{\mathbb{C}}$, with weight space $\mathfrak{a}^{\mathbb{C}}$. By Proposition 7.15, $\mathcal{F}_{G/K}$ cannot be projected under any Hopf map.

Assume that G/K is inner. The weights of $\rho_*^{\mathbb{C}}$ are the noncompact roots of $\mathfrak{g}$. Again by Proposition 7.15, $\mathrm{ad}(T)|_{\mathfrak{p}}$ is a complex structure in $\mathfrak{p}$ if and only if $\alpha(T) \in \{\pm 1\}$ for all noncompact roots of $\mathfrak{g}$ or, equivalently, for all positive noncompact roots (for a fixed ordering of $\Delta_{\mathfrak{g}}$). We still have to show that such $T \in \mathfrak{t}$ exists if G/K is inner. According to Proposition 7.14(iii), it is enough to show this for irreducible symmetric pairs (G, K).

So let G/K be inner and irreducible. If $\alpha, \beta \in \Delta_{\mathfrak{g}}$ are such that $\alpha + \beta \neq 0$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, then $\alpha + \beta$ is compact if and only if α and β are both compact or both noncompact.

Hence, the positive noncompact roots are exactly those roots $\alpha = \sum_{j=1}^{p} m_j \alpha_j$, with odd m_ν (recall the notation introduced just before this subsection). But since the highest noncompact root $\lambda = \sum_{i=1}^{p} z_i \alpha_i$ satisfies $z_\nu = 1$, for positive noncompact roots we always have $m_\nu = 1$. Taking $T = h_\nu$, then $\alpha(T) = 1$ for every noncompact positive root, and hence $\mathrm{ad}(T)|_{\mathfrak{p}}$ is a complex structure preserving $\mathcal{F}_{G/K}$. □

Fix now an irreducible inner compact symmetric pair (G, K) satisfying the maximality property and take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$. As defined in §7.5.2, let $\mathcal{J}$ be the subset of those $X \in \mathfrak{k}$ such that $\rho_*(X) = \mathrm{ad}(X)|_{\mathfrak{p}}$ is a complex structure on $\mathfrak{p}$, and let $\bar{C} \subset \mathfrak{t}$ be defined by the inequalities $\alpha \geq 0$, for every $\alpha \in \Pi_{\mathfrak{k}}$. We can now provide a complete description of the set $\mathcal{J} \cap \bar{C}$.

Lemma 7.23. *In the conditions above, let $\mu = \sum_{i=1}^{p} y_i \alpha_i$ be the highest root and $\lambda = \sum_{i=1}^{p} z_i \alpha_i$ the highest noncompact root. We have:*

(i) *If G/K is not Hermitian, then $\mathcal{J} \cap \bar{C} = \{-h_\nu\} \cup \{-h_\nu + 2h_i : i \in \mathcal{I}\}$, where $\mathcal{I}$ is the set of indices $i \in \{1, \ldots, p\} \setminus \{\nu\}$ such that $y_i = z_i = 1$.*

(ii) *If G/K is Hermitian, then $\mathcal{J} \cap \bar{C} = \{\pm h_\nu\} \cup \{-h_\nu + 2h_i : i \in \mathcal{I}\}$, where $\mathcal{I}$ is the set of indices $i \in \{1, \dots, p\} \setminus \{\nu\}$ such that $y_i = 1$.*

Proof. Let us first prove some auxiliary results. For that, let $T = \sum_{i=1}^{p} x_i h_i \in \mathcal{J} \cap \bar{C}$.

The condition $T \in \mathcal{J}$ implies that $\alpha(T) = \pm 1$ for all positive noncompact roots α. Since α_ν is noncompact, we get that $x_\nu = \alpha_\nu(T) = \pm 1$.

As $T \in \bar{C}$ and λ is noncompact, we must have $x_i = \alpha_i(T) \geq 0$ for all $i \in \{1, \dots, p\} \setminus \{\nu\}$ and $\lambda(T) = \pm 1$. Hence $x_\nu = 1$ implies that $x_i = 0$ for every $i \in \{1, \dots, p\} \setminus \{\nu\}$.

From the information in [84, Appendix C.1,2,4]), one can carry out a case-by-case analysis that shows the following. There exists a sequence of positive noncompact roots $\beta_1, \dots, \beta_p$, with $\beta_1 = \alpha_\nu$ and such that, if we express each β_i as a linear combination of the simple roots, $\beta_i = \sum_{j=1}^{p} m_{ij} \alpha_j$, then each coefficient m_{ij} is either 1 or 0, and the number of 1-coefficients increases by one from β_i to β_{i+1}, for every $i \in \{1, \dots, p-1\}$ (in particular, the coefficients of β_p are all 1).

The assumptions $x_\nu = -1$, $x_i \geq 0$ for every $i \neq \nu$ and $\beta_i(T) = \pm 1$ for every i imply then that $x_i = 0$ for all $i \neq \nu$ except for at most one index j, for which $x_j = 2$.

Now we prove (i). Assume that G/K is not Hermitian and, thus, $y_\nu = 2$. Since in the non-Hermitian case $-\mu \in \Pi_{\mathfrak{k}}$, we get that $-h_\nu \in \mathcal{J} \cap \bar{C}$ and $h_\nu \notin \bar{C}$. As λ is the highest noncompact root, we see that $-h_\nu + 2h_i \in \mathcal{J}$ if and only if $z_i = 1$. Moreover, $-h_\nu + 2h_i \in \bar{C}$ if and only if $-y_\nu + 2y_i \leq 0$. Since $y_\nu = 2$, this condition is equivalent to $y_i = 1$.

In the Hermitian case $\pm h_\nu \in \mathcal{J} \cap \bar{C}$. Since now $\lambda = \mu$ and the simple roots for $\Delta_{\mathfrak{k}}$ are just $\{\alpha_1, \dots, \alpha_p\} \setminus \{\alpha_\nu\}$, then $-h_\nu + 2h_i \in \mathcal{J} \cap \bar{C}$ if and only if $y_i = 1$, and (ii) follows. □

7.6.2 Congruence of the projections of homogeneous polar foliations

For the case of homogeneous polar foliations it is possible to refine the results of §7.5.2. This is the aim of this subsection. The criteria developed will be used in §7.6.3 to obtain the classification of isoparametric foliations on $\mathbb{C}P^n$ obtained from homogeneous polar foliations.

According to Proposition 7.16 and Remark 7.9, it is impossible that different compact orthogonal symmetric pairs satisfying the maximality property give rise to congruent foliations on a complex projective space. That is why we will focus on analyzing the congruence of foliations arisen from a fixed symmetric space.

Therefore, we fix a compact symmetric pair (G, K) satisfying the maximality property and with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In view of Theorem 7.22, we can assume that G/K is inner. We also fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$ and we let $\mathcal{J}$, $\sim$ and $\bar{C}$ be as in §7.5.2. Our aim is to determine the $\sim$-equivalence classes of $\mathcal{J}$.

Theorem 7.8 implies that $\mathrm{Aut}(\mathfrak{k}, \mathcal{F})$ is precisely the group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ of automorphisms of $\mathfrak{g}$ that restrict to automorphisms of $\mathfrak{k}$. Therefore, Proposition 7.17 now reads as follows.

Proposition 7.24. *Let $T_1, T_2 \in \mathcal{J} \cap \mathfrak{t}$. Then $T_1 \sim T_2$ if and only if there exists an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ leaving $\mathfrak{t}$ invariant and such that $\varphi T_1 = \pm T_2$.*

The negative $-\mathcal{B}_{\mathfrak{g}}$ of the Killing form of $\mathfrak{g}$ is a positive definite $\mathrm{Aut}(\mathfrak{g},\mathfrak{k})$-invariant inner product on $\mathfrak{g}$, so it can play the role of the inner product $\langle\cdot,\cdot\rangle$ considered in §7.5.2. The set $\Delta_V = \Delta_{\mathfrak{p}}$ of weights of the adjoint $\mathfrak{k}^{\mathbb{C}}$-representation on $\mathfrak{p}^{\mathbb{C}}$ is precisely $\Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{k}}$. Hence, the group $\mathrm{Aut}(\Delta_{\mathfrak{k}},\Delta_V)$ defined in §7.5.2 is now the group of automorphisms of the root system $\Delta_{\mathfrak{g}}$ that are automorphisms of the root subsystem $\Delta_{\mathfrak{k}}$. In this section, we denote this group by $\mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$. Then we have:

Proposition 7.25. *The restriction to* $\mathfrak{t}$ *of every element of* $\mathrm{Aut}(\mathfrak{g},\mathfrak{k})$ *preserving* $\mathfrak{t}$ *yields an element of* $\mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$. *Conversely, every element of* $\mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$ *can be extended to an element of* $\mathrm{Aut}(\mathfrak{g},\mathfrak{k})$ *preserving* $\mathfrak{t}$.

Proof. The first claim follows from Proposition 7.18. For the converse, let $\varphi \in \mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$. The second assertion in [76, Chapter IX, Theorem 5.1] affirms that $\varphi \in \mathrm{Aut}(\Delta_{\mathfrak{g}})$ can be extended to an automorphism $\widetilde{\varphi}$ of $\mathfrak{g}$. Let $\phi = \widetilde{\varphi}^{\mathbb{C}}$. Arguing as in Proposition 7.18, if $\alpha \in \Delta_{\mathfrak{k}}$, then $\phi(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta$ and $\varphi H_\alpha = \phi H_\alpha = H_\beta$, where $\beta = \alpha \circ \phi^{-1}|_{\mathfrak{t}^{\mathbb{C}}} \in \Delta_{\mathfrak{g}}$. Since φ sends compact coroots to compact coroots, we have that $\beta \in \Delta_{\mathfrak{k}}$. Hence $\phi(\mathfrak{g}_\alpha) \subset \mathfrak{k}^{\mathbb{C}}$ for every $\alpha \in \Delta_{\mathfrak{k}}$. Since $\phi(\mathfrak{t}^{\mathbb{C}}) = \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$ as well, we get $\phi(\mathfrak{k}^{\mathbb{C}}) = \mathfrak{k}^{\mathbb{C}}$ and, due to the invariance of $\mathfrak{g}$ under ϕ, we have that $\phi(\mathfrak{k}) = \mathfrak{k}$. Therefore, $\widetilde{\varphi}$ is the desired extension of φ. □

As if $\varphi \in \mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$, then also $-\varphi \in \mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$, the previous two propositions imply:

Corollary 7.26. *Let* $T_1, T_2 \in \mathcal{J} \cap \mathfrak{t}$. *Then* $T_1 \sim T_2$ *if and only if there exists* $\varphi \in \mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$ *such that* $\varphi T_1 = T_2$.

Henceforth we will further assume that the compact inner symmetric pair (G,K) satisfying the maximality property is irreducible. The classification of the complex structures preserving foliations induced by reducible symmetric spaces follows from the classification in the irreducible case, in view of Proposition 7.14(iii) and Corollary 7.26.

We will denote by $\mathrm{Out}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$ the subgroup of $\mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$ of automorphisms leaving $\bar{C}$ invariant, or equivalently, leaving invariant the simple compact coroots. Now the groups $\mathrm{Out}(\Delta_{\mathfrak{k}},\Delta_V)$ and $\mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}},\Delta_V)$ introduced in §7.5.2 are exactly $\mathrm{Out}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$. We have:

Proposition 7.27. *Let* $T_1, T_2 \in \mathcal{J} \cap \bar{C}$. *Then* $T_1 \sim T_2$ *if and only if there exists* $\varphi \in \mathrm{Out}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$ *such that* $\varphi(T_1) = T_2$. *If moreover* G/K *is Hermitian, we have:*

(i) *If* $T_1 \notin (\mathfrak{t}')^\perp$ *and* $T_2 \in (\mathfrak{t}')^\perp$, *then* $T_1 \nsim T_2$.

(ii) *If* $T_1, T_2 \in (\mathfrak{t}')^\perp$, *then* $T_1 \sim T_2$.

Proof. The first claim is a rewriting of Proposition 7.20. Let G/K be Hermitian. The fact that every element of $\mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$ is an orthogonal transformation of $\mathfrak{t}$ which leaves $\mathfrak{t}'$ invariant implies (i). Since $-\mathrm{Id}_{\mathfrak{t}} \in \mathrm{Aut}(\Delta_{\mathfrak{g}},\Delta_{\mathfrak{k}})$, and since the intersection of $\mathcal{J}$ with each 1-dimensional subspace of $\mathfrak{t}$ is either empty or a pair of opposite vectors, we obtain (ii). □

We now need to introduce an important notion for our work. First recall that the extended Dynkin diagram of $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{g}$ together with the extra node α_0, which is joined to the other nodes according to the usual rules. Thus, we define the *extended Vogan diagram* of $(\mathfrak{g}, \mathfrak{k})$ as the extended Dynkin diagram of $\mathfrak{g}$, where the nodes corresponding to noncompact roots are painted while the other nodes remain unpainted. This definition depends in principle on the maximally compact abelian subalgebra $\mathfrak{t}$ and on the chosen set of simple roots $\Pi_{\mathfrak{g}}$. However, by the Borel-de Siebenthal theorem and the choices made at the beginning of this section, we can and will assume that every extended Vogan diagram has either exactly one or exactly two painted nodes (α_ν and, maybe, α_0). The first case happens when G/K is not Hermitian and hence the adjoint $\mathfrak{k}^{\mathbb{C}}$-representation on $\mathfrak{p}^{\mathbb{C}}$ is irreducible. The second case occurs when G/K is Hermitian, so the adjoint $\mathfrak{k}^{\mathbb{C}}$-representation on $\mathfrak{p}^{\mathbb{C}}$ decomposes into the sum of two irreducible representations. In both cases, the roots corresponding to the painted nodes in the extended Vogan diagram (that is, the roots in $\Pi_{\mathfrak{g}} \setminus \Pi_{\mathfrak{k}}$) are exactly the lowest weights of the adjoint $\mathfrak{k}^{\mathbb{C}}$-representation on $\mathfrak{p}^{\mathbb{C}}$. Therefore, extended Vogan diagrams represent a very particular case of lowest weight diagrams. The extended Vogan diagrams of irreducible inner symmetric spaces can be obtained from Table 7.3. Analogously as in §7.5.2, we define an automorphism of an extended Vogan diagram as a permutation of its nodes preserving the graph and the painted nodes. For details, references and recent applications of extended Vogan diagrams, see [39].

We can now improve Proposition 7.21 for s-representations of inner symmetric spaces.

Proposition 7.28. *Every automorphism in* $\operatorname{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$ *determines an automorphism of the extended Vogan diagram of* $(\mathfrak{g}, \mathfrak{k})$ *in a unique natural way, and conversely.*

Proof. The first claim follows directly from Proposition 7.21. Let us show the converse. Every automorphism of the extended Dynkin diagram of $\mathfrak{g}$ defines an automorphism $\varphi \in \operatorname{Aut}(\Delta_{\mathfrak{g}})$ (see [95, Chapter VII, Proposition 1.4(a)]). Since every automorphism of the extended Vogan diagram of $(\mathfrak{g}, \mathfrak{k})$ preserves the unpainted nodes, the induced automorphism $\varphi \in \operatorname{Aut}(\Delta_{\mathfrak{g}})$ leaves invariant the simple compact coroots, and hence $\varphi \in \operatorname{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$. ☐

We finish this subsection by proving a rather useful technical lemma.

Lemma 7.29. *Let* $\mu = \sum_{i=1}^{p} y_i \alpha_i$ *be the highest root. Given* $\varphi \in \operatorname{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$*, let* σ *be the permutation of the set of indices* $\{0, 1, \dots, p\}$ *that defines the automorphism of the extended Vogan diagram associated with* φ*. We have:*

(i) *If* $\sigma(0) = 0$*, then* $\varphi(h_i) = h_{\sigma(i)}$ *for all* $i \in \{1, \dots, p\}$*.*

(ii) *If* $\sigma(0) = \nu$*, then* $\varphi(h_i) = h_{\sigma(i)} - y_i h_\nu$ *for* $i \in \{1, \dots, p\} \setminus \{\nu\}$ *and* $\varphi(h_\nu) = -h_\nu$*.*

(iii) *If* σ *interchanges* 0 *and* k*, with* $k \in \{1, \dots, p\} \setminus \{\nu\}$*, then* $\varphi(h_i) = h_{\sigma(i)} - y_i h_k$ *for* $i \in \{1, \dots, n\} \setminus \{k\}$ *and* $\varphi(h_k) = -y_k h_k$*.*

Proof. Fix $i \in \{1, \ldots, p\}$. For every $j \in \{0, 1, \ldots, p\}$ we have:

$$\alpha_j(\varphi(h_i)) = \mathcal{B}_{\mathfrak{g}}(H_{\alpha_j}, \varphi(h_i)) = \mathcal{B}_{\mathfrak{g}}(\varphi^{-1}(H_{\alpha_j}), h_i) = \mathcal{B}_{\mathfrak{g}}(H_{\alpha_{\sigma^{-1}(j)}}, h_i) = \alpha_{\sigma^{-1}(j)}(h_i),$$

which is equal to the Kronecker delta $\delta_{\sigma(i),j}$ if $\sigma^{-1}(j) \neq 0$ and is equal to $-y_i$ if $\sigma^{-1}(j) = 0$.

Assume that $\sigma(0) = 0$. Then σ leaves $\{1, \ldots, p\}$ invariant, and hence $\alpha_j(\varphi(h_i)) = \delta_{\sigma(i),j}$ for all $j \in \{1, \ldots, p\}$. Then (i) follows.

Assume now that $\sigma(0) = \nu$. Then α_0 is noncompact, σ interchanges 0 and ν and preserves $\{1, \ldots, p\} \setminus \{\nu\}$. Consider first that $i \neq \nu$. Then $\alpha_\nu(\varphi(h_i)) = \alpha_0(h_i) = -y_i$, $\alpha_{\sigma(i)}(\varphi(h_i)) = 1$, and $\alpha_j(\varphi(h_i)) = 0$ if $j \in \{1, \ldots, p\} \setminus \{\nu, \sigma(i)\}$. Therefore, if $i \neq \nu$, then $\varphi(h_i) = h_{\sigma(i)} - y_i h_\nu$. Since $\alpha_j(\varphi(h_\nu)) = 0$ for all $j \in \{1, \ldots, p\} \setminus \{\nu\}$, and $\alpha_\nu(\varphi(h_\nu)) = -y_\nu = -1$, we get $\varphi(h_\nu) = -h_\nu$.

Finally let σ be as in (iii). Then α_0 is compact and hence $\sigma(\nu) = \nu$. If $i \neq k$, we have that $\alpha_{\sigma(i)}(\varphi(h_i)) = 1$, $\alpha_k(\varphi(h_i)) = -y_i$, and $\alpha_j(\varphi(h_i)) = 0$ for all $j \in \{1, \ldots, p\} \setminus \{k, \sigma(i)\}$. It follows $\varphi(h_i) = h_{\sigma(i)} - y_i h_k$ if $i \neq k$. Since $\alpha_k(\varphi(h_k)) = -y_k$, and $\alpha_j(\varphi(h_k)) = 0$ for all $j \in \{1, \ldots, p\} \setminus \{k\}$, we obtain $\varphi(h_k) = -y_k h_k$. □

7.6.3 Classification of the complex structures

We are now in position to get the case-by-case classification of the complex structures that preserve homogeneous polar foliations, up to congruence of the projected foliations on the complex projective space. As in the previous subsection, we will consider compact irreducible inner symmetric pairs (G, K) satisfying the maximality property. According to the results above in this section, the set $\mathcal{J} \cap \bar{C}$ and the cardinality $N = N(\mathcal{F}_{G/K})$ of the quotient $(\mathcal{J} \cap \bar{C})/\!\sim$ can be calculated from the following data: μ, λ and the symmetries of the extended Vogan diagram. All this information can be extracted from Table 7.3, where the resulting value of N is also shown.

In each case, we begin by indicating the corresponding orthogonal symmetric pair $(\mathfrak{g}, \mathfrak{k})$ and the possible values of p and ν. Then we specify the Hermitian or non-Hermitian character and the set $\mathcal{J} \cap \bar{C}$. If needed, we give a set of generators of $\mathrm{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$ (defined by means of symmetries of the extended Vogan diagram) and their action on (maybe only some elements of) $\mathcal{J} \cap \bar{C}$. Finally, we specify the value of N.

Type A III: $(\mathfrak{su}(p+1), \mathfrak{s}(\mathfrak{u}(\nu) \oplus \mathfrak{u}(p-\nu+1)))$, $\quad p \geq 3$, $\quad 2 \leq \nu \leq p-1$.

- Hermitian.
- $\mathcal{J} \cap \bar{C} = \{\pm h_\nu\} \cup \{-h_\nu + 2h_i : i = 1, \ldots, p; i \neq \nu\}$.
- Generators of $\mathrm{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$:
  - φ_1: $\alpha_i \leftrightarrow \alpha_{\nu-i}$ for all $i = 0, \ldots, \nu$, and $\alpha_i \leftrightarrow \alpha_{p+\nu-i+1}$ for all $i = \nu+1, \ldots, p$.
  - φ_2 (only if $2\nu = p+1$): $\alpha_i \leftrightarrow \alpha_{p-i+1}$ for $i = 1, \ldots, p$, and fixes α_0 and α_ν.
- Action on $\mathcal{J} \cap \bar{C}$:

- ◦ φ_1: $h_\nu \leftrightarrow -h_\nu$, $-h_\nu + 2h_i \leftrightarrow -h_\nu + 2h_{\nu-i}$ for $i = 1, \dots, \nu - 1$, and $-h_\nu + 2h_i \leftrightarrow -h_\nu + 2h_{p+\nu-i+1}$ for $i = \nu + 1, \dots, p$.
- ◦ φ_2 (only if $2\nu = p + 1$): $-h_\nu + 2h_i \leftrightarrow -h_\nu + 2h_{p-i+1}$ for $i \in \{1, \dots, p\} \setminus \{\nu\}$, and fixes $\pm h_\nu$.

- $N = 1 + \left[\frac{\nu}{2}\right] + \left[\frac{p-\nu+1}{2}\right]$ if $2\nu \neq p+1$, and $N = 1 + \left[\frac{\nu}{2}\right]$ if $2\nu = p+1$ (where $[\cdot]$ denotes the integer part of a real number).

Type B I: $(\mathfrak{so}(2p+1), \mathfrak{so}(2\nu) \oplus \mathfrak{so}(2p-2\nu+1))$, $\quad p \geq 3$.

- Hermitian if and only if $\nu = 1$.
- $\mathcal{J} \cap \bar{C} = \{\pm h_1\}$ if $\nu = 1$, and $\mathcal{J} \cap \bar{C} = \{-h_\nu, -h_\nu + 2h_1\}$ otherwise.
- Generator of $\operatorname{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$: φ interchanges $\alpha_0 \leftrightarrow \alpha_1$, and fixes α_i for all $i \geq 2$.
- Action on $\mathcal{J} \cap \bar{C}$: φ interchanges $-h_\nu \leftrightarrow -h_\nu + 2h_1$ if $\nu \neq 1$.
- $N = 1$.

By the maximality property, the rank-one case is of type **B I**, with $\nu = p \geq 1$. Then $N = 1$ holds trivially.

Type C I: $(\mathfrak{sp}(p), \mathfrak{u}(p))$, $\quad \nu = p \geq 2$.

- Hermitian.
- $\mathcal{J} \cap \bar{C} = \{\pm h_\nu\}$.
- $N = 1$.

Type C II: $(\mathfrak{sp}(p), \mathfrak{sp}(\nu) \oplus \mathfrak{sp}(p-\nu))$, $\quad p \geq 4$, $\quad 2 \leq \nu \leq p-2$.

- Non-Hermitian.
- $\mathcal{J} \cap \bar{C} = \{-h_\nu, -h_\nu + 2h_p\}$.
- Generator of $\operatorname{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$: φ (only if $2\nu = p$) interchanges $\alpha_i \leftrightarrow \alpha_{p-i}$ for all $i \in \{0, \dots, p\}$.
- Action on $\mathcal{J} \cap \bar{C}$: φ (only if $2\nu = p$) interchanges $-h_\nu \leftrightarrow -h_\nu + 2h_p$.
- $N = 1$ if $2\nu = p$, and $N = 2$ otherwise.

Type D I: $(\mathfrak{so}(2p), \mathfrak{so}(2\nu) \oplus \mathfrak{so}(2p-2\nu))$, $\quad p \geq 4$, $\quad \nu \leq p-2$.

- Hermitian if and only if $\nu = 1$.
- $\mathcal{J} \cap \bar{C} = \{\pm h_1, -h_1 + 2h_{p-1}, -h_1 + 2h_p\}$ if $\nu = 1$, and $\mathcal{J} \cap \bar{C} = \{-h_\nu, -h_\nu + 2h_1, -h_\nu + 2h_{p-1}, -h_\nu + 2h_p\}$ otherwise.
- Generators of $\operatorname{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$:

- φ_1: $\alpha_{p-1} \leftrightarrow \alpha_p$ and fixes α_i for all $i \in \{0, \dots, p-1\}$.
- φ_2: $\alpha_0 \leftrightarrow \alpha_1$ and fixes α_i for all $i \in \{2, \dots, p\}$.
- φ_3 (only if $2\nu = p$): $\alpha_i \leftrightarrow \alpha_{p-i}$ for all $i \in \{0, \dots, p\}$.

- Action on $\mathcal{J} \cap \bar{C}$:
  - φ_1: $-h_\nu + 2h_{p-1} \leftrightarrow -h_\nu + 2h_p$ and fixes the other elements.
  - φ_2: $-h_\nu \leftrightarrow -h_\nu + 2h_1$ and fixes the other elements, if $\nu \neq 1$.
  - φ_3 (only if $2\nu = p$): $-h_\nu \leftrightarrow -h_\nu + 2h_p$.
- $N = 1$ if $2\nu = p$, and $N = 2$ otherwise.

Type D III: $(\mathfrak{so}(2p), \mathfrak{u}(p))$, $\nu = p \geq 4$.

- Hermitian.
- $\mathcal{J} \cap \bar{C} = \{\pm h_p, -h_p + 2h_1, -h_p + 2h_{p-1}\}$.
- Generator of $\mathrm{Out}(\Delta_\mathfrak{g}, \Delta_\mathfrak{k})$: φ interchanges $\alpha_i \leftrightarrow \alpha_{p-i}$ for all $i \in \{0, \dots, p\}$.
- Action on $\mathcal{J} \cap \bar{C}$: φ interchanges $-h_p + 2h_1 \leftrightarrow -h_p + 2h_{p-1}$.
- $N = 2$.

Type E II: $(\mathfrak{e}_6, \mathfrak{su}(6) \oplus \mathfrak{su}(2))$, $\nu = 2$.

- Non-Hermitian.
- $\mathcal{J} \cap \bar{C} = \{-h_2, -h_2 + 2h_1, -h_2 + 2h_6\}$.
- Generator of $\mathrm{Out}(\Delta_\mathfrak{g}, \Delta_\mathfrak{k})$: φ switches $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$, and α_0, α_2, α_4 stay fixed.
- Action on $\mathcal{J} \cap \bar{C}$: φ interchanges $-h_2 + 2h_1 \leftrightarrow -h_2 + 2h_6$, and fixes $-h_2$.
- $N = 2$.

Type E III: $(\mathfrak{e}_6, \mathfrak{so}(10) \oplus \mathfrak{so}(2))$, $\nu = 6$.

- Hermitian.
- $\mathcal{J} \cap \bar{C} = \{\pm h_6, -h_6 + 2h_1\}$.
- $N = 2$.

Type E V: $(\mathfrak{e}_7, \mathfrak{su}(8))$, $\nu = 2$.

- Non-Hermitian.
- $\mathcal{J} \cap \bar{C} = \{-h_2, -h_2 + 2h_7\}$.
- Generator of $\mathrm{Out}(\Delta_\mathfrak{g}, \Delta_\mathfrak{k})$: φ switches $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$, $\alpha_0 \leftrightarrow \alpha_7$, and fixes α_2, α_4.
- Action on $\mathcal{J} \cap \bar{C}$: φ interchanges $-h_2 \leftrightarrow -h_2 + 2h_7$.
- $N = 1$.

Type E VI: $(\mathfrak{e}_7, \mathfrak{so}(12) \oplus \mathfrak{su}(2))$, $\nu = 1$.

- Non-Hermitian.
- $\operatorname{Out}(\Delta_{\mathfrak{g}}, \Delta_{\mathfrak{k}})$ is trivial.
- $\mathcal{J} \cap \bar{C} = \{-h_1, -h_1 + 2h_7\}$.
- $N = 2$.

Type E VII: $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{so}(2))$, $\nu = 7$.

- Hermitian.
- $\mathcal{J} \cap \bar{C} = \{\pm h_7\}$.
- $N = 1$.

Type E VIII: $(\mathfrak{e}_8, \mathfrak{so}(16))$, $\nu = 1$.

- Non-Hermitian.
- $\mathcal{J} \cap \bar{C} = \{-h_1\}$.
- $N = 1$.

Type E IX: $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}(2))$, $\nu = 8$.

- Non-Hermitian.
- $\mathcal{J} \cap \bar{C} = \{-h_8\}$.
- $N = 1$.

Type F I: $(\mathfrak{f}_4, \mathfrak{sp}(3) \oplus \mathfrak{su}(2))$, $\nu = 4$.

- Non-Hermitian.
- $\mathcal{J} \cap \bar{C} = \{-h_4\}$.
- $N = 1$.

Type G: $(\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$, $\nu = 2$.

- Non-Hermitian.
- $\mathcal{J} \cap \bar{C} = \{-h_2\}$.
- $N = 1$.

7.7 Projecting FKM-foliations

The purpose of this section is analogous to that of Section 7.6, but here we will deal with FKM-foliations instead of homogeneous polar foliations. Our goal will be to classify the complex structures preserving FKM-foliations under the hypothesis $m_1 \leq m_2$. For the notation concerning FKM-foliations, we refer the reader to §7.4.2.

In view of Theorem 7.22, among all homogeneous polar foliations, only those arisen from inner symmetric spaces descend to the corresponding complex projective spaces. However, the behaviour of FKM-foliations is different: all of them can be projected. The idea behind the proof of this fact is very simple and already appeared in the original paper [63, §6.2].

Theorem 7.30. *Each FKM-foliation $\mathcal{F}_{\mathcal{P}}$ admits a complex structure preserving $\mathcal{F}_{\mathcal{P}}$.*

Proof. Let $(P_0, \ldots, P_m)$ be a Clifford system defining $\mathcal{F}_{\mathcal{P}} \subset S^{2n+1}$. Let F be its Cartan-Münzner polynomial. Define $J = P_0 P_1$, which is a complex structure on $\mathbb{R}^{2n+2}$. Then

$$F(\cos(t)x + \sin(t)Jx) = F\left((\cos(t)P_1 + \sin(t)P_0)P_1 x\right) = F(P_1 x) = F(x)$$

for all $x \in \mathbb{R}^{2n+2}$, since F is invariant under $S(\mathcal{P})$. Hence, J preserves $\mathcal{F}_{\mathcal{P}}$. $\square$

Let $(P_1,\dots,P_m)$ be a Clifford system on $V=\mathbb{R}^{2n+2}$ defining $\mathcal{F}_\mathcal{P}$ and satisfying $m_1\le m_2$. In view of Theorems 7.10 and 7.12, there is an effective representation $\rho\colon K\to \mathrm{O}(V)$ such that $\rho(K)$ is the maximal connected group of automorphisms of $\mathcal{F}_\mathcal{P}$ preserving the leaves (in fact, $\rho(K)$ is the connected component of the identity of $\mathrm{Aut}(\mathcal{F}_\mathcal{P})$), $K=\mathrm{Spin}(m+1)\cdot H$ is a direct product modulo a finite subgroup and H is the connected component of the identity of the corresponding group in Theorem 7.11. The explicit description of ρ as a tensor product of the spin representation of $\mathrm{Spin}(m+1)$ and the standard representation of the classical group H is given in Theorem 7.12.

In this situation, the results of Section 7.5 are applicable. Thus, fix a maximal abelian subalgebra $\mathfrak{t}$ of the compact Lie algebra $\mathfrak{k}$. Let $\Delta_\mathfrak{k}$ be the set of roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$ and Δ_V the set of weights of $\rho_*^\mathbb{C}$. We have that $\mathfrak{k}=\mathfrak{so}(m+1)\oplus\mathfrak{h}$, where $\mathfrak{h}$ is equal to

$$\begin{aligned}
&\mathfrak{so}(k),\ \text{if } m\equiv 1,7\ (\mathrm{mod}\,8), \\
&\mathfrak{u}(k),\ \text{if } m\equiv 2,6\ (\mathrm{mod}\,8), &&\mathfrak{so}(k_+)\oplus\mathfrak{so}(k_-),\ \text{if } m\equiv 0\ (\mathrm{mod}\,8),\\
&\mathfrak{sp}(k),\ \text{if } m\equiv 3,5\ (\mathrm{mod}\,8), &&\mathfrak{sp}(k_+)\oplus\mathfrak{sp}(k_-),\ \text{if } m\equiv 4\ (\mathrm{mod}\,8).
\end{aligned}$$

We present now some well-known information about the roots of $\mathfrak{k}$ and the weights of $\rho_*^\mathbb{C}$. It can be obtained for example from [127, Chapter IX] (cf. [84, p. 683–685]).

Let p, q be the ranks of the Lie algebras $\mathfrak{so}(m+1)$ and $\mathfrak{h}\in\{\mathfrak{so}(k),\mathfrak{u}(k),\mathfrak{sp}(k)\}$, respectively, and $q_\pm$ the rank of $\mathfrak{so}(k_\pm)$ or $\mathfrak{sp}(k_\pm)$ as appropriate. Set $\mathfrak{t}=\mathfrak{t}_s\oplus\mathfrak{t}_\mathfrak{h}$, where $\mathfrak{t}_s$ and $\mathfrak{t}_\mathfrak{h}$ are maximal abelian subalgebras of $\mathfrak{so}(m+1)$ and $\mathfrak{h}$, respectively. Some of the following assertions may need a rescaling of the inner product on the irreducible factors of $\mathfrak{k}$.

Let $\{\alpha_1^s,\dots,\alpha_p^s\}$ and $\{\alpha_1,\dots,\alpha_q\}$ be systems of simple roots for $\mathfrak{so}(m+1)$ and $\mathfrak{h}\in\{\mathfrak{so}(k),\mathfrak{u}(k),\mathfrak{sp}(k)\}$, respectively. There is an orthonormal basis $\{\omega_1^s,\dots,\omega_p^s\}$ of $\mathfrak{t}_s^*$ such that $\alpha_i^s=\omega_i^s-\omega_{i+1}^s$ for $i=1,\dots,p-1$, and $\alpha_p^s=\omega_p^s$ if m is even, or $\alpha_p^s=\omega_{p-1}^s+\omega_p^s$ if m is odd. The weights of the spin representation ρ_s of $\mathfrak{so}(m+1)$ are $\frac{1}{2}(\pm\omega_1^s\pm\cdots\pm\omega_p^s)$.

If $\mathfrak{h}=\mathfrak{so}(k)$, there is an orthonormal basis $\{\omega_1,\dots,\omega_q\}$ of $\mathfrak{t}_\mathfrak{h}^*$ such that $\alpha_i=\omega_i-\omega_{i+1}$ for $i=1,\dots,q-1$, and $\alpha_q=\omega_q$ if k is odd, or $\alpha_q=\omega_{q-1}+\omega_q$ if k is even. The weights of the standard representation $\rho_{\mathfrak{so}(k)}$ of $\mathfrak{so}(k)$ are $\pm\omega_1,\dots,\pm\omega_q$ if k is even, or $\pm\omega_1,\dots,\pm\omega_q,0$ if k is odd. Then $\rho_*^\mathbb{C}\cong\rho_s\otimes_\mathbb{C}\rho_{\mathfrak{so}(k)}$, so its weights are all possible sums of weights of ρ_s with weights of $\rho_{\mathfrak{so}(k)}$.

If $\mathfrak{h}=\mathfrak{u}(k)$, then $k=q$. There exists an orthonormal basis $\{\omega_1,\dots,\omega_k\}$ of $\mathfrak{t}_\mathfrak{h}^*$ such that $\alpha_i=\omega_i-\omega_{i+1}$ for $i=1,\dots,k-1$, and such that the weights of the standard representation $\rho_{\mathfrak{u}(k)}$ of $\mathfrak{u}(k)$ are $\omega_1,\dots,\omega_k$. Let $(\cdot)^\mathbb{R}$ denote realification. Since $\rho_*\cong\eta^\mathbb{R}$ with $\eta=\rho_s\otimes_\mathbb{C}\rho_{\mathfrak{u}(k)}$, then $\rho_*^\mathbb{C}\cong(\eta^\mathbb{R})^\mathbb{C}\cong\eta\oplus\bar\eta$, where $\bar\eta$ stands for the complex conjugate or contragredient representation of η. The weights of $\rho_*^\mathbb{C}$ are then all possible sums of weights of ρ_s with weights of $\rho_{\mathfrak{u}(k)}$, and the negatives of these sums.

If $\mathfrak{h}=\mathfrak{sp}(k)$, then $k=q$ and there is an orthonormal basis $\{\omega_1,\dots,\omega_k\}$ of $\mathfrak{t}_\mathfrak{h}^*$ such that $\alpha_i=\omega_i-\omega_{i+1}$ for $i=1,\dots,k-1$, and $\alpha_k=2\omega_k$. The weights of the standard representation $\rho_{\mathfrak{sp}(k)}$ are $\pm\omega_1,\dots,\pm\omega_k$. In this case, $\rho_*^\mathbb{C}$ is equivalent to $\rho_s\otimes_\mathbb{C}\rho_{\mathfrak{sp}(k)}$, so its weights are the sums of weights of ρ_s with weights of $\rho_{\mathfrak{sp}(k)}$.

If $\mathfrak{h} \in \{\mathfrak{so}(k_+) \oplus \mathfrak{so}(k_-), \mathfrak{sp}(k_+) \oplus \mathfrak{sp}(k_-)\}$, one can consider a basis $\{\omega_1^+, \dots, \omega_{q_+}^+\} \cup \{\omega_1^-, \dots, \omega_{q_-}^-\}$ of $\mathfrak{t}_{\mathfrak{h}}^*$ so that one can express the roots of $\mathfrak{h}$ and the weights of $\rho_{\mathfrak{so}(k_\pm)}$ or $\rho_{\mathfrak{sp}(k_\pm)}$ in terms of the $\omega_i^\pm$ in a completely analogous way as above. Moreover, $\rho_*^{\mathbb{C}}$ is isomorphic to $(\rho_s \otimes_{\mathbb{C}} \rho_{\mathfrak{so}(k_+)}) \oplus (\rho_s \otimes_{\mathbb{C}} \rho_{\mathfrak{so}(k_-)})$ or to $(\rho_s \otimes_{\mathbb{C}} \rho_{\mathfrak{sp}(k_+)}) \oplus (\rho_s \otimes_{\mathbb{C}} \rho_{\mathfrak{sp}(k_-)})$ accordingly.

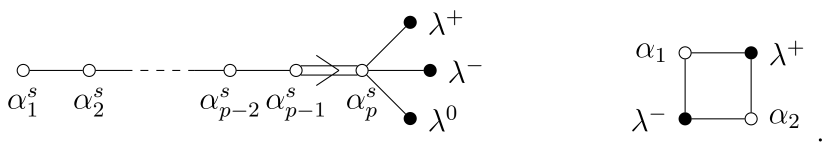
Let us consider the bases of $\mathfrak{t}$ dual to the bases of $\mathfrak{t}^*$ that we have defined. Let them be $\{e_1^s, \dots, e_p^s\} \cup \{e_1, \dots, e_q\}$ or $\{e_1^s, \dots, e_p^s\} \cup \{e_1^+, \dots, e_{q_+}^+\} \cup \{e_1^-, \dots, e_{q_-}^-\}$ depending on $\mathfrak{h}$. By definition, $\omega_i^s(e_j^s) = \delta_{ij}$, $\omega_i(e_j) = \delta_{ij}$ and $\omega_i^\pm(e_j^\pm) = \delta_{ij}$.

It follows from the above discussion that the lowest weights of $\rho_*^{\mathbb{C}}$ are:

- If $m \equiv 0 \,(\mathrm{mod}\, 8)$: $\lambda^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_p^s) - \omega_1^\pm$ if $k_\pm \geq 2$, and also $\mu^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_p^s) + \omega_1^\pm$ if $k_\pm = 2$, or $\lambda^0 = -\frac{1}{2}(\omega_1^s + \dots + \omega_p^s)$ if $k_\pm = 1$.
- If $m \equiv 1, 7 \,(\mathrm{mod}\, 8)$: $\lambda^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_{p-1}^s \pm \omega_p^s) - \omega_1$ if $k \geq 3$, while $\lambda^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_{p-1}^s \pm \omega_p^s) - \omega_1$ and $\mu^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_{p-1}^s \pm \omega_p^s) + \omega_1$ if $k = 2$, or $\lambda^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_{p-1}^s \pm \omega_p^s)$ if $k = 1$.
- If $m \equiv 2, 6 \,(\mathrm{mod}\, 8)$: $\lambda^+ = -\frac{1}{2}(\omega_1^s + \dots + \omega_p^s) - \omega_1$ and $\lambda^- = -\frac{1}{2}(\omega_1^s + \dots + \omega_p^s) + \omega_k$.
- If $m \equiv 3, 5 \,(\mathrm{mod}\, 8)$: $\lambda^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_{p-1}^s \pm \omega_p^s) - \omega_1$.
- If $m \equiv 4 \,(\mathrm{mod}\, 8)$: $\lambda^\pm = -\frac{1}{2}(\omega_1^s + \dots + \omega_p^s) - \omega_1^\pm$ if $k_\pm \neq 0$.

At this point, it is probably convenient to have in mind how the lowest weight diagrams associated with FKM-foliations look like. These can be obtained easily from the information above. We have included a generic picture of them in Table 7.4.

Remark 7.31. As follows from the above description, the number of lowest weights associated to FKM-foliations satisfying $m_1 \leq m_2$ may vary from one to four depending on m, k and $k_\pm$. Although generically the number is one or two and the lowest weight diagram takes the corresponding form shown in Table 7.4, for small values (i.e. up to 2) of k or $k_\pm$ there can be three or four lowest weights, and the diagram may adopt a somewhat different form (this happens also if $m = 1$). Just to show a pair of examples: the diagrams of the cases $m \equiv 0 \,(\mathrm{mod}\, 8)$, $k_+ = 2$, $k_- = 1$ and $m = 1$, $k = 4$ are, respectively



These pecularities should be taken into account in what follows.

The last ingredient to address the classification is the following result.

Theorem 7.32. *Let $\mathcal{F}$ be an FKM-foliation satisfying $m_1 \leq m_2$. Then $\mathrm{Out}_{\mathcal{F}}^{\pm}(\Delta_{\mathfrak{k}}, \Delta_V)$ is isomorphic to the group of automorphisms of the lowest weight diagram of $\mathcal{F}$. The correspondence is the natural one given in Proposition 7.21.*

Proof. In view of Proposition 7.21, it suffices to prove that every symmetry of the lowest weight diagram induces an element in $\mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$. We show this by cases, depending on the shape of the diagram. For the sake of clarity, we do the proof for diagrams with generic shape. Only minor changes are needed to deal with low values of k, $k_{\pm}$.

If $m = 2p - 1$ is odd, then there is an automorphism σ of the diagram that switches both lowest weights, also the roots α^s_{p-1} and α^s_p, and fixes the other roots. We can assume that $\mathfrak{t}_s = \mathrm{span}\{P_0P_1, P_2P_3, \dots, P_{2p-2}P_{2p-1}\}$ and $e^s_i = P_{2i-2}P_{2i-1}$, $1 \le i \le p$, where $(P_0, \dots, P_{2p-1})$ is a Clifford system defining $\mathcal{F}$. Then $\mathrm{Ad}(P_{2p-1}) \in \mathrm{Aut}(\mathfrak{so}(m+1))$ acts as a reflection on $\mathfrak{t}_s$: it fixes e^s_i for $1 \le i \le p-1$ and changes the sign of e^s_p. Then the trivial extension of $\mathrm{Ad}(P_{2p-1})|_{\mathfrak{t}_s}$ to $\mathfrak{t}$ belongs to $\mathrm{Aut}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ by Theorem 7.10, and since it is precisely induced by σ, it also belongs to $\mathrm{Out}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$.

If $m \equiv 1, 7 \,(\mathrm{mod}\, 8)$ and $k = 2q$ is even, there is a symmetry φ of the diagram that fixes all roots and lowest weights, except α_{q-1} and α_q, which are interchanged. It is not restrictive to assume that each generator e_i of $\mathfrak{t}_{\mathfrak{h}}$ is the matrix of $\mathfrak{so}(k)$ with 1 in the position $(2i, 2i-1)$, -1 in the position $(2i-1, 2i)$, and all other entries vanish. Then the diagonal matrix $A \in \mathrm{O}(k)$ with entries $(1, \dots, 1, -1)$ satisfies that $\mathrm{Ad}(A)e_i = e_i$, $1 \le i \le q-1$, and $\mathrm{Ad}(A)e_q = -e_q$. By Theorem 7.10, the trivial extension of $\mathrm{Ad}(A)|_{\mathfrak{t}_{\mathfrak{h}}}$ to $\mathfrak{t}$, which is precisely induced by φ, belongs to $\mathrm{Out}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$. The cases $m \equiv 0 \,(\mathrm{mod}\, 8)$, with $k_{\pm}$ even, can be tackled with a similar argument.

If $m \equiv 2, 6 \,(\mathrm{mod}\, 8)$, the diagram has only one symmetry σ, which fixes all roots α^s_j, interchanges both lowest weights and switches the roots α_i and α_{q-i}, for $1 \le i \le q-1$ (equivalently $\sigma(e_i) = -e_{q+1-i}$, $1 \le i \le q$). The opposition element *op* of the Weyl group of $\mathfrak{u}(k)$ sends each e_i to e_{q+1-i} (see [124, p. 88]). Denoting also by *op* its trivial extension from $\mathfrak{t}_{\mathfrak{h}}$ to $\mathfrak{t}$, we have that $op \in W(\Delta_{\mathfrak{k}}) \subset \mathrm{Aut}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$, so $\sigma = -op \in \mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$.

Let now $m \equiv 0 \,(\mathrm{mod}\, 4)$ with $k_+ = k_-$. After a suitable choice of $\mathfrak{t}$, the automorphism $\tau \in \mathrm{Aut}(\mathcal{F})$ defined in Proposition 7.13 determines an element $\varphi_\tau \in \mathrm{Out}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$. This φ_τ comes induced by the symmetry of the diagram that switches both lowest weights, interchanges α^+_i with α^-_i for $1 \le i \le q_+ = q_-$ and fixes the other roots α^s_j.

Finally, note that the automorphisms of the lowest weight diagrams that we have considered generate the symmetry group of such diagrams, from where the result follows. □

7.7.1 Classification of the complex structures

We can now address the case-by-case classification of the complex structures preserving FKM-foliations $\mathcal{F} = \mathcal{F}_{\mathcal{P}}$ satisfying $m_1 \le m_2$. Propositions 7.15, 7.20 and Theorem 7.32 make this work straightforward. For each case, we provide the following information: a set of simple roots for $\mathfrak{k}$ and the set of weights Δ_V (in accordance with the notation introduced above in this section); the algebraic conditions for a generic element $T = \sum_{i=1}^p x^s_i e^s_i + \sum_{j=1}^q x_j e_j \in \mathfrak{t}$ (or $T = \sum_{i=1}^p x^s_i e^s_i + \sum_{j=1}^{q_+} x^+_j e^+_j + \sum_{j=1}^{q_-} x^-_j e^-_j$ if $m \equiv 0 \,(\mathrm{mod}\, 4)$) to belong to $\mathcal{J} \cap \bar{C}$; the set $\mathcal{J} \cap \bar{C}$; a set of generators of $\mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$ (if needed); the action of these generators on $\mathcal{J} \cap \bar{C}$ (if needed); and, finally, the value $N = N(\mathcal{F}_{\mathcal{P}})$ of different complex structures up to congruence of the corresponding projected foliations on the complex projective space.

Type $m \equiv 0 \pmod 8$ **with** $k_+ = 2q_+$**,** $k_- = 2q_-$ **even**

- Simple roots: $\omega_1^s - \omega_2^s, \ldots, \omega_{p-1}^s - \omega_p^s, \omega_p^s$; $\omega_1^+ - \omega_2^+, \ldots, \omega_{q_+-1}^+ - \omega_{q_+}, \omega_{q_+-1}^+ + \omega_{q_+}$; $\omega_1^- - \omega_2^-, \ldots, \omega_{q_--1}^- - \omega_{q_-}, \omega_{q_--1}^- + \omega_{q_-}$.
- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^+$ for $j = 1, \ldots, q_+$, and $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^-$ for $j = 1, \ldots, q_-$.
- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_p^s \geq 0$, $x_1^+ \geq \cdots \geq x_{q_+-1}^+ \geq |x_{q_+}^+|$, $x_1^- \geq \cdots \geq x_{q_--1}^- \geq |x_{q_-}^-|$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j^\pm \in \{\pm 1\}$ for all combination of signs and all possible j.
- $\mathcal{J} \cap \bar{C} = \{2e_1^s, \sum_{j=1}^{q_+} e_j^+ + \sum_{j=1}^{q_-} e_j^-, -e_{q_-}^- + \sum_{j=1}^{q_+} e_j^+ + \sum_{j=1}^{q_--1} e_j^-, -e_{q_+}^+ + \sum_{j=1}^{q_+-1} e_j^+ + \sum_{j=1}^{q_-} e_j^-, -e_{q_+}^+ - e_{q_-}^- + \sum_{j=1}^{q_+-1} e_j^+ + \sum_{j=1}^{q_--1} e_j^-\}$.
- Generators of $\mathrm{Out}_{\mathcal{F}}^{\pm}(\Delta_{\mathfrak{k}}, \Delta_V)$:
  - $\varphi^\pm$ (only if $k_\pm \geq 2$): $e_{q_\pm}^\pm \leftrightarrow -e_{q_\pm}^\pm$, and fixes the other e_j^+, e_j^-, and e_i^s.
  - τ (only if $k_+ = k_-$): $e_j^+ \leftrightarrow e_j^-$, and fixes the e_i^s.
- Action on $\mathcal{J} \cap \bar{C}$: the group generated by φ^+, φ^-, and τ (some of these may not exist) fixes $2e_1^s$ and acts transitively on the other elements.
- $N = 2$.

Type $m \equiv 0 \pmod 8$ **with** $k_+ = 2q_+$ **even,** $k_- = 2q_- + 1$ **odd**

- Simple roots: $\omega_1^s - \omega_2^s, \ldots, \omega_{p-1}^s - \omega_p^s, \omega_p^s$; $\omega_1^+ - \omega_2^+, \ldots, \omega_{q_+-1}^+ - \omega_{q_+}, \omega_{q_+-1}^+ + \omega_{q_+}$; $\omega_1^- - \omega_2^-, \ldots, \omega_{q_--1}^- - \omega_{q_-}, \omega_{q_-}$.
- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^+$ for $j = 1, \ldots, q_+$, $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^-$ for $j = 1, \ldots, q_-$, and $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s)$.
- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_p^s \geq 0$, $x_1^+ \geq \cdots \geq x_{q_+-1}^+ \geq |x_{q_+}^+|$, $x_1^- \geq \cdots \geq x_{q_-}^- \geq 0$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \in \{\pm 1\}$, $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j^\pm \in \{\pm 1\}$ for all combination of signs and all possible j.
- $\mathcal{J} \cap \bar{C} = \{2e_1^s\}$, and hence $N = 1$.

Type $m \equiv 0 \pmod 8$ **with** $k_+ = 2q_+ + 1$**,** $k_- = 2q_- + 1$ **odd**

- Simple roots: $\omega_1^s - \omega_2^s, \ldots, \omega_{p-1}^s - \omega_p^s, \omega_p^s$; $\omega_1^+ - \omega_2^+, \ldots, \omega_{q_+-1}^+ - \omega_{q_+}, \omega_{q_+}$; $\omega_1^- - \omega_2^-, \ldots, \omega_{q_--1}^- - \omega_{q_-}, \omega_{q_-}$.
- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^+$ for $j = 1, \ldots, q_+$, $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^-$ for $j = 1, \ldots, q_-$, and $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s)$.

- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_p^s \geq 0$, $x_1^+ \geq \cdots \geq x_{q_+}^+ \geq 0$, $x_1^- \geq \cdots \geq x_{q_-}^- \geq 0$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \in \{\pm 1\}$, $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j^\pm \in \{\pm 1\}$ for all combination of signs and all possible j.

- $\mathcal{J} \cap \bar{C} = \{2e_1^s\}$, and hence $N = 1$.

Type $m \equiv 1, 7 \pmod 8$ **with** $k = 2q$ **even**

- Simple roots: $\omega_1^s - \omega_2^s, \ldots, \omega_{p-1}^s - \omega_p^s, \omega_{p-1}^s + \omega_p^s$; $\omega_1 - \omega_2, \ldots, \omega_{q-1} - \omega_q, \omega_{q-1} + \omega_q$.

- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j$, for all $j = 1, \ldots, q$.

- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_{p-1}^s \geq \left|x_p^s\right|$, $x_1 \geq \cdots \geq x_{q-1} \geq |x_q|$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j \in \{\pm 1\}$ for all combination of signs and all $j = 1, \ldots, q$.

- $\mathcal{J} \cap \bar{C} = \{2e_1^s, e_1 + \cdots + e_q, e_1 + \cdots + e_{q-1} - e_q\}$, and also $-2e_1^s$ if $m = 1$.

- Generators of $\mathrm{Out}_{\mathcal{F}}^{\pm}(\Delta_{\mathfrak{k}}, \Delta_V)$:

  - σ: $e_p^s \leftrightarrow -e_p^s$, and the other e_i^s and e_j stay fixed.
  - φ: $e_q \leftrightarrow -e_q$, and the other e_i^s and e_j stay fixed.

- Action on $\mathcal{J} \cap \bar{C}$: σ fixes all elements if $m \neq 1$ (if $m = 1$, then $2e_1^s \leftrightarrow -2e_1^s$), while φ interchanges $e_1 + \cdots + e_q \leftrightarrow e_1 + \cdots + e_{q-1} - e_q$ and fixes the other elements.

- $N = 2$.

Type $m \equiv 1, 7 \pmod 8$ **with** $k = 2q + 1$ **odd**

- Simple roots: $\omega_1^s - \omega_2^s, \ldots, \omega_{p-1}^s - \omega_p^s, \omega_{p-1}^s + \omega_p^s$; $\omega_1 - \omega_2, \ldots, \omega_{q-1} - \omega_q, \omega_q$.

- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j$, for all $j = 1, \ldots, q$, and $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s)$.

- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_{p-1}^s \geq \left|x_p^s\right|$, $x_1 \geq \cdots \geq x_q \geq 0$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \in \{\pm 1\}$, $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j \in \{\pm 1\}$ for all combination of signs and all $j = 1, \ldots, q$.

- $\mathcal{J} \cap \bar{C} = \{2e_1^s\}$ if $m \neq 1$, or $\mathcal{J} \cap \bar{C} = \{\pm 2e_1^s\}$ if $m = 1$.

- $N = 1$.

Type $m \equiv 2, 6 \pmod 8$

- Simple roots: $\omega_1^s - \omega_2^s, \dots, \omega_{p-1}^s - \omega_p^s, \omega_p^s$; $\omega_1 - \omega_2, \dots, \omega_{k-1} - \omega_k$.
- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j$, for all $j = 1, \dots, k$.
- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_p^s \geq 0$, $x_1 \geq \cdots \geq x_k$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j \in \{\pm 1\}$ for all combination of signs and all $j = 1, \dots, k$.
- $\mathcal{J} \cap \bar{C} = \{2e_1^s\} \cup \{\sum_{j=1}^k \epsilon_j e_j : \epsilon_j = \pm 1 \text{ for all } j, \epsilon_1 \geq \cdots \geq \epsilon_k\}$.
- Generator of $\mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$: σ switches $e_j \leftrightarrow -e_{k+1-j}$ for all j, and fixes the e_i^s.
- Action on $\mathcal{J} \cap \bar{C}$: σ switches $\sum_{j=1}^k \epsilon_j e_j \leftrightarrow -\sum_{j=1}^k \epsilon_{k-j+1} e_j$, and fixes $2e_1^s$.
- $N = 2 + \left[\frac{k}{2}\right]$.

Type $m \equiv 3, 5 \pmod 8$

- Simple roots: $\omega_1^s - \omega_2^s, \dots, \omega_{p-1}^s - \omega_p^s, \omega_{p-1}^s + \omega_p^s$; $\omega_1 - \omega_2, \dots, \omega_{k-1} - \omega_k, 2\omega_k$.
- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j$, for all $j = 1, \dots, k$.
- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_{p-1}^s \geq \left|x_p^s\right|$, $x_1 \geq \cdots \geq x_k \geq 0$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j \in \{\pm 1\}$ for all combination of signs and all $j = 1, \dots, k$.
- $\mathcal{J} \cap \bar{C} = \{2e_1^s, e_1 + \cdots + e_k\}$.
- Generator of $\mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$: σ switches $e_p^s \leftrightarrow -e_p^s$, and fixes the other e_i^s and e_j.
- Action on $\mathcal{J} \cap \bar{C}$: both elements are fixed by σ.
- $N = 2$.

Type $m \equiv 4 \pmod 8$

- Simple roots: $\omega_1^s - \omega_2^s, \dots, \omega_{p-1}^s - \omega_p^s, \omega_p^s$; $\omega_1^+ - \omega_2^+, \dots, \omega_{k_+-1}^+ - \omega_{k_+}^+, 2\omega_{k_+}^+$; $\omega_1^- - \omega_2^-, \dots, \omega_{k_--1}^- - \omega_{k_-}^-, 2\omega_{k_-}^-$.
- Weights: $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^+$ for $j = 1, \dots, k_+$, and $\frac{1}{2}(\pm\omega_1^s \pm \cdots \pm \omega_p^s) \pm \omega_j^-$ for $j = 1, \dots, k_-$.
- Conditions for $\mathcal{J} \cap \bar{C}$: $x_1^s \geq \cdots \geq x_p^s \geq 0$, $x_1^+ \geq \cdots \geq x_{k_+}^+ \geq 0$, $x_1^- \geq \cdots \geq x_{k_-}^- \geq 0$, and $\frac{1}{2}(\pm x_1^s \pm \cdots \pm x_p^s) \pm x_j^{\pm} \in \{\pm 1\}$ for all combination of signs and all possible j.
- $\mathcal{J} \cap \bar{C} = \{2e_1^s, e_1^+ + \cdots + e_{k_+}^+ + e_1^- + \cdots + e_{k_-}^-\}$.

- Generator of $\mathrm{Out}^{\pm}_{\mathcal{F}}(\Delta_{\mathfrak{k}}, \Delta_V)$, only if $k_+ = k_-$: τ switches $e_j^+ \leftrightarrow e_j^-$ for all j, and fixes the e_i^s.
- Action on $\mathcal{J} \cap \bar{C}$: both elements are fixed by τ.
- $N = 2$.

7.8 Inhomogeneous isoparametric foliations

Here we analyze when the projection of an isoparametric foliation to the complex projective space gives rise to a homogeneous foliation. Some curious consequences are also derived.

Let us start with an elementary consideration.

Remark 7.33. The pullback $\mathcal{F} = \pi^{-1}\mathcal{G}$ of any homogeneous foliation $\mathcal{G}$ on $\mathbb{C}P^n$ under the Hopf map π is homogeneous. Indeed, consider the maximal connected subgroup $\widetilde{K}$ of $\mathrm{U}(n+1)$ preserving the leaves of $\mathcal{G}$; by homogeneity, the orbit foliation of $\widetilde{K}$ on $\mathbb{C}P^n$ is $\mathcal{G}$. It follows that the orbit foliation of the action of $\widetilde{K}$ on $S^{2n+1} \subset \mathbb{C}^{n+1}$ is $\mathcal{F}$.

Therefore, every homogeneous isoparametric foliation on $\mathbb{C}P^n$ must be the projection of a homogeneous polar foliation. Then our aim reduces to deciding when the projection to $\mathbb{C}P^n$ of a homogeneous polar foliation $\mathcal{F}_{G/K}$ on S^{2n+1} is homogeneous. The following subtle improvement of [123, Theorem 3.1] gives us the solution.

Theorem 7.34. *Let (G, K) be a compact inner symmetric pair that satisfies the maximality property, $(G, K) = \Pi_{i=1}^r (G_i, K_i)$ its decomposition in irreducible factors and $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ the Cartan decomposition of G_i/K_i. Let $J = \mathrm{ad}(X)|_{\mathfrak{p}}$ be a complex structure on $\mathfrak{p} = \bigoplus_{i=1}^r \mathfrak{p}_i$ that preserves the foliation $\mathcal{F}_{G/K}$ and put $X = X_1 + \ldots + X_r$, with $X_i \in \mathfrak{k}_i$. Then the following conditions are equivalent:*

(i) *The projection of $\mathcal{F}_{G/K}$ to the complex projective space $\mathbb{C}P^n$ determined by J is a homogeneous foliation.*

(ii) *The irreducible factors G_i/K_i of G/K are Hermitian or of rank one and, for each Hermitian irreducible factor G_i/K_i, X_i belongs to the center of $\mathfrak{k}_i$.*

Proof. First assume (ii). If an irreducible factor (G_i, K_i) is Hermitian and X_i belongs to the center $Z(\mathfrak{k}_i)$ of $\mathfrak{k}_i$, then the adjoint K_i-action on $\mathfrak{p}_i$ commutes with J, that is, $\mathrm{Ad}(K_i)|_{\mathfrak{p}_i}$ consists of unitary transformations with respect to J. The irreducible factors of rank one are of the type $(\mathrm{SO}(2p_i+1), \mathrm{SO}(2p_i))$, with $2p_i = \dim \mathfrak{p}_i$. For these factors, the group $\mathrm{U}(p_i)$ of unitary transformations with respect to $\mathrm{ad}(X_i)|_{\mathfrak{p}_i}$ acts on $\mathfrak{p}_i$ with the same orbits as $\mathrm{SO}(2p_i)$. It follows that there exists a group $\widetilde{K}$ of unitary transformations with respect to J that acts on $\mathfrak{p}$ with the same orbits as $\mathrm{Ad}(K)|_{\mathfrak{p}}$. Therefore $\widetilde{K}$ induces an action on $\mathbb{C}P^n$ whose orbits coincide with those of the projection of $\mathcal{F}_{G/K}$.

Assume now (i). Then, there exists a group K' acting polarly on $\mathbb{C}P^n$ and with the leaves of the projection of $\mathcal{F}_{G/K}$ as orbits; the existence of sections intersecting all orbits is a consequence of the polarity of the s-representation of G/K (see Proposition 7.5). It was shown in [123, Theorem 3.1] that there exists a connected group $\widetilde{K}$ acting on $\mathfrak{p}$ effectively, unitarily with respect to J and polarly, and such that the projection of its orbits yields the orbits of K'. Hence, $\widetilde{K}$ and $\mathrm{Ad}(K)|_{\mathfrak{p}}$ act on $\mathfrak{p}$ with the same orbits. By maximality, we can identify $\widetilde{K}$ with a subgroup of $\mathrm{Ad}(K)|_{\mathfrak{p}}$.

Let us suppose that G/K is irreducible. Take a subgroup H of K such that $\mathrm{Ad}(H)|_{\mathfrak{p}} = \widetilde{K}$. Then [123, Lemma 3.2] implies that G/K is Hermitian or $(G,K) = (\mathrm{SO}(2p+1), \mathrm{SO}(2p))$. If G/K is Hermitian, then either G/K is of rank greater than one or $G = \mathrm{SO}(3)$, $K = \mathrm{SO}(2)$; in this last case, $X \in \mathfrak{k} = Z(\mathfrak{k})$. Assume that G/K is Hermitian of rank greater than one. Since $\widetilde{K} = \mathrm{Ad}(H)|_{\mathfrak{p}}$ acts on $\mathfrak{p}$ by unitary transformations with respect to $J = \mathrm{ad}(X)|_{\mathfrak{p}}$, and since the s-representation of (G,K) is effective, we have that $X \in Z(\mathfrak{h})$. Then, the center of $\mathfrak{h}$ cannot be trivial. The main theorem of [59] implies that $H = K$ unless:

- $G = \mathrm{SO}(9)$, $K = \mathrm{SO}(2) \times \mathrm{SO}(7)$, $H = \mathrm{SO}(2) \times \mathrm{G}_2$, or
- $G = \mathrm{SO}(10)$, $K = \mathrm{SO}(2) \times \mathrm{SO}(8)$, $H = \mathrm{SO}(2) \times \mathrm{Spin}(7)$.

In both cases, $\dim Z(\mathfrak{h}) = 1$ and $Z(\mathfrak{h}) = Z(\mathfrak{k})$. In any case, $X \in Z(\mathfrak{h}) = Z(\mathfrak{k})$.

Assume now that G/K is reducible. Then $\mathrm{Ad}(K)|_{\mathfrak{p}}$ (and hence $\widetilde{K}$) acts irreducibly on each $\mathfrak{p}_i$, $i = 1, \dots, r$; by [42, Theorem 4] each one of these actions is polar. For each i, let $\widetilde{K}_i$ be a quotient of $\widetilde{K}$ that acts irreducibly, polarly and effectively on $\mathfrak{p}_i$, and with the same orbits as the action of $\widetilde{K}$ on $\mathfrak{p}_i$. The actions of $\widetilde{K}_i$ and of $\mathrm{Ad}(K_i)|_{\mathfrak{p}_i}$ on $\mathfrak{p}_i$ have the same orbits; by maximality we can find a subgroup H_i of K_i such that $\mathrm{Ad}(H_i)|_{\mathfrak{p}_i} = \widetilde{K}_i$.

Every $\mathrm{Ad}(H_i)|_{\mathfrak{p}_i}$ acts unitarily on $\mathfrak{p}_i$ with respect to the complex structure $\mathrm{ad}(X_i)|_{\mathfrak{p}_i}$, since $\widetilde{K}$ acts unitarily on $\mathfrak{p}$ with respect to J. We can then apply the argument above for the irreducible case. We obtain that for each $i \in \{1, \dots, r\}$, G_i/K_i is Hermitian with $X_i \in Z(\mathfrak{k}_i)$, or $(G_i, K_i) = (\mathrm{SO}(2p_i+1), \mathrm{SO}(2p_i))$. □

We now obtain some consequences of this result. The first one is straightforward.

Corollary 7.35. *Let (G,K) be an irreducible compact inner symmetric pair of rank greater than one satisfying the maximality property and let $N(\mathcal{F}_{G/K})$ be as in §7.6.3 (see Table 7.3 for its concrete value depending on (G,K)).*

Then, among the $N(\mathcal{F}_{G/K})$ noncongruent irreducible isoparametric foliations of the complex projective space obtained by projecting $\mathcal{F}_{G/K}$, exactly $N(\mathcal{F}_{G/K}) - 1$ of them are inhomogeneous if G/K is Hermitian, whereas all of them are inhomogeneous if G/K is non-Hermitian.

The following result focuses on the existence of inhomogeneous isoparametric foliations depending on the dimension on the ambient complex projective space. Although Theorem 7.36(i) has been recently proved in [66, Theorem 1.1] by Ge, Tang and Yan, for the sake of completeness we include here a slightly different proof. The other claims in Theorem 7.36 are new.

Theorem 7.36. *We have:*

(i) $\mathbb{C}P^n$ *admits an inhomogeneous isoparametric foliation of codimension one if and only if* n *is an odd number greater or equal than* 3.

(ii) *Let* $q \in \mathbb{N}$, $q \geq 2$. *Then* $\mathbb{C}P^n$ *admits an irreducible inhomogeneous isoparametric foliation of codimension* q *if and only if* $(q+1)^2 \leq 2(n+1)$ *and* $q+1$ *divides* $2(n+1)$.

In particular, every irreducible isoparametric foliation on $\mathbb{C}P^n$ *is homogeneous if and only if* $n+1$ *is a prime number.*

Proof. We start with the necessity of (i). Clearly $\mathbb{C}P^1 \cong S^2$ only admits homogeneous isoparametric foliations. Let n be even, $\mathcal{F} \subset S^{2n+1}$ an isoparametric foliation of codimension one and M an arbitrary hypersurface of $\mathcal{F}$ with $g \in \{1,2,3,4,6\}$ principal curvatures with multiplicities $m_1, \ldots, m_g$. Recall that $m_i = m_{i+2}$ (indices modulo 2).

If $g \in \{1,3\}$, a standard argument involving the Coxeter group of $\mathcal{F}$ (see for example [146, p. 359]) implies that a generic M is not invariant under the antipodal map, so M is not foliated by Hopf circles. If $g = 2$, $\mathcal{F}$ is the orbit foliation of the s-representation of a product of two spheres, so its projection to $\mathbb{C}P^n$ is homogeneous, according to Theorem 7.34.

If $g = 4$ then $m_1 + m_2 = n$ is even. Hence a result of Abresch [1] implies that either $\min\{m_1, m_2\} = 1$ or $m_1 = m_2 = 2$. If $g = 6$, then again by [1] we have $m_1 = m_2 \in \{1,2\}$; since $3(m_1 + m_2) = 2n$ in this case, $m_1 = m_2 = 1$ is impossible.

If $g = 4$ and $\min\{m_1, m_2\} = 1$, according to Takagi's result [134], $\mathcal{F}$ is the orbit foliation of the s-representation of the symmetric pair **B I**, with $\nu = 1$. By virtue of Corollary 7.35, the projection of $\mathcal{F}$ onto $\mathbb{C}P^n$ is homogeneous. We are left with the cases $(g, m_1, m_2) \in \{(4,2,2), (6,2,2)\}$, for which Proposition 7.7 shows that M is not foliated by Hopf circles.

For the proof of (ii) and the sufficiency of (i), we will need the concrete values of the rank and dimension of the different symmetric spaces [76, p. 518].

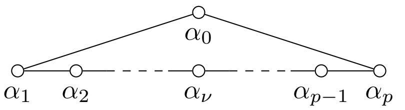
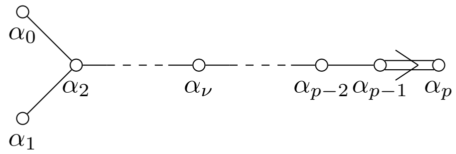
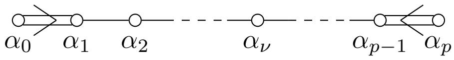
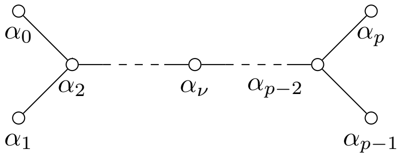
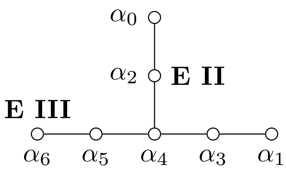
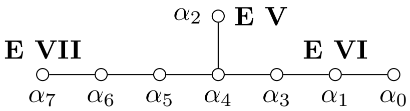
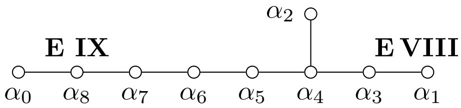
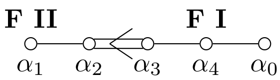
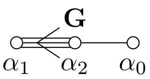
Assume that $\mathbb{C}P^n$ admits an irreducible inhomogeneous isoparametric foliation $\mathcal{G}$ of codimension $q \geq 2$. Then $\mathcal{G}$ is the projection of the foliation $\mathcal{F}_{G/K}$ of S^{2n+1} defined by certain irreducible symmetric space G/K. According to Table 7.3 and Corollary 7.35, the only possible cases for G/K are: **A III**, **BD I**, **C II**, **D III**, **E II**, **E III**, **E V**, **E VI**, **E VIII**, **E IX**, **F I** and **G**. One can easily check that, for all these cases, we have that $(\operatorname{rank} G/K)^2 \leq \dim G/K$ and that $\operatorname{rank} G/K$ divides $\dim G/K$. Since $\operatorname{rank} G/K = \operatorname{codim} \mathcal{F}_{G/K} + 1 = q + 1$ and $\dim G/K = 2(n+1)$, we get that $(q+1)^2 \leq 2(n+1)$ and $q+1$ divides $2(n+1)$.

If, conversely, these two conditions hold, then $G/K = \mathrm{SO}(q+r+1)/\mathrm{SO}(q+1) \times \mathrm{SO}(r)$, with $r = 2(n+1)/(q+1)$, defines a foliation $\mathcal{F}_{G/K}$ of codimension q on S^{2n+1}. Since moreover $q+1$ or r is even and $q+1 > 2$, then $N(\mathcal{F}_{G/K}) \geq 1$ and G/K is non-Hermitian. Hence, one can project $\mathcal{F}_{G/K}$ to an irreducible inhomogeneous isoparametric foliation with codimension q on $\mathbb{C}P^n$. An analogous argument proves the sufficiency of (i), if one considers $G/K = \mathrm{SU}(2+r)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(r))$ with $r = (n+1)/2$.

The last claim of the theorem follows easily from (i) and (ii). □

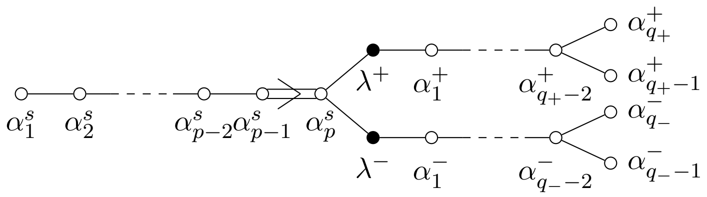
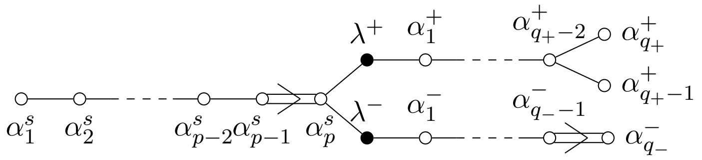
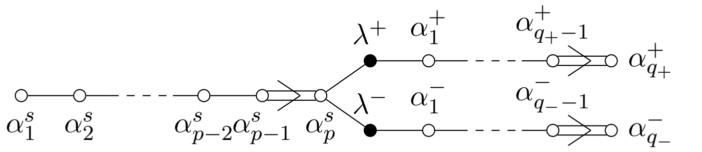
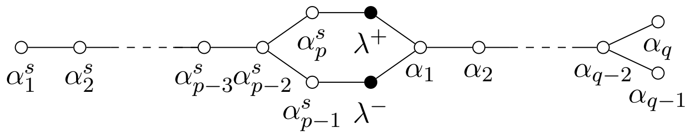
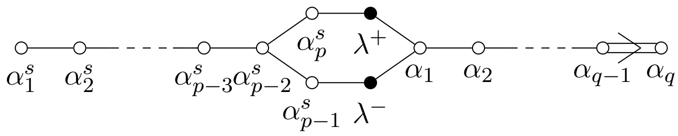
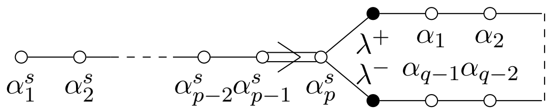
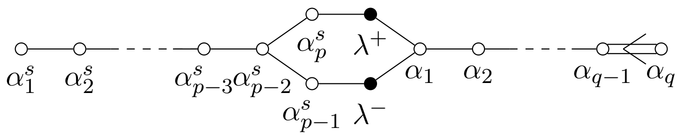
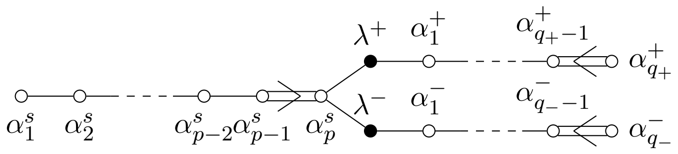
Remark 7.37. The assumption of the irreducibility in Theorem 7.36 (except in part (i)) is essential. For example, **D III** with $p = 6$ and **G** define a reducible inhomogeneous isoparametric foliation of codimension 4 on $\mathbb{C}P^{18}$.

Table 7.3: Extended Vogan diagrams of irreducible compact inner symmetric spaces

	Extended Dynkin diagram	μ	G/K	λ	$N(\mathcal{F}_{G/K})$
A_p		$(1\ldots1)$	**A III**	$(1\ldots1)$	$1+\left[\frac{\nu}{2}\right]+\left[\frac{p-\nu+1}{2}\right]$ (if $2\nu \neq p+1$) $1+\left[\frac{\nu}{2}\right]$ (if $2\nu = p+1$)
B_p		$(122\ldots2)$	**B I**	$(11\ldots\overset{\nu}{1}2\ldots2)$	1
C_p		$(2\ldots221)$	**C I** ($\nu = p$)	$(2\ldots221)$	1
			C II ($\nu < p$)	$(1\ldots\overset{\nu}{1}2\ldots21)$	2 (if $2\nu \neq p$) 1 (if $2\nu = p$)
D_p		$(12\ldots211)$	**D I** ($\nu \leq p-2$)	$(1\ldots\overset{\nu}{1}2\ldots211)$	2 (if $2\nu \neq p$) 1 (if $2\nu = p$)
			D III ($\nu \geq p-1$)	$(12\ldots211)$	2
E_6		(122321)	**E II**	(112321)	2
			E III	(122321)	2
E_7		(2234321)	**E V**	(1123321)	1
			E VI	(1234321)	2
			E VII	(2234321)	1
E_8		(23465432)	**E VIII**	(13354321)	1
			E IX	(23465431)	1
F_4		(2432)	**F I**	(2431)	1
			F II	(1321)	$-$ (rank one)
G_2		(32)	**G**	(31)	1

For each extended Dynkin diagram, we provide the maximal root μ and the associated symmetric spaces G/K using Cartan's notation. For every such G/K, we show the corresponding maximal noncompact root λ and the number $N(\mathcal{F}_{G/K})$ of noncongruent isoparametric foliations on the complex projective space induced by $\mathcal{F}_{G/K}$. Roots are specified in coordinates with respect to $\Pi_{\mathfrak{g}}$.

Table 7.4: Lowest weight diagrams of FKM-foliations with $m_1 \leq m_2$

m	Lowest weight diagram	$\mathfrak{h}$	$N(\mathcal{F}_\mathcal{P})$
0		$\mathfrak{so}(k_+) \oplus \mathfrak{so}(k_-)$ $k_\pm = 2q_\pm$	2
		$\mathfrak{so}(k_+) \oplus \mathfrak{so}(k_-)$ $k_+ = 2q_+,$ $k_- = 2q_- + 1$	1
		$\mathfrak{so}(k_+) \oplus \mathfrak{so}(k_-)$ $k_\pm = 2q_\pm + 1$	1
$1, 7$		$\mathfrak{so}(k)$ $k = 2q$	2
		$\mathfrak{so}(k)$ $k = 2q + 1$	1
$2, 6$		$\mathfrak{u}(k)$ $k = q$	$2 + \left[\frac{k}{2}\right]$
$3, 5$		$\mathfrak{sp}(k)$ $k = q$	2
4		$\mathfrak{sp}(k_+) \oplus \mathfrak{sp}(k_-)$ $k_\pm = q_\pm$	2

The following data are provided for each value of $m \pmod 8$: the corresponding lowest weight diagrams (see Remark 7.31 for exceptional cases with low k, $k_\pm$), the Lie algebra $\mathfrak{h}$ such that $\mathfrak{so}(m+1) \oplus \mathfrak{h}$ is the Lie algebra of $\mathrm{Aut}(\mathcal{F}_\mathcal{P})$, and the value of $N(\mathcal{F}_\mathcal{P})$.

Conclusions and open problems

This thesis has addressed the study of isoparametric foliations and polar actions, with focus on nonflat complex space forms. The investigation carried out allows us to present the following conclusions.

- Any real hypersurface with constant principal curvatures in a complex hyperbolic space $\mathbb{C}H^n$ satisfying the condition $h = 2$ (i.e. the number of nontrivial projections of its Hopf vector field onto the principal curvature spaces is two along the hypersurface) must be a homogeneous hypersurface (Theorem 3.1).
- Theorem 3.1 also shows that complex projective spaces do not admit real hypersurfaces in the conditions above.
- Every isoparametric hypersurface in a complex hyperbolic space has $g \in \{2, 3, 4, 5\}$ principal curvatures at each point, and $h \in \{1, 2, 3\}$ nontrivial projections of the Hopf vector field onto the principal curvature spaces (Theorem 4.5). These restrictions coincide with those of the homogeneous examples in $\mathbb{C}H^n$.
- A real hypersurface in a complex projective or hyperbolic space satisfying $h \leq 2$ at any point is isoparametric if and only if it has constant principal curvatures. In such a case, all examples are homogeneous (Theorem 4.11).
- Theorem 5.8 shows that Damek-Ricci spaces admit a large number of isoparametric families of hypersurfaces that were unknown up to now.
- There are uncountably many noncongruent inhomogeneous isoparametric families of hypersurfaces in complex hyperbolic spaces. All these examples have nonconstant principal curvatures (Theorem 5.9).
- There are uncountably many cohomogeneity one actions on quaternionic hyperbolic spaces apart from the examples given by Berndt and Brück in [10] (Theorem 5.11).
- The Cayley hyperbolic plane admits an inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures (Theorem 5.13).
- Theorem 6.4 classifies polar actions on complex hyperbolic spaces up to orbit equivalence. There are uncountably many new examples.

- Theorem 7.2 classifies irreducible isoparametric foliations of codimension greater than one on complex projective spaces. There are many inhomogeneous examples. As far as we know, these are the first such examples in Riemannian symmetric spaces.
- Theorem 7.3 classifies isoparametric hypersurfaces in complex projective spaces $\mathbb{C}P^n$ with $n \neq 15$. Again, there are many inhomogeneous examples.
- The complex projective space $\mathbb{C}P^n$ admits an irreducible inhomogeneous isoparametric foliation if and only if $n+1$ is not a prime number (Theorem 7.36).

In view of these results, there are still many open problems to be solved concerning isoparametric foliations and polar actions on rank one symmetric spaces. We emphasize the following ones, which are directly related to the investigation carried out in this memoir.

- Complete the classification of isoparametric hypersurfaces in spheres. It would be very interesting to find a more conceptual or geometrical argument that shows that every such hypersurface must be either homogeneous or of FKM-type.
- Classify real hypersurfaces with constant principal curvatures in nonflat complex space forms. Finding a bound on g or, even better, on h could help in the solution to this problem.
- Complete the classification of isoparametric hypersurfaces in complex hyperbolic spaces. Determine if there are more examples than the homogeneous ones and those constructed in Theorem 5.9.
- Extend the construction proposed in Chapter 5 to symmetric spaces of noncompact type.
- Classify polar actions on quaternionic hyperbolic spaces and on the Cayley hyperbolic plane. Proving that sections must be totally real would help. An important subproblem is the classification of cohomogeneity one actions on quaternionic hyperbolic spaces.
- Extend the methods developed in Chapter 7 to the case of quaternionic projective spaces. Dealing with the classification problem of isoparametric foliations on the Cayley projective plane will require new ideas, because of the inexistence of a Hopf fibration that is a Riemannian submersion for this space.

Resumo en galego

O concepto de simetría subxace a un elevado número de novas ideas e importantes avances en Ciencia, Enxeñaría e Arte. Desde o punto de vista matemático, a idea intuitiva de simetría como correspondencia equilibrada da forma ó longo do espazo tradúcese na existencia dun grupo de transformacións actuando en tal espazo. O primeiro campo natural para o estudo da simetría é, polo tanto, a xeometría. Reciprocamente, no seu influente Erlanger Programm, Felix Klein describiu a xeometría coma o estudo daquelas propiedades dun espazo que son invariantes baixo un grupo de transformacións. En consecuencia, a simetría atópase xa na propia esencia da xeometría.

En xeometría de Riemann, o grupo natural a considerar é o grupo de isometrías, é dicir, o grupo daquelas transformacións do espazo ambiente que preservan as distancias. Á acción dun subgrupo do grupo de isometrías dunha variedade dada chámaselle *acción isométrica.* A súa *cohomoxeneidade* é a menor codimensión das súas órbitas. Cada unha das órbitas dunha acción isométrica recibe o nome de subvariedade extrinsecamente homoxénea ou, simplemente, *subvariedade homoxénea*, e á colección de todas as órbitas chámaselle a *familia de órbitas* da acción.

Os principais obxectos de estudo nesta tese son certos tipos de subvariedades cun grao particularmente elevado de simetría. O noso obxectivo final é decidir se a noción intuitiva de simetría se reflicte no concepto matemático de simetría. Noutras palabras, preténdese analizar ata que punto a correspondencia de forma xeométrica en diferentes partes da subvariedade implica que a subvariedade é homoxénea.

Por unha banda, o estudo das accións isométricas en toda a súa xeneralidade resulta ser un problema moi difícil. Isto motivou a introdución de tipos especiais de accións isométricas cuxo estudo fose máis manexable. Este é o caso das *accións polares*, que son aquelas accións isométricas que admiten subvariedades totalmente xeodésicas intersecando ortogonalmente todas as órbitas.

Por outra banda, téñense proposto varias nocións xeométricas con vistas a intentar caracterizar as subvariedades homoxéneas. Así, nesta tese, consideramos os conceptos de *hipersuperficie con curvaturas principais constantes* e *subvariedade isoparamétrica.* Con todo, nalgúns casos estas nocións admiten *exemplos non homoxéneos.* Ata o de agora, críase que este fenómeno era moi pouco frecuente. O presente traballo amosa que é moito máis común do que se pensaba.

As familias de órbitas de accións isométricas constitúen os exemplos estándar da noción máis xeral de *foliación Riemanniana singular*. Unha foliación Riemanniana singular é un tipo de descomposición dunha variedade ambiente en subvariedades equidistantes (chamadas follas) de dimensións posiblemente diferentes. Este concepto foi introducido por Molino [103] e, nos últimos anos, ten atraído a atención de numerosos matemáticos, véxase [5]. A razón disto é que proporciona o marco adecuado para o estudo unificado de diferentes clases de obxectos xeométricos, non só das familias de órbitas de accións isométricas. Por exemplo, as subvariedades isoparamétricas definen localmente e, ás veces globalmente, unha foliación Riemanniana singular que denominamos *foliación isoparamétrica*.

A investigación das subvariedades homoxéneas e as súas xeneralizacións vén dando lugar a unha ampla área de investigación ó longo das últimas décadas. Historicamente, o caso de codimensión un foi o primeiro en ser abordado. O problema de clasificación das hipersuperficies homoxéneas en espazos de curvatura constante remóntase ós traballos de Somigliana [128], Levi-Civita [91], Segre [125] e Cartan [28], [30] nas primeiras décadas do século XX. En realidade, estes matemáticos estudaron as chamadas *hipersuperficies isoparamétricas*, é dicir, aquelas hipersuperficies cuxas hipersuperficies equidistantes suficientemente próximas teñen curvatura media constante. Á foliación isoparamétrica de codimensión un determinada por unha hipersuperficie isoparamétrica chámaselle a miúdo *familia isoparamétrica de hipersuperficies*. Estes obxectos xorden de modo natural en certo problema de óptica xeométrica: as frontes de onda daquelas ondas estacionarias e con frontes paralelas resultan ser hipersuperficies isoparamétricas.

Cartan probou que, nos espazos de curvatura constante, unha hipersuperficie é isoparamétrica se e só se ten curvaturas principais constantes. Esta condición resulta ser moi forte, e de feito caracteriza as hipersuperficies homoxéneas nos espazos euclidianos e hiperbólicos reais. Así, as únicas familias isoparamétricas de hipersuperficies nos espazos euclidianos veñen dadas por hiperplanos paralelos, por esferas concéntricas, e por cilindros coaxiais. No caso dos espazos hiperbólicos reais $\mathbb{R}H^n$, os exemplos son as horosferas, os tubos ó redor de subespazos hiperbólicos $\mathbb{R}H^k$, $k = 0, \ldots, n-2$, mergullados de xeito totalmente xeodésico, e os subespazos hiperbólicos totalmente xeodésicos $\mathbb{R}H^{n-1}$ xunto coas súas hipersuperficies equidistantes.

O problema en esferas é moito máis complicado. A clasificación das hipersuperficies homoxéneas (ou, equivalentemente, das accións isométricas de cohomoxeneidade un) en esferas tivo que esperar ata o traballo de Hsiang e Lawson [77]. Takagi e Takahashi [135] comprobaron que tales hipersuperficies teñen $g \in \{1, 2, 3, 4, 6\}$ curvaturas principais constantes. Empregando métodos xeométricos e topolóxicos, Münzner [108] amosou que esta restrición tamén se ten máis xeralmente para hipersuperficies isoparamétricas, e que toda hipersuperficie isoparamétrica nunha esfera define unha foliación isoparamétrica que enche toda a esfera e cuxas follas son variedades alxébricas. Non obstante, no caso das esferas non toda hipersuperficie isoparamétrica é homoxénea. Todos os contraexemplos coñecidos ata o de agora foron construídos por Ferus, Karcher e Münzner [63] empregando representacións de álxebras de Clifford. Estes exemplos incrementaron considerablemente o interese por este tema, ata o punto que o problema de clasificación foi incluído na lista de Yau de problemas abertos en xeometría [160]. Diversos matemáticos contribuíron nesta tarefa de clasificación, pero nos últimos cinco anos o avance foi particularmente notable. Os traballos de Cecil, Chi e Jensen [34], Immervoll [78], Chi [36], [37] e Miyaoka [101] completan a clasificación das hipersuperficies isoparamétricas nas esferas, coa única excepción do caso correspondente a aquelas hipersuperficies con catro curvaturas principais con multiplicidades $(7, 8)$.

Cando o espazo ambiente ten curvatura non constante, a complexidade do problema aumenta, polo cal a maioría dos resultados só puideron ser obtidos máis recentemente e, salvo contadas excepcións, unicamente cando a variedade ambiente é un espazo simétrico Riemanniano. A investigación nesta tese céntrase maioritariamente nos *espazos simétricos de rango un*, onde aínda permanecen abertos numerosos problemas. Prestamos especial atención ó caso dos *espazos de curvatura holomorfa constante* non nula, isto é, ós espazos proxectivo e hiperbólico complexos. A continuación presentamos as principais aportacións deste traballo.

Hipersuperficies reais con curvaturas principais constantes nos espazos proxectivo e hiperbólico complexos

O primeiro resultado orixinal desta tese fai relación a certa clasificación parcial de hipersuperficies reais con curvaturas principais constantes nos espazos de curvatura holomorfa constante. A constancia das curvaturas principais dunha hipersuperficie parece ser unha condición bastante restritiva, aínda máis que a condición de ser isoparamétrica en caso de que o espazo ambiente teña curvatura non constante. Ademais dos exemplos en esferas debidos a Ferus, Karcher e Münzner, e dun exemplo no plano hiperbólico de Cayley que se constrúe no Capítulo 5 (véxase máis abaixo), semella que todas as hipersuperficies con curvaturas principais constantes coñecidas ata o de agora en espazos simétricos son partes abertas de hipersuperficies homoxéneas. Deste xeito, o estudo de hipersuperficies con curvaturas principais constantes ten o seu propio interese.

O principal resultado do Capítulo 3 enmárcase dentro da liña iniciada por Kimura [83] e Berndt [7], quen clasificaron as hipersuperficies reais Hopf con curvaturas principais constantes nos espazos proxectivo e hiperbólico complexos, respectivamente. Dada unha hipersuperficie real M con campo de vectores normal unitario ξ, o seu campo de Hopf é o campo tanxente $J\xi$, onde J é a estrutura complexa do espazo ambiente. Definamos tamén h como a función en M con valores enteiros dada polo número de proxeccións non triviais de $J\xi$ sobre os espazos de curvaturas principais de M (isto é, os autoespazos do operador de configuración da hipersuperficie). Entón M dise que é Hopf se $h = 1$ ó longo de toda a hipersupericie, é dicir, se $J\xi$ é un autovector do operador de configuración en todo punto.

Así pois, *clasificamos as hipersuperficies reais con curvaturas principais constantes satisfacendo* $h = 2$ *nos espazos proxectivos e hiperbólicos complexos,* $\mathbb{C}P^n$ *e* $\mathbb{C}H^n$. Este é, polo tanto, o paso natural a dar despois das clasificacións de Kimura e Berndt. Próbase que os espazos proxectivos complexos $\mathbb{C}P^n$ non admiten tales hipersuperficies, pero si os espazos hiperbólicos complexos $\mathbb{C}H^n$. Neste caso, todos os exemplos son homoxéneos e, máis concretamente, partes abertas das hipersuperficies homoxéneas construídas por Lohnherr [92], Berndt e Brück [10].

Convén dicir que as hipersuperficies homoxéneas en $\mathbb{C}H^n$ foron clasificadas por Berndt e Tamaru [21], mentres que a súa xeometría extrínseca estudiárona Berndt e Díaz-Ramos en [15]. Destes traballos séguese que as hipersuperficies homoxéneas en $\mathbb{C}H^n$ teñen $g \in \{2,3,4,5\}$ curvaturas principais constantes e satisfán $h \in \{1,2,3\}$.

Os resultados do Capítulo 3 recóllense tamén nos artigos [47] e [54].

Hipersuperficies isoparamétricas nos espazos hiperbólicos complexos

Motivados polo noso obxectivo de caracterizar as hipersuperficies homoxéneas e, en particular, as hipersuperficies de Lohnherr-Berndt-Brück, no Capítulo 4 abordamos o estudo das hipersuperficies isoparamétricas nos espazos hiperbólicos complexos.

Primeiro, probamos que as hipersuperficies isoparamétricas compórtanse ben respecto da fibración de Hopf asociada ó espazo hiperbólico complexo $\mathbb{C}H^n$, isto é, unha hipersuperficie en $\mathbb{C}H^n$ é isoparamétrica se e só se a súa imaxe recíproca baixo a aplicación de Hopf é unha hipersuperficie isoparamétrica lorentziana no espazo de anti-De Sitter H_1^{2n+1}. Dado que a curvatura seccional do espazo de anti-De Sitter é constante, o estudo das hipersuperficies isoparamétricas en H_1^{2n+1} é entón equivalente ó estudo das hipersuperficies con curvaturas principais constantes en H_1^{2n+1}. Esta condición resulta ser máis doada de manexar.

A continuación, analizando cada unha das catro formas posibles que o operador de configuración da hipersuperficie levantada en H_1^{2n+1} pode adoptar, somos quen de atopar unha *restrición puntual no número g de curvaturas principais dunha hipersuperficie isoparamétrica en $\mathbb{C}H^n$, $g \in \{2,3,4,5\}$.* Tamén probamos que o *número de proxeccións non triviais do campo de Hopf sobre os espazos de curvaturas principal debe cumprir $h \in \{1,2,3\}$ en cada punto.*

Estas restricións coinciden coas dos exemplos homoxéneos en $\mathbb{C}H^n$. A maiores, pódese probar que as curvaturas principais das hipersuperficies isoparamétricas de $\mathbb{C}H^n$ son puntualmente as mesmas que aquelas dos exemplos homoxéneos, cunhas poucas excepcións. De feito, se $h \leq 2$ en todo punto, entón a hipersuperficie é isoparamétrica se e só se ten curvaturas principais constantes e é, polo tanto, unha parte aberta dunha hipersuperficie homoxénea (o mesmo resultado é válido nos espazos proxectivos complexos $\mathbb{C}P^n$). Isto faría plausible a posibilidade de probar un resultado de homoxeneidade. Non obstante, unha das consecuencias do Capítulo 5 é que tal resultado de homoxeneidade non pode existir.

O preprint [51] contén boa parte dos resultados do Capítulo 4.

Novas hipersuperficies isoparamétricas nos espazos de Damek-Ricci

No Capítulo 5 presentamos un método de construción de hipersuperficies isoparamétricas nos denominados espazos de Damek-Ricci. Estes espazos son variedades harmónicas entre as que se atopan os espazos simétricos de tipo non compacto e rango un; os outros espazos non simétricos son contraexemplos para a denominada conxectura de Lichnerowicz.

O noso método permite construír unha grande cantidade de *exemplos de familias isoparamétricas de hipersuperficies.* Cada unha destas familias vén definida polo *conxunto de tubos ó redor dun certo tipo de subvariedade focal minimal.* A nosa idea, que xeneraliza un método previamente proposto por Berndt e Brück [10], fai uso da estrutura alxébrica dos espazos de Damek-Ricci, da teoría de campos de vectores de Jacobi e, de xeito máis crucial, da introdución do concepto de *ángulo de Kähler xeneralizado* dun subespazo dun módulo de Clifford, que xeneraliza conceptos previos de ángulo de Kähler e de ángulo de Kähler cuaterniónico. Cando restrinximos a nosa atención ós espazos simétricos non compactos de rango un e curvatura non constante, é dicir, ós espazos hiperbólicos sobre as álxebras de división dos complexos $\mathbb{C}$, dos cuaternios $\mathbb{H}$ e dos octonios $\mathbb{O}$, pódense extraer algunhas consecuencias importantes.

En primeiro lugar, amosamos a existencia dunha *cantidade non numerable de familias isoparamétricas de hipersuperficies non homoxéneas con curvaturas principais non constantes nos espazos hiperbólicos complexos e cuaterniónicos* $\mathbb{C}H^n$ *e* $\mathbb{H}H^n$. Entre os exemplos construídos en $\mathbb{C}H^n$, a noción de ángulo de Kähler permítenos distinguir entre hipersuperficies homoxéneas e non homoxéneas.

En segundo lugar, construímos unha *cantidade non numerable de novos exemplos de accións de cohomoxeneidade un en espazos hiperbólicos cuateniónicos* $\mathbb{H}H^n$. É importante salientar que estas variedades son os únicos espazos simétricos de rango un para os cales a clasificación das accións de cohomoxeneidade un é aínda un problema aberto. Tal clasificación nos espazos hiperbólicos complexos $\mathbb{C}H^n$ e no plano hiperbólico de Cayley $\mathbb{O}H^2$ foi acadada por Berndt e Tamaru [21], mentres que o caso compacto fora resolto por Takagi [131] para os espazos proxectivos complexos $\mathbb{C}P^n$, e por Iwata [79], [80] para os espazos proxectivos cuaterniónicos $\mathbb{H}P^n$ e o plano proxectivo de Cayley $\mathbb{O}P^2$.

Finalmente, atopámonos cun exemplo aínda máis curioso. A nosa técnica proporciona unha *familia de hipersuperficies non homoxéneas con curvaturas principais constantes no plano hiperbólico de Cayley* $\mathbb{O}H^2$. Este é o único exemplo coñecido dunha familia de hipersuperficies isoparamétricas non homoxéneas pero con curvaturas principais constantes nun espazo simétrico distinto dunha esfera.

Os resultados do Capítulo 5 recóllense tamén nos artigos [48], [49] e [55].

Clasificación das accións polares nos espazos hiperbólicos complexos

No Capítulo 6 (ó igual ca no Capítulo 7) a nosa atención trasládase do caso de codimensión un ó de codimensión arbitraria, onde a complexidade deste tipo de problemas é normalmente maior. Os obxectos da nosa investigación no Capítulo 6 son as accións polares. Este tipo de accións isométricas xeneralizan dun modo natural a noción de acción de cohomoxeneidade un, permitindo cohomoxeneidades superiores pero requirindo que exista unha subvariedade que corte a todas as órbitas e sempre de xeito perpendicular. Tal subvariedade, que resulta ser totalmente xeodésica, denomínase sección. Unha acción polar tal que as súas seccións son chás dise hiperpolar.

As accións polares nos espazos euclídeos e nas esferas foron clasificadas por Dadok [42]. Tales accións coinciden coas representacións de isotropía dos espazos simétricos, salvo equivalencia de órbitas. O problema de clasificación foi entón abordado nos espazos simétricos de tipo compacto. Así, Podestà e Thorbergsson [123] clasificaron as accións polares nos espazos simétricos compactos de rango un e amosaron que existen accións polares que non son hiperpolares. Unha clasificación completa das accións hiperpolares nos espazos simétricos irreducibles de rango maior ca un acadouna Kollross [85]. A seguir, Kollross centrouse no problema de clasificación de accións polares nos espazos simétricos irreducibles de tipo compacto [86], [87]. Este problema foi resolto hai pouco por Kollross e Lytchak [89], quen amosaron que tales accións polares son sempre hiperpolares. Un resultado tamén recente debido a Fang, Grove e Thorbergsson [61] mostra que toda acción polar de cohomoxeneidade maior ou igual ca dous nunha variedade de curvatura positiva, simplemente conexa e compacta é equivariantemente equivalente a unha acción polar nun espazo simétrico de rango un.

Ó igual que sucede coas accións de cohomoxeneidade un, os resultados no caso non compacto son máis escasos. Wu clasificou as accións polares nos espazos hiperbólicos reais [155]. Moi recentemente, Berndt e Díaz-Ramos clasificaron as accións polares no plano hiperbólico complexo $\mathbb{C}H^2$ en [16], mentres que en [17] determinaron aquelas accións polares que inducen foliacións regulares en $\mathbb{C}H^n$.

O Capítulo 6 completa a *clasificación das accións polares nos espazos hiperbólicos complexos* $\mathbb{C}H^n$ *salvo congruencia de órbitas*. Este resultado, que inclúe a construción de numerosos novos exemplos de accións polares en $\mathbb{C}H^n$, constitúe a única clasificación coñecida de accións polares nunha familia enteira de espazos simétricos de tipo non compacto e curvatura non constante.

A proba deste resultado ten dúas partes principais, dependendo de se o grupo que actúa polarmente deixa invariante un subespazo totalmente xeodésico ou ben está contido nun subgrupo parabólico maximal de $\mathrm{SU}(1,n)$, o grupo de isometrías de $\mathbb{C}H^n$. Un papel importante nos nosos argumentos xógano a descomposición de Iwasawa asociada ó espazo simétrico $\mathbb{C}H^n$, a utilización de certo criterio de polaridade debido a Berndt e Díaz-Ramos [16], e o feito de que as seccións das accións polares en $\mathbb{C}H^n$ son subvariedades totalmente reais.

O artigo [50] recolle os resultados orixinais do Capítulo 6.

Foliacións isoparamétricas nos espazos proxectivos complexos

A introdución do concepto de subvariedade isoparamétrica de codimensión arbitraria nos espazos de curvatura constante é debida a Harle [72], Carter e West [32] e, principalmente, a Terng [140]. A motivación inicial consistía en atopar unha propiedade xeométrica que permitise caracterizar as órbitas de accións polares. Toda subvariedade isoparamétrica nun espazo forma real determina unha foliación isoparamétrica que enche todo o espazo. Contrariamente ó que sucede no caso de codimensión un, onde existen exemplos non homoxéneos, toda foliación isoparamétrica irreducible con codimensión polo menos dous nunha esfera é homoxénea e, de xeito máis específico, coincide coa familia de órbitas da representación de isotropía dun espazo simétrico. Este importante resultado foi obtido por Thorbergsson [143]. Máis recentemente, Heintze, Liu e Olmos [74] propuxeron unha definición de subvariedade isoparamétrica dun espazo ambiente Riemanniano arbitrario. Así, unha subvariedade M dise isoparamétrica se: (i) M ten fibrado normal chan, (ii) localmente, o campo curvatura media das subvariedades paralelas suficientemente próximas a M é paralelo con respecto á conexión normal, e (iii) a subvariedade M admite seccións, isto é, subvariedades totalmente xeodésicas intersecando M perpendicularmente. Esta definición xeneraliza, por un lado, a noción de hipersuperficie isoparamétrica dunha variedade de Riemann e, polo outro, a de subvariedade isoparamétrica dun espazo forma real.

O noso obxectivo no Capítulo 7 é investigar as subvariedades isoparamétricas nos espazos proxectivos complexos $\mathbb{C}P^n$. Sucede que, ó igual ca nos espazo forma reais, toda subvariedade isoparamétrica determina unha foliación isoparamétrica globalmente definida que enche todo o espazo. O Capítulo 7 contén unha análise pormenorizada do comportamento das foliacións isoparamétricas con respecto da fibración de Hopf de $\mathbb{C}P^n$. Sorprendentemente, como detectara Xiao en [158], unha hipersuperficie isoparamétrica M nunha esfera S^{2n+1} pode ser proxectada a $\mathbb{C}P^n$ dando lugar a hipersuperficies isoparamétricas *non congruentes* entre si, que poden ser non homoxéneas con independencia da homoxeneidade de M. Amosamos que este fenómeno ten lugar tamén en codimensións superiores. Así, obtemos a *clasificación das foliacións isoparamétricas irreducibles de codimensión maior que un nos espazos proxectivos complexos* e amosamos que a maioría dos exemplos son *foliacións non homoxéneas*. Ata onde sabemos, trátase dos primeiros exemplos de foliacións isoparamétricas (irreducibles) non homoxéneas con codimensión maior ca un en espazos simétricos.

Tamén estudamos o caso de codimensión un, que vén a estar relacionado co problema aberto da clasificación das hipersuperficies isoparamétricas nas esferas. Investigando as posibles proxeccións mediante a aplicación de Hopf dos exemplos construídos por Ferus, Karcher e Münzner, somos capaces de *clasificar as hipersuperficies isoparamétricas nos espazos proxectivos complexos* $\mathbb{C}P^n$*, para todo* $n \neq 15$. Deste modo, esta clasificación xeneraliza resultados de Takagi [131], Wang [150], Xiao [158] e Ge, Tang e Yan [66]. De novo, atópanse numerosos exemplos non homoxéneos.

O estudo da homoxeneidade das hipersuperficies isoparamétricas proporciónanos o seguinte resultado adicional: *toda foliación isoparamétrica irreducible en* $\mathbb{C}P^n$ *é homoxénea se e só se* $n+1$ *é un número primo.*

A principal ferramenta que desenvolvemos no Capítulo 7 é un método para o estudo de foliacións Riemannianas singulares con todas as follas pechadas nos espazos proxectivos complexos. Para este estudo, introducimos un certo tipo de grafo que chamamos *diagrama do peso mínimo* e que xeneraliza os denominados como diagramas de Vogan estendidos dos espazos simétricos interiores.

Os contidos do Capítulo 7 recóllense tamén no artigo [56].

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